Essays in general equilibrium with borrowing constraints, optimal growth, and FDI

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Résumé

La thèse se compose de 5 articles.

Le premier article considère une économie monétaire à horizon infini avec actifs financiers collatéralisés. La Banque Centrale fait des prêts à court et à long terme aux ménages. Les agents peuvent déposer ou/et emprunter à court ou à long terme. Néanmoins un plafond est imposé sur les emprunts de long terme. Tous les agents ont accès aux marchés financiers. Toutefois les agents doivent posséder suffisamment de collatéral en biens de consommations pour vendre un actif financier. Les agents font face à des contraintes de liquidité aussi bien lorsqu'ils achètent des biens de consommation que des actifs financiers. Sous des hypothèses de "Gains à l'échange", l'existence de l'équilibre est démontrée. Dans un tel cadre, plusieurs propriétés des équilibres sont démontrées, notamment l'existence d'une trappe à liquidité.

Le deuxième considère un modèle d'équilibre général à la Ramsey avec agents hétérogènes, contraintes d'emprunt, et offre de travail exogène. D'abord, l'existence d'un équilibre est démontrée même si les capitaux ne sont pas bornés uniformément et si les fonctions de production ne sont pas stationnaires. Ensuite (i) nous définissons la bulle du capital physique comme la différence strictement positive entre son prix et sa valeur fondamentale (ii) nous montrons qu'une bulle existe si, et seulement si, la somme des rendements du capital est finie. Enfin, lorsque les fonctions de production sont linéaires, tout équilibre intertemporel est efficient. De plus, on peut avoir des équilibres à la fois efficients et avec bulle.

Le troisième étudie la nature de la bulle financière dans un modèle d'équilibre général à l'horizon infini avec agents hétérogènes, contraintes d'emprunt endogènes. Nous démontrons l'existence d'un équilibre sans aucune condition sur des dotations initiales des agents. Nous disons qu'il y a une bulle financière à l'équilibre si le prix d'actif financier est supérieur à sa valeur fondamentale. Nous démontrons que les trois conditions suivantes sont équivalentes : (i) Il y a une bulle, (ii) le coût d'emprunt est strictement positif, (iii) les taux d'intérêt sont bas, i.e., la somme des taux d'intérêt au cours du temps est finie.

Nous donnons aussi une condition sur les variables exogènes pour que la bulle financière apparaisse à l'équilibre.

Le quatrième concerne l'interaction entre le marché financier et le secteur productif. Pour étudier cela, nous construisons un modèle d'équilibre général à horizon infini avec agents hétérogènes, contraintes d'emprunt endogènes dans lequel les agents investissent en actif financier ou/et en capital physique. Il y a une firme qui maximise son profit. D'abord, l'existence d'un équilibre est démontrée. Nous montrons que si la productivité est suffisamment élevée, l'économie ne tombe jamais en récession. [1] Si la productivité est basse, l'économie va tomber en récession avec un nombre infini de fois. Cependant, dans certains cas, l'actif financier pourrait bénéficier à l'économie en finançant l'achat du capital physique. Grâce à cela, une récession économique pourrait être évitée.

Dans notre modèle, l'actif financier pourrait non seulement créer des fluctuations du stock de capital physique agrégé mais aussi, dans certains cas, le rendre efficient pour l'économie.

Le dernier article porte sur l'analyse de la relation entre la croissance optimale et l'investissement direct à l'étranger (IDE), la compétition entre les firmes domestiques et étrangères et la stratégie optimale d'un pays recevant l'IDE.

Notre approche consiste à considérer une petite économie ouverte avec deux secteurs productifs : un vieux secteur produisant le bien de consommation avec le capital physique et un nouveau secteur produisant un nouveau bien en utilisant à la fois le capital physique et un travail spécifique. Il existe deux types de firmes dans la nouvelle industrie : une firme multinationale déjà implantée et une firme domestique potentielle. Notre cadre de travail met en évidence un certain nombre de résultats.

Tout d'abord, dans un pays pauvre avec un faible rendement de la formation et des retombées de l'IDE, aucune firme domestique ne peut être créée dans la nouvelle industrie qui exige un coût fixe élevé.

Deuxièmement, une fois que le pays d'accueil a la capacité de créer des firmes domestiques dans la nouvelle industrie, la compétitivité de l'entreprise domestique est le facteur clé qui lui permet d'entrer dans la nouvelle industrie, et potentiellement d'éliminer la firme multinationale. Il est intéressant de noter que dans certains cas où les retombées des IDE sont fortes, le pays devrait investir dans la nouvelle industrie, mais ne pas avoir besoin de former les travailleurs spécifiques. Ce résultat nous permet d'expliquer pourquoi dans certains pays, une nouvelle industrie pourrait être créée même si ces pays ne font aucune formation pour les travailleurs spécifiques.

Enfin, les contraintes de crédit, l'élasticité entre le capital et le travail jouent des rôles importants dans la compétition entre la firme multinatio-

1. Nous disons qu'une récession économique apparaît s'il n'y a pas d'investissement dans le secteur productif.

nale et la firme domestique.

Table des matières

Chapitre I

Introduction

The dissertation consists of two main parts. The fisrt part focuses on Intertemporal General Equilibrium with Incomplete Markets : monetary equilibrium and liquidity trap, bubbles and efficiency of equilibria, the interaction between financial and production sectors. In the second part, we focus on some concerns in Development Economics. More precisely we study the link among multinational firms, FDI spillovers and economic growth.

1 Monetary equilibrium

Dubey and Geanakoplos (2003a,b) proved the existence of equilibrium for 2-period monetary economies by using Gains to Trade Hypothesis (GTH). Conversely, they also proved that monetary equilibrium for pure exchange economies with money does not exist unless there are sufficient gains to trade (Theorem 6 and 7 in Dubey and Geanakoplos (2003a)). Dubey and Geanakoplos (2006b) then constructed a two-period monetary economy with production. They imposed that firms sell all goods at hand in period 2, and then there is always a strictly positive quantity of commodity which is sold in the economy for that holds the existence of an equilibrium.

Chapter 2 gives a full general equilibrium approach to money by constructing an infinite horizon model with consumptions, fiat money and collateralized financial assets. In this framework, there are a Central Bank and heterogeneous households. The Central Bank lends money to households by creating short- and long-term loans. Households can deposit and borrow money. A short (resp., long)-term loan will be paid back with interest rate at the end of the same (resp., the next) period. For each agent, there are borrowing constraints when she wants to borrow long-term loans. If agents want to sell a financial asset, they are required to hold certain commodities as collateral. When buying a commodity or an asset, tradings face cash-in-advance constraints. At each period, all trade is voluntary and trading timing of each agent is the following :

(i) trade bank loans

(ii) trade commodities and assets

(iii) deliver on assets

(iv) pay back bank loans.

Agents face a liquidity constraint at each sub-period : all purchases must be paid for in money. With this setup, fiat money which is the stipulated medium of exchange plays a central role in trading.

The first contribution concerns the existence of monetary equilibrium. The technique we use to proof the existence of equilibrium consists of three steps : (1) prove that there exists an equilibrium for each $T-$truncated economy $\mathcal{E}^T$, (2) prove that the sequence (depending in T) of equilibria has a limit, (3) prove that such limit will be an equilibrium for the monetary economy.

In standard general equilibrium models (Levine (1989), Levine and Zame (1996), Magill and Quinzii (1994, 1996), Araujo, Pascoa, Torres-Martinez (2002), Kubler and Schmedders (2003)), the budget constraints are homogeneous. Thanks to that, we can normalize prices by setting the sum of prices at each date to equal 1. As a consequence, we can easily prove that the sequence of equilibria of truncated economies has a limit. However, when introducing fiat money in our model, cash-in-advance constraints are no longer homogeneous, and hence we cannot assume that the sum of prices equals 1. Hence, the methods used in these papers are no longer valid. Moreover, since nominal interest rates may be zero, agents may also keep money on hand. Because of that, cash-in-advance constraints are generally not binding, which implies that each choice of agents at date t will appear in every their cash-in-advance constraints from date $t+1$. Therefore, we cannot easily apply Kuhn-Tucker Theorem as used in Araujo, Pascoa, Torres-Martinez (2002), Kubler and Schmedders (2003).

To overcome all difficulties discussed above, we introduce Uniform and Sequential Gains to Trade Hypotheses (for short, we write UGTH, SGTH, respectively). Thanks to these hypotheses, a strictly positive amount of commodities will be traded at each period. As a result, commodity prices are determined (cannot be either zero or infinity). Collateral constraints then ensure that asset prices and traded quantities of financial assets are bounded from above. UGTH and SGTH, and collateral constraints allow us to pass to the limit after having proved that there exists an equilibrium for each truncated economy. We give two versions of equilibrium : (i) under UGTH, prices are uniformly bounded from above and away from zero, (ii) under SGTH, prices are only bounded for the product topology.

Our framework is also different from Bloise, Dreze and Polemarchakis (2005) on two points. In Bloise, Dreze and Polemarchakis (2005), the financial market is sequentially complete and they assume that interest rates are exogenous. On the contrary, the financial market is incomplete and interest rates are endogenous in my paper.

The second contribution is about liquidity trap.

Definition 1.1 *At equilibrium, we say there is a liquidity trap if the interest rate of short-term loan at some date equals zero.*

In a two-period model, Dubey and Geanakoplos (2006a) assume that asset payoffs are linearly independent for that liquidity trap may exist in the first period at equilibrium. In Dubey and Geanakoplos (2006b), they consider a two-period model with production. The existence of liquidity trap in the first period in this model is based on the assumption that firms have strictly positive endowment and sell all in period 2.

In our model, without these assumptions, we show that liquidity trap may appear at any date in an infinite horizon economy. Our result says that : at some node, say ξ, if money supply for short-term loans, i.e., $M(\xi)$, is very high with respect to money supply which agents expect to be available at some date in the future, the economy will fall into a liquidity trap at node ξ. Different from Dubey and Geanakoplos (2006a,b), my argument is based on SGTH, UGTH and collateral constraints.

2 Bubbles

Let us start by the following comment :

> *"However, despite the widespread belief in the existence of bubbles in the real world, it is difficult to construct model economies in which bubbles exist in equilibrium."*
>
> Kocherlakota (2008)

We focus here on rational bubbles by constructing 2 types of model where there are bubbles. We give conditions under which bubbles appear at equilibrium. We also point out the relationships between bubbles, credit constraints and interest rates.

2.1 Financial asset bubble

Since there are many kinds of bubbles,[1] we start by defining bubbles. Consider a long-lived asset whose price at date t is q_t, ξ_t is its dividend at

1. Araujo, Pascoa, Torres-Martinez (2011) study rational bubbles in a model with durable goods and collateral constraints by focusing on individual deflators. Martin and Ventura (1912), Ventura (2012) define bubble as a short-lived asset. A survey on bubbles in asymmetric information, overlapping generation or heterogeneous-beliefs models can be found in Brunnermeier, Oehmke (2012).

date t. Asset pricing is the following

$$q_t = \gamma_{t+1}(q_{t+1} + \xi_{t+1}). \tag{I.1}$$

where γ_t is the discount factor of the economy from date t to date $t+1$. We define the discount factor, Q_t, of the economy from date 1 to date t

$$Q_t := \gamma_1 \cdots \gamma_t. \tag{I.2}$$

We then have

$$Q_t q_t \quad = \quad Q_{t+1}q_{t+1} + Q_{t+1}\xi_{t+1} \tag{I.3}$$

We see that

1. At date 1, one unit (from date 0) of this asset will give back 1 units of the same asset and ξ_1 units of consumption good as its dividend. This is represented by $q_0 = Q_1\xi_1 + Q_1 q_1$
2. At date 2, one unit of long lived asset will give one unit of the same asset and ξ_2 units of consumption good. This is represented by $Q_1 q_1 = Q_2\xi_2 + Q_2 q_2$, and so on.

This leads us to define the fundamental value of financial asset

$$FV_0 \quad := \quad \sum_{t=1}^{+\infty} Q_t \xi_t. \tag{I.4}$$

Definition 2.1 *There is a bubble if and only if the price of the asset is greater than its fundamental value :* $q_0 > FV_0$.

Remark 2.1 *There is a bubble if and only if* $\lim_{t\to\infty} Q_t q_t > 0$.

Bubbles in general equilibrium models : We now explain why we have (I.1). Let denote p_t the price of consumption good at date t, q_t the price of financial asset at date t. These prices are endogenous. There are m households. Each household has $e_{i,t}$ units of consumption good as endowment. Household i takes sequences of prices $(p, q) = (p_t, q_t)_{t=0}^{\infty}$ as given and maximizes her utility $\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})$, where β_i is the discount factor of agent i and u_i is the utility function of agent i.

Her budget constraint at date t

$$p_t c_{i,t} + q_t a_{i,t} \quad \leq \quad p_t e_{i,t} + (q_t + p_t\xi_t)a_{i,t-1}. \tag{I.5}$$

There are many types of constraints by which the financial market is incomplete. A type of exogenous borrowing constraint is $a_{i,t} \geq \bar{a}_t$ where $\bar{a}_t$ is an exogenous constant. Kocherlakota (1992) studies bubble in the model with $\bar{a}_t = \bar{a}$ for any t, Le Van and Vailakis (2012) work with $\bar{a}_t = 0$.

There is also endogenous borrowing constraints, which can be represented by

$$function(a_{i,t}, (p_t)_t, (q_t)_t, (e_{i,t})_t) \geq 0.$$

Chapter 4 considers the following type of endogenous borrowing constraint :

$$-(q_{t+1} + p_{t+1}\xi_{t+1})a_{i,t} \quad \leq \quad f_i p_{t+1} e_{i,t+1}. \tag{I.6}$$

This constraint means that the payment of agent i cannot exceed a fraction of her endowments. f^i is the borrowing limit which is set by law. Market clearing conditions are the following :

$$\text{Consumption good :} \quad \sum_{i=1}^{m} \bar{c}_{i,t} = \sum_{i=1}^{m} e_{i,t} + \xi_t, \tag{I.7}$$

$$\text{Financial asset :} \quad \sum_{i=1}^{m} \bar{a}_{i,t} = 1. \tag{I.8}$$

Under standard conditions, equilibrium is proved to exist. At equilibrium, we define

$$\gamma_{t+1} \quad := \quad \max_{i\in\{1,\ldots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \forall t \geq 0 \tag{I.9}$$

$$Q_0 \quad := \quad 1, \quad Q_t := \prod_{s=1}^{t} \gamma_s, \forall t \geq 1. \tag{I.10}$$

Q_t is the discount factor of the economy from initial period to period t.

Lemma 2.1 *For each $t \geq 0$ we have*

$$\frac{q_t}{p_t} \quad = \quad \gamma_{t+1}(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}) \tag{I.11}$$

$$Q_t\frac{q_t}{p_t} \quad = \quad Q_{t+1}(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}). \tag{I.12}$$

Note that $\frac{q_t}{p_t}$ is the asset price (in term of consumption good) at date t.

Remark 2.2 *We can see that borrowing constraint (I.6) is equivalent to*

$$Q_t\frac{q_t}{p_t}a_{i,t} + f^i Q_{t+1} e_{i,t+1} \geq 0.$$

This means that borrowing value of agent i does not exceed a fraction value of its endowments.

We then define bubbles as in Definition 2.1.

Remark 2.3 *It is easy to see that there is no bubble in finite horizon models or in models with complete financial market (i.e., there is no constraint on $a_{i,t}$).*

On the relationship between bubble and borrowing constraints, Kocherlakota (1992) suggests that borrowing constraints are binding infinitely often if bubbles exists. Actually, what he proved was that $\liminf_{t\to\infty}(a_{i,t}-a^*) = 0$ for any i. Our contribution is the following :

Proposition 2.1 *(***Borrowing constraint is binding at infinitely many date***)*
If bubble occurs, there exists an agent i and an infinite sequence $(t_n)_{n\geq 1}$ such that borrowing constraint of agent i is binding at each date t_n.

The following result shows that at bubble equilibirum, there is a fluctuation in financial asset volume of some agent.

Proposition 2.2 *If bubble occurs, there exists i such that the sequence $(a_{i,t})$ has no limit.*

We continue by pointing out the relationship between bubbles and interest rates.
In Alvarez and Jermann (2000), they define high implied interest rates as a situation in which present value of aggregate endowments is finite, i.e.,

$$\sum_{t=0}^{\infty} Q_t e_t < \infty,$$

where $e_t := \sum_{i=1}^{m} e_{i,t}$.

Proposition 2.3 *(Santos and Woodford (1997), Huang and Werner (2000))*
If an equilibrium has high implied interest rates, there is no bubble at this equilibrium.

Note that Tirole (1982) studies bubbles in a model in which $e_{i,t} = 0$ for every i and t. Therefore, his no-bubble result can be viewed as a particular case of Proposition 2.3.

Note that high implied interest rates is only an sufficient condition for no bubble. [2] Moreover, it seems that high implied interest rates is an abstract concept. For these two reasons, we will introduce another concept called "low (high) interest rates", and then show that there is a bubble if and only

2. See Example 3.1 in Le Van and Vailakis (2012).

if interest rates are low. To do this, we recall budget constraint of agent i at date $t-1$ and t.

$$\begin{aligned} p_{t-1}c_{i,t-1} + q_{t-1}a_{i,t-1} &\leq p_{t-1}e_{i,t-1} + (q_{t-1} + p_{t-1}\xi_{t-1})a_{i,t-1} \\ p_t c_{i,t} + q_t a_{i,t} &\leq p_t e_{i,t} + q_t(1 + \frac{p_t\xi_t}{q_t})a_{i,t-1}. \end{aligned}$$

One can interpret that if agent i buys $a_{i,t-1}$ units of financial asset at date $t-1$ with price q_{t-1}, she will receive $(1 + \frac{p_t\xi_t}{q_t})a_{i,t-1}$ units of financial asset with price q_t at date t. Therefore, $\frac{p_t\xi_t}{q_t}$ can be viewed as the interest rate of the financial asset at date t.

Definition 2.2 *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty} \frac{p_t\xi_t}{q_t} < \infty. \tag{I.13}$$

Otherwise, we say that interest rates are high.

We now present relationship between financial bubble and low interest rates.

Proposition 2.4 *There is a bubble if and only if interest rates are low.*

Although there are some examples of bubbles (Kocherlakota (1992), Huang and Werner (2000), Le Van and Vailakis (2012)), no one gives conditions of exogenous variables under which there is a bubble at equilibrium. Our contribution is to give an exogenous condition for bubble.

Theorem 2.1 *(***An exogenous sufficient condition for financial asset bubble***)*
Assume that $f^i = 0$ for every i. We normalize by setting $p_t = 1$ for every t. Denote

$$D_t := \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(e_{i,t})}{u_i'(W_{t-1})}, A_t := \min_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(W_{i,t})}{u_i'(\frac{W_{t-1}}{m})}, W_t := \sum_{i=1}^{m} e_{i,t} + \xi_t.$$

There is a financial asset bubble at equilibrium if the following conditions hold :

(i) $B := \sum_{t=1}^{\infty} B_t\xi_t < \infty$, *where* $B_t := \prod_{k=1}^{t} D_k$.

(ii) There exists i such that

$$u_i'\big(e_{i,0} + \xi_0 a_{i,-1} - B(1 - a_{i,-1})\big) \leq \beta_i \frac{A + \xi_1}{B} u_i'(W_1), \tag{I.14}$$

where $A := \sum_{t=2}^{\infty}(\prod_{s=2}^{t} A_s)\xi_s$.

Note that these conditions is satisfied if $\xi_1, \xi_2, \ldots,$ are small.

2.2 Physical capital bubble

Becker, Bosi, Le Van, Seegmuller (2014) is the first paper introducing the concept "bubble of physical capital". Following Becker, Bosi, Le Van, Seegmuller (2014), Chapter 3 considers physical capital bubbles in a Ramsey model with heterogeneous agents. There are m households. Each household i takes sequences of prices $(p, r) = (p_t, r_t)_{t=0}^{\infty}$ as given and solves

$$
\begin{aligned}
(P_i(p,r)) : \quad & \max_{\left((c_{i,t},k_{i,t+1})_{i=1}^{m}\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\right] && \text{(I.15)}\\
\text{subject to : } \quad & k_{i,t+1} \geq 0 && \text{(I.16)}\\
& p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t(p_t, r_t), && \text{(I.17)}
\end{aligned}
$$

where $(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^{m} \theta^i = 1$.

For each period, there is a representative firm who takes prices (p_t, r_t) as given and maximizes its profit.

$$
(P(r_t)) : \quad \pi_t(p_t, r_t) := \max_{K_t \geq 0} \left[p_t F_t(K_t) - r_t K_t\right]
$$

Note that we allow non-stationary technologies. At equilibrium, markets clear : for each $t \geq 0$,

$$
\begin{aligned}
\text{consumption good} \quad &: \quad \sum_{i=1}^{m} [c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] = F_t(K_t)\\
\text{physical capital} \quad &: \quad K_t = \sum_{i=1}^{m} k_{i,t}.
\end{aligned}
$$

Equilibrum is proved to exist. As in previous section, we define

$$
\begin{aligned}
\gamma_{t+1} \quad &:= \quad \max_{i \in \{1,\ldots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \forall t \geq 0 && \text{(I.18)}\\
Q_0 \quad &:= \quad 1, \quad Q_t := \prod_{s=1}^{t} \gamma_s, \forall t \geq 1. && \text{(I.19)}
\end{aligned}
$$

Q_t is the discount factor of the economy from initial period to period t.

Lemma 2.2 *We have*

$$
\begin{aligned}
1 \quad &= \quad (1 - \delta + \rho_{t+1})\gamma_{t+1} && \text{(I.20)}\\
Q_t \quad &= \quad (1 - \delta + \rho_{t+1})Q_{t+1}, && \text{(I.21)}
\end{aligned}
$$

where $\rho_t = \dfrac{r_t}{p_t}$ *is the return (in term of consumption good) of the physical capital at date* t.

In this framework, physical capital can be viewed as a long-lived asset whose price at initial date equals 1.

1. At date 1, one unit (from date 0) of this asset will give $(1-\delta)$ units of physical capital and ρ_t units of consumption good as its dividend. This argument is formalized by $1 = (1-\delta)Q_1 + \rho_1 Q_1$.
2. At date 2, $(1-\delta)$ units of physical capital will give $(1-\delta)^2$ units of physical capital and $(1-\delta)\rho_2$ units of consumption good. This argument is formalized by $(1-\delta)Q_1 = (1-\delta)^2 Q_2 + (1-\delta)\rho_2 Q_2$.

Therefore, the fundamental value of physical capital at date 0 can be defined by

$$FV_0 = \sum_{t=1}^{\infty}(1-\delta)^{t-1}\rho_t Q_t. \tag{I.22}$$

Definition 2.3 *We say that there is a capital asset bubble if* $1 > \sum\limits_{s=0}^{\infty}(1-\delta)^{t-1}\rho_t Q_t$.

We can see that there is a bubble on capital asset if and only if $\lim\limits_{t\to\infty}(1-\delta)^t Q_t > 0$.

Remark 2.4 *There is no physical capital bubble in the case of full depreciation of the capital, i.e., when* $\delta = 1$.

Definition 2.4 *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty}\rho_t < \infty. \tag{I.23}$$

Otherwise, we say that interest rates are high.

We state our main result in this section.

Proposition 2.5 *There is a bubble if and only if interest rates are low*

Corollary 2.1 *Assume that* $F_t = F$ *for every* t*,* F *is strictly increasing, concave. Then there is no bubble at equilibrium.*

Note that we do not require any condition on $F'(\infty)$ in Corollary 2.1. In Becker, Bosi, Le Van, Seegmuller (2014), they work with a endogenous labor supply model and assume that $\dfrac{\partial F}{\partial K}(\infty, m) = \dfrac{\partial F}{\partial L}(1,\infty) = 0$.

Corollary 2.2 *Assume that* $F_t(K) = a_t K$ *for each* t*. Then there is a bubble at equilibrium if and only if* $\sum\limits_{t=1}^{\infty} a_t < \infty$.

This result shows that if the productivity decrease to zero with high speed, a bubble in physical capital will appear.

3 Financial market vs productive sector

The financial market has been considered as one of main causes of economic recession or/and fluctuation. But, does financial market always cause an economic recession ? What is the role of financial market on the productive sector ? Chapter 5 explores the interaction between the financial market and the productive sector.

Definition 3.1 *We say there is an economic recession if there is no investment in the productive sector.*

To present our ideas in a simple way, consider an agent whose initial endowment is S. Agent has two choices to invest : to produce or to invest in financial asset. She may produce $AF(K)$ units of consumption good by using K units of physical capital. If she buys a units of financial asset with price q, she will receive ξa units of consumption good, where ξ is the dividend of the financial asset.

$$\max_{K,a\geq 0} AF(K) + \xi a \tag{I.24}$$

$$K + qa \leq S \tag{I.25}$$

Proposition 3.1 *(i) If $AF'(0) \leq \frac{\xi}{q}$, agent does not produce, i.e., $K = 0$ and $a = S$.*

(ii) If $AF'(S) \geq \frac{\xi}{q}$, agent does not invest in financial asset, i.e., $a = 0$ and $K = s$.

(iii) If $AF'(S) \leq \frac{\xi}{q} \leq AF'(0)$, agent produces and invests in financial asset. K is determined by $AF'(K) = \frac{\xi}{q}$ and $a = S - K$.

The intuition is very clear : We invest in the highest return asset. Point (i) says that we do not produce if the maximum return of the productive sector is less than the return of the financial sector. The main implication of Proposition 3.1 is that the productive sector will disappear if its productivity is low.

In Chapter 5, we embed this idea in an infinite horizon dynamic general equilibrium model. In such framework, the sequence of dividend $(\xi_t)_t$ is exogenous, but the sequences of consumption good prices $(p_t)_t$, asset prices $(q_t)_t$, capital return $(r_t)_t$ are endogenous. There are m households. Each household i takes sequences of prices $(p, r, q) = (p_t, r_t, q_t)_{t=0}^{\infty}$ as given and maximizes her intertemporal utility $\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})$ by choosing consumption $(c_{i,t})_t$, physical capital $(k_{i,t+1})$, and financial asset volume $(a_{i,t})_t$. There is a physical constraint $k_{i,t+1} \geq 0$. The budget constraint at date t :

$$p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + p_t\xi_t)a_{i,t-1} + \theta^i \pi_t.$$

Borrowing constraint at date t :

$$(q_{t+1} + p_{t+1}\xi_{t+1})a_{i,t} \geq -f^i(p_{t+1}(1-\delta) + r_{t+1})k_{i,t+1}$$

where $f^i \in (0,1)$ is borrowing limit of agent i. f^i is exogenous and set by law.

For each period, there is a representative firm who takes prices (p_t, r_t) as given and maximizes its profit.

$$(P(p_t, r_t)) : \qquad \max_{K_t \geq 0} \Big[p_t F(K_t) - r_t K_t\Big] \tag{I.26}$$

$(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^m \theta^i = 1$.
At equilibrium, every market clear : at each $t \geq 0$,

$$\begin{aligned}
\text{good} : \quad & \sum_{i=1}^m (c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) = F(K_t) + \xi_t, \\
\text{capital} : \quad & K_t = \sum_{i=1}^m k_{i,t}, \\
\text{financial asset} : \quad & \sum_{i=1}^m a_{i,t} = 1.
\end{aligned}$$

First, we prove the existence of equilibrium. We then explain when does an economic recession appear ? and point out the role of financial dividend. Our finding is summarized as follows.

Proposition 3.2 *Assume that there exists $\xi > 0$ such that $\xi_t \geq \xi$ for every $t \geq 0$ and $F'(0) \leq \delta$. Then there is an infinite sequence $(t_n)_{n=0}^\infty$ such that $K_{t_n} = 0$ for every $n \geq 0$.*

However, even when the productivity is low, a recession may be avoided thanks to financial asset. This ideas is formalized by the following result.

Proposition 3.3 *Assume that for every i,*

$$\beta_i(F'(0) + 1 - \delta)u_i'(\xi_{t+1}) > u_i'(\frac{\xi_t}{m}).$$

We have $K_{t+1} > 0$.

This result also shows that if $F'(0) = \infty$, we have $K_{t+1} > 0$.

In our framework, a fluctuation of (ξ_t) may create a fluctuation of (K_t).

Proposition 3.4 ***(Fluctuation of the capital stocks)***
Assume that

(i) $\beta_i = \beta$, $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$, *and* $F'(0) \leq \delta$.

(ii) $\xi_{2t} \to \xi^e$, $\xi_{2t+1} \to \xi^o$ *when* $t \to \infty$.

(iii) $\xi^e > \dfrac{m\xi^o}{\big(\beta(F'(0) + 1 - \delta)\big)^{\frac{1}{\sigma}}}$.

We have

(i) *There is an infinite sequence* $(t_n)_{n=0}^{\infty}$ *s.t.* $K_{t_n} = 0$ *for every* $n \geq 0$.

(ii) $\limsup_{t\to\infty} K_t > 0$.

4 Efficiency of equilibria

In the dissertation (Chapter 3 and Chapter 5), we are interested by the efficiency of intertemporal equilibria in Ramsey models with heterogeneous agents. Let us begin by defining the efficiency of a capital path.

Definition 4.1 *(Malinvaud (1953))*
Let F_t *be the production function at date* t, δ *be capital depreciation rate. A feasible path of capital is a positive sequence* $(K_t)_{t=0}^{\infty}$ *such that* $0 \leq K_{t+1} \leq F_t(K_t) + (1-\delta)K_t$ *for every* $t \geq 0$ *and* K_0 *is given.*
A feasible path is efficient if there is no other feasible path (K'_t) *such that*

$$F_t(K'_t) + (1-\delta)K'_t - K'_{t+1} \geq F_t(K_t) + (1-\delta)K_t - K_{t+1}$$

for every t *with strict inequality for some* t.

Here, aggregate feasible consumption at date t is defined by $C_t := F_t(K_t) + (1-\delta)K_t - K_{t+1}$.

We have some classical results :

Theorem 4.1 *(Malinvaud (1953))*
Assume that $F_t = F$ *for every* t, *where* F *is strictly increasing, strictly concave, twice continuously differentiable, and* $F(0) = 0$, $F'(\infty) = 0$, $F'(0) = \infty$.

A feasible path (K_t) *is efficient if*

$$\lim_{t\to\infty} \frac{K_t}{\prod_{s=0}^{t-1}(1-\delta+F'(K_s))} = 0. \quad \text{(I.27)}$$

Theorem 4.2 *(Cass (1972))*
Consider capital paths with $K_t \geq \underline{k} > 0$ *for every* t. *Assume that* $F_t = F$ *for every* t, *where* F *is strictly increasing, strictly concave, twice continuously*

differentiable, and $F(0) = 0$, $0 \le F'(\infty) < \delta < F'(\underline{k})$. *A feasible path* (K_t) *is inefficient if and only if*

$$\sum_{t=1}^{\infty} \prod_{s=0}^{t-1} (1 - \delta + F'(K_s)) < \infty. \tag{I.28}$$

Theorem 4.3 *(Cass and Yaari (1971))*
Assume that for each t, F_t *is strictly increasing, strictly concave, continuously differentiable, and* $F(0) = 0$. *The feasible path* (K_t) *is efficient if and only if*

$$\liminf_{T \to \infty} \sum_{t=0}^{T} \frac{c'_t - c_t}{\Pi_t} \le 0 \tag{I.29}$$

for every feasible capital path (K'_t), *where* $\Pi_t := \prod_{s=0}^{t-1} (1 - \delta + F'_s(K_s))$.

We now define the efficiency of intertemporal equilibrium.

Definition 4.2 *We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path* (K_t) *is efficient in sense of Malinvaud (1953).*[3]

Our first finding can be stated as follows :

Proposition 4.1 *Consider the Ramsey model used in Section 2.2. Assume that the production functions are linear. Then every equilibrium path is efficient.*

Note that this result does not require any conditions about the convergence or boundedness of the capital path as in previous literatures.

Corollary 2.2 and Proposition 4.1 indicate that with linear production functions, there exists an equilibrium the capital path of which is efficient and a bubble may arise at this equilibrium.

The following result shows the role of financial dividend on the efficiency of equilibrium.

Proposition 4.2 *Consider the Ramsey model used in Section 3 We assume that the production function* F *is strictly concave,* $F'(\infty) < \delta$, *and* $\limsup_{t \to \infty} \xi_t < \infty$. *If* $\limsup_{t \to \infty} \xi_t > 0$, *every equilibrium is efficient.*

3. Another concept of efficiency is *constrained efficiency*. About the constrained efficiency in general equilibrium models with financial asset, see Kehoe and Levine (1993), Alvarez and Jermann (2000), Bloise and Pietro (2011). About the constrained efficiency in the neoclassical growth model, see Davila, Hong, Krusell and Rios-Rull (2012).

Our results are related to Becker and Mitra (2012) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. However, their result is based on the fact that consumption of each household is uniformly bounded from below. In Proposition 4.1, we do not need this condition. Instead, the efficient capital path in our model may converge to zero. Mitra and Ray (2012) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid. However, their results are no longer valid without the convergence or the boundedness of capital paths.

Becker, Dubey, and Mitra (2014) give an example of inefficient Ramsey equilibrium in a model with only physical capital. The production function in their model satisfies $F'(\infty) = 0$ and they consider full depreciation of the capital. The following result shows that financial dividends, for such models, may make production paths efficient. Actually, our result is more general.

5 FDI, new industry and optimal growth

Almost economists agree that economic development needs the competitiveness of productive sectors, a well-functioning financial system and the political stability,... In developping coutries, FDI has been also viewed as an important factor in the economic growth.[4] However, is attracting FDI spillovers the key to developing of their own industries ? If not, what is the optimal policy of the host country ? More precisely, should the host country develop a new industry or continue to focus on already developed ones ? What are the roles of different macroeconomic variables such as development level, FDI spillovers, return of training, and heterogeneity of firms.

Let us start by considering a very simple situation in an industry. Assume that we have L units of labor. We get salary with wage w (in term of consumption good) if we work for multinational firm. There is a fixed cost $\bar{L}$ if we want to create a new firm in this industry. For simplicity, we begin by assuming that labor is the unique input and the production function of our firm is $F(L_d) = A_d(L_d - \bar{L})^+$, where A is the productivity. we formalize the problem by the following simple model

$$\max_{L_d, L_e} A_d(L_d - \bar{L})^+ + wL_e \tag{I.30}$$

$$L_d + L_e \leq L, \tag{I.31}$$

4. See Harrison, Rodriguez-Clare (2010) for a complete review.

where L_d is labor utilized to produce the consumption good, L_e is labor working for the multinaional firm.

We have the following result showing the optimal quantities of labor L_e, L_d.

Proposition 5.1 *(i) Assume that $L \leq \bar{L}$, we have $L_d = 0$ for every A_d.*

(ii) Assume that $L > \bar{L}$.

(ii.a) If $A_d(L - \bar{L}) \leq wL$, $L_d = 0$.

(ii.b) If $A_d(L - \bar{L}) > wL$, we have $L_d > \bar{L}$ and $L_e = 0$.

Point (i) proves that if the initial labor cannot cover the fixed cost, no domestic firm cannot be created in this industry for every level of the productivity A_d. Point (ii) says that even the initial labor is greater than the fixed cost, we invest in this industry if and only if the productivity A_d reachs a critical threshold $\frac{wL}{L - \bar{L}}$. Moreover, the multinational firm may be eliminated by the domestic one.

In chapter 6, we construct a full model with two industries (an old and a new industries), heterogeneous firms, two inputs (physical capital and speficic labor), and endogenous specific labor supply (which is from the investment in training of the host country). The old sector produces consumption good by using physical capital as the sole input. There is a unique representative domestic firm in this sector. The new sector produces a new good by using physical capital and a specific labor. There are two types of firm in the new sector : an already planted multinational firm and a potential domestic one. As in Proposition 5.1, the potential domestic firm cannot be created if it holds less than $\bar{L}$ units of specific labor.

In this economy, consumption good, physical capital, and new good can be freely exchanged with the rest of the world while the specific labor is not mobile. There are two agents : a social planner maximizing the GNP of the country and the multinational firm maximizing its profit. The prices (in term of consumption good) of physical capital and new good are assumed to be exogenous. However, wage is endogenous and determined by specific labor market clearing condition. Specific labor supply is also endogenous and from three sources : initial specific labor of the country, FDI spillovers effects, and investment in training of the host country.

We consider a two-period model and an infinite horizon model as well. The intuition of Proposition 5.1 will be explored in these models.

The two-period model allows us to analyze the roles of many different macroeconomic variables such as development level, FDI spillovers, return of training, and heterogeneity of firms, fixed cost.

First, point (i) of Proposition will be extended : to invest in the new industry, the country must hold one of the following conditions : (1) it is rich enough, (2) its return of training is high enough, (3) FDI spillovers are strong.

Second, once the country holds the above conditions, the productivity of the potential domestic firm is the key factor deciding the optimal choice of the country. If the old sector is competitive (i.e., high productivity), the country should focus on this sector. If the multinational firm is competitive, no domestic firm can be created in the new industry ; in this case all specific workers will be hired by the multinational firm. The host country should invest in the new industry if and only if the productivity of the domestic firm in this industry reaches a critical threshold ; in this case, although the domestic firm must pay the entry cost, it can dominate, even eliminate, the well-planted multinational firm. Moreover, when FDI spillovers are strong, the domestic firm may be created without training specific workers.

We also point out that in some cases, credit constraints of the domestic firm may prevent it to enter the new industry even if its productivity is greater than that of multinational firms.

The infinite horizon model based on the optimal growth theory gives us dynamic analysis. Consider a poor or developing country (i.e., initial capital stock is low) but its productive sectors are competitive. This country should train specific workers who will be hired by multinational firms. These workers get favorable salaries and, by the way, contribute to the GNP of the host country. There exists a date at which the GNP reaches a critical threshold and new domestic firms are created in the new industry. These domestic firm may eventually dominate multinational firms.

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Chapitre II

Collateral monetary equilibrium with liquidity constraints in an infinite horizon economy

Abstract : This paper considers an infinite horizon monetary economy. There is a Central Bank lending money to households by creating short- and long-term loans. Heterogeneous households can deposit and borrow money on both short- and long-term maturity loans. If households want to sell a financial asset, they are required to hold certain commodities as collateral. There is also a borrowing constraint when households want to borrow a long-term loan. Moreover, they face cash-in-advance constraints when buying commodities and financial assets. I introduce Uniform (resp., Sequential) Gains to Trade Hypothesis under which the existence of collateral monetary equilibrium is ensured and prices are uniformly (resp., only for the product topology) bounded. I also provide some properties of monetary equilibria, for example, the structure of interest rates, the liquidity trap.

Keywords : Monetary economy, cash-in-advance constraints, borrowing constraints, collateralized assets, infinite horizon, liquidity trap.
JEL Classifications : C62, D91, E40, E50, G10.

1 Introduction

The paper gives a full general equilibrium approach to money by studying the monetary equilibrium in an infinite horizon model with consumptions, fiat money and incomplete markets. In this framework, there are a Central Bank and heterogeneous households. The Central Bank lends money to households by creating short- and long-term loans. Households can deposit and borrow money in both short and long term by trading short-term and long-term loans. A short (resp., long)-term loan will be paid back with interest rate at the end of the same (resp., the next) period. For each agent, there is a borrowing constraint in long-term loan : the net repayment does not exceed the market value of his endowments at next period. If agents want to sell a financial asset, they are required to hold certain commodities as collateral. When buying a commodity or an asset, tradings face cash-in-advance constraints. At each period, all trade is voluntary and trading timing of each agent is the following :

(i) trade bank loans

(ii) trade commodities and assets

(iii) deliver on assets

(iv) pay back bank loans.

Agents face a liquidity constraint at each sub-period : all purchases must be paid for in money. With this setup, fiat money which is the stipulated medium of exchange plays a central role in trading.

The first contribution of this paper concerns the existence of monetary equilibrium.

Dubey and Geanakoplos (2003a,b) proved the existence of equilibrium for 2-period monetary economies by using Gains to Trade Hypothesis (GTH). Conversely, they also proved that monetary equilibrium for pure exchange economies with money does not exist unless there are sufficient gains to trade (Theorem 6 and 7 in Dubey and Geanakoplos (2003a)). Dubey and Geanakoplos (2006b) then constructed a two-period monetary economy with production. They imposed that firms sell all goods at hand in period 2, and then there is always a strictly positive quantity of commodity which is sold in the economy for that holds the existence of an equilibrium.

In our infinite horizon model, the technique we use for the proof of equilibrium existence consists of three steps : (1) prove that there exists an equilibrium for each $T-$truncated economy $\mathcal{E}^T$, (2) prove that the sequence (depending in T) of equilibria has a limit, (3) such limit will be proved to be an equilibrium for the monetary economy. Here, the difficulty is to show that the sequence of prices converges and actuallly are equilibrium prices.

In standard general equilibrium models (Levine (1989), Levine and Zame (1996), Magill and Quinzii (1994, 1996), Araujo, Pascoa, Torres-

Martinez (2002), Kubler and Schmedders (2003)), the budget constraints are homogeneous. Thanks to that, we can normalize prices by setting the sum of prices at each date to equal 1. As a consequence, we can easily prove that the sequence of equilibria of truncated economies has a limit. However, when introducing fiat money in our model, cash-in-advance constraints are no longer homogeneous, and hence we cannot assume that the sum of prices equals 1. As a result, the methods used in these papers are no longer valid. Moreover, since nominal interest rates may be zero, agents may also keep money on hand. Because of that, cash-in-advance constraints are generally not binding, which implies that each choice of agents at date t will appear in every their cash-in-advance constraints from date $t+1$. Therefore, we cannot easily apply Kuhn-Tucker Theorem as used in Araujo, Pascoa, Torres-Martinez (2002), Kubler and Schmedders (2003).

To overcome all difficulties discussed above, I introduce Uniform and Sequential Gains to Trade Hypotheses (for short, we write UGTH, SGTH, respectively). Thanks to these hypotheses, a strictly positive amount of commodities will be traded at each period. As a result, commoditiy prices are bounded from above and away from zero. Collateral constraints then ensure that asset prices and traded quantities of financial assets are bounded from above.

UGTH and SGTH, and collateral constraints allow us to pass the limit after having proved that there exists an equilibrium for each truncated economy This paper gives two versions of equilibrium : (i) under UGTH, prices are uniformly bounded from above and away from zero, (ii) under SGTH, prices are only bounded for the product topology.

The second contribution is about liquidity trap. At equilibrium, we say there is a liquidity trap if the interest rate of short-term loan at some date equals zero. In a two-period model, Dubey and Geanakoplos (2006a) assume that asset payoffs are linearly independent for that liquidity trap may exist in the first period at equilibrium. In Dubey and Geanakoplos (2006b), they consider a two-period model with production. The existence of liquidity trap in the first period in this model is based on the assumption that firms have strictly positive endowment and sell all in period 2.

In my paper, without these assumptions, I show that liquidity trap may appear at any date in an infinite horizon economy. Our result says that : at some node, say ξ, if money supply for short-term loans, i.e., $M(\xi)$, is very high with respect to money supply which agents expect to be available at some date in the future, the economy will fall into a liquidity trap at node ξ. Different from Dubey and Geanakoplos (2006a,b), my argument is based on SGTH, UGTH and collateral constraints. Indeed, if the interest rate of short-term loan, i.e., $r_s(\xi)$, is strictly positive, agents spend all money borrowed from the Central Bank through short-term loan. Under SGTH or SGTH, commodity prices, and so asset prices, are bounded independently from $M(\xi)$. By collateral constraints, sales of assets must also be bounded. As a consequence, expenditures at node ξ must be bounded. Therefore, when $M(\xi)$ is high enough, $r_s(\xi)$ must be zero.

Since liquidity trap may occur at any period, there may exist a sequence of dates such that liquidity trap appears at each date of this sequence.

Related literature : Bloise, Dreze and Polemarchakis (2005) also studied the monetary equilibrium in an infinite horizon model. However, they only considered the case where the financial market is sequentially complete and they assume that interest rates are exogenous. On the contrary, the financial market is incomplete and interest rates are endogenous in my paper. An excellent introduction to incomplete markets with infinite horizon can be found in Magill and Quinzii (2008).

On intertemporal equilibrium with production. Becker, Bosi, Le Van, Seegmuller (2014) proved the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical asset. Le Van and Pham (2013) proved the existence of intertemporal equilibrium in an infinite horizon model with physical capital, endogenous labor supply and financial asset with borrowing constraint, in which aggregate capital and consumption may be not uniformly bounded.

More about *liquidity trap*, see Krugman (1998), Eggertsson (2008) for a complete reviews. See Werning (2012) and Cochrane (2014) for monetary and fiscal policy in liquidity trap scenarios in a continuous-time version of the standard New-Keynesian model.

The remainder of the paper is organized as follows. Section 2 describes the model. In section 3, we discuss the existence of monetary equilibrium. Section 4 provides some properties of equilibria : the structure of interest rates, the liquidity trap. Section 5 comprising conclusion and some open questions. Most of the formal proofs are given in Appendix.

2 Model

I extend the model in Dubey and Geanakoplos (2003b) and Araujo, Pascoa, Torres-Martinez (2002) to the case of infinite horizon and add collateral constraints to financial assets.

2.1 The underlying economy

I consider an infinite horizon model with uncertainty. Time runs from $t = 0$ to $+\infty$.
At each date, there are S possible exogenous states (or shocks)

$$\mathcal{S} := \{s_1, \dots, s_S\}.$$

A node ξ is charaterized by $\xi = (t, a_0, a_1, \dots, a_t)$ where $t = t(\xi)$ is the date of node ξ and $a_0, \dots, a_t \in \mathcal{S}$. The unique previous node of ξ is denoted by ξ^-. For each T and ξ, we denote

- $\mathcal{D}$ is the set of all nodes. $\mathcal{D}(\xi)$ is the subtree with root ξ,
- $\mathcal{D}_T := \{\xi : t(\xi) = T\}$ is the family of nodes with date T.
- $D^T(\xi) := \bigcup\limits_{t=t(\xi)}^{T} D_t(\xi)$, where $D_t(\xi) := \mathcal{D}_T \cap \mathcal{D}(\xi)$.
- $\xi^+ := \{\mu \in \mathcal{D}(\xi) : t(\mu) = t(\xi) + 1\}$.

A path of nodes is a sequence of nodes $(\xi_n)_{n=0}^T$ such that $\xi_{n+1} \in \xi_n^+$ for every $n \geq 0$. Note that, given ξ, there is a unique path from ξ_0 to ξ, which is denoted by $(\xi_0, \xi_1, \dots, \xi)$.

The set of commodities is $\mathcal{L} := \{1, \dots, L\}$.

There are H types of consumers, $h \in \mathcal{H} = \{1, \dots, H\}$. Each agent h is equipped with an initial vector endowment $e^h(\xi) \in \mathbb{R}_+^L$ of goods at each node ξ. We denote $e^h := (e^h(\xi)_{\xi \in \mathcal{D}})$.

Assumption (H1) : *For each node ξ and each $h \in \mathcal{H}$, $\|e^h(\xi)\| > 0$, where $\|e^h(\xi)\| := \sum\limits_{\ell=1}^{L} e_\ell^h(\xi)$. For each node ξ and each commodity $\ell \in \mathcal{L}$, $e_\ell(\xi) := \sum\limits_{h=1}^{H} e_\ell^h(\xi) > 0$.*

Assumption (H1*) : *There exists $\bar{e}, \underline{e} \in (0, +\infty)$ such that $\underline{e} \leq e_\ell(\xi) \leq \bar{e}$ for every node ξ and for every commodity ℓ .*

Each agent h has the utility function $U^h(\cdot) = \sum\limits_{t=0}^{\infty} \sum\limits_{\xi \in \mathcal{D}_t} u_\xi^h(x_\xi^h)$.

Assumption (H2) : *$u_\xi^h : \mathbb{R}_+^L \to \mathbb{R}_+$ is concave, strictly increasing*[1] *and*

$$u_\xi^h(0) = 0, \quad \forall \xi \in \mathcal{D}, \tag{II.1}$$

$$\lim_{x_\ell(\xi) \to \infty} u_\xi^h(x_1, \dots, x_{\ell-1}, x_\ell(\xi), x_{\ell+1}, \dots, x_L) = +\infty, \quad \forall \xi \in \mathcal{D}, \tag{II.2}$$

$$\sum_{t=0}^{\infty} \sum_{\xi \in \mathcal{D}_t} u_\xi^h(x^h) < +\infty, \tag{II.3}$$

for each $x^h \in \mathbb{R}_+^L$.

Assumption (H2*) : *$u_\xi^h : \mathbb{R}_+^L \to \mathbb{R}_+$ has all conditions as in Assumption (H2) except that condition (II.3) is replaced by the following condition*

$$\sum_{t=0}^{\infty} \sum_{\xi \in \mathcal{D}_t} u_\xi^h(\bar{e}(\xi)) < +\infty, \tag{II.4}$$

where $\bar{e}(\xi) := (e_1(\xi), \dots, e_L(\xi))$.

1. i.e., for each note ξ, $u_\xi(x) > u_\xi(y)$ for $x \geq y$ and $x \neq y$. Here, $x \geq y$ means that $x_\ell \geq y_\ell$ for every $\ell = 1, \dots, L$.

Conditions (II.3) and (II.4) ensure that the utility of each household is finite. Note that, a standard example of the utility function is given by

$$\sum_{t=0}^{\infty} \beta_h^t \sum_{\xi \in \mathcal{D}_t} P_h(\xi) u_h(x^h(\xi)) = \mathbb{E}_0^h \sum_{t=0}^{\infty} \beta_h^t u_h(x_t^h),$$

where $u^h : \mathbb{R}_+ \to \mathbb{R}_+$, and $P_h(\xi)$ is the probability of node ξ under agent h's belief.

2.2 Money

As in Dubey and Geanakoplos (2006b), money is fiat and enters the economy in two ways. At each node ξ, each agent has endowment of money $m^h(\xi) \geq 0$. Denote $m^h := m^h(\xi)_{\xi \in \mathcal{D}}$. We call this outside money. We assume that total outside money at initial date is strictly positive.

Assumption (H3) :

$$\underline{m} := \sum_h m^h(\xi_0) > 0 \tag{II.5}$$

A Central Bank can make short loans totalling $M(\xi) > 0$ dollars for one period at node ξ, and also make long loans totalling $N(\xi) > 0$ for two periods at node ξ.[2] Each agent can borrow money from the bank by promising to pay back the loans with interest. If the interest rate for short loans is r_s, one can borrow $\mu/(1+r_s)$ dollars by promising to repay μ dollars at the end of the same period. If the interest rate for long loans is r_ℓ, one can borrow $\nu/(1+r_\ell)$ dollars by promising to repay ν dollars at the end of the next period. There is a borrowing constraint in long-term loan, which will be presented in section 2.5.

For each node ξ, denote

$$m(\xi) := \sum_{h=1}^{H} m^h(\xi), \quad \hat{m}(\xi) := \sum_{n=0}^{t(\xi)} m(\xi_n)$$

where $(\xi_0, \xi_1, \ldots, \xi)$ is the finite path whose terminal node is ξ. We can see that $\hat{m}(\xi)$ may tend to infinity if $t(\xi)$ tends to infinity. As a consequence, prices may tend to infinity. Therefore, we need the following assumption if we want to prove that prices are uniformly bounded at equilibrium.

Assumption (H4) : *For each path of nodes* $(\xi_n)_{n=0}^{\infty}$, *we have*

$$\sum_{n=0}^{\infty} m(\xi_n) \quad < \quad \infty, \tag{II.6}$$

2. With the long loans of maturity of T, we refer to Magill an Quinzii (2012).

and there exists $\bar{M}$ such that $M(\xi) + N(\xi) \leq \bar{M}$ for every node ξ.

This assumption requires that the quantity of outside and inside money in the economy is uniformly bounded. Note that our paper gives two types of equilibrium : (i) with Assumption (H4), equilibrium prices are uniformly bounded ; (ii) without Assumption (H4), we can still prove the existence of equilibrium at which prices are only bounded for the product topology.

2.3 Fundamental Macrovariables

The fundamental macrovariables are

$$\bar{\eta} = (\eta(\xi))_{\xi \in \mathcal{D}} = (r_s(\xi), r_\ell(\xi), p(\xi))_{\xi \in \mathcal{D}}$$

where, at each node ξ

– $r_s(\xi)$ is the interest rate on short-term bank loan.
– $r_\ell(\xi)$ is the interest rate on long-term bank loan.
– $p(\xi) \in \mathbb{R}^L$: commodity prices.

Denote $\bar{\eta}(0, \xi) = (\bar{\eta}(s_0), \ldots, \bar{\eta}(\xi^-), \bar{\eta}(\xi))$.

2.4 Collateralized assets

There are K types of financial assets. The set of financial assets is denoted by $\mathcal{K} = \{1, \ldots, K\}$. A collateralized security is a pair (A, c), where $A = (A(\bar{\eta}(0, \xi)))_{\xi \in \mathcal{D}}$, with $A(\bar{\eta}(0, \xi)) \in \mathbb{R}^K_+$, $A(\cdot)$ depends continuously on $\bar{\eta}(0, \xi)$, and $c = (c^k_\ell)_{k \in \mathcal{K}, \ell \in \mathcal{L}} \in \mathbb{R}^{K \times L}_+$. If one agent wants to sell one unit of financial asset k, she is required to hold $(c^k_\ell)_{\ell \in \mathcal{L}}$ units of goods. We assume that collateral is non null.

Assumption (H5) :

$$\sum_{\ell \in \mathcal{L}} c^k_\ell > 0, \qquad \textit{for all } k. \tag{II.7}$$

As in Geanakoplos and Zame (2002, 2010), the collateral requirement is the only means of enforcing promises. Therefore, the delivery per share of security (A, c) at node ξ will be the minimum of the face value and the value of the collateral :[3]

$$d_k(\xi) := \min \Big\{ A_k(\bar{\eta}(0, \xi)), p(\xi) \cdot c_k \Big\}. \tag{II.8}$$

The delivery of a portfolio $\alpha = (\alpha_1, \cdots, \alpha_K) \in \mathbb{R}^K$ at note ξ is

$$\sum_{k \in \mathcal{K}} \alpha_k d_k(\xi). \tag{II.9}$$

3. We denote here $x \cdot y = x_1 y_1 + \ldots + x_n y_n$ for $x, y \in \mathbb{R}^n$.

We now define the fundamental macrovariable η by adding financial asset prices into the fundamental macrovariable.

$$\eta := (\bar{\eta}, \pi).$$

2.5 Liquidity constraints for households

Given macrovariable η, we will define $\sum_\eta^h = (\sum_\xi^h)_{\xi\in D}$ the set of feasible choices of $h \in H$. Each choice $\sigma^h(\xi) \in \sum_\xi^h$ of agent h at node ξ

$$\sigma^h(\xi) := (\mu^h, \tilde{\mu}^h, \nu^h, \tilde{\nu}^h, q^h, \tilde{q}^h, \alpha^h, \tilde{\alpha}^h)(\xi) \geq 0$$

is described as follow
$\mu^h(\xi)$: short-term bank loans sold by h at node ξ,
$\tilde{\mu}^h(\xi)$: short-term money deposited by h at node ξ,
$\nu^h(\xi)$: long-term bank loans sold by h at node ξ,
$\tilde{\nu}^h(\xi)$: long-term money deposited by h at node ξ,
$\alpha_k^h(\xi)$: financial asset $k \in \mathcal{K}$ sold by h at node ξ (recall that selling a security is borrowing),
$\tilde{\alpha}_k^h(\xi)$: money spent by h on asset k at node ξ,
$q_\ell^h(\xi)$: quantity of commodity ℓ sold by h at node ξ,
$\tilde{q}_\ell^h(\xi)$: bid on h on commodity $\ell \in \mathcal{L}$ at node ξ.

The timing of trade is as follows : first, household h buys and sells bank loans ; second, she buys and sells financial assets, commodities ; third, she delivers on financial assets ; finally, she repays on loans. $\sigma^h(\xi)$ must satisfy the following liquidity constraints.

(i) deposited money $\leq$ money on hand :

$$\tilde{\mu}^h(\xi) + \tilde{\nu}^h(\xi) \leq m^h(\xi) + \tilde{m}^h(\xi^-) \qquad (1)^h(\xi)$$

where $\tilde{m}^h(\xi^-)$ is non-negative and represents the cash hold by household h at the end of node ξ^-.

(ii) expenditures of financial assets and commodities $\leq$ money unspent in $(1)^h(\xi)$ plus money borrowed via short- and long-term loans :

$$\sum_{k\in\mathcal{K}} \tilde{\alpha}_k^h(\xi) + \sum_{\ell\in\mathcal{L}} \tilde{q}_\ell^h(\xi) \leq \Delta(1^h(\xi)) + \frac{\mu^h(\xi)}{1 + r_s(\xi)} + \frac{\nu^h(\xi)}{1 + r_\ell(\xi)}, \qquad (2)^h(\xi)$$

where $\Delta(a)$ is the difference between the right-hand side and the left-hand side of inequality (a).

(iii) Delivery on assets of the previous period $\leq$ money left in $(2)^h(\xi)$ plus money obtained from sales of commodities, and assets

$$\begin{aligned} d(\xi) \cdot \alpha^h(\xi^-) \quad &\leq \quad \Delta(2^h(\xi)) + d(\xi) \cdot \frac{\tilde{\alpha}^h(\xi^-)}{\pi(\xi^-)} \\ &+ q^h(\xi) \cdot p(\xi) + \alpha^h(\xi) \cdot \pi(\xi). \qquad (3)^h(\xi), \end{aligned}$$

(iv) Repayments on loans

$$\begin{aligned}\mu^h(\xi)+\nu^h(\xi^-) \quad \leq \quad & \Delta(3^h(\xi))+(1+r_s(\xi))\tilde{\mu}^h(\xi)\\ & +(1+r_\ell(\xi^-))\tilde{\nu}^h(\xi^-) \qquad (4)^h(\xi),\end{aligned}$$

Moreover, $\sigma^h(\xi)$ must satisfy collateral and borrowing constraints. *Collateral constraint* (or *physical constraints*) requires that the agent h's commodity used as collateral cannot exceed its amount of commodity ℓ after trading within node plus the collateral from previous period, i.e.,

$$\sum_{k\in\mathcal{K}} c_\ell^k \alpha_k^h(\xi) \leq e_\ell^h(\xi)+\frac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)}-q_\ell^h(\xi)+\sum_{k\in\mathcal{K}} c_\ell^k\alpha_k^h(\xi^-) \qquad (pc)^h(\xi).$$

The collateral constraint allows us to ensure that the amount of financial asset sold by each agent is bounded.

Borrowing constraints[4] on bank loans requires the repayment on long-term loan of each agent cannot exceed its value of endowment :

$$\nu^h(\xi^-) \leq \sum_\ell e_\ell^h(\xi)p_\ell(\xi) \qquad (b)^h(\xi)$$

Liquidity, collateral and borrowing constraints define the feasible set $\sum_\eta^h = (\sum_\xi^h)_{\xi\in D}$. Note that given η, we see that $\sum_\eta^h$ is convex. The consumption of household h is given by

$$x_\ell^h(\xi) := e_\ell^h(\xi)-q_\ell^h(\xi)+\frac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)}-\sum_{k\in\mathcal{K}} c_\ell^k\alpha_k^h(\xi)+\sum_{k\in\mathcal{K}} c_\ell^k\alpha_k^h(\xi^-). \qquad \text{(II.10)}$$

And agent h's money at the end of this node is

$$\tilde{m}^h(\xi) := \Delta(4)^h(\xi). \qquad \text{(II.11)}$$

Here, we can see that $\tilde{m}^h(\xi)$ may be strictly positive. Because of this fact, the Lagrange multiplier techniques in Araujo, Pascoa, Torres-Martinez (2002) cannot be used to prove the existence of equilibrium. The reason is that each variable within node ξ will appear in every constraint after this node.

2.6 Monetary equilibrium

Definition 2.1 *The collection* $(\eta,(\sigma^a)_{a\in\mathcal{H}})$ *is a collateral monetary equilibrium (CME) for the monetary economy* $\mathcal{E}=\Big((u^h,e^h,m^h)_{h\in\mathcal{H}},(A,c),(M,N)\Big)$, *where* $M=(M(\xi))_{\xi\in\mathcal{D}}, N=(N(\xi))_{\xi\in\mathcal{D}}$, *are stocks of money from the Central Bank in short- and long-term loans, respectively, if*

4. If we collateralize long-term loans as we did with financial assets, the existence of equilibrium is still ensured.

(i) All agents maximize their utility

$$\sigma^t \in \arg\max_{\sigma^h_\eta \in \Sigma^h} U^h(x^h(\eta, \sigma^h)), \forall h \in \mathcal{H} \tag{II.12}$$

(ii) All markets clear : loans, derivatives and commodities

$$\frac{1}{1+r_s(\xi)} \sum_{h\in\mathcal{H}} \mu^h(\xi) = M(\xi) + \sum_h \tilde{\mu}^h(\xi) \tag{II.13}$$

$$\frac{1}{1+r_\ell(\xi)} \sum_{h} \nu^h(\xi) = N(\xi) + \sum_h \tilde{\nu}^h(\xi), \tag{II.14}$$

$$\pi_k(\xi) \sum_h \alpha^h_k(\xi) = \sum_h \tilde{\alpha}^h_k(\xi) \tag{II.15}$$

$$p_l(\xi) \sum_{h\in\mathcal{H}} q^h_\ell(\xi) = \sum_{h\in\mathcal{H}} \tilde{q}^h_\ell(\xi). \tag{II.16}$$

3 The existence of equilibrium

We prove the existence of collateral monetary equilibrium by using the standard approach.

(i) Step 1 : prove that there exists an equilibrium for each $T-$truncated economy $\mathcal{E}^T$.

(ii) Step 2 : prove that the sequence of equilibria has a limit.

(iii) Step 3 : prove that such limit will be an equilibrium for the monetary economy $\mathcal{E}$:

The main difficulty is to bound all prices. Different from standard general equilibrium models, there is the fiat money in the present model, which makes liquidity constraints become non-homogeneous. Because of that, we cannot normalize prices by setting the sum of prices to equals 1.

For truncated economies, by adopting Dubey and Geanakoplos (2003b), I introduce Uniform and Sequential Gains to Trade Hypotheses (for short, UGTH, SGTH, respectively), and then use them to prove that commodity prices are bounded. The boundedness of prices of financial assets can be proved thanks to collateral constraints and the boundedness of commodities prices.

Therefore, the sequence of equilibria has a limit. It is clear that market clearing conditions are satisfied. We prove the optimality of this limit of allocations by using borrowing constraints on long-term loans and collateral constraints on financial assets.

3.1 Gains to Trade Hypotheses

First, by following Dubey and Geanakoplos (2003b), we define non $\gamma-$ Pareto optimal allocation. Denote $\mathbb{R}^{L\times\mathcal{D}} := \{x = (x_\ell(\xi))_{\ell\in\in\mathcal{L},\xi\in\mathcal{D}} : x_\ell(\xi)\in\mathbb{R}\}$.

Definition 3.1 *$(x^1,\dots,x^h)\in\left(\mathbb{R}^{L\times\mathcal{D}}\right)^H$ is called non γ-Pareto optimal at node $\xi\in\mathcal{D}$ if there exists $\tau^1(\xi),\cdots,\tau^H(\xi)\in\mathbb{R}^L$ such that*

$$\begin{aligned}
&\sum_{h=1}^{H}\tau^h(\xi)=0,\\
&\tau^h(\xi)\neq 0,\quad ,x^h(\xi)+\tau^h(\xi)\in\mathbb{R}^L_+,\ \textit{for all } h\in\mathcal{H}\\
&U^h(\bar{x}^h(\gamma,\tau^h(\xi)))>U^h(x^h),\quad \forall h,
\end{aligned}$$

$$\textit{where } \bar{x}^h(\gamma,\tau^h(\xi))_\ell(\mu)=\begin{cases} x^h_\ell(\mu) \textit{ if } \mu\neq\xi,\\ x^h_\ell(\xi)+\min\{\tau^h_\ell(\xi),\dfrac{\tau^h_\ell(\xi)}{1+\gamma}\} \textit{ if } \mu=\xi.\end{cases}$$

Definition 3.2 *For $x=(x^1,\dots,x^h)\in\left(\mathbb{R}^{L\times\mathcal{D}}\right)^H$, we define $\gamma(x):=\sup\{\gamma :$ x is not $\gamma-$ Pareto optimal at node ξ $\}$*

Second, for each $a\geq 0$, $\xi\in\mathcal{D}$, we define the set $X^a(\xi)$ of allocations such that the level of trade does not exceed a.

$$\begin{aligned}
X^a(\xi):=\Big\{\quad & (x^1,\dots,x^h)\in\left(\mathbb{R}^{L\times\mathcal{D}}_+\right)^H \text{ such that there exists } (p,r)\in\mathbb{R}^{(L+2)\times\mathcal{D}}_+ \text{ and}\\
& (\alpha^1,\dots,\alpha^H)\in\left(\mathbb{R}^{K\times\mathcal{D}}_+\right)^H \text{ such that}\\
\forall\mu,\ell\quad & \sum_{h=1}^{H}x^h_\ell(\mu)+\sum_{h=1}^{H}\sum_{k=1}^{K}c^k_\ell\alpha^h_k(\mu)=\sum_{h=1}^{H}e^h_\ell(\mu)+\sum_{h=1}^{H}\sum_{k=1}^{K}c^k_\ell\alpha^h_k(\mu^-)\\
\forall\ell\quad & |x^h_\ell(\xi)-e^h_\ell(\xi)-\sum_{h=1}^{H}\sum_{k=1}^{K}c^k_\ell\alpha^h_k(\xi^-)\leq a\quad\Big\}.
\end{aligned}$$

For each node ξ, denote

$$\nu_\xi(m,M):=\frac{\hat{m}(\xi)+N(\xi^-)-M(\xi)}{M(\xi)}.$$

Assumption (H6) : (Sequential Gains to Trade Hypothesis)
At each node ξ, there exists $a(\xi)>0$ such that at each node $\xi\in\mathcal{D}$, $\gamma_\xi(x)>\nu_\xi(m,M)$ for all $x\in X^{a(\xi)}(\xi)$.

This hypothesis requires that at each node ξ, there exists a trade level $a(\xi)>0$ at which there are gains to trade. This hypothesis allows us to prove that there is a strictly positive amount of commodities will be traded. As a consequence, commodities prices are bounded. Formal proof is presented in Appendix 6.1.

We also define, for each path $(\xi_0, \xi_1, \dots, \xi)$,

$$\bar{m}(\xi) := \max_{\mu \in (\xi^-)^+} \Big(\hat{m}(\xi) - \frac{\hat{m}(\mu)}{N(\xi^-) + M(\mu)} N(\xi^-) \Big), \quad \text{(II.17)}$$

$$\mu_\xi(m, M) := \frac{\bar{m}(\xi)}{M(\xi)}. \quad \text{(II.18)}$$

Assumption (H6*) : (Uniform Gains to Trade Hypothesis)
There exists $a > 0$ such that at each node $\xi \in \mathcal{D}$, $\gamma_\xi(x) > \mu_\xi(m, M)$ for all $x \in X^a(\xi)$.

This hypothesis requires that there exists a trade level $a > 0$ at which there are gains to trade at every node.

3.2 Existence of equilibrium in the economy $\mathcal{E}^T$

We first define of $T-$ truncated economy $\mathcal{E}^T$.

Definition 3.3 (T-truncated economy $\mathcal{E}^T$) *We define $\mathcal{E}^T$ as $\mathcal{E}$ but for all $t > T$, $\eta_t = \sigma_t = 0$ and, at period T, there are neither trades in loans nor trades in financial assets, i.e. $\nu^h(\xi) = \tilde{\nu}^h(\xi) = \tilde{\alpha}_k^h(\xi) = \tilde{\alpha}_k^h(\xi) = 0$ for every h, k, and $\xi \in \mathcal{D}_T$.*

The following results claim that there exists an collateral monetary equilibrium for each T-truncated economy.

Theorem 3.1 *Under Assumptions (H1*), (H2*), (H3), (H4), (H5) and (H6*), there exists a collateral monetary equilibrium for $\mathcal{E}^T$. Moreover, at equilibrium, all prices are uniformly bounded.*

Proof: See Appendix 6.1. ■

Theorem 3.2 *Under Assumptions (H1), (H2), (H3), (H5) and (H6), there exists a collateral monetary equilibrium for $\mathcal{E}^T$.*

Proof: See Appendix 6.2. ■

Note that Assumption (H4) ensures that quantity of money at every node is uniformly bounded, hence so are prices. But without Assumption (H4), prices are only bounded for the product topology.

3.3 The existence of equilibrium : infinite horizon

Theorem 3.3 *Under Assumptions in Theorem 6.1 (or Theorem 3.2), there exists a collateral monetary equilibrium for the infinite horizon economy.*

Proof: By observing the proof of Theorem 6.1 (or Theorem 3.2), we see that allocations and prices are bounded for the product topology. Therefore, we can assume that the sequence of equilibra $(\eta^{*,T}, (\sigma^{*h,T})_{h\in\mathcal{H}})$ converges to $(\eta^*, (\sigma^{*h})_{h\in\mathcal{H}})$ when T tends to ∞.
We will prove that $(\eta^*, (\sigma^{*h})_{h\in\mathcal{H}})$ is an equilibrium of $\mathcal{E}$. It is clear that condition (ii) in Definition 2.1 hold. We only need to prove the optimality of choice $(\sigma^{*h})_{h\in\mathcal{H}}$.

Denote

$$\begin{aligned}\Sigma^{h,T} = \{\sigma^h : \quad & \sigma^h(\xi) = 0 \quad \forall \xi \notin \mathcal{D}^T(\xi_0), \\ \text{and} \quad & \nu^h(\xi) = \tilde{\nu}^h(\xi) = \tilde{\alpha}^h_k(\xi) = \tilde{\alpha}^h_k(\xi) = 0 \quad \forall \xi \notin \mathcal{D}_T(\xi_0)\}\end{aligned}$$

For a choice $\sigma^h \in \Sigma^h_{\eta^*}$, denote $U^{h,T}(\sigma^h) := \sum_{\xi\in\mathcal{D}^T} u^h_\xi(x^h(\eta^*, \sigma^h))$.
Suppose that there exists a choice $\bar{\sigma}^h \in \Sigma^h_{\eta^*}$, $\epsilon > 0$, and $T_1 \in \mathbb{N}$ such that for all $T \geq T_1$

$$U^{h,T}(\bar{\sigma}^h) - U^{h,T}(\sigma^{*h}) > 3\epsilon, \forall T \geq T_1.$$

Since the utility of household h is finite, there exists [5] $T_2 > T_1, \hat{\sigma}^h \in \Sigma^h_{\eta^*} \cap \Sigma^{h,T_2}$ such that $U^{h,T}(\hat{\sigma}^h) - U^{h,T}(\bar{\sigma}^h) > -\epsilon$ and $U^{h,T}(\sigma^{*h}) - U^h(\sigma^{*h}) > -\epsilon$ for all $T \geq T_2$. Hence, there exists $\hat{\sigma}^h \in \Sigma^h_{\eta^*} \cap \Sigma^{h,T_2}$ such that

$$U^{h,T}(\hat{\sigma}^h) - U^h(\sigma^{*h}) > \epsilon, \forall T \geq T_2.$$

We define $\psi^h : \Sigma^h_\eta \to \Sigma^{h,T_2}_\eta$ by $\psi(\sigma^h) := \{\hat{\sigma}^h \in \Sigma^{h,T_2} : U^{h,T_2}(\hat{\sigma}^h) - U^h(\sigma^h) > \epsilon\}$.
Denote Θ is the space of prices that is compact in Theorem 6.1. Define F^h is a correspondence from $\Theta \times \Sigma^h$ to Σ^{h,T_2} by

$$F^h(\eta, \sigma^h) = \Sigma^{h,T_2}_\eta \cap \psi^h(\sigma^h).$$

So F^h is lower semi-continuous with respect to the product topology.[6]
By definition of $\hat{\sigma}^h$, we see that $\hat{\sigma}^h \in F^h(\eta^*, \sigma^{*h})$.

5. Note that thanks to collateral and borrowing constraint, if σ^h is a feasible choice, the following choice σ_T is also feasible

$$\sigma_T(\xi) = \begin{cases} \sigma_T(\xi) \text{ for every } \xi \text{ such that } t(\xi) \leq T \\ 0, \text{ otherwise.} \end{cases}$$

6. See Pascoa and Seghir (2009).

Combining with $\lim_{T\to\infty}(\eta^{*T},(\sigma^{*h,T})_h) = (\eta^*,(\sigma^{*h})_h)$, there exists a sequence $(\hat{\sigma}^h_T)_{T\geq T_0} \subset \Sigma^{h,T_2}$ such that $\lim_{T\in\infty} \hat{\sigma}^h_T = \hat{\sigma}^h$ and $\hat{\sigma}^h_T \in F^h(\eta^{*T},\sigma^{*h,T})$ for all $T \geq T_0$.

Without the generality, we can assume that $T_0 \geq T_2$. Therefore, $\hat{\sigma}^h_{T_0} \in \Sigma^{h,T_0}_{\eta^{*T_0}}$ and

$$U^{h,T_2}(\hat{\sigma}^h_{T_0}) - U^h(\sigma^{h,T_0}) > \epsilon.$$

As a consequence, we obtain

$$U^{h,T_0}(\hat{\sigma}^h_{T_0}) \geq U^{h,T_2}(\hat{\sigma}^h_{T_0}) > U^h(\sigma^{h,T_0}) + \epsilon > U^{h,T_0}(\sigma^{h,T_0}).$$

This is a contradiction with the optimality of the truncated economy $\mathcal{E}^{T_0}$. ■

4 Properties of equilibria

In this section, we study properties of equilibria. Let's start by the following result.

Lemma 4.1 *At any equilibrium, we have*

$$\begin{aligned} &\sum_{h\in\mathcal{H},\ell\in\mathcal{L}} p_\ell(\xi)q^h_\ell(\xi) + \sum_{h\in\mathcal{H},k\in\mathcal{K}} \pi_k(\xi)\alpha^h_k(\xi) \\ \leq\ &\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + M(\xi) + N(\xi), \end{aligned} \tag{II.19}$$

and

$$\begin{aligned} &(1 + r_\ell(\xi^-))N(\xi^-) + (1 + r_s(\xi))M(\xi) \\ \leq\ &\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + M(\xi) + N(\xi). \end{aligned} \tag{II.20}$$

Proof: To prove (II.19), we use (2^h_ξ) and the fact that markets are clear. (II.20) is proved by using (4^h_ξ) and the fact that markets are clear. ■

The inequality (II.19) in Lemma 4.1 is a version of the Quantity Theory of Money with velocity of money is equal to 1. The inequality (II.20) indicates that the quantity of money paid back by the Central Bank at the end of node ξ is not greater than the total of outside money at node ξ, money unspent from previous node ξ^- and money injected by the Central Bank at node ξ.

The inequality (II.20) means that the quantity of money turned back to the Central Bank at the end of period ξ does not exceed the total money circulated at the same period.

Proposition 4.1 *At any equilibrium*

(i) $r_s(\xi), r_\ell(\xi) \geq 0$ *for all* t.

(ii) $1 + r_\ell(\xi) \geq \min_{\xi^+}\{(1 + r_s(\xi))(1 + r_s(\xi^+))\}$.[7] *A direct consequence of this result is that* $r_\ell(\xi) \geq r_s(\xi) + \min_{\xi^+}\{r_s(\xi^+)\}$.

(iii) $$r_s(\xi) \leq \frac{\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + N(\xi) - (1 + r_\ell(\xi^-))N(\xi^-)}{M(\xi)},$$

and $$r_\ell(\xi^-) \leq \frac{\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + N(\xi) - r_s(\xi)M(\xi) - N(\xi^-)}{N(\xi^-)}.$$

(iv) **Public debt :** *the following result can explain the fact in monetary policy : for every path* $(\xi_0, \xi_1, \ldots, \xi_n)$, *we have*

$$\sum_{k=0}^{n-1} \Big[r_\ell(\xi_k)N(\xi_k) + r_s(\xi_k)M(\xi_k)\Big] + r_s(\xi_n)M(\xi_n) \leq \hat{m}(\xi_n) + N(\xi_n). \quad \text{(II.21)}$$

where $\hat{m}(\xi) = \sum_{h=1}^{H}(m^h(\xi_0) + \cdots + m^h(\xi))$.

Proof: (i) and (ii) are standard. (iii) is proved by using (II.20) in Lemma 4.1. ■

Corollary 4.1 *Under Assumptions in Theroem 6.1, at equilibrium we have*

$$\lim_{t(\xi)\to\infty} r_\ell(\xi)N(\xi) + r_s(\xi)M(\xi) = 0. \quad \text{(II.22)}$$

This result implies that if the Central Bank can control all money in long run, and the quantity of money injected by the Central Bank is uniformly bounded, all interest rates tend to a very low level. Indeed, if the Central Bank can control all money in long run, Assumption 2.2 is hold. Hence for every infinite path, we have

$$\sum_{k=0}^{\infty} \Big[r_\ell(\xi_k)N(\xi_k) + r_s(\xi_k)M(\xi_k)\Big] + r_s(\xi_n)M(\xi_n) < \infty. \quad \text{(II.23)}$$

Consequently, $\lim_{n\to\infty} r_\ell(\xi_n)N(\xi_n) + r_s(\xi_n)M(\xi_n) = 0$.

7. Generally, if we consider the long-term loans of t-period with $t = 1, \ldots, T$,

$$1 + r_{L(T)}(\xi) \geq \min_{\xi^{t_1},\ldots,\xi^{t_a}:\sum_{i=1}^{a} t_i = T} \prod_{i=1}^{a} \Big(1 + r_{L(t_i)}(\xi^{t_1+\cdots+t_i})\Big).$$

As in Dubey and Geanakoplos (2006a), the following result shows that if agents borrow money from the bank and do not use all the money to purchase, liquidity trap occurs.

Lemma 4.2 *If $\mu^h(\xi) > 0$ and $r_s(\xi) > 0$, $\Delta(2^h(\xi)) = 0$.*

Proof: Consider at node ξ. Assume that $\mu^h(\xi) > 0$ and $r_s(\xi) > 0$. Let h borrow ϵ less on short-term loan $r_s(\xi)$. This action will leave agent h with $\epsilon r_s(\xi)$ more money after trading in node ξ. Agent h can spend this amount to buy more consumption good in next node, a contradiction. ∎

For each monetary economy $\mathcal{E} = \Big((u^h, e^h, m^h)_{h\in\mathcal{H}}, A, c, M, N\Big)$ we denote $M^b(\xi) := m(\xi_0) + \cdots + m(\xi) + N(\xi^-) + N(\xi) + M(\xi)$. Note that the total money at node ξ is smaller than $M^b(\xi)$ and that $M^b(\xi)$ does not depend on $M(\xi^-)$.

The following result indicates that if the Central Bank injects a quantity of money which is larger than the expected quantity of money in the future, the interest rate of short-term loan is zero. This result is consistent with those in Dubey and Geanakoplos (2006b).

Theorem 4.1 *(**Liquidity Trap Theorem**)*
Consider a monetary economy $\mathcal{E} = \Big((u^h, e^h, m^h)_{h\in\mathcal{H}}, A, c, M, N\Big)$. Consider node ξ. There exists a finite constant B and $T > t(\xi)$ such that : if $\dfrac{M(\xi)}{M^b(\xi')} > B$ for each $\xi' \in D_T \cap D(\xi)$, $r_s(\xi) = 0$.

Proof: Our proof is based on Theorem 3's proof and Lemma 4.2. By market clearing condition, we have $M(\xi) \leq \dfrac{1}{1 + r_s(\xi)} \sum\limits_{h=1}^{H} \mu^h(\xi)$ which implies that there exists h such that $\dfrac{\mu^h(\xi)}{1 + r_s(\xi)} \geq M(\xi)/H$. We will prove that $\Delta(2^h(\xi)) > 0$ when $M(\xi)$ is large enough.

On the one hand, we have

$$\sum_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi) \;\leq\; \sum_{h\in\mathcal{H}}\sum_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi) = \sum_{h\in\mathcal{H}}\sum_{\ell\in\mathcal{L}} p_\ell(\xi) q^h_\ell(\xi) \leq \sum_{\ell\in\mathcal{L}}\sum_{h\in\mathcal{H}} p_\ell(\xi) e_\ell(\xi).$$

Lemma 6.8 implies that $p_\ell(\xi)$ does not exceed a bound which does not depend on $M(\xi)$, but this bound depends on $M^b(\xi')$. Consequently, so is $\sum\limits_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi)$.

On the other hand, we have

$$\sum_{k\in\mathcal{K}} \tilde{\alpha}^h_k(\xi) \;\leq\; \sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}} \tilde{\alpha}^h_k(\xi) \leq \sum_{k\in\mathcal{K}}\sum_{h\in\mathcal{H}} \pi_k(\xi) \alpha^h_k(\xi),$$

By collateral constraints, $\alpha_k^h(\xi)$ are bounded. Lemma 6.9 implies that asset prices are also bounded.[8] Therefore, $\sum_{k\in\mathcal{K}} \tilde{\alpha}_k^h(\xi)$ does not exceed a bound which does not depend on $M(\xi)$. As a consequence, if $M(\xi)$ is large enough, $\Delta(2^h(\xi)) > 0$, therefore $r_s(\xi) = 0$. ■

Definition 4.1 *(***Constant of Liquidity Trap***)*
$A(\xi, T) := \inf\{A : \textit{ if } M(\xi) > AM^b(\xi') \quad \forall \xi' \in D_T \cap D(\xi) \textit{ then } r_s(\xi) = 0\}$ *is called constant of liquidity trap.*

Theorem 4.1 says that this constant exists and is finite. However, another question appears : how can we estimate this constant ? In practice, we face some challenges when we want to estimate this constant. For example, it is hard to know quantity of outside money $m(\xi)$, and then $M^b(\xi^+)$.

5 Conclusion

We have constructed an infinite horizon monetary economy model with incomplete financial market and liquidity constraints. By introducing Sequential (reps., Uniform) Gains to Trade Hypothesis, we proved the existence of equilibrium. At equilibrium, liquidity trap rises at one date if the Central Bank injects a quantity of money which is larger than the expected quantity of money at some date in the future.

It would be interesting in future work to introduce productive sectors into model and to study the impacts of liquidity constraints on strategy of firm as well as on aggregate economic activities.

6 Appendix

6.1 Proof of Theorem

We follows Dubey and Geanakoplos (2003b). For $\epsilon > 0, h \in \mathcal{H}$, define the ambient strategy space of an agent of type h :

$$\Sigma_\epsilon^h = \Big\{(\sigma^h(\xi))_{\xi\in\mathcal{D}} : 0 \leq \sigma^h(\xi) \leq \frac{1}{\epsilon}\Big\}.$$

These spaces are clearly convex and compact.
Given choices $\sigma \in \prod_{h\in\mathcal{H}} \Sigma_\epsilon^h$, define macrovariables $\eta_\epsilon(\sigma) = (r, \pi, p)(\sigma)$ as

8. Note that Lemma 6.9 is based on SGTH (UGTH) and collateral constraints.

follows

$$\frac{1}{1+r_s^\epsilon(\xi)} = \frac{\epsilon + M(\xi) + \sum_h \tilde{\mu}^h(\xi)}{\epsilon + \sum_h \mu^h(\xi)}, \quad \frac{1}{1+r_\ell^\epsilon(\xi)} = \frac{\epsilon + N(\xi) + \sum_h \tilde{\nu}^h(\xi)}{\epsilon + \sum_h \nu^h(\xi)},$$

$$\pi_k^\epsilon(\xi) = \frac{\epsilon + \sum_h \tilde{\alpha}_k^h(\xi)}{\epsilon + \sum_h \alpha_k^h(\xi)}, \quad p_\ell^\epsilon(\xi) = \frac{\epsilon + \sum_{h\in\mathcal{H}} \tilde{q}_\ell^h(\xi)}{\epsilon + \sum_{h\in\mathcal{H}} q_\ell^h(\xi)}.$$

And the delivery is defined

$$d_k^{\epsilon,\sigma}(\xi) = \frac{\epsilon^2}{\epsilon^2 + \sum_h \alpha_k^h(\xi)} \min\left(d_k(\xi), \frac{1}{\epsilon}\right) + \frac{\sum_h \alpha_k^h(\xi)}{\epsilon^2 + \sum_h \alpha_k^h(\xi)} d_k(\xi). \quad \text{(II.24)}$$

The payoff to any player of type $h \in \mathcal{H}$

$$\Pi^h(\sigma, \sigma^h) = u^h(x^h). \quad \text{(II.25)}$$

$\tilde{\Sigma}^h_{\eta_\epsilon(\sigma)}$ is defined in the same manner as $\Sigma^h_{\eta_\epsilon(\sigma)}$, but replacing d^k by $d_k^{\epsilon,\sigma}$

$$\tilde{\Sigma}^h_{\eta_\epsilon(\sigma)} = \Sigma^h_{\eta_\epsilon(\sigma)}(d_k^{\epsilon,\sigma}).$$

Define $\psi : \Sigma_\epsilon \to \Sigma_\epsilon$, where $\Sigma_\epsilon := \prod_{h\in\mathcal{H}} \Sigma_\epsilon^h$, by the follows

$$\psi_\epsilon^h(\sigma) = \underset{\bar{\sigma}^h \in \tilde{\Sigma}^h_{\eta_\epsilon(\sigma)} \cap \Sigma_\epsilon^h}{\arg\max} \Pi^h(\sigma, \bar{\sigma}^h). \quad \text{(II.26)}$$

We see that all the standard assumptions are satisfied, hence there exists an ϵ - collateral monetary equilibrium (i.e. type-symmetric Nash equilibrium for Γ^ϵ) for every $\epsilon > 0$.
We will prove that $\lim_{\epsilon\searrow 0}(\eta_\epsilon(\sigma^\epsilon), \sigma^\epsilon) = (\eta, \sigma)$ is an equilibrium of $\mathcal{E}^T$.

Lemma 6.1 *at every node ξ, the total commodity is uniformly bounded by a constant C which does not depend on the quantity of money for sufficiently small ϵ.*

Proof: Clear. ■

Lemma 6.2 *for every $h \in \mathcal{H}, \xi \in \mathcal{D}^T$; $\mu^{t,\epsilon}(\xi), \nu^{t,\epsilon}(\xi), r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ are bounded for sufficiently small ϵ.*

Proof: Notice that the total amount of money at a node is bounded by some constant D, so $\mu^{h,\epsilon}(\xi), \nu^{h,\epsilon}(\xi) \leq D$ for all small enough ϵ.
The fact that $r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ are bounded from above is easily proved by the boundedness of $\mu^{t,\epsilon}(\xi), \nu^{t,\epsilon}(\xi)$.. ■

Remark 6.1 *The constant D does not depend on T, it only depends on node ξ.*

Lemma 6.3 $r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ *are non negative for sufficiently small ϵ.*

Proof: Clear. ■

Lemma 6.4 *There exists $\underline{p}$ such that $p_\ell^\epsilon(\xi) \geq \underline{p}$ for sufficiently small ϵ.*

Proof: We choose H^* such that $u_\xi^h(0,\dots,0,H^*,0,\dots,0) > U^h(C)$ for all h. By assumption on outside money, there exists $h \in \mathcal{H}$ such that $m_0^{h,\epsilon} \geq \underline{\mathrm{m}}/H$, so we have

$$p_\ell^\epsilon(\xi) \geq \frac{\underline{\mathrm{m}}}{H.H^*}. \tag{II.27}$$

Indeed, if $p_\ell^\epsilon(\xi) < \dfrac{\underline{\mathrm{m}}}{H.H^*}$, let h spend $\underline{\mathrm{m}}/H$ to buy H^* units of commodity l at node ξ, so h obtain a final utility $\geq u^h(0,\dots,0,H^*,0,\dots,0) > u^h(C)$. Contradiction ! ■

Lemma 6.5 *We have $1 + r_\ell^\epsilon(\xi) \geq \min\limits_{\mu\in\xi^+}(1 + r_s^\epsilon(\xi))1 + r_s^\epsilon(\mu)$ at each node ξ.*

Proof: Clear. ■

Lemma 6.6 *At each final node $\xi \in D_T(\xi_0)$, for path $(\xi_0, \xi_1, \dots, \xi)$ we have*

$$\begin{aligned} r_s^\epsilon(\xi_0)M(\xi_0) + r_\ell^\epsilon(\xi_0)N(\xi_0) + \cdots + r_s^\epsilon(\xi^-)M(\xi^-) \\ r_\ell^\epsilon(\xi^-)N(\xi^-) + r_s^\epsilon(\xi)M(\xi) \quad &\leq \quad \hat{m}(\xi) + B\epsilon, \end{aligned}$$

where $\hat{m}(\xi) = \sum\limits_{h=1}^{H}(m^h(\xi_0) + \cdots + m^h(\xi))$, and B is a constant depending on ξ.

Proof: Clear. ■

Denote $\mu_\xi(m, M) := \dfrac{1}{M(\xi)}\big(\hat{m}(\xi) - \min\limits_{\mu\in(\xi^-)^+} \dfrac{\hat{m}(\mu)}{M(\mu) + N(\xi^-)}N(\xi^-)\big)$.

Lemma 6.7 *Given $\xi^- \in D_{T-1}(\xi_0)$, there exists a final node $\xi \in D_T(\xi_0)$ such that*

$$r_s(\xi) \leq \frac{\hat{m}(\xi)}{M(\xi) + N(\xi^-)} \leq \mu_\xi(m, M). \tag{II.28}$$

Proof: Given $\xi^- \in D_{T-1}(\xi_0)$. According to Lemma 6.5, there exists $\xi \in (\xi^-)^+$ such that $1 + r_l^\epsilon(\xi) \geq (1 + r_s^\epsilon(\xi^-))(1 + r_s^\epsilon(\xi))$. Hence $r_l^\epsilon(\xi) \geq r_s^\epsilon(\xi^-) + r_s^\epsilon(\xi)$. Lemma 6.6 implies that

$$r_s^\epsilon(\xi^-)(N(\xi^-) + N(\xi)) + r_s^\epsilon(\xi)(N(\xi^-) + M(\xi)) \leq \hat{m}(\xi) + B\epsilon.$$

Consequently, $r_s^\epsilon(\xi) \leq \dfrac{\hat{m}(\xi) + B\epsilon}{N(\xi^-) + M(\xi)}$.

Let $\epsilon \to 0$, we obtain $r_s^\epsilon(\xi) \leq \dfrac{\hat{m}(\xi)}{N(\xi^-) + M(\xi)}$.

On the other hand, definition of $\mu_\xi(m, M)$ implies that

$$\begin{aligned}
\mu_\xi(m, M) &= \frac{1}{M(\xi)}\big(\hat{m}(\xi) - \min_{\mu \in (\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu) + N(\xi^-)} N(\xi^-)\big) \\
&\geq \frac{1}{M(\xi)}\big(\hat{m}(\xi) - \frac{\hat{m}(\xi)}{M(\xi) + N(\xi^-)} N(\xi^-)\big) \\
&= \frac{\hat{m}(\xi)}{M(\xi) + N(\xi^-)}.
\end{aligned}$$

■

Lemma 6.8 *At final node $\xi \in D_T(\xi_0)$, if $r_s(\xi) \leq \mu_\xi(m, M)$, we have*

$$p_\ell(\mu) < \frac{M^b(\xi)}{a} \max(1, \frac{H^* H}{\underline{e}(\mu)}), \tag{II.29}$$

for each μ in the path $(\xi_0, \xi_1, \ldots, \xi)$, and $\ell \in \mathcal{L}$, where $\underline{e}(\mu) := \min_{\ell \in \mathcal{L}} \underline{e}_\ell(\mu)$.

Proof: Consider a final node $\xi \in D_T(\xi_0)$. Suppose that $r_s(\xi) \leq \mu_\xi(m, M)$. Assume for each $\ell \in \mathcal{L}$, $p_\ell(\xi) \geq \dfrac{M^b(\xi)}{a}$ then $a \geq \dfrac{M^b(\xi)}{p_\ell(\xi)} \geq \dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi)$.

If $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) < -a$, we have $q_\ell^h(\xi) > a$, so $p_\ell(\mu) < \dfrac{M^b(\xi)}{a}$, contradiction!

If $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) \geq -a$, we imply $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) \in [-a, a]$. Consequently,

$$|x_\ell^h(\xi) - e_\ell^h(\xi) - \sum_k c_\ell^k \alpha_k^h(\xi^-)| = |\frac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi)| \leq a.$$

It means that $x = (x^1, \ldots, x^h) \in \left(\mathbb{R}_+^{L \times D^T}\right)^H$ belongs to $X^a(\xi)$.

Uniform Gains to Trade Hypothesis implies that $\gamma_\xi(x) > \mu_\xi(m, M)$. Combining with Lemma 6.7, we get $\gamma_\xi(x) > r_s(\xi)$. By definition of $\gamma_\xi(x)$, there exists $\gamma > r_s(\xi)$ such that x is not γ - Pareto optimal at node ξ. Therefore, there exists $\tau(\xi) = (\tau^h(\xi))_{h\in\mathcal{H}} \in \mathbb{R}^{L\times H}$ such that

$$\tau^h(\xi) \neq 0, \quad \text{for all } h \in \mathcal{H}, \text{ and } \sum_{h=1}^{H} \tau^h(\xi) = 0, \tag{II.30}$$

$$x^h(\xi) + \tau^h(\xi) \in \mathbb{R}^L_+, \quad \text{for all } h \in \mathcal{H}, \tag{II.31}$$

$$U^h(\bar{x}^h(\gamma, \tau^h(\xi))) > U^h(x^h), \text{ for all } h \in \mathcal{H}. \tag{II.32}$$

Since $\sum\limits_{h=1}^{H} \tau^h(\xi) = 0$, there exists $i \in \mathcal{H}$ such that $p(\xi)\tau^i(\xi) \leq 0$. Without the generality, we can assume that $\tau^i(\xi) = (\tau^i_1(\xi), \ldots, \tau^i_m(\xi), -\tau^i_{m+1}(\xi), \ldots, -\tau^i_L(\xi))$, with $\tau^i_\ell(\xi) \geq 0$. We have $\sum\limits_{\ell=1}^{m} p_\ell(\xi)\tau^i_\ell(\xi) \leq \sum\limits_{\ell=m+1}^{L} p_\ell(\xi)\tau^i_\ell(\xi)$.

We construct a new strategy $(\hat{\sigma}^i(\mu))_{\mu\in\mathcal{D}^T}$ of agent i as follows : $\hat{\sigma}^i(\mu) = \sigma^i(\mu), \quad \forall \mu \neq \xi$ and at node ξ

$$\begin{aligned}
\hat{\mu}^i(\xi) &:= \mu^i_(\xi) + \sum_{\ell=1}^{m} p_\ell(\xi)\tau^i_\ell(\xi), \\
\hat{\tilde{q}}^i_\ell(\xi) &:= \tilde{q}^i_\ell(\xi) + p_\ell(\xi)\frac{\tau^i_\ell(\xi)}{1+\gamma}, \quad \forall \ell = 1, \ldots, m \\
\hat{q}^i_\ell(\xi) &:= q^i_\ell(\xi) + \tau^i_\ell(\xi), \quad \forall \ell = m+1, \ldots, L.
\end{aligned}$$

Since $x^i(\xi) + \tau^i(\xi) \in \mathbb{R}^L_+$, this new strategy satisfies the physical constraint $(pc)^h(\xi)$. Thanks to $\gamma^a_\xi(x) > r_s(\xi)$, we obtain $\dfrac{1}{1+\gamma} < \dfrac{1}{1+r_s(\xi)}$, hence liquidity constraint $(2^i(\xi))$ is hold. Liquidity constraint $(4^i(\xi))$ is satisfied because $\sum\limits_{\ell=1}^{m} p_\ell(\xi)\tau^i_\ell(\xi) \leq \sum\limits_{\ell=m+1}^{L} p_\ell(\xi)\tau^i_\ell(\xi)$.

On the other hand $U^i(\bar{x}^i(\gamma, \tau^i(\xi))) > U^i(x^i)$, contradiction to the optimality of σ^i.

We now consider a node μ in the path $(\xi_0, \xi_1, \ldots, \xi)$. Let $k \in \mathcal{L}$. Since $\sum\limits_{h=1}^{H} e^h_k(\mu) \geq \underline{e}(\mu)$, there exists h such that $e^h_k(\xi) \geq \underline{e}(\mu)/H$.
If $\dfrac{p_k(\mu)}{p_\ell(\xi)} > \dfrac{2H^*H}{\underline{e}(\mu)}$. Let h do nothing, just sell $\underline{e}(\mu)/(2H)$ units of commodity k, obtain $p_k(\mu)\underline{e}(\mu)/(2H)$ dollars. Hence h can buy at leastH^* units of commodity ℓ, contradiction.

Therefore, $\dfrac{p_k(\mu)}{p_\ell(\xi)} \leq \dfrac{2H^*H}{\underline{e}(\mu)}$, so we obtain (II.29) ■

Lemma 6.9 *Price of financial asset k at node ξ : $\pi_k(\xi)$ is bounded from above if commodity prices $p_\ell(\xi)$ are bounded from above.*

Proof: Choose h with $m_0^h > \underline{m}/h$. Let h uses m_0^h to buy a vector of commodities $(\frac{m_0^h}{p_\ell(\xi)L})_{\ell\in\mathcal{L}}$, at initial node in order to use these commodities as collateral at node ξ. So at note ξ, h can sell at least $w = \min_{\ell: c_\ell^k>0}\left\{\frac{m_0^h}{\bar{p}c_\ell^k L}\right\} > 0$ units of asset k and obtain $w\pi_k^\epsilon(\xi)$ dollars. Of course, $w\pi_k^\epsilon(\xi) \le \bar{p}H^*$. Thus $\pi_k^\epsilon(\xi) \le \bar{p}H^*/w$. ∎

Lemma 6.10 *Let ξ as in Lemma 6.7, we have*

$$\begin{aligned} r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) + \cdots + r_s(\xi^-)M(\xi^-) & \\ + r_\ell(\xi^-)N(\xi^-) + r_s(\xi)M(\xi) &= \hat{m}(\xi). \end{aligned}$$

Proof: Similarly Claim V in Dubey and Geanakoplos (2003b). ∎

Lemma 6.11 *At each final node ξ, we have $r_s(\xi) \le \mu_s(m, M)$*

Proof: Assume there is a final node ξ at which $r_s(\xi) > \mu_s(m, M)$. Then

$$M(\xi)r_s(\xi) + \min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) > \hat{m}(\xi).$$

Lemma 6.6 implies that

$$\begin{aligned} r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) + \cdots + r_s(\xi^-)M(\xi^-) & \\ + r_\ell(\xi^-)N(\xi^-) + r_s(\xi)M(\xi) &\le \hat{m}(\xi). \end{aligned}$$

Hence

$$\begin{aligned} \min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) &> r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) \\ &+ \cdots + r_s(\xi^-)M(\xi^-) + r_\ell(\xi^-)N(\xi^-). \end{aligned}$$

Let $\mu \in (\xi^-)^+$ be as in Lemma 6.7, i.e, $r_s(\mu) \le \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}$ then

$$\begin{aligned} &\min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + r_s(\mu)M(\mu) \\ \le\ & \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + r_s(\mu)M(\mu) \\ \le\ & \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}M(\mu) = \hat{m}(\mu). \end{aligned}$$

Consequently, we get

$$\begin{aligned} \hat{m}(\mu) &> r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) \\ &+ \cdots + r_s(\xi^-)M(\xi^-) + r_\ell(\xi^-)N(\xi^-) + r_s(\mu)M(\mu), \end{aligned}$$

contradiction to Lemma 6.10. ∎

Lemma 6.12 *All prices are bounded from above.*

Proof: This is a direct sequence of the above results. ■

Lemma 6.13 $d_k^{\epsilon,\sigma}(\xi,\eta) = d_k(\xi,\eta)$ *for sufficiently small* ϵ.

Proof: By collateral constraints, $d_k(\xi)$ is bounded. Consequently we get $d_k^{\epsilon,\sigma}(\xi,\eta) = d_k(\xi,\eta)$ for sufficiently small ϵ. ■

Lemma 6.14 *We have* $\alpha_k^h(\xi)$ *is bounded from above.*

Proof: By collateral constraints, we get

$$\underline{c}\sum_k \alpha_k^{h,\epsilon}(\xi) \le \sum_k \big[\sum_\ell c_\ell^k\big]\alpha_k^{h,\epsilon}(\xi) \le \sum_\ell e_\ell^h(\xi).$$

■

6.2 Proof of Theorem 3.2

We use the same method in the proof of Theorem 6.1. However, in order to prove that all prices are bounded when $\epsilon \to 0$, we use Sequential Gains to Trade Hypothesis.
At each node $\xi \in D^T(\xi_0)$, consider path $(\xi_0, \xi_1, \ldots, \xi)$. Let denote $\tilde{m}(\xi) := \sum_{h=1}^{h} \tilde{m}^h(\xi)$ be the total stock of money unspent at the end of node ξ. Then we have

$$\begin{aligned}
\tilde{m}(\xi) &:= \tilde{m}(\xi^-) + m(\xi) + N(\xi) + M(\xi) \\
&\quad -(1 + r_\ell(\xi^-))N(\xi^-) - (1 + r_s(\xi))M(\xi) \\
&= \cdots \\
&= \hat{m}(\xi) + N(\xi) - r_s(\xi)M(\xi) \\
&\quad -r_s(\xi^-)M(\xi^-) - r_\ell(\xi^-)N(\xi^-) - \cdots - r_s(\xi_0)M(\xi_0) - r_\ell(\xi_0)N(\xi_0)
\end{aligned}$$

Since markets clear, we have

$$\begin{aligned}
r_s(\xi)M(\xi) &\le \sum_h \mu^h(\xi) - M(\xi) \\
&\le m(\xi) + \tilde{m}(\xi^-) - M(\xi) \\
&\le \hat{m}(\xi) + N(\xi) - M(\xi).
\end{aligned}$$

Therefore, $r_s(\xi) \le \dfrac{\hat{m}(\xi) + N(\xi) - M(\xi)}{M(\xi)} < \gamma_\xi(x)$ for all $x \in X^{a(\xi)}(\xi)$. By using the same argument in Lemma 6.8, we obtain that all commodity prices are bounded from above, so are financial asset prices.

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Chapitre III

Intertemporal equilibrium with production : bubbles and efficiency

Abstract : We consider a general equilibrium model with heterogeneous agents, borrowing constraints, and exogenous labor supply. First, the existence of intertemporal equilibrium is proved even if the aggregate capitals are not uniformly bounded above and the production functions are not time invariant. Second, (i) we say that, at an equilibrium, there is a physical capital bubble if the fundamental value of physical capital is lower than its market price at this equilibrium, (ii) We say that an equilibrium has low interest rates if the sum (over time) of capital returns is finite. We show that there is a physical capital bubble at an equilibrium if and only if this equilibrium has low interest rates. Last, we prove that with linear technologies, every intertemporal equilibrium is efficient. Moreover, there is a room for both efficiency and bubble at equilibrium.

Keywords : Intertemporal equilibrium, physical capital bubble, efficiency, infinite horizon.
JEL Classifications : C62, D31, D91, G10

1 Introduction

Following Becker, Bosi, Le Van and Seegmuller (2014), we consider a dynamic general equilibrium model with heterogeneous agents. However, our framework is different from their model in three points : (i) for simplicity, we consider exogenous labor suply, (ii) our technology is not stationary, (iii) aggregate capital stock is not necessarily uniformly bounded from above. Heterogeneous agents decide to invest and consume. If they invest in physical capital, this asset will not only give them return in term of consumption good at the next period but also give back a fraction of the same asset (after being depreciated). Agents cannot borrow.

Our first contribution is to prove the existence of intertemporal general equilibrium. To do so, we firstly prove the existence of equilibrium for each $T-$truncated economy. Hence, we have a sequence of equilibria which depend on T. We then prove that this sequence has a limit (for the product topology) which is an equilibrium for the infinite horizon economy.

We say that a physical capital bubble (for short, *bubble*) occurs at equilibrium if the market price of the physical asset is greater than its fundamental value. We say that interest rates are low at equilibrium (for short, *low interest rates*) if the sum of capital returns is finite. Our second contribution is to prove that bubble is equivalent to low interest rates.

The no-bubble result in Becker, Bosi, Le Van and Seegmuller (2014) can be viewed as a particular case of our result. Indeed, in Becker, Bosi, Le Van and Seegmuller (2014), the aggregate capital stock is uniformly bounded, and then real return of the physical capital is uniformly bounded away from zero. Therefore, the sum of returns equals infinity. According to our result, the physical capital bubble is ruled out.

However, when we allow for non-stationary production functions, there may be a bubble at equilibrium. To see the point, take linear production functions whose productivity at date t is denoted by a_t. At equilibrium, real return of physical capital at date t must be a_t. As mentioned above, there is a bubble if and only if $\sum\limits_{t=0}^{\infty} a_t < \infty$. We can now see clearly that there is a bubble if productivities decrease with sufficiently high speed.

Our third contribution is about the efficiency of intertemporal equilibrium. An intertemporal equilibrium is called efficient if its aggregate capital path is efficient in sense of Malinvaud (1953). We prove that with linear production functions, every intertemporal equilibrium is efficient. However, as we mentioned above, this efficient intertemporal equilibrium may have bubble if productivities decrease with sufficiently high speed. Therefore, we have both efficient and bubble at equilibrium with such technologies. Note that our result does not require any conditions about the convergence or boundedness of the capital path as in previous literature.

Related literature

(1) *On bubbles.* Tirole (1982) proved that there is no financial asset bubble in a rational expectation model without endowment. A survey on bubble in models with asymmetric information, overlapping generation, heterogeneous-beliefs can be found in Brunnermeier and Oehmke (2012). Doblas-Madrid (2012) presented a model of speculative bubbles where rational agents buy an overvalued asset because given their private information, they believe they have a good chance of reselling at a profit to a greater fool. Martin and Ventura (1953), Ventura (2012) did not define bubble as we do. Instead, they defined bubble as a short-lived asset.

(2) *On the efficiency of a capital path.* Malinvaud (1953) introduced the concept of efficiency of a capital path and gave a sufficient condition for the efficiency : $\lim_{t\to\infty} P_t K_t = 0$, where (P_t) is a sequence of competitive prices, (K_t) is the capital path.[1] Following Malinvaud, Cass (1972) considered capital path which is uniformly bounded from below. Under the concavity of a stationary production function and some mild conditions, he proved that a capital path is inefficient if and only if the sum (over time) of future values of a unit of physical capital is finite. Cass and Yaari (1971) gave a necessary and sufficient condition for a consumption plan (C) to be efficient : the inferior limit of differences between the present value of any consumption plan and the plan (C) is negative.

Our paper is also related to Becker and Mitra (2012) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. However, their result is based on the fact that the consumption of each household is uniformly bounded from below. In our paper, we do not need this condition. Instead, the efficient capital path in our model may converge to zero. Mitra and Ray (2012) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid. However, their results are no longer valid without the convergence or the boundedness of capital paths.

(3) Another concept of efficiency is *constrained efficiency.* Constrained inefficiency occurs when there exists a welfare improving feasible redistribution subject to constraints (these constraints depends on models). About the constrained efficiency in general equilibrium models with financial asset, see Kehoe and Levine (1993), Alvarez and Jermann (2000), Bloise and Pietro (2011). About the constrained efficiency in the neoclassical growth model, see Davila, Hong, Krusell and Rios-Rull (2012).

The remainder of the paper is organized as follows. Section 2 describes the model. In section 3, existence of equilibrium is proved. Section 4 studies physical capital bubble. Section 5 explores our results on the efficiency of

1. See Malinvaud (1953), Lemma 5, page 248.

equilibria. Conclusion will be presented in Section 6. Technical details are gathered in Appendix.

2 Model

We follow Becker, Bosi, Le Van and Seegmuller (2014), but we consider : (i) exogenous labor supply, (ii) non-stationary production functions.

Consumption good : at each period $t = 0, 1, 2, \ldots, \infty$, the price of consumption good is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Physical capital : at time t, if agent i decides to buy $k_{i,t+1} \geq 0$ units of new capital, then at period $t+1$, after being depreciated, agent i will receive $(1-\delta)k_{i,t+1}$ units of old capital and a return on capital $k_{i,t+1}$ at the rate r_{t+1}. Here, δ is the capital depreciation rate.

Each household i takes the sequence of prices and capital returns $(p, r) = (p_t, r_t)_{t=0}^{\infty}$ as given and solves

$$(P_i(p,r)) : \quad \max_{\left((c_{i,t},k_{i,t+1})_{i=1}^m\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\right] \tag{III.1}$$

$$\text{subject to} : \quad k_{i,t+1} \geq 0 \tag{III.2}$$

$$p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t(p_t, r_t), \tag{III.3}$$

where $(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^{m} \theta^i = 1$.

Firm : For each period, there is a representative firm which takes prices (p_t, r_t) as given, and maximizes its profit.

$$(P(r_t)) : \quad \pi_t(p_t, r_t) := \max_{K_t \geq 0} \left[p_t F_t(K_t) - r_t K_t\right]$$

We write π_t instead of $\pi_t(p_t, r_t)$ if there is no confusion.

Definition 2.1 *A sequence of prices and quantities* $\left(\bar{p}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1})_{i=1}^m, \bar{K}_t\right)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\mathcal{E} = \left((u_i, \beta_i, k_{i,0}, \theta_i)_{i=1}^m, F,\right)$ *if the following holds.*

(i) Price positivity : $\bar{p}_t, \bar{r}_t > 0$ *for* $t \geq 0$.

(ii) All markets clear : at each $t \geq 0$,

$$\textit{consumption good} \quad : \quad \sum_{i=1}^{m} [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F_t(\bar{K}_t) \tag{III.4}$$

$$\textit{physical capital} \quad : \quad \bar{K}_t = \sum_{i=1}^{m} \bar{k}_{i,t}. \tag{III.5}$$

(iii) Optimal consumption plans : for each i, $\left((\bar{c}_{i,t}, \bar{k}_{i,t+1})_{i=1}^{m}\right)_{t=0}^{\infty}$ is a solution to problem $(P_i(\bar{p}, \bar{r}))$.
(iv) Optimal production plan : for each $t \geq 0$, $(\bar{K}_t)$ is a solution to problem $(P(\bar{r}_t))$.

3 The existence of equilibrium

The following result proves that the feasible aggregate capital and the feasible consumption are bounded from above for the product topology.

Lemma 3.1 *Feasible individual and aggregate capitals and feasible consumptions are in a compact set for the product topology. Moreover, they are uniformly bounded if there exists t_0 and an increasing, concave function G such that : (i) for every $t \geq t_0$ we have $F_t(K) \leq G(K)$ for every K, (ii) there exists $x > 0$ such that $G(y) + (1-\delta)y \leq y$ for every $y \geq x$.*

Proof: Denote

$$\begin{aligned} D_0 := D_0(F_0, \delta, K_0) &:= F_0(K_0) + (1-\delta)K_0, \\ D_t := D_t((F_s)_{s=0}^{t}, \delta, K_0) &:= F_t(D_{t-1}((F_s)_{s=0}^{t-1}, \delta, K_0)) \\ &\quad +(1-\delta)D_{t-1}((F_s)_{s=0}^{t-1}, \delta, K_0),\ \forall t \geq 0. \end{aligned}$$

Then $\sum_{i=1}^{m} c_{i,t} + K_{t+1} \leq D_t$ for every $t \geq 0$.

We now assume t_0 and the function G (as in Lemma 3.1) exist. We are going to prove that $0 \leq K_t \leq \max\{D_0, ..., D_{t_0-1}, x\} =: M$. Indeed, $K_t \leq D_{t-1} \leq M$ for every $t \leq t_0$. For $t \geq t_0$, we have

$$K_{t+1} = \sum_{i=1}^{m} k_{i,t+1} \quad \leq \quad G(K_t) + (1-\delta)K_t.$$

Then $K_{t_0+1} \leq G(K_{t_0}) + (1-\delta)K_{t_0} \leq G(M) + (1-\delta)M \leq M$. Iterating the argument, we obtain $K_t \leq M$ for each $t \geq 0$.
Feasible consumptions are bounded because $\sum_{i=1}^{m} c_{i,t} \leq G(K_t) + (1-\delta)K_t$. ■

We need the following assumptions.

Assumption (H1) : For each i, the utility function u_i of agent i is strictly increasing, strictly concave, continuously differentiable, and $u(0) = 0, u'(0) = \infty$.

Assumption (H2) : $F_t(\cdot)$ is continuously differentiable, strictly increasing, concave, the input is essential ($F_t(0) = 0$) and $F_t(\infty) = \infty$.

Assumption (H3) : $\delta \in (0,1)$ and $k_{i,0} > 0$ for every i.[2]

Assumption (H4) : For each i, the utility of agent i is finite

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t) < \infty.$$

3.1 Existence of equilibrium in $\mathcal{E}^T$

We define a $T-$ truncated economy $\mathcal{E}^T$ as the econonomy obtained from $\mathcal{E}$ by imposing that there are no activities from period $T+1$ to infinity, i.e., $c_{i,t} = k_{i,t} = 0$ for every $i = 1, \dots, m$, and for every $t \geq T+1$.

In the economy $\mathcal{E}^T$, agent i takes the sequence of prices $(p,r) = (p_t, r_t)_{t=0}^T$ as given and maximizes her intertemporal utility by choosing consumption and investment levels.

$$\begin{array}{rl}
(P_i(p,r)) : & \max\limits_{(c_{i,t},k_{i,t+1})_{t=0}^T} \left[\sum\limits_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \right] \\
\text{subject to :} & k_{i,t+1} \geq 0, \\
\text{(budget constraints)} & p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t,
\end{array}$$

where $k_{i,T} = 0$.

We then define the bounded economy $\mathcal{E}_b^T$ as obtained from $\mathcal{E}^T$ by assuming all variables are bounded in the following compact sets :

$$\begin{array}{rcl}
(c_{i,t})_{t=0}^T & \in & \mathcal{C}_i := [0, B_c]^{T+1} \\
(k_{i,t})_{t=1}^{T+1} & \in & \mathcal{K}_i := [0, B_k]^T \\
K := (K_t)_{t=1}^{T+1} & \in & \mathcal{K} := [0, B]^T,
\end{array}$$

where $B_c > \max\limits_t F_t(B) + (1-\delta)B$, $B > mB_k$.

Proposition 3.1 *Under Assumptions* $(H1)-(H3)$, *there exists an equilibrium for* $\mathcal{E}_b^T$.

Proof: See Appendix. ■

Proposition 3.2 *An equilibrium of the economy* $\mathcal{E}_b^T$ *is also an equilibrium of the unbounded economy* $\mathcal{E}^T$.

Proof: Similar to the one in Becker, Bosi, Le Van and Seegmuller (2014). ■

2. Becker, Bosi, Le Van and Seegmuller (3) weekens H3 by assuming $\sum_{i=1}^m k_{i,0} > 0$ because they assume that every agent has 1 unit of labor.

3.2 Existence of equilibrium in $\mathcal{E}$

Theorem 3.1 *Under Assumptions (H1)-(H4), there exists an equilibrium.*

Proof of Theorem 3.1:

We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. We denote by $(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T)$ an equilibrium of $T-$ truncated economy $\mathcal{E}^T$. We can normalize by setting $\bar{p}_t^T + \bar{r}_t^T = 1$ for every $t \leq T$. We see that

$$0 < \bar{c}_{i,t}^T, \bar{K}_t^T \quad \leq \quad D_t.$$

Without loss of generality, we can assume that

$$(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T) \xrightarrow{T\to\infty} (\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})$$

for the product topology.

We are going to prove that : (i) all markets clear, (ii) at each date t, $\bar{K}_t$ is a solution to the firm's maximization problem, (iii) $\bar{r}_t > 0$ for each $t \geq 0$, (iv) $(\bar{c}_i, \bar{k}_i)$ is a solution to the maximization problem of agent i for each $i = 1, \ldots, m$, (v) $\bar{p}_t > 0$ for each t. Consequently, we obtain that $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})$ is an equilibrium for the economy $\mathcal{E}$.

(i) By taking the limit of market clearing conditions for the truncated economy, we obtain the market clearing conditions for the economy $\mathcal{E}$.

(ii) Take $K \geq 0$ arbitrary. We have $\bar{p}_t^T F_t(K) - \bar{r}_t^T K \leq \bar{p}_t^T F_t(\bar{K}_t^T) - \bar{r}_t^T \bar{K}_t^T$. Let T tend to infinity, we obtain that $\bar{p}_t F_t(K) - \bar{r}_t K \leq \bar{p}_t F_t(\bar{K}_t) - \bar{r}_t \bar{K}_t$. Therefore, the optimality of $\bar{K}_t$ is proved.

(iii) If $\bar{r}_t = 0$ then $\bar{p}_t = 1$ (since $\bar{r}_t^T + \bar{p}_t^T = 1$). The optimality of $\bar{K}_t$ implies that $\bar{K}_t = \infty$. This is a contradiction, because we have $\bar{K}_t = \lim\limits_{T\to\infty} \bar{K}_t^T \leq D_t < \infty$.

(iv) First, we give some notations. For each i and t, we define $B_i^T(\bar{p}, \bar{r})$ and $C_i^T(\bar{p}, \bar{r})$ as follows

$$\begin{aligned} B_i^T(\bar{p}, \bar{r}) := \Big\{ & (c_{i,t}, k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1} = 0, \text{(b) } \forall t = 0, \ldots, T, \\ & k_{i,t+1} > 0, \quad \bar{p}_t[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] < \bar{r}_t k_{i,t} + \theta^i \pi_t(\bar{p}_t, \bar{r}_t) \Big\}, \\ C_i^T(\bar{p}, \bar{r}) := \Big\{ & (c_{i,t}, k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1} = 0, \text{(b) } \forall t = 0, \ldots, T, \\ & k_{i,t+1} \geq 0, \quad \bar{p}_t[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] \leq \bar{r}_t k_{i,t} + \theta^i \pi_t(\bar{p}_t, \bar{r}_t) \Big\}. \end{aligned}$$

Since $\bar{r}_t > 0$ for every t, it is easy to prove that $B_i^T(\bar{p}, \bar{r}) \neq \emptyset$.

Let (c_i, k_i) be a feasible allocation of the problem $P_i(\bar{p}, \bar{r})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

We define $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ as follows : $c'_{i,t} = c_{i,t}$ for every $t \leq T$, $= 0$ if $t > T$; $k'_{i,t+1} = k_{i,t+1}$ for every $t \leq T-1$, $= 0$ if $t \geq 0$. We see that $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ belongs to $C_i^T(\bar{p}, \bar{r})$. Since $B_i^T(\bar{p}, \bar{r}) \neq \emptyset$, there exists a sequence $\left((c^n_{i,t}, k^n_{i,t+1})_{t=0}^T\right)_{n=0}^{\infty} \in B_i^T(\bar{p}, \bar{r})$ with $k^n_{i,T+1} = 0$, and this sequence converges to $(c'_{i,t}, k'_{i,t+1})_{t=0}^T$ when n tends to infinity. We have

$$\bar{p}_t(c^n_{i,t} + k^n_{i,t+1} - (1-\delta)k^n_{i,t}) < \bar{r}_t k^n_{i,t} + \theta^i \pi_t(\bar{p}_t, \bar{r}_t)$$

We can chose $s_0 > T$, high enough, such that : for every $s \geq s_0$, we have

$$\bar{p}^s_t(c^n_{i,t} + k^n_{i,t+1} - (1-\delta)k^n_{i,t}) < \bar{r}^s_t k^n_{i,t} + \theta^i \pi_t(\bar{p}^s_t, \bar{r}^s_t).$$

It means that $(c^n_{i,t}, k^n_{i,t+1})_{t=0}^T \in C_i^T(\bar{p}^s, \bar{r}^s)$. Therefore, we get $\sum_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum_{t=0}^{s} \beta_i^t u_i(\bar{c}^s_{i,t})$. Let s tend to infinity, we obtain $\sum_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Let n tends to infinity, we have $\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$ for every T.

Let T tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

(v) p_t is strictly positive thanks to the strict increasingness of the utility functions.

■

4 Physical asset bubble

Let $\left(p_t, r_t, (c_{i,t}, k_{i,t})_{i=1}^m, K_t\right)_{t=0}^{+\infty}$ be an equilibrium.

Lemma 4.1 *For each t, we have*

$$1 = (1 - \delta + \frac{r_{t+1}}{p_{t+1}})\gamma_{t+1} \tag{III.6}$$

where $\gamma_{t+1} := \max_{i \in \{1,\ldots,m\}} \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}$.

Proof: Firstly, we write all FOCs for the economy $\mathcal{E}$. Denote by $\lambda_{i,t}$ the multiplier with respect to the budget constraint of agent i and by μ_{t+1} the multiplier with respect to the borrowing constraint (i.e., $k^T_{i,t+1} \geq 0$) of agent i.

$$\begin{aligned}\beta_i^t u_i'(c_{i,t}) &= \lambda_{i,t}p_t\\ \lambda_{i,t}p_t &= \lambda_{i,t+1}(r_{t+1}+p_{t+1}(1-\delta))+\mu_{i,t+1}\\ \mu_{i,t+1}k_{i,t+1} &= 0.\end{aligned}$$

Therefore, we have $\dfrac{p_{t+1}}{r_{t+1}+p_{t+1}(1-\delta)} \geq \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$ for every i.
Since $K_t > 0$ at equilibrium, there exists i such that $k_{i,t+1} > 0$. For such agent, we have $\mu_{i,t+1} = 0$. Thus, $\lambda_{i,t}p_t = \lambda_{i,t+1}(r_{t+1}+p_{t+1}(1-\delta))$. Consequently, we get (III.6) ■

Definition 4.1 *We define the discount factor of the economy from initial date to date t as follows*

$$Q_0 := 1, \quad Q_t := \prod_{s=1}^{t} \gamma_s, \quad t \geq 1. \tag{III.7}$$

According to Lemma 4.1, we have $Q_t = (1-\delta+\frac{r_{t+1}}{p_{t+1}})Q_{t+1}$ for every $t \geq 0$. As a consequence, we can write

$$\begin{aligned}1 &= (1-\delta+\frac{r_1}{p_1})Q_1 = (1-\delta)Q_1 + \frac{r_1}{p_1}Q_1\\ &= (1-\delta)(1-\delta+\frac{r_2}{p_2})Q_2 + \frac{r_1}{p_1}Q_1 = (1-\delta)^2 Q_2 + (1-\delta)\frac{r_2}{p_2}Q_2 + \frac{r_1}{p_1}Q_1\\ &= \cdots\\ &= (1-\delta)^T Q_T + \sum_{t=1}^{T}(1-\delta)^{t-1}\frac{r_t}{p_t}Q_t. \end{aligned}\tag{III.8}$$

Interpretation. In this model, physical capital is viewed as a long-lived asset.

1. At date 1, one unit (from date 0) of this asset will give $(1-\delta)$ units of physical capital and $\frac{r_1}{p_1}$ units of consumption good as its dividend.
2. At date 2, $(1-\delta)$ units of physical capital will give $(1-\delta)^2$ units of physical capital and $(1-\delta)\frac{r_2}{p_2}$ units of consumption good ...

Therefore, the fundamental value of physical capital at date 0 can be defined by

$$FV_0 = \sum_{t=1}^{\infty}(1-\delta)^{t-1}\frac{r_t}{p_t}Q_t.$$

Definition 4.2 *We say that there is a capital asset bubble if physical capital's price is greater that its fundamental value, i.e.,* $1 > \sum_{t=1}^{\infty}(1-\delta)^{t-1}\frac{r_t}{p_t}Q_t$.

From (III.8), we can see that there is a bubble on capital asset if and only if $\lim_{t\to\infty}(1-\delta)^t Q_t > 0$.

Definition 4.3 *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty} \frac{r_t}{p_t} < \infty. \tag{III.9}$$

Otherwise, we say that interest rates are high.

We now state our main result in this section.

Proposition 4.1 *There is a bubble if and only if interest rates are low*

Proof: According to (III.6), we see that $Q_t = (1-\delta+\frac{r_{t+1}}{p_{t+1}})Q_{t+1}$. Hence, we have

$$\begin{aligned} 1 &= (1-\delta+\frac{r_1}{p_1})Q_1 = (1-\delta+\frac{r_1}{p_1})(1-\delta+\frac{r_2}{p_2})Q_2 \\ &= \ldots = Q_T\prod_{t=1}^{T}(1-\delta+\frac{r_t}{p_t}) = Q_T(1-\delta)^T\prod_{t=1}^{T}[1+\frac{r_t}{(1-\delta)p_t}]. \end{aligned}$$

Consequently, $\lim_{t\to\infty}(1-\delta)^t Q_t > 0$ if and only if $\prod_{t=1}^{\infty}[1+\frac{r_t}{(1-\delta)p_t}] < +\infty$. This condition is equivalent to

$$\sum_{t=1}^{\infty} \frac{r_t}{p_t} < \infty. \tag{III.10}$$

It means that interest rates are low. ■

We point out some consequences of Proposition 4.1.

Corollary 4.1 *Assume that $F_t = F$ for every t, F is strictly increasing, strictly concave. Then there is no bubble at equilibrium.*

Proof: Case 1 : $F'(\infty) \geq \delta$. Therefore, we have $\frac{r_t}{p_t} \geq F'(K_t) \geq \delta$ for every t. As a result, $\sum_{t=1}^{\infty}\frac{r_t}{p_t} = \infty$ which implies that bubble is ruled out.

Case 2 : $F'(\infty) < \delta$. Since F is strictly increasing and strictly concave, aggregate capital stock is uniformly bounded, i.e., there exists $0 < K < \infty$ such that $K_t \leq K$. Consequently, $\frac{r_t}{p_t} = F'(K_t) > F'(K) > 0$ for every t. This implies that $\sum_{t=1}^{\infty}\frac{r_t}{p_t} = \infty$. According to Proposition 4.1, there is no bubble. ■

Note that we do not require any condition on $F'(\infty)$ in Corollary 4.1. In Becker, Bosi, Le Van, Seegmuller (2014), they work with a endogenous labor supply model and assume that $\frac{\partial F}{\partial K}(\infty, m) = \frac{\partial F}{\partial L}(1,\infty) = 0$.

Corollary 4.2 *Assume that $F_t(K) = a_t K$ for each t. Then there is a bubble at equilibrium if and only if* $\sum_{t=1}^{\infty} a_t < \infty$.

Proof: This is a direct consequence of Proposition 4.1 . ■

This result shows that if the productivity decreases to zero with high speed, a bubble in physical capital will appear.

5 On the efficiency of equilibria

In this section, we study the efficiency of intertemporal equilibrium. Following Malinvaud (1953), we define the efficiency of a capital path as follows.

Definition 5.1 *Let F_t be a production function, δ be the capital depreciation rate. A feasible path of capital is a positive sequence $(K_t)_{t=0}^{\infty}$ such that $0 \leq K_{t+1} \leq F_t(K_t) + (1-\delta)K_t$ for every $t \geq 0$ and K_0 is given.*
A feasible path is efficient if there is no other feasible path (K'_t) such that

$$F_t(K'_t) + (1-\delta)K'_t - K'_{t+1} \geq F_t(K_t) + (1-\delta)K_t - K_{t+1}$$

for every t with strict inequality for some t.

Here, aggregate feasible consumption at date t is defined by $C_t := F_t(K_t) + (1-\delta)K_t - K_{t+1}$.

Definition 5.2 *We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path (K_t) is efficient.*

Our main result in this section requires some intermediate steps. First, we have, as in (Malinvaud (1953)).

Lemma 5.1 *An equilibrium is efficient if* $\lim_{t\to\infty} Q_t K_{t+1} = 0$.

Proof: Let (K'_t, C'_t) be a feasible sequence. We have just to show that

$$\liminf_{T\to+\infty} \sum_{t=0}^{T} Q_t\left(C_t - C'_t\right) \geq 0. \tag{III.11}$$

It is enough to prove that feasibility and first-order conditions imply

$$\sum_{t=0}^{T} Q_t\left(C_t - C'_t\right) \geq -Q_T K_{T+1} \tag{III.12}$$

Let us prove inequality (III.12). We have

$$
\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^{T} Q_t\left(C_t - C_t'\right) \\
&= \sum_{t=0}^{T} Q_t\left[F_t\left(K_t\right) - F_t\left(K_t'\right) + (1-\delta)\left(K_t - K_t'\right) - \left(K_{t+1} - K_{t+1}'\right)\right] \\
&\geq \sum_{t=0}^{T} Q_t\left[F_t'\left(K_t\right)\left(K_t - K_t'\right)\right] + (1-\delta)\left(K_t - K_t'\right)\right] - \sum_{t=0}^{T} Q_t\left(K_{t+1} - K_{t+1}'\right) \\
&= \sum_{t=0}^{T} Q_t\left(1-\delta+\frac{r_t}{p_t}\right)\left(K_t - K_t'\right) - \sum_{t=0}^{T} Q_t\left(K_{t+1} - K_{t+1}'\right)
\end{aligned}
$$

By noticing that $K_0 = K_0'$ and $Q_{t+1}\left(1-\delta+\dfrac{r_{t+1}}{p_{t+1}}\right) - Q_t = 0$, we then get :

$$
\begin{aligned}
\Delta_T &\geq \sum_{t=1}^{T} Q_t\left(1-\delta+\frac{r_t}{p_t}\right)\left(K_t - K_t'\right) - \sum_{t=0}^{T} Q_t\left(K_{t+1} - K_{t+1}'\right) \\
&= \sum_{t=0}^{T-1}\left[Q_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) - Q_t\right]\left(K_{t+1} - K_{t+1}'\right) - Q_T\left(K_{T+1} - K_{T+1}'\right) \\
&\geq \sum_{t=0}^{T-1}\left[Q_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) - Q_t\right]\left(K_{t+1} - K_{t+1}'\right) - Q_T K_{T+1} \\
&= -Q_T K_{T+1}.
\end{aligned}
$$

■

We also have the transversality condition of each agent.

Lemma 5.2 *At any equilibrium, we have* $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) k_{i,t+1} = 0$ *for every* i.

Proof: See Theorem 2.1 in Kamihigashi (2002). ■

The following result shows the impact of borrowing constraints on the efficiency of an intertemporal equilibrium.

Lemma 5.3 *Consider an equilibrium. If there exists a date such that, from this date on, the borrowing constraints of agents are not binding at this equilibrium, then it is efficient.*

Proof: Assume that there exists t_0 such that $k_{i,t} > 0$ for every i and for every $t \geq t_0$. Then we have : for every $t \geq t_0$

$$\frac{Q_t}{Q_{t_0}} = \beta_i^{t-t_0} \frac{u_i'(c_{i,t})}{u_i'(c_{i,t_0})}.$$

According to Lemma 5.2, we have $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t})k_{i,t+1} = 0$. Then $\lim_{t\to\infty} Q_t k_{i,t+1} = 0$ for every i. This implies that $\lim_{t\to\infty} Q_t K_{t+1} = 0$. Therefore, this equilibrium is efficient. ■

We now state our main finding in this section.

Proposition 5.1 *Assume that the production functions are linear. Then every equilibrium path is efficient.*

Proof: Since production functions are linear, profit equals to zero. Recall that we have $c_{i,t} > 0$ for every i and every t. This implies that $k_{i,t} > 0$ at equilibrium. According to Lemma 5.3, every equilibrium path is efficient. ■

Corollary 4.2 and Proposition 4.1 indicate that with linear production functions, there exists an equilibrium the capital path of which is efficient and a bubble may arise at this equilibrium.

6 Conclusion

We build infinite-horizon dynamic deterministic general equilibrium models in which heterogenous agents invest in physical capital and consume. We proved existence of equilibrium in this model, even if technologies are not stationary and aggregate capital is not uniformly bounded.

We say there is a bubble of physical capital at equilibrium if the physical capital's price is greater than its fundamental value. We point out that bubbles exist if and only if the sum (over time) of capital returns is finite.

With linear technologies, every intertemporal equilibrium is efficient. Interestingly, it is possible to have both bubble and efficient at equilibrium.

7 Appendix : Existence of equilibrium for the truncated economy

Proof of Proposition 3.1: Denote $\Delta := \{z_0 = (p, r) : 0 \leq p_t, r_t \leq 1, p_t + r_t = 1 \quad \forall t = 0, \ldots, T\}$,

$$B_i(p, r) := \Big\{ \quad (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\ p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) < r_t k_{i,t} + \theta^i \pi_t \Big\},$$

and

$$C_i(p, r) := \Big\{ \quad (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\ p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t \Big\},$$

Denote by $\bar{B}_i(z_0)$ the closure of $B_i(z_0)$.

Lemma 7.1 *For every $(p,r) \in \mathcal{P}$, we have $B_i(p,q) \neq \emptyset$ and $\bar{B}_i(p,q) = C_i(p,q)$.*

Proof: We rewrite $B_i(p,r)$ as follows

$$B_i(p,r) := \big\{(c_i,a_i) \in \mathcal{C}_i \times \mathcal{A}_i \text{ such that } : \forall t = 0,\dots,T \\ 0 < p_t((1-\delta)k_{i,t} - c_{i,t} - k_{i,t+1}) + r_t k_{i,t} + \theta^i \pi_t\big\}.$$

Since $(1-\delta)k_{i,0} > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $k_{i,1} \in (0, B_k)$ such that

$$0 < p_0((1-\delta)k_{i,0} - c_{i,0} - k_{i,1}) + r_0 k_{i,0} + \theta^i \pi_0.$$

By induction, we see that $B_i(p,r)$ is not empty. ■

Lemma 7.2 *$B_i(p,r)$ is a lower semi-continuous correspondence on $\mathcal{P} := \Delta^{T+1}$. And $C_i(p,r)$ is upper semi-continuous on $\mathcal{P}$ with compact convex values.*

Proof: Clearly, since $B_i(p,r)$ is empty and has an open graph. ■

We define $\Phi := \Delta \times \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K}$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m+1}$ where $z_0 := (p,r), z_i := (c_i,k_i)$ for each $i = 1,\dots,m$, and $z_{m+1} = K$.

We now define correspondences. First, we define φ_0 (for additional agent 0)

$$\varphi_0 : \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K} \to 2^{\Delta}$$

$$\varphi_0((z_i)_{i=1}^{m+1}) := \operatorname*{arg\,max}_{(p,r)\in\Delta} \Big\{ \sum_{t=0}^{T} p_t\big(\sum_{i=1}^{m}[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] - F_t(K_t)\big) \\ + \sum_{t=0}^{T} r_t\big(K_t - \sum_{i=1}^{m} k_{i,t}\big)\Big\}.$$

For each $i = 1,\dots,m$, we define

$$\begin{aligned} \varphi_i : \quad & \Delta \to 2^{\mathcal{C}_i \times \mathcal{K}_i} \\ \varphi_i(p,r) \quad := \quad & \operatorname*{arg\,max}_{(c_i,k_i)\in C_i(p,r)} \Big\{\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t})\Big\}. \end{aligned}$$

For each $i = m+1$, we define

$$\begin{aligned}\varphi_{m+1}: \quad & \Delta \to 2^{\mathcal{K}} \\ \varphi_i(p,r) \quad := \quad & \arg\max_{K \in \mathcal{K}} \Big\{ \sum_{t=0}^{T} p_t F_t(K_t) - r_t K_t \Big\}.\end{aligned}$$

Lemma 7.3 *φ_i is upper semi-continuous convex-valued correspondence for each $i = 0, 1, \ldots, m+1$.*

Proof: This is a direct consequence of the Maximum Theorem. ■

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})$ such that

$$(\bar{p}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K}) \tag{III.13}$$

$$(\bar{c}_i, \bar{k}_i) \in \varphi_i(\bar{p}, \bar{r}) \tag{III.14}$$

$$\bar{K} \in \varphi_{m+1}(\bar{p}, \bar{r}). \tag{III.15}$$

Denote by $\bar{X}_t := \sum_{i=1}^m [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] - F_t(\bar{K}_t)$ and $\bar{Y}_t = \bar{K}_t - \sum_{i=1}^m \bar{k}_{i,t}$ the excess demands for goods and capital respectively. For every $(p, r) \in \Delta^{T+1}$, we have

$$\sum_{t=0}^{T} (p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T} (r_t - \bar{r}_t)\bar{Y}_t \leq 0. \tag{III.16}$$

By summing the budget constraints, for each t, we get

$$\bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \leq 0. \tag{III.17}$$

Hence, we have : for every $(p_t, r_t) \in \Delta$

$$p_t \bar{X}_t + q_t \bar{Y}_t \leq \bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \leq 0. \tag{III.18}$$

Therefore, we have $\bar{X}_t, \bar{Y}_t \leq 0$, which implies that

$$\sum_{i=1}^m \bar{c}_{i,t} + \bar{k}_{i,t+1} \leq (1-\delta) \sum_{i=1}^m \bar{k}_{i,t} + F_t(\bar{K}_t) \tag{III.19}$$

$$\bar{K}_t \leq \sum_{i=1}^m \bar{k}_{i,t}. \tag{III.20}$$

Lemma 7.4 $\bar{p}_t, \bar{r}_t > 0$ *for* $t = 0, \ldots, T$.

Proof: If $\bar{p}_t = 0$ then $\bar{c}_{i,t} = B_c > (1-\delta)B + F_t(B)$. Therefore, we get $\bar{c}_{i,t} + \bar{k}_{i,t+1} > (1-\delta)\sum_{i=1}^{m}\bar{k}_{i,t} + F_t(\bar{K}_t)$ which is a contradiction. Hence, $\bar{p}_t > 0$. If $\bar{r}_t = 0$, then the optimality of $\bar{K}$ implies that $K_t = B$. However, we have $\bar{k}_{i,t} \leq B_k$ for every i, t. Consequently, $\sum_{i=1}^{m}\bar{k}_{i,t} \leq mB_k < B = K_t$, contradiction to (III.20). Therefore, we get $\bar{r}_t > 0$. ■

Lemma 7.5 $\sum_{i=1}^{m}\bar{k}_{i,t} = \bar{K}_t$ *and* $\sum_{i=1}^{m}[\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F(\bar{K}_t)$

Proof: Since prices are strictly positive and the utility functions are strictly increasing, all the budget constraints are binding and, summing them across the individuals, we get

$$\bar{p}_t\bar{X}_t + \bar{r}_t\bar{Y}_t = 0. \tag{III.21}$$

We know that $\bar{X}_t, \bar{Y}_t \leq 0$ and $\bar{p}_t, \bar{r}_t > 0$. Then, $\bar{X}_t = \bar{Y}_t = 0$. The optimality of $(\bar{c}_i, \bar{k}_i)$ and $\bar{K}$ comes from (III.14) and (III.15). ■ ■

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Chapitre IV

Financial asset bubble with heterogeneous agents and endogenous borrowing constraints

Abstract : This paper studies the root of financial asset bubble in an infinite horizon general equilibrium model with heterogeneous agents and borrowing constraints. We say that there is a bubble at equilibrium if the price of the financial asset is greater than its fundamental value. First, we found that bubble can occur only if there exists an agent and an infinite sequence of date, (t_n), such that borrowing constraint of this agent are binding at each date t_n. Second, we prove that there is a bubble if and only if interest rates are low, which means that the sum (over time) of interest rates (in term of financial asset) is finite. Last, we give a condition on exogenous variables, under which a financial asset bubble occurs at equilibrium.

Keywords : Financial asset bubble, intertemporal equilibrium, infinite horizon, borrowing constraints.

JEL Classifications : C62, D5, D91, G10

1 Introduction

This paper is to study fundamental questions about rational bubbles : What is an asset price bubble ? what is the root of bubbles ? We also give an answer for the following debate :

> *"However, despite the widespread belief in the existence of bubbles in the real world, it is difficult to construct model economies in which bubbles exist in equilibrium."*
> Kocherlakota (2008)

To do this, we contruct an infinite horizon general equilibrium model with heterogeneous agents and endegenous borrowing constraints. There is a single consumption good and a finanical asset. On the one hand, the financial asset will give dividends in term of consumption good. On the other hand, agents can resell it. There is an endogenous borrowing constraint when agents want to borrow : at each date, each agent can borrow an amount but the delivery of this amount at next date cannot be greater than a fraction of the endowment of this agent. Because of the borrowing constraints, the financial market is dynamically incomplete.

Before studying the bubbles, we have to prove the existence of equilibrium. We do so by proving the existence of equilibrium for each truncated economy, and then pass to the limit. The existence of equilibrium for each truncated economy is proved by two steps : (i) consider bounded truncated economy and prove that there exists equilibrium for each bound, (ii) let bound tend to infinity to obtain an equilibrium for truncated economy. Our proof is crucial because we do not require any condition on endowments of agents as in Levine (1989), Levine and Zame (1996), Magill and Quinzii (1994), Araujo, Pascoa, Torres-Martinez (2002). Instead, endowments of agents may be zero in our paper. We overcome this difficulty by two steps : (1) we prove that there exists an equilibrium for the economy in which every agent has $\epsilon > 0$ units of endowment ; as a result we obtain a sequence of equilibria parameterized by ϵ, (2) let ϵ tend to zero, this sequence has a limit ; we prove that such limit is an equilibrium by using the positivity of financial dividend (since the financial dividend is in term of consumption good and can be consummed).

Second, we move to study bubble of financial asset. We say that a financial asset bubble occurs at an equilibrium (for short, *bubble*) if the price of the financial asset is greater than its fundamental value. Some significant papers studied rational asset bubbles. A well known result is that if the present value of aggregate endowment is finite, there is no bubble (Santos and Woodford (1997), Huang and Werner (2000)). However, the present value of aggregate endowments is endogenously determined. Why the present value of aggregate endowment is finite ?

Montrucchio (2004), Le Van and Vailakis (2012) have given conditions on endogenous variables, under which there is a bubble. Unfortunately, they did not explain the nature of these conditions. Although there are some examples of bubbles (Kocherlakota (1992), Huang and Werner (2000), Le Van and Vailakis (2012)), no one gives conditions of exogenous variables under which there is a bubble at equilibrium. Our paper will fill these gaps.

We begin by pointing out that, at equilibrium, individual transversality condition of each agent is satisfied but real transversality condition of each agent may be not held. If the real transversality condition of each agent is satisfied, there is no bubble.

We also find that if a bubble appear, there exists an agent i and an infinite sequence of date $(t_n)_{n=0}^{\infty}$ such that borrowing constraint of this agent is binding at each date t_n. This finding complements the one in Kocherlakota (1992) where he wanted to claim that borrowing constraint is binding infinitely often. However, he only proved that the inferior limit of difference between asset amount of each agent and exogenous borrowing bound equals zero.

We then define new concepts : *low interest rates* and *high interest rates*. An equilibrium is said to have low interest rates if the sum (over time) of interest rates (in term of financial asset) is finite, otherwise we say interest rates are high. A novel result is that interest rates are low if and only if bubbles exist. Our definition of low interest rates is different from the one in Alvarez and Jermann (2000) where implied interest rates are called to be high if the present value of aggregate endowment is finite. We proved that if equilibrium is high implied interest rates, it will be high interest rates.

Our last contribution is to give a condition of exogenous variables, under which bubble occurs at equilibrium. The intuition of our condition is the following : if there exists an agent whose highest subjective interest rate is less than the interest rate of the economy, this agent accepts to buy financial asset with a price which is greater than its fundamental value. Consequently, there is a bubble.

Related literature : A survey on bubble in models as asymmetric information, overlapping generation, heterogeneous-beliefs can be found in Brunnermeier and Oehmke (2012). Doblas-Madrid (2012) presents a model of speculative bubbles where rational agents buy an overvalued asset because given their private information, they believe they have a good chance of reselling at a profit to a greater fool. Martin and Ventura (2012), Ventura (2012) do not define bubble as our definition. They define bubble as a short-lived asset.

In a rational expectation model without endowment, Tirole (1982) proved that there is no financial asset bubble. His result can be viewed as a particular case of our model.

Our paper is also related to physical capital bubble. In the standard Ramsey model with heterogeneous agents and stationary concave technology, Becker, Bosi, Le Van and Seegmuller (2014) prove that physical capital bubble does not exist. Bosi, Le Van and Pham (2014) allow non-stationary technologies and prove that physical capital bubble exists if and only if the sum (over time) of capital return is finite. In Becker, Bosi, Le Van and Seegmuller (2014), physical capital can be viewed as a long-lived asset whose price (in term of consumption goood) is 1 and capital return can be viewed as dividend. However, r_t is endogenous in Becker, Bosi, Le Van and Seegmuller (2014).

The remainder of the paper is organized as follows. Section 2 decribes the model. In section 3, the existence of equilibrium is proved. Section 4 studies the root of financial asset bubble. Conclusion will be presented in Section 5. Technical details are gathered in Appendix.

2 Model

We consider a standard exchange economy in an infinite horizon model. At each date, agents are endowed with an amount of consumption good. p_t is price of consumption good at date t.

Financial market : there is one long-lived financial asset. At date t, if agent i buys $a_{i,t} \geq 0$ units of financial asset with price q_t, at the next date (date $t+1$), this agent will receive ξ_{t+1} units of consumption good as dividend and she will able to sell $a_{i,t}$ units of financial asset with price q_{t+1}.

Each household i takes sequences of prices $(p, q) = (p_t, q_t)_{t=0}^{\infty}$ as given and maximizes her utility :

$$(P_i(p,q)) : \quad \max_{\left((c_{i,t},a_{i,t})_{i=1}^m\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\right] \tag{IV.1}$$

$$\text{subject to} \quad p_t c_{i,t} + q_t a_{i,t} \leq p_t e_{i,t} + (q_t + p_t \xi_t) a_{i,t-1} \tag{IV.2}$$

$$-(q_{t+1} + p_{t+1}\xi_{t+1}) a_{i,t} \leq f_i p_{t+1} e_{i,t+1}, \tag{IV.3}$$

where β_i is the discount factor of agent i and u_i is the utility function of agent i. $f^i \in [0,1]$ is the borrowing degree of agent i. Borrowing constraint (IV.3) means that the payment of agent i cannot exceed a fraction of her endowments. If $f^i = 0$, agent i cannot borrow. Our setup is different from the one in Kocherlakota (1992) where he considers that borrowing constraints are exogenous, which is given by $a_{i,t} \geq a^*$ where a^* do not depend on time. Borrowing constraints in our framework allow us to bound the trade volume of asset by an exogenous bound. Indeed, assume that prices are strictly positive, we observe that (IV.3) implies that

$$-a_{i,t} \leq \frac{f_i p_{t+1} e_{i,t+1}}{q_{t+1} + p_{t+1}\xi_{t+1}} \leq \frac{f_i e_{i,t+1}}{\xi_{t+1}}. \tag{IV.4}$$

It means that agent i cannot borrow more than $\dfrac{f_i e_{i,t+1}}{\xi_{t+1}}$ which is exogenous but varies in time t. Although $\dfrac{f_i e_{i,t+1}}{\xi_{t+1}}$ is exogenous, it may tend to infinity.

Definition 2.1 *A sequence of prices and quantities* $\left(\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{a}_{i,t})_{i=1}^m\right)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\mathcal{E} = \left((u_i, \beta_i, a_{i,-1}, f^i)_{i=1}^m\right)$ *if*

(i) Price positivity : $\bar{p}_t, \bar{q}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing : at each $t \geq 0$,

$$\textit{Consumption good}: \quad \sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m e_{i,t} + \xi_t, \tag{IV.5}$$

$$\textit{Financial asset}: \quad \sum_{i=1}^m \bar{a}_{i,t} = 1. \tag{IV.6}$$

(iii) Optimal consumption plans : for each i,

$$(\bar{c}_{i,t}, \bar{a}_{i,t})_{t=0}^\infty$$

is a solution of the problem $(P_i(\bar{p}, \bar{q}))$.

3 The existence of equilibrium

We need some standard assumptions.

Assumption (H1) : The utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ is C^0, strictly increasing, concave and $u_i(0) = 0$, $u'(0) = +\infty$.

Assumption (H2) : $e_{i,t} \geq 0$ such that for every $t \geq 0$ and $i = 1, \ldots, m$.

Assumption (H3) : For each $i = 1, \ldots, m$, $a_{i,-1} \geq 0$ and $\sum\limits_{i=1}^m a_{i,-1} = 1$.

Assumption (H4) : $\xi_t > 0$ for every t.

Assumption (H5) : For each i, utility of agent i is finite

$$\sum_{t=0}^\infty \beta_i^t u_i(W_t) < \infty, \tag{IV.7}$$

where $W_t := \sum\limits_{i=1}^m e_{i,t} + \xi_t$.

Theorem 3.1 *Assume that Assumptions* $(H1) - (H5)$ *are satisfied, there exists an equilibrium in the infinite-horizon economy if* $(e_{i,0}, a_{i,-1}) \neq (0, 0)$.

We see that even if agents do not hold endowment, i.e., $e_{i,t} = 0$, there may be an equilibrium. This is the case where $a_{i,-1} > 0$ for every i. In Le Van and Vailakis (2012), they required there exists $e > 0$ such that $e_{i,t} > e$

for every i and for every t. This result also gives a foundation in order to study financial bubble in Tirole (1982) where he assumes that households do not hold endowments.

We prove Theorem 3.1 by two main steps : (1) we prove the existence of equilibrium for each $T-$truncated economy, we have a sequence of equilibria which depend on T ; (2) we prove that this sequence has a limit (for the product topology) which is an equilibrium for the infinite horizon economy.

3.1 Existence of equilibrium for truncated economy

For each $T \geq 0$, we define $T-$ truncated economy $\mathcal{E}^T$ as $\mathcal{E}$ but there are no activities from period $T+1$, i.e., $c_{i,t} = a_{i,t-1} = 0$ for every $i = 1, \ldots, m$, $t \geq T+1$.

1 Existence of equilibrium for bounded economy

We define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all variables (consumption demand, asset investment) are bounded.

$$\begin{aligned} \mathcal{C}_i &:= [0, B_c]^{T+1}, \quad B_c > 1 + \max_{t \leq T} W_t \\ \mathcal{A}_i &:= [-B_a, B_a]^T, \quad B_a > 1 + B, \end{aligned}$$

where B is satisfied $B > \max\{\max_t \frac{1+W_t}{\xi_t}, 1 + m \max_t \frac{1+W_t}{\xi_t}\}$.

Denote $\Delta := \{z_0 = (p, q) : 0 \leq p_t, q_t \leq 1, p_t + q_t = 1 \quad \forall t = 0, \ldots, T\}$.

For each $\epsilon > 0$ such that $2m\epsilon < 1$, we define $\epsilon-$economy $\mathcal{E}_b^{T,\epsilon}$ by adding ϵ units of each asset (consumption good and financial asset) at date 0 for each agent in the bounded economy. More presise, the feasible set of agent i is given by

$$\begin{aligned} C_i^{T,\epsilon}(p,q) := \big\{ &(c_{i,t}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } a_{i,T} = 0, \\ &\text{(b) } p_0 c_{i,0} + q_0 a_{i,0} \leq p_0(e_{i,0} + \epsilon) + (\bar{q}_0 + \bar{p}_0 \xi_0)(a_{i,t-1} + \epsilon) \\ &\text{(c) for each } 1 \leq t \leq T : \\ &\quad 0 \leq (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t(e_{i,t} + \epsilon) \\ &\quad p_t c_{i,t} + q_t a_{i,t} \leq p_t(e_{i,t} + \epsilon) + (q_t + p_t \xi_t) a_{i,t-1} \big\}. \end{aligned}$$

Definition 3.1 *A sequence of prices and quantities* $\Big(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{a}_{i,t}, \bar{k}_{i,t})_{i=1}^m, \bar{K}_t\Big)_{t=0}^T$ *is an equilibrium of the economy* $\mathcal{E}_b^{T,\epsilon}$ *if the following conditions are satisfied*

(i) Price are strictly positive, i.e., $\bar{p}_t, \bar{q}_t > 0$ *for* $t \geq 0$.

(ii) All markets clear :
Consumption good

$$\begin{aligned}\sum_{i=1}^{m}\bar{c}_{i,0} &= \sum_{i=1}^{m}\bar{e}_{i,0} + 2m\epsilon + \xi_0\\ \sum_{i=1}^{m}\bar{c}_{i,t} &= \sum_{i=1}^{m}\bar{e}_{i,t} + m\epsilon + \xi_t\end{aligned}$$

Financial asset

$$\sum_{i=1}^{m}\bar{a}_{i,0} = \sum_{i=1}^{m}(\bar{a}_{i,-1}+\epsilon), \quad \sum_{i=1}^{m}\bar{a}_{i,t+1} = \sum_{i=1}^{m}\bar{a}_{i,t}, \forall t \geq 0.$$

(iii) Optimal consumption plans : for each i, $\left(\bar{c}_{i,t}, \bar{a}_{i,t}\right)_{t=0}^{\infty}$ *is a solution of the maximization problem of agent i with the feasible set* $C_i^{T,\epsilon}(p,q)$.

The first step is to prove the existence of equilibrium for each $\epsilon-$ economy when ϵ is small. We then take a sequence (ϵ_n) converging to zero. When n tends to infinity, the sequence of equilibria depending on ϵ_n has a limit who will be proved to be an equilibrium for bounded economy $\mathcal{E}_b^T$. Formal proofs ars presented in Appendix.

2 Existence of equilibrium for unbounded economy

We claim that an equilibrium of $\mathcal{E}_b^T$ is also an equilibrium for $\mathcal{E}^T$. Indeed, let $\left(\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{a}_{i,t})_{i=1}^m\right)_{t=0}^{T}$ is an equilibrium of $\mathcal{E}_b^T$. Note that $a_{i,T} = 0$ for every $i = 1, \ldots, m$. It is easy to see that prices are strictly positive and all markets clear. We will prove the optimality of allocation.

Let $z_i := \left(c_{i,t}, a_{i,t}\right)_{t=0}^{T}$ be a feasible plan of household i. Assume that $\sum\limits_{t=0}^{T}\beta_i^t u_i(c_{i,t}) > \sum\limits_{t=0}^{T}\beta_i^t u_i(\bar{c}_{i,t})$. For each $\gamma \in (0,1)$, we define $z_i(\gamma) := \gamma z_i + (1-\gamma)\bar{z}_i$. By definition of $\mathcal{E}_b^T$, we can choose γ sufficiently close to 0 such that $z_i(\gamma) \in \mathcal{C}_i \times \mathcal{A}_i$. It is clear that $z_i(\gamma)$ is satisfied budget constraints.

By the concavity of the utility function, we have

$$\begin{aligned}\sum_{t=0}^{T}\beta_i^t u_i(c_{i,t}(\gamma)) &\geq \gamma\sum_{t=0}^{T}\beta_i^t u_i(c_{i,t}) + (1-\gamma)\sum_{t=0}^{T}\beta_i^t u_i(\bar{c}_{i,t})\\ &> \sum_{t=0}^{T}\beta_i^t u_i(\bar{c}_{i,t}).\end{aligned}$$

Contradiction to the optimality of $\bar{z}_i$.

3.2 Existence of equilibrium in the infinite horizon economy

We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. We denote by $(\bar{p}^T, \bar{q}^T, (\bar{c}_i^T, \bar{a}_i^T)_{i=1}^m)$ an equilibrium of $T-$ truncated economy $\mathcal{E}^T$.

We can normalize prices by setting $\bar{p}_t^T + \bar{q}^T = 1$ for every $t \leq T$.
It is easy to see that

$$0 < \bar{c}_{i,t}^T < D_t$$
$$-\bar{a}_{i,t}^T \leq \frac{W_{t+1}}{\xi_{t+1}} \quad \forall i, \text{ and } \quad \sum_{i=1}^m \bar{a}_{i,t}^T = 1.$$

Therefore, endogenous variables are bounded for the product topology. Therefore, we can assume that

$$(\bar{p}^T, \bar{q}^T, (\bar{c}_i^T, \bar{a}_i^T)_{i=1}^m)$$
$$\xrightarrow{T\to\infty} (\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^m) \quad \text{(for the product topology)}.$$

In Appendix, we prove that this limit is an equilibrium for the economy $\mathcal{E}$.

3.3 Individual behavior

We consider an equilibrium $(p, q, (c_i, a_i)_{i=1}^m)$. Let $\lambda_{i,t}$ denote the multiplier associated with budget constraints of agent i at period t, and the multiplier of borrowing constraint is denoted by $\mu_{i,t}$ ($\mu_{i,t} \geq 0$). We have

$$\beta_i^t u_i'(c_{i,t}) = p_t \lambda_{i,t} \quad \text{(IV.8)}$$
$$\lambda_{i,t} q_t = (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + p_{t+1}\xi_{t+1}) \quad \text{(IV.9)}$$
$$\mu_{i,t+1}\Big((q_{t+1} + p_{t+1}\xi_{t+1})a_{i,t} + f^i p_{t+1} e_{i,t+1}\Big) = 0. \quad \text{(IV.10)}$$

For each i, we define $S_{i,0} = 1$, $S_{i,t} := \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$ is the agent i's discount factor from initial period to period t.

Lemma 3.1 *We have*

$$\lim_{t\to\infty} S_{i,t} \frac{q_t}{p_t} a_{i,t} = 0 \quad \text{(IV.11)}$$
$$\sum_{t=0}^{\infty} S_{i,t} c_{i,t} = (\frac{q_0}{p_0} + \xi_0) a_{i,-1} + \sum_{t=0}^{\infty} S_{i,t} e_{i,t} + \sum_{t=1}^{\infty} f^i \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} e_{i,t} < \infty.$$
$$\text{(IV.12)}$$

Proof: Indeed, (IV.11) is proved by using the result in Kamihigashi (2002). At equilibrium, we have

$$\infty > \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \geq \sum_{t=0}^{\infty} \beta_i^t u_i'(c_{i,t}) c_{i,t}.$$

As a consequence, there exists $\sum_{t=0}^{\infty} S_{i,t} c_{i,t}$. Therefore, we obtain (IV.12). ■

We now define the discount factor of the economy. Since $\sum_{i=1}^{m} a_{i,t} = 1$, there exists $i(t)$ such that $a_{i(t),t} > 0$, hence $\mu_{i(t),t} = 0$. Therefore, we get

$$\frac{q_t}{q_{t+1} + p_{t+1}\xi_{t+1}} = \frac{\bar{\lambda}_{i,t+1}}{\bar{\lambda}_{i,t}} = \max_{i \in \{1,\dots,m\}} \frac{\bar{\mu}_{i,t+1}}{\bar{\mu}_{i,t}} = \gamma_{t+1} \frac{\bar{p}_t}{\bar{p}_{t+1}}, \qquad \text{(IV.13)}$$

where $\gamma_{t+1} := \max_{i \in \{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$. For each $t \geq 0$ we have

$$\frac{q_t}{p_t} = \gamma_{t+1}\left(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}\right). \qquad \text{(IV.14)}$$

We define $Q_0 := 1$, and for each $t \geq 1$, $Q_t := \prod_{s=1}^{t} \gamma_s$ is the discount factor of the economy from initial period to period t. Since $Q_{t+1} = \gamma_{t+1} Q_t$, we have

$$Q_t \frac{q_t}{p_t} = Q_{t+1}\left(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}\right). \qquad \text{(IV.15)}$$

Remark 3.1 *It is clear that $Q_t \geq S_{i,t}$ for every t, i.e., the market discount factor is greater individual discount factors. One can see that if borrowing constraint of agent i is not binding from initial date until date t, we have $Q_t = S_{i,t}$. .*

Remark 3.2 *We can see that borrowing constraint $(q_{t+1} + p_{t+1}\xi_{t+1})a_{i,t} \geq -f^i p_{i,t+1} e_{i,t+1} \geq 0$ is equivalent to $Q_{t+1}(q_{t+1}+p_{t+1}\xi_{t+1})a_{i,t} \geq -f^i Q_{t+1} p_{i,t+1} e_{i,t+1}$. According to (IV.15), this can be rewritten as*

$$Q_t \frac{q_t}{p_t} a_{i,t} + f^i Q_{t+1} e_{i,t+1} \geq 0$$

This means that borrowing value of agent i does not exceed a fraction value of its endowments.

We state the a fundamental result showing the information of borrowing constraints.

Proposition 3.1 *(***Fluctuation of borrowing constraints***)*

1. *For each i, there are only 2 cases*
   (a) *there does not exist* $\lim_{t\to\infty}\big(Q_t\frac{q_t}{p_t}a_{i,t}+f^iQ_{t+1}e_{i,t+1}\big)$.
   (b) $\lim_{t\to\infty}\big(Q_t\dfrac{q_t}{p_t}a_{i,t}+f^iQ_{t+1}e_{i,t+1}\big)=0$.
2. *We have, for each i,*

$$\liminf_{t\to\infty}\Big(Q_t\frac{q_t}{p_t}a_{i,t}+f^iQ_{t+1}e_{i,t+1}\Big)=0 \tag{IV.16}$$

Proof: Assume that there exists $\lim_{t\to\infty} Q_t\frac{q_t}{p_t}a_{i,t}+f^iQ_{t+1}e_{i,t+1} =: Q_i$.

If $Q_i>0$, there exists t_0 such that $Q_t\frac{q_t}{p_t}a_{i,t}+f^iQ_{t+1}e_{i,t+1}>0$ for each $t\geq t_0$. However, this condition is equivalent to $Q_{t+1}(\frac{q_{t+1}}{p_{t+1}}+\xi_{t+1})a_{i,t+1}+f^iQ_{t+1}e_{i,t+1}>0$. It means that the borrowing constraints of agent i are not binding from date t_0. This implies that $\frac{Q_t}{Q_{t_0}}=\frac{S_{i,t}}{S_{i,t_0}}$ for every $t\geq t_0$. According to Lemma 3.1, we get $\lim_{t\to\infty}Q_t\frac{q_t}{p_t}a_{i,t}=0$, and $\lim_{t\to\infty}Q_{t+1}e_{i,t+1}=0$, contradiction!

(IV.16) is proved by using the same argument. ■

4 Financial asset bubble

According to (IV.14), we have $\frac{q_t}{p_t}=\gamma_{t+1}(\frac{q_{t+1}}{p_{t+1}}+\xi_{t+1})$. Therefore, for each $t\geq 1$, we have

$$\begin{aligned}\frac{q_0}{p_0} &= \gamma_1(\frac{q_1}{p_1}+\xi_1)=Q_1\xi_1+\gamma_1q_1=Q_1\xi_1+\gamma_1\gamma_2(\frac{q_2}{p_2}+\xi_2)\\ &= Q_1\xi_1+Q_2\xi_2+Q_2\frac{q_2}{p_2}\\ &= \ldots=\sum_{s=1}^{t}Q_s\xi_s+Q_t\frac{q_t}{p_t}.\end{aligned}$$

Interpretation : In this model, financial asset is a long-lived asset whose price at date 0 is q_0.

1. At date 1, one unit (from date 0) of this asset will give back 1 units of the same asset and ξ_1 units of consumption good as its dividend.
2. At date 2, one units of long lived asset will give one unit of the same asset and ξ_2 units of consumption good ...

This leads us to have the following concept.

Definition 4.1 *The fundamental value of financial asset*

$$FV_0:=\sum_{t=1}^{+\infty}Q_t\xi_t \tag{IV.17}$$

Denote $b_0 := \lim_{t\to+\infty} Q_t \frac{q_t}{p_t}$, b_0 is called financial asset bubble. We have

$$q_0 = b_0 + FV_0. \tag{IV.18}$$

It means that the price of the financial asset equals its fundamental value plus its bubble.

Definition 4.2 *We say there is a bubble on financial asset if the price of financial asset is greater than its fundamental value : $q_0 > FV_0$.*

On the relationship between bubble and borrowing constraints, Kocherlakota (1992) suggests that borrowing constraints are binding infinitely often if bubbles exists.[1] However, what he proved was that $\liminf_{t\to\infty}(a_{i,t} - a^*) = 0$. We now prove that borrowing constraint is binding.

Proposition 4.1 *(***Borrowing constraint is binding at infinitely many date***)*
If bubble occurs, there exists i and an infinite sequence $(t_n)_{n\geq 1}$ such that borrowing constraint of agent i is binding at each date t_n.

Proof: Assume that for each i, there exists $t_i \geq 0$ such that borrowing constraints of agent i are not binding from t_i. We define $t_0 = \max_{i=1,\dots,m} t_i$. Hence, borrowing constraints of all agents are not binding from date t_0. By using the same argument in the proof of Proposition 3.1, we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} a_{i,t} = 0$. As a consequence, we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0$. ■

This result says that, if there is a bubble, there exists an agent whose borrowing constraints are binding at infinitely many dates. Our finding complements the one in Kocherlakota (1992).

The following result (as the one in Kocherlakota (1992)) shows that at bubble equilibirum, there is a fluctuation in financial asset volume of some agent.

Proposition 4.2 *If bubble occurs, there exists i such that the sequence $(a_{i,t})$ has no limit.*

Proof: If the sequence $(a_{i,t})$ converges for every i, there exists i such that $\lim_{t\to\infty} a_{i,t} > 0$. Borrowing constraint of this agent are not binding from some date t_i. Therefore, by using the same argument as in the proof of Proposition 3.1, we have $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} a_{i,t} = 0$. Hence, $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0$. ■

1. Recall that in Kocherlakota (1992), borrowing constraint is $a_{i,t} \geq a^*$.

4.1 Rational bubble and low interest rates

We firstly define what does low interest rates mean. We recall budget constraint of agent i at date $t-1$ and t.

$$\begin{aligned} p_{t-1}c_{i,t-1} + q_{t-1}a_{i,t-1} &\leq p_{t-1}e_{i,t-1} + (q_{t-1} + p_{t-1}\xi_{t-1})a_{i,t-1} \\ p_t c_{i,t} + q_t a_{i,t} &\leq p_t e_{i,t} + q_t(1 + \frac{p_t\xi_t}{q_t})a_{i,t-1}. \end{aligned}$$

One can interpret that if agent i buys $a_{i,t-1}$ units of financial asset at date $t-1$ with price q_{t-1}, she will receive $(1+\frac{p_t\xi_t}{q_t})a_{i,t-1}$ units of financial asset with price q_t at date t. Therefore, $\frac{p_t\xi_t}{q_t}$ can be viewed as the interest rate of the financial asset at date t.

Definition 4.3 *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty} \frac{p_t\xi_t}{q_t} < \infty. \tag{IV.19}$$

Otherwise, we say that interest rates are high.

Remark 4.1 *In Alvarez and Jermann (2000), they define high implied interest rates as a situation in which the present value of aggregate endowments is finite, i.e.,*

$$\sum_{t=0}^{\infty} Q_t e_t < \infty,$$

where $e_t := \sum_{i=1}^{m} e_{i,t}$*. We will compare these two concepts (high interest rates and high implied interest rates) at the end of this subsection.*

We now present relationship between financial bubble and low interest rates.

Proposition 4.3 *There is a bubble if and only if interest rates are low.*

Proof: According to (IV.15), we imply that

$$\frac{q_0}{p_0} = Q_T \frac{q_T}{p_T} \prod_{t=1}^{T}(1 + \frac{p_t\xi_t}{q_t}). \tag{IV.20}$$

Since $q_0 > 0$, we see that $\lim_{t\to+\infty} Q_t q_t > 0$ if and only if

$$\lim_{t\to\infty} \prod_{t=1}^{T}(1 + \frac{\xi_t}{q_t}) < \infty.$$

It is easy to prove that this condition is equivalent to (IV.19). ■

Proposition 4.3 is closed to the one in Montrucchio (2004). Note that Le Van and Vailakis (2012) give an example in which $p_t = 1$ and $q_t = 1 - \sum_{s=1}^{t} \xi_t$ for every t. Recall that their model can be viewed as a particular case of our model with $f^i = 0$ for every i. In their example, there is a bubble if and only if $\sum_{t=1}^{\infty} \xi_t < 1$. It is easy [2] to see that this condition is satisfied if and only if

$$\sum_{t=1}^{\infty} \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s} < \infty.$$

This is exactly our theoretical condition $\sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} < \infty$.

We give some consequences of condition (IV.19).

Corollary 4.1 *(i) Assume that* $\sum_{t=1}^{\infty} \xi_t < \infty$. *If* $\frac{q_t}{p_t}$ *is bounded from below, there is a bubble.*

(i) Assume that $\sum_{t=1}^{\infty} \xi_t = \infty$. *If there is a bubble, we have* $\lim_{t \to +\infty} \frac{q_t}{p_t} = \infty$.

(iii) Assume that there exists $x, X \in (0, \infty)$ *such that* $x \leq \frac{\xi_t}{\xi_{t+1}} \leq X$. *If bubble occurs; the real return* $\frac{\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}}{\frac{q_t}{p_t}}$ *is bounded from below.*

Proof: Point (i) and (ii) are clear. Let us prove point (iii). Since bubble occurs, condition IV.19 is held, thus there exists $M \in (0, \infty)$ such that

$$\frac{\frac{p_{t+1}\xi_{t+1}}{q_{t+1}}}{\frac{p_t \xi_t}{q_t}} \leq M.$$

By combining with the fact that rate of growth of financial dividend is bounded, we implies that rate of growth of financial asset prices $\frac{\frac{q_{t+1}}{p_{t+1}}}{\frac{q_t}{p_t}}$ is bounded from below. Consequently, the real return is bounded from below. ■

We now study the relationship between bubble and present values of agents. Recall that the present value of agent i is given by $\sum_{t=0}^{\infty} Q_t e_{i,t}$.

2. Indeed, $\sum_{t=1}^{\infty} \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s} < \infty$ if and only if $\lim_{T \to \infty} \prod_{t=1}^{T} (1 + \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s}) < \infty$ if and only if $\lim_{T \to \infty} \frac{1}{1 - \sum_{t=1}^{T} \xi_t} < \infty$ if and only if $\sum_{t=1}^{\infty} \xi_t < 1$.

Proposition 4.4 *If a bubble occurs, there exists i such that $\sum_{t=1}^{+\infty} Q_t e_{i,t} = \infty$, and then* $\limsup_{t\to\infty} \frac{Q_t}{S_{i,t}} = +\infty$.

Proof: Suppose that a bubble occurs. Assume that for every i, $\sum_{t=1}^{+\infty} Q_t e_{i,t} < \infty$. Thus, $\lim_{t\to\infty} Q_t e_{i,t} = 0$. We observe that

$$\begin{aligned}(\frac{q_0}{p_0} + \xi_0)a_{i,-1} + \sum_{t=0}^{T} Q_t e_{i,t} + f^i Q_{t+1} e_{i,t+1} &= \sum_{t=0}^{T} Q_t c_{i,t} + \Big(Q_t \frac{q_t}{p_t} a_{i,t} + f^i Q_{t+1} e_{i,t+1}\Big) \\ &\geq \sum_{t=0}^{T} Q_t c_{i,t}.\end{aligned}$$

Therefore, there exists $\sum_{t=1}^{+\infty} Q_t c_{i,t}$. Consequently, there exists $\lim_{t\to\infty} \Big(Q_t \frac{q_t}{p_t} a_{i,t} + f^i Q_{t+1} e_{i,t+1}\Big)$. According Proposition 3.1, we have $\lim_{t\to\infty} \Big(Q_t \frac{q_t}{p_t} a_{i,t} + f^i Q_{t+1} e_{i,t+1}\Big) = 0$ which implies that $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0$. ■

Interpretation : We define the gross interest rate $(1 + R_t)$ of the economy and gross interest rate $(1 + R_{i,t})$ of agent i at date t as follows

$$\frac{1}{1 + R_t} = \frac{q_{t-1}}{q_t + \xi_t} = \gamma_t = \max_{i\in\{1,\ldots,m\}} \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})} \tag{IV.21}$$

$$\frac{1}{1 + R_{i,t}} = \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})}. \tag{IV.22}$$

Proposition 4.4 indicates that if bubble occurs, there exists an agent i and an infinite sequence date (t_n) such that

$$\frac{Q_{t_n}}{S_{i,t_n}} = \frac{(1 + R_{i,1})\ldots(1 + R_{i,t_n})}{(1 + R_1)\ldots(1 + R_{t_n})} \tag{IV.23}$$

tends to infinity. It means that the individual interest rate is greater than the interest rate of the economy. We will come back to this point in next section.

We point out some consequences of Lemma 4.4.

Corollary 4.2 *If there exists $\alpha > 0$ such that $\xi_t \geq \alpha \sum_{i=1}^{m} e_{i,t}$, there is no bubble. Therefore, interest rates are high*

Proof: Note that $\sum_{t=1}^{+\infty} Q_t \xi_t \leq \frac{q_0}{p_0} < \infty$, we implies that $\sum_{t=1}^{+\infty} Q_t e_{i,t} < +\infty$ for every i. According to Proposition 4.4, there is no bubble. ■

Condition $\xi_t \geq \alpha \sum_{i=1}^{m} e_{i,t}$ is not only a sufficient condition ruling out bubble but also a sufficient condition under which interest rates are high at equilibirum. We can see that high financial dividend implies that interest rates are high, and then bubble is ruled out.

Corollary 4.3 *Assume that $e_{i,t} = 0$ for every t and every i. There is no bubble at equilibrium.*

This result is in line with Tirole (1982) where he proved that bubble does not exist in a fully dynamic rational expectation equilibrium model without endowments. However, he did not prove the existence of equilibrium.

Corollary 4.4 *(Santos and Woodford (1997), Huang and Werner (2000)) Assume that $\sum_{t=0}^{\infty} Q_t e_t < \infty$. There is no sequential price bubble.*

Proof: Clear. ■

Note that high implied interest rates is only an sufficient condition for no bubble.[3] We end this subsection by making clear the difference between high interest rates and high implied interest rates.

Proposition 4.5 *At equilibrium, if $\sum_{t=0}^{\infty} Q_t e_t < \infty$, we have $\sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} = \infty$. It means that*

Proof: According to Corollary 4.4, if the present value of aggregate endowment is finite, there is no bubble. As a consequence of Proposition 4.3, there is no bubble if and only if $\sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} = \infty$. Therefore, we obtain the result. ■

This result shows that if an equilibrium has high implied interest rates, it has high interest rates.

4.2 An exogenous sufficient condition for bubble

We have so far given necessary conditions or some sufficient (on endogenous variables) of bubble. Although there are some examples of bubble (Kocherlakota (1992), Huang and Werner (2000), Le Van and Vailakis (2012)), no one gives conditions of exogenous variables under which there is a bubble at equilibrium.

Our novel contribution is to give a sufficient condition (on exogenous parameters) under which a financial asset bubble occurs.

3. See Example 3.1 in Le Van and Vailakis (2012).

For simplicity, we assume that $f^i = 0$ for every i. Let us begin by the following result.

Lemma 4.1 *At each date t, there exists i such that $a_{i,t} \geq a_{i,t+1}$ and $a_{i,t} > 0$.*

Proof: Define i_0 such that

$$a_{i_0,t} - a_{i_0,t+1} = \max_i \{a_{i,t} - a_{i,t+1}\}.$$

Then $a_{i_0,t} - a_{i_0,t+1} \geq 0$.
Case 1 : $a_{i_0,t} - a_{i_0,t+1} > 0$ then $a_{i_0,t} > a_{i_0,t+1} \geq 0$.
Case 2 : $a_{i_0,t} - a_{i_0,t+1} = 0$ then $a_{i,t} - a_{i,t+1} \leq 0$ for every i. Since $\sum\limits_i (a_{i,t} - a_{i,t+1}) = 0$, we imply that $a_{i,t} - a_{i,t+1} = 0$ for every i. Choose i_1 such that $a_{i_1,t} > 0$, we have $a_{i_1,t} = a_{i_1,t+1}$ and $a_{i_1,t} > 0$. ■

We now bound the size of the discount rate γ_t by exogenous bounds.

Lemma 4.2 *(**Size of discount rate** γ_t)*
We have

$$A_t < \gamma_t < D_t, \tag{IV.24}$$

where $D_t := \max\limits_{i\in\{1,\ldots,m\}} \frac{\beta_i u_i'(e_{i,t})}{u_i'(W_{t-1})}$, $A_t := \min\limits_{i\in\{1,\ldots,m\}} \frac{\beta_i u_i'(W_{i,t})}{u_i'(\frac{W_{t-1}}{m})}$, *and* $W_t := \sum\limits_{i=1}^{m} e_{i,t} + \xi_t$.

Note that A_t, D_t are exogenous.

Proof: Recall that $\gamma_t := \max\limits_{i\in\{1,\ldots,m\}} \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})}$. By using Lemma 4.1, there exists i such that $a_{i,t-1} \geq a_{i,t}$ and $a_{i,t-1} > 0$. Since $a_{i,t-1} > 0$, we get $\mu_{i,t-1} = 0$, and then

$$\gamma_t = \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})}. \tag{IV.25}$$

On the one hand, we have $c_{i,t-1} < W_{t-1}$, so $u_i'(c_{i,t-1}) > u_i'(W_{t-1})$. On the other hand, we have

$$c_{i,t} + q_t a_{i,t} = e_{i,t} + (q_t + \xi_t)a_{t-1} \geq e_{i,t} + (q_t + \xi_t)a_t, \tag{IV.26}$$

hence $c_{i,t} \geq e_{i,t}$. Therefore, we get that

$$\gamma_t = \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,t-1})} \leq \frac{\beta_i u_i'(e_{i,t})}{u_i'(W_{t-1})} \leq D_t \tag{IV.27}$$

It is easy to see that there exists j such that $c_{j,t-1} \geq \frac{W_{t-1}}{m}$, so

$$\gamma_t \geq \frac{\beta_j u'_j(c_{j,t})}{u'_j(c_{j,t-1})} > \frac{\beta_j u'_j(W_t)}{u'_j(\frac{W_{t-1}}{m})} \geq A_t. \tag{IV.28}$$

■

Interest rates : Before give a sufficient condition for financial asset bubble, let us study the gross interest rate. We define r_t by $\frac{1}{1+r_t} = D_t$. It is easy to see that

$$r_t < R_t = \min_i \{R_{i,t}\}.$$

We recall that FV_t is the fundamental value of financial asset at date t. We have

$$q_0 = b_0 + \sum_{t=1}^{+\infty} Q_t \xi_t = b_0 + FV_0 \tag{IV.29}$$

$$q_1 = b_1 + \sum_{t=1}^{+\infty} Q_t^1 \xi_t = b_1 + FV_1, \tag{IV.30}$$

where $Q_t^1 := \dfrac{Q_t}{\gamma_1}$, and $b_1 = \dfrac{b_0}{\gamma_1}$.

Lemma 4.3 *We have*

$$\frac{q_1 + \xi_1}{q_0} = \frac{FV_1 + \xi_1}{FV_0}. \tag{IV.31}$$

Proof: We have $\dfrac{q_0}{q_1 + \xi_1} = \dfrac{b_0 + FV_0}{b_1 + FV_1 + \xi_1}$. Note that $b_0 = b_1\gamma_1 = \dfrac{q_0}{q_1 + \xi_1}$, and then we get (IV.31). ■

According to Condition (IV.31), the real return of an asset can be computed by using its prevent value : it equals the ratio between the sum its dividend and its fundamental value at date 1 and its fundamental value at date 0.

We now state our main result in this subsection.

Theorem 4.1 *(***An exogenous sufficient condition for financial asset bubble***)*
We normalize by setting $p_t = 1$ for every t.
There is a financial asset bubble at equilibrium if the following conditions hold :

(i) $B := \sum_{t=1}^{\infty} B_t \xi_t < \infty$, *where* $B_t := \prod_{k=1}^{t} D_k$.

(ii) There exists i such that

$$u_i'\big(e_{i,0}+\xi_0 a_{i,-1}-B(1-a_{i,-1})\big)\leq \beta_i\frac{A+\xi_1}{B}u_i'(W_1), \qquad \text{(IV.32)}$$

where $A:=\sum_{t=2}^{\infty}(\prod_{s=2}^{t}A_s)\xi_s$.

Note that these conditions is satisfied if $\xi_1,\xi_2,\dots,$ are small.

Proof: First, according to Lemma 4.2, we have

$$\begin{aligned} q_0 &= b_0+\sum_{t=1}^{+\infty}Q_t\xi_t < b_0+\sum_{t=1}^{+\infty}B_t\xi_t = b_0+B && \text{(IV.33)}\\ q_1 &\geq \sum_{t=2}^{\infty}(\prod_{s=2}^{t}\gamma_s)\xi_s > \sum_{t=2}^{\infty}(\prod_{s=2}^{t}A_s)\xi_s = A. && \text{(IV.34)} \end{aligned}$$

We now rewrite Condition (IV.32) as follows

$$\frac{\beta_i u_i'(W_1)}{u_i'\big(e_{i,0}+\xi_0 a_{i,-1}-B(1-a_{i,-1})\big)}\geq \frac{B}{A+\xi_1}. \qquad \text{(IV.35)}$$

This implies that

$$\frac{\beta_i u_i'(W_1)}{u_i'\big(e_{i,0}+\xi_0 a_{i,-1}-B(1-a_{i,-1})\big)}\geq \frac{B}{q_1+\xi_1}=\frac{q_0-b_0}{q_1+\xi_1}. \qquad \text{(IV.36)}$$

We also have

$$\begin{aligned} e_{i,0}+\xi_0 a_{i,-1}-B(1-a_{i,-1}) &< e_{i,0}+\xi_0 a_{i,-1}-(q_0-b_0)(1-a_{i,-1})\\ &= e_{i,0}+(q_0+\xi_0)a_{i,-1}-q_0+b_0(1-a_{i,-1})\\ &\leq e_{i,0}+(q_0+\xi_0)a_{i,-1}-q_0 a_{i,0}+b_0(1-a_{i,-1})\\ &= c_{i,0}+b_0(1-a_{i,-1}). \end{aligned}$$

Hence,

$$\frac{\beta_i u_i'(c_{i,1})}{u_i'(c_{i,0})}\leq \max_{j\in\{1,\dots,m\}}\frac{\beta_j u_j'(c_{j,1})}{u_j'(c_{j,0})}=\frac{q_0}{q_1+\xi_1}<\frac{b_0}{q_1+\xi_1}+\frac{\beta_i u_i'(c_{i,1})}{u_i'\big(c_{i,0}+b_0(1-a_{i,-1})\big)}.$$

Since the function $f(x)=\dfrac{x}{q_1+\xi_1}+\dfrac{\beta_i u_i'(c_{i,1})}{u_i'\big(c_{i,0}+x(1-a_{i,-1})\big)}$ is increasing in x, we obtain $b_0>0$. ■

Interpretation : We rewrite Condition (IV.32) as follows

$$\frac{\beta_i u_i'(W_1)}{u_i'\big(e_{i,0}+\xi_0 a_{i,-1}-B(1-a_{i,-1})\big)}\geq \frac{B}{A+\xi_1}. \qquad \text{(IV.37)}$$

B is an upper bound of the fundamental value FV_0 of financial asset at initial date 0. A is a lower bound of the fundamental value FV_1 of financial asset at date 1. We define $\bar{r}_1$ by $\frac{1}{1+\bar{r}_1} = \frac{B}{A+\xi_1}$. We see that $\bar{r}_1$ is a lower bound of the interest rate R_1. Indeed,

$$\frac{1}{1+R_1} = \frac{FV_0}{FV_1+\xi_1} \leq \frac{B}{A+\xi_1} = \frac{1}{1+\bar{r}_1}, \tag{IV.38}$$

So, we get

$$\bar{r}_1 < \bar{R}_1.$$

Define a bound $\bar{r}_{i,1}$ of interest rate of agent i by

$$\frac{1}{1+\bar{r}_{i,1}} = \frac{\beta_i u_i'(W_1)}{u_i'(e_{i,0}+\xi_0 - B(1-a_{i,-1})}.$$

We now see that Condition (IV.32) is equivalent to

$$1+\bar{r}_1 \geq 1+\bar{r}_{i,1}. \tag{IV.39}$$

This implies that the agent i's highest subjective interest rate $\bar{r}_{i,1}$ is less than the interest rate of the economy $\bar{R}_1$. Therefore, agent i accepts to buy financial asset with a price which is greater than the fundamental value. Consequently, there is a bubble.

5 Conclusion

We considered an infinite horizon general equilibrium asset pricing model with heterogeneous agents and endogenous borrowing constraints. We proved the existence of equilibrium in this model without any condition about endowments.

At equilibrium, if the market price of the financial asset is greater than its fundamental value, we say that there is a bubble. Borrowing constraints play an important role on bubble : a bubble can occur only if there exists an agent whose borrowing constraints are binding in infinitely many times. We prove that the existence of a bubble is equivalent to low interest rates. We also give an sufficient condition (on exogenous variables) for the existece of bubble : the highest subjective interest rate of agent is smaller than the interest rate of the economy.

6 Appendix

6.1 Existence of equilibrium for ϵ-economy

We also define $B_i^{T,\epsilon}(p, q,)$ as follows.

$$\begin{aligned} B_i^{T,\epsilon}(p,q) := \big\{ & (c_{i,t}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } a_{i,T} = 0, \\ & \text{(b) } p_0 c_{i,0} + q_0 a_{i,0} < p_0(e_{i,0} + \epsilon) + (q_0 + p_0 \xi_0)(a_{i,t-1} + \epsilon) \\ & \text{(c) for each } 1 \leq t \leq T : \\ & \quad 0 < (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t (e_{i,t} + \epsilon) \\ & \quad p_t c_{i,t} + q_t a_{i,t} < p_t(e_{i,t} + \epsilon) + (q_t + p_t \xi_t) a_{i,t-1} \big\}. \end{aligned}$$

We write $C_i^T(p,q), B_i^T(p,q)$ instead of $C_i^{T,0}(p,q), B_i^{T,0}(p,q)$.

Lemma 6.1 $B_i^{T,\epsilon}(p,q) \neq \emptyset$ *and* $\bar{B}_i^{T,\epsilon}(p,q) = C_i^{T,\epsilon}(p,q)$.

Proof: We write

$$\begin{aligned} B_i^{T,\epsilon}(p,q) := \big\{ & (c_{i,t}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : a_{i,T} = 0, \\ & 0 < p_0(e_{i,0} + \epsilon + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} + \epsilon - a_{i,0}) \\ & \text{and for each } 1 \leq t \leq T : \\ & 0 < q_t a_{i,t-1} + p_t(\xi_t a_{i,t-1} + f^i(e_{i,t} + \epsilon)) \\ & 0 < p_t(e_{i,t} + \epsilon + \xi_t a_{i,t-1} - c_{i,t}) + \bar{q}_t(a_{i,t-1} - a_{i,t}). \end{aligned}$$

Since $e_{i,0} + \epsilon + \xi_0 a_{i,t-1} > 0$ and $a_{i,t-1} + \epsilon > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $a_{i,0} \in (0, B_a)$ such that

$$0 < p_0(e_{i,0} + \epsilon + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} + \epsilon - a_{i,0}).$$

By induction, we see that $B_i(p, q,)$ is not empty. ■

Lemma 6.2 $B_i(p,r)$ *is lower semi-continuous correspondence on* $\mathcal{P}$*. And* $C_i(p,r)$ *is upper semi-continuous on* $\mathcal{P}$ *with compact convex values.*

Proof: Clearly, since $B_i(p,r)$ is empty and has open graph. ■

We define $\Phi := \Delta \times \prod_{i=1}^{m} (\mathcal{C}_i \times \mathcal{A}_i$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^m$ where $z_0 := (p, q, r), z_i := (c_i, a_i, k_i)$ for each $i = 1, \ldots, m$.

We now define correspondences. First, we define φ_0 (for additional agent 0)

$$\varphi_0 : \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{A}_i) \to 2^{\Delta}$$

$$\varphi_0((z_i)_{i=1}^m) := \arg\max_{(p,q)\in\Delta} \Big\{ p_0\big(\sum_{i=1}^{m}(c_{i,0}-e_{i,0}) - 2m\epsilon - \xi_0\big) + q_0 \sum_{i=1}^{m}(a_{i,0}-a_{i,-1}-\epsilon)$$
$$+ \sum_{t=1}^{T} p_t\big(\sum_{i=1}^{m}(c_{i,t}-e_{i,t}) - m\epsilon - \xi_t\big) + \sum_{t=1}^{T-1} q_t \sum_{i=1}^{m}(a_{i,t}-a_{i,t-1})\Big\}.$$

For each $i = 1, \ldots, m$, we define

$$\varphi_i : \quad \Delta\times \to 2^{\mathcal{C}_i\times\mathcal{A}_i}$$
$$\varphi_i(p,q) \quad := \quad \arg\max_{(c_i,a_i)\in C_i(p,q)} \Big\{ \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \Big\}.$$

Lemma 6.3 *The correspondence φ_i is lower semi-continuous and non-empty, concex, compact valued for each $i = 0, 1, \ldots, m+1$.*

Proof: This is a direct consequence of the Maximum Theorem. ■

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{q}) \in \varphi_0((\bar{c}_i, \bar{a}_i)_{i=1}^m) \tag{IV.40}$$
$$(\bar{c}_i, \bar{a}_i) \in \varphi_i((\bar{p}, \bar{q})). \tag{IV.41}$$

Denote

$$\bar{X}_0 := \sum_{i=1}^{m}(c_{i,0}-e_{i,0}) - 2m\epsilon - \xi_0) \tag{IV.42}$$

$$\bar{X}_t := \sum_{i=1}^{m}(c_{i,t}-e_{i,t}) - m\epsilon - \xi_t, \quad t \geq 1 \tag{IV.43}$$

$$\bar{Z}_0 = \sum_{i=1}^{m}(\bar{a}_{i,0} - \epsilon - \bar{a}_{i,-1}), \quad \bar{Z}_t = \sum_{i=1}^{m}(\bar{a}_{i,t} - \bar{a}_{i,t-1}), \quad t \geq 1. \tag{IV.44}$$

For every $(p,q) \in \Delta$, we have

$$\sum_{t=0}^{T}(p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T-1}(q_t - \bar{q}_t)\bar{Z}_t \leq 0. \tag{IV.45}$$

By summing the budget constraints, we get that : for each t

$$\bar{p}\bar{X}_t + \bar{q}\bar{Z}_t \leq 0. \tag{IV.46}$$

Hence, we have : for every $(p,q)\in\Delta$

$$p_t\bar{X}_t + q_t\bar{Z}_t \le \bar{p}\bar{X}_t + \bar{q}\bar{Z}_t \le 0. \tag{IV.47}$$

Therefore, we have $\bar{X}_t, \bar{Z}_t \le 0$, which implies that

$$\sum_{i=1}^{m}\bar{c}_{i,0} \le \sum_{i=1}^{m}\bar{e}_{i,0} + 2m\epsilon + \xi_0 \tag{IV.48}$$

$$\sum_{i=1}^{m}\bar{c}_{i,t} \le \sum_{i=1}^{m}\bar{e}_{i,t} + m\epsilon + \xi_t, \quad t\ge 1 \tag{IV.49}$$

$$\sum_{i=1}^{m}\bar{a}_{i,0} \le \sum_{i=1}^{m}(\bar{a}_{i,-1}+\epsilon), \quad \sum_{i=1}^{m}\bar{a}_{i,t} \le \sum_{i=1}^{m}\bar{a}_{i,t-1}, \quad t\ge 1. \tag{IV.50}$$

Lemma 6.4 *$\bar{p}_t > 0$ and $\bar{q}_t \ge 0$ for $t = 0,\dots,T$. Moreover, $\bar{q}_t > 0$ if $\xi_{t+1} > 0$.*

Proof: If $\bar{p}_t = 0$, the optimality implies that $\bar{c}_{i,t} = B_c > 1 + W_t$. Therefore, we get $\bar{c}_{i,t} > \sum\limits_{i=1}^{m} e_{i,t} + 2m\epsilon + \xi_t)$, contradiction. Hence, $\bar{p}_t > 0$
We now assume that $\xi_{t+1} > 0$. If $\bar{q}_t = 0$, we have $\bar{a}_{i,t} = B_a$ for each i. Thus, $\sum\limits_{i=1}^{m}\bar{a}_{i,t} \ge mB_a > 1 + B_a$. However, we have $\sum\limits_{i=1}^{m}\bar{a}_{i,t} \le \sum\limits_{i=1}^{m}\bar{a}_{i,-1} + m\epsilon = 1 + m\epsilon < 1 + B_a$, contradiction!
■

Lemma 6.5 *$\bar{X}_t = \bar{Z}_t = 0$.*

Proof: Since consumption good prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing budget constraints at date t we have which implies that.

$$\bar{p}_t\bar{X}_t + \bar{q}_t\bar{Z}_t = 0. \tag{IV.51}$$

By combining with the fact that $\bar{X}_t, \bar{Z}_t \le 0$, we obtain $\bar{X}_t = \bar{Z}_t = 0$. ■

The optimality of (c_i, a_i) is from (IV.41).

6.2 When ϵ tends to zero

We have so far proved that for each $\epsilon_n = 1/n > 0$, where n is interger number and high enough, there exists an equilibrium, say

$$equi(n) := \Big(\bar{p}_t(n), \bar{q}_t(n), (\bar{c}_{i,t}(n), \bar{a}_{i,t}(n))_{i=1}^{m}\Big)_{t=0}^{T},$$

for the economy $\mathcal{E}_b^{T,\epsilon_n}$.
By using borrowing constraint, we get that : for every n,

$$-\bar{a}_{i,t}(n) \leq \frac{f^i \bar{p}_{t+1}(n)(e_{i,t+1}+\epsilon_n)}{\bar{q}_{t+1}(n)+\xi_{t+1}\bar{p}_{t+1}(n)} \leq \frac{f^i(e_{i,t+1}+\epsilon_n)}{\xi_{t+1}} \leq \frac{1+W_{t+1}}{\xi_{t+1}}.$$

By combining with $\sum_{i=1}^m a_{i,t}(n) = 1$, we see that $a_{i,t}(n)$ is uniformly bounded when n tends to infinity. Moreover, we have $\bar{p}_t(n) + \bar{q}_t(n) = 1$. Therefore, we can assume that [4]

$$\begin{aligned}&(\bar{p}(n), \bar{q}(n), (\bar{c}_i(n), \bar{a}_i^T(n))_{i=1}^m) \\ &\qquad \xrightarrow{n\to\infty} (\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^m).\end{aligned}$$

Markets clearing conditions : By taking limit of market clearing conditions for economy $\mathcal{E}_b^{T,\epsilon_n}$, we obtain market clearing conditions for the economy $\mathcal{E}_b^T$.

Lemma 6.6 *$B_i^T(\bar{p}, \bar{q}) \neq \emptyset$ if $(e_{i,0}, a_{i,-1}) \neq (0,0)$.*

Proof: Recall that

$$\begin{aligned}B_i^{T,\epsilon}(\bar{p}, \bar{q}) := \big\{ &(c_{i,t}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : a_{i,T} = 0, \\ &0 < \bar{p}_0(e_{i,0} + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} - a_{i,0}) \\ &\text{ and for each } 1 \leq t \leq T : \\ &\ 0 < \bar{q}_t a_{i,t-1} + \bar{p}_t(\xi_t a_{i,t-1} + f^i e_{i,t}) \\ &0 < \bar{p}_t(e_{i,t} + \xi_t a_{i,t-1} - c_{i,t}) + \bar{q}_t(a_{i,t-1} - a_{i,t}).\end{aligned}$$

If $(e_{i,0}, a_{i,-1}) \neq (0,0)$, we have $e_{i,0} + \xi_0 a_{i,t-1} > 0$. By combining with $\bar{p}_t + \bar{q}_t = 1$, we can choose $c_{i,0} \in (0, B_c)$ and $a_{i,0} \in (0, B_a)$ such that

$$\begin{aligned}&0 < \bar{p}_0(e_{i,0} + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} - a_{i,0}) \\ &0 < \bar{q}_1 a_{i,0} + \bar{p}_1(\xi_1 a_{i,0} + f^i e_{i,1}.\end{aligned}$$

■

Lemma 6.7 *We have $\bar{p}_t, \bar{q}_t > 0$.*

Proof: Since $\sum_{i=1}^m a_{i,-1} = 1 > 0$, there exists an agent i such that $a_{i,-1} > 0$. According Lemma 6.6, we have $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$. We are going to prove that the optimality of allocation $(\bar{c}_i, \bar{a}_i)$.

4. In fact, since prices and allocations are bounded, there exists a subsequence $(n_1, n_2, \ldots,)$ such that $equi(n_s)$ converges. However, without loss of generality, we can assume that $equi(n)$ converges.

Let (c_i, a_i) be an feasible allocation of the maximization problem of agent i with the feasible set $C_i^T(\bar{p}, \bar{q})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Since $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$, there exists sequences $(h)_{h\geq 0}$ and $(c_i^h, a_i^h) \in B_i^T(\bar{p}, \bar{q})$ such that (c_i^h, a_i^h) converges to (c_i, a_i). We have

$$\bar{p}_t c_{i,t}^h + \bar{q}_t a_{i,t}^h < \bar{p}_t e_{i,t} + (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h$$
$$0 < (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h + f^i \bar{p}_t e_{i,t}.$$

Fixe h. Let n_0 (n_0 depends on h) be high enough such that for every $n \geq n_0$, $(c_i^h, a_i^h) \in C_i^{T,1/n}(\bar{p}(n), \bar{q}(n))$. Therefore, we have $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}^h) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t}(n))$.

Let n tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}^h) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Let h tend to infinity, we have $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. It means that we have just proved the optimality of $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$.

We now prove $\bar{p}_t > 0$ for every t. Indeed, otherwise we have $c_{i,t} = B_c > 1 + W_t$, contradiction. $\bar{q}_t > 0$ is from the positivity of financial dividend. ■

Lemma 6.8 *For each i, $(\bar{c}_i, \bar{a}_i)$ is optimal.*

Proof: Consider agent i. Since $(e_{i,0}, a_{i,-1}) \neq (0, 0)$, we have $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$. By using the same argument as in Lemma 6.7, we obtain the optimality of $(\bar{c}_i, \bar{a}_i)$.
■

6.3 The existence of equilibrium for the economy $\mathcal{E}$

It is easy to see that all markets clear.

Lemma 6.9 *We have $\bar{p}_t > 0$ for each $t \geq 0$.*

Proof: There exists i such that $a_{i,-1} > 0$. By using the same argument in Lemma 6.6, we see that $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$.

Let (c_i, a_i) be an feasible allocation of the problem $P_i(\bar{p}, \bar{q})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Note that, without loss of generality, we can only consider feasible allocations such that $\bar{p}_T c_{i,T} + \bar{q}_{i,T} a_{i,t} \geq 0$. We define $(c'_{i,t}, a'_{i,t})_{t=0}^T$ as follows :

$$\begin{aligned} a'_{i,t} &= a_{i,t}, \text{ if } t \leq T-1, = 0 \text{ if } t \leq T \\ c'_{i,t} &= c_{i,t}, \text{ if } t \leq T-1; \bar{p}_T c'_{i,T} = \bar{p}_T c_{i,T} + \bar{q}_{i,T} a_{i,t}; c_{i,t} = 0 \text{ if } t > T. \end{aligned}$$

We see that $(c'_{i,t}, a'_{i,t})_{t=0}^T$ belongs to $C_i^T(\bar{p}, \bar{q})$. Since $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$, there exists a sequence $\Big((c^n_{i,t}, a^n_{i,t})_{t=0}^T\Big)_{n=0}^{\infty} \in B_i^T(\bar{p}, \bar{q})$ with $a^n_{i,T} = 0$, and this sequence converges to $(c'_{i,t}, a'_{i,t})_{t=0}^T$ when n tends to infinity. We have

$$\bar{p}_t c^n_{i,t} + \bar{q}_t a^n_{i,t} < \bar{p}_t e_{i,t} + (\bar{q}_t + \bar{p}_t \xi_t) a^n_{i,t-1}.$$

We can chose s_0 high enough and $s_0 > T$ such that : for every $s \geq s_0$, we have

$$\bar{p}^s_t c^n_{i,t} + \bar{q}^s_t a^n_{i,t} < \bar{p}^s_t e_{i,t} + (\bar{q}^s_t + \bar{p}^s_t \xi_t) a^n_{i,t-1}.$$

It means that $(c^n_{i,t}, a^n_{i,t})_{t=0}^T \in C_i^T(\bar{p}^s, \bar{q}^s)$. Therefore, we get $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{s} \beta_i^t u_i(\bar{c}^s_{i,t})$. Let s tend to infinity, we obtain $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. Let n tends to infiniy, we have $\sum\limits_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$ for every T. As a consequence, we have : for every T

$$\sum_{t=0}^{T-1} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity, we obtain $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.
Therefore, we have proved that the optimality of $(\bar{c}_i, \bar{a}_i)$.
Prices $\bar{p}_t, \bar{q}_t$ is strictly positive since the utility function of agent i is strictly increasing and $\xi_t > 0$ for every t. ■

Lemma 6.10 *For each i, $(\bar{c}_i, \bar{a}_i)$ is optimal.*

Proof: Since $\bar{p}_t, \bar{q}_t$ and $(e_{i,0}, a_{i,-1}) \neq (0, 0)$, we get that $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$. By using the same argument in Lemma 6.9, we can prove the optimality of $(\bar{c}_i, \bar{a}_i)$. ■

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Chapitre V

Intertemporal equilibrium with financial asset and physical capital

Abstract : We build an infinite-horizon dynamic deterministic general equilibrium model with imperfect market (because of borrowing constraints) in which heterogeneous agents invest in capital or/and financial asset, and consume. There is a representative firm who maximizes its profit. Firstly, the existence of intertemporal equilibrium is proved even if aggregate capital is not uniformly bounded. Secondly, we study the interaction between the financial market and the productive sector.

Keywords : Infinite horizon, intertemporal equilibrium, financial friction, productivity, efficiency, fluctuation.

JEL Classifications : C62, D31, D91, E44, G10.

1 Introduction

The recent financial crisis requires us to reconsider the role of the financial market on aggregate economic activity. The financial market has been considered as one of the main causes of economic recession or/and fluctuation. But, does financial market always cause an economic recession ? What is the role of financial market on the productive sector ?

To answer these questions, our approach is to construct a dynamic deterministic general equilibrium with heterogenous agents, capital accumulation, and imperfect financial market. In our model, consumers differ in discount factors, reward functions and initial wealths.[1] Heterogeneous consumers invest, borrow, and consume. They have two choices to invest : in productive sector and in financial sector. At date t, if one invests in the physical capital, he (or she) will receive a return that depends on the marginal productivity of the economy at next date. In the financial market, if he (or she) buys one unit of financial asset at date t, he (or she) will be able to resell this asset and also receive ξ_{t+1} units of consumption good as dividend. When agents want to borrow, they are required to hold some amounts of the physical capital as collateral. The market value of collateral must be greater than the value of debt. Because of this constraint, the financial market is imperfect.

The first contribution of our paper concerns the existence of intertemporal equilibrium. Becker, Boyd III, Foias (1991) demonstrated the existence of intertemporal equilibrium under borrowing constraints with inelastic labor supply. Kubler and Schmedders (2003) constructed and proved the existence of Markov equilibrium in an infinite-horizon asset pricing model with incomplete market and collateral constraint, but without capital accummulation. Such a Markov equilibrium was also proved to be competitive equilibrium. Becker, Bosi, Le Van, Seegmuller (2014) proved the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical capital ; however, they only considered an implicit financial market and assumed that no one can borrow. In these papers, they needed some assumptions (about endowments as in Kubler and Schmedders (2003), and about production function as in Becker, Boyd III, Foias (1991), Becker, Bosi, Le Van, Seegmuller (2014)) to ensure that aggregate capital and consumption stocks are uniformly bounded. Here we allow growth for the physical quantities (consumption, capital stocks, outputs). Our framework is rich enough to cover both productive sector and imperfect financial market.[2] Moreover, in our proof of the existence of an intertemporal equilibrium, we allow non-stationary and even linear pro-

1. A detailed survey on the effects of heterogeneity in macroeconomics can be found in Guvenen (2012)

2. However, for simplicity, we assume exogenous supply of labour.

duction functions and do not need that aggregate capital and consumption stocks be uniformly bounded. We firstly prove that there exists an equilibrium for each $T-$truncated economy. We then obtain a sequence of equilibria (indexed by T) which will be proved to have a limit for the product topology. Last, we prove that such limit is an intertemporal equilibrium.

Analyzing the relationship between the financial market and the productive sector is our second contribution. We explore three important points.

The first one concerns the economic recession by which we mean a situation where no one invests in the productive sector. Although there are many sources for economic recession as war, policy shocks, financial shocks..., we focus on productivity of the productive sector. Our finding is summarized as follows.

(i) When the productivity is high enough, the economy never falls in recession.

(ii) When the productivity is low, the economy will fall in recession at infinitely many dates (not necessary at all dates) because the agents prefer financial assets to physical capital.

(iii) However, at some dates, even when the productivity is low, financial assets may be beneficial to the economy by providing financial support for the purchase of the physical capital. Thanks to that, a recession may be avoided. Moreover, when the productivity is lower than the depreciation rate, introducing dividends may prevent the economy to collapse, i.e., to converge to zero when time goes to infinity.

The second point concerns fluctuations of the aggregate capital path (K_t). We prove that, under some mild conditions, there exists an infinite sequence of time (t_n) such that $K_{t_n} = 0$ for every n, but $\limsup\limits_{t\to\infty} K_t > 0$.

Third, we study the efficiency and the existence of bubbles of intertemporal equilibrium. An intertemporal equilibrium is called to be efficient if its aggregate capital path is efficient in the sense of Malinvaud (1953). When the production technology is stationary, we give *exogenous* conditions on the financial dividends and the marginal productivity to obtain that, at equilibrium, there is no bubble on the financial or/and the physical assets markets and efficiency of any equilibrium path.

Related literature : Our paper is related to several strands of research.

(i) The first strand concerns *General equilibrium with incomplete markets*. An excellent introduction to asset pricing models with incomplete markets and infinite horizon can be found in Magill and Quinzii (2008). On collateral equilibrium, Geanakoplos, Zame (2002) proved the existence of collateral equilibrium in a two-period models that incorporates durables gooods and collateralized securities. By extending Geanakoplos, Zame

(2002), Araujo, Pascoa, Torres-Martinez (2002) proved the existence of equilibrium for an infinite horizon models with collateral requirement on selling financial assets. Pham (2013) proved the existence of collateralized monetary equilibrium in an infinite horizon monetary economy. Note that, in these papers, they did not take into account the role of the productive sector.

(ii) *Credit market frictions and aggregate economic activity* : Our paper is also related to Kiyotaki, Moore (1997). However, they did not take into account the existence of intertemporal equilibrium. Some other significant researchs (Scheinkman, Weiss (1986), Bernanke, Gertler, Gilchrist (1999), Matsuyama (2007), Gertler, Kiyotaki (2010), Christiano, Motto, Rostagno (2010)) have explained why credit market frictions can make impact on aggregate econnomic activity. Gabaix (2011) proposes that idiosyncratic firm-level shocks can explain an important part of aggregate movements. Basu, Pascali, Schiantarelli, Serven (2012) show that aggregate TFP, appropriately defined, and the capital stock can be used to construct sufficient statistics for the welfare of a representative consumer. Brunnermeier, Sannikov (2014) incorporated financial sector in a macroeconomic model with continuous time. See Brunnermeier, Eisenbach, Sannikov (2012) for a complete review on macroeconomics with financial frictions.

(iii) *On the efficiency of capital paths.* Malinvaud (1953) introduced the concept of efficiency of a capital path and gave a sufficient condition of the efficiency : $\lim\limits_{t\to\infty} P_tK_t = 0$, where (P_t) is a sequence of competitive prices, (K_t) is the capital path.[3] Following Malinvaud, Cass (1972) considered capital path which is uniformly bounded from below. Under the concavity of a stationary production function and some mild conditions, he proved that a capital path is inefficient if and only if the sum (over time) of future values of a unit of physical capital is finite. Cass and Yaari (1971) gave a necessary and sufficient condition for a consumption plan (C) to be efficient, which can be stated that the inferior limit of differences between the present value of any consumption plan and the plan (C) is negative. Our paper is also related to Becker and Mitra (2012) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. Mitra and Ray (2012) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid.

Our finding is different from their result because we introduce another long-lived asset into a standard Ramsey model with heterogeneous agents. Financial dividends play an important role on the efficiency of capital paths. It may make aggregate capital paths efficient. Interestingly, thanks to financial dividend, an efficient capital path may have zero capital stocks at

3. See Malinvaud (1953), Lemma 5, page 248.

some dates.

(iv) Another concept of efficiency is *constrained efficiency*. Constrained inefficiency occurs when there exists a welfare improving feasible redistribution subject to constraints (these constraints depends on models). About the constrained efficiency in general equilibrium models with financial asset, see Kehoe and Levine (1993), Alvarez and Jermann (2000), Bloise and Pietro (2011). About the constrained efficiency in the neoclassical growth model, see Davila, Hong, Krusell and Rios-Rull (2012).

The remainder of the paper is organized as follows. Section 2 presents the structure of economy. In Section 3, we discuss about the existence of intertemporal equilibrium. Section 4 studies the interaction between the financial market and the productive sector. The efficiency of intertemporal equilibrium is presented in Section 5. Section 6 gives conditions to have no bubble on both markets. Section 7 concludes. Technical proofs can be found in Appendix.

2 Model

The model is an infinite-horizon general equilibrium model without uncertainty, $t = 0, \ldots, \infty$. There are two types of agents : a representative firm without market power and m households. Each household invests in physical asset and/or financial asset, and consumes.

Consumption good : there is a single consumption good. At each period t, the price of consumption good is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Physical capital : at time t, if agent i buys $k_{i,t+1} \geq 0$ units of new capital, agent i will receive $(1-\delta)k_{i,t+1}$ units of old capital at period $t+1$, after being depreciated, and $k_{i,t+1}$ units of old capital can be sold at price r_{t+1} .

Financial assets : at period t, if agent i invests $a_{i,t}$ units of financial asset with price q_t, she will receive ξ_{t+1} units of consumption good as dividend and she will be able to resell $a_{i,t}$ units of financial asset with price q_{t+1}. These assets may be lands, houses...

Table 1 **: Household i's balance sheet at date** t

	Expenditures	Revenues	
Consumption	$p_t c_{i,t}$	$\theta^i \pi_t$	share of profit
Capital investment	$p_t(k_{i,t+1} - (1-\delta)k_{i,t})$	$r_t k_{i,t}$	capital return from date $t-1$
Financial asset	$q_t a_{i,t}$	$(q_t + p_t \xi_t) a_{i,t-1}$	financial delivery from date $t-1$

Each household i takes the sequence of prices $(p,r,q)=(p_t,r_t,q_t)_{t=0}^{\infty}$ as given and solves the following problem

$$\begin{aligned}
(P_i(p,r,q)) \quad &: \quad \max_{(c_{i,t},k_{i,t},a_{i,t})_{t=0}^{+\infty}} \Big[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\Big] \\
\text{subject to :} \quad & k_{i,t+1} \geq 0 \\
\text{budget constraint :} \quad & p_t(c_{i,t}+k_{i,t+1}-(1-\delta)k_{i,t}) + q_t a_{i,t} \\
& \leq r_t k_{i,t} + (q_t + p_t\xi_t)a_{i,t-1} + \theta^i \pi_t \\
\text{borrowing constraint :} \quad & (q_{t+1}+p_{t+1}\xi_{t+1})a_{i,t} \geq -f^i(p_{t+1}(1-\delta)+r_{t+1})k_{i,t+1},
\end{aligned}$$

where $f^i \in (0,1)$ is borrowing limit of agent i. f^i is exogenous and set by law.

In our setup, the borrowing constraint is endogenous. Agent i can borrow an amount but the repayment of this amount does not exceed a fraction of the market value of his physical capital. This fraction is less than 1, i.e., the market value of collateral of each agent is greater than its debt.

For each period, there is a representative firm which takes prices (p_t,r_t) as given and maximizes its profit by choosing physical capital amount K_t.

$$(P(p_t,r_t)) : \qquad \max_{K_t \geq 0} \Big[p_t F_t(K_t) - r_t K_t\Big] \tag{V.1}$$

$(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^{m} \theta^i = 1$.

2.1 Equilibrium

We define an infinite-horizon sequence of prices and quantities by

$$(p,r,q,(c_i,k_i,a_i)_{i=1}^m, K, L)$$

where, for each $i=1,\ldots,m$,

$$\begin{aligned}
(c_i,k_i,a_i) &:= \Big((c_{i,t})_{t=0}^{+\infty},(k_{i,t})_{t=0}^{+\infty},(a_{i,t})_{t=0}^{+\infty}\Big) \in \mathbb{R}_+^{+\infty}\times\mathbb{R}^{+\infty}\times\mathbb{R}_+^{+\infty}\times\mathbb{R}^{+\infty}, \\
(p,r,q) &:= \Big((p_t)_{t=0}^{+\infty},(r_t)_{t=0}^{+\infty},(q_t)_{t=0}^{+\infty}\Big) \in \mathbb{R}^{+\infty}\times\mathbb{R}_+^{+\infty}\times\mathbb{R}^{+\infty}, \\
(K) &:= \Big((K_t)_{t=0}^{+\infty}\Big) \in \mathbb{R}_+^{+\infty}.
\end{aligned}$$

We also denote $z_0 := (p,r,q)$, $z_i := (c_i,k_i,a_i)$ for each $i=1,\ldots,m$, $z_{m+1} = (K)$ and $z=(z_i)_{i=0}^{m+1}$.

Definition 2.1 *A sequence of prices and quantities* $\Big(\bar{p}_t,\bar{r}_t,\bar{q}_t,(\bar{c}_{i,t},\bar{k}_{i,t},\bar{a}_{i,t})_{i=1}^m,\bar{K}_t\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy*

$$\mathcal{E} = \Big((u_i,\beta_i,k_{i,0},a_{i,-1},f^i,\theta^i)_{i=1}^m,(F_t,\xi_t)_{t=0}^{\infty},\delta\Big).$$

if the following conditions are satisfied :

(i) Price positivity : $\bar{p}_t, \bar{r}_t, \bar{q}_t > 0$ *for* $t \geq 0$.
(ii) Market clearing : at each $t \geq 0$,

$$\begin{aligned} \textit{good}: &\quad \sum_{i=1}^{m}(\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) = F_t(\bar{K}_t) + \xi_t, \\ \textit{capital}: &\quad \bar{K}_t = \sum_{i=1}^{m} \bar{k}_{i,t}, \\ \textit{financial asset}: &\quad \sum_{i=1}^{m} \bar{a}_{i,t} = 1. \end{aligned}$$

(iii) Optimal consumption plans : for each i, $\Big((\bar{c}_{i,t}, \bar{k}_{i,t}, \bar{a}_{i,t})_{i=1}^m\Big)_{t=0}^{\infty}$ *is a solution of the problem* $(P_i(\bar{p}, \bar{r}, \bar{q}))$.
(iv) Optimal production plan : for each $t \geq 0$, $(\bar{K}_t)$ *is a solution of the problem* $(P(\bar{r}_t))$.

The following result proves that aggregate capital and consumption are bounded for the product topology.

Lemma 2.1 *Capital and consumption stocks are in a compact set for the product topology. Moreover, they are uniformly bounded if* $(\xi_t)_t$ *are uniformly bounded and there exists* t_0 *and an increasing, concave function* G *such that the two following conditions are satisfied : (i) for every* $t \geq t_0$ *we have* $F_t(K) \leq G(K)$ *for every* K*, (ii) there exists* $x > 0$ *such that* $G(y) + (1-\delta)y \leq y$ *for every* $y \geq x$.

Proof: Denote

$$\begin{aligned} D_0(F, \delta, K_0, \xi_0) &:= F_0(K_0) + (1-\delta)K_0 + \xi_0, \\ D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) &:= F_t(D_{t-1}(F, \delta, K_0, \xi_0, \dots, \xi_{t-1}), m) \\ &\quad +(1-\delta)D_{t-1}(F, \delta, K_0, \xi_0, \dots, \xi_{t-1}) + \xi_t \quad \forall t \geq 0. \end{aligned}$$

Then $\sum_{i=1}^{m} c_{i,t} + K_{t+1} \leq D_t$ for every $t \geq 0$.

We now assume that time t_0 and the function G (in the statement of Lemma 2.1) exist. We are going to prove that $0 \leq K_t \leq \max\{D_{t_0}, x\} =: K$. Indeed, $K_t \leq K$ for every $t < t_0$. For $t \geq t_0$, we have

$$K_{t+1} = \sum_{i=1}^{m} k_{i,t+1} \quad \leq \quad G(K_t) + (1-\delta)K_t.$$

Then $K_{t_0} \leq G(K_{t_0-1}) + (1-\delta)K_{t_0-1} \leq G(K) + (1-\delta)K \leq K$. Iterating the argument, we find $K_t \leq K$ for each $t \geq 0$.
Consumptions are bounded because $\sum_{i=1}^{m} c_{i,t} \leq F_t(K_t) + (1-\delta)K_t + \xi$. ■

3 The existence of equilibrium

Standard assumptions are required.

Assumption (H1) : u_i is in C^1, $u_i(0) = 0$, $u_i'(0) = +\infty$, and u_i is strictly increasing, concave, continuously differentiable.

Assumption (H2) : $F_t(\cdot)$ is strictly increasing, concave, continuously differentiable, $F_t(0) = 0$.

Assumption (H3) : For every $t \geq 0, 0 < \xi_t < \infty$.

Assumption (H4) : At initial period 0, $k_{i,0}, a_{i,-1} \geq 0$, and $(k_{i,0}, a_{i,-1}) \neq (0,0)$ for $i = 1, \dots, m$. Moreover, we assume that $\sum\limits_{i=1}^{m} a_{i,-1} = 1$ and $K_0 := \sum\limits_{i=1}^{m} k_{i,0} > 0$.

Remark 3.1 *Here we differ from Becker, Bosi, Le Van, Seegmuller (2014) by do not imposing $F'(\infty) < \delta$. We accept the AK production technology.*

First, we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$. Second, we show that this sequence of equilibriums converges for the product topology to an equilibrium of our economy $\mathcal{E}$. The added value in our proof is that we do not need that aggregate capital stocks are uniformly bounded, and we allow non-stationary technologies. Moreover, incorporating financial market with borrowing constraints also requires some new techniques in order to prove the existence of intertemporal equilibrium.

To prove the existence of equilibrium for $T-$ truncated economy $\mathcal{E}^T$, we prove the existence of the bounded economy $\mathcal{E}_b^T$ and then by using the concavity of the utility function, we will prove that such equilibrium is also an equilibrium of $\mathcal{E}^T$.

3.1 The existence of equilibrium for $T-$ truncated economy $\mathcal{E}^T$

We define $T-$ truncated economy $\mathcal{E}^T$ as $\mathcal{E}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t} = a_{i,t-1} = k_{i,t} = K_t = 0$ for every $i = 1, \dots, m$, $t \geq T+1$.

Then we define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all variables (consumption demand, capital supply, asset investment, capital demand) are bounded. See Appendix for details.

Lemma 3.1 *Under Assumptions (H1)-(H4), there exists an equilibrium for $\mathcal{E}_b^T$.*

Proof: See Appendix 8.1. ■

Lemma 3.2 *An equilibrium of $\mathcal{E}_b^T$ is an equilibrium for $\mathcal{E}^T$.*

Proof: See Appendix 8.2. ■

3.2 The existence of an equilibrium in $\mathcal{E}$

To take the limit of sequence of equilibria, we need the following assumption.

Assumption (H5) : For each i, utility of agent i is finite

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) < \infty. \tag{V.2}$$

Remark 3.2 *With stationary technology, condition V.2 holds if there exists $b < \infty$ such that, for every $i \in \{1, \dots, m\}$,*

$$\sum_{t=0}^{\infty} \beta_i^t \max_{s \leq t}\{\xi_s\} < \infty, \tag{V.3}$$

$$\sum_{t=0}^{\infty} \beta_i^t (F'(b) + 1 - \delta)^t \max_{s \leq t}\{\xi_s, 1\} < \infty. \tag{V.4}$$

Proof: See Appendix 8.4. ■

Note that there exist some cases where although $F'(\infty) > \delta$ and $(\xi_t)_t$ are not uniformly bounded, but conditions (V.3) and (V.4) still hold. For example, if there exist $b < \infty$ and $\alpha > 1$ such that $\xi_t \leq \alpha^t$ and $\alpha\beta_i(F'(b) + 1 - \delta) < 1$, conditions (V.3) and (V.4) hold.

Theorem 3.1 *Under Assumptions (H1)-(H5), there exists an equilibrium in the infinite-horizon economy $\mathcal{E}$.*

Proof: See Appendix 8.3. We consider the limit of sequences of equilibria in $\mathcal{E}^T$, when $T \to \infty$. We use convergence for the product topology. ■

4 Financial market vs productive sector

In this section, we will study the interaction between the financial market and the productive sector. For simplicity, we only consider stationary technology.

Let $\big(p_t, q_t, r_t, (c_{i,t}, a_{i,t}, k_{i,t})_{i=1}^m, K_t\big)_t$ be an equilibrium. Denote $\mu_{i,t}, \nu_{i,t+1}$ the multiplier associated with (V.1), (V.1) respectively. Denote $\lambda_{i,t+1}$ the multiplier associated with borrowing constraint of agent i at date t. We have

$$\beta_i^t u_i'(c_{i,t}) = p_t\mu_{i,t} \tag{V.5}$$
$$p_t\mu_{i,t} = (r_{t+1} + (1-\delta)p_{t+1})(\mu_{i,t+1} + f^i\nu_{i,t+1}) + \lambda_{i,t+1} \tag{V.6}$$
$$q_t\mu_{i,t} = (q_{t+1} + p_{t+1}\xi_{t+1})(\mu_{i,t+1} + \nu_{i,t+1}). \tag{V.7}$$

Note that $k_{i,t+1}\lambda_{i,t+1} = 0$ and

$$\nu_{i,t+1}\big((q_{t+1} + p_{t+1}\xi_{t+1})a_{i,t} + f^i(p_{t+1}(1-\delta) + r_{t+1})k_{i,t+1}\big) = 0.$$

Lemma 4.1 *We have, for each t,*

$$\frac{q_t}{q_{t+1} + p_{t+1}\xi_{t+1}} = \max_i\Big\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\Big\} \leq \frac{p_t}{r_{t+1} + (1-\delta)p_{t+1}}. \tag{V.8}$$

Moreover, the equality holds if there exists i such that $k_{i,t+1} > 0$.

Proof: Since $\sum_{i=1}^m a_{i,t} = 1$, there exists i such that $a_{i,t} > 0$, and then $\nu_{i,t+1} = 0$. As a consequence, we get

$$\frac{q_t}{q_{t+1} + p_{t+1}\xi_{t+1}} = \max_i\Big\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\Big\}.$$

It is easy to see that $\frac{p_t}{r_{t+1} + (1-\delta)p_{t+1}} \geq \max_i\Big\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\Big\}$. Assume that $k_{i,t+1} > 0$, we have $\lambda_{i,t+1} = 0$, and then

$$\frac{p_t}{r_{t+1} + (1-\delta)p_{t+1}} = \frac{\mu_{i,t+1} + f^i\nu_{i,t+1}}{\mu_{i,t}} \leq \frac{\mu_{i,t+1} + \nu_{i,t+1}}{\mu_{i,t}} = \frac{q_t}{q_{t+1} + p_{t+1}\xi_{t+1}}.$$

Therefore, we have $\frac{p_t}{r_{t+1}+(1-\delta)p_{t+1}} = \max_i\big\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\big\}$. ■

Normalization : Since $p_t > 0$, without loss of generality, we assume that $p_t = 1$.

In our framework, consumers have two possibilities to invest : in the financial asset and/or in the physical capital. We would like to know when consumers invest in the physical capital and/or in the financial asset. Note that the return of the physical capital is $r_{t+1} + 1 - \delta$, and the return of the financial asset is $\frac{q_{t+1}+\xi_{t+1}}{q_t}$.

Economic recession : We say that there is an economic recession at date t if no one invests in the productive sector, i.e., the aggregate capital equals zero, $K_t = 0$. We will point out the role of the competitiveness of the productive sector. We say that that the productive sector is competitive if

$\min_i \beta_i(F'(0)+1-\delta) > 1$. It is extremely competitive if $F'(0) = +\infty$. Note that $(F'(0)+1-\delta)\min_i \beta_i > 1$ is equivalent to $F'(0) > \dfrac{1}{\min_i \beta_i} - 1 + \delta$, where $\dfrac{1}{\min_i \beta_i} - 1 + \delta$ is the highest investment cost if we use some interest rates to define the discount factors β_i.

We have the following results showing the respective roles of the productivity and the financial dividends.

Lemma 4.2 *If* $\dfrac{q_{t+1}+\xi_{t+1}}{q_t} \geq (F'(0)+1-\delta)$ *then consumers do not invest in the physical capital, i.e.,* $K_{t+1} = 0$.

Proof: Suppose that $\dfrac{q_{t+1}+\xi_{t+1}}{q_t} \geq (F'(0)+1-\delta)$. If $K_{t+1} > 0$, there exists $i \in \{1,\cdots,m\}$ such that $k_{i,t+1} > 0$. On the one hand, according to Lemma 4.1, we have $k_{i,t+1}$ gives us : $\max\limits_i\{\dfrac{\mu_{i,t+1}}{\mu_{i,t}}\} = \dfrac{1}{r_{t+1}+1-\delta}$. FOC of K_{t+1} implies that $r_{t+1} = F'(K_{t+1}) < F'(0)$, hence $\max\limits_i\{\dfrac{\mu_{i,t+1}}{\mu_{i,t}}\} > \dfrac{1}{F'(0)+1-\delta}$. On the other hand, we have $\max\{\dfrac{\mu_{i,t+1}}{\mu_{i,t}}\} = \dfrac{q_t}{q_{t+1}+\xi_{t+1}}$. This implies that $\dfrac{q_{t+1}+\xi_{t+1}}{q_t} < (F'(0)+1-\delta)$, contradiction. ■

Lemma 4.2 says that if the maximum real return of the physical capital is less than the financial asset's return, households do not invest in the physical capital.

Proposition 4.1 *Assume that there exists* $\xi > 0$ *such that* $\xi_t \geq \xi$ *for every* $t \geq 0$ *and* $F'(0) \leq \delta$. *Then*[4] *there is an infinite sequence* $(t_n)_{n=0}^{\infty}$ *such that* $K_{t_n} = 0$ *for every* $n \geq 0$.

Proof: We claim that there exists an infinite increasing sequence $(t_n)_{n=0}^{\infty}$ such that $q_{t_n} + \xi_{t_n} > q_{t_n-1}$ for every $n \geq 0$.
Indeed, if not, there exists t_0 such that $q_{t+1}+\xi_{t+1} \leq q_t$ for every $t \geq t_0$. Combining with $\xi_t \geq \xi$ for every $t \geq 0$, we can easily prove that $q_{t+t_0} + t\xi \leq q_{t_0}$ for every $t \geq 0$. Let $t \to \infty$, we have $q_{t_0} = \infty$, contradiction !
Therefore, there exists a sequence (t_n) such that for every $n \geq 0$, $\dfrac{q_{t_n}+\xi_{t_n}}{q_{t_n-1}} > 1 \geq F'(0)+1-\delta$. Lemma 4.2 implies that $K_{t_n} = 0$ for every $n \geq 0$. ■

4. By using the same argument, we can prove this result if Assumption "$\xi_t \geq \xi > 0$ for every $t \geq 0$" is replaced by Assumption "$\sum\limits_{t=0}^{\infty} \xi_t = \infty$".

Proposition 4.1 shows that if the productivity is low, economic recession will appear at infinitely many dates. Since the bound ξ does not depend on the technology, we see that economic recession is not from the financial market, but from the fact that the productive sector is not competitive. This result suggests that we should invest in technology to improve the competitiveness of productive sector in order to avoid economic recession.

We illustrate our finding by the following example.

Example 4.1 *($K_t = 0$ for every $t \geq 1$)*
Consider an economy with two agents i and j such that

$$\begin{aligned}
&\beta_i = \beta_i = \beta \in (0,1), \quad u_i(x) = u_j(x) = \frac{x^{1-\sigma}}{1-\sigma},\\
&K_0 > 0, \quad \beta(F'(0) + 1 - \delta) \leq 1,\\
&a_{i,-1} = \theta^i = \frac{k_{i,0}}{K_0} = a \in (0,1),\\
&\xi_0; \quad \xi_t = \xi \quad \forall t \geq 1,
\end{aligned}$$

where q_0, ξ_0, ξ, K_0 are such that

$$\begin{aligned}
1 &\geq \beta(F'(0) + 1 - \delta)\Big(\frac{F(K_0) + (1-\delta)K_0 + \xi_0}{\xi}\Big)^\sigma,\\
\Big(\frac{F(K_0) + (1-\delta)K_0 + \xi_0}{\xi}\Big)^\sigma &= \frac{q_0}{\xi}\frac{1-\beta}{\beta}.
\end{aligned}$$

An equilibrium is given by the following

$$\begin{aligned}
\textit{Allocations :}\quad & a_{i,t} = a, \quad a_{j,t} = 1 - a \quad \forall t \geq 1,\\
& k_{i,t} = k_{j,t} = 0 \quad \forall t \geq 1,\\
& c_{i,0} = a(F(K_0) + (1-\delta)K_0 + \xi_0), \quad c_{i,t} = a\xi, \quad \forall t \geq 1,\\
& c_{j,0} = (1-a)(F(K_0) + (1-\delta)K_0 + \xi_0), \quad c_{j,t} = (1-a)\xi, \quad \forall t \geq 1,\\
\textit{Prices :}\quad & r_0 = F'(K_0), \quad r_t = F'(0) \quad \forall t \geq 1,\\
& q_0, \quad q_t = \xi\frac{\beta}{1-\beta} \quad \forall t \geq 1.
\end{aligned}$$

Proof: See Appendix 8.4. ■

In Proposition 4.2 and its two corollaries, the competitiveness of the productive sector is high.

Proposition 4.2 *Assume that there exist $t \geq 0, T \geq 1$ such that $\xi_t \geq \xi_{t+T}$. If $(F'(0) + 1 - \delta)\min_i \beta_i > 1$, there exists $1 \leq s \leq T$ such that $K_{t+s} > 0$.*

Proof: See Appendix 8.4. ■

Corollary 4.1 *Assume that there exists an infinite decreasing sequence $(\xi_{t_n})_{n=0}^{\infty}$, i.e., $\xi_{t_n} \geq \xi_{t_{n+1}}$ for every $n \geq 0$. If $(F'(0)+1-\delta)\min_i \beta_i > 1$, there exists an infinite sequence $(\tau_n)_{n\geq 0}$ such that $K_{\tau_n} > 0$ for every $n \geq 0$ at any equilibrium.*

Corollary 4.2 *Assume that $\xi_t = \xi > 0$ for every $t \geq 0$. If $(F'(0)+1-\delta)\min_i \beta_i > 1$, we have $K_t > 0$ for every $t \geq 1$ at any equilibrium.*

We continue our exposition by the following result :

Proposition 4.3 *If $\beta_i(F'(0)+1-\delta)u_i'(\xi_{t+1}) > u_i'(\dfrac{F(K_t)+(1-\delta)K_t+\xi_t}{m})$ for every $i = 1,\ldots,m$, we have $K_{t+1} > 0$.*

Proof: See Appendix 8.4. ■

On the one hand, Proposition 4.3 proves that if the productivity $F'(0) = \infty$ then $K_{t+1} > 0$ at equilibrium. On the other hand, Proposition 4.3 also shows that the financial market plays an important role in the productive sector. Indeed, consider an equilibrium where $K_t = 0$. Assume also that ξ_t is high enough such that, for every i,

$$\beta_i(1-\delta)u_i'(\xi_{t+1}) > u_i'(\frac{\xi_t}{m}).$$

According to Proposition 4.3, we obtain that $K_{t+1} > 0$. This is due to the fact that part of the financial dividend is used to buy the physical capital.

A natural question is whether the aggregate capital stock K_t is bounded away from zero. To answer this question, the following result is useful :

Proposition 4.4 *Given $K \geq 0, \xi > 0$, let $G_i(K,\xi)$ be defined by*

$$u_i'(G_i(K,\xi)) = \big(F'(K)+1-\delta\big)\beta_i u_i'(F(K)+(1-\delta)K+\xi).$$

At equilibrium, there exists i such that

$$\xi_t \leq K_{t+1} + mG_i(K_{t+1},\xi_{t+1}), \text{ for any } t. \tag{V.9}$$

Proof: See Appendix 8.4. ■

Since u', F' are decreasing and F is increasing, we can see that $G_i(K_{t+1},\xi_{t+1})$ is increasing in K_{t+1} and ξ_{t+1}. We point out some consequences of Proposition 4.4.

we start by the following result, the proof of which is trivial by Proposition 4.4.

Corollary 4.3 *Assume that $\xi_t > mG_i(0,\xi_{t+1})$ for every i. At equilibrium, we have*

$$K_{t+1} \geq \min_i \Big\{ B_i(\xi_t,\xi_{t+1}) \Big\} > 0$$

where $B_i(\xi_t,\xi_{t+1})$ is defined[5] *by*

$$\xi_t = B_i(\xi_t,\xi_{t+1}) + mG_i\Big(B_i(\xi_t,\xi_{t+1}),\xi_{t+1}\Big). \quad \text{(V.10)}$$

This result shows that under condition $\xi_t > mG_i(0,\xi_{t+1})$, we have K_{t+1} is bounded from below by an exogenous constant which is strictly positive. Note that we have $\xi_t > mG_i(0,\xi_{t+1})$ when ξ_t is high enough.

We end this section by the following result showing that a fluctuation of (ξ_t) may create a fluctuation of (K_t).

Corollary 4.4 *[**Fluctuation of the capital stocks**]*

Assume that

(i) $\beta_i = \beta$, $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$, *and* $F'(0) \leq \delta$.

(ii) $\xi_{2t} \to \xi^e$, $\xi_{2t+1} \to \xi^o$ *when* $t \to \infty$.

(iii) $\xi^e > \dfrac{m\xi^o}{\big(\beta(F'(0)+1-\delta)\big)^{\frac{1}{\sigma}}}$.

We have

(i) There is an infinite sequence $(t_n)_{n=0}^{\infty}$ *s.t.* $K_{t_n} = 0$ *for every* $n \geq 0$.

(ii) $\limsup_{t\to\infty} K_t > 0$.

Proof: The first point is a direct consequence of Proposition 4.1. Let us prove the second point. Assume that $\limsup_{t\to\infty} K_t = 0$. According to (V.9), we have, for each t,

$$\xi_{2t} \leq K_{2t+1} + mG_i(K_{2t+1},\xi_{2t+1}) \quad \text{(V.11)}$$

Let t tend to infinity, we get $\xi^e \leq mG_i(0,\xi^o)$. Under Assumptions of Corollary 4.4, it can be computed that

$$G_i(0,\xi^o) = \frac{\xi^o}{\big(\beta(F'(0)+1-\delta)\big)^{\frac{1}{\sigma}}}.$$

As a result, we obtain

$$\xi^e \leq \frac{m\xi^o}{\big(\beta(F'(0)+1-\delta)\big)^{\frac{1}{\sigma}}},$$

which is an contradiction. ■

5. Note that $B_i(\xi_t,\xi_{t+1})$ is increasing in ξ_t and decreasing in ξ_{t+1}.

5 On the efficiency of equilibria

In this section, we study the efficiency of intertemporal equilibrium. We continue to assume that $p_t = 1$ for every t.

Following Malinvaud (1953), we define the efficiency of a capital path as follows.

Definition 5.1 *Let F_t be a production function, δ be capital depreciation rate. A feasible path of capital is a positive sequence $(K_t)_{t=0}^{\infty}$ such that $0 \leq K_{t+1} \leq F_t(K_t) + (1-\delta)K_t + \xi_t$ for every $t \geq 0$ and K_0 is given.*
A feasible path is efficient if there is no other feasible path (K'_t) such that

$$F_t(K'_t) + (1-\delta)K'_t - K'_{t+1} \geq F_t(K_t) + (1-\delta)K_t - K_{t+1}$$

for every t with strict inequality for some t.

The aggregate feasible consumption at date t is defined by $C_t := F_t(K_t) + (1-\delta)K_t + \xi_t - K_{t+1}$.

Definition 5.2 *We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path (K_t) is efficient.*

Definition 5.3 *We define the discount factor of the economy from initial date to date t as follows*

$$Q_0 := 1, \quad Q_t := \prod_{s=1}^{t} \gamma_s, \quad t \geq 1 \tag{V.12}$$

where $\gamma_{t+1} := \max_{i \in \{1,\dots,m\}} \dfrac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}$.

Lemma 5.1 *An equilibrium is efficient if* $\liminf_{t\to\infty} Q_t K_{t+1} = 0$.

Proof: See Malinvaud (1953) and Bosi, Le Van and Pham (2014). ■

Becker, Dubey, and Mitra (2014) give an example of inefficient Ramsey equilibrium in a model with only physical capital. The production function in their model satisfies $F'(\infty) = 0$ and they consider full depreciation of the capital. The following result shows that financial dividends, for such models, may make production paths efficient. Actually, the result is more general.

Proposition 5.1 *We assume that the production functions are stationary, strictly concave, $F'(\infty) < \delta$, and $\limsup_{t\to\infty} \xi_t < \infty$. If $\limsup_{t\to\infty} \xi_t > 0$, every equilibrium is efficient.*

Proof: Since technologies are stationary and $\limsup_{t\to\infty} \xi_t < \infty$, we easily see that that (K_t) is bounded. Indeed, there exists $\xi > 0$ such that $\xi_t \leq \xi$ for every t. We are going to prove that $0 \leq K_t \leq \max\{K_0, x\} =: K$ where x such that[6] $F(x) + (1-\delta)x + \xi = x$. Note that if $y \geq x$ then $F(y) + (1-\delta)y + \xi \leq y$.
We have

$$\begin{aligned} K_{t+1} = \sum_{i=1}^{m} k_{i,t+1} &\leq F(K_t) + (1-\delta)K_t + \xi_t \\ &\leq F(K_t) + (1-\delta)K_t + \xi. \end{aligned}$$

Then $K_1 \leq F(K_0) + (1-\delta)K_0 + \xi \leq F(K) + (1-\delta)K + \xi \leq K$. Iterating the argument, we find $K_t \leq K$ for each $t \geq 0$.

Since $\limsup_{t\to\infty} \xi_t > 0$, there exists a constant Λ and a sequence (t_n) such that $K_{t_{n+1}} \leq \Lambda \xi_{t_n}$ for every n large enough.

According to Lemma 4.1, we have $q_t Q_t = (q_{t+1} + \xi_{t+1})Q_{t+1}$. As a consequence, we obtain

$$q_0 = \sum_{t=1}^{\infty} Q_t \xi_t + \lim_{t\to\infty} q_t Q_t.$$

Recall that $q_0 \in (0, \infty)$, hence $\sum_{t=1}^{\infty} Q_t \xi_t < \infty$. Therefore, we have $\lim_{t\to\infty} Q_t \xi_t = 0$ which implies that $\lim_{n\to\infty} Q_{t_n} K_{t_n+1} = 0$. According to Lemma 5.1, the capital path is efficient. ■

6 Bubbles and Efficiency

In this section we will assume that production technology is time invariant. We will first consider the productive sector. We will define physical asset bubble. Let $\big(p_t, q_t, r_t, (c_{i,t}, a_{i,t}, k_{i,t})_{i=1}^m, K_t\big)_t$ be an equilibrium. According to Lemma 4.1, we have :

Lemma 6.1 *For each t, we have*

$$1 \geq (1 - \delta + \frac{r_{t+1}}{p_{t+1}})\gamma_{t+1} \tag{V.13}$$

where $\gamma_{t+1} := \max_{i \in \{1,\dots,m\}} \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$. We have equality if $K_{t+1} > 0$.

6. since $F(\cdot)$ is concave, x is unique. The existence of x is ensured by $F(0) + \xi > 0$ and $\lim_{x\to\infty} F(x) - \delta x + \xi < 0$

Definition 6.1 *The rate of return* ρ_{t+1} *is defined by*

$$\max_i \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} = \frac{1}{1-\delta+\rho_{t+1}} \tag{V.14}$$

So, we have $1 = (1-\delta+\rho_t)\gamma_t$ and so $Q_t = (1-\delta+\rho_{t+1})Q_{t+1}$ for each $t \geq 0$. By interating, we get

$$\begin{aligned} 1 &= (1-\delta+\rho_1)Q_1 = (1-\delta)Q_1 + \rho_1 Q_1 \\ &= (1-\delta)(1-\delta+\rho_2)Q_2 + \rho_1 Q_1 = (1-\delta)^2 Q_2 + (1-\delta)\rho_2 Q_2 + \rho_1 Q_1 \\ &= \cdots \\ &= (1-\delta)^t Q_t + \sum_{s=1}^{t} (1-\delta)^{t-1} \rho_t Q_t. \end{aligned}$$

The fundamental value of the physical capital at date 0 can be defined by

$$\sum_{t=1}^{\infty} (1-\delta)^{t-1} \rho_t Q_t.$$

Definition 6.2 *We say that there is a capital asset bubble if* $1 > \sum\limits_{s=1}^{\infty} (1-\delta)^{t-1} \rho_t Q_t$.

We can see that there is a bubble on capital asset if and only if $\lim\limits_{t\to\infty} (1-\delta)^t Q_t > 0$.
We state the necessary and sufficient to have bubbles on the physical asset market.

Proposition 6.1 *There is a physical capital bubble if and only if* $\sum_{t=1}^{\infty} \rho_t < +\infty$.

Proof: The proof is similar to the one in Bosi, Le Van and Pham (2014). ■

Let us now move to financial asset market. It is easy to obtain the following relation

$$Q_t \frac{q_t}{p_t} = Q_{t+1}\left(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}\right) = Q_{t+1} \frac{q_{t+1}}{p_{t+1}} \left(1 + \frac{p_{t+1}\xi_{t+1}}{q_{t+1}}\right)$$

We also have $\frac{q_t}{p_t} = \gamma_{t+1}(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1})$. Therefore, for each $t \geq 1$, we have

$$\begin{aligned}
\frac{q_0}{p_0} &= \gamma_1(\frac{q_1}{p_1} + \xi_1) = Q_1\xi_1 + \gamma_1\frac{q_1}{p_1} = Q_1\xi_1 + \gamma_1\gamma_2(\frac{q_2}{p_2} + \xi_2) \\
&= Q_1\xi_1 + Q_2\xi_2 + Q_2\frac{q_2}{p_2} \\
&= \ldots = \sum_{s=1}^{t} Q_s\xi_s + Q_t\frac{q_t}{p_t}.
\end{aligned}$$

This leads us to have the following concept.

Definition 6.3 *The fundamental value of financial asset*

$$FV_0 := \sum_{t=1}^{+\infty} Q_t\xi_t \tag{V.15}$$

Denote $b_0 := \lim_{t\to+\infty} Q_t\frac{q_t}{p_t}$, b_0 is called financial asset bubble. We have

$$q_0 = b_0 + FV_0. \tag{V.16}$$

It means that the price of the financial asset equals its fundamental value plus its bubble.

Definition 6.4 *We say there is a bubble on financial asset if the price of financial asset is greater than its fundamental value : $q_0 > FV_0$.*

We give another definition of low interest rates for financial asset market. We recall budget constraint of agent i at date $t-1$ and t.

$$\begin{aligned}
p_{t-1}(c_{i,t-1} + k_{i,t} - (1-\delta)k_{i,t-1}) + q_{t-1}a_{i,t-1} &\leq r_{t-1}k_{i,t-1} + (q_{t-1} + p_{t-1}\xi_{t-1})a_{i,t-1} + \theta^i\pi_{t-1} \\
p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} &\leq r_t k_{i,t} + q_t(1 + \frac{p_t\xi_t}{q_t})a_{i,t-1} + \theta^i\pi_t
\end{aligned}$$

One can interpret that if agent i buys $a_{i,t-1}$ units of financial asset at date $t-1$ with price q_{t-1}, she will receive $(1 + \frac{p_t\xi_t}{q_t})a_{i,t-1}$ units of financial asset with price q_t at date t. Therefore, $\frac{p_t\xi_t}{q_t}$ can be viewed as the real interest rate of the financial asset at date t.

Definition 6.5 *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty} \frac{p_t\xi_t}{q_t} < \infty. \tag{V.17}$$

Otherwise, we say that interest rates are high.

We now give a relationship between financial bubble and low interest rates of the financial asset market.

Proposition 6.2 *There is a bubble if and only if interest rates are low.*

Proof: See Le Van, Pham and Vailakis (2014). ■

We also have

Lemma 6.2 *(1) For each i, we define $S_{i,0} = 1$, $S_{i,t} := \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$ is the agent i's discount factor from initial period to period t. Then $\lim_{t\to\infty} S_{i,t}\frac{q_t}{p_t}a_{i,t} = 0$.*
(2) If the borrowing constraints of agent i are not binding from t_0 to t then $\frac{Q_t}{Q_{t_0}} = \frac{S_{i,t}}{S_{i,t_0}}$.

Proof: (1) Use Theorem 2.1 in Kamihigashi (2002).
(2) See Le Van, Pham and Vailakis (2014). ■

The main result of this section is stated as follows.

Proposition 6.3 *(1) Assume that the production functions are stationary. Then at equilibrium there exists no bubble on the physical asset market.*
(2) Assume that the production functions are stationary, $F'(\infty) < \delta$ and $0 < \limsup_{t\to\infty} \xi_t < \infty$. Then any equilibrium is efficient and at equilibrium there exists no bubble on the physical asset market.
(3) We assume that the production functions are stationary, $F'(\infty) < \delta$ and $0 < \liminf_{t\to\infty} \xi_t \leq \limsup_{t\to\infty} \xi_t < \infty$. Then any equilibrium is efficient and at equilibrium there exists no bubble on the financial asset market.
(4) Assume that the production functions are stationary, $F'(\infty) < \delta$ and $0 < \liminf_{t\to\infty} \xi_t \leq \limsup_{t\to\infty} \xi_t < \infty$. Then any equilibrium is efficient and at equilibrium there exist no bubble on the financial asset market and no bubble on the physical asset market.

Proof: (1) We have

$$\frac{r_{t+1}}{p_{t+1}} + 1 - \delta \leq \frac{Q_t}{Q_{t+1}} = \rho_{t+1} + 1 - \delta.$$

If $F'(\infty) \geq \delta$ then for any t, $\rho_{t+1} \geq \frac{r_{t+1}}{p_{t+1}} \geq F'(\infty) \geq \delta$. This implies $\sum_{t=0}^{\infty} \rho_{t+1} = +\infty$. From Proposition 6.1, there is no physical capital bubble. If $F'(\infty) < \delta$, by Lemma 2.1, (K_t) is bounded uniformly by some constant K, which implies that there $F'(K_t) \geq F'(K)$. For any t, $\rho_{t+1} \geq \frac{r_{t+1}}{p_{t+1}} \geq F'(K)$. This implies $\sum_{t=0}^{\infty} \rho_{t+1} = +\infty$. From Proposition 6.1, there is no physical capital bubble.
(2) No bubble follows statement (1). Efficiency follows Proposition 5.1.

(3) (ii) follows Proposition 5.1.

Let us prove (i). First, since $\liminf_{t\to\infty} \xi_t > 0$ and $\sum_{t=1}^{+\infty} Q_t\xi_t < \infty$, we have $\sum_{t=1}^{+\infty} Q_t < \infty$. Since (K_t) is bounded uniformly, we also have $\lim_{T\to+\infty} Q_T k_{i,T+1} = 0$ for any i, and

$$\sum_{t=1}^{+\infty} F(K_t)Q_t \leq \sum_{t=1}^{+\infty} F(K)Q_t < \infty.$$

We claim that we always have

$$Q_T k_{i,T+1} = (1-\delta+\frac{r_{t+1}}{p_{t+1}})Q_{T+1}k_{i,T+1}$$

Indeed, the claim is trivially true if $k_{i,T+1}=0$. If $k_{i,T+1}>0$ then $K_{T+1}>0$ and $Q_T = (1-\delta+\frac{r_{t+1}}{p_{t+1}})Q_{T+1}$ (see Lemma 6.1).

For any agent i, consider her/his budget constraints. We have

$$\begin{aligned}
\sum_{t=0}^{T} Q_t c_{i,t} + Q_T k_{i,T+1} + Q_T\frac{q_T}{p_T}a_{i,T} &= (\frac{r_0}{p_0}+1-\delta)k_{i,0} + (\frac{q_0}{p_0}+\xi_0)a_{i,-1} + \theta^i\sum_{t=0}^{T}\frac{\pi_t}{p_t}Q_t \\
&\leq (\frac{r_0}{p_0}+1-\delta)k_{i,0} + (\frac{q_0}{p_0}+\xi_0)a_{i,-1} + \theta^i\sum_{t=0}^{T}F(K_t)Q_t \\
&< +\infty.
\end{aligned}$$

Now, from the borrowing constraint we get :

$$0 \leq Q_T\frac{q_T}{p_T}a_{i,T} + f^i(1-\delta+\frac{r_{t+1}}{p_{t+1}})Q_{T+1}k_{i,T+1} = Q_T\frac{q_T}{p_T}a_{i,T} + f^iQ_Tk_{i,T+1}$$

But

$$f^iQ_Tk_{i,T+1} \leq Q_Tk_{i,T+1}$$

We then obtain

$$0 \leq Q_T\frac{q_T}{p_T}a_{i,T} + Q_Tk_{i,T+1}$$

Therefore, $\sum_{t=0}^{\infty} Q_t c_{i,t} < +\infty$, and then $\lim_{T\to+\infty} Q_Tk_{i,T+1} + Q_T\frac{q_T}{p_T}a_{i,T}$ exists. Since $\lim_{T\to+\infty} Q_Tk_{i,T+1} = 0$, we have $\lim_{T\to+\infty} Q_T\frac{q_T}{p_T}a_{i,T}$ exists. If there exists a bubble then $\lim_{T\to+\infty} a_{i,T}$ exists. This property holds for any i. As a consequence, there exists i such that $\lim_{T\to+\infty} a_{i,T} > 0$. For this agent, there exists T such that the borrowing constraints will not bind for $t \geq T$. We have from Lemma 6.2 that $\frac{Q_t}{Q_T} = \frac{S_{i,t}}{S_{i_T}}$, for any $t \geq T$. We then have, by using Lemma 6.2

$$\lim_{t\to+\infty} Q_t\frac{q_t}{p_t}a_{i,t} = \frac{Q_T}{S_{i,T}}\lim_{t\to+\infty} S_{i,t}\frac{q_t}{p_t}a_{i,t} = 0$$

which is a contradiction. We conclude that there is no bubble on the financial asset market.

(4) The statement follows statements (2) and (3). ■

7 Conclusion

We build an infinite-horizon dynamic deterministic general equilibrium model in which heterogeneous agents invest in capital and/or financial asset, and consume. We proved the existence of intertemporal equilibrium in this model, even if aggregate capital is not uniformly bounded and technologies are not stationary.

By using this framework, we studied the relationship between the financial market and the productive sector.

When productivity is low and financial dividends are bounded away from zero, the productive sector will disappear at infinitely many dates. We illustrate by an example in which $F'(0) \leq \delta$ and there is no investment in the productive sector. However, when productivity is low and the financial dividend is high, there is investment in the productive sector. This is due to the fact that part of the financial dividend is used for the purchase of the physical capital. When productivity is high enough, the economy will produce at any period.

Fluctuations on financial dividend (ξ_t) can create fluctuations on the aggregate capital path (K_t). However, when the marginal productivity at infinity is lower than the depreciation rate, any equilibrium associated with a sequence of financial dividends (ξ_t) which satisfies $\limsup_{t\to\infty} \xi_t < +\infty$ and $\liminf_{t\to\infty} \xi_t > 0$ is efficient and there exist neither bubble on the financial asset market nor bubble on the physical asset market.

8 Appendix

8.1 Existence of equilibrium for truncated bounded economy

We define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all variables (consumption demand, capital supply, asset investment, capital demand) are bounded.

$$\begin{aligned}
\mathcal{C}_i &:= [0, B_c]^{T+1}, \quad B_c > 1 + \max_{t\leq T} F_t(B_K) + (1-\delta)B_k + \max_{t\leq T} \xi_t, \\
\mathcal{K}_i &:= [0, B_k]^T, \quad B_k > D^T := 1 + \max_{t\leq T} D_t(K_0, \xi_0, \ldots, \xi_t) \\
\mathcal{A}_i &:= [-B_a, B_a]^T, \quad B_a > 1 + B \\
\mathcal{K} &:= [0, B_K]^{T+1}, \quad B_K > 1 + mB_k,
\end{aligned}$$

where B is satisfied $B > \max\{\max_t \frac{D_t}{\xi_t}, 1 + m \max_t \frac{D_t}{\xi_t}\}$.

3 Existence of equilibrium for ϵ-economy

For each $\epsilon > 0$ such that $2m\epsilon < 1$, we define $\epsilon-$economy $\mathcal{E}_b^{T,\epsilon}$ by adding ϵ units of each asset (consumption good, physical capital, and financial asset) for each agent at date 0 . More presisely, the feasible set of agent i is given by

$$\begin{aligned}C_i^{T,\epsilon}(p,q,r) := \Big\{&(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1}, a_{i,T} = 0,\\ &\text{(b) } p_0(c_{i,0} + k_{i,1} - (1-\delta)(k_{i,0}+\epsilon)) + q_0 a_{i,0}\\ &\quad \leq p_0\epsilon + r_0(k_{i,0}+\epsilon) + (q_0 + p_0\xi_0)(a_{i,t-1}+\epsilon) + \theta^i\pi_0\\ &\text{(c) for each } 1 \leq t \leq T:\\ &\quad 0 \leq (q_t + p_t\xi_t)a_{i,t-1} + f^i(r_t + (1-\delta)p_t)k_{i,t}\\ &\quad p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} \leq (q_t + p_t\xi_t)a_{i,t-1} + r_t k_{i,t} + \theta^i\pi_t\Big\}.\end{aligned}$$

We also define $B_i^{T,\epsilon}(p,q,r)$ as follows.

$$\begin{aligned}B_i^{T,\epsilon}(p,q,r) := \Big\{&(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^T \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1}, a_{i,T} = 0,\\ &\text{(b) } p_0(c_{i,0} + k_{i,1} - (1-\delta)(k_{i,0}+\epsilon)) + q_0 a_{i,0}\\ &\quad < p_0\epsilon + r_0(k_{i,0}+\epsilon) + (q_0 + p_0\xi_0)(a_{i,t-1}+\epsilon) + \theta^i\pi_0\\ &\text{(c) for each } 1 \leq t \leq T:\\ &\quad 0 < (q_t + p_t\xi_t)a_{i,t-1} + f^i(r_t + (1-\delta)p_t)k_{i,t}\\ &\quad p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} < (q_t + p_t\xi_t)a_{i,t-1} + r_t k_{i,t} + \theta^i\pi_t\Big\}.\end{aligned}$$

We write $C_i^T(p,q,r), B_i^T(p,q,r)$ instead of $C_i^{T,0}(p,q,r), B_i^{T,0}(p,q,r)$.

Definition 8.1 *A sequence of prices and quantities* $\Big(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{a}_{i,t}, \bar{k}_{i,t})_{i=1}^m, \bar{K}_t\Big)_{t=0}^T$ *is an equilibrium of the economy* $\mathcal{E}_b^{T,\epsilon}$ *if the following conditions are satisfied :*

(i) Price positivity : $\bar{p}_t, \bar{r}_t, \bar{q}_t > 0$ *for* $t \geq 0$.

(ii) All markets clear :

Consumption good

$$\begin{aligned}\sum_{i=1}^m (\bar{c}_{i,0} + \bar{k}_{i,1} - (1-\delta)(\bar{k}_{i,0}+\epsilon)) &= 2m\epsilon + F_0(\bar{K}_0) + \xi_0\\ \sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) &= F_t(\bar{K}_t) + \xi_t\end{aligned}$$

Physical capital

$$\bar{K}_0 = \sum_{i=1}^m (\bar{k}_{i,0}+\epsilon), \quad \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t}.$$

Financial asset

$$\sum_{i=1}^{m} \bar{a}_{i,0} = \sum_{i=1}^{m} (\bar{a}_{i,-1} + \epsilon), \quad \sum_{i=1}^{m} \bar{a}_{i,t+1} = \sum_{i=1}^{m} \bar{a}_{i,t}.$$

(iii) Optimal consumption plans : for each i, $\left(\bar{c}_{i,t}, \bar{a}_{i,t}, \bar{k}_{i,t}\right)_{t=0}^{\infty}$ *is a solution of the maximization problem of agent* i *with the feasible set* $C_i^{T,\epsilon}(p,q,r)$.

(iv) Optimal production plan : for each $t \geq 0$, $(\bar{K}_t)$ *is a solution of the problem* $(P(\bar{r}_t))$.

Lemma 8.1 $B_i^{T,\epsilon}(p,q,r) \neq \emptyset$ *and* $\bar{B}_i^{T,\epsilon}(p,q,r) = C_i^{T,\epsilon}(p,q,r)$.

Proof: We rewrite

$$\begin{aligned}
B_i^{T,\epsilon}(p,q,r) := \Big\{ & (c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{T} \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : k_{i,T+1}, a_{i,T} = 0, \\
& 0 < p_0(\epsilon + (1-\delta)(k_{i,0} + \epsilon) + \xi_0(a_{i,t-1} + \epsilon) - c_{i,0} - k_{i,1}) \\
& \quad + r_0(k_{i,0} + \epsilon) + \bar{q}_0(a_{i,t-1} + \epsilon - a_{i,0}) + \theta^i \pi_0 \\
& \quad \text{and for each } 1 \leq t \leq T : \\
& \quad 0 < q_t a_{i,t-1} + f^i \bar{r}_t k_{i,t} + p_t(\xi_t a_{i,t-1} + f^i (1-\delta) k_{i,t}) \\
& 0 < p_t((1-\delta)k_{i,t} + \xi_t a_{i,t-1} - c_{i,t} - k_{i,t+1}) + r_t k_{i,t} + \bar{q}_t(a_{i,t-1} - a_{i,t}) + \theta^i \pi_t.
\end{aligned}$$

Since $\epsilon, k_{i,0} + \epsilon, a_{i,t-1} + \epsilon > 0$, we can choose $c_{i,0} \in (0, B_c)$, $k_{i,1} \in (0, B_k)$, and $a_{i,0} \in (0, B_a)$ such that

$$\begin{aligned}
& 0 < p_0(\epsilon + (1-\delta)(k_{i,0} + \epsilon) + \xi_0(a_{i,t-1} + \epsilon) - c_{i,0} - k_{i,1}) \\
& \quad + r_0(k_{i,0} + \epsilon) + \bar{q}_0(a_{i,t-1} + \epsilon - a_{i,0}) + \theta^i \pi_0.
\end{aligned}$$

By induction, we see that $B_i(p,q,r)$ is not empty. ■

Lemma 8.2 $B_i(p,r)$ *is lower semi-continuous correspondence on* $\mathcal{P}$*. And* $C_i(p,r)$ *is upper semi-continuous on* $\mathcal{P}$ *with compact convex values.*

Proof: Clearly, since $B_i(p,r)$ is empty and has open graph. ■

We define $\Phi := \Delta \times \prod_{i=1}^{m} (\mathcal{C}_i \times \mathcal{A}_i \times \mathcal{K}_i) \times \mathcal{K}$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m+1}$ where $z_0 := (p,q,r)$, $z_i := (c_i, a_i, k_i)$ for each $i = 1, \ldots, m$, and $z_{m+1} = (K)$.

We now define correspondences. First, we define φ_0 (for additional agent 0)

$$\varphi_0 : \prod_{i=1}^{m} (\mathcal{C}_i \times \mathcal{A}_i \times \mathcal{K}_i) \times \mathcal{K} \to 2^{\Delta}$$

$$\begin{aligned}
\varphi_0((z_i)_{i=1}^{m+1}) := \arg\max_{(p,q,r)\in\Delta} \Big\{ & p_0\big(\sum_{i=1}^{m}(c_{i,0} + k_{i,1} - (1-\delta)(k_{i,0}+\epsilon) - m\epsilon - F_0(K_0) - \xi_0)\big) \\
& + q_0 \sum_{i=1}^{m}(a_{i,0} - a_{i,-1} - \epsilon) + r_0(K_0 - \sum_{i=1}^{m}(k_{i,0}+\epsilon)) \\
& + \sum_{t=1}^{T} p_t\big(\sum_{i=1}^{m}(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} - F_t(K_t) - \xi_t\big) \\
& \sum_{t=0}^{T} r_t\big(K_t - \sum_{i=1}^{m} k_{i,t}\big) + \sum_{t=1}^{T-1} q_t \sum_{i=1}^{m}(a_{i,t} - a_{i,t-1})\Big\}.
\end{aligned}$$

For each $i = 1, \ldots, m$, we define

$$\begin{aligned}
\varphi_i : \quad & \Delta \times \mathcal{K} \to 2^{\mathcal{C}_i \times \mathcal{A}_i \times \mathcal{K}_i} \\
\varphi_i((p,q,r),K) \quad := \quad & \arg\max_{(c_i,a_i,k_i)\in C_i(p,q,r)} \Big\{ \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \Big\}.
\end{aligned}$$

For each $i = m+1$, we define

$$\begin{aligned}
\varphi_{m+1} : \quad & \Delta \to \mathcal{K} \\
\varphi_i((p,r)) \quad := \quad & \arg\max_{(K)\in\mathcal{K}} \Big\{ \sum_{t=0}^{T} p_t F_t(K_t) - r_t K_t \Big\}.
\end{aligned}$$

Lemma 8.3 *The correspondence φ_i is lower semi-continuous and non-empty, convex, compact valued for each $i = 0, 1, \ldots, m+1$.*

Proof: This is a direct consequence of the Maximum Theorem. ■

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{a}_i, \bar{k}_i)_{i=1}^{m}), K$ such that

$$(\bar{p}, \bar{q}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{a}_i, \bar{k}_i)_{i=1}^{m}) \tag{V.18}$$

$$(\bar{c}_i, \bar{a}_i, \bar{k}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r})) \tag{V.19}$$

$$(K) \in \varphi_{m+1}((\bar{p}, \bar{q}, \bar{r})). \tag{V.20}$$

Denote

$$\bar{X}_0 := \sum_{i=1}^{m}(c_{i,0} + k_{i,1} - (1-\delta)(k_{i,0}+\epsilon) - F_0(K_0) - \xi_0) \tag{V.21}$$

$$\bar{X}_t := \sum_{i=1}^{m}(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} - F_t(K_t)), \quad t \geq 1 \tag{V.22}$$

$$\bar{Y}_0 = \bar{K}_0 - \sum_{i=1}^{m}(\bar{k}_{i,0}+\epsilon), \quad \bar{Y}_t = \bar{K}_t - \sum_{i=1}^{m}\bar{k}_{i,t}, \quad t \geq 1 \tag{V.23}$$

$$\bar{Z}_0 = \sum_{i=1}^{m}(\bar{a}_{i,0} - \epsilon - \bar{a}_{i,-1}), \quad \bar{Z}_t = \sum_{i=1}^{m}(\bar{a}_{i,t} - \bar{a}_{i,t-1}), \quad t \geq 1. \tag{V.24}$$

For every $(p,q,r) \in \Delta$, we have

$$\sum_{t=0}^{T}(p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T-1}(q_t - \bar{q}_t)\bar{Z}_t + \sum_{t=0}^{T}(r_t - \bar{r}_t)\bar{Y}_t \leq 0. \tag{V.25}$$

By summing the budget constraints, we get that, for each t,

$$\bar{p}\bar{X}_t + \bar{q}\bar{Z}_t + \bar{r}\bar{Y}_t \leq 0. \tag{V.26}$$

As a consequence, we have, for every $(p,q,r) \in \Delta$,

$$p_t\bar{X}_t + q_t\bar{Z}_t + r_t\bar{Y}_t \leq \bar{p}\bar{X}_t + \bar{q}\bar{Z}_t + \bar{r}\bar{Y}_t \leq 0. \tag{V.27}$$

Therefore, we have $\bar{X}_t, \bar{Z}_t, \bar{Y}_t \leq 0$, which mean that

$$\sum_{i=1}^{m}\bar{c}_{i,0} + \bar{k}_{i,1} \leq m\epsilon + (1-\delta)\sum_{i=1}^{m}(\bar{k}_{i,0}+\epsilon) + F_0(\bar{K}_0) + \xi_0 \tag{V.28}$$

$$\sum_{i=1}^{m}\bar{c}_{i,t} + \bar{k}_{i,t+1} \leq (1-\delta)\sum_{i=1}^{m}\bar{k}_{i,t} + F_t(\bar{K}_t), \quad t \geq 1$$

$$\bar{K}_0 \leq \sum_{i=1}^{m}(\bar{k}_{i,0}+\epsilon), \quad \bar{K}_t \leq \sum_{i=1}^{m}\bar{k}_{i,t}, \quad t \geq 1 \tag{V.29}$$

$$\sum_{i=1}^{m}\bar{a}_{i,0} \leq \sum_{i=1}^{m}(\bar{a}_{i,-1}+\epsilon), \quad \sum_{i=1}^{m}\bar{a}_{i,t} \leq \sum_{i=1}^{m}\bar{a}_{i,t-1}, \quad t \geq 1. \tag{V.30}$$

Lemma 8.4 *$\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ for $t = 0, \ldots, T$.*

Proof: If $\bar{p}_t = 0$, we imply that $\bar{c}_{i,t} = B_c > n + (1-\delta)B_k + F_t(B_k) + \xi_t$. Therefore, we get $\bar{c}_{i,t} + \bar{k}_{i,t+1} > (1-\delta)\sum_{i=1}^{m}\bar{k}_{i,t} + F_t(\bar{K}_t)$, contradiction. As a result, $\bar{p}_t > 0$.

If $\bar{r}_t = 0$, the optimality of $(\bar{K})$ implies that $K_t = B_K$. However, we have $\bar{k}_{i,t} \leq B_k$ for every i, t. Consequently, $\sum_{i=1}^{m} \bar{k}_{i,t} \leq mB_k + n < B_K = K_t$, contradiction to (V.29). Therefore, we get $\bar{r}_t > 0$.

If $\bar{q}_t = 0$, we have $\bar{a}_{i,t} = B_a$ for each i. Thus, $\sum_{i=1}^{m} \bar{a}_{i,t} \geq mB_a > 1 + B_a$. However, we have $\sum_{i=1}^{m} \bar{a}_{i,t} \leq \sum_{i=1}^{m} \bar{a}_{i,-1} + m\epsilon = 1 + m\epsilon < 1 + B_a$, contradiction!
■

Lemma 8.5 $\bar{X}_t = \bar{Z}_t = \bar{Y}_t = 0.$

Proof: Since prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing budget constraints at date t we have.

$$\bar{p}_t \bar{X}_t + \bar{q}_t \bar{Z}_t + \bar{r}_t \bar{Y}_t = 0. \tag{V.31}$$

By combining this with the fact that $\bar{X}_t, \bar{Z}_t, \bar{Y}_t \leq 0$, we obtain $\bar{X}_t = \bar{Z}_t = \bar{Y}_t = 0$. ■

The optimalities of (c_i, a_i, k_i) and (K) are from (V.19) and (V.20).

4 *When ϵ tends to zero*

We have so far proved that for each $\epsilon_n = 1/n > 0$, where n is interger number and high enough, there exists an equilibrium, say

$$equi(n) := \Big(\bar{p}_t(n), \bar{q}_t(n), \bar{r}_t(n), (\bar{c}_{i,t}(n), \bar{a}_{i,t}(n), \bar{k}_{i,t}(n))_{i=1}^{m}, \bar{K}_t(n)\Big)_{t=0}^{T},$$

for the economy $\mathcal{E}_b^{T,\epsilon_n}$. Note that $\bar{p}_t(n) + \bar{q}_t(n) + \bar{r}_t(n) = 1$, we can assume that [7]

$$\begin{aligned} &(\bar{p}(n), \bar{r}(n), \bar{q}(n), (\bar{c}_i(n), \bar{k}_i(n), \bar{a}_i^T(n))_{i=1}^{m}, \bar{K}^T(n)) \\ &\xrightarrow{n\to\infty} (\bar{p}, \bar{r}, \bar{q}, (\bar{c}_i, \bar{k}_i, \bar{a}_i)_{i=1}^{m}, \bar{K}). \end{aligned}$$

Markets clearing conditions : By taking limit of market clearing conditions for economy $\mathcal{E}_b^{T,\epsilon_n}$, we obtain market clearing conditions for the economy $\mathcal{E}_b^{T}$.
Optimality of $\bar{K}_t$: Take $K \geq 0$. We have $\bar{p}_t(n)F_t(K) - \bar{r}_t(n)K \leq \bar{p}_t(n)F_t(\bar{K}_t(n)) - \bar{r}_t(n)\bar{K}_t(n)$. Let n tend to infinity, we obtain that $\bar{p}_tF_t(K) - \bar{r}_tK \leq \bar{p}_tF_t(\bar{K}_t) - \bar{r}_t\bar{K}_t$. Therefore, the optimality of $\bar{K}_t$ is proved.

7. In fact, since prices and allocations are bounded, there exists a subsequence $(n_1, n_2, \ldots,)$ such that $equi(n_s)$ converges. However, without loss of generality, we can assume that $equi(n)$ converges.

Lemma 8.6 *If $\bar{p}_t > 0$, we have $\bar{r}_t > 0$ for each $t \geq 0$.*

Proof: Assume that $\bar{r}_t = 0$. According to the optimality of $\bar{K}_t$, we have $F_t(K) \leq F_t(\bar{K})$ for every $K \geq 0$. Then $\bar{K}_t = B_K > D_{t-1}$. However, according to market clearing condition, we have

$$\bar{K}_{t+1} \leq (1-\delta)\bar{K}_t + F_t(\bar{K}_t) + \xi_t. \tag{V.32}$$

As a consequence, $\bar{K}_t < D_{t-1}$, contradiction. ■

Corollary 8.1 *We have $\bar{q}_t + \bar{r}_t > 0$ for each $t \geq 0$.*

Lemma 8.7 *We have $\bar{p}_0 + \bar{q}_0 > 0$.*

Proof: If $\bar{p}_0 + \bar{q}_0 = 0$, we get $\bar{p}_0 = 0$, $\bar{r}_0 = 1$. According to the optimality of K_0, we have $K \geq K_0$ for every $K \geq 0$. Then, $K_0 = 0$, contradiction. ■

Lemma 8.8 *$B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$ if $\bar{p}_0 + \bar{q}_0 > 0$, $\bar{q}_t + \bar{r}_t > 0$ for each $t \geq 0$, and one of the following condition is satisfied*

1. *$\bar{q}_0 = 0$.*
2. *$\bar{q}_0 > 0$ and $a_{i,-1} > 0$.*

Proof: In the case when $\bar{q}_0 = 0$, we imply that $\bar{p}_0, \bar{r}_0 > 0$. Since $(k_{i,0}, a_{i,-1}) \neq (0,0)$, we can use the same argument in Lemma 8.1 to prove that $B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$.

If $\bar{q}_0 > 0$ and $a_{i,-1} > 0$, we get $\xi_0 a_{i,-1} > 0$. $B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$ is also proved by using the same argument. ■

Lemma 8.9 *We have $\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$.*

Proof: Since $\sum_{i=1}^{m} a_{i,-1} = 1 > 0$, there exists an agent i such that $a_{i,-1} > 0$. According to Lemma 8.8, we have $B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$. We are going to prove the optimality of allocation $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$.

Let (c_i, k_i, a_i) be a feasible allocation of the maximization problem of agent i with the feasible set $C_i^T(\bar{p}, \bar{q}, \bar{r})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Since $B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$, there exists $(h)_{h \geq 0}$ and $(c_i^h, k_i^h, a_i^h) \in B_i^T(\bar{p}, \bar{q}, \bar{r})$ such that (c_i^h, k_i^h, a_i^h) converges to (c_i, k_i, a_i). We have

$$\bar{p}_t(c_{i,t}^h + k_{i,t+1}^h - (1-\delta)k_{i,t}^h) + \bar{q}_t a_{i,t}^h < \bar{r}_t k_{i,t}^h + (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h + \theta^i \pi_t$$
$$0 < (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h + f^i(\bar{r}_t + (1-\delta)\bar{p}_t) k_{i,t}^h.$$

Fixe h. Let n_0 (n_0 depends on h) be high enough such that for every $n \geq n_0$, $(c_i^h, k_i^h, a_i^h) \in C_i^{T,1/n}(\bar{p}(n), \bar{q}(n), \bar{r}(n))$. Therefore, we have $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}^h) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t}(n))$.

Let n tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}^h) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Let h tend to infinity, we have $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. It means that we have just proved the optimality of $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$.

We now prove $\bar{p}_t > 0$ for every t. Indeed, if $\bar{p}_t = 0$, the optimality of $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$ implies that $\bar{c}_{i,t} = B_c > (1-\delta)\bar{K}_t + F_t(\bar{K}_t) + \xi_t$, contradiction.

Therefore, it is easy to prove that $\bar{q}_t > 0, \bar{r}_t > 0$. ■

Lemma 8.10 *For each i, $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$ is optimal.*

Proof: By using the same argument in Lemma 8.9. ■

8.2 Existence of equilibrium for truncated unbounded economy

Proof of Lemma 3.2 : Let $\Big(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{a}_{i,t}, \bar{k}_{i,t},)_{i=1}^m, \bar{K}_t, \bar{L}_t\Big)_{t=0}^T$ be an equilibrium of $\mathcal{E}_b^T$. Note that $k_{i,T+1} = a_{i,T} = 0$ for every $i = 1, \dots, m$. We can see that conditions (i) and (ii) in Definition 8.1 are hold. We will show that conditions (iii) and (iv) in Definition 8.1 are hold too.

For Condition (iii), let $z_i := \big(c_{i,t}, a_{i,t}, k_{i,t}\big)_{t=0}^T$ be a feasible plan of household i.

Assume that $\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) > \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{i,t})$. For each $\gamma \in (0,1)$, we define $z_i(\gamma) := \gamma z_i + (1-\gamma)\bar{z}_i$. By definition of $\mathcal{E}_b^T$, we can choose γ sufficiently close to 0 such that $z_i(\gamma) \in \mathcal{C}_i \times \mathcal{A}_i \times \mathcal{K}_i$. It is clear that $z_i(\gamma)$ is a feasible allocation.

By the concavity of the utility function, we have

$$\begin{aligned}\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}(\gamma)) &\geq \gamma \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) + (1-\gamma) \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{i,t}) \\ &> \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{i,t}).\end{aligned}$$

Contradiction to the optimality of $\bar{z}_i$. So, we have shown that conditions (iii) in Definition 8.1 is hold. A similar proof for conditions (iv) in Definition 8.1 permits us to finish our proof. ■

8.3 Existence of equilibrium for the infinite horizon economy

Proof of Theorem 3.1:

We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. We denote $(\bar{p}^T, \bar{q}^T, \bar{r}^T, (\bar{c}_i^T, \bar{a}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T)$ is an equilibrium of $T-$ truncated economy $\mathcal{E}^T$.
We can normalize by setting $\bar{p}_t^T + \bar{q}^T + \bar{r}_t^T = 1$ for every $t \leq T$.
We see that

$$\begin{aligned} 0 < \bar{c}_{i,t}^T, \bar{K}_t^T &\leq D_t \\ -\bar{a}_{i,t}^T &\leq \frac{D_{t+1}}{\xi_{t+1}}, \quad \sum_{i=1}^m \bar{a}_{i,t}^T = 1. \end{aligned}$$

Therefore, we can assume that

$$\begin{aligned} &(\bar{p}^T, \bar{r}^T, \bar{q}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{a}_i^T)_{i=1}^m, \bar{K}^T) \\ &\xrightarrow{T \to \infty} (\bar{p}, \bar{r}, \bar{q}, (\bar{c}_i, \bar{k}_i, \bar{a}_i)_{i=1}^m, \bar{K}) \quad \text{(for the product topology).} \end{aligned}$$

It is easy to see that all markets clear, and at each date t, $\bar{K}_t$ is a solution of the firm's maximization problem. As in proof of the existence of equilibrium for bounded $T-$truncated economy, we have

(i) $\bar{r}_t > 0$ if $\bar{p}_t > 0$.

(ii) $\bar{r}_t + \bar{q}_t > 0$ for each $t \geq 0$.

(iii) $\bar{p}_0 + \bar{q}_0 > 0$.

Lemma 8.11 *We have $\bar{p}_t > 0$ for each $t \geq 0$.*

Proof: There exists i such that $a_{i,-1} > 0$. By using the same argument in Lemma 8.8, we see that $B_i^T(\bar{p}, \bar{r}, \bar{q}) \neq \emptyset$.

Let (c_i, k_i, a_i) be a feasible alloation of the problem $P_i(\bar{p}, \bar{r}, \bar{q})$. We have to prove that $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. Note that, without loss of generality, we can only consider feasible allocations such that $\bar{p}_T(c_{i,T} + k_{i,T+1} - (1-\delta)k_{i,t}) + \bar{q}_{i,T} a_{i,t} \geq 0$. We define $(c'_{i,t}, k'_{i,t+1}, a'_{i,t})_{t=0}^T$ as follows :

$$\begin{aligned} c_{i,t} &= k_{i,t+1} = a_{i,t} = 0 \text{ if } t > T \\ a'_{i,t} &= a_{i,t}, \text{ if } t \leq T-1, a_{i,T} = 0 \\ c'_{i,t} &= c_{i,t}, \text{ if } t \leq T-1, \quad k'_{i,t+1} = k_{i,t+1}, \text{ if } t \leq T-1 \end{aligned}$$

$$\bar{p}_T(c'_{i,T} + k'_{i,T+1} - (1-\delta)k_{i,t}) = \bar{p}_T(c_{i,T} + k_{i,T+1} - (1-\delta)k_{i,t}) + \bar{q}_{i,T} a_{i,t}$$

We see that $(c'_{i,t}, k'_{i,t+1}, a'_{i,t})_{t=0}^T$ belongs to $C_i^T(\bar{p}, \bar{r}, \bar{q})$.

Since $B_i^T(\bar{p},\bar{r},\bar{q}) \neq \emptyset$, there exists a sequence $\Big((c^n_{i,t}, k^n_{i,t+1}, a^n_{i,t})_{t=0}^T\Big)_{n=0}^{\infty} \in B_i^T(\bar{p},\bar{r},\bar{q})$ with $k^n_{i,T+1} = 0$, $a^n_{i,T} = 0$, and this sequence converges to $(c'_{i,t}, k'_{i,t+1}, a'_{i,t})_{t=0}^T$ when n tends to infinity. We have

$$\bar{p}_t(c^n_{i,t} + k^n_{i,t+1} - (1-\delta)k^n_{i,t}) + \bar{q}_t a^n_{i,t} < \bar{r}_t k^n_{i,t} + (\bar{q}_t + \bar{p}_t\xi_t)a^n_{i,t-1} + \theta^i \pi_t(\bar{p}_t,\bar{r}_t)$$

We can chose s_0 high enough such that $s_0 > T$ and for every $s \geq s_0$, we have

$$\bar{p}^s_t(c^n_{i,t} + k^n_{i,t+1} - (1-\delta)k^n_{i,t}) + \bar{q}^s_t a^n_{i,t} < \bar{r}^s_t k^n_{i,t} + (\bar{q}^s_t + \bar{p}^s_t\xi_t)a^n_{i,t-1} + \theta^i \pi_t(\bar{p}^s_t,\bar{r}^s_t).$$

It means that $(c^n_{i,t}, k^n_{i,t+1}, a^n_{i,t})_{t=0}^T \in C_i^T(\bar{p}^s,\bar{r}^s,\bar{q}^s)$. Therefore, we get $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{s} \beta_i^t u_i(\bar{c}^s_{i,t})$. Let s tend to infinity, we obtain $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. Let n tend to infiniy, we have $\sum\limits_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$ for every T. As a consequence, we have : for every T

$$\sum_{t=0}^{T-1} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity, we obtain $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.
Therefore, we have proved that the optimality of $(\bar{c}_i, \bar{k}_i, \bar{a}_i)$.
Prices $\bar{p}_t, \bar{q}_t$ are strictly positive since the utility function of agent i is strictly increasing. $\bar{r}_t > 0$ is implied by $\bar{p}_t > 0$. ■

Lemma 8.12 *For each i, $(\bar{c}_i, \bar{k}_i, \bar{a}_i)$ is optimal.*

Proof: Since $\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ and $(k_{i,0}, a_{i,-1}) \neq (0,0)$, we get that $B_i^T(\bar{p},\bar{r},\bar{q}) \neq \emptyset$. By using the same argument in Lemma 8.11, we can prove that $(\bar{c}_i, \bar{k}_i, \bar{a}_i)$ is optimal. ■ ■

8.4 Other formal proofs

Proof of Remark 3.2: Indeed, assume that there exists $b < \infty$ such that (V.3) and (V.4) for every i. Denote $A = F'(b), B = F(b)$. Since $F(\cdot)$ is increasing and concave, we obtain $F(x) \leq Ax + B$ for every $x \geq 0$. Since definition of $D_t(K_0, \xi_0, \dots, \xi_t)$, we have

$$\begin{aligned} D_t(F,\delta,K_0,\xi_0,\dots,\xi_t) &\leq (A+1-\delta)^{t+1}K_0 + (A+1-\delta)^t(B+\xi_0) + \cdots + (B+\xi_t) \\ &\leq (A+1-\delta)^{t+1}K_0 + (B + \max_{s\leq t}\{\xi_s\})\sum_{s=0}^{t}(A+1-\delta)^s \end{aligned}$$

Since u_i is concave, there exists $a_i > 0, b_i > 0$ such that $u_i(x) \leq a_i x + b_i$ for every $x \geq 0$. Then

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) \leq \sum_{t=0}^{\infty} \beta_i^t \Big(a_i D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) + b_i).$$

Case 1 : $A \leq \delta$ then $D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) \leq K_0 + (t+1)(B + \max_{s \leq t} \xi_s)$. Combining with (V.3), (V.4), and (V.33) we obtain (V.2).
Case 2 : $A > \delta$, then

$$\begin{aligned} D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) &= (A + 1 - \delta)^{t+1} K_0 + (A + 1 - \delta)^t (B + \xi_0) + \cdots + (B + \xi_t) \\ &\leq (A + 1 - \delta)^{t+1} K_0 + (B + \max_{s \leq t} \{\xi_s\}) \frac{(A + 1 - \delta)^{t+1} - 1}{A - \delta}. \end{aligned}$$

Combining with (V.3), (V.4), and (V.33) we obtain (V.2). ■

Proof for Example 4.1: It is easy to see that all markets clear and the optimal problem of firm is solved.

Let's check the optimality of household's optimization problem by verifying the FOCs.

FOCs of allocation are hold because of the choises of multipliers.

FOCs of $a_{h,t}$ with $h \in \{i, j\}$. We have $\frac{\mu_{h,t+1}}{\mu_{h,t}} = \beta$ for every $t \geq 1$. Since $q_t = \xi \frac{\beta}{1 - \beta}$ for every $t \geq 1$, we have $\frac{q_{t+1} + \xi}{q_t} = \frac{\mu_{h,t+1}}{\mu_{h,t}}$ for every $t \geq 1$. At initial date, we have to prove that $\frac{q_0}{q_1 + \xi} = \frac{\mu_{h,1}}{\mu_{h,0}}$, i.e.,

$$\frac{q_0}{\xi}(1 - \beta) = \frac{\mu_{h,1}}{\mu_{h,0}}.$$

FOC of $k_{h,t}$ with $h \in \{i, j\}$.
For $t \geq 2$, we have to prove that $\frac{1}{F'(0) + 1 - \delta} \geq \max_i \frac{\mu_{i,t+1}}{\mu_{i,t}}$. This is true because $\frac{\mu_{h,t+1}}{\mu_{h,t}} = \beta$ for every $t \geq 1$ and $\beta(F'(0) + 1 - \delta) \leq 1$.
At date 1, we have to prove that

$$1 \geq \frac{\mu_{h,1}}{\mu_{h,0}}(F'(0) + 1 - \delta) \quad \forall h \in \{i, j\}.$$

Therefore, we have only to check the following system

$$\begin{aligned} 1 &\geq \frac{\mu_{h,1}}{\mu_{h,0}}(F'(0) + 1 - \delta) \quad \forall h \in \{i, j\}. \\ \frac{q_0}{\xi}(1 - \beta) &= \frac{\mu_{h,1}}{\mu_{h,0}}, \quad \forall h \in \{i, j\}. \end{aligned}$$

We have

$$\frac{\mu_{i,1}}{\mu_{i,0}} = \beta\frac{u_i'(a\xi)}{u_i'(a(F(K_0)+(1-\delta)K_0+\xi_0))} = \beta\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma},$$
$$\frac{\mu_{j,1}}{\mu_{j,0}} = \beta\frac{u_j'(a\xi)}{u_j'(a(F(K_0)+(1-\delta)K_0+\xi_0))} = \beta\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma}.$$

Our system becomes

$$1 \geq \beta(F'(0)+1-\delta)\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma},$$
$$\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma} = \frac{q_0}{\xi}\frac{1-\beta}{\beta}.$$

Choose ξ, ξ_0, k_0, q_0 to be satisfied this system. ■

Proof for Proposition 4.3 : Assume that $\beta_i(F'(0)+1-\delta)u_i'(\xi_{t+1}) > u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})$ for every $i = 1, \dots, m$.
If $K_{t+1} = 0$, the market clearing conditions imply that

$$\sum_{i=1}^{m} c_{i,t} = F(K_t)+(1-\delta)K_t+\xi_t \tag{V.33}$$
$$\sum_{i=1}^{m} c_{i,t+1} + K_{t+2} = \xi_{t+1}. \tag{V.34}$$

Therefore, there exists $i \in \{1, \dots, m\}$ such that $c_{i,t} \geq \frac{F(K_t)+(1-\delta)K_t+\xi_t}{m}$, so $u_i'(c_{i,t}) \leq u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})$.
Moreover, FOC of K_{t+1} implies that $r_{t+1} \geq F'(K_{t+1}) = F'(0)$. FOC of $k_{i,t+1}$ implies that $\frac{1}{r_{t+1}+1-\delta} \geq \max_j \frac{\mu_{j,t+1}}{\mu_{j,t}}$. Therefore, we imply that

$$\frac{1}{F'(0)+1-\delta} \geq \max_j \frac{\mu_{j,t+1}}{\mu_{j,t}} \geq \frac{\mu_{i,t+1}}{\mu_{i,t}} = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \geq \frac{\beta_i u_i'(\xi_{t+1})}{u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})},$$

contradicting our assumption. ■

Proof for Proposition 4.2: Assume that there exists $t \geq 0, T \geq 1$ such that $\xi_t \geq \xi_{t+T}$. If $(F'(0)+1-\delta)\beta_i > 1$ for every $i = 1, \dots, m$.
If $K_{t+s} = 0$ for every $s = 1, \dots, T$, we have

$$\sum_{i=1}^{m} c_{i,t} = F(K_t)+(1-\delta)K_t+\xi_t,$$
$$\sum_{i=1}^{m} c_{i,t+s} + K_{t+s+1} = \xi_{t+s}, \quad \forall s = 1, \dots, T.$$

Therefore, we have

$$\sum_{i=1}^{m} c_{i,t} \geq \xi_t \geq \xi_{t+T} \geq \sum_{i=1}^{m} c_{i,t+T}. \tag{V.35}$$

Consequently, there exists $i \in \{1, \cdots, m\}$ such that $c_{i,t} \geq c_{i,t+T}$, hence $u_i'(c_{i,t+T}) \geq u_i'(c_{i,t})$.

FOC of K_{t+s} implies that $r_{t+s} \geq F'(K_{t+s}) = F'(0)$. FOC of $k_{i,t+s}$ implies that $\frac{1}{r_{t+s}+1-\delta} \geq \max_j \frac{\mu_{j,t+s}}{\mu_{j,t+s-1}}$. Hence

$$\Big(\frac{1}{F'(0)+1-\delta}\Big)^T \geq \prod_{s=1}^{T} \max_j \frac{\mu_{j,t+s}}{\mu_{j,t+s-1}} \geq \prod_{s=1}^{T} \frac{\mu_{i,t+s}}{\mu_{i,t+s-1}} = \frac{\beta_i^T u_i'(c_{i,t+T})}{u_i'(c_{i,t})} \geq (\beta_i)^T.$$

So $1 \geq (F'(0)+1-\delta)\beta_i$, contradiction! ■

Proof for Lemma 4.4: Market clearing conditions imply that

$$\sum_{i=1}^{m} c_{i,t} + K_{t+1} = F(K_t) + (1-\delta)K_t + \xi_t \tag{V.36}$$

$$\sum_{i=1}^{m} c_{i,t+1} + K_{t+2} = F(K_{t+1}) + (1-\delta)K_{t+1} + \xi_{t+1}. \tag{V.37}$$

Therefore, there exists $i \in \{1, \ldots, m\}$ such that $c_{i,t} \geq \frac{\xi_t - K_{t+1}}{m}$.

Moreover, FOC of K_{t+1} implies that $r_{t+1} \geq F'(K_{t+1})$. FOC of $k_{i,t+1}$ implies that $\frac{1}{r_{t+1}+1-\delta} \geq \max_j \frac{\mu_{j,t+1}}{\mu_{j,t}}$. Therefore

$$\frac{1}{F'(K_{t+1})+1-\delta} \geq \max_j \frac{\mu_{j,t+1}}{\mu_{j,t}} \geq \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \geq \frac{\beta_i u_i'(F(K_{t+1}) + (1-\delta)K_{t+1} + \xi_{t+1})}{u_i'(c_{i,t})}.$$

This can be rewritten as

$$u_i'(c_{i,t}) \geq \big(F'(K_{t+1}) + 1 - \delta\big)\beta_i u_i'(F(K_{t+1}) + (1-\delta)K_{t+1} + \xi_{t+1})$$

Hence, we have $c_{i,t} \leq G_i(K_{t+1}, \xi_{t+1})$, where $G_i(K_{t+1}, \xi_{t+1})$ is defined by

$$u_i'(G_i(K_{t+1}, \xi_{t+1})) = (F'(K_{t+1}) + 1 - \delta\big)\beta_i u_i'(F(K_{t+1}) + (1-\delta)K_{t+1} + \xi_{t+1}).$$

As a result, we obtain

$$\xi_t \leq K_{t+1} + mG_i(K_{t+1}, \xi_{t+1})$$

■

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Chapitre VI

Multinational firms, FDI, and economic growth

Abstract : We consider a small open economy with two productive sectors : an old sector producing consumption good by using physical capital and a new sector producing new good by using physical capital and specific labor. There are two types of firms in the new industry : a well planted multinational firm and a potential domestic firm. Our framework highlights a number of results. First, in a poor country with low return of training and weak FDI spillovers, the domestic firm does not exist in the new industry requiring a high fixed cost. Second, once the host economy has the capacity to create the new firm, the competitivity of the domestic firm is the key factor allowing it to enter into the new industry, and even eliminate the multinational firm. Interestingly, in some cases where FDI spillovers are strong, the country should invest in the new industry, but not train specific workers. Last, credit constraint and labor/capital shares play important roles in the competition between the multinational and the domestic firms.

Keywords : FDI spillovers, investment in training, heterogeneous firms, entry cost, optimal growth.
JEL Classification Numbers : F23, F4, F62, O3.

1 Introduction

Over the last five decades, operations of multinational firms have made a significant influence on developing economies. Multinational firms may generate FDI spillovers to domestic firms by transferring advanced technologies or training workers. Thanks to that, some countries can promote the development of their industries and particularly, encourage the entry of domestic firms into these industries.[1] However, is attracting FDI spillovers the key to developing of their own industries ? If not, what is the optimal policy of the host country ? More precisely, should the host country develop these new industries, or continue to focus on already developed ones ? What are the roles of different macroeconomic variables such as development level, FDI spillovers, return of training, and heterogeneity of firms.

To answer these questions, we consider a small open economy model (two-period and infinite horizon models) with two sectors and heterogeneous firms. The first, called old sector, produces consumption good by using physical capital as the sole input. There is a unique representative domestic firm in this sector. The second (new sector) produces a new good by using physical capital and a specific labor. There are two types of firm in the new sector : an already planted multinational firm and a potential domestic one. The potential domestic firm cannot be created if it holds less than $\bar{L}$ units of specific labor. These two firms differ not only in productivity but also in labor and capital shares.

In this economy, consumption good, physical capital, and new good can be freely exchanged with the rest of the world while the specific labor is not mobile. There are two agents : a social planner maximizing the GNP of the country and the multinational firm maximizing its profit. The prices (in term of consumption good) of physical capital and new good are assumed to be exogenous. However, the wage is endogenous and determined by labor market clearing condition. Specific labor supply is also endogenous and arises from three sources : initial specific labor of the country, FDI spillovers effects, and investment in training.

Our framework provides a number of results. First, to invest in the new industry, the country must hold one of the following conditions : (i) it is rich enough, (ii) its return of training is high enough, (iii) FDI spillovers are strong. This result is due to the existence of fixed cost $\bar{L}$ which prevents the domestic firm's entry. Our finding indicates that in a poor country with low FDI spillovers, the domestic firm cannot exist in the new industry even if its productivity is high.

Second, once the country satisfies the above conditions, the productivity of the potential domestic firm is the key factor deciding the optimal choice

1. See Harrison, Rodriguez-Clare (2010) for a complete review.

of the country. Our framework shows that if the old sector is competitive (i.e., high productivity), it will attract all investment. Hence, the country should focus on this sector, but neither develop the new sector nor invest in training. We prove that the host country should invest in the new industry if and only if the productivity of the domestic firm in this industry reaches a critical threshold. Moreover, although the domestic firm must pay an entry cost, it can dominate, even eliminate, the multinational firm. We also make clear the role of the entry cost by showing that the mentioned critical threshold of productivity increases when the entry cost increases. One may ask if training of specific labor is essential to create a new domestic firm in the new industry. Not always ! Indeed, in the case where FDI spillovers are strong and the domestic firm's productivity is high, the host country should invest in the new industry but not in training.

Third, we study the competition between the multinational firm and the domestic one in the new industry by analyzing heterogeneities of firms and the roles of exogenous prices, return of training. Since the wage is endogenous, specific workers will be hired by the more competitive firms.[2] Does the domestic firm benefit from high return of training/low physical capital price/high new good price in order to compete with the multinational firm ? Our model shows that, with high returns of training, the host country will not invest in the new industry when the physical capital share of the potential domestic firm is not too low. The main reason comes from credit capacity of firms. Indeed, high returns implies a high number of specific workers. If the potential domestic firm has a weak credit capacity, it cannot buy an arbitrary quantity of physical capital when its capital share is not too low and the number of workers is high. Therefore, its production process will be inefficient. By contrast, the multinational firm can get financing from its parent company and, thanks to that, when specific labor supply is high, it can buy an arbitrary amount of physical capital to make its production process efficient. As a consequence, all specific workers will be hired by the multinational firm, which implies that the domestic one cannot enter the new industry even if the country has a high return of training. A similar argument can be used when physical capital price is low. It seems that the host country may more likely invest in the new industry if the new good price is high. Unfortunately, this argument is not always true. Our framework points out that even if the new good price is very high, the domestic firm cannot enter this new industry because of its weak credit capacity. However, we should make clear that with middle level of return of training/physical capital price/new good price, productivities of firms play the most important role in their competition.

2. In our model, the competitiveness of firm is characterized by four factors : productivity, labor and capital share, and credit capacity.

Our paper is related to several strands of research. The first strand studies the fixed entry cost of firms and economic growth. Smith (1987) and Markusen (1995) point out that a potential domestic firm has to invest in a firm-specific fixed cost in order to be able to produce. By contrast, Smith (1987) considers that the multinational firm has a plant in its home country where this investment has been already realized, and then does not suffer it by producing in the host country. Fosfuri, Motta, Ronde (2001) indicate that a domestic firm may gain from new technologies thanks to the mobility of worker who initially worked for multinational firms. However, to do that, the domestic firm must to pay a fixed cost which may be interpreted as its absorptive capability. In our framework, we assume that the domestic firm must utilize a fixed number of skilled workers to ensure that its production process functions. We also make clear the impact of this fixed cost on the competition of firms, and then on the economic growth. In optimal growth context, Bruno, Le Van, Masquin (2009) prove that a poor country cannot invest in new technology. However, they consider do not take into account the impact of multinational firms. In our paper, multinational firms can generate FDI spillovers and may eventually help the country to invest in new technology.

The second concerns FDI spillovers and training of skilled workers. The literature shows the existence of four types of FDI spillovers. [3] First, FDI spillovers may be created via vertical linkages between foreign affiliates and local suppliers (Rodriguez-Clare , 1996; Markusen, Venables , 1995; Carluccio, Fally , 2013). Second, multinational firms can improve productivity of domestic firms through demonstration/imitation effects. Export is the third channel through which domestic firms can benefit from multinational firms (Aitken, Hanson, Harrison , 1997; Greenaway, Sousab, Wakelin , 2004). Last, FDI spillovers may arise due to the mobility of workers who have been trained by multinational firms (Ethier, Markusen , 1996; Fosfuri, Motta, Ronde , 2001; Poole , 2013). FDI spillovers in our paper are generated through the last form. By contrast, in our paper, specific workers are not only trained by multinational firms (through specific communication or learning by doing effects), but also by the government ; thanks to that, the host country gaining low FDI spillovers can still develop the new industry.

The last strand is the link between credit constraints and trade. Kletzer, Bardhan (1987) theoretically show how comparative advantage depends on credit market imperfections. By using a 30-year panel for 65 countries, Beck (2002) finds that financial development exerts a causal impact on exports and trade balance of manufactured goods. Manova (2008) studies the impact of equity market liberalizations on trade by giving empirical

3. See Blomstrom, Kokko (1998); Gorg, Greenaway (2004); Crespo, Fontoura (2007) for a substantial review of FDI spillovers.

evidence (with 91 countries), and then shows that credit constraints play an important role on international trade flows. Manova (2013) incorporates credit constraints and firm heterogeneity into Melitz (2003) and studies the impact of financial frictions not only on producers's entry into exporting but also on exporters' foreign sales. Different from theses papers, we focus on the impact of credit constraints on the competition between the domestic firm and the multinational one in the host country's market.

The remainder of the paper is organized as follows. Section 2 presents the structure of economy. In section 2, we explore the optimal strategy of the host country at equilibrium in a two-period model by analyzing roles of all factors of the economy. Section 3 gives dynamic analysis in an infinite horizon model. Section 4 concludes. All formal proofs can be found in Appendices.

2 A two-period model

We start by a two-period model. We consider a small open economy having two productive sectors. The first produces the consumption good by using physical capital good. We call it the old sector. There is a unique representative domestic firm (called consumption good firm) in this sector and its production function is given by

$$F^c(K_c) = A_c K_c^{\alpha_c} \quad \text{(VI.1)}$$

where $A_c > 0$ and $\alpha_c \in (0,1)$.[4]

The second sector produces a new good by using physical capital good and a specific labor. It is called new sector or new industry. In this sector, there are two types of firm : a multinational firm (or foreign firm) and a potential domestic one. The foreign firm is well planted in the country and its production function is

$$F^e(K_e, L_e) = A_e K_e^{\alpha_e} L_e^{\beta_e} \quad \text{(VI.2)}$$

where $A_e > 0$ and $\alpha_e, \beta_e \in (0,1)$, $\alpha_e + \beta_e \leq 1$.

The potential domestic firm's production function is given by

$$F^d(K_d, L_d) = A_d K_d^{\alpha_d} \big((L_d - \bar{L})^+\big)^{\beta_d} \quad \text{(VI.3)}$$

4. The reader may ask why there is only one input to produce consumption good. We can introduce labor into the production process of the consumption good by taking the production function as $F^c(K_c, L_c) = A_c K_c^{\alpha_c} L_c^{\beta_c}$ where L_c is low-skilled labor. If we assume that low-skilled labor is exogenous and there is no possible transfer between high-skilled and low-skilled workers, this setup becomes exactly our framework with the unique input.

where $A_d > 0$ and $\alpha_d, \beta_d \in (0,1)$, $\alpha_d + \beta_d \leq 1$. To enter the new industry, the domestic firm must make an initial investment. We model this investment by the fixed cost, $\bar{L}$, representing the number of specific workers needed to ensure that the production process functions. Thanks to the parent company, the foreign firm does not need to pay this investment.

Interpretation of $\bar{L}$: In general, we can assume that the production functions of firms are

$$F^d(K_d, L_d) = A_d K_d^{\alpha_d}\big((L_d - \bar{L}_d)^+\big)^{\beta_d}$$
$$F^e(K_e, L_e) = A_e K_e^{\alpha_e}\big((L_d - \bar{L}_e)^+\big)^{\beta_e}.$$

In the new industry, firms need some technical experts to set up the production process in order to be able to produce. The parent company of the foreign firm has such experts in the home country and sends them to host countries for new production plants. Once this setup is finished, the technical experts will come back to their home country. Hence, we can assume that $\bar{L}_d > \bar{L}_e$. Without loss of generality, we assume that $\bar{L}_e = 0$, and in this case we write $\bar{L}$ instead of $\bar{L}_d$.

We would like to make distinction between our threshold $\bar{L}$ and others in literature. Azariadis, Drazen (1990) consider the following production function $F = A_t F(K_t, L_t)$, where the scale factor A_t may depend functionally on a vector of social inputs that are not controlled by any one producer.

In Bruno, Le Van, Masquin (2009), A_t is endogenized : the social planner chooses physical capital $K_{e,t}$ and high-skilled labor $L_{e,t}$ in order to produce new technology, which enter the formula of A_t as follows $A_t := x_0 + a(F(K_{e,t}, L_{e,t}) - \bar{X})^+$. $\bar{X}$ a minimum level of adoption of new technologies which is necessary for them in order to impact the economy.

In Smith (1987), the potential domestic firm has the following problem

$$\max P(X)X - cost(X) - (\text{fixed cost}). \tag{VI.4}$$

In Melitz (2003), the production is given by $F(L) = \phi(L - f)^+$. The threshold $f > 0$ represents a fixed entry cost (measured in units of labor.

In Fosfuri, Motta, Ronde (2001), fixed cost arises in the local firm's valuation of the worker $v_l = N_2\Pi_d(\phi) - k$ where k indicates the cost that local firm has to pay in order to gain from new technology received by the trained worker.

The economy takes place into two periods : date 0 and date 1. All consumption good, physical capital, and new good can be freely exchanged with the rest of the world, but the specific labor is not mobile. Let consumption good price be numeraire. Denote p (resp., p_n) the international real prices of capital good (resp., new good) in term of consumption good. Prices p, p_n are exogenous. The initial endowment of the host country is S, $(S > 0.)$

Let L_0 be the initial specific labor, T_0 be the specific workers generated by the foreign firm at the first period.[5] We assume that if the country invests an amount H_1 in education, it will get ϵH_1 specific workers, where ϵ is the return of training. Hence the specific labor supply of the country after receiving FDI and training will be

$$L_0 + T_0 + \epsilon H_1.$$

Note that specific labor supply in this economy is endogenous.

Let denote this economy by

$$\mathcal{E} := (F^c, F^d, F^e, S, p, p_n, \epsilon, L_0, T_0, \bar{L}).$$

Denote by w_1 the real wage in term of consumption good at date 1, the wage is endogenous in our model. For simplicity, we assume that the depreciation rate of physical capital equal 1.

The foreign firm (without market power) maximizes its profit.

$$(F): \qquad \max_{K_{e,1}, L^D_{e,1} \geq 0} \Big[p_n F^e(K_{e,1}, L^D_{e,1}) - pK_{e,1} - w_1 L^D_{e,1} \Big].$$

The social planner takes prices as given and chooses $c_1, K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}$ to maximize GNP of the economy at the second period :[6]

$$(P): \qquad \max_{\big(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}\big)} \Big[U := F^c(K_{c,1}) + w_1 L_{e,1} + p_n F^d(K_{d,1}, L_{d,1}) \Big]$$

5. We note that FDI spillovers T_0 in our framework arise through workers mobility. We refer to Fosfuri, Motta, Ronde (2001); Gorg, Strobl (2005); Poole (2013) for the existence of such FDI spillovers.

6. In general, the social planner' problem is to choose $c_0, c_1, S, K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}$ that maximizes the utility

$$\max_{\big(c_0, c_1, S, K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}\big)} \Big[\beta_0 U(c_0) + \beta_1 U(c_1) \Big]$$

$$\text{subject to } c_0 + S \leq S_0, \quad c_1 \leq F(S)$$

where $S_0 > 0$ is given, β_i is the time preference at date $i = 0, 1$, and $F(S)$ is defined by

$$\begin{aligned} & \max \Big[F^c(K_{c,1}) + w_1 L_{e,1} + p_n F^d(K_{d,1}, L_{d,1}) \Big] \\ \text{subject to} \quad & H_1 + p(K_{c,1} + K_{d,1}) \leq S \\ & L_{d,1} + L_{e,1} \leq L_0 + T_0 + \epsilon H_1 \\ & K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1} \geq 0. \end{aligned}$$

For simplicity, we assume that S is exogenous. Then the problem of the social planner is equivalent to the problem (P).

subject to

$$H_1 + p(K_{c,1} + K_{d,1}) \leq S \tag{VI.5}$$
$$L_{d,1} + L_{e,1} \leq L_0 + T_0 + \epsilon H_1, \tag{VI.6}$$
$$K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1} \geq 0. \tag{VI.7}$$

At the first period (date 0), the social planner uses H_1 units of consumption good to train specific labor. She also buys $K_{c,1}$, (resp. $K_{d,1}$) units of physical capital as input for the consumption sector (resp. the new sector).

At the second period (date 1), an amount of specific labor $L_{e,1}$ is used by the multinational firm and another amount of specific labor $L_{d,1}$ is used by the domestic firm. The GNP of the economy (in term of consumption good) has three parts

(i) $F^c(K_{c,1})$: consumption good from the consumption sector.
(ii) $w_1 L_{e,1}$: salary in term of consumption good paid by the multinational firm.
(iii) $p_n F^d(K_{d,1}, L_{d,1})$: production value of the domestic firm.

Note that, if a specific worker works for the multinational firm, she only contributes to the GNP by her salary because the multinational firm takes away its profits. However, if she works for the domestic firm, the GNP is improved in two ways, salary of the worker and profit of the domestic firm.

Remark 2.1 *The constraint $H_1 + p(K_{c,1} + K_{d,1}) \leq S$ means that the host country cannot borrow from abroad. As a consequence, the potential domestic firm faces a credit constraint. By contrast, the multinational firm does not face credit constraint because it can get financing from its parent company.*

Definition 2.1 *Consider the economy $\mathcal{E} := (F^c, F^d, F^e, S, p, p_n, \epsilon, L_0, T_0, \bar{L})$. An equilibrium is a list $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}, K_{e,1}, w_1)$ such that*

(i) Given labor price w_1, $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1})$ is a solution of problem $(P0)$.
(ii) Given labor price w_1, $(L^D_{e,1}, K_{e,1})$ is a solution of problem (F).
(iii) Labor market clears : $L^D_{e,1} = L_{e,1}$.

In what follows, in order to avoid confusion, we present our findings in the case where the production functions of the firms are strictly decreasing returns to scale, i.e. $\alpha_d + \beta_d, \alpha_e + \beta_e < 1$. Note that, most of our findings are also valid for constant returns to scale technologies (Theorem 2.2 and Proposition 2.12).

As the wage is endogenous in our model, we firstly provide relations among exogenous prices, return of training and wage.

Proposition 2.1 *(i) w_1 decreases if p or ϵ increases, increases if p_n increases.*

(ii) Denote $\tilde{w}_1$ (resp. $\hat{w}_1$) the wage in the case $Y_{d,1} > 0$ (resp. $Y_{d,1} = 0$). Then we have $\hat{w}_1 \leq \tilde{w}_1$.

Proof: These are direct consequences of the equation determining wage. See (VI.49), (VI.49), (VI.53) and (VI.57) in Appendix 5 . ■

Point (i) is clear. For example, a rising of physical capital price makes induces a decrease of the production level in the new sector. Consequently, demand of specific labor decreases. Therefore, the wage will decrease.

Proposition 2.1 also points out that the entry of domestic firm into the new industry leads to a greater wage than that in the case without the domestic firm.

2.1 FDI spillovers, optimal shares, and GNP

In this section, we consider an equilibrium in which $H_1 > 0$. Denote θ_h, θ_d the optimal share of investment in training and in new sectors, respectively, i.e.,

$$pK_{c,1} = (1 - \theta_d - \theta_h)S, \quad pK_{d,1} = \theta_d S \quad and \quad H_1 = \theta_h S \tag{VI.8}$$

First, we focus on direct FDI spillovers T_0.

Proposition 2.2 *(i) When T_0 increases, GNP increases.*

(ii) When T_0 increases, θ_h decreases

Proof: See Appendix 5.3. ■

This result confirms the positive impact of direct FDI spillovers T_0 on GNP of the host country, as it is shown in the literature. Nevertheless, point (ii) shows that such positive externalities lowers the share of investment in training of specific labor. Indeed, an increase of T_0 will improve the specific workers supply in the host country, then lower wage, finally decrease investment in training.

On the other hand, there exists an indirect spillover effect generated by the multinational firm in our model. Indeed, we have

$$L_{d,1} + L_{e,1} = L_0 + T_0 + \epsilon H_1. \tag{VI.9}$$

Assume that $L_0 + T_0 = 0$ then $L_{e,1}$ (resp., $L_{d,1}$) represents the level of specific labor generated in the new sector by the multinational (resp., domestic) firm. Following Rodriguez-Clare (1996), they are called the *linkage* *coefficient* of the multinational (resp., domestic) firm. He says that the multinational firm has a negative (resp., positive) linkage effect if $L_{e,1}-L_{d,1}<0$ (resp., $L_{e,1}-L_{d,1}>0$). In our framework, the linkage coefficient is given by

$$L_{e,1}-L_{d,1}=\sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}-(\bar{L}+\frac{\beta_d}{\alpha_d+\beta_d}\sigma_d w_1^{\frac{-1}{1-\alpha_d-\beta_d}}). \qquad \text{(VI.10)}$$

We can see that the linkage effect is increasing in A_e, decreasing in A_d or $\bar{L}$.

Rodriguez-Clare (1996) give a condition (condition (14), page 862) under which the difference between the linkage coefficient of the multinational firm and the domestic firm is positive. This condition depends on labor shares of firms, the communication cost, the level of the underdeveloped economy.[7] In our framework, the condition for a positive linkage coefficient depends on different parameters such as (i) productivities A_e, A_d of the foreign firm and domestic firm, (ii) the entry cost $\bar{L}$, (iii) prices of physical capital and new good, and (iv) shares of physical capital and specific labors α_d, β_d (resp. α_e, β_e) of the domestic firm (resp. multinational firm).

2.2 Diversification or specification ?

We now study the roles of different factors on the optimal strategy of the social planner. We says that the country *invests in the new industry* if the domestic firm exists in this sector, i.e., $Y_{d,1}>0$.

Let us start by two cases : the entry cost $\bar{L}$ is very low and the initial endowment S is very high.

Proposition 2.3 *There exists $\bar{L}^*$ depending on other parameters such that if $\bar{L}<\bar{L}^*$ then $Y_{d,1}>0$.*

Proof: See Appendix 5.3. ■

This result shows that when the entry cost is small enough, the host country should invest in the new industry.

For the initial endowment S, we have the following result.

Proposition 2.4 *There exists $\bar{S}$ such that if $S>\bar{S}$ then $H_1>0$ and $Y_{d,1}>0$ at equilibrium.*

Proof: See Appendix 5.3 ■

7. In his paper, the development level of each country is defined by the number of variety.

In our model, the country's initial endowment S can be viewed as an index of the development level of the country. Proposition 2.4 shows that, when the country has a high development level (i.e, S reaches a critical level), it will easily cover the entry cost $\bar{L}$. Therefore, this result is in line with Proposition 2.3. Moreover, since firms have a decreasing return to scales technology, the country also invests in training of specific labor to get a favorable salary.

Roles of productivities : We now observe the impact of the multinational firm's productivity.

Proposition 2.5 *There exists* $\bar{A}_e > 0$ *such that if* $A_e \geq \bar{A}_e$*, we have* $Y_{d,1} = 0$ *and* $H_1 > 0$.

Proof: See Appendix 5.3. ■

This result shows that, if the foreign firm's productivity is so high, the host country should not invest in the new industry. It is optimal to train specific workers, however, and then let them work for the foreign firm in order to get a favorable salary.

We now see the important role of the old sector's productivity.

Proposition 2.6 *There exists* $\bar{A}_c > 0$ *such that if* $A_c \geq \bar{A}_c$*, we have* $Y_{d,1} = 0$ *and* $H_1 = 0$.

Proof: See Appendix 5.3. ■

This result implies that, whenever the old sector is highly competitive, the country should focus on this sector and should not invest in the new industry. Moreover, by contrast to Proposition 2.5, the country do not invest, in this case, in training of specific workers. Indeed, the goal of this investment is to provide specific labor for the new sector, but the domestic firm in this sector is less competitive than the one in the old sector. Consequently, investing in training is not the best choice.

Propositions 2.5 and 2.6 also indicate that even if A_c or A_e are very small, we do not know whether the host country will invest in the new industry. Indeed, if A_c is small, but A_e is high enough, this country will focus on training of specific labor and let them work for the foreign firm. Similarly, when A_e is low, but A_c is high, a sole investment in the consumption sector is more relevant than both investing in the new sector and in training.

It is now interesting to study how the productivity of the domestic firm and the development level of the host country affect the optimal strategy of the social planner.

Let us begin by the following result.

Proposition 2.7 *If $\epsilon S + L_0 + T_0 \leq \bar{L}$ then $Y_{d,1} = 0$.*

Proof: Since $H_1 \leq S$ then $L_0 + T_0 + \epsilon H_1 \leq \epsilon S + L_0 + T_0$. Consequently, $L_{d,1} - \bar{L} \leq 0$, hence $L_{d,1} = K_{d,1} = 0$. ■

Proposition 2.7 shows that if a country invests in the new industry, it must satisfy one of the following conditions : (i) its development level is high enough, (ii) the return of training is high enough, (iii) FDI spillovers are strong enough.

The Proposition also allows us to capture some important points.

(i) Consider a poor country : S, L_0 are small so that $L_0 + \epsilon S < \bar{L}$. If $T_0 \leq \bar{L} - L_0 - \epsilon S$, i.e., FDI spillovers are low, the new industry cannot be created in this poor country. This result suggests that a poor country receiving a low FDI spillovers should focus on other sectors and not on the new industry requiring a high entry cost.

(ii) Consider a country whose FDI spillovers are not so high : $T_0 < \bar{L}$. In this case, it cannot invest in the new industry if $L_0 + \epsilon S \leq \bar{L} - T_0$. This means that to be able to benefit from FDI, this country must reach a certain development level.

We now consider a host country such that the maximum specific labor supply is greater than the entry cost but specific labor supply without training is not.

$$\epsilon S + L_0 + T_0 > \bar{L} \quad and \quad L_0 + T_0 \leq \bar{L} \tag{VI.11}$$

We have the following result.

Proposition 2.8 *Assume that $\epsilon S > \bar{L} - (L_0 + T_0) \geq 0$. We have*
(i) there exists $\bar{A}_1 > 0$ such that $Y_{d,1}(A_d) > 0$ if and only if $A_d \geq \bar{A}_d$. In this case $H_1 > 0$.
(ii) there exists $\tilde{A}_1 \geq \bar{A}_1$ such that if $A_d > \tilde{A}_1$ then $Y_{d,1} > Y_{e,1}$.
(iii) both $\bar{A}_1$ and $\tilde{A}_1$ are increasing in $\bar{L}$.

Proof: See Appendix 5.3 ■

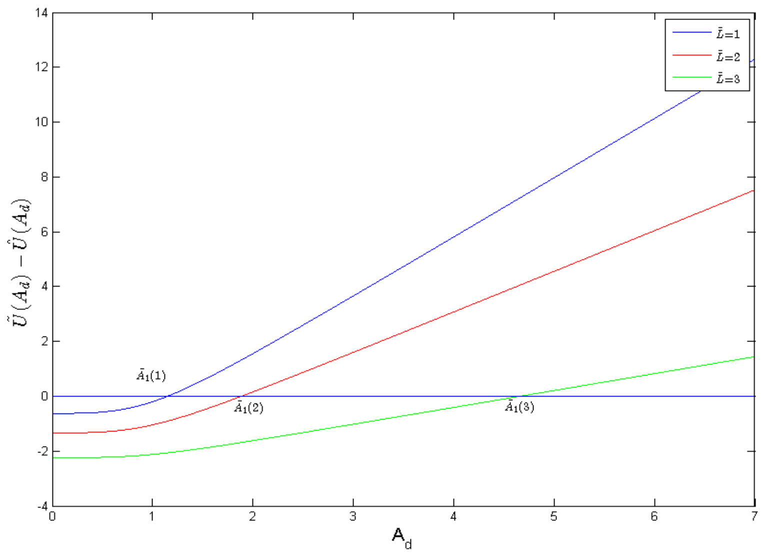
Proposition 2.8 indicates that, in a host country such as the one studied in this case, the productivity of the domestic firm is the key factor determining the optimal strategy of the social planner. If this firm is sufficiently competitive, it is optimal to invest in the new sector. However, since $L_0 + T_0 \leq \bar{L}$, training of specific workers is required to cover the entry cost of the domestic firm. That is why we have a strictly positive amount H_1 when A_d is high enough. Inversely, if the domestic firm has a low productivity, the social planner should not invest in the new industry. In this case, we do not have enough information to know whether the country invests in training of specific labor.

Proposition 2.8 also shows that the domestic firm can even dominate the foreign one, when the productivity of the former is very high, $A_d > \tilde{A}_1$. Our result is related to Markusen, Venables (1995) since these authors prove that in some countries, domestic firms may become sufficiently strong such that local production overtakes and carries out foreign one.

We clearly see how the fixed cost $\bar{L}$ prevent the host country invests in the new industry. The higher level of $\bar{L}$, the higher level of productivity the domestic firm must have to enter the market.

Let us show an example. Denote $\tilde{U}$ (resp. $\hat{U}$) the GNP in the case $Y_{d,1} > 0$ (resp. $Y_{d,1} = 0$). Figure VI.1 gives the path of the difference $\tilde{U} - \hat{U}$ as a function of A_d for three values of $\bar{L} = 1, 2, 3$. Note that $A_d > A_e$ (or $A_d < A_e$) is not sufficient to ensure the domestic firm's entry. We also see that the threshold $\bar{A}_1 > 0$ is increasing in $\bar{L}$.

FIGURE *VI.1: The graph of* $(\tilde{U} - \hat{U})$ *as a function of the domestic firm's productivity* A_d



$$A_c = A_e = 1.2; \epsilon = 1.2; S = 2; L_0 = 0.5; T_0 = 0.5; p = 1; p_n = 2;$$
$$\alpha_c = 0.7; \alpha_d = 0.3; \beta_d = 0.4; \alpha_e = 1/3; \beta_e = 7/15.$$

We have a direct consequence of Proposition 2.8.

Corollary 2.1 *Assume that* $L_0 = T_0 = 0$ *and* $\epsilon S > \bar{L}$. *We have* $Y_{d,1}, H_1 > 0$ *when* A_d *is high enough.*

This is the case where there is neither FDI spillovers effects nor initial specific labor. Our result gives an answer for the question : when the host country create a new industry ? A new industry can only be created under two conditions : (1) return of training is high, (2) the potential domestic firm is competitive enough.

We now study the case of a host country in which specific labor supply without training is high enough.

$$L_0 + T_0 > \bar{L} \tag{VI.12}$$

The optimal strategy of the social planner depends on different factors.

Proposition 2.9 *Assume that $L_0 + T_0 > \bar{L}$.*

1. *If $\epsilon S < \frac{\alpha_d}{\beta_d}(L_0+T_0-\bar{L})$ then there exists $A_2 > 0$ such that $Y_{d,1} > 0$ and $H_1 = 0$ if and only if $A_d \geq A_2$. Moreover, when A_d is high enough, we have $Y_{d,1} > Y_{e,1}$.*
2. *If $\epsilon S > \frac{\alpha_d}{\beta_d}(L_0+T_0-\bar{L})$ then there exists $A_3 > 0$ such that $Y_{d,1} > 0$ and $H_1 > 0$ if and only if $A_d \geq A_3$. Moreover, when A_d is high enough, we have $Y_{d,1} > Y_{e,1}$.*
3. *If $\epsilon S = \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$ then when A_d is high enough, we have $Y_{d,1} > 0$.*[8]

Proof: See Appendix 5.3. ■

Proposition 2.9 indicates that sole conditions on specific labor and entry cost are not sufficient to ensure the existence of the domestic firm. Once again, we observe the decisive role of its productivity A_d. The host country should invest in the new industry if and only if this productivity is high enough. This explains why in some rich countries, although there are sufficiently workers required to create a new firm, they do not choose to do it.

The first point of Proposition 2.9 shows us an interesting scenario : the host country can create a new firm *i.e.*, $Y_{d,1} > 0$ without training of specific labor (i.e., $H_1 = 0$). This is the case where the potential domestic firm's productivity is high and the condition $\epsilon S < \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$ holds, i.e., when

(i) ϵ is low (see Proposition 2.10 for further discussions.)

(ii) or/and the ratio α_d/β_d of capital share over specific labor share of the potential domestic firm is high

(iii) or/and the difference $L_0 + T_0 - \bar{L}$ is high. This means that the entry cost is relatively lower than FDI spillovers T_0 and/or the initial specific labor L_0.

8. $\epsilon S = \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$. We do not enough information to confirm $H_1 > 0$. It depends on other factors. We will answer this question in Section 2.1.

Some empirical studies are likely to support our finding. Gershenberg (1987) argues that in Kenya, some local managers usually started their career in multinational firms before creating their own firm. By using a sample of firm-level data in Ghana, Gorg, Strobl (2005) state that there exist some domestic firms whose entrepreneurs (owner or chairman) worked for a multinational firm before joining or setting up their own domestic firm.[9] These managers/entrepreneurs can be represented by parameter T_0 in our model.

We summarize our result in the following theorem.

Theorem 2.1 *We have the following properties at equilibrium*

1. *If* $L_0 + \epsilon S + T_0 \leq \bar{L}$ *then* $Y_{d,1}(A_d) = 0$ *for all* A_d.
2. *If* $L_0 + \epsilon S + T_0 > \bar{L}$ *then*
   - 2.1. *If* $L_0+T_0 \leq \bar{L}$ *then when* A_d *is high enough, we have* $Y_{d,1} > 0$ *and* $H_1 > 0$. *Moreover, when* A_d *is very high, we have* $Y_{d,1} > Y_{e,1}$.
   - 2.2. *If* $L_0 + T_0 > \bar{L}$ *then*
     - 2.2.1. *if* $\epsilon S < \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$ *then when* A_d *is high enough, we have* $Y_{d,1} > 0$ *and* $H_1 = 0$. *Moreover, when* A_d *is very high, we have* $Y_{d,1} > Y_{e,1}$.
     - 2.2.2 *if* $\epsilon S > \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$ *then when* A_d *is high enough, we have* $Y_{d,1} > 0$ *and* $H_1 > 0$. *Moreover, when* A_d *is very high, we have* $Y_{d,1} > Y_{e,1}$.
     - 2.2.3 *If* $\epsilon S = \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$ *then when* A_d *is high enough, we have* $Y_{d,1} > 0$.

Here, the multinational firm exists thanks to its decreasing return to scale technology. However, in the case of constant return to scale production functions, we have the following result.

Theorem 2.2 *We assume that* $\alpha_d + \beta_d = \alpha_e + \beta_e = 1$. *We have the following properties at equilibrium*

1. *If* $L_0 + \epsilon S + T_0 \leq \bar{L}$ *then* $Y_{d,1} = 0$, $Y_{e,1} > 0$.
2. *If* $L_0 + \epsilon S + T_0 > \bar{L}$ *then*
   - 2.1. *If* $L_0+T_0 \leq \bar{L}$ *then when* A_d *is high enough, we have* $Y_{d,1}, H_1 > 0$, *and* $Y_{e,1} = 0$.
   - 2.2. *If* $L_0 + T_0 > \bar{L}$ *then*

9. Among 228 domestic firms in the sample, the number of domestic firms whose the entrepreneur worked for a multinational firm is about 32. Most of them is in metals and machinery (34.4%), followed by furniture (31.3%), textiles (18.8%), and wood products (9.4%).

2.2.1. if $\epsilon S \leq \frac{\alpha_d}{1-\alpha_d}(L_0 + T_0 - \bar{L})$ *then when* A_d *is high enough, we have* $Y_{d,1} > 0$*,* $H_1 = 0$ *and* $Y_{e,1} = 0$*.*

2.2.2 if $\epsilon S > \frac{\alpha_d}{1-\alpha_d}(L_0 + T_0 - \bar{L})$ *then when* A_d *is high enough, we have* $Y_{d,1}, H_1 > 0$*, and* $Y_{e,1} = 0$*.*

Proof: See Appendix 6. ■

On the one hand, Theorem 2.2 shares the main point with Theorem 2.1. On the other hand, it indicates an interesting scenario in which the well planted foreign firm may be eliminated. There are two main conditions for such scenario : (i) the maximum specific labor supply is high enough to cover the entry cost, (ii) the domestic firm's productivity is high. In this scenario, although the domestic firm has to pay an entry cost, it may not only enter into the new industry but also eliminate the well planted multinational firm.

Roles of returng of training and credit constraints : We are now interested in the role of return of training of qualified workers on the optimal choices.

Proposition 2.10 *Then there exists* $\bar{\epsilon}, \underline{\epsilon}$ *depending on the other parameters such that : (i) if* $\epsilon > \bar{\epsilon}$ *then* $H_1(\epsilon) > 0$ *at equilibrium, (ii)* $\epsilon < \underline{\epsilon}$ *then* $H_1(\epsilon) = 0$*.*

Proof: See Appendix 5.3. ■

Proposition 2.10 shows that the host country will invest in training of specific labor if its return exceeds a threshold. But if return of training is low, the country should not invest in this sector.

In our framework, investment in training is to provide specific labors for the new industry. A natural question appearing is that when return ϵ is very high, will investing in the new industry be optimal for the host country ?

The answer is the following.

Proposition 2.11 *(***High return of training with DRS technologies***)*

(i) If $\frac{\beta_d}{\alpha_d+\beta_d} > \frac{\beta_e}{1-\alpha_e}$ *then when* ϵ *is high enough, the country should invest in both training and the new industry, i.e.,* $H_1, Y_{d,1} > 0$*. Moreover, we have* $\lim_{\epsilon\to\infty} \tilde{U} - \hat{U} = +\infty$*.*

(ii) If $\frac{\beta_d}{\alpha_d+\beta_d} < \frac{\beta_e}{1-\alpha_e}$*, we have* $\lim_{\epsilon\to\infty} \tilde{U} - \hat{U} = 0$ *and*

(ii.a) if $\frac{\beta_d}{\alpha_d+\beta_d} < \frac{\beta_e}{1-\alpha_e} < 1-\alpha_d$ *then when* ϵ *is high enough, the country should invest in both training and the new industry.*

(ii.b) if $1-\alpha_d < \frac{\beta_e}{1-\alpha_e}$ *then when* ϵ *is high enough, the country should invest in training, but not invest in the new industry.*

With constant return to scale (CRS) technologies, we have.

Proposition 2.12 *(***High return of training with CRS technologies***). Assume that* $\alpha_d + \beta_d = \alpha_e + \beta_e = 1$*. When* ϵ *is high enough, the country should invest in training, but not invest in the new industry.*

Proof: See Appendix 6. ■

Proposition 2.12 can be viewed as a consequence of point (ii.b) of Proposition 2.11. Indeed, let $\alpha_e + \beta_e$ tend to 1, then $\frac{\beta_e}{1-\alpha_e}$ tends to 1 which is greater that $1-\alpha_d$. According to point (ii.b) of Proposition 2.11, $H_1 > 0$, $Y_{d,1} = 0$ when return of training ϵ is high enough. Although CRS technologies would simplify computations, it may make a misunderstanding about the optimal strategy of the country. We can see here that if we only considered CRS technologies, we could not know the roles of labor/capital shares. That is why we need to analyze both technologies : CRS and DRS.

We now can give some implications of Propositions 2.11 and 2.12 by considering a country in which specific labor can be easily trained (i.e., ϵ is high).

First, as stated in Proposition 2.11, this country should focus on the new industry if the potential domestic firm has a high labor share. Indeed, on the one hand, high value of ϵ allows the domestic firm to cover more easily the entry cost $\bar{L}$. On the other hand, high labor share of the domestic firm make it be more competitive than the foreign firm. Consequently, the country should invest in the new industry.

Second, we discuss *credit constraints* of firms. The asymmetry of conditions in Propositions 2.11 and 2.12 is from the structural difference between the multinational firm and the domestic one. In our framework, the host country cannot borrow from abroad, therefore the potential domestic firm faces a credit constraint $pK_{d,1} < S$, i.e., $K_{d,1} < \bar{K} := S/p$. Therefore, if capital share reaches a critical threshold, $\alpha_d > \frac{1-\alpha_e-\beta_e}{1-\alpha_e}$, the production process will be inefficient when the number of workers is high. However, since the multinational firm is not credit constrained, [10] it can buy arbitrary high quantity of input $K_{e,1}$ to be consistent with high quantity of specific labor. Hence, in the environment of the multinational firm, specific workers have enough physical capital in order to produce efficiently the new good. As a consequence, when return ϵ is high, all specific workers will be hired by the multinational firm even if the domestic firm has higher productivity. [11]

10. it can get financing from the parent company.

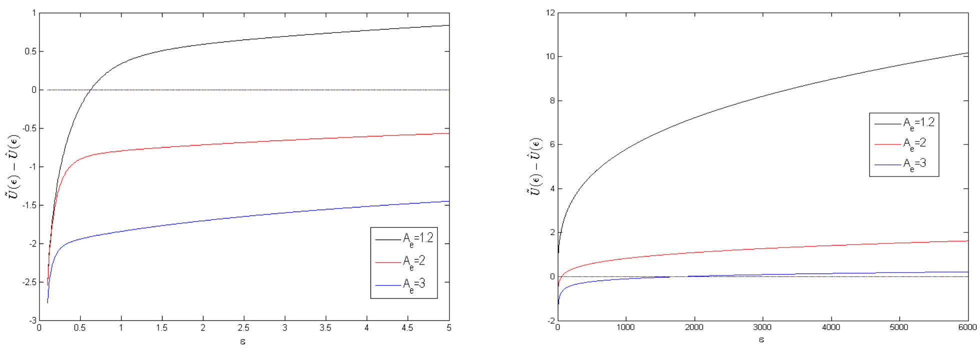
11. Indeed, consider the case CRS technologies with $A_d > A_e, \alpha_d = \alpha_e, \beta_d = \beta_e$, and there is no entry cost ($\bar{L} = 0$). Proposition 2.12 proves that $L_{d,1} = 0$ when ϵ is high

It means that credit constraints may prevent the domestic firm to entry in the new industry.

By analyzing the impact of credit constraints on the competition between the domestic firm and the multinational one, our result contributes to the literature about the impact of credit constraints on international trade (Kletzer, Bardhan , 1987; Beck , 2002; Alfaro, Chanda, Kalemli-Ozcan, Sayek , 2004; Manova , 2008, 2013).

We illustrate our finding by some examples. The first concerns point (i) of Proposition 2.11 (cf. Figure VI.2). We consider 3 cases : $A_e = 1.2$, $A_e = 2$, and $A_e = 3$.

Figure *VI.2: The graph of $(\tilde{U} - \hat{U})$ as a function of ϵ*



(a) ϵ : 0.1 to 5 (b) ϵ : 1 to 6000

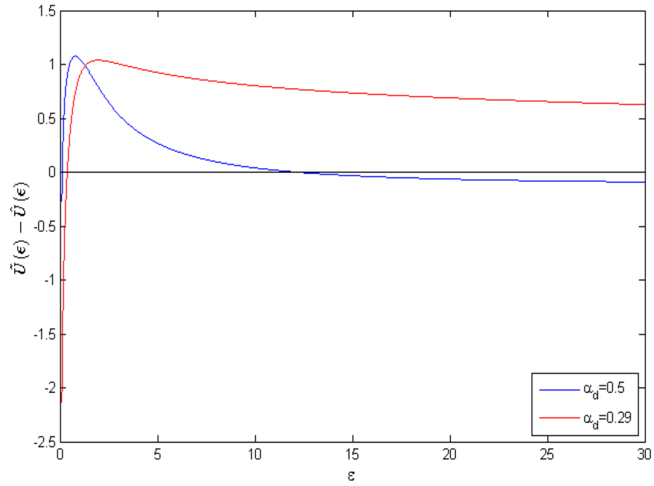
$A_c = A_d = 1; S = 1; L_0 = 2; T_0 = 1; \bar{L} = 1; p = 1; p_n = 2; \alpha_c = 0.7$
$\alpha_d + \beta_d = 0,7; \alpha_d = 0.12; \alpha_e = 1/3; \beta_e = 7/15;$ (i.e. $\frac{\beta_d}{\alpha_d+\beta_d} > \frac{\beta_e}{1-\alpha_e}$)

Figure VI.2 gives the path of the difference $\tilde{U} - \hat{U}$ as a function of ϵ for these three cases. We see that even if $A_d < A_e$, $\tilde{U} - \hat{U}$ will be positive (i.e., investing in the new industry is optimal) when ϵ is greater than a threshold. When $A_e = 1.2$, this threshold is just 0.63. However, when $A_e = 2$ (resp., $A_e = 3$), return of training must be greater than 56 (resp., 1600) in order to encourage investing in the new industry. Once again, we see the key role of productivity A_d.

The second example is to illustrate point (ii) of Proposition 2.11 (cf. Figure VI.3). We consider 2 cases : $\alpha_d = 0.50$ $(1-\alpha_d < \frac{\beta_e}{1-\alpha_e})$ and $\alpha_d = 0.29$ $(\frac{\beta_d}{\alpha_d+\beta_d} < \frac{\beta_e}{1-\alpha_e} < 1 - \alpha_d)$. First, we see that $\tilde{U} - \hat{U}$ is not monotonic with respect to ϵ and it tends to zero when ϵ tends to infinity. Secondly, when ϵ is high enough, $\tilde{U} - \hat{U}$ will be negative (resp., positive) in the first (resp., second) case.

enough.

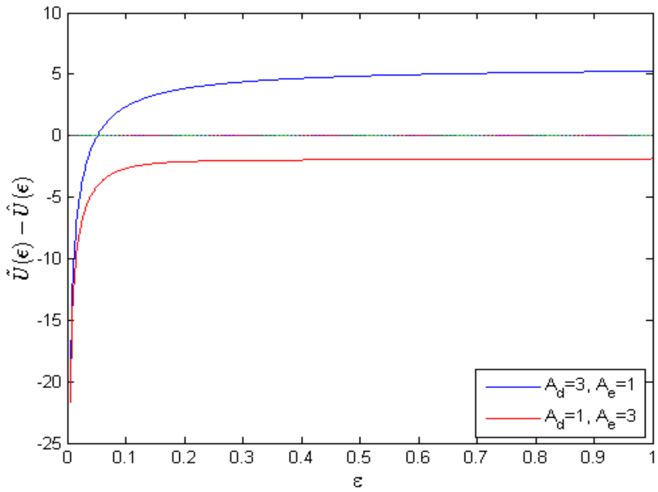
FIGURE *VI.3: The graph of* $(\tilde{U} - \hat{U})$ *as a function of* ϵ



$$S = 1; L_0 = 2; T_0 = 1; \bar{L} = 1p = 1; p_n = 2$$
$$A_d = 1.3; A_c = A_e = 1; \alpha_c = 0.7 = \alpha_d + \beta_d = 0.4; \alpha_e = 1/3; \beta_e = 7/15.$$

We note that in the case $\frac{\beta_e}{1-\alpha_e} = 1 - \alpha_d$, when ϵ is high, we must to have information of other factors, specially A_d, A_e, to know the optimal strategy of the country. Figure VI.4 gives us the answer. We also consider 2 cases : $3 = A_d > A_e = 1$ and $1 = A_d < A_e = 3$.

FIGURE *VI.4: The graph of* $(\tilde{U} - \hat{U})$ *as a function of* ϵ *with* $\frac{\beta_e}{1-\alpha_e} = 1 - \alpha_d$



$$A_c = 1; S = 1; L_0 = 2; T_0 = 1; \bar{L} = 1; p = 1; p_n = 2$$
$$\alpha_c = \alpha_d + \beta_d = 0.7; \alpha_e = 1/3; \beta_e = 7/15 \text{ (i.e. } \frac{\beta_e}{1-\alpha_e} = 1 - \alpha_d)$$

The figure indicates that, when ϵ is high enough, the host country will invest in the new industry $(\tilde{U} - \hat{U} > 0)$ if the productivity of the domestic firm is high ($3 = A_d > A_e = 1$). Conversely, such investment will not be done if this productivity is low ($1 = A_d < A_e = 3$). This result is totally consistent with Theorem 2.1.

Roles of exogeneous prices : In this section, we focus on the role of physical capital and new good prices. We analyze two cases, new good price p_n is high and physical capital price p is low. First, we study what happens when new good price p_n is very high.

Proposition 2.13 *(i) We have* $\lim_{p_n \to +\infty} \frac{w_1(p_n)}{p_n} = +\infty$.

(ii) When p_n is high enough, the host country invests in training, but not in the new industry.

Proof: See Appendix 5.3. ■

Point (i) of Proposition 2.13 implies that when the new good price increases, wage not only increases, but increases faster than the new good price (because the multinational firm is not credit constrained). Point (ii) then shows that when new good price is very high, the host country should invest in training, but not in the new industry, whatever the productivity of the domestic firm. The main reason is the following. When price p_n of new good increases, wage w_1 consequently increases. This encourages the host country to invest in training. Moreover, since wage increases faster than new good price and it must to pay an entry cost to invest in the new industry, it will be optimal to let all specific workers work for the foreign firm in order to get a favorable salary.

Let us give an example where the domestic firm's productivity is greater than that of the multinational firm (cf. Figure VI.5).

When p_n is high enough, we see that $\tilde{U} < \hat{U}$, i.e., the country should not invest in the new industry even $A_d > A_e$. Note that when p_n is low or medium, we need more informations of other factors in order to confirm $\tilde{U} < \hat{U}$.

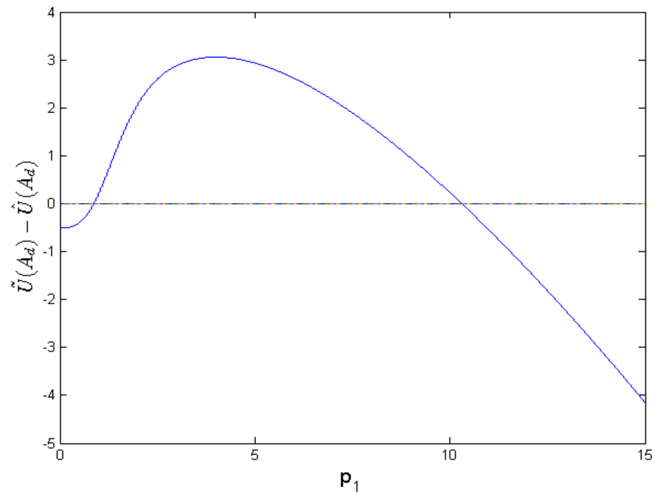
Second, we consider the case where physical capital price p is low. In this case, capital shares play an important role.

Proposition 2.14 *(i) Assume that $\frac{\alpha_e}{1-\alpha_e} > \max(\alpha_c, \alpha_d)$. The host country will invest in training, but not in the new industry when p is low enough.*

(ii) Assume that $\alpha_d > \max(\alpha_c, \frac{\alpha_e}{1-\alpha_e})$, $\epsilon S + L_0 + T_0 > \bar{L}$

(ii.a) If $\epsilon S > \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$. The host country will invest both in training and in the new industry when p is low enough.

FIGURE *VI.5: The graph of* $(\tilde{U} - \hat{U})$ *as a function of* p_n



$$\epsilon = 1.2; S = 1; L_0 = 2; T_0 = 1; \bar{L} = 1; p = 1;$$
$$A_c = 1.2; \alpha_c = 0.7; \alpha_d = 0.3; \beta_d = 0.4; \alpha_e = 1/3; \beta_e = 7/15;$$
$$2 = A_d > A_e = 1.2$$

(iii.b) If $\epsilon S < \frac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$*. The host country will invest in the new industry, but not in training when* p *is low enough.*[12]

Proof: See Appendix 5.3. ■

The reason for point (i) in Proposition 2.14 is similar to that of Proposition 2.13. Indeed, when p decreases, demand for specific labor increases and so is wage w_1. This incites the host country to invest in training. High capital share of the foreign firm makes it to be more competitive than the domestic firm. Therefore, the host country will not invest in the new industry. In this case, all specific workers work for the multinational firm.

One may ask why there are two possibilities in Proposition 2.14, but there is a unique in Proposition 2.13. The reason is from the fact that new good price p_n does not enter in the budget constraint of the social planner while physical capital price p makes influence not only in the new industry, but also in the old industry.

We now explore some implications of point (ii) of Proposition 2.14.

(a) First, as in point (i), if the domestic firm's capital share is high, the host country invests in the new sector when physical capital price is low.

12. We note that in the case where $\alpha_c > \max(\alpha_d, \frac{\alpha_e}{1-\alpha_e})$, we do not know if the country invest in the new industry. Because, this condition does not make a strong influence on the competition between firms in the new industry.

(b) Second, when physical capital price p is low, the country also invests in training if one of the following conditions holds : (1) return of training ϵ is high, (2) the ratio β_d/α_d of specific labor share over capital share of the potential domestic firm is high, (3) the difference $L_0 + T_0 - \bar{L}$ is low.

Let us finish this section by considering a specific case where we can give explicit conditions under which the host country invests in training of specific labor and in the new industry.

Example 2.1 *We assume that*

$$\frac{\alpha_c}{1-\alpha_c} = \frac{\beta_e}{1-\alpha_e-\beta_e} = \frac{\alpha_d+\beta_d}{1-\alpha_d-\beta_d}. \tag{VI.13}$$

(i) There exists an equilibrium with $H_1 > 0$ and $Y_{d,1} > 0$ if and only if the two following conditions hold

$$\epsilon S + L_0 + T_0 \geq \frac{\bar{L}}{1-\Omega} \tag{VI.14}$$

$$\epsilon S(\sigma_c+\sigma_d+\sigma_e) > (\epsilon S + L_0 + T_0 - \bar{L})(\sigma_c + \frac{\alpha_d}{\alpha_d+\beta_d}\sigma_d), \tag{VI.15}$$

where $\Omega := \frac{\alpha(\gamma_c+\gamma_d)+\gamma_e}{\alpha\gamma_c+\gamma_e}\Big(\frac{\gamma_c+\gamma_e}{\gamma_c+\gamma_e+\gamma_d}\Big)^{\frac{1}{\alpha}} < 1$, *with* $\alpha := \alpha_c = \alpha_d + \beta_d$ *and*

$$\sigma_c := \alpha_c\gamma_c, \qquad \sigma_d := (\alpha_d+\beta_d)\gamma_d, \qquad \sigma_e := \gamma_e \tag{VI.16}$$

$$\gamma_c := \alpha_c^{\frac{\alpha_c}{1-\alpha_c}} A_c^{\frac{1}{1-\alpha_c}} \big(\epsilon p\big)^{\frac{-\alpha_c}{1-\alpha_c}} \tag{VI.17}$$

$$\gamma_d := \alpha_d^{\frac{\alpha_d}{1-\alpha_d-\beta_d}} \beta_d^{\frac{\beta_d}{1-\alpha_d-\beta_d}} \big(A_d p_n\big)^{\frac{1}{1-\alpha_d-\beta_d}} \big(\epsilon p\big)^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}} \tag{VI.18}$$

$$\gamma_e := \alpha_e^{\frac{\alpha_e}{1-\alpha_e-\beta_e}} \beta_e^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \big(A_e p_n\big)^{\frac{1}{1-\alpha_e-\beta_e}} p^{\frac{-\alpha_e}{1-\alpha_e-\beta_e}}. \tag{VI.19}$$

(ii) There exists an equilibrium with $H_1 > 0$ and $Y_{d,1} = 0$ if and only if the two following conditions hold

$$\epsilon S + L_0 + T_0 < \frac{\bar{L}}{1-\Omega} \tag{VI.20}$$

$$\epsilon S\gamma_e > \alpha_c\gamma_c(L_0+T_0). \tag{VI.21}$$

Proof: See Appendix 5.3. ■

In point (i), condition (VI.42) is to ensure that the GNP in case $Y_{d,1} > 0$ is greater than the GNP in case $Y_{d,1} = 0$. $H_1 > 0$ is ensured by Condition (VI.43). In point (ii), condition (VI.20) is to ensure that the GNP in case $Y_{d,1} > 0$ is smaller than the GNP in case $Y_{d,1} = 0$, $H_1 > 0$ is ensured by Condition (VI.21).

Example 2.1 shows the complexity of solution and indicates that the optimal strategy of the host country depends on all parameters of the economy. However, with the assumption in Equation (VI.13), the example cannot illustrate all our results in previous section.

3 FDI in optimal growth context

In this section, we consider a small open economy in an infinite horizon model with FDI. The consumption good is taken as numeraire. Price (in term of consumption good) of physical capital (resp., new good) is denoted by p (resp., p_n) which is assumed to be exogenous.

At each date t, there is a multinational firm (without market power) who maximizes its profit

$$(F_t): \quad \pi_{e,t} = \max_{K_{e,t}, L^D_{e,t} \geq 0} \Big[p_n F^e(K_{e,t}, L^D_{e,t}) - pK_{e,t} - w_t L^D_{e,t} \Big], \tag{VI.22}$$

where, wage w_t (in term of consumption good) is endogenous and determined by market clearing condition as we will see below.

If the country chooses to buy $K_{c,t+1}$ units of physical capital at date t, it will produce $A_c K^{\alpha_c}_{c,t+}$ units of consumption good at date $t+1$. If the host country invests H_{t+1} units of consumption good to train specific labor at date t, it will receive $A_h H^{\alpha}_{t+1}$ units of specific labors. The host country may also create a new firm in the new industry.

For simplicity, we assume that the depreciation rate of physical capital equals 1. The social planner solves the dynamic growth problem.

$$(PO): \quad \max_{\left(c_t, K_{c,t}, K_{d,t}, L_{d,t}, L_{e,t}, H_t\right)_{t=0}^{+\infty}} \Big[\sum_{t=0}^{+\infty} \beta^t u(c_t) \Big] \tag{VI.23}$$

subject to, for every $t \geq 1$.

$$\begin{aligned}
0 &\leq K_{c,t}, K_{d,t}, L_{d,t}, L_{e,t}, H_t && \text{(VI.24)}\\
c_t + S_{t+1} &\leq A_c K^{\alpha}_{c,t} + w_t L_{e,t} + p_n F^d_t(K_{d,t}, L_{d,t}) && \text{(VI.25)}\\
S_{t+1} &= p(K_{c,t+1} + K_{d,t+1}) + H_{t+1} && \text{(VI.26)}\\
L_{e,t} &\leq A_h H^{\alpha}_t && \text{(VI.27)}\\
L_{d,t} &\leq L_t := \big(A_h H^{\alpha}_t - L_{e,t}\big) + Spillovers(A_e, L_{e,t}, S_t). && \text{(VI.28)}
\end{aligned}$$

At initial date, constraint is given by $c_0 + S_1 \leq S_0$, where S_0 is given.

We make the following assumptions.

Assumption 3.1 *The utility function $u(.)$ is strictly increasing, strictly concave, continuously differentiable, and satisfies $u(0) = 0$, $u'(0) = +\infty$.*

Assumption 3.2

$$Spillovers(A_e, L_{e,t}, S_t) = \frac{BA_eL_{e,t}}{1+S_t} \tag{VI.29}$$

$$F_t^d(K_{d,t}, L_{d,t}) = \begin{cases} A_dK_{d,t}^{\alpha}\big((L_{d,t}-\bar{L})^+\big)^{1-\alpha} & \text{if } \forall s \leq t-1 : Y_{d,s} = 0 \\ A_dK_{d,t}^{\alpha}L_{d,t}^{1-\alpha} & \text{otherwise.} \end{cases} \tag{VI.30}$$

As in the two-period case, to enter the new industry, the domestic firm must make a fixed cost $\bar{L}$ representing the number of specific workers needed to ensure that the production process functions. Once the domestic firm is created, it does not pay this cost.

Different from the two-period case, FDI spillovers are endogenous in this section. Each unit of specific labor hired by the multinational firm generates $\frac{BA_e}{1+S_t}$ units of specific labor. Specific labor generated by FDI spillovers can be hired by a domestic firm. By our assumption, FDI spillovers are decreasing in the development level of the host country S_t and increasing in the technology level of the multinational firm (see Crespo, Fontoura (2007) for detailed discussions). The coefficient B can be viewed as the absorbability of specific labors (or the level of learing by doing effects).

Definition 3.1 *An intertemporal equilibrium is a list*

$$(c_t, K_{c,t}, H_t, K_{d,t}, L_{d,t}, L_{e,t}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^{\infty}$$

such that

(i) Given $(w_t)_{t=0}^{\infty}$, $(c_t, K_{c,t}, H_t, K_{d,t}, L_{d,t}, L_{e,t})_{t=0}^{\infty}$ is a solution of the problem (p_n).

(ii) Given w_t, $(L_{e,t}^D, K_{e,t}^D)$ is a solution of the problem (F_t).

(iii) Labor market clears $L_{e,t}^D = L_{e,t}$.

The labor market clearing condition means that specific labor supplied by the host country equals specific labor demanded by the multinational firm.

Remark 3.1 *If there does not exist the new industry, we return to a closed economy. In this case, the problem (p_n) becomes the standard Ramsey optimal growth model.*

$$(p): \quad \max_{\left(c_t, K_{c,t}\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t)\right] \tag{VI.31}$$

$$\text{subject to} \quad c_t + pK_{c,t+1} \leq A_cK_c^{\alpha}. \tag{VI.32}$$

In this case, we know that $\lim_{t\to\infty} S_t = S_a$, where $S_a^{1-\alpha} = \alpha\beta(\frac{A_c}{p^{\alpha}})$.

Lemma 3.1 *Assume that $Y_{d,t} = 0$ for every t, we have, for every t,*

$$w_t = w := \Big(\alpha^\alpha (1-\alpha)^{1-\alpha} \frac{p_n A_e}{p^\alpha}\Big)^{\frac{1}{1-\alpha}}.$$

Wa also have $\lim_{t\to\infty} S_t = S_b$, where S_b is defined by $S_b^{1-\alpha} = \alpha\beta A$ and $A := \big((\frac{A_c}{p^\alpha})^{\frac{1}{1-\alpha}} + (wA_h)^{\frac{1}{1-\alpha}}\big)^{1-\alpha}$.

Proof: The problem $PO)$ can be rewritten as follows

$$\max_{\left(c_t, S_t\right)_{t=0}^{+\infty}} \Big[\sum_{t=0}^{+\infty} \beta^t u(c_t)\Big] \tag{VI.33}$$

$$\text{subject to} \quad c_t + S_{t+1} \leq f(S_t) \tag{VI.34}$$

$$c_t, S_{t+1} \geq 0, \tag{VI.35}$$

where, the function f is defined by

$$\begin{aligned} f(S) := & \max_{K_c, K, L \geq 0} A_c k_c^\alpha + wL_e \\ \text{subject to}: \quad & pK_c + H \leq S \\ & L_{e,t} \leq A_h H_t^\alpha. \end{aligned}$$

It can be established that $f(S) = AS^\alpha$. By using the well known result in the standard Ramsey model, we have $\lim_{t\to\infty} S_t = S_b$. ■

Interpretation : It is easy to see that $S_b > S_a$. Lemma 3.1 shows that with FDI, the economy's investment stock converges to a steady state which is greater than the steady state when the economy is closed. If $A_h = 0$ or $A_e = 0$, specific labors play no role and the economy is closed. In this case, our model becomes the standard Ramsey model.

3.1 Static problem

We will solve a general equilibrium model at each date. The social planner maximizes the GNP.

$$\text{(G)} \qquad G(S) = \max_{K_c, K_d, L_d, L_e, H \geq 0} A_c K_c^\alpha + wL_e + p_n A_d K_d^\alpha \big((L_d - \bar{L})^+\big)^{1-\alpha}$$

subject to

$$\begin{aligned} & p(K_c + K_d) + H \leq S \\ & L_e \leq A_h H^\alpha \\ & L_d \leq A_h H^\alpha - L_e + \frac{BA_e}{1+S} L_e. \end{aligned}$$

It is easy to see that the function G is continuous and increasing in S.

The multinational firm (without market power) maximizes profit

$$\pi_e = \max_{K_e, L_e^D \geq 0} \Big[p_n A_e K_e^\alpha (L_e^D)^{1-\alpha} - pK_e - wL_e^D\Big] \quad \text{(VI.36)}$$

Wage w is determinied by labor marker clearing condition : $L_e^D = L_e$.

Denote $Y_d := F^d(K_d, L_d)$ and $Y_e := F^e(K_e, L_e)$.

It is easy to see that

Lemma 3.2 *If* $\frac{BA_e}{1+S} > 1$, *we have* $L_e = A_h H^\alpha$.

This result shows that if FDI spillovers is high enough, all specific labor of the host country will be hired by the foreign firm in order to get salary and high spillover effects on high-skilled labor supply. The country may use labor generated by spillovers effects to create a new firm.

Lemma 3.3 *Assume that* $\max(\frac{BA_e}{1+S}, 1)A_h S^\alpha \leq \bar{L}$. *We have* $Y_d = 0$.

Proof: If $\frac{BA_e}{1+S} \leq 1$, we have $L_d \leq A_h H_t^\alpha \leq A_h S^\alpha \leq \bar{L}$. As a consequence, $Y_d = 0$.

If $\frac{BA_e}{1+S} \geq 1$, we have

$$\begin{aligned} L_d &\leq A_h H_t^\alpha - L_e + \frac{BA_e}{1+S} L_e \\ &\leq A_h H_t^\alpha + (\frac{BA_e}{1+S} - 1) A_h H_t^\alpha = A_h H_t^\alpha \leq A_h S^\alpha \leq \bar{L}. \end{aligned}$$

As a consequence, $Y_d = 0$. ∎

This result is consistent with the one in the two-period model where we have proved that if the entry cost is high, the poor country which does not have a strong FDI spillovers cannot invest in the new industry.

Lemma 3.4 *Assume that* $BA_e < 1$ *and* $A_h S^\alpha > \bar{L}$. *There exists* $\bar{A}$ *which does not depend on* S *such that for every* $A_d \geq \bar{A}$, $Y_d > 0$ *and* $Y_e = 0$.

Proof: Since $A_h S^\alpha > \bar{L}$, there exists A_1 such that $Y_d > 0$ for every $A_d > A_1$. Indeed, when $Y_d = 0$, $G(S)$ does not depend on A_d. Since $A_h S^\alpha > \bar{L}$, we can choose H closed to S such that $A_h H^\alpha > \bar{L}$. Then we chose $L_d = A_h H^\alpha > \bar{L}$. We now see that $G(S)$ tends continuously to infinity when A_d tends to infinity. This contradicts the optimality.

Let $A_d > A_1$, we have $Y_d > 0$. Assume that $Y_e > 0$. We can compute that

$$w = \Big(\alpha^\alpha (1-\alpha)^{1-\alpha} \frac{p_n A_e}{p^\alpha}\Big)^{\frac{1}{1-\alpha}}.$$

Denote $\lambda, \lambda_1, \lambda_2, \lambda_l$ Lagrange multipliers associated to constraints (VI.36), (VI.36), (VI.36), and $L_e \geq 0$ respectively. We have

$$\begin{aligned}
K_c: &\quad \alpha A_c K_c^{\alpha-1} = \lambda p && \text{(VI.37)}\\
K_d: &\quad \alpha p_n A_d K_d^{\alpha-1}(L_d - \bar{L})^{1-\alpha} = \lambda p && \text{(VI.38)}\\
L_d: &\quad (1-\alpha) p_n A_d K_d^{\alpha}(L_d - \bar{L})^{-\alpha} = \lambda_2 && \text{(VI.39)}\\
H: &\quad \lambda = (\lambda_1 + \lambda_2)\alpha A_h H^{\alpha-1} && \text{(VI.40)}\\
L_e: &\quad w = \lambda_1 + (1 - \frac{BA_e}{1+S})\lambda_2 \geq (1 - \frac{BA_e}{1+S})\lambda_2. && \text{(VI.41)}
\end{aligned}$$

FOCs of K_d, L_d imply that

$$\frac{p_n A_d}{p}\alpha^\alpha (1-\alpha)^{1-\alpha} = \lambda^\alpha \lambda_2^{1-\lambda} \leq \lambda \Big(\frac{w}{1 - \frac{BA_e}{1+S}}\Big)^{1-\alpha}.$$

As a consequence, we obtain $\lambda^\alpha \geq \frac{A_d}{A_e}\big(1 - \frac{BA_e}{1+S}\big)^{1-\alpha}$. (VI.41) implies that $w \geq (\lambda_1 + \lambda_2)(1 - \frac{BA_e}{1+S})$. Therefore, we have

$$\begin{aligned}
H^{1-\alpha} &= \alpha A_h \frac{\lambda_1 + \lambda_2}{\lambda} \leq \alpha A_h \Big(\frac{A_e}{A_d(1 - \frac{BA_e}{1+S})^{1-\alpha}}\Big)^{\frac{1}{\alpha}} \frac{w}{1 - \frac{BA_e}{1+S}}\\
&\leq \alpha A_h \Big(\frac{A_e}{A_d(1 - BA_e)^{1-\alpha}}\Big)^{\frac{1}{\alpha}} \frac{1}{1 - AB_e}\Big(\alpha^\alpha (1-\alpha)^{1-\alpha}\frac{p_n A_e}{p^\alpha}\Big)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

We define $\bar{H}$ by

$$\bar{H}^{1-\alpha} = \alpha A_h \Big(\frac{A_e}{A_d(1 - BA_e)^{1-\alpha}}\Big)^{\frac{1}{\alpha}} \frac{1}{1 - AB_e}\Big(\alpha^\alpha (1-\alpha)^{1-\alpha}\frac{p_n A_e}{p^\alpha}\Big)^{\frac{1}{1-\alpha}}.$$

Since $1 > BA_e$, we have $\bar{L} < L_d \geq A_h H^\alpha \leq A_h \bar{H}^\alpha$.

We define $\bar{A}$ by $\bar{L} = A_h \bar{H}^\alpha$. A contradiction will appear when $A_d \geq \bar{A}$. As a result, $Y_e = 0$. ■

3.2 Dynamic analysis

Let us define the sequence (x_t) as follows : $x_0 = S_0, x_{t+1} = f(x_t)$, where the function f was defined in Lemma 3.1. Define x^* by $f(x^*) = x^*$. It means that

$$x^* = (\frac{A_c}{p^\alpha})^{\frac{1}{1-\alpha}} + (wA_h)^{\frac{1}{1-\alpha}}.$$

We note that $f(x) \leq f(x^*) \leq x^*$ for every $x \leq x^*$ and $f(x) \leq x$ for every $x \geq x^*$.

We define $\bar{S} := \max\{S_0, x^*\}$. It is easy to prove that $x_t \leq \bar{S}$ for every t.

Proposition 3.1 *(***Poverty Trap***)*
Assume that $\max(BA_e, 1)A_h\bar{S}^\alpha \leq \bar{L}$. *Then we have* $Y_{d,t} = 0$ *for every* t. *And in this case* $\lim_{t\to} S_t = S_b$ *for every* t.

Proof: Assume that $S_0 \leq x^*$.

At initial date, we have $S_1 \leq S_0$ which implies that $L_{d,1} \leq \max(BA_e, 1)A_hS_0^\alpha \leq \bar{L}$. Therefore, $Y_{d,1} = 0$.

At date 2, we have $S_2 \leq f(S_1) \leq f(x^*) = x^* = \bar{S}$. Hence, we have $L_{d,1} \leq \max(BA_e, 1)A_h\bar{S}^\alpha \leq \bar{L}$. Therefore, $Y_{d,2} = 0$. By induction argument, we obtain that $Y_{d,t} = 0$ for every t. ■

This results shows that no domestic firm can be created in a new industry in a country if the country' inital endowment S_0, the technology of the old sector A_c, the productivity of training sector A_h, FDI spillovers effects BA_e are low.

Proposition 3.2 *Assume that*

$$\max(\frac{BA_e}{1+S_0}, 1)A_hS_0^\alpha < \bar{L} \tag{VI.42}$$

$$\max(\frac{BA_e}{1+S_b}, 1)A_hS_b^\alpha > \bar{L}. \tag{VI.43}$$

There exists $\bar{A}_d > 0$ *such that for each* $A_d > \bar{A}_d$, *there exists a date* $t_d > 1$ *such that* $Y_{d,t} = 0$ *for every* $t < t_d$ *and* $Y_{d,t_d} > 0$.

Proof: Assume that $Y_{d,t} = 0$ for every t, the welfare of the country W does not depend on A_d. We also have that $\lim_{t\to} S_t = S_b$. As a consequence, there exists t such that $\max(\frac{BA_e}{1+S_t}, 1)A_hS_t^\alpha > \bar{L}$.

If $\frac{BA_e}{1+S_t} < 1$. Let $L'_{e,t} = 0$ and $L'_{d,t} = A_hH'_t$. Choose H'_t is closed to S_t such that $\max(\frac{BA_e}{1+S_t}, 1)A_h(H'_t)^\alpha > \bar{L}$. Let A_d be high enough, the new welfare of the country will be greater than W. This violates the optimality of the country's choice.

If $\frac{BA_e}{1+S_t} \geq 1$. Let $L'_{e,t} = A_hH_t'^\alpha$ and $L'_{d,t} = \frac{BA_e}{1+S_t}A_hH_t^\alpha$. Choose H'_t is closed to S_t such that $\max(\frac{BA_e}{1+S_t}, 1)A_h(H'_t)^\alpha > \bar{L}$. Let A_d be high enough, the new welfare of the country will be greater than W. This violates the optimality of the country's choice.

$t_d > 1$ because of the assumption $\max(\frac{BA_e}{1+S_0}, 1)A_hS_0^\alpha < \bar{L}$. ■

Interpretation : Condition (VI.42) means that the host country's initial stock S_0 is low and with this initial stock, the country cannot invest in the new industry at the initial date.

Recall that $S_b = (\alpha\beta)^{\frac{1}{1-\alpha}}\big((A_cp^{-\alpha})^{\frac{1}{1-\alpha}} + (wA_h)^{\frac{1}{1-\alpha}}\big)$. Therefore, condition (VI.43) means that the host country may cover fixed cost $\bar{L}$ if it has high level of A_c, A_h or strong spillovers (B or A_e are high).

Proposition 3.2 gives us an interesting implication : Consider a poor or developing country (characterized by condition (VI.42)) but its productivities A_c, A_h, A_d are high enough (characterized by condition (VI.43)). Its optimal strategy should be the following :

(i) Stage 1 : It should train specific workers.

(ii) Stage 2 : These workers work for the multinational firm in the new industry to get favorable salary and working experiences or learning by doing effects in order to improve the GNP of the country.

(ii) Stage 3 : Once the GNP reaches a critical threshold, the country creates new firms in the new industry.

A natural question appears : Can the domestic firm eliminate the multinational firm ? The following result answers this question.

Proposition 3.3 *Assume that*

$$BA_e < 1, \quad A_h S_0^\alpha < \bar{L} < A_h S_b^\alpha \tag{VI.44}$$

(i) There exists $A^* > 0$ and t^* such that for each $A > A^*$, we have $Y_{d,t} = 0, Y_{e,t} > 0$ for every $t < t^*$ and $Y_{d,t} > 0, Y_{e,t} = 0$ for every $t \geq t^*$.

(ii) $\lim\limits_{t\to\infty} c_t = c$, $\lim\limits_{t\to\infty} S_t = S$, $\lim\limits_{t\to\infty} K_{c,t} = K_c$, $\lim\limits_{t\to\infty} K_{d,t} = K_d$, $\lim\limits_{t\to\infty} H_t = H$. Moreover, $S > S_b$.

Proof: Under condition (VI.44), assumptions in Proposition 3.3 are satisfied. Therefore, there exist A_1 and t_1 such that $Y_{d,t_1} > 0$. As a consequence, for every $t > t_1$, we have

$$F_t^d = A_d K_{d,t}^\alpha L_{d,t}^{1-\alpha}.$$

According to Lemma 3.4, we can choose $A^* > A_1$ such that $Y_{d,t} > 0$, $Y_{e,t} = 0$ for every $A \geq A^*$. Choose $t^* = t_1 + 1$.

We rewrite the social planner's problem from date t^*

$$(P): \qquad \max_{\left(c_t, S_{t+1}\right)_{t=0}^{+\infty}} \Big[\sum_{t=0}^{+\infty} \beta^t u(c_t)\Big] \tag{VI.45}$$

$$c_t + S_{t+1} \leq F(S_t) \tag{VI.46}$$

where, F is defined by

$$\begin{aligned} F(S) \quad &= \max_{K_c, K_d, H \geq 0} A_c K_c^\alpha + p_n A_d K_d^\alpha (A_h H^\alpha)^{1-\alpha} \\ \text{subject to} \quad & p(K_c + K_d) + H \leq S \end{aligned}$$

Since A_d is high, we have $F(S) \geq f(S)$. As a result, we obtain $S > S_b$.

It is easy to see that $F(S)$ is dominated by some function $\bar{F}S^{\bar{\alpha}}$, where $\bar{\alpha} < 1$, therefore we obtaint that S is finite. ■

Interpretation : Proposition 3.3 shows that if the domestic firm in a poor country has high productivity, this poor country should follow the strategy mentioned above. And the domestic firm will not only be created but also eliminate the multinational firm.

4 Conclusion

We have constructed a two-period small open economy model with multi-sector, heterogeneous firms, and then used it to study the optimal strategy of a country and analyze roles of all factors of the economy. Our finding indicates that the country's optimal strategy depends on its development level.

First, poor countries with low FDI spillovers cannot invest in a new industry that requires a high entry cost. In this case, all specific workers in this sector will work for multinational firms.

Second, the FDI spillovers can improve the GNP and help poor or developing countries to create a new firm, but it does decrease the optimal share of high-qualified labor. We proved that if FDI spillovers are high, these country may create a new firm without training of qualified workers. But if FDI spillovers are not high, these countries must train qualified workers in order to invest in this new industry.

Third, our model shows that once the host country has a sufficient high-skilled labor to cover the fixed cost in the new industry, the efficiency of domestic firm is necessary and sufficient to ensure its entry. This explains why developed countries do not invest in some new industries.

The competition between the multinational and the domestic firms depends on many factors. The most important factors are their productivities A_d, A_e. However, credit constraint also plays an important role. Because of credit constraint, the domestic firm may be eliminated even if it has a higher productivity.

We also give dynamic analysis by embedding the two-period model in an infinite horizon model. Our result suggests that a poor or developing countries holding a hig should follow the following strategy : It first trains specific workers. These workers work for the multinational firm in the new industry, and then improve the GNP of the country. Once the GNP reaches a critical threshold, domestic firms will enter in the new industry and potentially eliminate the multinational firm.

5 Appendix : Decreasing return to scale

We assume that $\alpha_d + \beta_d, \alpha_e + \beta_e < 1$.

At equilibrium, since the production functions of the foreign firm and the consumption good producer are decreasing return to scale, we always have $K_{c,1}, K_{e,1}, L_{e,1} > 0$.

We now write first order conditions (FOCs) for foreign firm.

$$\begin{aligned} L_{e,1}: &\quad p_n \beta_e A_e K_{e,1}^{\alpha_d} L_{e,1}^{\beta_e - 1} = w_1 \\ K_{e,1}: &\quad p_n \alpha_e A_e K_{e,1}^{\alpha_e - 1} L_{e,1}^{\beta_e} = p. \end{aligned}$$

Therefore, we get that

$$\begin{aligned} K_{e,1} &= \frac{\alpha_e}{\beta_e}\frac{w_1}{p} L_{e,1}, \quad L_{e,1} = \sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}, \\ \text{where } \sigma_e &:= \alpha_e^{\frac{\alpha_e}{1-\alpha_e-\beta_e}} \beta_e^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \big(A_e p_n\big)^{\frac{1}{1-\alpha_e-\beta_e}} p^{\frac{-\alpha_e}{1-\alpha_e-\beta_e}} \end{aligned}$$

Denote λ, μ Lagrange multipliers associated to conditions (VI.5), (VI.6), respectively, and λ_h is Lagrange multiplier with respect to condition $H_1 \geq 0$. We write FOCs for the social planner for variables $K_{c,1}, H_1, L_{e,1}$

$$\begin{aligned} K_{c,1}: &\quad \alpha_c A_c K_{c,1}^{\alpha_c - 1} = \lambda p \\ L_{e,1}: &\quad w_1 - \mu = 0 \\ H_1: &\quad -\lambda + \mu\epsilon + \lambda_h = 0, \text{ where } \lambda_h \geq 0, H_1\lambda_h = 0. \end{aligned}$$

Note that to solve social planner's optimization problem, we must consider two cases : $Y_{d,1} = 0$ and $Y_{d,1} > 0$. Then, we compare welfares in these cases in order to know what is the optimal strategy.

5.1 Equilibrium with $H_1 = 0$

(i) : If $L_0 + T_0 \leq \bar{L}$. We have $L_{e,1} + L_{d,1} \leq \epsilon H_1 + L_0 + T_0 = L_0 + T_0 \leq \bar{L}$, thus $L_{d,1} \leq \bar{L}$ then $Y_{d,1} = 0$.

(ii) : If $L_0 + T_0 > \bar{L}$. We have to consider two cases : $Y_{d,1} = 0$ and $Y_{d,1} > 0$.

Case 1 : $Y_{d,1} = 0$. In this case $K_{d,1} = L_{d,1} = 0$. Therefore, we get that $K_{c,1} = \dfrac{S}{p}$ and $L_{e,1} = L_0 + T_0$. By using FOCs of firm's maximization, we have $L_{e,1} = \sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}$. Hence, wage is computed by

$$w_1 = \Big(\frac{\sigma_e}{L_0 + T_0}\Big)^{\frac{1-\alpha_e-\beta_e}{1-\alpha_e}}. \tag{VI.47}$$

In this case, we have

$$\text{Welfare} = A_c\Big(\frac{S}{p}\Big)^{\alpha_c} + \beta_e \alpha_e^{\frac{\alpha_e}{1-\alpha_e}} A_e^{\frac{1}{1-\alpha_e}} \Big(\frac{p_n}{p^{\alpha_e}}\Big)^{\frac{1}{1-\alpha_e}} (L_0 + T_0)^{\frac{\beta_e}{1-\alpha_e}}.$$

Note that we have to justify the following condition

$$\text{FOC of } H_1 \ : \epsilon p w_1 \leq \lambda p = \alpha_c A_c K_{c,1}^{\alpha_c - 1}.$$

This condition is equivalent to the following condition under which ϵ is low enough.

$$\epsilon S^{1-\alpha_c}\beta_e \alpha_e^{\frac{\alpha_e}{1-\alpha_e}} A_e^{\frac{1}{1-\alpha_e}} p^{\alpha_c}\big(\frac{p_n}{p^{\alpha_e}}\big)^{\frac{1}{1-\alpha_e}} \leq \alpha_c A_c\big(L_0+T_0\big)^{\frac{1-\alpha_e-\beta-e}{1-\alpha_e}}. \quad \text{(VI.48)}$$

Case 2 : $Y_{d,1}>0$. In this case $L_{d,1}>\bar{L}$. We write FOCs for the social planner

$$\begin{aligned} K_{d,1}: &\quad p_n\alpha_d A_d K_{d,1}^{\alpha_d-1}(L_{d,1}-\bar{L})^{\beta_d} - \lambda p = 0\\ L_{d,1}: &\quad p_n\beta_d A_d K_{d,1}^{\alpha_d}(L_{d,1}-\bar{L})^{\beta_d-1} - \mu = 0 \end{aligned}$$

Since we are considering the case $H_1=0$, labor market clearing condition implies that $L_{d,1} = L_0+T_0-L_{e,1} = L_0+T_0-\sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}$.
By using FOC for variable $L_{d,1}$, we get that $K_{d,1} = \big(\frac{w_1}{\beta_d p_n A_d}\big)^{\frac{1}{\alpha_d}}\big(L_{d,1}-\bar{L}\big)^{\frac{1-\beta_d}{\alpha_d}}$.
On the other hand, we have $\alpha_c A_c K_{c,1}^{\alpha_c-1} = \lambda p = \alpha_d p_n A_d K_{d_1}^{\alpha_d-1}(L_{d,1}-\bar{L})^{\beta_d}$. Therefore

$$\begin{aligned} K_{c,1} &= \Big(\frac{\alpha_c A_c}{p_n\alpha_d A_d}\Big)^{\frac{1}{1-\alpha_c}} K_{d,1}^{\frac{1-\alpha_d}{1-\alpha_c}}\big(L_{d,1}-\bar{L}\big)^{\frac{-\beta_d}{1-\alpha_c}}\\ &= \Big(\frac{\alpha_c A_c}{\alpha_d}\Big)^{\frac{1}{1-\alpha_c}}\Big(\frac{1}{p_n A_d}\Big)^{\frac{1}{\alpha_d(1-\alpha_c)}}\Big(\frac{w_1}{\beta_d}\Big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}}\big(L_{d,1}-\bar{L}\big)^{\frac{1-\alpha_d-\beta_d}{\alpha_d(1-\alpha_c)}}. \end{aligned}$$

According $K_{c,1}+K_{d,1} = \frac{S}{p}$, we get that w_1 is a solution of the equation $G_2(x)=0$, where

$$\begin{aligned} G_2(x) := -\frac{S}{p} + \big(\frac{x}{\beta_d p_n A_d}\big)^{\frac{1}{\alpha_d}}\Big(L_0+T_0-\bar{L}-\sigma_e x^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}\Big)^{\frac{1-\beta_d}{\alpha_d}} \qquad \text{(VI.49)}\\ +\Big(\frac{\alpha_c A_c}{\alpha_d}\Big)^{\frac{1}{1-\alpha_c}}\Big(\frac{1}{p_n A_d}\Big)^{\frac{1}{\alpha_d(1-\alpha_c)}}\Big(\frac{x}{\beta_d}\Big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}}\Big(L_0+T_0-\bar{L}-\sigma_e x^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}\Big)^{\frac{1-\alpha_d-\beta_d}{\alpha_d(1-\alpha_c)}}. \end{aligned}$$

It is easy to see that the function G_2 is increasing. Moreover, $\inf_x G_2(x) = -\frac{S}{p}$ and $\sup_x G_2(x) = +\infty$. Therefore, the equation $G_2(x)=0$ has the unique solution, called w_1.
By observing the equation $G_2(w_1)=0$, we see that when A_d tends to infinity then $w_1(A_d)$ tends to infinity.
FOCs give us $\beta_d p_n Y_{d,1} = w_1(A_d)(L_{d,1}(A_d)-\bar{L}) = w_1(A_d)(L_0+T_0-\bar{L}-$

$\sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}$). Consequently, $\lim_{A_d\to+\infty} p_n Y_{d,1} = +\infty$, then the welfare in this case is greater than the welfare in the first case, which does not depend on A_d. Moreover, $\lim_{A_d\to+\infty} Y_{d,1} = +\infty$ implies that $Y_{d,1} > Y_{e,1}$ with A_d is high enough. It means that we have just proved the result mentioned in Proposition 2.9.

Remark 5.1 *Assume that $L_0 + T_0 > \bar{L}$ and ϵ is low enough such that $\epsilon\beta_d S < \alpha_d(L_0 + T_0 - \bar{L})$. When A_d is high enough then the list*

$$(K_{c,1}, K_{d,1}, L_{d,1}, L_{e,1}, K_{e,1}, w_1)$$

given in case 2 above is the unique equilibrium.

Proof: Indeed, labor market clearing condition is satisfied. All FOCs of the foreign firm hold. It remains to justify the FOC with respect to variable $H_1 = 0$, which is $\lambda \geq \epsilon w_1$, i.e., $\epsilon p w_1 K_{c,1}^{1-\alpha_c} \leq \alpha_c A_c$. We have

$$\frac{w_1 K_{c,1}^{1-\alpha_c}}{\alpha_c A_c} = \frac{w_1}{\alpha_d}\Big(\frac{1}{p_n A_d}\Big)^{\frac{1}{\alpha_d}}\Big(\frac{w_1}{\beta_d}\Big)^{\frac{1-\alpha_d}{\alpha_d}}\Big(L_0 + T_0 - \bar{L} - \sigma_e w_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}\Big)^{\frac{1-\alpha_d-\beta_d}{\alpha_d}}.$$

Since $G_2(w_1(A_d)) = 0$ and $\lim_{A_d\to+\infty} w_1(A_d) = +\infty$, we have

$$\frac{S}{p} \geq \big(\frac{1}{\beta_d p_n}\big)^{\frac{1}{\alpha_d}}(L_0 + T_0 - \bar{L})^{\frac{1-\beta_d}{\alpha_d}}\big(\limsup_{A_d\to+\infty}\frac{w_1(A_d)}{A_d}\big)^{\frac{1}{\alpha_d}}$$

Hence, $\limsup_{A_d\to+\infty}\frac{w_1(A_d)}{A_d} < +\infty$ then $\lim_{A_d\to+\infty}\frac{(w_1(A_d))^{1-\alpha_d}}{A_d} = 0$. Again, by using $\lim_{A_d\to+\infty} G_2(w_1(A_d)) = 0$ which implies that

$$\frac{S}{p} = \big(\frac{1}{\beta_d p_n}\big)^{\frac{1}{\alpha_d}}(L_0 + T_0 - \bar{L})^{\frac{1-\beta_d}{\alpha_d}}\lim_{A_d\to+\infty}\big(\frac{w_1(A_d)}{A_d}\big)^{\frac{1}{\alpha_d}}. \qquad \text{(VI.50)}$$

We now assume that $\epsilon\beta_d S < \alpha_d(L_0 + T_0 - \bar{L})$. We have

$$\begin{aligned}\lim_{A_d\to+\infty}\epsilon p\frac{w_1 K_{c,1}^{1-\alpha_c}}{\alpha_c A_c} &= \epsilon p \lim_{A_d\to+\infty}\big(\frac{w_1(A_d)}{A_d}\big)^{\frac{1}{\alpha_d}}\Big(\frac{1}{p_n}\Big)^{\frac{1}{\alpha_d}}\frac{1}{\alpha_d\beta_d^{\frac{1-\alpha_d}{\alpha_d}}}\Big(L_0 + T_0 - \bar{L}\Big)^{\frac{1-\alpha_d-\beta_d}{\alpha_d}} \\ &= \frac{\epsilon\beta_d S}{\alpha_d(L_0 + T_0 - \bar{L})} < 1.\end{aligned}$$

Consequently, if A_d is high enough then $\epsilon p w_1 K_{c,1}^{1-\alpha_c} \leq \alpha_c A_c$, i.e., FOC of H_1 is satisfied.
When A_d tends to infinity then the welfare tends to infinity. ■

5.2 Equilibrium with $H_1 > 0$

Let denote $\mathcal{L}_0 := L_0 + T_0 + \epsilon S$ and $\tilde{U}$ (resp. $\hat{U}$) the GNP in case $Y_{d,1} > 0$ (resp. $Y_{d,1} = 0$).
Recall that we are considering equilibrim with $H_1 > 0$, so $\lambda_h = 0$.

We will consider 2 cases : $Y_{d,1} > 0$ and $Y_{d,1} = 0$. Let denote $\tilde{U}$ and $\hat{U}$ the welfare value of problem (P) with $Y_{d,1} > 0$ and with $Y_{d,1} = 0$, respectively.

Case 1. Assume that $(\tilde{K}_{c,1}, \tilde{K}_{d,1}, \tilde{H}_1, \tilde{L}_{d,1}, \tilde{K}_{e,1}\tilde{L}_{e,1}, \tilde{w}_1)$ with $\tilde{L}_{d,1} > \bar{L}$, $\tilde{K}_{d,1} > 0$ is an equilibrium. We have

$$\begin{aligned}
\tilde{H}_1 : &\quad -\lambda + \mu\epsilon = 0 \\
\tilde{L}_{d,1} : &\quad p_n \beta_d A_d \tilde{K}_{d,1}^{\alpha_d}(\tilde{L}_{d,1} - \bar{L})^{\beta_d - 1} - \mu = 0 \\
\tilde{K}_{d,1} : &\quad p_n \alpha_d A_d \tilde{K}_{d,1}^{\alpha_d - 1}(\tilde{L}_{d,1} - \bar{L})^{\beta_d} - \lambda p = 0.
\end{aligned}$$

We get that

$$\tilde{K}_{c,1} = \Big[\frac{\alpha_c A_c}{\epsilon p \tilde{w}_1}\Big]^{\frac{1}{1-\alpha_c}} \tag{VI.51}$$

$$\tilde{L}_{d,1} - \bar{L} = \frac{\beta_d}{\alpha_d}\epsilon p \tilde{K}_{d,1}, \quad \tilde{K}_{d,1} = \Big[\frac{p_n \alpha_d A_d}{p\epsilon \tilde{w}_1}(\frac{\beta_d \epsilon p}{\alpha_d})^{\beta_d}\Big]^{\frac{1}{1-\alpha_d-\beta_d}}. \tag{VI.52}$$

By combining the budget constraint of the social planner and labor market clearing condition, we imply that

$$\Big((\tilde{L}_{d,1} - \bar{L}) + \epsilon p \tilde{K}_{d,1}\Big) + \epsilon p \tilde{K}_{c,1} + \tilde{L}_{e,1} = \epsilon S + L_0 + T_0 - \bar{L}.$$

It means that $\tilde{w}_1$ is a solution of the equation $G(x) = 0$, where we define

$$\begin{aligned}
G(x) &:= \sigma_c x^{\frac{-1}{1-\alpha_c}} + \sigma_e x^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \sigma_d x^{\frac{-1}{1-\alpha_d-\beta_d}} \\
&\quad -(\epsilon S + L_0 + T_0 - \bar{L}) && \text{(VI.53)} \\
\sigma_c &:= (\alpha_c A_c)^{\frac{1}{1-\alpha_c}}\big(\epsilon p\big)^{\frac{-\alpha_c}{1-\alpha_c}} && \text{(VI.54)} \\
\sigma_d &:= (\alpha_d + \beta_d)\alpha_d^{\frac{\alpha_d}{1-\alpha_d-\beta_d}}\beta_d^{\frac{\beta_d}{1-\alpha_d-\beta_d}}\big(A_d p_n\big)^{\frac{1}{1-\alpha_d-\beta_d}}\big(\epsilon p\big)^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}} && \text{(VI.55)}
\end{aligned}$$

We see that $\lim_{\tilde{w}_1 \to 0^+} G(\tilde{w}_1) = +\infty$, $\lim_{w_1 \to +\infty} G(\tilde{w}_1) = \bar{L} - \mathcal{L}_0$ and $G'(\tilde{w}_1) < 0$ for every $\tilde{w}_1 \in (0, +\infty)$. Therefore if $\bar{L} - \mathcal{L}_0 < 0$ then the equation (VI.53) has the unique solution in $(0, +\infty)$.

Condition $H_1 > 0$ is equivalent to

$$\frac{S}{p} > \Big[\frac{\alpha_c A_c}{\epsilon p \tilde{w}_1}\Big]^{\frac{1}{1-\alpha_c}} + \Big[\frac{p_n \alpha_d A_d}{p\epsilon \tilde{w}_1}(\frac{\beta_d \epsilon p}{\alpha_d})^{\beta_d}\Big]^{\frac{1}{1-\alpha_d-\beta_d}}.$$

This condition can be rewritten as follows

$$\epsilon S > \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d+\beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}. \tag{VI.56}$$

We now compute the welfare in this case. The welfare is given by

$$\begin{aligned}
\tilde{U} &= F^c(\tilde{K}_{c,1}) + \tilde{w}_1\tilde{L}_{e,1} + p_n F^d(\tilde{K}_{d,1}, \tilde{L}_{d,1}) \\
&= A_c\Big[\frac{\alpha_c A_c}{\epsilon p \tilde{w}_1}\Big]^{\frac{\alpha_c}{1-\alpha_c}} + \tilde{w}_1\sigma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} \\
&\quad + \frac{\epsilon p}{\alpha_d}\Big[\big(\frac{\beta_d \epsilon p}{\alpha_d}\big)^{\beta_d}\big(\frac{\alpha_d A_d p_n}{\epsilon p}\big)^{\frac{1}{1-\alpha_d-\beta_d}}\Big]\Big[\frac{1}{\tilde{w}_1}\Big]^{\frac{\alpha_d+\beta_d}{1-\alpha_d-\beta_d}} \\
&= \gamma_c \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} + \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}},
\end{aligned}$$

where, we define

$$\gamma_c := \frac{\sigma_c}{\alpha_c}, \quad \gamma_d := \frac{\sigma_d}{\alpha_d+\beta_d}, \quad \gamma_e := \sigma_e.$$

Case 2. Assume that $(\hat{K}_{c,1}, \hat{K}_{d,1}, \hat{H}_1, \hat{L}_{d,1}, \hat{K}_{e,1}\hat{L}_{e,1}, \hat{w}_1)$ with $\hat{L}_{d,1} = \hat{K}_{d,1} = 0$ is an equilibrium. In this case, we note that $\hat{L}_{e,1} + \epsilon p \hat{K}_{c,1} = \mathcal{L}_0$. As in the case 1, we have

$$\begin{aligned}
\hat{K}_{c,1} &= \Big[\frac{\alpha_c A_c}{\epsilon p \hat{w}_1}\Big]^{\frac{1}{1-\alpha_c}} \\
\hat{K}_{e,1} &= \frac{\alpha_e}{\beta_e}\frac{\hat{w}_1}{p_n}\hat{L}_{e,1}, \quad \hat{L}_{e,1} = \sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}.
\end{aligned}$$

We get that $\hat{w}_1$ is a solution of the following equation

$$Q(\hat{w}_1) := \sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} - \mathcal{L}_0 = 0 \tag{VI.57}$$

We see that $\lim_{\hat{w}_1 \to 0^+} Q(\hat{w}_1) = +\infty$, $\lim_{\hat{w}_1 \to +\infty} Q(\hat{w}_1) = -\mathcal{L}_0$ and $G'(\hat{w}_1) < 0$ for every $\hat{w}_1 \in (0, +\infty)$. Therefore the equation (VI.57) has the unique solution in $(0, +\infty)$. This solution is denoted by $\hat{w}_1$.

We now compute the welfare. The welfare is given by

$$\begin{aligned}
\hat{U} &= F^c(\hat{K}_{c,1}) + \hat{w}_1\hat{L}_{e,1} = A_c\Big[\frac{\alpha_c A_c}{\epsilon p \hat{w}_1}\Big]^{\frac{\alpha_c}{1-\alpha_c}} + \hat{w}_1\sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} \\
&= \gamma_c(\hat{w}_1)^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e(\hat{w}_1)^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}.
\end{aligned}$$

Condition $H_1 > 0$ is equivalent to $\dfrac{S}{p} > \Big[\dfrac{\alpha_c A_c}{\epsilon p \hat{w}_1}\Big]^{\frac{1}{1-\alpha_c}}$, i.e., $\epsilon S > \sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}}$..

Lemma 5.1 *If* $(1-\alpha_d-\beta_d)\gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} \geq \tilde{w}_1 \bar{L}$ *then* $\tilde{U} > \hat{U}$.

This result means that if the potential domestic firm's profit $(1-\alpha_d-\beta_d)p_nY_{d,1}$ can cover the value of the entry costs $\tilde{w}_1\bar{L}$, the country should invest in the new industry. But the inverse is not true. Because the domestic firm's entry increases the GNP by making a positive profit and increasing of wage.

Proof: We observe that

$$\hat{U} = \big(\epsilon S + L_0 + T_0 + (1-\alpha_c)\gamma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}}\big)\hat{w}_1 \tag{VI.58}$$

$$\begin{aligned}\tilde{U} &= \big(\epsilon S + L_0 + T_0 - \bar{L} + (1-\alpha_c)\gamma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} \\ &\quad + (1-\alpha_d-\beta_d)\gamma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta-d}}\big)\tilde{w}_1.\end{aligned} \tag{VI.59}$$

Therefore, we get

$$\begin{aligned}\tilde{U}-\hat{U} &= (\epsilon S + L_0 + T_0)(\tilde{w}_1 - \hat{w}_1) + (1-\alpha_c)\gamma_c(\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) \\ &\quad + \big(\frac{1-\alpha_d-\beta_d}{\alpha_d+\beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} - \bar{L}\big)\tilde{w}_1.\end{aligned} \tag{VI.60}$$

Consider the function $f(x) := (\epsilon S + L_0 + T_0)x + \frac{1-\alpha_c}{\alpha_c}\sigma_c \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}$. We have $f'(x) > 0$ for every x such that $\epsilon S + L_0 + T_0 > \sigma_c x^{\frac{-1}{1-\alpha_c}}$. Therefore, $\tilde{w}_1 > \hat{w}_1$ implies that $f(\tilde{w}_1) > f(\hat{w}_1)$, i.e.,

$$\tilde{U}-\hat{U} > \big(\frac{1-\alpha_d-\beta_d}{\alpha_d+\beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} - \bar{L}\big)\tilde{w}_1.$$

■

5.3 Formal proofs

Proof of Proposition 2.2:

The first statement is clear. We will prove the second one. It is trivial if $H_1 = 0$. Hence, we assume that $H_1 > 0$. We consider 2 cases : $Y_{d,1} = 0$ and $Y_{d,1} > 0$.

Case 1 : $Y_{d,1} = 0$. It is easy to see that when T_0 increases, $\hat{w}_1$ decreases. Therefore $\frac{pK_{c,1}}{S} = \frac{\sigma_c}{\epsilon S}\hat{w}_1^{\frac{-1}{1-\alpha_c}}$ will increases which implies θ_h decreases.
The same argument can be used to prove our result in the case $Y_{d,1} > 0$. ■

Proof of Proposition 2.3: **Case 1** : $H_1 > 0$. Let $\bar{L} \to 0$, we have $\lim_{\bar{L}\to 0} \hat{w}_1(\bar{L}) = \hat{w}_1$, $\lim_{\bar{L}\to 0} \tilde{w}_1(\bar{L}) = \tilde{w}_1$, where $\hat{w}_1, \tilde{w}_1$ such that

$$\sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 \tag{VI.61}$$

$$\sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \epsilon S + L_0 + T_0. \tag{VI.62}$$

We also have $\lim_{\bar{L}\to 0} \tilde{U}(\bar{L}) = \tilde{U}$, $\lim_{\bar{L}\to 0} \hat{U}(\bar{L}) = \hat{U}$, and

$$\begin{aligned}\tilde{U} - \hat{U} &= (\epsilon S + L_0 + T_0)\tilde{w}_1 - (\epsilon S + L_0 + T_0)\hat{w}_1 \\ &+(1-\alpha_c)\gamma_c(\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) + (1-\alpha_d-\beta_d)\gamma_d\tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta-d}}.\end{aligned} \tag{VI.63}$$

Consider the function $f(x) := (\epsilon S + L_0 + T_0)x + \frac{1-\alpha_c}{\alpha_c}\sigma_c\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}$. We have $f'(x) > 0$ for every x such that $\epsilon S + L_0 + T_0 > \sigma_c x^{\frac{-1}{1-\alpha_c}}$. Therefore, $\tilde{w}_1 > \hat{w}_1$ implies that $f(\tilde{w}_1) > f(\hat{w}_1)$, i.e., $\tilde{U} - \hat{U} > 0$. Since $\tilde{U}(\bar{L}) - \hat{U}(\bar{L})$ is continuous, there exists $\bar{L}^* > 0$ such that $\tilde{U}(\bar{L}) > \hat{U}(\bar{L})$ for every $\bar{L} \geq \bar{L}^*$.

Case 2 : When $H_1 = 0$. It is clear. ∎

Proof of Proposition 2.4.: It is easy to see that $\lim_{S\to+\infty} \hat{w} = \lim_{S\to+\infty} \tilde{w} = 0$. Therefore, for S is high enough, we have $(1-\alpha_d-\beta_d)\gamma_d\tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} > \bar{L}\tilde{w}_1$. Consequently, we obtain

$$\tilde{U}(S) - \hat{U}(S) > (\epsilon S + L_0 + T_0)(\tilde{w}_1 - \hat{w}_1) + \frac{1-\alpha_c}{\alpha_c}\sigma_c(\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}).$$

By using the same argument in the proof of Theorem, we have $\tilde{U}(S) - \hat{U}(S) > 0$ for S high enough. ∎

Proof of Proposition 2.5:

Let A_e tend to infinity, we have γ_e and so $\hat{w}_1, \tilde{w}_1$ will tend to infinity. Consequently, we have

$$\lim_{A_e\to+\infty} \gamma_e\hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 \tag{VI.64}$$

$$\lim_{A_e\to+\infty} \gamma_e\tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 - \bar{L} \tag{VI.65}$$

$$\lim_{A_e\to+\infty} \left(\frac{\tilde{w}}{\hat{w}_1}\right)^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = \frac{\epsilon S + L_0 + T_0}{\epsilon S + L_0 + T_0 - \bar{L}} > 1. \tag{VI.66}$$

Therefore, we get that

$$\lim_{A_e\to+\infty} \frac{\tilde{U}}{\hat{U}} = \left(\frac{\tilde{w}}{\hat{w}_1}\right)^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} < 1. \tag{VI.67}$$

Thus, there exists $A_e > 0$ such that $\hat{U} > \tilde{U}$, i.e., $Y_{d,1} = 0$. It remains to verify that $H_1 > 0$, i.e., $\epsilon\hat{w}_1 > \frac{\alpha_c A_c}{p^{\alpha_c}S^{1-\alpha_c}}$. This condition holds with A_c is high enough. ∎

Proof of Proposition 2.6: Assume that $H_1 > 0$. Like proof of Proposition 2.5, we have $Y_{d,1} = 0$. But in this case, condition $H_1 > 0$ is not satisfied. Hence, we have $H_1 = 0$ at equilibrium.

By using the similar argument in Remark 5.1, we get that $Y_{d,1} = 0$ and FOC with respect to H_1 is satisfied. ■

Proof of Proposition 2.8:

Assume that $\mathcal{L}_0 > \bar{L} \geq L_0 + T_0$. Wage $\tilde{w}_1$ is the unique solution of the following equation

$$\sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \epsilon S + L_0 + T_0 - \bar{L}). \qquad \text{(VI.68)}$$

We see that $\tilde{w}_1$ depends on A_d, we can write $\tilde{w}_1 = \tilde{w}_1(A_d)$. It is easy to see that $\tilde{w}_1(\cdot)$ is increasing in $(0, +\infty)$. Since $\lim\limits_{A_d \to \infty} \sigma_d = +\infty$ and $\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} < \mathcal{L}_0 - \bar{L}$, we obtain $\lim\limits_{A_d \to +\infty} \tilde{w}_1(A_d) = +\infty$. By combining with (VI.68), we have

$$\lim_{A_d \to +\infty} \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \mathcal{L}_0 - \bar{L} > 0.$$

Consequently, we obtain

$$\begin{aligned}
\lim_{A_d \to +\infty} \sigma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} &= \lim_{A_d \to +\infty} \sigma_d^{1-\alpha_d-\beta_d} \left[\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}\right]^{\alpha_d+\beta_d} \\
&= (\mathcal{L}_0 - \bar{L})^{\alpha_d+\beta_d} \lim_{A_d \to +\infty} \sigma_d^{1-\alpha_d-\beta_d} = +\infty.
\end{aligned}$$

Therefore $\lim\limits_{A_d \to +\infty} \tilde{U}(A_d) = +\infty$. By combining with $\hat{w}_1$ does not depend on A_d, we have $\lim\limits_{A_d \to +\infty} \tilde{U}(A_d) > \hat{U}$, then critical level $\bar{A}_1$ in Proposition 2.8 exists.

Since $\bar{L} \geq L_0 + T_0$, we have

$$\begin{aligned}
\epsilon S &\geq \mathcal{L}_0 - \bar{L} = \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} && \text{(VI.69)} \\
&> \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d + \beta_d} \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}. && \text{(VI.70)}
\end{aligned}$$

Therefore, we have $H_1 > 0$. We can now define $\bar{A}_1$ and $\tilde{A}_1$.

$$\begin{aligned}
\bar{A}_1 &:= \inf\{A_d : \tilde{U}(A_d) \geq \hat{U}\} && \text{(VI.71)} \\
\tilde{A}_1 &:= \inf\{A_d : Y_{d,1} \geq Y_{e,1}\}. && \text{(VI.72)}
\end{aligned}$$

We now prove that $\bar{A}_1$ increases if $\bar{L}$ increases.

For each $\bar{L}, A_d$, we write $\tilde{w}_1(\bar{L}, A_d)$ meaning that wage depends on $\bar{L}, A_d$. Then $\bar{A}_1$ is the unique level of productivity such that $\tilde{U}(\bar{A}_1) = \hat{U}$ which can be rewritten as

$$\gamma_c\big(\tilde{w}_1(\bar{L}, \bar{A}_1)\big)^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e\big(\tilde{w}_1(\bar{L}, \bar{A}_1)\big)^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} + \gamma_d\big(\tilde{w}_1(\bar{L}, \bar{A}_1)\big)^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} = \hat{U}.$$

Note that $\hat{U}$ does depend neither on $\bar{L}$ nor on A_d. Since $\tilde{w}_1$ is increasing in the first variable, decreasing in the second variable, and γ_d is increasing in A_d then we have $\bar{A}_1$ is increasing in $\bar{L}$.

Similarly, $\tilde{A}_1$ is increasing in $\bar{L}$. ■

Proof of Proposition 2.9:

Case 1 : $\epsilon S < \dfrac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$. This case is a direct consequence of Remark 5.1.

$\bar{A}_2$ can be defined as follows

$$\bar{A}_3 := \inf\{A_d : \tilde{U}(A_d) \geq \hat{U} \text{ and } \epsilon p K_{c,1}^{1-\alpha_c} \leq \alpha_c A_c\}, \tag{VI.73}$$

where $K_{c,1}$ is defined as in the case 2 of Appendix B.

Case 2 : $\epsilon S > \dfrac{\alpha_d}{\beta_d}(L_0 + T_0 - \bar{L})$. We get

$$\epsilon S > (\epsilon S + L_0 + T_0 - \bar{L})\frac{\alpha_d}{\alpha_d + \beta_d}.$$

Condition $H_1 > 0$ is equivalent to $\epsilon S > \epsilon p(K_{c,1} + K_{d,1})$, i.e.,

$$\epsilon S > \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d + \beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}.$$

As in Proposition 2.8, we have $\lim_{A_d \to +\infty} \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \epsilon S + L_0 + T_0 - \bar{L} > 0$. Thus

$$\lim_{A_d \to +\infty} \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d + \beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \frac{\alpha_d}{\alpha_d + \beta_d}(\epsilon S + L_0 + T_0 - \bar{L}) < \epsilon S.$$

This implies that $H_1 > 0$ if A_d is high enough. Other statements in this case are proved in Proposition 2.8.

$\bar{A}_3$ can be defined as follows

$$\begin{aligned}\bar{A}_3 := \inf\{A_d : \quad & \tilde{U}(A_d) \geq \hat{U} \text{ and} \\ & \epsilon S > \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d + \beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}\}.\end{aligned} \tag{VI.74}$$

■

Proof of Proposition 2.10 :

We have $H_1 = S - p(K_{c,1} + K_{d,1})$. Hence $H_1 > 0$ if and only if

$$\frac{S}{p} \quad > \quad K_{c,1} + K_{d,1}. \tag{VI.75}$$

If $Y_{d,1} = 0$. In this case, (VI.75) is equivalent to $\epsilon S > \sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}}$. Recall that $\hat{w}_1$ is the unique solution of the following equation

$$Q(\hat{w}_1) \ := \ \sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}} + \sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} - (\epsilon S + L_0 + T_0) = 0.$$

Consequently, $H_1 > 0$ if and only if $\sigma_e \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} > L_0 + T_0$. Since $\tilde{w}_1$ is decreasing in ϵ, this condition is equivalent to $\epsilon > \epsilon_1$, where ϵ_1 is the unique solution of the equation $\sigma_e x^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} = L_0 + T_0$.

If $Y_{d,1} > 0$. In this case, (VI.75) is equivalent to

$$\gamma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \alpha_d \gamma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} > L_0 + T_0 - \bar{L}. \tag{VI.76}$$

Since γ_d is decreasing and $\tilde{w}_1$ is increasing in ϵ, condition (VI.76) is equivalent to $\epsilon > \epsilon_2$, where ϵ_1 is the unique solution of the equation

$$\gamma_e \tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \alpha_d \gamma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = L_0 + T_0 - \bar{L}.$$

Therefore, $H_1 > 0$ if $\epsilon > \bar{\epsilon} := \max\{\epsilon_1, \epsilon_2\}$; $H_1 = 0$ if $\epsilon < \underline{\epsilon} := \min\{\epsilon_1, \epsilon_2\}$.

■

Proof of Proposition 2.11: We denote X_c, X_d, X_e such that

$$\sigma_c \ = \ X_c \epsilon^{\frac{-\alpha_c}{1-\alpha_c}}, \quad \sigma_d = X_d \epsilon^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}}, \quad \sigma_e = X_e.$$

By definition of $\hat{w}_1$, we have

$$\frac{X_c}{(\epsilon \hat{w}_1)^{\frac{1}{1-\alpha_c}}} + \frac{X_e}{\epsilon \hat{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}}} = S + \frac{L_0 + T_0}{\epsilon}. \tag{VI.77}$$

Let ϵ tend to infinity, thus $\hat{w}_1(\epsilon)$ will tend to zero. Moreover, we have $\lim_{\epsilon \to +\infty} \epsilon \hat{w}_1(\epsilon) = +\infty$. Indeed, (VI.77) implies that $\liminf_{\epsilon \to +\infty} \epsilon \hat{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \geq \frac{X_e}{S}$. Therefore

$$\epsilon \hat{w}_1 = \epsilon \hat{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} \to +\infty \text{ when } \epsilon \to +\infty.$$

By combining with (VI.77), we obtain $\lim_{\epsilon \to +\infty} \frac{X_e}{\epsilon \hat{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}}} = S$.

Definition of $\tilde{w}_1$ implies that

$$\frac{X_c}{(\epsilon \tilde{w}_1)^{\frac{1}{1-\alpha_c}}} + \frac{X_e}{\epsilon \tilde{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}}} + \frac{X_d}{(\epsilon \tilde{w}_1^{\frac{1}{1-\beta_d}})^{\frac{1-\beta_d}{1-\alpha_d-\beta_d}}} = S + \frac{L_0 + T_0 - \bar{L}}{\epsilon}. \tag{VI.78}$$

By the same argument, we obtain $\lim_{\epsilon\to+\infty} \epsilon\tilde{w}_1(\epsilon) = +\infty$.

We now compare the welfare between two cases : $Y_{d,1} > 0$ and $Y_{d,1} = 0$.

$$\begin{aligned}
\frac{\hat{U}(\epsilon)}{\tilde{U}(\epsilon)} &= \frac{\gamma_c \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e \hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}}{\gamma_c \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} + \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}} \\
&= \frac{\frac{X_c}{\alpha_c}\epsilon^{\frac{-\alpha_c}{1-\alpha_c}} \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + X_e \hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}}{\frac{X_c}{\alpha_c}\epsilon^{\frac{-\alpha_c}{1-\alpha_c}} \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + X_e \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} + \frac{X_d}{\alpha_d+\beta_d}\epsilon^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}} \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}} \\
&= \frac{\frac{X_c}{\alpha_c}(\epsilon\hat{w}_1)^{\frac{-\alpha_c}{1-\alpha_c}} \epsilon^{\frac{-\beta_e}{1-\alpha_e}} + X_e\big(\epsilon \hat{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}}\big)^{\frac{-\beta_e}{1-\alpha_e}}}{\frac{X_c}{\alpha_c}(\epsilon\tilde{w}_1)^{\frac{-\alpha_c}{1-\alpha_c}} \epsilon^{\frac{-\beta_e}{1-\alpha_e}} + X_e\big(\epsilon \tilde{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}}\big)^{\frac{-\beta_e}{1-\alpha_e}} + \frac{X_d}{\alpha_d+\beta_d}\epsilon^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}-\frac{\beta_e}{1-\alpha_e}} \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}}.
\end{aligned}$$

Case 1 : $\beta_d > \dfrac{\beta_e}{1-\alpha_e}$**.** This condition is equivalent to $\dfrac{1}{1-\beta_d} > \dfrac{1-\alpha_e}{1-\alpha_e-\beta_e}$ which implies that $\lim_{\epsilon\to+\infty} \epsilon(\tilde{w}_1(\epsilon))^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = +\infty$. Consequently, we obtain $\lim_{\epsilon\to+\infty} \dfrac{X_d}{\big(\epsilon\tilde{w}_1^{\frac{1}{1-\beta_d}}\big)^{\frac{1-\beta_d}{1-\alpha_d-\beta_d}}} = S$.

Denote $N := \dfrac{\frac{\alpha_d+\beta_d}{1-\alpha_d-\beta_d}}{\frac{\alpha_d}{1-\alpha_d-\beta_d}+\frac{\beta_e}{1-\alpha_e}}$. We observe that $\beta_d > \dfrac{\beta_e}{1-\alpha_e}$ is equivalent to $N > \dfrac{1}{1-\beta_d}$. Hence $\epsilon\tilde{w}_1^N = \epsilon\tilde{w}_1^{\frac{1}{1-\beta_d}}\tilde{w}_1^{N-\frac{1}{1-\beta_d}} \to 0$ when $\epsilon \to +\infty$. Moreover, we can write

$$\epsilon^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}-\frac{\beta_e}{1-\alpha_e}} \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} = \left(\epsilon\tilde{w}_1^N\right)^{-\frac{\alpha_d}{1-\alpha_d-\beta_d}-\frac{\beta_e}{1-\alpha_e}}.$$

Thus, we get that

$$\lim_{\epsilon\to+\infty} \frac{\hat{U}(\epsilon)}{\tilde{U}(\epsilon)} = 0.$$

Therefore, $Y_{d,1} > 0$ at equilibrium when ϵ is high enough.

Case 2 : $\beta_d = \frac{\beta_e}{1-\alpha_e}$**.** By using the same argument in previous case, and note that $\lim_{\epsilon\to+\infty} \frac{\hat{w}_1}{\tilde{w}_1} < 1$, we get

$$\lim_{\epsilon\to+\infty} \frac{\hat{U}(\epsilon)}{\tilde{U}(\epsilon)} < 1.$$

Therefore $Y_{d,1} > 0$ at equilibrium when ϵ is high enough.

Case 3 : $\beta_d < \frac{\beta_e}{1-\alpha_e} < \frac{\beta_d}{\alpha_d+\beta_d}$**.** In this case, we write

$$\sigma_d(\tilde{w}_1)^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} = X_d\big(\epsilon\tilde{w}_1^{1+\frac{\beta_d}{\alpha_d}}\big)^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}}.$$

Since $\beta_d < \frac{\beta_e}{1-\alpha_e} < \frac{\beta_d}{\alpha_d+\beta_d}$, we have $1 + \frac{\beta_d}{\alpha_d} - \frac{1-\alpha_e}{1-\alpha_e-\beta_e} > 0$, and we get

$$\lim_{\epsilon\to\infty} \epsilon\tilde{w}_1^{1+\frac{\beta_d}{\alpha_d}} = \lim_{\epsilon\to\infty} \epsilon\tilde{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \tilde{w}_1^{1+\frac{\beta_d}{\alpha_d}-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = 0.$$

Consequently, $\lim\limits_{\epsilon\to\infty} \sigma_d(\tilde{w}_1)^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} = \infty$. According Lemma 5.1, we have $\lim\limits_{\epsilon\to\infty} \tilde{U} - \hat{U} = +\infty$.

Note that, if $\beta_d < \frac{\beta_e}{1-\alpha_e} = \frac{\beta_d}{\alpha_d+\beta_d}$, we have $\lim\limits_{\epsilon\to\infty} \tilde{U} - \hat{U} > 0$, when $\epsilon \to +\infty$.

Case 4 : $\frac{\beta_e}{1-\alpha_e} > \frac{\beta_d}{\alpha_d+\beta_d}$. By using the same argument of Case 3, we have $\lim\limits_{\epsilon\to\infty} \sigma_d(\tilde{w}_1)^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} = 0$. On the one hand, we have

$$\begin{aligned} \tilde{U} - \hat{U} &= (\epsilon S + L_0 + T_0)(\tilde{w}_1 - \hat{w}_1) + (1-\alpha_c)\gamma_c(\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) \\ &\quad +\Big(\frac{1-\alpha_d-\beta_d}{\alpha_d+\beta_d}\sigma_d\tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} - \bar{L}\Big)\tilde{w}_1 \\ &\leq \Big(\frac{1-\alpha_d-\beta_d}{\alpha_d+\beta_d}\sigma_d\tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} - \bar{L}\Big)\tilde{w}_1. \end{aligned} \tag{VI.79}$$

Hence, $\liminf\limits_{\epsilon\to+\infty} \tilde{U} - \hat{U} \geq= 0$. On the other hand, we also have

$$\begin{aligned} \tilde{U} - \hat{U} &= \gamma_c(\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) + \gamma_e(\tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} - \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}) + \gamma_d\tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} \\ &\leq \gamma_d\tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}. \end{aligned}$$

Hence, $\limsup\limits_{\epsilon\to+\infty} \tilde{U} - \hat{U} \leq 0$. Therefore, we obtain $\lim\limits_{\epsilon\to+\infty} \tilde{U} - \hat{U} = 0$.

Case 4.1 : $1-\alpha_d > \frac{\beta_e}{1-\alpha_e} > \frac{\beta_d}{\alpha_d+\beta_d}$. We have $\frac{1}{\alpha_d} - \frac{1-\alpha_e}{1-\alpha_e-\beta_e} > 0$, and then

$$\epsilon\tilde{w}_1^{\frac{1}{\alpha_d}} = \epsilon\tilde{w}_1^{\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \tilde{w}_1^{\frac{1}{\alpha_d}-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} \to 0$$

when $\epsilon \to \infty$. Consequently, $\lim\limits_{\epsilon\to\infty} \epsilon\tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = \lim\limits_{\epsilon\to\infty} X_d\big(\epsilon\tilde{w}_1^{\frac{1}{\alpha_d}}\big)^{\frac{-\alpha_d}{1-\alpha_d-\beta_d}} = +\infty$. According (VI.79), $\tilde{U} - \hat{U} > 0$ when ϵ is high enough.

Case 4.2 : $1-\alpha_d < \frac{\beta_e}{1-\alpha_e}$. In this case, we have $\lim\limits_{\epsilon\to\infty} \epsilon\tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = 0$.

First, we have a remark that for $\gamma \in (0,1)$, we have $y^\gamma - z^\gamma > \gamma y^{\gamma-1}(y-z)$ for every $y, z > 0$. Therefore, we have

$$\begin{aligned} \hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) &= (\hat{w}_1^{\frac{-1}{1-\alpha_c}})^{\alpha_c} - (\tilde{w}_1^{\frac{-1}{1-\alpha_c}})^{\alpha_c} \geq \alpha_c\hat{w}_1(\hat{w}_1^{\frac{-1}{1-\alpha_c}} - \tilde{w}_1^{\frac{-1}{1-\alpha_c}}) \\ \hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} - \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} &= (\hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}})^{\frac{\beta_e}{1-\alpha_e}} - (\hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}})^{\frac{\beta_e}{1-\alpha_e}} \\ &\geq \frac{\beta_e}{1-\alpha_e}\hat{w}_1(\hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} - \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}). \end{aligned}$$

We now write

$$\begin{aligned}
\hat{U} - \tilde{U} &= \gamma_c(\hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) + \gamma_e(\hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} - \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}) - \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} \\
&= \frac{\beta_e}{1-\alpha_e}\Big((\frac{1-\alpha_e}{\beta_e} - 1)\frac{\sigma_c}{\alpha_c}(\hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) + \frac{\sigma_c}{\alpha_c}(\hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}})\Big) \\
&\quad + \frac{\beta_e}{1-\alpha_e}\frac{1-\alpha_e}{\beta_e}\gamma_e(\hat{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} - \tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}) - \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} \\
&\geq \frac{1-\alpha_e-\beta_e}{1-\alpha_e}\frac{\sigma_c}{\alpha_c}(\hat{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} - \tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}}) - \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}} \\
&\quad + \frac{\beta_e}{1-\alpha_e}\hat{w}_1\Big(\sigma_c(\hat{w}_1^{\frac{-1}{1-\alpha_c}} - \tilde{w}_1^{\frac{-1}{1-\alpha_c}}) + \sigma_e(\hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} - \hat{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}})\Big) \\
&> \frac{\beta_e}{1-\alpha_e}\hat{w}_1(\bar{L} - \sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}) - \gamma_d \tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}.
\end{aligned}$$

Therefore, we have

$$\hat{U} - \tilde{U} > \hat{w}_1\Big(\frac{\beta_e}{1-\alpha_e}\bar{L} - \frac{\beta_e}{1-\alpha_e}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} - \frac{\tilde{w}_1}{(\alpha_d+\beta_d)\hat{w}_1}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}\Big).$$

Recall that in this case, we have $\bar{L} > 0$, $\lim\limits_{\epsilon\to\infty} \frac{\tilde{w}_1}{\hat{w}_1} = 1$ and $\lim\limits_{\epsilon\to\infty} \epsilon \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}} = 0$. So, we imply that $\hat{U} - \tilde{U} > 0$ when ϵ is high enough. ■

Proof of Proposition 2.13: If $\epsilon S + L_0 + T_0 \leq \bar{L}$, we have $Y_{d,1} = 0$. We now consider the case $\epsilon S + L_0 + T_0 > \bar{L}$.
We assume that $H_1 > 0$. By observing equation determining wage, we see that wage increases when selling physical capital price increases. Moreover, $\lim\limits_{p_n\to+\infty} \tilde{w}_1 = \lim\limits_{p_n\to+\infty} \hat{w}_1 = +\infty$. VI.57 implies that

$$\lim_{p_n\to+\infty} \sigma_e \hat{w}_1^{-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0.$$

Equation determining $\tilde{w}$ is equivalent to

$$\sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\sigma_e}{p_n^{\frac{1}{1-\alpha_e-\beta_e}}}\Big(\frac{p_n}{\tilde{w}_1^{1-\alpha_e}}\Big)^{\frac{1}{1-\alpha_e-\beta_e}} + \frac{\sigma_e}{p_n^{\frac{1}{1-\alpha_d-\beta_d}}}\Big(\frac{p_n}{\tilde{w}_1}\Big)^{\frac{1}{1-\alpha_d-\beta_d}} = \epsilon S + L_0 + T_0 - \bar{L},$$

where we note that $\frac{\sigma_e}{p_n^{\frac{1}{1-\alpha_e-\beta_e}}}$ and $\frac{\sigma_e}{p_n^{\frac{1}{1-\alpha_d-\beta_d}}}$ do not depend on p_n.
Therefore, we have $\frac{p_n}{\tilde{w}_1}\tilde{w}^{\alpha_e} = \frac{p_n}{\tilde{w}_1^{1-\alpha_e}}$ is bounded. This implies that $\lim\limits_{p_n\to+\infty} \frac{p_n}{\tilde{w}_1} = 0$. Consequently, we get

$$\lim_{p_n\to+\infty} \sigma_e \tilde{w}_1^{-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 - \bar{L}.$$

Thus, we obtain $\lim_{p_n \to +\infty} \frac{\hat{w}_1}{\tilde{w}_1} = (\frac{\epsilon S + L_0 + T_0}{\epsilon S + L_0 + T_0 - \bar{L}})^{-\frac{1-\alpha_e-\beta_e}{1-\alpha_e}}$.
We now compare welfares

$$\begin{aligned}\frac{\hat{U}}{\tilde{U}} &= \frac{\gamma_c(\hat{w}_1)^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e(\hat{w}_1)^{\frac{-\beta_e}{1-\alpha_e-\beta_e}}}{\gamma_c\tilde{w}_1^{\frac{-\alpha_c}{1-\alpha_c}} + \gamma_e\tilde{w}_1^{\frac{-\beta_e}{1-\alpha_e-\beta_e}} + \gamma_d\tilde{w}_1^{\frac{-\alpha_d-\beta_d}{1-\alpha_d-\beta_d}}} \\ &= \frac{\gamma_c(\hat{w}_1)^{\frac{-1}{1-\alpha_c}} + \gamma_e(\hat{w}_1)^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}}}{\gamma_c\tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \gamma_e\tilde{w}_1^{\frac{-(1-\alpha_e)}{1-\alpha_e-\beta_e}} + \gamma_d\tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}}\frac{\hat{w}_1}{\tilde{w}_1}\end{aligned}$$

Hence, we obtain

$$\lim_{p_n \to +\infty} \frac{\hat{U}}{\tilde{U}} = (\frac{\epsilon S + L_0 + T_0}{\epsilon S + L_0 + T_0 - \bar{L}})^{\frac{\beta_e}{1-\alpha_e}} > 1.$$

This implies that when p_n is high enough, $\hat{U} > \tilde{U}$.
We can also see that condition (VI.75) is satisfied when p_n is high enough. So, when p_n is high enough, we have $Y_{d,1} = 0$ and $H_1 > 0$ at equilibrium. ■

Proof of Proposition 2.14:

Case (i) : $\frac{\alpha_e}{1-\alpha_e} > \max(\alpha_c, \alpha_d)$. As in proof of Proposition 2.13, we obtain

$$\begin{aligned}&\lim_{p\to 0}\hat{w}_1 p^{\alpha_c} = \lim_{p\to 0}\tilde{w}_1 p^{\alpha_c} = \lim_{p\to 0}\tilde{w}_1 p^{\alpha_d} = +\infty \\ &\lim_{p\to 0}\sigma_e \hat{w}_w^{-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 \\ &\lim_{p\to 0}\sigma_e \hat{w}_w^{-\frac{1-\alpha_e}{1-\alpha_e-\beta_e}} = \epsilon S + L_0 + T_0 - \bar{L}.\end{aligned}$$

Consequently, we get

$$\lim_{p\to 0}\frac{\hat{w}_1}{\tilde{w}_1} = (\frac{\epsilon S + L_0 + T_0}{\epsilon S + L_0 + T_0 - \bar{L}})^{-\frac{1-\alpha_e-\beta_e}{1-\alpha_e}} \quad \text{(VI.80)}$$

$$\lim_{p\to 0}\frac{\hat{U}}{\tilde{U}} = (\frac{\epsilon S + L_0 + T_0}{\epsilon S + L_0 + T_0 - \bar{L}})^{\frac{\beta_e}{1-\alpha_e}}. \quad \text{(VI.81)}$$

Therefore, when p is low enough, we have $\hat{U}_1 > \tilde{U}$.
We have to now check that $H_1 > 0$ when p is low enough. We will check that $\epsilon S > \sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}}$. As in proof of Proposition 2.13, we obtain $\lim_{p\to 0}\sigma_c \hat{w}_1^{\frac{-1}{1-\alpha_c}} = 0$. Hence, $H_1 > 0$ when p is low enough.
Case (ii.a) : The proof is similar to point (i) of Proposition 2.4.

Case (ii.b) : Assume that $H_1 = 0$, we write the equation of w_1

$$
\begin{aligned}
S = {} & \Big(\frac{1}{\beta_d p_n A_d}\Big)^{\frac{1}{\alpha_d}} (p w_1^{\frac{1}{\alpha_d}}) \Big(L_0 + T_0 - \bar{L} - \frac{M_e}{(p w_1^{\frac{1-\alpha_e}{\alpha_e}})^{\frac{\alpha_e}{1-\alpha_e-\beta_e}}}\Big)^{\frac{1-\beta_d}{\alpha_d}} \\
& + \Big(\frac{\alpha_c A_c}{\alpha_d}\Big)^{\frac{1}{1-\alpha_c}} \Big(\frac{1}{p_n A_d \beta_d^{1-\alpha_d}}\Big)^{\frac{1}{\alpha_d(1-\alpha_c)}} p w_1^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} \Big(L_0 + T_0 \\
& \qquad - \bar{L} - \frac{M_e}{(p w_1^{\frac{1-\alpha_e}{\alpha_e}})^{\frac{\alpha_e}{1-\alpha_e-\beta_e}}}\Big)^{\frac{1-\alpha_d-\beta_d}{\alpha_d(1-\alpha_c)}},
\end{aligned}
$$

where $M_e := \sigma_e p^{\frac{\alpha_e}{1-\alpha_e-\beta_e}}$ which does not depend on p.
First, it is easy to see that w_1 increases if p decreases. Moreover, $\lim_{p\to 0} w_1(p) = +\infty$.
If there is a sequence $(p(n))_{n=1,2,...,}$ converging to zero such that $p(n)\big(w_1(n)\big)^{\frac{1-\alpha_e}{\alpha_e}}$ is bounded from above.[13] Since $\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)} < \frac{1}{\alpha_d} < \frac{1-\alpha_e}{\alpha_e}$, We have

$$
\begin{aligned}
p(n)\big(w_1(n)\big)^{\frac{1}{\alpha_d}} &= p(n)\big(w_1(n)\big)^{\frac{1-\alpha_e}{\alpha_e}} \big(w_1(n)\big)^{\frac{1}{\alpha_d} - \frac{1-\alpha_e}{\alpha_e}} \to 0 \text{ when } n \to 0 \\
p(n)\big(w_1(n)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} &= p(n)\big(w_1(n)\big)^{\frac{1}{\alpha_d}} \big(w_1(n)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)} - \frac{1}{\alpha_d}} \to 0 \text{ when } n \to 0.
\end{aligned}
$$

Therefore, we get a contradiction to the equation of w_1. So, we have $\lim_{p\to 0} p\big(w_1(p)\big)^{\frac{1-\alpha_e}{\alpha_e}} = +\infty$.
We now prove that $\lim_{p\to 0} p\big(w_1(p)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} = 0$. Indeed, if there is a sequence $(p(n))_{n=1,2,...}$ converging to zero such that $p(n)\big(w_1(n)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}}$ is bounded from below. We get that

$$
p(n)\big(w_1(n)\big)^{\frac{1}{\alpha_d}} = p(n)\big(w_1(n)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} \big(w_1(n)\big)^{\frac{1}{\alpha_d} - \frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} \to +\infty
$$

when n tends to infinite. This implies a contradiction to to the equation determining wage.
We have proved that $\lim_{p\to 0} p\big(w_1(p)\big)^{\frac{1-\alpha_e}{\alpha_e}} = +\infty$ and $\lim_{p\to 0} p\big(w_1(p)\big)^{\frac{1-\alpha_d}{\alpha_d(1-\alpha_c)}} = 0$.
The equation determining wage implies that

$$
S = \Big(\frac{1}{\beta_d p_n A_d}\Big)^{\frac{1}{\alpha_d}} (L_0 + T_0 - \bar{L})^{\frac{1-\beta_d}{\alpha_d}} \lim_{p\to 0} p w_1^{\frac{1}{\alpha_d}}. \qquad \text{(VI.82)}
$$

By using the same argument in Remark 5.1, we see that the first order condition of H_1 is satisfied. ■

Proof of Example 2.1: We prove point (i). Point (ii) can be proved by the same argument.

13. We write $w_1(n)$ instead of $w_1(p(n))$.

Assume that there exists an equilibrium with $H_1, Y_{d,1} > 0$.
We find conditions in which $\tilde{U} \geq \hat{U}$ under Assumption VI.13.
By using Assumption VI.13, we have

$$\begin{aligned}\tilde{w}_1^x &= \frac{\mathcal{L}_0 - \bar{L}}{\sigma_c + \sigma_e + \sigma_d}\\ \tilde{U} &= (\gamma_c + \gamma_e + \gamma_d)\tilde{w}_1^{x+1}\\ (\hat{w}_1)^x &= \frac{\mathcal{L}_0}{\sigma_c + \sigma_e}\\ \hat{U} &= (\gamma_c + \gamma_e)(\hat{w}_1)^{x+1},\end{aligned}$$

where $x := -1/(1-\alpha_c)$. Therefore we see that

$$\begin{aligned}\tilde{U} \geq \hat{U} \quad &\Leftrightarrow \quad \frac{\gamma_c + \gamma_e + \gamma_d}{\gamma_c + \gamma_e} \geq \Big(\frac{\mathcal{L}_0}{\mathcal{L}_0 - \bar{L}} \frac{\sigma_c + \sigma_e + \sigma_d}{\sigma_c + \sigma_e}\Big)^{\alpha_c}\\ &\Leftrightarrow \quad \mathcal{L}_0 - \bar{L} \geq \Omega\mathcal{L}_0 \Leftrightarrow (1-\Omega)\mathcal{L}_0 \geq \bar{L}.\end{aligned}$$

Note that $\Omega < 1$. Indeed,

$$\begin{aligned}\Omega \quad &:= \quad \frac{\alpha(\gamma_c + \gamma_d) + \gamma_e}{\alpha\gamma_c + \gamma_e}\Big(\frac{\gamma_c + \gamma_e}{\gamma_c + \gamma_e + \gamma_d}\Big)^{\frac{1}{\alpha_c}}\\ &< \quad \frac{\alpha(\gamma_c + \gamma_d) + \gamma_e}{\alpha\gamma_c + \gamma_e}\frac{\gamma_c + \gamma_e}{\gamma_c + \gamma_e + \gamma_d}.\end{aligned}$$

On the other hand,

$$\frac{\alpha(\gamma_c + \gamma_d) + \gamma_e}{\alpha\gamma_c + \gamma_e} = 1 + \frac{\alpha\gamma_d}{\alpha\gamma_c + \gamma_e} < 1 + \frac{\gamma_d}{\gamma_c + \gamma_e} = \frac{\gamma_c + \gamma_e + \gamma_d}{\gamma_c + \gamma_e}.$$

Therefore $\Omega < 1$. Consequently, $\tilde{U} \geq \hat{U}$ if and only if $\mathcal{L}_0 \geq \dfrac{\bar{L}}{1-\Omega}$.
Condition $H_1 > 0$ is equivalent to

$$\epsilon S > \sigma_c \tilde{w}_1^{\frac{-1}{1-\alpha_c}} + \frac{\alpha_d}{\alpha_d + \beta_d}\sigma_d \tilde{w}_1^{\frac{-1}{1-\alpha_d-\beta_d}}.$$

Under Assumption 3, we have $\hat{w}^{\frac{1}{1-\alpha_c}} = \hat{w}^{\frac{1}{1-\alpha_d-\beta_d}} = \dfrac{\epsilon S + L_0 + T_0 - \bar{L}}{\sigma_c + \sigma_d + \sigma_e}$. Therefore, H_1 is equivalent to condition (VI.43).

We now assume that condition (VI.42) and (VI.43) hold. Then $\epsilon S + L_0 + T_0 - \bar{L} > 0$ then equation (VI.53) has a unique solution who is equilibrium wage. We see that all first order conditions hold. Condition (VI.43) ensures that $H_1 > 0$ in this case. Thus, the list $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}, K_{e,1}, w_1)$ given in proof of Example 2.1 in case $Y_{d,1} > 0$ is the unique equilibrium.

We now have

$$\frac{Y_{d,1}}{Y_{e,1}} = \beta_e \frac{\gamma_d}{\gamma_e} = \frac{\alpha_d^{\frac{\alpha_d}{1-\alpha_d-\beta_d}} \beta_d^{\frac{\beta_d}{1-\alpha_d-\beta_d}} A_d^{\frac{1}{1-\alpha_d-\beta_d}}}{\alpha_e^{\frac{\alpha_e}{1-\alpha_e-\beta_e}} \beta_e^{\frac{\beta_e}{1-\alpha_e-\beta_e}} A_e^{\frac{1}{1-\alpha_e-\beta_e}}} \frac{p_n^{\frac{1}{1-\alpha_d-\beta_d} - \frac{1}{1-\alpha_e-\beta_e}}}{p^{\frac{\alpha_d}{1-\alpha_d-\beta_d} - \frac{\alpha_e}{1-\alpha_e-\beta_e}}} \frac{1}{\epsilon^{\frac{\alpha_d}{1-\alpha_d-\beta_d}}}.$$

■

6 Appendix : Constant return to scale

We assume that $\alpha_d + \beta_d = \alpha_e + \beta_e = 1$. We write FOC for the multinational firm. If $K_{e,1}, L_{e,1} > 0$, we have

$$\alpha_e p_n A_e K_{e,1}^{\alpha_e - 1} L_{e,1}^{1-\alpha_e} = p \tag{VI.83}$$

$$(1-\alpha_e) p_n A_e K_{e,1}^{\alpha_e} L_{e,1}^{\alpha_e} = w_1. \tag{VI.84}$$

Consequently, we get $w_1^{1-\alpha_e} = \alpha_e^{\alpha_e}(1-\alpha_e)^{1-\alpha_e} A_e p_n p^{-\alpha_e}$. In this case, the multinational firm's profit equals zero. This implies that the multinational firm's profit equals zero in any case. Note that $K_{e,1} = L_{e,1} = 0$ is a solution of this firm's maximization problem.

Denote λ_h, λ_ℓ Lagrange multipliers associated to conditions (VI.5), (VI.6), $H_1 \geq 0$, and $L_{e,1} \geq 0$, respectively. We have

$$\begin{aligned} K_{c,1} : &\quad \alpha_c A_c K_{c,1}^{\alpha_c - 1} = \lambda p \\ L_{e,1} : &\quad w_1 - \mu + \lambda_\ell = 0, \text{ where } \lambda_\ell \geq 0, L_{e,1}\lambda_\ell = 0. \\ H_1 : &\quad -\lambda + \mu\epsilon + \lambda_h = 0, \text{ where } \lambda_h \geq 0, H_1\lambda_h = 0. \end{aligned}$$

If $Y_{d,1} > 0$, we have

$$\begin{aligned} K_{d,1} : &\quad p_n \alpha_d A_d K_{d,1}^{\alpha_d - 1}(L_{d,1} - \bar{L})^{1-\alpha_d} = \lambda p \\ L_{d,1} : &\quad p_n(1-\alpha_d) A_d K_{d,1}^{\alpha_d}(L_{d,1} - \bar{L})^{-\alpha_d} = \mu. \end{aligned}$$

Proof of Theorem 2.2: The first case is clear. We assume that $L_0 + T_0 + \epsilon S > \bar{L}$. It is easy to see that when A_d tends to infinity, the GNP tend to infinity if the host country invest in the new industry. So, when A_d is high enough, we have $Y_{d,1} > 0$ at equilibrium. Therefore, $L_0 + T_0 + \epsilon H_1 > \bar{L}$ at equilibrium.
If $L_0 + T_0 \leq \bar{L}$ then we have $H_1 > 0$. The second statement of Theorem 2.2 is proved.

Case (2.2.1) $L_0 + T_0 > \bar{L}$ and $\epsilon S < \frac{\alpha_d}{1-\alpha_d}(L_0 + T_0 - \bar{L})$. Assume that $L_{e,1} = H_1 = 0$. FOCs of $K_{c,1}$ and $K_{d,1}$ imply that

$$\alpha_c A_c K_{d,1}^{1-\alpha_d}(L_0 + T_0 - \bar{L})^{-(1-\alpha_d)} = p_n \alpha_d A_d K_{c,1}^{1-\alpha_c}.$$

We then get an equation determining $K_{c,1}$

$$S = pK_{c,1} + \left(\frac{p_n \alpha_c A_c}{\alpha_d A_d}\right)^{\frac{1}{1-\alpha_d}}(L_0 + T_0 - \bar{L})K_{c,1}^{\frac{1-\alpha_c}{1-\alpha_d}}.$$

This equation has a unique solution. It is easy to see that when A_d increases, $K_{c,1}$ decreases, and $\lim\limits_{A_d \to +\infty} K_{c,1} = 0$, $\lim\limits_{A_d \to +\infty} K_{d,1} = S/p$.

We now check FOC of H_1 : $\lambda \geq \mu\epsilon$ will be satisfied when A_d is high enough. This condition can be written as $\frac{\lambda p}{\mu} \geq \epsilon p$ which is equivalent to $\frac{\alpha_d}{1-\alpha_d}\frac{L_{d,1}-\bar{L}}{K_{d,1}} \geq \epsilon p$. This condition is satisfied when A_d is high enough since $\lim\limits_{A_d \to +\infty} K_{d,1} = S/p$.

Case (2.2.2) $L_0 + T_0 > \bar{L}$ and $\epsilon S > \frac{\alpha_d}{1-\alpha_d}(L_0 + T_0 - \bar{L})$. We will check that $L_{e,0} = 0$ and $H_1 > 0$ at equilibrium. Assume that $L_{e,1} = 0$ and $H_1 > 0$. Then we have $L_{d,1} = L_0 + T_0 + \epsilon H_1$ and $\lambda = \mu\epsilon$. FOCs of $K_{d,1}, L_{d,1}$ implies that $\frac{L_{d,1}-\bar{L}}{K_{d,1}} = \frac{1-\alpha_d}{\alpha_d}\epsilon p$. Hence, we get $pK_{d,1} = \frac{\alpha_d}{1-\alpha_d}(L_0 + T_0 + \epsilon H_1 - \bar{L})$ and

$$\alpha_c A_c K_{c,1}^{\alpha_c - 1} \;=\; \lambda p = p_n \alpha_d A_d K_{d,1}^{\alpha_d - 1}(L_{d,1} - \bar{L})^{1-\alpha_d} = p_n \alpha_d A_d \Big(\frac{1-\alpha_d}{\alpha_d}\epsilon p\Big)^{1-\alpha_d}.$$

Hence, we can compute $K_{c,1}$ in oder to get that

$$\begin{aligned} S &= pK_{c,1} + pK_{d,1} + H_1 \\ &= \Big(\frac{\alpha_c A_c}{p_n \alpha_d A_d}\Big)^{\frac{1}{1-\alpha_c}} \Big(\frac{\alpha_d}{(1-\alpha_d)\epsilon}\Big)^{\frac{1-\alpha_d}{1-\alpha_c}} p^{\frac{\alpha_d - \alpha_c}{1-\alpha_c}} + \frac{H_1}{1-\alpha_d} + \frac{\alpha_d(L_0 + T_0 - \bar{L})}{1-\alpha_d)\epsilon}. \end{aligned}$$

Since $S > \frac{\alpha_d(L_0+T_0-\bar{L})}{1-\alpha_d)\epsilon}$, this equation has a unique solution $H_1 > 0$ when A_d is high enough. It is easy to check all FOCs. Therefore, $L_{e,1} = 0, Y_{d,1}, H_1 > 0$ given as above is an equilibrium. ■

Proof of Proposition 2.12: When ϵ is high enough, it is clear that the country should invest in training.
Assume that the country also invests in the new industry, i.e., $Y_{d,1} > 0$.
According the computation in the proof of Theorem 2.2, we have

$$\alpha_c A_c \;\geq\; p_n \alpha_d A_d \Big(\frac{1-\alpha_d}{\alpha_d}\epsilon p\Big)^{1-\alpha_d} K_{c,1}^{1-\alpha_c} \geq p_n \alpha_d A_d \Big(\frac{1-\alpha_d}{\alpha_d}\epsilon p\Big)^{1-\alpha_d} \big(\frac{S}{p}\big)^{1-\alpha_c}.$$

This condition will be violated when ϵ is high enough. ■

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Chapitre VII

Conclusion

This dissertation discusses some issues which seem important for us.

1. By introducing Uniform and Sequential Gains to Trade Hypotheses, we prove the existence of monetary equilibrium in an infinite horizon model wih fiat money and collateralized assets and point out that the liquidity trap may appear in any period.
2. The existence of intertemporal equilibrium in infinite horizon models with incomplete financial markets (because of borrowing constraints) and capital accumulation is also proved. When there is a long-lived asset with positive dividend, equilibrium exists even agents do not have endowment. In our frameworks, technologies may be non-stationary and the aggregate capital stock are not necessarily uniformly bounded.
3. The dissertation contributes to the understanding of rational bubbles. Rational bubbles of financial asset are studied in two types of model : (1) general equilibrium model with incomplete financial market and without production, (2) Ramsey model with heterogeneous agents and incomplete financial market. Rational bubbles exist only if there exists an agent whose borrowing constraint is binding at infinitely many dates. We prove that there exists a bubble if and only if interest rates (in term of financial asset) are low, which means that the sum (over time) of interest rates is finite. We next give an exogenous condition under which bubbles appear. Bubbles of physical capital are also studied. Physical capital bubbles exist if and only if the sum of returns on capital is finite. As a consequence, if technologies are stationary and under standard assumption, physical capital bubbles are ruled out. In a standard Ramsey model with heterogeneous agents, with non-stationary linear technologies, bubbles exist if and only if the sum over time of productivities is finite (which means that productivities decrease with sufficiently high speed).

4. We study the efficiency of intertemporal equilibrium. An intertemporal equilibrium is called to be efficient if its aggregate capital path is efficient in sense of Malinvaud (1953). In standard Ramsey models with heterogeneous agents and linear production functions, every intertemporal equilibrium is efficient ; interestingly, efficiency and bubbles may co-exist at equilibrium with such technologies. In a general equilibrium model with physical capital and financial asset, financial dividends may make production paths efficient.
5. We study the relationship between the financial market and the productive sector in a dynamic deterministic general equilibrium model with heterogeneous agents, capital accumulation, and imperfect financial market. By economic recession we mean a situation at which no one invests in the productive sector. We prove that when the productivity is slow, the economy will fall in recession because the agents prefer financial assets to physical capital. However, in some cases, financial assets may be benefit to the economy by providing financial support for the purchase of the physical capital. We also point out that, a fluctuation of financial dividend (ξ_t) may create a fluctuation of the aggregate capital path (K_t).
6. The last contribution of the dissertation is to point out the link among multinational firms, FDI spillovers and economic growth. Consider a poor or developing country having competitive productive sectors. Its optimal strategy should be the following :
   (i) Stage 1 : It should train specific workers.
   (ii) Stage 2 : These workers work for the multinational firm in the new industry to get favorable salary and working experiences or learning by doing effects in order to improve the GNP of the country.
   (ii) Stage 3 : Once the GNP reaches a critical threshold, the domestic firm will enter in the new industry and potentially eliminate the multinational firm.

For the future, we would like to study the following problems :

1. We need a "general" theory of bubbles, which allows us to study bubbles not only in general equilibrium models but also in asymmetric information and overlapping generation models.
2. We feel that we still are far from a well-understanding of the efficiency of capital path and of intertemporal equilibrium as well. To solve this problem, we need a new version of Cass Theorem with non-concave, non-stationary technologies and without the boundedness of capital paths. Then, we apply this to understand the efficiency of

intertemporal equilibrium in models with non-stationary technologies and/or without discounting.

3. It would also be interesting to study the relationship between rational bubbles and the efficiency of capital paths.
4. Fluctuations of the aggregate capital path (K_t) is a promising topic. Can a fluctuation of (K_t) appear even when financial dividend is zero? Do credit constraints create an endogenous fluctuation of (K_t)?
5. We need to understand the impact of borrowing limits f^i not only on asset prices but also on the efficiency of intertemporal equilibrium.
6. By interpreting heterogeneous agents as heterogeneous countries and financial market (with financial frictions) as international financial market, we should develop the model in Chapter 5 to analyze capital flows and global imbalances.