Anosov flows and Birkhoff sections

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Flots d'Anosov et sections de Birkhoff

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Résumé

Nous étudions les flots d'Anosov en dimension 3. Ces flots ont des dynamiques chaotiques intéressantes, plus précisément ils ont des comportements hyperboliques aux voisinages de leurs orbites. Pour comprendre ces flots, nous utilisons des surfaces transverses aux flots, appelées sections de Birkhoff. Via l'application dite de premier retour sur une section de Birkhoff, la dynamique du flot est partiellement encodée par la dynamique d'un homéomorphisme d'une surface. Cette dynamique discrète est alors en dimension 2.

Dans une première partie, nous calculons explicitement les applications de premier retour d'une famille de sections de Birkhoff à bord fixé. Cela permet de comparer ces applications de premier retour sur plusieurs sections de Birkhoff. Dans une seconde partie nous étudions le bord des sections de Birkhoff et leurs orientations. Nous interprétons une section de Birkhoff comme un cobordisme transverse au flot, de son bord positif vers son bord négatif. Deux notions naturelles apparaissent alors : les flots vrillés (qui admettent un cobordisme transverse nul) et les orbites primitives de ces flots (qui ne sont pas des bords positifs de cobordisme transverse). Ces notions que nous allons étudier contiennent des informations sur la topologie du flot et de la variété ambiante.

Abstract

We study the Anosov flows, in dimension 3. These flows have interesting chaotic behaviors, more precisely they have hyperbolic behaviors on the neighborhood of there orbits. To understand these flows, we use some surfaces transverse to the flows, called Birkhoff sections. Thanks to the so called first return map on one Birkhoff section, the dynamic of the flow is partially encoded by the dynamic of a homeomorphism of a surface. This discrete dynamic being in dimension 2.

In a first part, we explicitly compute the first return maps of a family of Birkhoff sections with a fixed boundary. It allows one to compare the first return maps of the flow on several Birkhoff sections. In a second part, we study the boundaries of the Birkhoff sections and their orientations. A Birkhoff section is interpreted as a transverse cobordism of the flow, from its positive boundary to its negative boundary. Two natural notions appear: the twisted flows (which admit some transverse null-cobordism) and the primitive orbits of these flows (which are the positive boundary of no transverse cobordism). We study these notions, which contain some information on the topology of the flow and of the global manifold.

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Contents

Introduction

Les flots d'Anosov forment une famille de flots intéressants pour leur propriétés dynamiques et topologiques. Ces flots satisfont une propriété hyperbolique : ils laissent invariants deux feuilletages transverses, dits stable et instable, sur lesquels ils ont un comportement respectivement contractant et dilatant. D.Anosov a formalisé cette notion dans les années 60, pour comprendre les flots géodésiques sur les surfaces hyperboliques (voir [Ano67]). La classification des flots d'Anosov et des variétés portant de tels flots sont des problèmes encore largement ouverts. En dimension au moins 4, si la direction stable (ou instable) est de dimension 1 et si le groupe fondamental de la variété est polycyclique, alors le flot est une suspension d'après J.F.Plante (voir [Pla81, Bar92]). Les flots géodésiques sont des contre-exemples à ce théorème en dimension 3, et nous allons nous concentrer sur les variétés orientables de dimension 3. Deux grandes familles de flots d'Anosov, dits flots algébriques, sont restées quelques années les seuls exemples connus : les suspensions de difféomorphisme Anosov, et les flots géodésiques sur une surface hyperbolique.

W.Thurston et M.Handel ont construit des flots d'Anosov transitifs non algébriques [HT80], en utilisant des techniques de chirurgie. D.Fried [Fri83] et S.Goodman [Goo83] généralisèrent ce résultat grâce à deux classes de chirurgies le long d'orbites fermées du flot. La question naturelle formulée indépendamment par E.Ghys et D.Fried est de savoir si tout flot d'Anosov transitif peut être changé en une suspension via une suite de chirurgies de Fried ou Goodman. M.Shannon montra récemment que si le flot est transitif, les chirurgies de Fried et Goodman produisent des flots Anosov orbitalement équivalents (voir [Sha20]). On sait désormais passer d'un flot géodésique à une suspension, et d'une suspension à feuilletages orientables à toute suspension à feuilletages orientables, en utilisant des chirurgies de Fried-Goodman (voir [Hir] pour les suspensions et [DS19] pour les flots géodésiques). Les questions de D.Fried et E.Ghys restent irrésolues à ce jour.

D'autres familles de flots d'Anosov ont été mieux comprises depuis. Un premier exemple de flot d'Anosov non transitif a été construit par F.Franks et B.Williams [FW80]. M.Brunella montra que dans une 3-variété orientable, les pièces basiques des flots d'Anosov non-transitif sont séparées par des tores incompressibles transverses aux flots [Bru93]. Bonatti-Langevin construisirent ensuite un exemple de flot d'Anosov transitif admettant un tore transverse, mais n'étant pas une suspension [BL94]. C.Bonatti, F.Beguin et B.Yu ont généralisé cette idée pour décomposer [BBY16] ou construire [BBY17] tout flot d'Anosov en pièces basiques, collées le long de tores transverses et de bouteilles de Klein transverses.

La topologie des flots peut aussi être étudiée à travers la théorie des feuilletages appliquée aux feuilletages stable et instable du flot. S.Fenley et T.Barbot ont pris cette approche dans les années 90 et ont étudié l'espace des orbites des flots d'Anosov [Fen94]. Cet espace est un plan topologique, muni de deux feuilletages transverses. La topologie de ce plan bi-feuilleté contient beaucoup d'informations sur la dynamique du flot. En particulier l'action du groupe fondamental de la variété ambiante, sur l'espace des orbites, caractérise complètement la variété ambiante et le flot d'origine, à équivalence orbitale près (voir Théorème 4.6 de [Bar95a]). Quand les feuilletages stable et instable vérifient la propriété d'être alignable ($\mathbb{R}$-covered en anglais), il existe deux modèles de plan bi-feuilleté pour l'espace des orbites (voir ref S.Fenley ou T.Barbot). De ces modèles de plan bi-feuilleté peuvent être extraites certaines propriétés sur les orbites fermées du flot. Par exemple dans le cas appelé alignable penché, T.Barthelmé et S.R.Fenley ont montré que deux orbites homotopes sont isotopes [BF13]. Les flots alignables penchés seront un des objets d'étude important pour la deuxième moitié de cette thèse.

Un flot dans une variété M peut être étudié via des sections globales ou des sections de Birkhoff. Une section globale est une sous-variété de codimension 1, transverse au flot et qui intersecte toutes les orbites du flot. Une telle section permet de définir une application de premier retour, qui est un homéomorphisme de la section. La dynamique du flot est alors grandement contenue dans la dynamique de cette application. D'après S.Schwartzman puis D.Fried, l'existence de sections globales peut être déterminée via une condition homologique, à coefficients entiers (voir [Sch57, Fri82b]). Plus précisément les sections globales sont classifiées par leurs classes d'homologie,

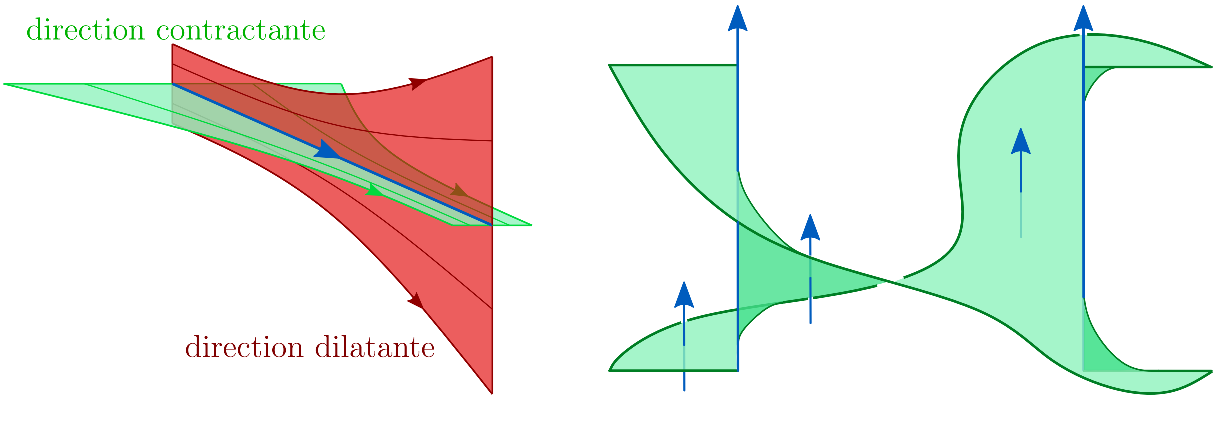


Figure 1: Illustration locale d'un flot d'Anosov à gauche, et d'une section de Birkhoff à droite.

qui vivent dans un certain cone convexe de $H_2(M, \mathbb{Z})$.

Pour étudier un flot d'Anosov, il est préférable d'étudier des sections de Birkhoff, qui sont un assouplissement de la notion précédente autorisant les sous-variétés à bord. La caractérisation des sections de Birkhoff par leurs homologies relatives se généralise. L'existence de sections de Birkhoff n'est pas automatique en général, cependant une construction dûe à D.Fried [Fri83] montre qu'un flot d'Anosov transitif admet toujours des sections de Birkhoff. On peut noter que pour un flot non transitif il n'y a pas de section de Birkhoff comme définie dans la Section 1.3 (en effet l'application de premier retour serait pseudo-Anosov, et tous les difféomorphismes pseudo-Anosov sont transitifs [FLP12, Corollaire 9.19]).

La dynamique du flot est aussi partiellement contenue dans les applications de premier retour sur les sections de Birkhoff, ces applications étant pseudo-Anosov d'après D.Fried. Le facteur d'expansion de ces applications a été étudié entre autres par D.Fried et C.McMullen. D.Fried montra une propriété de convexité de ce facteur d'expansion (une fois renormalisé), ainsi que sa divergence vers $+\infty$ quand l'homologie de la section de Birkhoff se rapproche du bord du cone convexe dans $H_2(M, \mathbb{Z})$. C.McMullen a construit [McM00] un polynôme à exposant dans l'homologie entière, tel qu'une fois évalué en l'homologie d'une section de Birkhoff, sa plus grande racine est le facteur d'expansion de l'application de premier retour de cette section. Cette construction a depuis été beaucoup utilisée pour l'étude des pseudo-

Anosov.

Tandis que les facteurs d'expansion sont compris de façon uniforme sur l'ensemble des sections de Birkhoff à bord fixé, leurs applications de premier retour ne le sont pas. Dans une partie de cette thèse, nous allons étudier les applications de premier retour pour certaines sections de Birkhoff, et relier les premiers retour à l'homologie de ces sections.

Trois questions ont été des motivations pendant ma thèse :

- Étant donné un flot d'Anosov et l'ensemble de ses sections de Birkhoff avec un bord fixé, peut-on calculer et comparer les applications de premier retour sur ces sections ?

- Étant donné un flot (préférablement d'Anosov), quelles informations sur la topologie du flot peut on obtenir en connaissant certaines de ces sections de Birkhoff ?

- Peut-on comprendre l'ensemble des flots d'Anosov à chirurgie de Fried-Goodman près ? (question de E.Ghys)

Nous allons décrire brièvement le contenu des chapitres, des dépendances entre les chapitres, et énoncer quelques théorèmes importants qui y sont démontrés.

Le Chapitre 1 regroupe principalement des notions et résultats classiques, ainsi que plusieurs constructions de surfaces transverses à un flot. Nous introduisons les flots géodésiques qui auront un rôle particulier pendant la suite de la thèse (le second chapitre étudie des sections de Birkhoff pour ces flots, et ils seront une source d'exemples pour les autres chapitres). L'espace des orbites d'un flot d'Anosov ainsi que les flots alignables seront ensuite introduits. Ces notions sont fortement liées à la topologie du flot et de la variété ambiante. Nous étudierons particulièrement ce lien dans les deux derniers chapitres.

Le Chapitre 1 introduit aussi plusieurs classes de surfaces transverses intéressantes, telles que les sections partielles et les sections de Birkhoff. Nous sommes particulièrement intéressés par les surfaces appelées sections partielles, dont l'intérieur est transverse au flot et le bord est tangent au flot. Les sections de Birkhoff sont celles qui de plus intersectent toutes les orbites en temps borné, l'application de premier retour étant alors bien définie. Le second chapitre étudie ces applications de premiers retour, et les deux derniers chapitres utilisent les sections partielles pour étudier la topologie des flots. Il sera donc important de pouvoir construire plusieurs familles de sections partielles : les sections à bord symétrique, les sections obtenues en désingularisant une géodésique, les anneaux de Birkhoff, les sections de Fried. Nous généralisons ensuite deux constructions de T.Barbot et D.Fried, en introduisant les domaines fondamentaux de sections partielles ramifiées. Cette construction rassemble les constructions de certains anneaux de Birkhoff et des sections de Fried.

Les chirurgies de Fried-Goodman seront introduites à la fin du chapitre 1 Ces chirurgies transforment une section de Birkhoff en une autre section de Birkhoff. La question de E.Ghys peut alors être exprimée en terme de chirurgie et en terme de section Birkhoff. Il y sera étudié l'action des chirurgies de Fried-Goodman sur les sections partielles et sur leurs bords. Ce lien sera important pour les deux derniers chapitres.

Dans le chapitre 2, nous fixons une surface hyperbolique S, et étudions le flot géodésique sur le fibré unitaire tangent T^1S, ainsi que les applications de premier retour sur certaines sections de Birkhoff. Une grande famille de sections de Birkhoff, dites à bord symétrique, a été introduite en 2016 par M.Cossarini et P.Dehornoy. Leur article contient la classification des sections de Birkhoff avec un certain bord fixé, chaque section correspondant à un point entier d'un polyèdre compact dans $H^1(S,\mathbb{Z})$. Nous calculons explicitement les applications de premier retour sur ces sections de Birkhoff, ce qui permet de les comparer. Cette partie est a été initialement pré-publiée dans l'article [Mar20].

Les sections de Birkhoff qui nous intéressent sont décrites via des données combinatoires, à savoir une famille finie de géodésiques et une coorientation dite Eulérienne de ce diagramme géodésique. Cette description nous permet d'avoir une approche algorithmique des applications de premier retour. Une idée importante est de définir une famille cyclique de sections de Birkhoff, telle que le flot induit des applications de retour partiel d'une section de Birkhoff vers la section suivante. Ces applications de retour partiel sont décrites combinatoirement, et sont simples à calculer. Après avoir détaillé la topologie et la combinatoire de ces sections de Birkhoff, nous prouverons le théorème suivant.

Théorème 1 (correspond au Théorème D)**.** *Soit S une surface fermée hyperbolique et considérons le flot géodésique sur T^1S. Pour une collection* *finie de géodésiques fermées* $\Gamma \subset S$, *notons* $\overleftrightarrow{\Gamma} \subset T^1S$ *la collection des* $2|\Gamma|$ *orbites du flot qui se projettent sur* Γ. *Il existe une surface à bord* Σ_Γ *et une collection de courbes simples* $\gamma_1, \cdots, \gamma_n$ *dans* Σ_Γ *qui satisfont la propriété suivante. Pour toute section de Birkhoff* Σ *à bord* $\overleftrightarrow{\Gamma}$ *et de multiplicité -1 le long de ses composantes de bord, il existe un difféomorphisme* $f : \Sigma \to \Sigma_\Gamma$ *tel que si* $r_\Sigma : \Sigma \to \Sigma$ *est l'application de premier retour sur* Σ, *alors* $f \circ r_\Sigma \circ f^{-1}$ *est isotope à un produit de twists de Dehn de la forme* $\tau^{-1}_{\gamma_{\sigma(1)}} \circ \cdots \circ \tau^{-1}_{\gamma_{\sigma(n)}}$, *pour une permutation* σ *de* $\{1, \cdots, n\}$. *De plus, les données* Σ_Γ, γ_i *et* σ *sont construites explicitement.*

Comme énoncé plus haut, D.Fried et C.McMullen ont décrit des propriétés uniformes sur le facteur d'expansion des sections de Birkhoff. Ce théorème permet de comprendre de façon uniforme les applications de premier retour, dans le cas des sections du flot géodésique.

Dans le Chapitre 3 nous étudions la nature de certains flots sur les 3-variétés orientées, en utilisant leurs sections partielles. La particularité de notre approche par rapport à la plupart des travaux antérieurs est de comparer les deux orientations sur les bords de ces sections: celles induites par le flot et celle induite par la coorientation de l'intérieur de la section par le flot. On parle de composante de bord positive ou négative selon que ces orientations coïncident ou non. Ce signe dépend de l'orientation de la variété ambiante. En particulier nous considérons les sections dites positives, c'est-à-dire dont toutes les composantes de bord sont positives, et les sections dites négatives dont toutes les composantes de bord sont négatives. Cette notion permet de décrire deux comportements topologiques de certains flots : plat et vrillé. Un flot sera dit topologiquement plat s'il admet une section globale, et topologiquement vrillé s'il admet une section de Birkhoff dont tous les bords ont la même orientation. Ces natures caractérisent quel type de sections partielles ces flots admettent.

Théorème 2 (correspond au Théoreme E)**.** *Soit* M *une* 3*-variété fermée et orientée, et* ϕ *un flot positivement topologiquement vrillé sur* M. *Alors* (M, ϕ) *n'admet pas de section partielle négative.*

Supposons que ϕ *est de plus Anosov, alors* (M, ϕ) *n'admet pas de section partielle sans bord.*

Soit ϕ *un flot topologiquement plat sur* M. *Alors* (M, ϕ) *n'admet pas de section partielle dont tous les bords ont la même orientation.*

Comme corollaire immédiat, être topologiquement plat, positivement topologiquement vrillé ou négativement topologiquement vrillé sont trois natures mutuellement exclues. Il y a ainsi une trichotomie pour les flots : plat / vrillé / autres flots. Nous montrons qu'elle reflète une trichotomie importante pour les flots d'Anosov : suspension / flot alignable penché / flot non alignable:

Théorème 3 (correspond au Théoreme G). *Soit M une 3-variété fermée orientée et ϕ un flot d'Anosov transitif sur M. Les propriétés suivantes sont équivalentes:*

- (M, ϕ) *est alignable et positivement penché.*
- (M, ϕ) *est positivement topologiquement vrillé.*
- (M, ϕ) *n'admet pas de section partielle négative ni de section partielle sans bord.*

Ce théorème est particulièrement intéressant pour comprendre la nature d'un flot obtenu par chirurgie de Fried-Goodman. En effet connaître assez de sections de Birkhoff du flot d'origine, et connaître l'action des chirurgies de Fried-Goodman sur ces sections peut permettre de déterminer la nature plate ou vrillée du flot après chirurgie. Cette idée sera explorée à la fin du dernier chapitre.

Le Chapitre 4 contient l'étude de certaines orbites d'un flot. On interprète une section partielle ayant un bord positif et un bord négatif comme un cobordisme transverse du bord positif vers le bord négatif. Deux familles d'orbites sont alors intéressantes : les orbites qui cobordent l'ensemble vide (ce qui est liée à la notion de flot vrillé), et les orbites qui ne sont le bord positif d'aucun cobordisme transverse. On appelle ces dernières orbites les orbites primitives. Nous étudierons deux variantes de cette notion : les orbites primitives et les familles d'orbites stablement primitives.

Théorème 4 (correspond au Théoreme I). *Soit M une 3-variété fermée orientée et ϕ un flot d'Anosov alignable et positivement penché. Alors pour toute orbite fermée γ, il existe un cobordisme transverse de γ vers une famille d'orbites primitives. Si de plus l'espace $H_1(M, \mathbb{Z})$ n'est pas réduit à zero, alors il existe une orbite primitive.*

Dans la suite du chapitre, nous caractérisons les orbites primitives pour les suspensions et les flots géodésiques. Pour un flot d'Anosov alignable positivement penché, aucune caractérisation simple des orbites primitives n'est espérée. Cependant pour ces flots, les familles d'orbites stablement primitives ont une caractérisation simple, qui utilise l'espace des orbites et les familles simples de losanges idéaux (définies dans les chapitres 1 puis 4). De plus, ces familles stablement primitives jouent un rôle important pour les chirurgies de Fried-Goodman.

Théorème 5 (correspond au Théoreme K). *Soit M une 3-variété fermée orientée, ϕ un flot d'Anosov alignable et positivement penché sur M dont les feuilletages stables et instables sont orientables, et Γ un ensemble d'orbites fermées de ϕ. On suppose que dans l'espace des orbites, il n'y a pas de losange idéal dont les deux coins sont des orbites de Γ. Alors il existe alors une équivalence:*

- $\Leftrightarrow$ Γ *est une famille stablement primitive,*
- $\Leftrightarrow$ $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ *est une famille simple de losanges idéaux.*
- $\Leftrightarrow$ $\{L^{-,-}(\gamma) | \gamma \in \Gamma\}$ *est une famille simple de losanges idéaux.*
- $\Rightarrow$ *Toute suite de chirurgies de Fried-Goodman sur Γ produit un flot alignable positivement penché.*

De plus la dernière implication est une équivalence quand $|\Gamma| = 1$.

L'étude des orbites primitives est encore en développement. En particulier leur lien avec la nature d'un flot après chirurgie devrait pouvoir être approfondie.

Chapter 1

Background and transverse surfaces

In this chapter, we review some classical notions and results about Anosov flows and Birkhoff sections. Suspensions flows and geodesic flows on hyperbolic surfaces will be introduced in Section 1.1.

We then introduce in Section 1.2 the orbit space of a flow, which is important to understand the topology of a flow. The connection between the orbit space and transverse surfaces is explained later in Section 1.7.

Several kinds of transverse surfaces will be used in later chapters, and are grouped in Section 1.3. We define the notions of partial sections, Birkhoff sections, global sections and transverse cobordisms. We also start developing the notions of linking number and of orientation of a boundary of a partial section, which will be important for the last two chapters. Global sections and Birkhoff sections satisfy a nice classification up to isotopy through the flow, using their homology with integer coefficients, it will be recalled in the same section. In Section 1.4, we study the behavior of the partial sections of Anosov flows, in a neighborhood of a boundary component.

In Section 1.5.1 we construct some partial sections and Birkhoff sections for the geodesic flows. Firstly we introduce the partial sections said to have symmetric boundaries, which are the main focus of the second chapter and are a source of examples for various phenomenons. Secondly we introduce some partial sections using multi-1-foliations on the surface, and especially some partial sections obtained by desingularising geodesics on the surface. This second kind of sections is a new construction, which will be used in the last chapter in order to characterize the closed orbits said to be primitive.

We introduce the Fried and the Goodman surgeries in Section 1.6. These surgeries change the flow and the underlying manifold, and they transform a Birkhoff section into another one, with a slightly different boundary. The orientation of the boundary of Birkhoff sections being the main focus of the last two chapters, Fried surgeries allow to construct new Anosov flows with some control on their topological nature.

Section 1.7 first reviews some properties of the trace of a partial section in the orbit space of an Anosov flow. Then we use this notion to construct several partial sections. We unify two constructions from T.Barbot and D.Fried, into the fundamental domain of a partial section in the orbit space. Then we apply it to construct immersed Birkhoff sections (construction of T.Barbot) and Fried sections.

1.1 Anosov flows

We introduce Anosov flows, some classical notions, and two important families of Anosov flows.

Definition 1.1.1. A flow ϕ in a closed 3-manifold M is said to be **Anosov** if there exists $A, B > 0$ and a splitting of TM into three line bundles that are ϕ-invariant $TM = E^{uu} \oplus X \oplus E^{ss}$, so that X generate the direction of the flow, and that for any metric on M, its norm $|.|$ satisfies $|(D\phi_t)_{|E^{ss}}| \leq A\exp(-Bt)$ for all $t \geq 0$ and $|(D\phi_t)_{|E^{uu}}| \leq A\exp(-B|t|)$ for all $t \leq 0$. This the manifold M compact, the choice of the norm $|.|$ does not affect that asymptotic property.

If ϕ is Anosov, the splitting is unique. Additionally $E^{ss} \oplus X$ and $E^{uu} \oplus X$ are integrable into two transverse 2-foliations denoted by $\mathcal{F}^s$ and $\mathcal{F}^u$, called the **stable foliation and unstable foliation**. For x a point or an orbit in M, we denote by $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ the stable and unstable leaves containing x. Each stable or unstable leaf is invariant under the flow. If a stable or unstable leaf admits a closed orbit, then it admits only one closed orbit and has the topology of either a cylinder or a Möbius strip. Otherwise it has the topology of a plane. Given an orientable manifold, an Anosov flow is easier to understand when its stable and unstable foliations are transversely orientable. When it is not the case, we lift the flow to the double covering manifold obtained as the orientations covering. Then the lifted flow admits transversely orientable stable and unstable foliations.

If (M, ϕ) and (N, ψ) are two flows, we call **orbital equivalence** any homeomorphism $f : M \to N$ sending the orbits of ϕ to the orbits of ψ, with the same orientations. We are mainly interested in topological information on Anosov flows which are invariant under orbital equivalence.

We focus on Anosov flows said to be **transitive**, that is which admit some dense orbits.

Example 1.1.2 (**Suspension flows**). *Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be a torus and $A \in GL_2(\mathbb{Z})$ a hyperbolic matrix with $|\det(A)| = 1$. That is either $\det(A) = 1$ and $|\operatorname{Tr}(A)| \geq 3$, or $\det(A) = -1$ and $\operatorname{Tr}(A) \neq 0$. Then A induces a diffeomorphism $A : \mathbb{T} \to \mathbb{T}$. We define the 3-manifold $\mathbb{T}_A = \mathbb{T}\times\mathbb{R}/(x, s+1) \simeq (Ax, s)$ with the Anosov flow ϕ given by $\phi_t(x, s) = (x, t + s)$. Because of the assumptions on $\det(A)$ and $\operatorname{Tr}(A)$, A has two eigenvalues λ and $\pm 1/\lambda$ so that $|\lambda| > 1$. Hence if E_λ and $E_{1/\lambda}$ are their two eigenspaces in $\mathbb{R}^2$, then the stable and unstable foliations of the suspension flow are made from parallel copies of the images of $E_{1/\lambda} \times \mathbb{R}$ and $E_\lambda \times \mathbb{R}$ in $\mathbb{T}_A$. If $f : \mathbb{T} \to \mathbb{T}$ is isotopic to the diffeomorphism $A : \mathbb{T} \to \mathbb{T}$ for such a hyperbolic matrix A and if the suspension flow induced by f is Anosov, then that suspension flow is orbitally equivalent to $\mathbb{T}_A$. Other suspensions flows can be defined, but only the ones presented here are Anosov.*

Example 1.1.3 (**Geodesic flows**). *Let S be a closed surface with a fixed hyperbolic metric, non necessarily orientable. We denote by $T^1S = \{u \in TS, \|u\| = 1\}$ the unitary tangent bundle over S, which is a closed orientable 3-manifold. The geodesic flow $\phi_t : T^1S \to T^1S$ is defined on a unitary vector $u \in T^1S$ by pushing u along the geodesic induced by u for a time t. The geodesic flow is Anosov. Up to orbital equivalence, it does not depend on the particular choice of hyperbolic metric (see the following Theorem). If H^2 is a hyperbolic plane given as the universal cover of S, a stable leaf of ϕ is given by the projection of the set of vectors in T^1H^2 whose induced oriented geodesic converges at plus infinity to a fixed point in ∂H^2. Similarly an unstable leaf of ϕ is given by the projection of the set of vectors in T^1H^2 whose induced oriented geodesic converges at minus infinity to a fixed point in ∂H^2.*

The geodesic flow is also well-defined over T^1S when S is an orbifold (the quotient of a surface by a discrete group, which is allowed to have some fixed points). We only mention these flows in some remarks.

A flow ϕ is said to be **topologically stable** if any small enough perturbation of ϕ induces a flow which is orbitally equivalent to ϕ. D.Anosov proved [Ano67] that the geodesic flows on hyperbolic manifolds are topologically stable. M.Gromov proved that the set of hyperbolic metrics of a given surface is connected (see [Gro00]). Additionally E.Ghys proved [Ghy84] that on the unitary bundle of a hyperbolic surface, if two metrics (non-necessarily hyperbolic) induce two Anosov flows, then these flows are orbitally equivalent. Hence on the unitary tangent bundle of a surface of genus at least 2, there is exactly one Anosov geodesic flow, up to orbital equivalence.

K.Kato, A.Morimoto generalised [KM73] the result of D.Anosov, by proving that all Anosov flows are topologically stable. Hence on a given manifold, the set of its Anosov flows is a discrete set, which justifies the focus on their topological properties.

1.2 Orbit space and R-covered flows

We review the concept of orbit space of an Anosov flow, the $\mathbb{R}$-covered property of an Anosov flow, and the ideal lozenges in an orbit space. These notions, as well as their first properties, were first considered by S.Fenley [Fen94] and T.Barbot [Bar95a].

Orbit space. Let ϕ be an Anosov flow on a 3-manifold M. We denote by $\tilde{M}$ the universal covering of M and $\tilde{\phi}$ the lift of ϕ to $\tilde{M}$. The **orbit space** of ϕ is defined as the set $\mathcal{O}(M) = \tilde{M}/\tilde{\phi}$ of orbits of $\tilde{\phi}$. Because the flow is Anosov, its orbit space is homeomorphic to $\mathbb{R}^2$ [Fen94]. We denote by $\pi : \tilde{M} \to \mathcal{O}(M)$ the projection. The natural action of $\pi_1(M)$ on $\tilde{M}$ commutes with the flow. So it induces an action of $\pi_1(M)$ on $\mathcal{O}(M)$.

The 2-foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ lift to foliations $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ in $\tilde{M}$, which then project to two transversal 1-dimensional foliations $\mathcal{L}^s$ and $\mathcal{L}^u$ in $\mathcal{O}(M)$. We also call **stable and unstable foliations** these 1-foliations. For a point x in the orbit space $\mathcal{O}(M)$, we denote by $\mathcal{L}^s(x)$ and $\mathcal{L}^u(x)$ the stable and unstable leaves containing x. Notice that if $o \in \mathcal{O}(M)$ represents a closed orbit γ, and if $g \in \pi_1(M)$ is the homotopy class of γ (such that $g.o = o$), then g acts on $\mathcal{O}(M)$ by contracting $\mathcal{L}^u$ and expanding $\mathcal{L}^s$ near o. The foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ on M are not necessarily transversely orientable. We only consider oriented 3-manifolds, so when one of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ is transversely orientable, the other is also transversely orientable. Plus in this case, the action of each element of $\pi_1(M)$ preserves the orientations of $\mathcal{L}^u$ and $\mathcal{L}^s$.

An Anosov flow is said to be **$\mathbb{R}$-covered** if the set of leaves of $\mathcal{L}^s$ (or $\mathcal{L}^u$) is homeomorphic to $\mathbb{R}$, or equivalently if this set of leaves is Hausdorff. The understanding of $\mathbb{R}$-covered Anosov flows is not complete, but some cases are well-understood. When the orbit space of ϕ is $\mathbb{R}^2$ bi-foliated horizontally and vertically by the stable and unstable foliations, Solodov proved that the flow is the suspension flow (see [Bar95a]). The orbit spaces (as bi-foliated planes) of $\mathbb{R}$-covered flows have been classified by S.Fenley into three types of orbit spaces, which are described in Figure 1.1.

Theorem 1.2.1 (S.Fenley [Fen94], T.Barbot [Bar95a])**.** *Let ϕ be an $\mathbb{R}$-covered Anosov flow on M. Then its orbit space $\mathcal{O}(M)$ is homeomorphic as a bi-foliated plane either to $\mathbb{R}^2$ or to $\{(x,y) \in \mathbb{R}^2, |y-x| \leq 1\}$, foliated by vertical and horizontal segments. The flow is a suspension flow in the first case, and it is said to be* ***skewed*** $\mathbb{R}$***-covered*** *in the second case.*

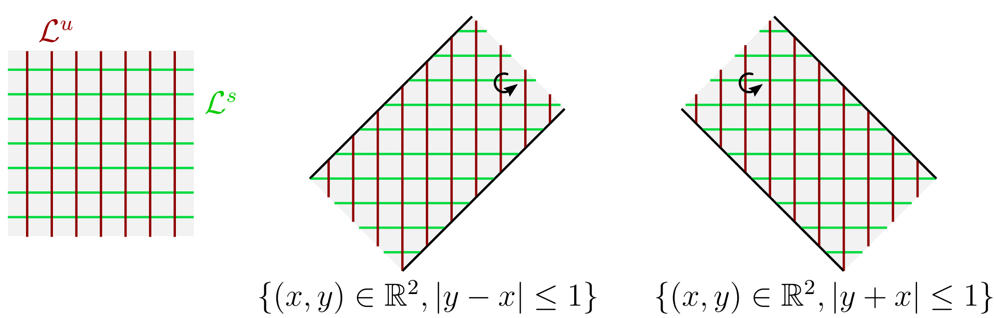


Figure 1.1: The transverse bi-foliation on the orbit space of a $\mathbb{R}$-covered Anosov flow. From left to right: suspension, positively skewed $\mathbb{R}$-covered and negatively skewed $\mathbb{R}$-covered. The vertical foliation is the unstable foliation, the horizontal foliation is the stable one. The difference between the two skewed cases is the orientation of the orbit space induced by the orientation of the 3-manifold.

When M is oriented, the orbit space $\mathcal{O}(M)$ inherits an orientation. Thus one can distinguish the positively skewed $\mathbb{R}$-covered flows and the negatively skewed $\mathbb{R}$-covered flows, depending on whether the orbit space is homeomorphic to $\{(x,y) \in \mathbb{R}^2, |y-x| \leq 1\}$ or to $\{(x,y) \in \mathbb{R}^2, |y+x| \leq 1\}$ with their canonical orientations. The geodesic flows on hyperbolic surfaces are negatively skewed $\mathbb{R}$-covered flows (with the canonical orientation we use in later chapters). T.Barbot proved [Bar95a] that an $\mathbb{R}$-covered Anosov flow is transitive. Additionally he proved that an $\mathbb{R}$-covered Anosov flow on a non-orientable 3-manifold is a suspension. Hence the $\mathbb{R}$-covered Anosov flows on 3-manifolds that are not yet understood are the skewed $\mathbb{R}$-covered flows.

For $X \subset M$ connected, we denote by $\tilde{X}$ one connected component of the lift of X in $\tilde{M}$, and by $\rho(X) = \pi(\tilde{X}) \subset \mathcal{O}(\Gamma)$. The sets $\tilde{X}$ and $\rho(X)$ are well-defined up to the action of an element of $\pi_1(M)$, and they usually correspond to a specific connected component that should be clear from the context.

Ideal lozenge. In the second half of this thesis, we use the notion of ideal lozenge to define some transverse surfaces with particular boundaries. We define the ideal lozenges below, and in Section 1.7.3 the Birkhoff annulus induced by an ideal lozenge.

An **ideal lozenge in** $\mathcal{O}(M)$ is a subset L of $\mathcal{O}(M)$ (represented in Figure 1.2) delimited by four half leaves of $A \subset \mathcal{L}^s(p)$, $B \subset \mathcal{L}^u(p)$, $C \subset \mathcal{L}^s(p')$ and $D \subset \mathcal{L}^u(p')$ for some $p, p' \in \mathcal{O}(M)$, so that

- for all $l \in \mathcal{L}^s$, one has $l \cap B \neq \emptyset$ if $l \cap D \neq \emptyset$,
- for all $l \in \mathcal{L}^u$, one has $l \cap A \neq \emptyset$ if $l \cap C \neq \emptyset$,
- one has $A \cap D = \emptyset$ and $B \cap C = \emptyset$.

The points p and p' are said to be the **corners of** L.

As explained later in Section 1.7.3, an ideal lozenge corresponds to the trace in the orbit space of some transverse surface to the flow. The sign of the boundaries of these surfaces are determined by the quadrant in which the ideal lozenge lies, which we introduce here. Fix an immersion $f : \mathcal{O}(M) \to \mathbb{R}^2$, so that the orientation of M induces on $\mathbb{R}^2$ the trigonometric orientation, and so that f sends the leaves $\mathcal{L}^s(p)$ and $\mathcal{L}^u(p)$ to the horizontal and vertical foliations on $\mathbb{R}^2$ respectively. Then for every $p \in \mathcal{O}(M)$, the leaves $\mathcal{L}^s(p)$ and $\mathcal{L}^u(p)$ delimit four connected regions in $\mathcal{O}(M)$. We refer to these regions as the **quadrants** $(\pm, \pm)$ of p. The first sign is positive for the two quadrants lying to the right of p, the second sign is positive for the two quadrants above p. The pairs of quadrants $((+,+),(-,-))$ and $((+,-),(-,+))$ do not depend on the choice of f.

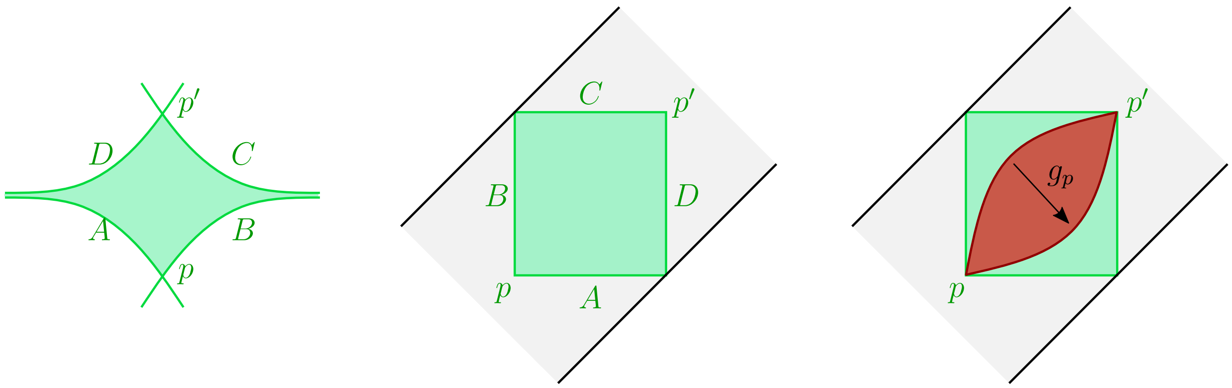


Figure 1.2: Ideal lozenges in the orbit space: a general ideal lozenge on the left, an ideal lozenge in a positively skewed $\mathbb{R}$-covered flow in the middle, and a fundamental domain of an immersed Birkhoff annulus on the right (used in Section 1.7.3).

For a positively skewed $\mathbb{R}$-Anosov flow, every point in the orbit space is the corner of two ideal lozenges, in its two quadrants $(+,+)$ and $(-,-)$. Another family of ideal lozenges for non $\mathbb{R}$-covered flows is given by the following theorem. Two ideal lozenges are said to be adjacent if they share a corner and one of their four sides.

Theorem 1.2.2 (S.Fenley [Fen99])**.** *Let l and l' be two different but not separated leaves of $\mathcal{L}^s$. Then there is an element $g \in \pi_1(M) \setminus \{1\}$ such that l and l' both contain a point invariant by g. Also there exists a finite sequence of at least two adjacent ideal lozenges so that l and l' have half leaves in the boundary of two ideal lozenges in that sequence, and all these ideal lozenges are stable by g.*

Period of an orbit. Let p be a point in $\mathcal{O}(M)$ that lifts to an orbit γ in M, and take x in γ. Then γ is a closed orbit if and only if there exists $g \in \pi_1(M,x)$ so that $g.p = p$. It is known that $Stab_{\pi_1(M,x)}(\gamma)$ is isomorphic to $\mathbb{Z}$ and generated by $[\gamma] \in \pi_1(M,x)$. One may want to work with $\pi_1(M)$ instead of $\pi_1(M,x)$, so we call $[\gamma] \in \pi_1(M)$ the **period of** p. We also refer to the first power of $[\gamma]$ that preserves the orientations of $\mathcal{L}^s(p)$ and $\mathcal{L}^u(p)$ as **oriented**

period of p. It is $[\gamma]^2$ if $[\gamma]$ inverts the orientations and $[\gamma]$ otherwise. Notice that the stable and unstable leaves of γ are open annuli when $[\gamma]$ preserves the orientation, and open Möbius strips otherwise.

1.3 Zoology of transverse surfaces

We define here all types of transverse surfaces that are used in the thesis. Take ϕ a flow on a closed 3-manifold M. An immersed compact surface Σ is called an **immersed partial section** of ϕ if $\mathring{\Sigma}$ is transverse to ϕ and $\partial\Sigma$ is a finite union of closed orbits of ϕ. When Σ is additionally embedded in its interior, it is called a **partial section**. A **Birkhoff section** is a partial section embedded in its interior that additionally intersects every orbit of ϕ in bounded time, that is, one has $\phi[0,T](\Sigma) = M$ for some $T > 0$. Similarly if an immersed partial section satisfies that additional property, it is called an immersed Birkhoff section. A **global section** is a Birkhoff section without boundary. For transverse closed surfaces, intersecting every orbit is equivalent to intersecting every orbit in bounded time. Several examples of immersed partial sections are given in the next two sections. Global sections and Birkhoff sections have particular properties which are detailed at the end of this section.

For Anosov flows, global sections rarely exist, and the existence of Birkhoff sections have been understood by D.Fried. When an Anosov flow is transitive D.Fried constructs a family of immersed partial sections [Fri83] (that we call Fried sections) and proves that the flow admits a Birkhoff section.

It is more convenient to work with partial sections that are embedded in their interior. For an Anosov flow, we describe in Section 1.4 a procedure which transforms an immersed partial section into another partial section embedded in its interior and relatively homologous to the original section. The desingularisation procedure is called a **Fried desingularisation** of Σ, and first appears in [Fri83]. We state in Proposition 1.4.13 the precise Fried-desingularisation that we use later.

Transverse surfaces can be used to express some topological properties transverse to the flow. Notice that the orbit space and the Markov partitions[1] also express transverse properties of the flow, and similar theories can be built

[1] We do not use them in this thesis. However they can be found in [Che02] for example

upon them.

Orientation of the boundary. In the last two chapters, the orientation of the boundary of some immersed partial sections plays an important role. We define the orientation of the boundary and the linking numbers here so they can be specified for the upcoming constructions of immersed partial sections. We denote by $\Cap$ the algebraic intersection between two sub-manifolds.

For orientations and co-orientations, we use the following conventions, illustrated in Figure 1.3. Fix an orientable closed 3-manifold M, an orientation o_M of M, and o_ϕ the orientation induced by the flow ϕ on its orbits. For a surface Σ transverse to the flow, we choose the orientation o_Σ on Σ so that $(o_\Sigma, o_\phi) = o_M$. Then we choose an orientation $o_{\partial\Sigma}$ of its boundary so that $(o_{\partial\Sigma}, X_{in}) = o_\Sigma$, where X_{in} is a vector field on $\partial\Sigma$, going inside Σ.

Let $\gamma \subset \partial\Sigma$ be a boundary component of an immersed partial section Σ. We say that γ is a **positive boundary** of Σ if the orientations of γ given by the flow and by Σ agree, otherwise γ is a **negative boundary** of Σ. We respectively denote by $\partial^+\Sigma, \partial^-\Sigma \subset \partial\Sigma$ the sets of positive and negative boundary components of Σ (which are not necessarily disjoint). Then we say that Σ is **positive** if $\partial^-\Sigma = \emptyset$ and $\partial^+\Sigma \neq \emptyset$, or that Σ is **negative** if $\partial^+\Sigma = \emptyset$ and $\partial^-\Sigma \neq \emptyset$. To compute the algebraic intersections, we orient $\partial\Sigma$ using the orientation of Σ, but when we write $\partial^+\Sigma$ and $\partial^-\Sigma$, we orient them accordingly to the flow, so that algebraically one has $\partial\Sigma = \partial^+\Sigma - \partial^-\Sigma$.

Consider an immersed partial section Σ as an abstract surface, and denote by $i : \Sigma \to M$ the immersion into the 3-manifold M. A boundary component γ' of Σ is immersed into a closed orbit γ of the flow, the algebraic degree of the immersion $(\gamma' \subset \partial\Sigma) \to (\gamma \subset M)$ is called the **multiplicity** of Σ along γ'. The sign of the multiplicity is the same sign that γ' has as boundary component of Σ. Generally the immersed partial section Σ can have several boundary components sent to the same closed orbit γ, so the multiplicity can be taken component by component. But we generally consider the global multiplicity, which is the sum of the multiplicities for all boundary components of Σ sent to γ. We sometime note $\partial^{+,-}\Sigma = a_1\gamma_1 \cup \ldots \cup a_n\gamma_n$ to say that the positive (or negative) boundary of Σ is topologically $\gamma_1 \cup \ldots \cup \gamma_n$ with multiplicity $a_1, \ldots, a_n \in \mathbb{N}$.

In the last chapter, we will additionally use **transverse cobordisms**, which are partial sections Σ embedded in their interiors so that $\partial^+\Sigma \cap \partial^-\Sigma = \emptyset$. Given two transverse cobordisms Σ_1 and Σ_2 with some common bound-

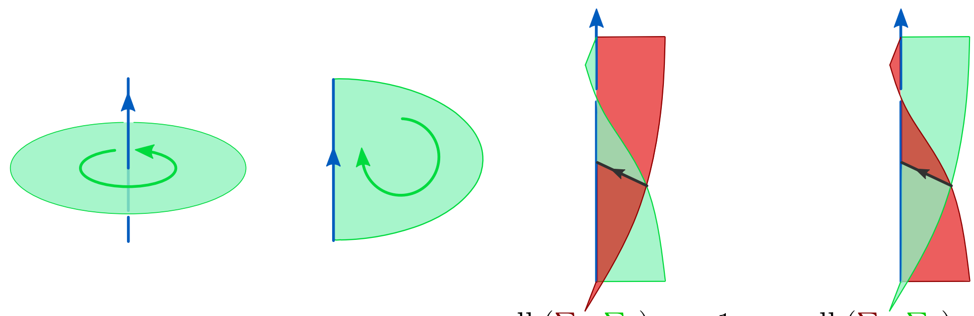


Figure 1.3: Orientation conventions. The flow orients the interior of the surface, which orient its boundary

ary components Γ, so that $\Gamma \subset \partial^+\Sigma_1$ with only multiplicity one along these orbits, and $\Gamma \subset \partial^-\Sigma_2$, the Fried-desingularisation Σ of $\Sigma_1 \cup \Sigma_2$ can be interpreted as the concatenation of the two transverse cobordisms along Γ. So we have $\partial^+\Sigma \cap \Gamma = \emptyset$.

We often distinguish the boundary of Σ in the manifold M, and the boundary of Σ as an abstract surface. Indeed two abstract boundary components of Σ can be immersed into the same closed orbit, even with different signs. Hence one closed orbit can be both a positive and a negative boundary of Σ. Examples of this phenomenon are easy to construct with immersed partial section, and also exist for partial sections embedded in their interiors.

Let Σ_1 and Σ_2 be two surfaces with a common boundary γ, assumed to be in general position. For $\epsilon > 0$ small enough, we denote by C_γ^ϵ the torus of points at distance ϵ from γ, and $\gamma_1^\epsilon = C_\gamma^\epsilon \cap \Sigma_1$ and $\gamma_2^\epsilon = C_\gamma^\epsilon \cap \Sigma_2$, obtained by pushing a multiple of γ into Σ_1 and Σ_2, at distance ϵ. We choose one orientation on γ, and choose coherent orientations on γ_1^ϵ and γ_2^ϵ. Then the **linking number** of Σ_1 and Σ_2 along γ, denoted by $\mathrm{lk}_\gamma(\Sigma_1, \Sigma_2)$, is the algebraic intersection $\gamma_1^\epsilon \pitchfork \gamma_2^\epsilon$ in C_γ^ϵ. Notice that the linking number along a common boundary component is anti-symmetric, and corresponds to the linking number of the two framings induced by the local surfaces. We consider the co-orientation on C_γ^ϵ going away from γ, so that the sign of the intersections are given as in Figure 1.3. When γ_1^ϵ and γ_2^ϵ are not single curves, one can consider the linking number for a single abstract boundary component, or the global linking number given by the sum of the linking number for all choices of single abstract boundary components.

Suppose that ϕ is an Anosov flow, with stable and unstable 2-foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. Let Σ be an immersed partial section and γ in $\partial\Sigma$. We can compute the linking number of Σ with $\mathcal{F}^s$ by using a connected component l^s of $\mathcal{F}^s(\gamma)\setminus\gamma$ in a small neighborhood of γ, which we denote by $\mathrm{lk}_\gamma(\Sigma) = \mathrm{lk}_\gamma(\Sigma, l^s)$. It does not depend on the choice of the connected component, and it equals the linking number with the unstable foliation $\mathcal{F}^u$.

Remark 1.3.1. Suppose that ϕ is Anosov, if $\mathrm{lk}_\gamma(\Sigma) > 0$ then γ is a negative boundary of Σ. If $\mathrm{lk}_\gamma(\Sigma) < 0$, then γ is a positive boundary of Σ. If $\mathrm{lk}_\gamma(\Sigma) = 0$, both cases are possible (depending on the relative position of Σ, $\mathcal{F}^s$ and $\mathcal{F}^u$).

Classification of global sections and Birkhoff sections. The global sections and the Birkhoff sections embedded in their interior have an additional role in the understanding of a flow. They are classified by their homology, and they induce a first-return map which contains parts of the dynamics of the flow. Until the end of this section, every Birkhoff section is supposed embedded in its interior.

We review the classification of global sections given by S.Schwartzman and D.Fried (see [Fri82b] for more details), and explain how it extends to Birkhoff sections. Let Σ a global section and γ be a closed orbit of the flow. Since Σ intersects every orbit in bounded time, the algebraic intersection $\Sigma \frown\!\!\!\!\cdot\,\gamma > 0$ is positive. It implies a natural condition $[\Sigma] \in \bigcap_\gamma \{c \in H_2(M,\mathbb{R}), c \frown\!\!\!\!\cdot\,\gamma > 0\}$ where the intersection is taken over of closed orbit γ of the flow. In general this condition is not sufficient, since some flows have no closed orbits. For any norm $\|.\|$ on $H_1(M,\mathbb{R})$, S.Schwartzman defined a set of asymptotic directions in $H_1(M,\mathbb{R})$ obtained by taking long closed almost-orbits, by re-normalizing their homologies in $\{x \in H_1(M,\mathbb{R}), \|x\| = 1\} \cup \{0\}$, and by taking the accumulation points when the length of the almost-orbits grow to $+\infty$. It defines a compact subset of $H_1(M,\mathbb{R})$ called **asymptotic directions**. Then the classification of global sections is given by the following two theorems. Notice that another choice of norm $\|.\|$ does not change the set of positive rays of the asymptotic directions.

Theorem 1.3.2 (S.Schwartzman [Sch57] (see also Fried [Fri82b]))**.** *Denote the set of asymptotic directions of ϕ by $D_\phi \subset H_1(M,\mathbb{R})$ and take an element u in $H_2(M,\mathbb{Z})/torsion$. Then ϕ admits a global section with homology u if and only if $u \frown\!\!\!\!\cdot\, d > 0$ for all $d \in D_\phi$.*

Hence ϕ admits a global section if and only if D_ϕ lies in an open half-space of $H_1(M, \mathbb{R})$.

The existence of a global section is given by a homology condition. Furthermore, the global section is completely determined by its homology.

Theorem 1.3.3 (D.Fried [Fri82b])**.** *Let Σ_1 and Σ_2 be two global sections of a flow ϕ. Then there exists an isotopy though the flow between Σ_1 and Σ_2 if and only if Σ_1 and Σ_2 are homologous in $H_2(M, \mathbb{Z})$.*

Similarly to global sections, Birkhoff sections are classified by their relative homology, that is, by an element of $H_1(M, \partial\Sigma, \mathbb{Z})$ for a Birkhoff section Σ. The asymptotic directions are less appealing to use for general Birkhoff sections, since they depend on the boundary of the section. An example of classification of some Birkhoff sections is described in the next section, for the geodesic flows. Also for the partial sections that are not Birkhoff sections, these results are wrong in general. We discuss some counter-examples of the second theorem in Section 2.3.3, by using some partial sections constructed in Section 1.5.1.

Given a Birkhoff section Σ, we have a well-defined **first-return map** $r : \mathring{\Sigma} \to \mathring{\Sigma}$, which to a point $x \in \mathring{\Sigma}$ associates the first point of the trajectory $\phi_{\mathbb{R}_+^\star}(x)$ that meets Σ again. The first-return map is a diffeomorphism of $\mathring{\Sigma}$ which can be extended to a homeomorphism of the surface obtain from Σ by retracting every boundary component into a point. When the flow is Anosov, the first-return map is Anosov on $\mathring{\Sigma}$ and pseudo-Anosov on Σ (see [Fri83] for the definition of pseudo-Anosov map and for the proof). We focus on the first-return map in Chapter 2.

1.4 Behavior of a partial section along a boundary component

In this section we consider an Anosov flow on an oriented 3-manifold. We study the behavior of a partial section along one boundary component. We prove that that two partial sections with the same data around a common boundary component can be isotoped one to the other along that boundary, only on the interior of the sections. Then we prove that a surface similar to a partial section but only continuous can be made smooth, under a tame condition. This is used later to define the Fried desingularisation.

1.4.1 Weakly tame local transverse section

In this section we define the notion representing local partial sections around a boundary. We give some local model for such local section. These models have good properties that are used later. We additionally prove some elementary properties of the local partial section.

Let Σ_1, Σ_2 be two partials sections with a common boundary component γ and with the same behavior along that boundary component. We want to find an isotopy from $\mathring{\Sigma}_1$ to $\mathring{\Sigma}_1$ along the flow inside a small neighborhood of γ. We will need to use this idea for surfaces that are piecewise smooth, so we consider the following notion. An open surface $\Sigma \subset M$ is said **topologically transverse** to the flow if for any $x \in \Sigma$, there is a small neighborhood in M around x on which every orbit arc of the flow intersects Σ exactly once. A transverse surface is topologically transverse, and a topologically transverse can be made transverse inside any sub-compact by a small isotopy along the flow. We describe below an isotopy lemma for local topologically transverse surfaces.

Let $\gamma \subset M$ be a closed orbit of the flow and $U \subset M$ be a closed tubular neighborhood of γ. Consider a closed annulus A whose boundary components are denoted by a and b. A **local transverse section** is the image $\mathrm{im}(f)$ of a function $f : (A, a) \to (U, \gamma)$ such that:

- $f^{-1}(\gamma) = a$ and $f^{-1}(\partial U) = b$,
- $f_{|A\setminus a}$ is injective and its image is topologically transverse to the flow.

When Σ is a local transverse section in a tubular neighborhood U of a closed orbit γ of ϕ, we denote by $\mathcal{F}^s(\gamma)$ and $\mathcal{F}^u(\gamma)$ the connected components of the stable and unstable leaves of γ that remain inside U and which contain γ. Notice that we do not require $f_a : a \to \gamma$ to be locally injective, nor to be of non-zero degree. We define the linking number of a local transverse section in the same way as we do for a partial section. Additionally given an orientation of the manifold M, we orient A using the coorientation $f(A \setminus a)$ and the orientation on M. Then the multiplicity of the local transverse section is the degree of the map $f_a : a \to \gamma$, which can be any integer in $\mathbb{Z}$. If Σ is a local transverse section, $\Sigma \setminus \gamma$ is topologically transverse to the flow, and $\mathcal{F}^s$ and $\mathcal{F}^u$ are invariant by the flow, so $\mathcal{F}^s$ and $\mathcal{F}^u$ restrict to 1-foliations on $\Sigma \setminus \gamma$.

Weakly tame property We consider a notion of tame local section, which is inspired by the tame notion appearing in [BG10].

A local transverse section Σ is said to be **weakly tame** if for any open curve α in $(\Sigma\setminus\gamma)\cap(\mathcal{F}^s(\gamma)\cup\mathcal{F}^u(\gamma))$, there exists a compact curve β in $\mathcal{F}^s(\gamma)\cup \mathcal{F}^u(\gamma)$ and a continuous function $T : \mathring{\beta} \to \mathbb{R}$ bounded such that the function $\phi_T : x \in \mathring{\beta} \mapsto \phi_{T(x)}(x)$ is a bijection from $\mathring{\beta}$ to $\mathring{\alpha}$. Informally, when looking in the universal covering of M, the intersections $(\Sigma \setminus \gamma) \cap \mathcal{F}^s(\gamma)$ and $(\Sigma \setminus \gamma) \cap \mathcal{F}^u(\gamma)$ is made of curves and each need to remain bounded for Σ to be weakly tame.

Example 1.4.1. *To understand an Anosov flow around a closed orbit, we define some local models. Take two real numbers λ and μ, such that $|\lambda| > 1$ and $0 < |\mu| < 1$. We consider the transformation $G : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $G(x, y, z) = (\lambda x, \mu y, z + 1)$ and the suspension manifold $N = N_{\lambda,\mu} = \mathbb{R}^3/G(p) \equiv p$, which is orientable when $\lambda\mu > 0$. We consider ψ the flow on N defined by $\psi_t(x, y, z) = (x, y, z + 1)$. We also define the two foliations $\mathcal{F}'^s$ and $\mathcal{F}'^u$ whose leaves are given respectively by the projections of $\mathbb{R} \times \{a\} \times \mathbb{R}$ and $\{b\} \times \mathbb{R} \times \mathbb{R}$ for $a, b \in \mathbb{R}$, they are both invariant by the flow. Finally we denote by γ_N the image of $\{0\} \times \{0\} \times \mathbb{R}$ in N, which is the only closed orbit of the flow ψ.*

Lemma 1.4.2. *We take $r \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ an integer. Let ϕ be a $\mathcal{C}^r$ Anosov flow on a 3-manifold M and γ be a closed orbit of ϕ. Denote by t the length of the orbit γ, that is $t = \min\{t > 0, (\phi_t)_{|\gamma} = \mathrm{id}_\gamma\}$, by λ and μ the two eigenvalues of $D\phi_t$ along γ such that $|\lambda| > 1$ and $0 < |\mu| < 1$, and by $(N_{\lambda,\mu}, \psi)$ the flow described in the previous example.*

Then there exist two small tubular neighborhoods $U \subset M$ and $V \subset N_{\lambda,\mu}$ of the orbits γ and $\gamma_{N_{\lambda,\mu}}$ such that the restrictions of the flows $\phi_{|U}$ and $\psi_{|V}$ are conjugated by a $\mathcal{C}^r$ diffeomorphism. Additionally the orbital equivalence sends the stable and unstable leaves of γ to the stable and unstable leaves of $\gamma_{N_{\lambda,\mu}}$, which are the projections of $\mathbb{R} \times \{0\} \times \mathbb{R}$ and $\{0\} \times \mathbb{R} \times \mathbb{R}$ inside $N_{\lambda,\mu}$.

Furthermore there exist two small tubular neighborhoods $U \subset M$ and $V \subset N_{\lambda,\mu}$ of the orbits γ and $\gamma_{N_{\lambda,\mu}}$ such that the restrictions of the flows $\phi_{|U}$ and $\psi_{|V}$ are orbitally equivalent (by only a continuous map), and the orbital equivalence sends the stable and unstable foliations of $\phi_{|U}$ to the foliations $\mathcal{F}'^s$ and $\mathcal{F}'^u$.

Proof. Take an embedding $f : \mathbb{R}^2 \to U$ transverse to the flow ϕ, such that $f(0,0)$ is in the orbit γ, and consider a neighborhood $D \subset \mathbb{R}^2$ of $(0,0)$ such that the first return map $r : f(D) \to f(\mathbb{R}^2)$ is well defined. Up to taking another $\mathcal{C}^r$ parametrization of the flow ϕ, we can suppose that the first return time on $f(D)$ is equal to 1, that is for all $t \in [0,1]$ and $x \in D$ we have $\phi_t(f(x)) \in f(\mathbb{R}^2)$ if and only if $t \in \{0,1\}$.

Since the flow is $\mathcal{C}^r$, the first return map $r : f(D) \to f(\mathbb{R}^2)$ is differential at $f(0,0)$, and since ϕ is Anosov, $D_{(0,0)}r$ has two real eigenvalues λ, μ such that $|\lambda| > 1$ and $0 < |\mu| < 1$. According to the linearisation Theorem of Sternberg, there exists a local diffeomorphism h of $\mathbb{R}^2$ on a neighborhood of $(0,0)$ such that the conjugation $h^{-1} \circ f^{-1} \circ r \circ f \circ h$ is well defined on a neighborhood of $(0,0)$, and coincide with a linear map $\tilde{r} : \mathbb{R}^2 \to \mathbb{R}^2$ with the same eigenvalues λ and μ. Hence on a small tubular neighborhood of γ, the flow ϕ is orbitally equivalent to the suspension of a linear map with eigenvalues λ and μ.

To prove the second point, one can use the previous idea together with a coordinate system on the surface D given by the two foliations $\mathcal{F}^s \cap D$ and $\mathcal{F}^u \cap D$ on D. The first return map on D contracts the stable direction and expands the unstable one, so it is topologically conjugated to the function $(x,y) \in \mathbb{R}^2 \mapsto (\lambda x, \mu y) \in \mathbb{R}^2$ restricted to a neighborhood of the point $(0,0)$. Hence one can build a $\mathcal{C}^0$ orbital equivalence as stated in the lemma. □

Examples 1.4.3. *Below we define some local transverse sections with the weakly tame property on a local model around a closed orbit with orientable stable and unstable foliations. We consider two real numbers $\lambda > 1$ and $0 < \mu < 1$. Denote by N the suspension manifold $\mathbb{R}^3/(x,y,t) \simeq (\lambda x, \mu y, t+1)$, by π the natural projection $\pi : \mathbb{R}^3 \to N$, by ψ_t the flow on N defined by $\psi_t(x,y,z) = (x,y,z+t)$ and by $\gamma \subset N$ the closed orbit $\psi_{\mathbb{R}}(0,0,0)$.*

- *Given $\epsilon_1, \epsilon_2 \in \{-1,1\}$, we define*

$$\Sigma_0 = \{(\epsilon_1 \lambda^t r, \epsilon_2 \mu^t r, t) \in \mathbb{R}^3 | t \in \mathbb{R}, r \in \mathbb{R}_+\}$$

  The surface Σ_0 is invariant by the transformation $(x,y,z) \in \mathbb{R}^3 \mapsto (\lambda x, \mu y, z+1)$, so its image $\pi(\Sigma_0) \subset N$ is a smooth annulus with a boundary component at γ (with multiplicity one). A direct computation shows that Σ_0 is transverse to the flow ψ in its interior and that it satisfies the weakly tame property. Additionally Σ does not intersect *the stable and unstable leaves* $0\times\mathbb{R}\times\mathbb{R}$ *and* $\mathbb{R}\times 0\times\mathbb{R}$*, so it has linking number zero along* γ*.*

By taking the appropriate values of ϵ_i*, one can make* Σ_0 *be in any quadrant delimited by the leaves* $\mathbb{R}\times 0\times\mathbb{R}$ *and* $0\times\mathbb{R}\times\mathbb{R}$

- *Consider two relatively prime integers* $p\neq 0$ *and* $q\in\mathbb{N}$ *relatively prime, which represent respectively the linking number and the multiplicity in absolute value of a local transverse section. We consider a map* $f:\mathbb{R}\to\mathbb{R}$ *such that*

  - f *is a smooth increasing diffeomorphism with* $f(0)=0$*,*
  - *for all* $t\in\mathbb{R}$*,* $f(t+\frac{1}{2p})=f(t)+\pi$*,*
  - *there exists* $\epsilon>0$ *such that* $f'_{|[\frac{1}{2p}-\epsilon,\frac{1}{2p}]}>q\frac{\ln(\lambda)-\ln(\mu)}{2}$*, and* $f([0,\frac{1}{2p}-\epsilon])\subset[0,\frac{\pi}{2}]$*.*

  Then we define the following map:

$$\begin{aligned} h:\mathbb{R}_+\times\mathbb{R} &\to \mathbb{R}^3 \\ (r,t) &\to (\lambda^{qt}\cos(f(t))r,\mu^{qt}\sin(f(t))r,qt) \end{aligned}$$

  We denote by $\Sigma_f=\pi(\mathrm{im}(h))$*. The image* $\mathrm{im}(h)$ *is a smooth and embedded surface with a boundary component at* $\pi^{-1}(\gamma)$*. If* $G:(x,y,z)\in\mathbb{R}^3\to(\lambda x,\mu y,t+1)$ *is the diffeomorphism used to construct quotient manifold* N*, then* $h(r,t+1)=G^q.h(r,t)$ *for all* $(r,t)\in\mathbb{R}_+\times\mathbb{R}$*. Also for all* $n\in\mathbb{Z}$ *and* $(r,t),(r',t')\in\mathbb{R}_+\times\mathbb{R}$*, one has* $G^n.h(r,t)=h(r',t')$ *if and only if*

  - $(r',qt')=(r,qt+n)$*,*
  - *if* $r\neq 0$ *then* $n\in q\mathbb{Z}$*.*

  Hence the surface Σ_f *inside* N *is an immersed annulus embedded in its interior and whose multiplicity along* γ *is* q *in absolute value. When* $q=0$*, the surface* Σ_f *can also be extended into a smooth and trasnverse surface, which intersects* γ *once in its interior. Because of the pseudo symmetric property of* f*, the linking number of* Σ_f *is* $\pm p$*. Additionally after fixing an orientation on* $\mathbb{R}^3$*, on can change the sign of the linking number by replacing* f *by* $-f$*.*

To prove that Σ_f is transverse to the flow in its interior, we compute the following determinant: $\det(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}, \frac{\partial \psi}{\partial t}) =$

$$\begin{aligned}
\det\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}, \frac{\partial \psi}{\partial t}\right) &= \begin{vmatrix} \lambda^{qt}\cos(f(t)) & \lambda^{qt} r(\ln(\lambda)\cos(f(t)) - f'(t)\sin(f(t))) & 0 \\ \mu^{qt}\sin(f(t)) & \mu^{qt} r(\ln(\mu)\sin(f(t)) + f'(t)\cos(f(t))) & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= r(\lambda\mu)^{qt} \begin{vmatrix} \cos(f(t)) & q\ln(\lambda)\cos(f(t)) - f'(t)\sin(f(t)) \\ \sin(f(t)) & q\ln(\mu)\sin(f(t)) + f'(t)\cos(f(t)) \end{vmatrix} \\
&= r(\lambda\mu)^{qt}\left(f'(t) + q\sin(2f(t))\frac{\ln(\lambda) - \ln(\mu)}{2}\right)
\end{aligned}$$

The third hypothesis on f guaranty that $f'(t) + q\sin(2f(t))\frac{\ln(\lambda)-\ln(\mu)}{2}$ is positive for all $t \in [0, \pi]$, and then for all $t \in \mathbb{R}$ by pseudo-periodicity of f. Hence the determinant is always non-zero when $r \neq 0$, and Σ_f is transverse to the flow in its interior. Futhermore $\frac{\partial h}{\partial r}_{r=0}$ is tangent to the stable or unstable foliation only when $f(t) \in \frac{\pi}{2}\mathbb{Z}$, so Σ_f is weakly tame.

- *To find local models for a closed orbit δ with non-orientable stable leaf, we additionally consider the diffeomorphism of $\mathbb{R}^3$ given by $I(x, y, z) = (-\sqrt{\lambda}x, -\sqrt{\mu}, z+\frac{1}{2})$. The diffeomorphism I induces an involution $N \to N$, whose quotient N/I is a local model of a neighborhood of the orbit δ. Denote by $\Sigma \subset N$ a local transverse section given in the first two examples, and consider Σ' the image of Σ in N/I. The surface Σ has good enough symmetries so that $\mathring{\Sigma}'$ is embedded and $\mathring{\Sigma} \to \mathring{\Sigma}'$ is a covering of degree 1 or 2. That degree with the double covering map $\gamma \subset N \to \delta \subset N/I$ allow one to determine the multiplicity and linking number of Σ' along δ.*

  Consider first $\Sigma = \Sigma_0$ given in the first example. Then the surface $I(\Sigma_0)$ intersect Σ_0 only along its boundary γ, so the image of Σ_0 inside N/I is a local transverse section with linking number zero and multiplicity 2 in absolute value.

  Consider $\Sigma = \Sigma_f$ given in the second example, with multiplicity in absolute value $q \in \mathbb{N}$ and linking number $p \neq 0$ such that $\gcd(p, q) = 1$. A direct computation on Σ_f shows that there exist $x \in \mathring{\Sigma}_f$ and $k \in \mathbb{N}$ such that $I^{2k+1}(x) \in \Sigma_f$ if and only if p, q are both odd. When it is the *case, the image of Σ_f in N/I is a local transverse section with absolute multiplicity q and linking number p. When p or q is even, the image of Σ_f has absolute multiplicity $2q$ and linking number $2p$.*

Notice that two local transverse section given above can be made to intersect in general position one to another, either by slightly pushing one along the flow or by choosing well the parameter f defining these surfaces. Then their intersection is a union (potentially empty) of properly embedded curves, that have an end inside the closed orbit γ.

All local transverse sections are not weakly tame. There are two convenient cases in which they are weakly tame, as we discuss in the two following lemmas.

Lemma 1.4.4. *Let U be a closed tubular neighborhood of a closed orbit γ such that ∂U intersects transversally and in exactly one closed curve every connected components of the leaves $\mathcal{F}^s(\gamma)\setminus\gamma$ and $\mathcal{F}^u(\gamma)\setminus\gamma$. Let $\Sigma \subset U$ be a local transverse section along the orbit γ with linking number zero. Then $(\Sigma \setminus \gamma)$ does not intersect the stable and unstable leaves $\mathcal{F}^s(\gamma)$ and $\mathcal{F}^u(\gamma)$. Furthermore Σ is weakly tame.*

Additionally every arc of orbit inside U intersects $\Sigma \setminus \gamma$ at most once.

Proof. Up to taking a double covering of U, we can suppose that the stable and unstable leaves of γ are orientable. First we prove that the multiplicity of Σ along γ is plus or minus one. The curve $\partial U \cap \Sigma$ is connected and simple, and has an algebraic intersection zero with $\mathcal{F}^s(\gamma)$, so it is either a parallel copy of one component of $\mathcal{F}^s(\gamma)$ or a trivial curve in homotopy. We need to prove that the second case is impossible, so that the multiplicity of Σ is plus or minus 1.

Suppose that $\partial U \cap \Sigma$ is homotopically trivial. Then Σ bound a region D inside U, which lifts to a compact region in the universal covering $\tilde{U}$ of U. We take a foliation $(E_t)_t$ of U with surfaces indexed by $t \in S^1$ and all positively transverse to the flow. Since the region D lifts to a compact region inside $\tilde{U}$, there exists a function $f : D \to \mathbb{R}$ such that for all point $x \in D$, $x \in E_{f(t)}$. We can isotope Σ along the flow so that f have its maximal and minimal points along $\Sigma \setminus \gamma$. Then the flow enter D at the minimal point of f and leaves D at the maximal point of f, both inside $\Sigma \setminus \gamma$. Since $\Sigma \setminus \gamma$ is topologically transverse, its is co-oriented by the flow, which contradicts the previous sentence. Hence this case is not possible and the multiplicity of Σ is plus or minus one.

We suppose that $\Sigma \setminus \gamma$ intersects a half stable leaf of γ and we eventually find a contradiction. Denote by l^s that half stable leaf minus the orbit γ, and by g the generator of $\pi_1(l^s)$. Since γ is not homotopically trivial inside U, g induces an non-zero element inside $\pi_1(U)$. Since Σ has a linking number zero and a multiplicity ± 1, g is also a generator of $\pi_1(\Sigma)$. We consider the infinite covering $\tilde{V}$ of $U \setminus \gamma$ obtained by quotient its universal covering by g. Alternatively we cut $U \setminus \gamma$ along the half stable leaves l^s, and glue a countable amount of copies of that along the copies of l^s. We denote by $\partial\tilde{V}$ the lift of ∂U inside $\tilde{V}$. By definition of the homotopy element g and the covering $\tilde{V}$, l^s and $\Sigma \setminus \gamma$ lift to two homotopic annulus $\tilde{l}^s$ and $\tilde{\Sigma}$ inside the covering $\tilde{V}$. There is a closed subset $C \subset \tilde{V}$ such that $\partial C = \tilde{l}^s \cup \Sigma$ and $\partial\tilde{V} \cap C$ is compact (bounded by $\tilde{l}^s \cap \partial\tilde{V}$ and $\tilde{\Sigma} \cap \partial\tilde{V}$). We also lift the Anosov flow to $\tilde{V}$.

Since l^s and Σ intersect, we can take the lifts $\tilde{l}^s$ and $\tilde{\Sigma}$ so that they also intersect. Additionally $\Sigma \setminus \gamma$ is topologically transverse to the flow, so we can take a point $y \in \tilde{\Sigma} \setminus \tilde{l}^s$ arbitrarily closed to the intersection $\tilde{l}^s \cap \tilde{\Sigma}$, and going inside C. Then the positive orbit of y does not intersect $\tilde{l}^s$ (or y would be inside $\tilde{l}^s$) nor $\tilde{\Sigma}$ (because $\tilde{\Sigma}$ is co-oriented by the flow so it cannot intersect twice an orbit with two opposite orientations). Since y is not in the stable leaf of γ, its positive orbit leaves $\tilde{V}$. We denote by $z(y) \in \partial\tilde{V} \cap C$ the intersection point of the positive orbit of y and of $\partial\tilde{V}$.

Because of the condition on the neighborhood U, for any $y \in U$ closed enough to the stable leaf of γ, its positive orbit inside U leaves U by a point arbitrarily close to the unstable leaf of γ. We consider a continuous curve $y : [0,1] \to \tilde{\Sigma}$ such that the only points inside $\tilde{l}^s$ is $y(1)$, that is $y^{-1}(\tilde{l}^s) = \{1\}$. Then the point $z(y(t))$ for $t < 1$ accumulates along an unstable leaf of γ. More precisely there exists a curve $\delta \subset \partial V$ in an unstable leaf of γ and $\epsilon \in (0,1]$ such that the adherence of $z \circ y([1-\epsilon, 1))$ is $z \circ y([1-\epsilon, 1)) \cup \delta$. Since C is closed and contain $z \circ y([1-\epsilon, 1))$, it also contains δ. Then inside the cylindre ∂V, the curve δ lies between the two curves $\tilde{\Sigma} \cap \partial S$ and $\tilde{l}^s$, which is not possible since $\tilde{l}^s$ intersects the curve $\tilde{\Sigma} \cap \partial V$ but not the curve δ. Hence l^s and $\Sigma \setminus \gamma$ do not intersect. Similarly $\Sigma \setminus \gamma$ does not intersect the other half stable and unstable leaves of γ.

Since $\Sigma \setminus \gamma$ is a proper surface of $U \setminus \gamma$, and do not intersect the stable leaf $\mathcal{F}^s(\gamma)$, the local transverse section Σ is weakly tame.

We prove the last sentence. Denote by Q the connected component of $U \setminus (\mathcal{F}^s(\gamma) \cup \mathcal{F}^u(\gamma))$ containing $\Sigma \setminus \gamma$, it has the topology of a cylinder. Since the multiplicity of Σ is ± 1, the inclusion from the fundamental group of $\Sigma \setminus \gamma$ to the fundamental group of Q is an isomorphism. Since $\Sigma \setminus \gamma$ is a proper surface inside Q, and because of the previous isomorphism, Σ bounds in Q two connected components. Hence since $\Sigma \setminus \gamma$ is co-oriented by the flow, an arc of orbit inside Q cannot intersect $\Sigma \setminus \gamma$ twice, or the two intersections would have opposite algebraic signs. □

Lemma 1.4.5. *Let Σ be a local transverse section along γ. We suppose that the surface Σ is piecewise of class $\mathcal{C}^1$. More precisely we suppose there exist a finite number of $\mathcal{C}^1$ curves $\delta_1, \dots, \delta_n$ inside Σ, disjoint in their interiors and such that for all i, δ_i is transverse to $\partial\Sigma$, and Σ is of class $\mathcal{C}^1$ inside all the adherence of the connected components of $\Sigma \setminus \cup_i \delta_i$. Then Σ is weakly tame.*

Proof. We suppose that the linking number of Σ along γ is not zero, for otherwise we can apply the previous lemma. Denote by $f : \Sigma \to U$ the inclusion of Σ, which is $\mathcal{C}^1$ inside all of the closure of the connected components of $\Sigma \setminus \cup_i \delta_i$. According to our hypothesis, for all x in the boundary component $f^{-1}(\gamma)$, there exists a tangent vector in $T_x\Sigma$ to Σ base at x, transverse to $\partial\Sigma$ and for which f is differentiable in that direction. Since there is only a finite number of number of such point x on which f is not differential, one can build a curve of tangent vectors $X : S^1 \to T_{\partial\Sigma}\Sigma$ along the curve $\partial\Sigma$, such that X is everywhere transverse to $\partial\Sigma$, f is differentiable in the direction given by X, and such that the base points of the curve X induce a degree 1 curve along the boundary component $f^{-1}(\gamma)$. We denote by $df(X)$ the differential of f along the vectors X, inside the tangent space $T_\gamma U$. Since Σ has a linking number non zero, there exists a point $x \in \partial\Sigma$ such that $df(X)(x)$ is not tangent to the stable and unstable leaves of γ in U. So for any curve $c : [0,1] \to \Sigma$ with an end point $c(0) = x$ tangent to the vector $X(x)$, $f \circ c_{|(0,\epsilon]}$ remains in $\Sigma \setminus (\mathcal{F}^s(\gamma) \cup \mathcal{F}^u\gamma)$ for some $\epsilon > 0$. Hence the intersection $f^{-1}(\mathcal{F}^s(\gamma) \cup \mathcal{F}^u(\gamma))$ does not contain any curves that accumulates and spiral along the boundary component $f^{-1}(\gamma)$.

Let α be a curve in $(\Sigma \setminus \gamma) \cap \mathcal{F}^s(\gamma)$. Since α does not accumulate and spiral along $\partial\Sigma$, if α has an end at infinity on γ, it must remain in a compact region of Σ. That is there is a compact curve $c : [0,1] \to \mathcal{F}^s(\gamma)$ and $T : [0,1) \to \mathbb{R}$ bounded and continuous such that $(x \in [0,1) \mapsto \phi_{T(x)}(x))$ is a parametrisation of α restricted to a neighborhood of that infinite end. If α has two ends at infinity along γ, then we can concatenate two such bounded parametrisation of the two ends of α. Hence Σ is weakly tame. □

Lemma 1.4.6. *Let ϕ be an Anosov flow on a closed orientable 3-manifold M, and $\Sigma \subset M$ a local transverse section along the closed orbit γ of ϕ. We denote by $p = \mathrm{lk}_\gamma(\Sigma)$ and $q = |\,\mathrm{mult}_\gamma(\Sigma)|$. Then:*

- *If the stable and unstable leaves of γ are orientable, then p and q are relatively prime integers.*
- *If the stable and unstable leaves of γ are not orientable, then p and q have the same parity, and $\gcd(p, q)$ is either 1 or 2, depending on the parity of p and q.*
- *There exists a smooth and weakly tame local transverse section along γ with the same linking number and multiplicity than Σ.*

Proof. The first two items can be proven by taking coordinates on the homology group $H_1(\partial U, \mathbb{Z})$ of the torus ∂U. We detail the computation in the following three paragraphs.

We denote by δ the curve $\Sigma \cap \partial U$. Up to smoothing the boundary of ∂U, we suppose that ∂U intersects transversally the stable and unstable leaves of γ inside U, and we denote by l^s a closed curved in the intersection of ∂U and of the stable leaf of γ.

When the stable and unstable leaves of γ are orientable, there exists a coordinate system on the homology $H_1(\partial U, \mathbb{Z}) \equiv \mathbb{Z}^2$ of ∂U such that $(1, 0)$ is the homology of a meridian of ∂U and $(0, 1)$ is the homology of l^s. We denote by (u, v) the coordinate in homology of δ, oriented such that $v \geq 0$. Since δ is a simple curve u and v are relatively prime. Also one has $p = \mathrm{lk}_\gamma(\Sigma) = \delta \Cap l^s = u$ and $q = |\,\mathrm{mult}_\gamma(\Sigma)| = |\delta \Cap (1, 0)| = v$, so p and q are relatively prime.

When the stable and unstable leaves of γ are not orientable, there exists a coordinate system on the homology of ∂U such that $(1, 0)$ is the homology of a homotopically trivial curve inside U and $(1, 2)$ is the homology of l^s. We denote by (u, v) the homology of δ, with $v \geq 0$ and $\gcd(u, v) = 1$ as in the previous case. Then one has $p = \mathrm{lk}_\gamma(\Sigma) = \delta \Cap l^s = 2u - v$ and $q = |\,\mathrm{mult}_\gamma(\Sigma)| = |\delta \Cap (1, 0)| = v$. Hence $\mathrm{lk}_\gamma(\Sigma) - \mathrm{mult}_\gamma(\Sigma) \in 2\mathbb{Z}$. Additionally $\gcd(p, q) = \gcd(2u, v) = (1 \text{ or } 2). \gcd(u, v) = (1 \text{ or } 2)$ depending on the parity of q.

We prove the last item. Fix p and $q \geq 1$ as in the lemma. According to Lemma 1.4.2, the flow ϕ is orbitally equivalent on a neighborhood of γ to a neighborhood of the closed orbit of the local model given in Example 1.4.1.

In Example 1.4.3, we defined two local transverse sections with multiplicity q in absolute value, and linking number p and $-p$. We push back these local transverse sections to M, and obtain two local transverse sections along γ with multiplicity q in absolute value, and linking number p and $-p$ (the sign of the linking number changes if the orbital equivalence reverses the orientation). □

1.4.2 Isotopy along the flow of local transverse sections

In this section, we explain how to find some isotopies along the flow between two weakly tame local transverse sections. We precise here the notions. Given two surfaces $S_1, S_2 \subset M$, a **homotopy along the flow** between S_1 and S_2 is a homotopy $\phi_T : [0,1] \times S_1 \to M$ given by $\phi_T(s,x) = \phi_{T(s,x)}(x)$ for a continuous function $T : [0,1] \times S_1 \to \mathbb{R}$, called time function, such that the function $T(0,.)$ is the zero function, and $(x \in S_1) \mapsto \phi_{T(1,x)}(x)$ is a homeomorphism between S_1 and S_2. We denote by $\phi_T(s,x) = \phi_{T(s,x)}(x)$. If one has a continuous function $T : S_1 \to \mathbb{R}$, we also denote by $\phi_T : S_1 \to \mathbb{R}$ the function which sends a point x to $\phi_{T(x)}(x)$.

We additionally say that a homotopy along the flow ϕ_T is an **isotopy along the flow** if for all $s \in [0,1]$, the function $(x \in S_1) \mapsto \phi_{T(s,x)}(x)$ is injective. We are interested by some homotopy, called **linear homotopy along the flow** (or linear isotopy along the flow) which are given by a continuous function $T : [0,1] \times S_1 \to \mathbb{R}$ which is linear in the first parameter, that is $T(s,x) = sT(1,x)$. In particular, given an isotopy ϕ_T along the flow, we consider the **linearisation** of ϕ_T, given by the homotopy $((s,x) \in [0,1] \times S_1) \mapsto \phi_{sT(1,x)}(x)$.

Lemma 1.4.7 (Isotopy characterisation)**.** *Take ϕ_T a homotopy along the flow from S_1 to S_2, given by a time function $T : [0,1] \times S_1 \to \mathbb{R}$. Then ϕ_T is an isotopy if and only if for all points $x \in S_1$ and all $t \in \mathbb{R}_+^\star$ such that $\phi_t(x)$ is in S_1, we have $T(s, \phi_t(x)) + t > T(s,x)$ for all $s \in [0,1]$.*

Proof. Suppose that ϕ_T is an isotopy. If two different points x and $y = \phi_t(x)$ of the surface S_1 are on the same orbit, then their relative order on that orbit must remains the same during the isotopy along the flow. Consider the case in which the orbit of x is not closed, otherwise the argument only need an adaptation. We endow that orbit with the coordinate system $s \in \mathbb{R} \to \phi_s(x)$. Since for all $s \in [0,1]$, the two points $\phi_{T(s,x)}(x)$ and $\phi_{T(s,y)}(y)$ are different, their coordinates which are $T(s,x)$ and $T(s,y)+t$ are different. Additionally when $s = 0$, we have $T(0, \phi_t(x)) + t = t > 0 = T(s, x)$, which implies the inequality for all $s \in [0, 1]$.

Similarly if the inequality is satisfied for all points $x, \phi_t(x) \in S_1$ with $t > 0$, then ϕ_T is an isotopy.

□

An immediate consequence of this lemma is that the linearisation of isotopy along the flow are also isotopy along the flow.

Lemma 1.4.8. *Suppose ϕ_T is an isotopy along the flow, then the linearisation $(s, x) \mapsto \phi_{sT(1,x)}(x)$ is an isotopy along the flow.*

Proof. It is enough to see for all pair of points $x, \phi_t(x)$ in S_1 with $t > 0$, the inequality $sT(1, \phi_t(x)) + t > sT(1, x)$ holds for all $s \in [0, 1]$ if and only it holds for $s = 1$. □

Lemma 1.4.9. *Let $S_1, S_2 \subset U$ be two closed surfaces inside a subset $U \subset M$. Let ϕ_T be an isotopy along the flow from S_1 to S_2, which remains inside U at each time. We also suppose that the set $Graph(\phi_T) = \{(s, \phi_T(s, x))|s \in [0, 1], x \in S_1\}$ is closed inside $[0, 1] \times U$. Consider two subsets U_1, U_2 of U and a real $\epsilon > 0$ such that the adherence $\bar{U}_2$ is inside the interior $\mathring{U}_1$ and such that $\phi_{[-\epsilon,\epsilon]}(U_1) \subset U$. We additionally consider a subset $S_1' \subset S_1$ such that for all point $x \in S_1'$ the orbit arc $\phi_{T([0,1],x)}(x)$ remains inside U_2. Then there exist a surface $S_2' \subset U$ and a global isotopy along the flow inside U, which sends the surface S_1 to S_2', such that $\phi_T(S_1') \subset S_2'$, and S_2' coincide with S_1 outside U_1.*

Proof. Since ϕ_T is an isotopy, there exists a continuous function a from $Graph(\phi_T)$ to $\mathbb{R}$ such that for all $s \in [0, 1]$ and $x \in S_1$, we have $a(s, x) = T(1, x)$. Since $Graph(\phi_T)$ is closed inside $[0, 1] \times U$, we can extend a to a continuous function $a : [0, 1] \times U \to \mathbb{R}$. We consider another continuous function $b : U \to [0, 1]$ such that b is equal to one inside U_2 and equal to zero outside U_1.

We consider the isotopy ψ on U given by the two equations $\frac{\partial \psi}{\partial t} = ab\frac{\partial \phi}{\partial t}$ and $\psi_{|s=0} = \mathrm{id}$. These conditions gives a well defined isotopy which preserve the orbit of the flow. Indeed we have $\phi_{[-\epsilon,\epsilon]}(U_1) \subset U$ and $b_{|U \setminus U_1} = 0$ so the solution of $\frac{\partial \psi}{\partial t}(s, x) = a(s, x)b(x)\frac{\partial \phi}{\partial t}(x)$ remains inside U.

Since U_2 contains the orbit arcs $\phi_{T([0,1],x)}(x)$ for any point x inside S_1', and that for all x in such an orbit arc, $a(s, x)b(x) = T(1, x)$, ψ coincide with ϕ_T on these orbit arc. Hence the image of S_1' after the isotopy is contained inside S_2'. Since b equal zero outside 0, the surface S_2' coincide with S_1 outside U_1. $\square$

Take two family of local transverse sections $(\Sigma_1, \dots, \Sigma_n), (\Sigma_1', \dots, \Sigma_n')$ along an orbit γ and inside the neighborhood U, such that in each family of local transverse sections, the surfaces are disjoint outside γ. We call **semi-isotopy along the flow** from $\cup_i \mathring{\Sigma}_i$ to $\cup_i \mathring{\Sigma}_i'$ any isotopy along the flow from the surface $\cup_i \mathring{\Sigma}_i$ to the interior of the surface $\cup_i \mathring{\Sigma}_i''$, for another family of local transverse sections $(\Sigma_1'', \dots, \Sigma_n'')$, such that for all index i the section Σ_i'' coincide with Σ_i' inside a neighborhood of γ, and with Σ_i inside a neighborhood of the other boundary component of Σ_i, that is $\Sigma_i \cap \partial U$.

Corollary 1.4.10. *Let $(\Sigma_1, \dots, \Sigma_n), (\Sigma_1', \dots, \Sigma_n')$ be 2 families of n local transverse sections along an orbit γ and in a neighborhood U of γ. Suppose that the surfaces $\Sigma_i \setminus \gamma$ are disjoint, and that the surface $\Sigma_i' \setminus \gamma$ are disjoint. Suppose additionally that for all i there exist two neighborhoods $V_i \subset \Sigma_i \setminus \gamma$ and $V_i' \subset \Sigma_i' \setminus \gamma$ of γ inside $\Sigma_i \setminus \gamma$ and $\Sigma_i' \setminus \gamma$, such that the union of the surface V_i is isotopic along the flow inside U to the union of the surface V_i'. Then the family $(\Sigma_1, \dots, \Sigma_n)$ is semi-isotopic along the flow to the family $(\Sigma_1', \dots, \Sigma_n')$.*

Proof. We take a closed neighborhood U_1 of γ inside U which intersects the surface V_i and V_i' only on the interiors. Then we take U_2 another closed neighborhood of γ inside the interior of U_2.

Denote by ϕ_T an isotopy from $\cup_i V_i$ to $\cup_i V_i'$, that we can consider linear according to Lemma 1.4.8. We consider the set $S \subset \cup_i V_i$ of points x such that the isotopy $s \mapsto \phi_T(s, x)$ remains inside U_2. Since $\phi_T(1, .)$ is a homeomorphism between the surface $\cup_i V_i$ and $\cup_i V_i'$, the set S contains a small neighborhood of γ inside the surface $\cup_i V_i$.

We prove that since ϕ_T is a linear isotopy along the flow from $\cup_i V_i$ to $\cup_i V_i'$, the graph $Graph(\phi_T) = \{(s, \phi_T(s, x)) | s \in [0, 1], x \in S_1\}$ is closed inside $[0, 1] \times U \setminus \gamma$. First take a point x is in a small enough neighborhood of γ inside $\cup_i V_i$, then $\phi_T(x)$ remains in a small neighborhood of γ inside $\cup_i V_i'$. Additionally the orbit arc $\phi_{[0,T(x)]}(x)$ remains in a small neighborhood of γ inside $U \setminus \gamma$. Then we consider a sequence $(s_i, \phi_T(s_i, x_i))_i$ converge to (s_∞, y_∞) inside $U \setminus \gamma$. Up to an extraction, s_i converge to s_∞. Since $\phi_T(s_i, x_i)$ converge to y_∞ inside $U \setminus \gamma$, the sequence x_i remains outside a small open neighborhood of γ inside $\cup_i V_i$, so by compactness the sequence x_i accumulates along a point x_∞. Then $y_\infty = \phi_T(s_\infty, x_\infty)$, so (s_∞, y_∞) is inside the graph of ϕ_T.

The previous Lemma 1.4.9 implies that there is an isotopy from $\cup_i V_i$ to a surface $\cup_i V_i''$, such that $\cup_i V_i''$ coincide with $\cup_i V_i$ outside U_1 and contains the surface $\phi_T(S)$. Since $\phi_T(S)$ is a small neighborhood of γ inside the local transverse section $\cup_i(\Sigma_i \setminus \gamma)$, the surface $\cup_i V_i'' \setminus \phi_T(S)$ remains out of a small neighborhood of γ. Hence the surfaces $\cup_i V_i''$ and $\cup_i V_i'$ coincide inside a small neighborhood of γ. Additionally we can extend that isotopy by a constant isotopy on $\cup_i(\Sigma_i \setminus (\gamma \cup V_i))$, which gives a semi-isotopy along the flow from $(\Sigma_1, \dots, \Sigma_n)$ to $(\Sigma_1', \dots, \Sigma_n')$. □

We need a last lemma to build the isotopies along the flow between the sets V_i and V_i' as described in the previous corollary.

Proposition 1.4.11. *Let ϕ be an Anosov flow on a closed orientable 3-manifold, $\gamma \subset M$ be an orbit of ϕ and $(\Sigma_1, \dots, \Sigma_n), (\Sigma_1', \dots, \Sigma_n')$ be $2n$ weakly tame local transverse sections along γ, such that for any indexes $i \neq j$, the surfaces Σ_i and Σ_j are disjoint in their interiors, and surfaces Σ_i' and Σ_j' are disjoint in their interiors. We suppose they all have the same multiplicities and linking numbers. If the linking number is zero, we additionally suppose that the interior of the surface Σ_i and Σ_i' remains inside the same connected component of $U \setminus (\mathcal{F}^s \cup \mathcal{F}^u)$. Then there exists a semi-isotopy from $(\Sigma_1, \dots, \Sigma_n)$ to $(\Sigma_1', \dots, \Sigma_n')$.*

This proposition allows one to isotope a local transverse section to another one, but only on their interior. A very similar result have already been proven in [BG10] but for local transverse section with linking number non-zero and under a tame condition.

Proof. We explain how to isotope one weakly tame local transverse section to another one. Then we construct a semi-isotopy between the two families.

According to Lemma 1.4.2, the flow is on a neighborhood of γ orbitally equivalent to a neighborhood of the closed orbit of the local model given in Example 1.4.1. Hence we can consider that U is a neighborhood of the closed orbit γ_N of the suspension flow ψ on the manifold N. Recall that N is obtained as the quotient of $\mathbb{R}^3$ by the function $G : (x, y, z) \in \mathbb{R}^3 \mapsto (\lambda x, \mu y, z)$ where λ and μ are two real numbers such that $|\lambda| > 1$, $0 < |\mu| < 1$ and $\lambda\mu > 0$. Also $N = \mathbb{R}^3/G$ and the flow ψ is generated by $\frac{\partial}{\partial z}$.

We first prove the proposition when the linking number is zero. Denote by Q the connected component $U \setminus (\mathcal{F}^s(\gamma) \cup \mathcal{F}^s(\gamma))$ containing the surfaces $\Sigma_1 \setminus \gamma$. Denote by Σ_0 the local transverse section given in the first item of Example 1.4.3, which we take inside $Q \cup \gamma$.

We denote by $R : N \to \mathbb{R}$ the function $R(x, y, z) = |\lambda|^{-2z}x^2 + |\mu|^{-2z}y^2$, which is well defined. Since $R \circ G = R$, R induces a function on N, still denoted by R. We fixe $\epsilon > 0$ such that $R^{-1}([0, \epsilon))$ is a small enough neighborhood of γ. By a direct computation, we notice that for $p \in Q$, the orbit $\psi_{|\mathbb{R}}(p)$ intersects only once Σ_0, at the point which minimises the function R along that orbit. We denote $T(p)$ the unique real number such that $\phi_{T(p)}(p)$ is in the surface $\Sigma_0 \setminus \gamma$. By what proceed, $R(p) \leq R \circ T(p)$. So $T_{|\Sigma_1 \setminus \gamma} : \Sigma_1 \setminus \gamma \to \mathbb{R}$ is a continuous function which satisfies that for all $p \in \Sigma_1 \setminus \gamma$, $\phi_{T(p)}(p)$ is in $\Sigma_0 \setminus \gamma$. Since $R(p) \leq R \circ T(p)$ for all p, the function $(x \in \Sigma_1 \setminus \gamma) \mapsto \phi_{T(p)}(p)$ converge to γ when p converge to γ. Hence its image contains a small neighborhood of γ inside $\Sigma_0 \setminus \gamma$. Additionally we can take a smaller tubular neighborhood U of γ which satisfies the hypotheses of Lemma 1.4.4, and such that the orbit arcs in N between two points inside U remain inside U. Then $(s, x) \in [0, 1] \times \Sigma_1 \setminus \gamma) \mapsto \phi_{sT(p)}(p)$ is injective (or there would exist an orbit that intersects Σ_1 twice inside U). Hence according to Lemma 1.4.10, Σ_1 and Σ_0 are semi-isotopic.

Hence every local transverse sections Σ_i and Σ'_i are semi-isotopic. Since the surface Σ_i intersects any orbit at most once, and intersects every orbit inside Q close enough to γ, we can find an orbit δ inside Q which intersects every surface Σ_i and Σ'_i only once. Up to changing the order of the surface, we can suppose that the orbit δ intersects $\Sigma_1, \ldots, \Sigma_n$ is that order, and $\Sigma'_1, \ldots, \Sigma'_n$ in that order. Then we consider a linear semi-isotopy ϕ_{T_i} from Σ_i to Σ'_i. Because of the discussion on the relative position of the surface, the union of the linear semi-isotopies gives an isotopy from two neighborhoods of γ inside of $\cup_i \Sigma_i \setminus \gamma$ two $\cup_i \Sigma'_i \setminus \gamma$. Hence according to Lemma 1.4.10, there exists a semi-isotopy from $(\Sigma_1, \ldots, \Sigma_n)$ to $(\Sigma'_1, \ldots, \Sigma'_n)$.

We now prove the proposition when the linking number is not zero. The ideas are the same, but some part of the proof need an adjustment. Take $\Sigma_f \subset N$ the second local transverse section defined in Example 1.4.3, with the same linking number and multiplicity than Σ_1. We consider $\widetilde{N \setminus \gamma}$ and $\widetilde{U \setminus \gamma}$ the universal covering spaces of $N \setminus \gamma$ and $U \setminus \gamma$, $\tilde{\phi}$ the lift of the flow ϕ to $\widetilde{N \setminus \gamma}$, and we lift the surface $\Sigma_1 \setminus \gamma$ to a connected surface $\widetilde{\Sigma_1 \setminus \gamma}$. Then $\widetilde{\Sigma_1 \setminus \gamma}$ is a proper surface inside the simply connected 3-manifold $\widetilde{U \setminus \gamma}$, so it bounds two connected components. Hence every orbit inside $\widetilde{U \setminus \gamma}$ intersects $\widetilde{\Sigma_1 \setminus \gamma}$ at most once (because the flow coorients the topologically transverse surfaces). Furthermore, by definition of Σ_f, the surface $\widetilde{\Sigma_f \setminus \gamma}$ intersects exactly once every orbit inside $\widetilde{N \setminus \gamma}$. Hence as previously, there exists a time function $\tilde{T} : \widetilde{\Sigma_1 \setminus \gamma} \to \mathbb{R}$ such that for all point x in $\widetilde{\Sigma_1 \setminus \gamma}$ we have $\tilde{\phi}_{T(x)}(x) \in \widetilde{\Sigma_f \setminus \gamma}$. Since Σ_1 and Σ_f have the same linking number and multiplicity, the function $\tilde{T}$ induces a function on the quotient $T : \Sigma \setminus \gamma \to \mathbb{R}$ such that the image of ϕ_T lies inside $\Sigma_f \setminus \gamma$. Additionally since the surface $\widetilde{\Sigma_1 \setminus \gamma}$ intersects at most once every orbit inside $\widetilde{N \setminus \gamma}$, the function ϕ_T is injective.

Since Σ_1 and Σ_f are weakly tame, the function T is bounded. Indeed there exists a compact set K and a surjection $u_1 : K \to \Sigma_1$ such that for every curve c inside $(\Sigma_1 \setminus \gamma) \cap (\mathcal{F}^s(\gamma) \cup \mathcal{F}^u(\gamma))$, there is a connected component of $u_1^{-1}(c)$ on which u_1 induces a homeomorphism with c. Denote by $D \subset N$ a compact disc transverse to the flow, such that every orbit intersecting Σ_1 also intersects D. Then there exists a continuous function $v_1 : K \to \mathbb{R}$ such that for all points $x \in K$, $\phi_{v_1(x)}(u_1(x))$ lies inside D. Since K is compact, the function v_1 is bounded. We defined similarly two functions $u_2 : K \to \Sigma_f$ and $v_2 : K \to \mathbb{R}$ such that for all $x \in K$, $\phi_{v_2(x)}(u_2(x))$ lies inside D, and such that $v_2 \circ \phi_T = v_1$. Since v_1 and v_2 are bounded and u_1 is surjective, T is also bounded. Hence the function $x \mapsto \phi_T(x)$ accumulates along γ when x accumulates along γ, so its image contains a small neighborhood of γ inside Σ_f. Hence according to Lemma 1.4.10, Σ_1 and Σ_f are semi-isotopic.

So all local transverse sections Σ_i and Σ_i' are semi-isotopic. Since the surfaces Σ_i are relatively homologous and disjoint in there interior, an orbit arc close enough to γ intersects the surfaces Σ_i in a cyclic order. Up to changing the order of the surface, we can suppose that an orbit arc closed enough to γ intersects all Σ_i and all Σ_i' is the same order. Then one can chose some semi-isotopy from Σ_i ato Σ_i' such that the union of that isotopy is injective inside a neighborhood of γ. Then according to Lemma 1.4.10, there exists a semi-isotopy along the flow from $(\Sigma_1, \dots, \Sigma_n)$ to $(\Sigma_1', \dots, \Sigma_n')$. $\square$

1.4.3 Smoothing and Fried-desingularisation

In this section, we use the notion studied above to smooth a topological surface into a partial section.

Lemma 1.4.12. *Let M be an oriented closed 3-manifold and ϕ be an Anosov flow on M. Let Σ be a compact surface, $f : \Sigma \to M$ be a continuous function* *and Γ be a finite union of closed orbits of the flow ϕ. We suppose that $\partial\Sigma = f^{-1}(\Gamma)$, and that the restriction $f_{|\mathring{\Sigma}}$ is topologically transverse to the flow ϕ and injective. We also suppose that $f(\Sigma)$ restricts to a union of weakly tame local transverse sections on some neighborhoods of each orbits of Γ. Then for all $\epsilon > 0$, there exists a smooth partial section Σ' and a function $T : \mathring{\Sigma} \to \mathbb{R}$ such that:*

- *Σ' is embedded in its interior and $\partial\Sigma' \subset \Gamma$,*
- *Σ' remains in an ϵ-neighborhood of $f(\Sigma) \cup \Gamma$,*
- *There exist a finite union of closed curves $\alpha \subset \Sigma'$ such that the function given by $(x \in \mathring{\Sigma}) \to \phi_{T(x)}(f(x))$ is a homeomorphism from $f(\Sigma) \setminus \Gamma$ to the surface $\Sigma' \setminus (\Gamma \cup \alpha)$,*
- *for all $s \in [0, 1]$, the function $(x \in \mathring{\Sigma}) \to \phi_{sT(x)}(f(x))$ is injective.*
- *the surfaces $f(\Sigma)$ and Σ' are relatively homologous in $H_2(M, \Gamma, \mathbb{Z})$.*

Proof. Take a closed orbit γ in Γ, we consider an ϵ-neighborhood U_γ of γ such that $f(\Sigma) \cap U_\gamma$ is a finite union of weakly tame local transverse sections, which we denote by $\Sigma_1, \dots, \Sigma_n$. Suppose for now that the linking number is non zero, then these local transverse section are disjoint on there inside, so they have the same linking number and multiplicity. According to Lemma 1.4.6 and Proposition 1.4.11, there exists a semi-isotopy along the flow from $(\Sigma_1, \dots, \Sigma_n)$ to a family of weakly tame and smooth local transverse section $(\Sigma'_1, \dots, \Sigma'_n)$ of γ, inside U_γ, with the same linking numbers and multiplicities.

If $f(\Sigma)$ has a linking number zero along a closed orbit γ in Γ, then we apply the previous argument for each quadrant of the flow around γ.

We apply that procedure on each closed orbit $\gamma \in \Gamma$ to obtain an isotopy along the flow from $f(\mathring{\Sigma})$ to $\mathring{\Sigma}'$ for another surface Σ' which is smooth on a small boundary of Γ. Additionally Σ' coincide on a small tubular neighborhood of Γ of a union of the standard topological transverse sections given in Example 1.4.3.

Out of Γ, the surface Σ' is topologically transverse to the flow, and is smooth inside a neighborhood of Γ. So we can find an ϵ-small isotopy along the flow from Σ' to a smooth surface Σ''. Which coincide with Σ' around Γ

It remains to see that Σ'' is a partial section. Along any closed orbit γ in Γ, there exists a small tubular neighborhood of γ on which Σ'' is conjugated to a union of standard topological transverse sections S given in Example 1.4.3. For each of these standard model, either γ is a boundary component of S and S is immersed along γ, or γ intersects geometrically S in only one point, and S is smooth and transverse to the flow in a neighborhood of that point. Hence Σ'' is a smooth partial section embedded in its interior. Each step of the proof preserve the relative homology of the surface inside (M, Γ),so $f(\Sigma)$ and Σ'' are relatively homologous in $H_2(M, \Gamma, \mathbb{Z})$. □

We can now study a precise statement for the Fried-desingularisation that we will use.

Proposition 1.4.13 (Fried-desingularisating)**.** *Let ϕ be an Anosov flow on an oriented closed 3-manifold M. Let Σ an immersed partial section, non necessarily connected. Then for all $\epsilon > 0$ there exists a partial section Σ' such that:*

- *Σ' is embedded in its interior, and $\partial\Sigma' \subset \partial\Sigma$,*
- *Σ' remains in an ϵ-neighborhood of Σ,*
- *the surfaces Σ and Σ' are relatively homologous in $H_2(M, \Gamma, \mathbb{Z})$.*
- *the boundary components $\partial^+\Sigma'$ and $\partial^-\Sigma'$ are disjoint.*

Proof. We consider $\epsilon > 0$ as in the hypothesis, and take $\epsilon' > 0$ small relatively to ϵ. The idea of the proof is to cut Σ along its self-intersection curves, and to use the previous lemma. First we use the Proposition 1.4.11 to isotope Σ on an ϵ'-small tubular neighborhood U of $\partial\Sigma$. After that isotopy, the surface Σ is given on a neighborhood of its boundary components by some union of the standard local model given in Example 1.4.3. Up to slightly isotopy along the flow on these models, we can suppose that the self intersection curves of Σ inside U are finitely many compact curves, and that $\mathring{\Sigma}$ intersects itself transversally inside U. Then the adherence of $\Sigma \setminus U$ inside Σ is a compact transverse to the flow, so there exists an ϵ'-small isotopy along the flow from $\Sigma \setminus U$ to a surface that intersects itself only transversally.

Hence after a small isotopy of $\mathring{\Sigma}$, Σ intersects itself along compact curves. We cut Σ along these curves and glue back the surface into a surface Σ' topologically transverse to the flow in its interior. We do that surgery operation such that for each two small open subsets $S_1, S_2 \subset \mathring{\Sigma}'$ which intersect at x, and for all $\mu > 0$, either $\phi_\mu(S_1) \cap S_2 = \emptyset$ or $\phi_\mu(S_2) \cap S_1 = \emptyset$. So there exists an ϵ'-small isotopy of Σ' such that the Σ' has no more self intersection outside $\partial\Sigma$. We can do this such that Σ' induces along $\partial\Sigma$ only remains weakly tame local transverse sections. Then according to the previous Lemma 1.4.12, there exists a partial section Σ'' in an ϵ'-neighborhood of Σ', which is embedded in its interior and is relatively homologous to Σ in $H_2(M, \partial\Sigma, \mathbb{Z})$. Additionally $\partial\Sigma'' \subset \partial\Sigma$.

At this point in the proof, we do not necessarily have $\partial^+\Sigma'' \cap \partial^-\Sigma'' = \emptyset$. Fix an orbit γ which is both a positive and a negative boundary of Σ''. Then Σ'' has linking number zero along γ. Otherwise take two neighborhoods of γ in Σ associated with two boundary components with opposite sign. Then these two neighborhoods have different linking numbers, so they intersects in their interior. Since Σ'' is embedded in its interior, the previous case is impossible, so the linking number of γ is zero.

Consider a small closed tubular neighborhoods U of γ such that Σ'' intersects U in a finite number of local transverse sections. By what proceed, there exist in $\Sigma'' \cap U$ two local transverse sections S_1 and S_2 with opposite multiplicity. The flow is Anosov so S_1 and S_2 are in two adjacents quadrants of γ. We can push $S_1 \cup S_2$ into a transverse surface in U, inside the union of these two quadrants, and outside γ.

To be precise, we consider the local model $N = \mathbb{R}^3/(x, y, z) \equiv (\lambda x, \mu y, z+1)$ with $\lambda\mu > 0$, $|\lambda| > 1$ and $0 < |mu| < 1$ given in Example 1.4.1. We suppose that S_1 and S_2 are on two quadrant adjacent along half a stable leaf of γ, the other case is similar. We take two functions $s_1 : \mathbb{R}^\star \to \mathbb{R}$, $s_2 : \mathbb{R}^\star \to \mathbb{R}$ which are equal outside a small open interval around zero, we take $y_0 \in \mathbb{R}^\star_+$ and the two surfaces given by $F_i : \{(\lambda^{t-s(x)}x, \mu^{t-s(x)}y_0, t) | x \in \mathbb{R}, t \in \mathbb{R}\}$. For F_1 to be well define, we require $s(x) \xrightarrow[x\to 0]{} +\infty$ and $s(x) = o_{x\to 0}(\ln(x))$ such that $\lambda^{-s(x)}x \xrightarrow[x\to 0]{} 0$. Then F_2 is transverse to the flow, F_1 is the union of two topologically local section along γ with two opposite sign along the boundary, and in two adjacent quadrants. According to Lemma 1.4.2 and Proposition 1.4.11, the union $S_1 \cup S_2$ is semi-isotopic to the two local transverse sections $F_1 \setminus \gamma$ along there interiors. Since F_1 and and F_2 are equal outside a small neighborhood of γ, we can remove the annulus $S_1 \cup S_2$ inside Σ'' and replace it with a smooth annulus transverse to the flow.

We do the previous procedure a finite amount of time until eventually obtaining a partial section Σ''' with $\partial^+\Sigma'''$ and $\partial^-\Sigma'''$ disjoint. The procedure may add some closed self intersection curves of Σ'''. If it is the case, we can apply the first part of the proof to remove these intersections curves.

Furthermore the relative homology in $H_2(M, \partial\Sigma, \mathbb{Z})$ of the partial section is preserved at each step. $\square$

Remark 1.4.14. Suppose we are given a surface Σ which is a pseudo-partial section with some ramification points. Under the condition that the surface admit a tangent plane at the ramification point and that the surface is transverse to the flow at that ramification point. We can apply the previous proof and find a partial section Σ' which satisfies the conclusion of the lemma.

1.5 Partial sections of the geodesic flows

In this section, we fix a hyperbolic surface S, and construct several immersed partial sections of the geodesic flow on T^1S. The first subsection introduces the main partial sections and Birkhoff sections studied in Chapter 2. The second subsection constructs some immersed partial sections, used in the last chapter to study the primitive orbits of the geodesic flows.

We fix the orientation on T^1M given by $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta})$, for a local map with coordinates (x, y) and $\frac{\partial}{\partial \theta}$ the trigonometric direction in this map, that is that goes from $(1, 0)$ to $(0, 1)$ through the shortest arc.

1.5.1 Partial section with symmetric boundary

In this subsection, we construct some partial sections said to have symmetric boundaries, and explain the classification of Birkhoff sections with symmetric boundaries. This construction comes from [CD16].

Let $\Gamma \subset S$ be a finite set of closed (non-oriented) geodesic that are in general position, which means that there is no point where three or more geodesic arcs intersect, as in Figure 1.4. We suppose most of the time that Γ is **filling**, that is, $S \setminus \Gamma$ is a union of disjoint discs. Given a geodesic γ of Γ and an orientation of γ, we can lift γ into $\vec{\gamma} = (\gamma, \gamma') \subset T^1S$, which is a closed orbit of the geodesic flow. We denote by $\overleftarrow{\gamma}$ the other closed orbit obtained by inverting the direction of γ. We also denote by $\overleftrightarrow{\Gamma} \subset T^1S$ the lift of Γ with both orientations, whose cardinality is twice the cardinality of Γ. We later construct partial sections with boundary $-\overleftrightarrow{\Gamma}$, that is whose boundary is topologically $\overleftrightarrow{\Gamma}$, but with multiplicity -1 along each orbit.

We see Γ as a graph (Γ_0, Γ_1) in S, were Γ_0 is the set of double points of Γ, and Γ_1 the set of edges bounded by Γ_0. We also denote by Γ_2 the

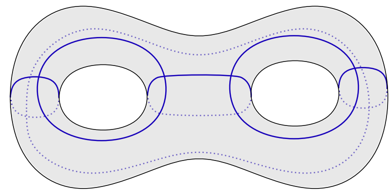


Figure 1.4: Filling geodesic multi-curve on a hyperbolic surface.

set of faces of S bounded by Γ. We consider a coorientation η of Γ, in the sense that η is the union of a transverse orientation for every edge in Γ_1. We are interested in **Eulerian coorientations** (illustrated in Figure 1.5), that is, around every vertex there are as many edges locally oriented clockwisely and anticlockwisely. In particular, around a vertex, there are two ways to coorient Γ up to rotation, that we call the alternating and non-alternating vertices. We denote by $\mathcal{E}ul\mathcal{C}o(\Gamma)$ the set of all Eulerian coorientations of Γ.

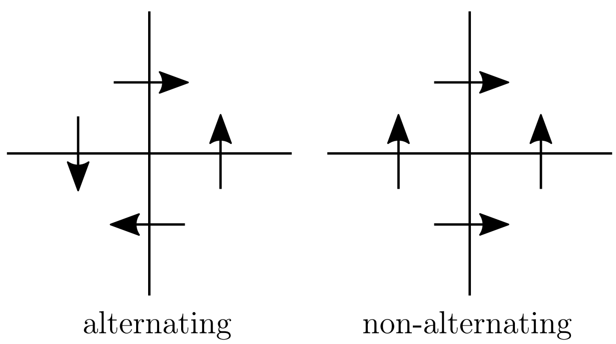


Figure 1.5: Eulerian coorientation around a vertex.

Examples 1.5.1.

- *When the surface S is orientable, we can coorient every geodesic of Γ and combine them into an Eulerian coorientation of Γ, with only non-alternating vertices.*

- *If $[\Gamma] \equiv 0 \in H^1(S, {}^{\mathbb{Z}}/_{2\mathbb{Z}})$, we can color the faces of $\Gamma \subset S$ in black and white with the condition that along every edge a black and a white face meet. This is sometimes also called a checkerboard coloring. Then we consider the coorientation that goes from white to black along every edge. It has only alternating vertices.*

We now fix an Eulerian coorientation η and construct the surface $\Sigma_\eta \subset T^1S$. The first step is to define a vertical 2-complex $\hat{\Sigma}_\eta$ in T^1S. For every edge $e \in \Gamma_1$, let $r_e = \{(x,v) \in T^1S | x \in \Gamma, v \text{ and } \eta_e \text{ are in the same direction}\}$ be a rectangle of the form (geodesic arc) $\times$ (half fiber) (see Figure 1.6). Then define the 2-complex $\hat{\Sigma}_\eta = \cup_{e\in\Gamma^1} r_e$. Apart from the fibers of the non-alternating vertices, it is a topological surface with boundary $\overleftrightarrow{\Gamma}$. Additionally it has multiplicity -1 along all its boundary components.

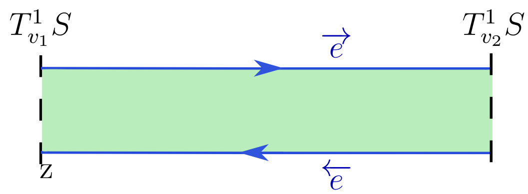


Figure 1.6: One rectangle r_e

Let v be an non-alternating vertex of Γ. On the fiber T^1_vS, the complex $\hat{\Sigma}_\eta$ admits a degree 4 edge, as a X-shape times $[0,1]$. We need to resolve this singularity. As explained in [CD16], there are two ways to desingularise and smooth $\hat{\Sigma}_\eta$ into a surface around T^1_vS, but only one is transverse to ϕ. We desingularise $\hat{\Sigma}_\eta$ into a topological surface topologically transverse to the flow. According to the following lemma, we can define a partial section Σ_η as the smoothing of $\hat{\Sigma}_\eta$. A local lift of Σ_η to $\mathbb{R}^2\times\mathbb{R}$ is represented in Figure 1.7. This surface is unique up to a small isotopy along the flow. To simplify forthcoming expressions, we denote by Σ_η the interior of the surface, but we still consider its boundary $\partial\Sigma_\eta = -\overleftrightarrow{\Gamma}$.

Lemma 1.5.2. *For all $\epsilon > 0$, there exists an embedded partial section Σ_η with boundary $-\overleftrightarrow{\Gamma}$ and a continuous function $T : \mathring{\hat{\Sigma}}_\eta \to [-\epsilon, \epsilon]$ such that the function $(x \in \mathring{\hat{\Sigma}}) \to (\phi_{T(x)}(x) \in T^1S)$ is a homeomorphism between the interior of $\hat{\Sigma}_\eta$ and the interior of Σ_η. Additionally Σ_η can be taken such that T is equal to zero outside a small neighborhoods of the fibers of the bundle $T^1S \to S$ above the intersection points of Γ.*

Proof. We explain how to desingularise around one vertex of Γ. We take two geodesic arcs γ_1 and γ_2 inside Γ, which intersects in a vertex v. We consider one of the two orientations of γ_1 in order to desingularise $\hat{\Sigma}_\eta$ on a neighborhood of $\vec{\gamma}_1$. We foliate a neighborhood of γ_1 with some geodesic arc $(\delta_t)_t$ transverse to the curve γ_1 such that $\gamma_2 = \delta_0$. We denote by $T^1_{\delta_t}S$ the fibers above the curve δ_t, and we lift the 1-foliation δ_t to a 2-foliation $(D_t)_t$ of a neighborhood of $\vec{\gamma}_1$, so that for all t, D_t is a neighborhood of $\vec{\gamma} \cap T^1_{\delta_t}S$ in the fibers above δ_t. One can find a small flow box $B \subset T^1S$ of the form $D^2 \times [-1,1]$ around $\vec{\gamma}_1$ where $D^2 \subset \mathbb{C}$ is the unitary disc in $\mathbb{C}$ and $a \in \mathbb{R}^\star_+$, such that the geodesic flow is given by $\phi_t(x,s) = (x, s+t)$ and for all $t \in [-1,1]$, $D^2 \times \{t\}$ coincide with the leaf D_t.

Additionally that flow box can be chosen such that the surface $\hat{\Sigma}_\eta \cap B$ is given by the image of the function $H : (r,\theta) \in [0,1] \times [-\theta_0, \theta_0] = (r\,\mathrm{e}^{i\theta}, h(\theta))$ for some $\theta_0 > 0$ and for a function $h : [-\theta_0, \theta_0] \to [-1,1]$. Notice that the disc D_t intersects $\hat{\Sigma}_\eta$ either along one compact curve, or along a sub-surface of D_0 when $t = 0$. Hence the function h is monotone and smooth outside two points. We can suppose that h is non-increasing, the other case is symmetric. We chose a function $h' : [-\theta_0, \theta_0] \to [-1,1]$ smooth, increasing, ϵ-close to h and equal to h on a small tubular neighborhood of $D^2 \times \{-\theta_0, \theta_0\}$. Then the image of the function $H' : (r,\theta) \in [0,1] \times [-\theta_0, \theta_0] \times = (r\,\mathrm{e}^{i\theta}, h'(\theta))$ is a smooth embedded surface tangent to $\vec{\gamma}$ and transverse to the flow outside of $\vec{\gamma}$, and it coincides with $\mathrm{im}(H)$ on a neighborhood of $D^2 \times \{-\theta_0, \theta_0\}$. Furthermore the surface $\mathring{H}'$ is the image of $\mathring{H}'$ by the isotopy along the flow given by the time function $(r,\theta) \mapsto (h'-h)(\theta)$.

According to what proceed, we can find an ϵ-small isotopy along the flow from $\hat{\Sigma}_\eta$ to a surface Σ', with a support in a small neighborhoods of its boundary, such that the surface Σ' is both embedded and smooth outside a small neighborhoods of $\partial\Sigma'$. Since the interior of Σ' is topologically transverse to the flow, we can find an ϵ-small isotopy of Σ' along the flow with a partial section Σ_η embedded in its interior. Since $\hat{\Sigma}_\eta$ has multiplicity -1 along all its boundary component, Σ is embedded. □

Remark 1.5.3. We will see that the diffeomorphism class of the surface Σ_η does not depend on the type of vertices induced by the coorientation η. Thus it does not depend on the coorientation η itself. However its isotopy type inside T^1S depends on η.

Classification of the Birkhoff sections with boundary $-\overset{\leftrightarrow}{\Gamma}$. Given a coorientation η and a generic oriented closed curve γ in S, we can count the algebraic intersection between (Γ, η) and a curve γ, which we denote by $\eta(\gamma)$.

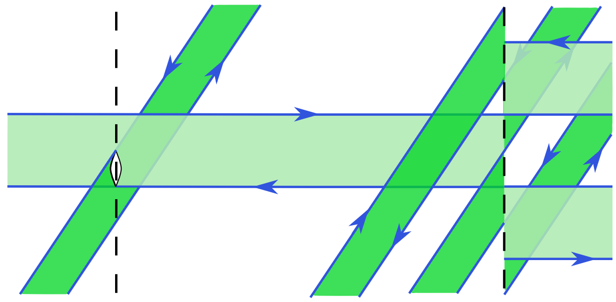


Figure 1.7: Local picture of a lift of $\Sigma_\eta \subset T^1S$ on $S \times \mathbb{R}$. On the left is represented the lift of a non-alternating vertex, and on the right of an alternating vertex.

Lemma 1.5.4. *[CD16] If η is Eulerian, $\eta(\gamma)$ depends only on the homology class $[\gamma] \in H_1(S, \mathbb{Z})$. Thus the coorientation η induces a cohomology class $[\eta] \in H^1(S, \mathbb{Z})$.*

Given an Eulerian coorientation η, its cohomology is used to classify the Birkhoff surfaces with symmetric boundary $-\overset{\leftrightarrow}{\Gamma}$ when Γ is filling [CD16].

Theorem 1.5.5 (Reformulation of Theorems C and D from [CD16])**.** *Let S be a hyperbolic closed surface with its geodesic flow on T^1S, and $\Gamma \subset S$ be a finite union of closed geodesics in general position. Then:*

- *For two Eulerian coorientation η and ν of Γ, the two partial sections Σ_η and Σ_ν are relatively homologous in $H_2(T^1S, \overset{\leftrightarrow}{\Gamma}, \mathbb{Z})$ if and only if η and ν are cohomologous in $H^1(S, \mathbb{Z})$.*

- *The partial section Σ_η is a Birkhoff section if and only $[\eta]$ lies in the interior of a polyhedra of $H^1(S, \mathbb{Z})$ (described as the unitary ball of an intersection norm).*

- *The relative homology class of every partial section bounded by $-\overset{\leftrightarrow}{\Gamma}$ in $H_2(T^1S, \overset{\leftrightarrow}{\Gamma}, \mathbb{Z})$ is obtained as one $[\Sigma_\eta]$ for some Eulerian coorientation η.*

- *In particular if Γ is filling, every Birkhoff section with boundary $-\overset{\leftrightarrow}{\Gamma}$ is isotopic through the flow to a Birkhoff section $[\Sigma_\eta]$ for some Eulerian coorientation η.*

- *If* Γ *is not filling, there is no Birkhoff section of the geodesic flow bounded by* $-\overset{\leftrightarrow}{\Gamma}$.

In Section 2.2.1, we explain that the partial section Σ_η is a Birkhoff section if and only the coorientation η is acyclic. It is a combinatorial property which is easy to check in practice. Futhermore we give below an algorithmic construction of a coorientation with fixed homology, which allows to describe algorithmically all Birkhoff sections with boundary $-\overset{\leftrightarrow}{\Gamma}$.

Construction of an explicit coorientation with fixed cohomology. An Eulerian coorientation η of Γ induces a cohomology element in $H^1(S, \mathbb{Z})$, that counts the algebraic intersection of a curve γ with (Γ, η). Notice that the parity of $[\eta]$ is fixed, since $[\eta] \equiv [\Gamma] \mod 2$. Here $[\Gamma] \in H^1(S, \mathbb{Z}/2\mathbb{Z})$ is the cohomology class that counts the geometric intersection between Γ and an oriented closed curve in generic position, modulo 2. Given $\omega \in H^1(S, \mathbb{Z})$ with the expected parity, when there exists a coorientation whose cohomology is ω, we can construct such a corientation. The ideas are already present in [CD16]. This part is not mandatory for the rest of the thesis.

We denote by $(\hat{S}, \hat{\Gamma})$ the universal covering of (S, Γ). In order to define an Eulerian coorientation, we use height functions. A **height function** is a function $h : \{\text{faces of } (\hat{S}, \hat{\Gamma})\} \to \mathbb{Z}$ so that for any two adjacent faces f_1, f_2, $|h(f_1) - h(f_2)| = 1$. For any $\omega \in H^1(S, \mathbb{Z})$, we say that a height function h is ω**-stable** if for any closed curve $\gamma : [0, 1] \to S$ and any lift $\hat{\gamma}$ of γ in $\hat{S}$, we have $h(\hat{\gamma}(1)) - h(\hat{\gamma}(0)) = \omega(\gamma)$.

Lemma 1.5.6. *[CD16] There is a* $1:1$ *correspondence between Eulerian coorientations of* Γ *with cohomology* ω*, and* ω*-stable height functions on the universal covering* $(\hat{S}, \hat{\Gamma})$ *of* (S, Γ)*, up to an additive constant.*

The proof consists in viewing an Eulerian coorientation η as the gradient of a height function, which is $[\eta]$-stable.

We fix $\omega \in H^1(S, \mathbb{Z})$ with $\omega \equiv [\Gamma] \bmod 2$ in $H_1(S, \mathbb{Z}/2\mathbb{Z})$. To construct a coorientation with cohomology ω, we use the notion of partial height functions, defined below. We recursively construct a decreasing finite sequence of ω-stable partial height function. Then either the last partial height function is a real height function, or the process gives an obstruction to the existence to an ω-stable height function.

Remark 1.5.7. For any closed curve $\delta \subset S$ and any Eulerian coorientation η of Γ, one has $|\eta(\delta)| \leq |\delta \cap \Gamma|$. Hence if a cohomology element $\omega \in H^1(S, \mathbb{Z})$ satisfies $|\omega(\delta)| > |\delta \cap \Gamma|$ for some closed curve δ, then ω is not the cohomology of any Eulerian coorientation.

Denote by $\pi : \hat{S} \to S$ the universal covering map, fix a face $\hat{f}$ in the universal cover $\hat{S}$ and $f = \pi(\hat{f})$. We denote by $\Gamma^\star$ the dual graph of Γ, so that every face of $S \setminus \Gamma$ corresponds to a vertex of $\Gamma^\star$. We also denote by $\Gamma^\star_0$ and $\Gamma^\star_1$ respectively the set of vertices and edges of $\Gamma^\star$. Let $T \subset \Gamma^\star$ be a sub-tree (a connected sub-graph without cycle) of $\Gamma^\star$, which contains all vertices of $\Gamma^\star$. We use T do define a partial height function $h : \Gamma^\star_0 \to \mathbb{Z}$, that is a function that satisfies $|h(v_1) - h(v_2)| = 1$ for all pair of faces v_1 and v_2 adjacent in T. We denote by $\hat{T}$ the connected component containing $\hat{f}$ of the lift of T to $\hat{\Gamma}$, which is an isomorphic graph to T. We define the function $h = h_{T,f} : \Gamma^\star_0 \to \mathbb{Z}$ as follows. If $g \in \pi_1(S)$ and $f' \in \hat{T}$, $h(g.f') = \omega(g) + \mathrm{d}(\hat{f}, f')_{\hat{T}} = \omega(g) + \mathrm{d}(f, \pi(f'))_T$, where $\mathrm{d}(\hat{f}, f')_{\hat{T}}$ is the distance between $\hat{f}$ and f' in the tree $\hat{T}$. Since T is a tree inside $\Gamma^\star$, there are no distinct tuples $(f_1, g_1), (f_2, g_2) \in \hat{T} \times \pi_1(S)$ such that $g_1.f_1 = g_2.f_2$, so h is well defined. Also the function h defined above is ω-stable. The function h is called the ω-stable **partial height function** supported by T. Notice that there is only a finite number of trees $T \subset \Gamma^\star$, so there is a finite number of ω-stable partial height functions.

Given a tree T containing all vertices of $\Gamma^\star$, its ω-stable partial height function $h_T : \hat{\Gamma} \to \mathbb{Z}$ is not a height function in general. The following procedure modify T to decrease h and make h closer to being a height function, and eventually detects when there is no ω-stable height function. Suppose that h is not a height function. Then there exists an edge $\hat{e}$ of $\hat{\Gamma}^\star$ whose ends $\hat{v}_1, \hat{v}_2 \in \hat{\Gamma}^\star_0$ satisfy $|h(\hat{v}_1) - h(\hat{v}_2)| \neq 1$. Since h is a ω-stable and $\omega \equiv [\Gamma] \bmod 2$, $|h(\hat{v}_1) - h(\hat{v}_2)|$ is an odd number greater than 2. We can suppose that $h(\hat{v}_1) < h(\hat{v}_2)$ and that $\hat{v}_2 \in T$ (up to exchanging $(\hat{v}_1, \hat{v}_2)$ with $(g.\hat{v}_1, g.\hat{v}_2)$ for some $g \in \pi_1(S)$). For the next few paragraphs, we denote by $g \in \pi_1(S)$ the unique element such that $g.\hat{v}_1 \in \hat{T}$, together with $v_1 = \pi(\hat{v}_1)$, $v_2 = \pi(\hat{v}_2)$ and $e = \pi(\hat{e})$. There are three cases to consider, depending of the relative positions of v_1, v_2 and f inside T.

Suppose first that $v_2 = f$, we prove that there is no coorientation whose cohomology is ω. We consider the unique path c in T from v_2 to v_1, and complete c into a cycle $c' = c \cup e$, which is also seen as a closed curve inside S.

By definition, $h(\hat{v}_1) = h(g^{-1}.(g.\hat{v}_1)) = \omega(g^{-1}) + \mathrm{d}(v_1, f)_T = -\omega(g) + |c'| - 1$ and $h(\hat{v}_1) \leq h(\hat{v}_2) - 3 = h(\hat{f}) - 3 = -3$. Therefore $\omega(g) \geq |c'| + 2$. By construction, g and c' are homotopic so $|\omega(c')| > [c']$. According to Remark 1.5.7, ω is not the cohomology of any Eulerian coorientation.

Suppose now that $v_2 \neq f$. Among the edges of T adjacent to v_2, only one is in the connected component of $T \setminus v_2$ containing f. We denote by e' that edge. Then either $(T \setminus e') \cup e$ is a connected sub-graph of $\Gamma^\star$, or it is not. We first consider that $(T \setminus e') \cup e$ is not connected. Then the geodesic segment $[v_1, f]_T$ inside T contains v_2, otherwise $[v_1, f]_T \cup e$ would be a path in $(T \setminus e') \cup e$ from f to v_2 and $(T \setminus e') \cup e$ would be connected. We denote by c the segment $[v_1, v_2]_{T\setminus e'}$, which does not contain f, and $c' = c \cup e$. Then c' is homotopic to g^{-1}, so one has $h(\hat{v}_1) = \omega(c') + \mathrm{d}(v_1, f)_T = \omega(c') + |c| + \mathrm{d}(v_2, f)_T = \omega(c') + |c'| - 1 + h(\hat{v}_2)$. Since $|h(\hat{v}_1) - h(\hat{v}_2)| \geq 3$, $|\omega(c')| \geq |c'| + 2$ so ω is not the cohomology of any Eulerian coorientation.

Suppose that $v_2 \neq f$ and that $(T \setminus e') \cup e$ (as defined in the previous paragraph) is connected. Then $T' = (T \setminus e') \cup e$ is a tree, otherwise it would have too much edges. Denote by h' the ω-stable partial height function supported by T'. We claim that $h' \lneq h$. Denote by T_1 the connected component of $T' \setminus e'$ containing f, and $T_2 = T' \setminus (T_1 \cup e')$. Since T_1 is a sub-tree of both T and T', the functions h and h' coincide on the vertices of $\pi^{-1}(T_1)$. We have $g.\hat{v}_2 \in T'$, so by definition one has $h'(\hat{v}_2) = h'(g^{-1}.(g.\hat{v}_2)) = \omega(g^{-1}) + \mathrm{d}(v_2, f)_{T'} = -\omega(g) + 1 + \mathrm{d}(v_1, f)_{T'} = 1 + h'(\hat{v}_1) = 1 + h(\hat{v}_1) < h(\hat{v}_2)$ since $h(\hat{v}_1) \leq h(\hat{v}_2) + 3$. Futhermore for any vertex $v \in T_2$, the shortest paths from v to f inside T and T' contain the segment $[v_1, v_2]$. Since this segment is shorter in T', one has $\mathrm{d}(v, f)_T - \mathrm{d}(v, f)_{T'} = \mathrm{d}(v_1, v_2)_T - \mathrm{d}(v_1, v_2)_{T'} = h(\hat{v}_2) - h'(\hat{v}_2) > 0$, so on the set V of vertices of $\pi^{-1}(T_2)$, one has $h'_{|V} < h_{|V}$. Hence h' is a ω-stable partial height function lower than h.

Since the number of ω-stable partial height functions is finite, one can apply this procedure a finite number of time until eventually obtaining either an ω-stable partial height function, or come across one of the first two first cases in which ω is not the cohomology of any Eulerian coorientation. Notice that each step of the procedure can easily be translated in an algorithm.

1.5.2 Partial sections given by multi-1-foliations.

In this part, we describe the relation between the immersed partial sections of the geodesic flow on T^1S and some multi-1-foliation on S. Then we build specific immersed partial sections obtained by desingularizing a geodesic, that will be used in Chapter 4.

Multi-1-foliation. We view an immersed partial section of the geodesic flow on T^1S as a multi-vector field over S, which can be integrated into a **multi-1-foliation**, that is collection of sub-surfaces of S together with 1-foliations on each sub-surface. This point of view has already been used multiple time. We denote by $\pi : T^1S \to S$ the bundle projection. We briefly describe some properties of these multi-1-foliations. Let $\Sigma \subset T^1S$ be an immersed partial section in general position, so that the projection $\pi_{|\Sigma}$ is an immersion outside a finite number of curves. Notice that since Σ is immersed, the function $\pi_{|\Sigma}$ has no ramification points.

Denote by U a connected sub-surface of Σ on which $\pi_{|\Sigma}$ is an embedding. We view U as the graph of vector field over $\pi(U)$, which can be integrated into an oriented 1-foliation F. In general, $\pi_{|\Sigma}$ is not an embedding, so one needs to replace the notions of vector field and 1-foliation with multi-vector field and multi-1-foliation. We don't give nor use precise definitions of these notions. We can relate the foliation F to the topologically transverse property. Recall that a surface $N \subset T^1S$ is said **topologically transverse** to the flow if for any $x \in N$, there is a small neighborhood in T^1S of x on which every orbit arc of the geodesic flow intersects N exactly once. Transverse surfaces are topologically transverse. Furthermore if N is a topologically transverse surface but not transverse, there are arbitrarily small isotopies of N that make it either transverse or not topologically transverse. We use the notion of topologically transverse surface, since it is more convenient to relate to the curvature of the leaves of the multi-1-foliation.

Lemma 1.5.8. *Let $\Sigma \subset T^1S$ be an embedded surface so that $\pi_{|\Sigma} : \Sigma \to S$ is an embedding, and denote by F the foliation on $\mathrm{im}(\pi_{|\Sigma})$ induced by the vector filed $\pi_{|\Sigma}^{-1}$. then*

- *Σ is topologically transverse to the geodesic flow if and only if the leaves of F do not contain any geodesic arc nor any inflection point.*

- *Σ is transverse to the geodesic flow if and only if the leaves of F are all curved in the same direction, with non-zero curvature everywhere.*

Proof. Denote by F the 1-foliation on $\pi(\Sigma)$ induced by Σ. First suppose that the leaves of F contain no geodesic arc nor inflection point. Then any small enough geodesic arc intersects any leaf of F at most once, so Σ is topologically transversal to the flow.

Now suppose that Σ is topologically transverse to the geodesic flow. If a leaf γ of F contains an oriented geodesic arc δ, then $\vec{\delta} \subset \vec{\gamma} \subset \Sigma$, so that Σ is not topologically transverse to the flow, which is impossible.

Suppose that a leaf l of F contains an inflection point $\pi(x)$ for a point x in Σ, and denote by $l' \subset U$ the geodesic arc going though $\pi(x)$ and tangent to l. There exists a geodesic arc $\delta \subset \pi(\Sigma)$ going through x and arbitrarily close to l', such that δ intersects l three times. Then $l \cup \delta$ bounds two disjoint 2-gons $P_1, P_2 \subset \pi(\Sigma)$. Since P_i is foliated by F and one of its two sides is a leaf of F, its other side is tangent to the foliation F at a point $z_i \in \delta$. Futhermore δ can be taken arbitrarily closed to the geodesic tangent of l at x, so that z_1 and z_2 are arbitrarily closed to x. Hence δ lifts to an orbit arc which intersects Σ in two points arbitrarily closed to x, which contradict its topologically transverse property. Hence all the leaves of F do not contain any inflection point nor geodesic arc. It finishes the proof of the first item.

Before proving the second item, notice that been a transverse surface and inducing leaves with non-zero curvatures are two properties which are stable by $\mathcal{C}^1$-small perturbation on a compact subset of Σ. Suppose that Σ is transverse to the flow but it induces a leaf with a curvature zero at some point x. An arbitrarily $\mathcal{C}^1$-small perturbation of Σ around x is still transverse, but if well chosen it induces a leaf containing a geodesic arc, which contradict the first point. Similarly if all curves of F have non-zero curvatures, Σ is transverse. □

When Σ is an immersed partial section, the previous lemma applies to all open subsets $U \subset \Sigma$ so that $\pi_{|U}$ is an embedding. Notice that if γ is an oriented geodesic so that $\vec{\gamma}$ is a boundary component of Σ, then $\gamma \cap \pi(U)$ is a leaf of the 1-foliation induced by $U \subset \Sigma$. Additionally the sign of a boundary component of Σ can be determined the following way. Let γ be a small geodesic arc such that $\vec{\gamma} \subset \partial\Sigma$. There is a small tubular neighborhood U of $\vec{\gamma}$ in Σ and a homomorphism $h : \pi(U) \to [0,1] \times [0,1]$ so that $h(\gamma) = [0,1] \times \{0\}$, and Σ induces on $\pi(U)$ the foliation whose leaves are $h^{-1}([0,1] \times \{t\})$. Then either $h^{-1}([0,1] \times \{1\})$ is a convex boundary component of $\pi(U)$, or a concave one. In the first case $\vec{\gamma}$ is an arc of a negative boundary component of Σ, in the second case, it is an arc of a positive boundary component.

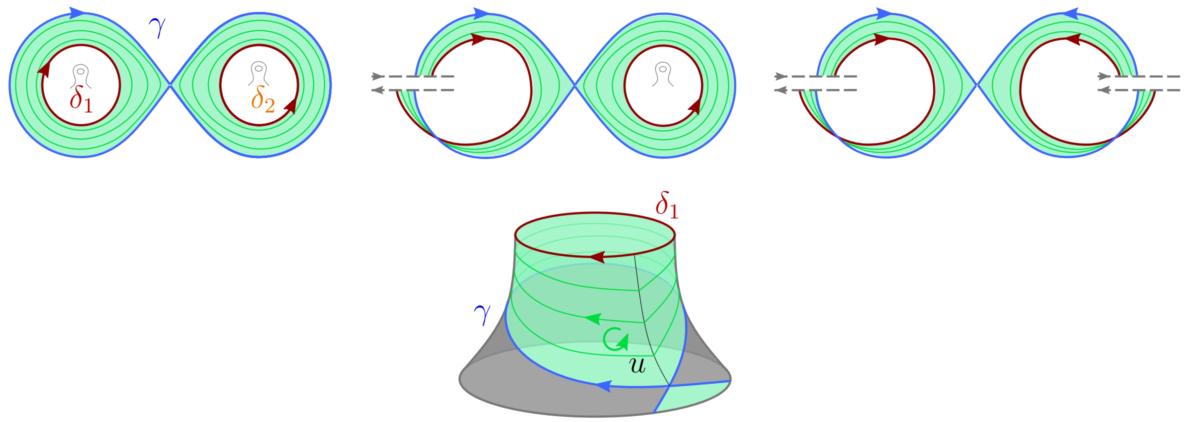


Figure 1.8: Construction of the transverse cobordism. Red geodesics are the projection of positive boundary components, blue geodesics are the projection of negative boundary component. The three surfaces illustrate the different cases for the region between δ_i and $\hat{c}_i$ in the following proofs (whether the region is orientable or not).

Desingularisation of a geodesic. Let $\gamma \subset S$ be an oriented closed geodesics. There is a nice way to create an immersed partial section Σ so that $\partial^- \Sigma = \gamma$. To do so we desingularise a self-intersecting point of γ, as in Figure 1.8.

Figure 1.8 shows three partial sections of the geodesic flow. We use immersions of these partial sections to define a cobordism for any desingularisation of one oriented geodesic. More details are given in the two following lemmas. Notice that for the geodesic flow on a hyperbolic orbifold, these immersed partial sections might not exist, since the desingularised curves (in red) might separate a cusp from the rest of the orbifold, and be homotopically trivial (so that they are not isotopic to any geodesic).

Lemma 1.5.9. *Let $\gamma \subset S$ be an oriented periodic geodesic with one and only one self-intersection point x. The two closed curves obtained by desingularising γ on x are isotopic to two geodesics δ_1 and δ_2. Furthermore there exists a transverse cobordism from $\vec{\delta_1} \cup \vec{\delta_2}$ to $\vec{\gamma}$ given by Figure 1.8.*

Proof. Denote by $\hat{c}_1$ and $\hat{c}_2$ the two closed curves obtained by desingularising γ at the double point x, and by c_1, c_2 the curves obtained by smoothing $\hat{c}_1$

and $\hat{c}_2$. The curves c_1 and c_2 are not null-homotopic, since γ is a geodesic which minimizes its length among the set of its homotopic closed curves. Thus there exist unique closed geodesics δ_1 and δ_2 isotopic respectively to c_1 and c_2. We orient δ_i accordingly to the orientation of γ. First we show that for $i = 1$ or 2, $\hat{c}_i \cup \delta_i$ co-bounds either an annulus or a triangle with a double corner. Then we lift this surface inside T^1S to half a partial section bounding by $\hat{c}_i \overrightarrow{\cup} \delta_i$ and by an arc of fibre. The sum of the two half partial sections is the required transverse cobordism.

We can suppose that c_1 and c_2 have been smoothed so that they intersect γ at most once, as in Figure 1.8. Then the geodesics δ_1 and δ_2 also intersect γ at most once (otherwise they would not minimize their length in the respective homotopy classes). Thus $\gamma \cup \delta_1 \cup \delta_2$ bounds two immersed surfaces, each of them being either an annulus with one corner, or a triangle with a double corner (which is not orientable). We will detail how to construct Σ for the annulus with one corner, the other case is similar.

Denote by A an annulus with one corner at x, bounded by $\hat{c}_1 \cup \delta_1$. The annulus A is convex, that is every two points in the universal covering $\tilde{A}$ of A are connected by a unique geodesic inside $\tilde{A}$. Thus one can fill A by oriented convexe closed curves, which are all curved in the same direction. Then this 1-foliation lifts inside T^1S to a surface Σ_1 whose interior is transverse to the flow.

By construction, the boundary of Σ_1 is the union of $\vec{\delta_1} \cup \vec{\hat{c}_1}$ and of an arc of the fiber $F_x := T^1_xS$. We can construct similarly Σ_2 with boundary $\vec{\delta_2} \cup \vec{\hat{c}_2}$ plus the same arc of fiber F_x, with the opposite direction. So after gluing and smoothing around F_x, the surface $\Sigma = \Sigma_1 \cup \Sigma_2$ is transverse to the geodesic flow, and as in Figure 1.8. An explicit computation shows that the γ is a negative boundary of Σ, and δ_1 and δ_2 are positive boundaries. □

Lemma 1.5.10. *Let $\gamma \subset S$ be an oriented periodic geodesic which is not simple, and x be a double point of γ. Then the two closed curves obtained by desingularising γ on x are isotopic to two geodesics δ_1 and δ_2, and there exists a transverse cobordism from $\vec{\delta_1} \cup \vec{\delta_2}$ to $\vec{\gamma}$.*

Proof. We consider the two curves $\hat{c}_1$ and $\hat{c}_2$ given by desingularising γ at the point x. Since γ is a geodesic, the curves $\hat{c}_1$ and $\hat{c}_2$ are non trivial in homotopy. So they are homotopic to two closed geodesics on S, which we denote by δ_1 and δ_2.

Let G be the sub-group of the fundamental group $\pi_1(S,x)$ generated by $\hat{c}_1$ and $\hat{c}_2$. We denote by $\tilde{S}_G$ the covering space induced by the group G, that is the quotient of the universal covering space of S by the group G. Then the geodesic γ lifts inside $\tilde{S}_G$ to a closed geodesic $\tilde{\gamma}$ with only one intersection point corresponding to x. Similarly the geodesic δ_i lift to a closed geodesic $\tilde{\delta}_i$ inside $\tilde{S}_G$. The proof of the previous lemma can be applied to this case, even if $\tilde{S}_G$ is not closed. So there exists a transverse cobordism $\tilde{\Sigma} \subset T^1\tilde{S}_G$ from the lifts of $\vec{\delta_1} \cup \vec{\delta_2}$ to the lift of $\vec{\gamma}$. Notice that the projections $\tilde{\gamma} \to \gamma$ and $\tilde{\delta}_i \to \delta_i$ are of degree one, so $\tilde{\Sigma}$ projects into T^1S to an immersed partial section Σ with $\partial^+\Sigma = \vec{\delta_1} \cup \vec{\delta_2}$ and $\partial^-\Sigma = \vec{\gamma}$. Then according to Proposition 1.4.13 the Fried-desingularisation of Σ is a transverse cobordism satisfying the conclusion of the lemma. $\square$

Remark 1.5.11. A similar construction may be done on two intersecting geodesics, to obtain an immersed partial section with one positive boundary and two negative boundaries. However this construction is not always possible, especially when S is not orientable. We do not use this construction later.

1.6 Blowing-up flows and Fried-Goodman surgeries

To understand a flow in a neighborhood of a closed orbit, it is convenient to blow-up the manifold along that orbit. Then we can study the induced flow on the boundary component created by the blowing-up operation. In this section, we study these flows, and give the definition of the Fried surgeries. The results of this section are already known, but they help understand some specific point needed for the last two chapters. In particular, we will see how to change the sign of one boundary of a partial section using a Fried surgery. We use an elementary approach, our arguments can be simplified using asymptotic directions, as described in [Fri83].

Fix a closed orbit $\gamma \subset M$ of a $\mathcal{C}^1$ flow ϕ. The **blowing-up of** M along γ is the manifold M_γ obtained by replacing γ by the set of rays in the normal bundle $\nu(\gamma) \to \gamma$ of γ in M. Then M_γ has a boundary component $\mathbb{T}_\gamma$ and a natural projection $\pi_\gamma : M_\gamma \to M$, which restricts to an embedding $M_\gamma \setminus \mathbb{T}_\gamma \to M \setminus \gamma$ and to a circle bundle $\mathbb{T}_\gamma \to \gamma$. When the flow ϕ is only of class $\mathcal{C}^0$, the flow $\phi_{M\setminus\gamma}$ does not necessarily extend to a flow on the compactification M_γ of $M\setminus\gamma$. Since we only consider orientable 3-manifolds M, the boundary $\mathbb{T}_\gamma$ is a torus. The flow ϕ on M lifts to a flow on M_γ that is tangent to $\mathbb{T}\gamma$, denoted by ψ. The flow $\psi_{\mathbb{T}_\gamma}$ on $\mathbb{T}_\gamma$ contains some topological information on the flow ϕ in a neighborhood of γ.

The fibers of the circle bundle $\pi_{\gamma|\mathbb{T}_\gamma}:\mathbb{T}_\gamma\to\gamma$ are global sections of $\psi_{\mathbb{T}_\gamma}$, that is transverse curves which intersect every orbit of $\psi_{\mathbb{T}_\gamma}$ in bounded time. A Fried surgery is a Dehn surgery along a closed orbit that uses the setup previously described. Let $\delta\subset\mathbb{T}_\gamma$ be a global section of $\psi_{\mathbb{T}_\gamma}$. We can push δ along the flow to define a foliation $\mathcal{F}_\delta$ of $\mathbb{T}_\gamma$ with parallel copies of δ, all transverse to the flow. We can contract every leaf of $\mathcal{F}_\delta$ to obtain a manifold M' with a projection $\pi'_\gamma:M_\gamma\to M'$. Up to continuous re-parametrization of the flow ψ, ψ induces a topological flow ϕ' on M'. The couple (M',ϕ') is said to be obtained using a **Fried surgery** on (M,ϕ).

On Anosov flows, there exists another type of surgery, called Goodman surgery. That surgery is obtained by cutting the manifold M along a small annulus transverse to the flow, and in a neighborhood of a closed orbit γ, and gluing back the two new boundary components using a good Dehn twist on that annulus (see [Goo83, Sha20] for more details). The manifold and the flow obtained are smooth. Under some good conditions, the new flow is additionally Anosov. If the flow ϕ is transitive and if the orbit γ has orientable stable and unstable foliations, M.Shannon proved [Sha20] that an Anosov flow obtain by a Goodman surgery on γ is orbitally equivalent to a flow obtain by a Fried surgery, along a curve δ which intersects twice the trace of the stable leaf of γ. Hence we speak about **Fried-Goodman** surgeries in this case, which transforms a smooth Anosov flow into another smooth Anosov flow, well-defined up to some orbital equivalence.

Consider a transitive Anosov flow on an orientable 3-manifold, and a closed orbit γ in the general case. One can still do a Fried surgery along γ and hope to construct a smooth Anosov flow, up to an orbital equivalence. Denote by M_γ the blowing up manifold of M along γ, by $\mathbb{T}_\gamma$ its boundary component and by $\pi:\mathbb{T}_\gamma\to S^1$ a bundle. Suppose that the fibers $\pi^{-1}(cste)$ are all transverse curves inside $\mathbb{T}_\gamma$, and all intersect the trace of the stable leaf of γ exactly twice. We do a Fried surgery by contracting the fibers of π and denote by M' and ϕ' the induced manifold and flow. By construction, the flow ϕ' is transitive, preserves two regular and transverse 2-foliations and satisfy an expanding property defined below.

Denote by $\tilde{M}'$ the universal covering space of M', and lift the flow ϕ' to a flow $\tilde{\phi}'$ on $\tilde{M}'$. The flow ϕ' is said to be orbitally expansive if for all $\alpha > 0$, for all pair of points x, y inside $\tilde{M}'$, if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that the points $\tilde{\phi}'_t(x)$ and $\tilde{\phi}'_{h(t)}(y)$ remain at most at a distance α, then x and y are on the same orbit. Since the flow $\tilde{\phi}'$ is obtained as the Fried surgery of an Anosov flow, it satisfies the orbitally expansive property. Under that two conditions: expansive properties and preserving two regular and transverse 2-foliations, M.Shannon proved [Sha20] that ϕ' is orbitally equivalent to a smooth Anosov flow.

Remark 1.6.1. As explained above, a Fried surgery transform a smooth transitive Anosov flow into a flow which is orbitally equivalent to another smooth transitive Anosov flow. We also call Fried-Goodman surgery this surgery.

Remark 1.6.2. Let Σ be a partial section for a transitive Anosov flow ϕ and ϕ' be an Anosov flow obtained by a Fried-Goodman surgery along a closed orbit γ. As explained in Section 1.4, one can isotope the interior of Σ into the interior of another partial section which given locally around γ by some local models with good properties (given in Example 1.4.3). In particular after the Fried-Goodman surgery, the surface induced by Σ is not necessarily a partial section, but there is an isotopy from its interior to a partial section Σ' of the flow ϕ' (as detailed in Lemma 1.4.12). We say that Σ' is the partial section induced by Σ after the Fried-Goodman surgery.

The surface Σ' can be chosen to be embedded in its interior if and only if Σ is embedded in its interior. Additionally Σ' is a Birkhoff section for ϕ' if and only if Σ is a Birkhoff section for ϕ.

Question (Ghys question)**.** Is any transitive flow equivalent to an Anosov suspension up to some Fried-Goodman surgeries ?

As suggested by what follows, the property of being equivalent to an Anosov suspension up to some Fried-Goodman surgeries is equivalent to admitting a Birkhoff section with genus 1 (for an Anosov flow). For now, it is only known that the suspension flows of all hyperbolic matrices that preserve the orientation, and the geodesic flows of all orientable hyperbolic surfaces (even orbifolds) are equivalent in this sense. But the Ghys' question remains open in general.

The surface $\mathbb{T}_\gamma$ admits local maps modeled by the trivial bi-foliation on $\mathbb{R}^2$, foliated horizontally by fibers of $\pi_{\gamma|\mathbb{T}_\gamma} : \mathbb{T}_\gamma \to \gamma$, and vertically by the orbits of $\psi_{|\mathbb{T}_\gamma}$. These maps can be used to compare the slopes of two simple curves δ_1 and δ_2 in T_γ without giving global coordinates on $\mathbb{T}_\gamma$: We pull tight δ_1 and δ_2 along the flow so that they always intersect with the same algebraic sign, and orient them using the coorientation by $\psi_{\mathbb{T}_\gamma}$. Then δ_1 has a **higher slope** than δ_2 in every local maps if and only if $\delta_1 \frown \delta_2 < 0$. Also the slope of δ_1 is said positive if δ_1 has a higher slope than any fiber of $\pi_{\gamma|\mathbb{T}_\gamma}$.

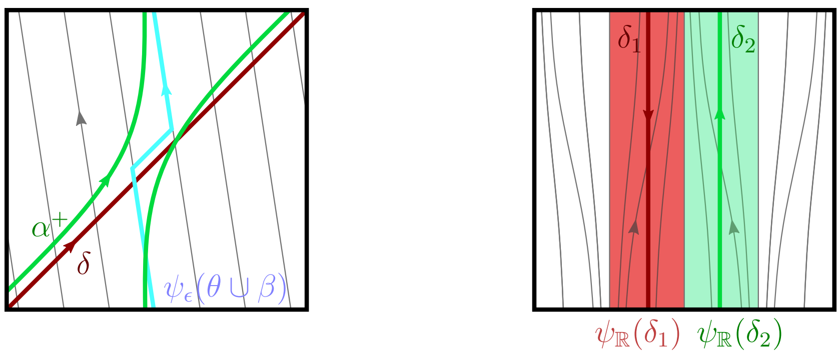


Figure 1.9: Transverse curves in a torus: comparaison of finite slopes on the left, two curves of infinite slopes on the right.

Thanks to the following lemma, we can interpret a global section of $\psi_{|\mathbb{T}_\gamma}$ as a curve with **finite slope**, and a transverse curve not intersecting all orbits of $\psi_{|\mathbb{T}_\gamma}$ as a curve with **infinite slope**. When δ has infinite slope, $\phi_\mathbb{R}(\delta) \subsetneq \mathbb{T}_\gamma$ is an annulus invariant by ψ, that can be thought of as the sector of δ. The behavior of transverse curves with finite and infinite slopes are detailed in the following statements.

Lemma 1.6.3. *Let δ be a transverse curve in T_γ. There exist transverse curves $\alpha_\pm$ with higher and lower slopes than δ if and only if δ intersects every orbit of $\psi_{|\mathbb{T}_\gamma}$. The curves $\alpha_\pm$ can be chosen to be global sections.*

When ϕ is Anosov, then $\alpha^\pm$ can additionally be taken so as to intersect $\mathcal{F}^s$ and $\mathcal{F}^u$ exactly twice.

Proof. Suppose that δ intersects every orbit of $\psi_{|\mathbb{T}_\gamma}$. Then by compactness of T_γ, δ intersects every orbit of $\psi_{|\mathbb{T}_\gamma}$ in bounded time. So there exists an arc θ of orbit of $\psi_{|\mathbb{T}_\gamma}$ that intersects δ exactly at its two ends. Let β be the arc of $\delta \setminus \partial\theta$ so that $\partial\beta = -\partial\theta$. Then for $\epsilon > 0$ small enough, $\psi_\epsilon(\theta \cup \beta)$

intersects δ only once, and positively. As in Figure 1.9, $\psi_\epsilon(\theta \cup \beta) \cup \delta$ can be smoothed into a curve $\alpha_+ \subset \mathbb{T}_\gamma$ transverse to $\psi_{|\mathbb{T}_\gamma}$ with higher slope than δ. Additionally α^+ intersects every orbit of $\psi_{|\mathbb{T}_\gamma}$, since δ does.

Suppose that δ does not intersect all the orbits of $\psi_{|\mathbb{T}_\gamma}$. Since the meridian m of $\pi : M_\gamma \to M$ intersects every orbit of $\psi_{|\mathbb{T}_\gamma}$, δ and m are not isotopic. Hence we can suppose that the projection $\pi(\delta)$ is a positive multiple of γ, up to changing all signs. Then suppose that there exists a transverse curve α with higher slope than δ. We pull α and δ tight so that they only intersect positively. Since δ does not intersect every orbit of $\psi_{|\mathbb{T}_\gamma}$, $\psi_\mathbb{R}(\delta)$ is an open subset invariant by $\psi_{|\mathbb{T}_\gamma}$, and not all $\mathbb{T}_\gamma$. So $\partial\psi_\mathbb{R}(\delta)$ contains a closed orbit δ_+ of $\psi_{|\mathbb{T}_\gamma}$, on which accumulates $\psi_t(\delta)$ for $t \to +\infty$. Also δ_+ does not intersect δ. Since δ_+ is isotopic to δ, $\delta_+ \cap\!\!\!\cdot\, \alpha = \delta \cap\!\!\!\cdot\, \alpha > 0$. It contradicts that δ_+ is an orbit of $\psi_{|\mathbb{T}_\gamma}$ and α is oriented by $\psi_{|\mathbb{T}_\gamma}$, so that $\delta_+ \cap\!\!\!\cdot\, \alpha \leq 0$.

Suppose that ϕ is Anosov. Let $l^s \in \mathcal{F}^s \cap \mathbb{T}_\gamma$ be a closed leaf of $\psi_{|\mathbb{T}_\gamma}$, and f be a fiber of $\mathbb{T}_\gamma \to \gamma$, which intersects $\mathcal{F}^s$ exactly twice. Then $\alpha^\pm$ can be taken as the transverse desingularisation of $f + n.l^s$ for $n \in \mathbb{Z}$, which all intersect $\mathcal{F}^s$ exactly twice and have higher and slower slopes than δ for a well-chosen n. $\square$

Lemma 1.6.4. *Let δ_1 and δ_2 be two simple closed curves transverse to $\psi_{|\mathbb{T}_\gamma}$, so that $\pi_\gamma(\delta_1)$ and $\pi_\gamma(\delta_2)$ are multiples of γ with opposite signs. Then either $\delta_1 \cap \delta_2 \neq \emptyset$ or $\psi_\mathbb{R}(\delta_1) \cap \psi_\mathbb{R}(\delta_2) = \emptyset$. In the second case, both curves have infinite slope.*

Figure 1.9 shows two isotopic curves in the case $\psi_\mathbb{R}(\delta_1) \cap \psi_\mathbb{R}(\delta_2) = \emptyset$.

Proof. Suppose that $\psi_\mathbb{R}(\delta_1) \cap \psi_\mathbb{R}(\delta_2) \neq \emptyset$ and $\delta_1 \cap \delta_2) = \emptyset$. Then there exists an orbit α of ψ that intersects δ_1 and δ_2. Also $\delta_1 \cap \delta_2 = \emptyset$, so δ_1 and δ_2 are parallel simple curves of $\mathbb{T}_\gamma$. Hence the orbit α co-orients δ_1 and δ_2 in the same way, that is the two curves are isotopic as oriented curves. This contradicts that $\pi_\gamma(\delta_1)$ and $\pi_\gamma(\delta_2)$ are multiples of γ with opposite signs. $\square$

Boundary of partial sections Let M be an orientable closed 3-manifold and ϕ be a smooth flow on M. We denote by M_γ the blowing up manifold of M along γ, by $\pi_\gamma : M_\gamma \to M$ the projection, by $\mathbb{T}_\gamma = \pi_\gamma^{-1}(\gamma)$ the boundary component of M_γ induced by γ, and denote by ψ the flow induced on M.

Consider an immersed partial section Σ inside M. We can lift the surface Σ to a surface inside the blowing up manifold M_γ. To do that we lift the interior of $\Sigma \setminus \gamma$ using the projection π_γ, and we lift the boundary of Σ on γ

using the direction tangent to Σ and transverse to γ. The immersed surface obtained that way is also equal to the adherence of $\pi_\gamma^{-1}(\mathring{\Sigma})$ inside M_γ. we say that this surface is **induced by** Σ inside the blowing up manifold M_γ. Notice that the surface induced by Σ is transverse to the flow in its interior, but non necessarily along its boundary components inside the torus $\mathbb{T}_\gamma \subset \partial M_\gamma$. We review below how the boundary of an immersed partial section inside $\mathbb{T}_\gamma$ can be related to some geometric properties of the immersed partial section.

Lemma 1.6.5. *Let Σ be an immersed partial section and γ be a boundary component of Σ. Then there is a smooth homotopy along the flow from Σ to an immersed partial section Σ' such that the boundary of $\pi_\gamma^{-1}(\mathring{\Sigma})$ inside $\mathbb{T}_\gamma$ is a union of closed leaf of $\psi_{|\mathbb{T}_\gamma}$ and of closed curves transverse to $\psi_{|\mathbb{T}_\gamma}$.*

Proof. Denote by δ a curve inside $\partial\pi_\gamma^{-1}(\mathring{\Sigma}) \cap \mathbb{T}_\gamma$, which is a smooth curve. When δ corresponds to an intersection between the orbit γ and the interior of Σ, δ is a fiber of $\mathbb{T}_\gamma \to \gamma$, so it is transverse to the flow $\psi_{|\mathbb{T}_\gamma}$. Now we suppose that the curve δ corresponds to the boundary of the immersed partial section Σ. In that case, since the boundary of Σ is immersed, the curve δ is transverse to the fibers of the bundle $(\pi_\gamma)_{|\mathbb{T}_\gamma} : \mathbb{T}_\gamma \to \gamma$.

Since the interior of Σ is transverse to the flow, it is oriented. Hence there is a natural orientation of δ induced by the orientation of Σ. Consider a local coordinate system $\mathbb{R}^2$ on $\mathbb{T}_\gamma$ such that the orbit arc of the flow are horizontal in the coordinate system, and the fibers of $\mathbb{T}_\gamma \to \gamma$ are vertical. Since δ is transverse to the fibers of $\mathbb{T}_\gamma \to \gamma$, δ corresponds in the local coordinate system to the graph of a smooth function $f : \mathbb{R} \to \mathbb{R}$. If δ is not monotone in the local coordinate system, then there exist two points on which f has a positive and a negative derivative. Then at these points the flow induces two opposite coorientations on δ. Consider one orbit arc which intersects δ negatively. Then arbitrarily close to that orbit arc there exists an other orbit arc which intersects the interior of Σ negatively, which is impossible. Hence δ corresponds in local coordinate system to the graph of a monotone smooth function.

If δ is not a closed leaf, we can find a second curve δ' inside $\mathbb{T}_\gamma$, which remains arbitrarily close to δ, transverse to the fibers of the bundle $\mathbb{T}_\gamma \to \gamma$, in a small neighborhood of δ and such that there exists a continuous function $T : \delta \to \mathbb{R}$ such that the function $(x \in \delta) \to \psi_{T(x)}(x)$ has image δ' and is of degree one. Then we extends T to a smooth function on a small neighborhood of δ inside $\pi_\gamma^{-1}(\mathring{\Sigma})$. On a smaller neighborhood of γ inside $\pi_\gamma^{-1}(\mathring{\Sigma})$, the image $(x \in$

$\delta) \to \psi_{T(x)}(x) \in M_\gamma$ is a smooth surface transverse to the flow, and its image in M is an immersed partial section. Hence we can find a homotopy along the flow from the interior of Σ to the interior of another immersed partial section whose boundary induces inside $\mathbb{T}_\gamma$ only closed leaf and transverse curves. $\square$

As discuss in Section 1.4, for Anosov flows we can isotope in its interior a partial section Σ to another partial section, which is given on small neighborhoods of it boundary by local models. In particular for these local models, their boundary in the blowing up manifold is always transverse to the flow.

Lemma 1.6.6. *Let Σ be an immersed Birkhoff section for ϕ. Denote by $\delta \subset \mathbb{T}_\gamma$ one boundary component of the surface induced by Σ in the blowing manifold M_γ. Then δ intersects every orbit of the flow $\psi_{|\mathbb{T}_\gamma}$.*

Proof. Denote by Σ_γ the surface induced by Σ inside the blowing up manifold M_γ, and let $x \in \mathbb{T}_\gamma$. Take $(x_n)_{n\in\mathbb{N}}$ be a family of points in $M_\gamma \setminus T_\gamma$ that converge to x. Since Σ is an immersed Birkhoff section, there exists $T > 0$ and for every $n \in \mathbb{N}$ there exists $0 \leq t_n \leq T$ so that $\psi_{t_n}(x_n)$ is inside Σ_γ. Up to taking a subfamily of $(x_n)_n$, we can suppose that $(t_n)_n$ converges to $0 \leq t \leq T$, so that $\psi_t(x) \in \Sigma_\gamma \cap \mathbb{T}_\gamma$. $\square$

Lemma 1.6.7. *Let γ be a closed orbit of ϕ, Σ be an immersed partial section for ϕ and Σ_γ be the surface induced by Σ inside the blowing up manifold M_γ. We suppose that Σ_γ has a boundary component δ immersed inside $\mathbb{T}_\gamma$, which is transverse to the flow. We still denote by $\delta \subset \mathbb{T}_\gamma$ the curve induced by this boundary component. Then any Fried surgery with slope higher (resp. lower) than δ makes δ a negative (resp. positive) boundary component of Σ after surgery. If δ is a global section of $\psi_{|T_\gamma}$, then the Fried surgery with meridian δ erases δ as boundary component of Σ after surgery.*

Now we suppose that Σ_γ has a boundary component δ immersed inside $\mathbb{T}_\gamma$, which is either a closed leaf of the flow $\psi_{|T_\gamma}$ or a transverse curve with infinite slope. Then for any Fried surgery along the orbit γ, the curve δ induces a boundary component after surgery which has the same sign than δ as boundary component of Σ.

Proof. Take δ a fiber of $\pi_{\gamma|\mathbb{T}_\gamma}$, and δ' a transverse curve in T_γ of finite slope, associated respectively to the projection $\pi'_\gamma : M_\gamma \to M'$. Denote by $\Sigma' = \pi'_\gamma \circ \pi_\gamma^{-1}(\Sigma)$ the immersed partial section in M' induced by Σ. We orient the curves in T_γ transversally to ψ using the coorientation by ψ and a fix orientation of T_γ. The sign of δ as boundary of Σ' is the sign of the algebraic intersection $\delta \bar{\cap} \delta'$. Hence if δ' has a higher slope than δ, then we have $\delta \bar{\cap} \delta' > 0$ and δ is a positive boundary of Σ'.

The same argument can be applied for the other cases. $\square$

One last lemma is needed to specify the linking numbers of Birkhoff sections along their boundary components.

Lemma 1.6.8. *Let Σ_1, Σ_2 be two partial sections embedded in their interiors, and $\gamma \in \partial^-\Sigma_1 \cap \partial^+\Sigma_2$ a common boundary component with opposite signs. If Σ_1 or Σ_2 is a Birkhoff section, then* $\mathrm{lk}_\gamma(\Sigma_1, \Sigma_2) \neq 0$.

This lemma is easy to prove for Anosov flows, by using the stable foliation, and needs the previous tools for general flows.

Proof. We define M_γ, π_γ and $\mathbb{T}_\gamma$ as above. As explained in Lemma 1.6.5, there exists two homotopies along the flow from the interior of Σ_1 and Σ_2 to the interior of two immersed partial sections Σ_1' and Σ_2', such that the every boundary component of the induced surfaces Σ_i' inside the torus $\mathbb{T}_\gamma$ is either a closed leaf of the flow $\psi_{|\mathbb{T}_\gamma}$ or is transverse to the flow $\psi_{|\mathbb{T}_\gamma}$. then we have $\mathrm{lk}_\gamma(\Sigma_1, \Sigma_2) = \mathrm{lk}_\gamma(\Sigma_1', \Sigma_2')$.

Up to exchanging all indices, we suppose that Σ_1 is an immersed Birkhoff section. Consider a boundary component $\delta_1 \subset \mathbb{T}_\gamma$ of the lift of Σ_1'. Since Σ_1 is an immersed Birkhoff section, Σ_1' is also an immersed Birkhoff section. So according to Lemma 1.6.6, the curve δ_1 is a global section of the flow inside $\mathbb{T}_\gamma$.

Similarly consider a boundary component $\delta_2 \subset \mathbb{T}_\gamma$ of the lift of Σ_2'. If δ_2 is a closed orbit of the flow, since δ_1 is a global section of the flow inside $\mathbb{T}_\gamma$, the curves δ_1 and δ_2 intersect, and all there intersections have the same algebraic sign (which depends only on the orientation of δ_2 relatively to the orientation of the flow). So the linking number between these boundary components is non-zero. If δ_2 is transverse to the flow, we follow the next argument.

Denote by $\delta^+, \delta^- \subset \mathbb{T}_\gamma$ the non-empty unions of curves induced by the boundary components in γ of Σ_1' and Σ_2' corresponding to positive and negative boundary components, and which are transverse to the flow inside the torus $\mathbb{T}_\gamma$. If we suppose that $\psi_\mathbb{R}(\delta^+)$ and $\psi_\mathbb{R}(\delta^-)$ are disjoint, and are open. Then, by connectedness of $\mathbb{T}_\gamma$, $\mathbb{T}_\gamma \setminus (\psi_\mathbb{R}(\delta^+) \cup \psi_\mathbb{R}(\delta^-))$ is not empty, which contradicts Lemma 1.6.6.

Hence $\psi_\mathbb{R}(\delta^+) \cap \psi_\mathbb{R}(\delta^-) \neq \emptyset$. By Lemma 1.6.4, the intersection $\delta^+ \cap \delta^-$ is not empty. We can isotope $\delta^+ \cup \delta^-$ along the flow such that if they contain two homologous curves inside $\mathbb{T}_\gamma$, then these homologous curves do not intersect. Then if two curves $a \subset \delta^+$ and $b \subset \delta^-$ have a geometric intersection, there algebraic intersections are non-zero.

Hence in both cases, the curves induced by Σ_1' and Σ_2' inside $\mathbb{T}_\gamma$ have a non-zero algebraic intersection. Since $\mathring{\Sigma}_1$ and $\mathring{\Sigma}_2$ are embedded, all curves in $\partial\pi_\gamma^{-1}(\Sigma_i')$ are homologous, for a fixed i. So the linking number $\mathrm{lk}_\gamma(\Sigma_1', \Sigma_2')$ is obtained is a positive multiple of the algebraic intersection of two curves δ_1, δ_2 in $\mathbb{T}_\gamma$ induced by the boundary of Σ_1' and Σ_2', which is not zero. $\square$

1.7 Partial sections of R-covered Anosov flows

In this section, we detail the construction of two types of immersed partial sections which will be used in the last two chapters: the Birkhoff annuli and the Fried sections. Before that we introduce the trace of a section in the orbit space, and give the general construction of a ramified partial section using its fundamental domain in the orbit space.

1.7.1 Trace of a section in the orbit space

Recall that M denotes a 3-manifold, $p : \tilde{M} \to M$ the canonical projection of its universal cover, ϕ an Anosov flow on M, and $\pi : \tilde{M} \to \mathcal{O}(M)$ the projection on the orbit space. Let $\Sigma \subset M$ be a connected immersed partial section and $\tilde{\Sigma}$ be a connected component of the lift of Σ to $\tilde{M}$. We call $\rho(\Sigma) = \pi(\tilde{\Sigma})$ the **trace** of Σ in $\mathcal{O}(M)$, also denoted by $\Theta_M(\Sigma)$. If Σ is a Birkhoff section, the trace of Σ is all of $\mathcal{O}(M)$. Hence the trace is interesting only for studying non-Birkhoff sections (that is immersed partial sections not intersecting all orbits). It is often convenient to take the trace of an immersed partial section Σ in $\mathcal{O}(M \setminus \partial\Sigma)$ instead, which is explained below. If $\Gamma \subset M$ is a finite set of closed orbits of the flow, we construct $M' = M \setminus \Gamma$, its universal covering $\tilde{M}'$ and its orbit space $\mathcal{O}(M \setminus \Gamma) = \tilde{M}'/(\text{orbits of the flow})$, which is also the universal covering space of $\mathcal{O}(M) \setminus (\pi(\text{pre-image of } \Gamma))$. The trace of Σ in $\mathcal{O}(M \setminus \Gamma)$, denoted by $\Theta_{M\setminus\partial\Sigma}(\Sigma)$, is obtained by lifting $\Sigma \setminus (\Sigma \cap \Gamma)$ to $\tilde{M}'$ and projecting it inside $\mathcal{O}(M \setminus \Gamma)$.

S.Fenley studied the boundary of the traces of transverse surfaces of a pseudo-Anosov flow [Fen99]. Given an immersed closed surface $\Sigma \subset M$ transverse to the flow, S.Fenley's question is whether Σ lifts to a global section inside some finite covering of M. This property is related to the topology of the foliations $\Sigma \cap \mathcal{F}^s$ and $\Sigma \cap \mathcal{F}^u$ of Σ, and with the trace of Σ inside $\mathcal{O}(M)$. We are interested in a similar question, but for partial sections with boundary and which are embedded in their interiors. We state below three lemmas and theorems from S.Fenley adapted to our hypotheses.

Theorem 1.7.1. *Let M be a closed 3-manifold and ϕ be an Anosov flow on M, and $\Sigma \subset M$ be a connected partial section embedded in its interior. Then:*

- *Consider $\tilde{\mathring{\Sigma}}$ a connected lift of $\mathring{\Sigma}$ in the universal covering of $M \setminus \partial\Sigma$. Then the map $\pi : \tilde{\mathring{\Sigma}} \to \mathcal{O}(M \setminus \partial\Sigma)$ is injective, equivalently, $\tilde{\mathring{\Sigma}}$ intersects at most once every orbit in the universal covering space of $M \setminus \partial\Sigma$.*

- *Σ is a Birkhoff section if and only if the trace of Σ on $\mathcal{O}(M \setminus \partial\Sigma)$ is all of $\mathcal{O}(M \setminus \partial\Sigma)$.*

- *When Σ is not a Birkhoff section then the boundary of the trace of Σ on $\mathcal{O}(M \setminus \partial\Sigma)$ is a union of stable and unstable leaves.*

In the third point of the theorem, the boundary component of the trace of Σ inside $\mathcal{O}(M \setminus \partial\Sigma)$ can correspond to half stable and unstable leaves of a closed orbit which is a boundary component of Σ. This phenomenon is precised further in the section.

We prove each point independently, starting with the first point proved below.

Proof of the first point. We prove first that $\pi : \tilde{\mathring{\Sigma}} \to \mathcal{O}(M \setminus \partial\Sigma)$ is injective. Suppose that it is not the case, that is there exist $x, y \in \tilde{\mathring{\Sigma}}$ such that $x \neq y$ and $\pi(x) = \pi(y)$. Since $\tilde{\mathring{\Sigma}}$ is connected, there exists a path $\delta_1 \subset \tilde{\mathring{\Sigma}}$ from y to x. Since $\pi(x) = \pi(y)$ and $x \neq y$, there exists a non-trivial orbit arc δ_2 between x and y. Up to inverting x and y, we can suppose that the flow orients δ_2 from x to y. By construction, for $\epsilon > 0$ small enough, $\phi_\epsilon(\delta_2) \cap \tilde{\mathring{\Sigma}}$ contains y so it is not empty. Also $\phi_\epsilon(\delta_1) \cap \tilde{\mathring{\Sigma}} = \emptyset$ since $\tilde{\mathring{\Sigma}}$ is embedded, transverse to the flow and $\delta_1 \subset \tilde{\mathring{\Sigma}}$. Hence $\delta_1 \cup \delta_2$ is a closed curve and its algebraic intersection with $\tilde{\mathring{\Sigma}}$ is $(\delta_1 \cup \delta_2) \pitchfork \tilde{\mathring{\Sigma}} = \tilde{\phi}_\epsilon(\delta_1 \cup \delta_2) \pitchfork \tilde{\mathring{\Sigma}} = |\tilde{\phi}_\epsilon(\delta_2) \cap \tilde{\mathring{\Sigma}}| \neq 0$. Furthermore $\delta_1 \cup \delta_2$ is a closed curve in the universal covering of $M \setminus \partial\Sigma$, so that its image inside $M \setminus \partial\Sigma$, denoted by δ is a null homologous closed curve. Hence algebraically $\delta \cap \mathring{\Sigma} = 0$ which is a contradiction. Hence $\pi : \tilde{\mathring{\Sigma}} \to \mathcal{O}(M \setminus \partial\Sigma)$ is injective. $\square$

S.Fenley used other arguments for this proof, which work for immersed transverse surfaces, but with no boundary.

Remark 1.7.2. Consider a stable leaf l^s in the boundary of the trace of Σ inside $\mathcal{O}(M \setminus \partial\Sigma)$. Suppose that there exist an element $g \in \pi_1(\Sigma)$ which is not trivial in homotopy inside M, and a point x inside l^s which satisfy $g.x = x$. S.Fenley proved that the curve l^s corresponds to a closed leaf of the foliation $\Sigma \cap \mathcal{F}^u$ on Σ. Indeed the unstable leaf l^u of x intersects the trace of Σ in a leaf arc, which is invariant by g by hypothesis. Hence that leaf arc corresponds inside the surface Σ to a close leaf of the foliation $\mathcal{F}^u \cap \mathring{\Sigma}$. We can erase that close leaf with the following procedure.

Let Σ be an immersed partial section, and let l be a closed stable or unstable leaf of $\mathcal{F}^s \cap \mathring{\Sigma}$. Then S.Fenley prove [Fen99] that the leaf of $\mathcal{F}^s$ which contains l also contains a closed orbit γ of the flow. Denote by the F the compact connected component of the stable leaf $\mathcal{F}^s(\gamma)$ minus the curves $\gamma \cup l$. Then one can cut Σ along the curve l and isotope it on a small neighborhood of F to obtain a new immersed partial section Σ' with two new boundary components $+\gamma$ and $-\gamma$ both with linking number zero. This idea is detailed at the end of the proof of Proposition 1.4.13. We obtain back Σ by desingularisating Σ' along these two new boundary components. Since the stable foliation has non trivial holonomy along a closed leaf of $\Sigma \cap \mathcal{F}^s$, the closed leaves of $\Sigma \cap \mathcal{F}^s$ are in finite quantity (and similarly for $\Sigma \cap \mathcal{F}^u$). Thus one can cut along a finite number of closed leaf to obtain an immersed partial section with no closed leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$ in its interior.

Trace of a partial section with linking number zero. Let Σ be a connected partial section embedded in its interior. We suppose that Σ has a boundary component γ with a linking number zero. Denote by U a connected component of a small closed tubular neighborhood of γ inside Σ. If Σ has several boundary components immersed into γ, we only consider one of them in this paragraph. Under some convenient hypothesis on U, we prove below that the trace of U inside $\mathcal{O}(M)$ is a triangle with two ideal points at infinity and whose edges are a half stable leaf l^s, a half unstable leaf l^u of $\rho(\gamma) \in \mathcal{O}(M)$ and a third curve, such that the interiors of l^s and l^u lift inside $\mathcal{O}(M \setminus \partial\Sigma)$ to two boundary components of the trace of Σ in $\mathcal{O}(M\setminus\Gamma)$. A precise statement is given in Proposition 1.7.7

We precise below some notations we use until the end of that subsection. The tubular neighborhood U of γ has the topology of an annulus, so it has another boundary component that we denote by $\delta \subset M$. Denote $\tilde{M}$ and $\widetilde{M \setminus \partial\Sigma}$ the universal covering of M and $M\setminus\partial\Sigma$. We lift the interior of Σ into a connected surface $\tilde{\Sigma}$ inside $\widetilde{M \setminus \partial\Sigma}$, we lift U into a connected strip $\tilde{U}$ inside $\tilde{M}$ and we denote by $\tilde{\gamma}$ and $\tilde{\delta}$ the boundary components of $\tilde{U}$ which respectively project into γ and δ inside M. Since the annulus U is a subset of Σ, we can suppose that the lifted surface $\tilde{\Sigma} \subset \widetilde{M \setminus \partial\Sigma}$ projects inside $\tilde{M}$ to a surface which contain the annulus $\tilde{U}\setminus\tilde{\gamma}$. We also denote by $\pi : \tilde{M} \to \mathcal{O}(M)$ the quotient map, so that $\pi(\tilde{U})$ is the trace of U inside $\mathcal{O}(M)$.

Lemma 1.7.3. *We can choose the neighborhood U arbitrarily small and such that its boundary component δ is transverse to the stable and unstable foliation of the flow, and such that the projection $\pi : \tilde{M} \to \mathcal{O}(M)$ is injective on $\tilde{U} \setminus \tilde{\gamma}$.*

Proof. We consider the neighborhood U of γ in Σ defined as above, and we find inside U a smaller annulus with the good property. We consider the suspension manifold $N = \mathbb{R}^3/(x, y, z) \equiv (\lambda x, \mu y, z + 1)$ for two real numbers λ and μ such that $\lambda\mu > 0$, $|\lambda| > 0$ and $0 < |\mu| < 1$. We also consider the flow ψ on N given by $\psi_t(x, y, z) = (x, y, z + 1)$, and its closed orbit $\gamma_N = (0, 0) \times (\mathbb{R}/\mathbb{Z})$. According to Lemma 1.4.2, the values of λ and μ can be chosen such that there exist a smooth orbital equivalence between the flow ϕ on a small neighborhood of γ in M and the flow ψ on a small neighborhood of γ_N in N. We consider the coordinate system around γ given by a neighborhood of γ_N in N.

We define the annulus $A_N = \{(\epsilon_1|\lambda|^t r, \epsilon_2|\mu|^t r, t) \in N | r \in \mathbb{R}_+^\star, t \in \mathbb{R}\}$ for some $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. According to Proposition 1.4.11, we can chose $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ such that there exists an isotopy along the flow from a tubular neighborhood of γ inside $U\tilde{\gamma}$ to a neighborhood of γ_N inside the annulus A_N. For $r_0 > 0$ small enough, the curve $(t \in \mathbb{R}) \to (\epsilon_1|\lambda|^t r_0, \epsilon_2|\mu|^t r_0, t)$ is transverse to the stable and unstable foliation, and bound a smaller annulus than U. Hence we can consider that small neighborhood of γ, that we denote by U'. We also denote by δ' its boundary component that is not γ.

The annulus $U' \setminus \gamma$ is given as a sub-annulus of the surface A, on which the trace of the stable and the unstable foliation can explicitly be computed. The foliations $U' \cap \mathcal{F}^s$ and $U' \cap \mathcal{F}^u$ are transverse and both homeomorphic to the foliation on $\mathbb{R}_+ \times S^1$ whose leaves are the curves $\mathbb{R}_+ \times \{cste\}$. Hence every pair of disjoint points $x, y \in \tilde{U} \setminus \tilde{\gamma}$ can be connected by an arc in $\tilde{U} \setminus \tilde{\gamma}$ transverse either to the foliation $U' \cap \mathcal{F}^s$ or to $U' \cap \mathcal{F}^u$. Since the map π is locally injective on $\tilde{U} \setminus \gamma$, the images $\pi(x)$ and $\pi(y)$ inside $\mathcal{O}(M)$ can be connected by an arc transverse to one of the two stable and unstable foliation $\mathcal{L}^s$ and $\mathcal{L}^u$. Since the orbit space is simply connected, there is no closed curve inside $\mathcal{O}(M)$ which are transverse to the stable or the unstable foliation. Hence $\pi(x)$ and $\pi(y)$ are disjoint. So $\pi : \tilde{M} \to \mathcal{O}(M)$ is injective on $\tilde{U} \setminus \tilde{\gamma}$. □

We consider $\pi(\tilde{\delta})$ the trace of δ inside $\mathcal{O}(M)$, and $\pi(\tilde{\gamma})$ which is a single point of $\mathcal{O}(M)$.

The partial section Σ has a linking number zero along γ, so according to 1.4.4 the germ of U at γ remains in a quadrant of γ delimited by the half stable leaf $\mathcal{F}^s(\gamma)$ and the half unstable leaf $\mathcal{F}^u$. We lift this quadrant to $\tilde{M}$ and projects it into $\mathcal{O}(M)$, and we denote by l^s and l^u the stable and unstable half leaves delimiting this quadrant inside $\mathcal{O}(M)$.

Recall that we denote by $\mathcal{L}^s$ and $\mathcal{L}^u$ the stable and unstable foliations on $\mathcal{O}(M)$. We fix an orientation on the foliations $\mathcal{L}^s$ and $\mathcal{L}^u$, and for any $x \in \mathcal{O}(M)$ we define $\mathcal{L}^s_+(x)$ and $\mathcal{L}^s_-(x)$ the half stable leaves respectively at the left of x and at the right of x, and by $\mathcal{L}^u_+(x)$ and $\mathcal{L}^u_-(x)$ the half unstable leaves respectively above x and below x. We consider the orientation on $\mathcal{L}^s$ and $\mathcal{L}^u$ such that $l^s = \mathcal{L}^s_+(\pi(\tilde{\gamma}))$ and $l^u = \mathcal{L}^u_+(\pi(\tilde{\gamma}))$. Then we consider the connected component of $\mathcal{O}(M) \setminus (l^s \cup l^u)$ containing the quadrant $(+,+)$ of $\pi(\tilde{\gamma})$, that we denote $Q_0^{+,+}$, and the subset $Q_1^{+,+} = \{x \in \mathcal{O}(M) | \mathcal{L}^s_-(x) \cap l^u \neq \emptyset \text{ and } \mathcal{L}^u_-(x) \cap l^s \neq \emptyset\} \subset Q_0^{+,+}$.

We denote by $g \in \pi_1(M)$ the homotopy class inside M of δ which satisfies $g.\tilde{\delta} = \tilde{\delta}$. Since δ is homotopic to either γ or 2γ, we have $g.\tilde{\gamma} = \tilde{\gamma}$. Then the element g also preserves the stable and unstable leaves $\mathcal{L}^s(\pi\tilde{\gamma})$ and $\mathcal{L}^u(\pi\tilde{\gamma})$, and it preserves the quadrant along $\tilde{\gamma}$ which contains $\tilde{\delta}$. Hence it preserves the half leaves l^s and l^u.

Lemma 1.7.4. *The set $Q_1^{+,+}$ is invariant by the action of g on $\mathcal{O}(M)$.*
If U is a small enough neighborhood of γ inside Σ, then $\pi(\tilde{U}) \subset Q_1^{+,+}$.

Proof. By definition of g, the leaves l^s and l^u are preserved by the action of g. So $Q_0^{+,+}$ is invariant by the action of g. Then by definition of $Q_1^{+,+}$, this set is also stabilised by the action of g.

First notice that since $\mathcal{L}^s, \mathcal{L}^u$ is a foliation locally trivial, for a small enough neighborhood N of the point $\pi(\tilde{\gamma})$ in the orbit space $\mathcal{O}(M)$, we have $N \cap Q_0^{+,+} \subset Q_1^{+,+}$. Denote by $\tilde{V} \subset \tilde{U}$ a compact fundamental domain of the covering map $\tilde{U} \to U$. The projection $\pi_{|\tilde{V}} : \tilde{V} \to \mathcal{O}(M)$ is continuous, so if U is taken small enough, then $\pi(\tilde{V})$ lies inside the neighborhood N. Additionally according to Lemma 1.4.4, if U is chosen small enough, $\pi(\tilde{V})$ remains in the quadrant $Q_0^{+,+}$. So $\pi(\tilde{V}) \subset Q_1^{+,+}$.

The fundamental group of U is generated by g, and $\tilde{V}$ is a fundamental domain of the covering map $\tilde{U} \to V$. So $\tilde{U}$ is equal to the union of the images of $\tilde{V}$ by g^n for all $n \in \mathbb{Z}$. That is $\cup_{n\in\mathbb{Z}} g^n.\tilde{V} = \tilde{U}$. Hence

$$\pi(\tilde{U}) \subset \cup_{n\in\mathbb{Z}}(g^n.\pi(\tilde{V})) \subset \cup_n(g^n.Q_1^{+,+}) \subset Q_1^{+,+}$$

□

For any point $x \in \mathcal{O}(M)$ in the quadrant $Q_1^{+,+}$, by definition the stable leaf $\mathcal{L}^s(x)$ and the unstable leaf $\mathcal{L}^u(x)$ intersect the two half leaves l^s and l^u. Since every pair of stable and unstable leaves in the orbit space intersect at most once, there exist two continuous projections $p^s : Q_1^{+,+} \to l^s$ and $p^u : Q_1^{+,+} \to l^u$ defined by taking the intersection of the stable and unstable leaves of a point x with the leaves l^s and l^u. We combine these projections to have a coordinate system on $Q_1^{+,+}$. The two half leaves l^s and l^u have each an end at infinity that we denote by $+\infty^s$ and $+\infty^u$. The two sets $l^s \cup \{+\infty^s\}$ and $l^u \cup \{+\infty^u\}$ are then homeomorphic to $[0,1]$. We pull back the order on $[0,1]$ to $l^s \cup \{+\infty^s\}$ and $l^u \cup \{+\infty^u\}$, such that $\pi(\tilde{\gamma})$ and $+\infty^{s,u}$ are respectively the minimal and maximal points of $l^s \cup \{+\infty^s\}$ and $l^u \cup \{+\infty^u\}$.

Lemma 1.7.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $Q_1^{+,+}$ such that $p^s(x_n) \xrightarrow[n\to+\infty]{} +\infty^s$ and $p^u(x_n) \xrightarrow[n\to+\infty]{} \pi(\tilde{\gamma})$, then $(x_n)_n$ is out of any compact for n large enough.*

The curve $\pi(\tilde{\delta})$ is embedded in $\mathcal{O}(M)$ such that we have $p^s(x) \xrightarrow[p^u(x)\to\pi(\tilde{\gamma})]{x\in\tilde{\delta}} +\infty^s$ and $p^s(x) \xrightarrow[p^u(x)\to+\infty^u]{x\in\tilde{\delta}} \pi(\tilde{\gamma})$.

Futhermore $\pi(\tilde{\delta})$ bounds two connected components in the orbit space $\mathcal{O}(M)$.

Proof. Suppose that the sequence $(x_n)_n$ admits an infinite amount of terms inside a compact K. Up to an extraction, we can suppose that $(x_n)_n$ converge to a point y in K and that $(p^s(x_n))_n$ and $(p^s(x_n))_n$ inside l^s and l^u are respectively increasing and decreasing. Then for n large enough, the half leaf $\mathcal{L}^u_-(x_n)$ intersects the half leaves $\mathcal{L}^s_-(y)$. Since $x_n \in Q^{+,+}$, there is an open arc of unstable leaf l^u_n whose ends are x_n and a point in l^s. There are three cases to consider: $\mathcal{L}^s_-(y) \subset \mathcal{L}^s(\pi(\tilde{\gamma}))$ or $\mathcal{L}^s_-(y)$ intersects l^u_n in its interior or $\mathcal{L}^s_-(y)$ does not intersect l^u_n. Since $p^s(x_n)$ grows to $+\infty^s$ we cannot have $y \in l^s$, so the first case is impossible. If $\mathcal{L}^s_-(y)$ intersects the unstable arc l^u_n in its interior, then that intersection point lies inside $Q_1^{+,+}$ so $\mathcal{L}^s_-(y)$ intersect the interior of $l^u \setminus \pi(\tilde{\gamma})$ in a point y^u. This is impossible since $p^u(x_n)$ converge to $\pi(\tilde{\gamma})$ but must remains above y^u.

Since $\mathcal{L}^s_-(y)$ intersects $\mathcal{L}^u_-(x_n)$ but not l^u_n, the stable leaf l^s separates y and x_n inside $\mathcal{O}(M)$. Hence y and x_n are not in the same connected component of $\mathcal{O}(M) \setminus \mathcal{L}^s(\pi(\tilde{\gamma}))$, which contradicts that x_n converges to y. Hence x_n leaves any compact in a finite step.

Now we prove the second point. Since $\tilde{\delta}$ is invariant by the homotopy g of δ, there exists a compact curve $c \subset \pi(\tilde{\delta})$ such that its image under $g^{\mathbb{Z}}$ recover all $\pi(\tilde{\delta})$. The flow is Anosov and g is the homotopy class of the closed curve γ (possibly with multiplicity 2), so g contracts l^s and expands l^u. Hence for any point x in $Q_1^{+,+}$, the sequence $(p^s(g^n.x))_n$ converges to $\pi(\tilde{\gamma})$ when n tends to $+\infty$ and converges to $+\infty^s$ when n tends to $-\infty$. Since the curve c is compact, the same convergence happens uniformly for x in c. Similarly for a point x in x, the sequences $(p^u(g^n.x))_n$ converges to $\pi(\tilde{\gamma})$ when n tends to $-\infty$ and converges to $+\infty^u$ when n tends to $+\infty$, and these convergence are uniform on c. These four convergences implies the second point of the lemma.

To prove the third point, we notice that $\pi(\tilde{\gamma})$ is the injective image of the curve $\tilde{\gamma}$ so that it is also a curve. According to the two first points, its intersection with any compact of $\mathcal{O}(M)$ is a compact. Also $\mathcal{O}(M)$ is a simple connected plane, these three properties imply that $\mathcal{O}(M) \setminus \pi(\tilde{\gamma})$ is made of two connected components. □

We consider $\tilde{V} \subset \tilde{U}$ a closed fundamental domain of the covering map $\tilde{U} \to U$, such that $\partial\tilde{V}$ is the union of two orbit arcs and of two open closed curves $\tilde{c}$ and $g.\tilde{c}$.

Lemma 1.7.6. *The curves $l^s \cup l^u \cup \pi(\tilde{\delta})$ cut the orbit space $\mathcal{O}(M)$ in three connected components, one of which, denoted by $T \subset \mathcal{O}(M)$, satisfies $\partial T =$*

$l^s \cup l^u \cup \pi(\tilde{\delta})$. Furthermore $\pi(\tilde{c})$ is a subset of T and $T \setminus \pi(\tilde{c})$ has two connected components.

Proof. We prove that $\pi(\tilde{\delta})$ delimits two connected component in $\mathcal{O}(M)$, one of which contains l^s and l^u. Since $l^s \cup l^u$ is a proper curve (it intersects any compact in a sub-compact) in a simply connected plane $\mathcal{O}(M)$, $(l^s \cup l^u)$ also delimits two connected components inside $\mathcal{O}(M)$. Since the two curves $(l^s \cup l^u)$ and $\pi(\tilde{\delta})$ are disjoint, their union bounds three connected components in $\mathcal{O}(M)$. We denote by T the unique connected component lying between $(l^s \cup l^u)$ and $\pi(\tilde{\delta})$, so that its boundary is $l^s \cup l^u \cup \pi(\tilde{\delta})$. Since both boundary components of T are non-compact curves inside the simply connected plane $\mathcal{O}(M)$, T is also simply connected.

Since the curve $\tilde{c}$ is inside the strip $\tilde{U} \cup \tilde{\gamma}$, and the projection $\pi : \tilde{M} \to \mathcal{O}(M)$ is injective on $\tilde{U} \cup \tilde{\gamma}$, the curve $\pi(\tilde{c})$ is disjoint from $l^s \cup l^u$ and $\pi(\tilde{\delta})$. But $\tilde{c}$ has an end inside $\tilde{\gamma}$ and an end inside $\tilde{\delta}$, so $\pi(\tilde{c})$ has exactly two accumulation points, one in both curves $l^s \cup l^u$ and $\pi(\tilde{\delta})$. Hence one has $\pi(\tilde{c}) \subset T$. Since $\pi(\tilde{c})$ is properly embedded inside T and T is simply connected, that curve cut T in two connected components. $\square$

Proposition 1.7.7. *Let Σ be a partial section embedded in its interior and with a linking number zero along a boundary component γ. Denote by $U \subset \Sigma$ an annulus in a small enough neighborhood of γ, such that $\gamma \subset \partial U$. Then there exists a closed ideal triangle T inside the orbit space $\mathcal{O}(M)$ with an end at $\rho(\gamma)$ and with two edges l^s and l^u which are half stable and unstable leaves of $\rho(\gamma)$ such that the trace of U inside $\mathcal{O}(M)$ is T minus the interior of l^s and l^u. Furthermore there exists a lift $s : T \setminus \rho(\gamma) \to \mathcal{O}(M \setminus \partial\Sigma)$ such that the image of $s(T \setminus (l^s \cup l^u))$ lies inside a trace of Σ, and $s(\mathring{l}^s)$ and $s(\mathring{l}^u)$ are boundary components of the trace of Σ inside $\mathcal{O}(M \setminus \partial\Sigma)$.*

Notice that the ideal triangle T in the previous lemma corresponds to the adherence of the set T used before.

Proof. We consider the notations defined above. We need to prove that $\pi(\tilde{U})$ is equal to $T \cup \tilde{\delta} \cup \pi(\tilde{\gamma})$, and relate this to the trace of Σ inside the orbit space $\mathcal{O}(M \setminus \partial\Sigma)$.

Consider the fundamental $\tilde{V}$ of $\tilde{U} \to U$, whose boundary are given by the union of two orbit arcs $\tilde{\gamma}_{\partial V} \subset \tilde{\gamma}$ and $\tilde{\delta}_{\partial V} \subset \tilde{\delta}$ with the two curves $\tilde{c}$ and $g.\tilde{c}$. According to Lemma 1.7.4, the image $\pi(\tilde{V})$ is included in the quadrant $Q_1^{+,+}$ of $\pi(\tilde{\gamma})$. Additionally if the neighborhood U of γ is taken small enough, then U does not intersects the germs of stable and unstable leaves of γ, so $\pi(\tilde{V}\setminus\tilde{\gamma}_{\partial V})\subset Q_1^{+,+}\setminus(l^s\cup l^u)$. Furthermore by definition of the fundamental domain, we have $\tilde{U}=\cup_{n\in\mathbb{Z}}(g^n\tilde{V})$ so $\pi(\tilde{U}\setminus\tilde{\gamma})\subset Q_1^{+,+}\setminus(l^s\cup l^u)$.

According to Lemma 1.7.3, the projection $\pi:\tilde{M}\to\mathcal{O}(M)$ is injective on $\tilde{U}\setminus\tilde{\gamma}$. Hence the image $\pi(\tilde{U}\setminus\tilde{\gamma})\setminus(l^s\cup l^u)$ is in the same quadrant $Q^{+,+}$ minus the curves $l^s\cup l^u\cup\pi(\tilde{\delta})$. According to 1.7.6 the curves $l^s\cup l^u\cup\pi(\tilde{\delta})$ delimit a connected component inside $\mathcal{O}(M)$, we denote by T its adherence inside $\mathcal{O}(M)$. Additionally $\pi(\tilde{c})\subset\pi(\tilde{U}\setminus\tilde{\gamma})$ is inside the interior $\mathring{T}$ and cut $\mathring{T}$ in two connected components. Hence Additionally the image $\pi(\tilde{U}\setminus\tilde{\gamma})$ lies inside $T\setminus(l^s\cup l^u)$.

We can now prove that $\pi(\tilde{U}\setminus\tilde{\gamma})=T\setminus(l^s\cup l^u)$. Let p be a point inside $T\setminus(l^s\cup l^u)$. Denote by $g\in\pi_1(M)$ the homotopy class of δ, such that g preserves T, contracts the leaf l^s and expands the leaf l^u. Then $g^n.p$ diverge at infinity to the end $+\infty^u$ when $n\to+\infty$, and to the end $+\infty^s$ when $n\to-\infty$. Since the curve $\pi(\tilde{c})$ delimits two connected components in $T\setminus(l^s\cup l^u)$, and separates the two infinity points $+\infty^u$ and $+\infty^s$, there exists $n\in\mathbb{N}$ such that $\pi(\tilde{c})$ separates the points $\{+\infty^s,g^n.p\}$ to the points $\{+\infty^u,g^{n+1}.p\}$ inside the set $T\setminus(l^s\cup l^u)$. Then $g.\pi(\tilde{c})$ separates $\{+\infty^s,g^{n+1}.p\}$ to the points $\{+\infty^u,g^{n+2}.p\}$. So $g^{n+1}.p$ is in a connected component of $(T\setminus(l^s\cup l^u))\setminus(\pi(\tilde{c})\cup g.\pi(\tilde{c}))$.

The projection π is injective on $\tilde{V}\setminus\tilde{\gamma}$, and $\pi(\partial\tilde{V})=\pi(\tilde{\gamma}_{\partial V}\cup\tilde{\delta}\partial V\cup\tilde{c}\cup g.\tilde{c})$, so $g^{n+1}.p$ lies inside $\pi(\tilde{V})$. Hence the point p is inside $g^{-n-1}.\pi(\tilde{V})\subset\pi(\tilde{U})$. Therefore $T\setminus(l^s\cup l^u)\subset\pi(\tilde{U}\setminus\tilde{\gamma})$, and according to what precedes, we have the equality $T\setminus(l^s\cup l^u)=\pi(\tilde{U}\setminus\tilde{\gamma})$.

We now prove that $T\setminus(l^s\cup l^u)$ lifts to a subset of the trace of Σ in $\mathcal{O}(M\setminus\partial\Sigma)$. We denote by $\widehat{M\setminus\partial\Sigma}$ the universal covering of $M\setminus\partial\Sigma$. We lift $U\setminus\gamma$ and $\Sigma\setminus\partial\Sigma$ into two connected surfaces $\widehat{U\setminus\gamma}$ and $\widehat{\Sigma\setminus\partial\Sigma}$ inside $\widehat{M\setminus\partial\Sigma}$, such that $\widehat{U\setminus\gamma}\subset\widehat{\Sigma\setminus\partial\Sigma}$ and that $\widehat{U\setminus\gamma}$ projects to $\tilde{U}\setminus\tilde{\gamma}$ with the projection $\widehat{M\setminus\partial\Sigma}\to\tilde{M}\setminus$(lifts of $\partial\Sigma$). We also denote by $\hat{\pi}:\widehat{M\setminus\partial\Sigma}\to\mathcal{O}(M\setminus\partial\Sigma)$ the projection. By construction there exists a lift $s:\pi(\tilde{U}\setminus\tilde{\gamma})\to\hat{\pi}(\widehat{U\setminus\gamma})$. Since $\pi(\tilde{U}\setminus\tilde{\gamma})$ is simply connected, the lift s is a homeomorphism.

By definition $\hat{\pi}(\widehat{U\setminus\gamma})$ is inside the trace $\hat{\pi}(\widehat{\Sigma\setminus\partial\Sigma})$ of $\Sigma\setminus\partial\Sigma$. So the leaves $s(\mathring{l}^s)$ and $s(\mathring{l}^u)$ are in the adherence of the trace $\hat{\pi}(\widehat{\Sigma\setminus\partial\Sigma})$. Suppose that there exists a point z in $\widehat{\Sigma\setminus\partial\Sigma}$ such that $\hat{\pi}(z)$ is inside $s(\mathring{l}^s)$. Since $s(\mathring{l}^s)$ is disjoint from of $s(\pi(\tilde{U}\setminus\gamma))$, the point z remains out of the annulus $\widehat{U\setminus\gamma}$.

And since $\hat{\pi}(z)$ is inside $s(\mathring{l}^s)$, there exists points in $\widehat{\Sigma \setminus \partial\Sigma}$ arbitrarily closed to z, which are also in $\widehat{U \setminus \gamma}$. Then z is in the boundary component of $\widehat{U \setminus \gamma}$ which is a lift of δ. So one has $\hat{\pi}(z) \in s(\pi(\tilde{\delta}))$, which is impossible since $\pi(\tilde{\delta})$ and l^s are disjoint. Hence $s(\mathring{l}^s)$ and similarly $s(\mathring{l}^u)$ are two boundary components of the trace of Σ. □

The last proposition is used to prove the second and third points of Theorem 1.7.1.

Pseudo-Anosov flows To finish the proof of Theorem 1.7.1, we need to introduce pseudo-Anosov flows. A pseudo-Anosov flow is informally a flow Anosov outside a finite number of singular orbits, these orbits being given by local model that we review below. We define the pseudo-Anosov flow as in [Fen99] and refer to it for more details.

Take an integer $n \geq 2$ and consider the quadratic differential $z^{n-2}dz^2$ on $\mathbb{C}$ (as defined in [Str84]). That quadratic form induces two singular 1-foliations f^s and f^u on $\mathbb{C}$, whose leaves satisfies $\arg(dz) = -\frac{1}{2}\arg(z^{n-2})$ (mod π) and $\arg(dz) = -\frac{1}{2}\arg(z^{n-2}) + \frac{\pi}{2}$ (mod π). These foliations admit a common n-prong singularity at the point zero, and are regular and transverses outside that singularity. Also f^s and f^u both admit n singular half leaves which contain the singular point.

For any $\lambda > 1$ and any integer $0 \leq k < n$, there exists homeomorphism $G_{\lambda,n,k} : \mathbb{C} \to \mathbb{C}$ preserving the foliations f^u and f^s, differentiable outside the singular point, expanding the leaves of l^u and contracting the leaves of l^s by factors λ and $1/\lambda$, and sending any singular leaf l to the singular leaves $R_{\frac{2k\pi}{n}}(l)$. Here $R_\theta : \mathbb{C} \to \mathbb{C}$ is the linear rotation of degree θ. We define the local model given by $N_{\lambda,n,k} = \mathbb{C} \times \mathbb{R}/(z, t+1) \equiv (G_{\lambda,n,k}(z), t)$ with the flow $\psi_s(x,t) = (x, t+s)$. The suspension of f^s and f^u are two singular 2-foliations of $N_{\lambda,n,k}$. We called strong foliations the foliation $f^s \times \{cste\}$ and $f^u \times \{cste\}$ on $N_{\lambda,n,k}$.

A pseudo-Anosov flow on a closed 3 manifold M is a $\mathcal{C}^0$ flow such that:

- For every point $x \in M$, the function $t \mapsto \phi_t(x)$ is of class $\mathcal{C}^1$.
- There is a finite number of orbits, called singular, out of which the flow is smooth.

- For every singular orbit γ, there exists a neighborhood of γ on which the flow is orbitally equivalent to a neighborhood of the closed orbit on the local model $N_{\lambda,n,k}$.

- Out of the singular orbits, the orbital equivalence mentioned above is smooth and its differential has a bounded norm, for some norm given by two metrics on M and $N_{\lambda,n,k}$.

- Out of the singular orbit, the flow is Anosov. That is for a norm $\|.\|$ given by a metric on M, the exists a splitting of the tangent bundle $TM \setminus$ singular orbits $= E^s \oplus E^u \oplus X$ and two constant $A, B > 0$ such that X is the direction of the flow, for every $u \in E^s$, $\|D\phi_t(u)\| \leq A\,\mathrm{e}^{-Bt}\,\|u\|$ for all $t \geq 0$, and for every $u \in E^u$, $\|D\phi_t(u)\| \leq A\,\mathrm{e}^{Bt}\,\|u\|$ for all $t \leq 0$.

- The orbital equivalences mentioned in the second item send the foliations E^s and E^u to the strong stable foliations and the strong unstable foliations of $N_{\lambda,n,k}$ (minus the singular orbit).

Notice that we only allowed n-prong singularities for $n \geq 2$. Three facts are important for now: a Fried surgery on an Anosov flow with oriented foliations induces a pseud-Anosov flow [Fri83], the theory of the orbit space of pseudo-Anosov flow is similar to the theory for Anosov flow, and S.Fenley [Fen99] characterizes the trace of a closed transverse surface in the orbit space of a pseudo-Anosov flow. Given a partial section in an Anosov flow, we use these facts to create a pseudo-Anosov flow and a partial section with less boundary components, and whose trace in the orbit space can be related to the original partial section.

We review the orbit space of a pseudo-Anosov flow. Denote by ϕ a pseudo-Anosov on a 3-manifold M. We defined similarly the universal covering $\tilde{M}$ of M, and the orbit space $\mathcal{O}(M)$ obtained as the quotient of $\tilde{M}$ by the orbits of the flow. The orbit space of a pseudo-Anosov flow is a topological plane [FM01], equipped with two singular 1-foliation $\mathcal{L}^s$ and $\mathcal{L}^u$ induced by the singular foliations of the flow. The singular foliations $\mathcal{L}^s$ and $\mathcal{L}^u$ have common n-prong singularity for $n \geq 3$, and are regular and transverse outside these prong singularities. Just for Anosov flow, we define the trace of a partial section Σ inside the orbit space $\mathcal{O}(M \setminus \partial\Sigma)$, which is the universal covering of the orbit space $\mathcal{O}(M)$ minus the discrete subset induced by the orbits $\partial\Sigma$.

Let M be a 3-oriented manifold, ϕ be an Anosov flow with orientable stable and unstable foliations, and $\Sigma \subset M$ be a partial section. Consider a finite number of closed orbits $\Gamma \subset M$ and consider a Fried surgeries along each components of Γ. These surgeries induces a manifold M', a set of closed orbits Γ' and a pseudo-Anosov flow ϕ' such that ϕ outside Γ and ϕ' outside Γ' are orbitally equivalent. Notice that we need the stable and unstable foliations to be orientable, otherwise the flow obtained by surgery may have 1-prong singularities around a singular orbit, which is not allowed by our definition of pseudo-Anosov flow. We now explain why these Fried surgeries also induce a partial section $\Sigma' \subset M'$ such that $\Sigma \setminus \Gamma$ and $\Sigma \setminus \Gamma'$ are isotopic along the flow.

Recall that the Fried surgeries are obtained using the following steps: we take the blowing up M_Γ of M along Γ together with the flow ϕ^Γ induced on M_Γ. On each boundary component $\mathbb{T}$ of the blowing up we consider a bundle $\mathbb{T} \to S^1$ whose fibers are all transverse to the flow $\phi^\Gamma_{|\mathbb{T}}$ and intersect every orbit of the flow $\phi^\Gamma_{|\mathbb{T}}$. Then we quotient each fibers inside $\mathbb{T}$ into one point. If Σ is a partial section of the flow ϕ, according to Lemma 1.4.11, there is an isotopy along the flow of the interior of Σ to the interior of a partial section Σ_0, which is given by local models given in Example 1.4.3. In particular all these local models lift to compact surfaces inside M_Γ, transverse to the flow. Hence Σ_0 lift to a compact surface Σ_0^Γ inside M_Γ, transverse to the flow, which then projects into a compact surface Σ' inside M'.

To make Σ' a partial section of the flow ϕ', we need to control how its boundary components are projected inside M'. Consider a boundary component $\mathbb{T}$ of M_Γ and $\mathbb{T} \to S^1$ the bundle mentioned above. One can isotope Σ_0^Γ such that the boundary component $\Sigma_0^\Gamma \cap \mathbb{T}$ of Σ_0^Γ are either all transverse to the fibers of the bundle $\mathbb{T} \to S^1$, or are all tangent to these fibers. Then Σ_0^Γ projects inside M' to a compact surface Σ_0' such that for any orbit γ in Γ', is either a smooth local transverse section around γ, or intersects γ in a finite number of points. In the last case, in a small neighborhood of γ, the surface Σ_0' can be isotope along the flow into a surface smooth and transverse to γ. Hence there exists a partial section Σ' isotopic to $\Sigma \setminus \Gamma$ outside Γ'.

Remark 1.7.8. We consider the notation of the three previous paragraphs for Σ a partial section of an Anosov flow ϕ (embedded in its interior). On can chose Γ to be the union of boundary components of Σ with linking number non-zero. Since Σ is embedded in its interior, up to an isotopy of $\mathring{\Sigma}$, Σ lifts inside M_Γ to a compact embedded surface Σ^Γ transverse to the flow. Additionally since Σ has linking number non-zero along Γ, for every boundary components $\mathbb{T}$ of the blowing up manifold M_Γ, Σ^Γ intersects every orbit of the flow inside $\mathbb{T}$. Hence one can chose a bundle $\mathbb{T} \to S^1$ whose fibers are transverse to the flow, intersect all orbits inside $\mathbb{T}$, and such that $\Sigma^\Gamma \cap \mathbb{T}$ is a union of fiber. Then the Fried surgeries given by these bundles induces a manifold M' with a flow ϕ', some closed orbits Γ' on which were made the surgeries, and a partial section Σ' such that:

- the flow $\phi_{M\setminus\partial\Sigma}$ and $\phi'_{M\setminus\Gamma}$ are canonically orbitally conjugated,
- the surfaces $\Sigma \setminus \Gamma$ and $\Sigma' \setminus \Gamma'$ are isotopic along the flow,
- all boundary components of Σ' have a linking number zero,
- the orbit spaces $\mathcal{O}(M \setminus \partial\Sigma)$ and $\mathcal{O}(M \setminus (\Gamma' \cup \partial\Sigma'))$ are canonically homeomorphic,
- the trace of Σ inside $\mathcal{O}(M \setminus \partial\Sigma)$ and the trace of Σ' inside $\mathcal{O}(M' \setminus (\Gamma' \cup \partial\Sigma'))$ are equal.

We use that remark to prove the last two items of Theorem 1.7.1.

Lemma 1.7.9. *Let ϕ be an Anosov flow on a 3-manifold and Σ be a partial section embedded in its interior. Take $f : M' \to M$ a finite covering space, ϕ' the flow induced on M', and Σ' be a connected component of $f^{-1}(\Sigma)$. Then Σ is a Birkhoff section if and only if Σ' is a Birkhoff section*

Lemma 1.7.10. *If Σ' is a Birkhoff section, then Σ intersects every orbit in bounded time, so it is also a Birkhoff section. We suppose that Σ is a Birkhoff section and prove that Σ' is also a Birkhoff section.*

The first case is when Σ has no boundary component. Then Σ is a global section of ϕ, so ϕ is the suspension of an Anosov diffeomorphism on Σ. In that case, ϕ' is also a suspension of an Anosov diffeomorphism on Σ', and Σ' is a finite covering space of Σ. Hence Σ' is a Birkhoff section.

In the other case, we can prove that the group morphisme $\pi_1(\Sigma) \to \pi_1(M)$ is surjective. Indeed take γ a boundary component of Σ and take a point x in γ. Take another close curve $\delta \subset M$ based at x. We first homotope δ relatively to x such that δ intersects $\partial\Sigma$ only at x. Since Σ is a Birkhoff section, we can homotope δ along the flow to a closed curve which is the union of two curve $\delta_1 \subset \mathring{\Sigma}$ and $\delta_2 \subset \gamma$. That curve is all inside Σ, so $\pi_1(\Sigma) \to \pi_1(M)$ is surjective. Hence the surface $f^{-1}(\Sigma)$ is connected, and it is clear that it intersects every orbit in bounded time. So Σ' is a Birkhoff section.

Proof of the second item of Theorem 1.7.1. Let M be a closed 3-manifold and ϕ be an Anosov flow on M, and $\Sigma \subset M$ be a connected partial section embedded in its interior. We need to prove that Σ is a Birkhoff section if and only if the trace of Σ in $\mathcal{O}(M \setminus \partial\Sigma)$ is not all the orbit space.

Suppose that Σ has a boundary component γ with linking number zero. Then according to Lemma 1.4.4, the section Σ does not intersect the germ of stable and unstable leaves of the orbit γ. Hence Σ does not intersect some half orbit inside the germ of the stable leaves at γ, so it is not a Birkhoff section. Additionally according to Proposition 1.7.7, the trace of Σ in $\mathcal{O}(M \setminus \partial\Sigma)$ is not all the orbit space, so the equivalence is satisfied in that case.

Suppose now that Σ has no boundary component with linking number zero. Because of the previous lemma, we can take a finite covering space $f : N \to M$ which is orientable such that the lifted flow $f^\star\phi$ on N is Anosov and has orientable stable and unstable foliations. This covering space can be defined as a connected component of the bundle given by the local orientations of M and the local orientations of its stable and unstable foliations. Then $\Sigma \subset M$ is a Birkhoff section if and only if any connected component of $f^{-1}(\Sigma)$ is a Birkhoff section. Additionally the trace of Σ in $\mathcal{O}(M \setminus \partial\Sigma)$ is equal to the trace of any connected component of $f^{-1}(\Sigma)$ in $\mathcal{O}(N \setminus f^{-1}(\partial\Sigma))$, since the trace is defined using the universal covering space of $M \setminus \partial\Sigma$. Hence up to changing the notation, we suppose that M is orientable and the flow has orientable stable and unstable foliations.

We consider the manifold M', the flow ϕ' and the partial section Σ' given in 1.7.8, obtained by Fried surgeries along the boundary components of Σ, such that Σ' has no boundary. Then S.Fenley proved in [Fen99] that Σ' is a global section of the flow ϕ' if and only if the trace of Σ' in $\mathcal{O}(M')$ is all the orbit space. We denote $\Gamma = \partial\Sigma$ and $\Gamma' \subset M'$ the set of closed orbits induced by Γ after surgery.

The orbit space $\mathcal{O}(M' \setminus \Gamma)$ is the universal covering of the space $\mathcal{O}(M')$ minus the points induced by Γ, so the trace of Σ' inside $\mathcal{O}(M' \setminus \Gamma')$ is the lift of the trace of Σ' inside $\mathcal{O}(M') \setminus (\text{lift of } \Gamma')$. Hence the trace of Σ inside $\mathcal{O}(M \setminus \partial\Sigma)$ is all the orbit space if and only Σ' is a Birkhoff section. Since $\Sigma \setminus \partial\Sigma$ and $\Sigma' \setminus \Gamma'$ are isotopic along the flow $\phi_{M\setminus\partial\Sigma}$, Σ is a Birkhoff section of ϕ if and only Σ' is a Birkhoff section of ϕ'. Which conclude the proof. □

We finish by proving the third point in Theorem 1.7.1. The proof starts with the original proof of S.Fenley, and finish by some precise considerations on the boundary of the partial section.

Proof of the third item of Theorem 1.7.1. Let M be a closed 3-manifold, ϕ be an Anosov flow on M, and $\Sigma \subset M$ be a connected partial section embedded in its interior. We will prove that the trace of Σ in $\mathcal{O}(M \setminus \partial\Sigma)$ is a union of stable and unstable leaves. We first consider an orientable finite covering space $f : M_1 \to M$ such that the flow ϕ lift to a flow ϕ^1 which is Anosov with oriented stable and unstable foliations. Notice that the trace of Σ is somewhat preserve by this covering, in a sense we are going to precise below.

We consider the sets of orbit $\Gamma = \partial\Sigma$ and $\Gamma_1 = f^{-1}(\Gamma)$, and the partial section Σ_1 obtained as one connected component of $f^{-1}(\Sigma_1)$. We consider the Fried surgeries described in Remark 1.7.8. More precisely we do the Fried surgeries on the connected components of Σ_1 with linking number non-zero, such that it induces a manifold M_2, pseudo-Anosov flow ϕ^2, a collection of orbits Γ_2 and a partial section Σ_2 such that:

- Σ_2 has no boundary components of linking number non-zero and $\partial\Sigma_2 \subset \Gamma_2$,
- $\phi^2 \setminus \Gamma_2$ is orbitally equivalent to $\phi^1 \setminus \Gamma_1$,
- we have the two equalities $\mathcal{O}(M \setminus \Gamma) = \mathcal{O}(M_1 \setminus \Gamma_1) = \mathcal{O}(M_2 \setminus \Gamma_2)$,
- the trace of Σ_2 inside $\mathcal{O}(M_2\setminus\Gamma_2)$ is equal to the trace of Σ_1 inside $\mathcal{O}(M_1\setminus \Gamma_1)$ and to the trace of Σ inside $\mathcal{O}(M \setminus \Gamma) = \mathcal{O}(M_1 \setminus \Gamma_1)$.

We rewrite below the proof of S.Fenley and we add some arguments for the boundary components of linking number. Denote by $\widetilde{M_2 \setminus \Gamma_2}$ the universal covering space of $M_2 \setminus \Gamma_2$, by $\tilde{\phi}^2$ the induced flow on $\widetilde{M_2 \setminus \Gamma_2}$, and by $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$ the trace of Σ_2 in $\mathcal{O}(M_2 \setminus \Gamma_2)$. Take a point $p \in \partial\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. We need to prove that inside $\mathcal{O}(M_2\setminus\Gamma_2)$, either the stable leaf or the unstable leaf of p is a boundary component of the trace of Σ_2.

Since ϕ^2 is pseudo-Anosov, so the orbit space $\mathcal{O}(M_2)$ may have singular stable and unstable leaves. However, all the singular orbits of M_2 are by construction inside the set of orbits Γ_2, so the manifold $\widetilde{M_2 \setminus \Gamma_2}$ and the orbit space $\mathcal{O}(M_2 \setminus \Gamma_2)$ have only regular stable and unstable leaves.

Take $(p_n)_{n\in\mathbb{N}}$ a sequence of point of $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$ so that $p_n \xrightarrow[n\to+\infty]{} p$. Let z in $\widetilde{M_2 \setminus \Gamma_2}$ be such that $\pi(z) = p \in \mathcal{O}(M_2 \setminus \Gamma_2)$ and y_i in $\mathring{\tilde{\Sigma}}_2 \subset \widetilde{M_2 \setminus \Gamma_2}$ for all i such that $\pi(y_i) = p_i \in \mathcal{O}(M_2 \setminus \Gamma_2)$. We take a small compact embedded disc $D \subset \widetilde{M_2 \setminus \Gamma_2}$ such that $z \in \mathring{D}$ and $\pi_{|D} \to \mathcal{O}(M_2 \setminus \Gamma_2)$ is injective.

Up to taking a subsequence of $(p_n)_{n\in\mathbb{N}}$, we can suppose that for all i, there exists $t_i \in \mathbb{R}$ such that $\tilde{\phi}_{t_i}(y_i) \in \mathring{\tilde{\Sigma}}_2$. If there exists a subsequence of $(t_i)_{i\in\mathbb{N}}$ which tends to $t \in \mathbb{R}$, then $\tilde{\phi}_t(z) \in \mathring{\tilde{\Sigma}}_2$, which contradict that $\pi(\tilde{\phi}_t(z)) = p \in \partial\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. Hence $|t_n| \xrightarrow[n\to+\infty]{} +\infty$.

Assume that $t_n \xrightarrow[n\to+\infty]{} +\infty$ (or that a subsequence of t_n tends to $+\infty$) and consider l^s the stable leaf of p inside $\mathcal{O}(M_2 \setminus \Gamma_2)$. We will prove that the stable leaf l^s is inside $\partial\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. Take $v \in \pi^{-1}(l^s) \subset \widetilde{M_2 \setminus \Gamma_2}$. For a point $x \in \widetilde{M_2 \setminus \Gamma_2}$, we denote by $\tilde{F}^s(x), \tilde{F}^u(x)$ the stable and unstable leaves of a point x, and $\tilde{F}^{ss}(x), \tilde{F}^{uu}(x)$ the strongly stable and strongly unstable leaves of x. For i big enough, the point $v_i = \tilde{\mathcal{F}}^{uu}(v) \cap \tilde{\mathcal{F}}^s(y_i)$ is well defined. For i big enough, there exists $s_i \in \mathbb{R}$ such that $\tilde{\phi}_{s_i}(v_i) \in \mathcal{F}^{ss}(y_i)$. Since $t_n \xrightarrow[n\to+\infty]{} +\infty$ one has $s_i \xrightarrow[i\to+\infty]{} +\infty$. Also $d(\tilde{\phi}_{s_i}(v_i), y_i) \xrightarrow[i\to+\infty]{} +\infty$.

Here we need to add some considerations on the boundary of Σ_2. We take U a the union of small tubular neighborhoods of all boundary components of Σ_2. Out of U, the partial section Σ_2 has an angle with the direction of ϕ which is bounded by below by a non zero angle. Hence for all $\epsilon > 0$ there exists $\mu > 0$ such that if a point x in M is at a distance at most μ from $\Sigma_2\setminus U$, then x is in an arc of orbit $\phi_{[-\epsilon,\epsilon]}(y)$ of one point $y \in \Sigma_2$. Suppose first that the projection of y_i inside $\Sigma_2 \setminus \partial\Sigma_2$ is always out of U. Then there exists a sequence $\epsilon_i \in \mathbb{R}$ such that $\tilde{\phi}_{s_i+\epsilon_i}(v_i) \in \mathring{\tilde{\Sigma}}_2$ for all i. Additionally the sequence $(\epsilon_i)_i$ converge to zero since the distance $d(\tilde{\phi}_{s_i}(v_i), y_i)$ converge to zero. So the image $\pi(v_i)$ inside $\mathcal{O}(M_2 \setminus \Gamma_2)$ is inside the trace of Σ_2. Since the sequence $(v_i)_i$ converge to v, the point $\pi(v)$ is in the adherence of the trace $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. Since the sequence $(s_i+\epsilon_i)_i$ diverge to $+\infty$, the point $\pi(v)$ is not in the adherence of the trace $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. Indeed if $\tilde{\phi}_t(v)$ were in the interior of $\widetilde{\Sigma_2 \setminus \partial\Sigma_2}$, then since the projection $\pi_{|\widetilde{\Sigma_2\setminus\partial\Sigma_2}}$ is injective, the points $\tilde{\phi}_{t+s_i+\epsilon_i}(v_i)$ converge to $\tilde{\phi}_t(v)$, which contradicts the fact that $(s_i+\epsilon_i)_i$ diverge to $+\infty$.

It remains to consider the following case: the projection of y_i inside $\Sigma_2 \setminus \partial\Sigma_2$ has a sub-sequence inside the neighborhood U of $\partial\Sigma_2$. Up to an extraction, we can consider that the projection of y_i inside $\Sigma_2 \setminus \partial\Sigma_2$ remains in a small tubular neighborhood U_γ of one boundary component γ of $\partial\Sigma_2$. We consider the trace of U_γ to prove that the stable leaf l^s is a boundary component of the trace $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$. According to Proposition 1.7.7, if the annulus U_γ is chosen closed enough to γ the trace of the annulus U_γ in the orbit space $\mathcal{O}(M_2 \setminus \Gamma_2)$ is an ideal annulus bounded by a half stable leaf l_i^s of γ, a half unstable leaves l_i^u of γ and by a third curve denoted by δ_i. The Proposition 1.7.7 cannot be applied yet on Σ_2, but one can apply it to a small annulus U'_γ inside Σ, then recover the trace of U_γ from the trace of U'_γ by using the definition of M_1 and M_2, and the equality $\mathcal{O}(M \setminus \Gamma) = \mathcal{O}(M_2 \setminus \Gamma_2)$.

Additionally the leaves l_i^s and l_i^u are boundary component of the trace of Σ_2. We want to prove that the point p lies inside one of the leaves $l_i^s \cup l_i^u$. To do so, we suppose that it is not the case.

For every point y_i, we denote by T_i the subset of $\Theta_{M_2\setminus\Gamma_2}(\Sigma_2)$ which correspond to the copy of the trace of U_γ containing $p_i = \pi(y_i)$. Also one of the ideal vertices of T_i projects inside $\mathcal{O}(M)$ to a lift $\rho_i(\gamma)$ of the orbit γ. Up to an extraction on p_i, we can suppose that T_i is always in a quadrant $(+,+)$ of its ideal points $\rho_i(\gamma)$.

We consider the relative position of the triangle T_i around the point p. Since the point p is in the boundary of the trace of Σ_2, it is not in the interior of T_i nor in δ_i for any i. Hence one of the edges inside ∂T_i separate the points p and p_i inside $\mathcal{O}(M_2 \setminus \Gamma_2)$. Additionally we have supposed that the point p is not in l_i^s and l_i^u. The trace of Σ_2 is connected, the leaves l_i^s and l_i^u are boundary components of the trace of Σ_2 and they both separate the orbit space $\mathcal{O}(M_2 \setminus \Gamma_2)$ in two connected component. So it must be the curve δ_i which separates p and p_i inside $\mathcal{O}(M_2 \setminus \Gamma_2)$. Since the sequence p_i converges to p, there exist two indexes $i, j \in \mathbb{N}$ such that $T_i \neq T_j$ and the curve δ_j separates p_i and p. The projection $\pi_{|\widetilde{\Sigma_2 \setminus \partial\Sigma_2}}$ is injective, so the triangle T_i and T_j are disjoint. It implies that $l_j^s \cup l_j^u$ separate p_i and p_j inside $\mathcal{O}(M_2 \setminus \Gamma_2)$, which contradicts the connectivity of the trace of Σ_2. Hence the point p lies inside one of the leaves $l_i^s \cup l_i^u$. It finishes the proof. $\square$

1.7.2 Fundamental domain of a ramified partial section

We can define and study immersed partial sections using some fundamental domains in the orbit space. We describe a general construction that produces ramified partial sections, that we only use to define the Birkhoff annuli and the Fried sections. We mainly need Lemma 1.7.12 to characterize the boundary of these sections.

Definition 1.7.11. A **fundamental domain** in the orbit space is the data of (P, F, f), so that:

1. $P : (P' \subset \mathbb{R}^2) \to \mathcal{O}(M)$ is an immersion, where P' is a n-gon with n even and $n \geq 2$. We confuse the immersion P and the polygon P'. The edges of P do not need to be arcs of stable and unstable leaves.

2. $F : Edge(P) \to Edge(P)$ is an involution with no fixed point,

3. $f : Edge(P) \to \pi_1(M)$ is so that for every edge e of P, $f(e).e = F(e)$.

4. for every edge e of P, $f(e)$ sends the coorientation on e going outside P to the coorientation on $F(e)$ going inside P.

Notice that these conditions imply that $f(F(e)) = f(e)^{-1}$ for every edge e of P. Also the data F is not necessary to define a fundamental domain, since it can be deduced from f. Let S_P be the quotient of P by F, which is a differential surface away from corners of P, and let $j : P \to S_P$ be the projection. The bi-foliation of $\mathcal{O}(M)$ by $\mathcal{L}^s, \mathcal{L}^u$ induces on S_P a transverse bi-foliation, possibly with p-prong singularities at the corners of P. The points of the form $j(\text{corner of } P) \in S_P$ that are singularities of index $+1$ (of the stable foliation) are considered as 0-prong singularities, and the points $j(\text{corner of } P) \in S_P$ where the bi-foliations could extend smoothly at that point are considered as 2-prong singularities. We will lift P into M and construct a ramified partial section whose interior has the topology of $\mathring{S}_P$.

Using a fundamental domain, we construct a **ramified partial section**, that is a surface which is immersed outside a finite number of points in its interior, which still admits a tangent plane at these points, and which is transverse in its interior and tangent to the flow along its boundary components.

The procedure is detailed below, and partially illustrated in Figure 1.10. Denote by $a_1, \dots, a_n$ the edges of P, ordered anti-clockwise. We define the $2n$-gon Q, whose edges are $b_1, a_1, \dots, b_n, a_n$, so that P is obtained from Q by contracting every edge b_k into a corner of P. Denote by $i : Q \to P$ the projection, and $c_k = i(b_k)$ the corner of P corresponding to c_k.

We recall the notation $\pi : \tilde{M} \to \mathcal{O}(M)$. We will construct a lift $s : Q \to \tilde{M}$ of Q, so that $\pi \circ s = i$ and that $\pi_1(M).s(Q)$ projects to a ramified partial section inside M, possibly with ramified points at the corners of P. In particular an edge b_k will be mapped by s into an orbit arc inside $\tilde{M}$.

We first choose an arbitrary lift $r : P \to \tilde{M}$ that we use to construct another lift $s : Q \to \tilde{M}$ which behaves well with respect to F. For every pair of edges $\{a_i, a_j\}$ of P so that $F(a_i) = a_j$ (or equivalently $F(a_j) = a_i$), we fix e one of these two edges and define the lift $s_{|e} = r_{|e}$. Then we define the lift $s_{|F(e)}$ by $s_{|F(e)}(x) = f(e).s_{|e}(f(e)^{-1}.x)$. Since $\pi \circ r$ and $\pi \circ s$ coincide on $F(e)$, $r_{|F(e)}$ and $s_{|F(e)}$ are isotopic along the flow. We can lift b_k as an arc of orbit, in the orbit of $r(c_k)$, so that $s_{|b_k}$ is either an embedding or reduced to a point. Since the lift $s : \partial Q \to \tilde{M}$ is isotopic along the flow to the lift $i \circ r_{|\partial Q}$, it extends to a lift $s : Q \to \tilde{M}$.

By construction of $s : Q \to \tilde{M}$, we have $s_{|F(e)} = f(e).s_{|e}(f(e)^{-1}.\,\mathrm{id})$ for all edges e of P. We can additionally choose s so that s and $f(e).s$ are glued smoothly along the interior of $F(e)$, and then $\pi_1(M).s(Q)$ is a surface smooth and transverse to the flow outside its boundary. Denote $\tilde{\Sigma} = \pi_1(M).s(Q)$ and $\Sigma = p(\tilde{\Sigma}) \subset M$, the latter being a surface whose interior is immersed and transverse to the flow. We also denote by Σ the abstract surface, obtained as the quotient of $\tilde{\Sigma}$ by the action of the elements $f(e) \in \pi_1(M)$ for all edges e of P. The boundary of Σ is the union of the arcs $p \circ s(b_k)$ for all $1 \leq k \leq n$. Notice that Σ is the blow-up of S_P along the images of the corners of P. Note that by construction (P, F, f) is a fundamental domain of Σ. One can choose the lift $r : P \to \tilde{M}$ so that every boundary component of Σ (as an abstract surface) is either immersed in M, or sent to a single point of M. In the second case, the image $\Sigma \subset M$ has a ramification at this point. However these ramifications are transverse to the flow, and a Fried-desingularisation erases these ramifications. Hence the surface Σ is only a ramified partial section. The following lemma will help to understand the boundary components of Σ.

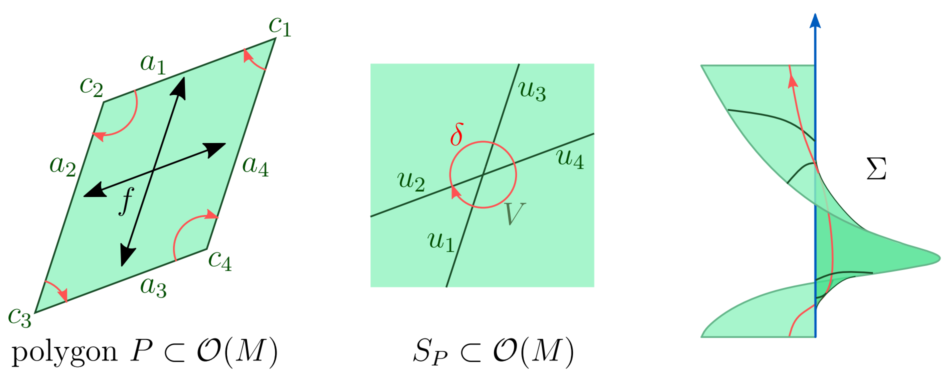


Figure 1.10: Fundamental domain of a ramified partial section.

Let c_k be a corner of P, $\gamma \subset M$ be the orbit induced by c_k and $\gamma_k \subset \partial\Sigma$

the abstract boundary component of Σ corresponding to c_k, that is $p \circ s \circ i^{-1}(\{c_k\})$. We denote by $V \subset S_P$ a small tubular neighborhood of $j(c_k) \in S_P$. Take $\Sigma' \subset \Sigma$ as the closure of $p \circ s \circ i^{-1} \circ j^{-1}(V) \subset \Sigma$, which is a small tubular neighborhood in Σ of γ_k. Let $\delta = \partial V$ be a small circle around $j(c_k) \in S_P$, oriented clockwise and let $u'_1, \ldots, u'_m$ be the images of the edges of P in s_P which intersect δ, ordered according to δ and possibly with two repetitions if δ intersects a given edge near its two ends. We can determine the behavior of Σ' along γ_k using the elements $f(j^{-1}(u_k)) \in \pi_1(M)$ that are used to glued P into S_P. For every $1 \leq k \leq m$ there are two edges of P whose image by $j : P \to S_P$ is u'_k. We take u_k to be the edge of P such that $j(u_k) = u'_k$ and $j^{-1}(\delta)$ is going inside P along u_k (or along the intersection of $j^{-1}(\delta) \cap u_k$ that is closed to $c_k \pmod f$)).

Lemma 1.7.12. *Keep the notations as above and take $x_0 \in \gamma$. The homotopy class of γ_k in M is $f(u_1)^{-1} \ldots f(u_m)^{-1} \in \pi_1(M, x_0)$. Also:*

- *Suppose that $f(u_1)^{-1} \ldots f(u_m)^{-1} = 1$. Then Σ can be isotoped along the flow so that the boundary component of Σ corresponding to c_k is sent to only one point. Then Σ is ramified along this point, denote by d its degree. Then c_k projects to a $2d$-prong singularity of S_P. When the degree d is 1, Σ can be made immersed in a neighborhood of this point.*

- *Suppose that $f(u_1)^{-1} \ldots f(u_m)^{-1} \neq 1$. Then Σ can be isotoped along the flow, so that Σ' is a partial section in a neighborhood of γ, with a boundary component at γ. Also γ is periodic, and if $g \in \pi_1(M, x_0)$ corresponds to the homotopy class of γ (so that $g.c_k = c_k$ in $\mathcal{O}(M)$), then one has $f(u_1)^{-1} \ldots f(u_m)^{-1} = g^{\mathrm{mult}_\gamma(\Sigma')}$.*

- *Suppose that $f(u_1)^{-1} \ldots f(u_m)^{-1} \neq 1$ as above, and denote by d the linking number of Σ along this boundary component. If γ has an orientable neighborhood, then c_k projects in a $2d$-prong singularity of S_P. Otherwise c_k projects in a d-prong singularity of S_P.*

Remark 1.7.13. By applying the previous argument on all boundary components of Σ, we can change it into an immersed partial section with ramification transverse to the flow. Then we can desingularise it into a partial section embedded in its interior.

Proof. Since Σ is the blow-up of S_P along the corners of P, δ induces in Σ a curve δ', freely homotopic in Σ to γ_k. So in $\pi_1(M, x_0)$, one has $[\gamma_k] = [\delta']$. We will lift δ into $\tilde{\Sigma}$ and compute its homotopy class.

Let $y_0' \in \mathring{\delta}$ be a point between u_m' and u_1'. We define $y_0 = s \circ i^{-1}(y_0')$ inside $s(Q) \subset \tilde{M}$, and we lift δ into a compact path $\tilde{\delta}$ in $\tilde{\Sigma}$ from y_0 to $[\delta'].y_0$. By construction, $j \circ p(\tilde{\delta}) = \delta$ intersects successively $u_1', \dots, u_n'$. So the curve $\tilde{\delta}$ intersects successively several images of $s(Q)$ under $\pi_1(M)$. The curve $\tilde{\delta}$ starts in $s(Q)$, then intersects the edge $s(u_1) = f(u_1)^{-1}s(F(u_1))$ and enters the polygon $f(u_1)^{-1}s(Q)$. Then it intersects the edge $f(u_1)^{-1}s(u_1) = f(u_1)^{-1}f(u_1)^{-1}s(F(u_1))$ and enters the polygon $f(u_1)^{-1}f(u_1)^{-1}s(Q)$. By induction, $\tilde{\delta}$ ends in $f(u_1)^{-1}\dots f(u_m)^{-1}s(Q)$. Hence the two points $[\delta'].y_0$ and $f(u_1)^{-1}\dots f(u_m)^{-1}.y_0$ are equal, and $[\gamma_k] = [\delta'] = f(u_1)^{-1}\dots f(u_m)^{-1}$.

Notice that the edges $F(u_k)$ and u_{k+1} are two edges of Q separated by an edge $b_{\sigma(k)}$ (where $F(u_k) = a_{\sigma(k)}$). Since F has no fixed point, we can successively choose the lift $s_{|u_k}$ of every $0 \leq k < n$, so that $s_{|F(u_k)}$ and $s_{|u_{k+1}}$ have the same value on their common ends with $b_{\sigma(k)}$. Then for $0 \leq k < n$, the lift $s_{|b_{\sigma(k)}}$ is constant, and the only edge that contributes of Σ to the boundary γ_k is $s_{|b_{\sigma(n)}}$. Hence $\pi(s_{|b_{\sigma(n)}}) \subset M$ is homotopic to $f(u_1)^{-1}\dots f(u_m)^{-1} = 1$.

Suppose that $f(u_1)^{-1}\dots f(u_m)^{-1} = 1$. Then $s_{|b_{\sigma(n)}}$ is constant, so Σ has a boundary whose image is reduced to one point, that we suppose to be x_0. We quotient this boundary component into one point, and still denote by Σ the new surface. We also denote by U a small neighborhood of x_0 in Σ. By construction, Σ is immersed and transverse to the flow inside $U \setminus \{x_0\}$. One can isotope Σ inside $U \setminus \{x_0\}$ such that the image of Σ admits a tangent plane at x_0, transverse to the flow. Hence Σ has a ramification at x_0, denote by d the degree of ramification of Σ at x_0. Then in a neighborhood of x_0, Σ intersects the germ of $\mathcal{F}^s(x_0)$ in $2d$ curves, each having an end at x_0. Hence the foliation $\Sigma \cap \mathcal{F}^s$ on Σ has a $2d$-pong singularity. But the interiors of S_P and Σ are homeomorphic as bi-foliated surfaces, so S_P has a $2d$-prong singularity on the point corresponding to x_0, which is the image of c_k.

The same argument can be used to prove that last statement on the linking number. The rest of the lemma is clear. □

1.7.3 Birkhoff annuli and Fried section.

We construct here some Birkhoff annuli and Fried sections using the fundamental domain.

Birkhoff annulus. An immersed Birkhoff annulus is an immersed partial section with the topology of an annulus. Thanks to a construction from T.Barbot, some Birkhoff annuli are easy to construct using the orbit space, which is mandatory for the last two chapters. We often distinguish immersed and embedded Birkhoff annuli.

Example 1.7.14. *Let S be an oriented hyperbolic surface, $\gamma \subset S$ be a simple geodesic, and η a coorientation of γ in S. Consider the geodesic flow on T^1S and the partial section Σ_η defined in Section 1.5.1 as the set of unitary vector based on γ in the half plane determined by η. We have a fibration $\Sigma_\eta \to \gamma$ whose fiber is half a circle, so Σ is an embedded Birkhoff annulus.*

When γ is not simple, one can similarly define an immersed Birkhoff annulus with a self intersection arcs in the fiber of each double point of γ.

When S is non-orientable and γ has a non-orientable tubular neighborhood, there is no coorientation of γ in S, but the set of fibers $T^1_\gamma S$ above γ is an immersed Birkhoff annulus whose boundary components both have multiplicity 2. If γ is simple, then $T^1_\gamma S$ is a Birkhoff annulus embedded in its interior.

Let $L \subset \mathcal{O}(M)$ be an ideal lozenge. We say that L is **simple** if there is no element $g \in \pi_1(M)$ such that g send a corner of the lozenge L inside the interior of L.

Theorem 1.7.15 (T.Barbot [Bar95b])**.** *Let L be an ideal lozenge in the orbit space $\mathcal{O}(M)$ whose corners induce closed orbits of the flow. There exists an immersed Birkhoff annulus A whose trace in $\mathcal{O}(M)$ is $\Theta_M(A) = L$. Furthermore, if there is no embedded Klein bottle in M and if L is simple, then A can be taken embedded.*

All Birkhoff annuli are not produced by this theorem, only the ones whose traces are ideal lozenges. The proof consists in giving a fundamental domain of the Birkhoff annuli in the orbit space, as in Figure 1.2, and lift it into M. Take c a curve in L between its two corners p and p', transverse to $\mathcal{L}^s$ and $\mathcal{L}^u$, and $g_p \in \pi_1(M) \setminus \{1\}$, so that $g_p.p = p$ and $g_p.p' = p'$. Then $c \cap g_p.c = \{p, p'\}$ and $c \cup g_p.c$ bounds a fundamental domain of the immersed Birkhoff annulus A, bounded by p and p'. When M is oriented, we can additionally recover the sign of ∂A. If L is in the quadrants $(+,+)$ and $(-,-)$ of its corners then the boundary components of A are positive, if L is in the other quadrants then the boundary components of A are negative.

Fried section. A Fried section is defined by a fundamental domain on a small 4-gon P as in Figure 1.11. Let R be a compact rectangle in the orbit space $\mathcal{O}(M)$ whose edges are arcs of two stable and two unstable leaves. Let $P \subset R$ be a 4-gon with corners a, b, c, d so that a and c are periodic and opposite corners of R. Denote by $g, h \in \pi_1(M) \setminus \{1\}$, the periods of a and c and take two positive integers n, m so that g^n and h^m preserve the orientations of $\mathcal{L}^s$ and $\mathcal{L}^u$. We suppose that the edges of P satisfy $g^n.[a, b] = [a, d]$ and $h^m.[c, d] = [c, b]$. Then there is a fundamental domain (P, F, f) given by $f([a, b]) = g^n$, $f([a, d]) = g^{-n}$, $f([c, d]) = h^m$ and $f([c, a]) = h^{-m}$, and F is defined using f. The immersed partial section it induces is called a **Fried section**. It corresponds to the construction of a transverse pair of pants proposed by Fried [Fri83].

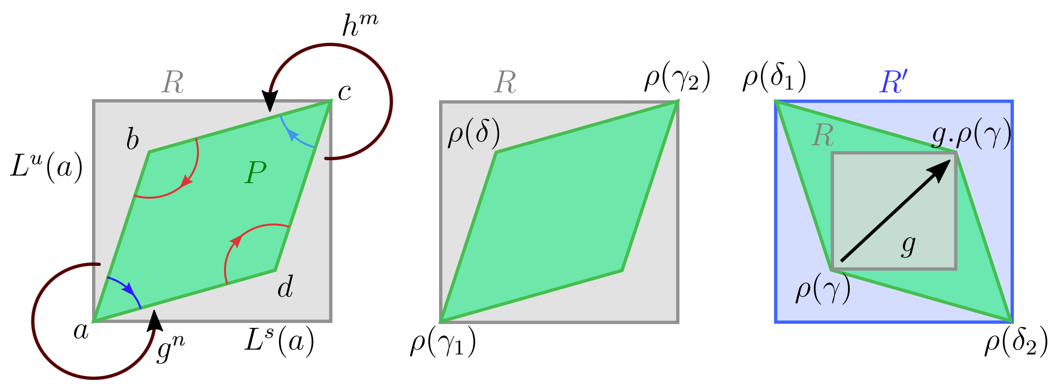


Figure 1.11: Fundamental domain of a Fried section. Its definition on the left and two existence Lemmas illustrated in the middle and right.

Lemma 1.7.16. *The surface induced by (P, F, f) is an immersed partial section Σ, so that Σ is an immersed pair of pants whose boundaries are the orbits induced by a, c and $b \simeq d$. The linking numbers of Σ along a and c are zero, and the linking number along b is one.*

If P is in the quadrants $(+,+)$ and $(-,-)$ of a and c, then a and c are positive boundary of Σ, and b is a negative boundary of Σ. Otherwise, all signs are reversed.

Proof. We suppose that a anb c are the corners of R given in Figure 1.11, the other case being similar with opposite signs. The surface $S_P = P/F$ is a sphere with three marked points a, c and $b \simeq d$, so Σ is an immersed sphere with at most three holes. According to Lemma 1.7.12, Σ has two boundary components on the orbits induced by a and c, with non-zero multiplicity n and m, and linking number zero.

According to the same Lemma, Σ has a potential boundary component on $b \simeq d$, whose homotopy class is given by $g^{-n}h^{-m}$. Since g and h cannot have other fixed points in R than a and c, g and h contract the unstable foliation inside R. Thus $h^m g^n$ also contracts the unstable foliation inside R. Thus $g^{-n}h^{-m} \neq 0$, and expands the unstable foliation inside R. So Σ has a negative boundary component on b. □

As discussed in Section 3.1, the partial sections of a flow contain information on the nature of this flow. The following Lemmas generate some Fried sections illustrated in Figure 1.11, that we use later to study the stably primitive orbits of skewed $\mathbb{R}$-covered Anosov flows.

Lemma 1.7.17. *Let γ_1 and γ_2 be two closed orbits, and suppose that their stable and unstable leaves $\mathcal{L}^{s,u}(\rho(\gamma_i))$ together bound a compact rectangle $R \subset \mathcal{O}(M)$. Then there exists an immersed Fried section Σ, bounded by γ_1, γ_2 and a third orbit δ. The boundaries of Σ in γ_1, γ_2 have linking numbers zero, and their signs are positive if and only if R lies in the quadrants $(+,+)$ and $(-,-)$ of $\rho(\gamma_i)$. The orbit δ is a boundary component with the opposite sign and with linking number one.*

If the leaf $\mathcal{F}^s(\gamma)$ is oriented, then Σ has multiplicity 1 along γ_i, otherwise it has multiplicity 2.

Proof. Denote by $o = \rho(\gamma_1)$ and $o' = \rho(\gamma_2)$ the two opposite corners of R. Denote by g and g' in $\pi_1(M)$ the first positive multiples of the periods of γ_1 and γ_2, so that $g.o = o$ and $g'.o' = o'$. Notice that $g \neq g'$, otherwise the leaves $\mathcal{L}^s(o)$ and $\mathcal{L}^u(o')$ would be fixed by g, so would be $\mathcal{L}^s(o) \cap \mathcal{L}^u(o') \neq o$. But there is at most one closed orbit in each stable leaf. The same argument proves that there is no element in R that is fixed by g. The flow is Anosov, so g and g' expand $\mathcal{L}^s$ and contract $\mathcal{L}^u$ inside R.

Hence the set $\mathcal{L}^u_R$ of those stable leaves intersecting R is a compact segment stable by $g'g$. By the Brouwer fixed-point Theorem, there exists a stable leaf intersecting R and fixed by $g'g$. With the same argument, there exists an unstable leaf intersecting R and fixed by $g'g$. Using the intersection of these stable and unstable leaves, there exists $p \in R$ so that $(g'g).p = p$. We denote $p' = g.p$, so that $p = g'.p'$. Notice that p and p' are in the interior of R. Take $c_1 \subset \mathring{R}$ an arc from o to p, and $c_2 \subset \mathring{R}$ an arc from o' to p', together with arcs $c_1' = g.c_1$ (from o to p') and $c_2' = g'.c_2$ (from o' to p). We can take these four arcs transverse to $\mathcal{L}^u$ and $\mathcal{L}^s$.

We claim that these four arcs are disjoints. Indeed suppose that there exists $x \in c_1 \cap c_1'$, and suppose that R is in the quadrant $(+,+)$ of o. Since c_1 is transverse to the bi-foliation, every point $y \in c_1$ must be in the quadrant $(+,+)$ or $(-,-)$ of x. But g contracts $\mathcal{L}^u$ and expands $\mathcal{L}^s$ inside R, so $g.x \in c_1$ is in the quadrant $(+,-)$ of x, contradiction. The other cases of intersection between c_1, c_1', c_2 and c_2' are impossible for similar reasons.

Hence $P \subset R$ delimited by $c_1 \cup c_1' \cup c_2 \cup c_2'$ is the desried 4-gon, and together with g and g' they form the fundamental domain of a Fried section. □

Lemma 1.7.18. *Let γ be a closed orbit, and suppose that there exists $g \in \pi_1(M)$ such that the stable and unstable leaves $\mathcal{L}^{s,u}(\rho(\gamma))$ and $\mathcal{L}^{s,u}(g.\rho(\gamma))$ bound a compact rectangle $R \subset \mathcal{O}(M)$. We also suppose that the action of g on $\mathcal{O}(M)$ preserves the orientations of the foliations $\mathcal{L}^s$ and $\mathcal{L}^u$. Then for all $n \in 2\mathbb{N}$ large enough, there exists an immersed Fried section Σ, bounded by $n\gamma$ and two other orbits δ_1 and δ_2. The boundaries of Σ in γ have linking number* 1 *and its sign is positive if and only if R is in the quadrants $(+,+)$ or $(-,-)$ of $\rho(\gamma)$. The signs of δ_1 and δ_2 as boundary components of Σ are opposite to the sign of γ.*

Proof. Denote by $o = \rho(\gamma)$ and by $h \in \pi_1(M)$ the oriented period of γ, so that $h.o = o$. We will construct points $p_n, q_m \in \mathcal{O}(M)$, so that $gh^n.p_n = p_n$ and $h^m g^{-1}.q_m = q_m$ for all $n, m \in \mathbb{N}$ large enough. Let $U \subset \mathcal{O}(M)$ be a small open neighborhood of R, also bounded by arcs of stable and unstable leaves. We denote by $\mathcal{L}^s_U \subset \mathcal{L}^s$ and $\mathcal{L}^u_U \subset \mathcal{L}^u$ the sets of leaves that intersect U (and similarly for $\mathcal{L}^{s,u}_R$). If U is close enough to R, then that h contracts $\mathcal{L}^u_U$ and expands $\mathcal{L}^s_U$.

Let $l \in \mathcal{L}^u_U \setminus \mathcal{L}^u_R$ so that $\mathcal{L}^u(g.o)$ is between $\mathcal{L}^u(o)$ and l. Since the flow is Anosov and h contracts $\mathcal{L}^u$, we have $h^n(l) \xrightarrow[n\to+\infty]{} \mathcal{L}^u(o)$. Thus $g.h^n(l)$ is between $\mathcal{L}^u(g.o)$ and l for $n > 0$ large enough. Also when it is the case, the set of unstable leaves in $\mathcal{L}^u$ between $\mathcal{L}^u(o)$ and l is a segment stable by gh^n, so by the Brouwer fixed point theorem, there exists a leaf $l^u_n \in \mathcal{L}^u_U$ fixed by gh^n. We can use the same argument on $\mathcal{L}^s_U$ and $h^{-n}g^{-1}$ to produce a leaf $l^s_n \in \mathcal{L}^s_U$ fixed by gh^n. Hence there exists $p_n = l^u_n \cap l^s_n \in U$ fixed by gh^n. Similarly, for all m large enough, the exists $q_m \in U$ so that $h^m g^{-1}.q_m = q_m$, and $\mathcal{L}^u(o)$ is between $\mathcal{L}^u(q_m)$ and $\mathcal{L}^u(g.o)$.

The points p_n and q_m in U are two opposite corners of a rectangle R' so that $R \subset R' \subset U$. According to Lemma 1.7.17 and to its proof, there exists a Fried section whose fundamental is a 4-gon in R' that admits p_n, o, q_m and $g.o$ as corners. We conclude with Lemmas 1.7.16 and 1.7.17.

$\square$

We give a last lemma that decomposes a Birkhoff annulus as the sum of two Fried sections. It is however not used later.

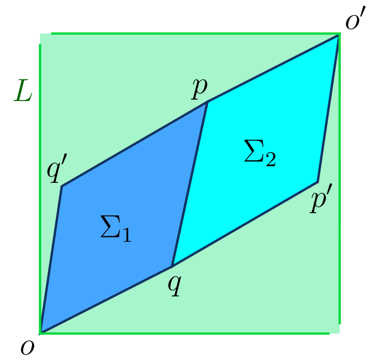
Lemma 1.7.19. *Let A be an immersed Birkhoff annulus whose trace is an ideal lozenge, and suppose that its interior intersects a closed orbit. Then there exist two Fried sections Σ_1, Σ_2, so that A is some Fried desingularisation of $\Sigma_1 \cup \Sigma_2$.*

When we say "some Fried desingularisation", it means that we desingularise only some self-intersection curves of $\Sigma_1 \cup \Sigma_2$.

Proof. We fix an orientation of M so that A has two positive boundaries. Let $L = \Theta_M(A)$ be the trace of A in $\mathcal{O}(M)$, which is an ideal lozenge with corners o and o', and $p \in \mathring{L}$ be the lift of a closed orbit intersecting $\mathring{A}$. Denote by γ, γ' and δ the orbits in M represented by o, o' and p, and by n the multiplicities of A along γ and γ' (which are equal).

We take $\gamma_A \subset A$ a simple curve staying at a small and constant distance from γ, γ_A is homotopic to $n\gamma$ inside a small neighborhood. Then we denote by $g \in \pi_1(M)$ the homotopy class of γ_A, which is the homotopy class of γ to the power n. We notice that $g.o = o$, $g.o' = o'$, and that g preserves the orientation of the stable and unstable foliations in $\mathcal{O}(M)$ (since A has linking number zero along γ, the curve γ_A stays in the same quadrant $(+,+)$ or $(-,-)$ around γ, and its homotopy class preserves the orientations of $\mathcal{L}^s$ and $\mathcal{L}^u$). Similarly we consider $h \in \pi_1(M)$ the double of the homotopy classes of o and p, such that $h.p = p$ and that h preserves the orientations of the stable and unstable foliations in $\mathcal{O}(M)$.

We consider the rectangle $R \subset L$ bounded by arcs of stable and unstable leaves, with two opposite corners o and p. By Lemma 1.7.17, there is an immersed Fried section Σ_1, whose fundamental domain is a 4-gon in L with corners o, q, p and q', where $q, q' \subset R$. Also $g.q = q'$ and $h.q' = q$. We denote by α the orbit represented by q and q'. Then by construction we have $\partial^+\Sigma_1 = n\gamma \cup 2\delta$ and $\partial^-\Sigma_1 = m\alpha$, where m is the positive integer so that hg is homotopic to $m\alpha$.



We define $p' = g^{-1}.p$, so that by construction we have $g.o' = o'$, $hg.q = q$, $g.p = p'$ and $hg.p' = h.p = p$. We can apply the same lemma to build an immersed Fried section Σ_2, whose fundamental domain is a 4-gon in L with corners o', p, q and p'. We can choose Σ_1 and Σ_2 so that their fundamental domains share the common segment $[p, q']$. By construction, $\partial^+\Sigma_2 = n\gamma' \cup m\alpha$, and $\partial^-\Sigma = 2\delta$ is induced by p.

Let Σ be the Fried desingularisation of $\Sigma_1 \cup \Sigma_2$ along the orbits of p and q, and along the segment $[p, q]$. Then $\partial^+\Sigma$ is the union of the orbits o and o', and $\partial^-\Sigma = \emptyset$. Also the union of the two fundamental domains of Σ_1 and Σ_2 can be described as a 2-gon bounded the curves $[o, q, p', o']$ and $[o, q', p, o'] = g.[o, q, p', o']$. Hence Σ is an immersed Birkhoff annulus, whose trace satisfies $\Theta_M(\Sigma) = L = \Theta_M(A)$. Futhermore, Σ has the same multiplicity than A along its boundary components, so A and Σ have the same fundamental domain in $\mathcal{O}(M)$, and they are isotopic along the flow. $\square$

Chapter 2

Explicit first-return maps

In the whole chapter, we fix an orientable hyperbolic surface S and one metric with negative curvature on S. We denote by ϕ the associated geodesic flow on T^1S and consider $\Gamma \subset S$ a filling union of finitely many closed geodesics, assumed to be in general position. We consider the Birkhoff sections Σ_η for some Eulerian coorientations $\eta \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ defined in Section 1.5.1. Our main goal is to compute the associated first-return maps as products of Dehn twists. First, we give some background about this question.

Given a 3-manifold M, W.Thurston defined a semi-norm on $H_2(M,\mathbb{R})$ which is a norm when the manifold is atoroidal [Thu86]. For an integral element $\omega \in H_2(M,\mathbb{R})$, the norm measures the minimal absolute value of the Euler characteristic of a surface without sphere components inside M whose homology class is ω. The unit ball of the Thurston norm is a polyhedron in $H_2(M,\mathbb{R})$, whose faces are dual of integer points in $H^2(M,\mathbb{Z})$. To a fibration $M \to S^1$ by compact surfaces homotopic to F corresponds a unique rational point in the unit sphere given by the homology ray containing the homology of F. W.Thurston and D.Fried have proved that to a flow admitting global sections in M corresponds a so-called fibered face in the unit sphere, given by all fibrations whose fibers are global sections for this flow. The flow also induces first-return maps on these global sections. The unit ball of the Thurston norm, its fibered faces, and the first-return map of its fibrations are not completely understood yet. The goal of this chapter is to understand how these first-return maps are all connected, in a specific case.

When the flow is of pseudo-Anosov type (Anosov outside finitely many closed orbits, these orbits being of prong type), the first-return map on a section S is pseudo-Anosov. In particular it has an expansion factor $K > 1$. One could desire to compare the expansion factors of the first-return maps given by several global sections on the same fibered face. Fried studied [Fri82a] the function $(S \mapsto \chi(S)\ln(K(S)))$, which is convex and tends to infinity on the boundary of the fibered face. McMullen defined [McM00] the Teichmüller polynomial in $\mathbb{Z}[H^1(M,\mathbb{Z})/torsion]$, whose specialization at an integral point ω gives as largest root the expansion K of a global section with homology $[\omega]$. Hence the expansion factor satisfies two uniform descriptions. This chapter goes in the same direction by giving, for one explicit family of fibered faces, a computation and a comparison of the first-return maps.

Since S is hyperbolic, the geodesic flow on T^1S is Anosov. Once one removes finitely many closed orbits, we obtain a pseudo-Anosov flow on a 3-manifold with toric boundary. For such flows, the fibered faces correspond to Birkhoff sections of the original flow, whose boundary components are in the set of removed closed orbits. Under a certain symmetry assumption, the fibered faces for these are rather well-understood.

Birkhoff sections with symmetric boundary. Given an Eulerian coorientation $\eta \in \mathcal{E}ul\mathcal{C}o(\Gamma)$, recall that we defined $\hat{\Sigma}_\eta$ as the set of vectors based on Γ and in the same direction that η, which we desingularise and smooth to obtain the surface Σ_η. Under a combinatorial condition on η, Σ_η is a Birkhoff section with boundary $-\overset{\leftrightarrow}{\Gamma}$. Furthermore every Birkhoff section with boundary $-\overset{\leftrightarrow}{\Gamma}$ is isotopic through the flow to one surface Σ_η for some coorientation $\eta \in \mathcal{E}ul\mathcal{C}o(\Gamma)$.

The surface Σ_η stays mainly in some specific fibers of $\pi : T^1S \to S$, and $\pi_{|\Sigma_\eta}$ is not an immersion. In order to make Σ_η easier to use, we deform it into a surface on which π is an immersion.

Theorem A. *Given a closed orientable hyperbolic surface S, a geodesic multi-curve $\Gamma \subset S$ supposed to be filling and in general position, and an Eulerian coorientation η of Γ, there exist a small smoothing of the associated surface $\Sigma_\eta \subset T^1S$ and a small isotopy $(f_t)_{t\in[0,1]}$ of the surface Σ_η such that $f_0 = \iota_{|\Sigma_\eta \hookrightarrow T^1S}$ and $\pi \circ f_1 : \Sigma_\eta \to S$ is an immersion.*

Figure 2.1 illustrates how the immersed surface in S looks like. We will study several representations of Σ_η in Section 2.1. We are primarily interested in the immersion of Theorem A, and in the ribbon graph representation it induces.

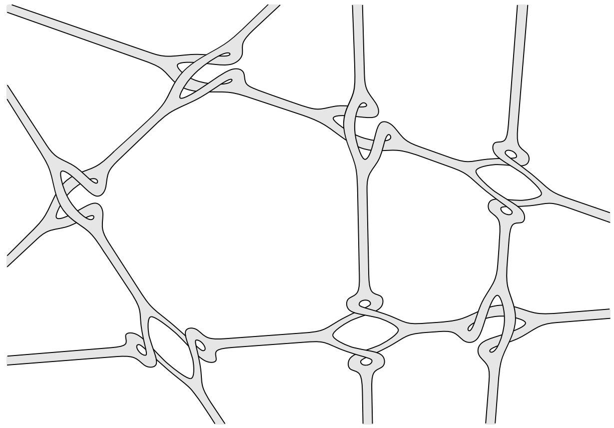


Figure 2.1: A small perturbation of Σ_η into a horizontal surface.

Partial return maps. The main idea for computing the first-return map on Σ_η is to define intermediate disjoint and homologous Birkhoff sections $(\Sigma_i)_i$, so that the first-return map r_{Σ_η} is the composition of some isotopies along the flow:

$$\Sigma_\eta = \Sigma_0 \to \Sigma_1 \to \ldots \to \Sigma_{n-1} \to \Sigma_n = \Sigma_\eta$$

We define the surfaces Σ_i by induction using elementary transformations, so that $r_i : \Sigma_{i-1} \to \Sigma_i$ is quite simple to compute. These elementary transformations have a combinatorial and a geometric version. The combinatorial version consists in taking an Eulerian coorientation η and modifying it around one specific face, thus obtaining a new coorientation η'. The surfaces Σ_η and $\Sigma_{\eta'}$ are isotopic and easy to compare. If we do this transformation around every face in the right order, we describe a cyclic family of Birkhoff sections $(\Sigma_i)_{0\leq i\leq n}$, pairwise easily comparable.

The geometric version of this transformation consists in taking η and η' that differ around a face f, and following the flow only in $T^1_f S$. It describes a map $\Sigma_\eta \to \Sigma_{\eta'}$ that we call **partial return map**. The partial return maps together with the family of Birkhoff sections $(\Sigma_i)_i$ allow us to reconstruct the first-return map.

Theorem (properly stated in Theorem B)**.** Let $\Gamma \subset S$ be a filling geodesic multi-curve of a hyperbolic oriented surface S and η an Eulerian coorientation of Γ. Suppose that the dual graph $\Gamma^\star$ does not admit any cycle oriented by η, and denote by $f_1, \ldots, f_n$ the faces of $S \setminus \Gamma$, ordered by η. Then the first-return map along the geodesic flow on the Birkhoff section Σ_η is given by $r_{\Sigma_\eta} = r_n \circ \ldots \circ r_1$, where r_i is the partial return map along the face f_i.

In this theorem, we order the faces so that if η goes from the face f_i to f_j around an edge, then $j < i$. We study this elementary transformation in Section 2.2, together with the combinatorial tools needed to express precisely Theorems B and C.

Given two homologous Birkhoff sections Σ_η and Σ_ν, we explain in Section 2.2.3 how to find a sequence of partial return maps from Σ_η to Σ_ν. Additionally, when Σ_η and Σ_ν are only partial sections, we study when they are isotopic through the flow using a combinatorial condition on η and ν.

Explicit first-return map. To compute the first-return map, we need to compute explicitly the partial return maps. Fix $r_i : \Sigma_{i-1} \to \Sigma_i$ a partial return map. We would like to compose r_i with a nice **correction** function c_i so that the composition $\Sigma_{i-1} \xrightarrow{r_i} \Sigma_i \xrightarrow{c_i} \Sigma_{i-1}$ is a Dehn twist. We will use the ribbon representation of Σ_{i-1} and Σ_i to compare them, especially around the vertices at which they differ. After defining c_i, the composition $c_i \circ r_i$ is isotopic to a negative Dehn twist along a curve γ_f.

We will define two families of curves γ_v and γ_f in Section 2.3, which are illustrated in Figure 2.2. The previous computation together with Theorem B, allows to compute the first-return map as a product of negative Dehn twists.

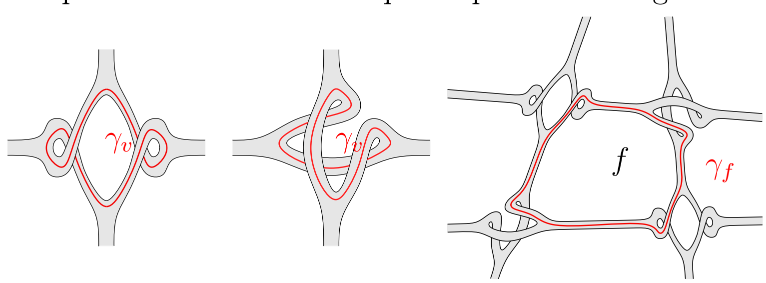


Figure 2.2: Curves γ_v for a vertex v and γ_f for a face f.

Theorem (properly stated in Theorem C)**.** Let η be an acyclic Eulerian coorientation and Σ_η its corresponding Birkhoff section. Then the first-return map $r : \Sigma_\eta \to \Sigma_\eta$ is the product of explicit negative Dehn twists along the explicit curves γ_v and γ_f for all $v \in \Gamma_0$ and $f \in \Gamma_0^\star$. The order of the Dehn twists is given by η.

A precise statement and a proof of this theorem are given in Section 2.3. This result is reminiscent of N.A'Campo's divide construction [A'C98], and of M.Ishikawa's generalization [Ish04]. They decompose a monodromy as an explicit product of three Dehn multi-twists. N.A'Campo's result was also recently generalized by P.Dehornoy and L.Liechti [DL19] who expressed the monodromy for divide links in the unit tangent bundle of arbitrary surfaces as products of two antitwists. Our results deal more generally with all integral points in the fibered face that corresponds to Birkhoff section with boundary $-\overset{\leftrightarrow}{\Gamma}$, instead of just the center point.

Theorem (properly stated in Theorem D)**.** Let S be a hyperbolic surface, Γ a finite collection of closed geodesics on S, and consider the geodesic flow on T^1S. There exists a common combinatorial model Σ_Γ for all Birkhoff sections with boundary $-\overset{\leftrightarrow}{\Gamma}$, and an explicit family $\gamma_1, \cdots, \gamma_n$ of simple closed curves in Σ_Γ such that the first-return maps for these Birkhoff sections are products of negative Dehn twists of the form $\tau^{-1}_{\gamma_{\sigma(1)}} \circ \cdots \circ \tau^{-1}_{\gamma_{\sigma(n)}}$ for some permutation σ of $\{1, \cdots, n\}$.

In Theorem C, the Birkhoff sections and the curves supporting the Dehn twists are explicit, and only depend on the choice of one coorientation. Also the ordering of the Dehn twists is almost canonical. In Theorem D, there are only one abstract Birkhoff surface and collection of curves, that are also explicit. But the ordering of the negative Dehn twists and the first-return map are less explicit, and need more work to be constructed by hand.

Example on a flat torus. On a flat torus, the classification of Birkhoff sections is different, and can be found in [Deh15]. However the surfaces Σ_η can be defined similarly and they are Birkhoff sections. Also Theorem A, B and C are still true for these surfaces. It is simpler to illustrate them on the torus.

In Figure 2.3, we briefly illustrate the theorems on the flat torus, given by a square whose opposite sides are identified. Let Γ and η be the multi-curve and the coorientation given on the picture. Theorem A gives an immersion of the Birkhoff section Σ_η into the torus, which is represented on the right. Four examples of the curves γ_f (in red) and γ_v (in blue and green) are also represented.

We order, from 1 to 12, the vertices of Γ and the faces it delimitates. For this, we complete the natural order given by the coorientation η of the faces, using additional rules explained in Section 2.2. Theorem C then states that the first-return map r on Σ_η is a product of negative Dehn twists, with the order previously chosen. So that if $T\gamma$ denotes the negative Dehn twist along γ, then:

$$r_{\Sigma_\eta} = T\gamma_{f_{12}} \circ T\gamma_{f_{11}} \circ T\gamma_{v_{10}} \circ T\gamma_{v_9} \circ T\gamma_{v_8} \circ T\gamma_{v_7} \circ T\gamma_{f_6} \circ T\gamma_{f_5} \circ T\gamma_{v_4} \circ T\gamma_{v_3} \circ T\gamma_{f_2} \circ T\gamma_{f_1}$$

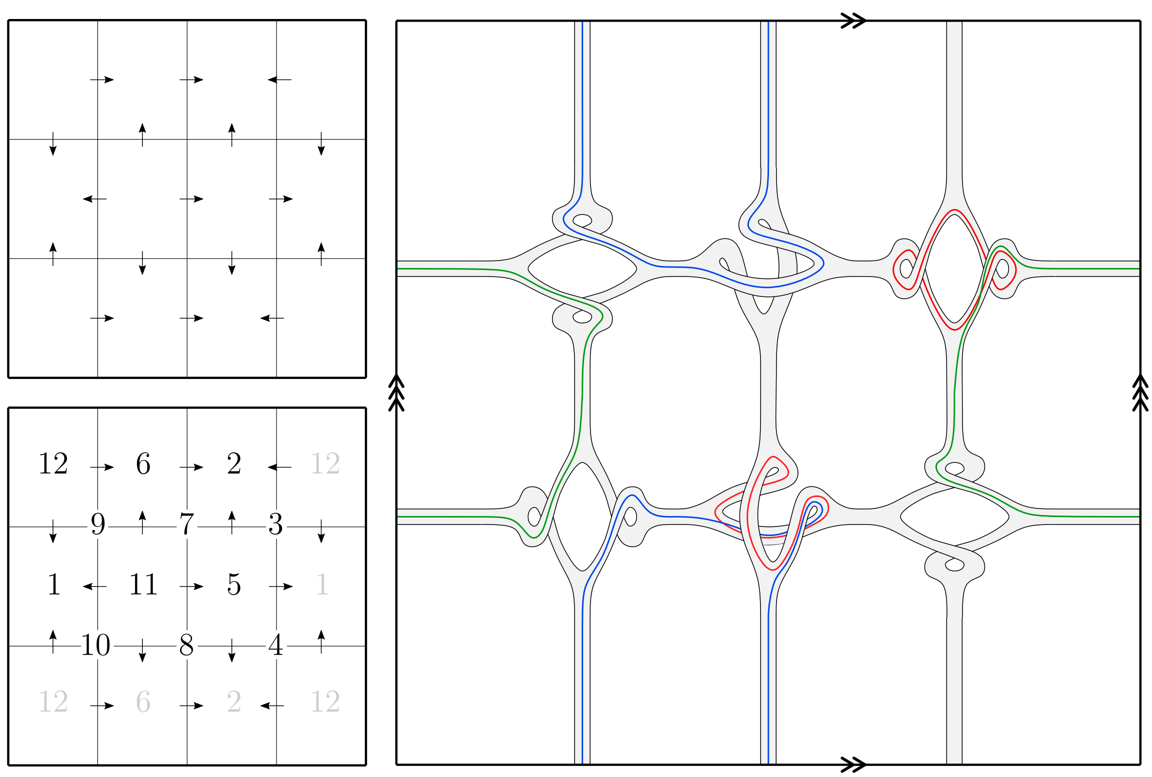


Figure 2.3: Example on a flat torus $\mathbb{R}^2/\mathbb{Z}^2$. A collection of geodesic and an Eulerian coorientation η are represented in the top left part. A corresponding ordering between the faces and the vertices is represented in the bottom left part. The gray indexes correspond to the faces which are not represented connected in the figure. The right part of the figure illustrates the surface Σ_η immersed inside the torus, and four of the curves γ_v and γ_f lying inside the surface Σ_η.

2.1 Representations of the Birkhoff sections Σ_η

In this section, we study the partial section Σ_η defined in Section 1.5.1. We find good representations of that partial section, including an immersion into S and a ribbon graph representation. It will later help us to do explicit computations. Our two goals are to prove Theorem A and to study some elementary properties of the ribbon representation.

2.1.1 Skeleton of Σ_η

We briefly describe a skeleton of Σ_η. Figure 2.4 shows a skeleton $\hat{X}$ of $\hat{\Sigma}_\eta$, that will be pushed into a skeleton X of Σ_η. Take an edge $e \in \Gamma_1$, it corresponds to a flat rectangle r_e in $\hat{\Sigma}_\eta$, that is isometric to $e \times [0, \pi]$. Denote by $\{v_1, v_2\} = \partial e$ and $a_1, a_2 \in (0, \pi)$ the angles between e and the intersecting geodesics on v_1 and v_2. The rectangle r_e is attached to four other rectangles (counting with multiplicity) on the four segments given by $v_i \times [0, a_i]$ and $v_i \times [a_i, \pi]$. So we put a vertex in the middle of each of these segments, and we connect them as in Figure 2.4. The union for every $e \in \Gamma_1$ defines a skeleton $\hat{X}$ of $\hat{\Sigma}_\eta$, that we push into Σ_η to define the skeleton X.

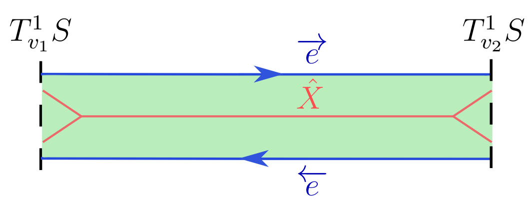


Figure 2.4: A rectangle r_e and a part of a skeleton $\hat{X}$ of $\hat{\Sigma}_\eta$

Notice that locally around every vertex $v \in \Gamma_0$, the skeleton X is homeomorphic to a circle glued once to four edges leaving the circle, independently of the nature of v. To be more precise, we describe $\pi(X)$ the projection of X into S. If the smoothing of Σ_η is well-chosen, $\pi(X)$ can be obtained from Γ by replacing each alternating vertex by a square, and each non-alternating vertex by an immersed 4-gon shaped as a twisted square, as in Figure 2.5.b.

Lemma 2.1.1. *There exists a deformation retract $(d_t)_{t\in[0,1]}$ of Σ_η onto the 1-skeleton X which satisfies that d_t is an immersion for all $t \in [0, 1)$.*

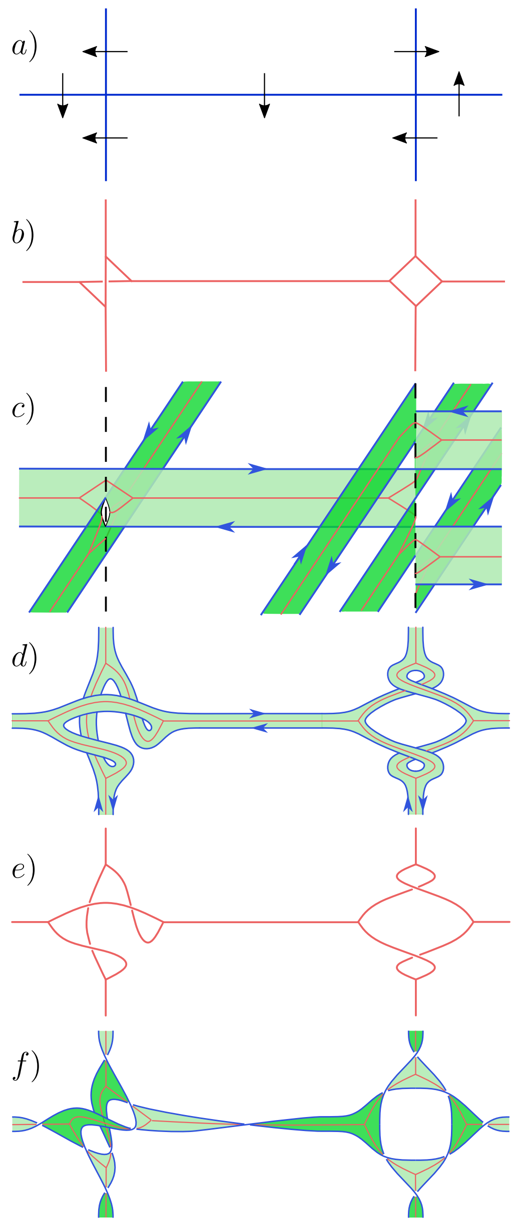


Figure 2.5: Local representations of Σ_η: *a*) a multi-geodesic Γ and a coorientation η of Γ, *b*) the projection $\pi(X) \subset S$ of the skeleton X, *c*) a local picture of the surface $\Sigma_\eta \subset T^1S$ locally identified with $\mathbb{R}^2 \times \mathbb{R}$, *d*) the isotope immersion $\Sigma_\eta \to S$ provided by Theorem A, *e*) the ribbon representation of the skeleton of Σ_η (see Section 2.1.3), and *f*) the twisted representation of Σ_η.

Proof. We consider a first deformation retract from Σ_η to Σ_η minus an ϵ-tubular neighborhood of its boundary. Inside a rectangle r_e, the image of that retract is the r_e minus an ϵ-tubular neighborhood of $\vec{e}$ and $\overleftarrow{e}$ and minus an ϵ-tubular neighborhood of the two points $(v_1, a_1), (v_2, a_2) \in T^1S$. Here $\vec{e}$ and $\overleftarrow{e}$ are the orbit arc of the geodesic flow which are lifts of e with the two possible orientations. One can find a deformation retract of each rectangle r_e minus these ϵ neighborhoods to the 1-skeleton $r_e \cap X$, such that the retract is given on ∂r_e by four homothéties centered at the four points of intersection in $\partial r_e \cap X$. For two edges e and e', these retracts are equal on $r_e \cap r_{e'}$, so we can glue all these retracts to a deformation retracts onto X. Furthermore each deformation retract given above can be chosen such that the final deformation $(d_t)_{t\in[0,1]}$ retract is an immersion for all $t \in [0, 1)$. □

2.1.2 Isotopies and immersion of Σ_η

The definition of Σ_η makes it a bit hard to compute algebraic and geometric intersections between explicit curves. We will give two other descriptions, obtained by isotopy, of Σ_η that will help us.

Isotopy with an immersion. Recall that the surface S is orientable, so that we can identify its tangent plans to $\mathbb{C}$. The isotopy of T^1S that interests us is the parallel transport that pushes (x, u) in the direction iu:

$$\big(f_t : (x, u) \in T^1S \mapsto \exp_{(x,u)}(tiu)\big)_{t\geq 0}$$

A first case is easy to understand. Considering one injective geodesic arc e, a coorientation η of e, and $r_e \subset T^1S$ the set of half fibers above e in the direction of η (which is used in Section 1.5.1 to construct the surface Σ_η). For $t > 0$ small enough, the function $\pi \circ f_t$ send half a fiber above a point x to half a circle with center x. Additionally $\pi \circ f_t$ is injective on r_e and its image has the shape of a thin rectangle, whose two opposites short sides are half circles.

Unfortunately the function $\pi \circ f_t$ has a more complicated behavior around the fiber of a vertex of Γ, and it does not induce an immersion on all of Σ_η. We will modify this isotopy to make it computable and prove Theorem A. First we study f_t in local flat models, then we will glue these local models. Let $S' = \mathbb{C}$ be the flat plane, $\gamma_1 = \mathbb{R} \times \{0\}$ and let $\gamma_2 = \{0\} \times \mathbb{R}$ represent two interesting geodesics. Let $\Gamma'_1 = \{\gamma_1\}$ and $\Gamma'_2 = \{\gamma_1, \gamma_2\}$ represent respectively an edge and a crossing. We define the function $g_t : T^1S' \to T^1S'$ given by $(z,u) \mapsto (z+tiu, u)$ similarly to f_t. It is enough to study g_t in this model. Let η be an Eulerian coorientation of Γ'_i and construct $\Sigma' \subset T^1S'$ in the same way as we construct Σ_η, for both alternating and non-alternating vertices.

Lemma 2.1.2. *Let $\mathcal{N} \subset \Sigma'$ be a neighborhood of $\partial\Sigma'$. Then, for every $T > 0$, there is a small smoothing of Σ' such that for all $t > T$, $(\pi \circ g_t)_{|(\Sigma' \setminus \mathcal{N})}$ is an immersion.*

Proof. We first prove the result for T arbitrary large and for a fixed smoothing of Σ'. We want to prove that $\pi \circ g$ is an immersion, so we compute the kernel of $d(\pi \circ g)$. Then we prove that that kernel is transverse to Σ_η on a good subset of Σ_η. We first consider Γ'_1. In this case we have $\Sigma' = \{(x, e^{i\theta}), x \in \mathbb{R}, \epsilon\theta \in [0,\pi]\}$, where $\epsilon = \pm 1$ depends on the coorientation η. The differential of $\pi \circ g$ is given by a complex 1-form $d(\pi \circ g_t)(x, \mathrm{e}^{i\theta}) = dx - \epsilon t\, \mathrm{e}^{i\theta}\, d\theta$, which is injective if $\theta \in (0,\pi)$. Thus $\pi \circ g_t$ is an immersion on the interior of Σ'.

Consider now Γ'_2. Fix $t > 0$ and take $(x,u) \in T^1S$. Then $\ker(d(\pi \circ g_t)(x,u))$ is spanned by $U_t = (u, \frac{1}{t}\frac{\partial}{\partial\theta})$. But $\lim\limits_{t \to +\infty} U_t = (u, 0)$ which generates the geodesic flow. Let $K \subset \Sigma' \setminus \partial\Sigma'$ be a compact sub-manifold. Then for $t > 0$ large enough, U_t is transverse to K, so $(\pi \circ g_t)_{|K}$ is an immersion. We can suppose that outside a compact $K' \subset S'$, Σ' has been smoothed so that the surface $\Sigma' \setminus \pi^{-1}(K')$ is contained in the set of fibers $\pi^{-1}(\Gamma' \setminus K')$. Then $\Sigma' \setminus (\pi^{-1}(K') \cup \partial\Sigma')$ is transverse to U_t for all $t > 0$, as in the first case.

We consider the surface $\Sigma'' \subset \Sigma' \setminus \partial\Sigma$ such that $\Sigma'' \cap \pi^{-1}(K') = K$ and such that Σ'' and $\Sigma' \setminus \partial\Sigma'$ coincide outside $\pi^{-1}(K')$. We can chose the compact K such that $\Sigma' \setminus \Sigma''$ remains in any neighborhood $\mathcal{N}$ of $\partial\Sigma'$. Then according to what precedes, for all $t > T$ the function $\pi \circ g_t$ induces an immersion on Σ''.

To prove that T can be made arbitrary small if we change the smoothing of Σ', it is enough to conjugate the previous isotopy with the diffeomorphism $h_s : (z,u) \in T^1\mathbb{C} \mapsto (sz, u)$ for $s > 0$ a fixed parameter small enough. This diffeomorphism makes the smoothing of Σ' smaller and satisfies $g_t \circ h_s(z,u) = h_s \circ g_{ts}(z,u)$. It finishes the proof. □

Proof of Theorem A. Let $\mathcal{N} \subset \Sigma_\eta$ be a small tubular neighborhood of $\partial\Sigma_\eta$ that does not intersect the skeleton X of Σ_η. Consider a flat metric $\tilde{g}$ on a small neighborhood of $\Gamma \subset S$, such that Γ stays geodesic for $\tilde{g}$. We consider a finite open cover $\mathcal{U}$ of a small neighborhood of $\Gamma \subset S$, such that each open $U \in \mathcal{U}$ intersects Γ in either on compact curve, or in two intersecting compact curves. Then each open $U \in \mathcal{U}$ is isometric to an open subset V

of the standard model S' containing 0. Then by using Lemma 2.1.2, we can chose a small smoothing of Σ_η and find an isotopy of $(\Sigma_\eta \setminus \mathcal{N}) \cap T^1U$ to an immersion that stays in any small thickening of V. Since the isotopies are parallel transports of the form $(x, u) \mapsto (x + \lambda iu, u)$, with the metric $\tilde{g}$, we can glue these isotopies for all $U \in \mathcal{U}$.

Therefore there exists an isotopy of $\Sigma_\eta \setminus \mathcal{N}$ whose composition with $\pi : T^1S \to S$ ends with an immersion. According to Lemma 2.1.1 we can compose this isotopy with a deformation retract from the surface Σ_η into $\Sigma_\eta \setminus \mathcal{N}$. Additionally that isotopy can be suppose small enough if the choice of the neighborhood $\mathcal{N}$ was initially made closed enough to $\partial\Sigma_\eta$. Then the image of the neighborhood of both kinds of vertices are obtained explicitly with the local model S'. □

The image of the immersion $\Sigma_\eta \to S$ is illustrated in Figure 2.5. It is obtained by taking a good smoothing of Σ_η such that the function $\pi \circ g_t$ (defined in the local models) is immersed for all $t > 0$ on a small enough neighborhood of the skeleton of Σ_η. Then the image represented is given by the image of a small tubular neighborhood of the skeleton. If one take a time $t > 0$ large enough such that $\pi \circ g_t$ is an immersion outside a small neighborhood of $\partial\Sigma$, the immersed surface obtained that way is isotopic though immersed surface to the surface illustrated in Figure 2.5. So up to an isotopy, we can work using the immersed surface illustrated in that figure.

Remarks 2.1.3. We could have chosen to take $t < 0$. The image of the immersion would be similar. Also the order of the self-intersections of the immersion does not matter since it comes from the projection of a S^1 fiber.

Definition 2.1.4. The immersion $\Sigma_\eta \to S$ thus constructed does not depend on the choices made (for $t > 0$) up to isotopy through immersion. It will be called the **isotope immersion**, and denoted by ψ_{im}.

The isotopy by twisted immersion. There exists another representation of Σ_η that might be interesting. Take X the skeleton of Σ_η as in Figure 2.5.b. Replace every vertex of X (which has degree 3) by a triangle and replace every edge by a twisted rectangle. Glue them along the triangle corresponding to the ends of the edges. We obtain the image of a twisted immersion of Σ_η as in Figure 2.5.f. There exists a small isotopy of Σ_η in T^1S that gives this representation when composed with π. One can prove this by using either the isotope immersion, or by understanding geometrically how to twist a rectangle r_e around a vertex v depending on the orientation of η around v. However we will not use this representation later.

2.1.3 The ribbon graph representation of Σ_η

In this subsection, we adapt and use the notion of ribbon graph to describe Σ_η and its skeleton X as combinatorial objects. We start by defining the combinatorial tools that interest us. Then we connect them with the isotope immersion of Σ_η. Eventually, this will make the study of isotopies and diffeomorphisms easier.

Definition 2.1.5. Let S be a surface, $X = (X_0, X_1)$ a graph and $\phi : X \to S$ a continuous map. We say that (X, ϕ) is a **ribbon graph** if

- $\phi_{|X_0}$ is injective,
- for all $e \in X_1$ (as closed segment), $\phi_{|e}$ is immersed,
- for all $v \in X_0$, the tangents to $\phi_{|e}$, for all $e \in X_1$ bounding v, are pairwise not positively collinear (and not zero).

Let (X, ϕ) be a ribbon graph on a surface S. We call **induced surface** of (X, ϕ) the thickened immersed surface obtained from the blackboard framing. More precisely, it corresponds to $(\Sigma_{X,\phi}, \iota, \pi)$, where $\Sigma_{X,\phi}$ is a smooth surface, $\iota : X \hookrightarrow \mathring{\Sigma}_{X,\phi}$ is an embedding and $\pi : \Sigma_{X,\phi} \looparrowright S$ is an immersion, such that $\Sigma_{X,\phi}$ deformation retracts to $\iota(X)$, and $\pi \circ \iota = \phi$.

Example 2.1.6. *The isotope immersion defined in Section 2.1.2 naturally yields a ribbon graph and its induced surface, as in Figure 2.5.*

Two ribbon graphs are said to be **weakly isotopic** if there exists a succession of isotopies of ribbon graphs, of twists, fusions and contractions that goes from one to the other. The twist, fusion and contraction moves are represented in Figure 2.6.

Notice that during such an isotopy, the order of the edges around a vertex do not change. Also a weak isotopy $(X, \phi) \to (Y, \psi)$ induces a canonical homotopy equivalence $X \to Y$. Examples of weak isotopies are given in Figure 2.7.

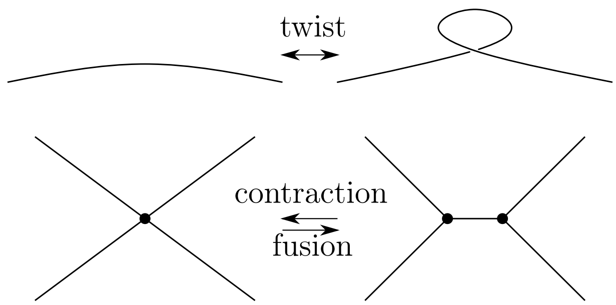


Figure 2.6: Weak isotopy of ribbon graph.

Proposition 2.1.7. *Let F be a weak isotopy between two ribbon graphs (X, ϕ) and (Y, ψ), $f : X \to Y$ the homotopy equivalence induced by F, and Σ_X, Σ_Y be the surfaces induced by (X, ϕ) and (Y, ψ). Then there exists a diffeomorphism $g : \Sigma_X \to \Sigma_Y$ so that the two maps $\pi_Y \circ g, f \circ \pi_X : \Sigma_X \to Y$ are homotopic.*

Proof. We gives some ideas of the proof. First notice that the twists, contraction and fusion moves induces a diffeomorphism g as in the proposition. Also a regular isotopy of ribbon graphs is locally induced by an ambiante isotopy, which induces an isotopy on the induced surface. So a regular isotopy of ribbon graphs also satisfies the property. Hence a finite concatenation of regular isotopies and of twists, contraction and fusion moves satisfies the conclusion of the property. □

Because of the twist move, this diffeomorphism does not always come from an isotopy of their immersed image in S.

The combinatorial representation of Σ_η . Theorem A gives a representation of Σ_η as a ribbon graph. In this paragraph, we use this representation to compare the alternating and non-alternating vertices, examples of which can be found in Figure 2.5.e. This will be useful in Section 2.2 for identifying the surface Σ_η one with another.

We will detail here how to compare two Birkhoff sections associated to two coorientations that differ around a specific vertex v of Γ. Figure 2.7 describes two weak isotopies of ribbon graphs that interest us. The idea of the isotopies is the following. Use the image $\pi(X) \subset S$ of the skeleton X. There is in $\pi(X)$ a 4-gon $\diamond$ associated to the vertex v. The shape of $\diamond$ is illustrated in Figure 2.5.b), as either a square or a twisted square. Fix e an edge of $\diamond$ which remains in the same quadrant around the vertex v. In the 4-gone $\diamond$, the edge e has two adjacent neighboring edges, that we move along e.

Definition 2.1.8. The isotopies described above and shown in Figure 2.7 are called **slide along** e, or slide along the quadrant containing e.

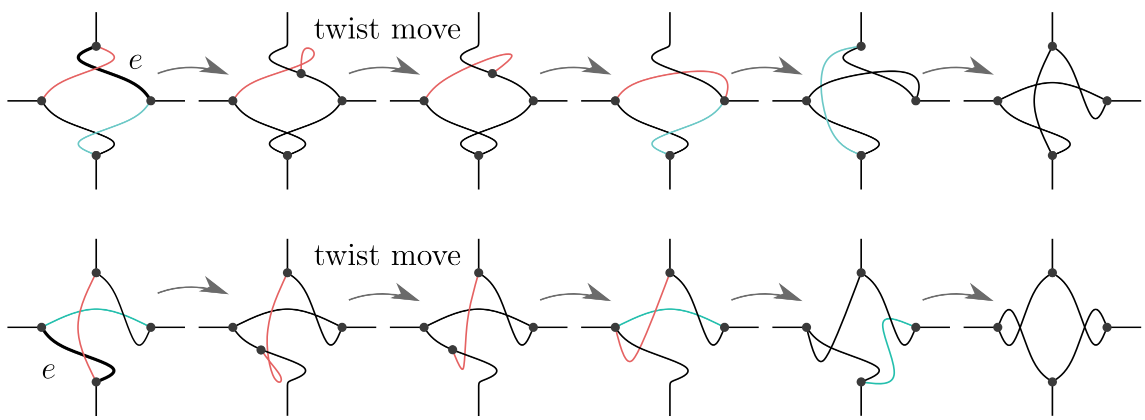


Figure 2.7: Isotopy of ribbon graphs (slide along e)

For more precision, for a vertex v one can do a slide in a quadrant around v if that quadrant is either a minimal or a maximal quadrant, for the coorientation η. In Figure 2.7 are represented respectively a slide along the top right edge of an alternating vertex and along the bottom left edge of a non-alternating vertex. All possible slides are obtained by doing rotation or symmetry of these two slides. Notice that these slides are pseudo involutions, in the sense that sliding along the same quadrant twice is isotopic to the identity. We are interested in compositions of slides.

Let v in Γ_0 and denote by c_1, c_2, c_3, c_4 the four quadrants around v, ordered according to an Eulerian coorientation, for example as in Figure 2.8. More precisely we require that the coorientation η is decreasing relatively to the chosen order on the four quadrant. Denote by sl_i the slide along c_i, which is well-defined when the skeleton X admits an edge e_i along c_i.

Lemma 2.1.9. *In the above context, the diffeomorphism of Σ_η induced by $sl_4 \circ sl_3 \circ sl_2 \circ sl_1$ is well-defined and isotopic to a negative Dehn twist along the curve γ_v, represented in Figure 2.9.*

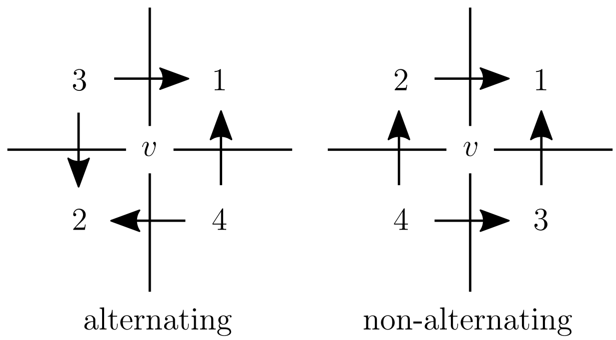


Figure 2.8: Ordering the slides around a vertex v.

Proof. Let $U \subset T^1S$ be a small tubular neighborhood of the fiber T^1_vS, so that $U \cap \Sigma_\eta$ is homeomorphic to an annulus, and let $\gamma_v \subset \Sigma_\eta$ be the skeleton of $\Sigma_\eta \cap U$. We prove the lemma when v is an alternating vertex. The other case only needs an adaptation of the diagram we will use. Let $\delta \subset \Sigma$ be a curve intersecting γ_v once, and with ends outside U, as in Figure 2.9. Denote by $f : \Sigma_\eta \to \Sigma_\eta$ the diffeomorphism induced by the isotopy $sl_4 \circ sl_3 \circ sl_2 \circ sl_1$.

In Figure 2.9, we give the diagrams of four isotopies of ribbon graphs, and we keep track of δ along these isotopies. It proves that the concatenation is well-defined, and that, in homology, $f_\star([\delta]) = [\delta] \pm [\gamma_v]$. Also the isotopy fixes the ribbon graph outside U. So the support of f is included in an annulus, and f acts in homology like a Dehn twist. Figure 2.9 gives the sign of the Dehn twist. Thus it is isotopic to the negative Dehn twist along γ_v.

□

2.2 Elementary flips and partial return maps

The main idea for computing the first-return map is to see it as a composition of partial return maps $\Sigma_\eta = \Sigma_0 \to \Sigma_1 \to \ldots \to \Sigma_n = \Sigma_\eta$. In this section, we study the combinatorics and the geometry of the partial return maps, in order to prove Theorem B. We also introduce tools needed to formulate Theorem C precisely.

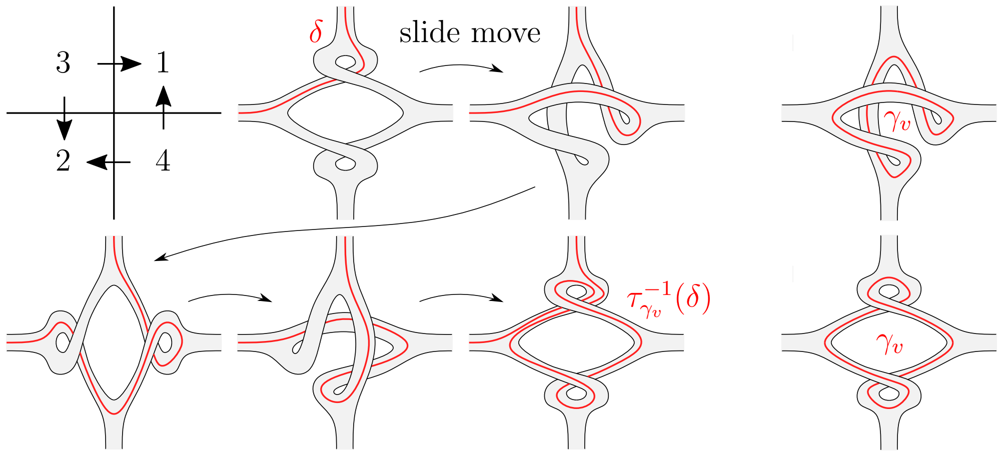


Figure 2.9: Action of four slides around an alternating vertex v and their traces on δ.

2.2.1 Combinatorial flip transformation

We introduce in this subsection the main combinatorial tool: the flip. We start by studying $\Gamma^\star$ the dual graph of $\Gamma \subset S$. In $\Gamma^\star$, every face f of $S \setminus \Gamma$ (diffeomorphic to B^2) is replaced by a vertex $f^\star$ inside the face. Every edge of $e \in \Gamma_1$ between two faces f_1 and f_2 (not necessarily different) is replaced by a transverse edge $e^\star$ from $f_1^\star$ to $f_2^\star$. And every vertex $v \in \Gamma_0$ is replaced by a face $v^\star$.

Let η be a coorientation of Γ. It naturally induces an orientation on $\Gamma^\star$, which will also be denoted by η. We are interested in geodesics in S that induce on $(\Gamma^\star, \eta)$ an oriented cycle. For a geodesic $\gamma \subset \Gamma$, pushing slightly γ, to its left or its right, induces two different cycles in $\Gamma^\star$, but they are simultaneously oriented or not-oriented for η (for homology reasons). We consider these curves for telling whether γ induces an oriented cycle in (Γ, η).

Lemma 2.2.1. *Let S be an orientable hyperbolic closed surface, Γ be a finite collection of closed geodesics which is filling, and take η in $\mathcal{E}ul\mathcal{C}o(\Gamma)$, then:*

- *For any curve $\gamma \subset S$ inducing an oriented cycle in $(\Gamma^\star, \eta)$, the geodesic homotopic to γ also induces an oriented cycle in $(\Gamma^\star, \eta)$.*
- *The surface Σ_η is a Birkhoff section if and only if the graph $(\Gamma^\star, \eta)$ has no oriented cycle. In this case, we say that η is an **acyclic** coorientation.*

- *If Γ admits an acyclic coorientation, then every edge in Γ_1 bounds two different faces of Γ.*

Proof. Let $\gamma \subset S$ be a curve inducing an oriented cycle in $\Gamma^\star$. Denote by $\tilde{\gamma}$ the unique geodesic of S homotopic to γ. We will prove that $\tilde{\gamma}$ induces an oriented cycle in $\Gamma^\star$. The curve γ induces an oriented cycle, so $\eta(\gamma) = \pm|\Gamma\cap\gamma|$. Since $\tilde{\gamma}$ is homotopic to γ, we have $|\eta(\gamma)| = |\eta(\tilde{\gamma})| \leq |\Gamma \cap \tilde{\gamma}|$, so γ minimises $|\Gamma \cap \gamma|$ in its homotopy class. Suppose that $\tilde{\gamma}$ is not inside Γ. Then since it is a geodesic, it also minimises the geometric intersection with Γ. Hence we have the equality $|\eta(\tilde{\gamma})| = |\Gamma \cap \tilde{\gamma}|$, so η induces an orientation on $\tilde{\gamma}$. Hence $\tilde{\gamma}$ induces an oriented cycle inside the oriented graph $(\Gamma^\star, \eta)$.

If $\tilde{\gamma}$ is a component of Γ, we can adapt the same argument. We push $\tilde{\gamma}$ slightly to its left and denote by $\tilde{\gamma}_\epsilon$ the curve obtained that way. The curve $\tilde{\gamma}_\epsilon$ is homotopic to γ and minimises the geometric intersection with Γ, so similarly η induces an orientation on $\tilde{\gamma}_\gamma$. Hence $\tilde{\gamma}$ induces an oriented cycle inside the oriented graph $(\Gamma^\star, \eta)$ as defined above.

We now prove the equivalence in the second point. Suppose that Σ_η is not a Birkhoff section. Then for arbitrarily large $T > 0$, there exists $(x, u) \in T^1S$ such that for $\forall 0 \leq t \leq T$, $\phi_t(x, u) \notin \Sigma_\eta$. Take $T > nd$ where $n = |\Gamma^\star|$ and d is the largest diameter of a face $f \in \Gamma^\star$. Then the geodesic arc $\phi_{[0,T]}(x, u)$ must travel through at least $n+1$ faces (counted with multiplicity). Thus it induces in $\Gamma^\star$ a path $\gamma^\star$ that admits self-intersections. Note that the orientation of $\gamma^\star$ in $\Gamma^\star$ is the opposite to the one provided by η. Hence a restriction of $\gamma^\star$ between two self-intersections, with the opposite orientation, is an oriented cycle in $(\Gamma^\star, \eta)$.

Suppose that there is an oriented cycle in $(\Gamma^\star, \eta)$. By the first point, there exists a closed geodesic γ inducing an oriented cycle. If $\gamma \not\subset \Gamma$, then the orbit $\overleftarrow{\gamma}$ of the geodesic flow given by the geodesic γ lifted with the opposite direction, satisfies $\overleftarrow{\gamma} \cap \Sigma_\eta = \emptyset$. Then Σ_η is not a Birkhoff section. Suppose that $\gamma \subset \Gamma$, and $\overleftarrow{\gamma} \subset \partial\Sigma_\eta$. Then every orbit in the stable leaf of $\overleftarrow{\gamma}$ stops intersecting Σ_η after a large enough time, since any slight push of γ in the appropriate direction induces an oriented cycle of η. Hence in both cases Σ_η is not a Birkhoff section.

For the last statement, it is enough to notice that an edge in Γ bounded twice by the same face is dual to a loop in $\Gamma^\star$. □

When Σ_η is a Birkhoff section, $(\Gamma^\star, \eta)$ is acyclic and η induces an order on the finite set $\Gamma^\star_0$. Thus η must have at least one **sink** face, that is, η is going inward f as in Figure 2.10.

Definition 2.2.2. Let η in $\mathcal{E}ul\mathcal{C}o(\Gamma)$ and f be a sink face. We define $I_f(\eta) \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ the coorientation obtained by flipping η along ∂f. We call I_f an **elementary flip** along f. We also define recursively $I_{(f_1,\dots,f_k)}(\eta) = I_{f_k}(I_{(f_1,\dots,f_{k-1})}(\eta))$, when recursively f_i is a sink face of $I_{(f_1,\dots,f_{i-1})}(\eta)$ for all $1 \leq i \leq k$.

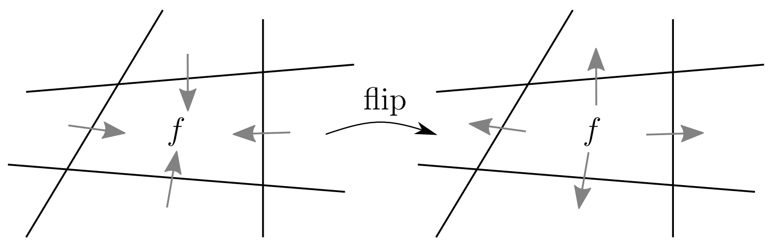


Figure 2.10: A sink face on the left, and a flip.

If η is Eulerian, $I_f(\eta)$ remains Eulerian and is cohomologous to η.

Representations. Around a vertex v, an Eulerian coorientation of Γ gives a partial ordering on the 4 adjacent faces (so that the coorientation is decreasing). We extend the ordering, by ordering v relatively to these faces using Figure 2.11. That is, if v is alternating, we set v bigger than the sink faces and smaller than the source faces. If v is not alternating, we set v smaller than the source face and bigger than the three other faces. We call this ordering on $\Gamma_2 \cup \Gamma_0$ the **coherent order**. These orderings represent the order of the Dehn twists in the product in Theorem C.

Remark 2.2.3. Suppose that Σ_η is a Birkhoff section. If one face f covers two quadrants around a vertex v, then by Lemma 2.2.1 it must be two opposite quadrants. Also Lemma 2.2.1 prevents f to be the sink and the source quadrants of a non-alternating vertex v. In the remaining cases, the coherent ordering is still well-defined on $\Gamma_2 \cup \Gamma_0$.

If there exist two faces such that both of them cover two opposite quadrants around v, the coherent ordering is still well-defined on $\Gamma_2 \cup \Gamma_0$ for the same reasons.

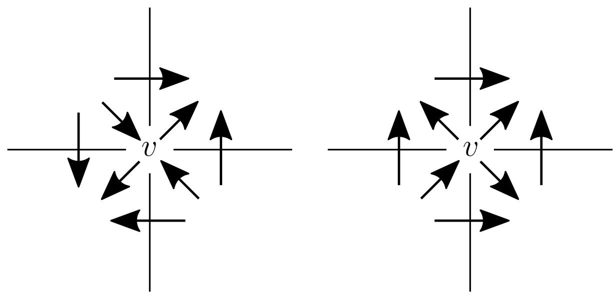


Figure 2.11: Coherent ordering of a vertex relatively to its adjacent faces.

Definition 2.2.4. Let η be an acyclic Eulerian coorientation of Γ. We call a **partial representation** of η a total order on Γ_2, which extends the coorientation η. We call a **representation** of η a total order on $\Gamma_0 \cup \Gamma_2$ which extends the coorientation η and the coherent order.

Thanks to acyclicity, representations always exist. There is no uniqueness in the representation, we will compare several choices of representation in Section 2.3.3.

Example 2.2.5. *If $\Gamma \equiv 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ as in Example 1.5.1, the faces of Γ can be colored in black and white, and consider the Eulerian coorientation η that goes from black to white. Then a representation can look like: white faces totally ordered $<$ vertices totally ordered $<$ black faces totally ordered. This choice of representation corresponds to the composition of three Dehn multi-twists studied by N.A'Campo and M.Ishikawa.*

The point is to use and deform a representation and its coorientation in order to represent the first-return map as a product of elementary diffeomorphisms. We have defined an elementary operation on coorientations, that we extend to representations.

Definition 2.2.6. Let $(\eta, \leq)$ be an acyclic Eulerian coorientation with a partial representation. We define $I(\eta, \leq)$ to be $(I_f(\eta), I_f(\leq))$, where f is the minimal face of $\leq$ and $I_f(\leq)$ is obtained from $\leq$ by setting f to be the maximum. It is called the **elementary flip** of $(\eta, \leq)$.

2.2.2 Algorithm for the first-return map

In order to describe the first-return map, we will first describe how it acts on the representations of acyclic Eulerian coorientations. Let η be such a coorientation and $\leq$ one partial representation. By iterating the flip I, we create a family of $\#\Gamma_2$ coorientations and partial representations, before looping to $(\eta, \leq)$. We will translate this geometrically later. For now let us detail a bit more how the coorientations obtained in this process look.

Lemma 2.2.7. *Let $(\eta, \leq)$ be a partial representation. Let $1 \leq k \leq n$, f be the k^{th} face for $\leq$ and $(\nu, \preceq) = I_{(f_1,\dots,f_k)}(\eta, \leq)$. For every $e \in \Gamma_1$ bounded by two faces f_1 and f_2, we have $\nu(e) = \eta(e)$ if and only if f_1 and f_2 are simultaneously greater or simultaneously smaller than f for $\leq$, that is, either $(f_1 > f$ and $f_2 > f)$ or $(f_1 \leq f$ and $f_2 \leq f)$.*

In particular $I_{(f_1,\dots,f_n)}(\eta, \leq) = (\eta, \leq)$.

Proof. The partial representation $\preceq$ differs from $\leq$ by moving the k lower faces on top. So we have $\nu(e) \neq \eta(e)$ if and only if one of the f_i is in this subset, and the other is not. □

The algorithm that consists in applying elementary flips I_f for successive minimal faces f will be called by the **flip algorithm**. This algorithm gives a way to compute the first-return map by computing the n elementary flips that correspond to the iteration of I.

2.2.3 Equivalence of cohomologous coorientations

We discuss a way to transform an acyclic Eulerian coorientation into its cohomologous coorientations by elementary flips I_f. The flip transformation and its combinatorics have already been studied by O.Pretzel [Pre86] and J.Propp [Pro20]. The following proposition is mainly a geometric reformulation. Only the first part of the proposition is mandatory for the following sections.

Proposition 2.2.8. *Let η, ν be in $\mathcal{E}ul\mathcal{C}o(\Gamma)$ be two cohomologous acyclic coorientations (so that Σ_η is a Birkhoff section). Then there exists a sequence of elementary flips that change η into ν.*

Let η, ν in $\mathcal{E}ul\mathcal{C}o(\Gamma)$ be two cohomologous coorientations that are not acyclic (so that Σ_η is not a Birkhoff section). Suppose that the union of *oriented cycles in* $(\Gamma^\star, \eta)$ *is connected. Then there exists a sequence of elementary flips that change* η *into* ν.

Let η *in* $\mathcal{E}ul\mathcal{C}o(\Gamma)$ *be not acyclic. Suppose that the union of oriented cycles in* $(\Gamma^\star, \eta)$ *is not connected. Then there exists* $\nu \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ *cohomologous to* η*, so that* Σ_η *and* Σ_ν *are not isotopic through the flow. In particular no sequence of flips can change* η *into* ν.

Notice that we are never allow to flip a face included in an oriented cycle, and oriented cycles remain oriented the same way after any sequence of flips.

Remark 2.2.9. O.Pretzel [Pre86] and J.Propp [Pro20] studied the set of orientations of a finite graph G. In our case, it corresponds to the Eulerian coorientations and the dual graph $\Gamma^\star$. They fix an orientation G and consider the strongly connected components of G, that is a set of vertices which are all pairwise connected by some oriented path. A strongly connected component corresponds either to one vertex which is not in any oriented cycle, or to a connected component of the union of oriented cycles. Then they study the flip transformation which transform a sink strongly connected component into a source strongly connected component, which only change the orientation of the edges in the boundary of that strongly connected component. The flip transformation preserves a function called circulation, which corresponds to the cohomology element in $H^1(G, \mathbb{Z})$ induced by an orientation of G.

O.Pretzel proved that given a strongly connected component H of G, the set of orientations of G with a fixed circulation is a distributive lattice, for the order given for two orientation o, o' by $o \leq o'$ if there exists a sequence of flips on the vertices of $G \setminus H$ which transforms o into o'. In particular the two first items of the previous proposition are direct consequences of that result.

Proof. We start with the case of η acyclic. Define $E = \{e \in \Gamma_1, \eta(e) \neq \nu(e)\}$ and notice that E is an embedded graph in S with degree 2 and 4 vertices. Also η and ν induce on E two opposite Eulerian coorientations.

The dual graph $E^\star$ is an acyclic oriented graph. Indeed take c a cycle in $E^\star$. The two coorientations are cohomologous, so $[\eta](c) = [\nu](c)$. Hence $2[E](c) = ([\eta] - [\nu])(c) = 0$ and c cannot be an oriented cycle in $E^\star$. Also every edge of E bounds two different connected components in $S \setminus E$, otherwise we would have a closed curve c intersecting E only once and with $\eta(c) \neq \nu(c)$.

Hence η restricted to E induces a partial ordering on $E_0^\star$. We will use this partial ordering to solve the problem by beginning with the local minimal elements. Let $F^\star \in E_0^\star$ be a local minimal vertex of η. On the boundary ∂F, η is going inward. Since η is acyclic, the faces $f \in \Gamma_0^\star$ with $f \subset F$ are partially ordered by η, and we can apply the flip algorithm to every sub-face of F, flipping once every sub-face of F. After this procedure, we obtain an acyclic Eulerian coorientation η' cohomologous to η, that differs only on ∂F. Indeed all edges in the interior of F have been reversed twice, and all edges in the boundary of F have been reversed only once. So the difference between η' and ν bounds less connected components. By applying this procedure at most a finite number of time, we describe a finite number of elementary flips that transform η into ν.

Suppose now that η is not acyclic. Denote by U the union of oriented cycles in $(\Gamma^\star, \eta)$, and suppose that U is connected, as a subgraph of $\Gamma^\star$. Notice that U does not admit any sink face. Since η and ν are cohomologous, an oriented cycle for η is also oriented for ν. So U is also the union of oriented cycles in $(\Gamma^\star, \nu)$. We also denote by $U \subset S$ the union of faces it induces. In $(\Gamma^\star, \eta)$, there is no oriented path outside U, starting and ending in U, otherwise this path would be a subset of an oriented cycle (since U is connected) and thus in U. So every oriented path starting at U and leaving U must be finite and end outside U.

Hence if η is not always going inside U along ∂U, then η admits a local minimal face outside U, and we flip η along any local minimal face. We can repeat this procedure a finite number of times until the coorientation is always going inside U along ∂U. Denote by η' the coorientation induced by the procedure, and ν' the coorientation induced by this procedure applied on ν. Then we compare η' and ν' on the connected components of $S \setminus U$. We will adapt the previous procedure to find a sequence of flips from η to ν. Define $E = \{e \in \Gamma_1, \eta(e) \neq \nu(e)\}$ the same way, then E delimitates connected regions. For every such region F, either U is outside F and we apply the flip algorithm on the sub-faces of F, or U is inside F, and we apply the flip algorithm on $S \setminus F$. Hence we can apply a sequence of elementary flips to eliminate ∂F from E, and successively transform η' into ν'.

Finally suppose that η is not acyclic and that U is not connected. We will construct ν cohomologous to η, and a non-closed geodesic that intersects finitely Σ_η and Σ_ν but with the same amount. Denote $U_1, \cdots, U_n$ the connected components of U. Notice that η partially orders $(U_i)_i$. Indeed suppose there is a finite sequence of oriented paths connecting U_{i_1} to $U_{i_1}, \cdots U_{i_k}$ and back to U_{i_1}, then it is included in an oriented cycle intersecting U_k, that must remain included in U_k.

We successively do every possible flip on sink faces in $S \setminus U$ (which terminates in a finite number of steps), to obtain η'. Let F be a connected component of $\Gamma^\star \setminus U$. Then every increasing path in $(F^\star, \eta')$ is finite, and ends at the boundary of F. Since the U_i are partially ordered, there is a maximal U_k. And since there is no path inside F starting and ending at U_k, η' is going inward U_k along its boundary. Let ν be the coorientation obtained from η' by changing the coorientation of U_k. Then ν is Eulerian and cohomologous to η' and η.

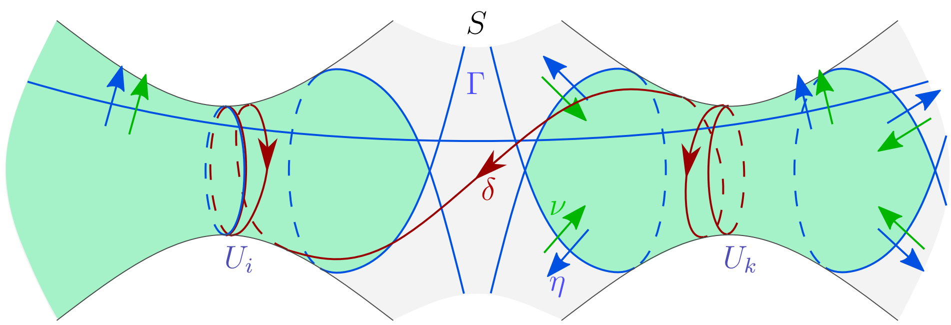


Figure 2.12: Non closed geodesic δ intersecting Σ_η and Σ_ν a different amount of times, but satisfying $[\eta] = [\nu]$.

By Lemma 2.2.1, for every $1 \leq i \leq n$, there exists a geodesic δ_i inside U_i (or on its boundary). Let $1 \leq i \leq n$ be different from k, and define a non-closed geodesic δ as in Figure 2.12, so that δ accumulates in the infinite past along $\bar{\delta}_i$ (δ_i with the opposite direction), and accumulates in the infinite future along $\bar{\delta}_k$. Since $i \neq k$, the algebraic intersection $\delta \cap \partial U_k$ is odd. We can do this so that δ remains inside the interior of $U_k \cup U_i$ outside a compact arc. But if $\delta_i \not\subset \partial U_i$ then $\overleftarrow{\delta_i} \cap \Sigma\eta = \emptyset$, and if $\delta_i \subset \partial U_i$ then $\overleftarrow{\beta_i} \cap \Sigma\eta = \emptyset$ where β_i is any slight push of δ_i inside U_i. Thus $\delta \cap \Sigma_{\eta'}$ and $\delta \cap \Sigma_\nu$ are finite, and differ by an odd integer that is not 0. Thus Σ_η and Σ_ν are not isotopic through the flow.

□

2.2.4 Partial return maps

The partial return maps are the geometric realisation of the combinatorial flip. We define the partial return maps and prove Theorem B in this subsection.

Let η be in $\mathcal{E}ul\mathcal{C}o(\Gamma)$ and f in $\Gamma_0^\star$ be a sink face for η. Write $\Sigma_1 = \Sigma_\eta$ and $\Sigma_2 = \Sigma_{I_f(\eta)}$. The elementary flip I_f acts geometrically by pushing Σ_1 along the geodesic flow only around the face f, as schematically depicted in Figure 2.13. Define $h : \Sigma_1 \to \mathbb{R}^+$ such that $h(x)$ is the smallest $t \geq 0$ such that $\phi_t(x,u)$ lies in Σ_2, and $r_f : \Sigma_1 \to \Sigma_2$ by $r_f(x,u) = \phi_{h(x)}(x,u)$.

Proposition 2.2.10. *There exist two smoothings of Σ_1 and Σ_2, $\epsilon > 0$ arbitrary small and U the complement of a small neighborhood of $T^1_f S$ such that :*

- *Σ_1 and Σ_2 are disjoint and $r_f : \Sigma_1 \to \Sigma_2$ is well-defined and smooth,*
- *$\phi_{-\epsilon}(\Sigma_2) \cap U = \Sigma_1 \cap U$ and $(\phi_{-\epsilon} \circ r_f)_{|\Sigma_1 \cap U} = \mathrm{id}$.*

We call r_f a **partial return map**. If Σ_1', Σ_2' are obtained using other smoothings and $r_f' : \Sigma_1' \to \Sigma_2'$ is the corresponding partial return map corresponding, there exists two unique small isotopies s_i along the flow for $i \in \{1,2\}$, from Σ_i to Σ_i'. Then r_f and r_f' are conjugated one to another by the two isotopies s_i. Hence r_f can be defined without precision on the choices of smoothing.

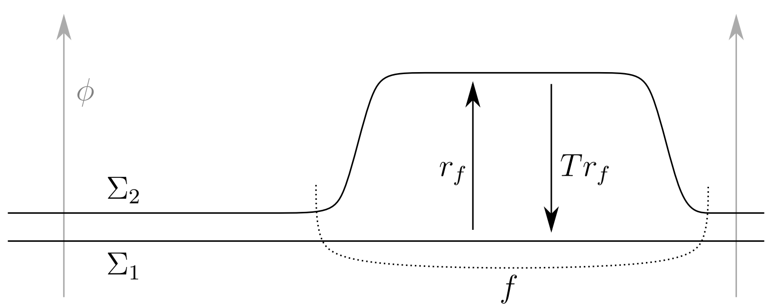


Figure 2.13: Relative positions of $\Sigma_1 = \Sigma_\eta$ and $\Sigma_2 = \Sigma_{I_f(\eta)}$ inside T^1S.

Proof. We write $\hat{\Sigma}_i$ for the 2-complex that we smooth for constructing Σ_i (without its boundary). First define $\hat{h} : \hat{\Sigma}_1 \to \mathbb{R}$ and $\hat{r} : \hat{\Sigma}_1 \to \hat{\Sigma}_2$ in the following way. Let (x,u) in $\hat{\Sigma}_1$ and not in $T^1_v \cap \hat{\Sigma}_1$ for any non-alternating vertex v. If x is in f and u goes inside f, define $\hat{r}(x,u)$ to be the first intersection of $\hat{\Sigma}_2$ and of the geodesic starting at (x,u), and $\hat{h}(x,u)$ to be the length of this geodesic arc. Elsewhere set $\hat{r}(x,u)=(x,u)$ and $\hat{h}(x,u)=0$.

Let v be a non-alternating vertex and take $(x,u)\in T^1_v\cap\hat{\Sigma}_1$. After the desingularisation of $\hat{\Sigma}_1$, two points of $\hat{\Sigma}_1$ correspond to (x,u) and we must define $\hat{r}$ and $\hat{h}$ for both points. One of them is adjacent to the two rectangles r_{e_1}, r_{e_2} for two edges e_1, e_2 bounding f, and we define $\hat{h}$ and $\hat{r}$ on it as if it was going inside f. The other point is adjacent to the two rectangles r_{e_3}, r_{e_4} for two edges e_3, e_4 not adjacent to f, and we define $\hat{h}$ and $\hat{r}$ on that point as it was outside of f.

Both functions $\hat{h}$ and $\hat{r}$ are well-defined and continuous. We smooth together $\hat{\Sigma}_1$, $\hat{\Sigma}_2$, $\hat{h}$ and $\hat{r}$ into Σ_1, Σ_2, h and r. We use smoothings smaller that $\epsilon/3$.

On a small neighborhood of each corner of f, h may be negative. To make h positive, take g a negative smoothing of $-\max(0,-h)$ and push Σ_1 with ϕ_{-g}. We suppose that $|g+\max(0,-h)|<\epsilon$ and that $g=-\max(0,-h)$ outside the tubular neighborhood $B(f,\epsilon)$ of f. Now $h-g\geq 0$.

Let $U = T^1S\setminus T^1_{B(f,\epsilon)}S$ be the complement of f in T^1S. By construction $\Sigma_1\cap U = \Sigma_2\cap U$ and $(r_f)_{|\Sigma_1\cap U} = \mathrm{id}$. We finish by replacing Σ_2 by $\phi_\epsilon(\Sigma_2)$. $\square$

Fix a representation of η. The flip algorithm generates a family of cohomologous Birkhoff sections, consecutively disjoint. The partial return maps describe how the flow moves one to the next one.

Theorem B. *Let $\Gamma\subset S$ be a filling geodesic multi-curve of a hyperbolic orientable surface, $\eta\in\mathcal{E}ul\mathcal{C}o(\Gamma)$ acyclic, $\leq$ a partial representation of η and denote the faces by $f_1\leq\cdots\leq f_n$. Denote by $\Sigma_0=\Sigma_\eta$ and successively by the partial return map $r_i:\Sigma_{i-1}\to\Sigma_i$ the partial return map along the face f_i. Then $\Sigma_n=\Sigma_\eta$ and the first-return map on Σ_η is the product of the partial return maps $r_{\Sigma_\eta}=r_n\circ\cdots\circ r_1$.*

The theorem is equivalent to having Birkhoff sections Σ_i pairwise disjoint, and ordered by the flow. So that the first-return map on Σ_0 is obtained by following the flow from Σ_0 to itself, crossing exactly once every other Birkhoff section Σ_i, in the order $i=1$ to $i=n-1$.

Proof. We will prove that $r=r_{\Sigma_\eta}=r_n\circ\cdots\circ r_1$ on a dense subset of Σ_0. Let x be in $\mathring{\Sigma}_\eta$ such that the geodesic starting at x intersects Σ_η again before intersecting $T^1_{\Gamma_0}S$. This represents a dense subset of Σ_η. We can suppose that the smoothing have been done away from the short geodesic starting at x and ending on Σ_η when it first intersects it. So for U a small neighborhood of the geodesic from x to $r(x)$, we have $\Sigma_i \cap U = \hat{\Sigma}_i \cap U$.

We denote by $x_0 = x, x_1, \dots, x_k$ the first k points in the intersection of the orbits $\phi_{\mathbb{R}_+}(x)$ and of the fibers $T^1_\Gamma S$ above Γ. We chose the integer $k \in \mathbb{N}$ such that for all $1 \leq i < k$, the point x_i is not in Σ_η, and x_k is inside Σ_η, that is $x_k = r_{\Sigma_\eta}(x)$. We also denote by $e_i \subset \Gamma$ the edge on which is base the vector x_i, and by f_i the face in which is pointing the vector x_i. Since for $1 \leq i < k$, the point x_i is not in Σ_η, the vector x_i induce the coorientation of e_i which is opposite to η. Hence by definition of the partial representation $\leq$ we have $f_{i-1} < f_i$. Similarly one has $f_{k-1} > f_k$. Since the partial return maps are obtain by flipping the faces from the minimal face to the maximal face, the composition of the partial return maps sends inductively the point x_i to the point x_{i+1} for all $0 \leq i < k$, and then remains constant equal to x_k since $f_{k-1} > f_k$. Hence $r_n \circ \cdots \circ r_1(x) = x_k = r_{\Sigma_\eta}(x)$, and by density $r_n \circ \cdots \circ r_1 = r_{\Sigma_\eta}$.

□

2.3 Explicit first-return maps

Let $r : \Sigma_\eta \to \Sigma_\eta$ be the first-return map along the geodesic flow. In Section 2.2 we have decomposed the first-return map as a product of partial return maps. In this section, we first compare these partial return maps to negative Dehn twists, along prescribed curves. Then we state and prove Theorem C. We finish by comparing several decompositions of first-return maps in Dehn twists, and prove Theorem D.

2.3.1 Explicit computation of partial return maps

Let f be a sink face of a coorientation $\eta \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ and denote $\Sigma_1 = \Sigma_\eta$ and $\Sigma_2 = \Sigma_{I_f(\eta)}$. We will compare the partial return map $r : \Sigma_1 \to \Sigma_2$ to a negative Dehn twist, but r is not an endomorphism. We need to correct it with a simple diffeomorphism $c : \Sigma_2 \to \Sigma_1$ so that $\Sigma_1 \xrightarrow{r} \Sigma_2 \xrightarrow{c} \Sigma_1$ can be expressed as a Dehn twist. In order to find c, we use the ribbon representation of Σ_i and the slides from Definition 2.1.8.

To simplify the computation of $c \circ r$, we need to precise which slides we use. Let $\{c_1, \ldots, c_k\}$ be the set of corners of f. If f has double corners, we consider them twice. For $1 \leq i \leq k$, the ribbon graph of Σ_2 around c_i as an edge corresponding to the vector based on c_i and going inside f. We denote by e_i this edge, as in Figure 2.14.

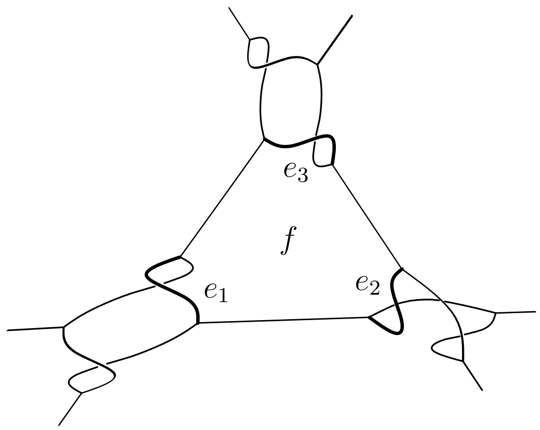


Figure 2.14: Edges used for the slide correction.

Definition 2.3.1. Let $r : \Sigma_1 \to \Sigma_2$ be a partial return map around f. We define $c : \Sigma_2 \to \Sigma_1$ the composition of slides along every e_i for $1 \leq i \leq k$. We call it the **slide correction** of r.

The diffeomorphism c is well-defined up to isotopy. Indeed f is a sink face so the slides are well-defined, and the slides on different corners can be done independently in a commutative way. The diffeomorphism $c \circ r$ will be compared to the Dehn twist along γ_f, for the curve γ_f represented in Figure 2.2. This curve does one turn around f, and follows the edge e_i for each corner c_i of f.

Proposition 2.3.2. *Let η and ν be two Eulerian coorientations that differ only by an elementary flip along a sink face f. Let $r : \Sigma_\eta \to \Sigma_\nu$ be the partial return map and $c_r : \Sigma_\nu \to \Sigma_\eta$ the corresponding slide correction. Then $c_r \circ r$ is isotopic to the negative Dehn twist along γ_f.*

Proof. We write $\Sigma_1 = \Sigma_\eta$ and $\Sigma_2 = \Sigma_\nu$. We start with an additional assumption on f: we suppose that f does not admit double corners as an immersed polygon. That is, we suppose that f is an embedded polygon. First we see that there is an annulus containing the support of $c_r \circ r$. Denote by U the union of the complement of a small neighborhood of f and of the opposite sides of e_i for every corner c_i of f (the opposite side in the ribbon graph of Σ_1 around a corner c_i). Denote $V = \Sigma_1 \setminus U$. We can do this choice so that V is homeomorphic to an annulus that retracts on γ_f, as in Figure 2.15.

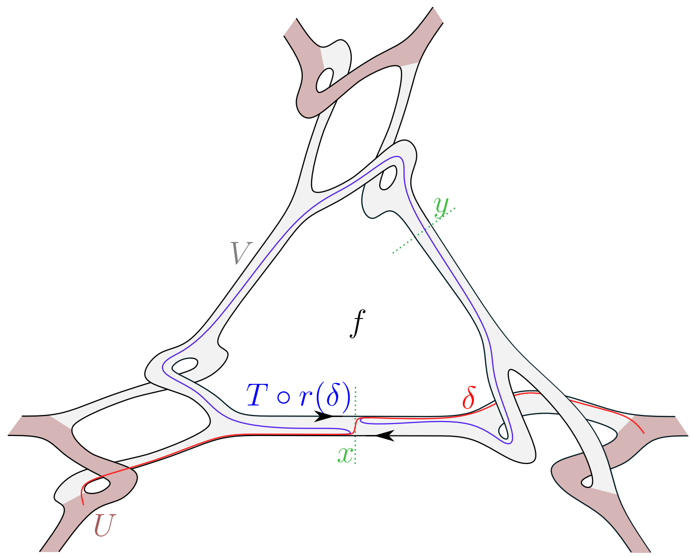


Figure 2.15: Action of the partial return map around an alternating vertex.

The fact that the support of $c_r \circ r$ is inside an annulus comes from the fact that both immersions $\pi \circ \psi_{im}$ and $\pi \circ \psi_{im} \circ c_r \circ r$ act as the identity on U. Thus they lift to the equality $(c_r \circ r)_{|U} = \mathrm{id}_U$ and $\mathrm{supp}(c_r \circ r) \subset V$. So $c_r \circ r$ is isotopic to a multiple of the negative Dehn twist along γ_f.

In order to understand which multiple it is, we use an arc δ transversal to V, and we compare δ to $c_r \circ r \circ \delta$. Let $x, y \in \partial f$ be two points that are not corners of f, and on different edges of f. Suppose that the smoothings of Σ_1 and Σ_2 have been done away from x and y, so that $\delta_x = T^1_x S \cap \Sigma_i$ and $\delta_y = T^1_y S \cap \Sigma_i$ are two arcs that do not depend on i. Once δ will be defined, since δ_y intersects the core of V only once, the multiplicity of the Dehn twist is equal to the algebraic intersection of $[c_r \circ r(\delta) - \delta].[\delta_y]$.

Take two arcs δ_x^+ and δ_x^- in $\partial\Sigma_1$, that start from the ends of δ_x and end in U. Then define δ to be an arbitrary closed smoothing of $\delta_x^- \cup \delta_x \cup \delta_x^+$ that remains in Σ_1, as in Figure 2.15. For $p \in \delta$ arbitrary close to $\partial\Sigma_1$, $r(p)$ is not in δ_y since x and y are on different sides of f. Indeed if the vector $p \in \delta$ is close enough to $\partial\Sigma_1$, then $r(p)$ is also close to the same boundary component of Σ, and remains in a small neighborhood of the fibers $T^1_f S$ above F. Hence $r(p)$ is in a small neighborhood of the fibers of the edge of f on which x lies. Since y is not on the same edge, $r(p)$ is not on the fiber $T^1_y S$ for $p \in \delta$ close enough to $\partial\Sigma$.

Thus $r(\delta)$ intersects δ_y only once, corresponding to the geodesic in f between x and y. Also by construction, $\delta \cap \delta_y = \emptyset$. Since c_r restricts to the identity outside a small neighborhood of $\Gamma_0 \cap S$, we have $[c_r \circ r(\delta) - \delta].[\delta_y] = \pm 1$, the multiplicity is ± 1. Figure 2.15 shows in blue $c_r \circ r(\delta)$, which helps finding the sign. To know more precisely the sign, one could detail how the orientation of $\partial\Sigma_1$ imposes to $c_r \circ r(\delta)$ to intersect $\Sigma \cap T^1 y$ with this sign. So $c_r \circ r$ is isotopic to a negative Dehn twist along γ_f.

To prove the property in the general case, we could either use a covering of S such that f lifts to an embedded polygon, or adapt the last argument around the double corners (and see why V is still an annulus). □

2.3.2 Reconstruction of the first-return map

We now understand the first-return map as product of simple maps. We first define the curves appearing in Theorem C. Then we restate and prove Theorem C.

Definition 2.3.3. For every vertex $v \in \Gamma_0$, define the curve γ_v as the skeleton of the annulus $\Sigma \cap T^1_B S$ for $B \subset S$ a small ball around v, as in Figure 2.16.

For each face $f \in \Gamma_2$, define the curve γ_f in Σ_η that does one turn around f, such that the behavior of γ_f around a corner of f is as in Figure 2.16. If needed, we denote by γ^η_x the curve along $x \in \Gamma_0 \cup \Gamma_2$ for the coorientation η.

An example of a full γ_f is presented in Figure 2.2. It is clear that a slide correction map send the curve γ^η_v to the curve γ^ν_v for any vertex $v \in \Gamma_0$ and for two Eulerian coorientation η and ν which differ by a flip. It is convenient to detail now how slide correction maps change the curve γ_f for a face $f \in \Gamma_2$. For the rest of the chapter, we will denote by $T\gamma$ or T_γ the negative Dehn twist along γ.

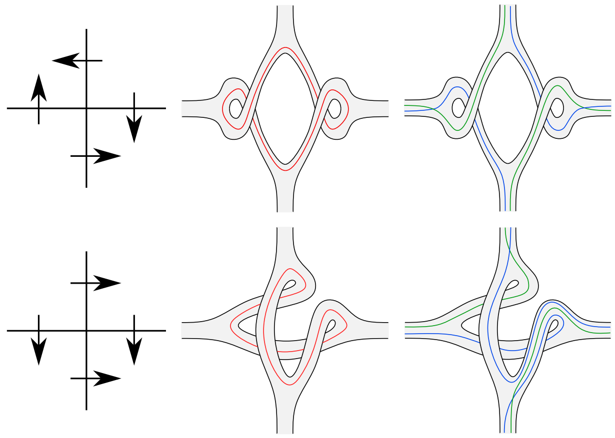


Figure 2.16: Curves γ_v for a vertex v (in the middle), and γ_f for a face f (in the right).

Lemma 2.3.4. *Let $\eta \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ be a coorientation with a sink face f, $\nu = I_f(\eta)$ the coorientation obtained by flipping η along f, and $c_f : \Sigma_\nu \to \Sigma_\eta$ be the correction map. Denote by $\prec_\eta$ and $\prec_\nu$ the coherent orderings for η and ν. Then for all faces $g \in \Gamma_2$, up to an isotopy in Σ_ν, $c_f^{-1}(\gamma_g^\eta) = \left(\prod_{v \in C_g} T_{\gamma_v^\nu}\right)(\gamma_g^\nu)$ where the product is taken over the set C_g of all corners v common to f and g so that $g \succ_\eta v$ and $g \prec_\nu v$. In particular $c_f^{-1}(\gamma_f^\eta) = \gamma_f^\nu$.*

Proof. The curves $c_f^{-1}(\gamma_g^\eta)$ and γ_g^ν coincide outside a small neighborhood of the corners of f. To compare them, we compute the image $c_f^{-1}(\gamma_g^\eta)$ using some diagrams of the ribbon representation. Figure 2.18 represents these curves on a neighborhoods of a corner v of f, which is non-alternating for η. It proves that on a neighborhood of v, $c_f^{-1}(\gamma_g^\eta)$ coincides with γ_g^ν.

For v an alternating vertex which is a simple or a double corner of f, the curves are represented Figure 2.17. When v is a double corner of f, one need to do two slides along the two quadrants corresponding to f, so there is an additional line in the figure, for that second slide. It proves that on a neighborhood of v, $c_f^{-1}(\gamma_g^\eta)$ coincides either with γ_g^ν, or with $T_{\gamma_v^\nu}(\gamma_g^\nu)$ if $g \succ_\eta v$ and $g \prec_\nu v$. $\square$

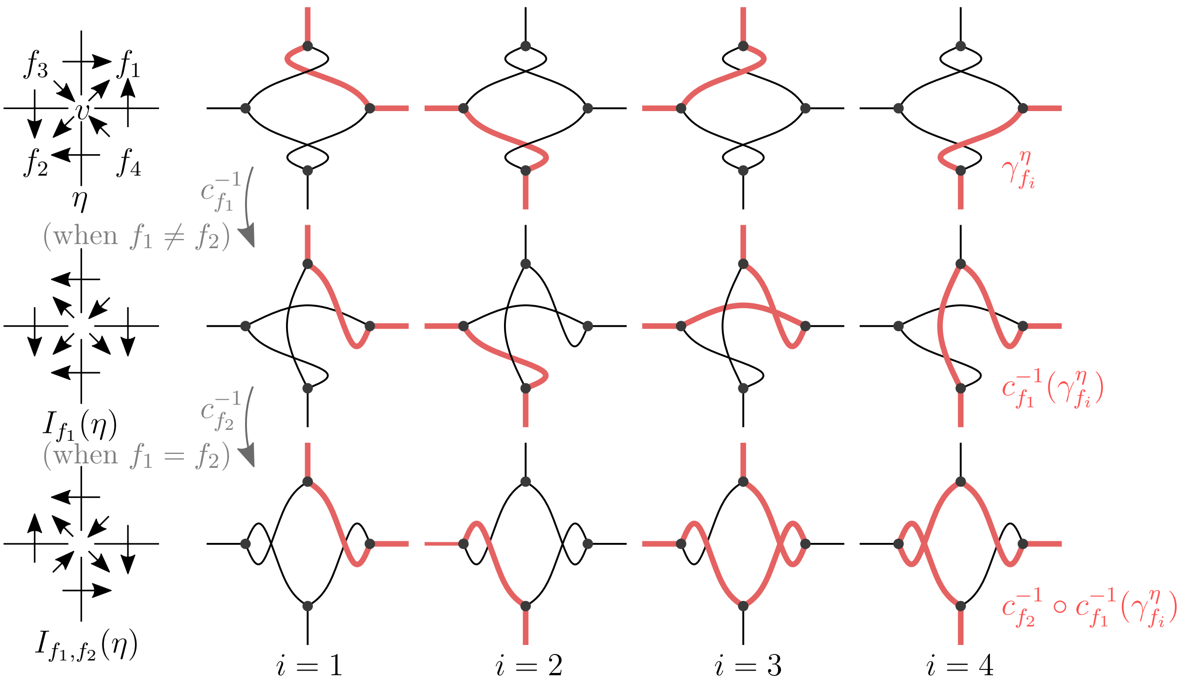


Figure 2.17: Action of a slide correction on the curves γ_f around one alternating vertex v of the face f_1. The thick curve in red correspond to the curve $\gamma_{f_i}^{\eta}$ together with its image by the slide correction c_f^{-1}. The four curves are indexed by the element $i = 1, \dots, 4$ given below the diagrams. Since the faces f_1 and f_2 can be equal, we need to do two slides in the two corresponding quadrants to compute the image of $\gamma_{f_i}^{\eta}$ under the slide correction c_f^{-1}.

Theorem C. *Let η be an acyclic Eulerian coorientation and Σ_η its corresponding Birkhoff section. Then the first-return map $r : \Sigma_\eta \to \Sigma_\eta$ is the product of negative Dehn twists along γ_v for all $v \in \Gamma_0$ and γ_f for all $f \in \Gamma_0^\star$. The product is ordered by any representation of η.*

Proof of Theorem C. Take a representation $\preceq_\eta$ of η and order the faces of Γ by $f_1 \preceq_\eta \dots \preceq_\eta f_n$. Denote by $\eta_0 = \eta$, successively $\eta_i = I_{f_i}(\eta_{i-1})$ the flip of η_{i-1} along f_i, so that $\eta_n = \eta$, and denote by $\Sigma_i = \Sigma_{\eta_i}$. According to Theorem B, the first-return map is a product of partial return maps $r_n \circ \dots \circ r_1$ for the partial return map $r_i : \Sigma_{i-1} \to \Sigma_i$ along the face f_i. Denote by $c_i : \Sigma_i \to \Sigma_{i-1}$ the slide correction of r_i, and for $1 \leq i \leq n$ define $g_i = c_1 \circ \dots \circ c_i$. According to Proposition 2.3.2, $c_i \circ r_i$ is isotopic to a negative Dehn twist

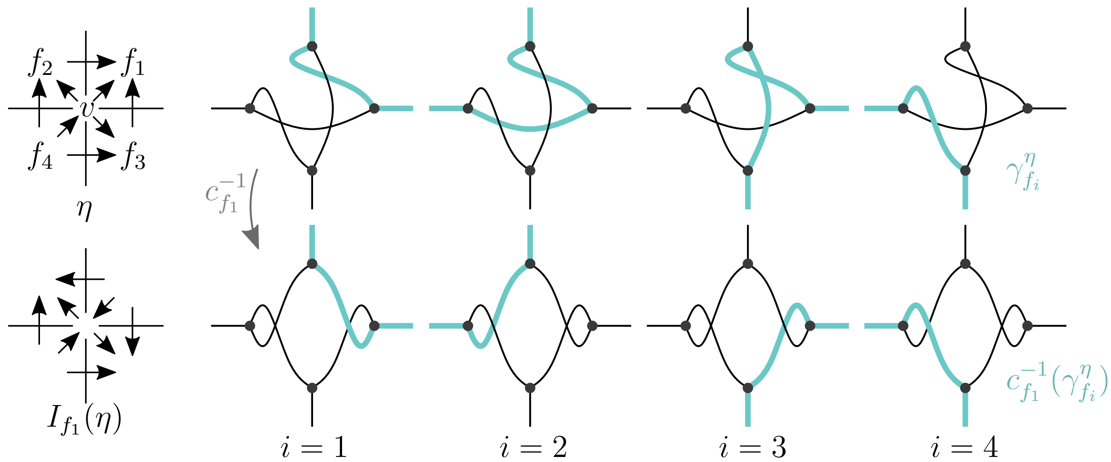


Figure 2.18: Action of a slide correction on the curves γ_f around one non alternating vertex v of the face f_1. The thick curve in light blue correspond to the curve $\gamma_{f_i}^{\eta}$ together with its image by the slide correction c_f^{-1}. The four curves are indexed by the element $i = 1, \ldots, 4$ given below the diagrams.

along $\gamma_{f_i}^{\eta_{i-1}}$, so that

$$\begin{aligned} r &= r_n \circ \ldots \circ r_1 \\ &= \prod_{i=n}^{1} r_i \\ &= \prod_{i=n}^{1} g_i^{-1} \circ g_{i-1} \circ c_i \circ r_i \\ &= g_n^{-1} \circ \prod_{i=n}^{1} g_{i-1} \circ c_i \circ r_i \circ g_{i-1}^{-1} \\ r &= g_n^{-1} \circ \prod_{i=n}^{1} g_{i-1} \circ T\gamma_{f_i}^{\eta_{i-1}} \circ g_{i-1}^{-1}. \end{aligned}$$

We will see that $g_n^{-1} : \Sigma_\eta \to \Sigma_\eta$ is a commutative product of Dehn twists, and then we will characterize the curve that support the Dehn twist $g_{i-1} \circ T\gamma_{f_i}^{\eta_{i-1}} \circ g_{i-1}^{-1}$. We will also change the order in which the Dehn twists appear.

We claim that we have

$$g_n^{-1} = \prod_{v \in \Gamma_0} T\gamma_v^\eta.$$

The diffeomorphism g_n^{-1} is a concatenation of slides. Since all faces appear in the product $g_n = c_1 \circ \ldots \circ c_n$, the slide for every corner of Γ appears in g_n^{-1}. If two slides appear on different vertices of Γ, they have disjoint supports. So we can rearrange the slides so that they appear by groups of four, one group for every vertex of Γ. The way we have constructed g_n^{-1} ensures that the slides in each group appear in the same order as in Lemma 2.1.9. Then this lemma implies that g_n^{-1} is a product of negative Dehn twists along the curves γ_v.

We proved that r is the product of negative Dehn twists given by the following product, where $g_{i-1} \circ T\gamma_{f_i}^{\eta_{i-1}} \circ g_{i-1}^{-1}$ is a pullback of a negative Dehn twist into Σ_η:

$$r = \prod_{v \in \Gamma_0} T\gamma_v^\eta \circ \prod_{i=n}^{1} (g_{i-1} \circ T\gamma_{f_i}^{\eta_{i-1}} \circ g_{i-1}^{-1})$$

Also $g_{i-1} \circ T\gamma_{f_i}^{\eta_{i-1}} \circ g_{i-1}^{-1} = T(g_{i-1}(\gamma_f^{\eta_{i-1}}))$ (see Remark 2.3.5). Here, f is the i-th face given by the representation, and γ_f is the curve of Σ_i along f.

Remark 2.3.5. Recall that for g an orientation-preserving diffeomorphism of S and γ a simple closed curve on S, we have $g \circ T\gamma \circ g^{-1} = Tg(\gamma)$ (Fact 3.7 of [FM12]). So if γ and δ are two simple closed curves on S, we have

$$\begin{aligned} T\delta \circ T\gamma &= T(T\delta(\gamma)) \circ T\delta \\ &= T\gamma \circ T(T^{-1}\gamma(\delta)) \end{aligned}$$

Instead of computing $g_{i-1}(\gamma_f^{\eta_{i-1}})$, we first change the order of Dehn twists in the product, so that they appear in the order prescribed by the representation. In order not to increase the notations, we will do an informal proof. We already know that r is a product of Dehn twists in curves corresponding to all vertices and all faces of Γ. Also the Dehn twists corresponding to the faces are already in order prescribed by the induced partial representation. We first change the order in the product to match the representation, then we compute the curves that support the Dehn twists.

A curve γ_v^η can only intersect $g_{i-1}(\gamma_{f_i}^{\eta_{i-1}})$ if f_i admits v as corner. So we can move γ_v^η to the right from the product defining g_n^{-1}, until $T\gamma_v^\eta \circ T(g_{i-1}(\gamma_{f_i}^{\eta_{i-1}}))$ appears in the product. According to Remark 2.3.5, we have:

$$T\gamma_v^\eta \circ T(g_{i-1}(\gamma_{f_i}^{\eta_{i-1}})) = T(T\gamma_v^\eta(g_{i-1}(\gamma_{f_i}^{\eta_{i-1}}))) \circ T\gamma_v^\eta$$

We use this equation for changing the position of the two curves in the product. We repeat this step until $T\gamma_v^\eta$ is at the place prescribed by the representation. Then we do the same procedure for all $v \in \Gamma_0$. The decomposition obtained is a product of Dehn twists, ordered accordingly to the representation $\preceq_\eta$, and along the curves γ_v^η for $v \in \Gamma_0$ and the curves $(\prod_{v\in C_i} T\gamma_v^\eta)(g_{i-1}(\gamma_{f_i}^{\eta_{i-1}}))$ for every $1 \leq i \leq n$, where $C_i = \{v \in \Gamma_0 | f_i \succeq_\eta v\}$. But g_{i-1} is a sequence of slides which sends the curve $\gamma_v^{\eta_{i-1}}$ in $\Sigma_{\eta_{i-1}}$ to the curve we have γ_v^η in Σ_η. Hence $(\prod_{v\in C_i} T\gamma_v^\eta)(g_{i-1}(\gamma_{f_i}^{\eta_{i-1}})) = g_{i-1}((\prod_{v\in C_i} T\gamma_v^{\eta_{i-1}})(\gamma_{f_i}^{\eta_{i-1}}))$.

Claim. *The curve $g_{i-1}((\prod_{v\in C_i} T\gamma_v^{\eta_{i-1}})(\gamma_{f_i}^{\eta_{i-1}}))$ is equal to $\gamma_{f_i^\eta}$.*

The theorem follows from the claim. To prove the claim, we use several times the Lemma 2.3.4 which details how a curve γ_f^η is transformed by a slide correction. We compute the curve $g_{i-1}^{-1}(\gamma_{f_i^\eta})$ which is equal to the curve $c_{i-1}^{-1} \circ \ldots \circ c_1^{-1}(\gamma_{f_i^\eta})$. By applying that lemma several times, we have:

$$g_{i-1}^{-1}(\gamma_{f_i^\eta}) = (\prod_{\substack{v\in\Gamma_0 \text{ and } 1\leq k\leq i-1 \\ f_i \succ_{\eta_{k-1}} v \text{ and } f_i \prec_{\eta_k} v}} T\gamma_v^{\eta_{i-1}})(\gamma_{f_i^{\eta_{i-1}}})$$

We need to determine which index $(v,i) \in \Gamma_0 \times [\![1,k]\!]$ satisfies the hypothesis in the index of the product. To do so, first we notice that the order between f_i and v for the orders $\prec_{\eta_0}, \ldots, \preceq_{\eta_n}$ changes exactly twice. It can be verified by considering one vertex together with the four different coherent orders around v that appears for the coorientations $\eta_0, \ldots, \eta_n$. In particular it changes at most twice when we only do the first $i-1$ slide corrections.

By definition of the flip operation along the i^{th} faces, the face f_i is minimal for the order $\prec_{\eta_{i-1}}$ and maximal for the order $\prec_{\eta_i}$. So one has $f_i \prec_{\eta_{i-1}} v$ and $f_i \succ_{\eta_i} v$. So the order between f and v changes at most once for the orders $\prec_{\eta_0}, \ldots, \preceq_{\eta_i}$. Since we have $f_i \prec_{\eta_{i-1}} v$, one has the two relations $f_i \succ_{\eta_{k-1}} v$ and $f_i \prec_{\eta_k} v$ for one index $1 \leq k \leq i-1$ if and only if $f_i \succ_{\eta_0} v$. Hence we have following equation, which concludes the proof:

$$g_{i-1}^{-1}(\gamma_{f_i^\eta}) = (\prod_{\substack{v\in\Gamma_0 \\ f_i\succ_{\eta_0} v}} T\gamma_v^{\eta_{i-1}})(\gamma_{f_i^{\eta_{i-1}}})$$

□

Remark 2.3.6. According to Remark 2.3.5, we could have taken other curves and make them appear in a different order. We took a convention that depends mainly on the choice of the Eulerian coorientation.

2.3.3 Comparison of different Eulerian coorientations

In this subsection, we compare the explicit products of negative Dehn twists for different representations or different acyclic Eulerian coorientations. We will in particular prove Theorem D.

Assume that η is an acyclic Eulerian coorientation, so that the surface Σ_η is a Birkhoff section. Given two representations, the curves γ_v and γ_f depend on η only.

Lemma 2.3.7. *Let $\preceq_1$ and $\preceq_2$ be two representations of η. The two products of negative Dehn twists in Theorem C can be changed one into another one by successively swapping the positions of consecutive commutating Dehn twists.*

Proof. Denote by $\preceq_0$ the coherent ordering, which is a partial ordering on $\Gamma_0 \cup \Gamma_2$. By definition, $\preceq_1$ and $\preceq_2$ agree with $\preceq_0$. Let x, y in $\Gamma_0 \cup \Gamma_2$ and suppose that γ_x and γ_y do not commute. Then by definition of the curves γ_x and γ_y, x and y must be either two adjacent faces or one face and an adjacent vertex, thus they are comparable under $\preceq_0$. So x and y have the same ordering under $\preceq_1$ and $\preceq_2$.

Now suppose that $\preceq_1$ and $\preceq_2$ are not equal, and let x and y be in $\Gamma_0 \cup \Gamma_2$ not ordered in the same way by $\preceq_1$ and $\preceq_2$. There is a pair of such elements (x, y) which are consecutive in $\preceq_2$, otherwise $\preceq_1$ and $\preceq_2$ would be equal. By what precedes, x and y are not adjacent, and the negative Dehn twists along γ_x and γ_y commute. So we can define $\preceq_3$ by only swapping in $\preceq_2$ the ordering of x and y, and $\preceq_3$ is a representation of η. This procedure can be recursively repeated to $\preceq_1$ and $\preceq_3$, and terminates in a finite number of steps. $\square$

Given two isotopic Birkhoff sections Σ_1, Σ_2 and two decompositions of the two first-return maps in Dehn twists, we can use an isotopy along the flow $\Sigma_2 \to \Sigma_1$ to compare the two decompositions. These decompositions can be compared using Hurwitz equivalences. Given a group G, two n-tuples of elements of G are said to be **Hurwitz equivalent** if we can change the first tuple to the second one by a sequence of transformations that changes a n-tuple $(g_1, \dots, g_n)$ into the n-tuple $(g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} \circ g_i \circ g_{i+1}, g_{i+2}, \dots, g_n)$

or $(g_1, \dots, g_{i-1}, g_i \circ g_{i+1} \circ g_i^{-1}, g_i, g_{i+2}, \dots, g_n)$ for some $i \in \{0, \dots, n-1\}$. In our case, G is the mapping class group $MCG(\Sigma_1)$ of Σ_1, that is the set of homeomorphisms of Σ_1 that preserve $\partial\Sigma_1$ component by component, up to isotopy. We denote by $\simeq$ the Hurwitz equivalence relation.

Lemma 2.3.8. *Let η and ν be two cohomologous acyclic Eulerian coorientations. Theorem C decompose the first-return map on Σ_η and Σ_ν into two products of elements from n-tuples g^η and g^ν of element of $MCG(\Sigma_\eta)$ and $MCG(\Sigma_\nu)$. Denote by $i : \Sigma_\eta \to \Sigma_\nu$ a diffeomorphism given by an isotopy along the flow. Then g^η and $i^{-1} \circ g^\nu \circ i = (i^{-1} \circ g_1^\nu \circ i, \dots, i^{-1} \circ g_n^\nu \circ i)$ are Hurwitz equivalent in $MCG(\Sigma_\eta)$.*

Proof. We first consider the case for η an acyclic coorientation, f a sink face of η and $\nu = I_f(\eta)$ the cohomologous coorientation obtained from η by flipping f. We also consider $r_f, c_f^{-1} : \Sigma_\eta \to \Sigma_\nu$ the partial return map along f and the inverse of the correction map defined in Section 2.3. By Lemma 2.3.2 $r_f = c_f^{-1} \circ T_{\gamma_f^\eta} = T_{c_f^{-1}(\gamma_f^\eta)} \circ c_f^{-1} = T_{\gamma_f^\nu} \circ c_f^{-1}$. We will determine $r_f(\gamma_x^\eta)$ for all $x \in \Gamma_1 \cup \Gamma_2$, then we detail the Hurwitz equivalence between g^ν and $r_f \circ g^\eta \circ r_f^{-1}$.

According to the previous Lemma, the Hurwitz classes of g^η and g^ν do not depend on the choices of representations for η and ν. We take two representations $\preceq_\eta$ and $\preceq_\nu$ of η and ν that differ only for f and for the alternating corners of f, and we take g^η and g^ν accordingly. For a vertex v of Γ, we have $c_f^{-1}(\gamma_v^\eta) = \gamma_v^\nu$. Let g be a face of Γ, according to Lemma 2.3.4, we have $c_f^{-1}(\gamma_g^\eta) = (\prod_{v \in C_g} T_{\gamma_v^\nu})(\gamma_g^\nu)$, where the product is taken over the set C_g of all vertices v common to f and g, so that $g \succeq_\eta v$ and $g \preceq_\nu v$.

We denote the element of $\Gamma_0 \cup \Gamma_2$ by $f = x_n \prec_\eta \dots \prec_\eta x_1$ so that $g^\eta = (T_{\gamma_{x_1}^\eta}, \dots, T_{\gamma_{x_n}^\eta})$. We have $r_f \circ g^\eta \circ r_f^{-1} = (r_f \circ T_{\gamma_{x_1}^\eta} \circ r_f^{-1}, \dots, r_f \circ T_{\gamma_{x_n}^\eta} \circ r_f^{-1}) = (T_{r_f(\gamma_{x_1}^\eta)}, \dots, T_{r_f(\gamma_{x_n}^\eta)}) = (T_{T_{\gamma_f^\nu} \circ c_f(\gamma_{x_1}^\eta)}, \dots, T_{T_{\gamma_f^\nu} \circ c_f(\gamma_{x_n}^\eta)})$. Additionally for all $1 \leq i \leq n$ we have

$$T_{\gamma_f^\nu} \circ c_f(\gamma_i^\eta) = \prod_{j \in D_i} T_{\gamma_j^\nu}(\gamma_i^\nu)$$

where $D_i = \{1 \leq j \leq n | x_i \preceq_\eta x_j \text{ and } x_i \succeq_\nu x_j\}$ ordered by $\preceq_\eta$. The first step of the Hurwitz equivalence, from $r_f \circ g^\eta \circ r_f^{-1}$ to g^ν, consists in moving $T_{\gamma_n^\nu} = T_{\gamma_f^\nu}$ to the left and using $\left(T_{(\prod_{j \in D_i} T_{\gamma_j^\nu})(\gamma_i^\nu)}, T_{\gamma_f^\nu}\right) \simeq \left(T_{\gamma_f^\nu}, T_{(\prod_{j \in D_i \setminus \{f\}} T_{\gamma_j^\nu})(\gamma_i^\nu)}\right)$. Then we proceed the same way for the vertices of f to move them from their positions in $\prec_\eta$ to their positions in $\prec_\nu$, eventually changing $T_{(\prod_{j\in D_i\setminus\{f\}} T_{\gamma_j^\nu})(\gamma_i^\nu)}$ into $T_{\gamma_i^\nu}$ for all i. The n-tuple thus obtained is g^ν.

Now consider η and ν any two cohomologous acyclic Eulerian coorientations. We will prove that $r\circ g^\eta\circ r^{-1}$ and g^ν are Hurwitz equivalent for one choice of $r:\Sigma_\eta\to\Sigma_\nu$, then for all choices. Thanks to Proposition 2.2.8 in Appendix 2.2.3, there exists a finite sequence of flips that change η into ν. If $r_1:\Sigma_\eta\to\Sigma_\nu$ is the composition of the partial return maps for these flips, according to what proceed, $r_1\circ g^\eta\circ r_1^{-1}$ and g^ν are Hurwitz equivalent. Let $r_2:\Sigma_\eta\to\Sigma_\nu$ be a diffeomorphism obtained by an isotopy along the flow, then there exists $p\in\mathbb{Z}$ such that $r_2\circ r_1^{-1}=r_{\Sigma_\nu}^p$, where r_{Σ_ν} is the first-return map on Σ_ν. Hence it is enough to prove that g^ν and $r_{\Sigma_\nu}\circ g^\nu\circ r_{\Sigma_\nu}^{-1}$ are Hurwitz equivalent. In general, since $(a,b)\simeq(aba^{-1},a)$, we have the sequence of Hurwitz equivalences:

$$\begin{aligned}(h_1,\dots,h_n)&\simeq(h_1h_2h_1^{-1},\dots,h_1h_nh_1^{-1},h_1)\\&\simeq h_1(h_2,\dots,h_n,h_1)h_1^{-1}\\&\simeq h_1h_2(h_3,\dots,h_n,h_1,h_1)h_2^{-1}h_1^{-1}\\&\ \ \vdots\\&\simeq(\prod_i h_i)(h_1,\dots,h_n)(\prod_i h_i)^{-1}\end{aligned}$$

But $r_f=\prod_{h\in g^\nu}h$ so $r_{\Sigma_\nu}\circ g^\nu\circ r_{\Sigma_\nu}^{-1}\simeq g^\nu$, which finishes the proof. □

Theorem D, proved below, proposes an alternative comparison, for cohomologous and non-cohomologous coorientations.

Theorem D. *Let S be a hyperbolic surface, Γ a finite collection of closed geodesics on S, and consider the geodesic flow on T^1S. There exists a common combinatorial model Σ_Γ for all Birkhoff sections with boundary $-\overset{\leftrightarrow}{\Gamma}$, and an explicit family of simple closed curves $\gamma_1,\cdots,\gamma_n$ in Σ_Γ such that the first-return maps for these Birkhoff sections are products of negative Dehn twists of the form $T\gamma_{\sigma(1)}\circ\cdots\circ T\gamma_{\sigma(n)}$ for some permutation σ of $\{1,\cdots,n\}$.*

Remark 2.3.9. We have $\chi(S)=|\Gamma_0|-|\Gamma_1|+|\Gamma_2|$ and $\chi(\Sigma)=-|\Gamma_1|$. Hence the number of Dehn twists appearing in the corollary is $\chi(S)-\chi(\Sigma)=$

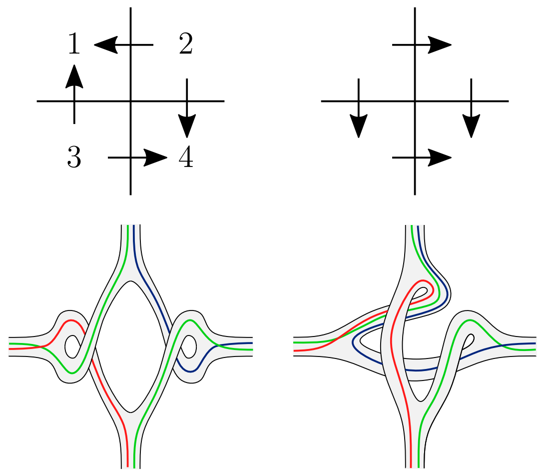


Figure 2.19: Action of a slide on the curves γ_f. The slide is done in the quadrant 1.

$2.genus(\Sigma) + |\partial\Sigma| - 2 + \chi(S)$. It is smaller but relatively closed to the minimal number $2.genus(\tilde{\Sigma}) + |\partial\Sigma|$ of Dehn twists generating the mapping class group of homeomorphisms of the surface Σ preserving every component of $\partial\Sigma$ one by one, up to isotopy (see in [FM12, Section 4.4.4]).

Proof. Let η be any acyclic Eulerian coorientation of Γ. Define $\Sigma_\Gamma = \Sigma_\eta$, and $\{\gamma_i\} = \{\gamma_f, f \in \Gamma_2\} \cup \{\gamma_v, v \in \Gamma_0\}$. As proven in [CD16], every Birkhoff section with boundary $-\overleftrightarrow{\Gamma}$ is isotopic to a Birkhoff section Σ_ν for $\nu \in \mathcal{E}ul\mathcal{C}o(\Gamma)$ acyclic. We will do a finite sequence of slides to compare Σ_ν to Σ_Γ, together with the curves they contain.

Let v be a vertex in Γ_0. Up to symmetry and rotation, there are nine configurations for (η, ν) around v. For each configuration, we can do one or two slides to isotope Σ_η to Σ_ν around v. Denote $sl : \Sigma_\eta \to \Sigma_\nu$ the diffeomorphism induced by the sequence of slides. We can compare γ_f^ν and $sl(\gamma_f^\eta)$. In each case $\gamma_v^\nu = sl(\gamma_v^\eta)$, and for any face f adjacent to v, either γ_f^ν and $sl(\gamma_f^\eta)$ are equal, or they differ by a Dehn twist along the curve γ_v (positive of negative). And in every case, we can apply Theorem C to ν, and obtain a product of negative Dehn twists along the curves γ_v^ν and γ_f^ν. We can swap positions of consecutive Dehn twists, including γ_v^ν and γ_f^ν, to obtain a product of Dehn twists along the curves $sl(\gamma_v^\eta)$ and $sl(\gamma_f^\eta)$. We will detail one case, the others being similar.

Consider the coorientation η (left) and ν (right) presented in Figure 2.19. In the figure, we represent γ_f^η on the left and $sl(\gamma_f^\eta)$ on the right, for the four faces f adjacent to v. We have $sl(\gamma_{f_i}^\eta) = \gamma_{f_i}^\nu$ for $i \in \{1, 4\}$, and $sl(\gamma_{f_j}^\eta) = T_{\gamma_v}(\gamma_{f_j}^\nu)$ for $j \in \{2, 3\}$, where T_{γ_v} is the negative Dehn twist along γ_v. Let $\preceq$ be a representation of ν, so that up to changing 2 and 3, we have $f_4 \preceq f_3 \preceq f_2 \preceq v \preceq f_1$. The first-return map of Σ_ν given by Theorem C contains a subproduct of the form $T\gamma_{f_1}^\nu \circ \cdots \circ T\gamma_v^\nu \circ \cdots \circ T\gamma_{f_2}^\nu \circ \cdots \circ T\gamma_{f_3}^\nu \circ \cdots \circ T\gamma_{f_4}^\nu$. But $T\gamma_v^\nu$ commutes with any Dehn twist that is not a $T\gamma_{f_i}$ for $1 \leq i \leq 4$. According to remark 2.3.5, for $i = 2, 3$ we have $T\gamma_v^\nu \circ T\gamma_{f_i}^\nu = T(T\gamma_v(\gamma_{f_i}^\nu)) \circ T\gamma_v^\nu = Tsl(\gamma_{f_i}^\eta) \circ T\gamma_v^\nu$.

So together with Remark 2.3.5, we can change the position of $T\gamma_v^\nu$ so that the Dehn twists appear in the order f_1, f_2, f_3, v, f_4, and are along the curve $sl(\gamma_{f_i}^\eta)$ and $sl(\gamma_v^\eta)$. We can do this procedure for all vertices $v \in \Gamma_0$, which prove that there exists a diffeomorphism $sl : \Sigma_\Gamma \to \Sigma_\nu$ so that the first-return map on Σ_ν is a product of negative Dehn twist along the curve $sl(\gamma_f^\eta)$ and $sl(\gamma_v^\eta)$, whose ordering depends on ν.

$\square$

Chapter 3

Topologically twisted flows

For transitive Anosov flows, there exists a trichotomy suspension flows / skewed $\mathbb{R}$-covered flows / non $\mathbb{R}$-covered flows. These classes of flows behave differently, and have different topological properties. In this chapter, we relate this trichotomy to the existence of some Birkhoff sections with certain sign on their boundaries, and extend it to non Anosov flows.

For a general flow ϕ in an oriented 3-manifold, we say that ϕ is **topologically flat** if it admits a global section, that is a Birkhoff section without boundary. We also say that ϕ is **topologically twisted** if it admits an oriented Birkhoff section, that is a Birkhoff section all of whose boundary components have the same sign (as defined in Section 1.3). Depending on the orientation of these boundary components, we additionally speak about positively topologically twisted flow and negatively topologically twisted flow. These properties are called the **nature of the flow**, and restrict the existence of some specific partial sections. Notice that the flow needs not to be smooth for defining these notions.

Theorem E. *Let M be an oriented closed 3-manifold, and ϕ be a positively (resp. negatively) topologically twisted flow on M. Then ϕ does not admit any negative partial section (resp. positive partial section).*

Suppose that ϕ is a topologically twisted Anosov flow on M, then ϕ does not admit any transverse section without boundary.

Suppose that ϕ is a topologically flat flow on M. Then ϕ does not admit any positive partial section, nor negative partial section.

The second point of this theorem is in fact already proven by T.Barbot, once Theorem G is understood. The third point can be summarized by a simple homological restriction. However it is the main idea to generalize this obstruction to twisted flows. To prove this theorem, we use a linking-number-like equality for two partial sections intersecting in their interior and along their boundaries. As a direct corollary, the nature of a flow is a well-defined notion.

Corollary F. *For a flow ϕ on an oriented 3-manifold M, being topologically flat, positively topologically twisted or negatively topologically twisted are mutually excluded.*

In Section 3.2, we relate this nature to the $\mathbb{R}$-covered notion for Anosov flows. Anosov flows that are not suspensions always admit some Birkhoff annulus (see Theorems 1.2.2 and 1.7.15). Using the orbit space and some Birkhoff annuli, we can build some partial sections and Birkhoff section with good boundaries. Hence we understand enough partial sections of Anosov flows to relate the topologically twisted property of general flows to the skewed $\mathbb{R}$-covered property of Anosov flows.

Theorem G. *Let M be an oriented closed 3-manifold and ϕ be a transitive Anosov flow on M. The following properties are equivalent:*

- *(M, ϕ) is positively skewed $\mathbb{R}$-covered.*
- *(M, ϕ) is positively topologically twisted.*
- *(M, ϕ) does not admit any partial section without boundary, or with only negative boundaries.*

The result holds when inverting all orientations.

The Birkhoff sections behave well under Fried-Goodman surgeries, namely one can compute the sign of a Birkhoff section after a surgery only by comparing the slope of the surgery to the slope of the Birkhoff section along this orbit. Hence the last theorem can be used to determine the nature of a flow after some Fried-Goodman surgeries.

Corollary H. *Let ϕ be a transitive Anosov flow on an oriented closed 3-manifold. There exists a sequence of Fried-Goodman surgeries on ϕ that induces a positive skewed $\mathbb{R}$-covered flow.*

Independently to my research, C.Bonatti found some results similar to Corollary H, using a different approach. For an Anosov flow, the nature of the orbit space of the flow is strongly related to the set of complete and incomplete quadrants. A quadrant on p is one of the four regions of the orbit space delimited by the stable and unstable leaf of p. A quadrant on p is said complete if every stable and unstable leaves l^s, l^u, entering the quadrant from its two boundaries, l^s and l^u intersect. By studying how a Fried-Goodman surgery changes the complete property of a quadrant, C.Bonatti proves [Bon21] two interesting results, about obtaining or losing the skewed $\mathbb{R}$-covered property by Fried-Goodman surgeries on one or two orbits only.

One can compare the topologically twisted property studied in this chapter with Reeb flows. A Reeb flow leaves invariant a contact structure, which by definition has a somehow twisted nature. Furthermore every contact structure admits an open book decomposition, that is a notion similar to Birkhoff sections with positive boundary. T.Barbot proved [Bar01] that all Anosov flows that leave invariant a topological contact structure are skewed $\mathbb{R}$-covered. The complete relation between Reeb flows and topologically twisted flows is not yet understood, even for Anosov flows.

3.1 Partial sections and topologically twisted flows

In this section, we study a linking-number-like equality for two intersecting partial sections. It is used to understand how the existence of one specific Birkhoff section restricts the existence of other partial sections. Then we prove Theorem E and Corollary F. We fix an oriented closed 3-manifold M and a flow ϕ of class $\mathcal{C}^1$ on M.

Let Σ_1 and Σ_1 be two immersed partial sections in M. We denote by $M_{\partial\Sigma_1\cup\partial\Sigma_2}$ the blowing up of M along $\partial\Sigma_1 \cup \partial\Sigma_2$. We still denote by Σ_1 and Σ_2 the induced partial sections in $M_{\partial\Sigma_1\cup\partial\Sigma_2}$. Generically, the intersection between Σ_1 and Σ_2 is represented in Figure 3.1. We orient $\partial\Sigma_1$ and $\partial\Sigma_2$ according to the orientations of Σ_1 and Σ_2. Recall that the linking numbers $\mathrm{lk}^+(\Sigma_1, \Sigma_2)$ and $\mathrm{lk}^-(\Sigma_1, \Sigma_2)$, defined in Section 1.3, refer to the linking numbers of Σ_1 and Σ_2 along their common boundary components whose orientations agree for lk^+ and disagree for lk^- (see Section 1.3).

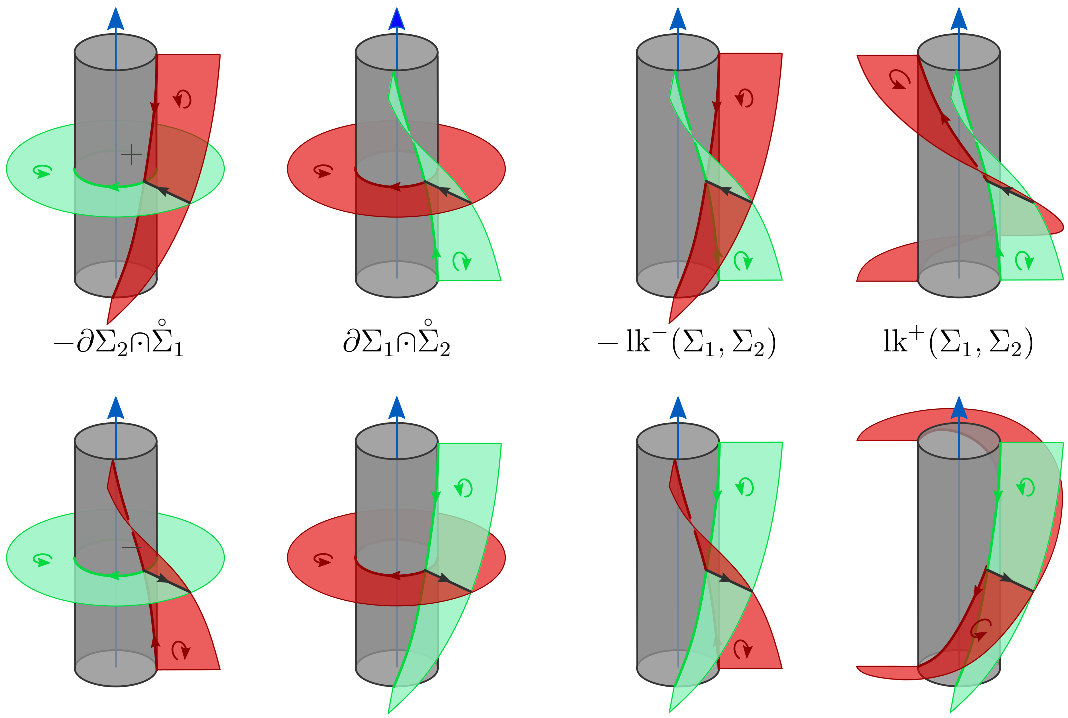


Figure 3.1: Generic intersection of immersed partial sections along a closed orbit of the flow, in the blowing-up manifold $M_{\partial\Sigma_1 \cup \partial\Sigma_2}$ (Σ_1 in light green, Σ_2 in dark red, and the flow direction in blue).

Lemma 3.1.1. *Let ϕ be a flow of an oriented 3-manifold M, and Σ_1 and Σ_2 be two immersed partial sections of ϕ. Then we have:*

$$(\partial\Sigma_2 \dot{\cap} \mathring{\Sigma}_1) - (\partial\Sigma_1 \dot{\cap} \mathring{\Sigma}_2) = \mathrm{lk}^+(\Sigma_1, \Sigma_2) - \mathrm{lk}^-(\Sigma_1, \Sigma_2)$$

The lemma remains true for non transverse surfaces, as long as two boundary components are either disjoint or equal.

Proof. Recall that the flow induces two coorientations on the interior of the surfaces Σ_1 and Σ_2, which induces two orientations on $\mathring{\Sigma}_1$ and $\mathring{\Sigma}_2$. We first do a small isotopy of Σ_1 and Σ_2 outside $\partial\Sigma_1 \cup \partial\Sigma_2$, so that $\Sigma_1 \cap \Sigma_2$ is a finite union of compact curves. We do not require the isotopic surfaces to remains immersed partial sections. We will orient them using the orientations of Σ_1, Σ_2 and M as in Figure 3.1. Let x be in $\mathring{\Sigma}_1 \cap \mathring{\Sigma}_2$. We choose a non-zero vector $u_1 \in T_x\Sigma_1$ such that u_1 coorients positively the surface Σ_2. Similarly we choose a non-zero vector $u_2 \in T_x\Sigma_2$ which coorients positively the surface Σ_1 the same coorientation that the flow. We finally choose a non-zero vector $v \in T_x(\Sigma_1 \cap \Sigma_2)$ so that the family (u_1, u_2, v) is a direct base on T_xM. Then v induces an orientation on $T_x(\Sigma_1 \cap \Sigma_2)$ that does not depend on the choices of u_1, u_2 and v, and is continuous on x.

Every non-closed curve of $\Sigma_1 \cap \Sigma_2$ has a positive and a negative end. Each end corresponds to an intersection of $\partial\Sigma_1$ and $\partial\Sigma_2$ in $\partial M_{\partial\Sigma_1\cup\partial\Sigma_2}$, and the sign of an end is the algebraic intersection of $\partial\Sigma_1$ and $\partial\Sigma_2$ at this point. We obtain the equation by adding the contributions of all ends to the terms $\partial\Sigma_2 \pitchfork \mathring{\Sigma}_1$, $\partial\Sigma_1 \pitchfork \mathring{\Sigma}_2$, $\mathrm{lk}^+(\Sigma_1, \Sigma_2)$ and $\mathrm{lk}^-(\Sigma_1, \Sigma_2)$. ∎

Corollary 3.1.2. *Let ϕ be a flow of an oriented 3-manifold M and Σ_1 and Σ_2 be a two partial sections of ϕ so that $\partial^-\Sigma_1 = \emptyset$ and $\partial^+\Sigma_1 \cap \partial^+\Sigma_2 = \emptyset$. Then we have $\partial\Sigma_2 \pitchfork \mathring{\Sigma}_1 \geq 0$, $\partial\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq 0$, $\mathrm{lk}^+(\Sigma_1, \Sigma_2) = 0$, $\mathrm{lk}^-(\Sigma_1, \Sigma_2) \leq 0$, and $(\partial\Sigma_2 \pitchfork \mathring{\Sigma}_1) - (\partial\Sigma_1 \pitchfork \mathring{\Sigma}_2) = -\mathrm{lk}^-(\Sigma_1, \Sigma_2)$.*

Hence $\partial^+\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq \partial^-\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq 0$.

When ϕ admits a positive Birkhoff section Σ, the intersection between a closed orbit $\gamma \notin \partial\Sigma$ and Σ is a good measure of the length of γ. One can extend this measure for $\gamma \in \partial\Sigma$ by the linking number $\mathrm{lk}_\gamma(\Sigma)$.

Proof. Since $\partial^-\Sigma_1 = \emptyset$ we have $\partial\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq 0$. Any common boundary component of Σ_1 and Σ_2 is a positive boundary component of Σ_1 and a negative boundary component of Σ_2. Hence $\mathrm{lk}^+(\Sigma_1, \Sigma_2) = 0$ and $\mathrm{lk}^-(\Sigma_1, \Sigma_2) \leq 0$. According to Lemma 3.1.1 we have $(\partial\Sigma_2 \pitchfork \mathring{\Sigma}_1) - (\partial\Sigma_1 \pitchfork \mathring{\Sigma}_2) = -\mathrm{lk}^-(\Sigma_1, \Sigma_2)$, so $\partial\Sigma_2 \pitchfork \mathring{\Sigma}_1 = \partial\Sigma_1 \pitchfork \mathring{\Sigma}_2 - \mathrm{lk}^-(\Sigma_1, \Sigma_2) \geq 0$.

The second point can be deduced from $\partial^+\Sigma_1 \pitchfork \mathring{\Sigma}_2 - \partial^-\Sigma_1 \pitchfork \mathring{\Sigma}_2 \partial\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq 0$, and $\partial^-\Sigma_1 \pitchfork \mathring{\Sigma}_2 \geq 0$ since $\partial^-\Sigma_1$ and $\mathring{\Sigma}_2$ are oriented accordingly to the flow. ∎

The previous lemma is enough to prevent two positive and negative partial sections to intersect on their boundaries.

Proof of Theorem E. Denote by S a positive Birkhoff section. Suppose there exists a negative partial section Σ. Then by Lemma 3.1.1, we have $(\partial\Sigma \pitchfork \mathring{S}) - (\partial S \pitchfork \mathring{\Sigma}) = \mathrm{lk}^+(S,\Sigma) - \mathrm{lk}^-(S,\Sigma)$.

Since $\partial^+\Sigma = \emptyset = \partial^- S$, we have $\partial\Sigma \pitchfork \mathring{S} \leq 0$, $\partial S \pitchfork \mathring{\Sigma} \geq 0$, $\mathrm{lk}^+(S,\Sigma) = 0$, and $\mathrm{lk}^-(S,\Sigma) = \mathrm{lk}_{\partial^+ S \cap \partial^- \Sigma}(S,\Sigma) \leq 0$. Hence $0 \geq (\partial\Sigma \pitchfork \mathring{S}) - (\partial S \pitchfork \mathring{\Sigma}) = -\mathrm{lk}^-(S,\Sigma) \geq 0$. So all four terms are zero, and $\partial\Sigma \cap S = \emptyset$. But S is a Birkhoff section which intersects every orbit, so $\emptyset \neq \partial^-\Sigma \subset \partial^+ S$. Then the linking number of S with Σ on a common boundary component must be zero, which contradicts Lemma 1.6.8. Hence such a partial section Σ cannot exist.

The third statement is proved the same way.

We now deal with the second point of the theorem. Suppose that the flow is Anosov, and admits a positive Birkhoff section S and a partial section Σ without boundary. For the same reason we have $\partial S \cap \mathring{\Sigma} = \emptyset$. The surface Σ is transverse to the stable foliation $\mathcal{F}^s$, so it is foliated by the trace of $\mathcal{F}^s$. Also Σ is co-oriented in an oriented 3-manifold, so Σ is orientable and admits a 1-foliation. Hence Σ is a torus. We isotope Σ through the flow so that it intersects transversally S. We will isotope Σ along the flow, so that it is in minimal position relatively to S.

First suppose that $S \cap \Sigma = \emptyset$. We define the first-return time from Σ to S, $T : \Sigma \to \mathbb{R}$ that sends $x \in \Sigma$ to $\min\{t \geq 0, \phi_t(x) \in S\}$. Since S is a Birkhoff section and $\partial S \cap \mathring{\Sigma} = \emptyset$, T is well-defined and continuous. But $\phi_T : \Sigma \to S$ is an immersion of Σ to S, which is not possible since Σ is closed and S has a non-empty boundary.

Secondly suppose that $S \cap \Sigma \neq \emptyset$. Then $S \cap \Sigma$ is a union of disjoint simple curves, that are closed since $\partial S \cap \mathring{\Sigma} = \emptyset$. Let U be a connected component of $\Sigma \setminus (S \cap \Sigma)$, so that the closure $\bar{U}$ intersects S only from below. We define similarly the first-return time that sends $x \in \bar{U}$ into $\min\{t \geq 0, \phi_t(x) \in S\}$. Then T is well-defined, continuous, zero on ∂U, and we can extend it by zero on $\Sigma \setminus U$. Then for $\epsilon > 0$ small enough, $t \in [0,1] \mapsto \phi_{t(T+\epsilon)}(\mathrm{id})$ isotopes Σ along the flow, and erases ∂U from $\Sigma \cap S$. If $\bar{U}$ intersects S only from above, we can also erase ∂U from $\Sigma \cap S$.

We can do this procedure until every connected component of $\Sigma \setminus (S \cap \Sigma)$ has its adherence that intersects S both from below and from above. Then there is no curve in $S \cap \Sigma$ that bounds a disc in Σ. Hence $S \cap \Sigma$ is a union of $n \geq 0$ parallel simple curves in Σ. We can suppose that $n \neq 0$ or the first case would apply. Denote by $c_1, \ldots, c_n$ the curves in $S \cap \Sigma$, with a cyclic order relatively to the torus Σ, and by $U_1, \ldots, U_n \subset \Sigma$ the annuli they delimit. We can also suppose that $\bar{U}_i$ intersects S along c_i from above, and intersects S along c_{i+1} from below.

Denote by $r : \mathring{S} \to \mathring{S}$ the first-return map, which is pseudo-Anosov since the flow is Anosov (see [Fri83]). We push U_i along the flow, using the first-return time from U_i to S. Then the image of U_i in S is an annulus bounded by $r(c_i)$ and c_{i+1}. Thus for all i, $r(c_i)$ is isotopic to c_{i+1}, so $r^n(c_i)$ is isotopic to c_i. Since $n > 0$ and r is pseudo-Anosov, it is impossible. □

The nature of a flow being well-defined, one can wonder how this nature changes when we do some Fried surgeries on a flow. It is hard to determine in general, but two specific cases are interesting.

Proposition 3.1.3. *Let ϕ be a topologically flat or positively topologically twisted flow on an oriented closed 3-manifold M. Any sequence of Fried surgeries with negative slopes induces a positively topologically twisted flow.*

Proof. By hypothesis, there exists a Birkhoff section Σ of ϕ such that $\partial^-\Sigma = \emptyset$. By Lemma 1.6.7, any Fried surgery with negative slope along a periodic orbit γ changes Σ into a Birkhoff section Σ', whose boundary is $\partial\Sigma \cup \gamma \neq \emptyset$, and with positive sign along all its boundary components. Hence after surgery, the flow is positively topologically twisted. □

Proposition 3.1.4. *Let M be an oriented 3-manifold. Any flow on M that admits a Birkhoff section can be made topologically flat, positively or negatively topologically twisted by a finite sequence of Fried surgeries.*

Proof. Let Σ be a Birkhoff section of ϕ, embedded in its interior. Denote by $M_{\partial\Sigma}$ the blowing-up of M along all boundary components of Σ, together with $\pi : M_{\partial\Sigma} \to M$ and ψ the induced flow on $\partial M_{\partial\Sigma}$. Then by Lemma 1.6.6, $\partial\pi^{-1}(\Sigma) \subset \partial M_{\partial\Sigma}$ is a union of global sections of ψ. Hence on each boundary component $\mathbb{T}$ of $M_{\partial\Sigma}$, one can choose a global section of $\psi_{|\mathbb{T}}$ with equal slope, higher slope or lower slope than $\partial\Sigma$. Then the Fried surgeries on $\partial\Sigma$ with these slopes induces a flow with a Birkhoff section whose boundaries and signs are given by Lemma 1.6.7. □

3.2 Topologically twisted Anosov flows

In this section, we prove Theorem G, which relates the topologically twisted property of a general flow to the skewed $\mathbb{R}$-covered property of an Anosov flow. Then we briefly see two consequences of this theorem.

Proof of Theorem G. Theorem E proves that the second point implies the third point. We will prove the two remaining implications.

Suppose that (M, ϕ) is positively skewed $\mathbb{R}$-covered, and take $K \subset \mathcal{O}(M)$ a compact containing a lift of every orbit of ϕ. Since (M, ϕ) is positively skewed $\mathbb{R}$-covered, for every p in K there exists an ideal lozenge $L_p \subset \mathcal{O}(M)$ containing p in it interior. Since ϕ is $\mathbb{R}$-covered, it is transitive, as proven by T.Barbot [Bar95b]. Then by transitivity of ϕ, the closed orbits of the flow are dense in M, so we can suppose that the corners of $L_p \subset \mathcal{O}(M)$ are $\rho(\gamma_p^1)$ and $\rho(\gamma_p^2)$ for two closed orbits $\gamma_p^{1,2}$. Then T.Barbot proves that there exists an immersed transverse annulus whose trace in $\mathcal{O}(M)$ is L_p (see Theorem 1.7.15). So, up to a Fried desingularisation as presented in Proposition 1.4.13, there exists a partial section Σ_p to the flow, bounded by γ_p^1 and γ_p^2. Since (M, ϕ) is positively skewed $\mathbb{R}$-covered, L_p is in the quadrant $(+,+)$ or $(-,-)$ of $\rho(\gamma_p^1)$ and $\rho(\gamma_p^2)$, so that $\partial^+\Sigma_p = \{\gamma_p^1, \gamma_p^2\}$ and $\partial^-\Sigma_p = \emptyset$.

For every $p \in K$, we have a positive partial section Σ_p intersecting every orbit that is close enough to p. So by compactness of K, there exist $p_1, \cdots, p_n \in K$ so that $K \subset \cup_i \mathring{L}_{p_i}$. We define Σ to be the Fried desingularisation of $\cup_i \Sigma_{p_i}$, which is a positive partial section thanks to Proposition 1.4.13. Then every orbit not in $\partial\Sigma$ intersects Σ. Also for every orbit $\gamma \subset \partial\Sigma$, γ intersects the interior of one of the Σ_{p_i}, so $\mathrm{lk}_\gamma(\Sigma, \mathcal{F}^s) \neq 0$. By a classical argument, Σ is a Birkhoff section, which then has only positive boundaries. Indeed suppose that for all $T > 0$, there is $q_T \in M$ so that $\phi_{[0,2T]}(q_T) \cap \Sigma = \emptyset$. By compactness of M, $\phi_T(q_T)$ accumulates on some $q_\infty \in M$ when T tends to $+\infty$. If $q_\infty \notin \partial\Sigma$ then there exists $q' \in \phi_\mathbb{R}(q_\infty) \cap \mathring{\Sigma}$ and $\mathring{\Sigma}$ is transverse to ϕ, which contradicts $\phi_{[-T,T]}(\phi_T(q_T)) \cap \Sigma = \emptyset$ for all $T > 0$. If $q_\infty \in \partial\Sigma$, the same argument holds. Indeed $\mathrm{lk}_\gamma(\Sigma, \mathcal{F}^s) \neq 0$ so there is a small neighborhood of q_∞ whose orbits all intersect Σ. Hence Σ is a positive Birkhoff section, and ϕ is positively topologically twisted.

Suppose that (M, ϕ) does not admit any partial section without boundary nor any negative partial section. Since ϕ has no partial section with empty boundary, it is not a suspension. Suppose that ϕ is not positively skewed $\mathbb{R}$-covered. Then it is either negatively skewed $\mathbb{R}$-covered, or not $\mathbb{R}$-covered. In the first case, there exists an ideal lozenge $L \subset \mathcal{O}(M)$ bounded by closed orbits. According to Theorem 1.7.15, there exists an immersed Birkhoff annulus A whose trace in $\mathcal{O}(M)$ is L. But the flow is negatively skewed $\mathbb{R}$-covered so A has only negative boundaries, which contradicts Theorem E. In the second case, ϕ is not $\mathbb{R}$-covered, so according to Theorem 1.2.2 there exists an ideal lozenge L bounded by two closed orbits, in the quadrant $(+,-)$ and $(-,+)$ of the two orbits. Then the same argument on L induces a contradiction. Hence ϕ is positively skewed $\mathbb{R}$-covered. □

The last step of the proof does not only give the existence of a positive Birkhoff section, but also gives a method to construct a large family of them with different boundaries. Given a Birkhoff section of an Anosov flow, one can produce a skewed $\mathbb{R}$-covered flow with Fried-Goodman surgeries along the boundary components of the Birkhoff section. Additionally the slopes of the surgeries needed are given by the multiplicities and the linking numbers of the Birkhoff section along each of its boundary components.

Proof of Corollary H. Let ϕ be a transitive Anosov flow on an oriented closed 3-manifold. D.Fried proved [Fri83] that ϕ admits a Birkhoff section. Then Corollary 3.1.4 adapted for Fried-Goodman surgeries implies that some Fried-Goodman surgeries on ϕ induce a positively topologically twisted Anosov flow, which is a positively skewed $\mathbb{R}$-covered flow according to Theorem G. □

Theorem G also gives a new way to construct some skewed $\mathbb{R}$-covered Anosov flows.

Corollary 3.2.1. *Let M be an oriented closed 3-manifold, and ϕ be an Anosov suspension or a positively skewed $\mathbb{R}$-covered transitive Anosov flow. Then any Fried-Goodman surgery with negative slope induces a positively skewed $\mathbb{R}$-covered transitive Anosov.*

The proof follows from Theorem G and Corollary 3.1.3. The corollary is a generalization of a Theorem from S.Fenley [Fen94, Theorem *D*], which states the result for the suspension flows and the geodesic flows.

Chapter 4

Primitive orbits of twisted flows

This chapter studies the notion of primitive orbits of twisted flows. A partial section Σ is viewed as a transverse cobordism from $\partial^-\Sigma$ to $\partial^+\Sigma$. In the previous chapter, we proved that a positively twisted flow does not admit a negative partial section, that is a transverse cobordism from $\emptyset$ to a non empty set of closed orbits.

Question. Given a set of closed orbits, is it the positive boundary of a transverse cobordism? If so, is it the positive boundary of a transverse cobordism with some restrictions on the multiplicities of the boundary components?

Orbits and sets of orbits that are not positive boundaries of some transverse cobordism are called primitive. In Section 4.1, we introduce two variations of this notion: primitive orbits and stably primitive orbits. One goal is to understand how the orbits of a flow can be decomposed using transverse cobordisms into some specific orbits, preferentially primitive. For that purpose, a family Γ of closed orbits is called **spanning** if every orbit γ not in Γ admits a transverse cobordism from γ to a subset of Γ.

Theorem I. *Let M be an oriented closed 3-manifold and ϕ be an Anosov flow on M which is supposed to be positively skewed $\mathbb{R}$-covered. Then the set $\mathrm{Prime}(\phi)$ of primitive orbits is spanning.*

If there exists a closed orbit which does not bound any partial section, or if $H_1(M,\mathbb{Z}) \neq \{0\}$, then there exists a primitive orbit.

In Sections 4.2 and 4.3, we study the primitive orbits for the algebraic flows, that is the suspension flows and the geodesic flows. For topologically flat flows, being a primitive orbit is mainly a homological property, but the primitive orbits of the geodesic flows are more interesting. We will see that for any transverse cobordism of a geodesic flow, its boundary components must project on the underlying surface into curves that admit some intersection points. Given a geodesic γ, we relate the self-intersection points of γ with the topology of the partial sections bounded by the lift $\vec{\gamma}$.

Theorem J. *Let S be a hyperbolic surface and consider the geodesic flow on T^1S and $\gamma \subset S$ an oriented closed geodesic. Then $\vec{\gamma}$ is a primitive orbit if and only if $\vec{\gamma}$ is a stably primitive orbit, if and only if γ is simple.*

Also the set of primitive orbits is spanning.

The stably primitive orbits play an important role in understanding the nature of a flow after some Fried surgeries. In Section 4.4, we study this role for the skewed $\mathbb{R}$-covered Anosov flows. The constructions of Birkhoff annuli and Fried sections, explained in the first Chapter, can be used to create interesting transverse cobordisms for these Anosov flows. Knowing enough partial sections allows us to relate the stably primitive orbits with some ideal lozenges in the orbit space. It is also useful for understanding the nature of the flow after surgeries.

Theorem (later properly stated as Theorem K)**.** Let ϕ be a positively skewed $\mathbb{R}$-covered Anosov flow with orientable stable and unstable foliations, on an oriented closed 3-manifold. Let Γ be a set of closed orbits of ϕ, such that there are no ideal lozenges inside the orbit space $\mathcal{O}(M)$ whose two corners are induced by two orbits inside Γ. Then the following implications hold:

$\Leftrightarrow$ Γ is a stably primitive set of orbits,

$\Leftrightarrow$ Γ bounds in the orbit space $\mathcal{O}(M)$ a set of ideal lozenges that will be called simple,

$\Rightarrow$ any finite sequence of Fried-Goodman surgery along orbits of Γ produces a positive skewed $\mathbb{R}$-covered flow.

Furthermore, the last implication is an equivalence when $|\Gamma| = 1$.

4.1 Zoology of primitive orbits

In this section, we define all variations of primitive orbits for flows, and prove Theorem I concerning the existence of some kind of primitive orbits for twisted flows.

We fix a 3-manifold endowed with an orientation and a flow ϕ.

Definition 4.1.1. Let γ be a closed orbit of the flow. We say that γ is a **primitive orbit** if there is no cobordism Σ so that $\partial^+\Sigma = \gamma$ with multiplicity one.

We say that γ is a **stably primitive orbit** if for all $n \in \mathbb{N}$, $n\gamma$ is primitive. By extension, we say that a family of orbits $\{\gamma_1, \ldots, \gamma_n\}$ is stably primitive if there is no cobordism Σ so that $\partial^+\Sigma = \gamma_1 \cup \ldots \cup \gamma_n$ with multiplicities in $\mathbb{N}$.

We can be more precise and call the previously defined orbits **positively primitive**, and define similarly **negatively primitive** orbits by reversing all signs in the previous definition (or by changing the orientation of the manifold). For any hyperbolic surface S, T^1S has a natural orientation which make negatively primitive orbits more interested. But for other flows, we only consider the positively primitive orbits.

Primitive orbits are mostly interesting for skewed $\mathbb{R}$-covered Anosov flows, as explained below. For a flow that admits a global section S, being primitive is a homological condition (see Section 4.2), and has few to do with the topology of the flow.

We denote by $\mathrm{Prime}(\phi)$ the set of primitive orbits of the flow ϕ.

Proposition 4.1.2. *The sets of primitive orbits and stably primitive orbits are invariant under orbital equivalence.*

Proof. Let $f : (M_1, \phi) \to (M_2, \phi_2)$ be an orbital equivalence, and $\Sigma \subset M_1$ a transverse cobordism for ϕ_1. Then $f(\Sigma)$ can be smoothed into a transverse cobordism with boundary $f(\partial\Sigma)$, with the same multiplicities. □

A set of closed orbits Γ is said to be **spanning** if for every closed orbit δ not in Γ there is a transverse cobordism from γ, with multiplicity one, to a finite subset (possibly empty) of Γ, with any multiplicities. Such a transverse cobordism from δ is called a decomposition of δ into Γ.

Proof of Theorem I. We uses several times an argument that we want to clarify here. If one is given a finite number of transverse cobordisms, their union may have self intersection. One can use Proposition 1.4.13 to desingularise and smooth the union into a new transverse cobordism. In that case we say that we concatenate the transverse cobordisms. Then the boundary of the new transverse cobordism is obtained algebraically as the sum of the boundary of the initial transverse cobordisms.

Thanks to Theorem G, the flow ϕ admits a Birkhoff section with only positive boundary components. Let S be such a section. To prove that the set $\mathrm{Prime}(\phi)$ of primitive orbits is spanning, we prove first that the set $\partial S \cup \mathrm{Prime}(\phi)$ is a spanning family. Let γ be an orbit not in $\partial S \cup \mathrm{Prime}(\phi)$. By definition, there exists a transverse cobordism Σ so that $\partial^+\Sigma = \gamma$. If $|\partial^-\Sigma| = 0$, then Σ is a transverse cobordism from γ to $\partial S \cup \mathrm{Prime}(\phi)$.

Suppose that $|\partial^-\Sigma| \geq 1$. We construct a transverse cobordism from the orbit γ to some other orbits, which either are inside $\partial S \cup \mathrm{Prime}(\phi)$ or intersect the surface $\mathring{S}$ less than the orbit γ. Then the proof follows by induction on the integer $\gamma \cap \mathring{S}$. We first consider the case $|\partial^-\Sigma| \geq 2$. By Corollary 3.1.2, we have $\partial\Sigma \dot{\cap} \mathring{S} = (\partial S \dot{\cap} \mathring{\Sigma}) + \mathrm{lk}^-(\Sigma, S)$ and the three terms are non negative. So

$$0 \leq \partial^-\Sigma \dot{\cap} \mathring{S} = (\partial^+\Sigma \dot{\cap} \mathring{S}) - (\partial S \dot{\cap} \mathring{\Sigma}) - \mathrm{lk}^-(\Sigma, S) \leq \partial^+\Sigma \dot{\cap} \mathring{S}$$

We will prove that for all $\delta \in \partial^-\Sigma \setminus \partial S$, we have $\delta \dot{\cap} \mathring{S} < \gamma \dot{\cap} \mathring{S}$, so that the algebraic intersection with $\mathring{S}$ is a function that decreases under transverse cobordisms. If we have $\partial^-\Sigma \cap \partial S \neq \emptyset$, then S is a Birkhoff section with non-zero linking number along its boundary (see Lemma 1.6.8) and $\mathrm{lk}^-_{\partial^-\Sigma \cap \partial S}(\Sigma, S) > 0$, so $\partial^-\Sigma \dot{\cap} \mathring{S} < \partial^+\Sigma \dot{\cap} \mathring{S}$.

Otherwise we denote the negative boundary components of Σ by $\delta_1, \ldots, \delta_n$, and by $a_1, \ldots, a_n$ the multiplicities of the curves δ_i as boundary components of Σ. Then we have $\delta_i \notin \partial S$, and $\partial^-\Sigma \dot{\cap} \mathring{S} = \sum_i a_i(\delta_i \dot{\cap} \mathring{S})$. Since S is a Birkhoff section, we have $\delta_i \dot{\cap} \mathring{S} \geq 1$ for every i. By hypothesis, we have either $n \geq 2$ or $a_1 \geq 2$, so in both cases we have $\delta_i \dot{\cap} \mathring{S} < \partial^-\Sigma \dot{\cap} \mathring{S}$ for every i. Thus in any case we have $\delta \dot{\cap} \mathring{S} < \gamma \dot{\cap} \mathring{S}$ for every orbit $\delta \in \partial^-\Sigma \setminus \partial S$.

Now consider the last case: $|\partial^-\Sigma| = 1$. Here Σ is a transverse cobordism from the orbit γ to another closed orbit δ. We claim that there is no infinite sequence of orbits γ_i and of transverse cobordisms Σ_i with $\partial^+\Sigma_i = \gamma_i$ and $\partial^-\Sigma_i = \gamma_{i+1}$. Assuming this claim and following the previous proof, if $|\partial^-\Sigma| = 1$, then we concatenate together transverse cobordisms satisfying $\partial^+\Sigma \dot{\cap} \mathring{S} = \partial^-\Sigma \dot{\cap} \mathring{S}$, until eventually obtaining a cobordism of γ with either $\partial^+\Sigma \cap \mathring{S} > \partial^-\Sigma \cap \mathring{S}$ or $|\partial^-\Sigma| = 0$ or $|\partial^-\Sigma| \geq 2$. Then we can apply the previous cases.

To prove the claim, suppose that there exists such a sequence of transverse cobordisms, starting at γ_0 and Σ_0. Then, for every $i \geq 0$, either one has $\mathring{\Sigma}_i \cap \partial S \neq \emptyset$, or γ_i and γ_{i+1} are homologous in $M \setminus \partial S$ and $|\gamma_i \cap \mathring{S}| = |\gamma_{i+1} \cap \mathring{S}|$. In the first case, by what precedes, the intersection with S decreases $|\gamma_i \cap \mathring{S}| < |\gamma_{i+1} \cap \mathring{S}|$, so it can only occur a finite number of time in the sequence. Therefore for all $i \in \mathbb{N}$ large enough we must be in the second case. Since the flow is Anosov, and S is a Birkhoff section, the first-return map is pseudo-Anosov (see [Fri83]). Hence there is only a finite number of orbits δ so that $|\delta \cap \mathring{S}|$ have a fixed value. Thus there exist $i_1 < i_2$ such that $\gamma_{i_1} = \gamma_{i_2}$. Then these transverse cobordisms Σ_i for all $i \in [\![i_1, i_2 [\![$ can be concatenate into a partial section without boundary. Since S is a positive Birkhoff section, it contradicts Theorem E. This proves the claim.

We proved that we can find a transverse cobordism from the orbit γ to some other orbits, which either are inside $\partial S \cup \mathrm{Prime}(\phi)$ or intersect the surface $\mathring{S}$ less than the orbit γ. We proceed by induction on $\gamma \cap \mathring{S}$. For every orbit $\delta \in \partial^-\Sigma \setminus (\partial S \cup \mathrm{Prime}(\phi))$ we construct a transverse cobordism Σ_δ from δ to some orbits of $\partial S \cup \mathrm{Prime}(\Gamma)$. Then we concatenate Σ to the transverse cobordisms Σ_δ for all $\delta \in \partial^-\Sigma \setminus \partial S$, which produces a transverse cobordism from γ to some orbits of $\partial S \cup \mathrm{Prime}(\Gamma)$.

Now we prove that any orbit of $\partial S \setminus \mathrm{Prime}(\phi)$ co-bounds a transverse cobordism with some orbits in $\mathrm{Prime}(\phi)$. Denote by $D = \{\delta_1, \dots, \delta_k\}$ the set of orbits in $\partial S \setminus \mathrm{Prime}(\phi)$ that do not co-bound a transverse cobordism with $\mathrm{Prime}(\phi)$, and suppose that $D \neq \emptyset$. For $1 \leq i \leq k$, δ_i is not primitive, so there exists a transverse cobordism Σ_i so that $\partial^+\Sigma_i = \delta_i$ with multiplicity one. We can use the above construction and suppose that $\partial^-\Sigma_i$ is in $\partial S \cup \mathrm{Prime}(\phi)$, and even in $D \cup \mathrm{Prime}(\phi)$ since the curves in $\partial S \setminus D$ co-bound transverse cobordisms with $\mathrm{Prime}(\phi)$. For any i, since δ_i does not co-bound a transverse cobordism with $\mathrm{Prime}(\phi)$, we have $\partial^-\Sigma_i \not\subset \mathrm{Prime}(\phi)$. So there exists an index $f(i)$ so that $\delta_{f(i)} \in \partial^-\Sigma_i$. Since D is finite, we can find a sequence $i_1, \dots i_p$ so that $f(i_k) = i_{k+1}$ and $f(i_p) = i_1$. We concatenate all Σ_i into one transverse cobordism Σ. Since δ_{i_k} is a positive boundary of only Σ_{i_k}, with multiplicity 1, and is a negative boundary of $\Sigma_{i_{k+1}}$ with multiplicity at least 1, Σ has no positive boundary. This is impossible by Theorem E. Thus D is empty and every non-primitive orbit co-bounds a transverse cobordism with Prime(ϕ).

To prove that the set of primitive orbits is not empty, we suppose that there exists a closed orbit γ which does not bound any partial section with multiplicity 1 along γ. Then the previous procedure applied on γ produces a transverse cobordism from γ to a subset Γ of primitive orbits. But γ does not bound any partial section, so $\Gamma \neq \emptyset$, and there exists a primitive orbit. Notice that since ϕ is Anosov, T.Adachi proved [Ada87] that the set of closed orbits of ϕ spans the homology $H_1(M, \mathbb{Z})$. Hence, if additionally one has $H_1(M, \mathbb{Z}) \neq \{0\}$, then there exists a non-null-homologous closed orbit of ϕ, which cannot bound a partial section on its own. $\square$

4.2 Primitive orbits of a topologically flat flow

We briefly study primitive orbits on topologically flat flows. Recall that a flow is said topologically flat if it admits a global section.

Proposition 4.2.1. *Let M be an oriented 3-manifold, ϕ a topologically flat flow on M, $\gamma_1, \dots, \gamma_p$ and $\delta_1, \dots, \delta_q$ two families of closed orbits of ϕ, and $a_1, \dots, a_p, b_1, \dots, b_q$ positive integers. Then there exists a transverse cobordism from $a_1\gamma_1 \cup \ldots \cup a_p\gamma_p$ to $b_1\delta_1 \cup \ldots \cup b_q\delta_q$ if and only if $\sum_i a_i[\gamma_i] = \sum_j b_j[\delta_j]$ in $H_1(M, \mathbb{Z})$.*

A corollary stated below is that for a closed orbit of a topologically flat flow, being primitive is mainly a homological condition. Also when the condition $\sum_i a_i[\gamma_i] = \sum_j b_j[\delta_j]$ in $H_1(M, \mathbb{Z})$ is satisfied, then there exists transverse cobordisms in both directions, that is from $a_1\gamma_1 \cup \ldots \cup a_p\gamma_p$ to $b_1\delta_1 \cup \ldots \cup b_q\delta_q$, and from $b_1\delta_1 \cup \ldots \cup b_q\delta_q$ to $a_1\gamma_1 \cup \ldots \cup a_p\gamma_p$.

Proof. The direct implication is clear.

We suppose $\sum_{1 \leq i \leq p} a_i[\gamma_i] = \sum_{1 \leq j \leq q} b_j[\delta_j]$ and we construct a transverse cobordism as stated in the proposition. We take a 2-cell Σ bounding $\sum_{1 \leq i \leq p} a_i[\gamma_i] - \sum_{1 \leq j \leq q} b_j[\delta_j]$. We can smooth Σ and use the non-empty boundary of Σ to erase its Whitney umbrellas. Hence we can suppose Σ to be a surface embedded in its interior. Denote by $S \subset M$ a global section for ϕ. We now prove that for n a large enough integer, the singular surface $\Sigma \cup nS$ is relatively homologous in $H_1(M, \partial\Sigma, \mathbb{Z})$ to a transverse cobordism from $a_1\gamma_1 \cup \ldots \cup a_p\gamma_p$ to $b_1\delta_1 \cup \ldots \cup b_q\delta_q$.

Since S is a global section, S is the fiber of a fibration $M \to S^1$ so there exists a foliation $\mathcal{F}$ of M made of parallel copies of S. We call *horizontal* a surface lying inside a leaf of $\mathcal{F}$, and *vertical* a surface foliated by orbit arcs of the flow. We can approximate Σ by a surface Σ', which is piecewise vertical and horizontal. That is there is a finite family of disjoint simple curves in Σ' so that every connected component of the complement of these curves is either vertical, or horizontal. If the approximation is good enough, Σ' does not self-intersect itself, and a small perturbation makes it smooth and embedded in its interior again. Every horizontal connected component in Σ' is either positively oriented by the flow, or negatively oriented. For every horizontal connected component c in Σ' which is negatively oriented, we take $S_c \in \mathcal{F}$ the leaf containing c, and replace in Σ the component c by $S_c \backslash c$. It corresponds to adding $[S_c]$ to $[\Sigma']$ in $H_2(M, \mathbb{Z})$, then erasing the component c that appears both in Σ' and in S_c with opposite orientations. We thus obtain a new surface with one less component negatively transverse to the flow, possibly with self-intersection. By doing this procedure a finite number of time, one obtains a self-intersecting surface homologous to $\Sigma \cup nS$ which is piecewise vertical and horizontal, with only positively transverse horizontal pieces. The latter can be smoothed into an immersed transverse cobordism, which can then be desingularised to obtain a transverse cobordism. It has the same boundary than Σ, so it is a transverse cobordism from $a_1\gamma_1 \cup \ldots \cup a_p\gamma_p$ to $b_1\delta_1 \cup \ldots \cup b_q\delta_q$. □

To express the set of primitive orbits of a topologically flat flow ϕ, we define the set of homology classes $C_\phi = \{[\gamma] \in H_1(M, \mathbb{Z}) | \gamma \text{ a closed orbit of } \phi\}$. Recall that for a global section S of ϕ and a closed orbit γ of ϕ, the algebraic intersection $\gamma \Cap [S]$ is positive. Hence $0 \notin C_\phi$.

Proposition 4.2.2. *Let M be an oriented 3-manifold and ϕ be a topologically flat flow on M. A closed orbit γ is primitive if and only if its homology class in $H_1(M, \mathbb{Z})$ is not the sum of at least two elements of C_ϕ, and if γ is the only closed orbit in its homology class.*

A closed orbit γ is stably primitive if and only there is no $n \geq 1$ such that the homology class of $n\gamma$ in $H_1(M, \mathbb{Z})$ is not the sum of at least two elements of C_ϕ, and if γ is the only closed orbit in the positive ray $\mathbb{R}_+^\star[\gamma]$ of its homology class.

This property makes the notion of primitive orbits not very interesting for topologically flat flows.

Proof. Let γ be a closed orbit of ϕ. Notice that since ϕ is topologically flat, there is no cobordism from γ to the empty set. Hence, by definition, γ

is primitive if and only there is no transverse cobordism from γ to some orbits $\delta_1, \dots \delta_q$ with multiplicity $b_1, \dots b_q \in \mathbb{N}_{>0}$, so that either $q \geq 1$. By Proposition 4.2.1, the existence of such a cobordism is equivalent to the equality $[\gamma] = \sum_i b_i[\delta_i]$ in $H_1(M, \mathbb{Z})$. So γ is primitive if and only if $[\gamma]$ is not the sum of at least two elements of C_ϕ and if γ is alone in its homology class.

The second statement is proven with the same arguments, by replacing γ by $n\gamma$ for any $n \geq 1$. $\square$

Suspension of an Anosov diffeomorphism. The case of the suspension of an Anosov map is even simpler. Let $A \in GL_2(\mathbb{Z})$ be a hyperbolic matrix as in Section 1.1, $\mathbb{T}_A = \mathbb{T}^2 \times \mathbb{R}/(p, t+1) \equiv (Ap, t)$ the suspension of $\mathbb{T}^2$ for the diffeomorphism induced by A, with its suspension flow ϕ.

Proposition 4.2.3. *Let $(\mathbb{T}_A, \phi)$ be the suspension flow of an Anosov diffeomorphism given by a matrix $A \in GL_2(\mathbb{Z})$. The suspension flow admits a primitive orbit in and only if either* $\det(A) = 1$ *and* $\mathrm{Tr}(A) = 3$, *or* $\det(A) = -1$ *and* $|\mathrm{Tr}(A)| = 1$. *In these cases, the primitive orbit is the orbit of the point $(0_{\mathbb{T}^2}, 0) \in \mathbb{T}_A$. There is no stably primitive orbit.*

Any primitive orbit of length one is on its own a spanning set.

Proof. First notice that $H_1(\mathbb{T}_A, \mathbb{Z}) = \mathbb{Z}$, that $C_\phi \subset \mathbb{Z}_{>0}$ and $1 \in C_\phi$ since the orbit of $(0_{\mathbb{R}^2}, 0)$ has homology 1. So to be primitive, a closed orbit γ need to have length one, that is its homology must be given by a generator of $H_1(\mathbb{T}_A, \mathbb{Z})$. Hence to prove the first point, we need to determine when there is another orbit of length one, that is when there exists a point $p \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that $(A - I_2)p \in \mathbb{Z}^2$. It is the case if and only if $|\det(A - I_2)| \neq 1$. But $\det(A - I_2) = \chi_A(1) = 1 - \mathrm{Tr}(A) + \det(A)$, which is enough to prove the point.

The second statement is a direct consequence of Lemma 4.2.1. $\square$

4.3 Primitive orbits of the geodesic flows

We fix a closed hyperbolic surface S, not necessarily orientable, and T^1S the unitary tangent bundle on S. In this section, we will study the primitive orbits of the geodesic flow, and prove Theorem J.

We fix the orientation on T^1M given by $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta})$, for any local map on S with coordinates (x, y), and where $\frac{\partial}{\partial \theta}$ is the trigonometric direction. This local orientation does not depend on the choice of coordinate on S, so it induces an orientation on T^1M. Recall that for this orientation, the geodesic flow is negatively skewed $\mathbb{R}$-covered. Hence we will study the negatively primitive orbits, which are the orbits that are not the negative boundary of some transverse cobordism.

Any immersion $\gamma : S^1 \to S$ lifts to a unique immersion $\vec{\gamma} : S^1 \to T^1S$, obtained by adding the renormalization of $\frac{\partial\gamma}{\partial\theta}$. We also denote by $\overleftarrow{\gamma} \subset T^1S$ the curve obtained by adding $-\frac{\partial\gamma}{\partial\theta}$ (or by composing γ with a symmetry on S^1). It induces a $1:1$-correspondance between oriented geodesics and orbits of the geodesic flow.

4.3.1 Immersed tubular neighborhood

To understand the transverse cobordisms of the geodesic flow, we use so-called immersed tubular neighborhoods. They relate the topology of the curves in S to the transverse cobordisms in T^1S. They will be used to prove that the primitive orbits of the geodesic flow are lifts of simple geodesics.

Definition 4.3.1. Let γ be a geodesic of S. We call **immersed tubular neighborhood** of γ a strip $\mathcal{N}$ (not necessarily oriented) with an embedding $i : \mathbb{S}^1 \to \mathcal{N}$ and an immersion $p : \mathcal{N} \to S$, so that $p \circ i : S^1 \to S$ is a parametrization of γ, and the image $\mathrm{im}(p)$ is a tubular neighborhood of γ as subspace of S. It induces an immersion $T^1p : T^1\mathcal{N} \to T^1S$. To simplify the notations, we also denote by γ the image $i(\mathbb{S}^1)$.

If Σ is a surface in T^1S whose boundary contains γ, we write $\Sigma_{\mathcal{N}}$ for the pre-image $(T^1p)^{-1}(\Sigma)$.

Depending on whether $\mathcal{N}$ is orientable or not, the boundary $\partial(T^1\mathcal{N}) = T^1_{\partial\mathcal{N}}\mathcal{N}$ is one or two torus. Also $\partial\mathcal{N}$ lifts to two or four curves $\overrightarrow{\partial\mathcal{N}} \cup \overleftarrow{\partial\mathcal{N}}$. If δ is a boundary component of $\mathcal{N}$ and $\mathbb{T} = T^1_\delta\mathcal{N}$, we denote by $\vec{\gamma}_{\mathbb{T}} = \vec{\delta}$ and $\overleftarrow{\gamma}_{\mathbb{T}} = \overleftarrow{\delta}$, as in Figure 4.1. Then $\mathbb{T}$ is transverse to the geodesic flow outside $\vec{\gamma}_{\mathbb{T}} \cup \overleftarrow{\gamma}_{\mathbb{T}}$. Notice that δ is not a geodesic, so $\overleftarrow{\gamma}_{\mathbb{T}}$ and $\vec{\gamma}_{\mathbb{T}}$ are not orbits of the flow.

Lemma 4.3.2. *Let Σ be a partial section with a boundary component at the orbit $\vec{\gamma}$, $\mathcal{N}$ be an immersed tubular neighborhood of γ such that $\Sigma_{\mathcal{N}}$ is transverse to $\partial(T^1\mathcal{N})$, $\mathbb{T}$ be a connected component of $\partial(T^1\mathcal{N})$, and δ be a component of $\Sigma_{\mathcal{N}} \cap \mathbb{T}$. Then δ intersects at most once $\vec{\gamma}_{\mathbb{T}} \cup \overleftarrow{\gamma}_{\mathbb{T}}$. Also, if it does, δ has two distinct ends.*

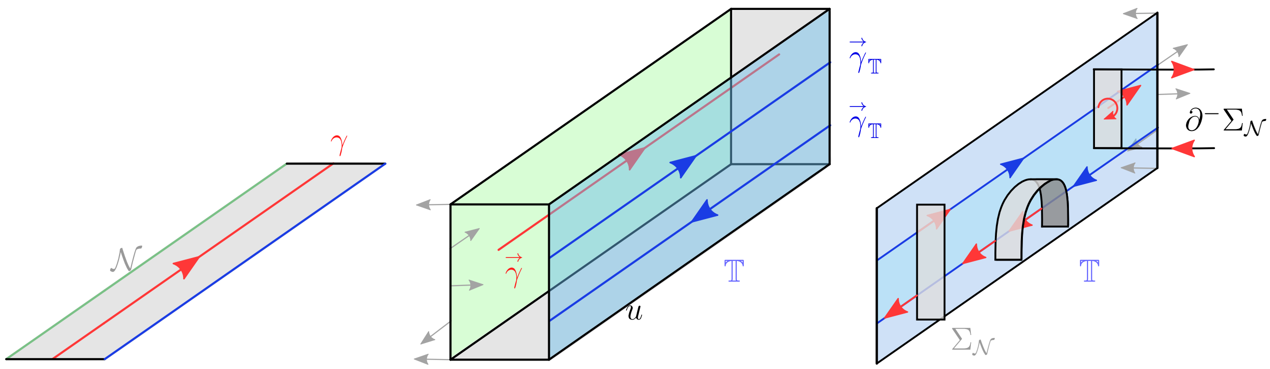


Figure 4.1: Immersed tubular neighborhood $\mathcal{N}$ of a geodesic γ. Is represented $\mathcal{N}$ on the left, $T^1\mathcal{N}$ on the middle, and the intersection of another surface with $\mathbb{T} \subset \partial(T^1\mathcal{N})$ on the right.

Proof. Since $\mathring{\Sigma}$ is transverse to the geodesic flow, an intersection of $\mathring{\Sigma}_{\mathcal{N}}$ and $\vec{\gamma}_{\mathbb{T}}$ determines the coorientation of $\mathring{\Sigma}$ by the geodesic flow. As shown in Figure 4.1, a second intersection would impose the opposite coorientation of $\mathring{\Sigma}$, which is impossible since $\mathring{\Sigma}$ is transverse to the flow. Also $\vec{\gamma}_T \cup \overleftarrow{\gamma}_T$ cuts $\mathbb{T}$ in two connected components, corresponding to vectors going inward and outward $T^1\mathcal{N}$ along $\mathbb{T}$. Thus if such an intersection exists, it cuts δ in two connected components. Since Σ is compact, δ has an end in each connected component of $\mathbb{T} \setminus (\vec{\gamma}_T \cup \overleftarrow{\gamma}_T)$. $\square$

Essential and alternating intersections We study the primitive orbits, that is orbits that are not the negative boundary of some transverse cobordism. So we are interested in the negative boundaries of the transverse cobordisms of the geodesic flow. Specifically we want to prove these cannot be lifts of simple closed geodesics. Thus it is necessary to relate the self-intersections of some geodesic with the cobordisms its lifts bound.

Definition 4.3.3. Let Σ be a partial section and γ an oriented closed geodesic, $\mathcal{N}$ be an immersed tubular neighborhood of γ such that $\Sigma_{\mathcal{N}}$ is transverse to $\partial(T^1\mathcal{N})$, and $\mathbb{T}$ be a connected component of $\partial(T^1\mathcal{N})$. Let δ be a component of $\Sigma_{\mathcal{N}} \cap \mathbb{T}$ that is a segment. We say that δ is an **essential intersection** on $\mathcal{N}$ if it intersects once $\vec{\gamma}_T \cup \overleftarrow{\gamma}_T$ and an **alternating intersection** if it does not intersect $\vec{\gamma}_T \cup \overleftarrow{\gamma}_T$.

Let Γ be a finite set of closed geodesics, and $\mathcal{N}$ an immersed tubular neighborhood of γ that is small enough, in the sense that no orbit arc of Γ enters $\mathcal{N}$ and leaves it from the same side without intersecting γ. Then any essential or alternating intersection has two ends whose orbits inside $T^1\mathcal{N}$ intersect $T^1_\gamma\mathcal{N}$ exactly once. Also notice that for a transverse cobordism Σ, any intersection of two boundary arcs of Σ induces two essential or two alternating intersections.

Lemma 4.3.4. *Let Σ be a partial section, γ be an oriented closed geodesic and $\mathcal{N}$ be an immersed tubular neighborhood of γ. Let δ be an essential intersection of Σ on $\mathcal{N}$, and denote by e_1 and e_2 its ends. Then the orbits of the geodesic flow going through the points e_1 and e_2 are negative boundaries of Σ.*

Let δ be an alternating intersection of Σ on $\mathcal{N}$, and denote by e_1 and e_2 its ends. Then one of the orbits of e_1 and e_2 is a positive boundary of Σ, the other is a negative boundary of Σ.

Proof. The proof is illustrated in Figure 4.1. Denote by $\mathcal{U}$ an immersed tubular neighborhood of δ in $\Sigma_{\mathcal{N}}$. The intersection $\delta \cap (\vec{\gamma}_{\mathbb{T}} \cup \vec{\gamma}_{\mathbb{T}})$ determines the coorientation of $\mathcal{U}$ by the geodesic flow. Thus it determines the signs of the orbits of e_1 and e_2 as boundary components of Σ. Also the direction of the flow on e_1 and e_2 is given by their relative position with respect to the curves $\vec{\gamma}_T$ (that delimit where the flow goes inward and outward $\mathcal{N}$). As shown in the figure, the two orientations on these orbits disagree. The second statement is proved in a similar way. $\square$

We have shown how essential and alternating intersections are connected to the boundary of Σ. The next statement gives an existence criterion.

Lemma 4.3.5. *Let Σ be a partial section of the geodesic flow, $\vec{\gamma}$ be a boundary component of Σ, and $\mathcal{N}$ be an immersed tubular neighborhood of γ that we suppose small enough. Then there are at least $|\operatorname{lk}_{\vec{\gamma}}(\Sigma)|$ essential intersections of Σ on $\mathcal{N}$.*

Proof. Denote $\Sigma_{\mathcal{N}}$ the boundary component of $(T^1p)^{-1}(\Sigma)$ containing $\vec{\gamma}$. If $\mathcal{N}$ is small enough then $\partial\Sigma$ intersects $T^1\mathcal{N}$ into orbit arcs, each of them intersecting exactly once $T^1_\gamma\mathcal{N}$. The boundary of $\Sigma_{\mathcal{N}}$ is made of curves in $\partial(T^1\mathcal{N})$ and of curves in the interior of $T^1\mathcal{N}$ that project into boundaries of Σ. By what precedes, each curve of the second type intersects $T^1_\gamma\mathcal{N}$.

Let $\mathbb{T}$ be a connected component of $\partial(T^1\mathcal{N})$ and denote by l^s the connected component of $\mathcal{F}^s(\gamma)\cap T^1\mathcal{N}$ containing $\vec{\gamma}$. Up to taking a smaller and closed tubular neighborhood than $\mathcal{N}$, we can suppose that $\vec{\gamma}_{\mathbb{T}}$ is closed to the curve $\mathbb{T}\cap l^s$, but disjoint from it. Then we denote by $V\subset\mathbb{T}$ the small region between $\vec{\gamma}_T$ and $\mathbb{T}\cap l^s$. For every $x\in V$, the arc of orbit through x enters and leaves $T^1\mathcal{N}$ without intersecting $T^1_\gamma\mathcal{N}$. Thus no boundary of $\Sigma_{\mathcal{N}}$ intersects V.

By definition of the linking number we have $|\operatorname{lk}(\Sigma)| = |(\mathbb{T}\cap l^s)\cap\Sigma_{\mathcal{N}}|$ and we can isotope Σ so that every intersection in $(\mathbb{T}\cap l^s)\cap\Sigma_{\mathcal{N}}$ has the same algebraic sign. So there are at least $|\operatorname{lk}(\Sigma)|$ curves in $\Sigma_{\mathcal{N}}\cap\mathbb{T}$ that intersect $\mathbb{T}\cap l^s$ only once. But these curves cannot have an end in V, so they must intersect $\vec{\gamma}_{\mathbb{T}}$, hence they are essential intersections. $\square$

4.3.2 Characterization of primitive orbits

In this subsection, we use the results of the previous subsection for proving Theorem J.

Lemma 4.3.6. *Let $\gamma\subset S$ be an oriented closed geodesic and $\Sigma\subset T^1S$ be a transverse cobordism of the geodesic flow, so that $\partial^-\Sigma=\vec{\gamma}$ with any multiplicity in $\mathbb{N}_{>0}$. Then γ is not simple.*

Proof. Suppose first that Σ is an annulus. Then Σ has two abstract boundary components $n\vec{\gamma}$ and $m\vec{\delta}$, with δ an oriented closed geodesic and $n,m\geq 1$. Then the image $\pi(\Sigma)\subset S$ describe a free homotopy between $n\gamma$ and $m\delta$. Since S is a hyperbolic surface, each non-trivial class of freely homotopic curve admits a unique oriented closed geodesic. So γ and δ induce the same non-oriented closed geodesic. Since Σ is a cobordism, $\partial^-\Sigma\cap\partial^+\Sigma=\emptyset$. Additionally $\partial^-\Sigma=\vec{\gamma}$, so either $\vec{\gamma}=\vec{\delta}$ and Σ has two negative boundary components immersed in $\vec{\gamma}$, or γ and δ have opposite orientations and $\partial^+\Sigma = m\vec{\delta}=m\overleftarrow{\gamma}$.

In the first case, Σ is a free homotopy between $n\vec{\gamma}$ and $-m\vec{\gamma}$. Then $n\gamma$ and $m\gamma$ traveled backward are homotopic inside S, and $(n+m)\gamma$ is homotopically trivial. It is impossible since S is a hyperbolic surface. The second case is impossible for the same reason. Hence Σ is not an annulus.

Since Σ is orientable, is not an annulus, and has a non-empty boundary, one has $\chi(\Sigma)\neq 0$. But $\mathring{\Sigma}$ is transverse to the geodesic flow, so $\Sigma\cap\mathcal{F}^s$ is regular on $\mathring{\Sigma}$. Thus by Poincaré-Hopf Theorem, the foliation $\Sigma\cap\mathcal{F}^s$ on Σ has a singularity on at least one of its boundary components, and Σ has non-zero linking number along this orbit.

Denote by $\mathcal{N}$ an immersed tubular neighborhood of γ, supposed to be close enough to γ. Suppose that Σ has a non-zero linking number along $\vec{\gamma}$. By Lemma 4.3.5, there exists an essential intersection of Σ on $\mathcal{N}$. Let e be one end of this essential intersection, and denote by $\vec{\delta}$ the orbit of e, where γ is an oriented closed geodesic. Then δ intersects transversely γ and, by Lemma 4.3.4, $\vec{\delta}$ is a negative boundary of Σ. But $\partial^-\Sigma = n\vec{\gamma}$, so $\delta = \gamma$ and γ admits self-intersection points.

Suppose that Σ has non zero linking number along a positive boundary δ. According to Lemma 4.3.5, there exists an essential intersection of Σ on an immersed tubular neighborhood of δ. By Lemma 4.3.4 and since Σ has only one negative boundary, γ intersects δ. Then the curve δ induces on γ either an alternating or an essential intersection. Using Lemma 4.3.4 and the previous argument, γ intersects itself.

Thus, in both cases, γ is not simple. $\square$

Proof of Theorem J. Let γ be an oriented closed geodesic of S so that $\vec{\gamma}$ is primitive. If γ was not simple, by Lemma 1.5.10, there would exist a transverse cobordism from $\vec{\gamma}$ to two other orbits. But $\vec{\gamma}$ is primitive, so γ is simple.

Let γ be an oriented simple closed geodesic. By the previous lemma, $\vec{\gamma}$ is stably primitive. Furthermore by definition, stably primitive orbits are primitive, which finishes to prove the equivalence.

To prove that the set of primitive orbit is spanning, one can take an oriented closed geodesic γ, and use the transverse cobordism from Lemma 1.5.10 to successively desingularise γ, and construct a transverse cobordism from $\vec{\gamma}$ to some primitive orbits. The process terminates since every desingularisation of a geodesic decreases the number of self-intersection points. $\square$

Remark 4.3.7. On a non-orientable hyperbolic surface and on a hyperbolic orbifold, some desingularisations of oriented geodesics do not induce a partial sections as discuss in Section 1.5.2. In particular, Theorem J is not true for hyperbolic orbifolds.

4.4 Stably primitive orbits of skewed R-covered Anosov flow.

In this section, we are interested in stably primitive set of orbits of a skewed $\mathbb{R}$-covered Anosov flow. We see that these orbits play an important role for changing the nature of a flow using Fried-Goodman surgeries, as stated in the following theorem. The main idea is to use the Birkhoff annuli whose trace in the orbit space are ideal lozenges. We are particularly interested in the **simple sets of ideal lozenges**, which are sets of ideal lozenges $\{L_i | i \in I\}$ so that there are no indices $i, j \in I$, and $g \in \pi_1(M)$ such that g sends one corner o of L_i to a point $g.o$ lying inside the interior $\mathring{L}_j$. Recall that for a positively twisted flow, any point in the orbit space is the corner of exactly two ideal lozenges, in its $(+,+)$- and $(-,-)$-quadrants.

Theorem K. *Let (M, ϕ) be a positively skewed $\mathbb{R}$-covered Anosov flow with orientable stable and unstable foliations, and et Γ be a set of closed orbits of ϕ, such that there are no ideal lozenges inside the orbit space $\mathcal{O}(M)$ whose two corners are induced by two orbits inside Γ. Then the following implications hold:*

$\Leftrightarrow$ Γ *is a stably primitive set of orbits,*

$\Leftrightarrow$ $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ *is a simple set of ideal lozenges,*

$\Leftrightarrow$ $\{L^{-,-}(\gamma) | \gamma \in \Gamma\}$ *is a simple set of ideal lozenges.*

$\Rightarrow$ *Any finite sequence of Fried-Goodman surgeries along orbits of Γ produces a positive skewed $\mathbb{R}$-covered flow.*

Furthermore, the last implication is an equivalence when $|\Gamma| = 1$.

The equivalence between the second and third point is in fact elementary. Indeed suppose that $g.\rho(\gamma)$ lies in the interior of $L^{+,+}(\rho(\delta))$ for some $g \in \pi(M)$. Since the stable and unstable foliations are orientable, one has $\rho(\gamma) \in g^{-1}.L^{+,+}(\rho(\delta)) = L^{+,+}(g^{-1}.\rho(\delta))$, so $g^{-1}.\rho(\delta)$ lies in the interior of $L^{-,-}(\rho(\gamma))$.

The proof of Theorem K is split in Proposition 4.4.2 and in Lemmas 4.4.4, 4.4.5 and 4.4.6 – one for each implication.

4.4.1 Nature of a flow after Fried-Goodman surgeries

If an Anosov flow is positively skewed $\mathbb{R}$-covered, we want to know whether we can remove this property by doing Fried-Goodman surgeries on some specific orbit. We give a partial answer to this question, a complete answer being yet to find. We recall that Theorem G gives a way to express the nature of the flow using oriented Birkhoff sections. Also in Section 1.6, we have detailed how a Fried-Goodman surgery affects a Birkhoff section and its boundary.

Lemma 4.4.1. *Suppose that (M, ϕ) is a positive skewed $\mathbb{R}$-covered transitive Anosov flow. Let Γ be a set of closed orbits. There exists a sequence of Fried surgeries along orbits of Γ that change the nature of the flow if and only if there is a transverse cobordism Σ with $\partial^+\Sigma \subset \Gamma$, and non-zero linking numbers along all orbits in $\partial^+\Sigma$.*

The condition on Γ in this lemma is similar to being stably primitive, but with an additional linking number condition. To work around this condition, we use several partial sections defined in Section 1.7.3 whose linking numbers are known.

Proof. Denote by $\gamma_1, \cdots, \gamma_n$ some elements of Γ and by ϕ' the flow obtained by some Fried-Goodman surgeries along $\gamma_1, \cdots, \gamma_n$. According to Corollary 3.1.3, we can suppose all slopes to be positive. Denote by γ' the orbit of ϕ' corresponding to an orbit γ.

Suppose that ϕ' is not positively twisted. According to Theorem G, ϕ' admits a partial section Σ' which is negative or without boundary. Then the corresponding partial section Σ in M is obtained by Fried surgeries with negative slopes along the orbits γ_i. Since ϕ is positively skewed $\mathbb{R}$-covered, Σ must have at least one positive boundary component. If a Fried surgery along γ_i changes the orientation of an orbit δ in $\partial\Sigma'$, then $\delta = \gamma_i$ and $\mathrm{lk}_\delta(\Sigma) \neq 0$. Thus $\partial^+\Sigma$ is a union of some γ_i, with non-zero linking numbers.

Conversely assume that such a partial section Σ exists. According to Lemma 1.6.7, any surgery with large enough positive slope along an orbit δ in $\partial^+\Sigma$ changes the sign of δ in $\partial\Sigma'$. So there is a sequence of Fried surgeries along some orbits of Γ that changes the nature of ϕ. ∎

This lemma is general, but not convenient unless we have a good understanding of the set of transverse cobordisms. In the remaining of the section, we will restrict our study to some sets of orbits Γ which we know to bound good partial sections.

Proposition 4.4.2. *Suppose that (M, ϕ) is a skewed $\mathbb{R}$-covered Anosov flow. Let $\{L_i | i \in I\}$ be a simple set of ideal lozenges bounded by closed orbits. Then any Fried-Goodman surgeries along the corners of the ideal lozenges L_i produces a skewed $\mathbb{R}$-covered Anosov flow with the same orientation.*

This proposition is a small generalization of a theorem [BI20, Corollary 4.1] of C.Bonatti and I.Iakovoglou, which requires the simplicity of more lozenges. Then in Theorem 1, C.Bonatti and I.Iakovoglou apply this result for the geodesic flow on a closed hyperbolic surfaces S (which is negatively skewed $\mathbb{R}$-covered for the orientation we chose). Take some closed geodesics $\gamma_1, \dots \gamma_n$ that are simple and pairwise disjoint, and L_i an ideal lozenges with corner $\rho(\overrightarrow{\gamma_i}) \cup \rho(\overleftarrow{\gamma_i})$. The set of $\{L_1, \dots L_n\}$ is simple so any sequence of Fried-Goodman surgeries along the orbits $\rho(\overrightarrow{\gamma_i}), \rho(\overleftarrow{\gamma_i})$ induces a negatively skewed $\mathbb{R}$-covered Anosov flow.

Additionally suppose that S is not orientable. We take an oriented closed geodesic γ with an orientable tubular neighborhood and a coorientation η. Suppose that at each self-intersection point of γ, the transverse orientation given by γ disagree with the coorientation η. Then the immersed Birkhoff annulus Σ_η (given in Section 1.5.1) has two negative boundary components $\overrightarrow{\gamma} \cup \overleftarrow{\gamma}$, is embedded in a neighborhood of $\overrightarrow{\gamma}$, and has linking number zero along $\overrightarrow{\gamma}$. Then, by using the technique developed in the proof below, any Fried-Goodman surgery along $\overrightarrow{\gamma}$ induces a negatively skewed $\mathbb{R}$-covered Anosov flow.

In Proposition 4.4.7, we give an example of two orbits of the geodesic flow for which all Fried-Goodman surgeries induce negatively skewed $\mathbb{R}$-covered Anosov flow.

Proof of Proposition 4.4.2. Suppose that ϕ is a positive skewed $\mathbb{R}$-covered flow, and denote by Γ the set of orbits that bound an ideal lozenge L_i for some $i \in I$. For $X \subset M$, we denote by X' the induced set after any Fried surgery. Take Σ a Birkhoff section embedded in its interior, and $\delta \subset \partial\Sigma$. According to Lemma 1.6.7 we can control the orientation of δ' as boundary of Σ' after the surgery, using the slope of the surgery and with the behavior of Σ on a small neighborhood of δ. We fix $\gamma_1, \dots, \gamma_n \subset \Gamma$ and some finite slopes on the curves γ_i, and consider the Fried-Goodman surgeries with these slopes. We will find a positive Birkhoff section whose slopes along $\gamma_1, \dots, \gamma_n$

are higher than the slopes of the surgeries, so that after the surgeries it induces another positive Birkhoff section. By Theorem G the induced flow will be positively skewed $\mathbb{R}$-covered.

Also, by Theorem G there exists Σ a positive Birkhoff section. We fix a family of ideal lozenges $L_1, \dots, L_p \subset \{L_i | i \in I\}$ so that every orbit γ_i is the boundary of one of these ideal lozenges. According to Theorem 1.7.15, for every $1 \leq i \leq p$, there exists an immersed Birkhoff annulus A_i in M whose trace in the orbit space $\mathcal{O}(M)$ is L_i. Since the flow is positively skewed, A_i has two positive boundary components and $\partial^- A = \emptyset$. Since L_i is simple, A_i is embedded on a neighborhood of its boundary. So we can desingularise A_i into a partial transverse section B_i with two positive boundary components and with linking number zero along these components. Also by hypothesis, for every $i \neq j$, we have $\partial B_i \cap \mathring{B}_j = \emptyset$. Hence, for large $k_1, \dots, k_p \in \mathbb{N}$, the Fried-desingularisation of $\Sigma \cup k_1 B_1 \cup \dots \cup k_p B_p$ is positive, and it has large multiplicities and fixed linking numbers along the orbits γ_i. Hence its slopes along the orbits γ_i can be taken higher that the slopes of the surgeries. Then after these Fried-Goodman surgeries, there exists a positive Birkhoff section and the flow is positively skewed $\mathbb{R}$-covered. $\square$

For proving the converse implication, we need more work. We state a converse for the case where $|\Gamma| = 1$ in the next section (Lemma 4.4.6) and give a counter-example in the general case.

4.4.2 Stably primitive orbits and simple ideal lozenges

In this section, we detail the equivalence between stably primitive sets of orbits and simple sets of ideal lozenges.

Lemma 4.4.3. *Let (M, ϕ) be a positively skewed $\mathbb{R}$-covered Anosov flow with orientable stable and unstable foliations, and let Γ be a stably primitive set of closed orbits. Then $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ and $\{L^{-,-}(\gamma) | \gamma \in \Gamma\}$ are simple sets of ideal lozenges.*

Proof. We prove the contraposition. Suppose that $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ is not simple. Then there exists two closed orbits $\gamma, \delta \in \Gamma$, a corner o of the ideal lozenge $L^{+,+}(\gamma)$ and $g \in \pi_1(M)$ such that $g.o$ lies inside the interior of $L^{+,+}(\delta)$. One has $L^{+,+}(g.\gamma) = g.L^{+,+}(\gamma)$ since the stable and unstable foliations of ϕ are orientable. We can suppose $o = \rho(\gamma)$, for otherwise $\rho(\delta)$ lies in the interior of $L^{+,+}(g.\gamma) = g.L^{+,+}(\gamma)$, so that $g^{-1}.\rho(\delta)$ lies in the interior of $L^{+,+}(\gamma)$, and we can exchange γ and δ. According to Lemma 1.7.17 there exists a transverse cobordism Σ with boundary $\partial^+\Sigma = \gamma \cup \delta$, so Γ is not a stably primitive set. $\square$

The converse implication needs more development. First we prove that under some hypothesis on a partial section Σ, we can substrat an immersed Birkhoff annulus to Σ.

Lemma 4.4.4. *Let (M, ϕ) be a positive skewed $\mathbb{R}$-covered Anosov flow. Let Σ be a transverse cobordism in M, $\gamma \subset \partial^+\Sigma$ be a closed orbit, and $L \subset \mathcal{O}(M)$ be an ideal lozenge with one corner at $\rho(\gamma)$, so that $\mathring{L} \cap \partial^+\Sigma = \emptyset$. We suppose that $\mathrm{lk}_\gamma(\Sigma) = 0$, and locally around $\rho(\gamma)$, L is in the same quadrant than the trace $\Theta_{M\setminus\partial\Sigma}(\Sigma)$. Then either Σ is a Birkhoff annulus with trace L or there is an Birkhoff annulus A with trace L and an immersed partial section Σ' so that Σ is the Fried desingularisation of $A \cup \Sigma'$.*

When $\mathrm{lk}_\gamma(\Sigma) = 0$, in a neighborhood of $\rho(\gamma)$, $\Theta_{M\setminus\partial\Sigma}(\Sigma)$ covers one of the two quadrants corresponding to $L^{+,+}(\gamma)$ and $L^{-,-}(\gamma)$. In this case we take L to be the corresponding ideal lozenge.

Proof. According to Theorem 1.7.15, there exists an immersed Birkhoff annulus A whose trace in $\mathcal{O}(M)$ is L. We define $\Gamma = \partial\Sigma \cup \partial A$. The general idea of the proof is to show that $\partial A \cap \mathring{\Sigma} = \emptyset = \partial\Sigma \cap \mathring{A}$, so that L lifts to an ideal lozenge in $\mathcal{O}(M \setminus \Gamma)$ (that we also denote by L). Then we prove that the trace of Σ in $\mathcal{O}(M \setminus \Gamma)$ contains L. Finally we use this trace to construct a copy of A inside Σ.

Denote the orbits bounding A by γ and δ, which may be equal. According to Lemma 3.1.1, we have $(\partial A \cdot\!\!\cap \mathring{\Sigma}) - (\partial\Sigma \cdot\!\!\cap \mathring{A}) = \mathrm{lk}^+(\Sigma, A) - \mathrm{lk}^-(\Sigma, A)$. We need to control each term to prove that $\partial\Sigma \cap \mathring{A} = \emptyset$.

We have $\partial A \cdot\!\!\cap \mathring{\Sigma} \geq 0$ since A has only positive boundaries, and $\partial\Sigma \cdot\!\!\cap \mathring{A} \leq 0$ since $\partial^+\Sigma$ does not intersect $\mathring{L}$. Hence $(\partial A \cdot\!\!\cap \mathring{\Sigma}) - (\partial\Sigma \cdot\!\!\cap \mathring{A}) \geq 0$.

The common boundary between Σ and A is γ plus possibly δ. By construction of the immersed Birkhoff annulus A (see Section 1.7.3), we have $\mathrm{lk}_\gamma(A) = 0$. By hypothesis, we have $\mathrm{lk}_\gamma(\Sigma) = 0$, and so $\mathrm{lk}_\gamma(\Sigma, A) = 0$. Also if $\delta \in \partial^+\Sigma$, then $\mathrm{lk}_\delta(\Sigma) - \mathrm{lk}_\delta(A) \leq 0$ so $\mathrm{lk}_\delta(\Sigma, A) \leq 0$. Since A and Σ have the same oriented boundary only on γ and potentially on δ, we have $\mathrm{lk}^+(\Sigma, A) \leq 0$. If $\delta \in \partial^-\Sigma$, then $\mathrm{lk}_\delta(\Sigma) - \mathrm{lk}_\delta(A) \geq 0$ so $\mathrm{lk}_\delta(\Sigma, A) \geq 0$. Hence in all cases we have $\mathrm{lk}^-(\Sigma, A) \geq 0$ and $\mathrm{lk}^+(\Sigma, A) - \mathrm{lk}^-(\Sigma, A) \leq 0$, but the latter is non negative by what precedes. Hence we have $\partial A \cdot\!\!\!\cap \mathring{\Sigma} = \partial\Sigma \cdot\!\!\!\cap \mathring{A} = 0 = \mathrm{lk}^+(\Sigma, A) = \mathrm{lk}^-(\Sigma, A)$, and $\partial A \cap \mathring{\Sigma} = \emptyset = \partial\Sigma \cap \mathring{A}$.

Since $\mathring{L} \cap \partial^+\Sigma = \emptyset$, γ belongs to $\partial^+\Sigma$ and $\rho(\gamma)$ is a corner of L, the orbit $\pi_1(M).\rho(\delta)$ does not intersect the interior L. So $\Gamma \cap \mathring{L} = \emptyset$, and L lifts to an ideal lozenge in $\mathcal{O}(M \setminus \Gamma)$, still denoted by L. We will prove that the trace of Σ in $\mathcal{O}(M\setminus\Gamma)$ contains $\mathring{L}$. Notice that since $\gamma \subset \partial^+\Sigma$ and $\mathrm{lk}_\gamma(\Sigma) = 0$, the trace of Σ in $\mathcal{O}(M \setminus \Gamma)$ contains the two half stable and unstable leaves that bound L and contains $\rho(\gamma)$. Denote by $l^s, l^u \subset \partial L$ these two half leaves.

If the trace of Σ does not contain L, then this trace must have a boundary component intersecting $\mathring{L}$. Either it is a stable leaf, an unstable leaf, or two half leaves based on a closed orbit. In the first two cases, the stable or unstable leaf would intersect either l^s or l^u. This is impossible, since the boundary of the trace of Σ contains the whole half leaves. In the third case, there exist two half leaves based on a point $\rho(\nu)$ for a closed orbit ν, which for the same reason must leave the ideal lozenge L by the two half leaves in $\partial L \setminus (l^s \cup l^u)$. Then ν is be a boundary of Σ inside $\mathring{L}$, which is impossible by hypothesis. Hence only the last case is possible: the trace of Σ in $\mathcal{O}(M\setminus\Gamma)$ contains L.

We can use the two previous results to construct a copy of A in Σ. Let γ^s be a copy of γ slightly pushed (or 2γ if the leaf $\mathcal{F}^s(\gamma)$ is not orientable) inside $M \setminus \Gamma$ along the stable foliation. We denote by $\tilde{g}$ its homotopy class in $\pi_1(M \setminus \Gamma)$, by a connected lift of Σ in $\tilde{M}$ and by $\pi(\tilde{\Sigma})$ its projection inside $\mathcal{O}(M \setminus \Gamma)$. Then we have $\tilde{g}.L = L$ inside $\mathcal{O}(M \setminus \Gamma)$.

Let $a \subset \mathring{L}$ be an open curve connecting the two corners of L and transverse to $\mathcal{L}^s$ and $\mathcal{L}^u$. By construction, $\tilde{g}$ respectively contracts and expands the unstable and stable leaves inside L, so for all $n \in \mathbb{Z}$ we have $\tilde{g}^n.a \cap a = \emptyset$.

We lift $a \subset \Theta_{M\setminus\partial\Sigma}(\Sigma)$ to a curve $\tilde{\alpha} \subset \tilde{\Sigma} \subset \tilde{M}$, and project it to the curve $\alpha \subset \Sigma$. We claim that $a \to \alpha$ is injective, so that α is an open simple curve in Σ. Indeed suppose that $a \to \alpha$ is not injective, so that there exists $h \in \iota(\pi_1(\mathring{\Sigma})) \subset \pi_1(M \setminus \partial\Sigma)$ such that $h \neq 1$ and $h.a \cap a \neq \emptyset$. We take $x \in h.a \cap a$ as in Figure 4.2. Then there is in $h.a$ an arc c bounded by x and $h.\rho(\gamma)$. But L is a simple ideal lozenge, so either $h.\rho(\gamma) = \rho(\gamma)$, or c intersects $l^u \cup l^s$. The first case is impossible since it would implies that $h = \tilde{g}^n$ for some $n \neq 0$, and $\tilde{g}^n.a \cap a = \emptyset$. But $l^u \cup l^s \subset \partial(\Theta_{M\setminus\Gamma}(\Sigma))$, and c must lift to a curve in $\tilde{\Sigma}$, which is impossible. Hence α is a simple curve in Σ.

Since $\gamma \subset \partial\Sigma$, we can choose a so that α admits an end in γ. Also Σ

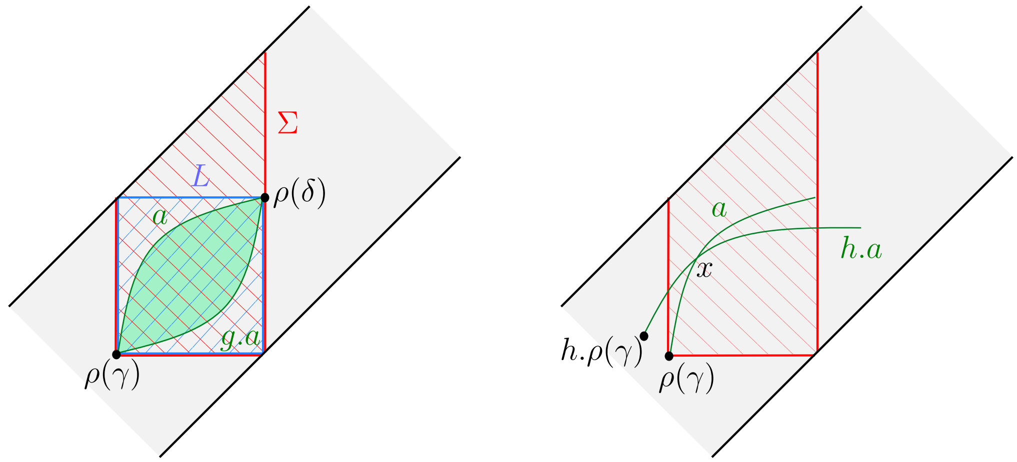


Figure 4.2: Trace of Σ and A in $\mathcal{O}(M)$, with a fundamental domain of A in green.

has linking number zero along γ, so γ^s is homotopic inside $M \setminus \Gamma$ to a closed curve lying inside Σ. Hence $\tilde{g}.\tilde{\Sigma} = \tilde{\Sigma}$ and $\tilde{g}.\tilde{\alpha}$ is another curve in $\tilde{\Sigma}$ with one end in a lift of γ.

We need to consider the other end of a. There are two cases: $\delta \subset \partial\Sigma$ or $\delta \not\subset \partial\Sigma$. In the second case, we have $\rho(\delta) \subset \partial\Theta_{M\setminus\Gamma}(\Sigma)$ since $\partial A \cap \mathring{\Sigma} = \emptyset$. Then $\mathcal{L}^s(\rho(\delta))$ or $\mathcal{L}^u(\rho(\delta))$ is a boundary component of $\Theta_{M\setminus\Gamma}(\Sigma)$. Thus according to Remark 1.7.2, we can cut Σ along a closed leaf of $\mathcal{F}^s \cap \Sigma$ or $\mathcal{F}^u \cap \Sigma$ and isotope the surface to obtain a new immersed partial section Σ' embedded in its interior, and with δ as negative boundary. Also the Fried-desingularisation of Σ' is Σ. Thus in all cases, we can consider an immersed partial section Σ' that either is equal to Σ or that we can desingularise into Σ, and with $\delta \subset \partial\Sigma'$. Then we can take a and α on Σ' so that α admits a second end in δ.

Thus $a \cup \tilde{g}.a$ bounds in L a 2-gon b, which lifts to a 4-gon $\tilde{\beta} \subset \tilde{\Sigma}$. Two opposite edges of $\tilde{\beta}$ embed in a and $\tilde{g}.a$, and the two other edges are orbit arcs which project to the ends of a. Furthermore $\tilde{\beta}$ projects to an immersed scare $\beta \subset \tilde{\Sigma}$, embedded in its interior, so that $\rho(\beta) = b$. The two opposite sides of β that correspond to a and $\tilde{g}.a$ are glued along α. Thus β is a connected component of Σ', and β is a Birkhoff annulus with trace L. If β is not all the surface Σ', then its complement is an immersed partial section Σ''.

And by construction Σ is the Fried desingularisation of $\beta \cup \Sigma''$. $\square$

Lemma 4.4.5. *Let (M, ϕ) be a positively skewed $\mathbb{R}$-covered Anosov flow and Γ be a set of closed orbits such that there are no ideal lozenges inside the orbit space $O(M)$ whose two corners are induced by two orbits inside Γ. If $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ and $\{L^{-,-}(\gamma) | \gamma \in \Gamma\}$ are simple sets of ideal lozenges, then Γ is a stably primitive set of orbits.*

Proof. Suppose that Γ is not a stably primitive set of orbits. Then there exists a cobordism Σ with $\partial^+\Sigma \subset \Gamma$. We would like to have non-zero linking numbers along all the negative boundary components of Σ. By Lemma 4.4.1, some Fried-Goodman surgeries on Γ would produce an Anosov flow which is not negatively skewed $\mathbb{R}$-covered. Since $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ is a simple set of ideal lozenges, it would contradict Lemma 4.4.2.

To obtain non-zero linking numbers along all the negative boundary components of Σ, we suppose that there is such a boundary component, and we erase this boundary component from Σ. Let γ be a negative boundary of Σ with linking number zero. Let L be the ideal lozenge $L^{+,+}$ or $L^{-,-}$ which is in the same quadrant than the trace $\Theta_{M \setminus \Gamma}(\Sigma)$ along $\rho(\gamma)$. Since $\{L^{+,+}(\gamma) | \gamma \in \Gamma\}$ and $\{L^{-,-}(\gamma) | \gamma \in \Gamma\}$ are simple sets of ideal lozenges, $\partial^+\Sigma \subset \Gamma$ does not intersect the interior of L. According to the hypothesis, Σ is not a Birkhoff annulus with trace L. So, according to Lemma 4.4.4, there is another immersed partial section Σ' so that Σ is a Fried desingularisation of Σ' plus an immersed Birkhoff annulus A with trace L. Hence $\partial^+\Sigma' = \partial^+\Sigma - \partial^+ A \subset \Gamma$, and the multiplicity of $\partial^+\Sigma'$ is less than the multiplicity of $\partial^+\Sigma$. The same property is satisfied by the Fried-desingularisation of Σ'. We can successively apply this claim a finite number of times to eventually erase all negative boundary components with zero linking number from Σ. Then the remark made above finishes the proof. $\square$

The previous theorem shows a key difference between primitive orbits and stably primitive orbits. By Theorem I, there exist primitive orbits on a skewed $\mathbb{R}$-covered Anosov flow, but they are hard do determine. The previous theorem shows that stably primitive orbits are easier to determine, but they may not exist. For example, take a hyperbolic orbifold S with 3 cusps and the topology of a sphere. There is no simple geodesic on S, so there is no stably primitive orbits of its geodesic flow (see Section 4.3 for this implication).

We finally prove the last equivalence in Theorem K, stated in a special case.

Lemma 4.4.6. *Let (M, ϕ) be a positively skewed $\mathbb{R}$-covered Anosov flow with transversely orientable stable and unstable foliations, and take a closed orbit γ which is not stably primitive. Then any Fried-Goodman along γ with high enough slope (which exists) induces an Anosov flow which is not positively skewed $\mathbb{R}$-covered.*

Proof of Lemma 4.4.6. According to Lemma 4.4.5, $L^{+,+}(\rho(\gamma))$ is not simple. So there exists g in $\pi_1(M)$ such that $g.\rho(\gamma)$ is in the interior of $L^{+,+}(\rho(\gamma))$. Also the stable and unstable foliations are transversely orientable, so g preserves the orientations of $\mathcal{L}^s$ and $\mathcal{L}^u$. According to Lemma 1.7.18 there is an immersed Fried section Σ with $\partial^+\Sigma = \gamma$ (not counted with multiplicity) and with non-zero linking number. Lemma 4.4.1 then implies that there exists a Fried-Goodman surgery on γ that induces an Anosov flow that is not positively skewed $\mathbb{R}$-covered. $\square$

Counter-example for more that one orbit Let S be an oriented hyperbolic surface and take γ_1 and γ_2 two simple geodesics that intersect only once, as in Figure 4.3. We consider the geodesic flow on T^1S and its periodic orbits $\vec{\gamma_1}$ and $\vec{\gamma_2}$. The geodesic flow on T^1S is negatively skewed $\mathbb{R}$-covered, so we use the variant of previous results that reverses the orientations. We refer to Section 1.5.2 for the construction of partial sections on T^1S, and to Section 4.3 for the definition of immersed tubular neighborhoods.

Proposition 4.4.7. *For any transverse cobordism Σ so that $\partial^-\Sigma \subset \gamma_1 \cup \gamma_2$ (with any multiplicity), we have $\mathrm{lk}_{\gamma_i}(\Sigma) = 0$ for $i = 1$ and 2. Hence no Fried-Goodman surgery on $\vec{\gamma_1}$ and $\vec{\gamma_2}$ changes the nature of the flow.*

Proof. We give a proof using an immersed tubular neighborhood. Alternatively one can prove the claim with the technique developed in the proof of Proposition 4.4.2.

Let Σ be a transverse cobordism with $\partial^-\Sigma \subset \gamma_1 \cup \gamma_2$. If $\mathrm{lk}_{\gamma_1}(\Sigma) \neq 0$, by Lemma 4.3.5, there is an essential intersection of Σ along γ_1. By Lemma 4.3.4, the ends of the essential intersection are on some negative boundary components of Σ, that is γ_2 since γ_1 is simple. Additionally these boundary components project to oriented geodesic arcs of S, that intersect γ_1 with opposite signs. This contradicts the fact that γ_1 is simple and that there is only one intersection of γ_1 and γ_2. Hence, by Lemma 4.4.1, no Fried-Goodman surgery on $\vec{\gamma_1}$ and $\vec{\gamma_2}$ changes the nature of the flow. $\square$

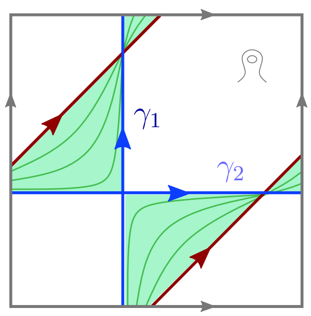


Figure 4.3: Transverse cobordism with linking number zero along its negative boundary.

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