Spatial dynamic of interfaces in ecology : deterministic and stochastic models

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Dynamique d'interfaces en écologie : modèles déterministes et stochastiques

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Abstract

Traveling fronts arising from reaction diffusion equations model various phenomena observed in physics and biology. From a biological standpoint, a traveling front can be seen as the invasion of an uninhabited environment by a species. Since biological systems are finite and thus undergo demographic fluctuations, these deterministic wavefronts only represent an approximation of the population dynamics, in which we presuppose that the local density of individuals is infinite so that the fluctuations self-average. In this sense, reaction diffusion equations can be seen as hydrodynamic limits of some individual based models. In this thesis, we investigate the long time behaviour of some finite microscopic systems modeling such front propagations and compare them to the one of their large population asymptotics.

The first part of this thesis is dedicated to the study of the dynamics of a population colonising a slowly varying environment. This question has been widely studied from the PDE point of view. However, the results given by the viscosity solutions theory turn out to be biologically unsatisfactory in some situations. We thus suggest to study an individual based model for front propagation in the limit, when the scale of heterogeneity of the environment tends to infinity. In this framework, we show that the spreading speed of the population may be much smaller than the speed of the front in the PDE describing the large population asymptotics of the system. This qualitative disagreement between the two behaviours is related to the so-called tail problem observed in PDE theory, due to the absence of local extinction in FKPP-type equations.

In a second part, we study the impact of the type of the deterministic limit waves on the related stochastic models to explain this drastic slow-down in the particle system. Indeed, wavefronts arising from monostable reaction diffusion PDEs are classified into two types: pulled and pushed waves. It is well-known that small perturbations have a huge impact on pulled waves. In sharp contrast, pushed waves are expected to be less sensitive. Nevertheless, some recent numerical experiments have suggested the existence of a third class of waves in stochastic fronts. It is a subclass of pushed fronts very sensitive to fluctuations. In this thesis, we investigate the internal mechanisms of such fronts to explain the transition between these three regimes.

Résumé en français

Les ondes progressives générées par les équations de réaction-diffusion peuvent modéliser divers phénomènes observés en physique et en biologie. Du point de vue biologique, une onde progressive peut être interprétée comme l'invasion d'un habitat par une espèce. Cependant, les systèmes biologiques étant finis, ils sont soumis à des fluctuations démographiques qui n'apparaissent pas dans ces équations de réaction-diffusion. Ainsi, les ondes progressives qui en résultent ne représentent qu'une approximation de la véritable dynamique de la population. Par ailleurs, la dynamique d'une population peut aussi être décrite par un modèle prenant en compte son caractère aléatoire grâce à des modèles individu-centrés, retraçant la vie de chaque particule. Contrairement aux ondes progressives, qui donnent une description macroscopique de la dynamique, les modèles individu-centrés en donnent une vision microscopique. Le lien entre ces deux représentations est obtenu en faisant tendre la densité locale de particules dans le modèle individu-centré vers l'infini de sorte que les fluctuations se compensent. En ce sens, les équations de réaction-diffusion peuvent être considérées comme les limites hydrodynamiques de modèles microscopiques.

Dans cette thèse, nous étudions le comportement en temps long de deux systèmes microscopiques modélisant des invasions biologiques et les comparons à celui de leurs limites en grande population. Plus précisément, nous nous intéressons à deux facteurs importants en écologie : l'impact d'habitats fragmentés sur les espèces ainsi que l'effet de la coopération entre individus sur la structure génétique d'une population. En effet, dans le contexte du changement climatique, il semble crucial de prendre en compte ces deux paramètres pour prédire l'évolution des espèces. D'une part, les habitats naturels se trouvent de plus en plus fragmentés par les aménagements humains et modifiés par le réchauffement climatique. Il est donc essentiel d'identifier des modèles permettant de mettre en lumière les conséquences de tels changements sur la dynamique des espèces. Par ailleurs, de nombreux travaux ont montré que la présence ou l'absence de coopération au sein d'une espèce affectait de manière radicale sa diversité génétique. Or, une grande diversité génétique se traduit par une plus grande capacité d'adaptation face aux changements environnementaux. Comprendre les mécanismes microscopiques menant à de telles différences de structure génétique permettrait donc de repérer les populations les plus sensibles aux changements d'environnement.

La première partie de cette thèse est consacrée à l'étude de l'impact des habitats présentant des hétérogénéités à grande échelle sur la vitesse de propagation des populations. Cette question a déjà été étudiée dans le domaine des EDP en généralisant des modèles d'invasion en milieu homogène tel que celui introduit dans les années 1930 par Fisher et par Kolmogorov, Petrovsky et Piskunov, l'équation de FKPP. Une théorie, dite des *solutions de viscosité*, répond à ce problème lorsque l'échelle d'hétérogénéité est très grande. Cependant, dans certaines situations, les solutions de viscosité fournissent des dynamiques irréalistes d'un point de vue biologique et donc inexploitables dans ce cadre. Nous proposons donc de revenir à un modèle individu-centré et d'étudier son comportement limite lorsque l'échelle d'hétérogénéité de l'environnement tend vers l'infini, avant de considérer des tailles de population arbitrairement grandes. Dans ce cadre, nous montrons que la vitesse de propagation de la population peut être beaucoup plus petite que la vitesse d'invasion donnée par l'équation déterministe décrivant la limite hydrodynamique du système microscopique. Par conséquent, la limite hydrodynamique ne fournit pas systéma-

tiquement la bonne dynamique macroscopique. Cette différence de comportements est liée au "problème des queues" dû à l'absence d'extinction locale dans les équations de type FKPP : des populations exponentiellement petites peuvent dicter la dynamique de l'invasion dans le modèle déterministe. De plus, cette étude nous renseigne sur les échelles d'hétérogénéité en jeu dans de tels problèmes de modélisation et nous donne une idée du modèle à adopter en fonction de la taille des populations étudiées.

Dans une seconde partie, nous étudions l'impact du type des fronts d'onde limites sur les modèles stochastiques associés pour expliquer ce net ralentissement des systèmes de particules. En effet, les dynamiques d'invasion modélisées par des équations de réaction-diffusion monostables sont divisées en deux classes : les fronts tirés et les fronts poussés. Cette classification est faite en fonction de la vitesse des fronts : ils sont dits tirés si leur vitesse correspond à la vitesse de l'équation linéarisée en 0 et poussés si leur vitesse est strictement plus grande. De nombreux travaux ont cherché à quantifier l'effet des fluctuations démographiques sur les fronts tirés. Il est désormais bien connu que les fluctuations ont un impact considérable sur ces fronts : la correction sur leur vitesse est bien plus grande que celle suggérée par le théorème central limite. En revanche, les fronts poussés sont supposées être moins sensibles aux perturbations. Néanmoins, de récentes études et simulations numériques sur des équations de réaction-diffusion bruitées suggèrent l'existence d'une troisième classe de fronts. Il s'agit d'une sous-classe de fronts poussés particulièrement sensibles aux fluctuations. Dans cette thèse, nous développons un système de particules capturant les mécanismes internes des fronts d'invasion afin d'expliquer la transition entre ces trois régimes ainsi que leurs conséquences sur la structure génétique des populations en expansion.

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CHAPTER I

Introduction

1 Modeling spatially structured populations

Biological invasions have been widely studied to understand, predict and control the spatial distribution of species with which we cohabit. The examples of such biological invasions are manifold and occur at different scales: they can be observed at a macroscopic level with plants, mammals, birds, insects, etc. as well as at microscopic levels (virus, bacteriae, etc.). Consequently, there also exists a variety of mathematical models to investigate biological invasions, each one depending on the specifications of the species studied. Nevertheless, one can identify two main classes of models: stochastic and deterministic ones. Stochastic models describe the population at a microscopic scale. Each individual evolves according to a series of rules, which are given by a list of conditional probabilities to give birth, migrate, reproduce, die... The population is thereupon seen as a system of interacting particles and naturally, the invasion speed of the species is determined by the velocity of the extremal particles in the particle system. From another perspective, the deterministic approach consists in assuming that the density of population is governed by a partial differential equation that combines two mechanisms: a diffusion, reporting the dispersal or migration of the particles, and a reaction, describing the births and deaths. As a result, *wavefront* or *traveling wave* solutions, which perform a switch from one state to another (*e.g.* uninhabited to colonised) at constant speed, often arise from this class of equations. The invasion speed of the population is then deduced from the speed of this wave. Prototypes of these two different approaches are described in the two following subsections (Sections 1.1 and 1.2).

In the context of climate change, there has been a renewed interest for population dynamics. Our own living conditions rely on ecosystem services (pollination by bees, carbon sequestration by trees, etc.) that are threatened by humans alterations. That is why understanding how these endangered species colonise their environment is an important challenge. From this perspective, we will focus on two major factors in ecological studies: large scale space-time heterogeneities in the environment and Allee effects.

The large space-time heterogeneities are at stake when it comes to model the effect of global warming on the ecosystem: the time scale at which we observe the temperature and therefore the climatic niches shift might be much larger than the typical life span of the individuals. On the other hand, spatial heterogeneities are ubiquitous on earth (mountains, oceans, human developments) and we have to take it into account to get a workable outcome. The first part of this thesis will be dedicated to the study of populations colonising slowly varying habitats.

The Allee effects depict, for instance, the effect of intraspecific cooperation. It is natural to consider that individuals struggle for resources since the food supply is always finite, but one should not overlook the fact that some individuals cannot reproduce without the help of their congeners. In other words, the presence of Allee effect in the system presupposes that the growth of the population is maximal at intermediate population densities. This phenomenon not only increases the invasion speed of the species but also drastically affects the genetic diversity of the population. Indeed, when the interactions are restricted to mere competition, only a few individuals give birth to a significant fraction of the population, which leads to a rapid diversity loss. The Allee effect might be the result of various biological mechanisms such as consanguinity, sexual reproduction, cooperation for food/against a predator, etc. In opposition, when the growth is maximal at low densities, we will say that there is no Allee effect.

1.1 Individual based models (IBM)

In this framework, the individuals are seen as particles. The first example of IBM is the birth and death process: it is a pure jump Markov process whose jumps are equal to ± 1. The transition rates of this process are the following. If the population size is equal to i, it jumps to $i+1$ with rate λ_i (birth) and to $i-1$ with rate μ_i (death). The sequences $(\lambda_i)_{i\in\mathbb{N}}$ and $(\mu_i)_{i\in\mathbb{N}}$ satisfy $\lambda_0 = \mu_0 = 0$ and may depict several situations. As an example, we cite the birth and death process with immigration ($\lambda_i = i\lambda + \rho$, and $\mu_i = i\mu$, with $\mu, \lambda, \rho > 0$) and the logistic birth and death process ($\lambda_i = i\lambda$ and $\mu_i = i\mu + ci(i-1)$ with $\mu, \lambda > 0$). See [Fel57] for further examples. These monotype processes has been widely studied. For instance, we know a simple necessary and sufficient condition on the sequences (λ_i) and (μ_i) for almost sure absorption at 0 (*i.e.* extinction) [KM57], and in the latter case, the average extinction time can expressed as a function of (μ_i) and (λ_i) [PK10].

The case of spatially structured populations can be seen as a generalisation of this toy model, in which individuals are characterised by a type. In this work, the type of the particles will correspond to their positions. The set of particles evolves in discrete or continuous time, in a continuous environment or on demes. In this thesis, we will only deal with the one-dimensional case so that the position of one particle x will be either a point of the real axis or of the rescaled lattice $\Delta x\mathbb{Z}$. In the microscopic framework, the invasion speed of the system is therefore investigated by tracking the displacement of its rightmost particle on this axis.

In discrete time t, the simplest example of microscopic system to model an expanding population is the branching random walk (a precise definition is given in the following subsection), in which there is no interaction between particles, apart from the one induced by the genealogy. In this model, at each generation, all the particles give birth independently to a random number of children, scattered around the parental location. Therefore, the population growth is Malthusian (the growth is exponential if the typical number of children is larger than 1), which is not realistic. On the other hand, one can assume that the individuals live on demes and that the population size is fixed on each site. Thus, a birth on one site has to coincide with a death on another site: this is the so-called *Stepping stone model* [KW64]. Obviously, this approach is not adapted to the study of biological invasions. Therefore, it is natural to consider arbitrary population sizes and to introduce regulation mechanisms. These mechanisms can be thought as the effect of competition for resources such as food, housing, etc. A legitimate question is to wonder if such systems survive or die out. Many models have been studied in the mathematical literature to answer this question [Eth04, BD07, DL94] but we will overlook this difficulty by considering simplifying assumptions that will ensure that the population does not go extinct.

In discrete time $t \in \{\Delta t, 2\Delta t, \dots\}$ and space $x \in \{\dots, -\Delta x, 0, \Delta x, 2\Delta x, \dots\}$, an individual based model can be described by its configuration ξ_t in such a way that $\xi_t(x)$ specifies the state of the system at time t on the site x. This state may refer to the number of particles living on a site or if the site is occupied or vacant. In this latter case, we mention the example of contact processes [DL94]. The configuration function ξ_t takes the value 1 at x if the site is occupied, 0 if it is vacant and evolves as follows. At each generation, a particle living on the site x dies with probability γ. If it survives, it gives birth to a particle on the site y with probability λ if y is a neighbour of x *i.e* if $|y - x| = \Delta x$. At the end of this step, if at least one particle lives on y, the site y is occupied. In addition, there can be at most one particle per site. This model is particularly well suited to annual plant populations (see [DL94] for examples) since it is reasonable to assume that the population evolves in generations and one can easily understand why a site can only be occupied by one individual (e.g. a tree). In this work, we will rather focus on the first representation, indicating the number of particles occupying each site.

The discrete time approach is often adopted when the population is supposed to evolve in generations. The dynamics of plant populations is often depicted by discrete models given that annual plants produce a random number of fertile seeds each year and that the individuals only migrate at their birth times. The discrete approach can also be seen as an approximation of a continuous time model if we let the time step between two generations Δt tend to zero.

In the continuous framework, particles live during a random lifespan. During its life time, a particle performs a random motion (e.g. a Brownian motion) before splitting in a random number of offspring or dying. Instead of describing these systems by their configurations, we will rather consider the set of particles alive at time t, denoted by $\mathcal{N}_t$, and for each particle $u \in \mathcal{N}_t$, we denote by $X_u(t)$ its position at time t. We refer to the following subsection on branching Brownian motion and to Section 2.2 for examples of processes evolving in continuous time and in a continuous state space ($\mathbb{R}$).

To conclude this paragraph, we point out that, in the continuous settings, the evolution of the system can also be represented by a measure valued process $(\rho_t)_{t \geqslant 0}$. Roughly speaking, an atom δ_x is added to the measure ρ_t when a particle is born at x. This strategy turns out to be very useful when it comes to investigate some scaling limits [CM07, FM04a].

We now introduce two individual based models which will serve as the basis for the two models investigated in this thesis: the branching random walk and the branching Brownian motion.

The Branching Random Walk

This particle system is governed by a reproduction law $(p_k)_{k\in\mathbb{N}}$ and a displacement law μ. It starts with a single particle at $x = 0$ at time $t = 0$. At time $t = 1$, this particle gives birth to a random number of children N, distributed according to (p_k), and die. The N newborns are respectively located at positions $\Xi_1, ..., \Xi_N$, where (Ξ_j) is a sequence of i.i.d. random variables of law μ. At time $t = n$, a particle living at position x_i is replaced by a random number of particles M of law (p_k), located at $x_i + \Xi^i_1, ..., x_i + \Xi^i_M$, where (Ξ^i_j) is a sequence of i.i.d random variables of law μ. Similarly, all the individuals of the n-th generation give birth independently of one another according to (p_k) and their children are independently distributed around the parental location according to μ.

Note that the reproduction law of the particles is the same at each generation and on each site. Moreover, all the particles evolve independently and there is no interaction between individuals in the system (except the one induced by the genealogy).

The invasion speed of this system can be deduced from Biggins' theorem [Big77]. Indeed, if we denote by M_n the position of the rightmost particle in the system at time $t = n$, we know that as $n \to \infty$,

$$\frac{M_n}{n} \to c \quad a.s. \tag{1.1}$$

The value of c depends on the laws (p_k) and μ through the rate function I. Let X be a random variable of law μ. Consider $m = \sum_{k=1}^{\infty} kp_k$ and $\Lambda(\theta) = \log \mathbb{E}\left[e^{\theta X}\right]$. The rate function I is then given by

$$I(x) = \sup_{\theta\in\mathbb{R}} \left(\theta x - \Lambda(\theta)\right),$$

and for $m > 1$, the speed c of the BRW is the unique positive solution of

$$I(c) = \log(m).$$

For instance, if μ is a standard normal distribution and $m = 1 + r$ for some $r > 0$, we have

$$c = \sqrt{2r}.$$

This remark will play a crucial role in the analysis made in Chapter II. For further details on the maximal displacement of a branching random walk, see Section A of Chapter II.

Branching Brownian Motion (BBM)

Branching Brownian motion is the continuous counterpart of the BRW. It starts with a single particle at time $t = 0$, located at the origin. Each particle moves independently according to a standard Brownian motion during its lifespan, which is exponentially distributed time with mean one. When it dies, a particle splits into two particles. Actually, the offspring distribution can be more general than dyadic branching but we will only consider this particular branching distribution in this work. We recall that we denote by $\mathcal{N}_t$ the set of particles alive in the BBM at time t and by $X_u(t)$ the position of the particle $u \in \mathcal{N}_t$ at time t.

As in the discrete case, the position M_t of the rightmost particle in this system at time t has already been well documented. Indeed, Bramson [Bra78] proved that under these assumptions, the rightmost particle sits close to $\sqrt{2}t$ for t large enough. More precisely, he studied the distribution function of the position of the random variable M_t,

$$u(t,x) = \mathbb{P}\left(\max_{u\in\mathcal{N}_t} X_u(t) \geqslant x\right) \tag{1.2}$$

and considered the position $m_\delta(t)$ such that $u(t, m_\delta(t)) = \delta$ for $0 < \delta < 1$. He proved that for fixed $\delta > 0$, we have the following asymptotic expansion of m_δ as $t \to \infty$:

$$m_\delta(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) + O(1). \tag{1.3}$$

1.2 Fisher's equation and wavefronts

From the PDE point of view, a population can be described by a space-time dependent density u governed by a reaction diffusion (RD) equation. Typically, the function u takes values in $[0, 1]$, the value 1 indicating the saturation state. This saturation state corresponds to a stable state in which the population has reached the carrying capacity of the environment. This density then evolves according to two mechanisms: the diffusion that accounts for the migrations and the reaction which characterises the births and deaths. A significant property of RD equations is the existence of wavefront solutions describing the switch from vacancy ($u \approx 0$) to saturation by a traveling front.

The prototypical model for front propagation is the Fisher-Kolmogorov-Petrovskii-Piskounov (FKPP) equation, a semi-linear parabolic partial different equation. It was first introduced by Fisher [Fis37] and independently by Kolmogorov, Petrovskii and Piskounov [KPP37] in the following form:

$$u_t = \frac{1}{2}u_{xx} + ru(1-u), \tag{1.4}$$

for some $r > 0$. Fisher related Equation (1.4) to the expansion of an advantageous gene in a linear environment as a shore line. What we call FKPP-type equations are the more general partial differential equations of the form

$$u_t = \frac{1}{2}u_{xx} + f(u), \tag{1.5}$$

where the forcing term f, which governs the population growth, satisfies the following conditions, referred to as the *KPP conditions*:

1. $f(0) = f(1) = 0$,

2. $f(u) > 0$ for all $u \in (0, 1)$,
3. $f'(0) > 0$ and $f(u) < f'(0)u$ for all $u \in (0, 1)$.

The first condition ensures that 0 and 1 are respectively sub-solution and super-solution of the Cauchy problem combining Equation (1.5) and an initial data u_0 such that $0 \leqslant u_0(x) \leqslant 1$, so that the solution u of this problem stays between 0 and 1 at all time. Biologically speaking, the third condition implies that the lower the population density, the faster the expansion. Throughout this thesis, we will assume that the forcing term f is continuously differentiable on $\mathbb{R}$ and negative on $(-\infty, 0) \cup (1, +\infty)$, even in the non-FKPP case (unless otherwise specified).

A crucial property of this equation is the existence [KPP37] of traveling wave solutions $u(t, x) = \varphi_c(x - ct)$ as long as $c \geqslant c^* := \sqrt{2f'(0)}$. These traveling fronts φ_c are solutions of the ordinary differential equation

$$\begin{cases} \frac{1}{2}\varphi_c'' + c\varphi' + f(\varphi_c) = 0 \\ \lim\limits_{x \to -\infty} \varphi_c(x) = 1, \ \lim\limits_{x \to +\infty} \varphi_c(x) = 0 \\ \varphi_c \geqslant 0. \end{cases} \tag{1.6}$$

Kolmogorov, Petrovskii and Piskounov [KPP37] proved that Equation (1.6) has a solution as long as $c \geqslant c^*$. For fixed $c \geqslant c^*$, the traveling front φ_c is unique up to translation and decreases from 1 to 0. In the case of Equation (1.4), note that

$$c^* = \sqrt{2r}.$$

The analysis of Equation (1.6) in a neighbourhood of the stable point 0 suggests that as $z \to \infty$ (or equivalently $\varphi_c \to 0$), the front $\varphi_c(z)$ decreases as $Ce^{-\lambda z}$ if $c > c^*$ and as $Cze^{-\lambda z}$ if $c = c^*$, where λ is a positive real number such that $-\lambda$ is the largest root of $\frac{1}{2}X^2 + cX + f'(0)$. Therefore, there is a bijection between the speed c of the traveling wave φ_c and the shape of its leading edge, called the *dispersion relation* and given by

$$c = \frac{f'(0)}{\lambda} + \frac{1}{2}\lambda. \tag{1.7}$$

Note that c is minimal and equal to c^* for $\lambda = \sqrt{2f'(0)} = c^*$.

The major interest of these traveling fronts lies in the fact that they provide the invasion speed of the population. Indeed, it was shown [KPP37] that for the solution u of (1.5) with initial condition $u(0, x) = \mathbb{1}_{x<0}$, there exists a centring term $m(t)$ such that

$$\sup_{x \in \mathbb{R}} |u(x - m(t), t) - \varphi_{c^*}(x)| = 0 \tag{1.8}$$

as $t \to \infty$ and satisfying

$$m(t) \sim \sqrt{2f'(0)}t.$$

In this thesis, we will distinguish two regions of this asymptotic shape φ_{c^*}: the *bulk* and the *leading edge*. In the leading edge, the population size is so small that the effect of the nonlinearities can be neglected ($u \approx 0$). The bulk can itself be divided into two zones: the first one corresponds to the sites where the population size is close to the equilibrium/saturation state ($u \approx 1$) and the second one is the transition region between the stable state 1 and the leading edge.

Another matter of interest is the asymptotic propagation speed of the solution u of the Cauchy problem (1.5) with compactly supported initial data. We say that the invasion speed of u is $c_0 > 0$ if for all $c < c_0$

$$\min_{|x|\leqslant ct} u(t,x) \to 1, \quad \text{as} \quad t \to \infty, \tag{1.9}$$

and for all $c > c_0$,

$$\sup_{|x|>ct} u(t,x) \to 0, \quad \text{as} \quad t \to \infty. \tag{1.10}$$

In what follows, when we talk about invasion speed, we implicitly assume that the initial condition u_0 is compactly supported and takes values in $[0,1]$. It was shown [AW78] that the invasion speed of the solution u of (1.5) is equal to c^*. In other words, if the population starts with a configuration u_0 such that $u_0(x) = 0$ for x large enough, the individuals successfully invade the right-hand side of the real axis at speed c^*. Therefore, under the KPP conditions, the upper bound and the lower bound on the invasion speed coincide. We mention that the dynamics arising from an initial data with tails $e^{-\alpha x}$ for some $\alpha < c^*$ has been investigated [Uch78], but this case is less relevant biologically. As a result, the latter invasion speed is larger than c^* and the decay of the transition front is given by the relation (1.7). For a study of FKPP-type equations with slowly decaying initial conditions, see [HR10].

If we then define the position of the invasion front x^* of a solution u of (1.5) by

$$x^*(t) = \sup\left\{x \in \mathbb{R} : u(t,x) \geqslant \frac{1}{2}\right\}, \tag{1.11}$$

in view of the above, we have

$$x^*(t) \sim c^* t \quad \text{as} \quad t \to \infty, \tag{1.12}$$

for any solution u arising from a compactly supported initial data.

While propagation speed seems well-understood in homogeneous FKPP equations, the actual expansion of biological species is much more complex [SK97]. The most obvious explanation is that the habitats are intrinsically heterogeneous and that we need to take this heterogeneity into account to design a realistic model. Various approaches were suggested to investigate the impact of space heterogeneity on the conditions of successful invasion and on the propagation speed of a species. Here, we only cite a simplified version of the SKT model [SKT86], which is a natural extension of the FKPP equation,

$$u_t = \frac{1}{2}u_{xx} + u(\mu(x) - \gamma u), \tag{1.13}$$

where μ and γ are respectively the intrinsic growth rate and γ the intraspecific competition coefficient. If the growth rate μ is a periodic function, the problem is well documented [SKT86]: the population will either becomes extinct or invade the real axis, in the latter case the spreading speed corresponds to the speed of the slowest *periodic traveling wave* [Wei02]. As in [HFR10], we will refer to the SKT patch model [SKT86] when μ is a 1−periodic step function such that

$$\mu(x) = \begin{cases} \mu^+ & x \in \left[0, \frac{1}{2}\right) \\ \mu^- & x \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{1.14}$$

for some $\mu^+ > \mu^-$. This presupposes that the environment can be seen as an alternation of favourable and unfavourable regions, which are broadly homogeneous. At first sight, one could expect the invasion front to behave as in the homogeneous settings in each patch. However, we will see in Section 3.1 that this is not necessarily the case. Heterogeneous FKPP equations will also be further discussed in Section 3.1.

A first relation between the stochastic and the deterministic approaches is the so-called McKean representation [McK75]. If X is a BBM with branching law $(p_k)_{k\in\mathbb{N}}$ and branching rate β and if $u : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$ solves

$$\begin{cases} u_t = u_{xx} + \beta(u - f(u)) \\ u(0,x) = g(x), \end{cases} \tag{1.15}$$

for some initial data g and $f(s) = \sum p_k s^k$, then u has the following representation:

$$u(t,x) = \mathbb{E}_x \left[\prod_{u \in \mathcal{N}_t} g\left(X_u(t)\right) \right].$$

In particular, if X is a dyadic branching Brownian motion with branching rate $\beta = r$, and g is the Heaviside function, Equation (1.15) is the FKPP equation introduced in Section 1.2:

$$\begin{cases} u_t = u_{xx} + r(u - u^2) \\ u(0,x) = \mathbb{1}_{x<0}. \end{cases} \tag{1.16}$$

The McKean representation then implies that the solution of this equation can be written as

$$u(t,x) = \mathbb{P}\left(\max_{u \in \mathcal{N}_t} X_u(t) \geqslant x \right),$$

which explains why the coefficient of the first term in (1.3) is given by the invasion speed of the FKPP equation. Actually, Bramson [Bra78, Bra83] was the first to prove, thanks to probabilistic arguments, the logarithmic correction of the invasion speed of the FKPP equation, in the sense that the centring term $m(t)$ from (1.8) has the same expansion as (1.3):

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t) + O(1). \tag{1.17}$$

2 Particle systems

In this thesis, we will focus on microscopic models for front propagation and compare their long time behaviour to the results obtained thanks to the deterministic approach. In Sections 2.1 and 2.2, we give an overview of the models studied in Chapters II and III.

2.1 A particle system with local competition and inhomogeneous branching rate modeling large space-time heterogeneities

In this section, we introduce a simplified version of the interacting particle system analysed in Chapter II, briefly explain the biological interpretation of its parameters and indicate the main ideas that allows us to compare this system to the branching random walk defined in Section 1.1.

The system evolves in generations and we assume that the time between two generations is given by a small parameter $\Delta t > 0$. We also discretise space: the population lives on demes $(x_i)_{i\in\mathbb{Z}}$, with $x_i = i\Delta x$ for some small space step $\Delta x > 0$. The state of the system is described by a sequence of configurations $(n_k)_{k\in\mathbb{N}}$, where $n_k(i)$ counts the number of particles living on the site x_i at time $t_k := k\Delta t$ (or equivalently after the k-th generation).

The dynamics of the particle system is governed by two parameters: its growth rate r and its local population density K. The first one is assumed to be a smooth function $r : (0,\infty)\times\mathbb{R} \to (0,\infty)$, $(t,x) \mapsto r(t,x)$ that models the heterogeneity of the environment. A favourable (resp. harsh) region or period corresponds to a large value or r (resp. close to 0). As discussed in Section 1.1, a regulation mechanism is introduced to prevent the population size from increasing exponentially. In practice, we assume that the maximal number of particles living on a site is approximately given by K. Consequently, this parameter can be seen as the local density of the population, the carrying capacity of the environment or the strength of the competition. Note that the larger the carrying capacity, the weaker the competition. Typically, we will be interested in the scaling limit of the system for large values of K (see Section 3.1). Finally, once in a generation, the particles migrate, according to a given law μ. To simplify the computations and get the expected large population limit, we assume in Chapter II that μ is a discretised Gaussian distribution. However, the same analysis could be conducted for any distribution exhibiting exponential tails.

Starting from a configuration $(n_k(i))_{i\in\mathbb{Z}}$ at time t_k, the system evolves as follows:

Step 1 At time t_k, on each site x_i, the $n_k(i)$ particles living on x_i duplicate independently with probability $r(t_k, x_i)\Delta t$.

Step 2 The population size on each site is truncated to K,

Step 3 All the particles migrate independently according to $(\mu_j)_{j\in\mathbb{Z}}$: a particle born at x_i jumps to x_{i+j} with probability μ_j.

The resulting configuration is n_{k+1}. We conclude the definition of this model with two remarks. First, the particles do not die, except trough the truncation that only affects large populations, so that the system never goes extinct. This assumption will drastically simplify the proof of the lower bound on the propagation speed, since there is always at least one particle in the system. Second, the spatial discretisation allows us to postulate that the competition is only local and that the particles do not interact with their neighbours. This might also be seen as the mean-field approximation of a more complex dynamic.

The main idea to study this system is to locally compare it with some BRWs, whose propagation speeds are well-known (see Equation (1.1)). Therefore, two difficulties arise. First, the reproduction law is not "constant" since we no longer assume that the environment is homogeneous. Second, the truncation (Step 2) breaks the "linearity" of the system: one has to keep in mind that for K large, the scaling limit of the particle system is not a system without interaction between the particles. In short, the analysis of this system relies on a *comparison principle* (or a coupling lemma) using the monotony of the system with respect to the growth rate r and the carrying capacity K.

Indeed, note that Step 1 implies that the system is monotone with respect to the growth rate r in the sense of stochastic domination. In other words, for any $m \in \mathbb{N}$, the probability to get more than m children on a site after Step 2 increases with the value of r. Thus, if there exist $r_1, r_2 > 0$ such that r takes values in the interval $[r_1, r_2]$, the system can be coupled with two other systems (n_k^1) and (n_k^2), evolving according to the same Steps 1,2 and 3, in which individuals duplicate with probability $r_1 \Delta t$ (resp. $r_2 \Delta t$), in such a way that

$$n_k^1(i) \leqslant n_k(i) \leqslant n_k^2(i), \quad \forall i \in \mathbb{Z},\ \forall k \in \mathbb{N}, \quad a.s. \tag{2.18}$$

This observation is the first element of proof. It allows to compare this system with a BRW of reproduction law $p_2 = r_0 \Delta t$, $p_1 = 1 - r_0 \Delta t$, in a cylinder $[t_{k_0}, t_{k_1}] \times [x_{i_0} - j\Delta x, x_{i_0} + j\Delta x]$, for some appropriate constant $r_0 > 0$.

Moreover, recall that thanks to the space discretisation, we can assume that the interactions between the individuals only take place locally, in such a way that the population size is regulated by a simple competition rule (Step 2). As a consequence, the system (n_k) is also monotone with respect to the local density K in the sense of coupling defined in (2.18). Moreover, remark that as long as the total population size does not exceed K, the competition step has no effect and the particles behave as if there was no interaction. These two remarks on the competition step constitute the second element of the proof.

As mentioned in Section 1, we are interested in the effect of a slowly varying environment on the invasion speed of a species. Hence, we will introduce an additional parameter $\varepsilon > 0$ that quantifies the scale of heterogeneity of the habitat and replace the probability to give birth in Step 1 by

$$r^{\varepsilon}(t_k, x_i)\Delta t := r(\varepsilon t_k, \varepsilon x_i)\Delta t. \tag{2.19}$$

Let X_k^* denote the position of the rightmost particle in this system at time t_k, that is

$$X_k^* = \max\{i \in \mathbb{Z} : n_k(i) > 0\}. \tag{2.20}$$

The goal of Chapter II is to establish the existence of a function x such that, for all $T > 0$, we have

$$(\varepsilon X_k^*,\ 0 \leqslant k \leqslant T/\varepsilon) \longrightarrow (x(t),\ t \in [0, T]), \tag{2.21}$$

when we let first $\varepsilon \to 0$, then $K \to \infty$. This function $t \mapsto x(t)$ will be the solution of a certain ordinary differential equation whose form depends on r and μ. The result of this chapter and some simulations of the dynamics are presented in Section 4.1.

2.2 A branching particle system modeling Allee effects

In this section, we define a version of the branching Brownian motion introduced in Section 1.1: we consider a BBM with absorption and inhomogeneous branching rate.

Branching Brownian motion with absorption was first introduced by Kesten in 1978 [Kes78]. This process evolves as the Branching Brownian motion (see Section 1.1) except that the particles move according to a Brownian motion with drift $-\mu$ and are killed (or absorbed) when they enter the interval $(-\infty, 0]$. Kesten proved the existence of a critical drift μ_c such that this process goes extinct almost surely as long as $\mu \geqslant \mu_c$. If $\mu < \mu_c$, the process survives forever with positive probability.

In Chapter III, we consider the following version of the BBM with absorption. We now assume that the particles branch at a space dependent rate $r(x)$, namely, we set

$$r(x) = \begin{cases} \frac{\rho}{2} & x \in [0,1] \\ \frac{1}{2} & x > 1, \end{cases} \tag{2.22}$$

for some $\rho \geqslant 1$. The drift μ is then chosen in such a way that the number of particles in the system (N_t) stays controlled. Actually, the process (N_t) will eventually die out but its fluctuations will report the mechanisms driving the invasion in the particle system.

The model introduced in this section is very largely inspired by the one analysed in [BBS13] to prove (under slightly different assumptions) the conjectures stated in [BDMM06b, BDMM07]. Their model corresponds to the special case where $\rho = 1$ in (2.22). In [BBS13], the drift is supercritical (in the case of dyadic branching Brownian motion and constant branching rate $r(x) \equiv 1$, $\mu_c = \sqrt{2}$). For each $N \in \mathbb{N}$, they consider a dyadic branching Brownian motion with absorption and drift $-\mu_N$, with

$$\mu_N = \sqrt{2 - \frac{2\pi^2}{(\log(N) + 3\log\log(N))^2}}, \tag{2.23}$$

starting, for instance, with $N \log(N)^3$ particles at $x = 1$, and denote by $M_N(t)$ the number of particles living in the process at time t. Their result is the following: as N tends to $+\infty$, $\left(\frac{2\pi}{N} M_N(\log(N)^3 t),\ t \geqslant 0\right)$ converges in law to a version of Neveu's continuous-state branching process (see below for a precise definition). Using the results from [BLG00], they deduce that the genealogy of the particle system is given by a Bolthausen-Sznitman coalescent.

As above-mentioned, this work was motivated by the conjectures from [BDMM06b, BDMM07] on a particle system with fixed population size N. Their system evolves in generations on a continuous state space, $\mathbb{R}$. At each generation, the individuals give birth to k children (*e.g.* $k = 2$), scattered around the parental location. Only the N rightmost individuals survive to the next generation. As discussed in [BDMM06b, BDMM07], this model can be seen as a model of biological population undergoing evolutionary selection. The position of a particle measures its fitness, which evolves because of successive mutations. When a particle gives birth, the dispersion of the offspring corresponds to the effect of mutations. The constant population size stands for the selection: the carrying capacity of the environment is finite so that only the N fittest individuals survive. In this work, we will rather consider the interpretation given in [BDMM06a]: this system is a stochastic model for front propagation, related to the fluctuating fronts arising from noisy FKPP-type equations. To this extent, the branching Brownian motion with absorption considered in [BBS13] can be seen as a toy model for what happens at the leading edge of a wavefront. See Sections 3.1 and 3.2 for further explanation on the link between FKPP-type equations, noisy FKPP-type equations and particle systems.

In chapter III, we consider a model similar to the one in [BBS13], where the saturation rule is no longer given by a fixed population size but modeled by a moving wall, that keeps the population size *approximately constant*. Biologically speaking, the parameter ρ scales the strength of the cooperation between individuals. Indeed, for a suitable choice of μ (typically the speed of the corresponding wavefront), the interval $[0,1]$ corresponds to the intermediate density region of the wavefront. In other words, this system is meant to depict the dynamics of a population undergoing Allee effects in a co-moving frame following the invasion front. Note that, in contrast to the previous model (Section 2.1), the space-dependent branching rate does not reflect spatial heterogeneities but the effect of cooperation.

The aim of Chapter III is first to determine the value of μ, to get the speed of the stochastic front, second, to study the microscopic internal dynamics of the front thanks to the particle system. The case $\rho = 1$ has already been dealt with in [BBS13, MS20]: μ is equal to 1 and the number of particles in the co-moving frame converges to a Neveu's CSBP so that the genealogy of the individuals is given by a Bolthausen-Sznitman coalescent. We will discuss the interpretation of this result later on (Sections 3.2 and 4.2).

Continuous-State Branching Processes (CSBP)

We now turn out our attention to the definition of the continuous state branching processes in order to motivate the results proved in Chapter III.

A continuous-state branching process is a Feller process $(X_t, t \geqslant 0)$, taking values in $[0, +\infty]$, satisfying the branching property:

$$\forall x, y \geqslant 0, \forall t \geqslant 0, \quad X_t(x+y) \overset{\mathcal{L}}{=} X_t(x) + \tilde{X}_t(y), \tag{2.24}$$

where $(X_t(x), t \geqslant 0)$ and $(\tilde{X}_t(y), t \geqslant 0)$ are two independent processes with the same law as $(X_t, t \geqslant 0)$, starting respectively from x and y. The states 0 and $+\infty$ are absorbing.

The branching property means that a population $Z_t(x+y)$ started with an initial population size $Z_0(x+y) = x+y$ can be seen as the sum of two subpopulations respectively starting from x and y and evolving independently. The branching and the Markov properties of the process ensures that there exists a map $\lambda \mapsto u_t(\lambda)$ such that for all $\lambda \geqslant 0$, $x \geqslant 0$ and $t, s \geqslant 0$, we have

$$\begin{cases} \mathbb{E}\left[e^{-\lambda X_t(x)}\right] = \exp\left(-x u_t(\lambda)\right) \\ u_{t+s}(\lambda) = u_s \circ u_t. \end{cases} \tag{2.25}$$

Each continuous-state branching process can be characterised by a function $\Psi : [0, +\infty) \to \mathbb{R}$, called the mechanism of the CSBP, such that if $(X_t, t \geqslant 0)$ is a Ψ-CSBP, the function $t \mapsto u_t(\lambda)$

from (2.25) is a solution of the differential equation

$$\begin{cases} \frac{\partial}{\partial_t} u_t(\lambda) = -\Psi(u_t(\lambda)) \\ u_0(\lambda) = \lambda. \end{cases} \tag{2.26}$$

We will be interested in α-stable CSBP for $\alpha \in [1, 2]$, for which the branching mechanism Ψ is of the form

$$\Psi(u) = \begin{cases} -au + bu^\alpha, & \text{if } \alpha \in (1, 2] \\ -au + bu \log u, & \text{if } \alpha = 1. \end{cases} \tag{2.27}$$

It is known that in this case, the CSBP does not explode in finite time, *i.e.* Grey's condition is satisfied. The 2-stable CSBP is also known as the *Feller diffusion* and the 1-stable CSBP as *Neveu's CSBP*.

As explained in [Ber09], a CSBP can be seen as the continuous analogous of a Galton-Watson process. In discrete time, let (Z_n) denote a Galton Watson process with offspring distribution (p_k). This process can be associated to a random walk on a random time scale. Indeed, if we consider (L_i), a sequence of i.i.d. random variables of law (p_k), $X_i = L_i - 1$, $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^{n-1} Z_i$, we have

$$Z_n = S_{T_n}, \quad \forall n \in \mathbb{N}.$$

Let us now consider a continuous time Galton-Watson process (Z_t), in which individuals branch at rate $a > 0$ and give birth to a random number of children L. This process can also be associated to a continuous time random walk (S_t), with jump rate a and increment distribution $X = L - 1$. Given that $Z_t = z$, the rate at which Z_t jumps to $z + x$ is the rate at which S_t jumps from z to $z + x$ times z. The continuous analogous of Z_t is the CSBP X_t and the analogous of the random walk S_t is a Levy process Y_t with non-negative jumps. The Laplace transform of this Levy process satisfies $\mathbb{E}\left[e^{-\lambda(Y_t - Y_0)}\right] = \exp\left(-t\Psi(\lambda)\right)$, where Ψ is the function from Equation (2.26).

Given this interpretation of the CSBP, the result in [BBS13] can be understood as follows. Over a time scale of order $\log(N)^3$, a particle reaches a position ahead of the front. Since this particle is far from 0, it produces in a short time a significant number of offspring, which causes, on the time scale $\log(N)^3$, a jump in the CSBP. The goal of Chapter III is to prove that on a suitable time scale, the number of particles in our system converges in law to an α-stable CSBP.

3 Motivations from PDE and biology literature

3.1 Hydrodynamic limit and tail problems

In Section 1.2, we pointed out a first relation between an individual based model, the branching brownian motion, and a certain class of FKPP-type equations through the McKean representation. A second connection between microscopic models and FKPP-type equations lies in what we call the *hydrodynamic limit* or the *large population asymptotics*. Essentially, this limit appears when we consider a particle system whose typical population size tends to $+\infty$. In practice, we consider a certain type of IBM with local density K (typically, K represents the average number of particles on a site in a stable phase of the system) and weight the individuals by $\frac{1}{K}$. The hydrodynamic limit of the process is thereupon the limit behaviour of the system as K tends to infinity: the fluctuations self-average and the system converges to a deterministic dynamic, governed by a PDE.

In the system presented in Section 2.1, the three steps to prove this convergence would be the following ones. We consider a finite number of sites (x_i), $i \in \{-M, ..., M\}$, and of generations (t_k), $k \in \{0, ..., L\}$.

1. Assume that the system starts with K individuals located at 0. Set $\xi_k^K(i) = \frac{n_k(i)}{K}$ and prove (using the law of large numbers) that as $K \to \infty$, we have
$$\left(\xi_k^K(i)\right)_{\substack{i=-M,...,M \\ k=1,...,L}} \Rightarrow \left(v_k(i)\right)_{\substack{i=-M,...,M \\ k=1,...,L}},$$
where (v_k) is a sequence on "densities" governed by the following deterministic dynamics:
$$\begin{cases} v_0(i) = \delta_0, \\ v_{k+1}(i) = \sum_j \mu_j \left(v_k(i-j) \wedge 1\right). \end{cases}$$
2. In the above-mentioned dynamics, let $\Delta x \to 0$. The sequence of configurations $(v_k)_{k=1,...,L}$ converges to a sequence of functions $(w_k)_{k=1,...,L}$ governed by
$$\begin{cases} w_0 = \delta_0 \\ w_{k+1}(x) = \left[\Gamma_{\Delta t} * (w_k \wedge 1)\right](x), \end{cases}$$
where $\Gamma_{\Delta t}(y) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{y^2}{2\Delta t}}$.
3. Finally, let $\Delta t \to 0$: one can show that (w_k) converges to the solution u of a FKPP-type equation (see (3.28)).

See [CM07] for another type of hydrodynamic limit, where the interactions between the particles are not replaced by a mean-field approximation.

The continuous limit u of the IBM introduced in Section 2.1 with duplication probability given by (2.19) would be solution of a PDE of the form
$$u_t = \frac{1}{2} u_{xx} + r(\varepsilon t, \varepsilon x) f(u), \tag{3.28}$$
with $f(u) = u \mathbb{1}_{u \leqslant 1}$. A first remark is that at order 1, the invasion speed of (3.28) is the same as if we replace the forcing term by $f(u) = u(1-u)$ [KPP37]. Equations of type (3.28) with a forcing term of the form $(x, u) \mapsto r(x) f(u)$, where f is a function satisfying the KPP conditions, has been broadly investigated in the PDE literature [ES89] but also with probabilistic arguments [Fre86].

The long time behaviour of this equation is studied by introducing an hyperbolic scaling
$$u^\varepsilon(t, x) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

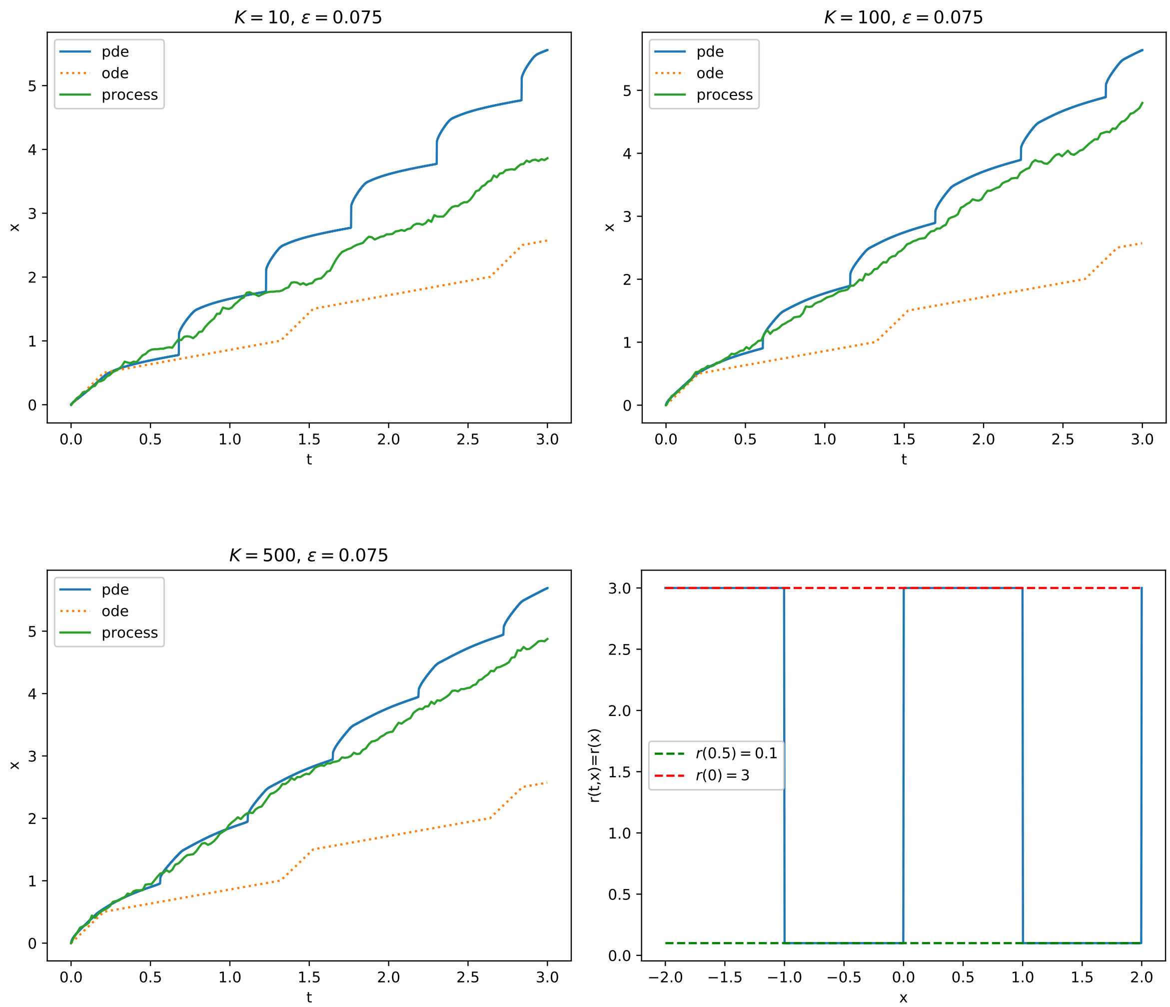

Figure I.1: Convergence of the IBM to its hydrodynamic limit. The fourth subfigure is the graph of the growth rate r: we chose a step function as in the SKT patch model (see Equations (1.13) and (1.14)). In the three first figures, the green line corresponds to the rescaled position of the rightmost particle (see (2.21)) in one simulation of the process introduced in Section 2.1. This process is run for $\varepsilon = 0.075$ and different values of K. The blue line corresponds to the position of the invasion front $x(t) = \sup\{x \in \mathbb{R} : u(t,x) > \frac{1}{K}\}$ for u solution of the limit PDE (3.28) with $f(u) = u(1-u)$ and $u(0,x) = \mathbb{1}_{x<0}$, on the same time scale as the process X^*. The orange dotted line represents the solution of the ordinary differential equation $\dot{x}(t) = \sqrt{2r(x(t))}$, $x(0) = 0$, which can be seen as the speed of the wave in the two equivalent homogeneous environments.

The idea (discussed in [Fre86]) behind this scaling is given by the convergence results in the homogeneous case (see Section 1.2). Indeed, let us consider u the solution of equation (1.5) with initial condition $u(0,x) = \mathbb{1}_{x<0}$. Since u converges to the travelling wave, we will get that for ε small enough, $u^\varepsilon(t,x) \approx \varphi_{c^*}\left(\frac{x-c^*t}{\varepsilon}\right)$. Yet, since φ_{c^*} decreases from 1 to 0, we expect that as $\varepsilon \to 0$,

$$u^\varepsilon(t,x) \to \mathbb{1}_{x<c^*t},$$

which provides the invasion speed of the solution. If we now go back to Equation (3.28) and assume that r does not depend on t, we obtain that u^ε is a solution of

$$u_t^\varepsilon = \frac{\varepsilon}{2} u_{xx}^\varepsilon + \frac{1}{\varepsilon} r(x) f(u^\varepsilon).$$

Note that the diffusion term vanishes as $\varepsilon \to 0$ and that the reaction term blows up. As a result, it was shown [ES89] that $v^\varepsilon = -\varepsilon \log(u^\varepsilon)$ converges to the viscosity solution v of an Hamilton-Jacobi equation of the form

$$\min\left(v_t + \frac{1}{2}(v_x(t,x))^2 + r(x), v(t,x) \right) = 0. \tag{3.29}$$

Again, if we denote by x^ε the position of the invasion front of the solution u^ε, for instance $x^\varepsilon(t) = \sup\left\{x \in \mathbb{R} : u^\varepsilon(t,x) \geqslant \frac{1}{2}\right\}$, this convergence result implies that for fixed $t \geqslant 0$,

$$x^\varepsilon(t) \to x^{HJ}(t) := \sup\left\{x : v(t,x) = 0\right\}.$$

This approach has been extensively employed so far to deal with different types of forcing terms or equivalently, of heterogeneous environments: periodic, random, patch models, etc. [Xin00, NR12, Nad16, Xin00, HNR11]. Note that this method hardly provides explicit propagation speeds except in very specific cases [HFR10, HNR11].

A major limit to this approach is the potential qualitative disagreement between the individual based model and its hydrodynamic limit. This is the so-called *tail problem* [Jab12]: artefacts may be generated by the deterministic equation, due to the lack of local extinction. Indeed, the solutions of FKPP-type equations exhibit exponential tails that is small populations sent to $+\infty$ by diffusion, at an *infinite speed* [HFR10] (for any compactly supported initial date u_0, the solution of (3.28) is positive for all $t > 0$). If these populations reach a favourable region before $x^\varepsilon(t)$, in which they expand fast enough, we may observe a jump in the position of the front: the tail *pulls* the invasion. The core of the problem is that these exponentially small populations, which can be meaningless in some biological way, drive the invasion. In the individual based model, there is no tail since the population is finite so that there is no ambiguity around the definition of the position of the rightmost particle.

This phenomenon can be observed in [HFR10, Fre85]. In the first article, the authors investigate the propagation speed of a population in a slowly varying environment by studying an SKT patch model (see Equation (1.13) and (1.14)) with a $L-$periodic reaction term ($\mu(\cdot)$ is replaced by $\mu(L\,\cdot)$ in (1.13)). Surprisingly, they numerically observe that the propagation speed of the system is an increasing function of L. Moreover, they remark a speeding up of the invasion front in the favourable patches, where the front can go faster than the corresponding propagation speed in the equivalent homogeneous environment. In [Fre85], the author gives a simple example of reaction term leading to a jump in x^{HJ}, namely a step function $f(x) = c_1 > 0$ if $x < 1$ and $f(x) = c_2 > 2c_1$ if $x \geqslant 1$. What we can derive from these two remarks is that in the case of patchy environments, if the ratio between the growth in the favourable and unfavourable patch is too large, one can observe a speed up of the invasion, generated by the tails. As a remark, the analysis conducted in [HFR10] explains how this phenomenon can be understood through the dispersion relation (1.7). With regard to the two above-mentioned examples, Freidlin [Fre85] claims that the Huygens principle does not hold: the propagation of the front cannot be described by a velocity field. A solution suggested from the PDE point of view is to cut the tails with a negative square root term and a threshold [Jab12, MBPS12]. This allows local extinction in the solution of the PDE: it can be positive in some regions and vanish in other zones. Another approach, advanced in [HFR10] consists in adding an *Allee effect* to the SKT patch model (see Section 3.2 for a precise definition). In both techniques, the idea is to add a small correction to the equation, to prevent meaningless populations from driving the invasion. As a consequence, the KPP conditions no longer hold.

A way to go back to the microscopic model in the case of heterogeneous environment, is to consider a noisy version of the Fisher-KPP equation. For instance [BDMM06a], the equation

$$u_t = \frac{1}{2}u_{xx} + u(1-u) + \sqrt{\frac{u(1-u)}{K}}\eta(t,x), \tag{3.30}$$

where η is a Gaussian white noise, can be seen as a microscopic model for front propagation. The deterministic part of (3.30) can be interpreted as the hydrodynamic limit of a *stochastic reaction-diffusion model* [Pan04] while the noise term depicts its microscopic fluctuations: the system is represented as a perturbation of its large population limit. As explained in [BDMM06a], the front of the noisy equation (3.30) vanishes with a faster decay than the exponential given by the deterministic FKPP equation in the region where $u \approx \frac{1}{K}$. Heuristically, one can understand this fact recalling that K is the average number of particle on a site at the equilibrium and there cannot be less than one individual on a site. Moreover, it was conjectured in [BD97, BDMM06a] that the velocity of the front in (3.30) admits a correction compared to the speed of the hydrodynamic limit, which depends on K and tends to 0 as the typical population size K tends to infinity. More precisely, they inferred that this correction is of order $\log(K)^{-2}$. Since then, this conjecture has been rigorously proved [MMQ10]. In this framework, the speed of the system converges to the speed of its hydrodynamic limit. In the light of the above, it is not always the case in heterogeneous media. A generalisation of Equation (3.30) will be further discussed in the following section.

3.2 Allee effects and pulled/pushed waves

The qualitative disagreement between the stochastic model and its hydrodynamic limit which can be observed under the KPP conditions comes from the fact that the wavefronts arising from the limit PDE (3.28) are *pulled*. The speed of pulled fronts is only governed by the value of the reaction term at low densities, which might generate jumps in the position of the front.

Recall from Section 1.2 that in the homogeneous settings, the KPP conditions ensure that the invasion speed is given by the linearised equation at zero. A wavefront solution φ_c traveling at speed c exists for all $c \geqslant c^* = \sqrt{2f'(0)}$ and any solution of (1.4) arising from a Heaviside initial data converges in shape and speed to the front φ_{c^*} [KPP37] (but it does not converge to a specific traveling wave solution φ_c, see (1.17)). A. N. Stokes [Sto76] interpreted this phenomenon as follows:

> "the speed of the wave is determined by the fecundity of their pioneers".

Two questions arise from this statement:

1. Is this assumption legitimate for all biological populations ?
2. What is the invasion speed of the population if the KPP conditions do not hold ?

To partially answer the second question, we first mention the case of *bistable* reaction terms f satisfying

- $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$,
- $\int_0^1 f(u)du > 0$,
- f has exactly 3 zeros in $[0, 1]$, namely 0, 1 and some $\alpha \in (0, 1)$.

Under these assumptions, Equation (1.6) has a wave solution φ_c traveling to the right for a unique value of $c > 0$ that not only depends on $f'(0)$, but on the functional form of f (this dependence on f is typically *integral-like* [Sto76]). We denote this speed by c_b. Moreover, under suitable assumptions on the initial condition (that ensure successful invasion), the solution of the PDE (1.5) arising from a compactly supported initial data converges uniformly to a pair of diverging fronts, traveling at speed c_b [FM77]. Consequently, the speed c_b is also the invasion speed of the solutions. In the latter case, Stokes calls the wavefront *pushed*.

More generally, the difference between pulled and pushed waves has been discussed in the case of propagation into unstable states [Van03, Rot81, Sto76]. Here, we only mention the case of a *monostable* reaction term, that is, satisfying the following conditions:

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad \text{and} \quad f(u) > 0, \ \forall u \in (0, 1). \tag{3.31}$$

The second condition ensures that 0 is a saddle point for the linearised equation so that it is an *unstable state* for the PDE: any small perturbation (*i.e.* introduction of a small population) results in a successful invasion. The *linear spreading speed* c^* of this nonlinear PDE is defined from its linearisation at zero:

$$u_t = \frac{1}{2}u_{xx} + f'(0)u. \tag{3.32}$$

If we denote by x^* the position of the invasion front (see Equation (1.11)) for a solution u of (3.32) arising from a compactly supported initial data, the linear spreading speed of (1.5) is defined as

$$c^* = \lim_{t\to\infty} \frac{x^*(t)}{t}. \tag{3.33}$$

Note that this definition does not depend on the threshold chosen for x^* since the equation is linear. Besides, the assumption $f'(0) > 0$ ensures that $c^* > 0$. It can be shown (*e.g.* using the Fourier transform [Van03] or comparison principles [AW78]) that in the monostable case

$$c^* = \sqrt{2f'(0)}.$$

On the other hand, it is a know fact [HR75] that under assumption (3.31), there exists a minimal speed $c_{\min}$ and a one-parameter family of traveling fronts $(\varphi_c)_{c \geqslant c_{\min}}$, solutions of Equation (1.6). The speed $c_{\min}$ plays a crucial role in the dynamics of the solutions of (1.5) arising from compactly supported data. Indeed, these solutions converge to a pair of diverging fronts at speed $c_{\min}$ (at least in shape and speed [Sto77]). Consequently, the invasion speed (see Equations (1.9) and (1.10)) is given by the minimal speed $c_{\min}$.

Based on these observations, pulled and pushed waves are defined as follows. A nonlinear wavefront arising from equations (1.5) and (3.31) is said to be "pulled" if its asymptotic invasion speed is equal to c^* and "pushed" if it is greater than c^*. Roughly speaking, this means that while the dynamics of pulled waves is only driven by their leading edge, the dynamics of pushed waves depends on the nonlinearities of the reaction term. For bistable reaction terms f, we have $f'(0) < 0$ so that 0 is a linearly stable state. Therefore, small perturbations are bound to die out so that the invasion is led by the nonlinearities. That is why bistable fronts are classified as pushed fronts. Another remark formulated in [Van03] is that nonlinear invasions can not be slower than c^* in the monostable case (3.31) since small populations go forward with speed c^*. Thus, the dynamics can only be accelerated by the advance of the bulk. Finally, in the light of the above, note that pushed waves are observed when $c^* \neq c_{\min}$.

Given these definitions, it is now clear that KPP waves are pulled and bistable fronts are pushed. An interesting question is then to determine when the transition between these two regimes appears, as well as the microscopic interpretation of this distinction. An enlightening example can be found in [BHK18]: the authors study an example of solvable model in which we can observe this transition from pulled to pushed fronts. They consider a reaction term of the form

$$f(u) = ru(1-u)(1+Bu) \tag{3.34}$$

for some parameter $B \geqslant 0$ (which is sometimes called the Fisher's Equation). For $B = 0$, the KPP conditions hold so that $c_{\min} = c^* = \sqrt{2r}$. Besides, the authors points out the fact that

$$\varphi_c(x) = \frac{1}{1+e^{\sqrt{rB}x}}, \tag{3.35}$$

$$c = \frac{1}{2}\sqrt{rB}\left(1+\frac{2}{B}\right), \tag{3.36}$$

is an explicit solution of Equation (1.6). Notice that necessarily, we have $c \geqslant c^*$ (one can easily prove this fact by computing $c - c^*$). At first sight, we do not know if this speed c corresponds to the minimal speed $c_{\min}$. Nevertheless, they claim that the transition between pulled and pushed regimes occurs when $c = c^*$ or equivalently when $B = 2$. This switch from one to another can be justified by analysing the decay of the function φ_c and proving that $c_{\min}$ is an increasing function of B. Indeed, we know [HR75] that the asymptotic behaviour of a traveling front $\varphi_{c'}$ as $z \to +\infty$ is given by

$$\varphi_{c'}(z) \sim Ce^{-\lambda(c')z}, \quad \text{with} \quad \lambda(c') = \begin{cases} c' + \sqrt{(c')^2 - 2f'(0)}, & \text{if } c' = c_{\min} \\ c' - \sqrt{(c')^2 - 2f'(0)}, & \text{if } c' > c_{\min}. \end{cases}$$

Equivalently, if we consider the two roots $0 > \lambda_1 > \lambda_2$ of $\frac{1}{2}X^2 + c'X + f'(0)$, we see that $\lambda(c') = -\lambda_2$ for $c' = c_{\min}$ and $\lambda(c') = -\lambda_1$ for $c' > c_{\min}$. A simple computation gives that for $B \geqslant 2$, $\lambda_2 = \sqrt{rB}$ so that $c' = c_{\min}$ and the traveling front (3.35) is the front associated to $c_{\min}$. Since $c_{\min} > c^*$ for $B > 2$ (because $c = c^*$ iff $B = 2$), the wavefronts are pushed for $B > 2$.

Then, note that f is increasing with respect to B. Yet, $c_{\min}$ is given by the following variational formulation [HR75]

$$c_{\min} = \inf \left\{ L(\rho) : \rho \in C^1([0,1]),\ \rho(0) = 0,\ \rho'(0) > 0 \right\}$$

with

$$L(\rho) = \sup_{u \in (0,1)} \rho'(u) + \frac{f(u)}{\rho(u)},$$

so that $c_{\min}$ is also increasing with respect to B. Therefore, $c_{\min} = \sqrt{2r} = c^*$ and the wavefronts are pulled for all $B \in [0, 2]$. This transition between pulled and pushed fronts is illustrated by Figure I.2.

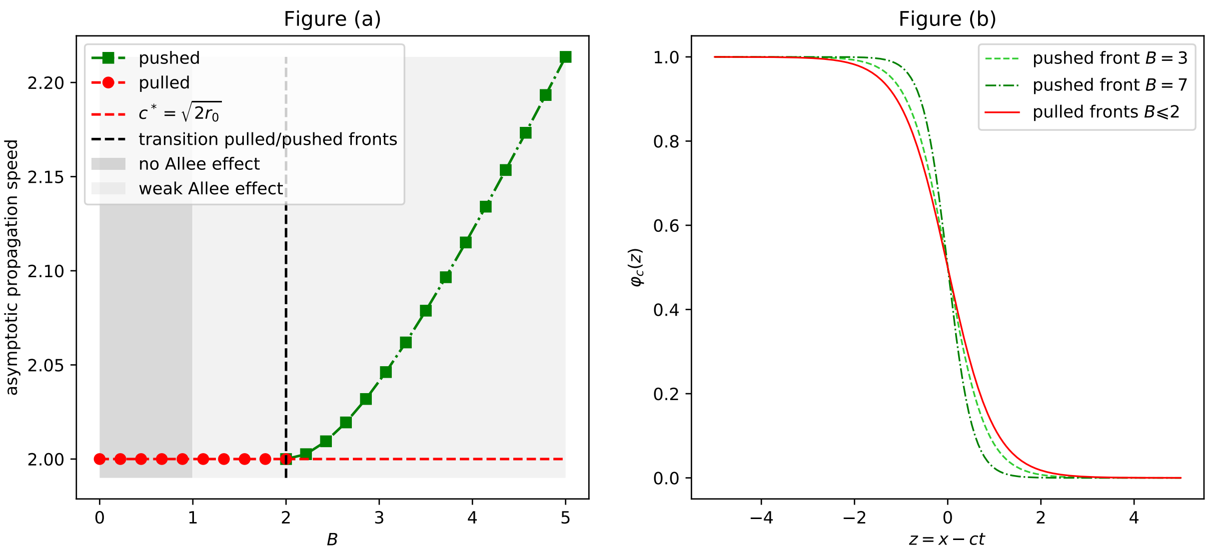

Figure I.2: Speed (a) and shape (b) of the fronts for a reaction term of the form (3.34) and different values of B for $r = 2$. Figure (a) The red dashed line corresponds to the linear spreading speed c^* from Equation (3.33). In the pulled regime ($B \leqslant 2$), it coincides with the asymptotic invasion speed of the solutions of the nonlinear equation (1.5). In the pushed regime ($B > 2$), this invasion speed is larger than the linear speed. It is given by Equation (3.36), which corresponds to the green square markers in the figure. This figure corresponds to Fig. 1.(A) from [BHK18]. Figure (b) Shape of the fronts in the corresponding co-moving frames at speed $c_{\min}$. For $B \leqslant 2$, we have $c_{\min} = c^*$ and the solutions of the nonlinear PDE converge in shape and speed to the front φ_{c^*} (red solid line). For $B > 2$, the larger B, the steeper the limit shape of the front. The green dotted/dashed lines correspond to the graph of φ_c for $c = c_{\min}$, for different values of B. In this case, one can prove that the solution of the nonlinear PDE converges uniformly to the front $\varphi_{c_{\min}}$

The aim of Chapter III is to study the internal mechanisms behind this changeover via a particle system. The reason why we take such an approach are the results of the simulations obtained in [BHK18] on a noisy version of Equation (3.34). Before we describe their observations, we give a first interpretation of the two regimes based on the deterministic equation (3.34). Indeed, the forcing term (3.34) might be interpreted as follows. While the first part of the equation is similar to the logistic growth from the FKPP equation, giving the saturation rule of the population, the second part can be seen as a cooperative term and B as the strength of this cooperation. Unlike the KPP conditions, this presumes that the per capita reproduction rate $r(u) = u^{-1}f(u)$ decreases at low densities and has a peak for some positive value of u. Consequently, the region of maximal growth in the front is shifted from the tip into the bulk of the wave when B increases (see Figure I.3). This is known as the Allee effect.

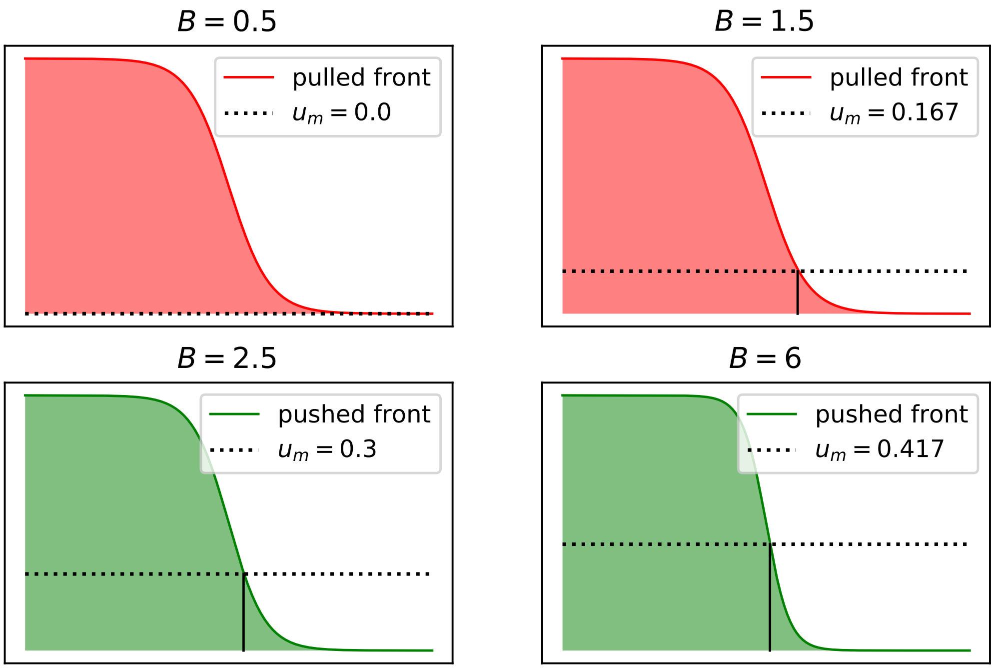

Figure I.3: Position of the region of maximal growth in the limit wavefront $\varphi_{c_{\min}}$ for different values of B (see Equation (3.34)). When the cooperativity is low ($B \leqslant 2$), the wavefront is pulled. For $B \leqslant 1$, the KPP conditions hold so that the growth is maximal at the leading edge. For $B \geqslant 1$, the region of maximal growth is shifted to the left when the strengh of cooperation increases. For $B > 2$, the growth in the bulk overtake the growth at the leading edge: the speed of the front is larger and the shape of the wavefront is steeper than in the pulled regime. The doted line corresponds to the value u_m of $u \in [0, 1]$ such that $uf(u)^{-1}$ is maximal.

There may be various biological interpretations for this mechanism: interspecific competition, reduced fitness due to consanguinity, difficulty to find a mate at low densities, etc. In the case of monostable reaction terms (3.31), we always have $r(0) > 0$. We say that the population undergo no Allee effect if $r(u) \leqslant r(0)$ for all $u \in (0, 1)$ and weak Allee effect otherwise. We speak of strong Allee effects when $r(0) < 0$ so that it is often modeled by bistable reaction diffusion equations. In Equation (3.34), there is no Allee effect for $B \leqslant 1$ (which coincides with the KPP conditions) and weak Allee effect for $B > 1$, so that the transition (B=2) between pulled and pushed fronts occurs in the region of weak Allee effect (see Figure I.2). From this point of view, pushed waves are called “pushed” because their are not pulled by their leading edge, but pushed by the growth in the bulk, that quickly overtakes the growth in the leading edge thanks to cooperation. See Figure I.4 for a different parametrisation of the forcing term (3.34) for which we can observe bistable waves and strong Allee effects.

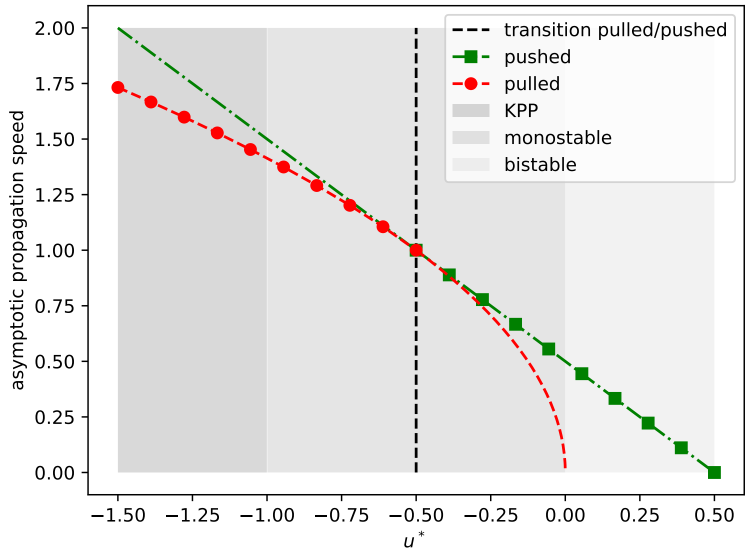

Figure I.4: Asymptotic spreading speed of the solutions of Equation (1.5) with a forcing term of the form $f(u) = r_0 u(1-u)(u-u^*)$. This forcing term can be obtained from (3.34) by setting $B = -\frac{1}{u^*}$ and $r_0 = -\frac{r}{B}$. This representation allows to plot the invasion speed of bistable reaction terms for $u \geqslant 0$, which corresponds to $B \to +\infty$ in (3.34). The main advantage of the first representation (with B) is that the growth rate per capita $u^{-1} f(u)$ at low densities is the same for all B. That is the reason why we can not observe bistable waves in (3.34). See [BHK18, SI] for a discussion on the two different parametrisations.

While the distinction between pushed and pulled fronts can be made thanks to the asymptotic invasion speed of the fronts, this is not the only consequence of cooperation on the population. Indeed, Allee effects also have a huge impact on the genetic diversity. A first way to see it [RGHK12, BHK18], is to track a neutral marker in the population (that does not interfere in the dynamics) to trace the location of the individuals that gave birth to a significant part of the population. In pulled fronts, they are positioned at the leading edge: only a few particles give birth to a significant part of the population so that a large fraction of the individuals shares the same ancestor. This leads to a drastic genetic loss. In sharp contrast, the ancestors are located in the interior of the front for pushed waves, which brings a wider genetic diversity.

Another standpoint developed in [BHK18] is the analysis of the transition between the two regimes in an "individual-based model". Actually, they consider the noisy equation

$$u_t = \frac{1}{2} u_{xx} + ru\,(1-u)\,(1+Bu) + \frac{1}{\sqrt{N}} \Gamma\,(u)\;\eta(t,x). \tag{3.37}$$

Here, N stands for the local number of particles at equilibrium, Γ is the strength of the demographic fluctuations and η a Gaussian white noise. As explained in Section 3.1, the noisy equation can be seen as a perturbation of the hydrodynamic limit of the individual based model, including the demographic fluctuations generated by the finite nature of the system. The effect of noise on the density n is the following: the speed and the shape are now fluctuating. Depending on the type of waves generated by the deterministic equation (without noise), the effect of these fluctuations on the system of the front will be drastically different. For instance, pulled waves are very sensitive to fluctuations. Indeed, recall from Section 3.1 that the difference between the velocity c of the deterministic front and the velocity c_N of the front in the noisy equation (3.37) is of order $\log(N)^{-2}$ [BDMM06a, MMQ10]:

$$c - c_N \sim \frac{C}{\log(N)^2}, \tag{3.38}$$

for some $C > 0$. This correction, that is much greater than expected $(1/\sqrt{N})$, underscores the large fluctuations in the pulled regime. In opposition, we expect pushed waves to be less sensitive.

This observation can be explained by considering the genealogies of such processes. In [BDMM06a], the authors describe the mechanisms driving the fluctuations of the noisy fronts for $B = 0$ by considering a particle system rather than a continuous stochastic model. Since the fluctuations develop at the leading edge, they do not need to introduce a regulation mechanism to infer the correction on the speed of the noisy wavefront. The addition of a saturation rule in this model then allows to describe the genealogies of the particles leaving close to the invasion front. In [BDMM06a], the authors suggest to analyse a branching random walk with a fixed population size N: at each generation, the individuals give birth to k children, scattered around the parental location. Only the N rightmost individuals survive to the next generation. This forms a cloud of particles that does not spread and can be described by a front [BDMM07]. It was shown that the same correction (3.38) holds for the speed of this cloud of particles [BG10] (for $k = 2$ and under suitable assumptions on the displacement law of the offspring). In this model, it was conjectured [BDMM06b, BDMM07] that the number of generations needed to reach the most recent common ancestor of two particles sampled in the population at a given generation is of order $\log(N)^3$. This implies that the genealogy of the particles is described by a Bolthausen-Sznitman coalescent, a coalescent with multiple mergers: the particles sent ahead of the front give birth to a large number of descendants. These offspring survive and expend exponentially fast since the effect of competition is negligible at the leading edge. As a consequence, a significant fraction of the population is replaced by these descendants. This result on the genealogies of the system has been proved under slightly different settings in [BBS13] (see Section 2.2). In opposition, for large values of B, the time scale at which we observe the fluctuations is of order N [BHK18], which suggest the presence of Kingman's coalescent with binary mergers (see [EP20] for a proof of this fact, in the framework of population genetics and in the case of bistable waves - see Figure I.4). Therefore, the population behaves like a neutral population and the system is far less sensitive to fluctuations.

However, the numerical experiments conducted in [BHK18] suggest the existence of a class of pushed fronts with large fluctuations. These fluctuations are observed in the position of the front as well as in the genetic diversity of the population. Besides, the numerical experiments conducted in [BHK18] suggests that the genealogies can be observed on a time scale of order N^{γ} for some $\gamma \in (0, 1)$ for intermediate values of B (that is close to 2). This leads to the distinction of two different classes in the pushed regime: weakly pushed (or semi pushed) waves and fully pushed waves. This semi pushed regime will be the object of Chapter III.

We end this section with two other remarks on the stability of pushed fronts. The first one concerns the asymptotic stability of the solutions in the pushed regime for monostable reaction terms. Indeed, Rothe [Rot81] showed the uniform convergence of solutions arising from compactly supported data to pairs of diverging fronts at speed $c_{\min}$ in the pushed regime. This is not the case in the pulled regime because of the logarithmic correction of the invasion speed (see Equation (1.17)). The second one can be connected to the tail problem. Indeed, it was observed that adding a strong Allee effect to an heterogeneous FKPP equation has the effect of cutting the tails [HFR10]. Similarly, it was shown by Freidlin [Fre85] that in the bistable case, the Huygens principle holds and that there is no jump in the position of the front.

4 Main results

4.1 Propagation speed of a population colonising a slowly varying environment

This section regards the result proved in Chapter II. It is based on the preprint [MRT21].

In this chapter, we are interested in the long time behaviour of the particle system described in Section 2.1 with a slowly varying growth rate of the form

$$(t,x) \mapsto r(\varepsilon t, \varepsilon x),$$

for some small parameter $\varepsilon > 0$. As mentioned in Section 3.1, considering the viscosity solutions of the hydrodynamic limit of this system may be unsatisfactory from a biological standpoint. Recall from Section 3.1 that some corrections on the PDE were suggested to tackle this issue. However, for biological concerns, we suggest to invert the two limits, that is to let first $\varepsilon \to 0$ in the microscopic system, then consider large population sizes (*i.e.* $K \to \infty$). In this framework, we are interested in the asymptotic behaviour of the position of the rightmost particle X_k^* (see Equation (2.20)) as $\varepsilon \to 0$. Basically, we expect that for all $T > 0$, we have

$$(\varepsilon X_k^*,\ 0 \leqslant k \leqslant T/\varepsilon) \longrightarrow (x(t),\ t \in [0,T]) . \tag{4.39}$$

when we let first $\varepsilon \to 0$, then $K \to \infty$.

The goal of Chapter II is to show that under appropriate assumptions on the reproduction and migration laws, the function x from Equation (4.39) satisfies an ODE, depending on the growth rate r and the dispersion law μ of the system. To simplify and clarify the proofs, we assume that μ is a discretised Gaussian distribution. The resulting ODE can therefore be expressed only in term of r. Moreover, we essentially assume that r is a smooth positive function and that the system is monotone with respect to r and K. This latter assumption allows us to locally compare the system with some branching random walks. This strategy can be thought as a *comparison principle*, widely used in PDE theory (see [AW78] for homogeneous FKPP equations, [BHN08] for recent developments in heterogeneous settings).

Recall from Section 2.1, that Δt refers to the time step between two generations and Δx denotes the space step between two sites. Roughly speaking, in Chapter II, we prove that for any $T > 0$ and $\delta > 0$, if Δt and Δx are small enough and K is sufficiently large (but fixed)

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\max_{k \in [\![0, \lfloor \frac{T}{\varepsilon \Delta t} \rfloor]\!]} |\varepsilon X_k^* - x(k\varepsilon \Delta t)| \leqslant \delta \right) = 1,$$

where $t \mapsto x(t)$ is the solution of the Cauchy problem

$$\begin{cases} x'(t) = \sqrt{2r(t, x(t))} \\ x(0) = 0. \end{cases} \tag{$\mathcal{C}$}$$

The proof of this result is divided into two main parts. We first establish an upper bound on the invasion speed of the interacting particle system (Section 4). Second, we establish a lower bound (Section 5), which concludes the proof. Both steps rely on a coupling lemma, stated in Section 3. The general assumptions under which each bound is proved as well as precise statements of the results are given in Section 2.

The main result of this asymptotic method is that the long time behaviour of the system satisfies the Huygens principle, without adding any correction. Moreover, the proof of this result suggests that it holds as long as $\varepsilon \log(K) \to 0$. On the other hand, we know that the large population asymptotics gives a proper estimate of the invasion speed when $\varepsilon \log(K) \to +\infty$. Therefore, this provides a good understanding of the scales at stake in each situation. If we refer to the datas collected in [SK97] on biological species, we can infer which approach will best fit each situation. Typically, our approach seems relevant when the population size is small compared to the scale of heterogeneity of the environment.

These two different limits are illustrated in Figures I.1 and I.5: in the first one, we fixed ε and observe the convergence to the hydrodynamic limit as K increases. In the second one, we fix K and remark that the rescaled position of the rightmost particle converges to the solution of the Cauchy problem $(\mathcal{C})$. In both cases, we simulate the simplified version of the system introduced in Section 2.1 for a space dependent growth rate r that is a $1-$periodic step function. For instance, this could depict the invasion of a species colonising a linear habitat, constituted of an alternation of favourable and unfavourable patches as in the SKT patch model [SKT86].

4.2 Particle systems and semi-pushed fronts

In Chapter III, we investigate the internal dynamic of a population undergoing moderate Allee effect. To this extent, we study the branching Brownian motion with absorption and accelerated branching rate in the interval $[0, 1]$ in Section 2.2. As above-mentioned, this system constitutes a toy model describing the dynamics of the particles ahead of the front and the barrier at zero combined with the drift $-\mu$ is seen as a moving frame. Thereupon, the effect of cooperativity on the particle system will be investigated by considering a space dependent branching rate $r(x)$, reaching its maximum where we expect to find an intermediate density of particles (that is between the bulk and the leading edge). For simplicity, we consider a step function of the following form:

$$r(x) = \frac{1}{2} + \frac{\rho - 1}{2} \mathbb{1}_{x \in [0,1]}. \tag{4.40}$$

This way, the intensity of the Allee effect can be easily strengthened by increasing ρ.

As explained in Section 3.2, this work was mainly inspired by the conjectures and the results in [BHK18, BDMM06a, BDMM06b, BBS13]. In [BDMM06a], the authors explain how to infer the speed correction of the front arising from (3.30) thanks to a particle system, in the pulled

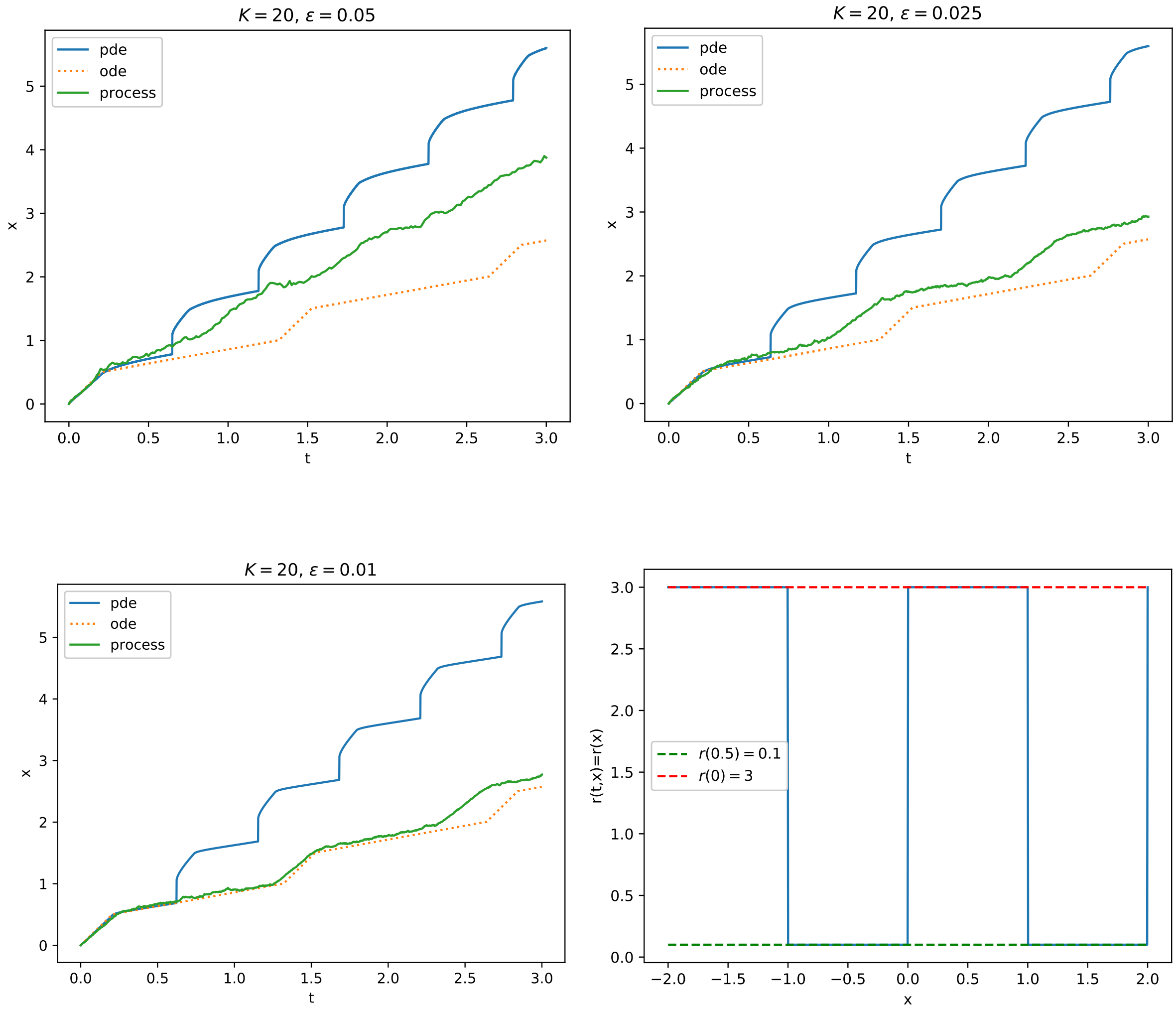

Figure I.5: Convergence of the position of the rightmost particle in the IBM to the solution of the ODE $(\mathcal{C})$. The green line corresponds to the rescaled position of the rightmost particle (see (2.21)) in one simulation of the process. This process is run for $K = 20$ and different values of ε. The orange dotted line corresponds to the graph of the solution of the Cauchy problem $(\mathcal{C})$ on a time scale of order 1 (and not ε^{-1} as in the process). The last subfigure is the graph of the growth rate r.

regime. The analysis carried out in [BDMM06b] is dedicated to the genealogies of a population undergoing selection. This selection is modeled by a constant population size: the saturation rule consists in keeping only the N rightmost individuals in the system at each time step. In our work, this cloud of particles will be seen as an IBM for front propagation. Under these assumptions, it was conjectured that the genealogy of the process is described by a Bolthausen-Snitzman coalescent in the pulled regime. In [BBS13] the authors prove this conjecture under slightly different settings: they investigate the genealogies of a branching Brownian motion with absorption. The drift μ_N (see Equation (1.29)) is chosen so that (3.38) holds (replacing c by $\mu = \sqrt{2}$). Instead of keeping a constant population size, they chose to kill the particles at a barrier to keep it *approximately* constant. Therefore, they can assume that particles evolve independently during their life span. Moreover, they also considered a continuous time model to simplify the analysis. Under this assumption, they show that the genealogy of this process is described on a time scale of order $\log(N)^3$ by a Bolthausen-Snitzman coalescent. Based on the observations made in [BHK18] (see Section 3.2), we study a BBM with absorption at 0 and branching rate r given by (4.40). The goal of Chapter III is to examine the effect of the parameter ρ on the genealogy of the process and identify the three regimes (pulled, semi-pushed and fully pushed) observed numerically in [BHK18].

In our system, the quantity of interest is the number N_t of particles alive at time t. As in [BBS13], we set μ in such a way that this number stays roughly constant. In our case, the population will eventually dye out, but the fluctuations of the process (N_t) during its life time will be extremely instructive and will provide some information on the genealogy of the particles as well as on the mechanisms driving the invasion.

First, we need to fix μ: it is given by the spectral analysis of the operator

$$Tv = v'' + r(x)v.$$

More precisely, it is related to the principal generalised eigenvalue of the operator T. For $\rho < \rho_1 = 1 + \frac{\pi^2}{4}$, we claim that $\mu = 1$. For $\rho > \rho_1$, it is the unique solution of

$$-\frac{\tan(\sqrt{\rho - \mu^2})}{\sqrt{\rho - \mu^2}} = \frac{1}{\mu^2 - 1}, \tag{4.41}$$

such that $\rho - \mu^2 \in \left[\frac{\pi}{2}, \pi\right]$. In this case, we have $\mu > 1$. Therefore, the critical value ρ_1 delineates the transition between pulled and pushed regimes in the particle system. The drift μ is plotted and compared with the speed of the deterministic limit waves from [BHK18] in Figure I.6.

As in the numerical experiments conducted in [BHK18], the semi pushed regime will emerge in the study of the fluctuations of the system for $\rho > \rho_1$. To observe this second transition, we assume that $\rho > \rho_1$ and set

$$\alpha = \frac{\mu + \sqrt{\mu^2 - 1}}{\mu - \sqrt{\mu^2 - 1}} > 1. \tag{4.42}$$

Suppose that the system starts with N particles at 1 and set $\bar{N}_t = \frac{N_t}{N}$. We claim that we can distinguish the weakly pushed and fully pushed regime thanks to the following observation:

1. If $\alpha \in (1, 2)$, $(\bar{N}_{N^{\alpha-1}t})$ converges in law to an $\alpha-$stable CSBP as $N \to \infty$,
2. If $\alpha > 2$, $(\bar{N}_{Nt})$ converges in law to a Feller diffusion as $N \to \infty$.

The transition at $\alpha = 2$ can also be seen in terms of ρ. Indeed, we have

$$\alpha > 2 \quad \Leftrightarrow \quad \mu_c > \frac{3}{4}\sqrt{2}, \tag{4.43}$$

which provides a second critical value of $\rho > \rho_1$ thanks to Equation (4.41). We denote by ρ_2 this second critical value and we claim that

$$\alpha > 2 \quad \Leftrightarrow \quad \rho > \rho_2.$$

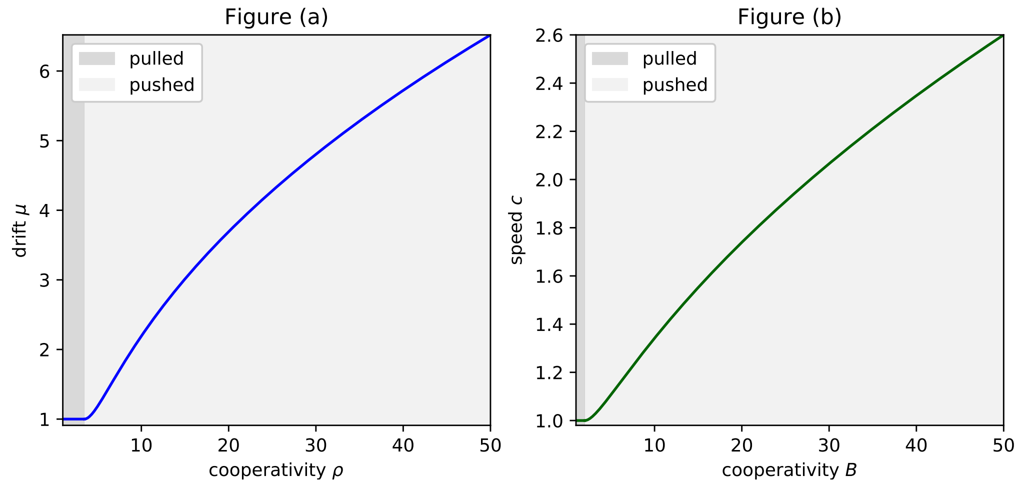

Figure I.6: The expansion velocity as a function of cooperativity. Figure (a): in the particle system. Graph of the drift μ as a function of ρ. For $\rho \in \left[\frac{1}{2}, \rho_1\right]$, $\mu = 1$, which corresponds to the pulled regime in the particle system. For $\rho > \rho_1$, the drift μ is given by Equation (4.41). In this case, $\mu > 1$, which corresponds to the pushed regime. Figure (b): in the PDE (1.5) with reaction term (3.34) for $r = \frac{1}{2}$. Graph of the speed $c_{\min}$ as a function of B. For $B \leqslant 2$ the invasion is pulled and its speed is given by the linear spreading speed *i.e.* $c_{\min} = c^* = \sqrt{2r} = 1$. For $B > 2$, it is pushed and the invasion speed is given by Equation (3.36).

Hence, the semi pushed regime occurs in the particle system when the strength of the cooperativity ρ takes values in the interval (ρ_1, ρ_2). This is illustrated by Figures I.7 and I.8.

The picture is then the following:

- If $\rho < \rho_1$, the Allee effect is not strong enough to let the growth in the bulk overtake the growth at the leading edge. This corresponds to the pulled regime. As in the deterministic framework, the speed of the wavefront μ is constant, equal to the linear spreading speed. An analysis of the fluctuations, similar to the ones carried in [BBS13, MS20], would prove that the genealogy of the population is described by a Bolthausen-Sznitman coalescent. Therefore, most particles descend from the particles who gave birth at the leading edge. In that sense, **the invasion is driven by the tip of the front**.

- If $\rho > \rho_2$, we claim that the fluctuations in the process occur on a larger time scale, of order N. This suggests that the genealogy of the process is given by a Kingman's coalescent. As expected, the particles evolve as a neutral population, because **the wave is pushed by the growth in the bulk**. The speed of the wave μ is an increasing function of ρ (as $c_{\min}$ is an increasing function of B for a forcing term of type (3.34)).

- If $\rho \in (\rho_1, \rho_2)$, the two dynamics coexist. Indeed, the wave is faster than the linear spreading speed ($\mu > 1$) so that the invasion is accelerated by the dynamic of the bulk. Nevertheless, the convergence of the rescaled process (N_t) to the $\alpha-$stable CSBP implies that the fluctuations occur at a shorter time scale than in the fully pushed regime. Besides, the genealogy associated to the $\alpha-$stable CSBP being given by a Beta$(2-\alpha, \alpha)$ coalescent,

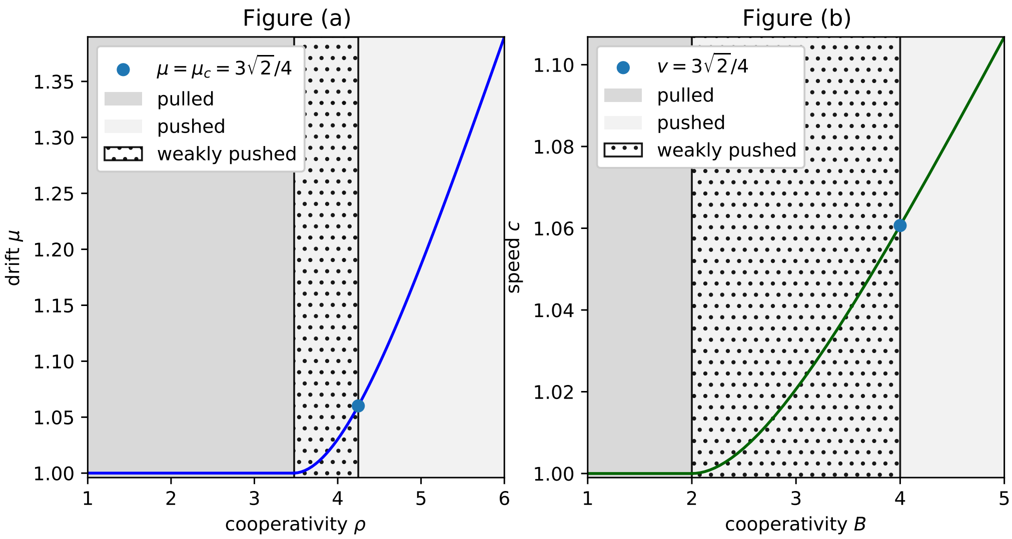

Figure I.7: The expansion velocity as a function of cooperativity. Figure (a): in the particle system. The weakly pushed regime is observed when $\mu \in (1, \mu_c)$, for μ_c defined in (4.43). Figure (b): in the PDE (1.5) with reaction term (3.34) for $r = \frac{1}{2}$. Graph of the speed $c_{\min}$ as a function of B.

we deduce that the invasion is still driven by the particles sent to the leading edge of the front.

This statement and the definitions of μ_c and α are consistent with the observations made in [BHK18, BHK20]. Indeed, the genealogy associated to the α-stable CSBP and the Feller diffusion are exactly the ones suggested for the semi and fully pushed regimes in [BHK20]. Besides, if we set $r = \frac{1}{2}$ in (3.37), the time scale $N^{\alpha-1}$ over which we observe the fluctuations in our process is the same as in [BHK18] (their definition of α is nothing else than $1-\alpha$ in our notations). In this case, the critical value of the speed c for which $\alpha = 1$ in [BHK18] is equal to μ_c from (1.14).

Chapter III is aimed at proving the convergence of the rescaled process N to the α-stable CSBP when $\rho \in (\rho_1, \rho_2)$. The proof of this fact is divided in two main parts. First, we estimate the first and second moments of several quantities in a process with an additional barrier (Sections 2, 3 and 4). Second, we gather these estimates and prove the convergence of the process to the CSBP by establishing the convergence of its Laplace transform (Sections 5, 6 and 7).

4.3 Open questions and conjectures

In this last section, we indicate the two main questions raised by the results mentioned in Sections 4.1 and 4.2. We then conclude with several other directions for future research.

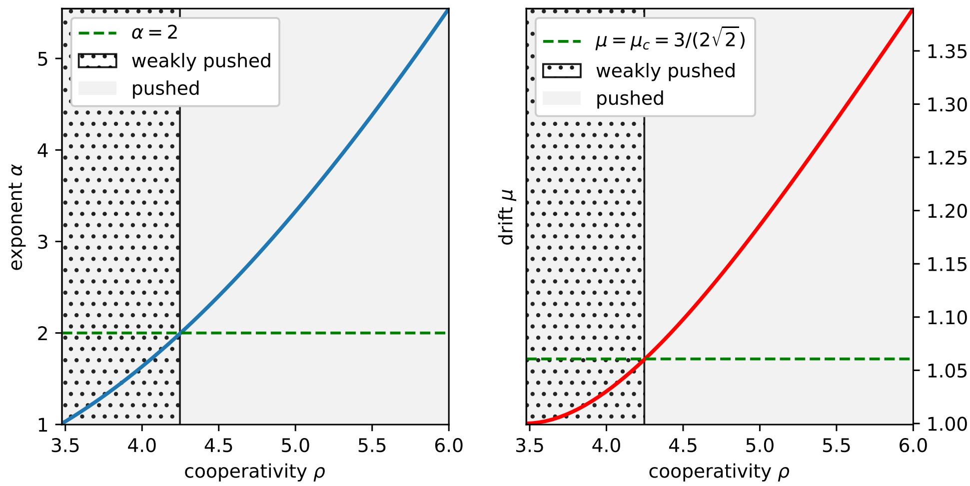

Figure I.8: The pushed regime in the particle system. Figure (a): Graph of α from Equation (4.42) as a function of ρ. The weakly pushed regime corresponds to the values of ρ such that $\alpha \in (1, 2)$. Figure (a): Graph of μ from Equation (4.42) as a function of ρ.

The interpolation regime between the two scales of heterogeneity

In order to determine which of the analysis of the individual based model from Section 2.1 or of its hydrodynamic limit (3.28) is better suited to a given biological population, one might consider our microscopic model in the limit when K and $1/\varepsilon$ go to infinity together. In chapter II, we deal with the case $\varepsilon \log(K) \to 0$ and prove the convergence to the ODE. This condition appears in the proof of the upper bound on the invasion speed of the particle system. Thanks to a first moment method, we obtain an upper bound on the probability that the rightmost particle moves faster than the solution of the ODE of the form

$$K \exp\left(-\frac{C}{\varepsilon}\right),$$

for some $C > 0$. Roughly speaking, this is the result of a union bound. Essentially, it corresponds to the probability that a descendent of a single particle goes too far to the right (which is given by the theory of large deviations), multiplied by the typical number of particles on a site. On the other hand, the analysis of the viscosity solutions given by the hydrodynamic limit described in Section 3.1 corresponds to the case $\varepsilon \log(K) \to +\infty$.

We conjecture that it is possible to interpolate between the two double limits in K and ε, when K and $1/\varepsilon$ go to infinity in such a way that $\log K$ is of the same order as $1/\varepsilon$. This relation is indeed suggested by the hyperbolic scaling (1.5). Precisely, setting $\log K = \kappa/\varepsilon$ for a constant $\kappa > 0$, we believe that the renormalised log-density is described in the limit by a solution v^κ to a variational inequality similar to the one satisfied by the solution v of (3.29) [Fre86, ES89], but with an additional constraint imposing that $v^\kappa(t, x) \geq \kappa$ implies $v^\kappa(t, x) = +\infty$, similarly to

[MBPS12]. The variational representation of the function v^κ would then be

$$v^\kappa(t,x) = \sup_\tau \inf_z \Big\{ \int_0^{t\wedge\tau(z)} \frac{z'(s)^2}{2} - r(t-s,z(s))\,ds \mid z(0)=x,\, z(t)=0,$$
$$\forall u \le \tau(z) : \int_u^{t\wedge\tau(z)} \frac{z'(s)^2}{2} - r(t-s,z(s))\,ds < \kappa \Big\}.$$

The fully pushed regime

In Chapter III, we study the long time behaviour of the particle system from Section 2.2 in the semi pushed regime $\rho \in (\rho_1, \rho_2)$. The logical follow-up question is to investigate the fully pushed regime $\rho > \rho_2$. As explained in Section 3.2, in the fully pushed regime, we expect the genealogy of the system to evolve on a time scale of order N. In Section 4.2, we claim that if we start with N particles at $x=1$, the rescaled number of particles in the process $\bar{N}_t = N_t/N$ at time t satisfies

$$\bar{N}_{Nt} \Rightarrow \Xi(t), \quad \text{as} \quad N \to \infty,$$

where Ξ is a Feller diffusion starting from 1.

To prove this result two methods can be employed:

- First, one can prove, as in Chapter III, that the Laplace transform of the process converges to the one of a Feller diffusion. Contrary to what is observed in [BBS13] and in Chapter III, the invasion is no longer driven by the particles far to the right of the front but by the ones living in the bulk. That is the reason why the proof of this point will mainly rely on first and second moments estimates on the branching Brownian motion in an interval.
- Second, one can prove that the limit of $(\bar{N}_{Nt})_{t\geqslant 0}$ as $N \to \infty$ is solution of a certain martingale problem whose unique solution is given by a Feller diffusion [Ber09, p.108]. Again, this proof relies on moments estimates.

An overview of the proof described in the first point is given at the end of Chapter III.

Other questions

In this section, we point out additional questions arising from the analyses conducted in Chapter II and III.

In Chapter II, we establish the convergence of the rightmost particle in the system from Section 2.1 for a particular reproduction law, ensuring that the population does not go extinct. First, we point out the fact that this assumption is not necessary to establish the upper bound on the invasion speed. Therefore, this upper bound is still valid in some situations where the population does not survive with probability one. However, one can also be interested in the propagation speed of our system under more general assumptions on the reproduction law. In this case, the invasion can be slower than the ODE.

In Chapter III, we investigated the fluctuations of a *dyadic* branching Brownian motion with absorption. As in [MS20], one could then investigate the behaviour of such process under more general assumptions on the branching law. For instance, one can consider that at each branching event, a particle gives birth to a random number of children N distributed according to a certain law $(p_k)_{k\in\mathbb{N}}$ satisfying $\sum kp_k > 1$.

We are also interested in the genealogy of the system studied in Chapter III. In [BBS13], the convergence of the genealogy to the Bolthausen–Sznitman coalescent ensues from the convergence of a certain process to Neveu's CSBP through a flow of bridges [BLG00, BLG06]. In their case, this convergence takes place on the same deterministic time scale $\log(N)^3$ as the convergence to the CSBP. A technical difficulty appears for the genealogy of the α-stable CSBP: the corresponding flow of bridges encoding the genealogy is defined thanks to a random time change, depending on the population size [BBC$^+$05]. Besides, we know that Neveu's CSBP never hits 0, which allows to consider the genealogies backwards in time starting from any time $t > 0$. The α-stable CSBP gets absorbed at 0 in finite time so that we can only sample individuals in the population before its extinction time.

CHAPTER II

Spatial dynamics of a population in a heterogeneous environment

We consider a certain lattice branching random walk with on-site competition and in an environment which is heterogeneous at a macroscopic scale $1/\varepsilon$ in space and time. This can be seen as a model for the spatial dynamics of a biological population in a habitat which is heterogeneous at a large scale (mountains, temperature or precipitation gradient…). The model incorporates another parameter, K, which is a measure of the local population density. We study the model in the limit when first $\varepsilon \to 0$ and then $K \to \infty$. In this asymptotic regime, we show that the rescaled position of the front as a function of time converges to the solution of an explicit ODE. We further discuss the relation with another popular model of population dynamics, the Fisher-KPP equation, which arises in the limit $K \to \infty$. Combined with known results on the Fisher-KPP equation, our results show in particular that the limits $\varepsilon \to 0$ and $K \to \infty$ do not commute in general. We conjecture that an interpolating regime appears when $\log K$ and $1/\varepsilon$ are of the same order.

Contents

1 Introduction

In this article, we are interested in the spatial propagation of a biological population in a heterogeneous environment, where the population lives on discrete sites or *demes*. Formally, the population is a system of interacting particles on the integers $\mathbb{Z}$ evolving in discrete time. At each generation, particles duplicate at a certain space- and time-depending rate, undergo a regulation step where roughly K particles at most survive at each site and jump (*migrate*) according to a discretized Gaussian distribution.

In this introductory section, we present a special case of the model and state our main results for this model. The general model is presented in Section 2.

1.1 Model and main result in a special case

We consider a particle system evolving on the rescaled lattice $\Delta x \cdot \mathbb{Z}$ at discrete time steps $0, \Delta t, 2\Delta t, \ldots$, where Δt, $\Delta x > 0$ are small parameters. The system depends furthermore on the following parameters:

- $\varepsilon > 0$ a small constant (with $1/\varepsilon$ being the space- and time-scale of interest)
- $K > 0$ a large constant (the *local population density*)
- $(t, x) \mapsto r(t, x)$ a smooth, bounded function (the *reproduction rate*, see Theorem 1.1 for precise assumptions).

We denote by $n_k(i)$ the number of particles on the site $i\Delta x$ at time $k\Delta t$. We assume that the initial condition satisfies $n_0(i) = 0$ for $i > 0$ and $n_0(0) \geq 1$, i.e. the right-most particle is at the origin. The configuration $(n_{k+1}(i))_{i\in\mathbb{Z}}$ is obtained from $(n_k(i))_{i\in\mathbb{Z}}$ through the following three consecutive steps:

1. *Reproduction step.* Each particle living on the i-th site at generation k duplicates with probability $r(\varepsilon k\Delta t, \varepsilon i\Delta x)\Delta t$. Hence the number of descendants (including the individual

itself) produced by a particle is then given by a random variable Y with law

$$\nu_{r(\varepsilon k\Delta t,\varepsilon i\Delta x)} = (1 - r(\varepsilon k\Delta t, \varepsilon i\Delta x)\Delta t)\,\delta_1 + r(\varepsilon k\Delta t, \varepsilon i\Delta x)\Delta t\,\delta_2. \tag{1.1}$$

2. *Competition step.* The number of particles at each site is truncated at K. In other words, the number of particles after the competition step is given by the truncated sum

$$\left(\sum_{m=1}^{n_k(i)} Y_m\right) \wedge K, \tag{1.2}$$

with (Y_m) a sequence of i.i.d. random variables of law $\nu_{r(\varepsilon k\Delta t,\varepsilon i\Delta x)}$.

3. *Migration step.* A particle on the i-th site jumps to the site $i+j$ with probability $\mu(j)$, where the *migration law* μ is a discretized normal distribution:

$$\mu = \sum_{j\in\mathbb{Z}} \left(\int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}}\,dx\right)\delta_j. \tag{1.3}$$

All particles jump independently.

The resulting configuration is n_{k+1}.

We denote by X_k^* the position of the rightmost particle at generation k in this system. Note that $X_0^* = 0$ by assumption. We investigate the long-time behaviour of the process $(\varepsilon X_k^*)_{k\in\mathbb{Z}}$. More precisely, we compare $(\varepsilon X_k^*)_{k\geq 0}$ and the solution x of the Cauchy problem

$$\begin{cases} x'(t) = & \sqrt{2r(t,x(t))} \\ x(0) = & 0. \end{cases} \tag{$\mathcal{C}$}$$

The result is the following.

Theorem 1.1. *Assume that $r \in C^1(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, that ∇r is bounded and that there exist $\underline{r}, \bar{r} > 0$,*

$$\forall (t,x) \in [0,\infty] \times \mathbb{R}, \quad \underline{r} < r(t,x) \leq \bar{r}.$$

Let $T > 0$ and $\delta > 0$. There exists $\Delta t_\delta > 0$ and $C_\delta > 0$ such that, if $\Delta t < \Delta t_\delta$ and $\Delta x < C_\delta \Delta t$, there exists K_0 such that, for all $K > K_0$

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\max_{k\in[\![0,\lfloor\frac{T}{\varepsilon\Delta t}\rfloor]\!]} |\varepsilon X_k^* - x(k\varepsilon\Delta t)| \leqslant \delta\right) = 1.$$

1.2 Discussion and comparison with deterministic models

In Theorem 1.1, we consider a double limit $\varepsilon \to 0$, $K \to \infty$. The order of the limit is to first let $\varepsilon \to 0$, then $K \to \infty$. A more classical large population asymptotics of *individual-based models* such as this one consists in exchanging the order of the two limits. If we let $K \to \infty$ and divide the population size by K (and let $\Delta x, \Delta t$ tend to 0 in a controlled way), we obtain a (deterministic) PDE ([FM04b], [CM07]). This PDE is a reaction-diffusion equation of Fisher-KPP type:

$$u_t^\varepsilon(t,x) = \frac{1}{2} u_{xx}^\varepsilon(t,x) + r(\varepsilon t, \varepsilon x) f(u^\varepsilon(t,x)), \tag{1.4}$$

with $f(u) = u\mathbf{1}_{u\le 1}$. The limiting behavior of u^ε as $\varepsilon \to 0$ has been widely investigated in the PDE literature [BES89, ES89, BN15] but also with probabilistic arguments [Fre86]. Introducing the change of variables $(t,x) \mapsto (t/\varepsilon, x/\varepsilon)$ and the WKB ansatz

$$u^\varepsilon(t,x) = e^{-v^\varepsilon(t/\varepsilon, x/\varepsilon)/\varepsilon}, \tag{1.5}$$

and assuming that, in a certain sense, $u^\varepsilon(t,x) \to \mathbf{1}_{x=0}$ as $t \to 0$, it can be shown that the function v^ε converges, when $\varepsilon \to 0$, to the viscosity solution v of the following Hamilton-Jacobi equation (or, more precisely, a variational inequality) [ES89]:

$$\begin{cases} \min\big(v_t(t,x) + \frac{1}{2}(v_x(t,x))^2 + r(t,x), v(t,x)\big) = 0 \\ v(t,x) \to \infty \mathbf{1}_{x\neq 0},\ t \to 0. \end{cases} \tag{1.6}$$

As a consequence, $u^\varepsilon(t/\varepsilon, x/\varepsilon)$ converges to 1 (resp. 0) uniformly on compact subsets of $\mathrm{int}(I)$ (resp. I^c), where

$$I = \{(t,x) \in [0,\infty) \times \mathbb{R} : v(t,x) = 0\}$$

In particular, if $x^\varepsilon(t)$ denotes the position of the front (for example, $x^\varepsilon(t) = \sup\{x : u^\varepsilon(t,x) \ge 1/2\}$), then, for fixed $t \ge 0$,

$$x^\varepsilon(t/\varepsilon) \to x^{HJ}(t) = \sup\{x : v(t,x) = 0\}, \quad \text{as } \varepsilon \to 0.$$

This approach has been extensively employed so far, to deal with different types of heterogeneous environments: periodic [SK97, SKT86, Xin91], random [BN15, Nad16], etc., but does not provide an explicit propagation speed, except in very specific situations [HNR11]. However, if $x(\cdot)$ denotes the solution of the ODE $(\mathcal{C})$, with initial condition $x(0) = 0$, we always have

$$x^{HJ}(t) \ge x(t) \qquad \text{for all } t \ge 0.$$

To see this, recall the variational representation of the function v [ES89]:

$$v(x,t) = \sup_\tau \inf_z \left\{ \int_0^{t\wedge\tau(z)} \frac{z'(s)^2}{2} - r(t-s, z(s))\, ds \mid z(0) = x,\ z(t) = 0 \right\}. \tag{1.7}$$

Here, the infimum is over all $z \in H^1_{\mathrm{loc}}([0,\infty);\mathbb{R})$ and the supremum is over all *stopping times*[1] τ, i.e. maps $\tau : H^1_{\mathrm{loc}}([0,\infty);\mathbb{R}) \to [0,\infty)$ satisfying for all $z, \tilde{z}$ and all $s \ge 0$:

$$\text{if } z \equiv \tilde{z} \text{ on } [0,s] \text{ and } \tau(z) \le s, \text{ then } \tau(\tilde{z}) = \tau(z).$$

In order to show that $x^{HJ}(t) \ge x(t)$, it suffices to show that $v(x(t),t) = 0$ for all $t \ge 0$. Fix $t \ge 0$. Define $z(s) = x(t-s)$ for $s \in [0,t]$. Then $z(0) = x(t)$, $z(t) = 0$ and for all $s \in [0,t]$,

[1] In fact, general theory of variational inequalities (see e.g. [BL82, p.6]) implies that for given (t,x), the optimal stopping time in (1.7) is given by $\tau_{t,x}(z) = \inf\{s \in [0,t] : v(t-s, z(s)) = 0\}$, but we don't make use of this fact.

$(z'(s))^2/2 = r(t-s, z(s))$. Hence, for every stopping time τ, the integral in (1.7) equals 0. This shows that $v(x(t), t) \le 0$ and thus $v(x(t), t) = 0$ by non-negativity of v.

It is easy to construct examples where $x^{HJ}(t) > x(t)$ for some or all $t > 0$. This is for example the case when $r(t, x) = r_0(x)$ for some strictly increasing function r_0. It is even possible to construct an example in which x^{HJ} has jumps: if we consider a function r such that $r(x) = c_1 > 0$ for $x < h$ and $r(x) = c_2 > 2c_1$ if $x \geqslant h > 0$, and an initial condition $\mathbb{1}_{(-\infty,0]}$, we observe a jump in the wavefront at time $T_0 := \frac{h}{c_2}\sqrt{2(c_2 - c_1)} < \frac{h}{\sqrt{2c_1}}$ (see Example 3 in [Fre85]). On the other hand, when r_0 is non-increasing, then $x^{HJ}(t) = x(t)$ for all $t \ge 0$. See [ES89, Fre85] for a detailed discussion and other sufficient conditions such that $x^{HJ}(t) = x(t)$ for all $t \ge 0$. In this case, one says that the *Huygens principle* is verified, in that the propagation of the front is described by a velocity field, see Freidlin [Fre86] for a discussion of this principle and its relation with the Hamilton-Jacobi limit, that he relates to geometric optics.

It has been observed previously that the viscosity solution method may be unsatisfactory, from a biological standpoint, in some situations [HFR10, Jab12]. This has been dubbed the "tail problem" [Jab12]: artifacts may be generated in the deterministic model by the absence of local extinction caused by the *infinite speed of propagation* [HFR10] of the solutions of (1.4), where meaningless, exponentially small "populations" are sent to favourable regions by diffusion before the invasion front $x(t)$, accelerating the speed of propagation and possibly causing jumps in the position of the invasion front. Some adjustments were suggested to "cut the tails" in the deterministic model. For instance, one can add a square root term with a survival threshold parameter in the F-KPP equation [Jab12, MBPS12]. Another correction suggested in [HFR10] consists in adding a strong Allee effect in Equation (1.4). Namely, they set the growth rate f to be negative at low densities, leading to a bistable reaction-diffusion equation. For such equations, the Huygens principle is verified, as shown by Freidlin [Fre86].

In this article, we propose to come back to the microscopic, or individual-based population model and to study it under a double limit, where we let first the space-time scale $1/\varepsilon$, then the population density K go to infinity. The discrete nature of our model has the effect of a "cutoff" which prevents the solution from being exponentially small in $1/\varepsilon$. In terms of the function v, which arises in the limit after a hyperbolic scaling, the cutoff prevents the function v from taking finite positive values and thus formally "pushes it up to ∞" whenever it is (strictly) positive. **The main conceptual advantage of this approach compared to the PDE approach is that our model naturally satisfies the Huygens principle, without the need of ad-hoc modifications.**

In order to determine which of the two models, with or without cutoff, is a better model for a given biological population, one might consider our microscopic model in the limit when K and $1/\varepsilon$ go to infinity together. Indeed, **we conjecture that it is possible to interpolate between the two double limits in K and ε, when K and $1/\varepsilon$ go to infinity in such a way that $\log K$ is of the same order as $1/\varepsilon$.** This relation is indeed suggested by the hyperbolic scaling (1.5). It also appears in the proof of Theorem 1.1, see for example Theorem 2.1. Precisely, setting $\log K = \kappa/\varepsilon$ for a constant $\kappa > 0$, we believe that the renormalized log-density is described in the limit by a solution v^κ to a variational inequality similar to (1.6), but with an additional constraint imposing that $v^\kappa(t, x) \ge \kappa$ implies $v^\kappa(t, x) = +\infty$, similarly to [MBPS12].

The variational representation of the function v^κ would then be

$$v^\kappa(t,x) = \sup_\tau \inf_z \Big\{ \int_0^{t\wedge\tau(z)} \frac{z'(s)^2}{2} - r(t-s, z(s))\, ds \mid z(0) = x,\, z(t) = 0,$$
$$\forall u \le \tau(z) : \int_u^{t\wedge\tau(z)} \frac{z'(s)^2}{2} - r(t-s, z(s))\, ds < \kappa. \Big\}$$

We leave the details for future work.

1.3 Explicit example and simulations

We now give an example for which the asymptotic speed of propagation of the Fisher-KPP equation (1.4) is strictly larger than the speed of the ODE ($\mathcal{C}$). Consider a reaction term of the form $r(\varepsilon t, \varepsilon x) f(u) = r(\varepsilon x) u(1-u)$ where r is a $1-$periodic function such that

$$\begin{cases} r(x) = \mu^+ & \forall x \in [0, \frac{1}{2}) \\ r(x) = \mu^- & \forall x \in [\frac{1}{2}, 1) \end{cases} \tag{1.8}$$

for some constants μ^+ and μ^- satisfying $0 < \mu^- < \mu^+$. For $\varepsilon > 0$, denote by c^*_ε the minimal speed such that a pulsating travelling front with speed c exists for all $c \geqslant c^*_\varepsilon$ [BHR05]. It has been shown [Nad10] that c^*_ε is nonincreasing with respect to ε and bounded. Therefore, it converges to some limit c^*_0 as $\varepsilon \to 0$. In the framework of slowly oscillating periodic media, an explicit formula of the asymptotic spreading speed c^*_0 has been computed in [HNR11] with the viscosity solution method. Under assumption (1.8), this expression is even more explicit and given [HFR10] by

$$\begin{aligned} c^*_0 &= 2\sqrt{2}\frac{(\mu^+)^2 + (\mu^-)^2 + (\mu^+ - \mu^-)\sqrt{\Delta}}{(\mu^+ + \mu^- + 2\sqrt{\Delta})^{3/2}} \\ \Delta &= (\mu^+)^2 + (\mu^-)^2 - \mu^-\mu^+. \end{aligned}$$

The limit speed of the ODE ($\mathcal{C}$) is the harmonic mean between the two speeds $\sqrt{2\mu^+}$ and $\sqrt{2\mu^-}$:

$$c^{ODE} = 2\frac{\sqrt{2\mu^+\mu-}}{\sqrt{\mu^-} + \sqrt{\mu^+}}.$$

In [HFR10], the authors point out that $c^*_0 > c^{ODE}$. On the one hand, they remark, thanks to a convexity argument, that c^{ODE} is strictly smaller than $\sqrt{\mu^+ + \mu^-}$, the spreading speed in the *averaged* environment. On the other hand, they claim that c^*_0 is larger that the homogenization limit $c^*_\infty = \sqrt{\mu^+ + \mu^-}$ of c^*_ε as $\varepsilon \to \infty$ [ESHR09]. The simulations plotted in Figure II.1 illustrate the behaviour of the process under the two double limits for a growth rate r of the form (1.8). We observe that when ε is fixed and K goes to infinity, the position of the rightmost particle in one simulation of the process tends to the solution of the PDE. When K is fixed and ε tends to 0, it tends to the solution of the ODE.

1.4 Relation with other stochastic models

The model we consider in this work is an example of a microscopic model for front propagation. Such models have seen considerable interest in the last two decades in mathematics, physics and

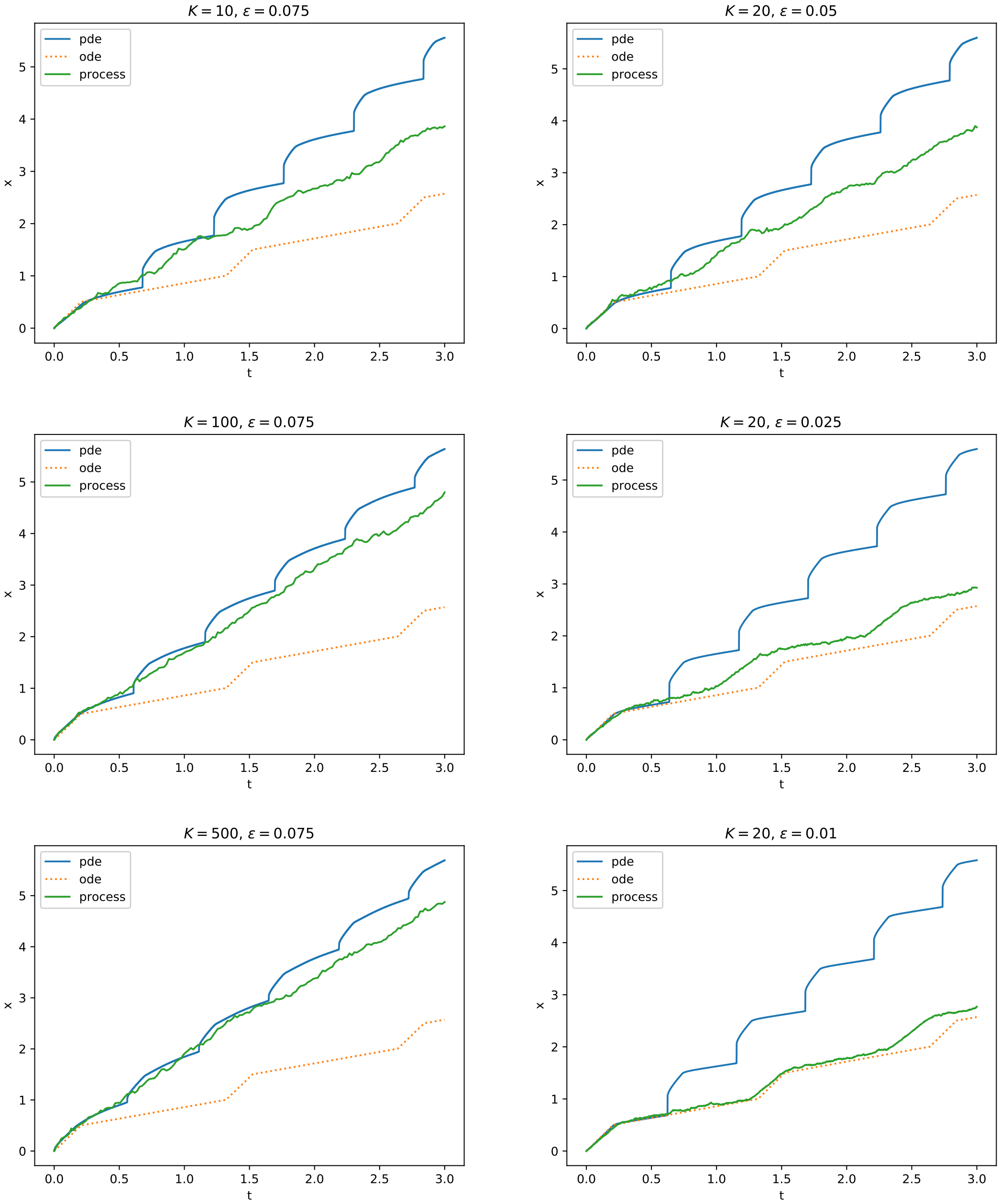

Figure II.1: Rescaled position of the rightmost particle in simulations of the process defined in Section 1.1 (green line) for different values of (K, ε). Left column: fixed ε, increasing K, right column: fixed K, decreasing ε. The growth rate r is a 1-periodic function of the form (1.8) with $\mu^+ = 3$ and $\mu^- = 0.1$, and the initial configuration is given by $n_k(i) = \mathbf{1}_{x_i<0}$. The orange dotted line is the graph of the solution of the ODE $(\mathcal{C})$. The blue solid line is the position of the front $x(t) = \sup\{x \in \mathbb{R} : u(t,x) > \frac{1}{K}\}$ for u solution of $u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + r(\varepsilon t, \varepsilon x)u(t,x)(1-u(t,x))$, with initial condition $\chi(x) = \mathbf{1}_{x<0}$.

biology. The prototypical model of front propagation is the Fisher-KPP equation, a semi-linear parabolic partial differential equation which admits so-called *travelling waves*, i.e. solutions which are stationary in shape and which travel at constant speed. Many microscopic models of front propagation (in homogeneous environment) can be seen as noisy versions of the Fisher-KPP equation, see e.g. the reviews [Pan04, Kue19]. A rich theory originating in the work of Brunet, Derrida and co-authors [BD97, BDMM06a, BDMM06b] has put forward some universal asymptotic behavior when the population density K goes to infinity. First, the speed of propagation of such systems admits a correction of the order $O((\log K)^{-2})$ compared to the limiting PDE. Second, the genealogy at the tip of the front is described by the Bolthausen–Sznitman coalescent over the time scale $(\log K)^3$, in stark contrast to mean-field models where the genealogy evolves over the much longer time scale K and is described by Kingman's coalescent. These facts have been proven rigorously for several models [BG10, MMQ10, BBS13, Mai16, Pai16, Cor16].

Compared to the case of homogeneous environment, the model in heterogeneous environment considered in this paper has a different speed of propagation than its continuous limit, *even in the limit of infinite population size.* A similar situation happens in homogeneous environment when the displacement is heavy-tailed. Such a microscopic model, with branching, competition and displacement with polynomial tails, was considered in [BDKT18]. For their model, the authors show the existence of a phase transition in the tail exponent of the displacement law: when the exponent is sufficiently large, the model grows linearly, whereas it grows superlinearly when the exponent is small. On the other hand, the continuous limit of the model, a certain integro-differential equation, always grows exponentially fast regardless of the exponent. This example, as well as the one considered in this paper, show that microscopic probabilistic models of front propagation or of spatial population dynamics can exhibit quite different qualitative behavior than their continuous limits. We believe this to be an exciting direction for future research.

Another body of literature is concerned with the behavior of locally regulated population models at equilibrium, i.e. in the bulk. Basic questions like survival and ergodicity are often studied using two methods stemming from interacting particle systems: duality and/or comparison with simpler models such as directed percolation [Eth04, HW07, BEM07, BD07, BEH09]. The genealogy of such systems is also of interest. Some related models from population genetics admit an explicit description of their genealogy in terms of coalescing, and sometimes branching random walks. Their behavior is therefore dimension-dependent, see e.g. [BEV13] for a survey. For the one-dimensional model considered here, we expect the same to happen: the genealogy should be described by random walks coalescing when they meet at a rate proportional to $1/K$, where K is the population density. In particular, on the time-scale K, its scaling limit should be a system of Brownian motions which coalesce at a rate proportional to their intersection local time, whereas on a larger time-scale, corresponding to small population density, it should be described by the Brownian web. See [SSS16, EFS17] for recent results on related models.

Finally, we point out that our model has been defined in such a way that it it is a *monotone* particle system. This property is crucial in order to compare the process to other, simpler processes. It is the analogue of the parabolic maximum principle for PDEs. Its absence causes significant technical difficulties, see for example [MP21] which studies (homogeneous) branching random walks with non-local competition.

1.5 Structure of the proof

In Section 2, we define the general model we study in this article and state Theorems 2.1 and 2.2, which taken together yield a more precise version of Theorem 1.1. In Section 3, we state a general coupling lemma allowing us to compare models with different reproduction rates r. In particular, this allows to compare the model to branching random walks.

Section 4 contains the proof of Theorem 2.1 (upper bound). The proof uses a Trotter-Kato-type scheme and local comparison with branching random walks. The proof of Theorem 2.2 (lower bound) appears in Section 5 and uses a martingale argument together with first and second moment estimates.

Appendix A recalls some known results on branching random walks and gives some explicit estimates on the rate functions of the branching random walks used in this article. Finally, Appendix B recalls known results on the Euler scheme for solutions to $(\mathcal{C})$.

2 General model: definition and results

2.1 Definition of the model, assumptions and notations

Consider a discrete-time system of interacting particles $\mathbf{X}$ on $\mathbb{Z}$, evolving in discrete time $(t_k)_{k\in\mathbb{N}}$. At each time step, the particles give birth to a random number of children and die. After their birth, the offspring migrate independently: they may settle on another deme or stay on their birth site.

The state of the system at time $t_k = k\Delta t$ (or equivalently at generation k) is described by its configuration $n_k : \mathbb{Z} \to \mathbb{N}$, where the integer $n_k(i)$ counts the number of particles living on the site $x_i = i\Delta x$. The intrinsic reproduction rate of the particles is governed by a function $r : [0,\infty) \times \mathbb{R} \to (0,\infty)$ and the intensity of the local competition, by a positive real number K, called the carrying capacity of the environment. In this paper, we investigate the impact of a long-scale heterogeneity on our system. The typical scale of this heterogeneity is of order ε^{-1}, for a small parameter $\varepsilon > 0$.

We will denote by $\nu_{r,n,K}$ the law of the number of offspring produced on a site inhabited by n parents, whose reproduction rate is r, given that the carrying capacity of the environment is K. The migration law of the particles will be given by

$$\mu = \left(\sum_{i\in\mathbb{Z}} \int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}}\, dx \right) \delta_i. \tag{2.9}$$

At time t_k, the $n_k(i)$ individuals living on the site x_i are replaced by a random number of offspring X distributed according to $\nu_{r(\varepsilon t_k, \varepsilon x_i), n_k(i), K}$. Once the population is renewed on all sites, the particles migrate independently according to μ. The resultant configuration is n_{k+1}.

Definition 1. *Let μ and ν be two probability distributions on $\mathbb{R}$. We say that ν stochastically dominates μ if $\nu([x,\infty)) \geqslant \mu([x,\infty))$ for all $x \in \mathbb{R}$.*

Definition 2. *A family of probability distributions* $(P_r)_{r>0}$ *on* $\mathbb{R}$ *is increasing with respect to* r *if for all* $r_1 < r_2$, P_{r_2} *stochastically dominates* P_{r_1}.

Assumption 1. *The function* $r : [0, \infty) \times \mathbb{R} \to (0, \infty)$ *is bounded and there exists* $\underline{r} > 0$ *such that*

$$r(s, z) \geqslant \underline{r}, \quad \forall (s, z) \in (0, \infty) \times \mathbb{R}.$$

Furthermore, the function r *is* $\mathcal{C}^1([0, \infty) \times \mathbb{R}))$ *and* ∇r *is bounded on* $[0, \infty) \times \mathbb{R}$.

Assumption 2.

1. *There exists a family of discrete probability distributions on* $\mathbb{N}$, *denoted by* $(P_r)_{r>0}$, *such that* (P_r) *is increasing with respect to* r, $\mathbb{E}[P_r] = 1 + r\Delta t$ *and, for all* $K > 0$, P_r^{*n} *stochastically dominates* $\nu_{r,n,K}$.
2. *There exists a probability distribution on* $\mathbb{N}$, *denoted by* $\bar{\nu}$, *with finite expectation and such that* $\bar{\nu}^{*K}$ *stochastically dominates* $\nu_{r,n,K}$, *for all* $n \in \mathbb{N}$.

Assumption 3. *There exists a family of probability distributions* $(\nu_r)_{r>0}$ *such that*

1. (ν_r) *is continuous and increasing with respect to* r,
2. $\mathbb{E}[\nu_r] = 1 + r\Delta t$,
3. $\nu_r(0) = 0$,
4. *if* X *is a random variable distributed according to* $\nu_{n,r,K}$ *and* $(Y_i)_{i\in\mathbb{N}}$ *is a sequence of i.i.d. random variables of law* ν_r,
$$X \overset{\mathcal{L}}{=} \left(\sum_{i=1}^{n} Y_i\right) \wedge K.$$

Notations We recall that $n_k \in \mathbb{N}^{\mathbb{Z}}$ describes the configuration of the process at time t_k and we define

$$X_k^* = \max\{i \in \mathbb{Z} : n_k(i) > 0\} \cdot \Delta x,$$

the position of the rightmost particle at time t_k. We denote by D_k the set of particles alive at time t_k and by X_u the position of the particle $u \in D_k$. For the sake of clarity, we define the following constants:

$$\gamma = \log(2), \quad C_0 = 16\gamma^{-\frac{1}{2}}, \text{and}, \quad L = \frac{\|\partial r/\partial t\|_\infty + \|\partial r/\partial x\|_\infty}{\sqrt{2\underline{r}}}. \tag{2.10}$$

Note that, for all $(t_1, x_1), (t_2, x_2) \in (0, \infty) \times \mathbb{R}$,

$$|\sqrt{2r(t_1, x_1)} - \sqrt{2r(t_2, x_2)}| \leqslant L(|t_1 - t_2| + |x_1 - x_2|), \tag{2.11}$$

under Assumption 1.

2.2 Main results

Theorem 2.1 (Upper bound on the propagation speed). *Assume that Assumptions 1 and 2 hold and that* $X_0^* = 0$. *Let* $T > 0$. *Let* $\Delta t \leqslant \|r\|_\infty^{-1}$ *and* $\Delta x \leqslant \frac{1}{80}\sqrt{2\gamma \underline{r}}\Delta t$. *There exists* $\alpha > 0$ *and* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon < \varepsilon_0$ *and* $K > 0$,

$$\mathbb{P}\left(\exists k \in \left[\!\left[0, \left\lfloor \frac{T}{\varepsilon \Delta t} \right\rfloor\right]\!\right] : \varepsilon X_k^* > x(k\varepsilon\Delta t) + C\left(\frac{\Delta x}{\Delta t} + \Delta t\right)\right) \leqslant \left(\frac{T+1}{\Delta t}\right)^2 K e^{-\frac{\alpha}{\varepsilon}}, \tag{2.12}$$

for some constant $C > 0$ *that only depends on* r *and* T, *and* x *the unique global solution of* $(\mathcal{C})$ *on* $[0, T]$.

Theorem 2.2 (Lower Bound on the propagation speed). *Assume that Assumptions 1 and 3 hold and that* $X_0^* = 0$. *Let* $\delta > 0$ *and* $T > 0$. *There exist* $\Delta t_\delta > 0$, $C_\delta > 0$ *such that, if*

$$\Delta t < \Delta t_\delta, \quad \Delta x < C_\delta \Delta t, \tag{H}$$

there exists $K_0 > 0$ *and* $\varepsilon_0 > 0$ *such that, for all* $K > K_0$ *and* $\varepsilon < \varepsilon_0$

$$\mathbb{P}\left(\exists k \in \left[\!\left[0, \left\lfloor \frac{T}{\varepsilon \Delta t} \right\rfloor\right]\!\right] : \varepsilon X_k^* < x(k\varepsilon\Delta t) - C\delta\right) \leqslant \delta.$$

for some constant $C > 0$ *that only depends on* r *and* T, *and* x *the unique global solution of* $(\mathcal{C})$ *on* $[0, T]$.

3 A coupling lemma

Let S^1 and S^2 be two systems of interacting particles on $\mathbb{Z}$ whose configurations, (n_k^1) and (n_k^2), evolve as follows. At time t_k, the particles of S^1 (resp. S^2) living on x_i, are replaced by a random number of offspring distributed according to $(p_l^1(n_k^1(i), x_i, t_k))_{l\in\mathbb{N}}$ (resp. $(p_l^2(n_k^2(i), x_i, t_k))_{l\in\mathbb{N}}$). Once the population is renewed on each site, the particles migrate independently according to μ in both processes. Furthermore, let τ be a stopping time for the process S^2 (which may be infinite). The following lemma establishes a coupling between S^1 and S^2 provided that the reproduction laws p^1 and p^2 meet certain conditions.

Lemma 3.1. *Assume that*

1. *The initial configurations satisfy* $n_0^1(i) \geqslant n_0^2(i)$, *for all* $i \in \mathbb{Z}$.
2. *For all* $(m, n) \in \mathbb{N}^2$ *such that* $n \geq m$,

$$\sum_{q\geqslant l} p_q^j(n, t_k, x_i) \geqslant \sum_{q\geqslant l} p_q^j(m, t_k, x_i), \quad \forall l \in \mathbb{N},\ j \in \{1, 2\}. \tag{3.13}$$

3. *Almost surely with respect to the process* S^2, *for every* $k < \tau$, $i \in \mathbb{N}$,

$$\sum_{q\geqslant l} p_q^1(n_k^2(i), t_k, x_i) \geqslant \sum_{q\geqslant l} p_q^2(n_k^2(i), t_k, x_i), \quad \forall l \in \mathbb{N},\ \forall i \in \mathbb{Z}. \tag{3.14}$$

Then, there exists two processes $\tilde{S}^1$ and $\tilde{S}^2$, distributed as S^1 and S^2, such that

$$\mathbb{P}(\forall i \in \mathbb{Z}, \forall k \leq \tau : \tilde{n}_k^1(i) \geqslant \tilde{n}_k^2(i)) = 1.$$

Proof. We first assume that $\tau = +\infty$. We construct a probability space supporting two processes $\tilde{S}^1$ and $\tilde{S}^2$, distributed as S^1 and S^2, such that $\tilde{S}^1$ dominates $\tilde{S}^2$. We first consider a set of particles organised according to n_0^1. On each site x_i, $n_0^2(i)$ of these particles are coloured blue. The remaining individuals are coloured red. The initial population of the process $\tilde{S}^2$ (resp $\tilde{S}^1$) is defined as the set of blue (resp. red and blue) particles.

We then construct the first generation ($k = 1$) as follows. Consider a site x_i such that $n_0^1(i) = \tilde{n}_0^1(i) \neq 0$: this site is inhabited by k_1 red particles, k_2 blue particles such that $k_1+k_2 > 0$. Draw a uniform random variable $\mathcal{U}$ on $[0,1]$ and consider $l_1(\omega)$ and $l_2(\omega)$, two integers defined by

$$l_1(\omega) = \max \left\{ n \in \mathbb{N} : \mathcal{U}(\omega) \geqslant \sum_{q=1}^{n-1} p_q^1(k_1 + k_2, t_1, x_i) \right\}$$

and likewise,

$$l_2(\omega) = \max \left\{ n \in \mathbb{N} : \mathcal{U}(\omega) \geqslant \sum_{q=1}^{n-1} p_q^2(k_2, t_1, x_i) \right\}.$$

According to equations (3.14) and (3.13), for all $l_2 \in \mathbb{N}$,

$$\begin{aligned} \sum_{q=1}^{l_2-1} p_q^2(k_2, t_1, x_i) &= 1 - \sum_{q \geqslant l_2} p_q^2(k_2, t_1, x_i) \geqslant 1 - \sum_{q \geqslant l_2} p_q^1(k_2, t_1, x_i) \geqslant 1 - \sum_{q \geqslant l_2} p_q^1(k_1 + k_2, t_1, x_i) \\ &\geqslant \sum_{q=1}^{l_2-1} p_q^1(k_1 + k_2, t_1, x_i). \end{aligned}$$

Hence, by definition of l_1 and l_2, if $l_1 \geqslant l_2 - 1$, Equation (3.13) implies that

$$\mathcal{U}(\omega) \geqslant \sum_{q=1}^{l_2-1} p_q^2(k_2, t_1, x_i) \geqslant \sum_{q=1}^{l_2-1} p_q^2(k_1 + k_2, t_1, x_i) \geqslant \sum_{q=1}^{l_1} p_q^1(k_1 + k_2, t_1, x_i) > \mathcal{U}(\omega).$$

Thus, we deduce that $l_1(\omega) \leq l_2(\omega)$. We then generate $l_2(\omega)$ individuals on x_i and $l_1(\omega)$ of them are coloured blue. The remaining ones are painted in red. We repeat this construction until the population is renewed on each non-empty site x_i. Then, all the particles (red and blue ones) migrate independently according to μ. After the migration phase, the first generation of $\tilde{S}^1$ (resp. $\tilde{S}^2$) is the set of blue (resp. red and blue) particles. The following generations are constructed similarly by induction on k.

If τ is an arbitrary stopping time, since it is a measurable function of the process S^2, it can be transferred to the probability space constructed above, to become a stopping time for the process $\tilde{S}^2$. The above chain of inequalities then still hold for every $k < \tau$ and the statement follows. $\square$

4 Proof of Theorem 2.1: Upper bound on the propagation speed

In this section, we give an upper bound on the invasion speed of the process $\mathbf{X}$ under Assumptions 1 and 2. The idea of the proof of Theorem 2.1 is to first establish a coupling between $\mathbf{X}$ and a process without competition. The absence of competition in this process then allows to compare it with several branching random walks, for which we can easily control the position of their rightmost particles (See Section 4.1).

4.1 An estimate on the branching random walk

Let X_1 be a random variable of law μ. We define the function Λ by

$$\mathbb{E}\left[e^{\lambda X_1}\right] = e^{\Lambda(\lambda)}, \quad \forall \lambda \in \mathbb{R}, \tag{4.15}$$

and denote by I its convex conjugate:

$$I(y) = \sup_{\lambda\in\mathbb{R}} \left(\lambda y - \Lambda(\lambda)\right), \quad \forall y \in \mathbb{R}. \tag{4.16}$$

Note that μ has super-exponential tails and that its support is unbounded both to the right and to the left, which implies that both Λ and I are finite and strictly convex on $\mathbb{R}$. Furthermore, $I(0) = I(\mathbb{E}[X_1]) = 0$ that I is decreasing on $(-\infty, 0)$ and increasing on $(0,\infty)$, as a consequence of strict convexity. We also define

$$\Lambda^0(\lambda) := \log \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\Delta t}} e^{\frac{-x^2}{2\Delta t}+\lambda x} dx = \frac{\Delta t}{2}\lambda^2, \quad \forall \lambda \in \mathbb{R} \tag{4.17}$$

and remark that $I^0(y) := \sup_{\lambda\in\mathbb{R}} \left(\lambda y - \Lambda^0(\lambda)\right)$ is given by

$$I^0(y) = \left(\frac{y}{\Delta t}\right) y - \Lambda^0\left(\frac{y}{\Delta t}\right) = \frac{y^2}{2\Delta t}. \tag{4.18}$$

Thus, for all $m > 1$, the equation $I^0(c) = \log(m)$ has a unique positive solution,

$$c_0 = \sqrt{2\Delta t \log m}. \tag{4.19}$$

Since I in increasing and strictly convex on $(0,\infty)$, the equation $I(c) = \log(m)$ also has a unique solution $c \in (0,\infty)$. In Appendix A, we state several results (Lemma A.2 to A.5) on the regularity of I and c. These results lead to a first rough estimate on c.

Lemma 4.1. *Let $\Delta t < \|r\|_\infty^{-1}$ and $\Delta x < \frac{1}{16}\sqrt{2\gamma \underline{r}}\Delta t$. Let $\bar{r} \in [\underline{r}, \|r\|_\infty]$ and $\bar{c}$ be the unique positive solution of $I(\bar{c}) = \log(1+\bar{r}\Delta t)$. Then,*

$$\frac{1}{2}\sqrt{2\gamma\underline{r}}\Delta t < \bar{c} < 2\sqrt{2\|r\|_\infty}\Delta t. \tag{4.20}$$

Proof. By concavity of the logarithm function,

$$(1-\bar{r}\Delta t)\log(1)+\bar{r}\Delta t\log(2)=\gamma\bar{r}\Delta t\leqslant\log(1+\bar{r}\Delta t). \tag{4.21}$$

This implies that $\Delta x<\frac{1}{16}\sqrt{2\Delta t\log(1+\underline{r}\Delta t)}$ and, according to Lemma A.3 and Equation (4.18), that

$$\frac{1}{2}\sqrt{2\Delta t\log(1+\bar{r}\Delta t)}<\bar{c}<2\sqrt{2\Delta t\log(1+\bar{r}\Delta t)}. \tag{4.22}$$

Finally, combining (4.21) and (4.22), we get that

$$\frac{1}{2}\sqrt{2\gamma\underline{r}}\Delta t\leqslant\frac{1}{2}\sqrt{2\gamma\bar{r}}\Delta t\leqslant\bar{c}\leqslant2\sqrt{2\bar{r}}\Delta t\leqslant2\sqrt{2\|r\|_\infty}\Delta t. \tag{4.23}$$

□

Lemma 4.2. *Under the same assumptions as in Lemma 4.1,*

$$\left|\bar{c}-\sqrt{2\Delta t\log(1+\bar{r}\Delta t)}\right|\leqslant a\Delta x,$$

with $a=16\gamma^{-\frac{1}{2}}\left(\frac{\bar{r}}{\underline{r}}\right)^{\frac{1}{2}}\leqslant16\gamma^{-\frac{1}{2}}\left(\frac{\|r\|_\infty}{\underline{r}}\right)^{\frac{1}{2}}$.

Proof. According to Lemma 4.1, $\bar{c}$ is located in a compact interval that does not depend on $\bar{r}$. Let $\bar{c}_0=\sqrt{2\Delta t\log(1+\bar{r}\Delta t)}$ and remark that the inequality (4.20) also holds when $\bar{c}$ is replaced by $\bar{c}_0$. Then, since $\Delta x<\frac{1}{4}\left(\frac{1}{2}\sqrt{2\gamma\underline{r}}\Delta t\right)$, Lemma A.5 applied with $\underline{y}=\frac{1}{2}\sqrt{2\gamma\underline{r}}\Delta t$ implies that

$$\frac{1}{8}\sqrt{2\gamma\underline{r}}|\bar{c}-\bar{c}_0|\leqslant|I(\bar{c})-I(\bar{c}_0)|. \tag{4.24}$$

In addition, note that $I(\bar{c})=I^0(\bar{c}_0)=\log(1+\bar{r}\Delta t)$ (see Equation (4.18)), so that

$$|I(\bar{c})-I(\bar{c}_0)|=|I^0(\bar{c}_0)-I(\bar{c}_0)|\leqslant2\bar{c}_0\frac{\Delta x}{\Delta t}\leqslant2\sqrt{2\bar{r}}\Delta x, \tag{4.25}$$

according to Lemma A.2. Then, thanks to Equations (4.24) and (4.25), we get that,

$$|\bar{c}-\bar{c}_0|\leqslant a\Delta x.$$

□

Let us now consider a branching random walk (BRW) of reproduction law $P_{\bar{r}}$, for some $\bar{r}\in[\underline{r},\|r\|_\infty]$, and migration law μ. In Lemma 4.3, we give an estimate on the speed of propagation of this BRW. For further details about the BRW, we refer to Appendix A.

Lemma 4.3. *Suppose the same assumptions as in Lemma 4.1 hold. Consider a branching random walk of reproduction law $P_{\bar{r}}$ and displacement law μ, starting with a single particle at 0, and denote by M_n the position of its rightmost particle at generation n. Then, for all $\eta>0$ and $A\geqslant0$,*

$$\mathbb{P}(\exists n\in\mathbb{N}:M_n>(1+\eta)n\bar{c}+A)\leqslant h(\eta)e^{-\frac{\sqrt{2\gamma\underline{r}}}{8}A},$$

with h defined by

$$h(\eta)=\frac{e^{-\frac{\gamma\underline{r}\Delta t}{8}\eta}}{1-e^{-\frac{\gamma\underline{r}\Delta t}{8}\eta}},\quad\forall\eta>0. \tag{4.26}$$

Proof. Let $\eta > 0$, $A \geqslant 0$. It will be enough to prove that

$$\mathbb{P}(\exists n \in \mathbb{N} : M_n > (1+\eta)n\bar{c} + A) \leqslant g(\eta)e^{-\frac{A\bar{c}}{4\Delta t}}, \tag{4.27}$$

where $\bar{c}$ refers to the unique positive solution of $I(\bar{c}) = \log(1+\bar{r}\Delta t)$ and

$$g(\eta) = \frac{e^{-\frac{\bar{c}^2}{4\Delta t}\eta}}{1 - e^{-\frac{\bar{c}^2}{4\Delta t}\eta}}. \tag{4.28}$$

Indeed, according to Lemma 4.1, $\bar{c} > \frac{1}{2}\sqrt{2\gamma \underline{r}}\Delta t$, so that

$$\frac{\bar{c}^2}{4\Delta t} \geqslant \frac{1}{8}\gamma \underline{r}\Delta t, \quad \text{and,} \quad \frac{\bar{c}}{4\Delta t} \geqslant \frac{1}{8}\sqrt{2\gamma \underline{r}}.$$

We now prove (4.27). For a particle v living in the process, we denote by Ξ_v its position and define $Z_n = \sum_{|v|=n} \mathbb{1}_{\Xi_v > (1+\eta)n\bar{c}+A}$. Markov's inequality implies that

$$\mathbb{P}(M_n > (1+\eta)n\bar{c} + A) = \mathbb{P}(Z_n \geqslant 1) \leqslant \mathbb{E}[Z_n], \tag{4.29}$$

and thanks to the many-to-one lemma (see Lemma A.1), we know that

$$\mathbb{E}[Z_n] = (1+\bar{r}\Delta t)^n \mathbb{P}(\Xi_v > (1+\eta)n\bar{c} + A), \tag{4.30}$$

for any particle v of the n-th generation. Besides, by Chernoff's bound,

$$\mathbb{P}(\Xi_v > (1+\eta)n\bar{c} + A) \leqslant e^{n(\Lambda(\theta) - \theta((1+\eta)\bar{c}+A/n))}, \quad \forall \theta \geqslant 0. \tag{4.31}$$

Remark that for $\theta < 0$,

$$\theta((1+\eta)\bar{c} + A/n) - \Lambda((1+\eta)\bar{c} + A/n) \leqslant -\Lambda((1+\eta)\bar{c} + A/n) \leqslant I(0) = 0.$$

Yet, $I((1+\eta)\bar{c}+A/n) \geqslant 0$, so that $I((1+\eta)\bar{c}+A/n)) = \sup_{\theta \geqslant 0} \theta((1+\eta)\bar{c}+A/n) - \Lambda((1+\eta)\bar{c}+A/n)$, and Equation (4.31) gives that

$$\mathbb{P}(\Xi_v > (1+\eta)n\bar{c} + A) \leqslant e^{-nI((1+\eta)\bar{c}+A/n)}. \tag{4.32}$$

Moreover, thanks to Lemma 4.1, we know that $\Delta x \leqslant \frac{\bar{c}}{8}$, therefore, according to Lemma A.4,

$$I(\bar{c}) + \frac{\bar{c}^2}{4\Delta t}\left(\eta + \frac{A}{n\bar{c}}\right) \leqslant I((1+\eta + A/(n\bar{c}))\bar{c}), \quad \forall n \in \mathbb{N}.$$

Thus, combining (4.29), (4.30) and (4.32), we get that

$$\mathbb{E}[Z_n] \leqslant e^{-\frac{n\bar{c}^2}{4\Delta t}\left(\eta + \frac{A}{n\bar{c}}\right)},$$

since $I(\bar{c}) = \log(1 + \underline{r}\Delta t)$. Finally, by a union bound,

$$\mathbb{P}(\exists n \in \mathbb{N} : M_n > (1+\eta)n\bar{c} + A) \leqslant \sum_{n=1}^{\infty} \mathbb{E}[Z_n] \leqslant e^{-\frac{A\bar{c}}{4\Delta t}} \sum_{n=1}^{\infty} e^{-\frac{n\bar{c}^2}{4\Delta t}\eta} = \frac{e^{-\frac{A\bar{c}}{4\Delta t} - \frac{\bar{c}^2}{4\Delta t}\eta}}{1 - e^{-\frac{\bar{c}^2}{4\Delta t}\eta}}.$$

This proves (4.27) and finishes the proof of the lemma. $\square$

4.2 Invasion speed estimate: small time steps

In this subsection, we bound the displacement of the rightmost particle in $\mathbf{X}$ after $\lfloor \varepsilon^{-1} \rfloor$ generations, i.e. after time $\Delta t \cdot \lfloor \varepsilon^{-1} \rfloor$.

Proposition 4.1. *Assume that Assumptions 1 and 2 hold. Let $\Delta t < \|r\|_\infty^{-1}$ and $\Delta x < \frac{1}{80}\sqrt{2\gamma \underline{r}}\Delta t$. There exist two positive constants α and ε_0 such that, for all $K > 0$, $\varepsilon < \varepsilon_0$, $(k_0, i_0) \in \mathbb{N} \times \mathbb{Z}$ and $k_1 = k_0 + \lfloor \varepsilon^{-1} \rfloor$,*

$$\mathbb{P}\left(\exists k \in [\![k_0, k_1]\!] : \varepsilon X_k^* > \varepsilon x_{i_0} + \sqrt{2r(\varepsilon t_{k_0}, \varepsilon x_{i_0})}\varepsilon \Delta t (k - k_0) + A(\Delta x + \Delta t^2) \big| X_{k_0}^* \leqslant x_{i_0} \right) \leqslant K e^{-\frac{\alpha}{\varepsilon}}, \quad (4.33)$$

for some constant $A > 0$ that only depends on $\|r\|_\infty$ and $\underline{r}$.

Proof. Let $(k_0, i_0) \in \mathbb{Z} \times \mathbb{N}$. Throughout the proof, we will assume that we start the process at generation k_0 with a deterministic initial condition n_{k_0} such that $X_{k_0}^* \leq x_{i_0}$. The estimates we obtain will not depend on this initial condition. Rewriting the statement slightly, it will therefore be enough to show the following:

$$\mathbb{P}\left(\exists k \in [\![k_0 + 1, k_1]\!] : X_k^* > a_k \right) \leqslant K e^{-\frac{\alpha}{\varepsilon}}, \quad (4.34)$$

where $a_k = x_{i_0} + \sqrt{2r(\varepsilon t_{k_0}, \varepsilon x_{i_0})}\Delta t (k - k_0) + A(\Delta x + \Delta t^2)/\varepsilon$ and α, ε_0, A, to be defined later, are as in the statement of the proposition.

The proof is divided into two steps. In the first step, we let the process run for one time step, after which the expected density of the process can be bounded by a constant multiple of K, thanks to Assumption 2. In the second step, we control the displacement of the rightmost particle in $\mathbf{X}$ between generations $k_0 + 1$ and k_1, thanks to several couplings with processes without competition and distinguishing the particles according to the position of their ancestor at generation $k_0 + 1$.

Step 1: Control of the population at generation $k_0 + 1$**.** In this step, we control the number of particles on each site in the process $\mathbf{X}$ after one generation. Recall that n_k denotes the configuration of the process $\mathbf{X}$ at generation k. We denote by N_i the number of individuals born on the site x_i during the first reproduction phase. In addition, for $\ell \in [\![1, N_i]\!]$, we denote by U_i^ℓ the displacement of the ℓ-th particle born on the site x_i during the first reproduction phase. Therefore, we have for every $i \in \mathbb{Z}$

$$n_{k_0+1}(i) = \sum_{j \leqslant i_0} \sum_{\ell=1}^{N_j} \mathbb{1}_{U_j^\ell = i - j}. \quad (4.35)$$

Recall that (U_i^ℓ) is a sequence of i.i.d. random variables of law μ and that the (N_i) are stochastically dominated by the sum of K i.i.d. random variables of law $\bar{\nu}$ of finite expectation m, by Assumption 2. Thus, we have that

$$\mathbb{E}[n_{k_0+1}(i)] \leqslant mK \sum_{j \leqslant i_0} \mathbb{P}(U = i - j), \quad (4.36)$$

where U is a random variable of law μ. In particular, we get

$$\mathbb{E}[n_{k_0+1}(i)] \leqslant mK. \tag{4.37}$$

We will also need a bound on the position of the maximal particle at generation k_0+1. Let $k \in \mathbb{N}_0$ and $i \in \mathbb{Z}$ such that $x_i - x_{i_0} \geqslant \frac{1}{2\sqrt{\varepsilon}} + (k+1)\Delta x$. Then, we have that

$$\begin{aligned}\sum_{j \leqslant i_0} \mathbb{P}(U = i-j) &= \int_{(i-i_0-\frac{1}{2})\Delta x}^{\infty} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx \leqslant \int_{\frac{1}{2\sqrt{\varepsilon}}+k\Delta x}^{\infty} \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2\Delta t}} dx \\ &\leqslant e^{-\frac{1}{2\Delta t}\left(\frac{1}{2\sqrt{\varepsilon}}+k\Delta x\right)^2} \leqslant e^{-\frac{1}{8\varepsilon\Delta t}} e^{-\frac{k\Delta x}{2\sqrt{\varepsilon}\Delta t}},\end{aligned}$$

where we have used the Gaussian tail estimate $\mathbb{P}(Z \geq x) \leq e^{-x^2/2}$ for a standard Gaussian r.v. Z. Using Equation (4.36), we have by Markov's inequality and a union bound,

$$\mathbb{P}\left(X^*_{k_0+1} > x_{i_0} + \frac{1}{2\sqrt{\varepsilon}} + \Delta x\right) \leqslant mKe^{-\frac{1}{8\varepsilon\Delta t}} \sum_{k=0}^{\infty} e^{-\frac{k\Delta x}{2\sqrt{\varepsilon}\Delta t}},$$

and therefore, as long as ε is small enough,

$$\mathbb{P}\left(X^*_{k_0+1} > x_{i_0} + \frac{1}{\sqrt{\varepsilon}}\right) \leqslant 2mKe^{-\frac{1}{8\varepsilon\Delta t}}. \tag{4.38}$$

Step 2: Between generations k_0+1 and k_1. As mentioned above, between generations k_0+1 and k_1, we control the process $\mathbf{X}$ by another process without competition between particles. More precisely, we denote by $\mathbf{X}^1$ the process defined as $\mathbf{X}$, but where for every $k \in [\![k_0+1, k_1]\!]$ the reproduction law on site x_i at time t_k is given by the probability distribution $P^{*n_k(i)}_{r(\varepsilon t_k, \varepsilon x_i)}$ instead of $\nu_{r(\varepsilon t_k, \varepsilon x_i), n, K}$ (the migration law is still μ). The position of its maximum at generation k is analogously denoted by X^{1*}_k. By the first part of Assumption 2 and Lemma 3.1, we can couple $\mathbf{X}$ and $\mathbf{X}^1$ such that $\mathbf{X}^1$ dominates $\mathbf{X}$. Hence, in what follows, it will be enough to prove (4.34) with $\mathbf{X}^1$ instead of $\mathbf{X}$. The advantage of working with $\mathbf{X}^1$ instead of $\mathbf{X}$ is the fact that $\mathbf{X}^1$ satisfies the branching property, i.e. the descendants of different individuals from the same generation evolve independently.

We first make use of the estimates from Step 1. Conditioning on the process at generation k_0+1, and using a union bound over the particles from that generation, with the notation $\mathbb{P}_{(\delta_i, k_0+1)}$ to mean that the process starts with one particle at site x_i at generation k_0+1, we get for sufficiently small ε,

$$\begin{aligned}&\mathbb{P}\left(\exists k \in [\![k_0+1, k_1]\!] : X^{1*}_k > a_k\right) \\ &\leqslant \sum_{i \in \mathbb{Z}: x_i \leq x_{i_0} + \frac{1}{\sqrt{\varepsilon}}} \mathbb{E}[n_{k_0+1}(i)] \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X^{1*}_k > a_k\right) + \mathbb{P}\left(X^*_{k_0+1} > x_{i_0} + \frac{1}{\sqrt{\varepsilon}}\right) \\ &\leqslant mK \sum_{i \in \mathbb{Z}: x_i \leq x_{i_0} + \frac{1}{\sqrt{\varepsilon}}} \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X^{1*}_k > a_k\right) + 2mKe^{-\frac{1}{8\varepsilon\Delta t}}.\end{aligned} \tag{4.39}$$

Here, we used (4.37) and (4.38) from Step 1 in the last line.

In what follows, we bound the probability appearing on the RHS of (4.39) for various values of x_i. The bound will depend on whether $x_i \geq x_{i_0} - R/\varepsilon$ or not, where

$$R \coloneqq 2\sqrt{2\|r\|_\infty}\Delta t. \tag{4.40}$$

We need a few more definitions. Define

$$\bar{r} = \max\left\{r(\varepsilon t, \varepsilon x);\ t_{k_0} \leqslant t \leqslant t_{k_1},\ |x - x_{i_0}| \leqslant \frac{2R}{\varepsilon}\right\}. \tag{4.41}$$

and $\bar{c}$ the unique positive solution of

$$I(\bar{c}) = \log(1 + \bar{r}\Delta t). \tag{4.42}$$

Note that thanks to Assumption 1 and Equation (2.11), with L defined there, we have that

$$|\sqrt{2\bar{r}} - \sqrt{2r(\varepsilon t_{k_0}, \varepsilon x_{i_0})}| \leqslant L(\varepsilon(t_{k_1} - t_{k_0}) + 2R) \leqslant L(1 + 4\sqrt{2\|r\|_\infty})\Delta t. \tag{4.43}$$

Denote by $\tilde{r}$ the function

$$\tilde{r}(t,x) = \begin{cases} \bar{r} & \text{if } |x - x_{i_0}| \leq \frac{2R}{\varepsilon} \\ \|r\|_\infty & \text{if } |x - x_{i_0}| > \frac{2R}{\varepsilon}. \end{cases} \tag{4.44}$$

Now introduce three more processes $\mathbf{X}^2$, $\mathbf{X}^3$ and $\mathbf{X}^4$. These processes are defined as $\mathbf{X}^1$, except that their reproduction law on site x_i at time t_k, $k \in [\![k_0+1, k_1]\!]$, is given by $P^{*n_k(i)}_{\tilde{r}(\varepsilon t_k, \varepsilon x_i)}$, $P^{*n_k(i)}_{\bar{r}}$ and $P^{*n_k(i)}_{\|r\|_\infty}$, respectively. In other words, $\mathbf{X}^3$ and $\mathbf{X}^4$ are BRW with reproduction laws $P_{\bar{r}}$ and $P_{\|r\|_\infty}$, respectively.

From the definition of $\tilde{r}$, we immediately get that $\tilde{r} \geq r$. Therefore, according to Lemma 3.1 and Assumption 2, there exists a coupling between $\mathbf{X}^1$ and $\mathbf{X}^2$ so that $\mathbf{X}^2$ dominates $\mathbf{X}^1$. Similarly, there exists a coupling between $\mathbf{X}^2$ and $\mathbf{X}^4$ so that $\mathbf{X}^4$ dominates $\mathbf{X}^2$. In order to construct a coupling between $\mathbf{X}^2$ and $\mathbf{X}^3$, define the stopping time τ as the first time at which a particle from the process $\mathbf{X}^2$ exits the interval $\left[x_{i_0} - \frac{2R}{\varepsilon}, x_{i_0} + \frac{2R}{\varepsilon}\right]$ before time t_{k_1}. By the definition of $\tilde{r}$, there exists then a coupling between $\mathbf{X}^2$ and $\mathbf{X}^3$, such that $\mathbf{X}^3$ dominates $\mathbf{X}^2$ until the time τ.

Let $i \in \mathbb{Z}$. As a consequence of the previous couplings, we have the following two bounds for the probability appearing on the RHS of (4.39). First,

$$\mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{1*} > a_k\right) \leq \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{4*} > a_k\right). \tag{4.45}$$

This bound will be used for $x_i \leq x_{i_0} - R/\varepsilon$. Second, denoting by $\bar{X}_k^{3*}$ the position of the *minimal* particle in the process $\mathbf{X}^3$ at generation k, we have

$$\begin{aligned} \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{1*} > a_k\right) &\leq \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{3*} > a_k\right) \\ &+ \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{3*} > x_{i_0} + 2R/\varepsilon \text{ or } \bar{X}_k^{3*} < x_{i_0} - 2R/\varepsilon\right) \\ &=: T_1 + T_2. \end{aligned} \tag{4.46}$$

This bound will be used for i such that $x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}]$.

Step 2a: particles close to the maximum. Let $i \in \mathbb{Z}$ such that $x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}]$. We bound the RHS of (4.46). Assume ε is small enough so that $1/\sqrt{\varepsilon} \leq R/\varepsilon$. Using first the assumption on x_i and then the symmetry and translational invariance of $\mathbf{X}^3$, we have

$$\begin{aligned} T_2 &\leq \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{3*} > x_i + R/\varepsilon \text{ or } \bar{X}_k^{4*} < x_i - R/\varepsilon\right) \\ &\leq 2\mathbb{P}_{(\delta_0, 0)}\left(\exists k \in [\![0, k_1 - (k_0+1)]\!] : X_k^{3*} > R/\varepsilon\right). \end{aligned}$$

We can now apply Lemma 4.3 with $A = \frac{R}{4\varepsilon}$ and $\eta = \frac{1}{4}$. Indeed, since $\Delta x < \frac{1}{80}\sqrt{2\gamma \underline{r}}\Delta t$, Lemma 4.2 applied to $\bar{c}$ (see Equation (4.42)) gives that

$$\bar{c} \leqslant \sqrt{2\Delta t \log(1 + \bar{r}\Delta t)} + a\Delta x \leqslant \sqrt{2\bar{r}}\Delta t + \frac{1}{5}\sqrt{2\bar{r}}\Delta t \leqslant \frac{6}{5}\sqrt{2\bar{r}}\Delta t \leq \frac{3}{5}R$$

so that, for $k \leqslant k_1$

$$(1+\eta)(k - k_0 - 1)\bar{c} + A \quad \leqslant \quad \frac{5}{4\varepsilon}\bar{c} + A \leqslant \frac{3R}{4\varepsilon} + \frac{R}{4\varepsilon} = \frac{R}{\varepsilon}.$$

Lemma 4.3 now gives that

$$T_2 \leqslant 2h(1/4)e^{-\frac{\sqrt{2\gamma \underline{r}}}{32}\frac{R}{\varepsilon}}, \tag{4.47}$$

with h as in the statement of Lemma 4.3.

We now bound the term T_1 on the RHS of (4.46). Let $i \in \mathbb{Z}$ such that $x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + 1/\sqrt{\varepsilon}]$. We then have by the definitions of (a_k) and $\bar{r}$

$$T_1 \leq \mathbb{P}_{(\delta_i, k_0+1)}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{3*} > x_{i_0} + \sqrt{2\bar{r}}\Delta t(k - k_0) + \frac{A}{\varepsilon}(\Delta x + \Delta t^2)\right). \tag{4.48}$$

Now, according to Lemma 4.3, we have, for some $C > 0$ not depending on ε,

$$\mathbb{P}\left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{3*} > x_i + (1+\Delta t)(k - k_0 - 1)\bar{c} + \frac{\Delta t^2}{\varepsilon}\right) \leqslant Ce^{-\frac{\sqrt{2\gamma \underline{r}}}{8}\frac{\Delta t^2}{\varepsilon}}. \tag{4.49}$$

Besides, recall from Lemma 4.2 that

$$\bar{c} \leqslant \sqrt{2\bar{r}}\Delta t + a\Delta x, \tag{4.50}$$

with some $a \leqslant 16\gamma^{-\frac{1}{2}}\left(\frac{\|r\|_\infty}{\underline{r}}\right)^{\frac{1}{2}}$. Therefore, combining (4.49) and (4.50) and using that $x_i \leq x_{i_0} + \frac{1}{\sqrt{\varepsilon}}$ and $\Delta t \leq \|r\|_\infty^{-1}$ and $k - k_0 \leqslant \frac{1}{\varepsilon}$ for $k \leqslant k_1$, we have

$$\begin{aligned} &\mathbb{P}\Bigg(\exists k \in [\![k_0+1, k_1]\!] : \\ &\qquad X_k^{3*} > x_{i_0} + \sqrt{2\bar{r}}\Delta t(k - k_0) + \frac{\sqrt{\varepsilon} + \sqrt{2\|r\|_\infty}\varepsilon + (1 + \sqrt{2\|r\|_\infty})\Delta t^2 + a(1 + \|r\|_\infty^{-1})\Delta x}{\varepsilon}\Bigg) \\ &\leqslant Ce^{-\frac{\sqrt{2\gamma \underline{r}}}{8}\frac{\Delta t^2}{\varepsilon}}. \end{aligned} \tag{4.51}$$

Combining (4.48), (4.51) and (4.43), it follows that for

$$A = 1 + \max(1 + \sqrt{2\|r\|_\infty} + L(1 + 4\sqrt{2\|r\|_\infty}), a(1 + \|r\|_\infty^{-1})),$$

we get for ε small enough,

$$T_1 \leq C e^{-\frac{\sqrt{2\gamma \underline{r}}}{8} \frac{\Delta t^2}{\varepsilon}}. \tag{4.52}$$

Combining (4.46), (4.47) and (4.52), we now get, using again that $1/\sqrt{\varepsilon} \leq R/\varepsilon$,

$$\sum_{i \in \mathbb{Z}: x_i \in (x_{i_0} - R/\varepsilon, x_{i_0} + \frac{1}{\sqrt{\varepsilon}}]} \mathbb{P}_{(\delta_i, k_0+1)} \left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{1*} > a_k \right)$$
$$\leq \frac{2R}{\Delta x \varepsilon} \left(2h(1/4) e^{-\frac{\sqrt{2\gamma \underline{r}}}{32} \frac{R}{\varepsilon}} + C e^{-\frac{\sqrt{2\gamma \underline{r}}}{8} \frac{\Delta t^2}{\varepsilon}} \right) \leq e^{-\alpha_1/\varepsilon}, \tag{4.53}$$

for some $\alpha_1 > 0$ and for ε sufficiently small, and with A as above.

Step 2b: particles far away from the maximum. Let $i \in \mathbb{Z}$ such that $x_i \leq x_{i_0} - R/\varepsilon$. We bound the RHS of (4.45). We have for every $A \geq 0$, using that $a_k \geq x_{i_0}$ for every $k \in [\![k_0+1, k_1]\!]$,

$$\mathbb{P}_{(\delta_i, k_0+1)} \left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{4*} > a_k \right) \leq \mathbb{P}_{(\delta_i, k_0+1)} \left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{4*} > x_{i_0} \right)$$
$$\leq \mathbb{P}_{(\delta_0, 0)} \left(\exists k \leq \lfloor \varepsilon^{-1} \rfloor : X_k^{4*} > x_{i_0} - x_i \right). \tag{4.54}$$

Denote by c_∞ the unique positive solution of $I(c_\infty) = \log(1 + \|r\|_\infty \Delta t)$. Following the same calculations as in Step 2a, we have $c_\infty \leq \frac{3}{5} R$. We again use Lemma 4.3 with $\eta = 1/4$ and $A = x_{i_0} - x_i - \frac{3R}{4\varepsilon}$ to get

$$\mathbb{P}_{(\delta_0, 0)} \left(\exists k \leq \lfloor \varepsilon^{-1} \rfloor : X_k^{4*} > x_{i_0} - x_i \right) \leq \mathbb{P}_{(\delta_0, 0)} \left(\exists k \leq \lfloor \varepsilon^{-1} \rfloor : X_k^{4*} > \frac{(1+\eta) c_\infty}{\varepsilon} + x_{i_0} - x_i - \frac{3R}{4\varepsilon} \right)$$
$$\leq h(1/4) \exp\left(-\frac{\sqrt{2\gamma \underline{r}}}{8} \left(x_{i_0} - x_i - \frac{3R}{4\varepsilon} \right) \right). \tag{4.55}$$

Combining (4.45) and (4.55), we now get

$$\sum_{i \in \mathbb{Z}: x_i \leq x_{i_0} - R/\varepsilon} \mathbb{P}_{(\delta_i, k_0+1)} \left(\exists k \in [\![k_0+1, k_1]\!] : X_k^{1*} > a_k \right)$$
$$\leq \sum_{j \geq 0} h(1/4) \exp\left(-\frac{\sqrt{2\gamma \underline{r}}}{8} \left(\frac{R}{4\varepsilon} + (\Delta x) j \right) \right) \leq \exp(-\alpha_2/\varepsilon), \tag{4.56}$$

for some $\alpha_2 > 0$ and for ε sufficiently small.

Combining (4.39), (4.53) and (4.56), and using the fact that $\mathbf{X}^1$ dominates $\mathbf{X}$, we obtain (4.34) for some $\alpha > 0$ and for ε sufficiently small, with A as above, depending only on $\|r\|_\infty$ and $\underline{r}$. This concludes the proof of the proposition. □

4.3 Comparison with the solution of $(\mathcal{C})$: proof of Theorem 2.1

Let $T > 0$, and for $\varepsilon > 0$, define $N = \lfloor T/\Delta t \rfloor$ and consider the subdivision $(s_i)_{i=0}^N$ of $[0, T]$:

$$s_0 = 0 < s_1 = \varepsilon \lfloor \varepsilon^{-1} \rfloor \Delta t < \ldots < s_N = N \varepsilon \lfloor \varepsilon^{-1} \rfloor \Delta t.$$

We denote by $A_\varepsilon = \frac{A}{\varepsilon\lfloor\varepsilon^{-1}\rfloor}$ where $A > 0$ is the constant from Proposition 4.1 and let $(\tilde{y}_j)_{i=1\ldots N}$ be a sequence defined by $\tilde{y}_0 = 0$ and,

$$\tilde{y}_{j+1} = \tilde{y}_j + \left(\sqrt{2r(s_j, \tilde{y}_j)} + A_\varepsilon\left(\frac{\Delta x}{\Delta t} + \Delta t\right)\right)\lfloor\varepsilon^{-1}\rfloor\varepsilon\Delta t, \quad \forall j \in [\![1, N-1]\!]. \tag{4.57}$$

For $j \in \mathbb{N}$, we define $\varphi(j) = j\lfloor\varepsilon^{-1}\rfloor$ and consider the following function:

$$f(t) = \tilde{y}_j + (\tilde{y}_{j+1} - \tilde{y}_j)\frac{t/(\varepsilon\Delta t) - \varphi(j)}{\varphi(j+1) - \varphi(j)}, \text{ if } t \in [\varepsilon\varphi(j)\Delta t, \varepsilon\varphi(j+1)\Delta t]. \tag{4.58}$$

First, recall that $X_0^* = 0$ and note that these three events are equal:

$$\begin{aligned}
B_0 :&= \left\{\exists k \in [\![\varphi(0), \varphi(1)]\!] : \varepsilon X_k^* > \tilde{y}_0 + (\tilde{y}_1 - \tilde{y}_0)\frac{k - \varphi(0)}{\varphi(1) - \varphi(0)}\right\} \\
&= \{\exists k \in [\![\varphi(0), \varphi(1)]\!] : \varepsilon X_k^* > f(k\varepsilon\Delta t)\} \\
&= \left\{\exists k \in [\![\varphi(0), \varphi(1)]\!] : \varepsilon X_k^* > \varepsilon X_0^* + \left(\sqrt{2r(0, \varepsilon X_0^*)} + A_\varepsilon\left(\frac{\Delta x}{\Delta t} + \Delta t\right)\right)\varepsilon\Delta t(k - \varphi(0))\right\}
\end{aligned}$$

Then, for all $j \in [\![0, N-1]\!]$, we define $B_j = \{\exists k \in [\![\varphi(j), \varphi(j+1)]\!] : \varepsilon X_k^* > f(k\varepsilon\Delta t)\}$. According to Proposition 4.1, there exists α and ε_0, that does not depend on $\tilde{y}_j$ nor on s_j, such that, if $\varepsilon < \varepsilon_0$, $K > 0$,

$$\mathbb{P}(B_j | X_{\varphi(j)}^* \leqslant \tilde{y}_j) \leqslant Ke^{-\frac{\alpha}{\varepsilon}}, \quad \forall j \in [\![0, N-1]\!].$$

Therefore, we get by induction on j that

$$\mathbb{P}(B_{j+1}) \leqslant \mathbb{P}(B_j) + \mathbb{P}(B_{j+1}|B_j^c) \leqslant (j+1)Ke^{-\frac{\alpha}{\varepsilon}}, \quad \forall j \in [\![1, N-1]\!],$$

and by union bound, we have

$$\mathbb{P}\left(\cup_{j=0}^{N-1} B_j\right) \leqslant N^2 K e^{-\frac{\alpha}{\varepsilon}}. \tag{4.59}$$

Then, let us consider the maximal solution $\tilde{x}$ of

$$\begin{cases} \dot{\tilde{x}}(t) &= \sqrt{2r(t, \tilde{x}(t))} + A_\varepsilon\left(\frac{\Delta x}{\Delta t} + \Delta t\right) \\ \tilde{x}(0) &= 0, \end{cases}$$

on [0,T]. Thanks to Equation (B.112) (see Appendix B), we have

$$\max_{j \in [\![0, N-1]\!]} |\tilde{x}(s_j) - \tilde{y}_j| \leqslant \frac{1}{2}e^{LT}\varepsilon\lfloor\varepsilon^{-1}\rfloor\Delta t \leqslant \frac{1}{2}e^{LT}\Delta t.$$

Thus, for all $j \in [\![0, N-1]\!]$ and $t \in [s_j, s_{j+1}]$,

$$\begin{aligned}
|\tilde{x}(t) - f(t)| &\leqslant |\tilde{x}(t) - \tilde{x}(s_j)| + |\tilde{x}(s_j) - \tilde{y}_j| + |f(t) - \tilde{y}_j| \\
&\leqslant 2\left(\sqrt{2\|r\|_\infty} + A_\varepsilon\left(\frac{\Delta x}{\Delta t} + \Delta t\right)\right)\Delta t + \frac{1}{2}e^{LT}\Delta t.
\end{aligned}$$

Let us now compare $\tilde{x}$ with the solution x of $(\mathcal{C})$ on $[0, T]$. According to Lemma B.1, we have

$$\max_{t\in[0,T]} |x(t) - \tilde{x}(t)| \leqslant A_\varepsilon \left(\frac{\Delta x}{\Delta t} + \Delta t\right) e^{LT}.$$

Thus, there exists a constant $B > 0$ that only depends on r and T such that, for all $\varepsilon < 1$

$$\sup_{t\in[0,T]} |x(t) - f(t)| \leqslant B \left(\Delta t + \Delta x + \frac{\Delta x}{\Delta t}\right). \tag{4.60}$$

Finally, combining (4.59) and (4.60) and choosing $T + 1$ instead of T, we get that for all $K > 0$ and $\varepsilon < \varepsilon_0$

$$\mathbb{P}\left(\exists k \in [\![0, \lfloor T/(\varepsilon\Delta t)\rfloor]\!] : \varepsilon X_k^* > x(k\varepsilon\Delta t) + B\left(\Delta t + \Delta x + \frac{\Delta x}{\Delta t}\right)\right) \leqslant \left(\frac{T+1}{\Delta t}\right)^2 K e^{-\frac{\alpha}{\varepsilon}},$$

which concludes the proof.

5 Proof of Theorem 2.2: Lower bound on the propagation speed

In this part, we establish a lower bound on the propagation speed of the process $\mathbf{X}$ under Assumptions 1 and 3. Note that Assumption 3 ensures that the process does not vanish and that the population size on each site is limited to K individuals, before the migration phase. The idea of the proof of Theorem 2.2 is to construct a minimising process $\mathbf{X}^0$ in which the effect of local competition is negligible (Section 5.1) so that it can be compared to a BRW (Section 5.2), and then to the solution of the ODE $(\mathcal{C})$ (Section 5.3). In contrast to Section 4, we can no longer compare the process $\mathbf{X}$ with several BRW over $\lfloor\varepsilon^{-1}\rfloor$ generations. That is why, we will consider smaller time intervals, of order $\log(K) \ll \lfloor\varepsilon^{-1}\rfloor$, during which the population size does not grow too much. Note that the length of the time steps considered in Section 4.2 was only constrained by the scale of heterogeneity of the function r and not by the carrying capacity of the environment.

In Section 5, we denote by c the unique positive solution of

$$I(c) = \log(1 + \|r\|_\infty \Delta t). \tag{5.61}$$

5.1 The rebooted process $\mathbf{X}^0$

As explained above, this subsection is aimed at constructing a minimising process $\mathbf{X}^0$ in which we can ignore the effect of local competition. By minimising process we mean a process that can be coupled with $\mathbf{X}$ in such a way that it constitutes a subtree of $\mathbf{X}$. We recall that X_k^{0*} denotes the position of the rightmost particle in the process $\mathbf{X}^0$ at generation k.

The idea of the following construction is to "reboot" the process $\mathbf{X}$ before its population size gets too large. Let $(\varphi(k))_{k\in\mathbb{N}}$ be a sequence of rebooting times *i.e.* an increasing sequence of integers. The process $\mathbf{X}^0$ starts with a single particle at X_0^* and has the same reproduction and migration laws as $\mathbf{X}$. At generation $\varphi(1)$, all the particles in $\mathbf{X}^0$ are killed, except one, located at $X^{0*}_{\varphi(1)}$. The process $\mathbf{X}^0$ then evolves as $\mathbf{X}$ until the following rebooting time $\varphi(2)$. Similarly, $\mathbf{X}^0$ is rebooted at each generation $\varphi(k)$ and is distributed as $\mathbf{X}$ between generations $\varphi(k)$ and $\varphi(k+1)$.

The goal of the following lemma is to show that for K large enough, the population size of $\mathbf{X}^0$ does not exceed K with high probability for

$$\varphi(k) = k\lfloor \log(K) \rfloor. \tag{5.62}$$

Lemma 5.1. *Let $\Delta t < \|r\|_\infty^{-1}$ and $\Delta x > 0$. Let $K > 0$. Consider a branching random walk of reproduction law $\nu_{\|r\|_\infty}$ and displacement law μ, starting with a single particle at 0. Let τ_0 be the the first generation during which the population size of the process exceeds K. Then,*

$$\mathbb{P}\left(\tau_0 \leqslant \lfloor \log(K) \rfloor\right) \leqslant K^{\log(1+\|r\|_\infty \Delta t)-1}.$$

Proof. Let N_k be the number of individuals alive during generation k in the BRW. Recall that (N_k) is a Galton-Watson process of reproduction law $\nu_{\|r\|_\infty}$. According to Assumption 3, the expectation of the reproduction law is equal to $1 + \|r\|_\infty \Delta t$. Thanks to basic results on Galton-Watson processes, we know that $((1 + \|r\|_\infty \Delta t)^{-k} N_k)_{k \geqslant 0}$ is a positive martingale of mean one. Thus, Doob's inequality implies that

$$\begin{aligned}
\mathbb{P}(\tau_0 \leqslant \lfloor \log(K) \rfloor) &= \mathbb{P}\left(\max_{l \leqslant \lfloor \log(K) \rfloor} N_l \geqslant K\right) \\
&\leqslant \mathbb{P}\left(\max_{l \leqslant \lfloor \log(K) \rfloor} \frac{N_l}{(1+\|r\|_\infty \Delta t)^l} \geqslant \frac{K}{(1+\|r\|_\infty \Delta t)^{\log(K)}}\right) \\
&\leqslant \frac{(1+\|r\|_\infty \Delta t)^{\log K}}{K} = e^{(\log(1+\|r\|_\infty \Delta t)-1)\log(K)}.
\end{aligned}$$

$\square$

Note that there exists a coupling between $\mathbf{X}^0$ and a BRW of reproduction law $\nu_{\|r\|_\infty}$, displacement law μ, starting with a single particle at $X^{0*}_{\varphi(k)}$ on each time interval $[t_{\varphi(k)}, t_{\varphi(k+1)}]$. Thus, if we consider a sequence of rebooting times $(\varphi(k))_{k \in \mathbb{N}}$ defined by (5.62), the probability that the population size of $\mathbf{X}^0$ exceeds K between generations $\varphi(k)$ and $\varphi(k+1)$ is bounded by $K^{\log(1+\|r\|_\infty \Delta t)-1}$, which tends to 0 as K tends to infinity if $\Delta t < \|r\|_\infty^{-1}$.

5.2 Comparison with a branching random walk

In this section, we bound the first (Lemma 5.5 and 5.3) and the second moment (Lemma 5.4) of the increments of the process $(X^{0*}_k)_{k \in \mathbb{N}}$ between generations $\varphi(k)$ and $\varphi(k+1)$, for φ defined by (5.62). Let $(\mathcal{F}_k)_{k \in \mathbb{N}}$ be the standard filtration associated with X^{0*}. In Lemma 5.2, we state a result on some stopping times, that will be needed to construct a coupling between $\mathbf{X}^0$ and a BRW between generations $\varphi(k)$ and $\varphi(k+1)$. In what follows, we denote by h the function defined by

$$h(x) = \frac{e^{-\frac{\gamma\sqrt{\|r\|_\infty}}{4\sqrt{2}}x}}{1-e^{-\frac{\gamma\sqrt{\|r\|_\infty}}{4\sqrt{2}}x}}, \quad \forall x > 0. \tag{5.63}$$

Note that $h(x) \to 0$ as $x \to +\infty$.

Lemma 5.2. *Let $\Delta t < \|r\|_\infty^{-1}$ and $\Delta x < \frac{1}{16}\sqrt{2\gamma \underline{r}\Delta t}$. Let $K > 0$ and $\varepsilon \leqslant (4\sqrt{2\|r\|_\infty}\varphi(1)\Delta t)^{-4}$. Let $r \in [\underline{r}, \|r\|_\infty]$. Consider a branching random walk of reproduction law ν_r, displacement law μ, starting with a single particle at 0. Denote by Ξ_v the position of a particule v living in the BRW and by N_k the size of the process at generation k, and define*

$$\tau_K = \inf\{k \in \mathbb{N} : N_k \geqslant K\} \text{ and } \tau_\varepsilon = \inf\left\{k \in \mathbb{N} : \exists v : |v| = k, |\Xi_v| > \varepsilon^{-1/4}\right\}.$$

Then,

$$\mathbb{P}(\tau_K \leqslant \varphi(1)) \leqslant K^{\log(1+\|r\|_\infty\Delta t)-1},$$

and,

$$\mathbb{P}(\tau_\varepsilon \leqslant \varphi(1)) \leqslant 2h\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right).$$

Proof. The branching random walk defined in Lemma 5.2 can be coupled with a BRW of reproduction law $\nu_{\|r\|_\infty}$, displacement law μ, starting with a single particle at 0. Thus, the estimate on τ_K directly ensues from Lemma 5.1 and it is sufficient to establish the result on τ_ε for $r = \|r\|_\infty$. Thanks to a similar argument than in Equation (4.32), one can prove that, for any particle v living during the n-th generation,

$$\mathbb{P}(\Xi_v > \varepsilon^{-1/4}) \leqslant e^{-nI\left(\frac{\varepsilon^{-1/4}}{n}\right)}.$$

Thus, by the many-to-one lemma (see Lemma A.1) and by symmetry of μ, we get that, for all $n \leqslant \varphi(1)$,

$$\begin{aligned}
\mathbb{P}(\exists v : |v| = n, |\Xi_v| > \varepsilon^{-1/4}) &\leqslant 2(1+\|r\|_\infty\Delta t)^n e^{-nI\left(\frac{\varepsilon^{-1/4}}{n}\right)} \\
&\leqslant 2e^{-n\left(I\left(\frac{\varepsilon^{-1/4}}{n}\right)-\log(1+\|r\|_\infty\Delta t)\right)} \\
&\leqslant 2e^{-n\left(I\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right)-\log(1+\|r\|_\infty\Delta t)\right)} \\
&= 2e^{-n\left(I\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right)-I(c)\right)}.
\end{aligned}$$

Besides, I is convexe, therefore $I\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right) \geqslant \frac{I(c)}{c}\frac{\varepsilon^{-1/4}}{\varphi(1)}$ as long as $\varepsilon^{-1/4} \geqslant c\varphi(1)$. Thus, if $\varepsilon^{-1/4} \geqslant 2c\varphi(1)$,

$$I\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right) - I(c) \geqslant I(c)\left(\frac{\varepsilon^{-1/4}}{c\varphi(1)} - 1\right) \geqslant \frac{I(c)}{2c\varphi(1)}\varepsilon^{-1/4} \geqslant \frac{\gamma\sqrt{\|r\|_\infty}}{4\sqrt{2}}\frac{\varepsilon^{-1/4}}{\varphi(1)}, \tag{5.64}$$

since $I(c) = \log(1+\|r\|_\infty \Delta t) \geq \gamma \|r\|_\infty \Delta t$ and $c \leqslant 2\sqrt{2\|r\|_\infty}\Delta t$ according to Lemma 4.1. Hence,

$$\begin{aligned}\mathbb{P}(\tau_\varepsilon \leqslant \varphi(1)) &= \mathbb{P}\left(\exists n \leqslant \varphi(1) : \exists v : |v| = n, |\Xi_v| > \varepsilon^{-1/4}\right)\\ &\leqslant 2\sum_{n=1}^{\varphi(1)} e^{-n\left(I\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right) - \log(1+\|r\|_\infty \Delta t)\right)}\\ &\leqslant 2h\left(\frac{\varepsilon^{-1/4}}{\varphi(1)}\right).\end{aligned}$$

□

Lemma 5.3 (Lower bound on the first moment)**.** *Let $\Delta t < \|r\|_\infty^{-1}$, $\Delta x < \frac{1}{16}\sqrt{2\gamma \underline{r}\Delta t}$ and $\eta > 0$. There exists $K_0 > 0$ such that, for all $K > K_0$, there exists ε_0 such that, for all $\varepsilon < \varepsilon_0$,*

$$\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)} | \mathcal{F}_{\varphi(k)}\right] \geqslant (c_{\varphi(k)} - \eta)\varphi(1),$$

with $c_{\varphi(k)}$ the unique positive solution of $I(c_{\varphi(k)}) = \log(1 + r_{\varphi(k)}\Delta t)$ for

$$r_{\varphi(k)} = \min\left\{r(\varepsilon t, \varepsilon x),\ (t,x) \in [\varphi(k)\Delta t, \varphi(k+1)\Delta t] \times [X^{0*}_{\varphi(k)} - \varepsilon^{-1/4}, X^{0*}_{\varphi(k)} + \varepsilon^{-1/4}]\right\}. \quad (5.65)$$

Proof. For the sake of simplicity, we assume that $\mathbf{X}^0$ starts at generation $\varphi(k)$ with a deterministic configuration $n^0_{\varphi(k)} = \delta_{X^{0*}_{\varphi(k)}}$. The estimates we obtain will not depend on this initial condition. The proof of the lemma relies on a coupling argument with a branching random walk of reproduction law $\nu_{r_{\varphi(k)}}$.

Consider a branching random walk Ξ of reproduction law $\nu_{r_{\varphi(k)}}$, displacement law μ, starting with a single particle at $X^{0*}_{\varphi(k)}$ at time $\varphi(k)\Delta t$. We denote by Ξ_v the position of a particule v living in this BRW and by N_k the size of the BRW at generation k, and define

$$\tau'_K = \inf\{l \geq \varphi(k) : N_l \geqslant K\} \text{ and } \tau'_\varepsilon = \inf\left\{l \geq \varphi(k) : \exists v : |v| = l, |\Xi_v - X^{0*}_{\varphi(k)}| > \varepsilon^{-1/4}\right\}.$$

In addition, for $n \geqslant \varphi(k)$, define

$$M_n = \max\{\Xi_v, |v| = n\}. \quad (5.66)$$

According to Lemma 3.1, one can couple $\mathbf{X}^0$ and Ξ such that Ξ is a subtree of $\mathbf{X}^0$ until generation $\varphi(k) + (\tau'_K \wedge \tau'_\varepsilon)$. Thus, we have

$$\begin{aligned}\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}\right] &= \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon \leqslant \varphi(k+1)}\right] + \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon > \varphi(k+1)}\right]\\ &\geqslant \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon \leqslant \varphi(k+1)}\right] + \mathbb{E}\left[(M_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon > \varphi(k+1)}\right].\end{aligned}$$

Note that $\mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon \leqslant \varphi(k+1)}\right] \geqslant 0$. Indeed, we know that for any particle v from generation $\varphi(k+1)$,

$$\mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon \leqslant \varphi(k+1)}\right] \geqslant \mathbb{E}\left[(X^0_v - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau'_K \wedge \tau'_\varepsilon \leqslant \varphi(k+1)}\right].$$

Therefore, using that the displacements of the particles and the population size are independent and that the displacement distribution μ is symmetric, we get that

$$\mathbb{E}\left[(X_v^0 - X_{\varphi(k)}^{0*})\mathbb{1}_{\tau_K' \wedge \tau_\varepsilon' \leqslant \varphi(k+1)}\right] = 0.$$

Besides,

$$\mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})\mathbb{1}_{\tau_K' \wedge \tau_\varepsilon' > \varphi(k+1)}\right] = \mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})\right] - \mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})\mathbb{1}_{\tau_K' \wedge \tau_\varepsilon' \leqslant \varphi(k+1)}\right], \tag{5.67}$$

and by the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})\mathbb{1}_{\tau_K' \wedge \tau_\varepsilon' \leqslant \varphi(k+1)}\right] \leqslant \sqrt{\mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})^2\right]}\sqrt{\mathbb{P}(\tau_K' \wedge \tau_\varepsilon' \leqslant \varphi(k+1))}. \tag{5.68}$$

Note that, by translational invariance, τ_K (resp. τ_ε) follows the same law as τ_K' (resp. τ_ε') from Lemma 5.2. Besides, according to Corollary A.1 and Lemma A.7, there exists $K_0 > 0$, that does not depend on $r_{\varphi(k)}$, nor on $X_{\varphi(k)}^{0*}$ such that, if $K > K_0$,

$$\mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})^2\right] \leqslant 4c^2\varphi(1)^2, \tag{5.69}$$

and,

$$\mathbb{E}\left[(M_{\varphi(k+1)} - X_{\varphi(k)}^{0*})\right] \geqslant (c_{\varphi(k)} - \eta)\varphi(1). \tag{5.70}$$

Hence, combining (5.67), (5.68), (5.69) and (5.70), we get that

$$\mathbb{E}\left[X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*}\right] \geqslant (c_{\varphi(k)} - \eta)\varphi(1) - 2c\varphi(1)\sqrt{\mathbb{P}(\tau_K \wedge \tau_\varepsilon \leqslant \varphi(1))}.$$

Finally, remark that $c_{\varphi(k)} \leqslant c$ since I is increasing on $(0, \infty)$ and that, according to Lemma 5.2, there exists $K_1 > 0$ such that, for all $K > K_1$, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$,

$$\mathbb{P}\left(\tau_k \wedge \tau_\varepsilon \leqslant \varphi(1)\right) \leqslant \frac{\eta^2}{c^2},$$

so that, for $K > \max(K_0, K_1)$ and $\varepsilon < \varepsilon_1$,

$$\mathbb{E}\left[X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*}\right] \geqslant (c_{\varphi(k)} - 3\eta)\varphi(1).$$

This proves the lemma. □

Lemma 5.4 (Upper bound on the second moment)**.** *Let $\Delta t < \|r\|_\infty^{-1}$, $\Delta x < \frac{1}{16}\sqrt{2\gamma \underline{r} \Delta t}$. There exists $K_0 > 0$ such that, for all $K > K_0$, for all $\varepsilon > 0$,*

$$\mathbb{E}\left[\left(X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*}\right)^2 |\mathcal{F}_{\varphi(k)}\right] \leqslant (4c^2 + \Delta t)\varphi(1)^2.$$

Proof. One can couple $\mathbf{X}^0$ and a BRW Ξ of reproduction law $\nu_{\|r\|_\infty}$, displacement law μ, such that $\mathbf{X}^0$ constitutes a subtree of Ξ until generation $\varphi(k+1)$. Then, if we denote by M_n the position of the rightmost particle in Ξ at generation n, we get that

$$\begin{aligned}\mathbb{E}\left[\left(\left(X_{\varphi(k+1)}^{0*} - X_{\varphi(k)}^{0*}\right) \vee 0\right)^2 |\mathcal{F}_{\varphi(k)}\right] &\leqslant \mathbb{E}\left[\left(M_{\varphi(1)} \vee 0\right)^2 |\mathcal{F}_{\varphi(k)}\right] \\ &\leqslant \mathbb{E}\left[M_{\varphi(1)}^2 |\mathcal{F}_{\varphi(k)}\right] \leqslant 4c^2\varphi(1)^2,\end{aligned}$$

for K large enough, according to Corollary A.1. Then, consider (Z_n) a random walk whose increments are distributed as μ and remark that

$$\begin{aligned}\mathbb{E}\left[\left(\left(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}\right)\wedge 0\right)^2 |\mathcal{F}_{\varphi(k)}\right] &\leqslant \mathbb{E}\left[\left(Z_{\varphi(1)}\wedge 0\right)^2 |\mathcal{F}_{\varphi(k)}\right]\\ &\leqslant \mathbb{E}\left[Z^2_{\varphi(1)}|\mathcal{F}_{\varphi(k)}\right] \leqslant \varphi(1)^2\Delta t,\end{aligned}$$

which concludes the proof of the lemma. □

Lemma 5.5 (Upper bound on the first moment)**.** *Let $\Delta t < \|r\|_\infty^{-1}$, $\Delta x < \frac{1}{16}\sqrt{2\gamma \underline{r}\Delta t}$ and $\eta > 0$. There exists $K_0 > 0$ such that, for all $K > K_0$, there exists ε_0 such that, for all $\varepsilon < \varepsilon_0$,*

$$\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}|\mathcal{F}_{\varphi(k)}\right] \leqslant (\bar{c}_{\varphi(k)} + \eta)\varphi(1)$$

with $\bar{c}_{\varphi(k)}$ the unique positive solution of $I(\bar{c}_{\varphi(k)}) = \log(1 + \bar{r}_{\varphi(k)}\Delta t)$ for

$$\bar{r}_{\varphi(k)} = \max\left\{r(\varepsilon t, \varepsilon x),\ (t,x) \in [\varphi(k)\Delta t, \varphi(k+1)\Delta t] \times [X^{0*}_{\varphi(k)} - \varepsilon^{-1/4}, X^{0*}_{\varphi(k)} + \varepsilon^{-1/4}]\right\}.$$

Proof. The proof is similar of the proof of Lemma 5.3 but some details are different, which is why we give a complete proof. Again, we assume that $\mathbf{X}^0$ starts at generation $\varphi(k)$ with a deterministic configuration $n^0_{\varphi(k)} = \delta_{X^{0*}_{\varphi(k)}}$. The estimates we obtain will not depend on this initial condition.

Consider a branching random walk $\tilde{\Xi}$ of reproduction law $\nu_{\bar{r}_{\varphi(k)}}$, displacement law μ, starting with a single particle at $X^{0*}_{\varphi(k)}$ at time $\varphi(k)\Delta t$. We denote by $\tilde{\Xi}_v$ the position of a particule v living in this BRW and define

$$\tau''_\varepsilon = \inf\left\{l \geq \varphi(k) : \exists v : |v| = l, |\tilde{\Xi}_v - X^{0*}_{\varphi(k)}| > \varepsilon^{-1/4}\right\}.$$

In addition, for $n \geqslant \varphi(k)$, we define

$$\tilde{M}_n = \max\{\tilde{\Xi}_v, |v| = n\}. \tag{5.71}$$

According to Lemma 3.1, one can couple $\mathbf{X}^0$ and $\tilde{\Xi}$ such that $\mathbf{X}^0$ constitutes a subtree of $\tilde{\Xi}$ until generation $\varphi(k) + \tau''_\varepsilon$. Therefore, we have

$$\begin{aligned}\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}\right] &= \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau''_\varepsilon \leqslant \varphi(k+1)}\right] + \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau''_\varepsilon > \varphi(k+1)}\right]\\ &\leqslant \mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau''_\varepsilon \leqslant \varphi(k+1)}\right] + \mathbb{E}\left[(M_{\varphi(k+1)} - X^{0*}_{\varphi(k)})\mathbb{1}_{\tau''_\varepsilon > \varphi(k+1)}\right]\\ &\leqslant \sqrt{\mathbb{E}\left[(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})^2\right]}\sqrt{\mathbb{P}(\tau''_\varepsilon \leqslant \varphi(k+1))}\\ &\quad + \sqrt{\mathbb{E}\left[(\tilde{M}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})^2\right]}\sqrt{\mathbb{P}(\tau''_\varepsilon > \varphi(k+1))},\end{aligned} \tag{5.72}$$

thanks to the Cauchy-Schwarz inequality. According to Lemma A.8, there exists $K_0 > 0$, that does not depend on $r_{\varphi(k)}$, nor on $X^{0*}_{\varphi(k)}$, such that, for all $K > K_0$,

$$\mathbb{E}\left[(\tilde{M}_{\varphi(k+1)} - X^{0*}_{\varphi(k)})^2|\mathcal{F}_{\varphi(k)}\right] \leqslant (\bar{c}_{\varphi(k)} + \eta)^2\varphi(1)^2. \tag{5.73}$$

Besides, according to Lemma 5.4, there exists $K_1 > 0$ such that for all $K > K_1$,

$$\mathbb{E}\left[\left(X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}\right)^2 |\mathcal{F}_{\varphi(k)}\right] \leqslant (4c^2 + \Delta t)\varphi(1)^2. \tag{5.74}$$

Let us now assume that $K > \max(K_0, K_1)$. By translational invariance, τ''_ε follows the same law as τ_ε from Lemma 5.2, so that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$,

$$\mathbb{P}\left(\tau_\varepsilon \leqslant \varphi(1)\right) \leqslant \frac{\eta^2}{(4c^2 + \Delta t)^2}. \tag{5.75}$$

Finally, combining Equations (5.72), (5.73), (5.74) and (5.75), we get that, for $\varepsilon < \varepsilon_1$,

$$\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}\right] \leqslant (\bar{c}_{\varphi(k)} + 2\eta)\varphi(1),$$

which concludes the proof of the lemma. $\square$

5.3 Comparison with the solution of $(\mathcal{C})$

Lemma 5.6. *Let $\delta \in (0,1)$. There exists $C_\delta > 0$ such that, if*

$$\Delta t < \left((1 \wedge C_\delta)\|r\|_\infty^{-1}\right), \quad \Delta x < \left(\frac{1}{16}\sqrt{2\gamma\underline{r}} \wedge \frac{\sqrt{2\underline{r}}}{3a}\delta\right)\Delta t, \tag{H$_\delta$}$$

then, for all $K > 0$, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$,

$$c_{\varphi(k)} \geqslant (1-\delta)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t, \quad \forall k \in \mathbb{N},$$

and

$$\bar{c}_{\varphi(k)} \leqslant (1+\delta)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t, \quad \forall k \in \mathbb{N},$$

for $c_{\varphi(k)}$ and $\bar{c}_{\varphi(k)}$ respectively defined in Lemmas 5.3 and 5.5, and a in Lemma 4.2.

Proof. Let $\Delta t < \|r\|_\infty^{-1}$ and $\Delta x < \frac{1}{16}\sqrt{2\gamma\underline{r}}\Delta t$. Thanks to Lemma 4.2, we know that

$$|c_{\varphi(k)} - \sqrt{2\Delta t \log(1 + r_{\varphi(k)}\Delta t)}| \leqslant a\Delta x. \tag{5.76}$$

Let $\delta \in (0,1)$ and and let $C_\delta > 0$ such that $\log(1+x) \geqslant (1-\delta/3)^2 x$, for all $x \leqslant C_\delta$. Let us now assume that $\Delta t < (1 \wedge C_\delta)\|r\|_\infty^{-1}$. Then, Equation (5.76) gives that

$$c_{\varphi(k)} \geqslant \sqrt{2\Delta t \log(1 + r_{\varphi(k)}\Delta t)} - a\Delta x \geqslant (1-\delta/3)\sqrt{2r_{\varphi(k)}}\Delta t - a\Delta x. \tag{5.77}$$

Besides, by definition of $r_{\varphi(k)}$ (see Equation (5.65)), Equation (2.11) gives that

$$|\sqrt{2r_{\varphi(k)}} - \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}| \leqslant L(\varepsilon\varphi(1)\Delta t + \varepsilon^{3/4}), \tag{5.78}$$

for L from Equation (2.10). Combining (5.77) and (5.78) we get that

$$c_{\varphi(k)} \geqslant (1-\delta/3)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t - (1-\delta/3)L(\varepsilon\varphi(1)\Delta t + \varepsilon^{3/4})\Delta t - a\Delta x$$

Remark that Assumption (H$_\delta$) implies that

$$a\Delta x \leqslant \frac{1}{3}\sqrt{2\underline{r}}\delta\Delta t \leqslant \frac{\delta}{3}\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t,$$

so that it is sufficient to choose ε such that

$$\|\nabla r\|_\infty(\varepsilon\varphi(1)\Delta t + \varepsilon^{3/4}) \leqslant \frac{2\underline{r}}{3}\delta,$$

to get

$$c_{\varphi(k)} \geqslant (1-\delta)\sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t.$$

Similarly, Equations (5.76) and (5.78) give that

$$\bar{c}_{\varphi(k)} \leqslant \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\Delta t + L(\varepsilon\varphi(1)\Delta t + \varepsilon^{3/4})\Delta t + a\Delta x,$$

and Assumption (H$_\delta$) implies that $a\Delta x \leqslant \frac{1}{2}\sqrt{2\underline{r}}\delta\Delta t$, so that it is sufficient to choose ε such that $\|\nabla r\|_\infty(\varepsilon\varphi(1)\Delta t + \varepsilon^{3/4})\Delta t \leqslant \underline{r}\delta$ to get the result. $\square$

Corollary 5.1. *Let $\delta > 0$. There exists $C_\delta > 0$ such that, if Δt and Δx satisfy (H$_\delta$), there exists $K_0 > 0$ such that, for all $K > K_0$, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$,*

$$\left|\mathbb{E}\left[X^{0*}_{\varphi(k+1)} - X^{0*}_{\varphi(k)}|\mathcal{F}_{\varphi(k)}\right] - \sqrt{2r(\varepsilon t_{\varphi(k)}, \varepsilon X^{0*}_{\varphi(k)})}\varphi(1)\Delta t\right| \leqslant 2\sqrt{2\|r\|_\infty}\delta\varphi(1)\Delta t.$$

Proof. We choose $\eta = \delta\sqrt{2\|r\|_\infty}\Delta t$ in Lemmas 5.3 and 5.5. $\square$

Proof of Theorem 2.2. Let $\delta \in (0,1)$ and assume that Δt and Δx satisfy (H$_\delta$). Let $T > 0$ and $N = \left\lfloor \frac{T}{\varepsilon\varphi(1)\Delta t} \right\rfloor$. We define the Euler scheme $(y_k)_{k=0}^N$ of Equation ($\mathcal{C}$) on $[0,T]$, of time step $\varepsilon\varphi(1)\Delta t$, by $y_0 = \varepsilon X^{0*}_0$ and

$$y_{k+1} = y_k + \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)}\varepsilon\varphi(1)\Delta t, \quad \forall k \in [\![0, N-1]\!].$$

We also consider the processes $(Y_k)_{k=0}^N$, $(Z_k)_{k=0}^N$ and $(W_k)_{k=0}^N$ such that $Y_k = \varepsilon X^{0*}_{\varphi(k)}$,

$$W_k = \mathbb{E}\left[Y_{j+1} - Y_j|\mathcal{F}_{\varphi(j)}\right], \text{ and } Z_k = Y_k - \sum_{j=0}^{k-1}\mathbb{E}\left[Y_{j+1} - Y_j|\mathcal{F}_{\varphi(j)}\right] = Y_k - W_k, \quad \forall k \in [\![0, N]\!].$$

Since (Z_k) is a martingale, we have

$$\mathrm{Var}(Z_k) = \sum_{j=0}^{k-1}\mathbb{E}[\mathrm{Var}(Y_{j+1} - Y_j|\mathcal{F}_{\varphi(j)})], \quad \forall k \in [\![0, N]\!]. \tag{5.79}$$

Besides, Lemma 5.4 implies that there exists $K_0 > 1$ such that, for all $K \geq K_0$,

$$\mathrm{Var}(Y_{j+1} - Y_j | \mathcal{F}_{\varphi(j)}) \leqslant (4c^2 + \Delta t)\varepsilon^2 \varphi(1)^2, \quad \forall j \in [\![0, N-1]\!]. \tag{5.80}$$

Thus,

$$\mathrm{Var}(Z_N) \leqslant (4c^2 + \Delta t)\varepsilon^2 \varphi(1)^2 N \leqslant (4c^2 + \Delta t)\varphi(1)\varepsilon T.$$

Moreover, let A_δ be the event

$$\left\{ \max_{k=0,\dots,N} |Z_k - Y_0| \leqslant \delta \right\},$$

then, by Doob's inequality, we know that

$$\mathbb{P}(A_\delta^c) \leqslant \frac{\mathrm{Var}(Z_N)}{\delta^2} \leqslant \frac{32 T \varphi(1)}{\delta^2} \varepsilon. \tag{5.81}$$

Besides, we know that on the event A_δ, $|Y_k - Y_0 - W_k| = |Z_k - Y_0| \leqslant \delta$ for all $k \in [\![0, N]\!]$ and, by triangle inequality

$$\begin{aligned} |Y_{k+1} - y_{k+1}| &\leqslant |Y_{k+1} - W_{k+1} - Y_0| + |W_{k+1} - Y_0 - y_{k+1}| \\ &\leqslant \delta + |W_{k+1} - Y_0 - y_{k+1}|, \end{aligned} \tag{5.82}$$

for all $k \in [\![0, N-1]\!]$. Moreover, using the definition of W_k and y_k, we get by triangle inequality that on the event A_δ, for $k \in [\![0, N-1]\!]$,

$$\begin{aligned} |W_{k+1} + Y_0 - y_{k+1}| &\leqslant |W_k + Y_0 - y_k| + \left| \mathbb{E}[Y_{k+1} - Y_k | Y_k] - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)}\varepsilon\varphi(1)\Delta t \right| \\ &\leqslant |W_k + Y_0 - y_k| + \left| \mathbb{E}[Y_{k+1} - Y_k | Y_k] - \sqrt{2r(\varepsilon t_{\varphi(k)}, Y_k)}\varepsilon\varphi(1)\Delta t \right| \\ &\quad + \left| \sqrt{2r(\varepsilon t_{\varphi(k)}, Y_k)}\varepsilon\varphi(1)\Delta t - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)}\varepsilon\varphi(1)\Delta t \right|. \end{aligned} \tag{5.83}$$

According to Corollary 5.1, there exists $K_1 > 0$ such that for all $K \geqslant K_1$, there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon < \varepsilon_1$, $k \in [\![0, N-1]\!]$,

$$|\mathbb{E}[Y_{k+1} - Y_k | Y_k] - \sqrt{2r(\varepsilon t_{\varphi(k)}, Y_k)}\varepsilon\varphi(1)\Delta t| \leqslant \sqrt{2\|r\|_\infty}\varepsilon\delta\varphi(1)\Delta t. \tag{5.84}$$

Besides, recall from Equation (2.11) that, for $k \in [\![0, N-1]\!]$, we have

$$\left| \sqrt{2r(\varepsilon t_{\varphi(k)}, Y_k)}\varepsilon\varphi(1)\Delta t - \sqrt{2r(\varepsilon t_{\varphi(k)}, y_k)}\varepsilon\varphi(1)\Delta t \right| \leqslant L|y_k - Y_k|\varepsilon\varphi(1)\Delta t. \tag{5.85}$$

Let us now assume that $K > \max(K_0, K_1)$ and $\varepsilon < \varepsilon_1(K)$. Combining Equations (5.83), (5.84) and (5.85), we obtain that, on A_δ, we have

$$|y_{k+1} - W_{k+1} - Y_0| \leqslant |y_k - W_k - Y_0| + \varepsilon\varphi(1)\Delta t \left(\sqrt{2\|r\|_\infty}\delta + L|Y_k - y_k| \right), \tag{5.86}$$

for all $k \in [\![0, N-1]\!]$. Thus, combining Equations (5.82) and (5.86), we get that, on A_δ, for all $k \in [\![0, N-1]\!]$,

$$\begin{aligned} |y_{k+1} - W_{k+1} - Y_0| &\leqslant |y_k - W_k - Y_0| + \varepsilon\varphi(1)\Delta t \left(\sqrt{2\|r\|_\infty}\delta + L\big(|y_k - W_k - Y_0| + \delta\big) \right) \\ &\leqslant (1 + L\varepsilon\varphi(1)\Delta t)\,|y_k - W_k - Y_0| + \varepsilon\delta\varphi(1)\Delta t \left(\sqrt{2\|r\|_\infty} + L \right) \\ &\leqslant e^{L\varepsilon\varphi(1)\Delta t}|y_k - W_k - Y_0| + \varepsilon\delta\varphi(1)\Delta t \left(\sqrt{2\|r\|_\infty} + L \right). \end{aligned}$$

Then, by induction, using that $Y_0 = y_0$ and $W_0 = 0$, we get that on A_δ,

$$\begin{aligned} |y_{k+1} - W_{k+1} - Y_0| &\leqslant \varepsilon\delta\varphi(1)\Delta t\left(\sqrt{2\|r\|_\infty} + L\right)\sum_{j=0}^{k} e^{jL\varepsilon\varphi(1)\Delta t} \\ &\leqslant N\varepsilon\varphi(1)\Delta t\delta\left(\sqrt{2\|r\|_\infty} + L\right)e^{NL\varepsilon\varphi(1)\Delta t} \\ &\leqslant \left(\sqrt{2\|r\|_\infty} + L\right)\delta T e^{LT}, \end{aligned} \tag{5.87}$$

for all $k \in [\![0, N-1]\!]$. If we set $\alpha = 1 + \left(\sqrt{2\|r\|_\infty} + L\right) T e^{LT}$, we obtain that

$$Y_k \geqslant y_k - \alpha\delta, \quad \forall k \in [\![0, N]\!], \quad \text{on } A_\delta, \tag{5.88}$$

thanks to (5.82) and (5.87). Moreover, note that

$$\begin{aligned} &\mathbb{P}\left(\exists k \in [\![0, N-1]\!] : \exists l \in [\![\varphi(k), \varphi(k+1)]\!] : \varepsilon X_l^{0*} < y_k - 2\alpha\delta\right) \\ &\leqslant \mathbb{P}\left(\exists k \in [\![0, N-1]\!] : \exists l \in [\![\varphi(k), \varphi(k+1)]\!] : \varepsilon X_l^{0*} < y_k - 2\alpha\delta | A_\delta\right) \\ &\quad + \mathbb{P}(A_\delta^c) \end{aligned} \tag{5.89}$$

To show that the first term in (5.89) is small, we prove that the probability of the event

$$\left\{\exists l \in [\![\varphi(k), \varphi(k+1)]\!] : X_l^{0*} - X_{\varphi(k)}^{0*} < -\frac{\alpha\delta}{\varepsilon}\right\}$$

decays exponentially as ε goes to zero, for all $k \in [\![0, N-1]\!]$. Let $l \in [\![\varphi(k), \varphi(k+1)]\!]$ and consider a sequence (Z_i) of *i.i.d.* random variables of law μ. Note that for any $M \in \mathbb{R}$, $\{X_k^{0*} < M\} \subset \{X_u^0 < M\}$ for any particle u living at generation k. Therefore,

$$\begin{aligned} \mathbb{P}\left(X_l^{0*} - X_{\varphi(k)}^{0*} < -\frac{\alpha\delta}{\varepsilon}\right) &\leqslant \mathbb{P}\left(\sum_{i=0}^{l-\varphi(k)} Z_i < -\frac{\alpha\delta}{\varepsilon}\right) \\ &= \mathbb{P}\left(\sum_{i=0}^{l-\varphi(k)} Z_i > \frac{\alpha\delta}{\varepsilon}\right) \\ &\leqslant e^{-(l-\varphi(k))I\left(\frac{\alpha\delta}{\varepsilon(l-\varphi(k))}\right)} \\ &\leqslant e^{-I\left(\frac{\alpha\delta}{\varepsilon\varphi(1)}\right)}, \end{aligned}$$

using a similar argument than in Equation (4.32). Finally, since I is convex, we know that $I\left(\frac{\alpha\delta}{\varepsilon\varphi(1)}\right) \geqslant \frac{I(\alpha\delta)}{\alpha\delta}\frac{\alpha\delta}{\varepsilon\varphi(1)} = \frac{I(\alpha\delta)}{\varepsilon\varphi(1)}$, as long as $\varepsilon\varphi(1) \leqslant 1$. Thus, by union bound, we get that, for $\varepsilon\varphi(1) \leqslant 1$,

$$\mathbb{P}\left(\exists k \in [\![0, N-1]\!] : \exists l \in [\![\varphi(k), \varphi(k+1)]\!] : X_l^{0*} - X_{\varphi(k)}^{0*} < -\frac{\alpha\delta}{\varepsilon}\right) \leqslant \frac{T}{\varepsilon\Delta t}e^{-\frac{I(\alpha\delta)}{\varepsilon\varphi(1)}}. \tag{5.90}$$

In addition, according to Equation (5.88),

$$\begin{aligned} &\mathbb{P}\left(\exists k \in [\![0, N-1]\!] : \exists l \in [\![\varphi(k), \varphi(k+1)]\!] : \varepsilon X_l^{0*} < y_k - 2\alpha\delta | A_\delta\right) \\ &\leqslant \mathbb{P}\left(\exists k \in [\![0, N-1]\!] : \exists l \in [\![\varphi(k), \varphi(k+1)]\!] : \varepsilon X_l^{0*} - \varepsilon X_{\varphi(k)}^{0*} < -\alpha\delta\right), \end{aligned} \tag{5.91}$$

and since the function x is a solution of $(\mathcal{C})$, Equation (B.112) and the mean value theorem imply that

$$\begin{aligned} |y_k - x\left(l\varepsilon\Delta t\right)| &\leqslant |y_k - x\left(k\varepsilon\varphi(1)\Delta t\right)| + |x\left(k\varepsilon\varphi(1)\Delta t\right) - x\left(l\varepsilon\Delta t\right)| \\ &\leqslant e^{LT}\varepsilon\varphi(1)\Delta t + \sqrt{2\|r\|_\infty}\varepsilon\varphi(1)\Delta t, \end{aligned} \tag{5.92}$$

for all $k \in [\![0, N-1]\!]$ and $l \in [\![\varphi(k), \varphi(k+1)]\!]$. Thus, if we choose ε small enough so that

$$\begin{aligned} \varepsilon\varphi(1)\Delta t(\sqrt{2\|r\|_\infty} + e^{LT}) &\leqslant \alpha\delta, \\ 32T\varphi(1)\varepsilon &\leqslant \frac{\delta^3}{2}, \\ \frac{T}{\varepsilon\Delta t}e^{-\frac{I(\alpha\delta)}{\varepsilon\varphi(1)}} &\leqslant \frac{\delta}{2}, \end{aligned}$$

we get by combining (5.81),(5.89), (5.90), and (5.91) that

$$\mathbb{P}\left(\exists n \in [\![0, \varphi(N)]\!] : X_n^{0*} < y_k - 2\alpha\delta\right) \leqslant \delta,$$

and finally, using Equation (5.92), we conclude that

$$\mathbb{P}\left(\exists n \in [\![0, \varphi(N)]\!] : X_n^{0*} < x(k\varepsilon\Delta t) - 3\alpha\delta\right) \leqslant \delta.$$

Again, choosing $T+1$ instead of T, we get that for all $K > \max(K_0, K_1)$, there exists $\varepsilon'(K) > 0$ such that for all $\varepsilon < \varepsilon'(K)$

$$\mathbb{P}\left(\exists n \in \left[\!\!\left[0, \left\lfloor\frac{T}{\varepsilon\Delta t}\right\rfloor\right]\!\!\right] : X_n^{0*} < x(k\varepsilon\Delta t) - 3\alpha\delta\right) \leqslant \delta. \tag{5.93}$$

We conclude the proof by remarking that for all $K > \max(K_0, K_1)$, one can couple $\mathbf{X}^0$ and a process rebooted every $\lfloor\log(\max(K_0, K_1))\rfloor$ generations in such a way that this process constitutes a subtree of $\mathbf{X}^0$. Besides, the previous computations hold for $K^* = \max(K_0, K_1)$ and yield the existence of an $\varepsilon'(K^*)$ such that (5.93) holds for all $\varepsilon < \varepsilon'(K^*)$. The coupling then implies that it also holds for all $K > K^*$ and $\varepsilon < \varepsilon'(K^*)$. □

A Appendix: the branching random walk

A branching random walk is a particle system governed by a reproduction law $(p_k)_{k\in\mathbb{N}}$ and a displacement law μ. The process starts with a single particle located at the origin. This particle is replaced by N new particles located at positions $(\Xi_1, ..., \Xi_N)$, where N is distributed according to $(p_k)_{k\in\mathbb{N}}$ and (Ξ_i) is an i.i.d. sequence of random variables of law μ, independent of N. These individuals constitute the first gereration of the branching random walk. Similarly, the individuals of the n-th generation reproduce independently of each other according to $(p_k)_{k\in\mathbb{N}}$ and their offspring are independently distributed around the parental location according to μ.

A.1 Many-to-one lemma

Let D_n denote the set of individuals living during the n-th generation of the BRW and remark that $(|D_n|)_{n\in\mathbb{N}}$ constitutes a Galton-Watson process. For an individual $u \in D_n$, we denote by $u_0, ..., u_n$ the set of its ancestors, ordered in the chronological order, so that $u_0 = \varnothing$ and $u_n = u$, and by Ξ_u its position. We assume that $1 < m = \sum_{i=0}^{\infty} kp_k < \infty$.

Lemma A.1 (Many-to-one Lemma). *Let $n \geqslant 1$ and $g : \mathbb{R}^n \to \mathbb{R}^+$ be a measurable function. Let $(Z_k)_{k\in\mathbb{N}}$ be a sequence of random variables such that $(Z_{k+1} - Z_k)$ is i.i.d. of law μ. Then,*

$$\mathbb{E}\left[\sum_{u\in D_n} g(\Xi_{u_1}, ..., \Xi_{u_n})\right] = m^n \mathbb{E}\left[g(Z_1, ..., Z_n)\right]. \tag{A.94}$$

A.2 Regularity of the rate function

In this subsection, we state several results on the function I defined by Equation (4.16) in Section 4. All the notations are introduced in Section 4.1

Lemma A.2. *Let $\Delta t > 0$ and $0 < \underline{y} < \bar{y}$. If $0 < \Delta x < \underline{y}$, then*

$$I^0(y) - y\frac{\Delta x}{\Delta t} \leqslant I(y) \leqslant I^0(y) + 2y\frac{\Delta x}{\Delta t}, \quad \forall y \in [\underline{y}, \bar{y}]. \tag{A.95}$$

Proof. Let $y \in [\underline{y}, \bar{y}]$. First, remark that, for $\lambda > 0$

$$\begin{aligned} \Lambda(\lambda) &= \log \sum_{i\in\mathbb{Z}} \left(\int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} \frac{1}{\sqrt{\Delta t}} \nu\left(\frac{z}{\sqrt{\Delta t}}\right) dz \right) e^{\lambda i \Delta x} \\ &\geqslant \log \sum_{i\in\mathbb{Z}} \left(\int_{(i-\frac{1}{2})\Delta x}^{(i+\frac{1}{2})\Delta x} \frac{1}{\sqrt{\Delta t}} \nu\left(\frac{z}{\sqrt{\Delta t}}\right) e^{\lambda(z-\Delta x)} dz \right) = \Lambda^0(\lambda) - \lambda\Delta x. \end{aligned}$$

Similarly, one can obtain an upper bound on Λ and get the following estimate for any $\lambda > 0$:

$$\Lambda^0(\lambda) - \lambda\Delta x \leqslant \Lambda(\lambda) \leqslant \Lambda^0(\lambda) + \lambda\Delta x. \tag{A.96}$$

Then,

$$\begin{aligned} I(y) = \sup_{\lambda\in\mathbb{R}} \left(\lambda y - \Lambda^{\Delta x}(\lambda)\right) &\geqslant \left(\frac{y}{\Delta t}\right) y - \Lambda\left(\frac{y}{\Delta t}\right) \\ &\geqslant \left(\frac{y}{\Delta t}\right) y - \Lambda^0\left(\frac{y}{\Delta t}\right) - \left(\frac{y}{\Delta t}\right)\Delta x \\ &\geqslant I^0(y) - y\frac{\Delta x}{\Delta t}, \end{aligned} \tag{A.97}$$

according to Equations A.96 and (4.18). Using (A.96) again, we get that for any $\lambda > 0$,

$$\lambda y - \Lambda(\lambda) \leqslant \lambda y - \Lambda^0(\lambda) + \lambda\Delta x \leqslant \sup_{\tilde{\lambda}\in\mathbb{R}} \left(\tilde{\lambda}(y + \Delta x) - \frac{\Delta t}{2}\tilde{\lambda}^2\right) = I^0(y + \Delta x).$$

Note that $\Lambda(\lambda) = \Lambda(-\lambda)$ for all $\lambda \in \mathbb{R}$. Hence, Equation (A.96) implies that

$$\Lambda^0(\lambda) - |\lambda|\Delta x \leqslant \Lambda(\lambda) \leqslant \Lambda^0(\lambda) + |\lambda|\Delta x, \quad \forall \lambda \in \mathbb{R}. \tag{A.98}$$

Then, if $\lambda < 0$, we get from (A.98) that

$$\lambda y - \Lambda(\lambda) \leqslant \lambda y - \Lambda^0(\lambda) - \lambda \Delta x \leqslant I^0(y - \Delta x), \quad \forall y \in \mathbb{R}.$$

Since I^0 is increasing on $(0, \infty)$, we know that $I(y - \Delta x) \leqslant I(y + \Delta x)$ as long as $\Delta x \leqslant y$ and

$$I(y) = \sup_{\lambda \in \mathbb{R}} (\lambda y - \Lambda(\lambda)) \leqslant I^0(y + \Delta x) = I^0(y) + y\frac{\Delta x}{\Delta t} + \frac{\Delta x^2}{2\Delta t}.$$

Since $0 < \Delta x < y$, we have that $\Delta x < 2y$ and

$$I(y) \leqslant I^0(y) + 2y\frac{\Delta x}{\Delta t}.$$

This estimate and Equation (A.97) conclude the proof. $\square$

Lemma A.3. *Let $\Delta t > 0$ and $1 < \underline{\rho} < \bar{\rho}$. If $\Delta x < \frac{1}{16}\sqrt{2\Delta t \log(\underline{\rho})}$, then*

$$\frac{c_0}{2} < c < 2c_0, \quad \forall \rho \in [\underline{\rho}, \bar{\rho}], \tag{A.99}$$

where c_0 is defined by (4.19) for $m = \log(\rho)$ and c is the unique solution of $I(c) = \log(\rho)$.

Proof. Let $\rho \in [\underline{\rho}, \bar{\rho}]$ and $y \in \left[0, \frac{c_0}{2}\right]$. The function I is increasing on $(0, \infty)$, therefore, $I(y) \leqslant I(\frac{c_0}{2})$. Besides, since $\Delta x \leqslant \frac{1}{16}\sqrt{2\Delta t \log(\underline{\rho})} \leqslant \frac{c_0}{16}$,

$$\begin{aligned} I\left(\frac{c_0}{2}\right) &\leqslant I^0\left(\frac{c_0}{2}\right) + c_0\frac{\Delta x}{\Delta t} \leqslant I^0\left(\frac{c_0}{2}\right) + \frac{1}{16}\frac{c_0^2}{\Delta t} \\ &\leqslant \frac{1}{4}I^0(c_0) + \frac{1}{8}\frac{c_0^2}{2\Delta t} = \frac{3}{8}I^0(c_0) = \frac{3}{8}\log(\rho), \end{aligned}$$

according to Lemma A.2 and Equations (4.18) and (4.19). Since I is increasing, this implies that $\frac{c_0}{2} < c$. Let us now consider $y > 2c_0$. Then, $I(y) \geqslant I(2c_0)$ and since $\Delta x \leqslant 2c_0$,

$$\begin{aligned} I(2c_0) &\geqslant I^0(2c_0) - c_0\frac{\Delta x}{\Delta t} = 4I^0(c_0) - c_0\frac{\Delta x}{\Delta t} \\ &\geqslant 4I^0(c_0) - \frac{1}{8}I^0(c_0) \geqslant 3I^0(c_0) = 3\log(\rho), \end{aligned}$$

according to Lemma A.2 and Equations (4.18) and (4.19). Since I is increasing, this estimate implies that $c < 2c_0$. $\square$

Lemma A.4. *Let $\Delta t > 0$, $y > 0$. If $\Delta x < \frac{y}{4}$, then*

$$I(y) + \frac{y^2}{4\Delta t}\eta \leqslant I((1 + \eta)y). \tag{A.100}$$

Proof. The function I is convex on $\mathbb{R}$. Thus,

$$\begin{aligned} I(y) &= I\left(\frac{1}{1+\eta}(1+\eta)y\right) \\ &\leqslant \frac{1}{1+\eta}I((1+\eta)y) + \frac{\eta}{1+\eta}I(0) = \frac{1}{1+\eta}I((1+\eta)y), \end{aligned} \tag{A.101}$$

which is equivalent to $I((1+\eta)y) \geqslant I(y) + \eta I(y)$. Besides, if $\Delta x < \frac{y}{4}$, Lemma A.2 implies that

$$I(y) \geqslant I^0(y) - y\frac{\Delta x}{\Delta t} \geqslant \frac{y^2}{2\Delta t} - \frac{y^2}{4\Delta t} = \frac{y^2}{4\Delta t}. \tag{A.102}$$

Combining (A.101) and (A.102), we get Equation (2.4). $\square$

Lemma A.5. *Let $\Delta t > 0$ and $\underline{y} > 0$. If $\Delta x < \frac{y}{4}$, then*

$$\frac{y}{4\Delta t}|y_1 - y_2| \leqslant |I(y_1) - I(y_2)|, \quad \forall (y_1, y_2) \in [\underline{y}, \bar{y}]^2. \tag{A.103}$$

Proof. Without loss of generality, assume that $\underline{y} < y_1 < y_2$. Since I is convex on $\mathbb{R}$, we know that

$$\frac{I(\underline{y})}{\underline{y}} = \frac{I(\underline{y}) - I(0)}{\underline{y} - 0} \leqslant \frac{I(y_1) - I(\underline{y})}{y_1 - \underline{y}} \leqslant \frac{I(y_2) - I(y_1)}{y_2 - y_1}. \tag{A.104}$$

Besides, we know thanks to Lemma A.2 that

$$I(\underline{y}) \geqslant I^0(\underline{y}) - \underline{y}\frac{\Delta x}{\Delta t} = \frac{\underline{y}^2}{2\Delta t} - \underline{y}\frac{\Delta x}{\Delta t} \geqslant \frac{\underline{y}^2}{4\Delta t}, \tag{A.105}$$

since $\Delta x \leqslant \frac{y}{4\Delta t}$. We conclude the proof by combining (A.104) and (A.105) and recalling that I is non-decreasing on $(0,\infty)$ so that $\frac{I(y_2)-I(y_1)}{y_2-y_1} = \frac{|I(y_2)-I(y_1)|}{|y_2-y_1|}$.

If $y_1 = \underline{y}$, we write (A.104) with $\underline{y} - \varepsilon$ instead of $\underline{y}$ and let ε tend to 0. The case $y_2 = \bar{y}$ can be dealt with similarly. $\square$

A.3 First and second moment of the maximum

For $n \in \mathbb{N}$, define M_n the position of the right-most particle in the branching random walk. In this section we study the asymptotic behaviour of the first and second moments of M_n. These are used for the proof of the lower bound in Section 5. We recall that the reproduction law of the BRW is denoted by $(p_k)_{k\in\mathbb{N}}$.

Lemma A.6 (Biggins' theorem [Big77])**.** *Let $\Delta t > 0$ and $\Delta x > 0$. Assume that $m > 1$ and $p_0 = 0$. Let c be the unique positive solution of $I(c) = \log(m)$. Then,*

$$\lim_{n\to 0} \mathbb{E}\left[\frac{M_n}{n}\right] = c.$$

In fact, Biggins [Big77] proves almost sure convergence of M_n/n. Lemma A.6 follows easily from the subadditive ergodic theorem as outlined in Zeitouni [Zei16].

Lemma A.7. *Let $\Delta t > 0$ and $\Delta x > 0$. Let $1 < \underline{\rho} < \bar{\rho}$. Consider a family of reproduction laws $(p_m)_{m>0}$ such that $\sum_{i=1}^{\infty} kp_{m,k} = m$ and (p_m) is increasing with respect to m (with respect to stochastic domination). Uniformly in $m \in [\underline{\rho}, \bar{\rho}]$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\frac{M_n}{n}\right] = c,$$

where c is the unique positive solution of $I(c) = \log(m)$.

Proof. Define

$$f_n : \begin{cases} (1,\infty) & \to \mathbb{R} \\ m & \mapsto \mathbb{E}\left[\frac{M_n}{n}\right] \end{cases}.$$

We claim that (f_n) is a sequence of increasing functions. Indeed, for any $1 < m_1 < m_2$, consider S^1 (resp. S^2) a branching random walk of reproduction law p_{m_1} (resp. p_{m_2}) and of displacement law μ. According to Lemma 3.1, we can construct a coupling between S^1 and S^2, such that S^1 is a subtree of S^2. Hence, $M_n^1 \leq M_n^2$, where M_n^i denotes the position of the maximal particle in the branching random walk S^i, $i = 1, 2$. It follows that f_n is increasing on $(1,\infty)$, for all $n \in \mathbb{N}$.

Let us now consider the function $c : (1,\infty) \to (0,\infty)$ that maps m to the unique positive solution of $I(c) = \log(m)$. According to Lemma A.5, the function c is continuous on $(1,\infty)$. Moreover, by Lemma A.6, $f_n \to c$ pointwise as $n \to \infty$. Using the monotonicity of f_n, Dini's theorem then yields uniform convergence on the compact sets of $[\underline{\rho}, \bar{\rho}]$. $\square$

Lemma A.8. *Let $\Delta t > 0$, $1 < \underline{\rho} < \bar{\rho}$ and $\Delta x < \frac{1}{16}\sqrt{2\Delta t \log(\underline{\rho})}$. Consider a reproduction law $(p_k)_{k\in\mathbb{N}}$ such that $\sum kp_k = m > 1$ and c the unique positive solution or $I(c) = \log(m)$. For any $\eta > 0$, uniformly in $m \in [\underline{\rho}, \bar{\rho}]$, there exists $N \in \mathbb{N}$ such that*

$$\forall n \geq N, \quad \mathbb{E}\left[\frac{M_n^2}{n^2}\right] \leqslant (c+\eta)^2.$$

Proof. Let us first remark that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\frac{1}{n^2}\left(\max_{|v|=n} \Xi_v\right)^2\right] \leqslant \mathbb{E}\left[\frac{1}{n^2}\left(\max_{|v|=n} |\Xi_v|\right)^2\right].$$

Then, we define $\xi_n = \frac{1}{n}\max_{|v|=n} |\Xi_v|$, and write that, for all $R > 0$,

$$\mathbb{E}[\xi_n^2] = \mathbb{E}[\xi_n^2 \mathbb{1}_{\xi_n^2 < R^2}] + \mathbb{E}[\xi_n^2 \mathbb{1}_{\xi_n^2 \geqslant R^2}]. \tag{A.106}$$

Besides, $\mathbb{E}[\xi_n^2 \mathbb{1}_{\xi_n^2 < R^2}] \leq R^2$ and,

$$\mathbb{E}[\xi_n^2 \mathbb{1}_{\xi_n^2 \geqslant R^2}] = \int_R^\infty 2u\mathbb{P}(\xi_n \geqslant u)du. \tag{A.107}$$

Thanks to the Many-to-one lemma (Lemma A.1) and Markov's inequality, we know that

$$\mathbb{P}(\xi_n \geqslant u) = \mathbb{P}\left(\frac{1}{n}\max_{|v|=n} |\Xi_v| \geqslant u\right) \leqslant \mathbb{E}\left[\sum_{|v|=n} \mathbb{1}_{\frac{|\Xi_v|}{n} \geq u}\right] \leqslant m^n \mathbb{P}\left(\frac{|Z_n|}{n} \geqslant u\right), \tag{A.108}$$

where $(Z_n)_{n\geq 0}$ is a random walk whose increments are distributed as μ. Moreover, by symmetry,

$$\mathbb{P}\left(\frac{|\Xi_v|}{n} \geqslant u\right) = 2\mathbb{P}\left(\frac{\Xi_v}{n} \geqslant u\right). \tag{A.109}$$

Equations (A.108) and (A.109), together with Chernoff's bound then give

$$\mathbb{P}(\xi_n \geqslant u) \leqslant 2m^n e^{-nI(u)} = 2e^{-n(I(c)-I(u))} \leqslant 2e^{-n\frac{I(c)}{c}(u-c)}. \tag{A.110}$$

According to Lemma A.3 and Equation (4.19), since $\Delta x < \frac{1}{16}\sqrt{2\Delta t \log(\underline{\rho})}$,

$$\frac{1}{2}\sqrt{2\Delta t\log(\underline{\rho})} < \frac{1}{2}\sqrt{2\Delta t\log(m)} < c < 2\sqrt{2\Delta t\log(m)} < 2\sqrt{2\Delta t\log(m)} < 2\sqrt{2\Delta t\log(\bar{\rho})},$$

so that

$$0 < \alpha := \frac{I\left(\frac{1}{2}\sqrt{2\Delta t\log(\underline{\rho})}\right)}{2\sqrt{2\Delta t\log(\bar{\rho})}} < \frac{I(c)}{c}. \tag{A.111}$$

Let $\eta > 0$ and consider $R = c + \eta$. Equations (A.107), (A.108), (A.110) and (A.111) give that

$$\mathbb{E}[\xi_n^2 \mathbb{1}_{\xi_n^2 \geqslant R^2}] \leqslant 4\int_{c+\eta}^{\infty} ue^{-n\alpha(u-c)}du = 4\int_{\eta}^{\infty}(u+c)e^{-n\alpha u}du \leqslant 4\int_{\eta}^{\infty}(u+\sqrt{2\Delta t\log(\bar{\rho})})e^{-n\alpha u}du.$$

Remark that the last integral tends to 0 as n tends to infinity, so that there exists $N_\eta \in \mathbb{N}$ such that, for all $n \geqslant N_\eta$,

$$\int_{\eta}^{\infty}(u+\sqrt{2\Delta t\log(\bar{\rho})})e^{-n\frac{\sqrt{2\Delta t\log(\underline{\rho})}}{4\Delta t}u}du \leqslant \eta,$$

and, thanks to Equation (A.106),

$$\mathbb{E}[\xi_n^2] = (c+\eta)^2 + \eta.$$

□

Corollary A.1. *Let $\Delta t > 0$, $1 < \underline{\rho} < \bar{\rho}$ and $\Delta x < \frac{1}{16}\sqrt{2\Delta t\log(\underline{\rho})}$. Let $\bar{c}$ be the unique positive solution of $I(\bar{c}) = \log(\bar{\rho})$. Consider a reproduction law $(p_k)_{k\in\mathbb{N}}$ such that $\sum kp_k = m > 1$ and reproduction law μ. Uniformly in $m \in [\underline{\rho}, \bar{\rho}]$, there exists $N \in \mathbb{N}$ such that*

$$\forall n \geq N, \quad \mathbb{E}\left[\frac{M_n^2}{n^2}\right] \leqslant 4\bar{c}^2.$$

B Appendix: stability of the solution of ($\mathcal{C}$) and convergence of the Euler scheme

B.1 Stability

Lemma B.1. *Let $\delta > 0$ and $T > 0$. Consider x the unique solution of*

$$\begin{cases} \dot{x}(t) &= \sqrt{2r(t,x(t))} \\ x(0) &= 0, \end{cases}$$

on $[0,T]$ *and* $\tilde{x}$ *the unique solution of*

$$\begin{cases} \dot{\tilde{x}}(t) &= \sqrt{2r(t,\tilde{x}(t))} + \delta \\ \tilde{x}(0) &= 0. \end{cases}$$

Then,

$$\sup_{t\in[0,T]} |x(t) - \tilde{x}(t)| \leqslant \delta e^{LT}.$$

Proof. Let $u(t) = \sqrt{(x(t) - \tilde{x}(t))^2 + \delta}$ for $t \in [0,T]$. Note that

$$|x(t) - \tilde{x}(t)| \leqslant u(t), \quad \forall t \in [0,T].$$

Besides, the function u is differentiable on $[0,T]$ and

$$\begin{aligned} u'(t) &= \frac{1}{2u(t)}\frac{d}{dt}((\tilde{x}(t) - x(t))^2 + \delta) = \frac{1}{u(t)}\left(\frac{d}{dt}x(t) - \frac{d}{dt}\tilde{x}(t)\right)(x(t) - \tilde{x}(t)) \\ &= \frac{1}{u(t)}\left(\sqrt{2(t,\tilde{x}(t))} - \sqrt{2r(t,x(t))} + \delta\right)(\tilde{x}(t) - x(t)) \\ &\leqslant \frac{1}{u(t)}(L|x(t) - \tilde{x}(t)| + \delta)(\tilde{x}(t) - x(t)) \leqslant \frac{1}{u(t)}(L|x(t) - \tilde{x}(t)| - \delta)u(t) \\ &\leqslant L(u(t) + \delta), \end{aligned}$$

where the third inequality is obtained thanks to Equation (2.11). Then, by the Gronwall's inequality, we have

$$u(t) \leqslant u(0)e^{Lt} \leqslant \delta e^{LT}, \quad \forall t \in [0,T],$$

which concludes the proof of the lemma. □

B.2 Euler scheme

Under Assumption 1, we know that each maximal solution y of $(\mathcal{C})$ is global *i.e.* defined on $[0,\infty)$. Besides, for any $T > 0$ and $h > 0$, we can define the Euler scheme of this solution by considering the sequence defined by

$$\begin{cases} y_0 &= y(0) \\ y_{i+1} &= y_i + \sqrt{2r(ih,y_i)}h. \end{cases}$$

Thanks to Equation (2.11) and convergence results on the Euler method (see Theorem 14.3 from [Hen65]), we know that

$$\max_{i\in[\![0,\lfloor T/h\rfloor]\!]} |y(t_i) - y_i| \leqslant e^{LT}\frac{h}{2}. \tag{B.112}$$

Remark B.1. *For any* $\delta > 0$*, Equation (B.112) still holds for the function* $\tilde{y}$ *solution of*

$$\dot{y} = \sqrt{2r(t,y(t))} + \delta,$$

and its Euler scheme

$$\begin{cases} \tilde{y}_0 &= \tilde{y}(0) \\ \tilde{y}_{i+1} &= \tilde{y}_i + (\sqrt{2r(ih,\tilde{y}_i)} + \delta)h, \end{cases}$$

CHAPTER III

Particle systems and semi-pushed fronts

We consider a system of particles performing a one-dimensional dyadic branching Brownian motion with space-dependent branching rate, negative drift $-\mu$ and killed upon reaching 0, starting with N particles. More precisely, particles branch at rate $\rho/2$ in the interval $[0, 1]$, for some $\rho > 1$, and at rate $1/2$ in $(1, +\infty)$. The drift $\mu(\rho)$ is chosen in such a way that, heuristically, the system is critical in some sense: the number of particles stays roughly constant before it eventually dies out. This particle system can be seen as an analytically tractable model for fluctuating fronts, describing the internal mechanisms driving the invasion of a habitat by a cooperating population. Recent studies from Birzu, Hallatschek and Korolev suggest the existence of three classes of fluctuating fronts: pulled, semi-pushed and pushed fronts. Here, we rigorously verify and make precise this classification and focus on the semi-pushed regime. More precisely, we prove the existence of two critical values $1 < \rho_1 < \rho_2$ such that for all $\rho \in (\rho_1, \rho_2)$, there exists $\alpha(\rho) \in (1, 2)$ such that the rescaled number of particles in the system converges to an α-stable continuous-state branching process on the time scale $N^{\alpha-1}$ as N goes to infinity. This complements previous results from Berestycki, Berestycki and Schweinsberg for the case $\rho = 1$.

Contents

1 Introduction

In this chapter, we are interested in the underlying dynamics of traveling wavefronts arising from certain reaction diffusion equations. Formally, the front is represented by a branching Brownian motion with absorption at zero and negative drift $-\mu$. This system can be seen as a co-moving frame following the particles located at the tip of the front. In this framework, the drift μ is interpreted as the speed of the wave.

In this introductory section, we first motivate our analysis with the results of some recent studies and state an informal version of the theorem in Section 1.1. In Section 1.2, we recall some well known facts on continuous-state branching processes. The model and the results are given in Section 1.3 and the sketch of the proof is outlined in Section 1.5. In Section 1.6, we explain the connection between the model defined in Section 1.1 and the generalised principle eigenvalue of the pertubated Laplacian on the half-line. We then discuss the link with previous work on pulled fronts in Section 1.7 and give a biological interpretation of the result in Section 1.8.

1.1 Noisy FKPP-type equations and semi-pushed fronts

This work is motivated by the results of recent work by Birzu, Hallatschek and Korolev [BHK18, BHK20] on the noisy FKPP-type equation

$$u_t = \frac{1}{2}u_{xx} + ru(1-u)(1+Bu) + \frac{1}{\sqrt{N}}\Gamma(u)W(t,x). \tag{1.1}$$

From a biological standpoint, Equation (1.1) models the invasion of an uncolonised habitat by a species: u corresponds to the population density, B is a positive parameter scaling the strength of cooperation between the individuals, N is the local number of particles at equilibrium, Γ stands for the strength of the demographic fluctuations and W is a Gaussian white noise. The numerical experiments and analytical arguments from [BHK18, BHK20] suggest the existence of three regimes in Equation (1.1): the *pulled* regime for $B \leqslant 2$, the *semi-pushed* or *weakly pushed* regime for $B \in (2, B_c)$, for some $B_c > 2$, and the *fully pushed* regime, for $B \geqslant B_c$.

The notion of pulled and pushed waves was first introduced by Stokes [Sto76] in PDE theory. The distinction between the pulled and pushed regime in Equation (1.1) relies on the asymptotic spreading speed v of the solutions of the limiting reaction-diffusion equation (that is when $N \to \infty$),

$$u_t = \frac{1}{2}u_{xx} + f(u), \tag{1.2}$$

where

$$f(u) = ru(1-u)(1+Bu). \tag{1.3}$$

It is a known fact (see *e.g.* [HR75]) that Equation (1.2) has a one-parameter family of front solutions $u(t,x) = \varphi_c(x-ct)$ for $c \geqslant c_{\min}$, for some $c_{\min} > 0$. Moreover, it was shown [Sto77] that the asymptotic spreading speed v of any solution to Equation (1.2) with compactly supported initial data is equal to the minimal speed $c_{\min}$. See Chapter I for further details on the convergence of such solutions. Thereupon, an invasion is said to be "pulled" if $c_{\min}$ coincides with the asymptotic speed $c_0 = \sqrt{2r}$ of the linearised equation

$$u_t = \frac{1}{2} u_{xx} + f'(0)u,$$

and "pushed" if $c_{\min} > c_0$. In Equation (1.2), the transition between pulled and pushed fronts occurs at $B = 2$ [HR75] since $c_{\min}$, and thus v, is given by

$$v(B) = \begin{cases} \sqrt{2r} & \textit{if} \quad B \leqslant 2 \\ \frac{1}{2}\sqrt{rB}\left(1 + \frac{2}{B}\right) & \textit{if} \quad B > 2. \end{cases} \tag{1.4}$$

As observed in [BHK18], the addition of demographic fluctuations in (1.2) uncover a third class of invasions: the *semi pushed* or *weakly pushed* regime. The effect of fluctuations on pulled fronts has already been widely studied in the literature. A rich theory based on the work of Brunet, Derrida and co-authors [BD97, BDMM06a, BDMM06b] describes the behaviour of the front solutions of (1.1) for $B = 0$. The spreading speed of these solutions admits a correction of order $\log(N)^{-2}$ compared to the one of the limiting PDE (1.2). In this sense, fluctuations have a huge impact on pulled fronts (see Section 1.7 for further details). Besides, the genealogy at the tip of the front is expected to be described by a Bolthausen–Sznitman coalescent over a time scale of order $\log(N)^3$, which suggests that the particles located at the tip of the front evolve as a population undergoing natural selection.

On the other hand, pushed fronts are expected to be less sensitive. In [BHK18], it is numerically observed that for $B > B_c$, the fluctuations in the position of the front and in the genetic drift occur on a time scale of order N, which may indicate the presence of Kingman's coalescent (a coalescent with binary mergers). This is consistent with the fact that the population in the bulk behaves like a neutral population. However, for intermediate values of B, that is $B \in (2, B_c)$, the fluctuations appear on a shorter time-scale, namely N^γ with $\gamma \in (0, 1)$. This intermediate region is defined as the semi pushed regime.

In this work, we propose an analytically tractable particle system to investigate the microscopic mechanisms leading to semi pushed invasions. This model is an extension of the one studied by Berestycki, Berestycki and Schweinsberg [BBS13] to prove the conjecture on the genealogy of pulled fronts. Similarly, we are able to exhibit the timescale and the structure of the genealogy of our particle system. Based on the branching particle system analysed in [BBS13], we consider a branching Brownian motion with absorption at 0, negative drift $-\mu$ and a space-dependent branching rate $r(x)$ of the form

$$r(x) = \frac{1}{2} + \frac{\rho - 1}{2}\mathbb{1}_{x \in [0,1]}, \tag{1.5}$$

for some $\rho \geqslant 1$. As mentioned above, this system is a toy model for what happens to the right of the front. Hence, the parameter ρ plays the same role as B in Equation (1.1) and thus scales the strengh of the cooperation between the particles.

We assume that the system starts with N particles located at 1. We denote by N_t the number of particles alive in the particle system at time t and consider the rescaled number of particles $\bar{N}_t = N_t/N$. Essentially, our result is the following:

Theorem 1.1 (informal version)**.** *There exists* $1 < \rho_1 < \rho_2$ *such that for all* $\rho \in (\rho_1, \rho_2)$*, there exists* $\mu(\rho) > 1$ *and* $\alpha = \alpha(\rho) \in (1, 2)$ *such that, if we consider the BBM with branching rate* (1.5) *and drift* $-\mu(\rho)$*, the process* $(\bar{N}_{N^{\alpha-1}t})_{t>0}$ *converges in law to an* α*-stable continuous-state branching process as* N *goes to infinity.*

This result seems consistent with the observations made on the fluctuations in [BHK18] and with the genealogical structure proposed in [BHK20] for semi pushed fronts. Indeed, it is known that the genealogy corresponding to an α-stable continuous-state branching process is given by a Beta$(2-\alpha, \alpha)$-coalescent [BBC$^+$05].

We refer to Section 1.3 for a precise statement of Theorem 1.1 and to Section 1.2 for a definition of the CSBP.

1.2 Continuous-state branching processes

We recall known facts about continuous-state branching processes (CSBP), and, more specifically, the family of α-stable CSBP, for $\alpha \in [1, 2]$ (see e.g. [Ber09, BBC$^+$05]). A continuous-state branching process is a $[0, \infty]$-valued Markov process $(\Xi(t), t \geq 0)$ whose transition functions satisfy the branching property $p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot)$, which means that the sum of two independent copies of the process starting from x and y has the same finite-dimensional distributions as the process starting from $x+y$. It is well-known that continuous-state branching processes can be characterised by their branching mechanism, which is a function $\Psi : [0, \infty) \to \mathbb{R}$. If we exclude processes that can make an instantaneous jump to ∞, the function Ψ is of the form

$$\Psi(q) = \gamma q + \beta q^2 + \int_0^\infty (e^{-qx} - 1 + qx\mathbb{1}_{x\leq 1})\, \nu(dx),$$

where $\gamma \in \mathbb{R}$, $\beta \geq 0$, and ν is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x^2)\nu(dx) < \infty$. If $(\Xi(t), t \geq 0)$ is a continuous-state branching process with branching mechanism Ψ, then for all $\lambda \geq 0$,

$$E[e^{-\lambda \Xi(t)} \,|\, \Xi_0 = x] = e^{-x u_t(\lambda)}, \tag{1.6}$$

where $u_t(\lambda)$ can be obtained as the solution to the differential equation

$$\frac{\partial}{\partial t} u_t(\lambda) = -\Psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda. \tag{1.7}$$

We will be interested in α-stable CSBP for $\alpha \in [1, 2]$, for which the branching mechanism Ψ is of the form

$$\Psi(u) = \begin{cases} -au + bu^\alpha, & \text{if } \alpha \in (1, 2], \\ -au + bu \log u, & \text{if } \alpha = 1. \end{cases} \tag{1.8}$$

It is known that in this case, the CSBP does not explode in finite time, i.e. Grey's condition is satisfied. The 2-stable CSBP is also known as the *Feller diffusion* and the 1-stable CSBP as *Neveu's CSBP*.

1.3 The model: assumptions and main result

We consider a dyadic branching Brownian motion with killing at zero, negative drift $-\mu$ and position-dependent branching rate

$$r(x) = \begin{cases} \rho/2 & x \in [0,1], \\ 1/2 & x > 1, \end{cases}$$

for some parameter $\rho \geqslant 1$. We denote by $\mathcal{N}_t$ the set of particles in the system at time t and for all $v \in \mathcal{N}_t$, we denote by $X_v(t)$ the position of the particle v at time t. Furthermore, we write $N_t = \#\mathcal{N}_t$ for the number of particles in the system at time t. The drift μ is chosen with respect to ρ in such a way that the number of particles in the system stays roughly constant. Depending on the value of ρ, μ is equal to 1 (*pulled regime*) or μ is strictly larger than 1 (*pushed regime*).

In practice, $\mu = \mu(\rho)$ is a function of ρ related to the principal generalised eigenvalue λ_1^∞ of a certain differential operator (see Section 1.5 for further details). More precisely, we have

- If $\rho < 1 + \frac{\pi^2}{4}$, then
$$\mu = 1. \tag{1.9}$$
- If $\rho > 1 + \frac{\pi^2}{4}$, then μ is the unique solution of
$$\frac{\tan(\sqrt{\rho-\mu^2})}{\sqrt{\rho-\mu^2}} = -\frac{1}{\sqrt{\mu^2-1}}, \quad \text{such that} \quad \rho - \mu^2 \in \left[\frac{\pi^2}{4}, \pi^2\right]. \tag{1.10}$$

In terms of λ_1^∞, we have that $\lambda_1^\infty = 0$ for $\rho < 1 + \frac{\pi^2}{4}$, $\lambda_1^\infty > 0$ for $\rho > 1 + \frac{\pi^2}{4}$ and that the definition of μ given by Equations (1.9) and (1.10) is equivalent to

$$\mu = \sqrt{1 + 2\lambda_1^\infty}, \tag{1.11}$$

so that

$$\mu > 1 \quad \text{for} \quad \rho > 1 + \frac{\pi^2}{4}.$$

This way, the branching Brownian motion with absorption at 0, branching rate $r(x)$ and drift $-\mu$ is fully defined. Let us define the exponent α: for $\mu > 1$, we set

$$\alpha = \frac{\mu + \sqrt{\mu^2-1}}{\mu - \sqrt{\mu^2-1}}. \tag{1.12}$$

We now define two regimes of interest for the parameter ρ. The first one corresponds to a (subset of) the *pushed* regime:

$$\rho > \rho_1, \quad \rho \notin \left\{1 + k^2\frac{\pi^2}{4}, k \in \mathbb{N}\right\}. \tag{H$_{psh}$}$$

where

$$\rho_1 = 1 + \frac{\pi^2}{4}. \tag{1.13}$$

It turns out that the transition between the *weakly pushed* and the *fully pushed* regimes occurs when $\alpha = 2$, which corresponds to the critical value of μ,

$$\mu_c = \frac{3}{4}\sqrt{2}. \tag{1.14}$$

Therefore, the *weakly pushed* regime corresponds to the following range of the parameter ρ:

$$\rho_1 < \rho < \rho_2, \tag{H_{wp}}$$

where ρ_2 is the unique solution of

$$\frac{\tan\left(\sqrt{\rho - \mu_c^2}\right)}{\sqrt{\rho - \mu_c^2}} = -\frac{1}{\sqrt{\mu_c^2 - 1}} \quad s.t. \quad \rho - \mu_c^2 \in \left[\frac{\pi^2}{4}, \pi^2\right]. \tag{1.15}$$

In this regime, we have the following result, which is the main result of this chapter:

Theorem 1.2. *Assume* (H_{wp}) *holds and suppose the system initially starts with N particles located at 1. Then, there exists an explicit constant $\sigma(\rho) > 0$ such that if we define $\bar{N}_t = \sigma(\rho)N_t/N$, as $N \to \infty$, the finite-dimensional distributions of the processes $(\overline{N}_{N^{\alpha-1}t})_{t>0}$ converge to the finite-dimensional distributions of an α-stable CSBP starting from 1, where α is given by Equation* (1.12).

A more general version of Theorem 1.2 is stated in Theorem 7.2. In addition, an explicit formula for $\sigma(\rho)$ is given in Section 7.2 (see Equation (7.15)). We strongly believe that this result can be completed with the study of the cases $\rho \in [1, \rho_1)$ and $\rho \in (\rho_2, +\infty)$. The expected convergence results are summarised in the following conjectures. This will be the subject of future work.

Conjecture 1. *If $\rho < \rho_1$, under suitable assumptions on the initial configurations, the finite-dimensional distributions of the processes $(\overline{N}_{(\log N)^3 t})_{t>0}$ converge to the finite-dimensional distributions of a 1-stable (Neveu's) CSBP starting from 1 as $N \to \infty$.*

Conjecture 2. *If $\rho > \rho_2$, under suitable assumptions on the initial configurations, the finite-dimensional distributions of the processes $(\overline{N}_{Nt})_{t>0}$ converge to the finite-dimensional distributions of a Feller diffusion starting from 1 as $N \to \infty$.*

The proof of Theorem 1.2 relies on the first and the second moment estimate of several processes. The assumptions (H_{psh}) and (H_{wp}) are used to estimate these moments in the *weakly pushed* regime. The first moments (see Sections 3.1 and 4.1) are computed under assumption (H_{psh}), so that they can also be used to investigate the *fully pushed* regime, whereas the upper bounds on the second moments require the assumption (H_{wp}).

The fact that $\rho \notin \left\{1 + k^2\frac{\pi^2}{4}, k \in \mathbb{N}\right\}$ is only a technical assumption and one could prove the result under more general assumptions with extra work. Moreover, one can also investigate systems with more general branching rates, of the form

$$r(x) = \frac{1}{2} + \frac{\rho - 1}{2}f(x),$$

for a function f that is compactly supported (or even a function that quickly converges to zero). In this case, the spectrum and eigenvectors are not necessarily explicit, but one can still analyse the system using spectral theoretic methods.

1.4 Comparison with results on fluctuating fronts

In the particle system, we say that the pulled regime corresponds to $\rho \in [1, \rho_1)$, the weakly pushed regime to $\rho \in (\rho_1, \rho_2)$ and the fully pushed regime to $\rho > \rho_2$. From a biological standpoint, the process N_t is related to *the number of descendants left by the early founders* mentioned in [BHK20]. Moreover, CSBPs can be seen as scaling limits of Galton-Watson processes, with associated genealogical structures [Ber09]. In this sense, the convergence results stated in Theorem 1.2 and in the two conjectures are consistent with the observations on the genealogical trees made in [BHK20]: in the pulled regime, the genealogy of the particles at the tip of the front is the one of a population undergoing selection, that is a Bolthausen–Sznitman coalescent. We know since the work of [BLG00] that it is precisely the genealogy associated with Neveu's CSBP. Similarly, we know that the genealogy associated to the α-stable process and the Feller diffusion are respectively the Beta$(2-\alpha, \alpha)$-coalescent and Kingman's coalescent [BBC+05]. Again, this is exactly what is observed [BHK20] on the genealogical structures of their model.

Besides, note that the transitions between the three regimes occur at the same critical values of μ and v. Indeed, consider Equation (1.1) with $r = \frac{1}{2}$. Therefore, $c_0 = 1$ and the invasion speed v is given by (see Equation (1.4))

$$v(B) = \begin{cases} 1 & \textit{if} \quad B \leqslant 2 \\ \frac{1}{2\sqrt{2}}\sqrt{B}\left(1+\frac{2}{B}\right) & \textit{if} \quad B > 2. \end{cases} \tag{1.16}$$

In the particle system, note that the drift is also equal to 1 in the pulled regime (see Equation (1.9)). In both cases, the definition of the transition between the pushed and the pulled regime happens when the propagation speed, μ or v, becomes larger than 1, that is when $\rho > 1 + \frac{\pi^2}{4}$ in the particle system and $B > 2$ in the noisy FKPP Equation (1.2). Similarly, the transition between weakly and fully pushed waves occurs for the same critical value of the invasion speed. Following [BHK18], consider $\tilde{\alpha}$ such that (see [BHK18, Equation (8)])

$$\tilde{\alpha} = \begin{cases} 1 - \frac{2\sqrt{1-c_0^2/v^2}}{1-\sqrt{c_0^2/v^2}} & \text{if} \quad \frac{v}{c_0} \in \left(1, \frac{3}{4}\sqrt{2}\right) \\ 2 & \text{if} \quad \frac{v}{c_0} \geqslant \frac{3}{4}\sqrt{2}. \end{cases} \tag{1.17}$$

They observe that the fluctuations in the pushed regime appear on a time scale $N^{\tilde{\alpha}-1}$, so that the transition between the weakly and fully pushed regimes occurs at $v = \frac{3}{4}\sqrt{2}c_0$. This is consistent with Theorem 1.2: if $r = \frac{1}{2}$, $c_0 = 1$ so that the transition occurs at $v = \frac{3}{4}\sqrt{2}$, which corresponds to the critical value μ_c from Equation (1.14), delineating the semi pushed and the pushed regime. Besides, note that for $c_0 = 1$, we have

$$\tilde{\alpha} = \begin{cases} \frac{v+\sqrt{v^2-1}}{v-\sqrt{v^2-1}}, & \text{if} \quad v \in \left(1, \frac{3}{4}\sqrt{2}\right) \\ 2 & \text{if} \quad v \geqslant \frac{3}{4}\sqrt{2}. \end{cases}$$

which seems to indicate the existence of a universality class given the definition of α (see Equation (1.12)). Besides, note that the exponent α (resp. $\tilde{\alpha}$) depends on ρ (resp. B) only through the drift μ (resp. the speed v). This can be explained by the fact that the particles which cause the jumps in the CSBP stay far away from the regions where the branching rate depends on ρ (see below for further explanations).

We now investigate the asymptotic behaviours of μ and v as the cooperation parameters ρ and B tend to their critical values. First, note that Equation (1.16) implies that for $r = \frac{1}{2}$,

$$v(B) \sim \frac{1}{2}\sqrt{\frac{B}{2}} \quad \text{as} \quad B \to +\infty.$$

On the other hand, by definition of μ (see Equation (1.10)), we have $\frac{\pi^2}{4} \leqslant \rho - \mu^2 \leqslant \pi^2$, so that

$$\mu \sim \sqrt{\rho} \quad \text{as} \quad \rho \to +\infty.$$

When $B \to 2$, $B > 2$, a second order Taylor expansion gives that

$$v(B) \sim 1 + \frac{(B-2)^2}{16}.$$

Besides, when $\rho \to \rho_1$, $\rho > \rho_1$, one can show that $\mu \to 1$ and the expansion of each term in Equation (1.10) gives

$$\mu^2 - 1 \sim \frac{1}{4}\left(\rho - 1 - \frac{\pi^2}{4}\right)^2,$$

so that we have

$$\mu \sim 1 + \frac{1}{8}\left(\rho - 1 - \frac{\pi^2}{4}\right)^2.$$

These similar asymptotic behaviours as well as the three regimes observed in the particle system support the hypothesis the existence of a universality class. This is illustrated in Figure III.1 and Figure III.2.

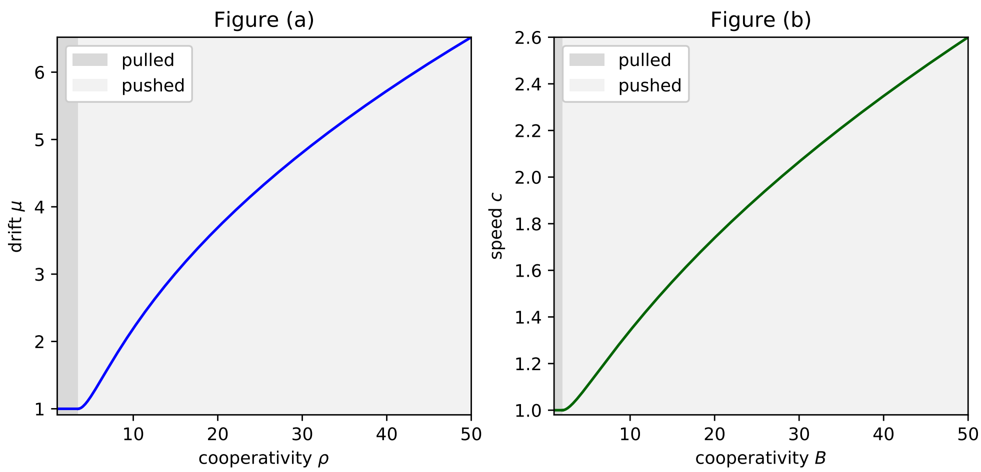

Figure III.1: The expansion velocity as a function of cooperativity. Figure (a): in the particle system. Graph of μ as a function of ρ (see Equations (1.9) and (1.10)). The transition between the pulled and the pushed regimes occurs at $\rho_1 = 1 + \frac{\pi^2}{4} \approx 3,47$. Figure (b): in the PDE (1.2). Graph of v as a function of B (see Equation (1.16)) for $r = \frac{1}{2}$. The transition between the pulled and the pushed regimes occurs at $B = 2$

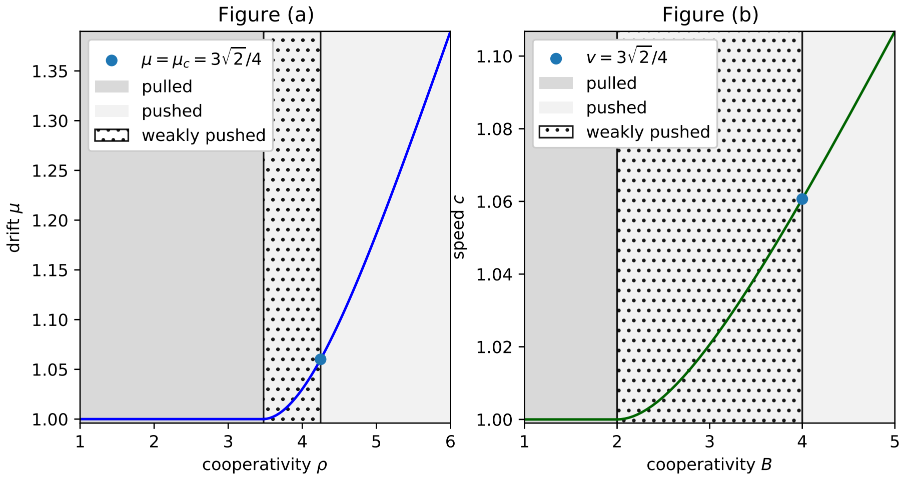

Figure III.2: The expansion velocity as a function of cooperativity. Figure (a): in the particle system. Graph of μ as a function of ρ (see Equation (1.10)). The weakly pushed regime is observed when $\mu \in (1, \mu_c)$. The transition between the weakly pushed and fully pushed regime occurs at $\rho = \rho_2$, given by Equation (1.15). Figure (b): in the PDE. Graph of v as a function of B (see Equation (1.16)) for $r = \frac{1}{2}$. In the noisy FKPP equation, the transition between weakly pushed and fully pushed waves occurs when $v = \mu_c$ (see (1.17)), which corresponds to $B = 4$.

1.5 Overview of the proof

The strategy of the proof is inspired by the work of Berestycki, Berestycki and Schweinsberg [BBS13], who treated the case of a constant branching rate r that is $\rho = 1$. The main idea is to introduce an additional barrier at a point L depending on N, in such a way that the jumps of the limit of the rescaled process $\bar{N}$ are caused by particles that reach L. In their case, one chooses $L = \log N + 3 \log \log N$, and it is reasonable to believe that this will be the case for $\rho < \rho_1$. If $\rho \in (\rho_1, \rho_2)$, we rather choose a barrier at $L = C \log N$ for some $C > 0$. In this section, we outline the main ideas used to choose this barrier and to prove the convergence to the α-CSBP in the case where $\rho \in (\rho_1, \rho_2)$.

As explained in [BBS13], the role of the barrier is to capture the particles which cause a jump in the CSBP, or, stated otherwise, which will have a number of descendants of order N at a later time. Hence, the barrier is chosen in such a way that the number of descendants at a later time (but shorter than the time-scale of the CSBP) is of order N. From this perspective, the behaviour of the particle system is the following:

1. Most of the time, the particles stay in the interval $[0, L]$. Therefore, the system is well approximated by a BBM with drift $-\mu$, branching rate $r(x)$, killed at 0 and at the additional barrier L.

2. From time to time, on the time-scale of the CSBP (which we expect to be $N^{\alpha-1}$) a particle reaches L. The barrier L is chosen in such a way that the number of descendant of a

particles hitting L is of order N after a short time (on the time scale of the CSBP).

3. In order to deal with these descendants, we let the particle reaching L evolve freely during a time period which is large but of order 1. Following [BBS13], one can for example fix some large constant y and track the descendants when they first reach $L-y$. The number of such descendants will be a random quantity with tail $1/x^\alpha$. This random quantity will be proportional to an *additive martingale* of the BBM rooted at the particle that reaches L.

4. After this large (but independent of L) relaxation time, all particles are again in the interval $[0, L]$ and the system can again evolve as before.

Thanks to this sketch of proof, one can infer a suitable value of L and justify the definition of the parameter μ. Indeed, the first step implies that most of the time, the system can be approximated by a heat equation in the interval $[0, L]$ with Dirichlet boundary conditions. In other words, if we denote by $\mathcal{N}_t^L$ the set of particles in the BBM at time t that have stayed in the interval $[0, L]$ until time t, the density of particles is given by the Many-to-one lemma:

Lemma 1.1 (Many-to-one lemma, see [Law18], p.88)**.** *Let $p_t(x, y)$ be the fundamental solution to the PDE*

$$\begin{cases} u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + \mu u_x(t,x) + r(x)u(t,x) \\ u(t,0) = u(t,L) = 0. \end{cases} \tag{A}$$

Then for every measurable, positive function $f : \mathbb{R}_+ \to \mathbb{R}$, we have[1]

$$\mathbf{E}_x \left[\sum_{v \in \mathcal{N}_t^L} f(X_v(t)) \right] = \int_0^L p_t(x,y) f(y)\, dy.$$

Therefore, the function p_t can be deduced from the Sturm–Liouville theory. Since (A) is not self-adjoint, we first set the function q_t in such a way that

$$p_t(x,y) = e^{\mu(x-y) + \left(\frac{1}{2} - \frac{\mu^2}{2}\right)t} q_t(x,y). \tag{1.18}$$

Then $q_t(x, y)$ is the fundamental solution to the self-adjoint PDE

$$\begin{cases} u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + \frac{\rho-1}{2}\mathbf{1}_{[0,1]}(x)u(t,x) \\ u(t,0) = u(t,L) = 0. \end{cases} \tag{B}$$

By the Sturm–Liouville theory, the eigenvalues of the operator

$$Tv = \frac{1}{2}v'' + \frac{\rho-1}{2}\mathbf{1}_{[0,1]}(x)v \tag{1.19}$$

are simple and can be numbered

$$\lambda_1^L > \lambda_2^L > \cdots > \lambda_n^L > \cdots \to -\infty.$$

[1] The notation $\mathbf{E}_x$ means that we start with one particle at position x.

As a result, we will prove that each λ_i^L is increasing with respect to L. If $v_1, v_2, \dots$ denote the corresponding eigenfunctions of unit L^2-norm, then the function q^t is given by [Zet10, p.87]

$$q_t(x,y) = \sum_{n=1}^{\infty} e^{\lambda_n^L t} v_n(x) v_n(y),$$

and hence,

$$p_t(x,y) = \sum_{n=1}^{\infty} e^{\mu(x-y)+(\lambda_n^L+\frac{1}{2}-\mu^2/2)t} v_n(x) v_n(y).$$

We say that p_t is the density of the BBM with branching rate $r(x)$, drift $-\mu$ and killed at 0 and L in the sense that, starting with a single particle at x, the expected number of particle in a Borel subset B a time t is given by $\int_B p_t(x,y)dy$. From these observations, μ will be chosen in such a way that the loss of mass in p_t is controlled. Yet, we will prove that for $\rho > \rho_1$, a positive and isolated generalised eigenvalue λ_1^∞ emerges as $L \to \infty$. Therefore, we will choose μ such that

$$\mu = \sqrt{1+2\lambda_1^\infty}, \tag{1.20}$$

as stated in (1.20). We will prove in Section 2.1 that this definition is equivalent to (1.10). In the case where $\rho < \rho_1$, the sequence (λ_i^L) converges to a non positive continuous spectrum, in particular $\lambda_1^\infty = 0$, so that $\mu = 1$.

For $\rho > \rho_1$ and t sufficiently large, we show that

$$p_t(x,y) \approx e^{\mu(x-y)+(\lambda_1^L-\lambda_1^\infty)t} v_1(x) v_1(y). \tag{1.21}$$

Note that the time parameter t only appears in the exponential factor in the formula (1.21) so that the population size should be roughly constant as long as $(\lambda_1^L - \lambda_1^\infty)t \ll 1$. Therefore, the time scale over which particles reach L is of order $(\lambda_1^\infty - \lambda_1^L)^{-1}$. Beside, the spectral analysis of the system (B) provides the existence of a constant $C > 0$ such that

$$\lambda_1^\infty - \lambda_1^L \sim Ce^{-2\sqrt{2\lambda_1^\infty}L}.$$

To simplify the notations, we set

$$\beta = \sqrt{2\lambda_1^\infty}. \tag{1.22}$$

As we expect the time scale of the CSBP (that is the time scale over which particles reach L) to be $N^{\alpha-1}$, the asymptotic behaviour of λ_1^L gives a first relation between α, N and L, that is

$$N^{\alpha-1} = e^{2\beta L}.$$

The eigenfunction associated to the principal eigenvalue λ_1^L will play a crucial role in this analysis. We denote by w_1 the eigenfunction which is of order 1 at $L-1$ and such that $w_1(1) = \sinh(\sqrt{2\lambda_1^L}(L-1))$. We then define as in [BBS13] the process

$$Z_t = \sum_{v \in \mathcal{N}_t} e^{\mu(X_v(t)-L)} w_1(X_v(t)) \mathbb{1}_{X_v(t)\in[0,L]}.$$

As long as the particles stay in $[0,L]$, this process coincides with

$$Z_t' = \sum_{v \in \mathcal{N}_t^L} e^{\mu(X_v(t)-L)} w_1(X_v(t)),$$

which is a supermartingale since, by the Many-to-one Lemma 1.1,

$$\mathbb{E}_x\left[Z'_t\right] = e^{(\lambda_1^L - \lambda_1^\infty)t} Z'_0. \tag{1.23}$$

The process Z_t, and thus Z'_t, governs the long time behaviour of the particle system. Indeed, for t large enough; the expected number of particles in the system starting with a single particle at x will be approximately given by

$$\mathbb{E}_x\left[N_t\right] \approx \int_0^L p_t(x,y)dy \approx e^{\mu x} v_1(x) e^{(\lambda_1^L - \lambda_1^\infty)t} \int_0^L e^{\mu y} v_1(y) dy.$$

We will show in Section 7.2 that the second integral is bounded by a constant and that $v_1(x) \approx Ce^{-\beta L} w_1(x)$ so that

$$\mathbb{E}_x\left[N_t\right] \approx Ce^{(\mu-\beta)L} Z'_0, \tag{1.24}$$

for $t \ll e^{2\beta L}$. Thus, we first prove Theorem 1.2 for Z_t instead of $\bar{N}_t$ and then deduce the result on $\bar{N}_t$.

Moreover, we claim that the barrier L has to be chosen so that

$$N = e^{(\mu-\beta)L}. \tag{1.25}$$

Indeed, L is fixed in such a way that the particles that reach L have a number of descendants of order N after a short time, on the time scale $e^{2\beta L}$ of the CSBP. Yet, if we consider the system starting with a single particle close to L, say at $x = L-1$, we get that Z'_0 is of order 1. Thus, the result ensues from Equation (1.24). Besides, we obtain that

$$\alpha = \frac{\mu+\beta}{\mu-\beta} \tag{1.26}$$

which is equivalent to the definition (1.12) given Equations (1.20) and (1.22).

In the light of Equations (1.24), (7.14) and (1.26), we claim that it is sufficient to prove that as $L \to \infty$,

$$Z_{e^{2\beta L}t} \Rightarrow \Xi(t), \tag{1.27}$$

where Ξ is an α-stable CSBP, starting with a suitable initial configuration.

As explained in [BHK20], the difference between the genealogical structures of the population for $\rho < \rho_1$, $\rho \in (\rho_1, \rho_2)$ and $\rho > \rho_2$ is explained by the *fluctuations in the total number of descendants left by the early founders*. In our particle system, this number of descendants is related to to the number of offspring of a particle hitting the barrier L. We prove that the number Z_y of these descendants reaching $L-y$ (for the first time) is such that

$$e^{(\mu-\beta)y} Z_y \Rightarrow W \quad \text{as} \quad y \to +\infty,$$

for some random variable W satisfying

$$\mathbb{P}(W > x) \sim \frac{C}{x^\alpha} \quad \text{as} \quad x \to +\infty.$$

The fact that α depends on ρ only through the drift μ can be explained by this barrier at $L-y$: it can be chosen in such a way that the particles are stopped before they reach 1 so that they behave as in a BBM with drift $-\mu$ and constant branching rate $\frac{1}{2}$.

The proof of the fact that the number of particles does not fluctuate too much when we add a barrier at L will rely on the estimate of the second moment of Z'_t. To this end, we will make use of the Many-to-two Lemma.

Lemma 1.2 (Many-to-two lemma [INW69], Theorem 4.15). *Let f and $p_t(x,y)$ be as in Lemma 1.1. Then*

$$\mathbf{E}_x\left[\left(\sum_{v\in\mathcal{N}_t^L} f(X_v(t))\right)^2\right]$$
$$= \int_0^L p_t(x,y)f(y)^2\,dy + \int_0^t\int_0^L p_s(x,y)2r(y)\mathbf{E}_y\left[\sum_{v\in\mathcal{N}_{t-s}^L} f(X_v(t-s))\right]^2 dy\,ds.$$

Actually, this result is more general and a stopping line version of this lemma holds. Once the result (1.27) is proved, one can deduce the same convergence on $\bar{N}_t$. Indeed, it will be sufficient to prove that over a short time, on the time-scale of the CSBP, Z does not vary much and that $\bar{N}$ is well approximated by Z (see (1.24)) as in [BBS13, Section 4.6].

We end this section with a reformulation of (H_{psh}) and (H_{wp}) in terms of λ_1^∞, α, μ and β (the first assertion will be proved in Section 2.1):

- Assume (H_{psh}) holds. Then, $\lambda_1^\infty > 0$ so that $\mu > 1$.
- Assume (H_{wp}) holds. Then, $\alpha \in (1,2)$, $\mu > 3\beta$ and $\lambda_1^\infty \in \left(0, \frac{1}{16}\right)$.

This second remark, which ensues from the definition of μ_c, will be useful in the estimate of the second moments.

1.6 Perturbation of the Laplacian on the half-line

A crucial role in the analysis will be played by the family of differential operators T_ρ, $\rho \in \mathbb{R}$, defined by

$$T_\rho u(x) = \begin{cases} \frac{1}{2}u''(x) + \frac{\rho}{2}\mathbf{1}_{[0,1]}(x)u(x), & x \in (0,1)\cup(1,\infty) \\ \lim_{z\to 1} T_\rho u(z), & x = 1. \end{cases}$$

with domain

$$\mathcal{D}_{T_\rho} = \{u \in C^1((0,\infty)) \cap C^2((0,1)\cup(1,\infty)) : \lim_{x\to 0} u(x) = 0,\ \lim_{x\to 1} T_\rho u(x) \text{ exists}\}.$$

The operator T_ρ is a perturbation of the Laplacian on the positive half-line by a function of compact support.

In this section, we recall a few well-known facts about such operators, based on Section 4.6 in [Pin95]. These results are only given for continuous perturbations but one can extend them to our particular perturbation by approximating the step function on $[0,1]$ by continuous functions. Actually, these facts will not be used in the following proofs, yet, they provide a better understanding of the three regimes in the particle system. To this end, we first define the generalised principal eigenvalue and the Green function of the operator T_ρ.

Define the *generalised principle eigenvalue* of the operator T_ρ by

$$\lambda_c(\rho) = \inf\{\lambda \in \mathbb{R} : \exists u \in \mathcal{D}_{T_\rho} : u > 0 \text{ on } (0,\infty),\ Tu = \lambda u\}.$$

Theorem 4.4.3 in [Pin95] implies that λ_c is a convex function of ρ and Lipschitz-continuous with Lipschitz constant $1/2$.

Let (B_t) be a standard Brownian motion starting at $x > 0$ and let $\tau = \inf\{t \in (0,\infty) : B_t \notin (0,\infty)\}$. The Green function G_ρ of the operator T_ρ is the unique function such that for all bounded measurable functions $g : \mathbb{R}^+ \to \mathbb{R}$, we have

$$\mathbb{E}\left[\int_0^\tau \exp\left(\int_0^t \frac{\rho}{2}\mathbb{1}_{[0,1]}(B_s)ds\right) g(B_t)dt\right] = \int_0^\infty G_\rho(x,y)g(y)dy.$$

Similarly, one can define the Green function of the operator $T_\rho - \lambda$, denoted by G_ρ^λ, such that

$$\mathbb{E}\left[\int_0^\tau \exp\left(\int_0^t \frac{\rho}{2}\mathbb{1}_{[0,1]}(B_s) - \lambda\, ds\right) g(B_t)dt\right] = \int_0^\infty G_\rho^\lambda(x,y)g(y)dy.$$

Recall from [Pin95], Section 4.3, that an operator is called

- *subcritical*, if its Green function is finite (and hence, positive harmonic functions, i.e. eigenfunctions of eigenvalue 0, exist),
- *critical*, if its Green function is infinite, but positive harmonic functions exist,
- *supercritical*, if no positive harmonic function exists.

It is well known that the Laplacian on the positive half-line, i.e. the unperturbed operator T_0, is subcritical in the sense of [Pin95], i.e. its Green function is finite, in fact, it is expressed by $G_0(x,y) = 2x \wedge y,\ x, y > 0$. Furthermore, its generalised principle eigenvalue is $\lambda_c(0) = 0$. It then follows from Theorem 4.6.4 in [Pin95] that there exists $\rho_c > 0$, such that $\lambda_c(\rho) = 0$ for all $\rho \le \rho_c$ and $\lambda_c(\rho) > 0$ for all $\rho > \rho_c$. Besides, T_ρ is subcritical for $\rho < \rho_c$, critical for $\rho = \rho_c$ and supercritical for $\rho > \rho_c$. In fact, Theorem 4.7.2 in [Pin95] implies that $T_\rho - \lambda_c(\rho)$ is critical for $\rho > \rho_c$.

These properties can be verified by elementary calculations, which also yield exact expressions of ρ_c and $\lambda_c(\rho)$. We summarise these calculations in the following proposition:

Proposition 1.1. *Define* $\rho_c = \pi^2/4$. *Define the function*

$$g(x) = \mathrm{sinc}(\sqrt{x})^{-2},\ x \in [\rho_c, \pi^2),$$

where $\mathrm{sinc}(z) = \sin(z)/z$. *Then* g *is an increasing and strictly convex function on* $[\rho_c, \pi^2)$ *with* $g(\rho_c) = \rho_c$, $g'(\rho_c) = 1$ *and* $g(x) \to +\infty$ *as* $x \to \pi^2$. *Denote by* g^{-1} *its inverse, defined on* $[\rho_c, \infty)$. *Then,*

$$\lambda_c(\rho) = \begin{cases} 0, & \rho \le \rho_c \\ \frac{1}{2}(\rho - g^{-1}(\rho)), & \rho > \rho_c. \end{cases}$$

The proof of this result is given in Appendix A. One could go on expliciting the positive eigenfunctions of the operator T_ρ for all ρ. One would see that there exists for every $\rho \in \mathbb{R}$ and every $\lambda \ge \lambda_c(\rho)$ a unique (up to a multiplicative constant) positive eigenfunction of eigenvalue λ. For $\lambda = \lambda_c(\rho)$, this function is affine on $[1, \infty)$ with positive slope for $\rho < \rho_c$, and exponentially decreasing, with exponent $-\sqrt{2\lambda_c(\rho)}$, on $[1, \infty)$, for $\rho > \rho_c$. In fact, in the latter case, an eigenfunction is

$$u(x) = \begin{cases} \sin(\sqrt{g^{-1}(\rho)}x), & x \in [0, 1] \\ \sin(\sqrt{g^{-1}(\rho)})e^{-\sqrt{2\lambda_c(\rho)}(x-1)}, & x \in [1, \infty). \end{cases}$$

This function will play a crucial role in the system. Indeed it corresponds to an harmonic function of the critical operator $T_\rho - \lambda_c(\rho)$. According to Theorem 8.6 in [Pin95], this function is the unique (up to positive multiples) *invariant function* for the transition measure associated to $T_\rho - \lambda_c(\rho)$. Roughly speaking, this means that u is a stable configuration in the particle system. On the other hand, for $\lambda > \lambda_c(\rho)$, the function grows exponentially on $[1, \infty)$ with exponent $\sqrt{2\lambda}$.

Let us now go back to the differential operator $L = \frac{1}{2}v'' + \mu\nabla + r(x)$ from Equation (A). Thanks to Equation (1.18), the Green function G of the operator L can be expressed thanks to $G^\lambda_{\rho-1}$ as follows:

$$G(x, y) = e^{\mu(x-y)}G^\lambda_{\rho-1}(x, y), \quad \text{for } \lambda = \frac{\mu^2 - 1}{2}.$$

The value of μ will be then chosen in such a way that the operator L has an harmonic function. Then, for $\rho - 1 < \rho_c$ it is sufficient to choose $\mu = 1$ since $T_{\rho-1}$ is subcritical. For $\rho - 1 > \rho_c$, we know that $T_{\rho-1} - \lambda_c(\rho - 1)$ is critical. Therefore, the corresponding Green function is infinite but the operator has harmonic functions. Hence, we will choose the drift μ such that

$$\mu(\rho) = \sqrt{1 + 2\lambda_c(\rho - 1)}.$$

Note that the limit λ^∞_1 of the maximal eigenvalues λ^L_1 and the generalised principal eigenvalue $\lambda_c(\rho - 1)$ coincide. This is a consequence of Theorem 4.1 in [Pin95].

1.7 Related models

A rich theory has been developed in the case where $B = 0$ in (1.1), which corresponds to a special case of the pulled regime. First, the equation

$$u_t = \frac{1}{2}u_{xx} + u(1-u) + \sqrt{\frac{u(1-u)}{N}}W(t, x), \tag{1.28}$$

was studied by [BDMM06a] to investigate the effect of demographic fluctuations on the FKPP equation. Indeed, if one removes the noise term in (1.28), one obtains the FKPP equation introduced by Fisher [Fis37] and independently by Kolmogorov, Petrovskii and Piskounov [KPP37], to describe the invasion of a stable phase ($u \approx 1$) in an unstable phase ($u \approx 0$). In this case, it is well-known [KPP37] that $c_{\min} = c_0 = \sqrt{2r}$ so that the invasion is pulled.

As explained in [Pan04], the FKPP equation can be seen as the hydrodynamic limit of many particle systems. However, the finite nature of the physical or biological systems induces fluctuations, that can be modeled by adding multiplicative square root noise to the FKPP equation. Heuristically, this correction corresponds to the rescaled *difference between the limiting PDE and the particle system in the style of a central limit theorem* [MT94]. The addition of this noise term in Equation (1.28) makes the shape and position of the front fluctuate.

In [BDMM06a], the authors explain how to infer the first order of the correction to the speed of the noisy fronts (compared to the deterministic fronts) thanks to a particle system. Since the fluctuations emerge at the leading edge of the front, they do not need to introduce a saturation rule in the particle system to deduce the correction to the velocity of the wave. Analysing the mechanisms driving the invasion, they conjecture that the fluctuations appear over a time scale of order $\log(N)^3$. They deduce from this fact that the correction of the speed c_0 is of order $\log(N)^{-2}$. This statement was then rigorously proved in [MMQ10] in the case of the SPDE (1.28). This correction, that is much greater than expected ($1/\sqrt{N}$) underscores the large fluctuations in the pulled regime.

In [BDMM06b, BDMM07], the authors analyse a particle system with a fixed population size to investigate the genealogy at the tip of the invasion front in the pulled regime. The particles evolve in discrete time and, at each generation, independently give birth to exactly k children, scattered around the parental location. At the end of each generation, only the N rightmost individual survive. This set of particles forms a cloud, that does not diffuse and can be described by a front governed by (1.28) [BDMM07]. In this framework [BDMM06b], they conjectured that the genealogy of the particles in the cloud is described by a Bolthausen–Sznitman coalescent. The fact that the correction on the speed of this system is the same as the one for solutions of (1.28), was rigorously proved in [BG10] (in the case $k = 2$).

The conjecture on the genealogy stated in [BDMM06b, BDMM07] was proved under slightly different assumptions in [BBS13]. Indeed, to simplify the analysis, it was proved for a continuous-time model, in which the constant population size is replaced by a moving wall. More precisely, they consider a branching Brownian motion with absorption for a suitable choice of drift $-\mu$ such that the population size in the system stays broadly constant. It is the branching property of the branching Brownian motion with absorption that makes this system analytically tractable. The drift is then chosen supercritical, matching the correction on the speed of the noisy front conjectured in [BDMM06a]. More precisely, for each integer N, they consider a dyadic BBM, with drift $-\mu_N$, with

$$\mu_N = \sqrt{1 - \frac{\pi^2}{(\log(N) + 3\log\log(N))^2}}, \tag{1.29}$$

starting, for instance, with $N\log(N)^3$ particles at $x = 1$. With the notations of Theorem 1.2, they obtain that as N goes to ∞, the processes $\left(\bar{N}_{\log(N)^3 t},\ t \geqslant 0\right)$ converge in law to Neveu's continuous-state branching process. Using the results from [BLG00], they deduce from this fact that the genealogy of their system is given by a Bolthausen–Sznitman coalescent.

In this work, we are interested in the genealogy of the particles at the tip of the front for a more general form of reaction term in the limiting PDE. While the study in [BDMM06a, BDMM06b] concerns FKPP fronts, that are classified as *pulled*, we focus on forcing terms of the form (1.3). In this case, the deterministic front of the limit PDE can be either *pulled* ($B \leqslant 2$) or *pushed* ($B > 2$).

1.8 Biological motivations : the Allee effects

In terms of population models, a front is pushed, for instance, in the presence of a sufficiently strong *Allee effect*, meaning that the particles near the front have a competitive advantage over particles far away from the front. The strength of the Allee effect is scaled by the parameter B in Equation (1.1) and by ρ in the particle system defined in Section 1.3.

Allee effects are well explained in [BHK18]: "The presence of conspecifics can be beneficial due to numerous factors, such as predator dilution, anti predator vigilance, reduction of inbreeding and many others. Then, the individuals in the very tip of the front do not count so much, because the rate of reproduction decreases when the number density becomes too small. Consequently, the front is pushed in the sense, that its time–evolution is determined by the behavior of an ensemble of individuals in the boundary region". In sharp contrast, pulled invasions are the one for which the growth is maximal at low densities so that the individuals located at the leading edge pull the invasion. As explained in [Sto76], the consequence of this fact is that "the speed of the wave is determined by the fecundity of their pioneers", or, in other words, it only depends on $f'(0)$ (see Equation (1.3)). Pushed waves are faster and *pushed*, or driven, by the nonlinear dynamic of the bulk (see Section 1.1). Consequently, the speed of the waves depends on the functional form of the reaction term f.

The shift in the invasion speed is not the only consequence of Allee effects on the population. Indeed, one can investigate the genealogies of a particle system governed by Equation (1.1). One expects them to evolve over larger time-scales for pushed fronts than for pulled fronts. In biological terms, this translates into larger genetic diversity [HN08]. For pulled fronts, the time-scale is logarithmic in N and the genealogy is described by the Bolthausen–Sznitman coalescent [BDS08]. If the Allee effect is sufficiently strong, it is natural to assume that the genealogy evolves over the timescale N and is described by Kingman's coalescent [BHK18]. This was proved in the case of strong Allee effects in the context of population genetics [EP20]. Strong Allee effects are often modeled by bistable reaction diffusion equations, which can not be considered with reaction terms of the form (1.3) (heuristically, it corresponds to $B \to \infty$). See Chapter 1 for further details on the classification of Allee effects. The simulations in [BHK18] and the analysis conducted here describe the intermediate regime between these two extremes: the genealogy is observed on a time scale $N^{\alpha-1}$ for some $\alpha \in (1,2)$ and its structure is given by a Beta-coalescent.

According to [BHK18], pulled and pushed fronts can also be distinguished by the spatial position of the ancestors of the particles. Taking a particle at random and looking at its ancestor at a time far in the past, this ancestor will sit at a position at the leading edge of the front (*i.e.* far to the right of the front) in pulled fronts, whereas it will be at the middle of the front in pushed fronts, where most particles lie [BHK18, Fig. 2]. One can consider the trajectory described by the ancestors of this particle as the path of an *immortal particle*, and thus conjecture the following two distinct behaviours: in pulled fronts, the path of an immortal particle typically spends most of its time far away from the front, whereas in pushed fronts, it spends most of its time near the middle of the front, in the vicinity of the other particles. Indeed, in the model studied in [BBS13], which can be seen as a simplification of the noisy FKPP equation, the prime example of a pulled front, the path of the immortal particle ressembles in the co-moving frame a Brownian motion constrained to stay in an interval of size of order $\log N$, and is thus typically a distance $\log N$ away from the front. On the other hand, for pushed fronts, one should expect that the path of an immortal particle is described in the co-moving frame by a positive recurrent Markov process independent of the population size.

Another distinction arises when one considers the events which drive the evolutionary dynamics, i.e. which cause mergers in the ancestral lines of individuals randomly sampled from the population. The authors of [BHK18] conjecture here that the distinction does not take place between pulled and pushed, but between pulled and semi-pushed on the one side and fully pushed on the other [BHK18, SI, p36]. In fully pushed fronts, the population can be approximated by a neutral population, with *all the organisms at the front.* In contrast, the particles located at the tip of the front that drive the evolutionary dynamics in semi-pushed and pulled waves. This is consistent with the genealogical structures introduced above. Indeed, in pulled and semi-pushed fronts we expect the genealogies to be described by coalescents with multiple mergers. In these coalescents, single individuals replace a fraction of the population during merge events. It is reasonable to think that for this to happen a particle has to move far away from the front in order to have time to produce a large number of descendants before being incorporated in the front again. On the other hand, in fully pushed fronts, we expect the genealogy to be described by Kingman's coalescent, indicating that the population behaves like a neutral population where particles are indistinguishable. Thus, typical particles, i.e. those which are near the front, should drive the evolutionary dynamics. Of course, it is still possible for particles to move far away from the front and replace a fraction of the population. But since Kingman's coalescent only consists of binary mergers, these events are not visible in the limit and thus have to happen on a longer time-scale than the time-scale N at which the genealogy evolves.

The characteristics of the three types of fronts are summarised in Table III.1.

1.9 Structure of the chapter

The proof of the result follows the steps detailed in Section 1.5.

First, we examine the density of particles p_t in Section 2. We give a detailed description of the eigenvalues of the PDE (B) for all $\rho \in (1, +\infty) \setminus \left\{1 + \frac{k^2\pi^2}{4}, k \in \mathbb{N}^2\right\}$. This first analysis motivates the distinction of the pushed and the pulled regime in the particle system. We also estimate the speed of convergence of the principle eigenvalue to its limit, which provides the time scale of interest. In Section 2.2, we prove that the density of particles p_t is well approximated by its first term for sufficiently large t. In Section 2.3, we control the Green function of Equation

	pulled	**pushed**	
		semi-pushed	fully pushed
cooperativity B	$B \in (0, 2]$	$B \in (2, B_c)$	$B \in (B_c, +\infty)$
Allee effect	$\leftarrow$ no Allee effect ($B = 0$)	**weak Allee effect**	strong Allee effect $\rightarrow$ ($B \to \infty$)
speed of front compared to linearized equation	same	faster	
path of an immortal particle	far to right of front	close to front	
time-scale of genealogy	polylog(N)	$N^{\alpha-1}$, $\alpha \in (1, 2)$	N
evolutionary dynamics driven by particles at positions...	...far to right of front		...close to front

Table III.1: Summary of the characteristics of pulled, semi-pushed and fully pushed fronts.

(B). These estimates are needed to bound the double integral given by the Many-to-two lemma 1.2.

In Section 3, we bound the first and second moments of several quantities (including Z'_t) which rule the long time behaviour of the system. The first moments is estimated under (H$_{psh}$), so that the results also hold in the fully pushed regime. The second moments are estimated under stronger assumptions, that is (H$_{wp}$).

In Section 4, we bound the number of particles that hit the additional barrier L. In Section, 5, we estimate the number of descendants of a particle that reach L, after a large time of order 1.

As explained in Section 1.5, we then prove the convergence of the process Z to the CSBP. Inspired by [MS20], we gather all the estimates established in Sections 3, 4 and 5 to estimate the Laplace transform of Z. In Section 6, we control the Laplace transform of Z on small time steps, on the time scale of the CSBP. In Section 7.1, we bring all these small time steps together to obtain the convergence of the process Z via its Laplace transform and the Euler scheme of the ODE satisfied by the branching mechanism Ψ of the CSBP. In Section 7.2, we give a more general version of Theorem 1.2 in the semi pushed regime and deduce it from the convergence of the process Z.

1.10 Notations

We recall in this section the definition of several quantities depending the parameter ρ of the model as well as their dependences. As outlined in Section 1.5, we denote by λ_1 the maximal eigenvalue of the operator

$$Tv = \frac{1}{2}v'' + \frac{\rho - 1}{2}\mathbb{1}_{[0,1]}(x)v,$$

on the domain $\mathcal{D}_L = \{v \in C^1((0, L)) \cap C^2((0, 1) \cup (1, L)) \,:\, v(0) = v(L) = 0,\, \lim_{x\to 1} Tu(x) \text{ exists}\}$. Hence, λ_1 depends on L and we prove (this is the object of Section 2.1) that for $\rho > \rho_1$, λ_1 increases with L and converges to a positive limit λ_1^∞ as L goes to ∞.

In this case, we write α, β, γ and μ to refer to the following quantities:

$$\mu = \sqrt{1+2\lambda_1^\infty}, \quad \beta = \sqrt{2\lambda_1^\infty}, \quad \alpha = \frac{\mu + \sqrt{\mu^2-1}}{\mu - \sqrt{\mu^2-1}} = \frac{\mu+\beta}{\mu-\beta}, \text{ and } \gamma = \sqrt{\rho - 1 - 2\lambda_1^\infty}, \tag{1.30}$$

to emphasize that they do not depend on L, but only on ρ.

2 BBM in an interval: the density of particles

2.1 Spectral analysis

Let $L > 1$, $\rho \in (1,\infty) \setminus \{1 + k^2\frac{\pi^2}{4}, k \in \mathbb{N}\}$ and consider the differential operator

$$Tv(x) = \frac{1}{2}v''(x) + \frac{\rho-1}{2}v(x)\mathbb{1}_{x\leqslant 1}, \tag{2.1}$$

on the domain $\mathcal{D}_L = \{v \in C^1((0,L)) \cap C^2((0,1)\cup(1,L)) \,:\, v(0) = v(L) = 0, \lim_{x\to 1} Tu(x) \text{ exists}\}$. We are interested in the spectrum of the operator T, that is the set

$$\{\lambda \in \mathbb{C} \,:\, Tv = \lambda v \text{ has a non-zero solution } v \in \mathcal{D}_L\}.$$

According to Sturm-Liouville theory (see Theorem 4.3.1 from [Zet10]), this set is infinite, countable and it has no finite accumulation point. Besides, it is upper bounded and all the eigenvalues are simple and real so that they can be numbered

$$\lambda_1 < \lambda_2 < \dots < \lambda_n \to -\infty, \text{ as } n \to +\infty.$$

The sequence $(\lambda_i)_{i\in\mathbb{N}}$ is described in Lemma 2.1. In Lemma 2.2, we compute an asymptotic expansion of the positive eigenvalues as $L \to +\infty$. For L sufficiently large, the set of positive eigenvalues is not empty as long as $\rho > 1 + \frac{\pi^2}{4}$. In Lemma 2.3 and Corollary 2.1, we give an explicit formula for the limit of the L^2-norm and the speed of convergence of the principal eigenvalue λ_1, which will be needed in Section 6.

We first introduce some notations that will be used throughout this section. Denote by

$$\mathcal{S}(x,\lambda) = \begin{cases} \frac{\sinh(\sqrt{\lambda}x)}{\sqrt{\lambda}} & \forall (x,\lambda) \in [0,+\infty)\times(0,+\infty) \\ \frac{\sin(\sqrt{-\lambda}x)}{\sqrt{-\lambda}} & \forall (x,\lambda) \in [0,+\infty)\times(-\infty,0) \\ x & \forall (x,\lambda) \in [0,+\infty)\times\{0\}. \end{cases} = \frac{\sinh(\sqrt{\lambda}x)}{\sqrt{\lambda}} \quad \forall (x,\lambda) \in [0,\infty)\times\mathbb{R}. \tag{2.2}$$

Similarly, we define $\mathcal{C}(x,\lambda) = \cosh\left(\sqrt{\lambda}x\right)$ and $\mathcal{T}(x,\lambda) = \frac{\mathcal{S}(x,\lambda)}{\mathcal{C}(x,\lambda)}$ for all $(x,\lambda) \in [0,+\infty)\times\mathbb{R}$.

Lemma 2.1. *There exists $L_0 = L_0(\rho)$, such that the following holds for all $L \geq L_0$: Let $K \in \mathbb{N}$ be the largest positive integer such that*

$$\rho - 1 > \left(K - \frac{1}{2}\right)^2 \pi^2, \tag{2.3}$$

and $K = 0$ otherwise. Then, for all $1 \leqslant k \leqslant K$, λ_k is the unique solution of

$$\mathcal{T}\,(1, 2\lambda + 1 - \rho) = \mathcal{T}\,(L - 1, 2\lambda)\,, \tag{$\square$}$$

such that,

$$(\rho - 1 - k^2\pi^2 \vee 0) < 2\lambda_k < \rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2, \tag{2.4}$$

Furthermore, $\lambda_k < 0$ for all $k > K$. More precisely, set for all $i \geq 0$:

$$A_i = \frac{1}{2}\left(\left(K + \frac{1}{2} + i\right)^2 \pi^2 + 1 - \rho\right) \tag{2.5}$$

$$N_i = \left\lfloor \frac{(L-1)}{\pi}\sqrt{A_i}\; - \frac{1}{2}\right\rfloor + i \tag{2.6}$$

and $A_{-1} = N_{-1} = 0$. Also, set $a_0 = 0$ and

$$a_j = \frac{\left(j - \frac{1}{2}\right)^2}{2(L-1)^2}\pi^2, \quad j \geq 1. \tag{2.7}$$

Then, for every $i \geq 0$ and every $j \in \mathbb{N}$ such that $N_{i-1} < j \leq N_i$, λ_{K+j} is the unique solution of ($\square$) in the interval

$$(-A_i, -A_{i-1}) \cap (-a_{j-i+1}, -a_{j-i}). \tag{2.8}$$

Finally, for all $k \in \mathbb{N}$, the eigenvector v_k associated with λ_k is unique up to a multiplicative constant and is given by

$$v_k(x) = \begin{cases} \mathcal{S}(x, 2\lambda_k - \rho - 1)/\mathcal{S}(1, 2\lambda_k - \rho - 1) & x \in [0,1], \\ \mathcal{S}(L - x, 2\lambda_k)/\mathcal{S}(L - 1, 2\lambda_k) & x \in [1, L]. \end{cases} \tag{2.9}$$

Proof. For $\lambda \in \mathbb{R}$, consider the system $(\mathcal{C})$

$$\begin{cases} v''(x) = (2\lambda + 1 - \rho)v(x) & x \in [0,1], \\ v''(x) = 2\lambda v(x) & x \in [1, L], \\ v(0) = v(L) = 0. \end{cases} \tag{$\mathcal{C}$}$$

First, note that if $(\mathcal{C})$ has a non-zero solution, then $2\lambda < \rho - 1$. Indeed, if there exists a function v solution of $(\mathcal{C})$ for some $\lambda > (\rho - 1)/2$, it is of the form

$$v(x) = \begin{cases} A \sinh\left(\sqrt{2\lambda + 1 - \rho}\, x\right) & x \in [0,1], \\ B \sinh\left(\sqrt{2\lambda}\,(L - x)\right) & x \in [1, L], \end{cases} \tag{2.10}$$

for some $A, B \in \mathbb{R}$. Thus, $\lambda_1 > \frac{\rho-1}{2}$ and v_1 satisfies (2.10). Yet, $v_1 > 0$ on $[0, L]$ (see Theorem 4.3.1 in [Zet10]), hence, A and B are both positive. Besides, since v_1 is continuous and differentiable at 1, we have

$$A\sqrt{2\lambda + 1 - \rho}\cosh(\sqrt{2\lambda + 1 - \rho}) = -B\sqrt{2\lambda}\cosh(\sqrt{2\lambda}(L - 1)).$$

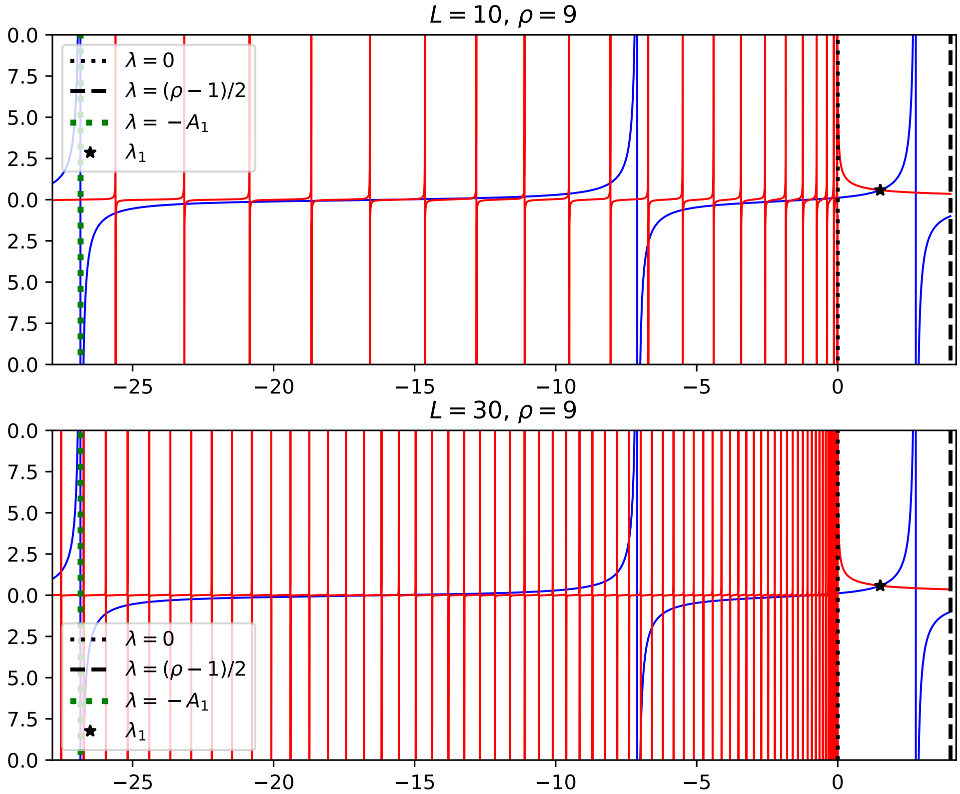

Figure III.3: Location of the eigenvalues of the differential operator T (see Equation (2.1)) for $\rho = 4$ and different values of L. The blue line represents the graph of the left-hand term in Equation (☐). The red line corresponds to the graph the right-hand term in Equation (☐). Thus, the eigenvalues are located at the intersections of the blue and red solid lines. The integer K corresponds to the number of positive eigenvalues. Their limits are isolated. Note that the negative eigenvalues tend to a continuous spectrum. For $\rho = 9$, we have $K = 1$.

Thus, $A = B = 0$ and v_1 is constant equal to 0. Similarly, one can prove that $\lambda_1 \neq \frac{\rho-1}{2}$. Therefore,

$$\lambda_k < \frac{\rho - 1}{2}, \quad \forall k \in \mathbb{N}.$$

If v is a solution of $(\mathcal{C})$ for some $0 < \lambda < \frac{\rho-1}{2}$, there exist $A, B \in \mathbb{R}$ such that

$$v(x) = \begin{cases} A \sin\left(\sqrt{\rho - 1 - 2\lambda}\, x\right) & x \in [0, 1], \\ B \sinh\left(\sqrt{2\lambda}\,(L - x)\right) & x \in [1, L]. \end{cases}$$

Since v is continuous and differentiable at 1, λ satisfies the following system:

$$\begin{cases} A \sin(\sqrt{\rho - 1 - 2\lambda}) = B \sinh(\sqrt{2\lambda}(L - 1)), \\ A\sqrt{\rho - 1 - 2\lambda} \cos(\sqrt{\rho - 1 - 2\lambda}) = -B\sqrt{2\lambda} \cosh(\sqrt{2\lambda}(L - 1)). \end{cases}$$

If $A = 0$ (resp. $B = 0$), the second equation of the system implies that $B = 0$ (resp. $A = 0$), so that v is not an eigenfunction. Hence, we can assume without loss of generality that $A \neq 0$ and

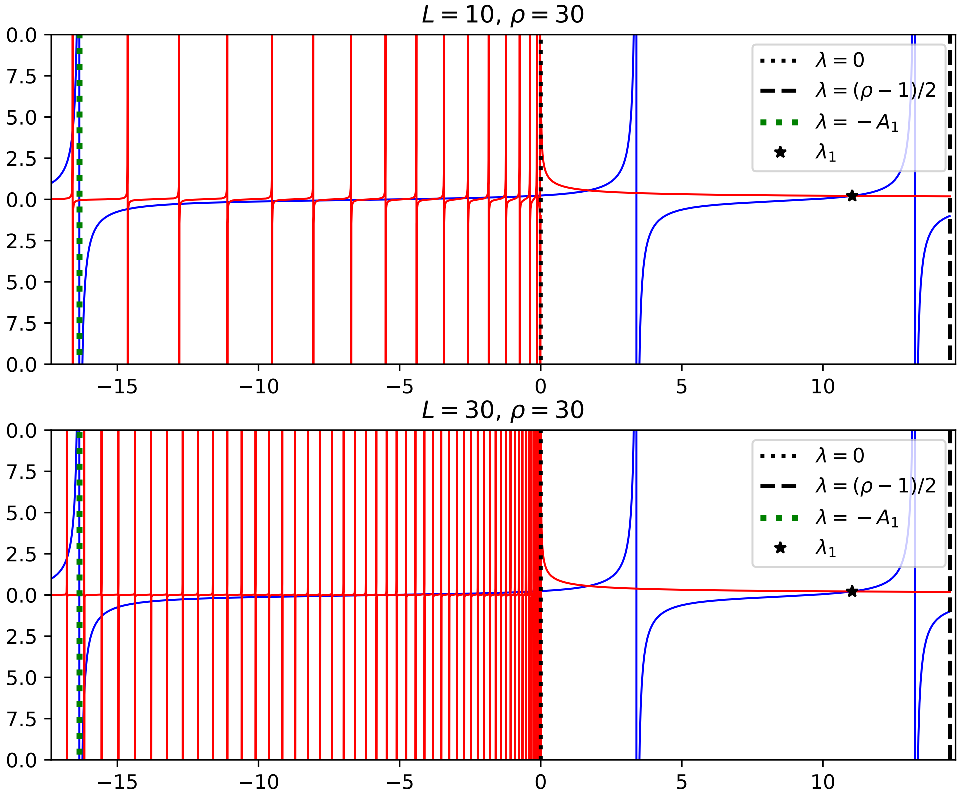

Figure III.4: Location of the eigenvalues of the differential operator T (see Equation (2.1)) for $\rho = 30$ and different values of L. The blue line represents the graph of the left-hand term in Equation (☐). The red line corresponds to the graph the right-hand term in Equation (☐). Thus, the eigenvalues are located at the intersections of the blue and red solid lines. The integer K corresponds to the number of positive eigenvalues. Their limits are isolated. Note that the negative eigenvalues tend to a continuous spectrum. For $\rho = 30$, we have $K = 2$.

$B \neq 0$. Therefore, λ is a solution of the equation

$$-\frac{\tan(\sqrt{\rho - 1 - 2\lambda})}{\sqrt{\rho - 1 - 2\lambda}} = \frac{\tanh(\sqrt{2\lambda}(L-1))}{\sqrt{2\lambda}}. \tag{2.11}$$

We now prove that Equation (2.11) has exactly K solutions in $\left(0, \frac{\rho-1}{2}\right)$, for L large enough. Define

$$f(x) = \frac{\tan(x)}{x}, \quad \text{and} \quad g(x) = \frac{\tanh(x)}{x}. \tag{2.12}$$

For $x \in (0, \frac{\pi}{2}) \cup \left(\cup_{k \in \mathbb{N}} (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})\right)$,

$$f'(x) = \frac{2x - \sin(2x)}{2x^2 \cos(x)^2} > 0$$

since $\sin(2x) < 2x$. Besides, $f(x) < 0$ if and only if $x \in \cup_{k \in \mathbb{N}} (k\frac{\pi}{2}, k\pi)$ and $f(x) \to 0$ as $x \to k\pi$ and $f(x) \to +\infty$ as $x \to (k+1)\frac{\pi}{2}^-$. Moreover, for $x \in (0, \infty)$,

$$g'(x) = \frac{2x - \sinh(2x)}{x^2 \cosh(x)^2} < 0$$

since $\sinh(2x) > 2x$. Note that $g(x) > 0$ for all $x \in (0, \infty)$ $g(x) \to 0$ as $x \to \infty$ and $g(x) \to 1$ as $x \to 0$.

Therefore, on each interval $\left(\left(\frac{1}{2}(\rho - 1 - (k+\frac{1}{2})^2\pi^2), \frac{1}{2}(\rho - 1 - (k-\frac{1}{2})^2\pi^2)\right)\right., k \in \{1, ..., K-1\}$, the function $\lambda \mapsto -f(\sqrt{\rho - 1 - 2\lambda})$ is increasing and

$$\begin{aligned} -f(\sqrt{\rho-1-2\lambda}) &= 0, \qquad \lambda = \frac{1}{2}(\rho - 1 - k^2\pi^2) \\ -f(\sqrt{\rho-1-2\lambda}) &\to +\infty, \qquad \lambda \to \frac{1}{2}\left(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2\right)^- . \end{aligned}$$

The function $\lambda \mapsto g(\sqrt{2\lambda}(L-1))$ is positive and decreasing on $(0, \infty)$. Hence, Equation (2.11) has a unique solution in each interval $\left(\left(\frac{1}{2}(\rho - 1 - k^2\pi^2), \frac{1}{2}(\rho - 1 - (k-\frac{1}{2})^2\pi^2)\right)\right., k \in \{1, ..., K-1\}$. It has no solution in $\cup_{k=1}^{K-1}(\left[\frac{1}{2}(\rho - 1 - (k+\frac{1}{2})^2\pi^2), \frac{1}{2}(\rho - 1 - k^2\pi^2)\right]$ since $\lambda \mapsto g(\sqrt{2\lambda}(L-1))$ is continuous and positive on each closed interval and for $k \in \{1, ..., K-1\}$, we have

$$\begin{aligned} -f(\sqrt{\rho-1-2\lambda}) &\leqslant 0, \qquad \lambda \in \cup_{k=1}^{K-1}\left(\frac{1}{2}\left(\rho - 1 - \left(k+\frac{1}{2}\right)^2\pi^2\right), \frac{1}{2}(\rho - 1 - k^2\pi^2)\right) \\ -f(\sqrt{\rho-1-2\lambda}) &\to -\infty, \qquad \lambda \to \frac{1}{2}\left(\rho - 1 - \left(k+\frac{1}{2}\right)^2\pi^2\right)^+, \\ -f(\sqrt{\rho-1-2\lambda}) &\to +\infty, \qquad \lambda \to \frac{1}{2}\left(\rho - 1 - \left(k+\frac{1}{2}\right)^2\pi^2\right)^- . \end{aligned}$$

Then, note that for sufficiently large L, Equation (2.11) has a unique solution in the interval $(0 \vee \frac{1}{2}(\rho - 1 - K^2\pi^2), \frac{1}{2}(\rho - 1 - (K - \frac{1}{2})^2\pi^2)$. Indeed, the function $\lambda \mapsto -f(\sqrt{\rho - 1 - 2\lambda})$ is positive, increasing and

$$\begin{aligned} -f(\sqrt{\rho-1-2\lambda}) &\to +\infty, \qquad \lambda \to \frac{1}{2}\left(\rho - 1 - \left(K - \frac{1}{2}\right)^2\pi^2\right), \\ -f(\sqrt{\rho-1-2\lambda}) &\to 0 \vee \left(-\frac{\tan\sqrt{\rho-1}}{\sqrt{\rho-1}}\right), \qquad \lambda \to 0 \vee \frac{1}{2}(\rho - 1 - K^2\pi^2). \end{aligned}$$

Besides, $\lambda \mapsto g(\sqrt{2\lambda}(L-1))$ is positive, decreasing and $g(\sqrt{2\lambda}(L-1)) \to L-1$ as $\lambda \to 0$. Therefore, if $L > 1 - \frac{\tan\sqrt{\rho-1}}{\sqrt{\rho-1}}$, Equation (2.11) has one solution in $(0 \vee \frac{1}{2}(\rho - 1 - K^2\pi^2), \frac{1}{2}(\rho - 1 - (K - \frac{1}{2})^2\pi^2))$. If it exists, this solution is unique.

There is no solution of (2.11) in $(0, \frac{1}{2}(\rho - 1 - K^2\pi^2)]$ since the left-hand side term in (2.11) is negative on this set. Therefore, for $L > 1 - \frac{\tan\sqrt{\rho-1}}{\sqrt{\rho-1}}$, we found exactly K solutions of (2.11) in $(0, \frac{\rho-1}{2})$. Conversely, one can check that these solutions are eigenvalues of T, corresponding to eigenvectors defined by (2.9).

Let us now consider the system $(\mathcal{C})$ for $\lambda = 0$. A solution of this system is of the form

$$v(x) = \begin{cases} C \sin\left(\sqrt{\rho-1}x\right) \sin\left(\sqrt{\rho-1}\right)^{-1} & x \in [0,1], \\ C\frac{L-x}{L-1} & x \in [1, L], \end{cases}$$

for some $C \in \mathbb{R}$. Remark that this function is not differentiable at 1 if $L > 1 - \frac{\tan(\sqrt{\rho-1})}{\sqrt{\rho-1}}$ so that 0 is not an eigenvalue of T.

If $(\mathcal{C})$ has a non-zero solution v associated with some $\lambda < 0$, there exist $A, B \in \mathbb{R}$ such that

$$v(x) = \begin{cases} A \sin\left(\sqrt{\rho - 1 - 2\lambda}\, x\right) & x \in [0,1], \\ B \sin\left(\sqrt{-2\lambda}\,(L - x)\right) & x \in [1, L]. \end{cases}$$

In the same way as for Equation (2.11), we get that λ has to satisfy the following equation:

$$-\frac{\tan(\sqrt{\rho - 1 - 2\lambda})}{\sqrt{\rho - 1 - 2\lambda}} = \frac{\tan(\sqrt{-2\lambda}(L-1))}{\sqrt{-2\lambda}}. \tag{2.13}$$

The function $\lambda \mapsto (L-1)f(\sqrt{-2\lambda}(L-1))$ is defined on $\cup_{j=0}^{\infty}(-a_{j+1}, -a_j)$, with (a_j) defined in (2.7). In view of the above, $\lambda \mapsto (L-1)f(\sqrt{-2\lambda}(L-1))$ is decreasing on each interval $(-a_{j+1}, -a_j)$. Similarly, the function $\lambda \mapsto -f(\sqrt{\rho - 1 - 2\lambda})$ is defined on $\cup_{i=0}^{\infty}(-A_i, -A_{i-1})$, where the sequence (A_i) is defined by Equation (2.5), and it is increasing on each interval $(-A_i, -A_{i-1})$, $i \in \mathbb{N}$. Besides, for all $i \geqslant 0$ and $j \geqslant 1$,

$$\begin{aligned} \lim_{\substack{\lambda \to -a_j \\ x > -a_j}} (L-1)f(\sqrt{-2\lambda}(L-1)) &= +\infty, \\ \lim_{\substack{\lambda \to -a_j \\ x < -a_j}} (L-1)f(\sqrt{-2\lambda}(L-1)) &= -\infty, \\ \lim_{\substack{\lambda \to -A_i \\ x < -A_i}} -f(\sqrt{\rho - 1 - 2\lambda}) &= +\infty, \\ \lim_{\substack{\lambda \to -A_i \\ x > -A_i}} -f(\sqrt{\rho - 1 - 2\lambda}) &= -\infty. \end{aligned}$$

Also note that $N_i - N_{i-1} + 1$ (see Equation (2.6)) is the number of complete intervals $((-a_{j+1}, -a_j))_{j \in \mathbb{N}}$ included in the interval $(-A_{i+1}, -A_i)$, $i \geqslant 0$. Therefore, Equation (2.13) has a unique solution in the interval $(-A_i, -A_{i-1}) \cap (-a_{j-i+1}, -a_{j-i})$, for all $N_{i-1} < j \leq N_i$, $i \in \mathbb{N}_0$.

Finally, remark that

$$\begin{aligned} \lim_{\substack{\lambda \to 0 \\ \lambda < 0}} (L-1)f(\sqrt{-2\lambda}(L-1)) &= L - 1, \\ \lim_{\substack{\lambda \to 0 \\ \lambda < 0}} -f(\sqrt{\rho - 1 - 2\lambda}) &= -\frac{\tan\sqrt{\rho-1}}{\sqrt{\rho-1}}. \end{aligned}$$

Hence, Equation (2.13) has no solution in $(-a_1, -a_0)$ if $L > 1 - \frac{\tan\sqrt{\rho-1}}{\sqrt{\rho-1}}$.

Again, one can check that all these solutions are eigenvalues, associated with the eigenvectors given by (2.9). □

The position of the eigenvalues for different values of L and ρ are illustrated in Figures III.3 and III.4. In the following lemma, we compute the speed of convergence of the positive eigenvalues to their limits. The speed of convergence of the maximal eigenvalue indicates the time scale of interest.

Lemma 2.2 (Asymptotic expansions of the positive eigenvalues)**.** *Assume (H_{psh}) holds. Let $K \in \mathbb{N}$ be the largest positive integer such that*

$$\rho - 1 > \left(K - \frac{1}{2}\right)^2 \pi^2.$$

Then, for all $1 \leqslant k \leqslant K$, λ_k is decreasing and tends to the unique solution λ_k^∞ of

$$-\frac{\tan(\sqrt{\rho - 1 - 2\lambda})}{\sqrt{\rho - 1 - 2\lambda}} = \frac{1}{\sqrt{2\lambda}} \tag{2.14}$$

located in the interval

$$\left(\frac{\rho - 1 - k^2\pi^2}{2} \vee 0, \frac{1}{2}\left(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2\right)\right),$$

as $L \to \infty$. Besides, for all $1 \leqslant k \leqslant K$, there exists a constant $C_k(\rho) > 0$ such that

$$\lambda_k = \lambda_k^\infty - C_k(\rho) e^{-2\sqrt{2\lambda_k^\infty} L} + o\left(e^{-2\sqrt{2\lambda_k^\infty} L}\right),$$

where β from Equation (1.30).

Proof. Recall from Lemma 2.1 that for $1 \leqslant k \leqslant K$, λ_k is the unique solution of Equation (□):

$$\frac{\tan(\sqrt{\rho - 1 - 2\lambda})}{\sqrt{\rho - 1 - 2\lambda}} = -\frac{\tanh(\sqrt{2\lambda}(L-1))}{\sqrt{2\lambda}}$$

located in the interval $(\frac{1}{2}(\rho - 1 - k^2\pi^2 \vee 0), \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2))$. Besides, $L \mapsto \lambda_k$ is an increasing function. Indeed, recall from the proof of Lemma 2.1 that the function $\lambda \mapsto -f(\sqrt{\rho - 1 - 2\lambda})$ is increasing on each interval $(\frac{1}{2}(\rho - 1 - k^2\pi^2 \vee 0), \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2))$ and does not depend on L. Yet, for $L > 1$, the function $\lambda \mapsto (L-1) g(\sqrt{2\lambda}(L-1))$ is decreasing on $(0, \infty)$ and for $\lambda > 0$, the function $L \mapsto (L-1) g(\sqrt{2\lambda}(L-1))$ is increasing on $[1, +\infty)$. Thus, $L \mapsto \lambda_k(L)$ is an increasing function for all $1 \leqslant k \leqslant K$. Since it is bounded by $\frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2)$, it converges to some limit $\lambda_k^\infty \in \left(\frac{1}{2}(\rho - 1 - k^2\pi^2 \vee 0), \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2)\right]$.

If $\lambda_k^\infty = \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2)$, the left-hand term in (2.11) tends to $+\infty$ as $L \to \infty$ whereas the right-hand term tends to $(2\lambda_k^\infty)^{-\frac{1}{2}}$. Thus,

$$\lambda_k^\infty \in \left(\frac{1}{2}(\rho - 1 - k^2\pi^2 \vee 0), \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2)\right).$$

Then, since the right-hand side and left-hand side terms in (2.11) are continuous on each interval $(\frac{1}{2}(\rho - 1 - k^2\pi^2 \vee 0), \frac{1}{2}(\rho - 1 - \left(k - \frac{1}{2}\right)^2 \pi^2))$, we obtain that λ_k^∞ is a solution of Equation (2.14). Moreover one can show that this solution is unique (see proof of Lemma 2.1).

Let us now compute an asymptotic expansion of λ_k as $L \to \infty$. From now, we assume that $k = 1$ but similar computations can be conducted for $k \in [\![2, K]\!]$. Let us recall the definitions of β and γ from Equation (1.30). First, note that $\beta > 0$ since $\lambda_1^\infty > 0$. Then, remark that $\cos(\gamma) < 0$. Indeed, we know thanks to Lemma 2.2 that $\gamma \in \left(\frac{\pi}{2}, \pi\right)$ and that

$$\sin(\gamma) = -\frac{\gamma}{\beta}\cos(\gamma).$$

Hence, we have $\cos(\gamma) = -\frac{\beta}{\gamma}\sin(\gamma) \leqslant -\frac{1}{2}\frac{\beta}{\gamma} < 0$ if $\gamma \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\cos(\gamma) \leqslant -\frac{1}{2}$ if $\gamma \in \left(\frac{3\pi}{4}, \pi\right)$.

Let us now rewrite ($\square$) as

$$\sqrt{\frac{2\lambda_1}{\rho - 1 - 2\lambda_1}}\tan\left(\sqrt{\rho - 1 - 2\lambda_1}\right) = -\tanh\left(\sqrt{2\lambda_1}(L-1)\right), \tag{2.15}$$

and define $h = \lambda_1 - \lambda_1^\infty$. As $L \to \infty$, $h \to 0$, so that

$$\begin{aligned}
\sqrt{\frac{2\lambda_1}{\rho - 1 - 2\lambda_1}} &= \sqrt{\frac{2(\lambda_1^\infty + h)}{\rho - 1 - 2(\lambda_1^\infty + h)}} = \left(\frac{2\lambda_1^\infty}{\rho - 1 - 2\lambda_1^\infty}\right)^{1/2}\left(\frac{1 + \frac{h}{\lambda_1^\infty}}{1 - \frac{2h}{\rho - 1 - 2\lambda_1^\infty}}\right)^{1/2} \\
&= \frac{\beta}{\gamma}\left(1 + \frac{2h}{\beta^2}\right)^{1/2}\left(1 - \frac{2h}{\gamma^2}\right)^{1/2} = \frac{\beta}{\gamma}\left(1 + \frac{h}{\beta^2} + o(h)\right)\left(1 + \frac{h}{\gamma^2} + o(h)\right) \\
&= \frac{\beta}{\gamma}\left(1 + \left(\frac{\gamma^2 + \beta^2}{\gamma^2\beta^2}\right)h + o(h)\right),
\end{aligned}$$

and

$$\tan(\sqrt{\rho - 1 - 2\lambda_1}) = \tan\left(\gamma - \frac{h}{\gamma} + o(h)\right) = \tan(\gamma) - \frac{h}{\gamma\cos(\gamma)^2} + o(h).$$

Then, since $\tan(\gamma) = -\frac{\gamma}{\beta}$, we have

$$\begin{aligned}
&\sqrt{\frac{2\lambda_1}{\rho - 1 - 2\lambda_1}}\tan(\sqrt{\rho - 1 - 2\lambda_1}) \\
&= \frac{\beta}{\gamma}\left(\tan(\gamma) + \left(\frac{\gamma^2 + \beta^2}{\gamma^2\beta^2}\tan(\gamma) - \frac{1}{\gamma\cos(\gamma)^2}\right)h + o(h)\right) \\
&= -1 - \frac{1}{\gamma^2\beta^2\cos(\gamma)^2}\left(\frac{\gamma}{\beta}(\gamma^2 + \beta^2)\cos(\gamma)^2 + \gamma\beta^2\right)h + o(h) \\
&= -1 - \frac{1}{\gamma^2\beta^2\cos(\gamma)^2}\left((\gamma^2 + \beta^2)\cos(\gamma)^2 + \beta^3\right)h + o(h) \\
&= -1 - \frac{(\rho - 1)\cos(\sqrt{\rho - 1 - 2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}}{2\lambda_1^\infty(\rho - 1 - 2\lambda_1^\infty)\cos(\sqrt{\rho - 1 - 2\lambda_1^\infty})^2}h + o(h).
\end{aligned}$$

Besides, as $x \to \infty$

$$\tanh(x) = 1 - 2e^{-2x} + o(e^{-2x}).$$

Thus,

$$\tanh(\sqrt{2\lambda_1}(L-1)) = 1 - 2e^{-2\sqrt{2\lambda_1}(L-1)} + o(e^{-2\sqrt{2\lambda_1}(L-1)}).$$

Combined with Equation (2.14), this implies that

$$2e^{-2\sqrt{2\lambda_1}(L-1)} + o(e^{-2\sqrt{2\lambda_1}(L-1)}) = \frac{(\rho-1)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}}{2\lambda_1^\infty(\rho-1-2\lambda_1^\infty)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2} h + o(h). \qquad (2.16)$$

Therefore we get that $Lh \to 0$ as $L \to +\infty$ and that

$$e^{-2\sqrt{2\lambda_1}(L-1)} = e^{-2\beta\left(1+\frac{1}{2}\frac{h}{\beta^2}+o(h)\right)(L-1)} = e^{-2\beta(L-1)}e^{o(1)} = e^{-2\beta(L-1)} + o(e^{-2\beta(L)}).$$

Finally, according to Equation (2.16), we have

$$\lambda_1 - \lambda_1^\infty = -2\frac{2\lambda_1^\infty(\rho-1-2\lambda_1^\infty)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2}{(\rho-1)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}} e^{-2\beta(L-1))} + o(e^{-2\beta L}),$$

with

$$\frac{2\lambda_1^\infty(\rho-1-2\lambda_1^\infty)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2}{(\rho-1)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}} > 0,$$

since $\cos(\sqrt{\rho-1-2\lambda_1^\infty}) = \cos(\gamma) > 0$. □

Lemma 2.3 (L^2-norm of the first eigenvector)**.** *Assume (H_{psh}) holds. As $L \to \infty$,*

$$\|v_1\|^2 \to \frac{1}{2}\frac{(\rho-1)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}}{\sqrt{2\lambda_1^\infty}(\rho-1-2\lambda_1^\infty)\cos(\sqrt{\rho-1-2\lambda_1^\infty})}.$$

Proof. The L^2 norm of the function v_1 is given by

$$\|v_1\|^2 = \int_0^L v_1(x)^2 dx = \frac{1-\frac{\sin(2\sqrt{\rho-1-2\lambda_1})}{2\sqrt{\rho-1-2\lambda_1}}}{2\sin(\sqrt{\rho-1-2\lambda_1})^2} + \frac{\frac{\sinh(2\sqrt{2\lambda_1}(L-1))}{2\sqrt{2\lambda_1}} - (L-1)}{2\sinh(\sqrt{2\lambda_1}(L-1))^2}.$$

One can show that the first term of this sum tends to $\frac{1}{2\sin(\gamma)^2}\left(1-\frac{\sin(2\gamma)}{2\gamma}\right)$ and the second one to $\frac{1}{2\beta}$ as $L \to \infty$. Besides, we know thanks to Equation (2.14) that $\sin(\gamma) = -\frac{\gamma}{\beta}\cos(\gamma)$. Therefore,

$$\begin{aligned}
\frac{1}{2\sin(\gamma)^2}\left(1-\frac{\sin(2\gamma)}{2\gamma}\right) + \frac{1}{2\beta} &= \frac{1}{2}\frac{\beta^2}{\gamma^2}\left(\frac{1-\frac{\sin(\gamma)\cos(\gamma)}{\gamma}}{\cos(\gamma)^2} + \frac{1}{2\beta}\right) \\
&= \frac{1}{2}\frac{\beta^2}{\gamma^2}\left(\frac{1+\frac{\cos(\gamma)^2}{\beta}}{\cos(\gamma)^2}\right) + \frac{1}{2\beta} \\
&= \frac{1}{2}\frac{\beta}{\gamma^2}\left(\frac{\beta+\cos(\gamma)^2}{\cos(\gamma)^2}\right) + \frac{1}{2\beta} \\
&= \frac{1}{2\beta\gamma^2\cos(\gamma)^2}\left((\gamma^2+\beta^2)\cos(\gamma)^2 + \beta^3\right) \\
&= \frac{1}{2}\frac{(\rho-1)\cos(\sqrt{\rho-1-2\lambda_1^\infty})^2 + (2\lambda_1^\infty)^{3/2}}{\sqrt{2\lambda_1^\infty}(\rho-1-2\lambda_1^\infty)\cos(\sqrt{\rho-1-2\lambda_1^\infty})}.
\end{aligned}$$

□

Corollary 2.1 (Asymptotic expansion of the maximal eigenvalue). *Assume (H_{psh}) holds. Therefore, we have*

$$\lambda_1 = \lambda_1^\infty - \frac{\beta}{\lim\limits_{L\to\infty} \|v_1\|^2} e^{-2\beta(L-1)} + o(e^{-2\beta L}), \tag{2.17}$$

as $L \to \infty$.

This expansion yields the following remark, that will be extensively used throughout this paper.

Remark 2.1. *Assume (H_{psh}) holds. There exist $C_1, C_2 > 0$ such that for L large enough,*

$$C_1 e^{\beta L} \leqslant \sinh(\sqrt{2\lambda_1}(L-1)) \leqslant C_2 e^{\beta L}.$$

More precisely, there exists a positive function η_L such that $\eta_L \to 0$ as $L \to \infty$ and

$$\sinh(\sqrt{2\lambda_1}(L-1)) = \frac{1}{2}(1+o(1))e^{\beta(L-1)}.$$

One can also deduce the following remark from the fact $\lambda_1 \nearrow \lambda_1^\infty$ and that $\gamma \in \left(\frac{\pi}{2}, \pi\right)$.

Remark 2.2. *Assume (H_{psh}) holds. Therefore, there exists a positive constant $C > 0$ such that, for L sufficiently large,*

$$\frac{\sin(\sqrt{\rho - 1 - 2\lambda_1})}{\sqrt{\rho-1-2\lambda_1}} > C.$$

Indeed, the left-hand side term increases to $\frac{\sin(\gamma)}{\gamma} = -\frac{\gamma}{\beta}\cos(\gamma)$ as $L \nearrow +\infty$. Besides, $\gamma \in (\frac{\pi}{2}, \pi)$ and $\cos(\gamma) = -\frac{\beta}{\gamma}\sin(\gamma) \leqslant -\frac{1}{2}\frac{\beta}{\gamma} < 0$ if $\gamma \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\cos(\gamma) \leqslant -\frac{1}{2}$ if $\gamma \in \left(\frac{3\pi}{4}, \pi\right)$.

The following lemma compares the eigenvectors $(v_k)_{k\geqslant 2}$ with the eigenvector associated with the maximal eigenvalue λ_1 in the *pushed regime*. The proof of this result is given in Appendix B.

Lemma 2.4. *Assume (H_{psh}) holds. Let K the largest integer such that*

$$\rho - 1 > \left(K - \frac{1}{2}\right)^2 \pi^2.$$

There exist $C_0, C_1, C_2, C_3, C_4 > 0$ such that for L large enough and $x \in [0, L]$, we have

$$\begin{aligned}
&\|v_1\| \geqslant C_0, \\
&\frac{|v_k(x)|}{\|v_k\|} \leqslant C_1 e^{\beta L}\, \frac{v_1(x)}{\|v_1\|}, && k \in \{2, ..., K\}, \\
&\|v_k\|^2 \geqslant C_2, && k \in \{2, ..., K\}, \\
&\frac{|v_k(x)|}{\|v_k\|} \leqslant C_3 \sqrt{\rho - 1 - 2\lambda_k}\; e^{\beta L}\, \frac{v_1(x)}{\|v_1\|}, && k > K, \\
&\|v_k\|^2 \geqslant \frac{C_4}{\sin(\sqrt{-2\lambda_k}(L-1))^2 \wedge \sin(\sqrt{\rho - 1 - 2\lambda_k})^2}, && k > K.
\end{aligned}$$

Remark 2.3. *If (H_{wp}) holds, then $K = 1$. Indeed, if (H_{wp}) holds, $\lambda_1^\infty < \frac{1}{16}$ (see Section 1.5) and according to Lemma 2.1, we have*

$$\rho < 1 + \pi^2 + 2\lambda_1 < 1 + \pi^2 + 2\lambda_1^\infty < \frac{9}{8} + \pi^2 \, (\approx 11),$$

since λ_1 increases with respect to L. Therefore, $\rho < \left(2 - \frac{1}{2}\right)^2 \pi^2 = \frac{9}{4}\pi^2 \, (\approx 22)$ and $K < 2$.

2.2 Heat kernel estimates

Consider a dyadic branching Brownian motion with branching rate $r(x)$ and particles killed at both 0 and L (and no drift). Suppose that there is initially a single particle at x in the system. Recall from the Many-to-one lemma 1.1 and the definition of q_t (see Equation (B)) that the density of particles in the system at time $t > 0$ is given by

$$q_t(x, y) = \sum_{k=1}^{\infty} \frac{1}{\|v_k\|^2} e^{\lambda_k t} v_k(x) v_k(y), \tag{2.18}$$

where the functions (v_k) are defined by Equation (2.9). We also recall that q_t is the density of the BBM in the sense that the expected number of particles in the Borel subset $B \subset (0, L)$ at t is $\int_B q_t(x, y) dy$. In Lemma 2.5, we prove that the first term of the sum in Equation (2.18) in dominant when $t \gg L$, using the estimates given in Lemma 2.4. From this perspective, we define

$$u_t(x, y) = \frac{1}{\|v_1\|^2} e^{\lambda_1 t} v_1(x) v_1(y). \tag{2.19}$$

In what follows, C is a positive constant whose value may change from line to line. The following result holds under assumption (H_{psh}) and the computations conducted in the proof are valid as long as $K \geqslant 1$.

Lemma 2.5. *Assume (H_{psh}) holds. Therefore, there exists a constant $C_0 > 0$ such that, for $t > 1$, L large enough and $x, y \in [0, L]$, we have*

$$|q_t(x, y) - u_t(x, y)| \leqslant M(t, L) u_t(x, y), \tag{2.20}$$

with

$$M(t, L) = C_0 \left(e^{2\beta L - \frac{5}{8}\pi^2 t} + L e^{2\beta L - \frac{1}{2}\lambda_1^\infty t} \right). \tag{2.21}$$

In particular, there exist $C_1, C_2 > 0$ such that for L sufficiently large and $t > C_2 L$,

$$|q_t(x, y) - u_t(x, y)| \leqslant e^{-C_1 L} u_t(x, y). \tag{2.22}$$

Proof. First, note that

$$q_t(x, y) - u_t(x, y) = \sum_{k=2}^{K} \frac{1}{\|v_k\|^2} e^{\lambda_k t} v_k(x) v_k(y) + \sum_{k=K+1}^{\infty} \frac{1}{\|v_k\|^2} e^{\lambda_k t} v_k(x) v_k(y) := S_1 + S_2. \tag{2.23}$$

Besides, recall from Lemma 2.1 that for L large enough and $k \geqslant 2$,

$$\lambda_1 - \lambda_k \geqslant \begin{cases} \frac{5}{8}\pi^2, & \text{if} \quad K \geqslant 2, \\ \frac{\lambda_1^\infty}{2} & \text{if} \quad K = 1. \end{cases} \tag{2.24}$$

According to Lemma 2.4, for L large enough and $k \in \{2, ..., K\}$, we have

$$\frac{1}{\|v_k\|^2} |v_k(x)v_k(y)| \leqslant Ce^{2\beta L - \lambda_1 t} u_t(x, y).$$

Therefore, using (2.24), we get that

$$|S_1| \leqslant CKe^{2\beta L - \frac{5}{8}\pi^2 t} u_t(x, y). \tag{2.25}$$

Let us now estimate the quantity S_2 from (2.23). According to Lemma 2.4, for L large enough and $k > K$, we have

$$\frac{1}{\|v_k\|^2} |v_k(x)v_k(y)| \leqslant Ce^{2\beta L - \lambda_1 t}(\rho - 1 - 2\lambda_k) u_t(x, y). \tag{2.26}$$

Besides, we know thanks to Lemma 2.1 that for $i \geq 0$ and $N_{i-1} < j \leqslant N_i$, we have

$$-A_i < \lambda_{K+j} < -A_{i-1}.$$

Hence,

$$\begin{aligned} S_3 \quad := \quad & \sum_{k=K+1}^{\infty} (\rho - 1 - 2\lambda_k) e^{(\lambda_k - \lambda_{K+1})t} = \sum_{i=0}^{\infty} \sum_{j=N_{i-1}+1}^{N_i} (\rho - 1 - 2\lambda_{K+j}) e^{(\lambda_{K+j} - \lambda_{K+1})t} \quad (2.27) \\ \leqslant \quad & \sum_{i=0}^{\infty} \sum_{j=N_{i-1}+1}^{N_i} (\rho + 2A_i) e^{0 \wedge (A_0 - A_{i-1})t} = \sum_{i=0}^{\infty} (N_i - N_{i-1})(\rho + 2A_i) e^{0 \wedge (A_0 - A_{i-1})t}. \end{aligned}$$

In addition, we know (see Lemma 2.1) that

$$\begin{aligned} \rho + 2A_i \quad &= \quad \left(K + \frac{1}{2} + i\right)^2, \\ N_i - N_{i-1} \quad &= \quad \left\lfloor \frac{(L-1)}{\pi} \sqrt{A_i} \; - \frac{1}{2} \right\rfloor + i - \left\lfloor \frac{(L-1)}{\pi} \sqrt{A_{i-1}} \; - \frac{1}{2} \right\rfloor - (i-1) \\ &\leqslant \quad \frac{L-1}{\pi} \left(\sqrt{A_i} - \sqrt{A_{i-1}}\right) + 2, \\ \sqrt{A_i} - \sqrt{A_{i-1}} \quad &= \quad \frac{A_i - A_{i-1}}{\sqrt{A_i} + \sqrt{A_{i-1}}} \leqslant \frac{A_i - A_{i-1}}{\sqrt{A_0}}, \\ A_i - A_{i-1} \quad &= \quad \frac{1}{2}(K + 1 + i) \leqslant C(i+1). \end{aligned}$$

for all $i \geqslant 1$. For $i = 0$, $N_0 - N_{-1} = \frac{L-1}{\pi}\sqrt{A_0}$. Therefore, for all $i \in \mathbb{N}_0$, we have

$$N_i - N_{i-1} \leqslant CL(i+1). \tag{2.28}$$

Moreover, for all $i \in \mathbb{N}$,

$$\begin{aligned} 2(A_0 - A_{i-1}) &= \left(K+\frac{1}{2}\right)^2 - \left(K+\frac{1}{2}+i-1\right)^2 = -2(i-1)\left(K+\frac{1}{2}\right) - (i-1)^2 \\ &\leqslant -(i-1)^2. \end{aligned} \tag{2.29}$$

Combining Equations (2.27), (2.28) and (3.12), we obtain that for sufficiently large L and $t > 1$,

$$S_3 \leqslant CL\left(1+\sum_{i=1}^{\infty}(i+1)\left(K+\frac{1}{2}+i\right)^2 e^{-\frac{1}{2}(i-1)^2 t}\right) \leqslant CL. \tag{2.30}$$

Therefore, this estimate combined with Equations (2.24) and (2.26) gives that for L large enough and $t > 1$, we have

$$|S_2| \leqslant Ce^{2\beta L}e^{(\lambda_{K+1}-\lambda_1)t}S_3 \leqslant CLe^{2\beta L-\frac{\lambda_1^\infty}{2}t}u_t(x,y). \tag{2.31}$$

Finally, Lemma 2.5 ensues from Equations (2.25) and (2.31). □

Remark 2.4. *Under Assumption (H_{wp}), the sum S_1 in Equation* (2.23) *is empty and one can conclude using* (2.31).

2.3 The Green function

In this section, we assume that (H_{psh}) holds and we consider a dyadic branching Brownian motion with branching rate $r(x)$, drift $-\mu$ with

$$\mu = \sqrt{1+2\lambda_1^\infty},$$

in which particles killed at 0 and L. We recall from Many-to-one lemma 1.1 and Equation (1.18) that the density of this BMM is given by

$$p_t(x,y) = e^{\mu(x-y)}e^{\frac{1}{2}-\frac{\mu^2}{2}t}q_t(x,y) = e^{\mu(x-y)}e^{-\lambda_1^\infty t}q_t(x,y), \tag{2.32}$$

where q_t is defined in Equation (2.18). This section is aimed at controlling the integral of the density

$$\int_0^t p_s(x,y)dy. \tag{2.33}$$

Indeed, the leading order term in the formula given by the Many-to-two Lemma 1.2 will be the double integral. More precisely, we will have to estimate the quantity

$$\int_0^t\int_0^L p_s(x,y)\left(e^{\mu y}v_1(y)\right)^2 dyds = \int_0^L \left(e^{\mu y}v_1(y)\right)^2\left[\int_0^t p_s(x,y)ds\right]dy.$$

To this end, we will introduce the Green function associated to the operator $L = \frac{1}{2}\Delta + \mu\nabla + r(x)$ on the domain $\mathcal{D}_L = \{v \in \mathcal{C}^1([0,L]) \,:\, v(0) = v(L) = 0\}$. Let us first introduce the Green function H associated to the operator T from (2.1). Following the definition given in [Pin95], if B_t is a one-dimensional Brownian motion without drift starting from x and $\tau = \inf\{t : B_t \notin (0, L)\}$, then for all bounded measurable functions g,

$$\mathbb{E}\left[\int_0^\tau \exp\left(\int_0^t r(B_u) - \frac{1}{2}\, du\right) g(B_t)\right] = \int_0^L H(x,y)g(y)dy.$$

By the Many-to-one Lemma 1.1, this definition is equivalent to

$$H(x,y) = \int_0^\infty q_t(x,y)\, dt.$$

Similarly, one can define the Green function associated to the operator $T - \lambda$,

$$H_\lambda(x,y) = \int_0^\infty e^{-\lambda t} q_t(x,y)\, dt.$$

Therefore, we know thanks to Equation (1.18), that the Green function associated to the operator $L - \xi$ is given by

$$\begin{aligned} G_\xi(x,y) &= \int_0^\infty e^{-\xi t} p_t(x,y)\, dt \\ &= e^{\mu(x-y)} \int_0^\infty e^{-(\lambda_1^\infty + \xi)t} q_t(x,y)\, dt \\ &= e^{\mu(x-y)} H_{\lambda_1^\infty + \xi}(x,y). \end{aligned}$$

A first idea to estimate (2.33) would be to bound it by $G_0(x,y)$ as in [BBS13]. However, to get a finer estimate of (2.33), that depends on t, we rather consider the function G_ξ and point out that

$$\int_0^t p_s(x,y)ds = \int_0^\infty p_s(x,y)\mathbb{1}_{s\in[0,t]}ds \leqslant \int_0^\infty e^{\frac{t-s}{t}} p_s(x,y)ds \leqslant e G_{\frac{1}{t}}(x,y). \tag{2.34}$$

Therefore, we will first explicitly compute the function G_ξ following the method introduced in [BS12, Chapter II,11.]. Then, in Lemmas 2.7 and 2.6, we bound G_ξ for different values of ξ. Equation (2.34) suggests that ξ is a small positive number, which tends to 0 as $t \to \infty$.

We now introduce some notations that will be used in the following lemmas. For $\xi > 0$, we set

$$\begin{aligned} f(\xi) &= \sqrt{2\xi}\sin(\sqrt{\rho - 1 - 2\xi}) + \sqrt{\rho - 1 - 2\xi}\cos(\sqrt{\rho - 1 - 2\xi}), \\ g(\xi) &= \sqrt{2\xi}\sin(\sqrt{\rho - 1 - 2\xi}) - \sqrt{\rho - 1 - 2\xi}\cos(\sqrt{\rho - 1 - 2\xi}), \\ \omega_\xi &= f(\xi)e^{\sqrt{2\xi}(L-1)} + g(\xi)e^{-\sqrt{2\xi}(L-1)}. \end{aligned} \tag{2.35}$$

We recall that λ_1^∞ is the unique solution of Equation (2.14) such that

$$\gamma = \sqrt{\rho - 1 - 2\lambda_1^\infty} \in \left(\frac{\pi}{2}, \pi\right).$$

Therefore, $f(\lambda_1^\infty) = 0$ and $g(\lambda_1^\infty) > 0$. Furthermore,

$$\begin{aligned} f'(\lambda_1^\infty) &= (1 + \sqrt{2\lambda_1^\infty})\left(\frac{\sin(\sqrt{\rho - 1 - 2\lambda_1^\infty})}{\sqrt{2\lambda_1^\infty}} - \frac{\cos(\sqrt{\rho - 1 - 2\lambda_1^\infty})}{\sqrt{\rho - 1 - 2\lambda_1^\infty}}\right) \\ &= -\frac{(1 + \sqrt{2\lambda_1^\infty})(\rho - 1)}{2\lambda_1^\infty\sqrt{\rho - 1 - 2\lambda_1^\infty}}\cos(\sqrt{\rho - 1 - 2\lambda_1^\infty}) > 0. \end{aligned}$$

Let φ_ξ and ψ_ξ be solutions of

$$Tu = \frac{1}{2}u'' + \left(r(x) - \frac{1}{2}\right) u = \xi u,$$

such that $\varphi_\xi(0) = 0$ and $\psi_\xi(L) = 0$. If $0 < \xi < \frac{\rho-1}{2}$, there exists some constants A, B, C, D such that

$$\varphi_\xi(x) = \begin{cases} \sin(\sqrt{\rho - 1 - 2\xi}\, x) & x \in [0, 1] \\ Ae^{\sqrt{2\xi}x} + Be^{-\sqrt{2\xi}x} & x \in [1, L], \end{cases}$$

and

$$\psi_\xi(x) = \begin{cases} C\cos(\sqrt{\rho - 1 - 2\xi}x) + D\sin(\sqrt{\rho - 1 - 2\xi}x) & x \in [0, 1] \\ \sinh(\sqrt{2\xi}(L - x)) & x \in [1, L]. \end{cases}$$

Since ψ_ξ and φ_ξ are continuous and differentiable at 1, the constants A and B satisfy

$$\begin{cases} \sin(\sqrt{\rho - 1 - 2\xi}) = Ae^{\sqrt{2\xi}} + Be^{-\sqrt{2\xi}} \\ \sqrt{\rho - 1 - 2\xi}\cos(\sqrt{\rho - 1 - 2\xi}) = \sqrt{2\xi}(Ae^{\sqrt{2\xi}} - Be^{-\sqrt{2\xi}}), \end{cases}$$

and similarly, the constants C and D are solutions of

$$\begin{cases} C\cos(\sqrt{\rho - 1 - 2\xi}) + D\sin(\sqrt{\rho - 1 - 2\xi}) = \sinh(\sqrt{2\xi}(L - 1)) \\ \sqrt{\rho - 1 - 2\xi}(-C\sin(\sqrt{\rho - 1 - 2\xi} + D\cos(\sqrt{\rho - 1 - 2\xi}) = -\sqrt{2\xi}\cosh(\sqrt{2\xi}(L - 1)). \end{cases}$$

Hence, we have

$$\begin{cases} A = \frac{e^{-\sqrt{2\xi}}}{2}\left(\sin(\sqrt{\rho - 1 - 2\xi}) + \frac{\sqrt{\rho-1-2\xi}}{\sqrt{2\xi}}\cos(\sqrt{\rho - 1 - 2\xi})\right) = \frac{1}{2\sqrt{\xi}}f(\xi)e^{-\sqrt{2\xi}} \\ B = \frac{e^{\sqrt{2\xi}}}{2}\left(\sin(\sqrt{\rho - 1 - 2\xi}) - \frac{\sqrt{\rho-1-2\xi}}{\sqrt{2\xi}}\cos(\sqrt{\rho - 1 - 2\xi})\right) = \frac{1}{2\sqrt{\xi}}g(\xi)e^{\sqrt{2\xi}} \\ C = \cos(\sqrt{\rho - 1 - 2\xi})\sinh(\sqrt{2\xi}(L - 1)) + \frac{\sqrt{2\xi}}{\sqrt{\rho-1-2\xi}}\sin(\sqrt{\rho - 1 - 2\xi})\cosh(\sqrt{2\xi}) \\ D = \sin(\sqrt{\rho - 1 - 2\xi})\sinh(\sqrt{2\xi}(L - 1)) - \frac{\sqrt{2\xi}}{\sqrt{\rho-1-2\xi}}\cos(\sqrt{\rho - 1 - 2\xi})\cosh(\sqrt{2\xi}). \end{cases}$$

Therefore, the functions φ_ξ and ψ_ξ are given (up to a multiplicative factor) by

$$\varphi_\xi(x) = \begin{cases} \sin(\sqrt{\rho - 1 - 2\xi}\, x) & x \in [0, 1] \\ \frac{1}{2\sqrt{2\xi}}\left(f(\xi)e^{\sqrt{2\xi}(x-1)} + g(\xi)e^{-\sqrt{2\xi}(x-1)}\right) & x \in [1, L], \end{cases} \tag{2.36}$$

and

$$\psi_\xi(x) = \begin{cases} \sinh(\sqrt{2\xi}(L - 1))\cos(\sqrt{\rho - 1 - 2\xi}(x - 1)) \\ -\frac{\sqrt{2\xi}}{\sqrt{\rho-1-2\xi}}\cosh(\sqrt{2\xi}(L - 1))\sin(\sqrt{\rho - 1 - 2\xi}(x - 1)) & x \in [0, 1], \\ \sinh(\sqrt{2\xi}(L - x)) & x \in [1, L]. \end{cases} \tag{2.37}$$

The Green function is then given by (see [BS12], Chapter II)

$$G_\xi(x, y) = \begin{cases} (\omega_{\lambda_1^\infty+\xi})^{-1}e^{\mu(x-y)}\psi_{\lambda_1^\infty+\xi}(x)\varphi_{\lambda_1^\infty+\xi}(y) & y \leqslant x \\ (\omega_{\lambda_1^\infty+\xi})^{-1}e^{\mu(x-y)}\psi_{\lambda_1^\infty+\xi}(y)\varphi_{\lambda_1^\infty+\xi}(x) & y \geqslant x, \end{cases} \tag{2.38}$$

with $\omega_\xi := \psi_\xi(x)\varphi'_\xi(x) - \psi'_\xi(x)\varphi_\xi(x) = f(\xi)e^{\sqrt{2\xi}(L-1)} + g(\xi)e^{-\sqrt{2\xi}(L-1)}$.

The two following lemmas will be used in the following sections to control the fluctuations of the system. Their proofs can be found in Appendix C.

Lemma 2.6. *Assume* (H_{psh}) *holds and let* $\xi : (1, \infty) \to (0, \infty)$ *be a function such that* $\xi(L) \to 0$ *and* $\xi(L)L \to 0$ *as* $L \to \infty$. *There exist some constants* $C, C' > 0$ *such that for* L *large enough, we have*

$$\omega_{\lambda_1^\infty + \xi} \geqslant C'\xi(L)e^{\beta L},$$

and for $x \in [0, L]$,

$$\begin{aligned} \varphi_{\lambda_1^\infty+\xi}(x) &\leqslant C(1 \wedge x)\left(\xi e^{\beta x} + e^{-\beta x}\right), \\ \psi_{\lambda_1^\infty+\xi}(x) &\leqslant C(1 \wedge (L-x))e^{\beta(L-x)}. \end{aligned}$$

Lemma 2.7. *Assume* (H_{psh}) *holds and let* $h > 0$. *Then, there exist some constants* $C, C' > 0$ *such that for* L *large enough and* $\xi = \frac{h}{L}$,

$$\omega_{\lambda_1^\infty+\xi} \geqslant \frac{C'h}{2L}e^{\beta L}.$$

and for $x \in [0, L]$,

$$\begin{aligned} \varphi_{\lambda_1^\infty+\xi}(x) &\leqslant C(1 \wedge x)\left(\frac{h}{L}e^{\beta x} + e^{-\beta x}\right), \\ \psi_{\lambda_1^\infty+\xi}(x) &\leqslant C(1 \wedge (L-x))e^{\beta(L-x)}. \end{aligned}$$

3 BBM in an interval: moment estimates

In this section, we assume that (H_{psh}) holds and we consider a dyadic branching Brownian motion with branching rate $r(x)$, drift $-\mu$ with

$$\mu = \sqrt{1 + 2\lambda_1^\infty},$$

and particles killed at 0 and L. We recall from Section 1.5 that the density of particles at y at time t is given by

$$p_t(x, y) = e^{\mu(x-y)+(\frac{1}{2}-\mu^2/2)t}q_t(x, y) = e^{\mu(x-y)-\lambda_1^\infty t}q_t(x, y),$$

where q_t is defined in Equation (2.18). We denote by $\mathcal{N}_t^L$ the set of particles alive in this system at time t and we define $N_t' = |\mathcal{N}_t^L|$. For a particle $v \in \mathcal{N}_t$, we denote by $X_v(t)$ its position at time t. We then define the processes Z_t', Y_t and $\tilde{Y}_t$ as follows:

$$Z_t' = \sum_{v \in \mathcal{N}_t^L} e^{\mu(X_v(t)-L)}w_1(X_v(t)), \tag{3.1}$$

$$Y_t = \sum_{v \in \mathcal{N}_t^L} (X_v(t) \wedge 1)e^{\mu(X_v(t)-L)}, \tag{3.2}$$

$$\tilde{Y}_t = \sum_{v \in \mathcal{N}_t^L} e^{\mu(X_v(t)-L)}, \tag{3.3}$$

with

$$w_1(x) = \sinh(\sqrt{2\lambda_1}(L-1))v_1(x), \tag{3.4}$$

and v_1 is the eigenvector associated with the principal eigenvalue λ_1 such that $v_1(1) = 1$, defined by Equation (2.9).

In Section 3.1, we estimate the first moments of the processes $(Z'_t)_{t>0}$ and $(\tilde{Y}_t)_{t>0}$ under (H_{psh}). In Section 3.2, we bound the second moment of $(Z'_t)_{t>0}$ under (H_{wp}). The main tools used in the proofs of the following lemmas are the estimates on the heat kernel, established in Lemma 2.5, and on the Green function, established in Lemma 2.6.

3.1 First moment

Lemma 3.1. *Assume (H_{psh}) holds. There exists $C > 0$ such that, for L large enough, we have*

$$w_1(x) \leqslant C(x \wedge 1)\sinh(\sqrt{2\lambda_1}(L - x)) \leqslant C(x \wedge 1)\sinh(\beta(L - x)),$$

for all $x \in [0, L]$. As a consequence, there exists $C' > 0$ such that, for L large enough, we have

$$v_1(x) \leqslant C(x \wedge 1)e^{-\beta x},$$

for all $x \in [0, L]$.

Proof. Remark that for $x \in [0, 1]$, we have $\sin(\sqrt{\rho - 1 - 2\lambda_1}x) \leqslant \sqrt{\rho - 1 - 2\lambda_1}x$, so that

$$w_1(x) \leqslant \frac{\sqrt{\rho - 1 - 2\lambda_1}}{\sin(\sqrt{\rho - 1 - 2\lambda_1})} x \sinh(\sqrt{2\lambda_1}(L - 1)) \leqslant \frac{\sqrt{\rho - 1 - 2\lambda_1}}{\sin(\sqrt{\rho - 1 - 2\lambda_1})} x \sinh(\sqrt{2\lambda_1}(L - x)).$$

Remark 2.2 combined with the fact that λ_1 is an increasing function of L then yields the first part of the lemma. The second part follows from Remark 2.1. □

Lemma 3.2 (First moment of Z'_t)**.** *Assume (H_{psh}) holds. Let $t > 0$. For sufficiently large L,*

$$\mathbb{E}[Z'_t] = e^{(\lambda_1 - \lambda_1^\infty)t} Z'_0,$$

Proof. By definition of Z'_t, we have

$$\begin{aligned}
\mathbb{E}_x[Z'_t] &= \int_0^L p_t(x, y)e^{\mu(y-L)} w_1(y)dy \\
&= e^{\mu(x-L)} e^{-\lambda_1^\infty t} \int_0^L q_t(x, y)w_1(y)dy.
\end{aligned}$$

Yet, w_1 is an eigenvector associated with the simple eigenvalue λ_1 of the operator T, and $q_t(x, y)$ is the fundamental solution of

$$(B)\begin{cases} \partial_t u = Tu \\ u(0) = u(L) = 0. \end{cases}$$

Therefore, $\int_0^L q_t(x, y)w_1(y) = e^{-\lambda_1 t} w_1(x)$, which concludes the proof of the lemma. □

Lemma 3.3 (First moment of $\tilde{Y}_t$)**.** *Assume (H_{psh}) holds. There exist $C_1, C_2 > 0$ such that, for L large enough, $t > C_2 L$, we have*

$$\mathbb{E}[\tilde{Y}_t] \leqslant C_1 e^{-\beta L} Z_0'.$$

Corollary 3.1 (First moment of Y_t)**.** *Assume (H_{psh}) holds. There exist $C_1, C_2 > 0$ such that, for L large enough, $t > C_2 L$, we have*

$$\mathbb{E}[Y_t] \leqslant C_1 e^{-\beta L} Z_0'.$$

Proof of Lemma 3.3. According to Lemmas 2.5 and 2.3, there exist $C_1, C_2 > 0$ such that for L large enough and $t > C_2 L$, we have

$$p_t(x, y) \leqslant C_1 e^{\mu(x-y)} e^{(\lambda_1 - \lambda_1^\infty)t} v_1(x) v_1(y).$$

Thus, we get that, for sufficiently large L and $t > C_2 L$,

$$\mathbb{E}_x[\tilde{Y}_t] \;=\; \int_0^L e^{\mu(y-L)} p_t(x,y) dy \leqslant C e^{\mu(x-L)} v_1(x) e^{(\lambda_1 - \lambda_1^\infty)t} \int_0^L v_1(y) dy.$$

Recalling from Remark 3.1 that $v_1(y) \leqslant C e^{-\beta y}$ for all $y \in [0, L]$, we get that $\int_0^L v_1(y)dy$ is bounded by a constant (the integral converge to a positive limit by the dominated convergence theorem). Therefore, using that $\lambda_1 \leqslant \lambda_1^\infty$, we get that for L large enough and $t > C_2 L$,

$$\mathbb{E}_x[\tilde{Y}_t] \leqslant C e^{\mu(x-L)} v_1(x).$$

Remark 2.1 then yields the lemma. $\square$

3.2 Second moment

Lemma 3.4 (Second moment of Z_t')**.** *Assume (H_{wp}) holds and let $u : (0, \infty) \to (0, \infty)$ be a function such that $u(L) \to \infty$ and $u(L)/L \to \infty$ as $L \to \infty$. There exists $C > 0$ such that for L large enough, we have*

$$\mathbb{E}[(Z_u')^2] \leqslant C \left(u e^{-2\beta L} Z_0' + Y_0 \right).$$

Proof. By the Many-to-two Lemma and Lemma 3.2, for all $t > 0$, we have

$$\begin{aligned} T &:= \mathbb{E}_x[(Z_u')^2] \\ &= \mathbb{E}_x\left[\sum_{v \in \mathcal{N}^L} e^{2\mu(X_v(u)-L)} w_1(X_v(u))^2\right] + 2\int_0^L \int_0^u r(y) p_s(x,y) \mathbb{E}_y[Z_{t-s}']^2 ds dy \\ &\leqslant \underbrace{\mathbb{E}_x\left[\sum_{v \in \mathcal{N}^L} e^{2\mu(X_v(u)-L)} w_1(X_v(u))^2\right]}_{=:T_1} + 2\rho \underbrace{\int_0^L \int_0^u p_s(x,y) e^{2\mu(y-L)} w_1(y)^2 ds dy}_{=:T_2}. \end{aligned} \tag{3.5}$$

We first bound T_1. According to Lemmas 2.3 and 2.5, there exist $C_1, C_2 > 0$ such that for all $L > 1$ large enough and $t > C_2 L$,

$$p_t(x,y) \leqslant C_1 e^{\mu(x-y)} e^{(\lambda_1 - \lambda_1^\infty)t} v_1(x) v_1(y).$$

Therefore, for L large enough and a time $u = u(L) > C_2 L$, we have

$$\begin{aligned} T_1 &= \int_0^L p_u(x,y) e^{2\mu(y-L)} w_1(y)^2 dy \\ &\leqslant C e^{\mu(x-L)+(\lambda_1-\lambda_1^\infty)u} v_1(x) \sinh(\sqrt{2\lambda_1}(L-1))^{-1} \int_0^L e^{\mu(y-L)} w_1(y)^3 dy, \\ &\leqslant C e^{\mu(x-L)} w_1(x) \sinh(\sqrt{2\lambda_1}(L-1))^{-2} \int_0^L e^{(3\beta-\mu)(L-y)} dy, \end{aligned}$$

where the last inequality comes from Lemma 3.1. Since $\mu > 3\beta$ the last integral is bounded by a constant. Finally, Remark 2.1 gives that

$$T_1 \leqslant C e^{\mu(x-L)} \sinh(\sqrt{2\lambda_1}(L-1))^{-2} w_1(x) \leqslant C e^{-2\beta L} e^{\mu(x-L)} w_1(x). \tag{3.6}$$

Let us now bound the quantity T_2. First, recall from Equation (2.38) the definition of the Green function:

$$G_{\frac{1}{u}}(x,y) = \begin{cases} \omega^{-1}_{\lambda_1^\infty+\frac{1}{u}} e^{\mu(x-y)} \psi_{\lambda_1^\infty+\frac{1}{u}}(x) \varphi_{\lambda_1^\infty+\frac{1}{u}}(y) & x \geq y \\ \omega^{-1}_{\lambda_1^\infty+\frac{1}{u}} e^{\mu(x-y)} \psi_{\lambda_1^\infty+\frac{1}{u}}(y) \varphi_{\lambda_1^\infty+\frac{1}{u}}(x) & y \geq x, \end{cases}$$

where $\varphi_{\lambda_1^\infty+\frac{1}{u}}$ and $\psi_{\lambda_1^\infty+\frac{1}{u}}$ are given by Equations (2.36) and (2.37) in Section 2.3, and from Equation (2.34) that

$$\int_0^u p_s(x,y) ds \leqslant e G_{\frac{1}{u}}(x,y).$$

Therefore, we have

$$\begin{aligned} T_2 &= \int_0^L \int_0^u p_s(x,y) e^{2\mu(y-L)} w_1(y)^2 ds dy \\ &\leqslant C \Bigg(e^{\mu(x-L)} \psi_{\lambda_1^\infty+\frac{1}{u}}(x) \underbrace{\omega^{-1}_{\lambda_1^\infty+\frac{1}{u}} \int_0^x e^{\mu(y-L)} w_1(y)^2 \varphi_{\lambda_1^\infty+\frac{1}{u}}(y) dy}_{=:A(x)} \\ &\qquad + e^{\mu(x-L)} \varphi_{\lambda_1^\infty+\frac{1}{u}}(x) \underbrace{\omega^{-1}_{\lambda_1^\infty+\frac{1}{u}} \int_x^L e^{\mu(y-L)} w_1(y)^2 \psi_{\lambda_1^\infty+\frac{1}{u}}(y) dy}_{=:B(x)} \Bigg). \end{aligned} \tag{3.7}$$

According to Lemma 2.7 (applied to $\xi = \frac{1}{u}$) and Lemma 3.1, there exist some constants (independent of x and L) such that,

$$\begin{aligned} \varphi_{\lambda_1^\infty+\frac{1}{u}}(x) &\leqslant C(1\wedge x)\left(\frac{1}{u}e^{\beta x}+e^{-\beta x}\right), &(3.8)\\ \psi_{\lambda_1^\infty+\frac{1}{u}}(x) &\leqslant C(1\wedge(L-x))e^{\beta(L-x)}, &(3.9)\\ \omega^{-1}_{\lambda_1^\infty+\frac{1}{u}} &\leqslant Cue^{-\beta L}, &(3.10)\\ w_1(x) &\leqslant C(1\wedge x)e^{\beta(L-x)}. &(3.11)\end{aligned}$$

Therefore, we get thanks to Equations (3.8), (3.10) and (3.11) that

$$\begin{aligned} A(x) &\leqslant Ce^{(\beta-\mu)L}\left(\int_0^x e^{(\mu-\beta)y}dy + u\int_0^x e^{(\mu-3\beta)y}dy\right)\\ &\leqslant C\left((1\wedge x)e^{(\beta-\mu)(L-x)} + u(1\wedge x)e^{(3\beta-\mu)(L-x)}e^{-2\beta L}\right)\\ &\leqslant C(1\wedge x)\left(e^{(\beta-\mu)(L-x)}+ue^{-2\beta L}\right), \qquad (3.12)\end{aligned}$$

since $\mu > 3\beta$. Equations (3.9), (3.10) and (3.11) give that

$$\begin{aligned} B(x) &\leqslant Cue^{(2\beta-\mu)L}\int_x^L e^{(\mu-3\beta)y}dy\\ &\leqslant Cue^{(2\beta-\mu)L}(1\wedge(L-x))e^{(\mu-3\beta)L} = Cu(1\wedge(L-x))e^{-\beta L}. \qquad (3.13)\end{aligned}$$

Therefore, combining Equations (3.9) and (3.12), we have

$$\psi_{\lambda_1^\infty+\frac{1}{u}}(x)A(x) \leqslant C(1\wedge x\wedge(L-x))\left(1+ue^{-2\beta L}e^{\beta(L-x)}\right), \qquad (3.14)$$

and Equations (3.8) and (3.13) implies that

$$\varphi_{\lambda_1^\infty+\frac{1}{u}}(x)B(x) \leqslant C(1\wedge x\wedge(L-x))\left(1+ue^{-2\beta L}e^{\beta(L-x)}\right) \qquad (3.15)$$

We now need to prove that

$$(1\wedge x\wedge(L-x))e^{\beta(L-x)} \leqslant Cw_1(x). \qquad (3.16)$$

To this end, we distinguish 3 cases:

- Suppose $x\in[0,1]$. Recall that for L large enough, $\sqrt{\rho-1-2\lambda_1}\in(\frac{\pi}{2},\pi)$, therefore, by concavity, we have $\sin(\sqrt{\rho-1-2\lambda_1}x) \geqslant \sin(\sqrt{\rho-1-2\lambda_1})x$. Thus, using Remark 2.1, we get that

$$\begin{aligned} w_1(x) &= \frac{\sin(\sqrt{\rho-1-2\lambda_1}x)}{\sin(\sqrt{\rho-1-2\lambda_1})}\sinh(\sqrt{2\lambda_1}(L-1) \geqslant x\sinh(\sqrt{2\lambda_1}(L-1) \geqslant Cxe^{\beta L}\\ &\geqslant Cxe^{\beta(L-x)},\end{aligned}$$

  so that (3.16) holds in [0,1].

- Suppose $x \in [1, L-1]$. Then, $w_1(x) = \frac{1}{2}\left(e^{\sqrt{2\lambda_1}(L-x)} - e^{-\sqrt{2\lambda_1}(L-x)}\right)$ and

$$\begin{aligned} w_1(x)e^{-\beta(L-x)} &= \frac{1}{2}\left(e^{(\sqrt{2\lambda_1}-\beta)(L-x)} - e^{-(\sqrt{2\lambda_1}+\beta)(L-x)}\right) \\ &\geqslant \frac{1}{2}\left(e^{(\sqrt{2\lambda_1}-\beta)L} - e^{-(\sqrt{2\lambda_1}+\beta)}\right). \end{aligned}$$

  Recall from Lemma 2.2 that $(\sqrt{2\lambda_1}-\beta)L \to 0$ as $L \to \infty$. Besides $e^{-(\sqrt{2\lambda_1}+\beta)} < 1$. Therefore, for L large enough, $e^{(\sqrt{2\lambda_1}-\beta)L} - e^{-(\sqrt{\lambda_1}+\beta)L} > C$ for some $C > 0$ and (3.16) holds in $[1, L-1]$.

- Suppose $x \in [L-1, L]$. By convexity, we have

$$\begin{aligned} w_1(x) &= \sinh(\sqrt{2\lambda_1}(L-x)) \geqslant \sinh(\sqrt{2\lambda_1})(L-x) \geqslant C(L-x) \geqslant C(L-x)e^{\beta} \\ &\geqslant C(L-x)e^{\beta(L-x)}, \end{aligned}$$

  which concludes the proof of (3.16).

Finally, combining Equations (3.7), (3.14), (3.15) and (3.16), we have

$$T_2 \leqslant Ce^{\mu(x-L)}\left((1 \wedge x) + ue^{-2\beta L}w_1(x)\right). \tag{3.17}$$

Equations (3.6) and (3.17) yield the lemma.

□

4 The particles hitting the right-boundary

In this section, we are interested in the number of particles killed at the right boundary in a BBM with branching rate $r(x)$, drift $-\mu$ and killed upon reaching 0 and L.

Let us denote by $R(S)$ the number of particles killed at the right border L in the time interval S, for a measurable subset $S \subset [0, \infty)$. In this section, we estimate the first moment and establish an upper bound on the second moment of the process $(R([0,t]))_{t>0}$, under the same assumptions as in Section 3, that is, under (H_{psh}) for the first moment and (H_{wp}) for the second moment.

In order to bound the expectation of $R(S)$ for a measurable subset $S \subset (0, \infty)$, we first consider the case of $S \subset (0, \infty)$ such that $\inf(S) > CL$ (Lemma 4.1) and use the estimates established in Lemma 2.5. We then bound the expected number of particles killed at the right boundary in a time interval of the form $[0, CL]$ (Lemma 4.2), thanks to the Green function and Lemma 2.7. We finally combine these two estimates in Lemma 4.4. The second moment of the process $(R([0,t]))_{t>0}$ is then upper bounded in Lemma 4.5 with a similar argument as for the process (Z'_t).

In order to apply these results, we define

$$I(x, S) = -\frac{1}{2}\int_S e^{-\lambda_1^\infty s}\frac{\partial}{\partial y}q_s(x,y)|_{y=L}ds,$$

and denote by $\ell(S)$ the following integral:

$$\ell(S) = \int_S e^{(\lambda_1-\lambda_1^\infty)s} ds. \tag{4.1}$$

First, remark that

$$\ell(S) \leqslant \lambda(S), \tag{4.2}$$

where $\lambda(S)$ refers to the Lebesgue measure of the set S.

4.1 First moment

First, we claim that

$$\mathbb{E}_x[R(S)] = -\frac{1}{2}\int_S \frac{\partial}{\partial y} p_s(x,y)|_{y=L} ds = e^{\mu(x-L)} I(x,S). \tag{4.3}$$

Indeed, this means that the density of particles hitting L at time s is equal to the heat flow of p_s out of the boundary L at time s. This result is classical but it is difficult to find a complete proof of this fact in the literature: see [IMJ74, p.154] for an early appearance of this fact or [Gar85, Section 5.2.1] for a non-rigorous proof of this result. Hence, it is sufficient to estimate the integral $I(x,S)$ to estimate the first moment of R.

Lemma 4.1. *Assume (H_{psh}) holds. Therefore, there exist $C_1, C_2 > 0$ such that for L large enough and* $\inf(S) > C_2 L$,

$$|I(x,S) - g(L)\ell(S)w_1(x)| \leqslant e^{-C_1 L} g(L) w_1(x),$$

with

$$g(L) = \frac{1}{2}\frac{\sqrt{2\lambda_1}}{\|v_1\|^2}\sinh(\sqrt{2\lambda_1}(L-1))^{-2}.$$

Remark 4.1. *According to Lemma 2.3 and Remark 2.1, for L large enough, we have*

$$g(L) \leqslant Ce^{-2\beta L}.$$

Proof. The proof of this result is very similar to the one of Lemma 2.5.

Since the sum is uniformly convergent for $s \geq \inf(S) > 0$, we have

$$\begin{aligned} U &:= I(x,S) - g(L)w_1(x)\int_S e^{(\lambda_1-\lambda_1^\infty)s} ds \\ &= \underbrace{\sum_{k=2}^{K} g_k \frac{v_k(x)}{\|v_k\|^2}\int_S e^{(\lambda_k-\lambda_1^\infty)s} ds}_{=: U_1} + \underbrace{\sum_{i=K+1}^{\infty} g_k \frac{v_k(x)}{\|v_k\|^2}\int_S e^{(\lambda_k-\lambda_1^\infty)s} ds}_{=: U_2}, \end{aligned}$$

with $\mathrm{g}_k := -\frac{1}{2}v_k'(L)$. Note that for all $k \geqslant 2$,

$$\int_S e^{(\lambda_k-\lambda_1^\infty)s} ds \leq \frac{e^{(\lambda_k-\lambda_1^\infty)\inf(S)}}{\lambda_1^\infty - \lambda_k}, \tag{4.4}$$

and recall from Lemma 2.1 and 2.2 that

$$\lambda_1^\infty - \lambda_k \geqslant \begin{cases} \frac{5}{8}\pi^2, & \text{if} \quad K \geqslant 2, \\ \lambda_1^\infty & \text{if} \quad K = 1. \end{cases} \tag{4.5}$$

According to Lemma 2.4, we know that for L large enough and $k \in \{2, ...K\}$, we have

$$\frac{|v_k(x)|}{\|v_k\|^2} \leqslant Ce^{\beta L}\frac{v_1(x)}{\|v_1\|^2}. \tag{4.6}$$

Then, remark that for $k \in \{2, ..., K\}$, we have

$$g_k = \frac{1}{2}\frac{\sqrt{2\lambda_k}}{\sinh(\sqrt{2\lambda_k}(L-1))} \leqslant C, \tag{4.7}$$

according to Lemma 2.1. Therefore, combining Equation (4.6), Equation (4.7) and Remark 2.1 that for L large enough, we have

$$\frac{g_k}{\|v_k\|^2}\,|v_k(x)| \leqslant Ce^{\beta L}\frac{v_1(x)}{\|v_1\|^2} \leqslant Ce^{2\beta L}g(L)w_1(x). \tag{4.8}$$

Combining Equations (4.5), (4.4) and (4.8), we get that for L large enough and $x \in [0, L]$, we have

$$|U_1| \leqslant Ce^{2\beta L - \frac{5}{8}\pi^2 \inf(S)}g(L)w_1(x). \tag{4.9}$$

We now bound the sum U_2. According to Lemma 2.4, we know that for L large enough and $k > K$, we have

$$\frac{|v_k(x)|}{\|v_k\|} \leqslant C\sqrt{\rho - 1 - 2\lambda_k}\, e^{\beta L}\frac{|v_1(x)|}{\|v_1\|}.$$

Besides, note that for $k > K$, we have

$$g_k = \frac{1}{2}\frac{\sqrt{-2\lambda_k}}{\sin\left(\sqrt{-2\lambda_k}(L-1)\right)} \leqslant C\frac{\sqrt{\rho - 1 - 2\lambda_k}}{\sin\left(\sqrt{-2\lambda_k}(L-1)\right)}$$

so that, according to Lemma 2.4, we have

$$\frac{g_k}{\|v_k\|} \leqslant C\sqrt{\rho - 1 - 2\lambda_k}.$$

Thus, using Remark 2.1 again, we get that

$$g_k\frac{|v_k(x)|}{\|v_k\|^2} \leqslant C(\rho - 1 - 2\lambda_k)e^{\beta L}\frac{|v_1(x)|}{\|v_1\|^2} \leqslant C(\rho - 1 - 2\lambda_k)e^{2\beta L}g(L)w_1(x).$$

Hence, we get thanks to Equation (4.4) that for L large enough,

$$|U_2| \leqslant Ce^{2\beta L}\left[\sum_{i=K+1}^{\infty}(\rho - 1 - 2\lambda_k)\frac{e^{(\lambda_k - \lambda_1^\infty)\inf(S)}}{\lambda_1^\infty - \lambda_k}\right]g(L)w_1(x).$$

We recall that the sum

$$\sum_{i=K+1}^{\infty}(\rho-1-2\lambda_i)e^{(\lambda_k-\lambda_{K+1})\inf(S)}$$

is bounded by a constant if $\inf(S)>1$ (see Equation (2.30) in the proof of Lemma 2.5). Therefore, we obtain thanks to Equation (4.5) that for L large enough and $\inf(S)>1$, we have

$$|U_2|\leqslant CLe^{2\beta L-\lambda_1^\infty\inf(S)}g(L)w_1(x). \tag{4.10}$$

Finally, we obtain the result by combining (4.9) and (4.10). □

Lemma 4.2. *Assume (H_{psh}) holds and let $a>0$. Therefore, there exists $C>0$ such that, for L large enough and $x\in[0,L]$, we have*

$$I(x,[0,aL])\leqslant C(1\wedge x).$$

Proof. Note that for all $x\in[0,L]$, $-\frac{\partial}{\partial y}q_s(x,y)|_{y=L}\geq 0$. Therefore, according to (2.34) we have,

$$I(x,[0,aL])=\int_0^{aL}e^{-\lambda_1^\infty s}\left(-\frac{\partial}{\partial y}q_s(x,y)|_{y=L}\right)ds\leqslant e\int_0^{\infty}e^{-\lambda_1^\infty s-s/aL}\left(-\frac{\partial}{\partial y}q_s(x,y)|_{y=L}\right)ds.$$

By definition of the Green function (see Section 2.3), we have

$$\frac{\partial}{\partial y}G_{\frac{1}{aL}}(x,y)|_{y=L}=e^{\mu(x-L)}\int_0^\infty e^{-\left(\lambda_1^\infty+\frac{\beta}{aL}\right)s}\left(-\frac{\partial}{\partial y}q_s(x,y)|_{y=L}\right)ds.$$

so that,

$$I(x,[0,aL])\leqslant Ce^{\mu(L-x)}\left(-\frac{\partial}{\partial y}G_{\frac{1}{aL}}(x,y)|_{y=L}\right).$$

We deduce from the explicit formula for the Green function given by Equation (2.38) that for $0\leqslant x\leqslant L$,

$$e^{\mu(L-x)}\frac{\partial}{\partial y}G_{\frac{1}{aL}}(x,y)|_{y=L}=(\omega_{\lambda_1^\infty+\frac{1}{aL}})^{-1}\psi'_{\lambda_1^\infty+\frac{1}{aL}}(L)\varphi_{\lambda_1^\infty+\frac{1}{aL}}(x),$$

with $\varphi_{\lambda_1^\infty+\frac{1}{aL}}$, $\psi_{\lambda_1^\infty+\frac{2}{aL}}$ and $\omega_{\lambda_1^\infty+\frac{1}{aL}}$ defined in Equations (2.36), (2.37) and (2.35) in Section 2.3. Then, remark that

$$-\psi'_{\lambda_1^\infty+\frac{1}{aL}}(L)=\sqrt{2\lambda_1^\infty+\frac{1}{aL}}\leqslant 2\sqrt{\lambda_1^\infty},$$

for L large enough. Lemma 2.7 applied to $h=\frac{1}{a}$ provides that for L large enough, we have

$$\begin{aligned}\omega^{-1}_{\lambda_1^\infty+\frac{1}{aL}}&\leqslant CLe^{-\beta L},\\ \varphi_{\lambda_1^\infty+\frac{1}{aL}}(x)&\leqslant C(1\wedge x)\left(\frac{1}{L}e^{\beta x}+e^{-\beta x}\right).\end{aligned}$$

Finally, we get that for L large enough,

$$I([0,aL])\leqslant C(1\wedge x)\left(e^{\beta(x-L)}+Le^{-\beta L}\right)\leqslant C(1\wedge x).$$

□

Lemma 4.3. *Assume (H_{psh}) holds. Therefore, there exists $C > 0$ such that for any measurable subset $S \subset [0, \infty)$, L large enough and $x \in [0, L]$, we have*

$$|I(x, S) - g(L)\ell(S)w_1(x)| \leqslant C((1 \wedge x) + g(L)w_1(x)).$$

Proof. Let $a > 0$. By definition of I,

$$I(x, S) = I(x, S \cap [0, aL]) + I(x, S \cap (aL, \infty)).$$

Then, by triangle inequality, we have

$$\begin{aligned}
& |I(x, S) - g(L)\ell(S)w_1(x)| \\
\leqslant\ & |I(x, S \cap [0, aL])| + |I(x, S \cap (aL, \infty)) - g(L)\ell(S)w_1(x)| \\
\leqslant\ & |I(x, S \cap [0, aL])| + |I(x, S \cap (aL, \infty)) - g(L)\ell(S \cap (aL, \infty))w_1(x)| \\
& + g(L)w_1(x)|\ell(S \cap (aL, \infty)) - \ell(S)|.
\end{aligned}$$

The second term can be bounded thanks to Lemma 4.1: there exist $C_1, C_2 > 0$ such that for L large enough and $a > C_2$,

$$|I(x, S \cap (aL, \infty)) - g(L)\ell(S \cap (aL, \infty))w_1(x)| \leqslant e^{-C_1 L} g(L)w_1(x).$$

For $a = C_2 + 1$, we know thanks to Lemma 4.2 that there exists a $C_3 > 0$ such that

$$|I(x, S \cap [0, aL])| \leqslant C_3(1 \wedge x).$$

Finally, remark thanks to Equation (4.2) that

$$g(L)w_1(x)|\ell(S \cap (aL, \infty)) - \ell(S)| \leqslant \ell([0, aL])g(L)w_1(x) \leqslant aLw_1(x)g(L),$$

and that according, to Lemmas 3.1 and Remark 4.1, there exists $C_4 > 0$ such that

$$Lw_1(x)g(L) \leqslant C_4(1 \wedge x).$$

This concludes the proof of the lemma. $\square$

Lemma 4.4. *Assume (H_{psh}) holds. Therefore, there exists $C > 0$ such that for L large enough and $s < t \in \mathbb{R}$, we have*

$$\left|\mathbb{E}[R([s, t])] - \ell([s, t])g(L)Z'_0\right| \leqslant C(Y_0 + g(L)Z'_0),$$

with

$$g(L) = \frac{1}{2}\frac{\sqrt{2\lambda_1}}{\|v_1\|^2}\sinh(\sqrt{2\lambda_1}(L-1))^{-2}.$$

Proof. The lemma follows directly from Equation (4.3) and Lemma 4.3. $\square$

4.2 Second moment

Lemma 4.5. *Assume (H_{wp}) holds and let $u:(0,\infty)\to(0,\infty)$ be a function such that $u(L)\to\infty$ and $u(L)/L\to\infty$ as $L\to\infty$. There exists a constant $C>0$ such that for L large enough,*

$$\mathbb{E}\left[R^2_{[0,u]}\right]-\mathbb{E}\left[R_{[0,u]}\right]\leqslant C\left(1+g(L)^2u^2\right)(Y_0+ue^{-2\beta L}Z_0'). \tag{4.11}$$

Proof. Standard moment calculations [INW69, Theorem 4.15] give

$$\begin{aligned}\mathbb{E}_x\left[R([0,u])^2\right] &= \mathbb{E}_x\left[R([0,u])\right]+2\int_{y=0}^{L}\int_{s=0}^{u} r(y)p_s(x,y)\mathbb{E}_y\left[R([s,u])\right]^2 dsdy\\ &\leqslant \mathbb{E}_x\left[R([0,u])\right]+\rho\int_{y=0}^{L}\int_{s=0}^{u} p_s(x,y)\mathbb{E}_y\left[R([s,u])\right]^2 dsdy.\end{aligned}$$

Besides, according to Lemma 4.4 and Equation (4.2), there exists $C>0$ such that for L large enough and $s\leqslant u$, we have

$$\mathbb{E}_y\left[R([s,u])\right]\leqslant Ce^{\mu(y-L)}((1\wedge y)+g(L)(1+(u-s))w_1(y))$$

Then, using that $(a+b)^2\leqslant 2a^2+2b^2$, $(1\wedge y)^2\leqslant 1$ and $1+(u-s)^2\leqslant 1+u^2\leqslant Cu^2$, we get that for L large enough, we have

$$\begin{aligned}\mathbb{E}_x\left[R([0,u])^2\right] &\leqslant \mathbb{E}_x\left[R([0,u])\right] \qquad (4.12)\\ &+ C\left(\underbrace{\int_0^L e^{2\mu(y-L)}\int_{s=0}^{u}p_s(x,y)dsdy}_{=:U_1}+g(L)^2u^2\underbrace{\int_0^L e^{2\mu(y-L)}w_1(y)^2\int_{s=0}^{u}p_s(x,y)dsdy}_{=:U_2}\right)\end{aligned}$$

Then, as in the proof of Lemma 3.4, we bound the integrals U_1 and U_2 using the estimates made on the Green function in Section 2.3. First, we recall that for all $u>0$, we have

$$\int_0^u p_s(x,y)ds\leqslant CG_{\frac{1}{u}}(x,y).$$

Therefore, as in Equation (3.7), we have

$$\begin{aligned}U_1 &:= \int_0^L e^{2\mu(y-L)}\int_{s=0}^{u}p_s(x,y)dsdy\\ &\leqslant Ce^{\mu(x-L)}\omega^{-1}_{\lambda_1^\infty+\frac{1}{u}}\left(\psi_{\lambda_1^\infty+\frac{1}{u}}(x)\int_0^x e^{\mu(y-L)}\varphi_{\lambda_1^\infty+\frac{1}{u}}(y)dy+\varphi_{\lambda_1^\infty+\frac{1}{u}}(x)\int_x^L e^{\mu(y-L)}\psi_{\lambda_1^\infty+\frac{1}{u}}(y)dy\right),\\ &:= B_1+B_2, \qquad (4.13)\end{aligned}$$

and,

$$\begin{aligned}U_2 &:= \int_0^L e^{2\mu(y-L)}w_1(y)^2\int_{s=0}^{u}p_s(x,y)dsdy\\ &\leqslant Ce^{\mu(x-L)}\omega^{-1}_{\lambda_1^\infty+\frac{1}{u}}\left(\psi_{\lambda_1^\infty+\frac{1}{u}}(x)\int_0^x e^{\mu(y-L)}w_1(y)^2\varphi_{\lambda_1^\infty+\frac{1}{u}}(y)dy\right.\\ &\qquad\left.+\varphi_{\lambda_1^\infty+\frac{1}{u}}(x)\int_x^L e^{\mu(y-L)}w_1^2(y)\psi_{\lambda_1^\infty+\frac{1}{u}}(y)dy\right).\end{aligned}$$

Note that U_2 has already been estimated in the proof of Lemma 3.4 (see Equations (3.7) and (3.17)). In particular, we know that for L large enough, we have

$$U_2 \leqslant Ce^{\mu(x-L)}\left((1\wedge x)+u^{-2\beta L}w_1(x)\right). \tag{4.14}$$

Then, following the proof Lemma 3.4, we use Equations (3.8), (3.9) and (3.10) to bound U_1. We get that for L large enough, we have

$$\begin{aligned}
B_1 &\leqslant C(1\wedge(L-x))e^{\mu(x-L)}ue^{-\beta x}\int_0^x e^{\mu(y-L)}\left(\frac{1}{u}e^{\beta y}+e^{-\beta y}\right)dy \\
&\leqslant C(x\wedge 1\wedge(L-x))e^{\mu(x-L)}\left(e^{\mu(x-L)}+ue^{(\mu-2\beta)x}e^{-\mu L}\right) \\
&\leqslant C(x\wedge 1\wedge(L-x))e^{\mu(x-L)}\left(1+ue^{-2\beta L}\right) \\
&\leqslant C(x\wedge 1\wedge(L-x))e^{\mu(x-L)}\left(1+ue^{-2\beta L}e^{\beta(L-x)}\right)
\end{aligned}$$

where the two last inequalities come from the fact that $(\mu-2\beta)x\leqslant(\mu-2\beta)L$ and $1\leqslant e^{\beta(L-x)}$. Similarly, we get that for L large enough,

$$\begin{aligned}
B_2 &\leqslant C(1\wedge x)e^{\mu(x-L)}e^{-\beta L}\left(e^{\beta x}+ue^{-\beta x}\right)\int_x^L e^{(\mu-\beta)(y-L)}dy \\
&\leqslant C(1\wedge x\wedge(L-x))e^{\mu(x-L)}e^{-\beta L}\left(e^{\beta x}+ue^{-\beta x}\right) \\
&\leqslant C(1\wedge x\wedge(L-x))e^{\mu(x-L)}\left(1+ue^{-2\beta L}e^{\beta(L-x)}\right)
\end{aligned}$$

We then recall from Equation (3.16) that $(1\wedge x\wedge(L-x))e^{\beta(L-x)}\leqslant Cw_1(x)$, so that

$$B_1+B_2\leqslant e^{\mu(x-L)}((1\wedge x)+ue^{-2\beta L}w_1(x)). \tag{4.15}$$

Therefore, combining Equation (4.12), (4.13), (4.14) and (4.15), we have

$$\mathbb{E}_x\left[R([0,u])^2\right]\leqslant\mathbb{E}_x\left[R([0,u])\right]+Ce^{\mu(x-L)}(1+g(L)^2u^2)((1\wedge x)+ue^{-2\beta L}w_1(x)),$$

which concludes the proof of the lemma. □

5 Descendants of a single particle

In this Section, we assume that (H_{psh}) holds and we consider a dyadic branching Brownian motion with branching rate $\frac{1}{2}$, drift $-\mu$ (see Equation (1.20)), starting with a single particle at 0 and killed at $-y$. Since $\mu>1$, we know that this process goes extinct almost surely [Kes78]. Therefore, we can define Z_y, the total number of particles killed at $-y$ in the BBM. It was shown [Nev88] that the process $(Z_y)_{y\geqslant 0}$ is a supercritical continuous time branching process. In this section, we investigate the asymptotic behaviour of the branching process (Z_y) as y goes to infinity. As explained in Section 1.5, this quantity is strongly related to the *number of descendants left by the early founders* from [BHK20] and dictates the genealogical structure of the population.

Lemma 5.1 follows from the uniqueness of the travelling wave solutions of Kolmogorov's equation and Karamata's Tauberian Theorem (see Theorem 8.1.6 of [BGT89]).

Lemma 5.1. *Assume* (H_{wp}) *holds. There exists a random variable W such that, as $y \to \infty$,*

$$e^{-(\mu-\beta)y} Z_y \to W \quad a.s. \tag{5.16}$$

Besides, we have $\mathbb{E}\left[\exp\left(-e^{-\mu u} W\right)\right] = \phi(u)$, *where* $\phi : \mathbb{R} \to (0,1)$ *solves Kolmogorov's equation*

$$\frac{1}{2}\phi'' - \mu\phi' = \frac{1}{2}\phi(1-\phi) \tag{5.17}$$

with $\lim_{x\to-\infty} \phi(x) = 1$ *and* $\lim_{x\to+\infty} \phi(x) = 0$. *Besides, there exists $b > 0$ such that as $\lambda \to 0$, we have*

$$\mathbb{E}\left[e^{-\lambda W}\right] = \exp\left(-\lambda + b\lambda^\alpha + o(\lambda^\alpha)\right), \tag{5.18}$$

with α from Equation (1.30).

Proof. First, note that Z_y has the same law as the number of "first crossings" of the line of equation $t \mapsto y + \mu t$ in a branching Brownian motion with branching rate $\frac{1}{2}$, no killing and no drift. For this BBM, we denote by $\mathcal{N}_t$ the set of individuals alive at time t and for $v \in \mathcal{N}_t$, by $X_v(t)$ its position at time t.

Equations (5.16) and (5.17) are a consequence of the uniqueness (up to translation) of the travelling wave solution of Equation (5.17). Indeed, let $\underline{\lambda} = \mu - \beta$ and define

$$W_t(\underline{\lambda}) = \sum_{v\in\mathcal{N}_t} e^{-\underline{\lambda}(X_v(t)+\mu t)}.$$

This process is a positive martingale so that it converges almost surely. We denote by $W(\underline{\lambda})$ its limit. It was shown [Nev88] that this convergence also holds in L^1.

We can then deduce from [Cha91] and [Nev88] that $W_t(\underline{\lambda})$ and $e^{-\underline{\lambda} y} Z_y$ have the same limit W in L^1 (up to a multiplicative constant) and that

$$\mathbb{E}\left[\exp\left(-e^{-\mu u} W\right)\right] = \phi(u).$$

Besides, note that we can also deduce from [Cha91] that the two limits are equal almost surely.

The second part of the lemma can be deduced from Theorem 2.2 of [Liu00]. We consider the branching Brownian motion at discrete times. This leads to a branching random walk and for all $\theta \in \mathbb{R}$, one can define the additive martingale $W_n(\theta)$ as follows:

$$W_n(\theta) = \sum_{v\in\mathcal{N}_n} e^{-\theta X_v(n) - n\varphi(\theta)},$$

with

$$\varphi(\theta) = \log \mathbb{E}\left[\sum_{v\in\mathcal{N}_1} e^{-\theta X_v(1)}\right] = \frac{\theta^2}{2} + \frac{1}{2}.$$

Theorem 2.2 of [Liu00] states that if for some $p > 1$,

$$\varphi(p\theta) = p\varphi(\theta), \qquad \mathbb{E}\left[\sum_{v\in\mathcal{N}_1}\left(e^{-\theta X_v-\varphi(\theta)}\right)^p \log^+\left(e^{-\theta X_v-\varphi(\theta)}\right)\right] < \infty,$$
$$\text{and} \qquad \mathbb{E}\left[\left(\sum_{v\in\mathcal{N}_1} e^{-\theta X_v-\varphi(\theta)}\right)^p\right] < \infty, \tag{5.19}$$

then $W(\theta) \overset{a.s.}{=} \lim_{n\to\infty} W_n(\theta)$ is such that as $x \to \infty$,

$$\mathbb{P}\left(W(\theta) > x\right) \sim \frac{l}{x^p}, \tag{5.20}$$

for some $l > 0$. Note that the first condition of (5.19) implies that

$$(p\theta)^2 + 1 = p(\theta+1) \Leftrightarrow p = \frac{1}{\theta^2}.$$

Besides, one can easily prove that the two following conditions hold. For $\theta = \underline{\lambda}$, one obtains that $p = 1/\underline{\lambda}^2$. Moreover, note that $\underline{\lambda}$ is the smallest root of $\frac{\theta^2}{2} - \mu\theta + \frac{1}{2}$. The second root of this polynomial function is $\bar{\lambda} = \mu + \beta$ and we have $\underline{\lambda}\bar{\lambda} = 1$. Therefore, for $\theta = \underline{\lambda}$, we have

$$p = \frac{1}{\underline{\lambda}^2} = \frac{\bar{\lambda}}{\underline{\lambda}} = \frac{\mu+\beta}{\mu-\beta} = \alpha > 1.$$

Finally, one can deduce from Equation (5.20) and Theorem 8.1.6 of [BGT89] that as $\lambda \to 0$,

$$\mathbb{E}\left[e^{-\lambda W(\underline{\lambda})}\right] = 1 - E[W(\underline{\lambda})]\,\lambda + b\lambda^\alpha + o(\lambda^\alpha),$$

with $b = -l\Gamma\left(-\frac{2\beta}{\mu-\beta}\right) > 0$. Hence, the same expansion holds for W and since $\mathbb{E}[W] = 1$ (the convergence (5.16) also holds in L^1) and $\alpha \in (1,2)$, we have

$$\mathbb{E}\left[e^{-\lambda W}\right] = \exp\left(-\lambda + b\lambda^\alpha + o(\lambda^\alpha)\right).$$

□

6 Small time steps

In this section, we assume that (H_{wp}) holds and consider the BBM absorbed at 0, with branching rate $r(x)$ and drift $-\mu$ (see Equation (1.20)). We recall that $\mathcal{N}_t$ is the set of particles alive at time $t > 0$ in the BBM with absorption at 0 and that $\mathcal{N}_t^L$ refers to the set of particles whose ancestors stay below L until time t. We also denote by $\mathcal{L}$ the set of particles that hit L. We then define the function z such that

$$z(x) = e^{\mu(x-L)}w_1(x), \quad \forall x \in [0, L],$$

where w_1 is defined in Equation (3.4) and recall the definitions of the processes defined in the previous sections:

$$\begin{aligned}
Z_t &= \sum_{v\in\mathcal{N}_t} z(X_v(t))\mathbb{1}_{X_v(t)\in[0,L]} = \sum_{v\in\mathcal{N}_t} e^{\mu(X_v(t)-L)} w_1(X_v(t))\mathbb{1}_{X_v(t)\in[0,L]},\\
Z'_t &= \sum_{v\in\mathcal{N}_t^L} z(X_v(t)) = \sum_{v\in\mathcal{N}_t^L} e^{\mu(X_v(t)-L)} w_1(X_v(t)),\\
Y_t &= \sum_{v\in\mathcal{N}_t^L} (X_v(t)\wedge 1)e^{\mu(X_v(t)-L)},\\
\tilde{Y}_t &= \sum_{v\in\mathcal{N}_t^L} e^{\mu(X_v(t)-L)}.
\end{aligned}$$

Morevover, we consider a quantity A that goes slowly to ∞ as L tends to infinity. In other words, we first let $L\to\infty$, then $A\to\infty$ and we denote by

- ε_L a quantity that is bounded in absolute value by a function $h(A,L)$ such that
$$\forall A \geqslant 1 : \lim_{L\to\infty} h(A,L) = 0,$$
- $\varepsilon_{A,L}$ a quantity that is bounded in absolute value by a function $h(A,L)$ such that
$$\lim_{A\to\infty}\limsup_{L\to\infty} h(A,L) = 0.$$

We set $\Lambda > 0$ and consider a function $\bar{\theta} = (0,\infty)\to(0,\infty)$ such that $\bar{\theta}(A)e^{4\beta A}\to 0$ as $A\to\infty$. We also fix a time $\tau > 0$ and consider $\theta\in(0,\bar{\theta}(A))$ such that $\tau\left(\theta e^{2\beta A}\right)^{-1}\in\mathbb{N}$. We then set $K\in\mathbb{N}$ such that $\tau = K\theta e^{2\beta A}$ and define a sequence of discrete times $(t_k)_{k=1}^K$ such that

$$t_k = k\theta e^{2\beta(L+A)}.$$

In what follows, the symbol $O(\cdot)$ denotes a quantity bounded in absolute value by a constant times the quantity inside the parentheses. The constant in this definition and the functions h defined above may only depend on Λ and $\bar{\theta}$.

The goal of this section is to prove the following proposition.

Proposition 6.1. *Assume (H_{wp}) holds and let $\Lambda > 0$. Uniformly in $\lambda\leqslant\Lambda$ and in $\theta\leqslant\bar{\theta}(A)$, on the event $\{\forall v\in\mathcal{N}_{t_k}, X_v(t_k)\leqslant L\}$, we have*

$$\mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}|\mathcal{F}_{t_k}\right] = \exp\left((-\lambda+\theta\left(C\lambda^\alpha+\varepsilon_{A,L}\right))e^{-(\mu-\beta)A}Z_{t_k} + O(e^{-(\mu-\beta)A}Y_{t_k})\right).$$

As outlined is Section 1.5, the proof of this result is divided into two steps: we first estimate the contribution of the particles that hit L (Lemma 6.2) and we then combine it with the estimates on the BBM killed at 0 and L (Section 6.2).

We end this paragraph by recalling two results from Sections 2 and 4 that will be needed in the following proofs. First, recall from Corollary 2.1 that

$$\lambda_1 - \lambda_1^\infty = -ae^{-2\beta(L-1)} + o(e^{-2\beta L}), \quad \text{with} \quad a = \frac{\beta}{\lim_{L\to\infty} \|v_1\|^2}, \tag{6.1}$$

and that $g(L)$ from Lemma 4.4 is such that

$$g(L) = \frac{1}{2}(a+\varepsilon_L)\sinh(\sqrt{2\lambda_1}(L-1))^{-2} = 2a(1+\varepsilon_L)e^{-2\beta(L-1)}, \tag{6.2}$$

according to Remark 2.1.

6.1 The particles hitting L

Lemma 6.1. *Assume (H_{wp}) holds and let $\Lambda > 0$ and $y : (0,\infty) \to (0,\infty)$ be a function such that $y(L) = o(L)$. Uniformly in $\lambda \leqslant \Lambda$, in $\theta \leqslant \bar{\theta}(A)$ and $u \in [t_k, t_{k+1}]$, we have*

$$\mathbb{E}_{(L-y,u)}{}^2\left[e^{-\lambda Z_{t_{k+1}}}\right] = \exp\left(-\frac{\lambda}{2}(1+O(\theta e^{2\beta A}) + \varepsilon_L)e^{-(\mu-\beta)y}\right).$$

Proof. On the event $\{R([u,t_{k+1}]) = 0\}$, $Z_t = Z_t'$ for all $t \in [u, t_{k+1}]$. Thus,

$$\begin{aligned}\left|\mathbb{E}_{(L-y,u)}\left[e^{-\lambda Z_{t_{k+1}}}\right] - \mathbb{E}_{(L-y,u)}\left[e^{-\lambda Z'_{t_{k+1}}}\right]\right| &\leqslant \mathbb{P}_{(L-y,u)}\left(R([u,t_{k+1}]) \geqslant 1\right) \\ &\leqslant \mathbb{E}_{(L-y,u)}\left[R([u,t_{k+1}])\right],\end{aligned} \tag{6.3}$$

by Markov's inequality. Thus, according to Lemma 4.4,

$$\mathbb{E}_{(L-y,u)}\left[R([u,t_{k+1}])\right] \leqslant C\left(g(L)\ell\left([0,t_{k+1}-u]\right)z(L-y) + e^{-\mu y}\right) \leqslant C(\theta e^{2\beta A}z(L-y) + e^{-\mu y}), \tag{6.4}$$

since $\ell([u,t_{k+1}]) \leqslant t_{k+1} - u \leqslant \theta e^{2\beta(L+A)}$ and $g(L) \leqslant Ce^{-2\beta L}$ according to Remark 4.1. Besides, Lemma 3.2 and Lemma 3.4 give that

$$\mathbb{E}_{(L-y,u)}\left[Z'_{t_{k+1}}\right] = e^{(\lambda_1-\lambda_1^\infty)(t_{k+1}-u)}z(L-y), \tag{6.5}$$

$$\begin{aligned}\mathbb{E}_{(L-y,u)}\left[(Z'_{t_{k+1}})^2\right] &\leqslant C((t_{k+1}-u)e^{-2\beta L}z(L-y) + (1\wedge(L-y))e^{-\mu y}) \\ &\leqslant C(\theta e^{2\beta A}z(L-y) + (1\wedge(L-y))e^{-\mu y}).\end{aligned} \tag{6.6}$$

According to Equation (6.1), $(\lambda_1 - \lambda_1^\infty)(t_{k+1}-u) = O(\theta e^{2\beta A})$, therefore, Equation (6.5) becomes

$$\mathbb{E}_{(L-y,u)}\left[Z'_{t_{k+1}}\right] = (1+O(\theta e^{2\beta A}))z(L-y). \tag{6.7}$$

[2]The notation $\mathbf{E}_{(x,t)}$ means that we start with one particle at position x at time t.

Besides, for L large enough, we have thanks to Equation (6.1) that

$$\begin{aligned} z(L-y) &= e^{-\mu y}w_1(L-y) = e^{-\mu y}\sinh(\sqrt{2\lambda_1}y) = e^{-\mu y}\left(\sinh(\beta y) + O(Le^{-\beta L})\right) \\ &= e^{-\mu y}\left(\frac{1}{2}e^{\beta y} + O(ye^{-\beta y})\right) = \frac{1}{2}e^{-(\mu-\beta)y}\left(1 + O(ye^{-2\beta y})\right) \\ &= \frac{1}{2}(1+\varepsilon_L)e^{-(\mu-\beta)y}, \end{aligned} \tag{6.8}$$

since $e^{-\beta y} \leqslant ye^{-\beta y}$ and $Le^{-\beta L} \leqslant ye^{-\beta y}$ for L large enough. Using that $e^{-\mu y} = e^{-(\mu-\beta)y}e^{-\beta y} = e^{-(\mu-\beta)y}\varepsilon_L$, we get from Equations (6.6), (6.7) and (6.8) that

$$\begin{aligned} \mathbb{E}_{(L-y,u)}\left[e^{-\lambda Z'_{t_{k+1}}}\right] &= 1 - \lambda\mathbb{E}_{(L-y,u)}\left[Z'_{t_{k+1}}\right] + O\left(\mathbb{E}_{(L-y,u)}\left[\left(Z'_{t_{k+1}}\right)^2\right]\right) \\ &= 1 - \frac{\lambda}{2}(1 + O(\theta e^{2\beta A}) + e^{-2\beta L} + \varepsilon_L)e^{-(\mu-\beta)y} \\ &= 1 - \frac{\lambda}{2}\left(1 + O(\theta e^{2\beta A}) + \varepsilon_L\right)e^{-(\mu-\beta)y}. \end{aligned}$$

Combining this with Equations (6.3) and (6.4), we get that

$$\mathbb{E}_{(L-y,u)}\left[e^{-\lambda Z_{t_{k+1}}}\right] = 1 - \frac{\lambda}{2}\left(1 + O(\theta e^{2\beta A}) + \varepsilon_L\right)e^{-(\mu-\beta)y}.$$

Finally, we use that $e^{-x+O(x^2)} = 1-x$ and obtain that

$$\mathbb{E}_{(L-y,u)}\left[e^{-\lambda Z_{t_{k+1}}}\right] = \exp\left(-\frac{\lambda}{2}(1 + O(\theta e^{2\beta A}) + \varepsilon_L)e^{-(\mu-\beta)y}\right),$$

which concludes the proof of the lemma. $\square$

Lemma 6.2. *Assume (H_{wp}) holds and let $\Lambda > 0$. There exist $C > 0$ and $b > 0$ such that, uniformly in $\lambda \leqslant \Lambda$, in $\theta \leqslant \bar{\theta}(A)$ and $u \in [t_k, t_{k+1} - CL]$, we have*

$$\mathbb{E}_{(L,u)}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}\right] = \exp\left(\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}\right) + e^{-(\mu+\beta)A}\varepsilon_{A,L}\right),$$

with $\psi_{1,b}(\lambda) = -\lambda + b\lambda^\alpha$.

Proof. Let $C > 0$ and $y : (0,\infty) \to (0,\infty)$ be a function such that $y(L) = o(L)$. We start with a single particle located at L at time $u \in [t_k, t_{k+1} - CL]$ and stop the particles as soon as they hit $L-y$. We denote by κ the number of particles that hit $L-y$ and $(w_i)_{i=1}^{\kappa}$ the times they hit it. Then, for sufficiently large L, $w_i \in [u, t_{k+1}] \subset [t_k, t_{k+1}]$ for all $i \in [\![1,\kappa]\!]$, with probability $1-\varepsilon_L$. Thus,

$$\begin{aligned} \mathbb{E}_{(L,u)}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}\right] &= \mathbb{E}_{(L,u)}\left[\prod_{i=1}^{\kappa}\mathbb{E}_{(L-y,w_i)}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}\right]\right] \\ &= \mathbb{E}_{(L,u)}\left[\prod_{i=1}^{\kappa}\mathbb{E}_{(L-y,w_i)}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}\right]\mathbb{1}_{w_i\in[t_k,t_{k+1}],\forall i=1\ldots\kappa}\right] + \varepsilon_L. \end{aligned}$$

Besides, according to Lemma 6.1, we have

$$\begin{aligned}
&\mathbb{E}_{(L,u)}\left[\prod_{i=1}^{\kappa}\mathbb{E}_{(L-y,w_i)}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_{k+1}}}\right]\mathbb{1}_{w_i\in[t_k,t_{k+1}],\forall i=1\ldots\kappa}\right]\\
=\;& \mathbb{E}_{(L,u)}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)\kappa e^{-(\mu-\beta)y}\right)\mathbb{1}_{w_i\in[t_k,t_{k+1}],\forall i=1\ldots\kappa}\right]\\
=\;& (1-\varepsilon_L)\mathbb{E}_{(L,u)}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)\kappa e^{-(\mu-\beta)y}\right)\right]\\
=\;& \mathbb{E}_{(L,u)}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)\kappa e^{-(\mu-\beta)y}\right)\right]+\varepsilon_L.
\end{aligned}$$

Yet, we know thanks to Lemma 5.1 that $\kappa e^{-(\mu-\beta)y}$ converges in law to a random variable W as $L\to\infty$. Therefore, using that $|e^{-x}-e^{-y}|<|x-y|\wedge 1$, for all $x,y>0$, we get that

$$\begin{aligned}
&\mathbb{E}_{(L,u)}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)\kappa e^{-(\mu-\beta)y}\right)\right]\\
&=\mathbb{E}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)W\right)\right]+\varepsilon_L
\end{aligned}$$

In addition, Lemma 5.1 gives the existence of a constant $b>0$ such that as $\lambda\to 0$,

$$\mathbb{E}\left[e^{-\lambda W}\right]=\exp\left(\psi_{1,b}(\lambda)+o(\lambda^\alpha)\right)=\exp\left(-\lambda+b\lambda^\alpha+o(\lambda^\alpha)\right).$$

Yet, $\alpha(\mu-\beta)=\mu+\beta$, hence , we have

$$\begin{aligned}
&\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)\right)\\
&=-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)+b\frac{\lambda^\alpha}{2^\alpha}e^{-(\mu+\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)^\alpha+e^{-(\mu+\beta)A}\varepsilon_{A,L}\\
&=\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}\right)+O(\theta e^{-(\mu-3\beta)A})+e^{-(\mu+\beta)A}\varepsilon_{A,L}\\
&=\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}\right)+e^{-(\mu+\beta)A}\varepsilon_{A,L},
\end{aligned}$$

since $\theta e^{-(\mu-3\beta)A}=\theta e^{4\beta A}e^{-(\mu+\beta)A}$ and $\theta e^{4\beta A}\to 0$ as $A\to\infty$. Therefore, we have

$$\mathbb{E}\left[\exp\left(-\frac{\lambda}{2}e^{-(\mu-\beta)A}(1+O(\theta e^{2\beta A})+\varepsilon_L)W\right)\right]\;=\;\exp\left(\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A)}\right)+e^{-(\mu+\beta)A}\varepsilon_{A,L}\right),$$

and finally, we get that

$$\begin{aligned}
\mathbb{E}_{(L,u)}\left[e^{-\lambda e^{-(\mu-\beta)\beta A}Z_{t_{k+1}}}\right]\;&=\;\exp\left(\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}\right)+e^{-(\mu+\beta)A}\varepsilon_{A,L}\right)+\varepsilon_L,\\
&=\;\exp\left(\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}+e^{-(\mu+\beta)A}\varepsilon_{A,L}\right)\right),
\end{aligned}$$

which concludes the proof of the lemma. □

6.2 Proof of Proposition 6.1

Starting with one particle at $x \leqslant L$, we stop the particles when they hit L. For each particle v hitting L at time u, we denote by $Z^{(v)}$ the contribution of its descendants to Z. We further define $\mathcal{L}$ the set of particles that hit L. Thus, we write the process Z as

$$Z_t = Z_t' + \sum_{v \in \mathcal{L}} Z_t^{(v)}.$$

Therefore, conditioning on $\mathcal{L}$, we obtain that

$$\mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_{k+1}}}\right] = \mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t'_{k+1}}} \prod_{v \in \mathcal{L}} \mathbb{E}_{(L,u)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z^{(v)}_{t_{k+1}}}\right]\right].$$

Since Lemma 6.2 holds for all $u \in [t_k, t_{k+1} - CL]$, we need to prove that only a few particles hit L between $t_{k+1} - CL$ and t_{k+1}. Set $s = t_{k+1} - CL$. Using Lemma 4.4, Corollary 3.1, Markov's inequality and conditioning on $\mathcal{F}_s$, we get that

$$\begin{aligned}\mathbb{P}_{(x,t_k)}\left(|\mathcal{L} \cap [s, t_{k+1}]| \geqslant 1\right) &\leqslant \mathbb{E}_{(x,t_k)}\left[R([s, t_{k+1}])\right] \leqslant \mathbb{E}_{(x,t_k)}\left[(\ell([0, CL]) + C) g(L) Z_s' + C Y_s'\right] \\ &\leqslant \varepsilon_L z(x). \end{aligned} \tag{6.9}$$

Besides, Lemma 6.2 implies that

$$\begin{aligned}&\mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z'_{t_{k+1}}} \prod_{v \in \mathcal{L} \cap [t_k, s]} \mathbb{E}_{(L,u)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z^{(v)}_{t_{k+1}}}\right]\right] \\ &= \mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z'_{t_{k+1}} + R([t_k, s])\left(\psi_{1,b}\left(\frac{\lambda}{2} e^{-(\mu-\beta)A}\right) + e^{-(\mu+\beta)A} \varepsilon_{A,L}\right)}\right].\end{aligned}$$

Thus, a Taylor expansion combined with Equation (6.9) give that

$$\begin{aligned}\mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_{k+1}}}\right] &= 1 - \lambda e^{-(\mu-\beta)A} \mathbb{E}_{(x,t_k)}\left[Z'_{t_{k+1}}\right] \\ &\quad + \left(\psi_{1,b}\left(\frac{\lambda}{2} e^{-(\mu-\beta)A}\right) + e^{-(\mu+\beta)A} \varepsilon_{A,L}\right) \mathbb{E}_{(x,t_k)}\left[R([t_k, s])\right] \\ &\quad + O\left(e^{-2(\mu-\beta)A} \mathbb{E}_{(x,t_k)}\left[\left(Z'_{t_{k+1}}\right)^2 + R([t_k, s])^2\right]\right) + \varepsilon_L z(x). \end{aligned} \tag{6.10}$$

Yet, thanks to Lemmas 3.2, 3.4, 4.4 and 4.5, we know that

$$\mathbb{E}_{(x,t_k)}\left[Z'_{t_{k+1}}\right] = e^{(\lambda_1 - \lambda_1^\infty)(t_{k+1} - t_k)} z(x), \tag{6.11}$$

$$\mathbb{E}_{(x,t_k)}\left[\left(Z'_{t_{k+1}}\right)^2\right] \leqslant C\left(\theta e^{2\beta A} z(x) + (1 \wedge x) e^{\mu(x-L)}\right), \tag{6.12}$$

$$\mathbb{E}_{(x,t_k)}\left[R([t_k, s])\right] = (\ell([0, s - t_k]) + C) g(L) z(x) + O\left((1 \wedge x) e^{\mu(x-L)}\right), \tag{6.13}$$

$$\mathbb{E}_{(x,t_k)}\left[(R([t_k, s]))^2\right] \leqslant C(1 + \theta^2 e^{4\beta A})\left(\theta e^{2\beta A} z(x) + (1 \wedge x) e^{\mu(x-L)}\right), \tag{6.14}$$

since $g(L)(t_{k+1}-t_k)=O(\theta e^{2\beta A})$. Besides,

$$\ell([0,s-t_k])=\frac{1}{\lambda_1^\infty-\lambda_1}\left(1-e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}\right)$$

and

$$(\lambda_1^\infty-\lambda_1)^{-1}=(1+\varepsilon_L)a^{-1}e^{2\beta(L-1)},\qquad g(L)=2a(1+\varepsilon_L)e^{-2\beta(L-1)}.$$

according to Equations (6.1) and (6.2). Therefore,

$$\mathbb{E}_{(x,t_k)}\left[R([t_k,s])\right]=2(1+\varepsilon_L)\left(1-e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}\right)z(x)+O\left((1\wedge x)e^{\mu(x-L)}\right),$$

and thus, using that $\psi_{1,b}(\lambda)=-\lambda+b\lambda^\alpha$ and that $\alpha(\mu-\beta)=\mu+\beta$, we obtain

$$\begin{aligned}&\left(\psi_{1,b}\left(\frac{\lambda}{2}e^{-(\mu-\beta)A}\right)+e^{-(\mu+\beta)A}\varepsilon_{A,L}\right)\mathbb{E}_{(x,t_k)}\left[R([t_k,s])\right] \\ &=\left(\left(1-e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}\right)\left(-e^{-(\mu-\beta)A}\lambda+\frac{b}{2^{\alpha-1}}e^{-(\mu+\beta)A}\lambda^\alpha\right)+e^{-(\mu+\beta)A}\varepsilon_{A,L}\right)z(x) \\ &\quad+O\left(e^{-(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)}\right).\end{aligned}\tag{6.15}$$

Then, note that

$$e^{-2(\mu-\beta)A}e^{2\beta A}=e^{-(\mu-\beta)A}e^{-(\mu-3\beta)A}=\varepsilon_{A,L}e^{-(\mu-\beta)A}.$$

Yet, (H_{wp}) holds, hence, $\mu>3\beta$ and according to Equations (6.13) and (6.14), we have

$$e^{-2(\mu-\beta)A}\mathbb{E}_{(x,t_k)}\left[\left(Z'_{t_{k+1}}\right)^2+R([t_k,s])^2\right]=O\left(\theta e^{-(\mu-\beta)A}\varepsilon_{A,L}z(x)+e^{-2(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)}\right)\tag{6.16}$$

Combining Equations (6.10), (6.11), (6.15) and (6.16), we get that

$$\begin{aligned}&\mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{(\mu-\beta)A}Z_{t_{k+1}}}\right] \\ &=1-\Big[\lambda-\lambda\left(e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}-e^{(\lambda_1-\lambda_1^\infty)(t_{k+1}-t_k)}\right) \\ &\quad+\left(1-e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}\right)\left(\frac{b}{2^{\alpha-1}}\lambda^\alpha+\varepsilon_{A,L}\right)e^{-2\beta A}+\theta\varepsilon_{A,L}\Big]e^{-(\mu-\beta)A}z(x) \\ &\quad+O\left(e^{-(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)}\right).\end{aligned}$$

Moreover, we know that

$$\left|e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}-e^{(\lambda_1-\lambda_1^\infty)(t_{k+1}-t_k)}\right|\leqslant(\lambda_1^\infty-\lambda_1)(t_{k+1}-s)=\varepsilon_L$$

and,

$$e^{(\lambda_1-\lambda_1^\infty)(s-t_k)}=\exp\left(-\theta(ae^{2\beta})e^{2\beta A}+\theta e^{2\beta A}\varepsilon_{A,L}\right)=1-(ae^{2\beta})\theta e^{2\beta A}+\theta e^{2\beta A}\varepsilon_{A,L},$$

so that,

$$\begin{aligned}&\mathbb{E}_{(x,t_k)}\left[e^{-\lambda e^{(\mu-\beta)A}Z_{t_{k+1}}}\right] \\ &=1-\left[\lambda-\frac{(ae^{2\beta})b}{2^{\alpha-1}}\theta\lambda^\alpha+\theta\varepsilon_{A,L}\right]e^{-(\mu-\beta)A}z(x)+O\left(e^{-(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)}\right).\end{aligned}\tag{6.17}$$

Finally, we use that $e^{-x+O(x^2)} = 1-x$ as $x \to 0$: according to Lemma 3.1,

$$\begin{aligned} e^{-2(\mu-\beta)A}z(x)^2 &= e^{-2(\mu-\beta)A}e^{2\mu(x-L)}w_1(x)^2 \leqslant Ce^{-2(\mu-\beta)A}e^{2\mu(x-L)}(1\wedge x)e^{2\beta(L-x)} \\ &\leqslant C\left(e^{-(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)}\right)\left(e^{-(\mu-\beta)A}e^{(2\beta-\mu)(L-x)}\right) \\ &\leqslant Ce^{-(\mu-\beta)A}(1\wedge x)e^{\mu(x-L)} \end{aligned}$$

since $\mu > 3\beta$. This remark combined with Equation (6.17) conclude the proof of Proposition 6.1.

7 Convergence to the CSBP

In this section, we prove the convergence of the rescaled process Z to the α-stable CSBP. In practice, we gather the estimates from Proposition 6.1 to control the Laplace transform of this process.

In what follows, we denote by $M(t)$ the position of the rightmost particle in the process at time t, we write $\Rightarrow$ to refer to the convergence in distribution and $\to_p$ for the convergence in probability. We also use all the notations defined at the beginning of Section 6.

7.1 The process Z_t

Theorem 7.1. *Assume (H_{wp}) holds and suppose the configuration of particles at time zero satisfies $Z_0 \Rightarrow Z$ and $M(0)-L \to_p -\infty$ as $L\to\infty$. There exists $b>0$ such that the finite-dimensional distributions of the processes*

$$(Z_{e^{2\beta L}t}, t \geqslant 0)$$

converges as $L\to\infty$ to the finite-dimensional distributions of a continuous-state branching process $(\Xi(t), t\geqslant 0)$ with branching mechanism $\psi_{0,b}(\lambda) = b\lambda^\alpha$, whose distribution at time zero is the distribution of Z.

The proof of Theorem 7.1 is inspired by the one of Theorem 2.1 in [MS20]. Likewise, we claim that it is sufficient to prove the one-dimensional convergence of the process. Indeed, Theorem 7.1 can be deduced from the one-dimensional convergence result and the Markov property of the process if we prove that, under the assumptions of Theorem 7.1, for any $t>0$, $Z(te^{2\beta L}) \Rightarrow Z'$ for some random variable $Z' \geqslant 0$ and that $L - M(te^{2\beta L}) \to_p \infty$ as L tends to ∞ (the result then follows by induction). Yet, the first point is a consequence of the one-dimensional convergence and the second follows from Lemmas 3.1 and 4.4.

More precisely, this second point can be proved by considering a branching Brownian motion killed at 0 and $L+y$ for some large $y>0$. In fact, it could be proved using estimates on the density of the branching Brownian motion killed at 0, but since we only established estimates on the BBM in an interval, we use the results from Sections 3 and 4 with an additional barrier larger than L.

Define $Z_{t,y}$, $Z'_{t,y}$, $Y_{t,y}$, $\tilde{Y}_{t,y}$, z_y, $w_{1,y}$ and R_y in the the same way as Z_t, Z'_t, Y_t, $\tilde{Y}_t$, z, w_1 and R from Section 6 but for $L-y$ instead of L. First, note that since $Z_0 \Rightarrow Z$ and $L - M(0) \to_p +\infty$, then

$$Z_{0,-y} \Rightarrow e^{-(\mu-\beta)y} Z, \quad L \to \infty. \tag{7.1}$$

Indeed, there exists a sequence a satisfying $a_L \to \infty$ as $L \to \infty$ and such that the event $A_L := \{L - M(0) \geqslant a_L\}$ occurs with high probability:

$$\mathbb{P}(A_L) = \mathbb{P}(L - M(0) \geqslant a_L) = 1 - \varepsilon_L. \tag{7.2}$$

Therefore, for any bounded and continuous test function f on $[0, \infty)$,

$$\mathbb{E}\left[f(Z_{0,-y}) - f(e^{-(\mu-\beta)y}Z)\right] \leqslant 2\|f\|_\infty \mathbb{P}(A_L^c) + \mathbb{E}\left[\left(f(Z_{0,-y}) - f(e^{-(\mu-\beta)y}Z)\right) \mathbb{1}_{A_L}\right].$$

On the event A_L, we know that for any particle $v \in \mathcal{N}_0$, we have $L - X_v(0) \geqslant a_L$. Therefore, on A_L, for all $v \in \mathcal{N}_0$, we get that

$$z_{-y}(X_v(0)) = (1+\varepsilon_L) e^{-(\mu-\beta)y} z(X_v(0)).$$

Indeed, one can show that for all $0 \leqslant x \leqslant L - a_L$, we have

$$z_{-y}(x) = e^{\mu(x-L-y)} w_{1,-y}(x) = (1+\varepsilon_L) e^{\mu(x-L-y)} e^{\beta y} w_1(x).$$

Hence, on the event A_L,

$$Z_{0,-y} = (1+\varepsilon_L) e^{-(\mu-\beta)y} Z_0,$$

which concludes the proof of (7.1).

Let us now prove that

$$L - M(te^{2\beta L}) \to_p +\infty \tag{7.3}$$

as $L \to \infty$, under the assumptions of Theorem 7.1. We know thanks to Corollary 3.1 and 4.4 and Equations (6.1) and (6.2) that for sufficiently large L, conditionally on $\mathcal{F}_0$, we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{Y}_{te^{2\beta L},-y} \mathbb{1}_{R_{-y}([0,te^{2\beta L}])=0} | \mathcal{F}_0\right] \mathbb{1}_{M(0)\leqslant L} &\leqslant C Z'_{0,-y} \mathbb{1}_{M(0)\leqslant L} \\
\mathbb{E}\left[R_{-y}([0, te^{2\beta L}]) | \mathcal{F}_0\right] \mathbb{1}_{M(0)\leqslant L} &\leqslant C(Y_{0,-y} + Z'_{0,-y}) \mathbb{1}_{M(0)\leqslant L}.
\end{aligned}$$

Therefore, conditioning on the number of particles that hit $L+y$ between times 0 and $te^{2\beta L}$, we get that for L large enough

$$\begin{aligned}
\mathbb{P}\left(L - M(te^{2\beta L}) \leqslant a_L | \mathcal{F}_0\right) &= \mathbb{P}\left(e^{\mu(L-M(te^{2\beta L}))} \leqslant e^{\mu a_L} | \mathcal{F}_0\right) \mathbb{1}_{M(0)\leqslant L} \\
&\quad + \mathbb{P}\left(e^{\mu(L-M(te^{2\beta L}))} \leqslant e^{\mu a_L} | \mathcal{F}_0\right) \mathbb{1}_{M(0)>L} \\
&\leqslant \mathbb{1}_{M(0)>L} + \Big[\mathbb{P}\left(\tilde{Y}_{te^{2\beta L},-y} \geqslant e^{-\mu a_L}, R_{-y}([0, te^{2\beta L}]) = 0 | \mathcal{F}_0\right) \\
&\quad + \mathbb{P}\left(R_{-y}([0, te^{2\beta L}]) \neq 0 | \mathcal{F}_0\right)\Big] \mathbb{1}_{M(0)\leqslant L} \\
&\leqslant \mathbb{1}_{M(0)>L} + C(Y_{0,-y} + Z'_{0,-y}) \mathbb{1}_{M(0)\leqslant L},
\end{aligned}$$

by Markov' inequality. Note that $Z'_{0,-y} \leqslant Z_{0,-y}$ and that we can deduce from (7.1) that $Z_{0,-y} \to_p 0$ as $y \to \infty$. We now have to show that $Y_{0,-y} \to_p 0$ to conclude this proof. First, we prove that under the assumptions of Theorem 7.1,

$$Y_0 \to_p 0, \text{ as } L \to \infty. \tag{7.4}$$

Indeed, for L sufficiently large and $x \in [0, L]$,

$$\frac{(1 \wedge x)e^{\mu(x-L)}}{z(x)} \leqslant \frac{1}{L-x},$$

which implies that on the event A_L,

$$Y_0 \leqslant \frac{1}{L - M(0)} Z_0 \leqslant \frac{1}{a_L} Z_0. \tag{7.5}$$

Since $\mathbb{P}(A_L^c) = \varepsilon_L$, $a_L \to \infty$ and $Z_0 \Rightarrow Z$ as $L \to \infty$, this concludes the proof of (7.4). Then, remark that $Y_{0,-y} = e^{-\mu y} Y_0$ so that $Y_{0,-y} \to_p 0$ as $L \to \infty$. Finally, if we choose $y = -L$, we see that $Z'_{0,-y} + Y_{0,-y} \to_p 0$ as $L \to \infty$, and since $\mathbb{P}(M(0) \geqslant L) = \varepsilon_L$, we get that

$$\mathbb{P}\left(L - M(te^{2\beta L}) \leqslant a_L\right) = \varepsilon_L.$$

Note that it is also sufficient to assume that $Z_0 \to_p z_0$ as $L \to \infty$ for some constant $z_0 > 0$. One can then deduce Theorem 7.1 thanks to a conditioning argument. We now prove the one dimensional convergence under this assumption: we fix $t > 0$ and prove that for any $\lambda > 0$,

$$\lim_{L \to \infty} \mathbb{E}\left[e^{-\lambda Z_{te^{2\beta L}}}\right] = e^{-z_0 u_t(\lambda)}, \tag{7.6}$$

where $u_t(\lambda)$ is the function from Equation (1.7) corresponding to the branching mechanism $\Psi(\lambda) = \psi_{0,b}(\lambda) = b\lambda^\alpha$. To prove Equation (7.6), we discretise time: we consider the same parameter A as in Section 6 and choose $\theta \leqslant \bar{\theta}(A)$ such that $t = K\theta e^{2\beta A}$, for some integer K, and consider the corresponding time discretisation $t_k = k\theta e^{2\beta(L+A)}$ for $k \in [\![0, K]\!]$. We recall the definition of a_L from Equation (7.2) and define $b_L = a_L - A$ and the events

$$G_k = \left\{\forall j \in [\![0, k]\!] : M(t_j) \leqslant L - A,\, Y_{t_j,A} \leqslant Z_{t_j,A}/b_L\right\}, \tag{7.7}$$

for all $k \in [\![0, K]\!]$. Note that $b_L \to \infty$ as $L \to \infty$ by definition of A.

Lemma 7.1. *We have* $\mathbb{P}(G_K) = 1 - \varepsilon_L$.

Proof. According to Equations (7.2) and (7.5), $P(G_0) = 1 - \varepsilon_L$. Let $k \in [\![1, K]\!]$. Again, one can prove that $\mathbb{P}(L - A - M(t_k) \geqslant b_L) = 1 - \varepsilon_L$. We then deduce from (7.5) that $\mathbb{P}(Y_{t_k,A} \leqslant Z_{t_k,A}/b_L) = 1 - \varepsilon_L$ and conclude the proof of the lemma using a union bound. □

We now fix $\lambda > 0$ and for $\delta \in \mathbb{R}$, we define the sequence $(\lambda_k^{(\delta)})_{k=0}^K$ as follows:

$$\begin{aligned} \lambda_K^{(\delta)} &= \lambda \\ \lambda_k^{(\delta)} &= \lambda_{k+1}^{(\delta)} - \theta(\psi_{0,b}(\lambda_{k+1}^{(\delta)}) - \delta). \end{aligned}$$

Lemma 7.2. *1. There exists $\Lambda > 1$ such that for $|\delta|$ small enough and for θ small enough, we have $\lambda_k^{(\delta)} \in [0, \Lambda]$ for all $k \in [\![0, K]\!]$.*

2. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for θ sufficiently small,

$$\lambda_0^{(\delta)}, \lambda_0^{-(\delta)} \in [u_t(\lambda) + \varepsilon, u_t(\lambda) - \varepsilon].$$

3. For every $\delta > 0$, we have for sufficiently large A and L, for every $k = 0, ..., K$,

$$\mathbb{E}\left[e^{-\lambda_k^{(\delta)} e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k}\right] - \mathbb{P}\left(G_K \setminus G_k\right) \leqslant \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_K,A}} \mathbb{1}_{G_K}\right], \tag{7.8}$$

and

$$\mathbb{E}\left[e^{-\lambda_k^{(-\delta)} e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k}\right] \geqslant \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_K,A}} \mathbb{1}_{G_K}\right]. \tag{7.9}$$

Proof. The proof of parts 1 and 2 relies on standard results on the Euler scheme and is similar to the proof of Theorem 2.1 in [MS20]. The only difference is that we do not need to modify the function Ψ at zero, since it is a Lipschitz function on any interval $[0, \Lambda]$, $\Lambda > 0$.

We prove part 3 of the lemma. According to Equation (7.1), $Z_{0,A} \to_p e^{(\mu-\beta)A} z_0$ as $L \to \infty$. Besides, we know that $L - A - M(0) \to_p \infty$ as $L \to \infty$.

Let $\Lambda > 0$ such that the first part of the lemma holds. By proposition 6.1, we know that for L and A sufficiently large, for all $\lambda' \in [0, \Lambda]$, and for all $k \in [\![0, K]\!]$,

$$e^{-\lambda' + \theta(\psi_{0,b}(\lambda') - \delta) e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k} \leqslant \mathbb{E}\left[e^{-\lambda' e^{-(\mu-\beta)A} Z_{t_k,A}} | \mathcal{F}_k\right] \mathbb{1}_{G_k} \leqslant e^{-\lambda' + \theta(\psi_{0,b}(\lambda') + \delta) e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k} \quad a.s.$$

Besides, for $\delta > 0$ small enough, $\lambda_k^{(\pm\delta)} \in [0, \Lambda]$ for all $k \in [\![0, K]\!]$. Therefore, we have for every $\delta > 0$ small enough, A and L large enough, that for all $k \in [\![0, K]\!]$,

$$\mathbb{E}\left[e^{-\lambda_{k+1}^{(\delta)} e^{-(\mu-\beta)A} Z_{t_{k+1},A}} | \mathcal{F}_k\right] \mathbb{1}_{G_k} \geqslant e^{-\lambda_k^{(\delta)} e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k}, \tag{7.10}$$

$$\mathbb{E}\left[e^{-\lambda_{k+1}^{(\delta)} e^{-(\mu-\beta)A} Z_{t_{k+1},A}} | \mathcal{F}_k\right] \mathbb{1}_{G_k} \leqslant e^{-\lambda_k^{(-\delta)} e^{-(\mu-\beta)A} Z_{t_k,A}} \mathbb{1}_{G_k}. \tag{7.11}$$

The third part of the lemma follows by induction as in [MS20]: for $k = K$, Equation (7.8) holds. Let $k \in [\![0, K-1]\!]$ and assume that (7.8) holds for $k+1$. By definition of the sequence (G_k), we have $G_k \subset G_{k+1}$, so that the induction hypothesis implies that

$$\mathbb{E}\left[e^{-\lambda_k^{(\delta)} e^{-(\mu-\beta)A} Z_{t_{k+1},A}} \mathbb{1}_{G_k}\right] - \mathbb{P}\left(G_K \setminus G_k\right) \leqslant \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_K,A}} \mathbb{1}_{G_K}\right]$$

$$\mathbb{E}\left[e^{-\lambda_k^{(-\delta)} e^{-(\mu-\beta)A} Z_{t_{k+1},A}} \mathbb{1}_{G_k}\right] \geqslant \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A} Z_{t_K,A}} \mathbb{1}_{G_K}\right]$$

These equations combined with Equations (7.10) and (7.11) concludes the proof of the third point. $\square$

We now prove Equation (7.6) using Lemmas 7.1 and 7.2. According to Lemma 7.1, we have

$$\mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_K,A}}\right] = \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_K,A}}\mathbb{1}_{G_K}\right] + \varepsilon_L. \tag{7.12}$$

Let $\varepsilon > 0$ and choose $\delta > 0$ such that the second part of Lemma 7.2 holds. Therefore, the third part of the lemma implies that for L and A large enough,

$$\mathbb{E}\left[e^{-(u_t(\lambda)+\varepsilon)e^{-(\mu-\beta)A}Z_{0,A}}\right] - \varepsilon_L \leqslant \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{t_K,A}}\right] \leqslant \mathbb{E}\left[e^{-(u_t(\lambda)-\varepsilon)e^{-(\mu-\beta)A}Z_{0,A}}\right] + \varepsilon_L.$$

Letting $\varepsilon \to 0$ and using that $Z_{0,A} \to_p e^{(\mu-\beta)A}z_0$ as $L \to \infty$, we get that

$$\lim_{L\to\infty} \mathbb{E}\left[e^{-\lambda e^{-(\mu-\beta)A}Z_{te^{2\beta L},A}}\right] = e^{-z_0 u_t(\lambda)}$$

Hence, if we denote by Ξ the CSBP with branching mechanism $\psi_{0,b}$, such that $\Xi_0 = z_0$, then, we get that $e^{-(\mu-\beta)A}Z_{te^{2\beta L},A} \Rightarrow \Xi_t$ as $L \to \infty$. Since we know that $L - A - M(te^{2\beta L}) \to_p +\infty$, we have (see Equation (7.1))

$$e^{-(\mu-\beta)A}Z_{te^{2\beta L},A} \Rightarrow Z_{te^{2\beta L},0} = Z_{te^{2\beta L}},$$

as $L \to \infty$. This concludes the proof of Theorem 7.1.

7.2 The number of particles N_t

In this section, we conclude the proof of Theorem 1.2 by deducing the result on the number of particles N_t from the convergence of the processe Z_t obtained in Theorem 7.1. For a sake of clarity, we first recall several definitions from previous sections. In Theorem 7.2, we state a version of Theorem 1.2, under more general assumptions on the initial configuration.

Recall from Section 1.3 that the exponent α is defined as

$$\alpha = \frac{\mu + \sqrt{\mu^2 - 1}}{\mu - \sqrt{\mu^2 - 1}}, \tag{1.12}$$

and from Section 6 that the process Z_t is given by

$$Z_t = \sum_{v\in\mathcal{N}_t} e^{\mu(X_v(t)-L)}w_1(X_v(t))\mathbb{1}_{X_v(t)\in[0,L]},$$

with

$$w_1(x) = \begin{cases} \sinh\left(\sqrt{2\lambda_1}(L-1)\right)\sin(\sqrt{\rho-1-2\lambda_1}\,x)/\sin(\sqrt{\rho-1-2\lambda_1}) & \text{if } \; x \in [0,1], \\ \sinh\left(\sqrt{2\lambda_1}(L-x)\right) & \text{if } \; x \in [1,L]. \end{cases}$$

In addition, we define the constant C_0 as

$$C_0 = \frac{1}{2}\left(\lim_{L\to\infty}\|v_1\|\right)^{-2}\left(\lim_{L\to\infty}\int_0^L e^{-\mu y}v_1(y)dy\right). \tag{7.13}$$

where

$$v_1 = \sinh(\sqrt{2\lambda_1}(L-1))^{-1} w_1.$$

Recall from Theorem 2.3 that the L^2-norm of the eigenvector v_1 converges to a positive limit as L goes to ∞. Besides, one can prove that $L \mapsto \int_0^L e^{-\mu y} v_1(y)dy$ also converges to a positive limit as $L \to \infty$, using the dominated convergence theorem combined with Lemma 3.1 and Remark 2.2, so that the constant C_0 is well defined.

Theorem 7.2. *Assume* (H_{wp}) *holds. In addition, suppose the configuration of particles at time zero satisfies* $Z_0 \to_p z_0$*, for some* $z_0 > 0$*, and* $M(0) - L \to_p -\infty$ *as* $L \to \infty$*. Then, there exists* $b > 0$ *such that the finite-dimensional distributions of the processes*

$$\left(\frac{1}{C_0 e^{(\mu-\beta)L} z_0} N_{e^{2\beta L} t}, t \geqslant 0 \right)$$

converge as $L \to \infty$ *to the finite-dimensional distributions of a continuous-state branching process* $(\Xi(t), t \geqslant 0)$ *with branching mechanism* $\psi_{0,b}(\lambda) = b\lambda^\alpha$ *starting from* 1.

Note that Theorem 1.2 can be deduced from Theorem 7.2 by computing Z_0 when the system starts with N particles located a 1. In this case, according to Remark 2.1, we have

$$Z_0 = N e^{\mu(1-L)} w_1(1) = \frac{1}{2} N (1+\varepsilon_L) e^{-(\mu-\beta)L} e^{\mu-\beta}.$$

Hence, we set

$$L = \frac{1}{\mu-\beta} \log(N),$$

so that

$$N = e^{(\mu-\beta)L} \quad \text{and} \quad N^{\alpha-1} = e^{2\beta L}. \tag{7.14}$$

Then, Theorem 7.2 yields the result and gives that

$$\sigma(\rho) = \frac{2}{C_0 e^{\mu-\beta}}, \tag{7.15}$$

where C_0 is defined by Equation (7.13).

As in the proof of Theorem 7.1, we define the process (N'_t), which corresponds to the number of particles alive in the system at time t, whose ancestors stay in $(0, L)$ until time t, and estimate its first and second moments.

As outlined in Section 1.5, there exists a constant $C > 0$ such that the number of particles N_t can be approximated by

$$N_t \approx C e^{(\mu-\beta)L} Z_t$$

for t and L large enough. Actually, this constant is equal to C_0 and a rigorous statement of this observation is given by the following lemma.

Lemma 7.3. *Assume* (H_{wp}) *holds and suppose the configuration of particles at time zero satisfies* $Z_0 \to_p z_0$*, for some* $z_0 > 0$*, and* $M(0) - L \to_p -\infty$ *as* $L \to \infty$*. Let* $t > 0$*. Then, we have*

$$\left| e^{-(\mu-\beta)L} N_{e^{2\beta L} t} - C_0 Z_{e^{2\beta L} t} \right| \to_p 0 \tag{7.16}$$

as L *goes to* $+\infty$.

As in [BBS13], we claim that one can deduce Theorem 7.2 from Theorem 7.1 combined with Lemma 7.3.

In what follows, we consider a time t and denote by u the corresponding time on the time scale of the CSBP:

$$u = te^{2\beta L}.$$

We also consider a small parameter $0 < \delta < 1$ and note that, according to Lemma 2.5, for all $(x, y) \in (0, L)^2$, we have

$$p_{\delta u} = (1 + \varepsilon_L)(1 + O(\delta))\|v_1\|^{-2} e^{\mu(x-y)} v_1(x) v_1(y), \tag{7.17}$$

where $O(\delta)$ is a quantity bounded $C\delta$ for some constant $C > 0$ that does not depend on L, x, and y. The proof of Lemma 7.3 is divided in three parts:

1. We first prove that N'_u is well approximated by $C_0 e^{(\mu-\beta)L} Z'_{(1-\delta)u}$ for large values of L, by controlling the first and second moments of N' (Lemma 7.4 and Lemma 7.5).
2. Then, we show that Z' does not vary two much between times $(1-\delta)u$ and u for δ small enough with a similar argument.
3. Finally, we recall why Z'_u (resp. N'_u) is a good approximation of Z_u (resp. N_u) for L large enough.

Let us first bound the first and second moment of N'. As in Section 3, we estimate its first moment in the pushed regime (that is under (H_{psh})) and the second moment in the weakly pushed regime (*i.e.* under (H_{wp})).

Lemma 7.4 (First moment of N'). *Assume (H_{psh}) holds. Therefore, we have*

$$\mathbb{E}[N'_u | \mathcal{F}_{(1-\delta)s}] = C_0(1 + \varepsilon_L)(1 + O(\delta)) e^{(\mu-\beta)L} Z'_{(1-\delta)u},$$

where C_0 is the constant given by Equation (7.13).

Proof. The Many-to-one Lemma 1.1 implies that for any $x \in (0, L)$, we have

$$\mathbb{E}_{(x,(1-\delta)u)}[N'_u] = \mathbb{E}_{(x,(1-\delta)u)}\left[\sum_{v \in \mathcal{N}^L} 1\right] = \int_0^L p_{\delta u}(x, y) dy.$$

Besides, we get thanks to Equation (7.17) that

$$\mathbb{E}_{(x,(1-\delta)u)}[N'_u] = C(1 + \varepsilon_L)(1 + O(\delta))\|v_1\|^{-2} e^{\mu x} v_1(x) \int_0^L e^{-\mu y} v_1(y) dy.$$

Moreover, Remark 2.1 implies that

$$e^{\mu x} v_1(x) = \frac{1}{2}(1 + \varepsilon_L) e^{(\mu-\beta)L} e^{\mu(x-L)} w_1(x). \tag{7.18}$$

Finally, we obtain thanks to Lemma 2.3 that for $x \in [0, L]$, we have

$$\mathbb{E}_{(x,(1-\delta)u)}[N'_u] = C_0(1+\varepsilon_L)(1+O(\delta))e^{(\mu-\beta)L}e^{\mu(x-L)}w_1(x),$$

which concludes the proof of the lemma. □

Lemma 7.5 (Second moment of N')**.** *Assume* (H_{wp}) *holds. Therefore, for* L *large enough, we have*

$$\mathbb{E}\left[(N'_u)^2|\mathcal{F}_{(1-\delta)u}\right] \leqslant Ce^{2(\mu-\beta)L}\left(Y_{(1-\delta)u}+\delta Z'_{(1-\delta)u}\right).$$

Proof. By the Many-to-two Lemma 1.2, we have for all $x \in (0, L)$,

$$\mathbb{E}_{(x,(1-\delta)u)}\left[(N'_u)^2\right] = \mathbb{E}_{(x,(1-\delta)u)}\left[N'_u\right] + \underbrace{\int_0^{\delta u}\int_0^L p_s(x,y)2r(y)\mathbb{E}_y\left[N'_{\delta u-s}\right]^2 dyds}_{=:U}.$$

The quantity U can be written as follows:

$$U = \int_0^L 2r(y)\int_0^{\delta u} p_s(x,y)\left(\int_0^L p_{\delta u-s}(y,z)dz\right)^2 dsdy.$$

To estimate this integral, we split the second integral into three parts.

- According to Lemma 2.5 and Lemma 2.3, there exists $C_1 > 0$ such that for $\delta u - s > C_1 L$ and L large enough,

$$p_{\delta u-s}(y,z) \leqslant Ce^{(\lambda_1-\lambda_1^\infty)(\delta u-s)}e^{\mu(x-y)}v_1(x)v_1(y) \leqslant Ce^{\mu(x-y)}v_1(x)v_1(y). \tag{7.19}$$

  Therefore, combining Equations (7.18) and (7.19), we obtain that

$$\begin{aligned} U_1 &:= \int_0^L 2r(y)\int_0^{\delta u-C_1L} p_s(x,y)\left(\int_0^L p_{\delta u-s}(y,z)dz\right)^2 dsdy \\ &\leqslant Ce^{2(\mu-\beta)L}\int_0^L e^{2\mu(y-L)}w_1^2(y)\int_0^{\delta u-C_1L} p_s(x,y)dsdy \\ &\leqslant Ce^{2(\mu-\beta)L}\int_0^L e^{2\mu(y-L)}w_1^2(y)\int_0^{\delta u} p_s(x,y)dsdy \\ &\leqslant Ce^{2(\mu-\beta)L}\left(e^{\mu(x-L)}\left((1\wedge x)+\delta u e^{-2\beta L}w_1(x)\right)\right) \end{aligned}$$

  where we use Equations (3.7) and (3.17) from the proof of Lemma 3.4 applied to $u(L) = \delta e^{2\beta L}$ to get the last line. Hence, we get that for L large enough

$$U_1 \leqslant Ce^{2(\mu-\beta)L}e^{\mu(x-L)}\left((1\wedge x)+\delta w_1(x)\right). \tag{7.20}$$

- For small values of $\delta u - s$, we bound the integral $\int_0^L p_{\delta u-s}(y,z)dz$ by the expected number of particles in a branching Brownian motion with no killing and constant branching rate $\rho/2$. This way, we get that

$$\int_0^L p_{\delta u-s}(y,z) \quad\leqslant\quad e^{\frac{\rho}{2}(\delta u-s)},$$

We use this upper bound on the last integral for $\delta u - s \in [0, C_2 L]$ for some constant C_2 to determine. Therefore, we define

$$U_2 := \int_0^L 2r(y) \int_{\delta u}^{\delta u - C_2 L} p_s(x,y) \left(\int_0^L p_{\delta u - s}(y,z) dz \right)^2.$$

Note that Equation (7.17) holds for all $s \in [\delta u, \delta u - C_2 L]$ instead of δu as long as L is large enough. Therefore, we get that for L large enough

$$\begin{aligned}
U_2 &\leqslant C \int_0^L \int_{\delta u - C_2 L}^{\delta u} p_s(x,y) e^{\rho(\delta u - s)} ds dy \\
&\leqslant C e^{(\mu-\beta)L} e^{\mu(x-L)} w_1(x) \int_0^L e^{-\mu y} v_1(y) \left(\int_{\delta u - C_2 L}^{\delta u} e^{\rho(\delta u - s)} ds \right) dy \\
&\leqslant C e^{(\mu-\beta)L} e^{\mu(x-L)} w_1(x) e^{\rho C_2 L} \int_0^L e^{-\mu y} v_1(y) dy \\
&\leqslant C e^{(\mu-\beta)L} e^{\mu(x-L)} w_1(x) e^{\rho C_2 L}.
\end{aligned}$$

Hence, we choose a constant $C_2 > 0$ satisfying $\rho C_2 < \mu - \beta$. This way, we get that

$$U_2 \leqslant C e^{2(\mu-\beta)L} \left(e^{(\rho C_2 - (\mu - \beta))L} \right) e^{\mu(x-L)} w_1(x). \tag{7.21}$$

- We now consider the remaining part of the time integral that is $s \in [\delta u - C_1 L, \delta u - C_2 L]$. To this end, we go back to the definition of the density p:

$$p_u(x,y) = e^{\mu(x-y)} \sum_{i=1}^{\infty} e^{(\lambda_i - \lambda_1^\infty)u} \frac{v_i(x) v_i(y)}{\|v_i\|^2}.$$

Recall from Remark 2.3 that in the semi pushed regime we have $K = 1$. Besides, by definition of the eigenfunction v_i (see Lemma 2.1) and using the lower bound on the L_2-norm of v_i given in Lemma 2.4 for $i > K$, there exists a constant $C > 0$ such that

$$\frac{v_i(x) v_i(y)}{\|v_i\|^2} \leqslant C,$$

for all $(x,y) \in (0,L)^2$ and $i > K$. Hence, recalling that $\|v_1\|$ converges to a positive limit as L goes to infinity, we get that

$$p_u(x,y) \leqslant C e^{\mu(x-y)} \left(v_1(x) v_1(y) + e^{(\lambda_2 - \lambda_1^\infty)u} \sum_{i=2}^{\infty} e^{(\lambda_i - \lambda_2)u} \right).$$

Note that, since $\lambda_i < 0$ for $i > K = 1$, the last sum is bounded by

$$\sum_{i=2}^{\infty} e^{(\lambda_i - \lambda_2)u} \leqslant (\rho - 1)^{-1} S_3 = (\rho - 1)^{-1} \sum_{i=2}^{\infty} e^{(\lambda_i - \lambda_2)t} (\rho - 1 - 2\lambda_i),$$

where S_3 corresponds to the sum defined in Equation (2.27). Yet, we proved (see Equation (2.30)) that $S_3 \leqslant CL$ for $t > 1$. Therefore, we get that for $C_2 L \leqslant u \leqslant C_1 L$, we have

$$p_u(x,y) \leqslant C e^{\mu(x-y)} \left(v_1(x) v_1(y) + L e^{-\lambda_1^\infty u} \right),$$

so that

$$\left(\int_0^L p_u(y,z)dz\right)^2 \leqslant Ce^{2\mu y}\left(v_1(y)^2 + L^2 e^{-2\lambda_1^\infty u}\right), \tag{7.22}$$

since $\int_0^L e^{-\mu z} v_1(z)dz$ converges to a positive limit as $L \to \infty$. Besides, note that Equation (7.17) holds for all $s \in [\delta u - C_1 L, \delta u - C_2 L]$ instead of δu as long as L is large enough. Therefore, one can write

$$\begin{aligned} U_3 &:= \int_0^L 2r(y) \int_{\delta u - C_1 L}^{\delta u - C_2 L} p_s(x,y) \left(\int_0^L p_{\delta u - s}(y,z)dz\right)^2 dsdy \\ &\leqslant U_{3,1} + U_{3,2}, \end{aligned}$$

with

$$U_{3,1} \leqslant Ce^{\mu x} v_1(x) \int_0^L \int_{\delta u - C_1 L}^{\delta u - C_2 L} e^{\mu y} v_1^3(y) dsdy$$

and

$$U_{3,2} \leqslant CL^2 e^{\mu x} v_1(x) \int_0^L \int_{\delta u - C_1 L}^{\delta u - C_2 L} e^{\mu y} v_1(y) e^{-2\lambda_1^\infty(\delta u - s)} dyds.$$

Since $\mu > 3\beta$ in the semi pushed regime, we have

$$\begin{aligned} U_{3,1} &\leqslant CLe^{\mu x} v_1(x) \int_0^L e^{\mu y} v_1^3(y) dy \\ &\leqslant CLe^{(\mu-\beta)L} e^{\mu(x-L)} w_1(x) e^{(\mu - 3\beta)L} \\ &\leqslant C\left(Le^{-2\beta L}\right) e^{2(\mu-\beta)L} e^{\mu(x-L)} w_1(x). \end{aligned} \tag{7.23}$$

On the other hand,

$$\begin{aligned} U_{3,2} &\leqslant CL^2 e^{\mu x} v_1(x) \left(\int_0^L e^{\mu y} v_1(y) dy\right)\left(\int_{\delta u - C_1 L}^{\delta u - C_2 L} e^{-2\lambda_1^\infty(\delta u - s)} ds\right) \\ &\leqslant Ce^{2(\mu-\beta)L} e^{\mu(x-L)} w_1(x) \left(L^2 e^{-2\lambda_1^\infty C_2 L}\right),. \end{aligned} \tag{7.24}$$

Finally, we obtain the lemma by combining equations (7.20), (7.21), (7.23) and (7.24). □

Proof of Lemma 7.3. Let $\gamma > 0$. Let us prove that for L large enough, we have

$$\mathbb{P}\left(|\bar{N}_u - C_0 Z_u| > \gamma\right) = \mathbb{P}\left(|N_u - C_0 e^{(\mu-\beta)L} Z_u| > \gamma e^{(\mu-\beta)L}\right) < \gamma.$$

As explained at the beginning of the section, we use that for $u > 0$,

$$\begin{aligned} |N_u - C_0 e^{(\mu-\beta)L} Z_u| &\leqslant |N_u - N'_u| + |N'_u - C_0 e^{(\mu-\beta)L} Z'_{(1-\delta)u}| \\ &\quad + C_0 e^{(\mu-\beta)L} |Z'_{(1-\delta)u} - Z'_u| + C_0 e^{(\mu-\beta)L} |Z'_u - Z_u|, \end{aligned}$$

and that each quantity in the right hand side term is small as long as L is large enough and δ is small enough.

First, we choose $\delta > 0$ of the form

$$\delta = \theta e^{2\beta A},$$

where A and θ are defined in the beginning of Section 6. Recall that for fixed θ and A, we have

$$\mathbb{P}(G_K) = 1 - \varepsilon_L,$$

where (G_k) is defined in Equation (7.7) and K is such that $t = K\theta e^{2\beta A}$. Note that with these notations, we have $u = t_K$ and $(1-\delta)u = t_{K-1}$.

Since the variances of N'_u and Z'_u are both bounded by a quantity that depends on $Z'_{(1-\delta)u}$, we first control $Z'_{(1-\delta)u}$ on G_K. To this end, we consider the martingale $U_j = e^{(\lambda_1^\infty - \lambda_1)j\theta e^{2\beta(L+A)}} Z'_{t_j}$ for $j = 1, ..., K$ (see Lemma 3.2). Using that $t_K = K\theta e^{2\beta(L+A)} = te^{2\beta L}$ and that $\lambda_1^\infty - \lambda_1 = a(1+\varepsilon_L)e^{-2\beta(L-1)} \geqslant \frac{a}{2}(1+\varepsilon_L)e^{-2\beta(L-1)}$ for L large enough (see Equation (6.1)), we get thanks to Doob's martingale inequality that for $B > 0$,

$$\begin{aligned}
\mathbb{P}\left(\max_{0\leqslant j\leqslant K} Z'_{t_j} > B\gamma^{-1}\right) &= \mathbb{P}\left(\max_{0\leqslant j\leqslant K} U'_j > B\gamma^{-1}e^{(\lambda_1^\infty-\lambda_1)j\theta e^{2\beta(L+A)}}\right) \\
&\leqslant \mathbb{P}\left(\max_{0\leqslant j\leqslant K} U'_j > B\gamma^{-1}e^{(\lambda_1^\infty-\lambda_1)K\theta e^{2\beta(L+A)}}\right) \\
&\leqslant \mathbb{P}\left(\max_{0\leqslant j\leqslant K} U'_j > B\gamma^{-1}e^{\frac{a}{2}e^{-\beta}t}\right) \\
&\leqslant \gamma\frac{\mathbb{E}[U_0]}{Be^{\frac{a}{2}e^{-\beta}t}} = \gamma\frac{\mathbb{E}[Z'_0]}{Be^{\frac{a}{2}e^{-\beta}t}} = \gamma\frac{\mathbb{E}[Z_0]}{Be^{\frac{a}{2}e^{-\beta}t}},
\end{aligned} \tag{7.25}$$

on G_K. Let $E_\gamma^K = G_K \cap \left\{\max_{0\leqslant j\leqslant K} Z'_{t_j} \leqslant B\gamma^{-1}\right\}$. Remark that we can choose B large enough in such a way that $\mathbb{P}(E_\gamma^K) \geqslant 1 - \gamma/4$. From now, we consider the event E_γ^K corresponding to this choice of B. Besides, note that on E_γ^K, we can assume without loss of generality that $Y_{(1-\delta)u} \leqslant \delta Z_{(1-\delta)u}$ (see Equation (7.5)).

We now bound the quantities $|N'_u - C_0e^{(\mu-\beta)L}Z'_{(1-\delta)u}|$ and $C_0e^{(\mu-\beta)L}|Z'_{(1-\delta)u} - Z'_u|$ with high probability. First, recall from Lemma 3.2 that

$$\mathbb{E}\left[Z'_u|\mathcal{F}_{(1-\delta)u}\right] = (1+O(\delta))Z'_{(1-\delta)u}$$

and that according to Lemma 3.4, we have

$$\mathbb{E}\left[\left(Z'_u\right)^2|\mathcal{F}_{(1-\delta)u}\right] \leqslant C\left(\delta Z'_{(1-\delta)u} + Y_{(1-\delta)u}\right)$$

Therefore, on the event E_γ^K, Chebyshev's inequality gives that for δ small enough and L large enough, we have

$$\mathbb{P}\left(\left|Z'_u - \mathbb{E}\left[Z'_u|\mathcal{F}_{(1-\delta)u}\right]\right| > \frac{\gamma}{2}|\mathcal{F}_{(1-\delta)u}\right) \leqslant C\delta B\gamma^{-2} \leqslant \frac{\gamma}{2},$$

and we know that, on the event E_γ^K, $\delta Z'_{(1-\delta)u} \leqslant \delta B\gamma^{-1} \leqslant \frac{\gamma}{2}$ for δ small enough and L large enough. Hence, we get that for sufficiently large L and sufficiently small δ, we have

$$\begin{aligned}
\mathbb{P}\left(\left|Z'_u - Z'_{(1-\delta)u}\right| > \gamma\right) &\leqslant \mathbb{P}\left(\left|Z'_u - Z'_{(1-\delta)u}\right| > \gamma, E_\gamma^K\right) + \mathbb{P}((E_\gamma^K)^c) \\
&\leqslant \frac{\gamma}{2} + \mathbb{P}((E_\gamma^K)^c) \leqslant \gamma.
\end{aligned} \tag{7.26}$$

Similarly, we get thanks to Lemma 7.4 and Lemma 7.5 that for δ small enough and L large enough,

$$\mathbb{P}\left(\left|N'_u - C_0 e^{(\mu-\beta)L} Z'_{(1-\delta)u}\right| > \gamma e^{(\mu-\beta)L}\right) \leqslant \gamma. \tag{7.27}$$

Besides, recall that if the process starts with all its particles to the right of L at time $(1-\delta)u$, then $Z'_{(1-\delta)u} = Z_{(1-\delta)u}$ and, $Z'_u = Z_u$ with high probability. This is a consequence of Lemma 4.4 combined with Markov's inequality. Indeed, on the event E_γ^K, we have

$$\mathbb{P}\left(\left|Z_u - Z'_u\right| > 0 \middle| \mathcal{F}_{(1-\delta)u}\right) \leqslant \mathbb{P}\left(R([0,\delta u]) \geqslant 1 \middle| \mathcal{F}_{(1-\delta)u}\right) \leqslant C\left(\delta Z'_{(1-\delta)u} + Y_{(1-\delta)u}\right) \leqslant \frac{\gamma}{2},$$

for L large enough and δ small enough. With a similar argument as in Equation (7.26), we get that

$$\mathbb{P}\left(\left|Z_u - Z'_u\right| > 0\right) \leqslant \gamma. \tag{7.28}$$

Similarly, we get that for sufficiently small δ and large L, we have

$$\mathbb{P}\left(\left|N'_u - N_u\right| > 0\right) \leqslant \gamma. \tag{7.29}$$

Combining (7.26), (7.27), (7.28) and (7.29), we get that for L large enough, we have

$$\mathbb{P}\left(|\bar{N}_u - C_0 Z_u| > 2\gamma\right) \leqslant 4\gamma,$$

which concludes the proof of the lemma. □

8 The case $\alpha > 2$: the fully pushed regime

In this section, we briefly outline the adjustments required to prove the second conjecture stated at the end of Section 1.3. In this case, we have $\rho > \rho_2$ so that $\mu < 3\beta$ (see Equation (1.14)).

In the fully pushed regime, we expect (see Equation (7.14)) the genealogy to evolve over the timescale

$$N = e^{(\mu-\beta)L}.$$

Therefore, we consider a small parameter $\theta > 0$ and a large A as in Section 6, but such that $\theta e^{(\mu-\beta)A} \to 0$ as $L \to \infty$. Similarly, we will consider a subdivision of the form

$$t_k = k\theta e^{(\mu-\beta)(L+A)}, \quad k \in \mathbb{N}.$$

For $t = \theta e^{(\mu-\beta)(L+A)}$, Equation (1.23) becomes

$$\mathbb{E}[Z'_t] = (1+\varepsilon_L) Z'_0,$$

since $\lambda_1^\infty - \lambda_1 \sim a e^{-2\beta L}$ and $\mu < 3\beta$. Essentially, this means that for L large enough, no particle exits the interval $(0, L)$ during the time interval $[0, t]$. Indeed, one can show that on the same time interval

$$\mathbb{E}[R([0,t])] = \varepsilon_L Z'_0.$$

On the other hand, an analysis similar to the one conducted in Section 5 would give that the number Z_y of descendants stopped at $L - y$ is such that

$$e^{-(\mu-\beta)L} Z_y \to W \quad a.s.$$

for some random variable W satisfying

$$\mathbb{E}\left[e^{-\lambda W}\right] = \exp\left(-\lambda + o(\lambda^2)\right).$$

Therefore, we would require a finer estimate of the second moment of Z' (not only an upper bound) to establish a result similar to Proposition 6.1 with $\alpha = 2$. Again, the second moments can be estimated thanks to the Green function, using analogous techniques.

The computations from Sections 6 and 7 would then be quite similar to prove the convergence of the process $(Z_{e^{(\mu-\beta)L}t}, t \geqslant 0)$ to a 2-stable CSBP.

From a biological standpoint, we see that for N large enough, the particles in the system do not reach the additional barrier L so that the invasion is no longer driven by the particles far to the right of the front (*i.e.* that hit L) but by the ones living in the bulk (*i.e.* that stay far from L).

A Appendix: proof of Proposition 1.1

The properties of g are easily checked. We only have to prove the expression of $\lambda_c(\rho)$.

We first show that $\lambda_c(\rho) \geq 0$ for all $\rho \in \mathbb{R}$. Let $\lambda < 0$. Assume $u \in \mathcal{D}_{T_\rho}$, $T_\rho u = \lambda u$. Then $u'' = 2\lambda u$ on $[1, \infty)$, and so $u = a\sin(\sqrt{2|\lambda|}x) + b\cos(\sqrt{2|\lambda|}x)$ on $[1, \infty)$ for some $a, b \in \mathbb{R}$. But then u changes sign on $[1, \infty)$ and so we cannot have $u > 0$ on $(0, \infty)$. Hence, $\lambda_c(\rho) \geq 0$ for all $\rho \in \mathbb{R}$.

We now claim that $\lambda_c(\rho) = 0$ for $\rho \leq \rho_c$. Since $\lambda_c(\rho)$ is increasing, it is enough to show that $\lambda_c(\rho_c) = 0$. Define

$$u(x) = \begin{cases} \sin(\frac{\pi}{2}x), & x \in [0, 1] \\ 1, & x \geq 1. \end{cases}$$

Then $u \in C^1((0, \infty)) \cap C^2((0, 1) \cup (1, \infty))$, $u(0) = 0$ and $u > 0$ on $(0, \infty)$. Moreover, $T_{\rho_c} u(x) = 0$ for $x \in (0, 1) \cup (1, \infty)$, hence, $u \in \mathcal{D}_{T_{\rho_c}}$ and $T_{\rho_c} u = 0$. It follows that $\lambda_c(\rho_c) = 0$.

Now let $\rho > \rho_c$ and $\lambda \in (0, \rho/2)$. Let $u \in \mathcal{D}_{T_\rho}$ such that $u > 0$ on $(0, \infty)$ and $T_\rho u = \lambda u$. Then $u'' = (2\lambda - \rho)u$ on $(0, 1)$ and so, since $\lim_{x\to 0} u(x) = 0$, we have $u(x) = C\sin(\sqrt{\rho - 2\lambda}x)$ for $x \in (0, 1)$ for some $C \in \mathbb{R}$. Since $u > 0$, we have $\rho - 2\lambda < \pi^2$ and $C > 0$. Suppose w.l.o.g. that $C = 1$, so that $u(x) = \sin(\sqrt{\rho - 2\lambda}x)$ for $x \in (0, 1)$.

For $x \in (1, \infty)$, we have $u''(x) = 2\lambda u(x)$, so that $u(x) = a\cosh(\sqrt{2\lambda}(x-1)) + b\sinh(\sqrt{2\lambda}(x-1))$ for some $a, b \in \mathbb{R}$. Since u and u' are continuous at 1, we must have $a = u(1) = \sin(\sqrt{\rho - 2\lambda})$

and $b = u'(1) = \sqrt{\rho - 2\lambda}\cos(\sqrt{\rho - 2\lambda})$. Furthermore, we have $u > 0$ on $(1, \infty)$ iff $a + b \geq 0$, which holds iff

$$\frac{\cos(\sqrt{\rho - 2\lambda})}{\sqrt{2\lambda}} \geq -\frac{\sin(\sqrt{\rho - 2\lambda})}{\sqrt{\rho - 2\lambda}} \tag{A.1}$$

Moreover, all of these conditions are also sufficient: if $\lambda \in J_\rho := (0 \vee \frac{1}{2}(\rho - \pi^2), \rho/2)$ satisfies (A.1), then the function u defined by

$$u(x) = \begin{cases} \sin(\sqrt{\rho - 2\lambda}x), & x \in [0, 1] \\ \sin(\sqrt{\rho - 2\lambda})\cosh(\sqrt{2\lambda}(x - 1)) + \sqrt{\rho - 2\lambda}\cos(\sqrt{\rho - 2\lambda})\sinh(\sqrt{2\lambda}(x - 1)), & x \geq 1 \end{cases}$$

satisfies $u \in C^1((0, \infty)) \cap C^2((0, 1) \cap (1, \infty))$, $u > 0$ on $(0, 1) \cup (1, \infty)$, $\lim_{x \to 0} u(x) = 0$ and $T_\rho u(x) = \lambda u(x)$ for $x \in (0, 1) \cup (1, \infty)$. Hence, by continuity of u, $T_\rho u(1) = \lim_{x \to 1} T_\rho u(x)$ exists and equals $\lambda u(1)$. Hence, $u \in \mathcal{D}_{T_\rho}$, $u > 0$ on $(0, \infty)$ (again by continuity) and $T_\rho u = \lambda u$.

Let Λ_ρ be the set of those $\lambda \in J_\rho$ such that (A.1) holds. It remains to show that $\inf \Lambda = \lambda_c(\rho)$, where $\lambda_c(\rho)$ is as in the statement of the theorem. Note that $\sqrt{\rho - 2\lambda} \in (0, \pi)$ for $\lambda \in J_\rho$, so that we can rewrite (A.1) as

$$\sqrt{2\lambda} \geq -\sqrt{\rho - 2\lambda}\cot(\sqrt{\rho - 2\lambda}). \tag{A.2}$$

Now the right-hand side is a decreasing function of λ on the interval $J'_\rho := (0 \vee \frac{1}{2}(\rho - \pi^2), \frac{1}{2}(\rho - \rho_c))$, and the left-hand side an increasing function of λ. Admit for the moment that $\lambda_c = \lambda_c(\rho)$ as defined in the statement of the theorem yields equality in (A.2) and that $\lambda_c \in J'_\rho$. It then follows that $\Lambda_\rho \cap J'_\rho = [\lambda_c, \frac{1}{2}(\rho - \rho_c)]$, and in particular, $\lambda_c = \inf \Lambda_\rho$, which was to be proven.

It remains to show that $\lambda_c \in J'_\rho$ and that λ_c yields equality in (A.2). Note that $g^{-1}(\rho) \in (\rho_c, \pi^2)$ for all $\rho > \rho_c$, whence $\lambda_c \in (\frac{1}{2}(\rho - \pi^2), \frac{1}{2}(\rho - \rho_c))$. Furthermore, $g^{-1}(\rho) < \rho$ for all $\rho > \rho_c$, since $g(\rho) > \rho$ for all $\rho > \rho_c$ by the properties of g stated in the theorem. Hence, $\lambda_c > 0$. It follows that $\lambda_c \in J'_\rho$.

On the interval J'_ρ, the right-hand side of (A.2) is positive, whence equality holds in (A.2) if and only if

$$2\lambda = (\rho - 2\lambda)\cot(\sqrt{\rho - 2\lambda})^2 = (\rho - 2\lambda)(\sin(\sqrt{\rho - 2\lambda})^{-2} - 1) = g(\rho - 2\lambda) - (\rho - 2\lambda),$$

whence, if and only if

$$g(\rho - 2\lambda) = \rho,$$

which is exactly satisfied for $\lambda = \lambda_c$. This finishes the proof.

B Appendix: estimates on the eigenvectors

In this section, we compare the eigenvectors and their L^2-norms under the assumption (H_{psh}). In this case we know that $K \geqslant 1$ in Lemma 2.1.

L^2-norms of the eigenvectors

Lemma B.1. *If $K \geqslant 1$, there exists $C > 0$, such that for L large enough,*

$$\|v_1\|_{L^2}^2 \leqslant C.$$

Proof. This result directly ensues from Lemma 2.3. □

Lemma B.2. *There exists $C > 0$ such that for L large enough and $k \in [\![1, K]\!]$,*

$$\|v_k\|^2 \geqslant C. \tag{B.1}$$

Proof. For $1 \leqslant k \leqslant K$,

$$\|v_k\|^2 = \frac{1 - \frac{\sin(2\sqrt{\rho-1-2\lambda_k})}{2\sqrt{\rho-1-2\lambda_k}}}{2\sin(\sqrt{\rho-1-2\lambda_k})^2} + \frac{\frac{\sinh(2\sqrt{2\lambda_k}(L-1))}{2\sqrt{2\lambda_k}} - (L-1)}{2\sinh(\sqrt{2\lambda_k}(L-1))^2}.$$

Remark that the second term of the sum is non negative by convexity of the function sinh. Besides, according to Lemma 2.1, $\sqrt{\rho-1-2\lambda_k} \in \left(\left(k-\frac{1}{2}\right)\pi, k\pi\right)$ for L large enough, which implies that

$$1 - \frac{\sin(2\sqrt{\rho-1-2\lambda_k})}{2\sqrt{\rho-1-2\lambda_k}} \geqslant 1 - \frac{1}{2\sqrt{\rho-1-2\lambda_k}} \geqslant 1 - \frac{1}{2(k-1/2)\pi} \geqslant 1 - \frac{1}{\pi},$$

so that

$$\|v_k\|^2 \geqslant \frac{1}{2}\left(1-\frac{1}{\pi}\right)\frac{1}{\sin(\sqrt{\rho-1-2\lambda_k})^2} \geqslant \frac{1}{2}\left(1-\frac{1}{\pi}\right).$$

□

Lemma B.3. *There exists a constants $C > 0$ such that for L large enough and $k > K$,*

$$\|v_k\|^2 \geqslant \frac{C}{\sin(\sqrt{-2\lambda_k}(L-1))^2 \wedge \sin(\sqrt{\rho-1-2\lambda_k})^2}.$$

Proof. For $k > K$

$$\|v_k\|^2 = \frac{1 - \frac{\sin(2\sqrt{\rho-1-2\lambda_k})}{2\sqrt{\rho-1-2\lambda_k}}}{2\sin(\sqrt{\rho-1-2\lambda_k})^2} + \frac{(L-1) - \frac{\sin(2\sqrt{-2\lambda_k}(L-1))}{2\sqrt{-2\lambda_k}}}{2\sin(\sqrt{-2\lambda_k}(L-1))^2}.$$

Both terms are non negative. Besides, since $\lambda_k < 0$, we have $\sqrt{\rho-1-2\lambda_k} > \sqrt{\rho-1} > \frac{\pi}{2} > 0$ and there exists $C > 0$ such that

$$1 - \frac{\sin(2\sqrt{\rho-1-2\lambda_k})}{2\sqrt{\rho-1-2\lambda_k}} \geqslant C,$$

for all $k > K$. Besides, recall from Lemma 2.1 that $\lambda_k < -a_1$ for all $k > K$. Therefore, we have

$$\left| \frac{\sin(2\sqrt{-2\lambda_k}(L-1))}{2\sqrt{-2\lambda_k}} \right| \leqslant \frac{1}{2\sqrt{a_1}},$$

and,

$$L - 1 - \frac{\sin(2\sqrt{-2\lambda_k}(L-1))}{2\sqrt{-2\lambda_k}} > L - 1 - \frac{1}{2\sqrt{a_1}} > L - 1 - \frac{1}{\pi}(L-1) = \left(1 - \frac{1}{\pi}\right)(L-1).$$

Thus, we have for L large enough and $k > K$,

$$\|v_k\|^2 \geqslant C \left(\frac{L-1}{2\sin(\sqrt{-2\lambda_k}(L-1))^2} \vee \frac{1}{2\sin(\sqrt{\rho - 1 - 2\lambda_k})^2} \right).$$

□

Upperbound on v_k for $k \leqslant K$

Lemma B.4. *There exists $C > 0$ such that for L large enough, $x \in [0,1]$ and $k \leqslant K$,*

$$|v_k(x)| \leqslant C v_1(x).$$

Proof. Let $x \in [0,1]$. Note that for $k \leqslant K$, $\sqrt{\rho - 1 - 2\lambda_k} \in \left((k - \frac{1}{2})\pi, k\pi\right)$ and

$$|v_k(x)| \leqslant \frac{\sqrt{\rho - 1 - 2\lambda_k}}{\sin(\sqrt{\rho - 1 - 2\lambda_k})}\, x.$$

Then, one can prove with a similar argument as in Remark 2.2 that $\frac{\sqrt{\rho-1-2\lambda_k}}{\sin(\sqrt{\rho-1-2\lambda_k})}$ is bounded by a positive constant. Finally, remark that

$$v_1(x) \geqslant x,$$

since $\sqrt{\rho - 1 - 2\lambda_1} \in (\frac{\pi}{2}, \pi)$ for L large enough. This concludes the proof of the lemma. □

Lemma B.5. *There exists $C > 0$ such that for L large enough, $x \in [1, L]$ and $k \leqslant K$,*

$$v_k(x) \leqslant C e^{\beta L} v_1(x).$$

Proof. Let $x \in [1, L]$. Since sinh is convex on $(0, \infty)$, we have

$$v_k(x) = \frac{\sinh\left(\sqrt{2\lambda_k}(L-x)\right)}{\sinh\left(\sqrt{2\lambda_k}(L-1)\right)} \leqslant \frac{L-x}{L-1} \leqslant L - 1.$$

Besides, we also have, for L large enough,

$$v_1(x) \geqslant \frac{\sqrt{2\lambda_1}(L-x)}{\sinh(\sqrt{2\lambda_1}(L-1))} \geqslant C e^{-\sqrt{2\lambda_1^\infty} L}(L-x), \tag{B.2}$$

according to Remark 2.1. Finally, we get that for L large enough,

$$v_k(x) \leqslant C e^{\beta L} v_1(x).$$

□

Corollary B.1. *There exists $C > 0$ such that for L large enough, $x \in [0, L]$ and $k \leqslant K$,*

$$|v_k(x)| \leqslant Ce^{\beta L} v_1(x).$$

Upperbound on v_k for $k > K$.

Lemma B.6. *For L large enough, $x \in [0, 1]$ and $k \in \mathbb{N}$, we have*

$$|v_k(x)| \leqslant \left| \frac{\sqrt{\rho - 1 - 2\lambda_k}}{\sin(\sqrt{\rho - 1 - 2\lambda_k})} \right| v_1(x).$$

Proof. As in the proof of Lemma B.4, we use that for $x \in [0, 1]$, we have $v_1(x) \geqslant x$ and that

$$|v_k(x)| \leqslant \frac{\sqrt{\rho - 1 - 2\lambda_k}}{|\sin(\sqrt{\rho - 1 - 2\lambda_k})|}\, x.$$

□

Lemma B.7. *For L large enough, $x \in [1, L]$ and $k > K$,*

$$|v_k(x)| \leqslant L \frac{\sqrt{\rho - 1 - 2\lambda_k}}{|\sin(\sqrt{-2\lambda_k}(L - 1))|} \sin(\pi x / L). \tag{B.3}$$

Proof. Let $x \in [1, L]$. Again, we use a convexity argument to claim that

$$|v_k(x)| \leqslant \frac{\sqrt{-2\lambda_k}}{|\sin(\sqrt{-2\lambda_k}(L - 1)|}\, (L - x),$$

and that

$$\sin(\pi x / L) = \sin(\pi (L - x) / L) \geqslant \sin(\pi / L)(L - x) \geqslant \frac{L - x}{L},$$

for L large enough. Therefore, we get that

$$|v_k(x)| \leqslant L \frac{\sqrt{-2\lambda_k}}{|\sin(\sqrt{-2\lambda_k}(L - 1)|} \sin(\pi x / L),$$

and we conclude the proof of the lemma by remarking that $\sqrt{-2\lambda_k} \leqslant \sqrt{\rho - 1 - 2\lambda_k}$ since $\lambda_k < 0$. □

Lemma B.8. *There exists $C > 0$ such that for all L large enough and $x \in [1, L]$,*

$$\sin(\pi x / L) \leqslant C L^{-1} e^{\beta L} v_1(x). \tag{B.4}$$

Proof. Let $x \in [1, L]$. Note that

$$\sin(\pi x / L) = \sin(\pi (L - x) / L) \leqslant \pi \frac{L - x}{L},$$

and recall from Equation (B.2) that for L large enough, we have

$$v_1(x) \geqslant Ce^{-\sqrt{2\lambda_1^\infty}L}(L-x).$$

Combining these two estimates, we get that

$$\sin(\pi x/L) \leqslant CL^{-1}e^{-\sqrt{2\lambda_1^\infty}L},$$

which concludes the proof of the lemma. $\square$

Corollary B.2. *There exists $C>0$ such that for L large enough, $x \in [1,L]$ and $k>K$,*

$$|v_k(x)| \leqslant C\frac{\sqrt{\rho-1-2\lambda_k}}{|\sin(\sqrt{-2\lambda_k}(L-1))|}e^{\beta L}v_1(x).$$

Corollary B.3. *There exists $C>0$ such that for L large enough, $x \in [0,L]$ and $k>K$,*

$$|v_k(x)| \leqslant \frac{C\sqrt{\rho-1-2\lambda_k}}{|\sin(\sqrt{\rho-1-2\lambda_k})| \wedge |\sin(\sqrt{-2\lambda_k}(L-1))|}e^{\beta L}v_1(x).$$

C Appendix: the Green function

Lemma C.1. *There exist $C>0$ and $\delta>0$ such that for L sufficiently large, $x \in [0,L]$ and $\xi \in (0,\delta)$,*

$$\psi_{\lambda_1^\infty+\xi}(x) \leqslant C\sinh\left(\sqrt{2(\lambda_1^\infty+\xi)}(L-x)\right), \tag{C.1}$$

$$\varphi_{\lambda_1^\infty+\xi}(x) \leqslant C(1\wedge x)\left(f(\lambda_1^\infty+\xi)e^{\sqrt{2(\lambda_1^\infty+\xi)}(x-1)} + g(\lambda_1^\infty+\xi)e^{-\sqrt{2(\lambda_1^\infty+\xi)}(x-1)}\right). \tag{C.2}$$

Proof. According to Equation (2.37), it is sufficient to prove that (C.1) holds in $[0,1]$. Yet, for $x \in [0,1]$, we have

$$\psi_{\lambda_1^\infty+\xi}(x) \leqslant |\psi_{\lambda_1^\infty+\xi}(x)| \leqslant \left(1+\frac{\sqrt{2(\lambda_1^\infty+\xi)}}{\sqrt{\rho-1-2(\lambda_1^\infty+\xi)}}\right)e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-1)}. \tag{C.3}$$

Since $\sqrt{\rho-1-2\lambda_1^\infty} \in (\pi/2,\pi)$, there exists $\delta>0$ such that

$$\pi/2 < \sqrt{\rho-1-2(\lambda_1^\infty+\xi)} < \pi, \tag{C.4}$$

and

$$\sqrt{0\wedge(\rho-1-\pi^2)} \leqslant \sqrt{2(\lambda_1^\infty+\xi)} \leqslant \sqrt{\rho-1-\pi^2/4},$$

for all $|\xi|<\delta$. Therefore, for all $\xi \in (-\delta,\delta)$, we have

$$\frac{\sqrt{2(\lambda_1^\infty+\xi)}}{\sqrt{\rho-1-2(\lambda_1^\infty+\xi)}} \leqslant 2\frac{\sqrt{\rho-1-\pi^2/4}}{\pi}. \tag{C.5}$$

Moreover, we know that for $x \in [0,1]$ and $\xi \in (0,\delta)$,

$$\begin{aligned}\sinh\left(\sqrt{2(\lambda_1^\infty+\xi)}(L-x)\right) &\geqslant \sinh\left(\sqrt{2(\lambda_1^\infty+\xi)}(L-1)\right)\\ &= \frac{1}{2}(1-e^{-2\sqrt{2\lambda_1^\infty+\xi}(L-1)})e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-1)}\\ &\geqslant \frac{1}{2}(1-e^{-2\sqrt{2\lambda_1^\infty}(L-1)})e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-1)}\\ &\geqslant \frac{1}{4}e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-1)},\end{aligned}$$

for L large enough (that does not depend on δ). This estimate, combined with Equations (C.3) and (C), concludes the proof of (C.1).

In order to prove (C.2), we use that $f(\lambda_1^\infty)=0$, $f'(\lambda_1^\infty)>0$ and $g(\lambda_1^\infty)>0$. Therefore, without loss of generality, we have $f(\lambda_1^\infty+\xi)>0$ and $g(\lambda_1^\infty+\xi)>\frac{1}{2}g(\lambda_1^\infty)$ for all $\xi \in (0,\delta)$. Thus, for $x\in[0,1]$ and $\xi\in(0,\delta)$, we have

$$f(\lambda_1^\infty+\xi)e^{\sqrt{2(\lambda_1^\infty+\xi)}(x-1)}+g(\lambda_1^\infty+\xi)e^{-\sqrt{2(\lambda_1^\infty+\xi)}(x-1)} \geqslant \frac{g(\lambda_1^\infty)}{2}. \tag{C.6}$$

Besides, combining Equations (2.36) and (C.4), we obtain that for $x\in[0,1]$ and $\xi\in(0,\delta)$,

$$\varphi_{\lambda_1^\infty+\xi}(x)=\sin(\sqrt{\rho-1-2(\lambda_1^\infty+\xi)}\,x)\leqslant\sqrt{\rho-1-2(\lambda_1^\infty+\xi)}x\leqslant\pi x.$$

This equation, combined with (C.6), implies that (C.2) holds on $[0,1]$ for any $C>\frac{2\pi}{g(\lambda_1^\infty)}$. Finally, note that for $\xi\in(0,\delta)$,

$$\frac{1}{\sqrt{2(\lambda_1^\infty+\xi)}}<\frac{1}{\sqrt{2\lambda_1^\infty}},$$

so that (C.2) holds on $[0,L]$ for any $C>\max\left(\frac{1}{\sqrt{2\lambda_1^\infty}},\frac{2\pi}{g(\lambda_1^\infty)}\right)$. □

Proof of Lemma 2.6. Since

$$\sqrt{2(\lambda_1^\infty+\xi)}-\sqrt{2\lambda_1^\infty}=\frac{2\xi}{\sqrt{2\lambda_1^\infty}+\sqrt{2(\lambda_1^\infty+\xi)}}\sim\frac{1}{\sqrt{2\lambda_1^\infty}}\xi,$$

as $L\to\infty$, we know that for L large enough (that does not depend on x), we have

$$e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-x)}\leqslant e^{\sqrt{2\lambda_1^\infty}(L-x)}e^{\frac{2}{\sqrt{2\lambda_1^\infty}}\xi(L-x)}.$$

Yet, $\xi(L-x)\leqslant\xi L$ uniformly tends to 0 as $L\to\infty$. Therefore, for L large enough (which does not depend on x), we have

$$e^{\sqrt{2\lambda_1^\infty}(L-x)}\leqslant e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-x)}\leqslant 2e^{\sqrt{2\lambda_1^\infty}(L-x)}. \tag{C.7}$$

We then use that $g(\lambda_1^\infty)>0$, $f(\lambda_1^\infty)=0$ and $f'(\lambda_1^\infty)>0$ to claim that

$$0<\frac{1}{2}g(\lambda_1^\infty)<\quad g(\lambda_1^\infty+\xi(L))\quad<2g(\lambda_1^\infty), \tag{C.8}$$

$$\frac{1}{2}f'(\lambda_1^\infty)\xi(L)<\quad f(\lambda_1^\infty+\xi(L))\quad<2f'(\lambda_1^\infty)\xi(L), \tag{C.9}$$

for L large enough. Thus, combining the definition of the Wronskian (2.35) and Equations (C.7), (C.8) and (C.9), we get that for L large enough,

$$\omega_{\lambda_1^\infty+\xi} > f(\lambda_1^\infty+\xi)e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-1)} > Cf'(\lambda_1^\infty)\xi(L)e^{\sqrt{2\lambda_1^\infty}L}.$$

Then, Equation (C.7) applied to $x=1$, divided by Equation (C.7) implies that for L large enough,

$$\frac{1}{2}e^{\sqrt{2\lambda_1^\infty}(x-1)} \leqslant e^{\sqrt{2(\lambda_1^\infty+\xi)}(x-1)} \leqslant 2e^{\sqrt{2\lambda_1^\infty}(x-1)}.$$

This inequality combined with Equations (C.2) from Lemma C.1, (C.9) and (C.8) yields the expected control on $\varphi_{\lambda_1^\infty+\xi}$.

The estimate on $\psi_{\lambda_1^\infty+\xi}$ can be easily deduced from Equations (C.1) from Lemma C.1 and from Equation (C.7) on $[0,L-1]$. For $x\in[L-1,L]$, we use that

$$\sinh\left(\sqrt{2(\lambda_1^\infty+\xi)}(L-x)\right) \leqslant \sinh\left(\sqrt{2(\lambda_1^\infty+\xi)}\right)(L-x) \leqslant C(L-x),$$

for L large enough. Therefore, we have according to Equation (C.1) that for L large enough and $x\in[L-1,L]$,

$$\psi_{\lambda_1^\infty+\xi}(x) \leqslant C(L-x) \leqslant C(L-x)e^{\sqrt{2\lambda_1^\infty}(L-x)},$$

which concludes the proof of the lemma. □

Proof of Lemma 2.7. The proof is similar to the one of Lemma 2.6 except that

$$\sqrt{2(\lambda_1^\infty+\xi)} - \sqrt{2(\lambda_1^\infty+\xi)} \sim \frac{1}{\sqrt{2\lambda_1^\infty}}\frac{h}{L},$$

as $L\to\infty$, so that for L large enough (that does not depend on x), we have

$$e^{\sqrt{2(\lambda_1^\infty+\xi)}(L-x)} \leqslant e^{\sqrt{2\lambda_1^\infty}(L-x)}e^{\frac{2}{\sqrt{2\lambda_1^\infty}}h}.$$

□

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List of Figures

ECOLE
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Titre : Dynamique d'interfaces en écologie : modèles déterministes et stochastiques

Mots clés : réaction-diffusion, modèle structuré en espace, modèle individu-centré, marche aléatoire branchante, mouvement brownien branchant

Résumé : Les ondes progressives générées par les équations de réaction-diffusion peuvent modéliser divers phénomènes observés en physique et en biologie. Du point de vue de la biologie, une onde progressive peut être interprétée comme l'invasion d'un habitat par une espèce. Les systèmes biologiques étant finis et donc soumis à des fluctuations démographiques, ces fronts d'onde déterministes ne représentent qu'une approximation de la véritable dynamique de la population, dans laquelle on postule que la densité locale d'individus est infinie de sorte que les fluctuations se compensent. En ce sens, certaines équations de réaction-diffusion peuvent être considérées comme les limites hydrodynamiques de modèles individus centrés. Dans cette thèse, nous étudions le comportement en temps long de deux systèmes microscopiques finis modélisant de telles invasions et les comparons à celui de leurs limites en grande population.

La première partie de cette thèse est consacrée à l'impact d'un milieu variant lentement sur la vitesse de propagation d'une population. Cette question a déjà été étudiée dans le domaine des EDP. Cependant, dans certaines situations, les résultats obtenus par la théorie des solutions de viscosité s'avèrent inexploitables du point de vue de la biologie. Nous proposons donc de revenir à un modèle individu-centré et d'étudier son comportement limite lorsque l'échelle d'hétérogénéité de l'environnement tend vers l'infini. Dans ce cadre, nous montrons que la vitesse de propagation de la population peut être beaucoup plus lente que la vitesse du front de l'EDP décrivant le comportement du système en grande population. Cette différence de comportements est liée au "problème des queues" observé en EDP, dû à l'absence d'extinction locale dans les équations de type FKPP.

Dans une seconde partie, nous étudions l'impact du type des fronts d'onde limites sur les modèles stochastiques associés pour expliquer ce net ralentissement des systèmes de particules. En effet, les fronts d'onde issus des équations de réaction-diffusion monostable sont divisés en deux types : les fronts tirés et les fronts poussés. Il est bien connu que les fluctuations ont un impact considérable sur les fronts tirés. En revanche, les ondes poussées sont supposées être moins sensibles aux perturbations. Néanmoins, de récentes simulations numériques sur des équations bruitées suggèrent l'existence d'une troisième classe de fronts. Il s'agit d'une sous-classe de fronts poussés particulièrement sensibles aux fluctuations. Dans cette thèse, nous étudions les mécanismes internes de ces fronts pour expliquer la transition entre ces trois régimes.

Title : Spatial dynamic of interfaces in ecology : deterministic and stochastic models

Keywords : reaction-diffusion equation, individual-based model, heterogeneous environment, branching random walk, branching Brownian motion

Abstract : Traveling waves arising from reaction diffusion equations explain various phenomena observed in physics and biology. From a biological standpoint, a traveling wavefront can be seen as the invasion of an uninhabited environment by a species. Since biological systems are finite and thus undergo demographic fluctuations, these deterministic wavefronts only represent an approximation of the population dynamics, in which we consider that the local density of individuals is infinite so that the fluctuations self-average. In this sense, reaction diffusion equations can be seen as hydrodynamic limits of some individual based models. In this thesis, we investigate the long time behaviour of some finite microscopic systems modeling such front propagations and compare them to the one of their large population asymptotics.

The first part of this thesis is dedicated to the impact of a slowly varying media on the propagation speed of a population. This question has been widely studied from the PDE point of view. However, the results given by the viscosity solutions theory turns out to be biologically unsatisfactory in some situations. We thus suggest to study an individual based model for front propagation in the limit when the scale of heterogeneity of the environment tends to infinity. In this framework, we show that the spreading speed of the population may be much slower than the speed of the front in the PDE describing the large population asymptotics of the system. This qualitative disagreement between the two behaviours is related to the so-called tails problem observed in PDE theory, due to the absence of local extinction in FKPP-types equations.

In a second part, we study the impact of the type of the deterministic waves on the related stochastic models to explain this drastic slow-down in the particle system. Indeed, wavefronts arising from monostable reaction diffusion PDE are classified in two types : pulled and pushed waves. It is well-known that small perturbations have a huge impact on pulled waves. In sharp contrast, pushed waves are expected to be less sensitive. Nevertheless, some recent numerical experiments have suggested the existence of a third class of waves in stochastic fronts. It is a subclass of pushed fronts very sensitive to fluctuations. In this thesis, we investigate the internal mechanisms of such fronts to explain the transition between these three regimes.

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