From norms to metrics in non-Archimedean geometry

Rémi Reboulet

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Rémi REBOULET

Thèse dirigée par **Catriona MACLEAN**,Maitre de conférences, Université Grenoble Alpes
et codirigée par **Sébastien BOUCKSOM**,DR, École polytechnique

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Des normes aux métriques en géométrie non-archimédienne

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devant le jury composé de :

Madame CATRIONA MACLEAN
MAITRE DE CONFERENCE HDR, UNIVERSITE GRENOBLE ALPES, Directrice de thèse
Monsieur DAVID WITT NYSTRÖM
MAITRE DE CONFERENCES, Chalmers University of Technology, Rapporteur
Monsieur JOSE IGNACIO BURGOS GIL
CHARGE DE RECHERCHE, Instituto de Ciencias Matematicas, Rapporteur
Monsieur PHILIPPE EYSSIDIEUX
PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES, Examinateur
Monsieur CHARLES FAVRE
DIRECTEUR DE RECHERCHE, CNRS DELEGATION ILE-DE-FRANCE SUD, Examinateur
Monsieur SEBASTIEN BOUCKSOM
DIRECTEUR DE RECHERCHE, CNRS DELEGATION ILE-DE-FRANCE SUD, Co-directeur de thèse
Monsieur EMMANUEL PEYRE
PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES, Président
Monsieur ANTOINE DUCROS
PROFESSEUR DES UNIVERSITES, SORBONNE UNIVERSITE, Examinateur

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> « Les sujets perdaient en leurs discussions le relief et le poids des choses véritables, autour desquelles se concentrent les faims d'argent et de bien-être et tous les durs intérêts des hommes. Ils s'idéalisaient dans l'air léger de la raison pure, qu'aucune expérience n'épaissit.
> Les meilleurs n'ignoraient pas ce paradoxe. Ni Largilier, ni Augustin, ni Bruhl ne s'y méprenaient. Mais précisément cet irréel les enchantait : "Dans notre enfance, disait Largilier, nous jouions à l'explorateur et au soldat. Nous continuons de jouer." »

Augustin, ou le maître est là, Joseph Malègue.

Résumé.

Nous étudions plusieurs aspects de la théorie du pluripotentiel sur un corps non-archimédien, en elle-même et à travers ses liens avec la géométrie complexe. Les objets centraux en sont les métriques plurisousharmoniques (ou psh) sur un fibré en droites L au-dessus d'une variété X sur un corps non-archimédien, dont la théorie globale a récemment été développée par Boucksom-Eriksson-Favre-Jonsson et al. Le cas le plus étudié est celui où le corps K est doté de la valeur absolue triviale ; dans cette thèse, nous nous concentrerons particulièrement sur les cas où la valeur absolue n'est pas triviale. Nous étudions dans un premier temps l'image de l'opérateur de Fubini-Study asymptotique sur des corps non-archimédiens généraux, qui permet d'approcher des métriques plurisousharmoniques sur un fibré en droites ample à l'aide de normes agissant sur les sections de ses puissances. Ensuite, nous construisons des géodésiques plurisousharmoniques dans les espaces de métriques non-archimédiennes psh d'énergie finie sur un fibré ample, et étudions leurs propriétés de régularité, ce qui étend des constructions classiques du monde complexe au monde non-archimédien. Enfin, nous considérons une dégénérescence analytique de variétés complexes X sur le disque unité, que nous identifions avec une variété X_K sur le corps non-archimédien K des séries de Laurent à coefficients complexes. Etant donné un fibré en droites relativement ample L sur X, nous construisons l'espace métrique géodésique des métriques relativement maximales d'énergie finie sur L. Nous montrons que l'espace des métriques d'énergie finie non-archimédiennes sur L_K (ayant également identifié L avec une variété sur le corps K) se plonge isométriquement et géodésiquement dans le précédent, ce qui permet de déduire la convexité d'incarnations non-archimédiennes de diverses fonctionnelles en lien avec la K-stabilité.

Abstract.

We study and develop pluripotential theory over a non-Archimedean field, in itself, and through its interactions with complex geometry. Its main objects are plurisubharmonic (or psh) metrics on a line bundle L over a variety X over a non-Archimedean field. The global theory of such psh metrics has recently been developed by Boucksom-Eriksson-Favre-Jonsson et al. The most well known case is that of a field K endowed with the trivial absolute value; in this thesis, we will focus on fields endowed with nontrivial absolute values. We first look into the image of the asymptotic Fubini-Study operator over general non-Archimedean fields, which allows us to approximate plurisubharmonic (psh) metrics on an ample line bundle L using norms acting on the sections of the tensor powers of L. Then, extending an important construction from complex geometry to the non-Archimedean world, we show that there exist plurisubharmonic geodesics in spaces of finite-energy psh metrics on an ample line bundle, and study their regularity properties with respect to the regularity of their endpoints. Finally, we consider an analytic degeneration of complex varieties, X, fibred over the unit disc, which we identify with a variety X_K over the non-Archimedean field of complex Laurent series. Given a relatively ample line bundle on X, we construct the geodesic metric space of relatively maximal finite-energy metrics on L. We show that the space of non-Archimedean finite-energy metrics on L_K (having once again identified L with a variety over the field K) embeds isometrically and geodesically into the former, allowing us to deduce convexity of some non-Archimedean versions of various functionals related to K-stability.

Contents

Introduction en français.

Le sujet d'étude principal de cette thèse est la théorie du pluripotentiel non-archimédien, dont les objets centraux sont les métriques (pluri)sousharmoniques non-archimédiennes. Nous expliquerons bientôt plus en détail ces termes. Nous développons cette théorie en suivant deux axes.

Premièrement, en elle-même: ses fondements sont récents ([BFJ15], [BJ21], [BE]), et particulièrement dans le cas de valuation non triviale, beaucoup d'aspects demeurent mystérieux. Il est donc de mise d'essayer de transposer divers concepts et énoncés du monde complexe au monde non-archimédien, ce qui est parfois très délicat. Dans plusieurs cas, le comportement de certains objets diverge complètement du cas complexe, ce qui donne de nouvelles perspectives intéressantes.

Ensuite, nous l'étudions à travers ses applications à la géométrie complexe, puisque dans certains cas, la géométrie non-archimédienne encode de manière très efficace des phénomènes singuliers ou asymptotiques liés à des objets complexes. Dans ce cadre, nous nous spécialiserons au cas où le corps de base est soit le corps $\mathbb{C}$ muni de la valeur absolue triviale, ou le corps $\mathbb{C}((t))$ des séries formelles à coefficients complexes, muni de sa valeur absolue t-adique usuelle.

De la géométrie complexe à la géométrie non-Archimédienne.

Le principe GAGA pour les variétés algébriques sur un corps non-archimédien.

Soit X une variété algébrique projective sur $\mathbb{C}$. Le célèbre théorème "GAGA" de Serre ([SerGAGA]) affirme que l'on peut lui associer une variété analytique projective complexe, X^{an}, en en préservant les propriétés topologiques essentielles (connexité, séparation, compacité), ainsi que les données des faisceaux cohérents. Cela permet d'importer des techniques analytiques puissantes, venues de l'analyse complexe et de la géométrie différentielle, afin d'étudier des problèmes de nature algébrique.

Soit maintenant X une variété algébrique projective sur un corps K non-archimédien, c'est-à-dire un corps muni d'une valeur absolue $|\cdot|$, complet pour la topologie induite par celle-ci, et dont l'inégalité triangulaire est raffinée en l'inégalité ultramétrique: pour tous x, y dans K,

$$|x+y| \leq \max(|x|, |y|).$$

Des exemples fondamentaux de tels corps sont:

1. $\mathbb{C}((t))$, le corps des séries de Laurent formelles sur $\mathbb{C}$, muni de la valeur absolue t-adique;

2. pour p premier, $\mathbb{Q}_p$, le corps des p-adiques, muni de la valeur absolue p-adique;

3. pour p premier, $\mathbb{F}_p((t))$, le corps des séries de Laurent formelles à coefficients dans le corps fini $\mathbb{F}_p$, que l'on munit également de la valeur absolue t-adique;

4. tout corps K peut être muni de la valeur absolue triviale, égale à 1 sur $\mathrm{K}^\times$, qui est non-archimédienne.

Ces exemples mettent en valeur la diversité et l'utilité du monde non-archimédien. Les corps p-adiques sont omniprésents en théorie des nombres (on notera des résultats classiques comme le principe de Hasse-Minkowski, le lemme d'Hensel, le théorème de Mahler, mais aussi des applications aux équations différentielles p-adiques, et à la théorie du potentiel p-adique) et leurs extensions *perfectoïdes* ont donné naissance à une théorie fertile ([Scholze], [FF]). Le corps $\mathbb{F}_p((t))$ permet de définir des voisinages formels de courbes arithmétiques. La classe des variétés sur le corps $\mathbb{C}((t))$, nous y reviendrons longuement au cours de cette thèse, contient comme cas particulier les dégénérescences de variétés analytiques complexes. Le cas de valuation triviale enfin, malgré sa dénomination, a un fort intérêt géométrique, sur lequel nous travaillerons également.

Il paraît souhaitable d'avoir à disposition un arsenal analytique similaire à celui du monde complexe pour les K-variétés. Cependant, des difficultés se manifestent rapidement. En premier lieu, l'inégalité ultramétrique a des conséquences notables sur la topologie de K : celle-ci est totalement discontinue, c'est-à-dire que ses seuls connexes sont les points et l'ensemble vide. Se pose ensuite le problème de définir une notion de variété analytique, et donc de fonction analytique, sur K. La manière naïve de procéder consiste à imiter le cas complexe, en définissant une fonction analytique sur K^d comme une série entière convergente à coefficients dans K, puis de procéder par recollements. Le résultat est catastrophique : Serre ([Ser65]) montre que toute variété compacte en ce sens est isomorphe à une union disjointe de boules $\{|x| \leq 1\}$. Il n'est pas acceptable qu'un tel objet puisse être l'analytification d'une variété algébrique sur K!

Nous observons à travers ces phénomènes, comme l'a fait Berkovich en son temps, qu'il convient de "rajouter des points" à un candidat pour la putative analytification X^{an} de X. Nous ne décrirons pas la procédure en détail dans cette introduction ; brièvement, l'analytifié dit *de Berkovich* X^{an} s'identifie à la compactification d'un ensemble de valuations d'origine géométrique. Les points cruciaux de cette construction sont les suivants :

1. elle préserve les propriétés topologiques de connexité, séparation, compacité;

2. elle réalise une équivalence de catégories entre les catégories de faisceaux cohérents sur X et X^{an};

3. elle s'étend aux variétés sur $\mathbb{C}$, auquel cas elle coïncide avec l'analytification usuelle au sens de Serre.

En particulier, si le corps K de base est de valuation triviale, alors X^{an} s'identifie à l'ensemble des valuations sur les corps de fractions $\mathrm{K}(Y)$, pour toutes les sous-variétés irréductibles $Y \subset X$.

Théorie du pluripotentiel géométrique.

Le point de départ est le suivant. Etant donné un domaine $\Omega \subset \mathbb{C}^d$, une fonction f lisse (ou simplement deux fois dérivable) à valeurs réelles sur Ω est dite plurisousharmonique si sa hessienne complexe est une matrice positive. Cette définition peut être comprise comme généralisant la notion de fonction convexe (lisse ou deux fois dérivable) de variables réelles, et de telles fonctions jouissent de propriétés similaires.

Plus généralement, une fonction f semi-continue supérieurement et localement intégrable sur Ω est dite plurisousharmonique si elle peut être écrite comme limite décroissante d'un filet de fonctions lisses plurisousharmoniques ; ou, de manière équivalente, si sa hessienne complexe au sens des distributions est une mesure positive.

De telles fonctions admettent une généralisation naturelle aux variétés complexes, et aux fibrés en droites sur celles-ci. Commençons par supposer qu'X est une variété projective complexe compacte, de dimension d, que l'on munit d'un fibré en droites L ample. Qu'L soit ample équivaut à demander que, pour tous entiers k assez grand, le produit tensoriel $L^{\otimes k}$ (que l'on notera additivement par kL) dispose d'une base $(s_i)_{i=1}^{\dim H^0(X,kL)}$ de l'espace $H^0(X, kL)$ de ses sections, tels que l'application

$$x \mapsto [s_1(x), \dots, s_{\dim H^0(X,kL)}(x)]$$

soit un plongement dans l'espace projectif $\mathbb{P}^{\dim H^0(X,kL)-1}$.

Soit τ_Ω une trivialisation $L|_\Omega \simeq \mathbb{C} \times \Omega$ de L. Une métrique singulière sur L est une métrique ϕ qui s'écrit, pour toutes telles trivialisations et pour $(\ell, x) \in \mathbb{C} \times \Omega$,

$$|\tau_\Omega(\ell)|\, e^{-\phi_\Omega(x)} = \ell,$$

où les *poids locaux* ϕ_Ω appartiennent à $L^1_{\mathrm{loc}}(\Omega)$. Le courant de courbure de L, qui est donné par $c_1(L) = dd^c\phi$ (qui est bien défini en vertu de la condition d'intégrabilité locale) et est en fait indépendant de ϕ, est positif (au sens français) si et seulement si les poids locaux ϕ_Ω sont tous *plurisousharmoniques*, auquel cas l'on dira que la métrique est elle-même plurisousharmonique, et l'on écrira $\phi \in \mathrm{PSH}(X, L)$.

Grâce aux résultats de Demailly, nous pouvons en fait caractériser la classe des métriques psh sur L comme la plus petite classe de métriques singulières sur L qui:

1. est stable par maxima finis;

2. est stable par limites décroissantes de filets;

3. est stable par addition de constantes réelles;

4. contient toutes les métriques de type Fubini-Study, c'est-à-dire les métriques de la forme
$$\phi = \frac{1}{2k} \log \sum_i |s_i|^2 e^{2\lambda_i},$$
où les s_i forment une base de sections d'une puissance kL, sans point-base, et les λ_i sont des constantes réelles.

Supposons maintenant qu'X est une variété sur un corps non-archimédien K. Sur un espace de Berkovich, il n'y a pas de calcul différentiel à proprement parler. Nous ne pouvons donc pas utiliser de caractérisations locales pour définir des métriques plurisousharmoniques sur L^{an}, comme nous l'avons fait dans le cas complexe.

Il faut donc être plus astucieux: nous allons imiter *a posteriori* comme définition d'une métrique psh non-archimédienne, l'énoncé du résultat précédent de Demailly. Le seul ingrédient manquant dans le cas non-archimédien est la notion de métrique Fubini-Study. Afin d'avoir les mêmes propriétés, il convient de remplacer les sommes de carrés par des maxima (un thème récurrent, comme nous le verrons).

Definition 0.0.0.1. Soit K un corps non-archimédien, X une K-variété projective compacte et L un fibré en droites ample sur X. Une métrique ϕ sur L^{an} est dite de Fubini-Study si elle peut s'écrire de la forme

$$\phi = \frac{1}{k} \log \max_i |s_i| e^{\lambda_i},$$

où les s_i forment une base de sections d'une puissance kL, sans point-base, et les λ_i sont des constantes réelles. Nous noterons $\mathrm{FS}(L^{\mathrm{an}})$ cette classe.

Definition 0.0.0.2. Avec les conventions de la précédente définition, une métrique ϕ sur L^{an} est dite plurisousharmonique si elle peut s'écrire comme la limite décroissante d'un filet de métriques Fubini-Study sur L^{an}. Nous noterons $\mathrm{PSH}(L^{\mathrm{an}})$ cette classe.

Il est alors possible de voir qu'est vraie la même caractérisation que dans le cas complexe: la classe des métriques psh sur L^{an} est la plus petite classe de métriques singulières sur L^{an} qui:

1. est stable par maxima finis;

2. est stable par limites décroissantes de filets;

3. est stable par addition de constantes réelles;

4. contient toutes les métriques de type Fubini-Study sur L^{an}.

Métriques d'énergie finie.

Un objet fondamental dans l'étude des métriques psh sur un fibré en droites holomorphe est l'*opérateur de Monge-Ampère*, qui associe à une métrique psh lisse ϕ le produit $(dd^c\phi)^d$. Dans le cas où ϕ n'est plus lisse (ou du moins C^2), $dd^c\phi$ est strictement un courant et non plus une forme différentielle : leur produit n'est plus défini. Le travail remarquable de Bedford-Taylor ([BT]) a permis d'étendre cet opérateur (ainsi que des produits plus généraux de la forme $(dd^c\phi_1) \wedge \cdots \wedge (dd^c\phi_d)$) à la classe des métriques localement bornées. Cela empêche toutefois de considérer des métriques singulières ; cependant, des décennies plus tard, cet opérateur sera étendu par Boucksom-Eyssidieux-Guedj-Zeriahi ([BEGZ]) à la classe $\mathcal{E}^1(X, L)$ de métriques *d'énergie finie*, que nous définissons maintenant.

Posons deux métriques ϕ_0 et ϕ_1, lisses, et définissons leur *énergie de Monge-Ampère* ainsi :

$$E(\phi_0, \phi_1) = (d+1)^{-1} \sum_{i=0}^{d} (\phi_0 - \phi_1)\,(dd^c\phi_0)^i \wedge (dd^c\phi_1)^{d-i}.$$

Il est remarquable qu'ayant fixé la métrique ϕ_1, l'expression

$$\phi \mapsto E(\phi, \phi_1)$$

est décroissante en ϕ. Par le théorème de régularisation de Demailly ([Dem92], [BK07]), toute métrique psh sur L peut être réalisée comme une suite décroissante de métriques psh lisses, et l'on peut définir une extension de E à $\mathrm{PSH}(X, L)$ en posant

$$E(\phi, \phi_1) = \lim_{k\to\infty} E(\phi^k, \phi_1)$$

pour $\phi \in \mathrm{PSH}(X, L)$, et $k \mapsto \phi^k$ une suite de métriques psh lisses sur L décroissant vers ϕ. Notons que cette énergie peut alors prendre la valeur $-\infty$.

Nous pouvons ainsi définir l'espace des métriques plurisousharmoniques d'énergie finie sur L:

$$\mathcal{E}^1(X, L) = \{\phi \in \mathrm{PSH}(X, L),\, E(\phi, \phi_1) > -\infty, \forall\, \phi_1 \in C^\infty \cap \mathrm{PSH}(X, L)\}.$$

Notons que E satisfait la propriété *de cocycle*

$$E(\phi_0, \phi_1) = E(\phi_0, \phi_2) + E(\phi_2, \phi_1),$$

comme il peut se voir sur des métriques lisses, puis par régularisation ; ce qui implique que la classe $\mathcal{E}^1$ est indépendante du choix de métrique de référence, ce qui est mis en évidence par notre notation.

La classe $\mathcal{E}^1$ contient strictement la classe des métriques localement bornées : en particulier, elle contient "beaucoup" de métriques psh singulières. C'est une classe qui est intrinsèquement intéressante, de surcroît grâce à [BEGZ], où l'opérateur de Monge-Ampère y est comme promis étendu.

Il a été observé par Darvas ([Dar17]) que cette classe pouvait être dotée d'une structure d'espace métrique complet, comme suit : l'on définit l'enveloppe "toit" (*rooftop*) de deux métriques ϕ_0, ϕ_1 dans $\mathcal{E}^1(X, L)$ comme la régularisation semi-continue supérieurement

$$P(\phi_0, \phi_1) = \mathrm{usc}\,\sup\{\psi \in \mathrm{PSH}(X, L),\, \psi \leq \phi_0, \phi_1\}.$$

C'est la généralisation de l'enveloppe convexe de deux fonctions convexes ; si les deux métriques sont d'énergie finie, alors leur enveloppe toit est également d'énergie finie, et l'on écrit

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1)).$$

Il est également possible de définir les opérateurs de Monge-Ampère associés à des métriques continues psh non-archimédiennes, par la théorie de l'intersection (suivant Gubler, Boucksom-Eriksson-Favre) ou bien la théorie des formes différentielles sur les espaces de Berkovich (d'après Chambert-Loir et Ducros, se basant sur les *superformes* de Lagerberg).

Cela nous permet de définir l'énergie de Monge-Ampère en imitant exactement le cas complexe, ainsi que la classe

$$\mathcal{E}^1(L^{\mathrm{an}})$$

des métriques psh d'énergie finie sur L^{an}. De la même manière, nous pouvons introduire la distance d_1 (comme fait dans [BJ21], [Reb20b]) sur $\mathcal{E}^1(L^{\mathrm{an}})$, en utilisant les enveloppes psh non-archimédiennes. Nous nous heurtons ici à un premier problème : celui de la *continuité des enveloppes*, c'est-à-dire le fait que l'enveloppe psh d'une métrique continue est elle-même continue. C'est une propriété de la paire (X, L), qui est classique dans le cas complexe si X est (par exemple) normale et L est ample ; dans le cas non-archimédien, cette propriété est partiellement conjecturale sous les mêmes hypothèses. Elle est connue, au moins, dans tous les cas pertinents aux applications à la géométrie complexe (c'est-à-dire dans le cas où le corps de base est le corps des complexes muni de la valeur absolue triviale, ou bien le corps des séries formelles à coefficients complexes). Nous passons en revue les cas connus dans la Section du Chapitre 2 dédiée aux enveloppes psh non-archimédiennes.

Pour conclure ces préliminaires, décrivons brièvement une application récente de la théorie du pluripotentiel non-archimédien à la géométrie complexe, suivant les travaux de Berman, Boucksom, et Jonsson : une approche variationnelle à la conjecture de Yau-Tian-Donaldson.

L'existence de métriques à courbure constante (en divers sens) est un fil rouge de la géométrie différentielle depuis de nombreuses décennies. Un problème particulièrement intéressant est celui de l'existence de métriques Kähler-Einstein sur une variété Kählérienne compacte X, c'est-à-dire une métrique

ϕ lisse et strictement psh sur le fibré canonique K_X, dont la $(1,1)$-forme ω associée est proportionnelle à sa courbure de Ricci $\mathrm{Ric}(\omega)$. L'existence d'une métrique Kähler-Einstein est une condition très forte, et nécessite un substrat topologique adéquat : elle nécessite que la classe $c_1(X)$ soit elle-même signée.

D'après les travaux d'Aubin et Yau, l'on sait que si X est canoniquement polarisée (i.e. K_X est ample), ou Calabi-Yau (K_X est trivial), il existe nécessairement une métrique Kähler-Einstein sur X. Dans le cas canoniquement polarisé, une telle métrique est de surcroît unique.

Le cas Fano (K_X antiample) est notoirement plus difficile.

Inspiré par diverses métaphores mathématiques (la théorie des quotients GIT, plus particulièrement le théorème de Kempf-Ness et ses conséquences en géométrie symplectique ; ainsi que l'alors récent résultat de Donaldson-Uhlenbeck-Yau, démontrant équivalence entre une notion algébrique de stabilité de fibrés vectoriels holomorphes et l'existence de connexions hermitiennes de Yang-Mills sur ceux-ci), Donaldson formule alors une conjecture, par la suite raffinée et désormais connue sous le nom de conjecture de Yau-Tian-Donaldson. Selon celle-ci, l'existence de métriques Kähler-Einstein sur une Fano X équivaudrait à la positivité de certaines quantités purement algébriques, les invariants de Futaki, de toutes les *configurations test* de X, c'est-à-dire des dégénérescences $\mathbb{C}^*$-équivariantes de X. Elle a été prouvée de nombreuses fois, en premier lieu par Chen-Donaldson-Sun ([CDS1], [CDS2], [CDS3]), puis dans [DZ], [CSW], [BBJ], [Zha].

Les travaux de Berman-Boucksom-Jonsson ([BBJ]), dont nous généralisons certaines lignes au cours de cette thèse, en donnent une formulation non-archimédienne ([BBJ, Theorem A, Corollary 5.2]) : l'existence d'une métrique Kähler-Einstein sur X est équivalente à la coercivité d'une fonctionnelle (la fonctionnelle de Ding) sur l'espace des métriques psh non-archimédiennes sur l'analytifié de K_X par rapport à la valeur absolue triviale sur $\mathbb{C}$. Notons que des généralisations de cette conjecture ont par la suite été étudiés, comme l'existence de métriques cscK qui a été récemment réduit par Li ([Li]) à une conjecture purement de pluripotentiel non-archimédien ; et le cas, plus général, des *solitons* a été étudié de manière similaire par Han-Li ([HL20]).

Résumé des résultats obtenus.

Quantification de métriques non-Archimédiennes par des espaces de normes.

Depuis les travaux de Bouche-Tian-Catlin-Zelditch, il est connu que l'on peut approximer uniformément une métrique continue psh ϕ (complexe) par des métriques Fubini-Study ϕ_k associées à ϕ.

Plus précisément, on peut construire pour tout k une norme hermitienne ζ_k sur l'espace des sections $H^0(kL + K_X)$, donnée par

$$\zeta_k(s)^2 = \int_X |s|^2 e^{-k\phi} :$$

en effet, $|s|^2 e^{-k\phi}$ définit bien une mesure sur X. Si l'on prend une base $(s_j)_j$ de cet espace de sections, qui est orthonormale pour ζ_k, on définit son noyau de Bergman

$$B_{\phi,k} := \sum_j |s_j|^2.$$

Ce noyau est en fait indépendant du choix d'une telle base, et la métrique

$$\phi_k = \frac{1}{k} \log B_{\phi_k}$$

est une métrique Fubini-Study. Nous avons alors que les ϕ_k convergent uniformément vers la métrique d'origine, ϕ.

Le but de l'article [Reb20a] est d'étudier des problèmes de nature similaire, dans le cas non-archimédien. Nous allons en fait, à l'aide d'une procédure de quantification comme au-dessus, caractériser une classe de métriques psh non-archimédiennes à l'aide de données purement algébriques. Nous rendons cette promesse plus précise sous peu.

Fixons un corps K non-archimédien et non trivialement valué. Au lieu d'admettre comme donnée initiale une métrique continue sur L^{an}, nous considérons plutôt une suite $\zeta_\bullet = (\zeta_k)_k$ de normes (ultramétriques) sur chaque pièce graduée $H^0(kL)$ de l'algèbre des sections de L, satisfaisant à une condition de compatibilité avec l'opération de multiplication des sections : notre suite de normes doit être *sous-multiplicative*, c'est-à-dire que pour toute paire de sections $s_k \in H^0(kL)$, $s_\ell \in H^0(\ell L)$, nous avons

$$\zeta_{k+\ell}(s_k \cdot s_\ell) \leq \zeta_k(s_k) \cdot \zeta_\ell(s_\ell).$$

A chacune de ces normes nous associons une métrique de Fubini-Study

$$\mathrm{FS}_k(\zeta_k) = \frac{1}{k} \log \sup_{s \in H^0(kL)} |s|/\zeta_k(s).$$

Si la norme graduée vérifie une condition de croissance (que nous n'expliciterons pas ici), auquel cas nous parlerons de norme graduée *bornée* (éventuellement sur L), la condition de sous-multiplicativité implique via le lemme de Fekete que la limite

$$\lim_k \mathrm{FS}_k(\zeta_k)$$

existe. Sa régularisation semi-continue supérieurement est alors une métrique psh non-archimédienne, que nous noterons

$$\mathrm{FS}_\bullet(\zeta_\bullet) = \mathrm{usc}\,(\lim_k \mathrm{FS}_k(\zeta_k)).$$

L'opérateur $\mathrm{FS}_\bullet$, que nous appellerons opérateur de Fubini-Study asymptotique, n'est pas injectif sur l'espace des normes graduées bornées sur L. Cependant, il devient injectif si l'on prend le quotient cet espace par une relation d'équivalence algébrique, que l'on décrit maintenant.

Tout espace de normes $\mathcal{N}(V)$ sur un K-espace vectoriel V de dimension finie peut être muni d'une distance d_1, modélisée sur la distance d_1 dans les espaces euclidiens. Si L possède de bonnes propriétés de positivité, alors la limite des distances $d_{1,k}$ sur les $\mathcal{N}(H^0(kL))$ "converge" en une semi-distance sur l'espace des normes graduées bornées sur L, c'est-à-dire que pour deux telles normes graduées $\zeta_\bullet$, $\zeta'_\bullet$, la limite

$$d_1(\zeta_\bullet, \zeta'_\bullet) := \lim_k d_{1,k}(\zeta_k, \zeta'_k)$$

existe, et satisfait l'inégalité triangulaire et la symétrie. Notons pour simplifier $\hat{\mathcal{N}}(L)$ l'espace quotient pour la relation

$$d_1(\zeta_\bullet, \zeta'_\bullet) = 0.$$

Nous prouvons alors au Chapitre 3 :

Theorem 0.0.0.3. *L'opérateur de Fubini-Study asymptotique réalise un bijection entre son image* $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ *et l'espace* $\hat{\mathcal{N}}(L)$*, qui est de surcroît une isométrie.*

Ici, les distances considérées sont la distance de Darvas héritée de $\mathcal{E}^1(L^{\mathrm{an}}) \supset \mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ à gauche, et la distance quotient de d_1 à droite. L'image $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ peut être caractérisée comme l'ensemble des métriques psh sur L^{an} *approchables par en-dessous*, soit les métriques qui sont limites *croissantes* de filets de métriques psh.

Ce résultat est la version non-trivialement valuée d'un résultat de Boucksom-Jonsson dans [BJ18a]. Nous nous appuyons fondamentalement sur l'article [CM15], en en utilisant les résultats concernant l'asymptotique de fonctions superadditives sur les corps d'Okounkov. Nous nous basons également sur le formalisme développé dans [BE].

Géodésiques et segments maximaux dans les espaces de métriques non-Archimédiennes.

Dans [Reb20b], ici le Chapitre 4, nous nous intéressons à la transposition dans le monde non-Archimédien d'objets essentiels en géométrie complexe : les géodésiques dans les espaces de métriques plurisousharmoniques.

Commençons par décrire la situation dans le monde complexe pour un instant. C'est encore une fois la recherche de métriques Kähler-Einstein qui est à l'origine de l'intérêt de la construction de géodésiques dans l'espace $\mathcal{H}(L)$ des métriques lisses et strictement psh sur L. L'équation Kähler-Einstein admet une certaine fonctionnelle, l'énergie K de Mabuchi, comme fonctionnelle d'Euler-Lagrange, c'est-à-dire que les points critiques de cette fonctionnelle sont les solutions de l'équation en question. Afin de poursuivre la stratégie variationnelle, il serait donc désirable que K soit convexe le long de certains rayons distingués de métriques dans $\mathcal{H}(L)$. Les candidats idéaux sont les droites affines dans $\mathcal{H}(L)$ (qui est un espace convexe), mais ceux-ci malheureusement n'ont pas cette propriété. Il se trouve que K est convexe pour une autre structure riemannienne sur $\mathcal{H}(L)$, découverte par Mabuchi ([Mab]), et dont les géodésiques sont données par une certaine équation de Monge-Ampère (d'après Semmes ([Sem]), Donaldson ([Don99])). X.X. Chen ([CX00]) a le premier montré l'existence de telles géodésiques dans $\mathcal{H}(L)$.

Ces géodésiques sont caractérisées par leur *maximalité*: fixons deux métriques lisses et strictement psh ϕ_0, ϕ_1 sur L. Nous noterons $[0,1] \ni t \mapsto \phi_t$ la géodésique de Mabuchi entre ces deux métriques. Il est possible de la voir comme une métrique psh, invariante par rotation, sur le produit $L \times A \to X \times A$, où A est l'anneau

$$A = \{e^{-1} \leq |z| \leq 1\}$$

en posant $t = -\log |z|$. Alors, pour toute métrique psh $A \ni z \mapsto \psi_z$ invariante par rotation sur $L \times A$, que l'on peut également voir comme un segment psh $t \mapsto \psi_t$, si l'on a $\psi_0 \leq \phi_0$ et $\psi_1 \leq \phi_1$, alors pour tout t dans $[0,1]$,

$$\psi_t \leq \phi_t.$$

Cette propriété caractérise les segments géodésiques de Mabuchi. Nous pouvons donc les réaliser comme l'enveloppe

$$\sup\{\psi \text{ segment psh}, \psi_0 \leq \phi_0, \psi_1 \leq \phi_1\}.$$

Il se trouve que cette définition a du sens plus généralement si les métriques au bord ne sont plus continues : c'est l'approche utilisée par Darvas pour parler de géodésiques faibles dans $\mathcal{E}^1(L)$. Notons qu'une caractérisation alternative est que ϕ est un segment psh le long duquel l'énergie de Monge-Ampère est affine.

Supposons maintenant que (X, L) est une variété compacte polarisée sur un corps K non-archimédien. Evidemment, il n'est plus possible de définir une notion de géodésique dans $\mathrm{PSH}(L^{\mathrm{an}})$ au sens riemannien. Cependant, nous pouvons définir une classe de segments Fubini-Study assez naturelle, qui sont des maxima finis de segments de la forme

$$[0,1] \ni t \mapsto \frac{1}{k} \max_i (\log |s_i| + t\lambda_i + (1-t)\lambda'_i),$$

où les $(s_i)_i$ forment encore une base de sections sans point-base d'une puissance kL, et les λ_i, λ'_i sont des constantes réelles. Ensuite, nous définissons un segment psh comme une limite décroissante de segments Fubini-Study.

En s'inspirant de la caractérisation extrémale du côté complexe, nous développons dans [Reb20b] (ici le Chapitre 4) une théorie des géodésiques dans les espaces de métriques d'énergie finie non-archimédiennes. On traite d'abord le cas de géodésiques entre deux métriques continues, où l'on définit les géodésiques comme un supremum, et où l'on montre via une transformée de Legendre que celles-ci restent également continues en temps et en espace.

On montre ensuite, en adaptant l'approche quantifiée de Darvas-Lu-Rubinstein à notre contexte, que l'énergie est affine le long de ces géodésiques, et qu'elles satisfont bien l'équation géodésique. Du côté algébrique, on obtient de manière équivalente des géodésiques au sens fort entre des normes graduées bornées.

Enfin, on étend la construction au cas de deux métriques ϕ_0, ϕ_1 générales d'énergie finie, en les approchant par des suites décroissantes de métriques continues ϕ_0^k, ϕ_1^k, et en définissant la géodésique entre ϕ_0 et ϕ_1 comme la limite (nécessairement décroissante, par la caractérisation extrémale mentionnée ci-dessus) des géodésiques $t \mapsto \phi_t^k$. Afin de montrer que cette limite existe et est unique, on développe certaines propriétés de l'espace $\mathcal{E}^1$ non-archimédien. Résumons nos résultats en un énoncé compact :

Theorem 0.0.0.4. *Etant donné $\phi_0, \phi_1 \in \mathcal{E}^1(L)$, nous posons*

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

. Alors,

1. *$(\mathcal{E}^1(L), d_1)$ est un espace métrique;*
2. *il existe un segment psh maximal $t \mapsto \phi_t$ joignant ϕ_0 et ϕ_1;*
3. *$\phi_t \in \mathcal{E}^1(L)$ pour tout $t \in [0, 1]$;*
4. *le segment ϕ_t est une géodésique (au sens métrique) pour la distance d_1, c'est-à-dire qu'il existe $c \geq 0$ satisfaisant*

   $$d_1(\phi_t, \phi_s) = c \cdot |t - s|$$

   pour tous $t, s \in [0, 1]$;
5. *l'énergie de Monge-Ampère est affine le long du segment ϕ_t, et ce segment est l'unique segment psh joignant ϕ_0 et ϕ_1 satisfaisant cette propriété.*

Si de surcroît les métriques ϕ_0 et ϕ_1 sont continues, alors le segment psh maximal les joignant est également continu, en temps et espace.

Espaces de métriques sur des dégénérescences de variétés complexes.

De nombreux travaux au cours de la précédente décennie, dus notamment à Berman-Boucksom-Favre-Hisamoto-Jonsson et al. ont été dédiés à l'étude de limites non-archimédiennes de rayons géodésiques de métriques plurisousharmoniques. Plus particulièrement, étant donné une variété projective compacte X munie d'un fibré en droites ample L, un rayon géodésique

$$(0, \infty] \ni t \mapsto \phi_t \in \mathrm{PSH}(X, L)$$

est identifié à une métrique psh Φ $\mathbb{S}^1$-invariante sur le tiré en arrière de L au produit trivial $\overline{\mathbb{D}}^* \times X$, où la variable z donnée par la première projection est identifiée à notre t via $t = -\log |z|$. La limite en $t \to \infty$ définit, via l'étude des singularités de Φ, une métrique non-Archimédienne sur l'analytifié de X par rapport à la valeur absolue triviale sur $\mathbb{C}$. Cette approche a culminé en la preuve variationnelle de la conjecture de Yau-Tian-Donaldson dans [BBJ].

Le but de mon article [Reb21], ici le Chapitre 5, est de pousser cette étude au cas plus général d'une dégénérescence arbitraire $\pi : X \to \overline{\mathbb{D}}^*$ de variétés projectives complexes, pas nécessairement $\mathbb{S}^1$-invariante ni même isotriviale, munie d'un fibré en droits relativement ample L. Il est toujours possible de réaliser X comme une variété algébrique sur le corps $\mathbb{C}((t))$, et un modèle analytique $\mathcal{X}$ (i.e. un espace analytique possiblement singulier relatif à $\overline{\mathbb{D}}$, isomorphe à X en-dehors de sa fibre centrale) avec un modèle algébrique, défini sur l'anneau de valuation $\mathbb{C}[[t]]$.

On définit premièrement une notion de métrique psh adaptée à ce contexte : on requiert que nos métriques admettent une extension psh à un certain modèle analytique $(\mathcal{X}, \mathcal{L})$ de (X, L). On notera $\mathrm{PSH}(L)$ cet espace. En parallèle avec les travaux de Berman-Darvas-Lu ([BDL]), qui définissent une distance d_1 radiale sur l'espace des rayons géodésiques d'énergie finie via

$$d_1(\phi, \psi) = \lim_t \frac{d_1(\phi_t, \psi_t)}{t}$$

(les d_1 à droite sont les distances d_1 "classiques" introduites par Darvas dans les espaces de métriques psh sur un fibré fixé), on montre que prendre un certain nombre de Lelong de l'application définie sur le disque $\overline{\mathbb{D}}^*$ par les distances d_1 fibre-à-fibre définit bien une distance sur la sous-classe de métriques relativement maximales (c'est-à-dire maximales sur toute préimage d'un ouvert relativement compact dans le disque épointé) et d'énergie finie de PSH(L). Notons $\hat{\mathcal{E}}^1(L)$ cette classe de métriques, et $\hat{d}_1$ cette distance.

Theorem 0.0.0.5. *L'espace $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ est un espace métrique géodésique et complet. Il admet de surcroît des segments $\hat{d}_1$-geodésiques maximaux parmi tous les segments géodésiques, qui peuvent être construits comme familles de segments géodésiques fibre à fibre.*

Ensuite, à toute métrique ϕ^{NA} d'énergie finie dans PSH(L), on associe une métrique non-Archimédienne sur l'analytifié L^{an} de L vu comme une variété sur $\mathbb{C}((t))$. Sur un point divisoriel de X^{an}, qui est associé à un diviseur D dans la fibre centrale d'un modèle de X, la valeur de ϕ^{NA} correspond au nombre de Lelong générique de ϕ le long de D. Elle est étendue de manière unique par une propriété générale des fonctions psh non-Archimédiennes: elles sont définies de manière unique par leurs valeurs aux points divisoriels.

On étudie ensuite l'évolution de l'énergie de Monge-Ampère le long d'une métrique $\phi \in \hat{\mathcal{E}}^1(L)$, particulièrement quand $z \to 0$. Plus précisément, on étudie la métrique
$$E(\phi) : z \mapsto \langle \phi_z^{d+1} \rangle = E(\phi_z)$$
sur le fibré $\langle L \rangle$ sur le disque épointé, qui peut être vue comme une famille d'énergies de Monge-Ampère intrinsèques. On montre que, dans le cas où ϕ s'étend de manière localement bornée sur un modèle analytique de (X, L), on a :
$$(E(\phi))^{\mathrm{NA}} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$
ce qui peut être interprété comme disant que le nombre de Lelong des énergies fibre à fibre coïncide avec l'énergie de Monge-Ampère non-archimédienne de ϕ^{NA}. Cela généralise le cas classique, puisqu'un nombre de Lelong n'est qu'une pente à l'infini généralisée.

Cette égalité n'est pas vraie dans le cas général. La classe de métriques de $\hat{\mathcal{E}}^1(L)$ satisfaisant à
$$(E(\phi))^{\mathrm{NA}} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$
est définie comme la classe des métriques hybridement maximales $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$, et généralise la classe des rayons géodésiques maximaux de [BBJ]. On obtient une caractérisation extrémale de ces métriques, et on montre que cette

classe est isométrique à la classe des métriques non-Archimédiennes $\mathcal{E}^1(L^{\mathrm{an}})$. On obtient comme conséquence un résultat heuristique sur la convexité de fonctionnelles non-Archimédiennes.

Theorem 0.0.0.6.
- *Il y a un plongement isométrique de* $(\mathcal{E}^1(L^{\mathrm{an}}), d_1^{\mathrm{NA}})$ *dans* $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ *avec pour image* $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$.

- *Un segment psh dans* $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ *est une géodésique psh si et seulement si son image dans* $\mathcal{E}^1(L^{\mathrm{an}}_{\mathrm{K}})$ *est une géodésique psh non-archimédienne au sens du Chapitre 4.*

- *L'on dispose d'une propriété générale d' "extension de plurifonctionnelles" : étant donné* $d+1$ *fibrés relativement amples* L_i *sur* X*, pour tout* $(d+1)$*-uplet de métriques* $\phi_i \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L_i)$*, on a*

$$(\langle \phi_0, \ldots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle \phi_0^{\mathrm{NA}}, \ldots, \phi_d^{\mathrm{NA}} \rangle.$$

Plan du manuscrit.

Au Chapitre 1, nous développons les préliminaires purement algébriques nécessaires à cette thèse. Nous commençons par des généralités sur les corps non-archimédiens. Ensuite, ayant fixé un tel corps K, nous étudions les propriétés métriques et spectrales des espaces de normes sur les K-espaces vectoriels de dimension finie. Enfin, nous étendons cette étude aux espaces de normes graduées sur des K-algèbres graduées engendrées en degré un. Cette dernière partie contient quelques résultats de [Reb20b], ainsi qu'un résultat non publié sur la complétude d'un certain espace de normes graduées pour une distance de type d_∞.

Au Chapitre 2, nous nous penchons sur les aspects géométriques des préliminaires. Après avoir brièvement rappelé la construction de Berkovich, nous nous penchons sur plusieurs types de métriques sur des analytifications de fibrés en K-droites. Nous expliquons également les bases de la théorie du pluripotentiel non-archimédien.

Les Chapitres 3, 4, et 5 consistent essentiellement des résultats de (respectivement) [Reb20a], [Reb20b], et [Reb21], comme expliqué précédemment.

Introduction.

The main subject of this thesis is non-Archimedean pluripotential theory, in which the role of central object is played by non-Archimedean (pluri)subharmonic metrics. Naturally, we will explain those terms in due time. We shall develop this theory along two axes.

We shall first study it in and for itself: the foundations are recent ([BFJ15], [BJ21], [BE]), and many aspects remain mysterious, especially in the case of nontrivial valuation. It seems therefore appropriate to try and transpose various concepts and results from the complex to the non-Archimedean world. This is often more delicate than it seems. In many cases, the behaviour of some objects is completely different from the complex case, opening interesting directions of research.

Secondly, we will study non-Archimedean pluripotential theory through its applications to complex geometry. Indeed, in certain cases, non-Archimedean geometry is a very efficient way to encode singular or asymptotic behaviours of complex objects. For this approach, we will specialize to the case where the base field is either $\mathbb{C}$ together with its trivial absolute value, or the field $\mathbb{C}((t))$ of complex Laurent series together with its t-adic absolute value.

From complex geometry to non-Archimedean geometry.

The GAGA principle for varieties over a non-Archimedean field.

Let X be a projective algebraic variety over the field $\mathbb{C}$. The now-famous GAGA principle of Serre ([SerGAGA]) asserts that one can associate to such X a projective complex manifold, X^{an}, preserving its essential topological properties (such as connectedness, separatedness, and compactness), as well as data given by coherent sheaves (in the sense that there is an equivalence of categories between the categories of coherent sheaves on X and X^{an}). This allows one to use powerful analytic methods coming from complex analysis and differential geometry in order to study problems of algebraic nature.

Consider now a projective algebraic variety X over a non-Archimedean field K, i.e. a field together with an absolute value $|\cdot|$, which is furthermore complete with respect to the topology induced by the latter, and satisfying the *ultrametric inequality* refining the triangle inequality: for all x, y in K,

$$|x+y| \leq \max(|x|, |y|).$$

Fundamental examples of such fields include:

1. $\mathbb{C}((t))$, the field of Laurent series with coefficients in $\mathbb{C}$, together with the t-adic absolute value;

2. given p prime, $\mathbb{Q}_p$, the field of p-adics, together with the p-adic absolute value;

3. given p prime, $\mathbb{F}_p((t))$, the field of Laurent series with coefficients in the finite field $\mathbb{F}_p$, with the t-adic absolute value;

4. any field K with the trivial absolute value, equal to 1 on $\mathrm{K}^\times$, which is non-Archimedean.

Such examples shed some light on the diversity of the non-Archimedean world. Uses for such fields are many: p-adic fields are ubiquitous in number theory (note the Hasse-Minkowski principle, Hensel's Lemma, Mahler's Theorem; as well as applications to p-adic differential equations and p-adic potential theory); and their *perfectoid* extensions have recently been at the heart of a very rich theory developed by Scholze and others ([Scholze], [FF]). The field $\mathbb{F}_p((t))$ is used to define formal neighbourhoods of arithmetic curves. The class of varieties over $\mathbb{C}((t))$, as we shall explain in much detail over the course of this manuscript, contains as a particular case degenerations of complex manifolds. Finally, the case of trivial valuation holds much geometric importance, in spite of its name; we will work on this side of the story as well.

It therefore seems appropriate to have at hand analytic tools the likes of which are available in the complex world. Many difficulties arise shortly. First of all, the ultrametric inequality has notable consequences on the topology of K, for it is totally discontinuous (i.e. its only connected sets are the null sets and singletons). One also needs to define a notion of an analytic space (and also of analytic functions) over K. Naively, one would mimick the complex case, by defining an analytic function on K^d to be a converging power series with coefficients in K; then, to define analytic spaces via gluing. The result is rather catastrophic: Serre ([Ser65]) shows that any compact analytic space in this sense is a disjoint union of unit balls; a notably poor candidate for the analytification of a K-algebraic variety!

As Berkovich in his time, we then notice that our putative analytification X^{an} of X needs "additional points". Although we will not explain the exact procedure here, we shall describe it briefly: X^{an} is roughly identified with the compactification of valuations of geometric origin on $\mathrm{K}(X)$. The points of utmost importance are as follows:

1. the analytification preserves topological properties such as connectedness, separatedness, and compactness;

2. it realizes an equivalence of categories between the categories of coherent sheaves on X and X^{an};

3. it is also defined for varieties over Archimedean fields - in particular $\mathbb{C}$, where it coincides with the usual analytification in the sense of Serre.

In particular, if the base field K is trivially valued, then X^{an} corresponds to the space of valuations on fields of fractions $\mathrm{K}(Y)$ for all irreducible subvarieties $Y \subset X$, endowed with the topology of pointwise convergence.

Geometric pluripotential theory.

Given a domain $\Omega \subset \mathbb{C}^d$, a smooth (or twice differentiable) function $f : \Omega \to \mathbb{R}$ is plurisubharmonic if its complex Hessian is a positive matrix. One can see this definition to generalize the notion of a convex (smooth or twice differentiable) function, and indeed plurisubharmonic functions share many properties with convex functions.

More generally, an upper semicontinuous and locally integrable real-valued function f on Ω is plurisubharmonic if it can be realized as a decreasing limit of a net of smooth plurisubharmonic metrics ; or equivalently, if its complex Hessian in the sense of distributions defines a positive measure.

The definition of such functions naturally extends to complex varieties, and to line bundles on them. Let us begin by fixing a complex compact projective manifold X, of dimension d, endowed with an ample line bundle L. We recall that L is ample if and only if, for all large enough integers k, the k-fold tensor product $L^{\otimes k}$ (denoted additively kL) admits a basis of sections $(s_i)_{i=1}^{\dim H^0(X,kL)}$ such that the map

$$x \mapsto [s_1(x), \ldots, s_{\dim H^0(X,kL)}(x)]$$

defines an embedding in projective space $\mathbb{P}^{\dim H^0(X,kL)-1}$.

Let τ_Ω be a trivialization $L|_\Omega \simeq \mathbb{C} \times \Omega$ of L. A singular metric on L is a metric ϕ on L which can be written, in all such trivializations and for all $(\ell, x) \in \mathbb{C} \times \Omega$, as

$$|\tau_\Omega(\ell)|\, e^{-\phi_\Omega(x)} = \ell,$$

where the *local weights* ϕ_Ω belong to $L^1_{\mathrm{loc}}(\Omega)$. The curvature current of L, given by $c_1(L) = dd^c\phi$ (which one notices to be well-defined due to the local integrability condition) turns out to be independent of ϕ, and is (frenchly) positive if and only if the local weights ϕ_Ω are all *plurisubharmonic*, in which case we shall say that ϕ is itself plurisubharmonic, and we shall write $\phi \in \mathrm{PSH}(X, L)$.

Due to results of Demailly, we may in fact characterize the class of plurisubharmonic (psh) metrics on L as the smallest class of singular metrics on L which:

1. is closed under finite maxima;

2. is closed under decreasing limits of nets;
3. is closed under addition of real constants;
4. contains all Fubini-Study metrics, i.e. metrics written as

$$\phi = \frac{1}{2k} \log \sum_i |s_i|^2 e^{2\lambda_i},$$

   where the s_i are a basepoint-free basis of sections of kL, and the λ_i are real constants.

Let us now assume X to be a variety over a non-Archimedean field K. On a Berkovich space, there is no notion of differential calculus. We thus cannot use local characterizations as before to define plurisubharmonic functions on X^{an} and metrics on L^{an}, as is done in the complex case.

We will thus mimick *a posteriori* the statement of Demailly's result to take as our definition of a non-Archimedean psh metric. The only missing ingredient remains the notion of a Fubini-Study metric. As we will notice to be a recurrent theme, we simply "replace sums of squares with maxima".

Definition 0.0.0.7. Let K be a non-Archimedean field, X a compact projective K-variety, and L an ample line bundle on X. A metric ϕ on L^{an} is Fubini-Study provided one can write it as

$$\phi = \frac{1}{k} \log \max_i |s_i| e^{\lambda_i},$$

where the s_i are a basepoint-free basis of some power kL, and the λ_i are real constants. We will denote this class of metrics by $\mathrm{FS}(L^{\mathrm{an}})$.

Definition 0.0.0.8. With the conventions of the previous Definition, a metric ϕ on L^{an} is plurisubharmonic if it can be written as the decreasing limit of a net of Fubini-Study metrics on L^{an}. We will denote this class of metrics by $\mathrm{PSH}(L^{\mathrm{an}})$.

One can then see that the same characterization of the class of psh metrics holds as in the complex case: it is the smallest class of singular metrics on L^{an} which:

1. is closed under finite maxima;
2. is closed under decreasing limits of nets;
3. is closed under addition of real constants;
4. contains all Fubini-Study metrics on L^{an}.

Finite-energy metrics.

A fundamental object in the study of psh metrics on a holomorphic line bundle is the *Monge-Ampère operator*, associating to a smooth metric ϕ the product $(dd^c\phi)^d$. In the case where ϕ is no longer smooth (or twice differentiable), $dd^c\phi$ is strictly a current and no longer a differential form: such a product is therefore no longer defined. The remarkable work of Bedford-Taylor ([BT]) allowed to extend this operator (as well as more general products of the form $(dd^c\phi_1)\wedge\cdots\wedge(dd^c\phi_d)$) to the class of locally bounded metrics. This still prevents us from considering possibly singular metrics; however, decades later, this operator will be extended by the work of Boucksom-Eyssidieux-Guedj-Zeriahi ([BEGZ]) to the class $\mathcal{E}^1(X,L)$ of *finite-energy* metrics, which we describe now.

Let us fix two smooth psh metrics ϕ_0 and ϕ_1, and let us define their *Monge-Ampère energy* as follows:

$$E(\phi_0,\phi_1) = (d+1)^{-1}\sum_{i=0}^{d}(\phi_0-\phi_1)\,(dd^c\phi_0)^i\wedge(dd^c\phi_1)^{d-i}.$$

Remarkably, having fixed the right-hand metric ϕ_1, the expression

$$\phi\mapsto E(\phi,\phi_1)$$

is decreasing in ϕ. By the Demailly regularization Theorem ([Dem92], [BK07]), any psh metric on L can be realized as a decreasing limit of smooth psh metrics, so that one can extend E to $\mathrm{PSH}(X,L)$ by setting

$$E(\phi,\phi_1) = \lim_{k\to\infty} E(\phi^k,\phi_1)$$

for $\phi\in\mathrm{PSH}(X,L)$, and $k\mapsto\phi^k$ a sequence of smooth psh metrics on L decreasing to ϕ. Let us note that this energy can possibly take the value $-\infty$.

We therefore define the space of finite-energy metrics on L:

$$\mathcal{E}^1(X, L) = \{\phi \in \mathrm{PSH}(X, L),\ E(\phi, \phi_1) > -\infty, \forall\, \phi_1 \in C^\infty \cap \mathrm{PSH}(X, L)\}.$$

Note that E satisfies the *cocycle* property:

$$E(\phi_0, \phi_1) = E(\phi_0, \phi_2) + E(\phi_2, \phi_1),$$

as one can see on smooth metrics, then by regularization; this implies the class $\mathcal{E}^1$ to be independent of the choice of a reference metric.

Darvas ([Dar17]) later observed that this class could be endowed with the structure of a complete metric space, as follows: we define the "rooftop" envelope of two metrics ϕ_0, ϕ_1 in $\mathcal{E}^1(X, L)$ as the usc regularization

$$P(\phi_0, \phi_1) = \text{usc}\ \sup\{\psi \in \mathrm{PSH}(X, L),\ \psi \leq \phi_0, \phi_1\}.$$

This generalizes the convex envelope of two convex functions; if both metrics have finite energy, then their envelope also has finite energy, and we write

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1)).$$

One may also define Monge-Ampère operators associated to continuous non-Archimedean psh metrics, using either intersection theory (Gubler, Boucksom-Eriksson-Favre) or differential forms on Berkovich spaces (Chambert-Loir and Ducros, building on the *superforms* of Lagerberg).

By mimicking exactly the definition in the complex case, we may then define the Monge-Ampère energy as well as the class

$$\mathcal{E}^1(L^{\mathrm{an}})$$

of finite-energy psh metrics on L^{an}. Similarly, we may introduce the distance d_1 (as in [BJ21], [Reb20b]) on $\mathcal{E}^1(L^{\mathrm{an}})$, using non-Archimedean psh envelopes. We run into a first problem: that of continuity of envelopes, that is: the fact that the psh envelope of a continuous metric is itself continuous. This is a property of the pair (X, L), classically true in the complex case if X is (say) normal and L is ample; in the non-Archimedean case, this property is partially conjectural, under similar hypotheses. It is currently known in all cases useful for applications to complex geometry, i.e. when the base field is either $\mathbb{C}$ with its trivial absolute value, of $\mathbb{C}((t))$ with its natural absolute value. We review the currently known cases in the Section in Chapter 2 dedicated to non-Archimedean psh envelopes.

Concluding our preliminaries, we briefly describe a recent application of non-Archimedean pluripotential theory to complex geometry, after Berman, Boucksom, and Jonsson: a variational approach to the Yau-Tian-Donaldson conjecture.

The existence of constant curvature metrics (in all possible flavours) has been a recurring theme in differential geometry for many decades. A problem of particular interest is that of existence of Kähler-Einstein metrics on a compact Kähler manifold X, i.e. a smooth strictly psh metric ϕ on the canonical bundle K_X, whose associated $(1,1)$-form ω is proportional to its Ricci curvature $\mathrm{Ric}(\omega)$. The existence of such a metric is a very strong condition, subordinate to topological conditions: the class $c_1(X)$ itself has to be signed.

By the works of Aubin and Yau, it is known that if X is canonically polarized (with ample canonical bundle) or Calabi-Yau (with trivial canonical bundle), there exists a Kähler-Einstein metric on X, which is furthermore unique in the former case.

The Fano case (with antiample canonical bundle) is notoriously more difficult.

Inspired by various ideas (from GIT quotient theory, in particular the Kempf-Ness Theorem and its consequences in symplectic geometry; as well as the then recent result of Donaldson-Uhlenbeck-Yau proving equivalence between an algebraic notion of stability for holomorphic vector bundles and the existence of Hermitian Yang-Mills connections on them), Donaldson formulates a conjecture, further and further refined, known by the name of Yau-Tian-Donaldson conjecture, according to which existence of a Kähler-Einstein metric on X is equivalent to the positivity of some purely algebraic quantities (so-called Futaki invariants). It has now been proven several times, first by Chen-Donaldson-Sun ([CDS1], [CDS2], [CDS3]), then in [DZ], [CSW], [BBJ], [Zha].

The works of Berman-Boucksom-Jonsson ([BBJ]), some parts of which we generalize over the course of this thesis, give a non-Archimedean formulation of the aforementioned conjecture ([BBJ, Theorem A, Corollary 5.2]): existence of a Kähler-Einstein metric on X is equivalent to coercivity of some functional (the Ding functional) on the space of non-Archimedean psh metrics on K_X^{an}, the analytification of K_X with respect to the trivial absolute value on $\mathbb{C}$. Generalizations of this conjecture have also been studied, most fundamentally the existence of cscK metrics which has recently been reduced by Li ([Li]) to a conjecture purely in the domain of non-Archimedean pluripotential theory; and the more general case of *solitons* has similarly been studied by Han-Li ([HL20]).

Summary of the main results.

Quantization of non-Archimedean metrics via spaces of norms.

Since the works of Bouche-Tian-Catlin-Zelditch, it is known that a (complex) continuous psh metric ϕ can uniformly be approximated by Fubini-Study metrics ϕ_k associated to ϕ.

Precisely, one can construct for all k a Hermitian norm ζ_k on the space of sections $H^0(kL + K_X)$, given by

$$\zeta_k(s)^2 = \int_X |s|^2 e^{-k\phi} :$$

indeed, $|s|^2 e^{-k\phi}$ does define a measure on X. Picking a basis $(s_j)_j$ of this space of sections, orthonormal for ζ_k, one defines its Bergman kernel

$$B_{\phi,k} := \sum_j |s_j|^2.$$

This kernel is in fact independent of the choice of such a basis, and the metric

$$\phi_k = \frac{1}{k} \log B_{\phi_k}$$

is Fubini-Study. We then have that the ϕ_k converge uniformly to the original metric, ϕ.

The purpose of the article [Reb20a] is to study problems of similar nature in the non-Archimedean setting. Using a "quantization" procedure as above, we will also characterize a class of non-Archimedean psh metrics using purely algebraic data.

Let us fix a non-Archimedean, nontrivially valued field K. Instead of having as initial data a continuous metric on L^{an}, we shall rather consider a sequence $\zeta_\bullet = (\zeta_k)_k$ of ultrametric norms on each graded piece $H^0(kL)$ of the section algebra of L, further satisfying a compatibility condition with respect to multiplication of sections: it must be *submultiplicative*, i.e. for any two sections $s_k \in H^0(kL)$, $s_\ell \in H^0(\ell L)$, we have that

$$\zeta_{k+\ell}(s_k \cdot s_\ell) \leq \zeta_k(s_k) \cdot \zeta_\ell(s_\ell).$$

To each such norm we associate a Fubini-Study metric by setting

$$\mathrm{FS}_k(\zeta_k) = \frac{1}{k}\log \sup_{s\in H^0(kL)} |s|/\zeta_k(s).$$

Assuming our graded norm to satisfy an additional growth condition (which we shall not make explicit here, but roughly ensuring that it does not "blow up"), in which case we shall speak of a *bounded* graded norm, the submultiplicativity condition implies via Fekete's Lemma that the limit

$$\lim_k \mathrm{FS}_k(\zeta_k)$$

exists. Its usc regularization is then a non-Archimedean psh metric which we denote

$$\mathrm{FS}_\bullet(\zeta_\bullet) = \mathrm{usc}\,(\lim_k \mathrm{FS}_k(\zeta_k)).$$

The operator $\mathrm{FS}_\bullet$ so defined, which we will call the asymptotic Fubini-Study operator, is not in general injective on the space of bounded graded norms on L. It however becomes injective by taking the quotient of this space under some algebraic equivalence relation, which we describe now.

Any space of norms $\mathcal{N}(V)$ on a K-vector space v of finite dimension can be endowed with a distance, d_1, modelled on the distance d_1 in Euclidean spaces. If L has sufficient positivity properties, the limit of the distances $d_{1,k}$ on each $\mathcal{N}(H^0(kL))$ "converges" to a pseudodistance on the space of bounded graded norms on L, in the sense that for any two such graded norms $\zeta_\bullet$, $\zeta'_\bullet$, the limit

$$d_1(\zeta_\bullet, \zeta'_\bullet) := \lim_k d_{1,k}(\zeta_k, \zeta'_k)$$

exists, and satisfies the symmetry property as well as the triangle inequality. Let us write for clarity $\hat{\mathcal{N}}(L)$ for the quotient space under the relation

$$d_1(\zeta_\bullet, \zeta'_\bullet) = 0.$$

In Chapter 3, we prove the following:

Theorem 0.0.0.9. *The asymptotic Fubini-Study operator realizes a bijection between its image* $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ *and the space* $\hat{\mathcal{N}}(L)$*, which is furthermore an isometry.*

Here, the distances in question in the isometry statement are the Darvas distance inherited from $\mathcal{E}^1(L^{\mathrm{an}}) \supset \mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ on the left, and the quotient d_1 distance on the right. The image $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ can be characterized as the set of psh metrics on L^{an} *approximable by below*, i.e. metrics that are limits of *increasing* nets of psh metrics.

This result is the nontrivially-valued version of a result of Boucksom-Jonsson in [BJ18a]. We extensively use the results of [CM15]concerning asymptotics of superadditive functions on Okounkov bodies. We also build on the formalism developed in [BE].

Geodesics in spaces of non-Archimedean metrics.

In [Reb20b], or in this thesis Chapter 4, we transpose essential objects from the complex to the non-Archimedean world: geodesics in spaces of plurisubharmonic metrics.

Let us begin with the complex case. The purpose of such geodesics is again sparked by the search for Kähler-Einstein metrics in the space $\mathcal{H}(K_X)$ of smooth and strictly psh metrics on the canonical bundle. The Kähler-Einstein equation admits a certain functional, the K-energy of Mabuchi, as its Euler-Lagrange equation; in other words, critical points of the K-energy are solutions to the Kähler-Einstein problem. In order to continue the variational approach, it would be desirable that K be convex along certain distinguished segments of metrics in $\mathcal{H}(K_X)$. The ideal candidates would be affine lines, for $\mathcal{H}$ is a convex set, but they do not satisfy that property. Instead, K is convex with respect to another Riemannian structure on $\mathcal{H}$, by the work of Mabuchi ([Mab]), whose geodesics are given by a Monge-Ampère equation (see Semmes ([Sem]), Donaldson ([Don99])). X.X. Chen ([CX00]) was the first to show existence of such geodesics in $\mathcal{H}$.

Such geodesics are characterized by their *maximality*, as we explain. Pick two smooth, strictly psh metrics ϕ_0, ϕ_1 on K_X, and denote by $[0,1] \ni t \mapsto \phi_t$ the Mabuchi geodesic joining them. One can see it as a rotation-invariant psh metric on the product $K_X \times A \to X \times A$, where A is the annulus

$$A = \{e^{-1} \leq |z| \leq 1\},$$

upon setting $t = -\log|z|$. We then have that, for any rotation-invariant psh metric $A \ni z \mapsto \psi_z$ on $K_X \times A$ (which we can similarly see as a psh segment

$t \mapsto \psi_t$), if $\psi_0 \leq \phi_0$ and $\psi_1 \leq \phi_1$, then for all t in $[0,1]$,

$$\psi_t \leq \phi_t.$$

This property completely characterizes Mabuchi geodesic segments, which are for this reason sometimes called "maximal psh segments". We can furthermore realize them as the envelope

$$\sup\{\psi \text{ segment psh}, \, \psi_0 \leq \phi_0, \, \psi_1 \leq \phi_1\}.$$

As it turns out, this definition also makes sense more generally on any line bundle L on X, and also if the endpoint metrics are no longer continuous: this is the approach of Darvas to define weak geodesics in $\mathcal{E}^1(L)$. Finally, a last characterization of such geodesics is that ϕ is a psh segment along which the Monge-Ampère energy is affine.

Let us now assume (X, L) to be a polarized variety over a non-Archimedean field K. Obviously, geodesics in $\mathrm{PSH}(L^{\mathrm{an}})$ in the Riemannian sense can no longer be defined. However, we may still define a rather natural class of Fubini-Study segments, as finite maxima of segments of the form

$$[0,1] \ni t \mapsto \frac{1}{k} \max_i (\log |s_i| + t\lambda_i + (1-t)\lambda_i'),$$

where $(s_i)_i$ is again a basepoint-free basis of some $H^0(kL)$, and the λ_i, λ_i' are real constants. One then defines psh segments as decreasing limits of nets of Fubini-Study segments. Then, inspired by the extremal characterization in the complex case, we develop in [Reb20b] (Chapter 4 of this manuscript) a theory for geodesics in spaces of non-Archimedean finite-energy metrics. We first treat the case where the endpoint metrics are continuous, where we define our geodesic as an envelope as above, and we show via a Legendre transform that such a geodesic remains continuous in time and space.

Then, adapting the quantization approach of Darvas-Lu-Rubinstein in our context, we show that the Monge-Ampère energy is affine along such geodesics, and that they do define metric geodesics with respect to the non-Archimedean d_1 distance. On the algebraic side, we construct geodesics between bounded graded norms.

Finally, we extend our construction to the case of finite-energy endpoints, via decreasing approximations. Our results may be summarized as follows:

Theorem 0.0.0.10. *Given $\phi_0, \phi_1 \in \mathcal{E}^1(L)$, and setting*

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1)),$$

we have that:

1. *$(\mathcal{E}^1(L), d_1)$ is a metric space;*
2. *there exists a unique maximal psh segment $t \mapsto \phi_t$ joining ϕ_0 and ϕ_1;*
3. *$\phi_t \in \mathcal{E}^1(L)$ for all $t \in [0,1]$;*
4. *the segment ϕ_t is a metric geodesic with respect to the d_1 distance defined above, i.e. there exists $c \geq 0$ with*

$$d_1(\phi_t, \phi_s) = c \cdot |t - s|$$

   for all $t, s \in [0,1]$;

5. *the Monge-Ampère energy is affine along the segment ϕ_t, and any psh segment satisfying such a property with endpoints ϕ_0 and ϕ_1 has to coincide with the geodesic segment joining ϕ_0 et ϕ_1.*

If the endpoint metrics ϕ_0 and ϕ_1 are furthermore continuous, then the maximal psh segment joining them is also continuous in time and space.

Spaces of metrics on degenerations of complex manifolds.

Many recent works over the course of the previous decade, notably by Berman-Boucksom-Favre-Hisamoto-Jonsson et al., have been dedicated to the study of non-Archimedean limits of plurisubharmonic geodesic rays. Precisely, given a compact projective complex manifold X with an ample line bundle L, a geodesic ray

$$[0, \infty) \ni t \mapsto \phi_t \in \mathrm{PSH}(X, L)$$

is identified with a rotation-invariant psh metric Φ on the pullback of L to the trivial product $\overline{\mathbb{D}}^* \times X$, where the variable z given by the first projection is identified with t via $t = -\log|z|$. The limit as $t \to \infty$ defines, on studying the singularities of Φ along certain compactifications of the product, a non-Archimedean metric on the analytification of X with respect to the trivial absolute value on $\mathbb{C}$, as explained before.

The purpose of [Reb21], Chapter 5 in this thesis, is to generalize such considerations to the case of an arbitrary degeneration $\pi : X \to \overline{\mathbb{D}}^*$ of complex projective manifolds without necessarily assuming $\mathbb{S}^1$-invariance, or even isotriviality. Non-Archimedean phenomena immediately arise: one can realize such a degeneration as a variety over the field $\mathbb{C}((t))$.

We first define a notion of a (family of) psh metrics adapted to this context. Let L be a relatively ample line bundle on X. We denote by $\mathrm{PSH}(L)$ the space of psh metrics which can be extended to some analytic model $(\mathcal{X}, \mathcal{L})$ of (X, L), i.e. to a polarized complex analytic space $(\mathcal{X}, \mathcal{L})$ fibred over the unpunctured unit disc, which is isomorphic to (X, L) outside of its fibre over zero. Generalizing the work of Berman-Darvas-Lu ([BDL]), who define a radial d_1 distance on the space of finite-energy psh rays via

$$d_1(\phi, \psi) = \lim_t \frac{d_1(\phi_t, \psi_t)}{t}$$

(the d_1 in the right-hand side being the "classical" Darvas distances), we show that taking a generalized Lelong number of the map defined on $\overline{\mathbb{D}}^*$ of fibrewise d_1 distances also defines a distance on the space of "relatively maximal" fibrewise finite-energy metrics in $\mathrm{PSH}(L)$. The notion of relative maximality essentially states that they are maximal in the usual pluripotential theory sense over all preimages of relatively compact open sets in the punctured disc. We will write $\hat{\mathcal{E}}^1(L)$ for this class of metrics, and $\hat{d}_1$ for this distance. We then show the following:

Theorem 0.0.0.11. *The space $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ is a complete, geodesic metric space. There furthermore exist distinguished $\hat{d}_1$-geodesic segments, maximal among all geodesic segments, which can be constructed as fibrewise families of geodesic segments in each $\mathcal{E}^1(L_z)$.*

Then, to any fibrewise finite-energy metric ϕ in $\mathrm{PSH}(L)$, we associate a non-Archimedean metric ϕ^{NA} on the analytification L^{an} of L seen as a variety over $\mathbb{C}((t))$. On a divisorial point of X^{an}, associated to a divisor D inside the central fibre of some analytic model of X, the value of ϕ^{NA} roughly corresponds to the generic Lelong number of ϕ along D.

We then study the singular limit of the fibrewise Monge-Ampère energy along a fibrewise finite-energy metric in $\mathrm{PSH}(L)$, i.e. the metric

$$E(\phi) : z \mapsto \langle \phi_z^{d+1} \rangle = E(\phi_z)$$

on the Deligne pairing bundle $\langle L \rangle$ over the punctured disc. We show that, if ϕ extends as a locally bounded metric to some analytic model of (X, L), then

$$(E(\phi))^{\mathrm{NA}} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$

i.e. the Lelong number of the fibrewise Monge-Ampère energies corresponds with the non-Archimedean Monge-Ampère energy of ϕ^{NA}.

Such an equality does not hold in general. The class of metrics inside $\hat{\mathcal{E}}^1(L)$ satisfying

$$(E(\phi))^{\mathrm{NA}} = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}),$$

is what we define to be the class of "hybrid maximal" metrics $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$. One can see this to generalize the maximal geodesic rays of Berman-Boucksom-Jonsson, as a Lelong number is merely a generalized slope at infinity. We then obtain an extremal characterization of such metrics, and show that this class is isometric to the class $\mathcal{E}^1(L^{\mathrm{an}})$: we have realized non-Archimedean metrics as purely complex geometric objects. As a consequence, we obtain a heuristic result regarding proving convexity of non-Archimedean energy functionals. Our second main result is the following:

Theorem 0.0.0.12.

- *There exists an isometric embedding $(\mathcal{E}^1(L^{\mathrm{an}}), d_1^{\mathrm{NA}})$ in $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ with image $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$.*

- *A psh segment $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ is geodesic if and only if its image $\mathcal{E}^1(L^{\mathrm{an}}_{\mathrm{K}})$ is a non-Archimedean geodesic in the sense of Chapter 4.*

- *One has a general "plurifunctional extension property": given $d+1$ relatively ample line bundles L_i on X, for all $(d+1)$-uple of metrics $\phi_i \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L_i)$, we have*

$$(\langle \phi_0, \dots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle \phi_0^{\mathrm{NA}}, \dots, \phi_d^{\mathrm{NA}} \rangle.$$

Organization of the manuscript.

In Chapter 1, we develop algebraic preliminaries necessary for the main results of this manuscript. We begin with generalities on non-Archimedean fields. Then, given such a field K, we study metric and spectral properties of spaces of norms on finite-dimensional K-vector spaces. Finally, we extend this study to the case of graded norms on K-graded algebras generated in degree one. This last part contains some results from [Reb20b], as well as an unpublished result regarding completeness of a certain space of graded norms with respect to a d_∞-type distance.

In Chapter 2, we look into geometric preliminaries. After briefly recalling Berkovich's construction, we focus on various types of metrics on analytifications of K-line bundles, and explain basic notions of non-Archimedean pluripotential theory.

Finally, Chapters 3, 4, et 5 essentially contain the results of respectively [Reb20a], [Reb20b], and [Reb21], as previously explained.

Chapter 1

Algebraic preliminaries.

As a convention throughout this manuscript, rings and fields shall be assumed to be commutative.

1.1 Non-Archimedean fields.

1.1.1 Basic definitions.

Definition 1.1.1.1. Let R be any ring. A **multiplicative seminorm** on R is a function

$$|\cdot| : \mathrm{R} \to \mathbb{R}_+$$

satisfying the following properties:

(i) $|0| = 0$, $|1| = 1$;

(ii) $|xy| = |x| \cdot |y|$, for x, $y \in \mathrm{R}$;

(iii) $|x + y| \leq |x| + |y|$, for x, $y \in \mathrm{R}$.

It is furthermore **ultrametric** or **non-Archimedean** if the triangle inequality (iii) is refined to the *ultrametric inequality*:

(iiib) $|x + y| \leq \max\{|x|, |y|\}$, for x, $y \in \mathrm{R}$.

If R is a field, in which case we will usually denote it by K, a multiplicative seminorm on K will be referred to as an **absolute value** on K, and an ultrametric multiplicative seminorm as an **ultrametric** or a **non-Archimedean absolute value** on K. A field K endowed with an absolute value shall be referred to as a **valued field**.

Example 1.1.1.2. An example to keep in mind is the **trivial absolute value**, defined as

$$|x|_0 = 1,\ x \neq 0,$$

which is non-Archimedean, and well-defined for *any* field K, by the field axioms.

Remark 1.1.1.3. To any absolute value $|\cdot|$ on a field K is associated its valuation $\nu(\cdot) := -\log|\cdot|$.

To make sense of the following definition, note that an absolute value on a field K always endows it with a topology, induced by the distance

$$d(x, y) := |x - y|,$$

for x, $y \in \mathrm{K}$.

Definition 1.1.1.4. We will define a **non-Archimedean field** to be a field K which is Cauchy-complete with respect to the topology induced by a non-Archimedean absolute value $|\cdot|$. We define an **Archimedean field** to be a field K which is Cauchy-complete with respect to the topology induced by an absolute value $|\cdot|$ which is not non-Archimedean.

1.1.2 Classification of complete valued fields.

For the remainder of this Section, we shall fix a valued field $(\mathrm{K}, |\cdot|)$.

As we see from the classical Gelfand-Mazur Theorem, the classification of Archimedean fields is very simple.

Theorem 1.1.2.1 ([AC5-7, VI, \$6, Théorème 1])**.** *If* $(\mathrm{K}, |\cdot|)$ *is an Archimedean field, then it is isomorphic to either* $\mathbb{R}$ *or* $\mathbb{C}$ *endowed with their usual Archimedean absolute values.*

The non-Archimedean case has more structure: they are classified by **bicharacteristic** and by their **value group**, as we now explain. We assume for now that K is non-Archimedean.

Consider the valuation $\nu(\cdot) = -\log|\cdot|$ associated to the absolute value on K. By the absolute value axioms, its possible values define an additive subgroup of $\mathbb{R}$, the **value group** of $(\mathrm{K}, |\cdot|)$. It is a well-known fact that such subgroups are either *trivial*, nontrivial *discrete*, or *dense*.

Definition 1.1.2.2. We say that K is:

- **trivially valued** if its value group is trivial (equivalently if it is endowed with the trivial absolute value);
- **discretely valued** if its value group is nontrivial and discrete;
- **densely valued** if its value group is dense.

We now look into the second way to classify non-Archimedean fields: the pair given by their characteristic, and the characteristic of their residue field.

Definition 1.1.2.3. To any non-Archimedean field K, one may associate the following objects:

- its **valuation ring** K°, defined as the set of elements in K with absolute value ≤ 1;
- the **maximal ideal** of K°, $\mathrm{K}^{\circ\circ}$, characterized as the set of elements in K° (or K) with absolute value < 1;
- its **residue field** $\tilde{\mathrm{K}} = \mathrm{K}^{\circ}/\mathrm{K}^{\circ\circ}$.

Only three cases are possible, by which we say that:

- K has **equicharacteristic** 0 if K and $\tilde{\mathrm{K}}$ are of characteristic 0;
- K has **mixed chracteristic** if K is of characteristic 0 and $\tilde{\mathrm{K}}$ is of characteristic p, p prime;
- K has **equicharacteristic** p if K and $\tilde{\mathrm{K}}$ are of characteristic p, p prime.

Remark 1.1.2.4. One can see that the valuation ring of a non-Archimedean field is Noetherian if and only if the field is discretely valued.

We are now equipped to understand the main examples of non-Archimedean fields.

Example 1.1.2.5. We first look at the discretely-valued case.

- the field $\mathbb{C}((t))$ of complex formal Laurent series is discretely valued, and has equicharacteristic 0, for its residue field is $\mathbb{C}$;

- the field $\mathbb{Q}_p$ of p-adics, p prime, is discretely valued and has mixed characteristic, for its residue field is $\mathbb{F}_p$;
- the field $\mathbb{F}_p((t))$ of formal Laurent series over the finite field $\mathbb{F}_p$, p prime, is discretely valued, and has equicharacteristic p, for its residue field is $\mathbb{F}_p$.

The densely valued world is "wilder". Fields of Puiseux series (i.e. series with exponents bounded below in $\mathbb{Q}$) over $\mathbb{C}$ and $\mathbb{F}_p$ are well-known examples, but they are not Cauchy complete! In the mixed characteristic case, the fundamental example is the field $\mathbb{C}_p$ of complex p-adics; though recently, much attention has been brought to *perfectoid* fields, i.e. densely valued mixed characteristic non-Archimedean fields with surjective Frobenius on K° mod p. We will give examples closer to our considerations in the next Subsection.

We finally mention a last classification result, of potential interest to the reader.

Theorem 1.1.2.6 (Cohen's structure Theorem, [SerBook, \$4, Théorème 2])**.** *If* K *is an equicharacteristic discretely valued non-Archimedean field, then it is isomorphic to the field $\tilde{K}((t))$, the field of formal Laurent series over its residue field.*

1.1.3 Maximal completeness.

We briefly review a property for non-Archimedean field which will be essential in our study of norms on K-vector spaces later on. For this section, our references are [Poo93] and [CS18].

Definition 1.1.3.1. Let $(\mathrm{K}, |\cdot|)$ be a non-Archimedean field. Let $(\mathrm{L}, |\cdot|')$ be a valued field extension of $(\mathrm{K}, |\cdot|)$. We say that it is an **immediate extension** if the value groups of K and L are isomorphic, and if $\tilde{\mathrm{K}}$ and $\tilde{\mathrm{L}}$ are isomorphic.

Definition 1.1.3.2. A non-Archimedean field which admits no immediate extension will be called a **maximally complete field**.

Example 1.1.3.3. Any equicharacteristic discretely valued field is maximally complete. The field $\mathbb{Q}_p$, p prime, is also maximally complete.

Our claim is then the following:

Proposition 1.1.3.4. *Any nontrivially valued non-Archimedean field can be embedded into a field which is:*

1. *densely valued;*
2. *Cauchy-complete;*
3. *algebraically closed;*
4. *maximally complete;*
5. *of the same bicharacteristic as the original field.*

Proof. Let K be as in the above statement. Let $\hat{\mathrm{K}}$ be the Cauchy completion of the algebraic closure of K. Taking algebraic closures in the discretely or densely valued case already yields a densely valued field. $\hat{\mathrm{K}}$ still has the same bicharacteristic as K. Then, [Kap42, Theorem 5] ensures that there exists a maximally complete immediate extension of $\hat{\mathrm{K}}$, which then has the desired properties by definition. □

Example 1.1.3.5. Let us begin with a field of complex formal Laurent series $\mathbb{C}((t))$. The field of formal Puiseux series is an algebraic closure thereof, but as mentioned before, it is not Cauchy-complete. The Levi-Civita field $L[\mathbb{Q}, \mathbb{C}]$ of formal complex power series P with $\mathrm{supp}(P) \cup (-\infty, k]$ finite for all $k \in \mathbb{Z}$ is then a Cauchy closure of the field of complex Puiseux series. Its maximally complete immediate extension is given by the field of Hahn series $\mathbb{C}((\mathbb{Q}))$ of formal complex power series P with $\mathrm{supp}(P)$ well-ordered in $\mathbb{Q}$. If one were to start from $\mathbb{F}_p$ instead, one would have to first pass to an algebraic closure of $\mathbb{F}_p$, and proceed as before ([CS18, Theorem 7.2]). If one started from $\mathbb{Q}_p$, one would obtain a p-adic Mal'cev-Neumann field as in [Poo93, Section 4].

1.2 Spaces of norms on vector spaces over non-Archimedean fields.

Throughout this Section, unless otherwise specified, we fix a non-Archimedean field K, with absolute value $|\cdot|$; and a finite-dimensional vector space V over K, of dimension d.

1.2.1 Spaces of norms.

Definition 1.2.1.1. A **norm** on V is a function

$$\zeta : V \to \mathbb{R}_+,$$

satisfying the following properties:

- $\zeta(v) = 0$ if and only if $v = 0_V$;
- $\zeta(\lambda \cdot v) = |\lambda| \cdot \zeta(v)$, for $\lambda \in \mathrm{K}$, $v \in V$;
- $\zeta(v + w) \leq \max\{\zeta(v), \zeta(w)\}$, for v, $w \in V$.

We denote by

$$\mathcal{N}(V)$$

the set of norms on V. This space is closed under the (pointwise) maximum operation, which we denote

$$\zeta \vee \zeta' = \max(\zeta, \zeta'),$$

for any two norms ζ, $\zeta' \in \mathcal{N}(V)$.

Definition 1.2.1.2. A norm $\zeta \in \mathcal{N}(V)$ is **diagonalizable** if there exists a basis $(e_1, \ldots, e_d)$ of V such that, for all

$$v = \sum v_i e_i,$$

with $v_i \in \mathrm{K}$ for all i, we have that

$$\zeta(v) = \max_i |v_i| \cdot \zeta(e_i).$$

We also say that this basis is **orthogonal** for ζ. We say that it is a **lattice norm**, or a **pure diagonalizable norm**, if, for all i, $\zeta(e_i) = 1$. We define

$$\mathcal{N}^{\mathrm{diag}}(V), \mathcal{N}^{\mathrm{latt}}(V)$$

as respectively the set of diagonalizable norms and the set of lattice norms on V.

Remark 1.2.1.3. In our conventions, we will define a **lattice** of V to be a submodule L of V of finite type over K°, such that $L \otimes_{\mathrm{K}^\circ} \mathrm{K} = V$. In particular, the unit ball of a lattice norm is always a lattice, justifying the terminology.

Remark 1.2.1.4. It is also common practice to use the terminology of **cartesian bases**, see e.g. [BGR, Ch. 2].

We note that, when K is maximally complete, we know from [BGR, 2.4.4] (see also [BE, Lemma 1.12]) that any norm ζ on V admits an orthogonal basis. Further emphasizing the difference between the possible value groups, we have:

Lemma 1.2.1.5 ([BE, L1.29])**.** Let K be nontrivially valued.

- if K is discretely valued, the unit ball of any diagonalizable norm is a lattice of V;
- if K is densely valued, the unit ball of a norm $\zeta \in \mathcal{N}(V)$ is a lattice if and only if ζ is a lattice norm.

There are many ways to construct norms from pre-existing ones. The two building blocks are the following:

- let $W \subset V$ be a subspace; any norm ζ induces a quotient norm $\zeta_{V/W}$ on V/W, as follows: given $[v] \in V/W$,
$$\zeta_{V/W}([v]) = \inf_{w \in W} \zeta(v+w);$$
- a norm ζ on V induces a norm $\zeta^{\otimes n}$ on any tensor power $V^{\otimes n}$ of V, by setting, for each $v \in V^{\otimes n}$,
$$\zeta^{\otimes n}(v) = \inf_{v=\sum_i v_1^i \otimes \cdots \otimes v_n^i} \max_i (\zeta(v_1^i) \times \cdots \times \zeta(v_n^i)),$$
where we take the infimum over all possible decompositions of v of the form $\sum_i v_1^i \otimes \cdots \otimes v_n^i$, where the sum over i is finite, and $v_k^i \in V$ for all i, k.

We may combine these constructions. Let n be an integer, λ be a partition of n, and let $\mathbb{S}^\lambda$ denote the Schur functor associated to λ. Then a norm $\zeta \in \mathcal{N}(V)$ defines a norm $\zeta^\lambda \in \mathcal{N}(\mathbb{S}^\lambda(V))$, as this vector space is a composition of quotients of tensor products. In particular, $\zeta \in \mathcal{N}(V)$ induces:

- a norm $\zeta^{\wedge n}$ on the n-fold exterior product $V^{\wedge n}$;
- a norm $\zeta^{\odot n}$ on the n-fold symmetric product $V^{\odot n}$.

1.2.2 Relative spectra.

Generalizing the set of eigenvalues of the transition matrix between to jointly orthonormalized complex Hermitian norms, we introduce the relative spectrum of two norms ζ, $\zeta' \in \mathcal{N}(V)$.

Definition 1.2.2.1. The **relative spectrum** of ζ and ζ' is the set (counting multiplicities) $\mathrm{Sp}(\zeta, \zeta')$ which contains all real numbers of the form

$$\lambda_i(\zeta, \zeta') = \sup_{W \in \bigcup_{i \leq r \leq \dim V} \mathrm{Gr}_{\mathrm{K}}(r,V)} \inf_{w \in W - \{0\}} [\log \zeta'(w) - \log \zeta(w)],$$

where

$$\mathrm{Gr}_{\mathrm{K}}(r, V)$$

denotes the r-th Grassmannian of V.

Remark 1.2.2.2. When K is maximally complete, again from [BGR, 2.4.4], for any two norms, there exists a basis diagonalizing both of them (we shall in that case speak of **codiagonalizing bases**).

By [BE, P2.24], if both norms are diagonalizable by a basis (s_i), ordered such that

$$i > j \Rightarrow \frac{\zeta'(s_i)}{\zeta(s_i)} \geq \frac{\zeta'(s_j)}{\zeta(s_j)},$$

then

$$\lambda_i(\zeta, \zeta') = \log \zeta'(s_i) - \log \zeta(s_i).$$

Due to the diagonalizability issues explained before in non-maximally complete fields, it will be desirable to pass to a maximally complete field extension, such as one of the lovely fields promised by Proposition 1.1.3.4. We now see how to extend norms, and how spectra behave under this operation.

Definition 1.2.2.3. Let L/K be a non-Archimedean field extension. Let ζ be a non-Archimedean norm on V. The **ground field extension** ζ_{L} on $V_{\mathrm{L}} = V \otimes_{\mathrm{K}} \mathrm{L}$ is defined as

$$\zeta_{\mathrm{L}}(v') = \inf \max_i |a_i'| \cdot \zeta(v_i),$$

for any $v' \in V_{\mathrm{L}}$, where the inf is defined over all representations

$$v' = \sum_i a_i' \cdot v_i,$$

with coefficients a_i' in L and $v_i \in V$.

This defines by [BE, Proposition 1.24(i)] a non-Archimedean norm on V_{L}, which coincides with the original norm ζ on V. Two essential results for us are as follows:

Proposition 1.2.2.4 ([BE, Lemma 1.25, Proposition 2.14(v)]). *Let* L/K *be a field extension. Let ζ be a norm on V, with ground field extension ζ_{L} on V_{L}. We then have:*

- *if ζ is diagonalizable with basis (e_i), then ζ_{L} is also diagonalizable with basis $(e_i \otimes 1)$;*
- *the relative spectra of ground field extensions of norms coincides with with the relative spectra of original norms: for any other norm ζ' with ground field extension ζ'_{L}, we have*

$$\mathrm{Sp}(\zeta, \zeta') = \mathrm{Sp}(\zeta_{\mathrm{L}}, \zeta'_{\mathrm{L}}).$$

The second point follows from the first, and the fact that the ground field extension of a norm coincides with the original norm on V.

1.2.3 Spectral measures, volumes, and metric structures on $\mathcal{N}(V)$

Definition 1.2.3.1. The **relative spectral measure**

$$\sigma(\zeta, \zeta')$$

of ζ and ζ' is defined to be the discrete probability measure supported on $\mathrm{Sp}(\zeta, \zeta')$, that is:

$$\sigma(\zeta, \zeta') = d^{-1} \sum \delta_{\lambda_i(\zeta, \zeta')},$$

where we recall that $d = \dim_{\mathrm{K}} V$.

Definition 1.2.3.2. Let $p \in [1, \infty)$. The d_p-distance of ζ and ζ' is defined by:

$$d_p(\zeta, \zeta')^p = \int_{\mathbb{R}} |\lambda|^p \, d\sigma(\zeta, \zeta').$$

We furthermore define

$$d_\infty(\zeta, \zeta') = \max_{\lambda \in \mathrm{Sp}(\zeta, \zeta')} |\lambda|.$$

Of utmost interest for our considerations are the cases d_1 and d_∞, which have more practical expressions:

$$d_\infty(\zeta, \zeta') = \sup_{v \in V - \{0\}} |\log \zeta'(v) - \log \zeta(v)|,$$

and

$$d_1(\zeta, \zeta') = d^{-1} \sum \lambda_i(\zeta, \zeta'),$$

where $d = \dim_{\mathrm{K}} V$. The distance d_2 also has some importance in the Euclidean picture, which we lightly touch on in the next Subsection.

Remark 1.2.3.3 (Important characterization of the distance d_∞)**.** The distance $d_\infty(\zeta, \zeta')$ is equivalently characterized as the maximal exponential distorsion between the two norms, or in other words **best constant** $C > 0$ such that for all $v \in V$,

$$e^{-C} \zeta'(v) \leq \zeta(v) \leq e^{C} \zeta'(v).$$

Closely related to the distance d_1 (see Theorem 1.2.3.6 below), the relative volume of norms generalizes ratios of volumes of balls of holomorphic sections, originally studied in [BB10].

Definition 1.2.3.4. The **relative volume** of ζ and $\zeta' \in \mathcal{N}(V)$ is defined as

$$\mathrm{vol}(\zeta, \zeta') = \int_{\mathbb{R}} \lambda \, d\sigma(\zeta, \zeta'),$$

that is: the mean value of the relative spectrum of those norms. Note that we normalize our spectral measure by d, while the authors in [BE, T2.25] do not.

Remark 1.2.3.5. In particular, if $\zeta \leq \zeta'$, then

$$\mathrm{vol}(\zeta, \zeta') = d_1(\zeta, \zeta'),$$

and, reversing the inequality, we obtain

$$-\mathrm{vol}(\zeta, \zeta') = d_1(\zeta, \zeta').$$

Theorem 1.2.3.6 ([BE, T2.25])**.** *Let* $\{e_1, \ldots, e_d\}$ *be a basis of* V*. We have that*

$$\mathrm{vol}(\zeta, \zeta') = \frac{1}{d} \left[\log \zeta'^{\wedge d}(e_1 \wedge \cdots \wedge e_d) - \log \zeta^{\wedge d}(e_1 \wedge \cdots \wedge e_d) \right]$$

Corollary 1.2.3.7. *Volumes satisfy a* ***cocycle property****: given a third norm* $\zeta'' \in \mathcal{N}(V)$,

$$\mathrm{vol}(\zeta, \zeta') = \mathrm{vol}(\zeta, \zeta'') + \mathrm{vol}(\zeta'', \zeta').$$

Proposition 1.2.3.8 ([BE, P1.8, T1.19, L1.29])**.** *With respect to the distance* d_∞ *on* $\mathcal{N}(V)$*, we have that:*

(i) $\mathcal{N}(V)$ *is complete;*

(ii) $\mathcal{N}^{\mathrm{diag}}(V)$ *is dense in* $\mathcal{N}(V)$*, with equality if* K *is discretely valued;*

(iii) if K *is discretely valued,* $\mathcal{N}^{\mathrm{latt}}(V)$ *is discrete and closed in* $\mathcal{N}(V)$*;*

(iv) if K *is densely valued,* $\mathcal{N}^{\mathrm{latt}}(V)$ *is dense in* $\mathcal{N}^{\mathrm{diag}}(V)$*.*

We note that relative volumes behave well with respect to the d_∞ distance.

Lemma 1.2.3.9 (Volumes are Lipschitz, [BE, P2.14])**.** The mapping vol is 1-Lipschitz in both variables, and thus Lipschitz on the product $\mathcal{N}(V) \times \mathcal{N}(V)$. In other words, given two pairs of norms (ζ_0, ζ_1) and (ζ_0', ζ_1') acting on V, we have

$$|\,\mathrm{vol}(\zeta_0, \zeta_1) - \mathrm{vol}(\zeta_0', \zeta_1')| \leq d_\infty(\zeta_0, \zeta_0') + d_\infty(\zeta_1, \zeta_1').$$

Finally, we see that taking quotients of norms is a contracting operation for the d_∞ distance.

Proposition 1.2.3.10 (Quotients decrease distance)**.** *Let* $W \subset V$ *be a proper linear subspace of* V*, let* ζ *and* ζ' *be norms on* V*. Denote* $\tilde{\zeta}$ *and* $\tilde{\zeta}'$ *the induced quotient norms on* V/W*. We then have that:*

$$d_\infty(\tilde{\zeta}, \tilde{\zeta}') \leq d_\infty(\zeta, \zeta').$$

Proof. Fix $a = d_\infty(\zeta, \zeta')$, and $\tilde{v} \in V/W$. It is enough to show that

$$e^{-a}\tilde{\zeta}'(\tilde{v}) \leq \tilde{\zeta}(\tilde{v}) \leq e^{a}\tilde{\zeta}(\tilde{v}).$$

We lift $\tilde{v}$ to a sum $v + w$ with $v \in V - W$ and $w \in W$. Note that

$$e^{-a}\zeta'(v + w) \leq \zeta(v + w),$$

for all such lifts, so that we can pass to the inf and get that

$$e^{-a}\tilde{\zeta}'(\tilde{v}) \leq \tilde{\zeta}(\tilde{v}).$$

Similarly, we get that

$$e^{-a}\tilde{\zeta}(\tilde{v}) \leq \tilde{\zeta}'(\tilde{v}).$$

The result follows. □

1.2.4 Apartments.

Norms oftentimes like to be roommates. This affinity gives insight into the building-like structure of the space of diagonalizable norms on the vector space V. Pick a basis $\mathfrak{s} = (s_i)_i$ of V, and the projection

$$\iota_{\mathfrak{s}} : \mathbb{R}^d \to \mathcal{N}(V) = \mathcal{N}^{\text{diag}}(V)$$

defined by sending a vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ to the unique norm ζ diagonalized in the basis $\mathfrak{s}$, and with

$$\zeta(s_i) = e^{-\alpha_i}$$

for all i. The image of this injection map is called the apartment $\mathbb{A}_{\mathfrak{s}}$ associated to the basis $\mathfrak{s}$. It inherits the geometry of $\mathbb{R}^d$, in the sense that for any distance d_p, $p \in [1, \infty]$, ι_S realizes an isometry onto its image for the distances d_p as defined in the previous section.

The space of diagonalizable norms $\mathcal{N}^{\text{diag}}(V)$, as the (non-disjoint!) union of all the apartments

$$\bigcup_{\mathfrak{s} \text{ basis of } V} \mathbb{A}_{\mathfrak{s}}$$

then inherits a complete Euclidean building structure for $p = 2$, in the sense of [RTW15]. As two diagonalizable norms may be diagonalized in the same basis, it follows that any pair of norms share some apartment in the building $\mathcal{N}^{\text{diag}}(V)$. If K is maximally complete, all norms are diagonalizable, so that $\mathcal{N}^{\text{diag}}(V) = \mathcal{N}(V)$.

As a Euclidean building, $\mathcal{N}(V)$ therefore has the nice property that d_p-geodesics always exist: given two norms ζ, ζ' codiagonalized by a basis $\mathfrak{s}$, a geodesic segment connecting them may be obtained as the image through $\iota_{\mathfrak{s}}$ of a d_p-geodesic segment connecting $\iota_{\mathfrak{s}}^{-1}(\zeta)$ and $\iota_{\mathfrak{s}}^{-1}(\zeta')$ in $\mathbb{R}^d$. We give an explicit description of such geodesics in what follows. The reader might be interested in consulting the article [Gér81].

1.2.5 Norm geodesics.

Pick a basis $\mathfrak{s} = (s_i)_i$ of V. In the apartment $\mathbb{A}_{\mathfrak{s}}$, there is what we call a norm geodesic

$$t \in [0, 1] \mapsto \zeta_t$$

defined as follows: for all i, $\zeta_t(s_i) = \zeta_0(s_i)^{1-t} \cdot \zeta_1(s_i)^t$. From an elementary computation it follows that it is geodesic for all distances d_p.

Remark 1.2.5.1. In the case $p > 1$, the norm geodesic is the only geodesic segment between two norms. However, if $p = 1$, there are infinitely many geodesic segments between two norms, reflecting the d_1 geometry of $\mathbb{R}^d$.

Consider now two non-necessarily diagonalizable norms ζ_0 and $\zeta_1 \in \mathcal{N}(V)$, and approximations by diagonalizable norms ζ_0^n, ζ_1^n such that

$$d_1(\zeta_i^n, \zeta_i) \leq \frac{1}{n}$$

for $i = 0$, 1. Define

$$\zeta_t = \lim_n \zeta_t^n,$$

where $t \mapsto \zeta_t^n$ is the norm geodesic defined above joining ζ_0^n and ζ_1^n. We make the following claim:

Proposition 1.2.5.2. *The limit ζ_t above exists, and satisfies the following properties:*

1. *it is independent of the approximation;*
2. *it is controlled uniformly by the endpoints: given approximations $(\zeta_i^m)_m$, $i = 0, 1$, of the bounds, we have*
$$d_1(\zeta_t, \zeta_t^m) \leq (1-t)d_1(\zeta_0, \zeta_0^m) + td_1(\zeta_1, \zeta_1^m);$$
3. *ζ_t is the pointwise limit of the approximating norms: for $s \in V$,*
$$\zeta_t(s) = \lim_m \zeta_t^m(s).$$

This claim relies on the diagonalizable case of Corollary 1.2.5.8, which has yet to be proven. However, all the results in this Section hold in the general non-necessarily diagonalizable case. In order to avoid writing each statement twice, we first prove Proposition 1.2.5.2 using the diagonalizable case of Corollary 1.2.5.8, and then state each result of this Section in the general case. Circular reasoning is avoided by following the logical order "proof of a result in the diagonalizable case" $\to$ "proof of Corollary 1.2.5.8 and Proposition 1.2.5.2" $\to$ "proof of a result in the non-diagonalizable case".

Proof. Pick m, $n \geq 0$, and write using Corollary 1.2.5.8:

$$d_1(\zeta_t^m, \zeta_t^n) \leq (1-t)d_1(\zeta_0^m, \zeta_0^n) + td_1(\zeta_1^m, \zeta_1^n)$$

which establishes that the sequence (ζ_t^n) is d_1-Cauchy, thus has a limit in $\mathcal{N}(V)$. This also establishes the statement about uniform approximation by passing to the limit in n in the previous inequality. Pick now a second pair of approximations $(\zeta_0'^n)$, $(\zeta_1'^n)$, and write $\zeta_t'^n$ for the norm geodesic between the adequate bounds for all n. Then,

$$d_1(\zeta_t^n, \zeta_t'^n) \leq (1-t)d_1(\zeta_0^n, \zeta_0'^n) + td_1(\zeta_1^n, \zeta_1'^n)$$

which vanishes as $n \to \infty$ since both pairs of approximations have the same limits. For the third point, we recall that all metric structures d_p, $p \in [1, \infty]$ are equivalent on spaces of norms (see [BE, 3.1]). In particular, d_1-convergence is equivalent to d_∞-convergence, where

$$d_\infty(\zeta, \zeta') = \log \sup_{s \in V} \left| \frac{\zeta'(s)}{\zeta(s)} \right|,$$

which gives pointwise convergence. □

A fundamental property of norm geodesics is the following:

Lemma 1.2.5.3 (Log-convexity of norm geodesics)**.** Let $t \in [0,1]$, let V be a d-dimensional K-vector space. Pick two norms ζ_0, $\zeta_1 \in \mathcal{N}(V)$, and let $t \mapsto \zeta_t$ denote the norm geodesic as defined above. Let $s \in V$. We then have that $\log \zeta_t(s)$ is a convex function of t:

$$\zeta_t(s) \leq \zeta_0(s)^{1-t}\zeta_1(s)^t.$$

Proof. Let (s_i) be a basis codiagonalizing the endpoints, and write s as $\sum a_i \cdot s_i$, so that

$$\begin{aligned}
\zeta_t(s) &= \max_i |a_i| \cdot \zeta_t(s_i) \\
&= \max_i |a_i| \cdot \zeta_0(s_i)^{1-t}\zeta_1(s_i)^t \\
&= \max_i |a_i|^{1-t}\zeta_0(s_i)^{1-t} \cdot |a_i|^t\zeta_1(s_i)^t \\
&\leq (\max_i |a_i| \cdot \zeta_0(s_i))^{1-t} \cdot (\max_i |a_i| \cdot \zeta_1(s_i))^t = \zeta_0(s)^{1-t}\zeta_1(s)^t.
\end{aligned}$$

The non-diagonalizable case follows upon approximation with diagonalizable norms and passing to the pointwise limit in the log-convexity inequalities

$$\zeta_t^m(s) \leq \zeta_0^m(s)^{1-t}\zeta_1^m(s)^t.$$

$\square$

We now prove an important comparison inequality concerning norm geodesics with comparable endpoints. They will be crucial in proving many later results, including the metric convexity of d_1 in spaces of norms.

Proposition 1.2.5.4 (Monotonicity of norm geodesics with respect to endpoints)**.** *Let $k \in \mathbb{N}$, and set two couples of norms (ζ_0, ζ_1) and (ζ_0', ζ_1') acting on $H^0(kL)$. If*

$$\zeta_0' \leq \zeta_0 \text{ and } \zeta_1' \leq \zeta_1,$$

we then have that for all t

$$\zeta_t' \leq \zeta_t.$$

Proof. Assume first all norms to be diagonalizable. Write then a section s of in$H^0(kL)$ as $s = \sum a_i s_i$ where (s_i) is a basis codiagonalizing ζ_0 and ζ_1 (hence all the ζ_t), so that

$$\begin{aligned}
\zeta_t'(s) &\leq \max_i |a_i| \cdot \zeta_t'(s_i) \\
&\leq \max_i |a_i| \cdot \zeta_0'(s_i)^{1-t}\zeta_1'(s_i)^t \\
&\leq \max_i |a_i| \cdot \zeta_0(s_i)^{1-t}\zeta_1(s_i)^t = \zeta_t(s),
\end{aligned}$$

where we have used the ultrametric inequality, log-convexity of ζ_t', the inequalities in the hypotheses, then the definition of a basis diagonalizing ζ_t. This concludes the proof. If we do not have diagonalizability, one uses approximations by diagonalizable norms and Proposition 1.2.5.2(iii) to conclude. $\square$

As explained before, one can take determinants of norms on V. We now look into the behaviour of geodesics under this operation, which will allow us to prove metric convexity of geodesics. We first recall the following result from [BE]:

Lemma 1.2.5.5. Let ζ be a norm on a d-dimensional K-vector space V, and let $\mathfrak{s} = (s_i)_i$ be a basis of V. If $\mathfrak{s}$ diagonalizes ζ, then

$$\det \zeta(s_1 \wedge \cdots \wedge s_d) = \prod_{i=1}^{d} \zeta(s_i).$$

We recall that the determinant of a norm ζ on V is the norm induced by ζ on $\det V = V^{\wedge d}$.

Lemma 1.2.5.6. Let $t \mapsto \zeta_t$ be a norm geodesic in V. Then,

$$t \mapsto \det \zeta_t$$

is the one-dimensional norm geodesic joining $\det \zeta_0$ and $\det \zeta_1$ in $\det V$.

Proof. By density, we may assume that there exists a basis $\mathfrak{s} = (s_i)_i$ of V diagonalizing the ζ_t for all t. We have:

$$\begin{aligned} \det \zeta_t(s_1 \wedge \cdots \wedge s_d) &= \prod \zeta_t(s_i) \\ &= \prod \zeta_0(s_i)^{1-t} \zeta_1(s_i)^t \\ &= (\det \zeta_0(s_1 \wedge \cdots \wedge s_d))^{1-t} (\det \zeta_1(s_1 \wedge \cdots \wedge s_d))^t, \end{aligned}$$

which by definition proves the statement. We have used Lemma 1.2.5.5 for the first equality, the diagonalizing property of (s_i) for the second, and Lemma 1.2.5.5 again for the third. □

This yields a proof that the relative volume of geodesics is affine:

Corollary 1.2.5.7. *Given two norm geodesics $t \mapsto \zeta_t$, $t \mapsto \zeta'_t$ in V, the function*

$$\operatorname{vol}(\zeta_t, \zeta'_t) = t \mapsto \log \frac{\det \zeta'_t}{\det \zeta_t}$$

is affine.

Proof. By density again, without loss of generality we can assume that we can pick a basis $\mathfrak{s} = (s_i)_i$ diagonalizing ζ_t for all t. We then have that for all

t,

$$\begin{aligned}\det\zeta_t(s_1\wedge\cdots\wedge s_d) &= \prod_{i=1}^d \zeta_t(s_i)\\ &= \prod_{i=1}^d \zeta_0(s_i)^{1-t}\zeta_1(s_i)^t\\ &= (\det\zeta_0(s_1\wedge\cdots\wedge s_d))^{1-t}(\det\zeta_1(s_1\wedge\cdots\wedge s_d))^t.\end{aligned}$$

Furthermore, from Lemma 1.2.5.3 and Lemma 1.2.5.6 we have that the functions $t\mapsto\det\zeta_t$ and $t\mapsto\det\zeta'_t$ are log-convex. Combining this with the equality above, we find

$$\log\frac{\det\zeta'_t}{\det\zeta_t}\leq\log\frac{(\det\zeta'_0(s_1\wedge\cdots\wedge s_d))^{1-t}(\det\zeta'_1(s_1\wedge\cdots\wedge s_d))^t}{(\det\zeta_0(s_1\wedge\cdots\wedge s_d))^{1-t}(\det\zeta_1(s_1\wedge\cdots\wedge s_d))^t},$$

i.e. the function $t\mapsto\mathrm{vol}(\zeta_t,\zeta'_t)$ is convex. Note that the argument applies symmetrically to show convexity of $t\mapsto\mathrm{vol}(\zeta'_t,\zeta_t)=-\mathrm{vol}(\zeta_t,\zeta'_t)$, i.e. that $t\mapsto\mathrm{vol}(\zeta_t,\zeta'_t)$ is also concave, proving affineness. □

We may now establish the following result, building on the proof of [BDL, Proposition 5.1]:

Corollary 1.2.5.8 (Metric convexity of norm geodesics)**.** *Given two norm geodesics $t\mapsto\zeta_t$, $t\mapsto\zeta'_t$ in V, we have*

$$d_1(\zeta_t,\zeta'_t)\leq(1-t)d_1(\zeta_0,\zeta'_0)+td_1(\zeta_1,\zeta'_1).$$

Proof. If the endpoints are comparable in the same order, i.e. $\zeta_0\geq\zeta'_0$ and $\zeta_1\geq\zeta'_1$; or $\zeta_0\leq\zeta'_0$ and $\zeta_1\leq\zeta'_1$, this follows immediately from the previous Corollary. In the general case, we have to be careful, as the maximum of norm geodesics is not a priori a norm geodesic. However, from Proposition 1.2.5.4, we have that the geodesic

$$t\mapsto\chi_t$$

joining $\zeta_0\vee\zeta'_0$ and $\zeta_1\vee\zeta'_1$ satisfies

$$\chi_t\geq\zeta_t,\zeta'_t$$

for all t, i.e.

$$\chi_t \geq \zeta_t \vee \zeta_t'.$$

Therefore, $\mathrm{vol}(\chi_t, \zeta_t \vee \zeta_t') \leq 0$, and we have

$$\begin{aligned} d_1(\zeta_t, \zeta_t') &= \mathrm{vol}(\zeta_t, \zeta_t \vee \zeta_t') + \mathrm{vol}(\zeta_t', \zeta_t \vee \zeta_t') \\ &= \mathrm{vol}(\zeta_t, \chi_t) + \mathrm{vol}(\chi_t, \zeta_t \vee \zeta_t') + \mathrm{vol}(\zeta_t', \chi_t) + \mathrm{vol}(\chi_t, \zeta_t \vee \zeta_t') \\ &\leq \mathrm{vol}(\zeta_t, \chi_t) + \mathrm{vol}(\zeta_t', \chi_t). \end{aligned}$$

Since $\zeta_t, \zeta_t' \leq \chi_t$, the statement of the corollary holds for the volumes above, and we have

$$\begin{aligned} d_1(\zeta_t, \zeta_t') &\leq (1-t)\,\mathrm{vol}(\zeta_0, \zeta_0 \vee \zeta_0') + t\,\mathrm{vol}(\zeta_1, \zeta_1 \vee \zeta_1') \\ &\quad + (1-t)\,\mathrm{vol}(\zeta_0', \zeta_0 \vee \zeta_0') + t\,\mathrm{vol}(\zeta_1', \zeta_1 \vee \zeta_1') \\ &= (1-t) d_1(\zeta_0, \zeta_0') + t d_1(\zeta_1, \zeta_1'), \end{aligned}$$

proving the general statement. □

Remark 1.2.5.9. From the general theory of metric spaces, a result such as Corollary 1.2.5.8 ensures existence and good properties of the "cone at infinity" or boundary at infinity of $\mathcal{N}(V)$, which can be described as equivalence classes of (norm) geodesic rays in $\mathcal{N}(V)$ staying at bounded distance.

The same result holds in d_∞ distance:

Lemma 1.2.5.10 (Convexity of d_∞ along norm geodesics)**.** Let ζ_0, ζ_1 and ζ_0', ζ_1' be four norms on V, and denote by ζ_t, resp. ζ_t' the norm geodesic joining the first two, resp. the last two. Then,

$$d_\infty(\zeta_t, \zeta_t') \leq (1-t) d_\infty(\zeta_0, \zeta_0') + t d_\infty(\zeta_1, \zeta_1').$$

Proof. As usual, the general case follows from the diagonalizable case by approximation, therefore we make this assumption. We have that

$$\begin{aligned} d_\infty(\zeta_t, \zeta_t') &= \max_i |\lambda_i(\zeta_t, \zeta_t')| \\ &= \max_i \, |(1-t)\lambda_i(\zeta_0, \zeta_0') + t\lambda_i(\zeta_1, \zeta_1')| \\ &\leq (1-t) \max_i |\lambda_i(\zeta_0, \zeta_0')| + t \max_i |\lambda_i(\zeta_1, \zeta_1')|, \end{aligned}$$

and the last term is simply $(1-t) d_\infty(\zeta_0, \zeta_0') + t d_\infty(\zeta_1, \zeta_1')$, which is the desired result. □

1.3 Spaces of norms on graded algebras over non-Archimedean fields.

Throughout this Section, $(\mathrm{K}, |\cdot|)$ will be a non-Archimedean field, and $V_\bullet$ will denote a graded K-algebra

$$V_\bullet = \bigoplus_{k \in \mathbb{N}} V_k,$$

such that $V_0 = \mathrm{K}$, each V_k is a finite-dimensional K-vector space. We assume that $V_\bullet$ is furthermore **generated in degree one** in that the multiplication morphisms

$$V_1^{\odot k} \to V_k$$

(where $\odot$ denotes the symmetric product) are surjective for all $k \in \mathbb{N}^*$.

1.3.1 Bounded graded norms.

An algebra norm on $V_\bullet$ compatible with the grading may be characterized as the data of norms $\zeta_\bullet = (\zeta_k)_k$ acting on each V_k, satisfying the following **submultiplicativity** condition: given $v_k \in V_k$ and $v_\ell \in V_\ell$, we must have that

$$\zeta_{k+\ell}(v_k \cdot v_\ell) \leq \zeta_k(v_k) \cdot \zeta_\ell(v_\ell).$$

A sequence of norms on $V_\bullet$ satisfying this condition is called a **graded norm**. In order to study asymptotic properties of graded norms, using e.g. Fekete's lemma, we will need a growth condition that is both natural and ensures that the graded norms does not "blow up". We explain here how to formulate such a condition algebraically. This requires an additional definition.

Definition 1.3.1.1. We shall say that a graded norm $\zeta_\bullet$ on $V_\bullet$ is **generated in degree one** if, for all $k \in \mathbb{N}^*$, ζ_k is the quotient norm induced by the surjective symmetry morphism $V_1^{\odot k} \to V_k$.

Example 1.3.1.2. If K is trivially valued, the most simple example of a graded norm generated in degree one is the **trivial graded norm** $\zeta_{\mathrm{triv},\bullet}$, given by $\zeta_{\mathrm{triv},k}(v_k) = 1$ for all $v_k \in V_k - \{0_{V_k}\}$. Later on, we will see how graded norms generated in degree one naturally arise in a geometric context.

Definition 1.3.1.3. We will say that a graded norm $\zeta_\bullet$ on $V_\bullet$ is a **bounded graded norm** if it has at most exponential distorsion with respect to a norm generated in degree one, i.e. there exists a graded norm $\zeta'_\bullet$ generated in degree one on $V_\bullet$, and a constant $C > 0$ with

$$e^{-kC}\zeta'_m \leq \zeta_k \leq e^{kC}\zeta'_m$$

for all $k \in \mathbb{N}^*$. We denote by $\mathcal{N}_\bullet(V_\bullet)$ the set of such graded norms.

1.3.2 Asymptotic spectral measures, volumes, and metric structures on $\mathcal{N}_\bullet(V_\bullet)$

We now see how to transpose the constructions of Section 1.2.3.

Theorem 1.3.2.1. *Fix two bounded graded norms $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(V_\bullet)$. Then, the sequence of rescaled measures*

$$(k \cdot \dim_{\mathrm{K}}(V_k))_*^{-1}\sigma(\zeta_k, \zeta'_k)$$

*weakly converges to a compactly supported probability measure. We call this limit measure the **relative spectral measure** of $\zeta_\bullet$ and $\zeta'_\bullet$:*

$$\sigma(\zeta_\bullet, \zeta'_\bullet) = \lim_k (k \cdot \dim_{\mathrm{K}}(V_k))_*^{-1}\sigma(\zeta_k, \zeta'_k).$$

Remark 1.3.2.2. This Theorem is proven in [CM15, Theorem 5.2], see also [BE, Theorem 9.5] and the similar statement in [Reb20a], building on ideas of [Nys09]. The trivially-valued case is proven in [BJ18a, Theorem 3.2]. We will state this result in larger generality in Chapter 3.

Definition 1.3.2.3. Fix two bounded graded norms $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(V_\bullet)$. We define:

- for $p \in [1, \infty)$, their **asymptotic d_p-distance** by:

$$d_1(\zeta_\bullet, \zeta'_\bullet) = \int_{\mathbb{R}} |\lambda| \, d\sigma(\zeta_\bullet, \zeta'_\bullet);$$

- their **asymptotic d_∞-distance** by:

$$d_\infty(\zeta_\bullet, \zeta'_\bullet) = \sup_{k \in \mathbb{N}^*} k^{-1} d_\infty(\zeta_k, \zeta'_k);$$

- their **asymptotic relative volume** by:

$$\mathrm{vol}(\zeta_\bullet, \zeta'_\bullet) = \int_{\mathbb{R}} \lambda \, d\sigma(\zeta_\bullet, \zeta'_\bullet).$$

One then sees that the asymptotic d_p-distances may be recovered as the limit of the finite-dimensional distances

$$k^{-1} d_p(\zeta_k, \zeta'_k)^p = k^{-1} \int_{\mathbb{R}} |\lambda|^p \, d\sigma(\zeta_k, \zeta'_k).$$

Similarly, the asymptotic relative volume is recovered as

$$\mathrm{vol}(\zeta_\bullet, \zeta'_\bullet) = \lim_k k^{-1} \, \mathrm{vol}(\zeta_k, \zeta'_k).$$

It therefore has the same algebraic properties of finite-dimensional volumes, i.e. the cocycle property and antisymmetry.

Note that the d_p "distances" above are merely pseudodistances: for example, since for any two norms ζ_k, ζ'_k on a fixed V_k,

$$d_1(\zeta_k, \zeta'_k) \leq d_\infty(\zeta_k, \zeta'_k),$$

then if two bounded graded norms have at most subexponential growth in k, we have

$$d_1(\zeta_\bullet, \zeta'_\bullet) \leq \lim_k k^{-1} d_\infty(\zeta_k, \zeta'_k) = 0.$$

Even worse: there exist bounded graded norms such that $k^{-1} d_\infty(\zeta_k, \zeta'_k) \to C > 0$ but $d_1(\zeta_\bullet, \zeta'_\bullet) = 0$, see e.g. [BJ18a, R3.8]. This justifies the following definition.

Definition 1.3.2.4. We say that two bounded graded norms $\zeta_\bullet$ and $\zeta'_\bullet$ on $V_\bullet$ are **asymptotically equivalent**, and we write

$$\zeta_\bullet \sim \zeta'_\bullet,$$

if and only if

$$d_1(\zeta_\bullet, \zeta'_\bullet) = 0.$$

We then have that

$$(\mathcal{N}_\bullet(L)/\sim, d_1)$$

with the induced d_1 distance, is a *bona fide* metric space.

Remark 1.3.2.5. One has that $\zeta_\bullet \sim \zeta'_\bullet$ if one of the three following equivalent conditions is realized:

- for some $p \in [1, \infty)$, $\int_{\mathbb{R}} |\lambda|^p \, d\sigma(\zeta_\bullet, \zeta'_\bullet) = 0$;
- for all $p \in [1, \infty)$, $\int_{\mathbb{R}} |\lambda|^p \, d\sigma(\zeta_\bullet, \zeta'_\bullet) = 0$;
- the asymptotic spectral measure $\sigma(\zeta_\bullet, \zeta'_\bullet)$ is the Dirac measure δ_0.

This is proven in the trivially valued case in [BJ18a, Section 3.6], and the proofs are identical in the nontrivially valued case. In particular, this justifies our choice of notation for $\sim$, which does not emphasize the choice of a $p \in [1, \infty)$.

Remark 1.3.2.6. Note that, since d_∞ is defined as a sup rather than as a limit, it defines a genuine distance on $\mathcal{N}_\bullet(L)$.

Finally, we check that bounded graded norms remain bounded graded after piecewise ground field extension. This is a result of Boucksom-Eriksson:

Lemma 1.3.2.7 ([BE, L9.4])**.** Set $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(V_\bullet)$. Let L/K be a complete field extension, and consider the sequences of ground field extensions

$$\zeta_{\mathrm{L},\bullet},\ \zeta'_{\mathrm{L},\bullet}.$$

Then, those sequences are bounded graded norms, and furthermore

$$\sigma(\zeta_{\mathrm{L},\bullet}, \zeta'_{\mathrm{L},\bullet}) = \sigma(\zeta_\bullet, \zeta'_\bullet),$$

which implies stability of the asymptotic volumes and d_p-distances under base change.

1.3.3 Geodesics between bounded graded norms.

In [Reb20b], we extend the classical results of Section 1.2.5 to the context of bounded graded norms. Namely, given two bounded graded norms $\zeta^0_\bullet$ and $\zeta^1_\bullet$, one wonders whether there exists a geodesic of graded norms in $\mathcal{N}_\bullet(V_\bullet)$ for the asymptotic d_1-distance (or d_p, $p < \infty$). It seems obvious to consider, for all k, the norm geodesic ζ^t_k joining ζ^0_k and ζ^1_k in V_k, and to set, for all t,

$$\zeta^t_\bullet = (\zeta^t_k)_k.$$

There are two points to show here: *submultiplicativity*, and *geodesicity*. The latter is rather simple: by definition of our norm geodesics, we have that, for t, $t' \in [0,1]$,

$$d_p(\zeta_k^t, \zeta_k^{t'}) = |t - t'| \cdot d_p(\zeta_k^0, \zeta_k^1),$$

so that geodesicity follows upon taking the limit in k. Showing submultiplicativity is a bit trickier.

Theorem 1.3.3.1. *For all $t \in [0,1]$, the sequence of norms $\zeta_\bullet^t$ defined above is submultiplicative.*

Proof of Theorem 1.3.3.1. We assume at first that all norms involved are diagonalizable. We start with the following case: let v_m belong to a basis of V_m orthogonal for ζ_m^0 and ζ_m^1. Define in in the same way $s_n \in V_n$. We then have that

$$\begin{aligned}
\zeta_{m+n}^t(v_m \cdot v_n) &\leq \zeta_{m+n}^0(v_m \cdot v_n)^{1-t}\zeta_{m+n}^1(v_m \cdot v_n)^t \\
&\leq \zeta_m^0(v_m)^{1-t}\zeta_n^0(v_n)^{1-t}\zeta_m^1(v_m)^t\zeta_n^1(v_n)^t \\
&= \zeta_m^t(v_m) \cdot \zeta_n^t(v_n),
\end{aligned}$$

where we have used log-convexity (Lemma 1.2.5.3) in the first inequality, submultiplicativity of the endpoints in the second inequality, and finally the fact that v_m and v_n belong to bases codiagonalizing the endpoints, so that $\zeta_m^t(v_m) = \zeta_m^0(v_m)^{1-t}\zeta_m^1(v_m)^t$, and the same holds for n.

To pass to the general case, write

$$v_m = \sum a_i v_{m,i},\ v_n = \sum b_j v_{n,j},$$

in their adapted bases, and note that

$$\begin{aligned}
\zeta_{m+n}^t(v_m \cdot v_n) &\leq \max_{i,j} |a_i| \cdot |b_j| \cdot \zeta_{m+n}^t(v_{m,i} \cdot v_{n,j}) \\
&\leq \max_{i,j} |a_i| \cdot |b_j| \cdot \zeta_m^t(v_{m,i}) \cdot \zeta_n^t(v_{n,j}) \\
&\leq (\max_i |a_i| \cdot \zeta_m^t(v_{m,i})) \cdot (\max_j |b_j| \cdot \zeta_n^t(v_{n,j})) \\
&\leq \zeta_m^t(v_m) \cdot \zeta_n^t(v_n).
\end{aligned}$$

The second inequality follows from the result we just proved, which applies to the $v_{m,i}$ and the $v_{n,j}$; the fourth inequality follows from the fact that those bases diagonalize ζ_m^t and ζ_n^t. This proves the desired result.

Finally, if the norms are not diagonalizable, pick, for $i = 0, 1$, $\varepsilon > 0$, and all $m \in \mathbb{N}^*$, diagonalizable norms $\zeta_m^{i,\varepsilon}$ such that

$$d_\infty(\zeta_m^{i,\varepsilon}, \zeta_m^i) < \varepsilon.$$

Those always exist by d_∞-density of the set of diagonalizable norms on a finite-dimensional K-vector space. By the distortion characterization of d_∞, we then have that, for all m:

$$e^{-\varepsilon}\zeta_m^i \le \zeta_m^{i,\varepsilon} \le e^{\varepsilon}\zeta_m^i. \tag{1.1}$$

Pick sections $v_m \in V_m$, $v_\ell \in V_\ell$. We then have that

$$\begin{aligned}\zeta_{m+\ell}^{i,\varepsilon}(v_m \cdot v_\ell) &\le e^{\varepsilon}\zeta_{m+\ell}^i(v_m \cdot v_\ell)\\ &\le e^{\varepsilon}\zeta_m^i(v_m) \cdot \zeta_\ell^i(v_\ell)\\ &\le e^{3\varepsilon}\zeta_m^{i,\varepsilon}(v_m) \cdot \zeta_\ell^{i,\varepsilon}(v_\ell).\end{aligned}$$

We have used the right-hand side of (1.1) for the first inequality; submultiplicativity of $\zeta_\bullet^i$ for the second inequality, and finally the left-hand side of (1.1) for the third one. Multiplying both sides by $e^{3\varepsilon}$ we then have that

$$e^{3\varepsilon}\zeta_{m+\ell}^{i,\varepsilon}(v_m \cdot v_\ell) \le e^{6\varepsilon}\zeta_m^{i,\varepsilon}(v_m) \cdot \zeta_\ell^{i,\varepsilon}(v_\ell),$$

i.e. the sequence of norms $e^{3\varepsilon}\zeta_\bullet^{i,\varepsilon}$ is submultiplicative. As

$$d_\infty(e^{3\varepsilon}\zeta_m^{i,\varepsilon}, \zeta_m^i) = 3\varepsilon + d_\infty(\zeta_m^{i,\varepsilon}, \zeta_m^i) < 4\varepsilon,$$

one can see the $e^{3\varepsilon}\zeta_m^{t,\varepsilon}$ to be the norm geodesics joining the $e^{3\varepsilon}\zeta_m^{i,\varepsilon}$, and also actually $e^{3\varepsilon}$ times the geodesic joining the $\zeta_m^{i,\varepsilon}$). We find

$$d_\infty(e^{3\varepsilon}\zeta_m^{t,\varepsilon}, \zeta_m^t) \le (1-t)d_\infty(e^{3\varepsilon}\zeta_m^{0,\varepsilon}, \zeta_m^0) + td_\infty(e^{3\varepsilon}\zeta_m^{1,\varepsilon}, \zeta_m^1) < 4\varepsilon,$$

thanks to metric convexity of d_∞ (Lemma 1.2.5.10). This states that $e^{3\varepsilon}\zeta_m^{t,\varepsilon}$ converges pointwise to ζ_m^t for all m. As $e^{3\varepsilon}\zeta_m^{i,\varepsilon}$ is diagonalizable for all m, by the previous case, $e^{3\varepsilon}\zeta_\bullet^{t,\varepsilon}$ is submultiplicative, and in particular we have

$$e^{3\varepsilon}\zeta_{m+\ell}^{t,\varepsilon}(v_m \cdot v_\ell) \le e^{6\varepsilon}\zeta_m^{t,\varepsilon}(v_m)\zeta_\ell^{t,\varepsilon}(v_\ell)$$

Using the pointwise convergence found above to pass to the limit as $\varepsilon \to 0$, this proves the Theorem. $\square$

We remark that the last part of the above proof also shows the following:

Proposition 1.3.3.2. *Let $\zeta_\bullet$ be a bounded graded norm on $V_\bullet$. Then, there exist bounded graded norms $\zeta_\bullet^\varepsilon$ on $V_\bullet$, for all $\varepsilon > 0$, satisfying the following properties:*

- *ζ_m^ε is diagonalizable for all m;*
- *$d_\infty(\zeta_m^\varepsilon, \zeta_m) < \varepsilon$ for all m.*

1.3.4 Completeness of $\mathcal{N}_\bullet$ with respect to the d_∞ distance.

To conclude this Chapter, we prove the following unpublished result.

Theorem 1.3.4.1. *The metric space $(\mathcal{N}_\bullet(V_\bullet), d_\infty)$ is complete.*

Proof. Let $k \mapsto \zeta_\bullet^k$ be a Cauchy sequence of bounded graded norms with respect to the d_∞ distance. Then, for all $\varepsilon > 0$, there exist an $k_\varepsilon > 0$ such that for all m, $n > k_\varepsilon$,

$$\sup_{p \in \mathbb{N}^*} k^{-1} d_\infty(\zeta_p^m, \zeta_p^n) < \varepsilon,$$

which implies that for all integers k,

$$d_\infty(\zeta_p^m, \zeta_p^n) < p \cdot \varepsilon.$$

In other words, the sequence $k \mapsto \zeta_p^k$ is Cauchy with respect to the d_∞ distance on $\mathcal{N}(V_p)$. Since spaces of norms on finite-dimensional vector spaces are complete with respect to d_∞, there exists a d_∞-limit $\zeta_p \in \mathcal{N}(V_p)$ to this sequence. The family of norms $\zeta_\bullet : p \mapsto \zeta_p \in \mathcal{N}(V_p)$ is then by definition a d_∞-limit to the sequence of bounded graded norms $(\zeta_\bullet^k)_k$.

We claim that $\zeta_\bullet$ is bounded graded. Boundedness follows from the triangle inequality, and we must therefore show that $\zeta_\bullet$ is graded. This will follow from the "submultiplicativity trick" from the proof of submultiplicativity of geodesics in $\mathcal{N}_\bullet(L)$. We thus first choose an $\varepsilon > 0$ and pick k large enough so that

$$d_\infty(\zeta_\bullet^k, \zeta_\bullet) < \varepsilon.$$

For all integers p, this yields

$$d_\infty(\zeta_p^k, \zeta_p) < p \cdot \varepsilon,$$

and using the fact that d_∞ is the maximal exponential distorsion between two norms, we find

$$e^{-p\cdot\varepsilon}\zeta_p^k \leq \zeta_k \leq e^{p\cdot\varepsilon}\zeta_p^k,$$

for all p uniformly. We now pick p, q integers, and elements $v_p \in V_p$, $v_q \in V_q$. Using submultiplicativity of $\zeta_\bullet^k$, we find:

$$\begin{aligned}\zeta_{p+q}(v_p \cdot v_q) &\leq e^{p\cdot\varepsilon}\zeta_{p+q}^k(v_p \cdot v_q)\\ &\leq e^{p\cdot\varepsilon}\zeta_p^k(v_p)\zeta_q^k(v_q)\\ &\leq e^{3p\cdot\varepsilon}\zeta_p(v_p)\zeta_q(v_q).\end{aligned}$$

Since, for all $\varepsilon > 0$, we can find a k such that this works, this proves that

$$\zeta_{p+q}(v_p \cdot v_q) \leq \zeta_p(v_p)\zeta_q(v_q).$$

□

Remark 1.3.4.2. It is expected that the quotient spaces $\mathcal{N}_\bullet(V_\bullet)/\sim$ are not complete with respect to any of the d_p distances, $p < \infty$. In the case where K is trivially valued and $p = 1$, the completion has been characterized by Boucksom-Jonsson ([BJ21]).

Chapter 2

Geometric preliminaries.

Conventions.

Throughout this Chapter, K will be a complete valued field. A **variety** X over K will be a geometrically integral, separated scheme, of finite type over K.

2.1 Analytic geometry over non-Archimedean fields.

2.1.1 The Berkovich analytification.

In analogy with the fundamental GAGA principle of Serre ([SerGAGA]), Berkovich defines in [BerkBook] an **analytification functor** sending a K-variety X to its **Berkovich analytification** X^{an}. It is a K-analytic space in the sense of [BerkBook], but we shall not develop this aspect here, and rather focus on the topological space X^{an}.

The affine case. Assume X to be the spectrum $\operatorname{Spec}\mathcal{A}$ of an algebra $\mathcal{A}$ of finite type over K. Then, the underlying set of $(\operatorname{Spec}\mathcal{A})^{\text{an}}$ is the set of multiplicative seminorms on $\mathcal{A}$ which extend the absolute value on K.

The topology on $(\operatorname{Spec}\mathcal{A})^{\text{an}}$ is the topology of pointwise convergence, that is: the coarsest topology such that, for all $a \in \mathcal{A}$, the evaluation map

$$X^{\text{an}} \ni |\cdot|_{\mathcal{A}} \mapsto |a|_{\mathcal{A}}$$

is continuous.

This construction gives a natural **kernel map** from $(\operatorname{Spec}\mathcal{A})^{\mathrm{an}}$ to $\operatorname{Spec}\mathcal{A}$, as follows. Let $|\cdot|$ be a point in $\operatorname{Spec}\mathcal{A}$. One associates to it the ideal $\mathfrak{a}_{|\cdot|} := \{a \in \mathcal{A},\ |a| = 0\}$. This is a prime ideal, and therefore defines a scheme point. The mapping

$$\ker(|\cdot|) = \mathfrak{a}_{|\cdot|}$$

is then well-defined, and continuous by definition.

The general case. If X is an abstract variety, we proceed by gluing the affine construction above. We cover X by affines $U_i = \operatorname{Spec}\mathcal{A}_i$. By [Berk93, Proposition 1.3.3], one can glue together the U_i^{an} provided each $U_{ij} := U_i \cap U_j$ is affine. This is ensured by separatedness of X. The kernel maps $\ker_i : U_i^{\mathrm{an}} \to U_i$ glue together as well to a global kernel map $\ker : X^{\mathrm{an}} \to X$.

We then have the following:

Theorem 2.1.1.1 ([BerkBook, Theorems 3.4.1, 3.4.8, 3.5.1, 3.5.3])**.** *Assume X to be a connected projective* K*-variety. The mapping $X \mapsto X^{\mathrm{an}}$ realizes an equivalence of categories between the categories of coherent sheaves on X and X^{an}. Furthermore, X^{an} is compact, connected, and Hausdorff.*

Example 2.1.1.2. Assume K to be $\mathbb{C}$ endowed with its Archimedean absolute value. Then, X^{an} is isomorphic to the usual analytification of the complex variety X in the sense of Serre ([SerGAGA]).

Example 2.1.1.3. Assume K to be trivially valued. Then, the points of X^{an} are identified with the set of **semivaluations** on X, i.e. the set of valuations on $\mathrm{K}(Y)$ for all irreducible subvarieties Y of X.

Example 2.1.1.4. If K is non-Archimedean, in general, one identifies X^{an} with pairs $(\mathfrak{a}, \nu_{\mathfrak{a}})$, where $\mathfrak{a}$ is a scheme point of X, and $\nu_{\mathfrak{O}}$ is a valuation on the residue field of that point, extending the valuation $-\log|\cdot|$ on K.

Example 2.1.1.5. If $X = \operatorname{Spec}\mathrm{K}$, then its analytification is again a point, corresponding to the absolute value on K.

This analytification is, as mentioned, *functorial*: this implies that, given a morphism of K-schemes $f : X \to Y$, one obtains a morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ compatible with the analytic structure morphisms.

2.1.2 Models and divisorial points.

We now assume K to be non-Archimedean, and X to be a connected projective K-variety.

Definition 2.1.2.1. A **model** of X is the data of:

1. a flat scheme $\mathcal{X}$ of finite type over the valuation ring K°;

2. an isomorphism $\mathcal{X} \times_{\mathrm{K}^\circ} \mathrm{K} \to X$ as schemes over K.

Example 2.1.2.2. If K is trivially valued, the only model of X is X itself, up to automorphisms. We will need an alternative definition, that of a *test configuration*. This will be considered in the next Subsection.

For the remainder of this Subsection, we assume K to be nontrivially valued.

Definition 2.1.2.3. As a scheme, $\operatorname{Spec} \mathrm{K}^\circ$ has two points: the **generic point** corresponding to the ideal $\{0\}$, with residue field K, and a closed point, the **special point**, corresponding to $\mathrm{K}^{\circ\circ}$, with residue field $\tilde{\mathrm{K}}$. Changing the base to K, resp. $\tilde{\mathrm{K}}$, amounts to taking the **generic fibre** $\mathcal{X}_{\mathrm{K}}$, resp. **special fibre** or **central fibre** $\mathcal{X}_s$ (we will sometimes also denote it by $\mathcal{X}_0$).

Remark 2.1.2.4. Note that, if K is discretely valued, K° is a discrete valuation ring. On the other hand, if K is densely valued, K° can never be Noetherian. This implies that, over densely valued fields, we will have to work with non-Noetherian schemes.

Definition 2.1.2.5. We say that a model $\mathcal{X}'$ **dominates** (resp. **properly dominates**) a model $\mathcal{X}$ if there exists a morphism (resp. a proper morphism) $\mathcal{X}' \to \mathcal{X}$. Any two models may always be jointly dominated by a third.

Definition 2.1.2.6. Let $\mathcal{X}$ be a model of X. Let $x = (\mathfrak{a}, \nu_{\mathfrak{a}})$ be a point of X^{an}, identified with a scheme point of X together with a valuation on the residue field at $\mathfrak{a}$ extending the underlying absolute value on K. Let $\mathrm{R}_{\mathfrak{a}}$ denote the associated valuation ring in the residue field $\mathrm{K}(\ker \mathfrak{a})$ at $\ker \mathfrak{a}$. The valuative criterion for properness ensures that there exists a unique lift of the map $\operatorname{Spec} \mathrm{R}_{\mathfrak{a}} \to \operatorname{Spec} \mathrm{K}^\circ$ to $\operatorname{Spec} \mathrm{R}_{\mathfrak{a}} \to \mathcal{X}$. We denote by $\mathrm{red}_{\mathcal{X}}(\mathfrak{a})$ the image of the closed point $\mathrm{K}^{\circ\circ}$ by this lift. The map $\mathrm{red}_{\mathcal{X}}$ so defined is called the **reduction map** associated to $\mathcal{X}$. We shall say that a point in $x \in X^{\mathrm{an}}$ is **divisorial** or **Shilov**, and we will write $x \in X^{\mathrm{div}}$, if there exists a model $\mathcal{X}$ such that $\mathrm{red}_{\mathcal{X}}(x)$ is a generic point of the special fibre $\mathcal{X}_s$. The set of divisorial points is dense in X^{an}.

Example 2.1.2.7. If K is discretely valued, and if we pick a normal model $\mathcal{X}$ of X, then $\mathcal{X}_s$ decomposes as a Weil divisor

$$\mathcal{X}_s = \sum a_i E_i,$$

each E_i an irreducible component of the special fibre; furthermore, the E_i each define a unique divisorial point x_{E_i}, such that $\mathrm{red}_{\mathcal{X}}(x_{E_i})$ the generic point of E_i, defined as follows. Let r be the smallest nonzero positive element in the value group of K. Then,

$$x_{E_i} = r^{\mathrm{ord}_{E_i}(\cdot)/a_i},$$

where a_i is the coefficient of E_i in the Weil decomposition of the special fibre.

We now assume K to be trivially valued. We then define the set X^{div} of divisorial points inside the analytification X^{an} of X with respect to the trivial absolute value on K, to be the set of points of the form

$$c \cdot \mathrm{ord}_E(\cdot),$$

where c is a positive rational constant, and E is a prime divisor in a projective, normal birational model

$$E \subset Y \to X$$

of X. It is again dense in X^{an}, provided X has nonzero dimension. We will not study here the notion of *test configuration*, which plays the role of models in the trivially-valued case.

2.2 Metrization of analytifications of line bundles.

2.2.1 Line bundles and models thereof.

We will fix X as usual, and let L be a line bundle on X. We will denote by

$$H^0(X, L),$$

or simply $H^0(L)$, its space of sections. Its dimension will be written as

$$h^0(X, L).$$

Finally, we use additive notation for tensor powers of line bundles, which means that we write

$$L^{\otimes k} \otimes M^{\otimes -1} = kL - M,$$

given another line bundle M on X.

We shall occasionally speak of $\mathbb{Q}$-**line bundles**, by which we mean formally an element of $\mathrm{Pic(X)} \otimes \mathbb{Q}$. For all divisible enough m, mL is therefore a genuine line bundle on X.

We will say that a line bundle L is:

1. **very ample** if its sections define a closed immersion of X into $\mathbb{P}^{h^0(X,L)-1}$;
2. **ample** if mL is very ample for large enough m;
3. **basepoint-free** if there is no point $x \in X$ such that all sections of L vanish at x;
4. **semiample** if mL is basepoint-free for large enough m.

We will say that a $\mathbb{Q}$-line bundle L is:

1. **ample** if mL is very ample for divisible enough m;
2. **semiample** if mL is basepoint-free for divisible enough m.

We now turn to polarized models.

Definition 2.2.1.1. Let L be a line bundle on X. A **model** $(\mathcal{X}, \mathcal{L})$ **of** (X, L) is the data of:

1. a model $\mathcal{X}$ of X, projective over K°;
2. a line bundle $\mathcal{L}$ on $\mathcal{X}$;
3. an isomorphism $\mathcal{L}_{\mathrm{K}} \simeq L$ compatible with the isomorphism $\mathcal{X}_{\mathrm{K}} \simeq X$.

One then says that $\mathcal{L}$ is a model of L **determined on** $\mathcal{X}$. Denote by $\pi : \mathcal{X} \to \operatorname{Spec} \mathrm{K}^\circ$ the structure morphism. We will say that a model $(\mathcal{X}, \mathcal{L})$ of (X, L) is:

1. **very ample** if $\mathcal{L}$ is π-very ample;

2. **ample** if $m\mathcal{L}$ is π-very ample for large enough m;

3. **basepoint-free** if the morphism $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective;

4. **semiample** if $m\mathcal{L}$ is basepoint-free for large enough m;

5. **nef** if $\mathcal{L} \cdot C \geq 0$ for all projective curves in the special fibre $\mathcal{X}_s$.

Definition 2.2.1.2. Let L be a $\mathbb{Q}$-line bundle on X. A **$\mathbb{Q}$-model** $(\mathcal{X}, \mathcal{L})$ is the data of:

1. a model $\mathcal{X}$ of X, projective over K°;

2. a $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $(\mathcal{X}, m\mathcal{L})$ is a model of (X, mL) for divisible enough m.

One then says that $\mathcal{L}$ is a $\mathbb{Q}$-model of L **determined on** $\mathcal{X}$. We will say that a $\mathbb{Q}$-model $(\mathcal{X}, \mathcal{L})$ of (X, L) is **ample** (resp. **semiample**) if for divisible enough m, $m\mathcal{L}$ is ample (resp. semiample).

Example 2.2.1.3. If $L = \mathcal{O}_X$, one can identify models of L determined on $\mathcal{X}$ with **vertical Cartier divisors** on $\mathcal{X}$, i.e. Cartier divisors supported in the special fibre of $\mathcal{X}$.

2.2.2 Metrics over analytifications of K-line bundles.

For $x \in X^{\mathrm{an}}$, we will denote by $\mathcal{H}(x)$ the completion of the residue field at x, endowed with its canonical absolute value.

Definition 2.2.2.1. A **continuous metric** ϕ on L is defined by the data of a collection of norms

$$|\cdot|_{\phi_x}$$

on each $L \otimes \mathcal{H}(x)$, indexed over $x \in X^{\mathrm{an}}$, such that for any section $s_U \in H^0(U, L)$ over a Zariski open set U, the composition

$$|s_U|_\phi : U^{\mathrm{an}} \xrightarrow{(s_U)^{\mathrm{an}}} (L|_U)^{\mathrm{an}} \xrightarrow{|\cdot|_\phi} \mathbb{R}_{\geq 0}$$

is continuous. We will denote the space of continuous metrics on L^{an} by $C^0(L^{\mathrm{an}})$. More generally, we can define a **singular metric** to be a sum of the form $\phi + u$, where ϕ is a continuous metric on L^{an}, and u is any function $u : X^{\mathrm{an}} \to [-\infty, \infty)$. Whenever we shall speak of metrics in the remainder of this thesis, we will assume that they can be singular. Similarly, we may define:

1. the spaces $L^\infty(L^{\mathrm{an}})$ and $L^\infty_{\mathrm{loc}}(L^{\mathrm{an}})$ of **bounded**, resp. **locally bounded metrics** on L^{an}, which are written $\phi + u$ for $\phi \in C^0(L^{\mathrm{an}})$ and u a function $X^{\mathrm{an}} \to \mathbb{R}$ which is bounded, resp. locally bounded;

2. given a Radon measure μ with finite total mass on X^{an}, the spaces $L^1(\mu, L^{\mathrm{an}})$ and $L^1_{\mathrm{loc}}(\mu, L^{\mathrm{an}})$ of **integrable**, resp. **locally integrable metrics** on L^{an} with respect to μ, which are written $\phi + u$ for $\phi \in C^0(L^{\mathrm{an}})$ and u a function $X^{\mathrm{an}} \to \mathbb{R}$ in $L^1(\mu, X^{\mathrm{an}})$, resp. $L^1_{\mathrm{loc}}(\mu, X^{\mathrm{an}})$.

Fix a reference continuous metric ϕ_{ref} on L^{an}. We also use additive conventions for metrics, which implies that, given two line bundles L and M on X,

- given metrics ϕ on L^{an} and ϕ' on M^{an}, the induced metric on $(kL-M)^{\mathrm{an}}$ is written as $k\phi - \phi'$;

- we may identify a metric ϕ with the (possibly singular) function $-\log |1|_{\phi-\phi_{\mathrm{ref}}}$ on X^{an}, since a metric on $(\mathcal{O}_X)^{\mathrm{an}}$ canonically identifies with a function on X^{an}.

In particular, by noticing that two metrics ϕ, ϕ' on L^{an} transform as $|\cdot|_\phi = |\cdot|_\psi e^{\phi'-\phi}$, we can see the spaces $C^0(L^{\mathrm{an}})$, $L^\infty(L^{\mathrm{an}})$, $L^\infty_{\mathrm{loc}}(L^{\mathrm{an}})$, $L^1(\mu, L^{\mathrm{an}})$, and $L^1_{\mathrm{loc}}(\mu, L^{\mathrm{an}})$ as affine spaces modelled on respectively $C^0(X^{\mathrm{an}})$, $L^\infty(X^{\mathrm{an}})$, $L^\infty_{\mathrm{loc}}(X^{\mathrm{an}})$, $L^1(\mu, X^{\mathrm{an}})$, and $L^1_{\mathrm{loc}}(\mu, X^{\mathrm{an}})$.

We shall write for the rest of this Chapter

$$|s| = |s|_{\phi_{\mathrm{ref}}}$$

for any section $s \in H^0(L)$.

2.2.3 Model metrics.

Given an ample line bundle L on X, we now define a class of metrics defined using very explicit algebraic data: that given by a model $(\mathcal{X}, \mathcal{L})$ of (X, L) - generalizing the idea that a K°-lattice in a K-vector space canonically defines a norm. Throughout this Subsection, we will use many results from (and follow) [BE, Section 5.3].

Definition 2.2.3.1. Let $(\mathcal{X}, \mathcal{L})$ be a model of (X, L). We define the **model metric** $\phi_{\mathcal{L}} \in C^0(L^{\mathrm{an}})$ as follows. Pick $x \in X^{\mathrm{an}}$, and recall that there is associated to $\mathcal{X}$ its *reduction map* $\mathrm{red}_{\mathcal{X}} : X^{\mathrm{an}} \to \mathcal{X}_s$. Pick a section s_U of $\mathcal{L}$ over a Zariski neighbourhood of $\mathrm{red}_{\mathcal{X}}(x)$, which does not vanish at the reduction of x. Using the identification $\mathcal{X}_{\mathrm{K}} \simeq X$, we may then consider the analytification $(X \cap U)^{\mathrm{an}} \ni x$, and notice the analytification $(s_U)^{\mathrm{an}}$ to be nonvanishing at x. We may therefore require that

$$|s_U|_{\phi_{\mathcal{L}}} \equiv 1$$

on $(X \cap U)^{\mathrm{an}}$. This gives a well-defined continuous metric on L^{an}: given another section s'_U of $\mathcal{L}$ over U nonvanishing at $\mathrm{red}_{\mathcal{X}}(x)$, then there exists a unit $u \in \mathcal{O}^{\times}_{\mathcal{X}}(U)$ such that $s'_U = u \cdot s_U$, which implies that $|s'_U|_{\phi_{\mathcal{L}}} = |s_U|_{\phi_{\mathcal{L}}} \equiv 1$.

While this definition seems rather abstract, we will see in the next Section that, if L has nice positivity properties, then model metrics on L^{an} can be recovered as more familiar objects.

Example 2.2.3.2. If K is trivially valued, and (X, L) has no automorphisms, so that the only model of (X, L) is itself, the associated model metric is the trivial metric on L^{an}. This gives some insight into how to generalize certain results from the trivially valued to the nontrivially valued case: model metrics can be seen as *playing the role of* the trivial metric in the case there is no canonical trivial model.

In order to define model metrics associated to $\mathbb{Q}$-models, we shall need the following result:

Lemma 2.2.3.3 ([BE, Lemma 5.10])**.** If $(\mathcal{X}, \mathcal{L})$ is a model of (X, L), then:

1. for all integers m, $\phi_{m\mathcal{L}} = m\phi_{\mathcal{L}}$;
2. if a model $(\mathcal{Y}, \mathcal{M})$ dominates a model $(\mathcal{X}, \mathcal{L})$ via $\pi : \mathcal{Y} \to \mathcal{X}$, then $\phi_{\pi^*\mathcal{L}} = \phi_{\mathcal{L}}$;
3. in particular, if $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are two models of (X, L), then $\phi_{\mathcal{L}} = \phi_{\mathcal{L}'}$ if and only if one can dominate both models by a third model $(\mathcal{Y}, \mathcal{M})$ with $\pi : \mathcal{Y} \to \mathcal{X}$, $\pi' : \mathcal{Y} \to \mathcal{X}'$, and such that $\pi^*\mathcal{L} = \pi'^*\mathcal{L}'$.

Definition 2.2.3.4. Let L be a $\mathbb{Q}$-line bundle, and $(\mathcal{X}, \mathcal{L})$ be a $\mathbb{Q}$-model of (X, L). We define $\phi_{\mathcal{L}} = m^{-1}\phi_{m\mathcal{L}}$ for any m divisible so that $m\mathcal{L}$ is a line bundle; this is well-defined by the first point of the above Lemma.

Given L, M two $\mathbb{Q}$-line bundles on X, and model metrics $\phi_{\mathcal{L}}$, $\phi_{\mathcal{M}}$ on L^{an} and M^{an}, as well as a rational $r \in \mathbb{Q}$, then $r\phi_{\mathcal{L}} - \phi_{\mathcal{M}}$ is a model metric on $rL - M$.

Remark 2.2.3.5. We may now begin to explain the sibylline remark at the beginning of our talk on models. Given a model $(\mathcal{X}, \mathcal{L})$ of (X, L), the space of sections $H^0(\mathcal{X}, \mathcal{L})$ determines a K°-lattice the K-vector space $H^0(X, L)$, and therefore a lattice norm on $\mathcal{N}(H^0(X, L))$. We will see shortly how to associate to such norms a metric on L^{an}, and we will in fact see that it will coincide with the metric $\phi_{\mathcal{L}}$ in nice cases.

2.2.4 Metrization of the canonical line bundle, after Temkin.

A surprising fact of life in the non-Archimedean world is that the (powers of the) canonical line bundle can always be endowed with a special metric, generalizing the log-discrepancy on the valuation space of a variety. This construction is due to Temkin ([Tem]). The details in the general case are rather complicated and out of the scope of this manuscript, and we therefore direct the reader to Temkin's original article (see also an exposition in Stevenson's thesis [StevThesis]). We will therefore focus on the special cases of a trivially-valued field, and a discretely-valued field of equal characteristic zero. Other references for this Subsection are [JM12], [BJ17], [BJ18a].

The trivially-valued case. If the base field is trivially-valued, any line bundle L on a projective K-variety X admits a *trivial metric* ϕ_{triv}, characterized as the model metric associated to the unique (trivial) model of (X, L), by which one may canonically identify metrics on L^{an} with functions on X^{an}. **Temkin's metric** ϕ_{Tem} on K_X^{an} is then characterized as

$$\phi_{\mathrm{Tem}} = \phi_{\mathrm{triv}} + A_X,$$

where ϕ_{triv} is thus the trivial metric on K_X^{an}, and A_X is the **log-discrepancy** function on X^{an}, whose definition we recall now. Given a divisorial point x on X^{an}, which is by definition an exponential of a valuation of the form

$$c \cdot \mathrm{ord}_E(\cdot),$$

where $c \in \mathbb{Q}_{>0}$, and E is a prime divisor in a projective, normal birational model $\pi : Y \to X$. Then, we set

$$A_X(x) = c \cdot (1 + \mathrm{ord}_E(K_Y - \pi^* K_X)).$$

This is extended to a lsc function on all of X^{an}, which is characterized as the largest lsc extension of A_X, as in [BJ18a, Theorem 2.1]. As we shall now see, a similar characterization holds in the discretely-valued case.

The discretely-valued, equal characteristic zero case. There are now no longer any "canonical" models of the canonical bundle, and therefore now longer a trivial metric. Pick therefore a proper model $\mathcal{X}$ of X, and denote by $s : \mathcal{X} \to \operatorname{Spec} \mathrm{K}^\circ$ its structure morphism.

Ingredient 1: the model metric. As in [MN12, (4.1.1)] and [BJ17, (5.3)], one can define the **relative canonical divisor** $K_{\mathcal{X}/\operatorname{Spec}\mathrm{K}^\circ}$ associated to the morphism s, and thus the **relative log canonical divisor**

$$K^{\log}_{\mathcal{X}/\operatorname{Spec}\mathrm{K}^\circ} = K_{\mathcal{X}/\operatorname{Spec}\mathrm{K}^\circ} + (\mathcal{X}_s)_{\mathrm{red}} - \mathcal{X}_s,$$

where $(\mathcal{X}_s)_{\mathrm{red}}$ denotes the reduction (in the sense of scheme theory) of the special fibre of $\mathcal{X}$. Those are Weil divisors; if we furthermore assume one (hence both) to be ($\mathbb{Q}$-)Cartier, the relative log canonical divisor is then a model of K_X, defining a model metric $\phi_{K^{\log}_{\mathcal{X}/\operatorname{Spec}\mathrm{K}^\circ}}$.

Ingredient 2: the log-discrepancy function. To $\mathcal{X}$ is also associated a log-discrepancy function $A_{\mathcal{X}}$, characterized on divisorial points as follows: let $\rho : \mathcal{Y} \to \mathcal{X}$ be some model dominating $\mathcal{X}$. Since any two models can be jointly dominated by a third model, any point in cX^{div} is a valuation ν_E associated to a divisor a the central fibre $\mathcal{Y}_s = \sum_i a_i E_i$ of such models. The value of the log discrepancy function against such a divisorial point is thus fully characterized via the formula

$$K_{\mathcal{Y}} + \mathcal{Y}_s = \rho^*(K_{\mathcal{X}} + \mathcal{X}_s) + \sum_i A_{\mathcal{X}}(\nu_{E_i}) a_i E_i.$$

This is then extended as before to a maximal lsc function $A_{\mathcal{X}}$ on X^{an}, as in [BJ17, (5.6)]. **Temkin's metric** is recovered as

$$\phi_{\mathrm{Tem}} = \phi_{K^{\log}_{\mathcal{X}/\operatorname{Spec}\mathrm{K}^\circ}} + A_{\mathcal{X}},$$

A striking observation is that this is *independent of the choice of a model* $\mathcal{X}$ (using the formula above characterizing $A_{\mathcal{X}}$, and the fact that the model metric of a dominating model is the model metric of the original model), thus giving a very natural new metric on K_X^{an}.

2.3 Pluripotential theory over nontrivially valued fields

2.3.1 Fubini-Study and plurisubharmonic metrics.

In the complex setting, due to the work of Demailly, the class of psh metrics on a holomorphic line bundle L is characterized as the smallest class stable under finite maxima, addition of constants, and decreasing limits, of metrics containing all Fubini-Study metrics, that is, metrics of the form

$$\phi = \log \frac{1}{2k} \sum_{j=1}^{h^0(kL)} |s_j|^2 e^{2\lambda_j},$$

where the s_j are a basepoint-free basis of sections of kL, and the λ_j are real constants. We will therefore define non-Archimedean metrics similarly, using an adequate "non-Archimedean version" of Fubini-Study metrics.

Definition 2.3.1.1. Let L be a $\mathbb{Q}$-line bundle on X. We define a **Fubini-Study metric** on L^{an} to be a metric of the form

$$\phi = \frac{1}{k} \log \max_{j=1,\dots,h^0(kL)} |s_j| e^{\lambda_j},$$

where the s_j are a basepoint-free basis of sections of kL, and the λ_j are real constants. We furthermore say that such a metric is:

- a $\mathbb{Q}$**-Fubini-Study metric** if the λ_j are all rational;
- a K**-rational Fubini-Study metric** if the λ_j belong to the value group of K;
- a **pure Fubini-Study metric** if the λ_j are all equal to zero.

We shall write $\mathrm{FS}(L^{\mathrm{an}})$, $\mathrm{FS}_{\mathbb{Q}}(L^{\mathrm{an}})$, $\mathrm{FS}_{\Gamma(\mathrm{K})}(L^{\mathrm{an}})$, and $\mathrm{FS}_0(L^{\mathrm{an}})$ for those classes, respectively. More generally, as in [BE], we can define the class $\mathrm{FS}_\Gamma(L^{\mathrm{an}})$ of Γ-Fubini-Study metrics, for any subgroup Γ of the real line.

We look at some immediate properties of Fubini-Study metrics.

Proposition 2.3.1.2 ([BE, Proposition 5.4])**.** *Given a subgroup Γ of $\mathbb{R}$, two line bundles L and M on X, and a positive integer k, then:*

1. $\mathrm{FS}_\Gamma(L^{\mathrm{an}}) + \mathrm{FS}_\Gamma(M^{\mathrm{an}}) \subset \mathrm{FS}_\Gamma(L^{\mathrm{an}} + M^{\mathrm{an}})$;
2. $\mathrm{FS}_\Gamma(mL^{\mathrm{an}}) = m\mathrm{FS}_\Gamma(L^{\mathrm{an}})$;
3. $\mathrm{FS}_\Gamma(L^{\mathrm{an}}) = \mathrm{FS}_{\mathbb{Q}(\Gamma+\Gamma(\mathrm{K}))}(L^{\mathrm{an}})$;
4. $\mathrm{FS}_\Gamma(L^{\mathrm{an}})$ *is stable under finite maxima.*

Remark 2.3.1.3. As a consequence of the third point above, we have

$$\mathrm{FS}_0(L^{\mathrm{an}}) = \mathrm{FS}_{\mathbb{Q}(\Gamma(\mathrm{K}))}(L^{\mathrm{an}}).$$

This implies that, after a large enough ground field extension, any Fubini-Study metric becomes pure.

As it turns out, "positive" model metrics are in correspondence with Fubini-Study metrics.

Theorem 2.3.1.4 ([BE, Theorem 5.14, Corollary 7.9])**.** *Let L be a $\mathbb{Q}$-line bundle. Set a metric $\phi \in C^0(L^{\mathrm{an}})$. We then have that:*

- *ϕ is a pure Fubini-Study metric if and only if it coincides with the model metric $\phi_{\mathcal{L}}$ associated to a semiample $\mathbb{Q}$-model $\mathcal{L}$ of L;*
- *ϕ is a Fubini-Study metric if and only if it coincides with the model metric $\phi_{\mathcal{L}}$ associated to a nef $\mathbb{Q}$-model $\mathcal{L}$ of L.*

We now take the statement of the complex regularization Theorem to be our definition of a non-Archimedean plurisubharmonic metric:

Definition 2.3.1.5. A metric ϕ on the analytification L^{an} a $\mathbb{Q}$-line bundle L is **plurisubharmonic** or **psh** if it can be written as a decreasing net of Fubini-Study metrics on L, and is not identically $-\infty$. We denote by $\mathrm{PSH}(L^{\mathrm{an}})$ the class of plurisubharmonic metrics on L^{an}.

Proposition 2.3.1.6 ([BJ21, Theorem 4.5])**.** *Let L, M be $\mathbb{Q}$-line bundles. The class* PSH *satisfies the following properties:*

1. *it contains all Fubini-Study metrics on L^{an};*
2. *it is stable under:*
    (a) *taking finite maxima;*

   (b) *addition of a real constant;*

   (c) *limits of decreasing nets;*

3. *the convex combination of two psh metrics on* L^{an} *is a psh metric on* L^{an}*;*

4. $\mathrm{PSH}(L^{\mathrm{an}}) + \mathrm{PSH}(M^{\mathrm{an}}) \subset \mathrm{PSH}(L^{\mathrm{an}} + M^{\mathrm{an}})$*;*

5. *if a net of psh metrics on* L^{an} *converges uniformly to a limit metric, then this limit metric is psh.*

Note that most of those properties follow directly from the definition, but $2(c)$ is not immediate, as we consider not only decreasing sequences, but decreasing limits of decreasing nets. We have cited [BJ21], which details the trivially valued case, but the proofs follow the same lines in the nontrivially valued case.

As desired, we then have a similar characterization as in the complex case:

Proposition 2.3.1.7 ([BJ21, Corollary 4.16])**.** *The class* $\mathrm{PSH}(L^{\mathrm{an}})$ *is the smallest class of metrics on* L *satisfying properties* 1. *and* 2. *above.*

We endow $\mathrm{PSH}(L^{\mathrm{an}})$ with the topology of pointwise convergence on the set of divisorial points $X^{\mathrm{div}} \subset X^{\mathrm{an}}$: a net ϕ_i in $\mathrm{PSH}(L^{\mathrm{an}})$ is said to converge to $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$ if and only if $\phi_i(x) \to \phi(x)$ for all $x \in X^{\mathrm{div}}$. It is important to note that psh metrics are uniquely determined by their restriction to X^{div}.

2.3.2 The Fubini-Study and supnorm operators.

We now introduce the *Fubini-Study operators*, which give a way to turn norms into Fubini-Study metrics:

Definition 2.3.2.1. We define the (m-th) **Fubini-Study operator** as follows:

$$\mathrm{FS}_m : \mathcal{N}(H^0(mL)) \to C^0(L^{\mathrm{an}}),$$
$$\zeta \mapsto \mathrm{FS}_m(\zeta) = \frac{1}{m} \log \sup_{s \in H^0(mL) - \{0\}} \frac{|s|}{\zeta(s)}.$$

The implicit claim that metrics in the image of FS_m are continuous is a consequence of [BE, Theorem 7.16].

Fubini-Study operators behave well under ground field extension:

Lemma 2.3.2.2 ([BE, Lemma 7.20])**.** Let K'/K be a complete field extension. Let $\zeta \in \mathcal{N}(H^0(kL))$ for some k. Let $L_{\mathrm{K}'}$ denote the base change of L to K', and $\zeta_{\mathrm{K}'}$ the ground field extension of ζ to the base change $H^0(kL_{\mathrm{K}'}) = H^0(kL)_{\mathrm{K}'}$. Then, $\mathrm{FS}_k(\zeta_{\mathrm{K}'})$ coincides with the pullback of $\mathrm{FS}_k(\zeta)$ to $L_{\mathrm{K}'}^{\mathrm{an}}$.

There is a nice way to compute the Fubini-Study operators restricted to each $\mathcal{N}^{\mathrm{diag}}(H^0(mL))$. Indeed, assume (s_i) diagonalizes a norm ζ on $H^0(mL)$ for some large m. We then have that:

$$\mathrm{FS}_m(\zeta) = \frac{1}{m} \log \max_i \frac{|s_i|}{\zeta(s_i)}.$$

This remark combined with the above Lemma allows us to ensure that our Fubini-Study operators work as intended:

Corollary 2.3.2.3. *A metric in the image of some Fubini-Study operator is Fubini-Study.*

Proof. This is only nontrivial if K is not maximally complete, since not all norms on K-vector spaces are diagonalizable. But we can pick a maximally complete extension L of K, where all norms on K-vector spaces are diagonalizable, and the above Lemma proves our result. □

Remark 2.3.2.4. It follows that pure Fubini-Study metrics are characterized as those metrics ϕ which belong to the image of some Fubini-Study operator restricted to the set of lattice norms on some space of plurisections of L.

We now construct operators going the other way around: from the space of continuous metrics on L^{an} to spaces of norms on the $H^0(mL)$.

Definition 2.3.2.5. The **m-th supnorm operator** N_m sends $\phi \in C^0(L^{\mathrm{an}})$ to the norm on $H^0(mL)$ defined as

$$\mathrm{N}_m(\phi)(s) = \sup_{X^{\mathrm{an}}} |s|_{m\phi},$$

for $s \in H^0(mL)$.

It it straightforward from the definitions to see the following:

Proposition 2.3.2.6 (Lipschitz-like properties of FS and N)**.** *Consider two continuous metrics ϕ, ϕ' on L, and two norms ζ and ζ' on some $H^0(mL)$. We then have that:*

- $d_\infty(\mathrm{N}_m(\phi), \mathrm{N}_m(\phi')) \leq m \cdot \sup_{X^{\mathrm{an}}} |\phi' - \phi|$;
- $\sup_X |\mathrm{FS}_m(\zeta') - \mathrm{FS}_m(\zeta)| \leq m^{-1} \cdot d_\infty(\zeta, \zeta')$.

2.3.3 Plurisubharmonic envelopes.

Given a function $f : \mathbb{R}^d \to \mathbb{R}$, one can construct its convex envelope as the largest convex function bounded above by f. Given a metric on a holomorphic line bundle, one can again construct its plurisubharmonic envelope; it is a classical result from pluripotential theory that this envelope is continuous provided the original metric is continuous. We look here into similar problems in the non-Archimedean world. Throughout this Subsection, L is a line bundle on a projective K-variety X, with K non-Archimedean.

Definition 2.3.3.1. Let ϕ be a bounded metric on L^{an}. The **psh envelope** of ϕ is defined as

$$P(\phi) = \sup\{\phi' \in \mathrm{PSH}(L^{\mathrm{an}}),\, \phi' \leq \phi\}.$$

The **regular psh envelope** of ϕ is defined as

$$Q(\phi) = \sup\{\phi' \in \mathrm{PSH}(L^{\mathrm{an}}) \cap C^0(L^{\mathrm{an}}),\, \phi' \leq \phi\}.$$

Definition 2.3.3.2 (Continuity of envelopes)**.** We say that the pair (X, L) admits **continuity of envelopes** if the following property holds true:

- if ϕ is a continuous metric on L, then $P(\phi)$ is continuous.

Example 2.3.3.3. By [BJ18b], a smooth, projective variety X defined over any field K which satisfies all of the following properties:

- K is of equal characteristic 0;
- K is either trivially or discretely valued,

admits continuity of envelopes for any ample line bundle L over X. Furthermore, by [GJKM19], continuity of envelopes also holds:

- for any line bundle on a curve, over any field (from the work of Thuillier);
- for all line bundles on a d-dimensional variety X over K, where K is a discretely valued field of positive characteristic p, *provided we have resolution of singularities* over K in dimension $d+1$.

It is expected in general that, for X unibranch (in particular, normal) and L ample, over any non-Archimedean field, continuity of envelopes holds for (X, L).

One can show the following:

Theorem 2.3.3.4 ([BE, Theorem 7.26])**.** *Let ϕ be a bounded on L^{an}. Then,*

$$\lim_m \mathrm{FS}_m(\mathrm{N}_m(\phi)) = Q(\phi).$$

Here, the limit is taken over a divisible sequence of integers so that the associated power of L is always globally generated. Furthermore, the convergence is uniform if and only if $Q(\phi)$ is continuous.

We then have that:

Proposition 2.3.3.5. *The following are equivalent:*

1. *continuity of envelopes holds for (X, L);*
2. *for all continuous metrics ϕ on L^{an}, $P(\phi) = Q(\phi)$;*
3. *for all continuous metrics ϕ on L^{an}, $\lim_m \mathrm{FS}_m(\mathrm{N}_m(\phi)) = P(\phi)$;*
4. *the sequence $m \mapsto \mathrm{FS}_m(\mathrm{N}_m(\phi))$ converges uniformly;*
5. *given any family $(\phi_i)_{i\in I}$ of psh metrics on L^{an} uniformly bounded above, the usc regularization of the upper envelope*

   $$\mathrm{usc}(\sup_{i\in I} \phi_i)$$

   is psh.

Proof. That (2) and (3) are equivalent follows from Theorem 2.3.3.4. That (1) and (2) are equivalent is immediate. That (1) and (4) are equivalent follows from Theorem 2.3.3.4 again and Dini's Lemma. Finally, equivalence of (1) and (5) is the statement of [BE, Lemma 7.29]. □

This is the foundation for the *quantization* principle: approximating bounded psh metrics from below with metrics coming from norms (our space of Fubini-Study metrics!), which has proven its usefulness multiple times in complex geometry.

Definition 2.3.3.6. The **asymptotic Fubini-Study operator** is defined on the set of bounded graded norms on L as the usc regularization

$$\mathrm{FS}(\zeta_\bullet) = \mathrm{usc}\left(\lim_m \mathrm{FS}_m(\zeta_m)\right).$$

Remark 2.3.3.7. The asymptotic Fubini-Study operator is well-defined and defines a bounded psh metric provided that (X, L) admits continuity of envelopes, as, by Fekete's lemma,

$$\lim_m \mathrm{FS}_m(\zeta_m) = \sup_m \mathrm{FS}_m(\zeta_m).$$

We shall also define its brother:

Definition 2.3.3.8. The **graded supnorm operator**

$$\mathrm{N}_\bullet$$

sends a bounded metric ϕ to the bounded graded norm $(\mathrm{N}_m(\phi))_m$.

Definition 2.3.3.9. We say that a plurisubharmonic metric ϕ on L^{an} is **regularizable from below**, and we write

$$\phi \in \mathrm{PSH}^{\uparrow}(L^{\mathrm{an}}),$$

if and only if ϕ is the pointwise limit on X^{div} of an increasing net of Fubini-Study metrics, equivalently of an increasing net of continuous, psh metrics.

Remark 2.3.3.10. We then have that ϕ is the usc regularized supremum of such a net. Furthermore, $\phi \in \mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ if and only if $\phi = \mathrm{usc}\, Q(\phi)$.

We then show that $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ coincides with the image of the asymptotic Fubini-Study operator.

Theorem 2.3.3.11. *A metric ϕ belongs to $\mathrm{PSH}^{\uparrow}(L^{\mathrm{an}})$ if and only if there exists a bounded graded norm $\zeta_\bullet$ such that $\mathrm{FS}(\zeta_\bullet) = \phi$.*

Proof. Assume ϕ is the image of some bounded graded norm $\zeta_\bullet$ by the asymptotic Fubini-Study operator, i.e. $\phi = \operatorname{usc}(\lim_m \mathrm{FS}_m(\zeta_m))$. In particular, ϕ is psh, by continuity of envelopes. By the remark above, it is then enough to show that $\phi = \operatorname{usc} Q(\phi)$, which is clear by construction.

We now assume ϕ to be regularizable from below. By Theorem 2.3.3.4, $Q(\phi) = \lim_m \mathrm{FS}_m(\mathrm{N}_m(\phi))$. Then, by definition, $\operatorname{usc} Q(\phi) = \mathrm{FS}(\mathrm{N}_\bullet(\phi))$. Since $\phi \in \mathrm{PSH}^\uparrow(L^{\mathrm{an}})$, we have that $\phi = \operatorname{usc} Q(\phi)$, thus

$$\phi = \mathrm{FS}(\mathrm{N}_\bullet(\phi)),$$

which proves the Theorem. □

The purpose of Chapter 3, and indeed the paper [Reb20a], is to reverse this characterization: we show that the asymptotic Fubini-Study operator in fact defines a bijection between $\mathrm{PSH}^\uparrow(L^{\mathrm{an}})$, and the space of bounded graded norms on L modulo asymptotic equivalence.

2.4 The space of finite-energy plurisubharmonic metrics.

We conclude this preliminary Chapter by studying the space of non-Archimedean finite-energy metrics, in analogy with the works of Darvas and Berman-Boucksom-Eyssidieux-Guedj-Zeriahi in the complex case ([Dar17], [BBEGZ], [BBGZ]). We will again focus on the case where the base field K is nontrivially valued; the trivially-valued situation has been treated in much detail in [BJ21]. We again assume, throughout this Section, that X is a projective K-variety.

2.4.1 Monge-Ampère operators and Deligne pairings.

We begin with a discussion of Monge-Ampère operators in non-Archimedean pluripotential theory. Using either intersection pairings ([Gub], [BE]), or the theory of differential forms on Berkovich spaces developed by A. Chambert-Loir and A. Ducros in [CLD, 5, 6], one may define a Radon probability measure associated to $d = \dim X$ continuous plurisubharmonic metrics $\phi_1, \dots, \phi_d$

on analytifications of ample line bundles $L_1, \dots, L_d$ over the analytic space X^{an}, denoted

$$\mathrm{MA}(\phi_1, \dots, \phi_d) = V^{-1} \cdot dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_d \wedge \delta_X,$$

with V the intersection number of the L_i. For short, if (e.g.) the metric ϕ_1 appears n times in the expression, we write

$$\mathrm{MA}(\phi_1^{(n)}, \dots) = V^{-1} \cdot (dd^c\phi_1)^n \wedge \cdots \wedge \delta_X,$$

and so on; and we set

$$\mathrm{MA}(\phi) = \mathrm{MA}(\phi^{(d)}) = V^{-1} \cdot (dd^c\phi)^d \wedge \delta_X.$$

Proposition 2.4.1.1 ([BE, P8.3(iv)])**.** *Let* K'/K *be a non-Archimedean field extension. Consider the cartesian diagram:*

$$\begin{tikzcd} X^{\mathrm{an}}_{\mathrm{K}'} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X^{\mathrm{an}} \arrow[d] \\ (\operatorname{Spec} \mathrm{K}')^{\mathrm{an}} \arrow[r] & (\operatorname{Spec} \mathrm{K})^{\mathrm{an}} \end{tikzcd}$$

We then have that:

$$\pi_{1*}\left(dd^c(\pi_1^*\phi_1) \wedge \cdots \wedge dd^c(\pi_1^*\phi_d)\right) = dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_d.$$

An elegant way to encode information given by mixed Monge-Ampère operators is through the (metrized) Deligne pairing construction. it has a long history, starting from the complex case in Deligne's original article, treating the case of relative dimension 1 ([Del]), further generalized by Elkik in [Elk89], [Elk90]. Its use to formulate functionals arising complex geometry has been popularized via [PRS], and recently, Deligne pairings have also been shown to be of great use in non-Archimedean geometry ([BHJ16], [BE], see also [PRS, Remark 6]). The non-Archimedean case over a point has been thoroughly developed in [BE]. This is the case which we review now.

Consider a $(d+1)$-uple $L_0, \dots, L_d$ of ample line bundles on X. To this data, we can associate a line bundle $\langle L_0, \dots, L_d \rangle$ on $\operatorname{Spec} \mathrm{K}$ (i.e. a K-line), given by

$$\langle L_0, \dots, L_d \rangle = \sum_{I \subset \{0, \dots, d\}} (-1)^{d+1-|I|} \sum_{j=1}^{d+1} (-1)^j \det H^j \left(\sum_{i \in I} L_i \right),$$

i.e. the top-iterated difference of the determinant of cohomology. This construction has the property that it is symmetric and multilinear; furthermore, given a regular section s of L_0, we have an isomorphism

$$\langle L_0, \ldots, L_d \rangle \simeq \langle L_1|_{\operatorname{div} s}, \ldots, L_d|_{\operatorname{div} s} \rangle.$$

By multilinearity and symmetry, the difference

$$\langle L_0, \ldots, L_d \rangle - \langle M_0, \ldots, M_d \rangle$$

can be identified with the trivial line $\langle \mathcal{O}_X, \ldots, \mathcal{O}_X \rangle$ on $\operatorname{Spec} \mathrm{K}$ as soon as there exists some j with $L_j = M_j$. Recalling that $(\operatorname{Spec} \mathrm{K})^{\mathrm{an}}$ is a point, a non-Archimedean metric on the analytification of such a difference can then be identified with a genuine real number. Consider now, for each i, a *continuous* psh metric ϕ_i on L_i^{an}.

Theorem 2.4.1.2 ([BE, Theorem 8.16])**.** *To the data above, one can associate a metric $\langle \phi_0, \ldots, \phi_d \rangle$ on $\langle L_0, \ldots, L_d \rangle^{\mathrm{an}}$, which is uniquely characterized by the following properties:*

1. *the association of $\langle \phi_0, \ldots, \phi_d \rangle$ to $(\phi_0, \ldots, \phi_d)$ is symmetric and multilinear;*

2. *if s is a regular section of L_0, then*

$$\langle \phi_0, \ldots, \phi_d \rangle = \langle \phi_1|_{\operatorname{div} s}, \ldots, \phi_d|_{\operatorname{div} s} \rangle - \int_X \log |s|_{\phi_0} \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_d.$$

Remark 2.4.1.3. In particular, one obtains the *change of metric formula*: given another continuous psh metric ϕ_0' on L_0^{an}, we have

$$\langle \phi_0, \ldots, \phi_d \rangle - \langle \phi_0', \ldots, \phi_d \rangle = \int_X (\phi_0 - \phi_0') \, dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_d.$$

2.4.2 The Monge-Ampère energy.

In the previous Section, we have seen that (continuous or bounded) psh metrics and bounded graded norms were closely related. It turns out that there are also natural objects mimicking the relative volumes and d_1-distances on spaces of norms.

Fix an ample line bundle L on X. We define, given two continuous psh metrics ϕ_0, ϕ_1 on L^{an}, their **relative Monge-Ampère energy**:

$$E(\phi_0,\phi_1) = \frac{1}{d+1}\sum_{i=0}^{d}\int_X (\phi_0-\phi_1)\,\mathrm{MA}(\phi_0^{(i)},\phi_1^{(d-i)}).$$

Note that this is always finite as the metrics are continuous. Of interest to us are the following properties: given ϕ_0, ϕ_1, ϕ_2 a triple of continuous psh metrics on L^{an}, we have

- antisymmetry: $E(\phi_0,\phi_1) = -E(\phi_1,\phi_0)$;
- a cocycle property: $E(\phi_0,\phi_1) = E(\phi_0,\phi_2) + E(\phi_2,\phi_1)$;
- increasingness in the first argument: if $\phi_0 \leq \phi_1$, then $E(\phi_0,\phi_2) \leq E(\phi_1,\phi_2)$.

Remark 2.4.2.1. We would like to briefly address the issue of conventions: we follow those of [BJ18a], wherein the Monge-Ampère energy is normalized by the volume of L. This is not the case in [BE].

The Monge-Ampère energy admits an extension to the class $\mathrm{PSH}(L^{\mathrm{an}})$ via

$$E(\phi,\phi_{\mathrm{ref}}) = \inf\{E(\psi,\phi_{\mathrm{ref}}),\ \psi\geq\phi,\ \psi\in C^0(L^{\mathrm{an}})\cap\mathrm{PSH}(L^{\mathrm{an}})\}$$

for a fixed continuous psh metric ϕ_{ref}. We can also partially extend the relative Monge-Ampère energy, by setting

$$E(\phi,\phi') = E(\phi,\phi_{\mathrm{ref}}) - E(\phi',\phi_{\mathrm{ref}})$$

for ϕ, $\phi'\in\mathrm{PSH}(L)$, which is defined whenever at least one of the two terms in the right-hand side is finite. This extended relative Monge-Ampère energy can therefore take $-\infty$ or ∞ as values. This makes E continuous along decreasing nets.

Definition 2.4.2.2. The class $\mathcal{E}^1(L^{\mathrm{an}})$ of **finite-energy plurisubharmonic metrics** is defined as the set of psh metrics ϕ on L^{an} satisfying

$$E(\phi,\phi_{\mathrm{ref}}) > -\infty$$

for a reference continuous psh metric ϕ_{ref}.

Due to the cocycle property of the energy, the class $\mathcal{E}^1(L)$ is in fact independent of the choice of a reference metric, justifying our choice of notation for this class of metrics of finite energy.

Remark 2.4.2.3. A strong motivation to study this class is that mixed Monge-Ampère operators can be extended to $\mathcal{E}^1$, by the work of Boucksom-Favre-Jonsson (see [BFJ15, Section 6.3]).

As one can see, the relative Monge-Ampère energy shares some properties with the relative volume of bounded graded norms. In fact, we have the following results:

Theorem 2.4.2.4 ([BE, Theorem 9.15], [Reb20a, Theorem B])**.** *Let L be a semiample $\mathbb{Q}$-line bundle on X. Let $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(L)$. We then have:*

$$\lim_m E(\mathrm{FS}_m(\zeta_m), \mathrm{FS}_m(\zeta'_m)) = \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

Conversely, given two continuous psh metrics ϕ and ϕ' on L^{an}, we have

$$E(\phi, \phi') = \mathrm{vol}(N_\bullet(\phi), N_\bullet(\phi')).$$

The first part of the statement will be proven in the next Chapter. It will be helpful in proving that the asymptotic Fubini-Study operator is injective on the space of bounded graded norms modulo asymptotic equivalence.

2.4.3 The metric space of finite-energy psh metrics.

From now on, unless stated otherwise, we will assume that continuity of envelopes holds for (X, L). The space $C^0(L^{\mathrm{an}}) \cap \mathrm{PSH}(L^{\mathrm{an}})$ can be endowed with a metric structure as follows:

Definition 2.4.3.1. Consider two metrics ϕ_0, $\phi_1 \in C^0(L) \cap \mathrm{PSH}(L)$. We define

$$d_1(\phi_0, \phi_1) = d_1(\mathrm{N}_\bullet(\phi_0), \mathrm{N}_\bullet(\phi_1)),$$

where the distance in the right-hand side is the distance d_1 on bounded graded norms.

Remark 2.4.3.2. Define for ease of notation

$$\mathrm{vol}(\phi_0, \phi_1) = \mathrm{vol}(\mathrm{N}_\bullet(\phi_0), \mathrm{N}_\bullet(\phi_1)).$$

It follows (see e.g. [Reb20b, Remark 5.4.5]) that we have the formula

$$d_1(\phi_0, \phi_1) = \mathrm{vol}(\phi_0, P(\phi_0, \phi_1)) + \mathrm{vol}(\phi_1, P(\phi_0, \phi_1)).$$

By the results from the previous Subsection, this is also equal to

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1)).$$

This distance is sometimes called the **Darvas distance**, as it was introduced in [Dar15] in the complex case. We will see that, as in [Dar15], it extends as a distance on the space of finite-energy metrics.

Proposition 2.4.3.3. *The d_1 distance defined above is indeed a distance on the set of continuous psh metrics.*

Proof. Symmetry is immediate. The triangle inequality follows from taking the limit in the finite-dimensional triangle inequalities

$$k^{-1} d_1(\zeta_k, \zeta_k') \leq k^{-1} d_1(\zeta_k, \zeta_k'') + k^{-1} d_1(\zeta_k'', \zeta_k')$$

for any three bounded graded norms $\zeta_\bullet, \zeta_\bullet', \zeta_\bullet'' \in \mathcal{N}_\bullet(R)$. If $d_1(\phi, \phi') = 0$, then $N_\bullet(\phi)$ and $\mathrm{N}_\bullet(\phi')$ belong by definition to the same equivalence class of bounded graded norms. Since $\mathrm{FS}_\bullet \circ \mathrm{N}_\bullet$ is the identity on continuous psh metrics and $\mathrm{FS}_\bullet$ factors through asymptotic equivalence, it follows that $\phi = \phi'$. Finally, if $\phi = \phi'$, then their distance is naturally zero. $\square$

We now turn to the case of finite-energy metrics.

Theorem 2.4.3.4. *Assume continuity of envelopes to hold for (X, L). Then, $(\mathcal{E}^1(L^{\mathrm{an}}), d_1)$ is a metric space.*

Remark 2.4.3.5. In [BJ21], Boucksom-Jonsson prove completeness of $(\mathcal{E}^1(L^{\mathrm{an}}), d_1)$ in the trivially valued case, if and only if continuity of envelope holds. Their proof relies on deep results concerning spaces of measures of finite-energy on Berkovich spaces. The author expects to help in developing similar theory for the nontrivially-valued setting.

In order to prove the Theorem, we first show that our distance is well-defined:

Proposition 2.4.3.6. *Given two metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, $P(\phi_0, \phi_1)$ belongs to $\mathcal{E}^1(L^{\mathrm{an}})$.*

Proof. Fix a continuous L^{an}-psh reference metric ϕ_{ref}. Let, for $i = 0, 1$, $k \mapsto \phi_i^k$ be sequences of continuous psh metrics decreasing to ϕ_i. Assuming $\phi_{\mathrm{ref}} \geq \phi_0^k$ for all (large enough) k, we then have from Lemma 2.4.3.7 and the fact that the distance of two comparable metrics is a volume:

$$\begin{aligned} 0 \leq \mathrm{vol}(P(\phi_0^k, \phi_1^k), P(\phi_{\mathrm{ref}}, \phi_1^k)) &= E(P(\phi_0^k, \phi_1^k), P(\phi_{\mathrm{ref}}, \phi_1^k)) \\ &\leq E(\phi_0^k, \phi_{\mathrm{ref}}), \end{aligned}$$

Since E and P are continuous along decreasing nets, this gives at the limit

$$0 \leq E(P(\phi_0, \phi_1), P(\phi_{\mathrm{ref}}, \phi_1)) \leq E(\phi_0, \phi_{\mathrm{ref}}) < \infty.$$

In particular, using the cocycle property, $E(P(\phi_0, \phi_1), \phi_{\mathrm{ref}})$ is finite for any continuous psh reference metric, hence $P(\phi_0, \phi_1) \in \mathcal{E}^1(L^{\mathrm{an}})$. □

The following Lemma was used in the proof of the previous Proposition.

Lemma 2.4.3.7. Let ϕ_0, ϕ_1 be continuous L^{an}-psh metrics. Then, for any continuous L^{an}-psh metric ϕ, we have

$$d_1(P(\phi_0, \phi), P(\phi_1, \phi)) \leq d_1(\phi_0, \phi_1).$$

Proof. This is essentially an asymptotic version of [BJ18a, Lemma 3.1]. By continuity of envelopes, the two metrics in the left-hand side are continuous (and psh), so that they define bounded graded supnorms via the $\mathrm{N}_\bullet$ operator. By [BE, Theorem 7.27],

$$\begin{aligned} P(\phi_0, \phi) &= \mathrm{FS}_\bullet(\mathrm{N}_\bullet(\phi_0 \wedge \phi)) \\ &= \mathrm{FS}_\bullet(\mathrm{N}_\bullet(\phi_0) \vee \mathrm{N}_\bullet(\phi)) \end{aligned}$$

(note that the statement of [BE, Theorem 7.27] uses the envelope Q which corresponds to the envelope defined by Fubini-Study metrics; but for continuous metrics, $P = Q$ by [BE, Proposition 7.26]). Similarly,

$$P(\phi_1, \phi) = \mathrm{FS}_\bullet(\mathrm{N}_\bullet(\phi_1) \vee \mathrm{N}_\bullet(\phi)),$$

i.e.

$$\mathrm{N}_\bullet(P(\phi_0, \phi)) = \mathrm{N}_\bullet(\phi_0) \vee \mathrm{N}_\bullet(\phi)$$

and

$$\mathrm{N}_\bullet(P(\phi_1, \phi)) = \mathrm{N}_\bullet(\phi_1) \vee \mathrm{N}_\bullet(\phi).$$

Now, for all m, by [BJ18a, Lemma 3.1],

$$d_1(\mathrm{N}_m(\phi_0) \vee \mathrm{N}_m(\phi), \mathrm{N}_m(\phi_1) \vee \mathrm{N}_m(\phi)) \leq d_1(\mathrm{N}_m(\phi_0), \mathrm{N}_m(\phi_1)),$$

which at the limit and using the equalities above yields

$$d_1(\mathrm{N}_\bullet(P(\phi_0, \phi)), \mathrm{N}_\bullet(P(\phi_1, \phi))) \leq d_1(\mathrm{N}_\bullet(\phi_0), \mathrm{N}_\bullet(\phi_1)),$$

i.e. by definition

$$d_1(P(\phi_0, \phi), P(\phi_1, \phi)) \leq d_1(\phi_0, \phi_1),$$

as promised. □

In order to prove that d_1 satisfies the triangle inequality, and also to make some later results easier to prove, we will approximate the d_1 distance as follows. We approximate two metrics ϕ_0 and ϕ_1 in $\mathcal{E}^1(L^{\mathrm{an}})$ by sequences (ϕ_0^k), (ϕ_1^k) in $C^0(L^{\mathrm{an}}) \cap \mathrm{PSH}(L^{\mathrm{an}})$. We will show that

$$d_1(\phi_0, \phi_1) = \lim_k d_1(\phi_0^k, \phi_1^k).$$

Proposition 2.4.3.8. *Given two metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, and nets (ϕ_0^k), (ϕ_1^k) in $C^0(L^{\mathrm{an}}) \cap \mathrm{PSH}(L^{\mathrm{an}})$ decreasing to ϕ_0, ϕ_1 we have*

$$d_1(\phi_0, \phi_1) = \lim_k d_1(\phi_0^k, \phi_1^k).$$

Proof. By [Reb20b, Remark 5.4.5], i.e. the Darvas formula for d_1 on continuous psh metrics, we have for all k

$$d_1(\phi_0^k, \phi_1^k) = E(\phi_0^k, P(\phi_0^k, \phi_1^k)) + E(\phi_1^k, P(\phi_0^k, \phi_1^k)).$$

P is continuous along monotone (hence decreasing) nets, so that $P(\phi_0^k, \phi_1^k)$ decreases to $P(\phi_0, \phi_1) \in \mathcal{E}^1(L^{\mathrm{an}})$, and the result follows by continuity of the energy along decreasing nets. □

We may now show that $\mathcal{E}^1(L^{\mathrm{an}})$, endowed with d_1, is a metric space.

Proof of Theorem 2.4.3.4. Symmetry is immediate. The triangle inequality follows from Proposition 2.4.3.8 and the triangle inequality of d_1 on continuous psh metrics, so that we only have to show that our distance does indeed separate points.

Assume first that $\phi_0 \geq \phi_1$, so that the distance is in fact a Monge-Ampère energy. Then, Proposition 4.2.6.2 gives $\phi_0 = \phi_1$.

In the general case, we use Corollary 2.4.3.8 to find

$$0 = d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1)).$$

Both quantities on the right-hand side are positive, which yields

$$\phi_0 = P(\phi_0, \phi_1) = \phi_1,$$

by the previous argument, proving our result. $\square$

Chapter 3

The range of the asymptotic Fubini-Study operator over general non-Archimedean fields.

The main result.

In this Chapter, we will concern ourselves with proving the following result:

Theorem 3.0.0.1. *Let X be a projective* K*-variety, endowed with a semi-ample line bundle L. Assume (X, L) to admit continuity of envelopes. The asymptotic Fubini-Study operator* FS *then defines a bijection:*

$$\mathrm{FS} : \mathcal{N}_\bullet(L)/\sim \to \mathrm{PSH}^\uparrow(L).$$

This is a generalization of [BJ18a, Theorem 4.16], which treats the trivially-valued case. The main ingredient in the proof is the following result:

Theorem 3.0.0.2. *Assume (X, L) to admit continuity of envelopes. Then, given two bounded graded norms $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(L)$, we have that*

$$E(\mathrm{FS}(\zeta_\bullet), \mathrm{FS}(\zeta'_\bullet)) = \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

The idea of proof is as follows. In the case where $\mathrm{FS}(\zeta_\bullet)$ and $\mathrm{FS}(\zeta'_\bullet)$ are model metrics, this is essentially proven in [BE]. If $\zeta_\bullet$ is merely finitely generated, we then show that it can be nicely approximated by graded norms giving rise to model metrics, thereby extending the Theorem to this case. In the general case, inspired by Fujita's lemma, we show that we can approximate the volume of $\zeta_\bullet$ by the volumes of the $\zeta_\bullet^{(r)}$, which are graded norms on $R(X, rL)$ generated in degree r by propagating ζ_r through the surjective multiplication morphisms. This result builds on the theory of superadditive functions on Okounkov bodies developed in [CM15].

3.1 Some preliminary results on approximations of bounded graded norms.

We begin with some preliminary results on bounded graded norms, focusing especially on finitely generated norms. As a convention, we will assume L to be a semiample $\mathbb{Q}$-line bundle.

3.1.1 Bounded graded norms on section rings of semiample $\mathbb{Q}$-line bundles.

Since L is a semiample $\mathbb{Q}$-line bundle, there always exists a r such that $R(X, rL)$ is generated in degree one. Thus, while norms generated in degree one may not necessarily exist (as they require by definition $R(X, L)$ to be generated in degree one), the following class is always nonempty:

Definition 3.1.1.1. We say that a graded norm $\zeta_\bullet$ on some subalgebra $R(X, kL)$ of $R(X, L)$ is **finitely generated** if it is generated in degree one on $R(X, kL)$.

We may therefore extend the definition of a bounded graded norm to graded norms on $R(X, L)$:

Definition 3.1.1.2. A **bounded graded norm** on $R(X, L)$ is a graded norm $\zeta_\bullet$ on $R(X, L)$ such that there exists a finitely generated graded norm $\zeta'_\bullet$, generated in degree one on $R(X, kL)$ for some k, such that

$$d_\infty(\zeta_{k\bullet}, \zeta'_\bullet) < \infty.$$

The proof of existence of a spectral measure , as we will show, still works in such generality, and we may similarly define the d_p distances.

3.1.2 Okounkov bodies associated to section rings.

Throughout this Subsection, we follow [Bou12] and [KK12]. We begin with the following definition.

Definition 3.1.2.1. A **valuation with one-dimensional leaves** with values in an totally ordered group $(G, <)$ on $R(X, L)$ is a valuation

$$\nu : R(X, L) \to G,$$

such that for all $\alpha \in G$, and all positive integers k, the quotient vector spaces (or **leaves**)

$$\mathrm{gr}_{k,\alpha}(R(X, L)) := \{s \in H^0(kL),\, \nu(s) \geq \alpha\} / \bigcup_{\alpha' > \alpha} \{s \in H^0(kL),\, \nu(s) \geq \alpha'\}$$

have either dimension zero or one. In fact ([KK12, Proposition 2.4]), each graded piece $H^0(kL)$ decomposes as a sum of finitely many such leaves.

We will mostly be interested in valuations with value group $\mathbb{Z}^d$.

Definition 3.1.2.2. Let ν be a valuation with one-dimensional leaves, taking values in $(\mathbb{Z}^d, <)$ for some total order $<$ on $\mathbb{Z}^d$. We define the sub-semigroup $\Gamma(H^0(kL)) \subseteq \mathbb{N}^d$ of possible values of ν:

$$\Gamma(H^0(kL)) = \nu(H^0(kL)),$$

and finally the **semigroup of integral points** of $(R(X, L), \nu)$, denoted by $\Gamma(R(X, L))$ as the graded sub-semigroup of $\mathbb{N}^{d+1}$ defined as follows:

$$\Gamma(R(X, L)) = \{(k, \alpha) \in \mathbb{N} \times \mathbb{N}^d,\, \alpha \in \Gamma(H^0(kL))\}.$$

It then follows that $(k, \alpha) \in \Gamma(R(X, L))$ if and only if $\dim \mathrm{gr}_{k,\alpha}(R(X, L)) = 1$. For brevity, we will write

$$\Gamma(R(X, L)) = \Gamma(L).$$

Note that it still depends on the choice of a valuation.

Lemma 3.1.2.3 ([Bou12, L2.11, P3.3])**.** The semigroup $\Gamma(L)$ satisfies the following properties:

(i) *linear growth:* $\Gamma(L)$ is contained within a finitely generated monoid $\langle a_1, \ldots, a_k \rangle$, $k < \infty$, where for all i, $a_i \in \{1\} \times \mathbb{N}^d$;

(ii) *bigness:* $\Gamma(L)$ generates $\mathbb{N}^{d+1}$ as a group.

Definition 3.1.2.4. We will define a **convex body** in $\mathbb{R}^d$ to be a subset of $\mathbb{R}^d$ which is compact, convex, and has nonempty interior.

Definition 3.1.2.5. As a consequence of the previous Lemma, one can define a convex body in $\mathbb{R}^d$ by projecting the base of the convex cone $\overline{\mathrm{Cone}}(\Gamma(L))$ to the last d variables:

$$\Delta(L) = \overline{\mathrm{Cone}}(\Gamma(L)) \cap (\{1\} \times \mathbb{R}^d).$$

This is the **Okounkov body** of $\Gamma(L)$. Again, $\Delta(L)$ depends on the choice of ν.

Remark 3.1.2.6. This construction generalizes the moment polytope of a polarized toric variety, as in [LM09, Section 6.1].

We now prove a result which is similar in spirit to Fujita's approximation Theorem, which will be of use in the third Chapter. We first start by recalling the following result:

Theorem 3.1.2.7 ([Bou12, L1.13])**.** *Let $\Gamma_\bullet$ be a sub-graded semigroup of $\mathbb{N}^{d+1}$ satisfying conditions (i)-(ii) of Lemma 3.1.2.3, and let K be a compact convex subset of $\mathbb{R}^d$ contained in the interior of the Okounkov body $\Delta(\Gamma_\bullet)$. For all large enough integers m, we then have that:*

$$K \cap \frac{\Gamma_m}{m} = K \cap \frac{\mathbb{Z}^d}{m},$$

where Γ_m is the m-th graded piece of $\Gamma_\bullet$.

We now prove the approximation result in question, which can also be seen as an *ad hoc* version of [Bou12, L1.21].

Lemma 3.1.2.8. Let $\Gamma^k_\bullet$ be a sub-graded semigroup of some sub-graded semigroup $\Gamma_\bullet$ of $\mathbb{R}^{d+1}$, such that:

- $\Gamma^k_1 = \Gamma_k$,
- $\Gamma^k_r \subseteq \Gamma_{kr}$ for all $r \geq 1$,

- $\Gamma_\bullet$ satisfies the properties (i) and (ii) of linear growth and bigness as in Section 4.1.

We then have that

$$k^{-d}\,\mathrm{vol}(\Delta(\Gamma^k_\bullet)) \to_{k\to\infty} \mathrm{vol}(\Delta(\Gamma_\bullet)).$$

Proof. First remark that, by the inclusion property

$$\Gamma^k_r \subseteq \Gamma_{kr},$$

we have that, for all $k \geq 1$,

$$\frac{\Delta(\Gamma^k_\bullet)}{k} \subseteq \Delta(\Gamma_\bullet).$$

If we can show that any compact (convex) subset K of $\Delta(\Gamma_\bullet)^o$ is also included in $\frac{\Delta(\Gamma^k_\bullet)}{k}$ for large enough k, then our assertion would be true. Pick such a compact K, and embed it into another compact convex subset $L \subset \Delta(\Gamma_\bullet)^o$ such that the number

$$d(K, \partial L) = \inf\{d(x,\ell),\, x \in K,\, \ell \in \partial L\}$$

is (strictly) positive. We then have compact inclusions

$$K \subset L \subset \Delta(\Gamma_\bullet)^o,$$

with K not "touching" the boundary of L.
By the bigness hypothesis, $\Gamma_\bullet$ generates $\mathbb{Z}^{d+1}$ as a group. Then, the regularization of Γ_k is $\mathbb{Z}^d$, whence, for all large enough k,

$$\left(L \cap \frac{\Gamma_k}{k}\right) = \left(L \cap \frac{\mathbb{Z}^d}{k}\right),$$

(by Theorem 3.1.2.7), so that the convex hull of $\left(\frac{\Gamma_k}{k}\right)$ naturally contains K. (It does not necessarily contain L.) Now, since

$$\Gamma^k_1 = \Gamma_k,$$

the convex hull of $\left(\frac{\Gamma_k}{k}\right)$ is contained in the scaled Okounkov body

$$\frac{\Delta(\Gamma^k_\bullet)}{k}.$$

To conclude, we have a chain of compact inclusions

$$K \subset \mathrm{Hull}\left(\frac{\Gamma_k}{k}\right) \subset \frac{\Delta(\Gamma^k_\bullet)}{k},$$

from which follows the desired inclusion of K. $\square$

3.1.3 Okounkov bodies and limit measures.

We now apply the constructions of the previous Subsection to a more precise geometric context.

Let x be a regular K-rational point of X, and pick a regular sequence $(z_1, \ldots, z_d)$ in the local ring $\mathcal{O}_{X,x}$. By Cohen's structure Theorem, any element $f \in \mathcal{O}_{X,x}$ may then be written as a formal power series

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha,$$

where the coefficients f_α belong to K. Pick a monomial valuation ν on $\mathrm{K}[[t_1, \ldots, t_d]]$. Given a section $s \in H^0(X, L)$, one may pick a trivialization of L at x, so that s defines an element $s_x \in \mathcal{O}_{X,x}$, and we proceed similarly for plurisections of L, allowing us to identify $R(X, L)$ with a subalgebra of $\mathrm{K}[[t_1, \ldots, t_d]]$. Note that this is independent of the choice of a trivialization. We may now apply the constructions of the previous Subsection provided we have a good choice of a valuation with one-dimensional leaves.

We first begin with the following definition.

Definition 3.1.3.1. A **monomial order** on $\mathbb{N}^d$ is defined to be a total order $\leq$ satisfying the following properties:

1. given any $\alpha \in \mathbb{N}^d$, $0_{\mathbb{N}^d} \leq \alpha$;
2. given any $\alpha \in \mathbb{N}^d$, for all $\alpha_0, \alpha_1 \in \mathbb{N}^d$ with $\alpha_0 \leq \alpha_1$, we have

$$\alpha_0 + \alpha \leq \alpha_1 + \alpha.$$

Note that such an order naturally extends to $\mathbb{Z}^d$.

Definition 3.1.3.2. A **monomial valuation** or **Gröbner valuation** on $R(X, L)$ is a valuation of the form

$$\nu\left(\sum_{\alpha \in \mathbb{Z}^d} f_\alpha z^\alpha\right) = \min{}_{\leq}\{\alpha \in \mathbb{Z}^d,\ f_\alpha \neq 0\},$$

with $\leq$ a monomial order on $\mathbb{N}^d$, and where we expand an element of the algebra $R(X, L)$ as a power series as above. Since the transcendence degree of the residue field of the valuation is 0, ν has one-dimensional leaves.

Since the leaves of a monomial valuation are one-dimensional, there exist elements s in each nonzero-dimensional $gr_{k,\alpha}$ which can be expanded as

$$s = z^{\alpha} + \sum_{\beta \geq \alpha} v_{\beta} z^{\beta},$$

i.e. with monomial first term. Given a bounded graded norm $\zeta_\bullet$ on L, the individual norm ζ_k induces a quotient norm $\zeta_{k,\alpha}$ on $\mathrm{gr}_{k,\alpha}(L)$. Given any s with a Taylor expansion as above, it is immediate that its class $[s]_{k,\alpha} =: s_{k,\alpha}$ in $\mathrm{gr}_{k,\alpha}(V_\bullet)$ contains all elements with such an expansion, and we define

$$\begin{aligned} \Phi : \Gamma(L) &\to \mathbb{R} \\ k, \alpha &\mapsto -\log\left[\, \zeta_{k,\alpha}\left(s_{k,\alpha}\right) \right]. \end{aligned}$$

By submultiplicativity of $\zeta_\bullet$ and the fact that

$$s_{k,\alpha} \cdot s_{\ell,\beta} = s_{k+\ell,\alpha+\beta}$$

in the algebra

$$\bigoplus_{k \in \mathbb{N}} \bigoplus_{\alpha \in \Gamma(H^0(kL))} \mathrm{gr}_{k,\alpha}(L),$$

the function Φ so defined is then superadditive on the semigroup $\Gamma(L)$.

Theorem 3.1.3.3. *Let $\zeta_\bullet$, $\zeta'_\bullet$ be bounded graded norms on $R(X, L)$. Then, the sequence of their relative spectral measures $\sigma_k(\zeta_k, \zeta'_k)$ converges weakly to a compactly supported measure on $\mathbb{R}$.*

Proof. By [CM15, T5.2], our desired limit measure exists provided the hypotheses of [CM15, T4.5] are verified, which means the following:

(1) $\zeta_\bullet$ and $\zeta'_\bullet$ are submultiplicative graded norms;

(2) $\lim_{m \to \infty} m^{-1} d_\infty(\zeta_m, \zeta'_m) < \infty$;

(3) there exists a uniform positive constant C such that

$$\inf_{\alpha \in \Gamma(H^0(kL))} -\Phi(k, \alpha) \geq -Cn,$$

where Φ is defined as above, and similarly for Φ' defined using the graded norm $\zeta'_\bullet$.

The criteria (1) and (2) are by definition true. What remains is to prove (3) which is equivalent to showing that $(k,\alpha) \mapsto \Phi(k,\alpha)$ is *linearly bounded above* in the first variable, i.e. there exists a uniform positive constant C such that

$$\Phi(k,\cdot) \leq C \cdot k.$$

By definition of the quotient norm $\zeta_{k,\alpha}$, it is enough to show that there exists C such that

$$-\log \zeta_k(v) \leq C \cdot k$$

holds for all s of the form

$$s = z^\alpha + \sum_{\beta \geq \alpha} s_\beta z^\beta. \tag{3.1}$$

By the finite growth property of $\Gamma(L)$, we know that there exists a uniform positive constant C' such that

$$\alpha \in \Gamma(H^0(kL)) \Rightarrow |\alpha| \leq C' \cdot k. \tag{3.2}$$

Thus, it is enough to show that

$$-\log \zeta_k(s) \leq C(k + |\alpha|) \tag{3.3}$$

for all such s. We first assume that $\zeta_\bullet = \zeta_{\bullet\phi}$ for some bounded metric ϕ on L. Now, we know that we can find a trivialization τ_x of L and analytic isomorphisms from a neighborhood U of a regular rational point $x \in X$ to an open polydisc $\mathbb{D} = \prod_1^d \mathbb{D}(r_i) \subset \mathrm{K}^d$, such that a section $s \in H^0(nL)$ satisfies

$$\log |s|_{n\phi} = \log |s_U| + n \log |\tau_x|_\phi,$$

for some analytic function s_U of the form

$$s_U(z) = z^\alpha + \sum_{\beta \geq \alpha} s_\beta z^\beta.$$

Since ϕ is bounded on U, so is the term $n \log |\tau_x|_\phi$, and by the maximum principle, applied in each variable, we have that

$$r^{|\alpha|} \leq \sup_U |s_U|,$$

and finally (3.3) follows. In general, since our norms are bounded graded, they are at linearly bounded distance from a norm of the form $\zeta_{\bullet\phi}$, thereby showing that Φ satisfies 3.3, and proving the Theorem. $\square$

Since we no longer restrict ourselves to algebras generated in degree one, it should be interesting to understand the behaviour of the asymptotic spectral measure restricted to subalgebras. We have the following results:

Proposition 3.1.3.4. *Recall that, given any measure μ on the reals and any μ-measurable function $f : \mathbb{R} \to \mathbb{R}$, $f_*\mu$ denotes the pushforward of μ by f. Set $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(L)$. We then have that:*

- $f(\lambda) = -\lambda \Rightarrow f_*\sigma(\zeta_\bullet, \zeta'_\bullet) = \sigma(\zeta'_\bullet, \zeta_\bullet)$;
- *for any* $c \in \mathbb{R}$, $f(\lambda) = \lambda + c \Rightarrow f_*\sigma(\zeta_\bullet, \zeta'_\bullet) = \sigma(e^{-c}\zeta_\bullet, \zeta'_\bullet)$;
- *for any* $r \in \mathbb{N}^*$, $f(\lambda) = r\lambda \Rightarrow f_*\sigma(\zeta_\bullet, \zeta'_\bullet) = \sigma(\zeta_{r\bullet}, \zeta'_{r\bullet})$,

where $\zeta_{r\bullet}$ denotes the restriction of $\zeta_\bullet$ to the subalgebra $R(X, rL)$.

This Proposition is the non-trivially valued equivalent of Propositions 3.4 and 3.5 of [BJ18a], and are proven in the same manner.

3.1.4 Equidistribution of Okounkov points of superadditive functions associated to bounded graded norms.

We now prove a "norm-volume" version of the Fujita approximation Theorem, i.e. that the asymptotic relative volumes between two bounded graded norms can be approximated using the asymptotic relative volumes between their finitely generated approximations. We begin by recalling the following Proposition:

Proposition 3.1.4.1 ([CM15, L4.1, T4.3])**.** *Assume Φ is a superadditive function*

$$\Phi : \Gamma(L) \to \mathbb{R},$$

such that $\Phi(0, 0_{\mathbb{N}^d}) = 0$. For any $t \in \mathbb{R}$, set

$$\Gamma^{\Phi, \geq t} = \{(n, \alpha) \in \Gamma(L),\, \Phi(n, \alpha) \geq n \cdot t\}.$$

Then, $\Gamma^{\Phi,\geq t}$ is a sub-semigroup of $\Gamma(L)$ satisfying properties (i)-(ii) of Lemma 3.1.2.3 whenever

$$t < \theta = \lim_{n\to\infty} \sup_{\alpha \in \Gamma_n(L)} n^{-1}\Phi(n, \alpha).$$

Remark 3.1.4.2. It is immediate that

$$l < t \Rightarrow \Gamma^{\Phi,\geq l} \subseteq \Gamma^{\Phi,\geq t}.$$

Definition 3.1.4.3. Let Φ be a superadditive function on $\Gamma(L)$. We set

$$G_\Phi : \Delta(\Gamma(L)) \to \mathbb{R} \cup \{-\infty\},$$

$$(n, \alpha) \mapsto \sup\{t \in \mathbb{R} \cup \{-\infty\}, (n, \alpha) \in \Delta(\Gamma^{\Phi,\geq t})\}.$$

The function G_Φ is the **Chebyshev function** of the semigroup $\Gamma(L)$ (associated to Φ). The term *concave transform* is also common in the literature, see e.g. [Nys09] and [KMR19].

Remark 3.1.4.4. By [BC11] this function is concave, hence continuous, on the interior of $\Delta(\Gamma(L))$.

The proof of our main result relies on the following equidistribution Theorem for the values of a superadditive function defined on an Okounkov body:

Theorem 3.1.4.5 ([CM15, T4.3, R4.4])**.** *Let Φ be a superadditive function from $\Gamma(L)$ to $\mathbb{R}$, whose* lim sup *we denote by θ as previously. Let $\mu(k)$ be the finitely supported probability measure on $\mathbb{R}$ defined as*

$$\mu(k) = \sum_{\alpha \in \Gamma_k(L)} \delta_{k^{-1}\Phi(k,\alpha)}.$$

This sequence then converges to a compactly supported probability measure μ on $\mathbb{R}$ satisfying

$$\mu([t, \infty)) = \frac{\mathrm{vol}(\Delta(\Gamma_\bullet^{\Phi,\geq t}))}{\mathrm{vol}(\Delta(\Gamma_\bullet))},$$

for any $t \leq \theta$. Furthermore, μ is equal to the pushforward of the normalized Lebesgue measure on the Okounkov body $\Delta(\Gamma(L))$ by the Chebyshev function G_Φ.

If Φ is a superadditive function defined on an Okounkov body, associated to a bounded graded norm $\zeta_\bullet$ as before, we denote the limit measure obtained in the previous Theorem by

$$\mu(\zeta_\bullet),$$

and the measures $\mu(k)$ as

$$\mu(\zeta_k).$$

Proposition 3.1.4.6. *Let L be such that $R(X, L)$ is generated in degree one. Let $\zeta_\bullet$ be a bounded graded norm on $R(X, L)$. Consider, for each $k \in \mathbb{N}^*$, the bounded graded norm $\zeta_\bullet^{(k)}$ on $R(X, kL)$ generated in degree one by ζ_k, i.e. the sequence of quotient norms induced by ζ_k and the symmetry morphisms*

$$H^0(kL)^{\odot r} \twoheadrightarrow H^0(rkL)$$

for all $r \in \mathbb{N}^*$. Set

$$\Gamma(kL) = \{(n, \alpha) \in \Gamma(L),\, k|n\}.$$

We then have

$$\mu(\zeta_\bullet^{(k)}) \rightharpoonup_{k\to\infty} \mu(\zeta_\bullet),$$

where $\rightharpoonup$ denotes weak convergence of measures, in particular: the sequence of functions $t \mapsto \int_{-\infty}^{t} d\mu(\zeta_\bullet^{(k)})$ converges pointwise to $t \mapsto \int_{-\infty}^{t} d\mu(\zeta_\bullet)$.

Proof. Let Φ and Φ_k be the superadditive functions associated to the norms $\zeta_\bullet$ and $\zeta_\bullet^{(k)}$. We first notice the following properties of Φ and the Φ_k:

(i) $\Phi_k(k, \alpha) = \Phi(k, \alpha)$, for all $(k, \alpha) \in \Gamma_k$;

(ii) $\Phi_k(kn, \alpha) \leq \Phi(kn, \alpha)$, for all $(kn, \alpha) \in \Gamma_{k\bullet}$;

(iii) if $d|k$, then $\Phi_d(kn, \alpha) \leq \Phi_k(kn, \alpha)$, for all $(kn, \alpha) \in \Gamma_{k\bullet}$.

Let θ_k and θ be the above bounds on the supports of the appropriate measures. We then show that

$$\mu(\zeta_\bullet^{(k)})([t, \theta]) \to \mu(\zeta_\bullet)([t, \theta]),$$

for all $t \in [-\infty, \theta]$.
Now, since

$$\mu(\zeta_\bullet^{(k)})([t, \theta]) = \frac{\mathrm{vol}\left(\Delta(\Gamma_{k\bullet}^{\Phi_k, \geq t})\right)}{\mathrm{vol}(\Delta(\Gamma_{k\bullet}))},$$

and

$$\mathrm{vol}(\Delta(\Gamma_{k\bullet}))^{-1} = \mathrm{vol}(\Delta(\Gamma_\bullet))^{-1},$$

the problem reduces to showing that the sequence of functions $(v_k)_k$, defined as

$$v_k : t \mapsto \mathrm{vol}\left(\Delta(\Gamma_{k\bullet}^{\Phi_k, \geq t})\right)$$

converges pointwise to

$$v : t \mapsto \mathrm{vol}(\Delta(\Gamma_\bullet^{\Phi,\geq t})).$$

Note that (ii), (iii), and the expressions

$$\theta = \lim_{n\to\infty} \sup_{\Gamma_n} \frac{\Phi(n,\alpha)}{n} < \infty,$$

and

$$\theta_k = \lim_{n\to\infty} \sup_{(kn,\alpha)\in\Gamma_{kn}} \frac{\Phi_k(kn,\alpha)}{n} < \infty$$

imply that $(\theta_k)_k$ is an increasing sequence converging to θ.
Finally, the semigroups $\Gamma_{(k)\bullet}^{\Phi_{(k!)},\geq t}$ and $\Gamma_\bullet^{\Phi,\geq t}$, satisfy the hypotheses of Lemma 3.1.2.8 (note (i)), which yields

$$v_k(t) \to v(t),$$

concluding the proof. □

We may now prove our desired Theorem.

Theorem 3.1.4.7. *Let L be such that $R(X,L)$ is generated in degree one. Let $\zeta_\bullet$, $\zeta'_\bullet$ be two bounded graded norms on L, and for each $k \in \mathbb{N}^*$, let $\zeta_\bullet^{(k)}$ and $\zeta_\bullet'^{(k)}$ denote the bounded graded norms on $R(X,kL)$ generated in degree one by ζ_k and ζ'_k respectively. Then, we have that:*

$$\mathrm{vol}(\zeta_\bullet^{(k)}, \zeta_\bullet'^{(k)}) \to_{k\to\infty} \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

Proof. Let Φ' and for all k, Φ'_k be the superadditive functions associated to the norms $\zeta'_\bullet$ and $\zeta'^{(k)}_\bullet)$ respectively.

Recall the identity

$$\mathrm{vol}(\zeta_\bullet, \zeta'_\bullet) = \lim_{m\to\infty} m^{-1}\,\mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

Note that

$$\int_{\mathbb{R}} \lambda\, d\mu(\zeta_m) - \int_{\mathbb{R}} \lambda\, d\mu(\zeta'_m) = m^{-1} \sum_{\alpha\in\Gamma_m(L)} \left[\Phi(m,\alpha) - \Phi'(m,\alpha)\right],$$

where $\mu(\zeta_m)$ and $\mu(\zeta'_m)$ are defined as the finitely supported measures as in Theorem 3.1.4.5. By [CM15, (29)], the quantity on the right is identified with

$$m^{-1}\operatorname{vol}(\zeta_m, \zeta'_m),$$

so that at the limit,

$$\int_{\mathbb{R}} \lambda \, d\mu(\zeta_\bullet) - \int_{\mathbb{R}} \lambda \, d\mu(\zeta'_\bullet) = \operatorname{vol}(\zeta_\bullet, \zeta'_\bullet).$$

Doing the same process with $\zeta_\bullet^{(k)}$ and ${\zeta'}_\bullet^{(k)}$, we then find that

$$\int_{\mathbb{R}} \lambda \, d\mu(\zeta_\bullet^{(k)}) - \int_{\mathbb{R}} \lambda \, d\mu({\zeta'}_\bullet^{(k)}) = \operatorname{vol}(\zeta_\bullet^{(k)}, {\zeta'}_\bullet^{(k)}).$$

An application of Theorem 3.1.4.6 then yields the desired convergence. □

3.1.5 Approximation of graded norms generated in degree one via graded norms coming from models.

It is important to emphasize that bounded graded norms $\zeta_\bullet$ generated in degree one are very easy to study: their asymptotic behaviour is heavily controlled by that of ζ_1 and of the asymptotic structure of the underlying algebra, as one can see from the following result:

Lemma 3.1.5.1. Let $V_\bullet$ be a graded K-algebra generated in degree one. Let $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(V_\bullet)$ be generated in degree one. We then have that:

$$d_\infty(\zeta_\bullet, \zeta'_\bullet) = d_\infty(\zeta_1, \zeta'_1).$$

Proof. This follows on repeatedly applying Proposition 1.2.3.10. Set $a = d_\infty(\zeta_1, \zeta'_1)$. For any $m > 1$, we have that

$$\phi_m : V_1^{\odot m} \to V_m$$

is surjective. Consider $v \in V_m$, and lifts $\tilde{v}$ of v in $V_1^{\odot m}$, which themselves lift to $\tilde{\tilde{v}} \in V_1^{\otimes m}$. We naturally have that

$$e^{-ma}(\zeta'_1)^{\otimes m}(\tilde{\tilde{v}}) \leq (\zeta_1)^{\otimes m}(\tilde{\tilde{v}}) \leq e^{ma}(\zeta'_1)^{\otimes m}(\tilde{\tilde{v}}),$$

so that, applying the above Proposition,

$$e^{-ma}(\zeta'_1)^{\odot m}(\tilde{v}) \leq (\zeta_1)^{\odot m}(\tilde{v}) \leq e^{ma}(\zeta'_1)^{\odot m}(\tilde{v}),$$

and finally, since a graded norm generated in degree one is a quotient,

$$e^{-ma}(\zeta'_m)(s) \leq (\zeta_m)(v) \leq e^{ma}(\zeta'_v)(s).$$

This establishes

$$d_\infty(\zeta_\bullet, \zeta'_\bullet) \leq d_\infty(\zeta_1, \zeta'_1),$$

and since the d_∞ distance is defined as a sup, we in fact have equality. □

The main result of this Subsection will then be a powerful approximation Theorem for graded norms generated in degree one. It requires specific constructions of certain bounded graded norms, which cannot in general be assumed to be being generated in degree one; however, they will coincide in all high enough degrees with one such norm. Hence, we introduce the following definition, to make our later statements lighter.

Definition 3.1.5.2. We say that a bounded graded norm $\zeta_\bullet$ is **eventually generated in degree one** if there exists a norm generated in degree one $\zeta^o_\bullet$, and a positive integer r, such that for all $m \geq r$,

$$\zeta_m = \zeta^o_m.$$

We will say that $\zeta_\bullet$ **eventually coincides** with $\zeta^o_\bullet$.

Let L be an ample line bundle on X whose algebra of sections is generated in degree one. We now describe how to construct, starting from a lattice norm on $H^0(L)$, a model $(\mathcal{X}, \mathcal{L})$ of (X, L), such that the bounded graded norm associated to the sections of $\mathcal{L}$ is eventually generated in degree one.

Let ζ thus be a lattice norm on $H^0(L)$, which as we recall means that there exists a basis of sections (s_i) of $H^0(L)$ which is orthonormal for ζ. Denote $\mathcal{V}_1$ the K°-submodule of $H^0(L)$ generated by this basis of sections, i.e. the unit ball of ζ. Then, the surjective symmetry morphisms $\phi_r : H^0(L)^{\odot r} \to H^0(rL)$ of $R(X, L)$ being surjective for all $r \geq 1$, $\mathcal{V}_1$ induces a K°-subalgebra $\mathcal{V}_\bullet$ of $R(X, L)$, which is furthermore generated in degree one, and torsion-free. The scheme

$$\mathcal{X} = \mathrm{Proj}\ \mathcal{V}_\bullet \tag{3.4}$$

is then flat and projective over K°. Let $\mathcal{L}$ be its twisting sheaf $\mathcal{O}_{\mathcal{X}}(1)$. $(\mathcal{X}, \mathcal{L})$ is a model of (X, L). Furthermore, for all m large enough, $H^0(m\mathcal{L})$ coincides with $\mathcal{V}_m$ (see [Har77, Ex. II-5.14]). In particular, the sequence of norms

$$(\zeta_{H^0(m\mathcal{L})})_m$$

is eventually generated in degree one, and the norm generated in degree one with which it eventually coincides is generated by ζ.

We may then prove the following result:

Proposition 3.1.5.3. *Assume* K *to be densely valued, and let* $\zeta_\bullet$ *be generated in degree one. Then, for all* $\varepsilon > 0$*, there exists a model* $(\mathcal{X}^\varepsilon, \mathcal{L}^\varepsilon)$ *of* (X, L)*, such that, for large enough* m*,*

$$d_\infty(\zeta_m, \zeta_{H^0(m\mathcal{L}^\varepsilon)}) < m\varepsilon.$$

Proof. Since K is densely valued, for all $\varepsilon > 0$, there exists a lattice norm ζ^ε with

$$d_\infty(\zeta_1, \zeta^\varepsilon) < \varepsilon.$$

Being a lattice norm, we associate to ζ^ε a model $(\mathcal{X}^\varepsilon, \mathcal{L}^\varepsilon)$ as in the construction (3.4) above, whose associated graded norm $\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}$ eventually coincides with the norm generated in degree one by ζ^ε.

Since $\zeta_\bullet$ and $\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}$ are both eventually generated in degree one, we have that

$$d_\infty(\zeta_m, \zeta_{H^0(m\mathcal{L}^\varepsilon)}) \leq m d_\infty(\zeta_1, \zeta^\varepsilon) < m\varepsilon$$

for all m large enough. $\square$

Finally, we may prove the main Theorem of this section.

Theorem 3.1.5.4. *Assume* K *to be densely valued, and assume* L *to be such that* $R(X, L)$ *is generated in degree one. Let* $\zeta_\bullet$, $\zeta'_\bullet$ *be bounded graded norms generated in degree one on* L*. Then, for all* $\varepsilon > 0$*, there exist models* $(\mathcal{X}^\varepsilon, \mathcal{L}^\varepsilon)$ *and* $(\mathcal{Y}^\varepsilon, \mathcal{M}^\varepsilon)$ *of* (X, L)*, such that:*

$$\mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}, \zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}) \to_{\varepsilon\to 0} \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

Proof. We pick sequences of models $(\mathcal{X}^\varepsilon, \mathcal{L}^\varepsilon)$ and $(\mathcal{Y}^\varepsilon, \mathcal{M}^\varepsilon)$ of (X, L) as in Proposition 3.1.5.3. Using the cocycle condition on volumes, we have that

$$\mathrm{vol}(\zeta_\bullet, \zeta'_\bullet) = \mathrm{vol}(\zeta_\bullet, \zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}) + \mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}, \zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}) + \mathrm{vol}(\zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}, \zeta'_\bullet),$$

so that it is then enough to prove

$$\mathrm{vol}(\zeta_\bullet, \zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}) \to_{\varepsilon\to 0} 0.$$

(The proof for $\mathcal{L}$ being also valid for $\mathcal{M}$.) Since volumes respect a Lipschitz property with respect to the d_∞-distance (Proposition 1.2.3.9), we have

$$|\operatorname{vol}(\zeta_\bullet, \zeta_{H^0(\bullet\mathcal{L}^\varepsilon)})| = |\operatorname{vol}(\zeta_\bullet, \zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}) - \operatorname{vol}(\zeta_\bullet, \zeta_\bullet)| \leq \limsup_m m^{-1} d_\infty(\zeta_m, \zeta_{H^0(m\mathcal{L}^\varepsilon)}).$$

In light of Proposition 3.1.5.3, we then have that

$$|\operatorname{vol}(\zeta_\bullet, \zeta_{H^0(\bullet\mathcal{L}^\varepsilon)})| \leq d_\infty(\zeta_1, \zeta_{H^0(\mathcal{L})}) < \varepsilon,$$

which concludes the proof. □

3.2 The range of the asymptotic Fubini-Study operator.

3.2.1 Relating asymptotic volumes and Monge-Ampère energies.

The goal of this Subsection is to prove the following Theorem, a generalization of [BJ18a, T4.13], where we consider a general non-Archimedean field, rather than one which is trivially valued.

Theorem 3.2.1.1. *Let L be a semiample $\mathbb{Q}$-line bundle on a projective* K-*variety X. Let $\zeta_\bullet$, $\zeta'_\bullet \in \mathcal{N}_\bullet(L)$. We then have:*

$$\lim_m E(\mathrm{FS}_m(\zeta_m), \mathrm{FS}_m(\zeta'_m)) = \operatorname{vol}(\zeta_\bullet, \zeta'_\bullet).$$

As a first reduction, note that we can assume L to be a globally generated (genuine) line bundle and the algebra of sections to be generated in degree one, thanks to Proposition 3.1.3.4.

We now show that we can reduce to the case where $\mathbb{K}$ is a maximally complete, algebraically closed, densely valued field, as in Proposition 1.1.3.4.

Lemma 3.2.1.2. Assume K to be any non-trivially valued, non-Archimedean field, and that Theorem 3.2.1.1 holds for the base change of $X_{\mathrm{K}'}$ to an algebraically closed extension K$'$ of K. Then, Theorem B holds for X.

Proof. By invariance of finite-dimensional volumes under ground field extension, the right-hand side is indeed invariant under ground field extension, so that we only have to take care of the energy side of the equation. Consider the base change $X_{\mathrm{K}'}$ and its pullback line bundle $L_{\mathrm{K}'}$. Note that the ground field extension $R(X, L)_{\mathrm{K}'}$ of the algebra of sections of L coincides with $R(X_{\mathrm{K}'}, L_{\mathrm{K}'})$. Consider the associated norms $\zeta_{\bullet,\mathrm{K}'}$ and $\zeta'_{\bullet,\mathrm{K}'}$.

- by Proposition 1.2.2.4, the Fubini-Study operators associated to each individual norm coincide with those associated to their ground field extension, and that (say)

$$\mathrm{FS}_m(\zeta_{m,\mathrm{K}'}) = {\pi_1}^*\mathrm{FS}_m(\zeta_m);$$

- by Proposition 2.4.1.1,

$$\pi_{1*}\,\mathrm{MA}(\mathrm{FS}_m(\zeta_{m,\mathrm{K}'}), \mathrm{FS}_m(\zeta'_{m,\mathrm{K}'})) = \mathrm{MA}(\mathrm{FS}_m(\zeta_m), \mathrm{FS}'_m(\zeta_m)),$$

  where $\mathrm{MA}(\phi, \phi')$ denotes any mixed Monge-Ampère measure involving only ϕ and ϕ'.

It follows that both quantities in the assertion of Theorem B are invariant under ground field extension. Using that the Theorem then holds over $X_{\mathrm{K}'}$, this finishes the proof. $\square$

From now on, assume K to be as in Proposition 1.1.3.4.

Lemma 3.2.1.3. Theorem 3.2.1.1 holds whenever $\zeta_\bullet$ and $\zeta'_\bullet$ are both graded norms generated in degree one.

Proof. Pick approximations $\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}$ and $\zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}$ as in Theorem 3.1.5.4. By Lemma 3.2.1.4 below, we have, for all $\varepsilon > 0$,

$$E(\phi_{\mathcal{L}^\varepsilon}, \phi_{\mathcal{M}^\varepsilon}) = \mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}, \zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}).$$

Now, the statement of Theorem 3.1.5.4 is that

$$\mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L}^\varepsilon)}, \zeta_{H^0(\bullet\mathcal{M}^\varepsilon)}) \to_{\varepsilon\to 0} \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet).$$

In particular, by construction, we have that

$$\lim_m \mathrm{FS}_m(\zeta_{H^0(m\mathcal{L}^\varepsilon)}) = \mathrm{FS}_1(\zeta_{H^0(\mathcal{L}^\varepsilon)}) = \phi_{\mathcal{L}^\varepsilon},$$

so that the Lemma is proven once we show that

$$\lim_m E(\mathrm{FS}_m(\zeta_{H^0(m\mathcal{L}^\varepsilon)}), \mathrm{FS}_m(\zeta_{H^0(m\mathcal{M}^\varepsilon)})) \to_{\varepsilon\to 0} \lim_m E(\mathrm{FS}_m(\zeta_m), \mathrm{FS}_m(\zeta'_m)).$$

But using the 1-Lipschitz property of the operator FS_m with respect to the sup norm of metrics and the d_∞-distance, we find that for all m, for all $\varepsilon > 0$,

$$\sup_X |\mathrm{FS}_m(\zeta_{H^0(m\mathcal{L}^\varepsilon)}) - \mathrm{FS}_m(\zeta_m)| \leq \varepsilon,$$

so that finally,

$$\lim_m \mathrm{FS}_m(\zeta_{H^0(m\mathcal{L}^\varepsilon)}) \to_{\varepsilon\to 0} \lim_m \mathrm{FS}_m(\zeta_m),$$

uniformly. Proceeding similarly for $\mathcal{M}$, and then using continuity of the Monge-Ampère energy along uniform limits, we find the desired result. □

We then have the following Lemma, as promised.

Lemma 3.2.1.4. Assume $(\mathcal{X}, \mathcal{L})$, $(\mathcal{Y}, \mathcal{M})$ to be semiample models of L defined on the same model $\mathcal{X}$ of X. Denoting $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{M}}$ their associated model metrics, we then have that

$$E(\phi_{\mathcal{L}}, \phi_{\mathcal{M}}) = \mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L})}, \zeta_{H^0(\bullet\mathcal{M})}).$$

Proof. We first start by stating the following equality ([BE, L9.17]):

$$\mathrm{vol}(\zeta_{H^0(\bullet\mathcal{L})}, \zeta_{H^0(\bullet\mathcal{M})}) = \mathrm{vol}(\mathrm{N}_\bullet(\phi_{\mathcal{L}}), \mathrm{N}_\bullet(\phi_{\mathcal{M}})).$$

Note that our conventions for the volume and energy are different from those of [BE], but as

$$\lim \frac{h^0(mL)}{m^{\dim X}} = \mathrm{vol}(L),$$

the changes cancel out. Furthermore, their notation

$$\mathrm{vol}(L, \phi, \psi)$$

corresponds to

$$\mathrm{vol}(\mathrm{N}_\bullet(\phi), \mathrm{N}_\bullet(\psi))$$

in our case.

The above equality follows from earlier results of [BE], wherein it is shown that

$$d_\infty(\zeta_{H^0(m\mathcal{L})}, \mathrm{N}_m(\phi_{\mathcal{L}})) = O(1),$$

([BE, T6.4]) so that Lipschitz continuity of the volume with respect to the d_∞-distance concludes. Then, the Lemma is proven by applying Theorem 9.15 of [BE] to $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{M}}$. □

We now prove the main Theorem.

Proof. Assume now that both norms are not necessarily finitely generated. By surjectivity of $H^0(kL)^{\odot m} \to H^0(kmL)$ for large and divisible enough k, $m > 0$, we may endow each $H^0(kmL)$ with the quotient norm induced by this morphism using the norms ζ_k, ζ'_k. We denote these norms $\zeta_m^{(k)}$, $\zeta_m'^{(k)}$. These define graded norms, generated in degree one, on $R(X, kL)$. Consider their associated Fubini-Study metrics:

$$\mathrm{FS}_k(\zeta_\bullet^{(k)})$$

and

$$\mathrm{FS}_k(\zeta_\bullet'^{(k)}).$$

Recall that the Theorem holds for those norms. Now, since $(\mathrm{FS}_k(\zeta_\bullet^{(k)}))_k$, resp. $(\mathrm{FS}_k(\zeta'^{(k)}_\bullet))_k$ are decreasing nets, by continuity of E along decreasing nets follows:

$$\lim_{k\to\infty} E\left(\mathrm{FS}_k(\zeta_\bullet^{(k)}), \mathrm{FS}_k(\zeta_\bullet'^{(k)})\right) = E(\lim_k \mathrm{FS}_k(\zeta_k), \mathrm{FS}_k(\zeta'_k)).$$

The right-hand side limit, that is,

$$\lim_{k\to\infty} \mathrm{vol}(\zeta_\bullet^{(k)}, \zeta'^{(k)}_\bullet) = \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet),$$

is the statement of Theorem 3.1.4.7. □

3.2.2 The asymptotic Fubini-Study operator descends to a bijection.

We now prove the following Theorem, which is a generalization of [BJ18a, T4.16] to the nontrivially-valued case:

Theorem 3.2.2.1. *Let (X, L) admit continuity of envelopes, with L ample. The asymptotic Fubini-Study operator* FS *then defines a bijection*

$$\mathcal{N}_\bullet(L)/\sim \to \mathrm{PSH}^\uparrow(L).$$

The proof follows that of the aforementioned theorem. We start with preparatory lemmas:

Lemma 3.2.2.2. Assume that $\zeta_\bullet \geq \zeta'_\bullet$ pointwise. Then,

$$d_1(\zeta_\bullet, \zeta'_\bullet) = 0 \Leftrightarrow \mathrm{FS}(\zeta_\bullet) = \mathrm{FS}(\zeta'_\bullet).$$

Proof. We first notice that, since $\zeta_\bullet \leq \zeta'_\bullet$ pointwise, the definition of d_1 using successive minima implies

$$d_1(\zeta_\bullet, \zeta'_\bullet) = \mathrm{vol}(\zeta_\bullet, \zeta'_\bullet),$$

and this volume is equal to 0 by our hypothesis. Using Theorem 3.2.1.1, we then have that

$$E(\mathrm{FS}(\zeta_\bullet), \mathrm{FS}(\zeta'_\bullet)) = 0.$$

But since $\zeta_\bullet \geq \zeta'_\bullet$, $\mathrm{FS}(\zeta_\bullet)$ and $\mathrm{FS}(\zeta'_\bullet)$ are comparable, and [Reb20b, P6.3.2] implies that $\mathrm{FS}(\zeta'_\bullet) = \mathrm{FS}(\zeta_\bullet)$. □

Lemma 3.2.2.3. Let $\zeta_\bullet$ be an element of $\mathcal{N}_\bullet(L)$, and assume continuity of envelopes to hold for (X, L). We then have that $\zeta_\bullet \geq \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet))$, and furthermore those norms are equivalent.

Proof. The first assertion follows from [BE, L7.23] (and its proof). To show asymptotic equivalence, by the previous lemma, it is thus enough to show that $\mathrm{FS}(\mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet))) = \mathrm{FS}(\zeta_\bullet)$. But, by [BE, T7.26],

$$\mathrm{FS}(\mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet))) = Q(\mathrm{FS}(\zeta_\bullet)),$$

which in turn is equal to $\mathrm{FS}(\zeta_\bullet)$ itself, since it is a limit of an increasing net of Fubini-Study potentials. □

We now prove Theorem 3.2.2.1.

Proof. Note that

$$d_1(\zeta_\bullet, \zeta'_\bullet) = \mathrm{vol}(\zeta_\bullet, \zeta_\bullet \vee \zeta'_\bullet) + \mathrm{vol}(\zeta'_\bullet, \zeta_\bullet \vee \zeta'_\bullet),$$

and by Theorem 3.2.1.1, the right-hand side is in fact equal to

$$E(\mathrm{FS}(\zeta_\bullet), \mathrm{FS}(\zeta_\bullet \vee \zeta'_\bullet)) + E(\mathrm{FS}(\zeta'_\bullet), \mathrm{FS}(\zeta_\bullet \vee \zeta'_\bullet)). \tag{3.5}$$

The trick is now to prove the following:

$$\mathrm{FS}(\zeta_\bullet \vee \zeta'_\bullet) = Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet)),$$

where $\wedge$ denotes the min operator. Since, by Lemma 3.2.2.3,

$$\zeta_\bullet \geq \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet)),$$

$$\zeta_\bullet \sim \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet)),$$

and the same holds for $\zeta'_\bullet$, then

$$\zeta_\bullet \vee \zeta'_\bullet \geq \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet)) \vee \mathrm{N}_\bullet(\mathrm{FS}(\zeta'_\bullet)),$$

$$\zeta_\bullet \vee \zeta'_\bullet \sim \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet)) \vee \mathrm{N}_\bullet(\mathrm{FS}(\zeta'_\bullet)),$$

and furthermore, by [BJ18a, (4.4)],

$$\mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet)) \vee \mathrm{N}_\bullet(\mathrm{FS}(\zeta'_\bullet)) = \mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet)).$$

Lemma 3.2.2.2 then implies

$$\mathrm{FS}(\mathrm{N}_\bullet(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))) = \mathrm{FS}(\zeta_\bullet \vee \zeta'_\bullet),$$

and the left-hand side is equal to $Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))$, by [BE, T7.26]. We may now rewrite (3.5) as:

$$d_1(\zeta_\bullet, \zeta'_\bullet) = E(\mathrm{FS}(\zeta_\bullet), Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))) + E(\mathrm{FS}(\zeta'_\bullet), Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))).$$

Now,

$$\mathrm{FS}(\zeta_\bullet) \geq Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))$$

and

$$\mathrm{FS}(\zeta'_\bullet) \geq Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet)),$$

so that the two energies above have the same sign. In particular, the distance $d_1(\zeta_\bullet, \zeta'_\bullet)$ vanishes if and only if the energies vanish, and we conclude using Lemma 3.2.2.2. $\square$

Remark 3.2.2.4. Note that the previous proof shows that there is an expression of the d_1 distance using Monge-Ampère energies, analogous to [BJ18a, C4.21]:

$$d_1(\zeta_\bullet, \zeta'_\bullet) = E(\mathrm{FS}(\zeta_\bullet), Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))) + E(\mathrm{FS}(\zeta'_\bullet), Q(\mathrm{FS}(\zeta_\bullet) \wedge \mathrm{FS}(\zeta'_\bullet))).$$

Remark 3.2.2.5. One can see the asymptotic Fubini-Study operator as giving an injective isometry with dense image of the space of finitely-generated graded norms modulo asymptotic equivalence, into the metric space $\mathcal{E}^1(L)$. Since it is not known whether the envelope P coincides with the envelope Q for general metrics approachable from below, such a result does not hold a priori for the entire space of bounded graded norm modulo asymptotic equivalence, although it is conjectured that this is true.

Chapter 4

Geodesics in non-Archimedean pluripotential theory.

Summary of the main results.

This Chapter focuses on the main results of [Reb20b]. Namely, we define a class of non-Archimedean plurisubharmonic segments, mimicking the *a posteriori* characterization thereof in the complex case, and we prove the following:

Theorem 4.0.0.1. *Given $\phi_0, \phi_1 \in \mathcal{E}^1(L)$, we set*

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

. Then,

1. *$(\mathcal{E}^1(L), d_1)$ is a metric space;*

2. *there exists a maximal psh segment $t \mapsto \phi_t$ joining ϕ_0 and ϕ_1;*

3. *$\phi_t \in \mathcal{E}^1(L)$ for all t;*

4. *the segment ϕ_t is a (constant speed) metric geodesic for d_1, i.e. there exists a real constant $c \geq 0$ such that*

   $$d_1(\phi_t, \phi_s) = c \cdot |t - s|$$

   for all $t, s \in [0, 1]$;

5. *the Monge-Ampère energy is affine along ϕ_t, and it is the unique psh segment joining ϕ_0 and ϕ_1 with this property.*

If the endpoints are continuous, this maximal psh segment is furthermore continuous in time and space.

4.1 Plurisubharmonic segments in non-Archimedean geometry.

4.1.1 Fubini-Study segments, plurisubharmonic segments.

The basic building block for our plurisubharmonic segments are what we call Fubini-Study segments, which we define as follows.

Definition 4.1.1.1. A Fubini-Study segment is a map

$$[0,1] \ni t \mapsto \phi_t \in \mathcal{H}(L)$$

such that there exist a finite basepoint-free collection of sections (s_i) of some $H^0(kL)$, and for each i, real constants λ_i and $\lambda'_i \in \mathbb{R}$ such that for all t,

$$\phi_t = k^{-1} \max_i \log |s_i| + (1-t)\lambda_i + t\lambda'_i.$$

Note the similarity with our definition of Fubini-Study metrics. Again, such segments are immediately seen to be convex in t, stable under finite maxima and addition of constants.

Remark 4.1.1.2. In particular, the image by the operator FS_k of some norm geodesic $t \mapsto \zeta_t$ in $H^0(X, kL)$, with $\zeta_{0,1}$ diagonalizable, defines a Fubini-Study segment: indeed, given a basis $\mathfrak{s} = (s_i)_i$ codiagonalizing the endpoints, we have for all t, i

$$\zeta_t(s_i) = \zeta_0(s_i)^{1-t}\zeta_1(s_i)^t$$

so that

$$FS_k(\zeta_t) = \max_i \left(\log |s_i| - (1-t)\log \zeta_0(s_i) - t \log \zeta_1(s_i)\right).$$

Then, following the idea that psh metrics are decreasing limits of Fubini-Study metrics, we define

Definition 4.1.1.3. A plurisubharmonic segment or psh segment is a map $[0,1] \to \mathrm{PSH}(L)$ which is a decreasing limit of a net Fubini-Study segments.

Proposition 4.1.1.4. *The class of psh segments is the smallest class of segments*

$$[0,1] \to \mathrm{PSH}(L)$$

which contains all segments of the form

$$t \mapsto k^{-1}(\log|s| + (1-t)\lambda + t\lambda'),$$

for s a section of some kL and $\lambda, \lambda' \in \mathbb{R}$, is stable under finite maxima, addition of constants, and decreasing limits of nets.

Proof. If we can show that the set of psh segments on L satisfies all those properties, then it will by definition be the smallest such class. As was the case for the proof of the same result for psh metrics rather than segments, only the property of being stable under decreasing limits is not immediate from the definition. However, using the trick from the proof of [BJ21, Proposition 5.6(vi)], we can reduce to the case of a decreasing net of Fubini-Study paths, and by stability under maximum we may also assume that our net $(\phi_{t,\alpha})_\alpha$ only contains segments of the form

$$t \mapsto \phi_{t,\alpha} = k_\alpha^{-1}(\log|s_\alpha| + (1-t)\lambda_\alpha + t\lambda'_\alpha).$$

Since our segments are assumed to be decreasing along the net, fixing t gives a decreasing net $(\phi_{t,\alpha})_\alpha$ of L-psh metrics, which then converges to a L-psh metric ϕ_t. Therefore, for all $t \in [0,1]$, ϕ_t is not identically $-\infty$. The problem is that we do not know whether the nets of constants converge to finite values. But taking $t = 0, 1$ yields in particular that the nets $k_\alpha^{-1}(\log|s_\alpha| + \lambda_\alpha)$ and $k_\alpha^{-1}(\log|s_\alpha| + \lambda'_\alpha)$ decrease to the L-psh metrics ϕ_0 and ϕ_1. Let x be a point on which ϕ_0 and ϕ_1 are nonsingular. Then,

$$\gamma = \phi_1(x) - \phi_0(x) = \lim_\alpha(\lambda'_\alpha - \lambda_\alpha)$$

is finite, and constant on the set of all such x. Performing this argument for all pairs $a < b \in [0,1]$ shows that ϕ_t corresponds to the segment

$$t \mapsto \phi_0 + \gamma \cdot t,$$

which is a psh segment, as desired. □

Finally, we show that our segments also satisfy the remaining properties of Proposition 2.3.1.6.

Proposition 4.1.1.5. *Plurisubharmonic segments satisfy the following properties:*

1. *the convex combination of two psh segments is a psh segment;*
2. *the addition of a L-psh segment and a M-psh segment is a $L+M$-psh segment;*
3. *if a net of psh segments converges uniformly to a limit segment, then this limit segment is psh.*

Proof. We start with (2). The statement follows from the case of Fubini-Study segments. Consider thus two such segments

$$t \mapsto \phi_t = k^{-1} \max_i \log |s_i| + (1-t)\lambda_i + t\lambda'_i$$

and

$$t \mapsto \psi_t = \ell^{-1} \max_j \log |t_j| + (1-t)\gamma_j + t\gamma'_j.$$

Then

$$\begin{aligned}\phi_t + \psi_t &= (k\ell^{-1})(\max_i(\log |s_i^\ell| + (1-t)\ell\lambda_i + t\ell\lambda'_i) \\ &\quad + \max_j(\log |t_j^k| + (1-t)k\gamma_j + tk\gamma'_j)) \\ &= (k\ell^{-1})(\max_{i,j} \log |s_i^\ell t_j^k| + (1-t)(\ell\lambda_i + k\gamma_j) + t(\ell\lambda'_i + k\gamma'_j)),\end{aligned}$$

which is a Fubini-Study segment on $(k\ell)(M+L)$.

The third point follows from noticing that we can use sequences rather than nets when dealing with uniform convergence, and then adding constants to reduce to the case of a decreasing limit of psh segments, which converges by definition to a psh segment. Finally, the first point follows again from the Fubini-Study case, from a simple computation similar to the proof of (2). $\square$

4.1.2 A maximum principle for Fubini-Study segments.

The Fubini-Study operators FS_k are not injective. Hence, it is pleasant to consider a "minimal" norm in the fibre of a Fubini-Study metric, corresponding to its image by N_k. The following result shows that Fubini-Study segments obtained as the image of a norm geodesic joining two such minimal norms is maximal (compare with [Berndt09, Proposition 3.1]):

Lemma 4.1.2.1 (Maximum principle for norm geodesics)**.** Set two metrics ϕ_0, ϕ_1 in $\mathcal{H}(L)$ defined by sections in $H^0(kL)$. Let $\tilde{\phi}_t$ be the Fubini-Study segment obtained as the image by FS_k of the norm segment joining $\mathrm{N}_k(\phi_0)$ and $\mathrm{N}_k(\phi_1)$. Then, for all t, and for all Fubini-Study segments in the image of FS_k joining ϕ_0 and ϕ_1, we have

$$\phi_t \leq \tilde{\phi}_t.$$

Proof. Note that $\mathrm{FS}_k(\mathrm{N}_k(\phi_i)) = \phi_i$ for $i = 0, 1$ by [BE, Lemma 7.23(ii)]. By definition of a Fubini-Study metric, we can write

$$\phi_0 = \max_i \log |s_i| + \lambda_i$$

and

$$\phi_1 = \max_i \log |t_i| + \lambda'_j$$

where (s_i) and (t_i) are basepoint-free bases of $H^0(kL)$. By monotonicity of norm geodesics in the form of Proposition 1.2.5.4, it is enough to show that

$$\mathrm{N}_k(\phi_{0,1}) \leq \zeta_{0,1},$$

but by [BE, Lemma 7.23(i),(iii)] and anti-monotonicity of the N_k operator, we have

$$\mathrm{N}_k(\phi_0) = \mathrm{N}_k(\mathrm{FS}_k(\zeta_0)) \leq \zeta_0,$$

and similarly for ζ_1, which proves the result. □

4.1.3 Maximal psh segments.

We conclude this section by stating a central Theorem in this article:

Theorem 4.1.3.1. *Let ϕ_0, ϕ_1 be any two psh metrics on L. Then,*

- *either there exists no psh segment between ϕ_0 and ϕ_1,*

- *or there exists a unique maximal psh segment $t \mapsto \phi_t$ between ϕ_0 and ϕ_1.*

We will prove this result in Section 4.3.2. In what follows, we will state and prove versions of this result in larger and larger classes of metrics, starting from the continuous psh case, then finite-energy metrics, and finally general psh metrics. In each case, we show that the maximal segment remains in the same class as the endpoints, for all t.

4.2 Geodesics between continuous psh metrics.

Throughout this Section, we assume that L is an ample line bundle over a projective K-variety X, K non-Archimedean, and that continuity of envelopes holds for (X, L).

4.2.1 Main Theorem for continuous psh metrics.

We start by studying maximal psh segments in the space of continuous psh metrics. The main Theorem of this section is then the following:

Theorem 4.2.1.1. *Let ϕ_0, ϕ_1 be two continuous psh metrics on L. Then,*

1. *there exists a (unique) maximal psh segment $(t, x) \mapsto \phi_t(x)$ joining ϕ_0 and ϕ_1;*
2. *this segment is continuous in both variables;*
3. *this segment is a geodesic segment for the distance d_1;*
4. *the Monge-Ampère energy is affine along this segment, and it is the unique psh segment joining ϕ_0 and ϕ_1 with this property.*

4.2.2 A non-Archimedean Kiselman minimum principle.

In this Subsection, we prove an auxiliary result, of independent interest, that will help us prove the first two points of Theorem 4.2.1.1.

Given a convex function $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R} \cup \{\infty\}$, it is well-known that the infimum of the marginals

$$\mathbb{R}^q \ni y \mapsto \inf_{x \in \mathbb{R}^p} f(x, y)$$

is also convex. This generalizes in multiple ways, as Prekopa's theorem ([Pré73]), but also to plurisubharmonic functions (independent of the imaginary part of the variable over which the infimum is taken). This is the well-known Kiselman minimum principle ([Kis78], [Kis94]), and a crucial tool in the study of plurisubharmonic functions. We propose here a non-Archimedean version of this result.

Lemma 4.2.2.1 (Non-Archimedean Kiselman minimum principle)**.** Let $[0, 1] \ni t \mapsto \phi_t$ be a psh segment in $\mathrm{PSH}(L)$. Then, for each $\tau \in \mathbb{R}$, the Legendre transform

$$\hat{\phi}_\tau : x \mapsto \inf_{t \in [0,1]} \phi_t(x) - t\tau$$

is in $\mathrm{PSH}(L)$. Furthermore, by Legendre duality,

$$\phi_t = \sup_{\tau \in \mathbb{R}} \hat{\phi}_\tau + t\tau.$$

Proof. Since a psh segment on L is a global decreasing limit of Fubini-Study segments, and one notices the map between segments of psh metrics

$$(t \mapsto \phi_t) \mapsto (\tau \mapsto \hat{\phi}_\tau)$$

to be continuous along decreasing sequences of segments (by its definition as an infimum over t of psh metrics), it is enough to consider the case where $t \mapsto \phi_t$ is a Fubini-Study segment, i.e. there exists a finite basepoint-free collection of sections $(s_i)_{i \in I}$ of $H^0(kL)$ for some k such that

$$\phi_t = \max_{i \in I} (\log |s_i| + t\lambda_i + c_i),$$

with fixed constants (λ_i) and (c_i).

Set $\tau \in \mathbb{R}$, and consider the functions

$$f : [0, 1] \times \mathbb{R}^{|I|} \to \mathbb{R}, \, (t, s) \mapsto k^{-1} \max_i s + t(\lambda_i - \tau) + c_i$$

and

$$g : \mathbb{R}^{|I|} \to \mathbb{R},\ s \mapsto \inf_{t \in [0,1]} f(t, s).$$

It is clear that $\hat{\phi}_\tau$ is the composition of g and the formal tropicalization map

$$\mathrm{trop} : X \ni x \mapsto \left(\log |s_1|(x), \ldots, \log \left|s_{|I|}\right|(x)\right) \in (\mathbb{R} \cup \{-\infty\})^{|I|}.$$

We first show that g is a piecewise-linear convex map, and we will explain how from this result we can prove that $\hat{\phi}_\tau$ is Fubini-Study.

The strict epigraph of g

$$E_g = \{(s, y) \in \mathbb{R}^{|I|} \times \mathbb{R},\ g(s) < y\}$$

is the image under the projection $p : [0,1] \times \mathbb{R}^{|I|} \times \mathbb{R} \to \mathbb{R}^{|I|} \times \mathbb{R}$ of the epigraph of f

$$E_f = \{(t, s, y) \in [0,1] \times \mathbb{R}^{|I|} \times \mathbb{R},\ f(t, s) < y\}.$$

One notices that, since f is piecewise-linear and convex in all variables, E_f is convex and its closure is a piecewise-linear set. Since both of those properties are preserved under linear maps, E_g is also convex and PL. This implies that the same holds for the function g. (In particular, putting aside the PL hypothesis, this is precisely the standard proof of the convex infimum principle for marginals.)

Note that our convex PL function g is increasing in each variable and satisfies, for any real constant C,

$$g(s_1 + C, \ldots, s_{|I|} + C) = g(s_1, \ldots, s_{|I|}) + k^{-1}C,$$

because those properties are satisfied by f in the $|I|$ last coordinates, and are preserved upon taking the infimum over the first coordinate. Composing such a function with our formal tropicalization map naturally yields a psh metric (by Remark 4.2.2.2 below), which proves our result. $\square$

Remark 4.2.2.2. We have claimed that, given a basepoint-free basis of sections $\mathfrak{s} = (s_i)_{i \in I}$ of $H^0(kL)$ and a convex PL function f of $p = |I|$ real variables, increasing in each variable, and satisfying

$$f(z_1 + C, \ldots, z_p + C) = f(z_1, \ldots, z_p) + k^{-1}C,$$

for all real constants C, then $f(\log|s_1|, \ldots, \log|s_p|)$ is a Fubini-Study metric. We will show this with $k = 1$ for clarity, and all the arguments below can be adapted for general k upon dividing where needed. Indeed, since f is convex, PL, and satisfies the property above, there exist finitely many affine functions f_j such that

$$f = \max_j f_j$$

and

$$f_j(z_1, \ldots, z_p) = \sum_i \alpha_{i,j} z_i + b$$

with $\sum_i a_{i,j} = 1$. Since a maximum of psh metrics is psh, it is enough to prove that $f_j(\log|s_1|, \ldots, \log|s_p|)$ is psh. We will therefore drop the subscript j and write a_i for the coefficients above.

Now, the monotonicity condition ensures that the a_i all belong to $[0, 1]$, i.e. the vector $(a_i)_i$ is in the p-dimensional simplex. We assume at first that $a_i = p_i/q_i \in \mathbb{Q} \cap [0, 1]$. Denote

$$\alpha_i = p_i \cdot q_i^{-1} \cdot \prod_j q_j$$

and remark that, by the simplex condition,

$$\sum_i \alpha_i = \prod_i q_i.$$

Then,

$$\begin{aligned} f(\log|s_1|, \ldots, \log|s_p|) &= b + \sum_i a_i \log|s_i| \\ &= \frac{b \cdot \prod_i q_i}{\prod_i q_i} + \frac{1}{\prod_i q_i} \log \prod_i |s_i^{\alpha_i}|. \end{aligned}$$

Now, $\prod_i s_i^{\alpha_i}$ is a section of $(\sum_i \alpha_i)L = (\prod_i q_i)L$. Therefore, f is in the image of $FS_{\prod_i q_i}$, i.e. it is a Fubini-Study metric. If some of the coefficients are irrational, then f can be uniformly approximated by a function with rational coefficients satisfying all the conditions above, i.e. $f(\log|s_1|, \ldots, \log|s_p|)$ can be uniformly approximated by Fubini-Study metrics, which shows that it is psh.

Remark 4.2.2.3. We have stated our minimum principle so as to match the form it will be used in, in the next subsection. A brief look at the proof shows that it can be generalized to the following statement: given a Fubini-Study "polyhedron" (or psh, upon taking decreasing limits) parameterized as

$$\phi_t : P \times X \ni (t,x) \mapsto \max_i \log|s_i| + \langle \alpha_i, t\rangle + b_i,$$

where P is a convex polyhedral subset of $\mathbb{R}^d$ for some d, $\alpha_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, and the s_i are sections of some $H^0(kL)$, we have that for all $\alpha \in \mathbb{R}^d$,

$$\inf_{t\in P} \phi_t(x) - \langle \tau, t\rangle$$

is psh for all $\tau \in \mathbb{R}^d$.

4.2.3 Proof of Theorem 4.2.1.1, (1) and (2).

With this principle in hand, we can now prove the first two points of Theorem 4.2.1.1, following ideas from e.g. [Dar19] and [RWN14] in the complex case. Consider the envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for $\tau \in \mathbb{R}$. By continuity of envelopes, this defines a continuous psh metric. The two next lemmas essentially prove our result:

Lemma 4.2.3.1. If $\phi_0, \phi_1 \in C^0(L) \cap \mathrm{PSH}(L)$, the map $t \mapsto \phi_t$ defined as the Legendre transform

$$\phi_t = \sup_{\tau\in\mathbb{R}} (t\tau + \hat{\phi}_\tau)$$

is a psh segment, which is continuous on $[0,1] \times X^{\mathrm{an}}$.

Proof. We start with continuity. Since ϕ_0, ϕ_1 are continuous, by continuity of envelopes $P(\phi_0, \phi_1 - \tau)$ is continuous for all τ as well. Start by choosing a compact interval

$$S = [a,b] \subset (0,1).$$

- for large positive τ, and for all $t \in S$, $\phi_1 - \tau \leq \phi_0$ (since ϕ_0, ϕ_1 are continuous, thus bounded) and

  $$t\tau + P(\phi_0, \phi_1 - \tau) = t\tau + P(\phi_1 - \tau) = P(\phi_1) + (t-1)\tau.$$

  Since $t - 1 < 0$, $(t-1)\tau$ is very negative while $P(\phi_1)$ is bounded, so that $t\tau + \hat{\phi}_\tau$ does not contribute to the supremum;

- for large negative τ, by boundedness again we have $\phi_0 \leq \phi_1 - \tau$ so that

$$t\tau + P(\phi_0, \phi_1 - \tau) = t\tau + P(\phi_0) \leq P(\phi_0).$$

Therefore, for some constant $C(S) > 0$, and for all $t \in S$,

$$\phi_t = \sup_{\tau \in [-C(S), C(S)]} t\tau + \hat{\phi}_\tau,$$

a supremum of continuous functions over a compact set, which is therefore continuous. We have proven that $(t, x) \mapsto \phi_t(x)$ is continuous on $(0, 1) \times X$.

We now prove that it is continuous up to the boundary. We start with the case $t = 0$. For very small values of t, very positive values of τ will never contribute to the supremum, so that we need only consider values of τ bounded above by some constant C_0. Since we always have

$$P(\phi_0, \phi_1 - \tau) \leq P(\phi_0) = \phi_0,$$

it follows that for very small values of t,

$$t\tau + \hat{\phi}_\tau \leq tC_0 + P(\phi_0, \phi_1 - \tau) \leq tC_0 + \phi_0.$$

Taking the supremum, we thus have

$$\phi_t - \phi_0 \leq tC_0. \tag{4.1}$$

By boundedness of ϕ_1, there exists a negative enough value of τ, say C_0', such that $\hat{\phi}_{C_0'} = P(\phi_0) = \phi_0$, i.e. for all small enough t, there exist some τ with

$$C_0' t + \phi_0 \leq t\tau + \hat{\phi}_\tau$$

which implies

$$C_0' t \leq \phi_t - \phi_0.$$

Combining this with (4.1) shows that ϕ_t converges uniformly to ϕ_0 for small enough t, which proves continuity at $t = 0$. If t is very close to 1, the argument proceeds in the same way, by noticing that

$$P(\phi_0, \phi_1 - \tau) = P(\phi_0 + \tau, \phi_1) - \tau.$$

To show that $t \mapsto \phi_t$ is a psh segment, we consider the net of psh segments

$$I \mapsto \max_{\tau \in I}(t \mapsto t\tau + \hat{\phi}_\tau),$$

where I belongs to the set of finite collections of elements in $\mathbb{R}$, directed by inclusion. This does indeed define a psh segment for all such I, as a finite maximum of psh segments. By definition, the limit of this net is $t \mapsto \phi_t$, and it is naturally increasing along inclusion. By Dini's Theorem, this gives a sequence of psh segments converging uniformly to $t \mapsto \phi_t$, which is equivalent to saying that it is a psh segment, proving our result. $\square$

Lemma 4.2.3.2. The curve

$$t \mapsto \phi_t = \sup_{\tau \in \mathbb{R}} t\tau + \hat{\phi}_\tau$$

is the largest psh segment joining ϕ_0 and ϕ_1.

Proof. We first show that ϕ_t bounds from above all psh segments between the endpoints. Plurisubharmonic segments are defined as decreasing limits of Fubini-Study segments. Therefore, at the endpoints, Dini's theorem gives uniform convergence, ensuring that if $k \mapsto \psi_t^k$ is a sequence of Fubini-Study segments decreasing to a psh segment ψ_t with $\psi_{0,1} \leq \phi_{0,1}$, then we can assume that for all large enough k, $\psi_{0,1}^k \leq \phi_{0,1}$. Therefore, it is enough to treat the case of Fubini-Study segments.

Consider therefore a Fubini-Study segment $t \mapsto \psi_t$ with $\psi_0 \leq \phi_0$, $\psi_1 \leq \phi_1$. Let

$$\mathbb{R} \ni \tau \mapsto \hat{\psi}_\tau = \inf_{t \in [0,1]} \psi_t - t\tau$$

be its Legendre transform. By the minimum principle Lemma 4.2.2.1, $\hat{\psi}_\tau \in \mathcal{H}(L)$ for all τ. Taking $t = 0, 1$ we have

$$\hat{\psi}_\tau \leq \psi_0 \leq \phi_0$$

and

$$\hat{\psi}_\tau \leq \psi_1 - \tau \leq \phi_1 - \tau.$$

As $\hat{\psi}_\tau$ is psh, we then have

$$\hat{\psi}_\tau \leq \hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau).$$

Taking the Legendre transform again, we find

$$\psi_t = \sup_{\tau \in \mathbb{R}} \hat{\psi}_\tau + t\tau \leq \sup_{\tau \in \mathbb{R}} \hat{\phi}_\tau + t\tau = \phi_t,$$

which establishes our first desired result: ϕ_t bounds all psh segments by above. By Lemma 4.2.3.1, ϕ_t is itself a psh segment, which concludes the proof. □

We then have all the tools in hand to prove the Theorem.

Proof of Theorem 4.2.1.1 (1)-(2). By Lemma 4.2.3.2, the curve

$$t \mapsto \phi_t = \sup_{\tau \in \mathbb{R}} t\tau + \hat{\phi}_\tau$$

is equal to

$$\sup\{\phi_t,\ \phi_t \text{ psh segment joining } \phi_0, \phi_1\},$$

and by Lemma 4.2.3.1, this segment is a psh segment joining ϕ_0 and ϕ_1. Therefore, it is a maximal psh segment, and hence is unique. This establishes (1). The continuity statement (2) also follows from Lemma 4.2.3.1. □

4.2.4 Quantization with geodesics of bounded graded norms.

We now turn to the statements (3) and (4) of Theorem 4.2.1.1. From our definition of the maximal psh segment as a Perron envelope, it is not obvious how to recover the desired properties. Instead, we will obtain a "quantized" characterization of that segment, using sequences of Fubini-Study segments.

Let ϕ_0, ϕ_1 be two continuous psh metrics as before. To those metrics, we can associate the bounded graded norms $\mathrm{N}_\bullet(\phi_i)$, $i = 0, 1$. They can be joined by the geodesic of graded norms $\zeta^t_\bullet$, where ζ^t_k is the norm geodesic joining the $\mathrm{N}_k(\phi_i)$, as in the first Chapter. By Theorem 1.3.3.1, for all t, $\zeta^t_\bullet$ is submultiplicative, so that the limit

$$\Phi_t : t \mapsto \lim_k \mathrm{FS}_k(\zeta^t_k)$$

exists for all t by Fekete's lemma. By the same lemma, this is in fact a supremum over k. We claim that this limit coincides with the maximal psh segment ϕ_t joining ϕ_0 and ϕ_1. To that end, we show that Φ_t bounds all psh segments by above.

Proposition 4.2.4.1. *Let ψ_t be a plurisubharmonic segment joining two metrics $\psi_0 \le \phi_0$ and $\psi_1 \le \phi_1$ in $C^0(L) \cap \mathrm{PSH}(L)$. We then have that*

$$\psi_t \le \Phi_t$$

for all $t \in [0, 1]$.

Proof. As in the proof of Lemma 4.2.3.2, Dini's theorem gives uniform convergence of a sequence $k \mapsto \psi_t^k$ of Fubini-Study segments decreasing to ψ_t, so that for all large enough k, $\psi_{0,1}^k \le \phi_{0,1}$, since we have assumed $\psi_{0,1} \le \phi_{0,1}$.

Therefore, it enough to prove the result for all Fubini-Study segments ψ_t with $\psi_{0,1} \le \phi_{0,1}$. The argument is similar to that of [DLR20, Proposition 2.12].

We start by fixing some notation. As we have just said, we can assume $t \mapsto \psi_t$ to be a Fubini-Study segment in the image of some FS_k, with $\psi_0 \le \phi_0$, $\psi_1 \le \phi_1$. Denote by:

- $t \mapsto \tilde{\psi}_t$ the image by FS_k of the norm geodesic in $H^0(kL)$ joining $\mathrm{N}_k(\psi_0)$ and $\mathrm{N}_k(\psi_1)$;
- $t \mapsto \Phi_t^k$ the image by FS_k of the norm geodesic in $H^0(kL)$ joining $\mathrm{N}_k(\phi_0)$ and $\mathrm{N}_k(\phi_1)$.

By definition, since $\Phi_t = \sup_m \Phi_t^m$, we have

$$\Phi_t^k \le \Phi_t. \tag{4.2}$$

Since $\psi_0 = \tilde{\psi}_0$, $\psi_1 = \tilde{\psi}_1$, by the maximum principle for norm geodesics Lemma 4.1.2.1, we have

$$\psi_t \le \tilde{\psi}_t. \tag{4.3}$$

Since the composition of $\mathrm{FS}_k \circ \mathrm{N}_k$ preserves inequalities while N_k and FS_k reverse them, and since we have

$$\tilde{\psi}_0 = \mathrm{FS}_k(\mathrm{N}_k(\psi_0)) \text{ and } \tilde{\psi}_1 = \mathrm{FS}_k(\mathrm{N}_k(\psi_1))$$

as well as

$$\Phi_0^k = \mathrm{FS}_k(\mathrm{N}_k(\phi_0)) \text{ and } \Phi_1^k = \mathrm{FS}_k(\mathrm{N}_k(\phi_1)),$$

we then have

$$\tilde{\psi}_0 \le \Phi_0^k \text{ and } \tilde{\psi}_1 \le \Phi_1^k,$$

so that, by monotonicity of norm geodesics in the form of Proposition 1.2.5.4 (applied to the $\mathrm{N}_k(\psi_{0,1}) \geq \mathrm{N}_k(\phi_{0,1})$), we have

$$\tilde{\psi}_t \leq \Phi_t^k. \tag{4.4}$$

Combining (4.3), (4.4), and (4.2), we finally have

$$\psi_t \leq \Phi_t,$$

as desired. □

Theorem 4.2.4.2. *Let ϕ_0, ϕ_1 be two continuous psh metrics. The segment*

$$\Phi_t : t \mapsto \lim_k \mathrm{FS}_k(\zeta_k^t),$$

where $t \mapsto \zeta_k^t$ is the norm geodesic joining $\mathrm{N}_k(\phi_0)$ and $\mathrm{N}_k(\phi_1)$, coincides with the maximal psh segment ϕ_t joining ϕ_0 and ϕ_1.

Proof. By Proposition 4.2.4.1, we have $\Phi_t \geq \phi_t$, since in particular ϕ_t is a psh segment joining ϕ_0 and ϕ_1. But, by Fekete's lemma, the limit of the sequence $k \mapsto \mathrm{FS}_k(\zeta_k^t)$ is in fact a supremum, i.e. Φ_t is a supremum of a subset of the set of psh segments below ϕ_0 and ϕ_1. This gives $\Phi_t \leq \phi_t$, which proves our result. □

4.2.5 Proof of Theorem 4.2.1.1, (3) and (4).

Using our newfound expression for ϕ_t, we may now finish the proof of the Theorem.

Proof of Theorem 4.2.1.1 (3)-(4). We start with (4). By the cocycle property, we can set $\phi_{\mathrm{ref}} = \phi_1$. By [BE, Theorem 9.15],

$$E(\phi_t, \phi_1) = \mathrm{vol}(\phi_t, \phi_1).$$

For any k, let ζ_k^t be the norm geodesic joining $\mathrm{N}_k(\phi_0)$ and $\mathrm{N}_k(\phi_1)$, and write

$$\begin{aligned}
\mathrm{vol}(\zeta_k^t, \zeta_k^1) &= h^0(kL)^{-1} \sum \lambda_{i,k}(\zeta_k^t, \zeta_k^1) \\
&= (1-t)\, h^0(kL)^{-1} \sum \lambda_{i,k}(\zeta_k^0, \zeta_k^1) \\
&= (1-t)\, \mathrm{vol}(\zeta_k^0, \zeta_k^1).
\end{aligned}$$

Taking the limit, we have

$$\mathrm{vol}(\zeta^t_\bullet, \zeta^1_\bullet) = \mathrm{vol}(\phi_t, \phi_1) = (1-t)\,\mathrm{vol}(\phi_0, \phi_1).$$

By [BE] again, this is equal to the energy:

$$E(\phi_t, \phi_1) = \mathrm{vol}(\phi_t, \phi_1).$$

The energy is then affine in the diagonalizable case. If the norms are not diagonalizable, we simply note that the geodesics ζ^t_k can be approximated by diagonalizable geodesics $\zeta^t_{k,\varepsilon}$ for which

$$\mathrm{vol}(\zeta^t_{k,\varepsilon}, \zeta^1_{k,\varepsilon}) = (1-t)\,\mathrm{vol}(\zeta^0_{k,\varepsilon}, \zeta^1_{k,\varepsilon})$$

so that at the limit

$$\mathrm{vol}(\zeta^t_k, \zeta^1_k) = (1-t)\,\mathrm{vol}(\zeta^0_k, \zeta^1_k)$$

still holds for all k, proving our result.

We now show that such a psh segment is unique. Fix a reference metric $\phi_{\mathrm{ref}} \in C^0(L) \cap \mathrm{PSH}(L)$. Assume $t \mapsto \psi_t$ is another such segment joining two metrics ϕ_0, $\phi_1 \in C^0(L) \cap \mathrm{PSH}(L)$, i.e. it is a psh segment along which the energy is affine. By the maximum principle Theorem 4.1.3.1, we then have

$$\psi_t \leq \phi_t$$

for all t, and $t \mapsto E(\phi_t, \phi_{\mathrm{ref}})$, $t \mapsto E(\psi_t, \phi_{\mathrm{ref}})$ are then affine functions with the same endpoints, hence for all t

$$E(\phi_t, \phi_{\mathrm{ref}}) = E(\psi_t, \phi_{\mathrm{ref}}).$$

By Proposition 4.2.6.2, since $\psi_t \leq \phi_t$ and the energies coincide, we have $\psi_t = \phi_t$.

Finally, for (3), the same idea as in (4) works: for all k, and for all t and t' in $[0,1]$ we find

$$d_1(\zeta^t_k, \zeta^{t'}_k) = |t - t'| d_1(\zeta^0_k, \zeta^1_k),$$

and we conclude by passing to the limit. □

Remark 4.2.5.1. However, reflecting the d_1-geometry of real Euclidean space, there are many more d_1-geodesics than just the unique psh geodesic (e.g., take the reparametrization of the concatenation of the geodesic joining ϕ_0 and $P(\phi_0, \phi_1)$, and the geodesic joining $P(\phi_0, \phi_1)$ and ϕ_1). The fact that our segment is maximal at least ensures that it is maximal in the set of d_1-geodesics which are also psh segments.

4.2.6 A result concerning comparable metrics with zero relative energy.

We have used, in the proof of Theorem 4.2.1.1, the fact that if two comparable metrics have the same Monge-Ampère energy, then they are equal. The most natural setting for this result is that of finite-energy metrics, and indeed we will use it in its full generality in the proof of Theorem 4.3.1.1. In this section, we prove this result. We will need another bifunctional acting on continuous psh metrics, the I energy.

Definition 4.2.6.1. Let ϕ_0, ϕ_1 be two continuous L^{an}-psh metrics. Their relative I-energy is defined as

$$I(\phi_0, \phi_1) = \int_X (\phi_0 - \phi_1)(\operatorname{MA}(\phi_1) - \operatorname{MA}(\phi_0)).$$

Their relative J-energy is defined as

$$J(\phi_0, \phi_1) = -E(\phi_0, \phi_1) + \int_X (\phi_0 - \phi_1) \operatorname{MA}(\phi_1).$$

Given a reference metric ϕ_{ref}, we write

$$I(\phi_0) = I(\phi_0, \phi_{\mathrm{ref}})$$

and

$$J(\phi_0) = J(\phi_0, \phi_{\mathrm{ref}}).$$

By [BJ21, Section 6.3], the functionals I and J also admit an extension to $\mathcal{E}^1(L^{\mathrm{an}})$, which is continuous along decreasing nets.

Note that we have by [BJ21, (3.15)]

$$\int_X (\phi_0 - \phi_1) \operatorname{MA}(\phi_0) \leq E(\phi_0, \phi_1) \leq \int_X (\phi_0 - \phi_1) \operatorname{MA}(\phi_0),$$

so that the I-energy is always nonnegative. This result relies on a special case of the local Hodge index theorem, as in [BJ21, Proposition 3.5]. That the J-energy is nonnegative follows from the very expression of the Monge-Ampère energy.

Proposition 4.2.6.2. *Let ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$. If $E(\phi_0, \phi_1) = 0$ and $\phi_0 \geq \phi_1$, then $\phi_0 = \phi_1$.*

Proof. The main argument has been communicated to the author by S. Boucksom and M. Jonsson, as part of works on finite-energy spaces currently in writing.

Approximate ϕ_0 and ϕ_1 by decreasing nets ϕ_0^k, $\phi_1^k \in C^0(L^{\mathrm{an}}) \cap \mathrm{PSH}(L^{\mathrm{an}})$. Up to taking the maximum of the two sequences, we can assume without loss of generality that for all k, $\phi_0^k \geq \phi_1^k$. We have that

$$E(\phi_0^k, \phi_1^k) = \frac{1}{\dim X + 1} \sum_i \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_0^{k(i)}, \phi_1^{k(\dim X - i)}).$$

Since $\phi_0 \geq \phi_1$, all of the terms in the above sum are integrals against positive measures of nonnegative functions, hence they are all positive. In particular,

$$0 \leq I(\phi_0^k, \phi_1^k) = \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_1^k) \leq (\dim X + 1) E(\phi_0^k, \phi_1^k) \to 0,$$

where the vanishing follows from continuity of E along decreasing nets, and the fact that $E(\phi_0, \phi_1) = 0$.

Pick any positive measure μ that can be expressed as $\mathrm{MA}(\phi)$ for some $\phi \in C^0(L^{\mathrm{an}}) \cap \mathrm{PSH}(L^{\mathrm{an}})$, and write for $x \in X^{\mathrm{an}}$

$$\begin{aligned}
&\mu(\{x\})(\phi_0^k(x) - \phi_1^k(x)) - \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_1^k) \\
&= \int_X \mu(\{x\})(\phi_0^k - \phi_1^k)\delta_x - \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_1^k) \\
&\leq \int_X (\phi_0^k - \phi_1^k) \,\mu - \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_1^k) \\
&\leq \int_X (\phi_0^k - \phi_1^k)(\mu - \mathrm{MA}(\phi_1^k)).
\end{aligned}$$

By [BJ21, Corollary 3.20], given four Fubini-Study metrics $\phi_i \in \mathcal{H}(L)$, $i \in \{0, 1, 2, 3\}$ there exists constants C, a, b depending only on $\dim X$ such that

$$\int_X (\phi_0 - \phi_1)(\mathrm{MA}(\phi_2) - \mathrm{MA}(\phi_3)) \leq C \cdot I(\phi_0, \phi_1)^a \cdot I(\phi_2, \phi_3)^a \cdot \max_i J(\phi_i)^b.$$

In our case, we then have

$$\int_X (\phi_0^k - \phi_1^k)(\mu - \mathrm{MA}(\phi_1^k)) \leq C \cdot I(\phi_0^k, \phi_1^k)^a \cdot I(\phi, \phi_1^k)^a \cdot \max(J(\phi_0^k), J(\phi_1^k), J(\phi))^b,$$

recalling that we have defined $\mu = \mathrm{MA}(\phi)$. Now, by continuity of the extensions of I and J along decreasing nets,

$$I(\phi, \phi_1^k)^a \cdot \max(J(\phi_0^k), J(\phi_1^k), J(\phi))^b \to I(\phi, \phi_1)^a \cdot \max(J(\phi_0), J(\phi_1), J(\phi))^b$$

while we have established before that

$$I(\phi_0^k, \phi_1^k) \to 0.$$

We then find that

$$\mu(\{x\})(\phi_0^k(x) - \phi_1^k(x)) - \int_X (\phi_0^k - \phi_1^k) \,\mathrm{MA}(\phi_1^k) \leq C \cdot I(\phi_0^k, \phi_1^k)^a \cdot I(\phi, \phi_1^k)^a \cdot \max(J(\phi_0^k), J(\phi_1^k), J(\phi))^b$$

and the right-hand side vanishes, while the left-hand side converges as $k \to \infty$ to the nonnegative quantity

$$\mu(\{x\})(\phi_0(x) - \phi_1(x)).$$

The key point now is to solve the non-Archimedean Monge-Ampère equation in order to find a measure $\mu = \mathrm{MA}(\phi)$ with positive Dirac mass at x. Now, we recall that a psh function is uniquely determined by its restriction to the set of divisorial points in X^{an}, and that for any such point x we may find a Monge-Ampère measure μ_x associated to a projective model of X which has an atom at x, as in [BE, Example 8.11]. As we then have

$$0 \leq \mu_x(\{x\})(\phi_0(x) - \phi_1(x)) = 0,$$

and $\mu_x(\{x\}) > 0$, we have that $\phi_0 = \phi_1$ on all divisorial points of X^{an}, hence on X^{an}. □

4.2.7 A simple example.

Consider the symmetric algebra $\mathrm{Sym}^\bullet(V)$ of some finite-dimensional K-vector space V. One can easily see that the k-th symmetric power of a geodesic segment in $\mathcal{N}(V)$ is a geodesic segment in $\mathcal{N}(V^{\odot k}) = \mathcal{N}(\mathrm{Sym}^k(V))$. Using this result together with our quantization results, we have a nice example: on projective spaces, Fubini-Study segments are in fact maximal psh segments.

Corollary 4.2.7.1. *Let $X = \mathbb{P}^n$ for some positive integer n and $L = \mathcal{O}_{\mathbb{P}^n}(m)$ for some positive integer m. Let ζ_t be a norm geodesic in $H^0(L)$. Then, the psh geodesic $t \mapsto \phi_t$ joining $\mathrm{FS}_1(\zeta_0)$ and $\mathrm{FS}_1(\zeta_1)$ corresponds to $t \mapsto \mathrm{FS}_1(\zeta_t)$.*

This does not hold in general: even if the boundary norms are generated in degree m for some m, the geodesic is not necessarily in degree m for all t, and thus the result only holds "at infinity".

Proof. Since $R(X, L) = \langle (x^I)_{I \in T^{\odot m}} \rangle_{\mathrm{K}}^{\odot \bullet}$, with $x = (x_0, \dots, x_n)$, we have that a geodesic $t \mapsto \zeta_t$ in $H^0(L)$ induces a geodesic $t \mapsto \zeta_t^{\odot k}$ in $H^0(kL)$ for all k through the k-th symmetric powers, as discussed above. Therefore, for all t, the geodesic is generated in degree one, and we have

$$\mathrm{FS}_k(\zeta_t^{\odot k}) = \mathrm{FS}_1(\zeta_t),$$

proving our result. □

4.3 Geodesics between finite-energy metrics.

4.3.1 Main Theorem for finite-energy metrics.

Our main Theorem in this Section is the following.

Theorem 4.3.1.1. *Given ϕ_0, ϕ_1 in the metric space $(\mathcal{E}^1(L), d_1)$, we have that:*

1. *there exists a maximal psh segment $t \mapsto \phi_t$ joining ϕ_0 and ϕ_1;*
2. *$\phi_t \in \mathcal{E}^1(L)$ for all t;*
3. *the segment ϕ_t is a (constant speed) metric geodesic for d_1, i.e. there exists a real constant $c \geq 0$ such that*

   $$d_1(\phi_t, \phi_s) = c \cdot |t - s|$$

   for all $t, s \in [0, 1]$;
4. *the Monge-Ampère energy is affine along ϕ_t, and it is the unique psh segment joining ϕ_0 and ϕ_1 with this property.*

4.3.2 Proof of Theorem 4.1.3.1.

Consider now two metrics ϕ_0, $\phi_1 \in \mathrm{PSH}(L)$, and pick decreasing nets ϕ_0^k, ϕ_1^k in $C^0(L) \cap \mathrm{PSH}(L)$ converging to ϕ_0, ϕ_1.

Lemma 4.3.2.1. Let $\phi_0 \leq \phi_0'$, $\phi_1 \leq \phi_1'$ be continuous psh metrics, and denote by ϕ_t, ϕ_t' the maximal psh segments joining them. Then, for all t, $\phi_t \leq \phi_t'$.

Proof. By definition, if ϕ_t' is maximal, it bounds from above all segments joining endpoints bounded above by ϕ_0', ϕ_1', and the result follows. □

Therefore, the net $k \mapsto \phi_t^k$ is monotonous, where ϕ_t^k is the maximal psh segment joining ϕ_0^k and ϕ_1^k. We claim that

$$\phi_t : t \mapsto \lim_k \phi_t^k$$

is our desired geodesic segment.

Proof of Theorem 4.1.3.1. If no psh segment exists between ϕ_0 and ϕ_1, the first statement of the Theorem is proven.

Assume now that there exist psh segments between ϕ_0 and ϕ_1. Let $t \mapsto \psi_t$ be such a segment. It is then a decreasing limit of a net of psh segments $t \mapsto \psi_t^k$. For all k, let $t \mapsto \psi_t'^k$ denote the maximal psh segment joining ψ_0^k and ψ_1^k. By maximality, we have for all t, k,

$$\psi_t^k \leq \psi_t'^k.$$

In particular, $\lim_k \psi_t^k \leq \lim_k \psi_t'^k$, and both are psh segments between ϕ_0 and ϕ_1. This shows that one needs only consider limits of maximal psh segments. Furthermore, by the same argument, one needs only consider sequences with endpoints equal to ϕ_0 and ϕ_1.

Therefore, we must show that given any two nets of maximal segments ϕ_t^k, ψ_t^k, such that the endpoints converge to ϕ_0 and ϕ_1, the limits are equal for all t:

$$\lim_k \phi_t^k = \lim_k \psi_t^k.$$

But

$$\lim_k \phi_t^k = \lim_n \sup\{\varphi_t^k \text{ psh segment between } \phi_0^k \text{ and } \phi_1^k\},$$

and similarly for ψ_t^k. Both nets converge to the limit

$$\sup\{\varphi_t \text{ psh segment between } \phi_0 \text{ and } \phi_1\},$$

which depends only of the endpoints ϕ_0 and ϕ_1, proving the Theorem. $\square$

4.3.3 Proof of Theorem 4.3.1.1.

From the previous section, the unique maximal psh segment joining two finite-energy metrics ϕ_0, ϕ_1 can be recovered as the limit ϕ_t of maximal segments ϕ_t^k joining decreasing approximations ϕ_0^k, ϕ_1^k of those metrics, in $C^0(L) \cap \mathrm{PSH}(L)$. It could a priori be the case that this leaves the class $\mathcal{E}^1(L)$. Theorem 4.3.1.1(3) will ensure that this is not the case.

Proof of Theorem 4.3.1.1 (2)-(5). As we have just discussed, existence (i.e. (2) in the Theorem) is ensured by Theorem 4.1.3.1. For all k, and for any reference metric $\phi_{\mathrm{ref}} \in C^0(L) \cap \mathrm{PSH}(L)$,

$$t \mapsto E(\phi_t^k, \phi_{\mathrm{ref}})$$

is affine, with coefficient equal to $E(\phi_1^k, \phi_0^k)$. By continuity of the energy along decreasing nets, the limit function

$$t \mapsto E(\phi_t, \phi_{\mathrm{ref}})$$

is therefore affine, with coefficient equal to the (finite) energy $E(\phi_0, \phi_1)$. This gives (3).

Furthermore, by Proposition 4.2.6.2, that ϕ_t is the only possible psh segment with the property that the Monge-Ampère energy is affine along it is proven using the same arguments as the proof of Theorem 4.2.1.1(4), establishing (5). To show that they are geodesics for our extended d_1 distance on $\mathcal{E}^1(L)$ again follows from the fact that the segments ϕ_t^k are d_1-geodesic, by Theorem 4.2.1.1: for all t, t',

$$d_1(\phi_t^k, \phi_{t'}^k) = |t - t'| d_1(\phi_0^k, \phi_1^k),$$

and taking the limit in k, using Proposition 2.4.3.8. $\square$

Chapter 5

The space of finite-energy metrics over a degeneration.

Summary of the main results.

This Chapter covers the results of [Reb21], which is concerned with generalizing the formalism of plurisubharmonic geodesic rays to the case of arbitrary (non-isotrivial) degenerations over the punctured unit disc, and studying non-Archimedean interpretations, building on the work of Darvas-Lu ([DL20]) and Berman-Boucksom-Jonsson ([BBJ]).

At least half of this Chapter takes place in the complex world; we have observed silence regarding complex pluripotential theory in this manuscript, rather focusing on the non-Archimedean case. Whenever necessary (and indeed this is where we shall begin), we will recall relevant notions in their right context.

5.1 Relative finite-energy spaces.

5.1.1 Reminders on finite-energy spaces in complex geometry.

We begin with some reminders concerning d_1-structures on spaces of finite-energy metrics in the classical setting. We thus consider a fixed *compact* Kähler manifold X, with $\dim X =: d$, endowed with an ample line bundle L.

Consider two metrics ϕ_0, $\phi_1 \in C^0 \cap \mathrm{PSH}(L)$. Their relative Monge-Ampère energy is the quantity

$$E(\phi_0, \phi_1) = \frac{1}{(L^d)(d+1)} \sum_{i=0}^{d} \int_X (\phi_0 - \phi_1)\, (dd^c\phi_0)^i \wedge (dd^c\phi_1)^{d-i}.$$

Note that we have a cocycle identity

$$E(\phi_0, \phi_1) = E(\phi_0, \phi') + E(\phi', \phi_1)$$

for any other continuous psh metric ϕ'. Having fixed a continuous psh metric ϕ_{ref} on the right, $E(\phi) := E(\phi, \phi_{\mathrm{ref}})$ can be seen as an operator on $C^0 \cap \mathrm{PSH}(L)$ which is also a primitive of the Monge-Ampère operator $\mathrm{MA} : \phi \mapsto (dd^c\phi)^d$. It admits a (possibly infinite) extension to $\mathrm{PSH}(L)$ via

$$E(\phi) = \lim_{k\to\infty} E(\phi_k),$$

where ϕ_k is a net of continuous psh metrics decreasing to ϕ, which always exists by [BK07], [Dem92]. The space of finite-energy metrics is the space

$$\mathcal{E}^1(L) = \{\phi \in \mathrm{PSH}(L),\ E(\phi) \text{ is finite}\}.$$

By the cocycle identity, this space does not depend on the choice of a reference metric. From the work of Darvas, we know this space to admit (amongst others) a d_1-type complete metric space structure via

$$d_1(\phi_0, \phi_1) = E(\phi_0) + E(\phi_1) - 2E(P(\phi_0, \phi_1)),$$

where $P(\phi_0, \phi_1)$ is the envelope

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L),\ \phi \leq \min(\phi_0, \phi_1)).$$

It will be more practical to use a different expression of the Monge-Ampère energy, just as we did in the non-Archimedean case, as a difference of absolute Deligne pairings.

5.1.2 Relative finite-energy metrics and extended Deligne pairings.

We consider a holomorphic submersion between complex manifolds

$$\pi : X \to Y$$

of relative dimension d. Pick $d+1$ pairs (L_i, ϕ_i), where L_i is a relatively ample line bundle over X, and ϕ_i is a continuous psh metric on L_i. To this data, one associates a line bundle over Y,

$$\langle L_0, \dots, L_d \rangle_{X/Y},$$

together with a metric

$$\langle \phi_0, \dots, \phi_d \rangle_{X/Y}$$

in a way that is multi-additive, symmetric; the construction furthermore commutes with base change (in particular, is stable upon restriction to an open set on the base), and satisfies

- the change of metric formula: given another continuous psh metric ϕ_0' on L_0, we have

$$\langle \phi_0, \dots, \phi_d \rangle_{X/Y} - \langle \phi_0', \dots, \phi_d \rangle_{X/Y} = \pi_* \left((\phi_0 - \phi_0')(dd^c\phi_1 \wedge \dots \wedge dd^c\phi_d) \right) \tag{5.1}$$

  (see [Elk90, Théorème I.1.1(d)]);

- the curvature formula

$$dd^c \langle \phi_0, \dots, \phi_d \rangle_{X/Y} = \pi_* (dd^c\phi_0 \wedge \dots \wedge dd^c\phi_d) \tag{5.2}$$

  (see [Elk90, Théorème I.1.1(d)]).

The last formula shows that the metric $\langle \phi_0, \dots, \phi_d \rangle_{X/Y}$ is positive. One also notices that a metric on a trivial Deligne pairing

$$\langle \mathcal{O}_X, L_1, \dots, L_d \rangle_{X/Y}$$

can be identified with a genuine function on the base Y, upon evaluation against the trivial section 1.

Assume for the moment that Y is a point. In that case, Deligne pairings can be seen as complex lines together with a Hermitian norm. In this setting, we will omit the subscript $\cdot_{X/Y}$. Using the change of metric formula, one can see the relative Monge-Ampère energy between two continuous psh metrics on a fixed line bundle L over X as a difference of Deligne pairings:

$$(d+1)E(\phi_0, \phi_1) = \langle \phi_0^{d+1} \rangle - \langle \phi_1^{d+1} \rangle,$$

which suggests that the Monge-Ampère energy should be seen intrinsically as a genuine (Hermitian) metric

$$(d+1)E(\phi) = \langle \phi^{d+1} \rangle$$

on the line $\langle L^{d+1} \rangle$.

We now return to arbitrary Y. The change of metric formula suggests that the Deligne pairing construction could possibly make sense in a larger class of metrics, where each ϕ_i has fibrewise finite energy. This motivates the following definition.

Definition 5.1.2.1. Let L be a relatively ample line bundle on X. We define the class of relative finite-energy metrics

$$\mathcal{E}^1_{X/Y}(L)$$

to be the class of plurisubharmonic metrics ϕ on L such that, for all $y \in Y$, $\phi_y \in \mathcal{E}^1(L_y)$. Here, L_y is the restriction of L to the fibre $\pi^{-1}(y)$.

Since we have required plurisubharmonicity on all of L, it follows that any metric in $\mathcal{E}^1_{X/Y}(L)$ can be approximated by a decreasing net of continuous psh metrics on L. In particular, such metrics admit Deligne pairings.

Theorem 5.1.2.2. *Let $\pi : X \to Y$ be a holomorphic submersion between complex manifolds of relative dimension d, and let $(L_i)_{i=0}^d$ be a collection of $d+1$ relatively ample line bundles on X. There exists a unique extension of the Deligne pairing construction to metrics in $\mathcal{E}^1(L_i)_{X/Y}$, which is multilinear, symmetric, stable upon restriction to a smaller open set on the base, and such that the change of metric formula* (5.1) *holds.*

Proof. We first restrict to an open set U on the base Y, so that we may apply Demailly regularization on $\pi^{-1}(U)$. Fix for each i a metric $\phi_i \in \mathcal{E}^1_{X/U}(L_i)$, and let $k \mapsto \phi_i^k$ be a sequence of continuous psh metrics on L_i decreasing to ϕ_i. We claim that the sequence

$$k \mapsto \langle \phi_0^k, \dots, \phi_d^k \rangle_{X/U}$$

decreases to a finite-valued metric on U, independent of the choices of approximating sequences, which defines our construction restricted to U. Assuming this convergence to hold, one sees that this construction is multilinear, symmetric, satisfies the change of metric formula. Uniqueness follows from the change of metric formula, which itself shows that the construction glues well over X.

That it would define a finite-valued metric on U follows from Lemma 5.1.2.3 below, so that all that is left in order to prove the Theorem is that the limit in question is decreasing. We proceed by induction on the number n of indices $i \in \{0, \dots, d+1\}$ such that ϕ_i belongs to $\mathcal{E}^1_{X/U}(L_i) - C^0 \cap \mathrm{PSH}(L_i)$. In the case $n = 0$, all metrics are continuous psh and this is the classical Deligne pairing, so that we have nothing more to prove.

Assume thus that the assertion holds for some $d+1 > n > 0$. Assume the metrics ϕ_i, $i = 1, \dots, d+1-n$ to belong to $C^0 \cap \mathrm{PSH}(L_i)$, and the $n+1$ other metrics ϕ_i to belong strictly to $\mathcal{E}^1_{X/U}(L_i)$, $i = 0$ or $i = d+2-n, \dots, d$. (We can do this without loss of generality, by symmetry and up to reordering the indices.) We approximate ϕ_0 and the $(\phi_i)_{i=d+2-n}^d$ by sequences $k \mapsto \phi_i^k$ of continuous psh metrics. For a fixed $\ell \in \mathbb{N}^*$ and by the induction assumption, the sequence

$$k \mapsto \langle \phi_0^\ell, \phi_1, \dots, \phi_{d+1-n}, \phi_{d+2-n}^k, \dots, \phi_d^k \rangle_{X/Y}$$

is decreasing and converges to a limit $\langle \phi_0^\ell, \phi_1, \dots, \phi_{d+1-n}, \phi_{d+2-n}, \dots, \phi_d \rangle_{X/U}$. This limit satisfies, for any fixed metric $\phi_0' \in C^0 \cap \mathrm{PSH}(L_0)$ the formula

$$\begin{aligned}&\langle \phi_0^\ell, \phi_1, \dots, \phi_{d+1-n}, \phi_{d+2-n}, \dots, \phi_d \rangle_{X/U} - \langle \phi_0', \phi_1, \dots, \phi_{d+1-n}, \phi_{d+2-n}, \dots, \phi_d \rangle_{X/U} \\ &= \int_{X/U} (\phi_0^\ell - \phi_0')\, dd^c\phi_1 \wedge \dots \wedge dd^c\phi_{d+1-n} \wedge dd^c\phi_{d+2-n} \wedge \dots \wedge dd^c\phi_d.\end{aligned}$$

Now, this expression yields a decreasing net as ℓ increases, and its limit is finite. In particular, it be seen to be the decreasing limit of

$$k \mapsto \langle \phi_0^\ell, \phi_1, \dots, \phi_{d+1-n}, \phi_{d+2-n}^k, \dots, \phi_d^k \rangle_{X/U},$$

which proves our desired statement by induction. □

Along the way, we have used the following Lemma regarding finiteness of products of absolute finite-energy classes. The proof follows from exactly the arguments of [BJ21, Theorem 5.8], therefore we leave the details to the interested reader.

Lemma 5.1.2.3. Let X be a compact Kähler manifold of dimension d, and let (L_i) be a collection of $d+1$ ample line bundles on X. Fix, for all $i = 0, \ldots, d$, a metric $\phi_i \in \mathcal{E}^1(L_i)$, and a continuous metric $\phi'_0 \in \mathcal{E}^1(L_0)$. Then, the integral

$$\int_X (\phi_0 - \phi'_0)\, dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_d$$

is finite.

5.1.3 Relative plurisubharmonic segments.

In the case where Y is a point (the absolute setting), it is well-known ([Dar15]) that any two metrics ϕ_0, ϕ_1 in $\mathcal{E}^1_{X/Y}(L) = \mathcal{E}^1(X, L)$ can be joined by a plurisubharmonic *geodesic* segment in $\mathcal{E}^1(X, L)$, in the following sense. There exists a $\mathbb{S}^1$-invariant plurisubharmonic metric Φ on the product $L \times \mathcal{A}$ (where $\mathcal{A}$ is the annulus $\{e^{-1} \leq |z| \leq 1\}$) identified via $t = -\log|z|$ with a segment

$$[0, 1] \ni t \mapsto \Phi_t \in \mathcal{E}^1(X, L),$$

such that Φ bounds by above all other such segments Ψ with $\Psi_0 \leq \phi_0$ and $\Psi_1 \leq \phi_1$. We now look at what happens when Y is no longer a point. Given ϕ_0, $\phi_1 \in \mathcal{E}^1_{X/Y}(L)$, and a point $y \in Y$ on the base, there exists by the previous discussion a plurisubharmonic geodesic segment $t \mapsto \phi_{t,y}$ joining $\phi_{0,y}$ and $\phi_{1,y}$ in $\mathcal{E}^1(X_y, L_y)$. Varying y, this gives a collection of plurisubharmonic geodesic segments $t \mapsto \phi_t$. It is not obvious that, for given t, ϕ_t has plurisubharmonic variation with respect to Y. We thus claim the following:

Theorem 5.1.3.1. *Given any ϕ_0, $\phi_1 \in \mathcal{E}^1_{X/Y}(L)$, the collection $[0, 1] \ni t \mapsto \phi_t$ of fibrewise psh geodesic segments belongs to $\mathcal{E}^1_{X/Y}(L)$. Furthermore, identifying the collection $t \mapsto \phi_t$ with a $\mathbb{S}^1$-invariant metric Φ on $L \times \mathcal{A}$, Φ is plurisubharmonic, and is the unique psh metric on $L \times \mathbb{D}^*$ such that for all $y \in Y$,*

$$dd^c_t (\pi_y)_* \langle \Phi^{d+1} \rangle_{X \times \mathcal{A}/Y \times \mathcal{A}} = 0,$$

where $\pi_y : X \times \mathcal{A} \to \{y\} \times \mathcal{A}$ is the projection to the point y. Such a segment will be called a psh geodesic segment in $\mathcal{E}^1_{X/Y}(L)$.

The last statement can be interpreted as saying that the Monge-Ampère energy along the psh geodesic segment is fibrewise affine.

This Theorem is proven via families of Bergman kernels. We recall some facts on this topic. For all z on the base, picking a smooth, strictly psh metric ϕ on L endows the $H^0(kL_z + K_{X_z})$ (for all positive integers k) with a Hermitian norm

$$\|s_z\|^2_{\phi,z,k} = \int_X |s_z|^2 e^{-k\phi_z},$$

for $|s_z|^2 e^{-k\phi_z}$ is indeed a measure on X_z. We may now pick a basis $(s_{j,z})$ which is orthonormal for $\zeta_{\phi,z,k}$, and define the Bergman kernel

$$B_{\phi,z,k} := \sum_j |s_{j,z}|^2.$$

There are two key points regarding this object. The first one, which is easier to see, is that $B_{\phi,z,k}$ is independent of the choice of such a basis. The second, much deeper point is that as we move *on the base*, the Fubini-Study metrics

$$\phi_k : z \mapsto FS_k(\zeta_{\phi,z,k}) := k^{-1} \log B_{t,z,k}$$

vary plurisubharmonically, i.e. define a globally psh metric on L. This is a particular case of [BP08, Theorem 0.1]. We will return to this construction shortly.

We now turn to some facts regarding spaces of norms. Given any two Hermitian norms ζ_0, ζ_1 acting on a complex finite-dimensional vector space V, it is a well-known fact that one may find a basis $(s_j)_j$ of V which is orthonormal for ζ_0 and also ortho*gonal* for ζ_1. One may then define a distinguished segment $t \mapsto \zeta_t$ of Hermitian norms by requiring ζ_t to be the unique norm orthogonal in the basis $(s_j)_j$, and such that for all j,

$$\|s_j\|_t = \|s_j\|_1^t.$$

Such segments are in fact geodesic for various d_p-type metric structures on the space of Hermitian norms on V, but we will not need that fact.

We can consider as before the psh geodesic $t \mapsto \phi_{t,z}$ joining $\phi_{0,z}$ and $\phi_{1,z}$ in $\mathcal{E}^1(L_z)$. Then, a result of Berndtsson ([Berndt09, Theorem 1.2]) states that the Fubini-Study metrics from before approximate the geodesic ϕ_t uniformly in t: there exists a constant $c = c(z)$ such that

$$|FS_k(\zeta_{t,k,z}) - \phi_{t,z}| \leq c(z) \cdot k^{-1} \log k. \tag{5.3}$$

One would like to have such an approximation to be (Y-locally) independent of the variable on the base. Firstly, on reading the proof of [Berndt09, Theorem 1.2], one notices that the constant c depends only on the endpoints $\phi_{0,z}$ and $\phi_{1,z}$, so that our problem reduces to knowing whether one can find c such that, for all z in some compact U in Y,

$$|FS_k(\zeta_{0,k,z}) - \phi_{0,z}| \leq c \cdot k^{-1} \log k$$

(and similarly for $t = 1$; the proof is symmetric.) This follows from adapting the general uniform Bergman kernel asymptotics result [MMBook, Theorem 4.1.1] to the case of varying complex structure, along the same lines as explained in the proof of [MZ, Theorem 1.6]. The proof of Theorem 5.1.3.1 is now a matter of adequately piecing together all the previous results.

Proof of Theorem 5.1.3.1. Let ϕ_0, $\phi_1 \in \mathcal{E}^1_{X/Y}(L)$, which we assume to be smooth and strictly psh (while the general case follows from regularization), and consider the families of fibrewise psh geodesics $t \mapsto \phi_t$. As explained before, for any compact U on the base, and all $z \in U$, there exists a c independent of z such that

$$|FS_k(\zeta_{t,k,z}) - \phi_{t,z}| \leq c \cdot k^{-1} \log k,$$

by [Berndt09, Theorem 1.2] and the uniform Bergman kernel asymptotics. Furthermore, as also discussed, the families

$$(t, z) \mapsto FS_k(\zeta_{t,k,z})$$

have plurisubharmonic variation in z and t. Combined with the above estimates, this means that, over $\pi^{-1}(U) \times \mathcal{A}$ (where $\pi : X \to Y$ is the structure morphism of X over Y, and $\mathcal{A}$ is the annulus corresponding to $[0, 1]$), the segment $t \mapsto \phi_t$ seen as a metric on $L \times \mathcal{A}$ can be uniformly approximated by continuous psh metrics on $L \times \mathcal{A}$, furthermore $\mathbb{S}^1$-invariant under the second variable. This settles the first statement of the Theorem.

That ϕ_t would be the unique segment such that

$$dd^c_t (\pi_y)_* \langle \Phi^{d+1} \rangle_{X \times \mathcal{A}/Y \times \mathcal{A}} = 0$$

for all $y \in Y$ follows by definition of the fibrewise psh geodesic segments (they are themselves characterized as the unique segments in each $\mathcal{E}^1(X_y, L_y)$ along which the Monge-Ampère energy is affine). $\square$

5.1.4 Relatively maximal metrics.

Definition 5.1.4.1. Let $\pi : X \to Y$ be a holomorphic submersion with compact Kähler fibres. Let L be a relatively ample line bundle on X. We say that a metric ϕ on L is relatively maximal if it is maximal in the usual sense of Sadullaev (e.g. [KliBook]) on the preimage of any relatively compact open subset of Y. In other words, ϕ is relatively maximal if and only if, for any relatively compact open subset U of Y, for any relatively compact open subset V of $\pi^{-1}(U)$, and for any psh metric ψ on the restriction of L to $\pi^{-1}(U)$ such that

$$\limsup \psi(z) - \phi(z) \leq 0$$

as z approaches the boundary of $\pi^{-1}(U)$, then

$$\psi(z) \leq \phi(z)$$

for all z in $\pi^{-1}(U)$.

Remark 5.1.4.2. One sees from this definition that a decreasing limit of relatively maximal psh metric is also relatively maximal.

Remark 5.1.4.3. Let M be a compact Kähler manifold together with an ample line bundle L_M. Let $[0, \infty) \ni t \mapsto \phi_t$ be a psh ray of psh metrics on L_M. Seen as a $\mathbb{S}^1$-invariant psh metric on the product $L \times \overline{\mathbb{D}}^*$, ϕ is relatively maximal in our sense if and only if it is "geodesic" in the sense of [BBJ].

A nice way to generate relatively maximal metrics is via Perron-Bremmermann envelopes, as we now prove. We extend our setting slightly, to allow for singular fibres, which will be useful later on. We state our result in maximal generality, but the case to keep in mind is that of a holomorphic submersion over the punctured disc with a singular fibre over zero.

Theorem 5.1.4.4. *Let $\pi : X \to Y$ be a holomorphic projective surjective morphism. Let Ω be a relatively compact, smooth open subset of Y, such that π is a submersion above (hence near) $\partial\Omega$. Let L be a π-ample line bundle on X. Let ϕ be a continuous collection of fibrewise psh metrics on $\pi^{-1}(\partial\Omega)$. We then have that:*

1. *if there exists a continuous psh extension of ϕ to all of $\pi^{-1}(\Omega)$, then there exists a (unique) relatively maximal continuous psh extension of ϕ to all of $\pi^{-1}(\Omega)$;*

2. *if Ω is defined as $\{\rho < 0\}$, where ρ is a smooth strictly psh function on Y, such that $\nabla\rho \neq 0$ whenever $\rho = 0$, then a continuous psh extension as above exists.*

Remark 5.1.4.5. An open subset that satisfies the second point above is sometimes called a *hyperconvex* open subset. In particular, $\mathbb{D}$ and annuli centered at zero are such open sets. The proof of the first point follows some ideas dating back to the work of Bedford-Taylor ([BT]), see e.g. [BBGZ, Proposition 6.3], [PS].

Proof of Theorem 5.1.4.4. The hypotheses in the Theorem give that $\pi^{-1}(\bar{\Omega})$ is a manifold with boundary, which we denote $M := \pi^{-1}(\bar{\Omega})$, and whose boundary is $\pi^{-1}(\partial\Omega)$, which we denote $\partial M := \pi^{-1}(\bar{\Omega})$. We will finally write $\bar{M} := \pi^{-1}(\bar{\Omega})$.

1. Existence of a continuous relatively maximal metric, assuming existence of a subsolution.
We claim that the envelope

$$\mathcal{P}\phi = \sup{}^*\{\psi \in C^0 \cap \mathrm{PSH}(L|_M),\ \psi \leq \phi \text{ on } L|_{\partial M}\}$$

is our desired relatively maximal, continuous metric on $L|_M$ which coincides with ϕ on $L|_{\partial M}$. By definition, $\mathcal{P}\phi$ is relatively maximal; furthermore, since there exists a continuous subsolution, i.e. a candidate ψ to the envelope which coincides with ϕ on $L|_{\partial M}$, $\mathcal{P}\phi$ also has the correct boundary values. We are therefore left to show continuity.

We begin with a continuity estimate near the boundary. Having fixed a reference smooth, strictly psh metric ϕ_{ref} on L, and setting $\omega = dd^c\phi_{\mathrm{ref}}$, we can see any candidate ψ for the envelope $\mathcal{P}\phi$ as a continuous ω-psh function $g = \psi - \phi_{\mathrm{ref}}$. Fix such a g, and set

$$f_0 = \phi - (\phi_{\mathrm{ref}})|_{\partial M}.$$

Since $dd^g \geq -\omega$, the Laplacian $\Delta_\omega g$ of g with respect to ω is bounded below by $-d-1$. Let f be the (continuous) solution on $\bar{M}$ to the Dirichlet problem

$$\Delta_\omega f + (d+1) = 0,\ f|_{\partial M} = f_0.$$

We then have that $\Delta_\omega(g - f) \geq 0$, which implies by the maximum principle that

$$\sup_M (g - f) = \sup_{\partial M} (g - f),$$

while this supremum is nonpositive since $\psi = g + \phi_{\text{ref}}$ is a candidate for the envelope $\mathcal{P}\phi$. We then have that

$$g \leq f$$

on all of $\bar{M}$, and this is true for any candidate ψ, so that

$$\mathcal{P}\phi \leq \phi_{\text{ref}} + f$$

on $\bar{X}$.

We now look at continuity on M. Let $\tilde{\phi}$ denote a continuous psh extension of ϕ to $L|_M$. We fix a very small $\varepsilon > 0$, and define

$$U = \{\mathcal{P}\phi^* < \tilde{\phi} + \varepsilon\},$$

which is the complement of a small compact set containing $\partial\Omega$. By regularization (e.g. [BouL2, Theorem 3.8]), we can find a sequence $\psi_k \in C^0 \cap \mathrm{PSH}(L|_U)$ which decreases to $\mathcal{P}\phi$. Now, by Dini, using compactness, we have that U is covered by finitely many of the

$$U_k = \{\psi_k < \tilde{\phi} + \varepsilon\}$$

(since such inequality holds, for all $z \in M$, and for all large enough k_z). In particular, for large enough k, one has $\psi_k < \tilde{\phi} + \varepsilon$. We now define $\tilde{\psi}_k := \max(\psi_k - \varepsilon, \tilde{\phi})$, which is defined on all of M. For all k, $\tilde{\psi}_k$ is continuous, as ψ_k is continuous away from the boundary and $\tilde{\phi}$ is continuous everywhere (in particular, near and up to the boundary). Furthermore, $\tilde{\psi}_k$ is equal to ϕ on the boundary, so that

$$\psi_k - \varepsilon \leq \tilde{\psi}_k \leq \mathcal{P}\phi \leq \mathcal{P}\phi^* \leq \psi_k.$$

This implies that ψ_k converges uniformly to $\mathcal{P}\phi$, i.e. $\mathcal{P}\phi$ is continuous on M. Furthermore, since:

1. $\mathcal{P}\phi \leq \phi_{\text{ref}} + f$, as at the end of the first point of the proof;

2. $\phi_{\text{ref}} + f$ converges continuously to ϕ near the boundary, and $\mathcal{P}\phi$ is continuous on M;

3. there exists a psh extension $\tilde{\phi}$, ensuring that $\mathcal{P}\phi = \phi$ on the boundary,

then $\mathcal{P}\phi$ is continuous up to the boundary.

2. Construction of a subsolution under the hyperconvexity assumption.
The second point of the Theorem will follow from a more general principle: consider the class $\mathcal{C}(L|_{\partial M})$ consisting of continuous, fibrewise psh metrics on $L|_{\partial M}$ admitting a continuous psh extension (hence a relatively maximal continuous extension, by the first point) to all of $L|_M$. Then, this class is stable under uniform limits, which follows from seeing that the mapping $\mathcal{C}(L|_{\partial M}) \ni \phi \mapsto \mathcal{P}\phi$ from the first part of the proof is continuous under uniform convergence.

To prove the second point, we therefore have to show that there exists a sequence $\phi_k \in \mathcal{C}(L|_{\partial M})$ converging uniformly to our boundary data ϕ. We proceed bu Bergman kernel approximation. Since L is π-ample, the sheaves $\pi_*(kL)$ are locally free for all k large enough, and correspond to the sections of a vector bundle E_k whose fibres are the $H^0(kL_z =, z \in M$. The collection of L^2-norms $N_k(\phi)$ associated to $k\phi$ then define a continuous collection of Hermitian metrics h_k on $E_k|_{\partial\Omega}$. We pick a sequence of smooth families of Hermitian metrics $(h_{k,j})_j$ on $\pi_*(kL)$ so that $h_{k,j} \to h_k$ uniformly on $\pi_*(kL)|_{\partial\Omega}$. The associated collection of metrics

$$\phi_{k,j,z} = FS_k(h_{k,j,z})$$

vary smoothly on L. Since they are fibrewise *smooth* and *strictly psh* (both of which are necessary conditions for the following argument), we may compensate for the lack of plurisubharmonicity in the direction of z, by pulling back a high enough multiple $m_{k,j}\pi^*\rho$ of the defining function ρ of Ω, which as we recall vanishes on the boundary of Ω. We therefore have a continuous psh extension

$$\phi_{k,j} + m_{k,j}\pi^*\rho$$

of $(\phi_{k,j})|_{\partial M}$. This implies that $\phi_{k,j} \in \mathcal{C}(L|_{\partial M})$; furthermore, $(\phi_{k,j})|_{\partial M} \to \phi_k$ uniformly, which implies that $\phi_k \in \mathcal{C}(L|_{\partial M})$. Now, by Bergman kernel approximation, the ϕ_k themselves converge uniformly *and increasingly* to ϕ, which implies $\phi \in \mathcal{C}(L|_{\partial M})$, concluding the proof. $\square$

Remark 5.1.4.6. By adapting classical arguments of pluripotential theory, one shows that a continuous psh metric on L is relatively maximal iff $(dd^c\phi)^{d+1} = 0$ on X, iff it coincides over each relative open subset with its Perron-Bremmermann envelope as in the above Theorem.

We now characterize relatively maximal metrics of relative finite energy.

Proposition 5.1.4.7. *Let ϕ be a metric in $\mathcal{E}^1_{X/Y}(L)$. Assume that Y is covered by relatively compact hyperconvex smooth open subsets. Then, ϕ is relatively maximal if and only if $\langle \phi^{d+1} \rangle_{X/Y}$ has zero curvature.*

Proof. Assume ϕ to be relatively maximal. Since the Deligne pairing construction is stable upon restriction to an open set, we will work over the preimage U of some smooth hyperconvex relatively compact open set in Y. If ϕ is continuous, we have just mentioned that $(dd^c\phi)^{d+1} = 0$ there, so that by (5.2), it follows that $dd^c E(\phi) \equiv 0$. The non-continuous case follows from regularization on U: pick a sequence of continuous metrics ϕ_k decreasing to ϕ on U; by Theorem 5.1.4.4, there exists a continuous, relatively maximal psh metric Φ_k coinciding with ϕ_k on $L|_{\pi^{-1}(\partial U)}$. By maximality, the sequence Φ_k necessarily converges to ϕ (since ϕ is assumed to be relatively maximal), and continuity of Deligne pairings along decreasing nets ensures $\langle \phi^{d+1} \rangle_{X/Y}$ to have zero curvature.

Conversely, assume $\langle \phi^{d+1} \rangle_{X/Y}$ to have zero curvature. In the continuous case, using (5.2) again, it follows that $(dd^c\phi)^{d+1} = 0$, as it is a nonnegative measure. In the general case, we again proceed base-locally, and approximate ϕ on the preimage of a relatively compact open subset U via a decreasing sequence of continuous psh metrics $k \mapsto \phi_k$. Let Φ_k be for each k the unique continuous and relatively maximal metric on U with prescribed boundary condition $\phi_k|_{\pi^{-1}(\partial U)}$, given by Theorem 5.1.4.4. Let Φ denote the limit of the decreasing sequence $k \mapsto \Phi_k$, which is relatively maximal. By continuity of the Deligne pairing along decreasing nets, this sequence also defines a decreasing sequence of zero curvature metrics $\langle \Phi_k^{d+1} \rangle_{X/U}$ which has to converge to the metric

$$\langle \Phi^{d+1} \rangle_{X/U},$$

which is a zero curvature metric $\tilde{\phi}$ on U, coinciding on ∂U with $\langle \phi^{d+1} \rangle_{X/U}$. Since $\langle \phi^{d+1} \rangle_{X/U}$ also has zero curvature, we have to have

$$\langle \Phi^{d+1} \rangle_{X/U} = \langle \phi^{d+1} \rangle_{X/U}$$

on all of U. Fix z in U, and note that this implies

$$E(\Phi_z) = E(\phi_z),$$

while by relative maximality of Φ, $\phi_z \leq \Phi_z$, which implies $\Phi_z = \phi_z$, thus concluding our proof. $\square$

5.2 Finite-energy metrics over degenerations.

5.2.1 Analytic models and degenerations.

We now turn to our main setting. We will consider the base Y to be the punctured unit disc, and we will assume that our family degenerates (meromorphically) as one approaches zero.

Definition 5.2.1.1. Consider a holomorphic submersion $\pi : X \to \overline{\mathbb{D}}^*$ with compact Kähler fibres, and a relatively ample line bundle L on X. An analytic model (or simply a model) of X is a normal complex analytic space $\mathcal{X}$, together with a flat, proper holomorphic morphism $\pi : \mathcal{X} \to \overline{\mathbb{D}}$, realizing an isomorphism $X \simeq \pi^{-1}(\overline{\mathbb{D}}^*)$. An analytic model of (X, L) is the data of an analytic model $\mathcal{X}$ on X, and an ample line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $\mathcal{L}$ restricted to $\pi^{-1}(\overline{\mathbb{D}}^*)$ is isomorphic to L. We define a degeneration (or a degeneration with meromorphic singularities) to be a morphism $\pi : X \to \overline{\mathbb{D}}^*$ as above, such that there exists an analytic model of (X, L).

Example 5.2.1.2. This construction specializes to the following well-known cases:

- if all the fibres of X are isomorphic to M, a model $\mathcal{X}$ can simply be viewed as a compactification of an isotrivial degeneration of M;

- if the above condition holds, and furthermore the isomorphism is generated by a $\mathbb{C}^*$-action, this is simply a (real) one-parameter degeneration of $(M, L|_M)$, i.e. a test configuration for $(M, L|_M)$.

The central fibre of a model of X is the space $\mathcal{X}_0 = \pi^{-1}(\{0\})$. If the degeneration $X \to \overline{\mathbb{D}}^*$ is isotrivial, we say that M, the fibre over 1, is the generic fibre of X.

5.2.2 Generalized slopes and Lelong numbers.

As we will be working with (generalized) subharmonic functions on the base $\overline{\mathbb{D}}^*$, we will often have to work with some notions of Lelong numbers. We review some (old and new) facts in this Section.

Definition 5.2.2.1. We say that a subharmonic function f on $\mathbb{D}^*$ has logarithmic growth (near zero) if there exists a real number a such that $f(z) + a \log |z|$ is bounded above near zero.

In particular, a subharmonic function f with logarithmic growth can be extended as a subharmonic function over the entire disc. In this case, one can define its generalized (in the sense that it is possibly signed) Lelong number, as follows. Pick a number a as above. The function

$$g(t) := \sup_{|z|=e^{-t}} (f(z) + a \log |z|),$$

for $t \in [0, \infty)$, is then a convex function of t, and the rate of change

$$\frac{g(t) - g(0)}{t}$$

is thus an increasing nonnegative function of t, which has a finite value as $t \to \infty$. This corresponds to saying that the limit

$$\lim_{r \to 0} \frac{\sup_{|z|=r} f(z) + a \log |z|}{-\log r}$$

exists and is finite.

Definition 5.2.2.2. Given a subharmonic function f with logarithmic growth on $\overline{\mathbb{D}}^*$, we define its generalized slope (or generalized Lelong number at zero) to be the value

$$\hat{f} := \lim_{r \to 0} \frac{\sup_{|z|=r} f(z) + a \log |z|}{-\log r} + a,$$

where a is a real such that $f + a \log |z|$ is bounded near zero. In particular, $\hat{f}$ is independent of the choice of such an a.

Example 5.2.2.3. In the case of an $\mathbb{S}^1$-invariant subharmonic function f, i.e. a convex function on $[0, \infty)$, this simply computes the slope at infinity

$$\lim_{t \to \infty} \frac{f(t)}{t}.$$

Remark 5.2.2.4. As a consequence of Harnack's inequality, $\hat{f}$ may equivalently be computed using the integrals

$$\fint_{|z|=r} f(z)\, dz$$

in place of the suprema.

The following estimate will be very useful later on.

Lemma 5.2.2.5. Let f a subharmonic function f with logarithmic growth on $\overline{\mathbb{D}}^*$. Then, for all z, we have

$$f(z) \leq \log(1/|z|) \cdot \hat{f} + c,$$

where $c = \sup_{|z|=1} f(z)$.

Proof. Define the function

$$g(t) := \sup_{|z|=e^{-t}} (f(z) + a \log |z|)$$

as before. Since the rate of change

$$\frac{g(t) - g(0)}{t}$$

is an increasing function of t, we have for all $s \in [0, \infty)$, all $|z| = e^{-s}$ and since, having fixed our boundary data, $v(0) = 0$, we have

$$\frac{f(z) + a \cdot \log |z| - g(0)}{-\log |z|} \leq \frac{g(s) - g(0)}{s} \leq \lim_{t \to \infty} \frac{g(t)}{t} = \hat{f} - a.$$

Adding a then concludes the proof. $\square$

5.2.3 Plurisubharmonic metrics on degenerations.

Fix now a degeneration $\pi : X \to \overline{\mathbb{D}}^*$, endowed with a relatively ample line bundle L. We will take our interest to plurisubharmonic metrics on L, and in particular their singularities. However, a general psh metric on a degeneration can behave very poorly near the singularity, even though we have assumed existence of an analytic model of X. Thus, we need to enforce a rather natural growth condition on such psh metrics, akin to that of linear growth for geodesic rays.

Definition 5.2.3.1. We say that a psh metric ϕ on L has *logarithmic growth* if there exists a model $(\mathcal{X}, \mathcal{L})$ of (X, L) such that ϕ extends as a psh metric on $\mathcal{L}$.

We will write PSH(L) for the space of psh metrics of logarithmic growth on L. If it comes to be necessary, we will rather write PSH(X, L) when considering the space of (non-necessarily of logarithmic growth) psh metrics on L. We will soon show that PSH(L) has many desirable properties. We will also shortly explain our terminology. We begin with the following result:

Lemma 5.2.3.2. Given a psh metric ϕ on L, the following are equivalent:

(i) ϕ has logarithmic growth, i.e. there exists a model $(\mathcal{X}, \mathcal{L})$ such that ϕ extends to a psh metric on $\mathcal{L}$;

(ii) for all models $(\mathcal{X}, \mathcal{L})$ of (X, L), there exists a constant $c = c(\mathcal{X}, \mathcal{L})$ such that $\phi + c \cdot \log |z|$ extends to a psh metric on $\mathcal{L}$;

(iii) there exists a model $(\mathcal{X}, \mathcal{L})$ and a smooth metric ϕ_{ref} on $\mathcal{L}$ such that

$$\rho^* \phi(z) \leq \phi_{\text{ref}}(z) + O(\log |z|)$$

as $z \to 0$, where ρ denotes the isomorphism between X and $\mathcal{X} - \mathcal{X}_0$;

(iv) for all models $(\mathcal{X}, \mathcal{L})$ of (X, L) and all smooth metrics ϕ_{ref} on $\mathcal{L}$, (iv) holds.

Proof. By classical results of pluripotential theory, (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iv). Since (iv)$\Rightarrow$(iii) is immediate, we only need to prove (iii)$\Rightarrow$(iv). Assume that

$$\rho^* \phi(z) \leq \phi_{\mathcal{L}}(z) + O(\log |z|)$$

for a smooth reference metric $\phi_{\mathcal{L}}$ on $\mathcal{L}$. Pick another model $(\mathcal{Y}, \mathcal{M})$ together with a smooth metric $\phi_{\mathcal{M}}$. Note that the equation above holds if and only if the same equation holds for the pullbacks of $\phi_{\mathcal{L}}$ and $\rho^*\phi$ to a higher model. Thus, we pick a model $(\mathcal{Z}, \mathcal{N})$ dominating both via $\pi_{\mathcal{X}} : \mathcal{Z} \to \mathcal{X}$, $\pi_{\mathcal{Y}} : \mathcal{Z} \to \mathcal{Y}$. There exists a unique Cartier divisor D supported on the special fibre $\mathcal{Z}_0$ such that

$$\pi_{\mathcal{X}}^* \mathcal{L} + D = \pi_{\mathcal{Y}}^* \mathcal{M},$$

and given a local equation f_D for D, we have

$$\pi_{\mathcal{X}}^* \phi_{\mathcal{L}} \leq \pi_{\mathcal{Y}}^* \phi_{\mathcal{M}} - \log |f_D| + O(1) \leq \pi_{\mathcal{Y}}^* \phi_{\mathcal{M}} + O(\log |z|).$$

Thus,

$$\pi_{\mathcal{X}}^* \rho^* \phi \leq \pi_{\mathcal{X}}^* \phi_{\mathcal{L}} + O(\log |z|) \leq \pi_{\mathcal{Y}}^* \phi_{\mathcal{M}} + O(\log |z|),$$

as desired. $\square$

Remark 5.2.3.3. The above result shows that one could equivalently define our growth condition using some fixed reference data $(\mathcal{X}_{\mathrm{ref}}, \mathcal{L}_{\mathrm{ref}})$, using e.g. point (ii). In the isotrivial case, there furthermore exists some very natural reference data: the "trivial model" given by the product family of the generic fibre with the whole disc.

Example 5.2.3.4. Let $[0,\infty) \ni t \mapsto \phi_t$ be a ray of psh metrics on an ample line bundle L over a fixed variety X. It may be identified as a psh metric Φ over the trivial model $(X \times \overline{\mathbb{D}}^*, L \times \overline{\mathbb{D}}^*)$, by setting $\Phi_z = \phi_{-\log|z|}$. In this case, the logarithmic growth condition is merely the usual linear growth condition on psh rays.

We then have as an immediate Corollary:

Corollary 5.2.3.5. *The space* $\mathrm{PSH}(L)$ *is stable under limits of decreasing nets, finite maxima, and addition of constants. It is furthermore the smallest such set containing all psh metrics on* L *which admit a locally bounded extension to some model* $(\mathcal{X}, \mathcal{L})$ *of* (X, L).

Proof. All of those properties are seen to preserve characterization (iv) above, having fixed some reference model. To show that it is the smallest set closed under those operations, only the statement about decreasing nets could *a priori* be delicate. Given a metric $\phi \in \mathrm{PSH}(L)$, (i) shows that it extends as a genuine metric on some model $(\mathcal{X}, \mathcal{L})$, and Demailly's regularization Theorem yields a decreasing sequence of smooth (in particular locally bounded) psh metrics decreasing to the extension of ϕ, which shows in particular that ϕ belongs to the closure of the set of locally bounded psh metrics on $\mathcal{L}$, proving our result. $\square$

5.2.4 The main setting, and some important examples.

We begin with some notation. Let $\pi : X \to \overline{\mathbb{D}}^*$ be a degeneration together with a relatively ample line bundle L. We now, and for the remainder of this article, fix some reference boundary data ϕ_∂, which is the restriction to the boundary $\pi^{-1}(\mathbb{S}^1)$ of a smooth psh metric on L. This is a minor distinction which will allow us to later obtain a genuine metric structure on a particular subspace of $\mathrm{PSH}(L)$, rather than a pseudometric structure, and therefore we will assume that a metric in $\mathrm{PSH}(L)$ has boundary data equal to ϕ_∂. We will define

$$\mathcal{E}^1(L) = \mathcal{E}^1_{X/\overline{\mathbb{D}}^*}(L) \cap \mathrm{PSH}(L)$$

to be the space of fibrewise finite-energy metrics in $\mathrm{PSH}(L)$. We also set

$$\hat{\mathcal{E}}^1(L) = \{\phi \in \mathcal{E}^1(L),\, \phi \text{ is relatively maximal}\}.$$

Example 5.2.4.1. Although those are seemingly restrictive conditions, they are in fact general enough to encompass the study of maximal geodesic rays. Let (X, L) be a product family $(M \times \overline{\mathbb{D}}^*, L_M \times \overline{\mathbb{D}}^*)$. For a $\mathbb{S}^1$-invariant metric ϕ on $L_M \times \overline{\mathbb{D}}^*$, seen as a ray $[0, \infty) \ni t \mapsto \phi_t$,

1. being in $\mathrm{PSH}(L)$ corresponds to the usual linear growth condition;
2. being relatively maximal corresponds to being a geodesic ray in the sense of [BBJ];
3. being in $\mathcal{E}^1(L)$ corresponds to having fibrewise finite-energy and linear growth;
4. therefore, belonging to $\hat{\mathcal{E}}^1(L)$ corresponds to being a fibrewise finite-energy geodesic rays with linear growth emanating from a given point - exactly the space of rays $\mathcal{R}^1(L)$ considered in [DL20].

Example 5.2.4.2 (Relative dimension zero, part 1)**.** Consider the case of relative dimension zero with a trivial line bundle L over $X \simeq \mathbb{D}^*$. Then,

1. $\mathrm{PSH}(L)$ corresponds to the set of subharmonic functions with logarithmic growth on $\mathbb{D}^*$;
2. the class of relatively maximal metrics in $\mathrm{PSH}(L)$ corresponds to the class of harmonic functions;
3. $\mathcal{E}^1(L)$ corresponds to finite-valued subharmonic functions with logarithmic growth;
4. finally, $\hat{\mathcal{E}}^1(L)$ corresponds to finite-valued harmonic functions with logarithmic growth.

It well-known that any harmonic function on the punctured disc decomposes as a sum of a multiple of $\log |z|$ and the real part of an analytic function. This is where our general setting starts diverging from the better-behaved $\mathbb{S}^1$-invariant. Indeed, by [BBJ, Proposition 4.1], for rays of metrics of finite energy, maximality implies linear growth. However, in our case, maximality plus finite energy no longer implies logarithmic growth, since there exist harmonic functions on the punctured disc that do not have logarithmic growth at zero (e.g. the real part of $z \mapsto e^{1/z}$)!

Assuming logarithmic growth, we then have a full description of $\hat{\mathcal{E}}^1(L)$ in relative dimension zero, since we then see that any (finite-valued) harmonic function with logarithmic growth has to be of the form $c \cdot \log|z| + H(z)$, where $H(z)$ is the solution of the generalized Dirichlet problem over the whole disc with the given boundary data. In particular, it is an affine space isomorphic to $\mathbb{R}$! This agrees with the radial case, where $\mathcal{E}^1(L)$ is simply the set of affine functions on $[0, \infty)$ emanating from the same point, which is isomorphic to the set of possible slopes.

Example 5.2.4.3 (Relative dimension zero, part 2)**.** We now consider what will be a model case for many future considerations: we still work in relative dimension zero over $\overline{\mathbb{D}}^*$, but we choose a nontrivial line bundle on $\overline{\mathbb{D}}^*$. The existence of a model for $(\overline{\mathbb{D}}^*, L)$ means that there is a relatively ample extension $\mathcal{L} \to \overline{\mathbb{D}}$. We can now pick a trivialization τ of $\mathcal{L}$ over $\overline{\mathbb{D}}$, which allows us to identify a metric $\phi \in \mathrm{PSH}(L)$ (extended to $\mathcal{L}$ via the logarithmic growth condition!) with the function

$$u = -\log|\tau|_\phi$$

on $\overline{\mathbb{D}}$. By the discussion above, if $dd^c\phi = 0$, then u decomposes as

$$u(z) = c \cdot \log|z| + H(z),$$

where H is bounded on $\overline{\mathbb{D}}$. This decomposition (in particular, c and H) depends on τ; but the fact that ϕ can be decomposed in any trivialization in such a way does not!

This is a nice model case for us, because the Deligne pairing construction (in our setting of fibrations over $\overline{\mathbb{D}}^*$) naturally gives line bundles over $\overline{\mathbb{D}}^*$, as we see in action now.

Corollary 5.2.4.4. *The relative maximality condition for metrics in $\mathcal{E}^1(L)$ can be pushed forward to the base via the Deligne pairing, i.e. we have a well-defined map*

$$\hat{\mathcal{E}}^1(L) \to \hat{\mathcal{E}}^1(\langle L^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}).$$

Furthermore, a metric $\phi \in \mathcal{E}^1(L)$ belongs to $\hat{\mathcal{E}}^1(L)$ if and only if, for any model $(\mathcal{X}, \mathcal{L})$ of (X, L) and any trivialization of the Deligne pairing $\langle \mathcal{L}^{d+1} \rangle_{\mathcal{X}/\overline{\mathbb{D}}}$, denoting $u = -\log |\tau|_{\langle \phi^{d+1} \rangle_{\mathcal{X}/\overline{\mathbb{D}}}}$, one has

$$u(z) = c \cdot \log |z| + H(z),$$

where c is a real constant and H *is a harmonic function on $\overline{\mathbb{D}}$ depending only on τ and the boundary data.*

Proof. The map above is naturally given by

$$\phi \mapsto \langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*},$$

in which case both statements are corollaries of Proposition 5.1.4.7 and the two examples above. □

5.2.5 Metrization.

As an important, and somewhat surprising consequence of our previous results, we may define a metric structure on the space $\hat{\mathcal{E}}^1(L)$. This generalizes e.g. [DL20], in which the authors endow the space of maximal psh rays with the distance

$$\hat{d}_1(\phi_0, \phi_1) = \lim_{t \to \infty} \frac{d_1(\phi_{0,t}, \phi_{1,t})}{t}.$$

In the next Sections, we will show that this structure furthermore satisfies some good properties, namely completeness and geodesicity.

Theorem 5.2.5.1. *The space $\hat{\mathcal{E}}^1(L)$ can be endowed with a metric space structure, defined by the generalized slope $\hat{d}_1(\phi_0, \phi_1)$.*

Naturally, this suggests that the d_1-distance is subharmonic with logarithmic growth along metrics in $\hat{\mathcal{E}}^1(L)$, a fact that we prove now.

Proposition 5.2.5.2. *Let ϕ_0, $\phi_1 \in \hat{\mathcal{E}}^1(L)$. Then, the map*

$$z \mapsto d_1(\phi_{0,z}, \phi_{1,z})$$

is subharmonic with logarithmic growth on $\mathbb{D}^*$.

Proof. By the formula for d_1,

$$d_1(\phi_{0,z}, \phi_{1,z}) = \langle \phi_{0,z}^{d+1} \rangle + \langle \phi_{1,z}^{d+1} \rangle - 2\langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle.$$

By Proposition 5.1.4.7, the first two metrics on the right-hand side have zero curvature, therefore we are left to show that the metric $\langle P(\phi_0, \phi_1)^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ is superharmonic. We pick any zero curvature metric ϕ_{ref} on $\mathbb{D}^*$, and note that $\langle P(\phi_0, \phi_1)^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ is superharmonic if and only if $\langle P(\phi_0, \phi_1)^{d+1} \rangle_{X/\overline{\mathbb{D}}^*} - \phi_{\mathrm{ref}}$ is a superharmonic function. Fix $a \in \mathbb{D}^*$ and let $r > 0$ be such that $\mathbb{D}(a, r) = \{|z - a| \leq r\} \subset \mathbb{D}^*$. Let ψ be the relatively maximal psh metric on $\mathbb{D}(a, r)$ and with boundary data

$$\phi(z) = P(\phi_{0,z}, \phi_{1,z}),\ \forall z \in S(a, r). \tag{5.4}$$

Such a metric is given by Theorem 5.1.4.4. We now deduce the two following facts:

(i) by maximality of ψ, if follows from Proposition 5.1.4.7 that

$$z \mapsto \langle \psi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$$

has zero curvature;

(ii) since on the boundary $S(a, r)$ we have $\psi(z) \leq \phi_{0,z}, \phi_{1,z}$, and ϕ_0, ϕ_1 are relatively maximal, we have

$$\psi_z \leq \phi_{0,z}, \phi_{1,z}$$

for all $z \in \mathbb{D}(a, r)$, thus $\psi_z \leq P(\phi_{0,z}, \phi_{1,z})$ and finally

$$\langle \psi_z^{d+1} \rangle \leq \langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle$$

by monotonicity of the Monge-Ampère energy.

Using (5.4), (i), and (ii) in order, we find:

$$\begin{aligned} ⨍_{S(a,r)} \langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle - \phi_{\mathrm{ref},z} &= ⨍_{S(a,r)} \langle \psi_z^{d+1} \rangle - \phi_{\mathrm{ref},z} \\ &= \langle \psi_a^{d+1} \rangle - \phi_{\mathrm{ref},a} \\ &\leq \langle P(\phi_{0,a}, \phi_{1,a})^{d+1} \rangle - \phi_{\mathrm{ref},a}. \end{aligned}$$

As the inequality is true for all a, our metric $\langle P(\phi_0, \phi_1)^{d+1}\rangle_{X/\overline{\mathbb{D}}^*}$ is then superharmonic.

We now show that there exists a real number $a \in \mathbb{R}$ such that

$$z \mapsto d_1(\phi_{0,z}, \phi_{1,z}) + a \log |z|$$

is bounded above. By Lemma 5.2.3.2(iv), for any model $(\mathcal{X}, \mathcal{L})$ of (X, L), fixing a reference metric $\phi_{\mathrm{ref}} \in \hat{\mathcal{E}}^1(L)$ which is locally bounded on $\mathcal{L}$, one has (up to adding large enough constants)

$$\phi_0 \leq \phi_{\mathrm{ref}} + c \cdot \log |z|$$

for some real constant c. In this case,

$$d_1(\phi_{0,z} - c \cdot \log |z|, \phi_{\mathrm{ref},z}) = \langle (\phi_{0,z} - c \cdot \log |z|)^{d+1}\rangle - \langle \phi_{\mathrm{ref},z}^{d+1}\rangle,$$

and the term on the right-hand side is a harmonic function with logarithmic singularities at the origin, so that substracting constants the result also holds for $z \mapsto d_1(\phi_{0,z}, \phi_{\mathrm{ref},z})$. Proceeding similarly for ϕ_1, our result then follows from the triangle inequality. □

Finally, we note an immediate consequence of Lemma 5.2.2.5 together with the previous Proposition 5.2.5.2.

Lemma 5.2.5.3. Let ϕ_0, $\phi_1 \in \hat{\mathcal{E}}^1(L)$. Then, for all z on the base, we have

$$d_1(\phi_{0,z}, \phi_{1,z}) \leq \hat{d}_1(\phi_0, \phi_1) \log(1/|z|).$$

Remark 5.2.5.4. Had we not fixed boundary data, we would have an additional error term in the above expression, corresponding exactly to the supremum of $z \mapsto d_1(\phi_{0,z}, \phi_{1,z})$ for $z \in \mathbb{S}^1$.

We are now equipped to endow the space $\hat{\mathcal{E}}^1(L)$ with a metric structure.

Proof of Theorem 5.2.5.1. That $\hat{d}_1(\phi, \phi) = 0$ and $\hat{d}_1(\phi_0, \phi_1) = \hat{d}_1(\phi_1, \phi_0)$ are immediate statements, and nonnegativity will follow from the triangle inequality and the former statement. Therefore, we must show that for any other $\phi_2 \in \hat{\mathcal{E}}^1(L)$, we have

$$\hat{d}_1(\phi_0, \phi_1) \leq \hat{d}_1(\phi_0, \phi_2) + \hat{d}_1(\phi_2, \phi_1).$$

Let a_{01} be such that $d_1(\phi_{0,z}, \phi_{1,z}) + a_{01} \log|z|$ is bounded above on the punctured disc, and define similarly a_{02}, a_{21}. We have by the triangle inequality of the fibrewise metric d_1

$$d_1(\phi_{0,z}, \phi_{1,z}) \leq d_1(\phi_{0,z}, \phi_{2,z}) + d_1(\phi_{2,z}, \phi_{1,z})$$

for all z in $\mathbb{D}^*$, and in particular

$$d_1(\phi_{0,z}, \phi_{1,z}) + (a_{02} + a_{21}) \log|z| \leq d_1(\phi_{0,z}, \phi_{2,z}) + d_1(\phi_{2,z}, \phi_{1,z}) + (a_{02} + a_{21}) \log|z|.$$

Upon taking (negative) Lelong numbers and adding constants, we find

$$\begin{aligned} & a_{02} + a_{21} - \nu_0(d_1(\phi_{0,z}, \phi_{1,z}) + (a_{02} + a_{21}) \log|z|) \\ & \leq a_{02} - \nu_0(d_1(\phi_{0,z}, \phi_{2,z}) + a_{02} \log|z|) + a_{21} - \nu_0(d_1(\phi_{2,z}, \phi_{1,z}) + a_{21} \log|z|). \end{aligned}$$

Since

$$z \mapsto d_1(\phi_{0,z}, \phi_{1,z}) + (a_{02} + a_{21}) \log|z|$$

is bounded above, the previous equation is by the very definition of d_1 equivalent to

$$\hat{d}_1(\phi_0, \phi_1) \leq \hat{d}_1(\phi_0, \phi_2) + \hat{d}_1(\phi_2, \phi_1),$$

as desired. Finally, assuming $\hat{d}_1(\phi_0, \phi_1) = 0$, Lemma 5.2.5.3 shows that we must have $\phi_0 = \phi_1$. $\square$

5.2.6 Completeness.

We now prove completeness of our space.

Theorem 5.2.6.1. *The metric space $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ is complete.*

In order to prove this, discuss possible topologies for $\mathcal{E}^1(L)$.

Remark 5.2.6.2 (Topologies on $\mathcal{E}^1(L)$)**.** We have already considered the topology of fibrewise d_1-convergence on $\mathcal{E}^1(L)$. There is a yet finer topology, that of locally uniform fibrewise d_1-convergence, by which ϕ_k converges to ϕ if, for all relatively compact open sets U in X, $d_1(\phi_{k,z}, \phi_z) \to 0$ uniformly in z on U. In between the two, there is the topology of "base-locally" uniform fibrewise d_1-convergence, which is the same but over the $\pi^{-1}(U)$ with U relatively compact open in $\mathbb{D}^*$. By the previous Lemma, the latter is equivalent to the topology induced by $\hat{d}_1$ on $\hat{\mathcal{E}}^1(L)$!

Proposition 5.2.6.3. *Let ϕ_k be a sequence of metrics in $\hat{\mathcal{E}}^1(L)$ converging to some metric ϕ on L for the topology of base-locally uniform fibrewise d_1 convergence. Then, ϕ belongs to $\hat{\mathcal{E}}^1(L)$.*

Proof. Pick a sequence $k \mapsto \phi_k \in \hat{\mathcal{E}}^1(L)$ and a fixed metric ϕ in $\mathcal{E}^1(L)$. Assume that, for a relatively compact open $U \subset \overline{\mathbb{D}}^*$ we have

$$d_1(\phi_{k,z}, \phi_z) \to 0$$

uniformly in $z \in \pi^{-1}(U)$. Since convergence in Monge-Ampère energy is subordinate to d_1-convergence we have that

$$\langle \phi_{k,z}^{d+1} \rangle \to \langle \phi_z^{d+1} \rangle$$

again uniformly in z; by maximality, the metrics $\langle \phi_k^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ are zero curvature, and an uniform limit of such is again zero curvature. As having zero curvature is a local property and the $\pi^{-1}(U)$ cover X, we then have that $\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ has zero curvature on all of X. By virtue of being in $\mathcal{E}^1(L)$, this implies ϕ to be relatively maximal by 5.1.4.7, as long as we can show that the limit is psh. On $\pi^{-1}(U)$, there exists $c > 0$ independent of $z \in U$ such that

$$\int (\phi_{k,z} - \phi_z)\, d\mu_z \leq c \cdot d_1(\phi_{k,z}, \phi_z) \leq c \cdot c'$$

against a fixed smooth family of volume forms $z \mapsto \mu_z$, so that uniform fibrewise d_1-convergence implies L^1 convergence of ϕ_k to ϕ on $\pi^{-1}(U)$, which establishes plurisubharmonicity of the limit there, hence on X. □

We may now prove completeness.

Proof of Theorem 5.2.6.1. Consider a Cauchy sequence $m \mapsto \phi_m \in \hat{\mathcal{E}}^1(L)$. For all ε and all large enough m, n,

$$\hat{d}_1(\phi_m, \phi_n) \leq \varepsilon,$$

which by Lemma 5.2.5.3 implies the individual sequences $m \mapsto \phi_{m,z}$ to be d_1-Cauchy. By completeness of the fibrewise $\mathcal{E}^1$ spaces ([Dar19, Theorem 3.36]), those sequences d_1-converge to a unique finite-energy metric $\phi(z)$, and in fact this convergence is seen to hold base-locally uniformly fibrewise. The mapping

$$z \mapsto \phi(z)$$

is therefore a metric in $\hat{\mathcal{E}}^1(L)$ by Proposition 5.2.6.3. □

5.2.7 Geodesics.

We now show that, much as in the absolute $\mathcal{E}^1$ setting, one can find geodesics in $\hat{\mathcal{E}}^1(L)$.

Theorem 5.2.7.1. *Given any ϕ_0, $\phi_1 \in \hat{\mathcal{E}}^1(L)$, the psh geodesic segment $t \mapsto \phi_t$ joining them, given by Theorem 5.1.3.1, is $\hat{d}_1$-geodesic in the metric sense, i.e.*

$$\hat{d}_1(\phi_t, \phi_s) = |t - s|\hat{d}_1(\phi_0, \phi_1).$$

Furthermore, given any model $(\mathcal{X}, \mathcal{L})$ of (X, L) and a trivialization τ of $\langle \mathcal{L}^{d+1} \rangle$ over $\overline{\mathbb{D}}$, setting

$$u_t := -\log |\tau|_{\phi_t},$$

the segment of generalized slopes

$$t \mapsto \hat{u}_t$$

is affine on $[0, 1]$; and $t \mapsto \phi_t$ is uniquely characterized by this property among psh segments.

Proof of Theorem 5.2.7.1. Let ϕ_0, $\phi_1 \in \hat{\mathcal{E}}^1(L)$. We consider as in Theorem 5.1.3.1 the family of fibrewise maximal geodesics

$$t \mapsto \phi_{t,z}.$$

To show that it belongs to $\hat{\mathcal{E}}^1(L)$, we must make sure that it has logarithmic growth and is relatively maximal. The former is due to Lemma 5.2.3.2(ii), since for fixed $x \in X$, $\phi_t(x) \leq (1 - t)\phi_0(x) + t\phi_1(x)$ by convexity of maximal segments, so that if there exist a_i, $i = 0, 1$ such that $\phi_i + a_i \log |z|$ are bounded above near the central fibre of some model, then so is $\phi_t + (1 - t)a_0 + a_1$. Regarding maximality, $\langle \phi_t^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ is a convex combination of zero curvature metrics with logarithmic growth, hence ϕ_t is also relatively maximal by Proposition 5.1.4.7.

That $t \mapsto \phi_t$ is $\hat{d}_1$-geodesic is a consequence of the fact that, for all z, $t \mapsto \phi_{t,z}$ is $d_{1,z}$-geodesic. Finally, the statement regarding the Monge-Ampère energy follows upon taking generalized slopes in the statement of Theorem 5.1.3.1. $\square$

5.2.8 Extension of the distance to $\mathcal{E}^1(L)$

In this Section, we construct a "maximal envelope" map, which will allow us to extend the d_1-distance as a pseudodistance to all of $\mathcal{E}^1(L)$.

Proposition 5.2.8.1. *For all $\phi \in \mathcal{E}^1(L)$, there exists a unique smallest relatively maximal metric $\hat{P}(\phi) \in \hat{\mathcal{E}}^1(L)$ with $\phi \leq \hat{P}(\phi)$ and*

$$d_1(\phi_z, \hat{P}(\phi)_z) = o\left(\log|z|\right)$$

as $z \to 0$. This defines a natural projection

$$\hat{P} : \mathcal{E}^1(L) \to \hat{\mathcal{E}}^1(L).$$

Before proving this result, we note this immediate Corollary:

Corollary 5.2.8.2. *The mapping*

$$\hat{d}_1(\phi_0, \phi_1) = \hat{d}_1(\hat{P}(\phi_0), \hat{P}(\phi_1))$$

defines a pseudodistance on $\mathcal{E}^1(L)$.

Proof of Proposition 5.2.8.1. Let $\phi \in \mathcal{E}^1(L) \cap C^0(L)$, and, for all $r \in (0,1)$, let U_r denote the annulus $\{r < |z| < 1\} \subset \overline{\mathbb{D}}^*$, and $V_r = \pi^{-1}(U_r) \subset X$. Let ϕ_r be the relatively maximal metric on V_r, coinciding with ϕ on ∂V_r, given by Theorem 5.1.4.4. Fixing z on the base, the sequence $r \mapsto \phi_{r,z}$ is an increasing sequence of psh metrics in $\mathcal{E}^1(L_z)$. We claim that the limit family

$$z \mapsto \left(\lim_{r\to 0}{}^* \phi_{r,z}\right)$$

is the desired envelope $\hat{P}(\phi)$. Denote this limit $\hat{\phi}$ for the moment. Fix some r. By construction, $\hat{\phi}$ restricted to V_r coincides everywhere with its Perron-Bremmermann envelope; furthermore, it is locally bounded (since it is approximable from below). By the discussion in Section 5.1.4, since this holds for all r, $\hat{\phi}$ is relatively maximal. Furthermore, by construction again, it satisfies $\phi \leq \hat{\phi}$ and is the smallest such relatively maximal metric. We are therefore only left to prove that $d_1(\hat{\phi}_z, \phi_z) = O(\log|z|)$ as $z \to 0$. As in Corollary 5.2.4.4, we pick a model $(\mathcal{X}, \mathcal{L})$ of (X, L), and we extend $\langle \phi^{d+1} \rangle$ to the trivializable line bundle $\langle \mathcal{L}^{d+1} \rangle$. Picking a trivialization τ allows us to identify the energies $\langle \hat{\phi}^{d+1} \rangle$ and the $\langle \phi_r^{d+1} \rangle$ with functions u and u_r on $\mathbb{D}$

and U_r respectively. By Proposition 5.1.4.7, those functions are harmonic, and for all $s \in (0,1)$, the functions u_r, $r > s$ increase over $\overline{U_s}$ to u, which implies the convergence to be uniform (as an increasing sequence of harmonic functions over a compact set). Now, by harmonicity, for $r > s$, the integrals

$$\fint_{|z|=r} u_s(z)\, dz$$

are affine functions of $\log r$. Writing

$$v = -\log |\tau|_{\langle \phi^{d+1} \rangle},$$

we then have

$$\fint_{|z|=r} u_s(z)\, dz = \frac{\log r}{\log s} \fint_{|z|=s} v(z)\, dz + \left(1 - \frac{\log r}{\log s}\right) \cdot \fint_{|z|=1} v(z)\, dz,$$

(recall how we have defined ϕ_s and u_s!). Taking the limit $s \to 0$ using the uniform convergence discussed above yields

$$\fint_{|z|=r} u_s(z)\, dz = -(\log r)\, \hat{v} + \fint_{|z|=1} v(z)\, dz,$$

where $\hat{v}$ denotes the generalized slope of the subharmonic function v. Taking slopes in this equality, one then finds

$$\hat{v} = \hat{u}.$$

Now, since $\phi \leq \hat{\phi}$, we have

$$d_1(\phi_z, \hat{\phi}_z) = u(z) - v(z),$$

whose slopes we have seen to coincide, proving our statement that $d_1(\hat{\phi}_z, \phi_z) = O(\log |z|)$. Therefore, $\hat{\phi}$ is our desired envelope $\hat{P}(\phi)$. Finally, if ϕ is not continuous, we extend it to some model $(\mathcal{X}, \mathcal{L})$, and a decreasing approximation by continuous metrics ϕ_i on $\mathcal{L}$ gives a sequence of relatively maximal metrics $\hat{\phi}_i$ decreasing to some relatively maximal metric $\hat{\phi}$ which has the desired properties, as we show now: define u^i, u^i_r and v^i as above for ϕ_i, and u, u_r and v for ϕ. By monotonicity of Deligne pairings along decreasing nets, we have that $u^i \to u$, $u^i_r \to u_r$ and $v^i \to v$ decreasingly; we then have for all $r > s \in (0,1)$ and all positive integers i that

$$\fint_{|z|=r} u^i_s(z)\, dz = \frac{\log r}{\log s} \fint_{|z|=s} v^i(z)\, dz + \left(1 - \frac{\log r}{\log s}\right) \cdot \fint_{|z|=1} v^i(z)\, dz;$$

furthermore, we may normalize all our sequences so that all the functions involved are nonpositive, thereby allowing us to use monotone convergence and find

$$\fint_{|z|=r} u_s(z)\,dz = \frac{\log r}{\log s}\fint_{|z|=s} v(z)\,dz + \left(1-\frac{\log r}{\log s}\right)\cdot\fint_{|z|=1} v(z)\,dz,$$

so that we may proceed using the same argument as before to show that $d_1(\hat{\phi}_z,\phi_z) = O(\log|z|)$; that $\hat{\phi}$ is the smallest relatively maximal metric bounded below by ϕ and satisfying this equality follows again by construction, since decreasing limits of relatively maximal metrics over annuli remain relatively maximal. $\square$

5.3 The non-Archimedean limit.

We move away from relatively maximal and finite-energy metrics for the moment, and focus on the space $\mathrm{PSH}(L)$. The purpose of this Section is to show that there is a natural map from this space to a certain space of non-Archimedean metrics.

5.3.1 Degenerations as varieties over discretely valued fields.

Dating back to ideas of Berkovich ([Berk94], [Berk09]), objects such as degenerations and analytic models thereof can be interpreted as varieties over the field $\mathbb{C}((t))$ (see also [Fav], [BJ17]). For clarity, we will from now on write $\mathrm{K} = \mathbb{C}((t))$ and $\mathrm{R} = \mathbb{C}[[t]]$.

Pick a degeneration $\pi : X \to \overline{\mathbb{D}}^*$ and an analytic model $\pi : \mathcal{X} \to \overline{\mathbb{D}}$ of X. As X is projective, it can be embedded in some $\mathbb{P}^n \times \mathbb{D}$, where it is presented by a finite number of homogeneous polynomials with coefficients in the set of holomorphic functions on $\overline{\mathbb{D}}^*$ that are meromorphic at zero. Since this set of functions can be identified with the field K of complex Laurent series, one can then view X as a variety X_{K} over the field K. Similarly, $\mathcal{X}$ can be presented by finitely many homogeneous polynomials with coefficients in $\mathcal{O}(\mathbb{D})$, i.e. holomorphic functions over the disc, so that it can be identified with a variety $\mathcal{X}_{\mathrm{R}}$ over R.

Example 5.3.1.1. In the case of an isotrivial degeneration $X \simeq M \times \mathbb{D}^*$ for some complex projective manifold M, X can be identified with the base change of M to the field K. In particular, there exists a "trivial" algebraic model, defined by taking the base change of M to R, which corresponds to the product analytic family over $\mathbb{D}$.

K is a (non-Archimedean) valued field, with valuation

$$\nu_0(\Sigma a_i t^i) = \min\{i,\, a_i \neq 0\}.$$

This also defines a valuation on the Noetherian ring R. From the general work of Berkovich ([BerkBook]), one can associate to a scheme X over a valued ring R, in a functorial way, its analytification X^{an} with respect to the given valuation on the base. The underlying points of this analytification roughly correspond to valuations on the function field $\mathrm{K}(X)$ extending the base valuation on K, and the topology is that of pointwise convergence.

In our setting, the Berkovich analytification $X_{\mathrm{K}}^{\mathrm{an}}$ of X_{K} contains an important dense subset: the set of divisorial points X^{div}. It is described as follows. Let $\mathcal{X}$ be an analytic model of X. By Noetherianity and normality, the fibre of $\mathcal{X}$ over 0 is then a Cartier divisor which decomposes as the Weil divisor

$$\mathcal{X}_0 = \sum_i a_i E_i,$$

with each E_i irreducible. Each component of such a decomposition defines a valuation ν_{E_i} on $\mathrm{K}(X)$ as follows: for all $f \in \mathrm{K}(X)$,

$$\nu_{E_i}(f) = \mathrm{ord}_{E_i}(f)/a_i.$$

All divisorial points of $X_{\mathrm{K}}^{\mathrm{an}}$ are then obtained in this manner.

5.3.2 Relating non-Archimedean psh functions and models.

Let X be a degeneration with a line bundle L on X. Let $(\mathcal{X}, \mathcal{L})$ be a model of (X, L). Recall that to $\mathcal{L}$ one can associate a model metric $\phi_{\mathcal{L}}$ on $L_{\mathrm{K}}^{\mathrm{an}}$, (as explained, for example, in detail in [BFJ16]). Such a metric is uniquely characterized as follows: given an open set $\mathcal{U} \subset \mathcal{X}$ and a nonvanishing section of the restriction of $\mathcal{L}$ to $\mathcal{U}$, then we require that $|s|_{\phi_{\mathcal{L}}} = 1$ on $(\mathcal{U}_{\mathrm{K}} \cap X_{\mathrm{K}})^{\mathrm{an}}$.

Fixing a psh model metric $\phi_{\mathcal{L}}$ on $L_{\mathrm{K}}^{\mathrm{an}}$, one can identify psh metrics on $L_{\mathrm{K}}^{\mathrm{an}}$ with "$\mathcal{L}$-psh" functions on $X_{\mathrm{K}}^{\mathrm{an}}$, via $\phi \leftrightarrow \phi - \phi_{\mathcal{L}}$. We define more generally the set of L-psh functions to be the reunions of all $\mathcal{L}$-psh functions for all nef models $\mathcal{L}$ of L.

Any vertical ideal sheaf $\mathfrak{a}$ on a model $\mathcal{X}$ of X defines a function $\log|\mathfrak{a}|$ on X, via

$$\log|\mathfrak{a}|(x) = \max\{\log|f(x)|\},$$

where the f run over a set of local generators for $\mathfrak{a}$. (In particular, any vertical Cartier divisor D on a model defines such a function.) We then have the following crucial result:

Lemma 5.3.2.1 ([BFJ16])**.** Let $(\mathcal{X}, \mathcal{L})$ be a model of (X, L). Let $\mathfrak{a}$ be a vertical ideal sheaf on $\mathcal{X}$, such that $\mathcal{L} \otimes \mathfrak{a}$ is globally generated. Then, $\phi_{\mathcal{L}} + \log|a|$ is a psh metric on $L_{\mathrm{K}}^{\mathrm{an}}$.

5.3.3 The main result.

We are now equipped to describe the main construction of this Section. We fix a metric $\phi \in \mathrm{PSH}(L)$. Given any divisorial point ν_E associated to the component E of a model $\mathcal{X}$ of X, we know that $\phi + a\log|z|$ extends to a metric over E for some $a \in \mathbb{R}$. Pick a psh metric ϕ_E with divisorial singularities of type E_i on $\mathcal{X}$, i.e. locally of the form

$$\phi_E = \log|f_E| + O(1),$$

where f_E is a local equation for E. We can then define a generic (signed) Lelong number

$$\varphi^{\mathrm{NA}}(\nu_E) = \mathrm{ord}_E(\phi) := -\sup\{c \geq 0,\, \phi + a\log|z| \leq c \cdot \phi_E + O(1) \text{ near E}\} + a. \tag{5.5}$$

By linearity, this is independent of the choice of such an a. Performing this construction over all possible E captures the singularities of ϕ along all possible models of $\mathcal{X}$. Our main result for this Section is then the following:

Theorem 5.3.3.1. *Let X be a degeneration together with a relatively ample line bundle L. The Lelong numbers of a metric $\phi \in \mathrm{PSH}(L)$ define a function on X^{div}, which admits a unique L-psh extension, giving a map*

$$(\cdot)^{\mathrm{NA}} : \mathrm{PSH}(L) \to \mathrm{PSH}(L_{\mathrm{K}}^{\mathrm{an}}),$$

which is furthermore lower semicontinuous and order-preserving.

5.3.4 Some preliminaries.

We now prove some auxiliary results that will be useful in the proof of Theorem 5.3.3.1. We first show that multiplier ideals of psh metrics on L give $L^{\mathrm{an}}_{\mathrm{K}}$-psh functions.

Lemma 5.3.4.1. Let ϕ be a metric in $\mathcal{E}^1(L)$. Let $(\mathcal{X}, \mathcal{L})$ be a model of (X, L) such that ϕ extends as a psh metric on $\mathcal{L}$. Then, up to restricting to a slightly smaller disc, for all m, the multiplier ideal

$$\mathfrak{a}_m = \mathcal{J}(m\phi)$$

is vertical, and there exists an integer m_0 (depending only on $\mathcal{L}$ and not on m or ϕ) such that $(m + m_0)\mathcal{L} \otimes \mathcal{J}(\phi)$ is globally generated on $\mathcal{X}$.

Proof. Since ϕ in particular has fibrewise finite energy, it has zero Lelong numbers on all fibres. As a consequence, ϕ has zero Lelong numbers on all of $\mathcal{X} - \mathcal{X}_0$, as Lelong numbers cannot increase upon evaluating them on a larger space. Skoda's integrability theorem ([Sko, Theorem 1], see also [Dem12, Lemma 5.6(a)]) then yields local L^1-integrability of $e^{-\phi}$, which in turn implies local L^p-integrability of $e^{-\phi}$ for all $\infty > p \geq 1$, and in particular, for all positive integers m, L^1-integrability of $e^{-m\phi}$. By [Dem12, Lemma 5.6(a)] again, the multiplier ideals satisfy

$$\mathfrak{a}_{m,x} = \mathcal{O}_{\mathcal{X},x}$$

for all m and for all x outside of the central fibre, i.e. $\mathfrak{a}_m$ is cosupported on the central fibre.

Now, the global generation statement, follows from a relative equivalent of [Dem12, Proposition 6.27]. We can in fact argue just as in [BBJ, Lemma 5.6]: we must prove that there exists m_0 such that the sheaf $(m+m_0)\mathcal{L}\otimes\mathcal{J}(\phi)$ is π-globally generated. By the relative Castelnuovo-Mumford criterion, having picked a relatively very ample line bundle V on X and an m_0 such that $m_0 \cdot \mathcal{L} - K_{\mathcal{X}} - (d+1)V$ is relatively ample (after possibly restricting to a smaller disc), it is enough to show that for all $j = 1, \ldots, d$,

$$R^j\pi_*(((m + m_0)\mathcal{L} - jV) \otimes \mathcal{J}(\phi)) = 0$$

on the disc, which follows from Kodaira and Nadel vanishing. $\square$

We thus obtain the following:

Corollary 5.3.4.2. *For any metric $\phi \in \mathrm{PSH}(L)$, and any model $(\mathcal{X}, \mathcal{L})$ of (X, L) such that ϕ extends as a psh metric on $\mathcal{L}$, there exists an integer m_0 such that the function*

$$(m + m_0)^{-1} \log |m\mathcal{J}(\phi)|$$

is $\mathcal{L}$-psh for all positive integers m.

Proof. In the case where ϕ also has fibrewise finite energy, this follows from the previous Lemma. In the general case, one can approximate ϕ on $\mathcal{L}$ by a decreasing sequence of (e.g.) locally bounded metrics ϕ_k. Since the integer m_0 depends only on $\mathcal{L}$, the sequence

$$k \mapsto \tilde{\phi}_k := (m + m_0)^{-1} \log |m\mathcal{J}(\phi_k)|$$

is then a sequence of $\mathcal{L}$-psh functions. Since the sequence ϕ_k is decreasing, we have for all k that $\mathcal{J}(\phi_{k+1}) \subseteq \mathcal{J}(\phi_k)$, i.e. the sequence $\tilde{\phi}_k$ is also decreasing, which implies its limit $(m + m_0)^{-1} \log |m\mathcal{J}(\phi)|$ to be $\mathcal{L}$-psh, as desired. □

5.3.5 Proof of Theorem 5.3.3.1.

We may now prove Theorem 5.3.3.1.

Proof. We fix a metric $\phi \in \mathrm{PSH}(L)$. We need to show that the function defined on X^{div} by

$$\phi^{\mathrm{NA}} : \nu_E \mapsto \mathrm{ord}_E(\phi),$$

where ν_E corresponds to a divisorial valuation and ord_E is defined as a generic Lelong number as in (5.5), admits a psh extension on $X^{\mathrm{an}}_{\mathrm{K}}$. Since a non-Archimedean psh function is uniquely defined on the set of divisorial points, it is then though to show that ϕ^{NA} can be approximated by a decreasing sequence of psh model functions on $X^{\mathrm{an}}_{\mathrm{K}}$. Note that, by construction, the map $\phi \mapsto \phi^{\mathrm{NA}}$ is lsc and order preserving.

By Corollary 5.3.4.2, the metric

$$\psi_m = (m + m_0)^{-1} u_m,$$

where u_m is the model function $\log |\mathcal{J}(m\phi)|$, is $\mathcal{L}^{\mathrm{an}}_{\mathrm{R}}$-psh. Pick a divisorial point $\nu_E \in X^{\mathrm{div}}$ associated to a component in the central fibre of an analytic model $(\mathcal{X}, \mathcal{L})$ of (X, L). Using a version of the estimate [BBJ, Lemma B.4] (which is proven exactly as in the trivially valued case), one has

$$m \cdot \varphi^{\mathrm{NA}}(\nu_E) \leq u_m(\nu_E) \leq m \cdot \varphi^{\mathrm{NA}}(\nu_E) + A_{\mathcal{X}}(\nu_E),$$

where $A_{\mathcal{X}}$ is the log discrepancy function as before. The sequence ψ_m is therefore a sequence of $\mathcal{L}_{\mathrm{R}}^{\mathrm{an}}$-psh functions converging pointwise on X^{div} to φ^{NA}. To show that φ^{NA} is $\mathcal{L}_{\mathrm{R}}^{\mathrm{an}}$-psh, it is then enough to prove that we can have this sequence be decreasing. By subadditivity of multiplier ideals we have

$$\mathcal{J}(2m\phi) \subseteq \mathcal{J}(m\phi)^2,$$

thus

$$\psi_{2m} \leq 2\psi_m,$$

and as $\phi_m \leq 0$,

$$\psi_{2m} \leq \frac{2(m+m_0)}{2m+m_0}\psi_m \leq \psi_m.$$

Picking the subsequence $i \mapsto \psi_{2^i}$ therefore yields a decreasing subsequence converging to φ^{NA}, as desired. We then set

$$\phi^{\mathrm{NA}} := \varphi^{\mathrm{NA}} + \phi_{\mathcal{L}},$$

which concludes our proof. □

5.3.6 Locally bounded metrics in the non-Archimedean limit.

We now begin studying the behaviour under the map $(\cdot)^{\mathrm{NA}}$ of the class of metrics ϕ, such that there exists a model $(\mathcal{X}, \mathcal{L})$ of (X, L) on which ϕ admits a locally bounded extension.

Proposition 5.3.6.1. *Let $\phi \in \mathrm{PSH}(L)$. Then,*

1. *ϕ extends to a psh metric on a model $(\mathcal{Y}, \mathcal{M})$ of (X, L) if and only if $\phi^{\mathrm{NA}} \leq \phi_{\mathcal{M}}$;*

2. *ϕ extends to a locally bounded psh metric on $(\mathcal{Y}, \mathcal{M})$ if and only if $\phi^{\mathrm{NA}} = \phi_{\mathcal{M}}$.*

Proof. Note that it is equivalent to show the following: given $(\mathcal{X}, \mathcal{L})$ an analytic model of (X, L) and ψ be a reference metric admitting a locally bounded extension to $\mathcal{L}$, (1) holds if and only if $\phi^{\mathrm{NA}} - \psi^{\mathrm{NA}} \leq \phi_{\mathcal{M}} - \phi_{\mathcal{L}}$, and (2) if and only if we have equality. This will allow us to work at the level of functions and relatively to another model, which is easier.

Assume first ϕ to extend to a psh metric on $\mathcal{M}$. Let $\mathcal{Z}$ dominate both models via $\pi_{\mathcal{X}} : \mathcal{Z} \to \mathcal{X}$ and $\pi_{\mathcal{Y}} : \mathcal{Z} \to \mathcal{Y}$, and we have

$$\pi_{\mathcal{Y}}^* \mathcal{M} = \pi_{\mathcal{X}}^* \mathcal{L} + D$$

for a unique Cartier divisor D supported in the special fibre $\mathcal{Z}_0$. Since ϕ extends to a psh metric on $(\mathcal{Y}, \mathcal{M})$ if and only if it extends to a psh metric on any model dominating $(\mathcal{Y}, \mathcal{M})$, we may without loss of generality focus on $\mathcal{Z}$. Picking a local equation f_D for the divisor D obtained as above, ϕ extends to $\pi_{\mathcal{Y}}^* \mathcal{M}$ if

$$\phi - \psi \leq -\log |f_D| + C$$

near $\mathcal{Z}_0$. Taking generic Lelong numbers with respect to the underlying divisor of a divisorial point ν gives

$$\nu(\phi) - \nu(\psi) \geq -\nu(D),$$

i.e.

$$\phi^{\mathrm{NA}}(\nu) - \psi^{\mathrm{NA}}(\nu) \leq \phi_{\mathcal{M}}(x) - \phi_{\mathcal{L}}(x).$$

In the case where ϕ admits a locally bounded extension, then there is also a lower bound, which shows by the same argument that $\phi^{\mathrm{NA}} = \phi_{\mathcal{M}} - \phi_{\mathcal{L}}$. The converse is obtained by uniqueness of the Siu decomposition of ϕ on $\mathcal{X}$. $\square$

5.4 Finite-energy spaces and the Monge-Ampère extension property.

5.4.1 The Monge-Ampère energy in the non-Archimedean limit.

In the trivially-valued setting, we have already seen that a metric in $\mathcal{E}^1(L)$ coincides with a finite-energy psh geodesic ray $t \mapsto \phi_t$. Two natural "asymptotic" energies arise:

1. the radial limit $\lim_t \frac{E(\phi_t)}{t}$;
2. the non-Archimedean energy of the non-Archimedean metric ϕ^{NA} associated to ϕ.

In [BBJ], it is established that if ϕ extends to a locally bounded metric on a test configuration, then those two quantities coincide. This is not the case in general, however. In this Section, we generalize those results to our relatively maximal psh metrics on degenerations. It will be clearer to express this using the relative dimension zero case of the construction from the previous Section.

Remark 5.4.1.1 (Relative dimension zero and the non-Archimedean limit)**.** As mentioned in Example 5.2.4.3 and Corollary 5.2.4.4, given a model $(\mathcal{X}, \mathcal{L})$ of (X, L) and a metric $\phi \in \hat{\mathcal{E}}^1(L)$, one can identify the Monge-Ampère energy $\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ of ϕ with a function on the punctured disc, by picking a trivialization τ of $\langle \mathcal{L}^{d+1} \rangle$ and setting $u = -\log|\tau|_\phi$. The function u then has a finite generalized slope (or Lelong number) at zero, but this Lelong number depends on the choice of a trivialization. A nice way of capturing all possible such Lelong numbers is by looking directly at the metric $(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}}$ on $\langle L_{\mathrm{K}}^{d+1} \rangle$! The Lelong number of u specifically is then recovered as the difference of Deligne pairings $(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} - \langle \phi_{\mathcal{L}}^{d+1} \rangle$, where $\phi_{\mathcal{L}}$ is the model metric associated to $\mathcal{L}$ on $L_{\mathrm{K}}^{\mathrm{an}}$.

Theorem 5.4.1.2. *For all $\phi \in \mathcal{E}^1(L)$ admitting a locally bounded extension to some model $(\mathcal{X}, \mathcal{L})$, we have*

$$(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle (\phi^{\mathrm{NA}})^{d+1} \rangle,$$

as non-Archimedean metrics on the Deligne pairing $\langle L^{\mathrm{an}} \rangle$ over $\operatorname{Spec}\mathrm{K}$.

Proof. Note that the metric $\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*}$ is subharmonic by (5.2), so that the left-hand side is well-defined (this is the relative dimension zero case of Example 5.2.4.3).

We pick a model $(\mathcal{X}, \mathcal{L})$ such that ϕ extends to a locally bounded metric on $\mathcal{L}$. By Proposition 5.3.6.1, we necessarily have $\phi^{\mathrm{NA}} = \phi_{\mathcal{L}}$, the model metric on $L_{\mathrm{K}}^{\mathrm{an}}$ associated to $\mathcal{L}$, so that we are left to show that, given a trivialization τ of $\langle \mathcal{L}^{d+1} \rangle_{\mathcal{X}/\mathbb{D}}$ and setting $u(z) = -\log|\tau(z)|_{\langle \phi_z^{d+1} \rangle}$, we have

$$\hat{u} = 0$$

(recall how we defined the model metric $\phi_{\mathcal{L}}$ in Section 5.3.2). But ϕ is locally bounded near the central fibre of $\mathcal{L}$, so that u is locally bounded near zero, which implies $\hat{u} = 0$ as desired. $\square$

Remark 5.4.1.3. We will occasionally refer to a metric satisfying the statement of Theorem 5.4.1.2 as satisfying the Monge-Ampère extension property. We also remark that the proof of the Theorem works more generally for arbitrary Deligne pairings(!): given $d+1$ pairs of relatively ample line bundles L_i on X and metrics $\phi_i \in \mathcal{E}^1(L_i)$ admitting locally bounded extensions to some model of L_i, one has

$$(\langle \phi_0, \dots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle \phi_0^{\mathrm{NA}}, \dots, \phi_d^{\mathrm{NA}} \rangle.$$

The fact that the slopes are well-defined follows as in the proof of the above Theorem from the general property (5.2) of Deligne pairings! In Section 5.4.4, we will show how to extend this result to the class of metrics satisfying the Monge-Ampère extension property.

5.4.2 Hybrid maximal metrics: existence and uniqueness.

We now study hybrid maximal metrics. Such metrics can be described as being relatively maximal, but with boundary values prescribed both at the complex boundary of X and at the "asymptotic" or non-Archimedean boundary. We will then see that they correspond exactly to metrics satisfying the Monge-Ampère extension property.

Definition 5.4.2.1. Let $\phi \in \hat{\mathcal{E}}^1(L)$. We say that ϕ is hybrid maximal if for any $\psi \in \mathcal{E}^1(L)$ such that $\psi^{\mathrm{NA}} \leq \phi^{\mathrm{NA}}$ and $\limsup(\psi - \phi) \leq 0$ near the boundary of X, we have $\psi \leq \phi$.

Remark 5.4.2.2. We show how to relate our terminology with that of [BBJ], which deals with special cases of our objects:

- a geodesic ray in [BBJ] is a relatively maximal $\mathbb{C}^*$-invariant (logarithmic growth) psh metric on a line bundle over a test configuration in our article;

- a maximal geodesic ray in [BBJ] is a hybrid maximal $\mathbb{C}^*$-invariant (logarithmic growth) psh metric on a line bundle over a test configuration in our article.

The "hybrid" refers to (e.g.) the work of Boucksom-Jonsson, in which a hybrid property is a property that passes well from the complex setting to the non-Archimedean limit. Other possible denominations could be "Lelong-maximal" or "maximal in the non-Archimedean limit", but both of those seem to focus more on the limit behaviour while we require our metric to also be maximal in the complex world.

Theorem 5.4.2.3. *For any $\Phi^{\mathrm{NA}} \in \mathcal{E}^1(L^{\mathrm{an}})$, there exists a unique metric $\phi \in \hat{\mathcal{E}}^1(L)$ such that $\phi^{\mathrm{NA}} = \Phi^{\mathrm{NA}}$.*

Proof. The proof is in two parts. We begin with the assumption that the non-Archimedean metric is a model metric, and construct the unique solution via adapted envelope techniques (inspired by [Berm16, Proposition 2.7]). Then, for the general case, we use properties of the Monge-Ampère energy.

First step: the model case. Assume thus Φ^{NA} to be a model metric corresponding to a model $(\mathcal{X}, \mathcal{L})$ of (X, L). Denote by ϕ the "Perron-Bremmermann-Lelong" envelope defined as the supremum of all metrics $\psi \in \mathcal{E}^1(L)$ with

$$\lim_{z \to \xi} \psi(z) \leq \phi_z$$

for all $\xi \in \partial X$, and

$$\psi^{\mathrm{NA}} \leq \phi^{\mathrm{NA}}.$$

We begin with a claim that ϕ so defined belongs to $\mathcal{E}^1(L)$. Note that if we can show that it is plurisubharmonic, then it necessarily has logarithmic growth, as the supremum of a family of metrics with logarithmic growth, and it is by definition relatively maximal. Furthermore, the fibrewise finite-energy condition will also immediately follow, so that we need to focus on the plurisubharmonicity. Let ψ be in the class of contributions to the supremum above. The hypothesis that $\psi^{\mathrm{NA}} \leq \Phi^{\mathrm{NA}}$ implies via Proposition 5.3.6.1 that ψ extends with at worst analytic singularities as a psh metric on $\mathcal{L}$. We therefore see ϕ to be the restriction of a metric $\phi_{\mathcal{X}}$ defined as the supremum of all metrics on $\mathcal{L}$, with the same boundary conditions as above on $\partial \mathcal{X}$, and extending with at worst analytic singularities over the central fibre of $\mathcal{X}$. That the envelope satisfies our claim is then a particular case of Theorem 5.1.4.4 (which allows singular fibres!).

Finally, the second case of Proposition 5.3.6.1 together with the non-Archimedean maximality assumption ensure that it is hybrid maximal, provided we can show that for any model metric Φ^{NA} there exists a metric $\psi^{\mathrm{NA}} \in \mathcal{E}^1(L)$ with $\psi^{\mathrm{NA}} = \Phi^{\mathrm{NA}}$. But this also follows from the same Lemma, since one only has to choose ψ to be a psh metric with a locally bounded extension to $\mathcal{L}$. That ϕ is the unique hybrid metric given our data follows again from the extremal characterization.

Second step: the general case. The general case again proceeds by approximation: we pick a sequence of model metrics ϕ_i^{NA} decreasing to Φ^{NA}, and their associated hybrid maximal metrics ϕ_i in $\hat{\mathcal{E}}^1(L)$, which exist and are unique by the first part of the proof. The ϕ_i then give a decreasing sequence of metrics by maximality. We write ϕ for their limit and ϕ^{NA} the non-Archimedean metric it defines. Since the mappings $\phi \mapsto \phi^{\mathrm{NA}}$ are order-preserving, we find

$$\phi^{\mathrm{NA}} \leq \phi_k^{\mathrm{NA}}$$

for all k, i.e.

$$\phi^{\mathrm{NA}} \leq \Phi^{\mathrm{NA}}. \tag{5.6}$$

Fix a model $(\mathcal{X}, \mathcal{L})$ of (X, L), so that ϕ and the ϕ_k extend to $\mathcal{L}$ (with singularities). Fix a trivialization τ of $\langle \mathcal{L}^{d+1} \rangle_{\mathcal{X}/\mathbb{D}}$, and set

$$E(\phi_z) := -\log |\tau(z)|_{\langle \phi_z^{d+1} \rangle},$$

$$E(\phi_{k,z}) := -\log |\tau(z)|_{\langle \phi_{k,z}^{d+1} \rangle}.$$

We will also denote as usual by $E^{\mathrm{NA}}(\phi^{\mathrm{NA}})$ the metric $\langle (\phi^{\mathrm{NA}})^{d+1} \rangle$. Now, by Corollary 5.2.4.4, and Theorem 5.4.1.2, we have for all k

$$E(\phi_{k,z}) = c_k \cdot \log |z| + H(z), \tag{5.7}$$

where $H = H(\phi_\partial)$ is some function bounded near zero and independent of k. In fact, one can see that

$$c_k = -(\langle (\phi_k^{\mathrm{NA}})^{d+1} \rangle - \langle \phi_{\mathcal{L}}^{d+1} \rangle) = -(E^{\mathrm{NA}}(\phi_k^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}})).$$

Since the (Archimedean and non-Archimedean) Monge-Ampère energies are continuous along decreasing nets, we have

$$E^{\mathrm{NA}}(\phi_k^{\mathrm{NA}}) \to E^{\mathrm{NA}}(\phi^{\mathrm{NA}})$$

while

$$E(\phi_{k,z}) \to E(\tilde{\phi}_z)$$

for all z. Combining those with (5.7), one finds

$$E(\phi_z) = -(E^{\mathrm{NA}}(\phi_k^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}})) \cdot \log |z| + H(z), \tag{5.8}$$

which by Corollary 5.2.4.4 shows that ϕ is a relatively maximal metric. Furthermore, we know that $\mathrm{PSH}(L)$ is closed under decreasing limits: ϕ thus has logarithmic growth. To establish existence, i.e. to show that ϕ is our desired solution, we now only have to show that $\phi^{\mathrm{NA}} = \Phi^{\mathrm{NA}}$. Using [Reb20b, Proposition 6.3.2], this is proven provided we can show that

$$E^{\mathrm{NA}}(\phi^{\mathrm{NA}}) = E^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) \tag{5.9}$$

by (5.6). One inequality is immediate from the same equation (5.6) and monotonicity of E^{NA}:

$$E^{\mathrm{NA}}(\phi^{\mathrm{NA}}) \leq E^{\mathrm{NA}}(\Phi^{\mathrm{NA}}).$$

From (5.8) we have

$$\hat{E}(\phi) = E^{\mathrm{NA}}(\Phi^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}}), \tag{5.10}$$

so that we have the other inequality (hence (5.9)), provided we can show that

$$\hat{E}(\phi) \leq E^{\mathrm{NA}}(\phi^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}}). \tag{5.11}$$

This inequality follows from a similar argument. Let ψ_k^{NA} be a decreasing sequence of model metrics approximating ϕ^{NA}. Let ψ_k denote their associated hybrid maximal metric, and define $E(\psi_k)$ as before. Now, since for all k $\phi^{\mathrm{NA}} \leq \psi_k^{\mathrm{NA}}$, by maximality, we have

$$\phi \leq \psi_k \tag{5.12}$$

whence

$$E(\phi_z) \leq E(\psi_{k,z}). \tag{5.13}$$

Taking negative Lelong numbers,

$$\hat{E}(\phi) \leq \hat{E}(\psi_k). \tag{5.14}$$

By Theorem 5.4.1.2 and the arguments above, $\hat{E}(\psi_k) = E^{\mathrm{NA}}(\psi_k^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}})$ which again upon taking the decreasing limit in the right-hand side (along which E^{NA} is continuous) establishes

$$\hat{E}(\phi) \leq \lim_k E^{\mathrm{NA}}(\psi_k^{\mathrm{NA}})) - E^{\mathrm{NA}}(\phi_{\mathcal{L}}) = E^{\mathrm{NA}}(\phi^{\mathrm{NA}}) - E^{\mathrm{NA}}(\phi_{\mathcal{L}}) \tag{5.15}$$

by definition of the ψ_k^{NA}. This establishes (5.11) as desired, hence existence of a hybrid maximal metric with non-Archimedean metric equal to Φ^{NA}. That such a segment is unique is then a consequence of the extremal definition of hybrid maximality. $\square$

Corollary 5.4.2.4. *The space $\mathcal{E}^1(L)$ is mapped by $(\cdot)^{\mathrm{NA}}$ to $\mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}})$; furthermore, for any $\phi \in \mathcal{E}^1(L)$, we have*

$$(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} \leq \langle (\phi^{\mathrm{NA}})^{d+1} \rangle.$$

In other words, we do not have non-Archimedean extension of the Monge-Ampère energy in $\mathcal{E}^1(L)$, but simply an inequality.

Proof. We start by picking a metric $\phi \in \mathcal{E}^1(L)$, and we define Φ to be the hybrid maximal metric with $\Phi^{\mathrm{NA}} = \phi^{\mathrm{NA}}$ obtained from Theorem 5.4.2.3. Then, since Φ is relatively maximal, $\phi \leq \Phi$, so that by monotonicity of $\phi \mapsto \phi^{\mathrm{NA}}$,

$$(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} \leq (\langle \Phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}}$$

while $(\langle \Phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle (\Phi^{\mathrm{NA}})^{d+1} \rangle$ by hybrid maximality, proving our inequality. To prove the first statement, it is enough to notice that the logarithmic growth condition built into $\mathcal{E}^1(L)$ forces $(\langle \phi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}}$ to be finite. $\square$

5.4.3 The isometric embedding.

We denote by

$$\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$$

the subspace of hybrid maximal metrics in $\hat{\mathcal{E}}^1(L)$. Our main Theorem is the following:

Theorem 5.4.3.1. *The inverse of the mapping $(\cdot) \mapsto (\cdot)^{\mathrm{NA}}$ given by Theorem 5.4.2.3 is an isometric embedding of $(\mathcal{E}^1(L^{\mathrm{an}}), d_1^{\mathrm{NA}})$ into $(\hat{\mathcal{E}}^1(L), \hat{d}_1)$ with image $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$. Furthermore, a psh segment in $\mathcal{E}^1(L^{\mathrm{an}})$ is a psh geodesic if and only if its image is a psh geodesic.*

Remark 5.4.3.2. The first statement of the Theorem can be thought of as saying that hybrid maximal metrics have the d_1-extension property. The whole of Theorem 5.4.3.1 essentially means that we realize the (non-Archimedean) space $\mathcal{E}^1(L^{\mathrm{an}})$ as a purely complex geometric object!

Proof. We thus first show that our mapping preserves psh geodesic segments. Pick a psh geodesic segment ϕ_t^{NA} between two metrics ϕ_0^{NA} and ϕ_1^{NA} in $\mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}})$, and consider for all $t \in [0,1]$ the hybrid maximal metric ϕ_t with $(\phi_t)^{\mathrm{NA}} = \phi_t^{\mathrm{NA}}$. By Theorem 5.2.7.1, it is enough to show that $t \mapsto (\langle \phi_t^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}}$ is affine; by the Monge-Ampère energy extension property, this is equivalent to asking that $t \mapsto \langle \phi_t^{\mathrm{NA}} \rangle$ is affine, which holds by [Reb20b]. The reverse implication is proved in the same way.

We now prove the isometry statement. Pick ϕ_0, ϕ_1 in $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$. We assume both metrics to be continuous, and the general result will proceed as usual from regularization. Using Theorem 5.4.2.3 together with the expressions of the distances and additivity of Lelong numbers,

$$d_1^{\mathrm{NA}}(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}}) = \langle (\phi_0^{\mathrm{NA}})^{d+1} \rangle + \langle (\phi_1^{\mathrm{NA}})^{d+1} \rangle - 2\langle (P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}})^{d+1} \rangle,$$

$$d_1(\phi_{0,z}, \phi_{1,z}) = \langle \phi_{0,z}^{d+1} \rangle + \langle \phi_{1,z}^{d+1} \rangle - 2\langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle,$$

we only have to show that

$$(-\langle P(\phi_0, \phi_1)^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = -\langle P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}})^{d+1} \rangle.$$

Recall that we have seen $z \mapsto \langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle$ to be superharmonic, so that the left-hand side is well-defined.

Set some $r \in (0,1)$. We consider the relatively maximal metric ψ_r on the preimage U_r of the annulus $\{r \leq z \leq 1\}$ with boundary data given by $z \mapsto P(\phi_{0,z}, \phi_{1,z})$ for $z \in \partial U_r$, which exists by Theorem 5.1.4.4. Having fixed $z \in X$, the sequence $r \mapsto \psi_{r,z}$, $r \leq |\pi(z)|$, is decreasing as r decreases, and therefore the limit $\lim_{r \to 0} \psi_r =: \psi$ is still a relatively maximal metric. As we have, for all $|\pi(z)| = r$,

$$\langle \psi_{r,z}^{d+1} \rangle = \langle P(\phi_{0,z}, \phi_{1,z})^{d+1} \rangle,$$

it follows that

$$-(\langle \psi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = (-\langle P(\phi_0, \phi_1)^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}}.$$

We must now prove that

$$(\langle \psi^{d+1} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}})^{d+1} \rangle.$$

We first claim that ψ realizes the supremum

$$\psi = \sup\{\varphi \in \mathrm{PSH}(L),\ \varphi \leq \phi_0, \phi_1\}.$$

Since ψ is itself such a metric, it is enough to show that for all candidates φ, we have $\varphi \leq \psi$. But for all $z \in X$, since $\varphi_z \leq \phi_{0,z}, \phi_{1,z}$, we have

$$\varphi_z \leq P(\phi_{0,z}, \phi_{1,z})$$

hence

$$\varphi_z \leq \psi_{r,z}$$

and finally

$$\varphi_z \leq \lim_r \psi_{r,z} = \psi_z.$$

We now conclude: by this extremal characterization of ψ, we have that $\varphi^{\mathrm{NA}} \leq \psi^{\mathrm{NA}}$ for all $\varphi \leq \phi_0, \phi_1$. In particular, since the construction is order-preserving, the hybrid maximal metric Ψ with $\Psi^{\mathrm{NA}} = P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}})$ satisfies $\Psi \leq \psi$, so that

$$P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}}) \leq \psi^{\mathrm{NA}},$$

while on the other hand, $\psi \leq \phi_0, \phi_1$, hence $\psi^{\mathrm{NA}} \leq \phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}}$ and finally

$$\psi^{\mathrm{NA}} \leq P(\phi_0^{\mathrm{NA}}, \phi_1^{\mathrm{NA}}).$$

$\square$

Remark 5.4.3.3. The proof of the above result in the case of geodesic rays, which does not appear explicitly in the literature (but is based on some ideas from [BDL]), was nicely explained to the author by Tamas Darvas.

Remark 5.4.3.4. In the above proof, we implicitly defined an envelope operator sending two metrics ϕ_0, ϕ_1 in $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ to the largest metric $\hat{P}(\phi_0, \phi_1)$ in $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ bounded above by ϕ_0 and ϕ_1. In [Xia, Example 3.3], this construction appears already in the case of geodesic rays, and Xia uses this envelope to define alternative distance

$$\hat{d}_1'(\phi_0, \phi_1) := \lim_t t^{-1}(E(\phi_{0,t}) + E(\phi_{1,t}) - 2E(\hat{P}(\phi_0, \phi_1)_t),$$

which (a specialization of) our proof shows to coincide with the usual distance $\hat{d}_1$. In fact, Xia defines this envelope more generally in [Xia, Example 3.2], in the radial equivalent of the space $\hat{\mathcal{E}}^1(L)$. It is likely that this construction generalizes to metrics in $\hat{\mathcal{E}}^1(L)$ in our setting, although this is outside the scope of the present article.

5.4.4 Non-Archimedean extension of generalized functionals.

In Theorem 5.1.2.2, we have seen that the fibrewise finite energy condition is the adequate condition for finiteness of fibrewise Deligne pairings. Further following the mantra that properties pertaining to the energy govern the same properties for more general Deligne pairings, we show that non-Archimedean extension of generalized energy functionals in the sense of Remark 5.4.1.3 holds for our class of hybrid maximal metrics, i.e. metrics satisfying the Monge-Ampère extension property.

Proposition 5.4.4.1. *Suppose given $d+1$ relatively ample line bundles L_i on X. Then, for any $(d+1)$-uple of metrics $\phi_i \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L_i)$, we have*

$$(\langle \phi_0, \dots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} = \langle \phi_0^{\mathrm{NA}}, \dots, \phi_d^{\mathrm{NA}} \rangle.$$

Proof. We approximate each of the ϕ_i^{NA} by a decreasing sequence of model metrics $\phi_{i,k}^{\mathrm{NA}}$, and denote by $\phi_{i,k}$ their associated hybrid maximal metrics. By our previous results, $\phi_{i,k}$ decreases to ϕ_i by hybrid maximality. Since Deligne pairings are decreasing along (mixed) decreasing limits, using the estimates [BBJ, Lemma A.1, Lemma A.2], we find for all z in X

$$0 \leq \langle \phi_{0,k,z}, \dots, \phi_{d,k,z} \rangle - \langle \phi_{0,z}, \dots, \phi_{d,z} \rangle \leq C(z) \cdot \max_i d_1(\phi_{i,k,z}, \phi_{i,z}),$$

where the slope of $C(z)$, $\hat{C}$, is a finite real constant. Indeed, $C(z)$ is a maximum of a collection of functionals expressed as Deligne pairings, which are all subharmonic along relatively maximal metrics. We take generalized slopes in the above inequality to find

$$0 \leq (\langle \phi_{0,k}, \dots, \phi_{d,k} \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} - (\langle \phi_0, \dots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} \leq \hat{C} \cdot \max_i d_1^{\mathrm{NA}}(\phi_{i,k}^{\mathrm{NA}}, \phi_i^{\mathrm{NA}}),$$

where we have used the d_1-extension property of hybrid maximal metrics. Using Remark 5.4.1.3, we then have that

$$0 \leq \langle \phi_{0,k}^{\mathrm{NA}}, \dots, \phi_{d,k}^{\mathrm{NA}} \rangle - (\langle \phi_0, \dots, \phi_d \rangle_{X/\overline{\mathbb{D}}^*})^{\mathrm{NA}} \leq \hat{C} \cdot \max_i d_1^{\mathrm{NA}}(\phi_{i,k}^{\mathrm{NA}}, \phi_i^{\mathrm{NA}}),$$

and taking the limit in k in the above inequality finally gives our result. $\square$

Example 5.4.4.2. Many functionals acting on $\mathrm{PSH}(L)$ satisfy the statement of the above Proposition. Having fixed some reference metric $\phi_{\mathrm{ref}} \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$, some among the most important are:

1. the I-functional, which appeared in the estimates mentioned in the above proof, is defined as

$$I(\phi) = \langle \phi - \phi_{\mathrm{ref}}, \phi_{\mathrm{ref}}^d \rangle_{X/\overline{\mathbb{D}}^*} - \langle \phi - \phi_{\mathrm{ref}}, \phi^d \rangle_{X/\overline{\mathbb{D}}^*},$$

which has many important norm-like properties and is commonly used to study properties of finite-energy spaces ([BBEGZ], [BHJ16], see also [BJ21] for the non-Archimedean trivially-valued case);

2. the J-functional, defined as

$$J(\phi) = \langle \phi, \phi_{\mathrm{ref}}^d \rangle_{X/\overline{\mathbb{D}}^*} - \langle \phi_{\mathrm{ref}}^{d+1} \rangle_{X/\overline{\mathbb{D}}^*} - (d+1)^{-1}(E(\phi) - E(\phi_{\mathrm{ref}})),$$

which can be seen as a corrected relative Monge-Ampère energy which is translation invariant;

3. the twisted energy functionals, defined as

$$E^{\psi}(\phi) = \langle \psi, \phi^d \rangle_{X/\overline{\mathbb{D}}^*},$$

for $\psi \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L')$, where L' is another line bundle on X. A special case of it appears in the expression of the Mabuchi K-energy, and the study of its slopes in the trivially-valued case is essential to establish the general (cscK) case of the Yau-Tian-Donaldson conjecture, as in [Li].

5.4.5 Test configurations and the trivially valued case.

All of our previous results encapsulate the trivially valued case, as we explain now. Let $\pi : X \to \mathbb{D}^*$ be now a polarized test configuration, i.e. a degeneration with relatively ample line bundle L such that π and L are equivariant under some $\mathbb{C}^*$-action (forcing all fibre pairs (X_z, L_z) to be isomorphic). One may then choose a reference continuous psh metric ϕ_{ref} on the fibre at 1 and require our psh metrics ϕ to satisfy $\phi_z = {i_z}^* \phi_{\mathrm{ref}}$ for $z \in \mathbb{S}^1$, and with

$$i_z : X_z \to X_1$$

the isomorphism as mentioned above. The authors in [BBJ] (e.g.) study the space $\mathcal{E}^1_0(X_1^{\mathrm{an}})$ of finite-energy metrics over the analytification of X_1 with respect to the trivial absolute value on $\mathbb{C}$. We denote by $\mathcal{R}^1(L_1)$ the space of hybrid maximal finite-energy rays in $\mathrm{PSH}(L_1)$ emanating from ϕ_{ref} (where, as mentioned before, a hybrid maximal ray corresponds in the terminology of [BBJ] to a maximal psh geodesic ray). We then claim the following:

Proposition 5.4.5.1. *There is a sequence of distance-preserving maps*

$$\mathcal{E}^1_0(L_1^{\mathrm{an}}) \simeq \mathcal{R}^1(L_1) \hookrightarrow \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L) \simeq \mathcal{E}^1(L^{\mathrm{an}}),$$

where the first and last maps are bijective (i.e. isometries), while the middle map is injective.

Proof. The case of the last map has been treated by Theorem 5.4.3.1. The rest of the proof is merely a matter of correctly defining our maps.

For the first map, the bijection is given by [BBJ, Theorem 6.6]. The metrization of the space $\mathcal{E}^1_0(L_1^{\mathrm{an}})$ is described in a [BJ21], but proceeds much as the metrization of $\mathcal{E}^1(L^{\mathrm{an}})$ in [Reb20b], while we recall that we metrize the space of maximal rays by

$$\hat{d}_{1,0}(\phi, \phi') = \lim_t t^{-1} d_1(\phi_t, \phi'_t)$$

and take equivalence classes to yield the space $\mathcal{R}^1(L_1)$. (We direct the reader to e.g. [BDL]. Note that in the cited article, the authors consider the space of all (non-necessarily hybrid) maximal psh rays.) Proving the distance-preservingness of the isomorphism is then essentially a simpler version of Theorem 5.4.3.1, which we leave to the interested reader.

We claim that the middle map, which we will denote ι_0, can be represented as follows: let $\phi : t \mapsto \phi_t$ be a hybrid maximal psh geodesic ray in X_1. Let i_z be as before the isomorphism $i_z : X_z \to X_1$, and define $\iota_0(\phi)$ to be the metric

$$z \mapsto {i_z}^*(\phi_{-\log|z|}).$$

The distance-preservingness is immediate(!), so that we are left to check that $\iota_0(\phi)$ is a hybrid maximal metric. By [BBJ, Corollary 6.7],

$$t \mapsto E(\phi_t)$$

is affine, which implies by invariance of the energy under polarized isomorphisms that

$$z \mapsto \iota_0(\phi)(z)$$

is harmonic on $\mathbb{D}^*$, proving maximality by Proposition 5.1.4.7, and hybrid maximality is given by construction. $\square$

Remark 5.4.5.2. One also notices (by mimicking our proofs in the discretely valued case) there to be under the above maps a correspondance

$$\{\text{non-Archimedean maximal psh segments in } \mathcal{E}^1_0(L_1^{\mathrm{an}})\}$$

$$\wr|$$

$$\{\text{rays of complex geodesics between two rays in } \mathcal{R}^1(L_1)\}$$

$$\downarrow$$

$$\{\text{discs of complex geodesics between two metrics in } \hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)\}$$

$$\wr|$$

$$\{\text{non-Archimedean maximal psh segments in } \mathcal{E}^1(L^{\mathrm{an}})\}.$$

We may state the most interesting part of this result as follows.

Proposition 5.4.5.3. *Let $t \mapsto \phi_{0,t}$, $t \mapsto \phi_{1,t}$ be maximal psh geodesic rays in the sense of [BBJ]. Let, for all t,*

$$[0,1] \ni s \mapsto \phi_{s,t}$$

be the maximal psh segment joining $\phi_{0,t}$ and $\phi_{1,t}$. Then, for all $s \in [0,1]$, $t \mapsto \phi_{s,t}$ is a maximal psh geodesic ray in the sense of [BBJ].

Furthermore, let for all $s \in [0,1]$, ϕ_s^{NA} be the non-Archimedean metric associated to the psh geodesic ray $s \mapsto \phi_{s,t}$. Then,

$$s \mapsto \phi_s^{\mathrm{NA}}$$

is the maximal non-Archimedean psh geodesic joining ϕ_0^{NA} and ϕ_1^{NA} in the sense of [Reb20b].

5.4.6 Convexity of non-Archimedean functionals.

Via the isometry ι given by Theorem 5.4.3.1, it is now clear what we mean by "a functional on the space of hybrid maximal metrics", since $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ inherits a K-vector space structure by setting, for all ϕ, $\psi \in \hat{\mathcal{E}}^1(L)$, and $\lambda \in \mathrm{K}$,

$$\phi + \lambda \cdot \psi = \iota^{-1}(\iota(\phi) + \lambda \cdot \iota(\psi)).$$

(In particular, one can see multiplication by a scalar in K as a fibrewise scaling of the metrics, varying meromorphically.)

Definition 5.4.6.1. Let $\hat{F}$ be a functional on $\hat{\mathcal{E}}^1_{\mathrm{hyb}}(L)$ and F^{NA} be a functional on $\mathcal{E}^1(L^{\mathrm{an}})$. We say that F^{NA} is a non-Archimedean extension of $\hat{F}$ if the diagram

$$\begin{tikzcd} \hat{\mathcal{E}}^1(L) \arrow[rr, "\iota"] \arrow[dr, "\hat{F}"'] & & \mathcal{E}^1(L^{\mathrm{an}}) \arrow[dl, "F^{\mathrm{NA}}"] \\ & \mathbb{R} & \end{tikzcd}$$

commutes.

Example 5.4.6.2. By construction, E^{NA} is a non-Archimedean extension of the "discal" energy $\hat{E}$. Furthermore, all "generalized energy" functionals of Proposition 5.4.4.1 and Example 5.4.4.2 admit non-Archimedean extensions.

By Theorem 5.4.3.1, we get "for free" a way to study convexity of non-Archimedean functionals.

Heuristic 5.4.6.3. *Let F be a functional that is convex along complex psh geodesics, $\hat{F}$ its "discal" version, and F^{NA} a non-Archimedean extension of $\hat{F}$. Then F^{NA} is convex along maximal non-Archimedean psh geodesics in $\mathcal{E}^1(L^{\mathrm{an}})$.*

Proof. Given a non-Archimedean maximal psh segment ϕ_t^{NA} joining ϕ_0^{NA} and $\phi_1^{\mathrm{NA}} \in \mathcal{E}^1(L^{\mathrm{an}})$, we can write using Theorem 5.4.3.1 $\phi_t := \iota^{-1}(\phi_t^{\mathrm{NA}})$ using the maximal psh segments $t \mapsto \phi_{t,z}$ joining the $\phi_{0,z}$ and $\phi_{1,z}$. We then simply write for all z

$$F(\phi_{t,z}) \leq (1+t)F(\phi_{0,z}) + tF(\phi_{1,z}),$$

and take the limit to find

$$\hat{F}(\phi_t) \leq (1+t)\hat{F}(\phi_0) + t\hat{F}(\phi_1),$$

which by the definition of a non-Archimedean extension together with $\iota(\iota^{-1}(\phi^{\mathrm{NA}})) = \phi^{\mathrm{NA}})$ gives

$$F^{\mathrm{NA}}(\phi_t^{\mathrm{NA}}) \leq (1+t)F^{\mathrm{NA}}(\phi_0^{\mathrm{NA}}) + tF^{\mathrm{NA}}(\phi_1^{\mathrm{NA}})$$

as desired. □

Remark 5.4.6.4. Using Proposition 5.4.5.3, the same results also hold *mutatis mutandis* in the trivially-valued case.

Example 5.4.6.5. This allows us to obtain convexity of the (trivially-valued) non-Archimedean K-energy modulo the entropy approximation conjecture, as follows. Pick a compact Kähler manifold X_1 together with an ample line bundle L_1, and consider the trivially-valued analytification $(X_1^{\mathrm{an}}, L_1^{\mathrm{an}})$ as before. One then introduces the non-Archimedean entropy as

$$H^{\mathrm{NA}}(\phi^{\mathrm{NA}}) = \int_X A_X \,\mathrm{MA}(\phi^{\mathrm{NA}}),$$

where A_X is the log-discrepancy function on X^{an} identified with a space of semivaluations ([BJ18a]), and $\phi \in \mathcal{E}^1_0(L_1^{\mathrm{an}})$. The entropy approximation conjecture states that, given $\phi^{\mathrm{NA}} \in \mathcal{E}^1_0(L_1^{\mathrm{an}})$, there exists a sequence ϕ_k^{NA} of model metrics converging to ϕ^{NA} such that $H^{\mathrm{NA}}(\phi_k^{\mathrm{NA}}) \to H^{\mathrm{NA}}(\phi)$.

Now, Chi Li ([Li, Conjecture 1.6]) shows that, assuming this conjecture, the entropy H^{NA} is exactly the non-Archimedean (radial) extension of the usual complex entropy functional. Adding the energy part that makes up the Mabuchi K-energy, which is convex along complex geodesics, and using Example 5.4.4.2, our previous result shows that the non-Archimedean K-energy is convex along non-Archimedean geodesics if the conjecture holds.

As things currently stand, extension of the K-energy is only known for rays that generate model metrics, via [BHJ16, Theorem 3.6]. However, the non-Archimedean geodesics of [Reb20b] do not remain in the space of model metrics even if the endpoints are, much as geodesics between Kähler potentials are merely $C^{1,\bar{1}}$.

Example 5.4.6.6. In [BLXZ], Blum-Liu-Xu-Zhuang prove, using algebraic techniques, convexity of the non-Archimedean Ding energy (and other functionals) along geodesics between test configurations ([BLXZ, Theorem 3.7]). Our heuristic allows us to also recover this result: convexity of the complex Ding energy is a result of Berndtsson ([Berndt15]) while the non-Archimedean extension of the Ding energy follows from [BHJ16].

Remark 5.4.6.7. As a Corollary, existence and uniqueness of minimizers for non-Archimedean energy functionals can also be detected strict convexity results in the complex setting, but some form of uniform strict convexity (in $|z|$) is required to ensure strict convexity of the limit.

5.4.7 Kähler-Einstein metrics in families.

Let M be a complex projective manifold with ample canonical bundle K_M. It is a consequence of the by now classical Aubin-Yau Theorem that M carries a Kähler-Einstein metric. If X is more generally a family of canonically polarized projective manifolds, the family $z \mapsto \phi_z$ of fibrewise Kähler-Einstein metrics is known to have plurisubharmonic variation (from the work of e.g. Schumacher [Sch12]) and to have logarithmic growth ([Sch08, Theorem 3]). In particular, ϕ defines a metric in our class $\mathcal{E}^1(K_X)$. Interpreting the work of Pille-Schneider through our lens, one can see [Pil-S, Theorem A] to essentially imply that ϕ^{NA} corresponds to a model metric in $\mathcal{H}(K_X^{\mathrm{an}})$ associated to a distinguished model of (X, K_X) (see e.g. [Tia93], [Song]).

An immediate question arises: how does the metric ϕ relate to our relatively maximal metrics framework? In particular, how does ϕ relate to the hybrid maximal metric Φ corresponding to ϕ^{NA}? Interestingly, ϕ is not even relatively maximal when the Kodaira-Spencer class of the family X is nontrivial, by [Sch12, Main Theorem], since ϕ will be strictly positive (in particular, cannot satisfy $\mathrm{MA}(\phi) = 0$). As a consequence, we have that $dd^c d_1(\phi, \Phi) = dd^c(E(\Phi) - E(\phi))$ is given explicitly by the formula of Schumacher, using the pushforward formula for Deligne pairings.

Naturally, it would be interesting to know whether one could detect via non-Archimedean tools the existence of a family of Kähler-Einstein metrics in the class of a hybrid maximal metric. This seems a bit ambitious, since one only captures the "asymptotic" behaviour of a family of metrics when considering non-Archimedean data. A more realistic (and perhaps just as interesting) problem would be solving the following hybrid "almost Kähler-Einstein" problem: to find $\phi \in \hat{\mathcal{E}}^1_{\mathrm{hyb}}(K_X)$, such that $\phi^{\mathrm{NA}} = \psi^{\mathrm{NA}}$ where ψ^{NA} is an "almost Kähler-Einstein metric":

$$(\omega_z + i\partial\bar{\partial}\psi_z)^d - e^{h_{\omega_z} + \psi_z} {\omega_z}^d \to_{z \to 0} 0.$$

The upshot is that this problem gives, intuitively, a purely non-Archimedean criterion for the existence of a family of complex manifolds degenerating to a Kähler-Einstein manifolds! (Of course, the same problem arises in the (possibly twisted) Fano case.)

Finally, we briefly mention an additional difficulty in the Calabi-Yau case. By a counterexample of Cao-Guenancia-Paun, we know that a family $\phi = (\phi_z)_z$

of Kähler-Einstein metrics on a degeneration of Calabi-Yau manifolds does not necessarily vary plurisubharmonically ([CGP19, Theorem 3.1]). One can however take the plurisubharmonic envelope $P(\phi)$ of ϕ, and then the hybrid maximal metric Φ with $\Phi^{\mathrm{NA}} = P(\phi)^{\mathrm{NA}}$. In [BJ17], Boucksom-Jonsson show that the family of measures $\mathrm{MA}(\phi_z)$ converge in a certain sense to the non-Archimedean Monge-Ampère measure of some metric ψ^{NA}. We therefore formulate the following result, which would connect our hybrid maximal setting with degenerations of Kähler-Einstein metrics on Calabi-Yau manifolds:

Conjecture 5.4.7.1. $\psi^{\mathrm{NA}} = P(\phi)^{\mathrm{NA}} = \Phi^{\mathrm{NA}}$.

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