Analytical methods for the study of the two-body problem, and alternative theories of gravitation

François Larrouturou

THÈSE DE DOCTORAT
DE L'UNIVERSITÉ SORBONNE UNIVERSITÉ

Spécialité : Physique

École doctorale nº564: Physique en Île-de-France

réalisée

sous la direction conjointe de Luc BLANCHET & Cédric DEFFAYET

présentée par

François LARROUTUROU

pour obtenir le grade de :

DOCTEUR DE L'UNIVERSITÉ SORBONNE UNIVERSITÉ

Sujet de la thèse :

Méthodes analytiques pour l'étude du problème à deux corps, et des théories alternatives de gravitation

soutenue le 18 juin 2021

devant le jury composé de :

Mme.	Danièle STEER	: Rapporteuse
M.	Clifford M. WILL	: Rapporteur
Mme.	Marie-Christine ANGONIN	: Examinatrice
M.	Éric GOURGOULHON	: Examinateur
M.	Shinji MUKOHYAMA	: Examinateur
M.	Luc BLANCHET	: Directeur de thèse
M.	Cédric DEFFAYET	: Membre invité

Καί με θεὰ πρόφρων ὑπεδέξατο, χεῖρα δὲ χειρί
δεξιτερὴν ἕλεν, ὧδε δ΄ ἔπος φάτο καί με προσηύδα·
ὦ κοῦρ΄ ἀθανάτοισι συνάορος ἡνιόχοισιν,
ἵπποις ταί σε φέρουσιν ἱκάνων ἡμέτερον δῶ
χαῖρ΄, ἐπεὶ οὔτι σε μοῖρα κακὴ προὔπεμπε νέεσθαι
τήνδ΄ ὁδόν ἦ γὰρ ἀπ΄ ἀνθρώπων ἐκτὸς πάτου ἐστίν,
ἀλλὰ θέμις τε δίκη τε. Χρεὼ δέ σε πάντα πυθέσθαι
ἠμὲν Ἀληθείης εὐκυκλέος ἀτρεμὲς ἦτορ
ἠδὲ βροτῶν δόξας, ταῖς οὐκ ἔνι πίστις ἀληθής.

Et la déesse en toute bienveillance m'accueillit, elle prit ma main droite
dans sa main, elle proféra ces paroles en s'adressant à moi :
Jeune homme, compagnon d'immortels cochers,
qui grâce aux juments qui te portent parviens à notre demeure,
bienvenue, car ce n'est pas un mauvais destin qui t'a conduit à prendre
cette voie, si loin des hommes qu'elle soit à l'écart du sentier battu,
c'est la règle, la justice. Il faut que tu sois instruit de tout,
et du cœur sans tremblement de la vérité bien persuasive,
et de ce qui paraît aux mortels, où n'est pas de croyance vraie.

Parménide,
Sur la nature Frag. I, v. 22–30, trad. B. Cassin.

Acknowledgment

The past three years have been a thrilling and fruitful journey, mostly thanks to all the encounters and discussions they were filled with. All those interactions constituted for me the living proof that doing science is a collaborative process. That is why I would like to thank all the people that made this work possible.

First of foremost, I would like to express my deepest gratitude to both my supervisors, Luc Blanchet and Cédric Deffayet. They were patient and inspiring guides in the learning and comprehension of post-Newtonian and solitonic defects frameworks, but also, and more importantly, in the practical understanding of the scientific method.

Regarding post-Newtonian theory, I am grateful to Guillaume Faye, for his patience in answering my questions, and tireless help with software-related issues; Tanguy Marchand for guiding my first steps with *Mathematica* and practical post-Newtonian computations; Sylvain Marsat for his help and valuable advice; and Quentin Henry, who has been far more a friend than a collaborator during those three years we shared inside and outside IAP. The fruitful and inspiring conversations I had with Gilles Esposito-Farèse, Eugeny Babichev and Sebastian Garcia-Saenz provided strong motivation and support during my discovery of scalar theories, and I would like to thank them warmly. I would also like to thank Stefano Foffa and Riccardo Sturani for introducing me to the EFT language and techniques, and Guillaume Hébrard and Jean-Philippe Beaulieu for showing me the way to the wonderful world of exoplanets.

Beyond those projects carried out with Luc and Cédric, the freedom they gave me allowed me to pursue other collaborations on my own. I would thus like to express all my gratitude to Shinji Mukohyama and Antonio De Felice, that opened me the doors of the fascinating realm of minimal theories, and hosted me for two weeks at Yukawa Institute for Theoretical Physics in Kyôto; and to Michele Oliosi and Özgün Mavuk with whom my discoveries were as much focused on science as on Kyôto and Naoshima lifestyles.

Those three years were also filled with inspiring scientific exchanges, and I would like to thank Laura Bernard, Giulia Cusin, Giulia Isabella, Éric Gourgoulhon, Otto Hannuksela, Rafael Porto for post-Newtonian and post-Minkowskian conversations; Shweta Dalal, Gwenaël Boué, Alain Lecavelier des Étangs and Jordan Philidet for stellar and exoplanetary discussions; Timothy Anson and Lucas Pinol for cosmological ones; and Catherine de Montety for being my guide through the ideas of Aristotle and Occam.

I am also extremely thankful to Danièle Steer and Clifford M. Will for having accepted to referee this dissertation, and to Marie-Christine Angonin, Éric Gourgoulhon and Shinji Mukohyama for being part of the thesis jury.

Those three years would have been much more difficult without the help of the efficient supporting staff of IAP, and I would like to express my gratitude to Valérie Bona, Roselys Rakotomandimby, Isabelle Coursimault, Sandy Artero and Chantal Le Vaillant for their patience and kindness; Carlos Carvalho, Laurent Domisse, Jean Mouette and Lionel Provost for their help with computer and other related issues; Cynthia Tshiela Kuyakula and Pierre Wachel for their unfailing morning smiles and jokes; and Valérie de Lapparent for taking care of the practical issues of PhD students.

One of the strength of IAP is that it is, and managed to remained under the reign of Covid, a place with a real and exciting “PhD vibe”. I would thus like to thank all the other students with whom I shared either coffee or beer, Alexandre, Aline, Amaël, Amaury, Amélie, Arno, Clément, David, Doogesh, Émile, Erwan, Étienne, Florian, Jacopo, Julie, Julien, Lukas, Marko, Martin, Nicolas, Oscar, Pierre, Raphaël, Sandrine, Simon, Valentin and Virginia.

I would like to thank all the teachers and professors that gave me a taste for physics and gravitation, and especially Xavier Ovido, Jack-Michel Cornil and Gilles Esposito-Farèse, who played each a major role in the path that led me to this thesis. I would also like to express my deepest gratitude to my family for their support and help during those years, and notably to my mother for her careful and valuable proofreading of this dissertation, and to Nathalie for her help with the subtelties of the English language.

And to finish *en beauté*, I am very grateful to my *fiancée*, Andréane Bourges, for her infallible encouragements and priceless support.

Contents

Chapter I

General Introduction

I.1 A brief history of gravitation

Before entering the core subject of this thesis, it appeared essential to briefly review the history of theories that were built to explain the gravitational phenomena. Indeed, if this work is firmly anchored in the most up-to-date theory, namely General Relativity, it aims to push it forward, and thereby, to enter a more general motion of building up new ways to understand gravitation. As this motion began roughly 25 centuries ago, it seemed important to have a look on the path that has been taken.

Note that, due to obvious biases from our part, but also to the lack of written records for some civilizations (notably precolumbian ones), we will focus on the "western" (*ie.* circum-Mediterranean) area.

I.1.1 The pre-Aristotelian conceptions

The first compiled observations of stars and planets that reached us were conducted by the Mesopotamian civilizations around the second millennium. The astonishing precision they attained (they localized celestial bodies with an arc-minute accuracy) allowed them to predict Moon's and Sun's eclipses [104]. Nevertheless, they were only "bookkeepers" of the astronomical events: they transcribed observations with great care, but without using them to build up a unified picture of the sky. The first representation of the cosmos by a *théôria* (*ie.* a global vision that explains the world as a whole) was elalorated by the Greeks thinkers of the "Milesian school" (Thales, Anaximander,...) around the sixth century BC. If Thales or Anaximenes imagined a flat Earth, floating on water or air inside a spherical sky, Anaximander argued that the Earth cannot be supported by anything else than itself, and thus has to be spherical. As remarkably described in the work of J.-P. Vernant (see notably [343, 344]), this revolution in the representation of the world has been the corollary of the birth of geometry: the Mesopotamian astronomy was an arithmetic one, when the Greek cosmology was of geometric nature.

Summarizing [344], it is fascinating to note that this fundamental difference can be linked to a not less profound discrepancy in the political organizations. The Mesopotamian system was purely pyramidal, with the King at the top and a complex system of hierarchical relations in the administration. Conversely, the Greeks were organized in cities, where important decisions were taken by an *agora*, *ie.* an assembly of citizen[1] gathered in a circular place, the speaker standing in the center. Moreover the Greek world experienced a crisis in the seventh century, due to an increasing commercial expansion and to the beginning of a monetary economics. The two main outcomes of this crisis were that

[1]Note that this relative equity was also taking place when the city was led by a king. It can be seen explicitly in the second song of the *Odyssey*, where Telemachus, that reigns upon Ithaca, summons the assembly of warriors to discuss important matters [234].

the administration of the cities became a secular matter, and the rise of a political awareness, the Greek citizens taking conscience of themselves as political beings, linked to other citizens[2] by equity relationships. This directly echoes the two main achievements of the Milesian school: demystifying the Nature by throwing away divine attributes, and producing a knowledge accessible to everyone, notably by the use of simple and common pictures. In a nutshell, the sixth century realized the transition from a mythical vision of the world to a geometrical and political global conception of the cosmos. This revolution in the representation of space was a necessary condition, and a prolog to the astonishing intellectual impetus of the fifth century.

Let us nevertheless note that those global representations are cosmologies, and do not mention the sector of gravitation, *ie.* they do not answer the question of why apples fall.

I.1.2 Rise and fall of the Aristotelian models

The first elaborated theory of gravitation *per se* is given by Aristotle's works. In his conception, each object is composed of four elements (earth, water, air and fire), and each of those elements has a *natural location*, to which it aims (up for the fire, and down for the earth, for instance). The relative abundance of those elements within the object will determine its natural location. Moreover, the natural state of any object is to be at rest, within its natural location, and a motion that is not forced by an eternal external action has to dissipate and thus stops at some point: Aristotle's conception of motion ignores inertia. For example the *natural location* of a rock is down, as it is mostly composed of earth. When launched upwards, it experiences a "violent" motion (it is propelled by an external action) and looses its "strength" till it stops and goes down, following its natural motion, propelled "by itself" [29].

This behavior is only valid in our common world. Indeed Aristotle differentiated the world under the Moon, that is composed of the four elements, and the world above the Moon, which is the location of eternal and harmonious celestial bodies, that follow circular motions (the circle being the "perfect" trajectory, as it is the only one that can be followed with an eternally constant speed). This supra-lunar world is filled by a fifth element [30], as Aristotle's Universe cannot be empty [28]. It is interesting to note that Aristotle's Universe is finite and composed of spheres, the outermost one being the one where the stars are fixed, and that the center of the Universe is the center of the (spherical) Earth.

At this point one can notice two main obstacles that stood in the way towards a "modern" theory of gravitation [302]. The first one is that the questions of physics were strongly subordinated to the questions of metaphysics. For instance the discussion on the falling rock takes places in the broader framework of a debate about the *substance* ($o\upsilon\sigma\iota\alpha$), and thus about the *being*. The second obstacle is the fact that motion was considered as the way an object naturally deploys itself, and therefore depending of the essence of the object. Due to the discrepancy in essence between the sub- and supra-lunar worlds, the motion of stars and falling apples were thus thought to be different in nature. No modern vision of physics could emerge from this discrepancy, that forbids to conceptualize an unified, global gravitational law.

Aristotle astronomical conceptions took their final shape in the famous Ptolemy's planetary system. If this system has been questioned during the medieval times, notably by Andalusian thinkers as Averroes [318], it survived until the seminal work of N. Copernicus [128] and the well-known development of modern astronomy.

I.1.3 The Newtonian theory of gravitation

The modern approach of gravitation was initiated by the conceptualization of inertia by G. Galilei and R. Descartes, which designates the resistance of a body to a change in its velocity (it requires

[2] This class of citizens was nevertheless restricted to free men of the city, excluding most of its population.

less energy to stop a fly than a truck, even if they move at the same speed: the truck's inertia is greater than the fly's one). This conceptualization of inertia led I. Newton to formulate his first law, also called *principle of inertia* [293], stating that the "natural" state of an isolated body (*ie.* without external forces applied on it) is to propagate in a rectilinear trajectory with a constant speed. A first consequence is that it always exists a frame of reference in which this body appears to be at rest (such frames are dubbed *inertial* or *Galilean*). This principle is a breakthrough as it implies the relativity of the observer's point of view: being "at rest" is meaningless and thus Aristotle's conception of the natural state became obsolete. A direct corollary is that the laws of gravitation should be independent of the inertial frame in which they are written, and thus can only depend on the acceleration of the body, or on relative quantities with respect to other bodies. This is formalized in Newton's second law (also named *fundamental principle of dynamics*), that states that a body responds to an external force by a change in its *acceleration*, the proportionality coefficient being called "inertial mass". Newton's third law, or *action–reaction principle*, states that if an object exerts a force on a second one, it will suffer in return a force with the same intensity, but reverse direction.

In addition to those three laws of mechanics, and strongly inspired by J. Kepler's and R. Hooke's works, I. Newton exposed the *law of universal gravitation* in its *Principia* [293]. As by the name, this law reconciliates the motion of falling apples with the motion of celestial bodies, by stating that they both obey the same underlying physics: the gravitational force exerted by one body on another one is attractive and proportional to the product of their respective *gravitational masses* divided by the square of their relative distance (the proportionality constant being now called *Newton's constant*). Note that, as expected, this law agrees with the principle of inertia, as it only depends upon relative distances, that are independent from the frame of observation. A major breakthrough, experimentally due to G. Galilei, has been to identify the inertial and gravitational masses, that are conceptually different: the inertial mass measures the resistance to a change in speed, whereas the gravitational mass describes how a body responds to gravitational attraction. This *equivalence principle* (that has been at the heart of the construction of General Relativity) is currently tested to astonishing precision: one part in ten trillions using torsion pendulum on Earth [347], and one part in a quadrillion using the MICROSCOPE instrument in space [336]. Why this principle holds, and thus those two conceptually different masses are identical, remains a mystery of Nature.

With the three laws of mechanics and the law of universal gravitation, I. Newton has laid in 1687 the foundations of the modern physics, that survived in this form until the dawn of the XX$^{\text{th}}$ century. The fame of this "Newtonian gravitation" culminated with the outstanding prediction of the existence of the planet Neptune by U. Le Verrier in 1846 [262]. Applying Newtonian celestial mechanics, he guessed that the small irregularities detected in the trajectory of Uranus were due to the presence of a disturbing body, of which he computed the mass and orbital parameters, indicating to J. Galle where he had to look at to observe the new planet. Newton's physics was so triumphant during the XIX$^{\text{th}}$ century that it ended with A. Michelson stating that "future truths of physical science are to be looked for in the sixth place of decimals" [283], implying that the era of constructing new theoretical frameworks was revolved, and that the only way forward in physics would consist in increasing the accuracy of the measures. Similarly, the XX$^{\text{th}}$ century opened with the famous speech of W. Thomson, stating in substance that the future of physics was bright, excepted for two "dark clouds" [334], that turned out to give birth to Quantum Mechanics and General Relativity, drastically changing our comprehension of Nature. One of the observational problems was the unexplained precession of Mercury's orbit (see chapter II for a detailed discussion on the precession of celestial bodies). Summing up the identified Newtonian causes for such a periastron advance (influence of Venus and Jupiter, non-sphericity of the Sun, *etc*), astronomers did not recover the observed value. Quite naturally, U. Le Verrier tried to reiterate the *tour de force* of the discovery of Neptune, by predicting the existence of a small companion to Mercury, that he baptized Vulcain. But this hypothetical planet has never been found, as the explanation for the abnormal periastron advance was due to a failure of Newtonian physics.

In addition to those concrete problems, Newtonian gravitation (as seen by a XXI$^{\text{th}}$ century physicist) also suffers from conceptual ambiguities, as the gravitational interaction is instantaneous: by moving your hand, you affect each body in the whole Universe, at once. Moreover it lacks a carrier for the information: to state it roughly, how does the apple "knows" where it has to fall ?

I.2 General Relativity as the current theory of gravitation

The spirit of this section is absolutely not to give a comprehensive view of General Relativity, as it has already been done (*eg.* in [285, 348]), and far better done than all what could be tempted in this dissertation. Actually, this section is devoted to highlight some specific features of General Relativity, that will be used in the core of the thesis. The notations that are used, and different technical aspects, are exposed in app. A.

I.2.1 A geometrical description of the gravitational phenomena

General Relativity (GR) depicts the gravitational phenomena as the effects of distortions of the space-time structure, and is thus a theory of geometrical nature. Given a manifold[3] $\mathcal{M}$, with a (at least locally defined) chart of coordinates $\{x^\mu\}$, the *metric* $g_{\mu\nu}$, is defined by the square of the distance between two infinitesimally nearby points: $\mathrm{d}s^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$, which is nothing but the generalization of Pythagoras theorem to non-flat manifolds. This metric encodes the whole local geometrical structure, and can be thought as a tensorial field living on the manifold. General Relativity is then simply a theory describing the interactions of the metric with itself, and with the matter fields[4] that live in the manifold. The self-interactions of the metric are governed by the so-called Ricci scalar, which express the curvature of the manifold at each point of it, and involves second derivatives of the metric, thus playing the role of a kinetic term in the action. As the matter content of the manifold is directly coupled to the metric, it will source such curvature term: the matter bends the space-time. Conversely, the matter will "feel" the space-time curvature, and freely falling particles follow *geodesics* of the manifold, *ie.* the shorter paths between two points. Let us also note that the curvature term is highly non-linear in the metric, which leads to remarkable effects that will be described in the bulk of this thesis (notably in chapters III and IV). Technical details together with the derivation of the field equations from the action are presented in app. A.2.

A remarkable feature of GR is that it is a "covariant" theory: under a change of coordinates, its components vary in such a way that physical (*ie.* observable) quantities do not depend on the system of coordinates. To state it roughly, the laws of Nature are independent of the observer and of its measure apparatus, which appears to be a reasonable attribute for a gravitational theory. This independence is mathematically translated as an invariance under *diffeomorphisms*, and is extremely useful in practical computation, as physical quantities will be the same whatever the coordinate system used to compute them. Thus one can always choose the most suitable coordinate system for the derivation, without worrying about the final result. We will make great use of this feature in the following.

Another fundamental consequence of this geometrical description is the existence of *gravitational waves* [177, 179]. As usual in field theory, the metric field can oscillate around a fixed background value determined by the matter content of the Universe. Those tiny oscillations propagate at the speed of light, and induce small perturbations in the manifold structure ("ripples in space-time" as usually stated). When passing through matter, they induce extremely small changes of the physical distances between two points, but preserve volumes: a rigid circle spanned by such a wave wobbles, but its area remains constant. The oscillations of objects spanned by gravitational waves can be

[3]Usually of dimension 4, but we will also use manifolds of arbitrary, complex dimensions for regularization purposes.

[4]By "matter", we encompass all non-gravitational forms of energy: baryons, light, neutrinos, *etc.*

decomposed in two independent motions, called *polarizations*: the first one, dubbed + polarization, corresponds to elongations in the "top-bottom" and "left-right" directions; the second, × polarization, corresponds to elongations on 45° directions. Those oscillations are depicted in fig. I.1.

Those waves are sourced by any motion that is not symmetrical under spatial rotations, as will be discussed in details in chapter IV. For instance, when raising your elbow to enjoy some wine, you emit gravitational waves. But such waves are so weak that they cannot have detectable effects (the preceding example will induce a change of $\sim 10^{-43}$ m in the size of your drinking fellows, located 1m away from you) and extreme events (such a supernovae, coalescences of compact binaries or the beginning of the Universe) are needed to produce gravitational radiations that can be detected by current human means.

Note that this geometrical conception clarifies the two conceptual ambiguities of Newtonian gravitation raised in the previous section: gravitational interaction is not instantaneous, but is carried by gravitational waves that propagate at the speed of light; bodies fall by following the local curvature of space-time.

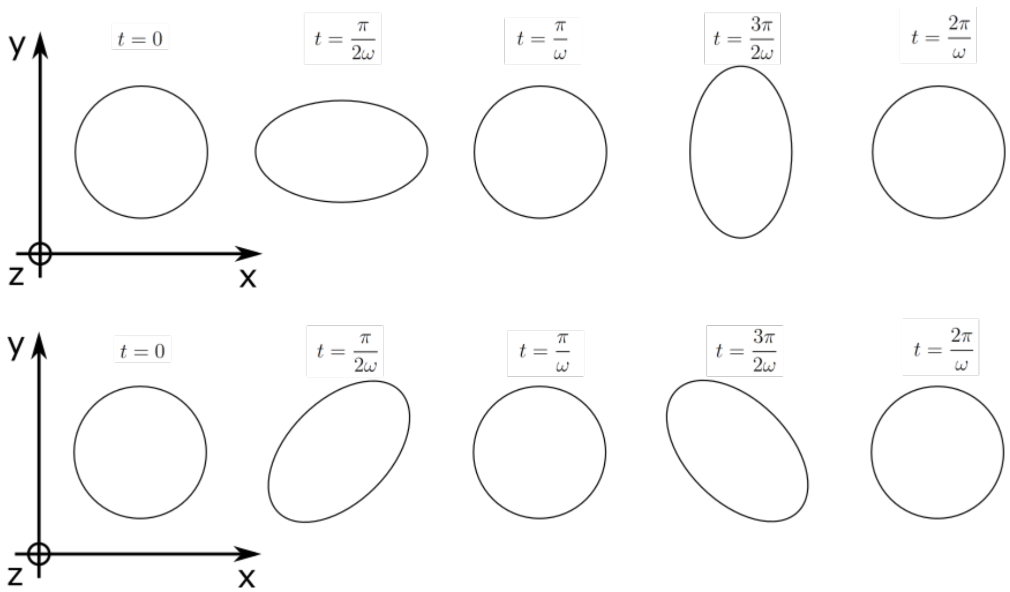


Figure I.1: Extreme magnification of the effects of a gravitational wave going through a rigid circle, lying in the (xy) plane. The wave is of frequency ω and propagates along the $\vec{z}$ direction. The upper panel shows the effect of the + polarization and the lower one, the × polarization. No deformations are to be observed in the $\vec{z}$ direction.

I.2.2 Lovelock's theorem and theories "beyond" General Relativity

The next feature of GR to be discussed in this section is a powerful uniqueness theorem due to D. Lovelock [271]. It states that, under a set of six hypothesis, the only possible equations of motion in vacuum are those of GR (with a cosmological constant). The hypothesis are that the only gravitational field is the metric, that the manifold is four-dimensional, that the equations of motion are at most second-order and that the theory is local, invariant under diffeomorphisms and derives from a least-action principle. In addition to be a remarkable property of GR *per se*, this theorem is extremely useful as it indicates the possible directions to seek for "beyond" GR theory, by breaking one (or more) of the hypothesis. It would be extremely presumptuous to pretend that we can list here all the attempts that have been made to construct "beyond" GR theories, and we will hereafter only present a selection of the major steps that led to the current profusion of models.

The most explored direction is to break the first hypothesis, *ie.* adding extra gravitational fields. The simplest configuration, by adding a single scalar field, has been initiated by the seminal works of P. Jordan, C. Brans and R. Dicke [110, 244] and has known many developments under the name of *scalar-tensor* theories. If the scalar sector of Brans and Dicke theory was quite simple, many efforts have been made in finding the most general scalar-tensor theory. Allowing second-order derivatives in the action (but keeping the equations of motion to be second-order), A. Nicolis, R. Rattazzi and E. Trincherini constructed the theory of "Galileons" (where the scalar field enjoys a Galilean symmetry) in flat space-time [294]. Those Galileons were then covariantized [163], *ie.* written in a generic, dynamical space-time, and finally generalized [164, 166] so to obtain the thought-to-be most general theory leading to second-order equations of motion. As a matter of fact, this theory has been shown to be equivalent to the model that G. Horndeski constructed 35 years ago, and that has been forgotten then [121, 236]. The following development has been to use non-invertible metric transformations to construct "beyond-Horndeski" theories [208], and even to use higher derivatives in the action, together with degeneracy conditions, to construct even more general scalar-tensor theories [46].

This conceptually simple framework of scalar-tensor theories is in fact extremely powerful, as it encompasses many other models, that are *a priori* fundamentally different. For example, H. Buchdahl constructed theories involving an arbitrary function of the Ricci scalar [114], naturally dubbed $f(\mathcal{R})$ theories. Due to their promising possible cosmological applications, those models have received a renewed interest starting from the 80's [331] and are still actively explored [141]. Interestingly, it can be shown that, under a particular change of variables, they can be rewritten as a special case of scalar-tensor theories.

Departing from the scalar-tensor framework, one can construct *vector-tensor* theories [224, 355], at the cost of creating preferred-frame effects. If less active than the scalar-tensor one, this framework is still explored, as proven by the recent work [159]. One can naturally also built tensor-tensor (dubbed bi-gravity) models [221], as we will discuss in more details in chapter VI, or combine everything with tensor-vector-scalar proposals [45].

Naturally, all other possible breakings of Lovelock's hypothesis have been investigated. For example a five-dimensional theory has been constructed by G. Dvali, G. Gabadadze and M. Porrati [171], where the matter fields live in a four-dimensional "brane" embedded in a five-dimensional space-time. This model was in fact the inspiration of the original Galileon theory, the scalar field being the bending of the brane. Other examples are given by non-local theories [274, 352, 359], diffeomorphism-breaking ones [31, 160, 189, 237] or emergent theories (that do not derive from a least-action principle) [320, 342].

I.2.3 The brilliant success and darkness of General Relativity

When formulated by A. Einstein in its final form [175], General Relativity differentiated itself from Newtonian gravity by four phenomenological consequences: a relativistic precession of planetary perihelia, a relativistic deflection of light by massive bodies, a gravitational redshift effect and the existence of gravitational radiation. Those four phenomena have been observed and tested to extremely high accuracy level, thus confirming the strong foundations of GR.

The existence of a relativistic precession of the perihelion of Mercury was a prerequisite to any "beyond" Newtonian theory of gravitation, as discussed in sec. I.1.3. The fact that A. Einstein found the good value with its theory [174][5] was thus only half a success for GR, as the whole effort in theory-building at this time was aimed at recovering this precise value. On the contrary, predicting

[5]To be exact, A. Einstein computed the value using the first formulation of its theory, that turned out to be incorrect. But, as this initial theory and GR do not differ in vacuum, he found the good value, even if he was using the wrong theory.

a relativistic deflection of light by massive bodies was a new feature of the theory, and thus led to a genuine prediction, that was confirmed during the 1919's Solar eclipse by a collaboration led by A. Eddington [172]. This observation, that rendered GR and thus A. Einstein extremely popular (notably by the symbol of a British astronomer confirming a German theory, six months after the end of the war), was a real confirmation for the young theory. Later on, other genuine GR effects taking place in the Solar system have been computed and tested, for instance the gravitational redshift effect, stating that a light ray escaping the gravitational field of a body looses energy, and thus its frequency decreases (it becomes redder). This effect is deeply linked to the fact that clocks measure different times, depending on their altitude, which is naturally primordial to take into account for the GPS system. The Earth-induced redshift has been measured in 1959 by R. Pound and G. Rebka [309], and the GR prediction has been recovered. All Solar system-based tests have been pushed to an astonishing precision (up to one part in hundred thousand), confirming with high accuracy the predictions of GR, and putting strong constraints on "beyond" GR theories [182, 354]. For a comprehensive review on those post-Newtonian tests, see [353].

As for gravitational radiation, it has first been indirectly detected by monitoring the motion of a binary system of compact objects. Such a system looses energy trough the emission of gravitational waves, and thus its orbital period decreases. The Nobel-winning detection of this effect in a system of two pulsars [238, 351] was the first of a series of tests of GR in strong-field regimes. The first direct detection of a gravitational wave event took place in 2015 and was performed by the two LIGO (Laser Interferometer Gravitational-Wave Observatory) instruments [3]. The delay of a century between the prediction and observations of gravitational radiation is due to the extreme precision that is needed to detect it: the relative change of length induced by the first event, provoked by the coalescence of two black holes roughly 400 Mpc away from us, was of order 10^{-21} (which corresponds to detect that the closest star oscillates by the thickness of a human hair). The European detector, Virgo, joined the observation campaign and gravitational radiation emitted by the coalescence of two neutron stars was detected in 2017 [4], together with an electromagnetic counterpart [5], allowing notably to localize precisely the event, and thus to conduct stringent tests on the emission and propagation of gravitational waves [6]. At the time being, a fourth detector, the Japanese KAGRA (KAmioka GRAvitational-wave detector), joined the collaboration and 39 gravitational wave events have been confirmed [7, 9]. This catalog of observations have naturally been used to perform tests on the emission and propagation of gravitational waves, as well as on the nature of the compact objects that emit the radiation. Those tests show perfect agreement with the predictions coming from GR [10]. Note that gravitational radiation is particularly powerful to constraint the presence of additional gravitational degrees of freedom, as those would induce additional polarization modes. The monitoring of pulsars has put a severe constraint on the presence of scalar fields, as the energy radiated through the additional polarization has to be less than 1% of the total energy radiated away [169, 354] (due to the configuration of the detector network, the direct observation of gravitational waves is not yet competitive for such tests [10][6], but the LISA (Laser Interferometer Space Antenna) gravitational-wave detector will be able to perform accurate tests of the presence of additional polarizations [44]).

The gravitational waves that we detect are emitted by binary systems of compact objects and, among the compact objects that exist in the Universe, black holes are also genuine predictions of GR. It is true that the idea of "black holes" also exists in Newtonian gravity, as celestial bodies that are massive enough so that their escape velocity[7] exceeds the speed of light. Such objects were conceived independently in 1783 by J. Michell [282] and in 1796 by P.-S. de Laplace [157], but were rapidly discarded by the community. But the story is different in GR and K. Schwarzschild noticed as soon

[6]They can only perform "GR vs only non-GR" polarization tests, but no modern theory contains exclusively non-GR polarizations.

[7]We recall that the escape velocity is the minimal velocity needed to escape from the gravitational influence of a celestial body. If it exceeds the speed of light, even light would be trapped in the vicinity of the body, which would thus appear black for observers at infinity.

as 1916 [325] that black holes appear quite naturally as vacuum solutions (contrarily to Newtonian black holes that are tangible bodies) possessing an *horizon*, *ie.* a surface that encloses a *trapped* region, from which no information can escape. This horizon can be formed dynamically, notably by the collapse of a massive star. Thus, and by definition, one cannot access what is "inside" a black hole, and the only relevant physics is the one that takes place from the horizon up to the spatial infinity. Note that the definition of this horizon can be non-trivial, as will be discussed in chapter VII. The static Schwarzchild solution has rapidly been generalized to bear an electric charge, by H. Reissner and G. Nordström [297, 313], but its generalization to rotating space-time had to wait for 1963 and the seminal work of R. Kerr [252]. A very powerful theorem, known under the name of the "no-hair theorem", states that black holes in GR only depend on three parameters : mass, spin and electric charge, and are thus entirely described by Kerr-Newman space-times [292]. This theorem is heavily tested by observing the coalescence of the black hole binaries, as a dependence to another parameter than the mass and spin would be an indication of some "beyond" GR phenomenology (the electric charge is irrelevant in an astrophysical context). By definition, black holes cannot be directly observed, but there are currently four ways to indirectly detect them: by the gravitational waves they emit when two of them coalesce; by observing the powerful electromagnetic emissions of the accreting matter that surrounds them; by directly observing this accreting matter [16]; or by monitoring the motion of stars in their vicinity [12]. Note that, excepted for the first technique, one can only detect objects with large masses (at least millions of solar masses), and that the direct observation of their surroundings (stars and accreting matter) gave excellent agreement with GR predictions [12, 16].

But despite passing all possible tests with flying colors, GR suffers from some "darkness", *ie.* yet ambiguous gravitational phenomena that can be explained either by non-gravitational ingredients, or by "beyond" GR completions. First of all, the stars living in periphery of the galaxies do not move accordingly to GR predictions, as observed by A. Bosma, V. Rubin and J. Ford [105, 317]. Instead, they behave as if the total mass of the observed stars was only a small fraction of the total mass of the galaxy, thus the name of the "missing mass" problem. The usual way to address this problem is to postulate the existence of a yet unknown form of matter that will account for the missing mass by forming large clusters encompassing the galaxies. This new particle has to be insensitive to electromagnetic forces (otherwise we would have already seen it), and be non-relativistic in order to pass galaxy formation constraints, thus the name "cold dark matter" [58]. If this paradigm is appealing as it naturally explains the missing mass problem, and is a cornerstone of current physics, there is still no evidence of this new particle, despite decades of tremendous theoretical and experimental efforts [2, 18, 27].

One can also imagine a population of small primordial black holes populating the galaxies: they would naturally explain the phenomena as they are massive, but invisible. Sadly this hypothesis has difficulties to pass other constraints, such as microlensing or gravitational waves emission [215, 295]: it seems quite unlikely that such objects account for the whole missing mass (but they still can account for some fraction of the missing mass).

Another way to address the missing mass problem is to consider that at low energy, the laws of gravity are not given by GR anymore. For example, in 1983 M. Milgrom found a very simple, but *ad hoc* modification of Newton's law that accounts for the motion of stars in galaxies [284] (at those energies, GR reduces to Newtonian gravity, up to definitively negligible corrections). This MONDian (for MOdified Newtonian Dynamics) framework is also successful in explaining the Tully-Fischer relation [338] (a proportionality relationship between the stellar mass of a galaxy and the speed of its peripheric stars), yet unexplained by the cold dark matter paradigm. Nevertheless, the MONDian approach is not yet a consistent theory, but rather a phenomenological paradigm, and many attempts to embed it within a complete theory have all failed till today [78, 113, 167]. For a comprehensive review on the MONDian approach, see [185].

The second sector of "darkness" is the accelerated expansion of the Universe, that has been first observed in 1998 [305, 314] and then monitored by the WMAP [233] and Planck [15] satellites. Indeed this acceleration cannot be explained by the (dark) matter only, there must be an additional "dark" energy that sources it. The simplest way to account for this dark energy is to simply add an *ad hoc cosmological constant* to GR, as first introduced by A. Einstein on very different motivation grounds [178]. The usual comprehension of such a cosmological constant is to associate it with the mean value of the vacuum's energy (non-vanishing due to natural oscillations of the quantum fields populating the Universe). But this perspective leads to a complication: the observed value of the cosmological constant is 120 orders of magnitude smaller than the computed mean energy of the vacuum [350]. Despite this issue, our current standard model for cosmology consists in a cosmological constant together with (dark) matter, and thus is dubbed ΛCDM (Λ denotes the cosmological constant, and CDM stands for cold dark matter). See [301] for a review on cosmological constant as the source of dark energy.

As in the case of dark matter, the standard model is not the only one, and there has been great efforts in explaining this late-time acceleration of the expansion by means of "beyond" GR theories. A first example is to promote the cosmological constant to a dynamical "dark fluid", modeled by a scalar field [33, 312] (thus in the class of scalar-tensor theories). Another path has been explored by the so-called massive gravities, where the metric acquires a mass term [158, 160, 221]. If such mass term is of order of 10^{-30} eV$/c^2$ (which is consistent with all current bounds), then it would be a natural explanation for the late-time acceleration. Note that some of those theories also contain a *degravitation* mechanism [158], *ie.* a machinery that makes the mean energy of vacuum gravitate less than (dark) matter, thus reducing its effects (but none have yet reached the Graal of reducing it by 120 orders of magnitude).

I.3 Motivations for this thesis

Those darkness point out that some sectors of the gravitational phenomenology remain mysterious to us. Even if those mysteries finally turn out to be of non-gravitational nature[8] it is vital to keep questioning our gravitational theory. This questioning has to be dual: an endogenous one, that seeks to deepen our knowledge of GR and to better understand its phenomenological implications; and an external one, that aims at building viable and distinguishable alternatives to GR. It is vital for those two inquiries to be carried out together, as only simultaneous progresses on those paths will lead to a better understanding of gravitational phenomena.

This thesis thus aims at following both directions, by deepening our comprehension of the relativistic two-body problem on the one hand; by investigating non-canonical scalar defects and by constructing and testing "minimal" theories of gravitation, on the other hand.

I.3.1 Pushing forward our comprehension of General Relativity

The first direction that will be explored in this thesis is to aim at deepening our knowledge of GR. This approach has two main goals: the first one is of epistemological nature and relies on the fact that, despite one century of progress, we still imperfectly master GR (note that the situation in this regard is quite different from what happened with Newtonian theory, that was well mastered one century after its release). Naturally, aiming at going "beyond" GR cannot be done without better understanding it. On this regard, the two-body problem is the perfect toy model: even if it is one of the simplest systems in Nature, and is easily solved in Newtonian theory, it remains unsolved in GR. Moreover, the main efforts to solve it rely on perturbative expansions, *ie.* by expanding physical quantities with respect to a "small" parameter. But this series is probably an *asymptotical* one,

[8] For example, a cold dark matter particle, and a dark fluid accounting for the late-time acceleration of the Universe.

meaning that there is no assurance that it does converge towards a finite result [1], thus questioning our own ability to access the whole comprehension of the two-body problem by those means.

The second goal is to have stronger tests, as a better knowledge of GR phenomenology allows to spot more easily deviations from it. In this regard also, the two-body problem offers many advantages. First of all, it is one of the most common systems in Nature, from the falling apple to the coalescence of two supermassive black holes, passing by the Earth-Moon system. The corollary of this abundance is that such systems are widely monitored (either by electromagnetic or gravitational radiations) and many tests have already been performed, as discussed in the previous section. Another advantage is that if some new physics shows up, it is quite likely to appear in the strong-field regimes, *eg.* by alternatives to black holes (indicating new gravitational physics) or exotic compact stars (indicating new states of matter). This new physics will leave faint imprints on the gravitational-wave signals (for example a non-canonical matter in neutron stars will show up at least at the $(\text{next})^5$-to-leading order in the gravitational waveform), and we should have a perfect knowledge of GR at those orders in order to discriminate such new phenomenologies.

As stated by the title of this thesis, we will only focus on analytic methods to iteratively solve the two-body problem, and will not discuss at all the great achievements of the numerical methods to solve GR (see *eg.* [66] and references therein). We will work within the Blanchet-Damour-Iyer formalism that relies on a small-velocity-weak-field expansion: for a binary of speed $v \ll c$, total mass M and separation r, the small parameter is $\epsilon \sim v^2/c^2 \sim GM/r/c^2$ (those two quantities being of the same order of magnitude for virialized bound systems that will be considered). A post-Newtonian order (PN order) is then an order in this expansion, so that nPN denotes the order $\epsilon^n \sim (v/c)^{2n}$.

I.3.2 Seeking for viable alternative

But this endogenous questioning of GR has to be accompanied by an outer one, *ie.* by seeking viable and distinguishable alternatives. This is naturally vital to perform *educated* tests of GR. For example, instead of looking for generic deviations in the gravitational waveforms, one can construct viable scalar-tensor theories, and subsequently derive the exact imprint that the scalar field would yield in the gravitational waveform. In the same spirit, constructing viable "massive" gravities allows to derive precisely the effect of a mass in the dispersion relation for gravitational waves, and thus to refine the algorithm that searches such deviations from GR predictions. But the construction of viable alternatives to GR is also important as it reinforces our knowledge of GR itself, by pointing out what GR is not. For instance, the first trials to construct a healthy "massive" gravity have revealed the presence of additional polarization states, one of which is a "ghost", *ie.* it interacts wrongly to matter, and thus renders the theory harmful. Understanding what such dreadful polarization state really implies, and more importantly, why it does not exist in GR, requires a careful analysis of GR, and it took roughly 40 years to construct a "massive" viable alternative free of this ghost.

I.3.3 Work done in this thesis

The rest of this dissertation is organized so that each chapter presents at least a new result obtained during this thesis.

The first part is devoted to the study of the relativistic two-body problem, within the post-Newtonian-multipolar-post-Minkowskian approximation scheme.

In chapter II, the Newtonian and relativistic two-body problems are introduced and the state-of-the-art methods and results are outlined. Focusing on post-Newtonian treatments, the split between a conservative and a radiative sector is exposed. To concretely illustrate this rather theoretical discussion, we investigate the case of a peculiar exoplanetary system, HD 80606b, and propose a realizable method to detect relativistic effects in the motion of the exoplanet. Such observation, that will be feasible around 2025, would be the first Solar system-like test of gravitation, performed in a distant stellar system, and thus a strong trial for GR.

The conservative sector of the two-body problem is presented in chapter III, and illustrated by the high precision (next-to-next-to-next-to-leading order) computation of the non-linear "tail" effects in the conserved energy. Such computation heavily relies on novel synergies between the traditional post-Newtonian formalism and the Effective Field Theory techniques, synergies that appear to be a very promising tool for efficient computations of high accuracy quantities.

Finally, chapter IV deals with the radiative sector, and the ongoing computation of the gravitational phase for non-spinning compact binaries, at the 4^{th} post-Newtonian order, which was the main goal of this thesis. Three steps towards such result are presented in details: the derivation of the Hadamard regularized source mass quadrupole, its infra-red dimensional regularization and the d-dimensional derivation of the non-linear interactions entering the radiative mass quadrupole. The two last ones are presented here for the first time.

Part B is dedicated to the construction and phenomenological study of some alternative theories.

Chapter V focus on scalar theories, and more precisely deals with the existence and stability of domain-wall solitons. By circumventing the usual topological argument used to prove their stability, a new class of theories supporting stable solitons is constructed. Moreover, we show that those non-canonical solitons can perfectly mimic the canonical ones, even at the linear perturbation level.

Chapter VI introduces a recent guideline for building alternative theories of gravitation: the principle of *minimalism*, requiring that the theory only contains the observationally required polarizations. Following this principle, we present in details the construction of a "minimal theory of bigravity", *ie.* a theory of two interacting metrics, that contains only four polarizations (twice the polarizations that are present in GR), instead of the usual seven that are present in bigravity theories. The cosmological phenomenology of this new theory is derived and its confrontation against observations is discussed.

Finally, chapter VII is devoted to another class of tests for alternative theories, by exploring their strong-field regimes. Black hole space-times are constructed in two *minimal* theories and their phenomenologies, as well as their distinguishability from GR ones, are discussed.

The conventions used within this dissertation, as well as the Hamiltonian formulation of General Relativity, are presented in app. A. App. B displays a "post-Minkowskian" toolkit, useful for the discussions and computations of chapters III and IV. As we will deal with lengthy post-Newtonian expressions, app. C collects those that are too long to appear in the core of the dissertation. The crucial derivation of the d-dimensional non-linear interactions appearing in the radiative mass quadrupole, which is one of the main results obtained during this thesis, is presented in app. D. Finally, app. E provides a summary in French of this dissertation.

The research work done during this thesis led to the following publications.

On the relativistic two-body problem:

- L. Blanchet, G. Hébrard, FL; *Detecting the General Relativistic Orbital Precession of the Exoplanet HD 80606b*, A&A **628**, A80 (2019). arXiv: 1905.06630 [astro-ph]. [95]
- L. Blanchet, S. Foffa, FL, R. Sturani; *Logarithmic tail contributions to the energy function of circular compact binaries*, Phys. Rev. D **101**, 084045 (2020). arXiv: 1912.12359 [gr-qc]. [96]
- T. Marchand, Q. Henry, FL, S. Marsat, G. Faye, L. Blanchet; *The mass quadrupole moment of compact binary systems at the fourth post-Newtonian order*, CQG **37**, 215006 (2020). arXiv: 2003.13672 [gr-qc]. [278]

On alternative theories:

- C. Deffayet, FL; *Domain walls without a potential*, Phys. Rev. D **103**, 036010 (2021). arXiv: 2009.00404 [hep-th]. [165]
- A. De Felice, FL, S. Mukohyama, M. Oliosi; *Minimal Theory of Bigravity: construction and cosmology*, JCAP **04**, 015 (2021). arXiv: 2012.01073 [gr-qc]. [155]
- A. De Felice, FL, S. Mukohyama, M. Oliosi; *Black holes and stars in the minimal theory of massive gravity*, Phys. Rev. D **98**, 104031 (2018). arXiv: 1808.01403 [gr-qc]. [148]
- A. De Felice, A. Doll, FL, S. Mukohyama; *Black holes in a type-II minimally modified gravity*, JCAP **03**, 004 (2021). arXiv: 2010.13067 [gr-qc]. [154]
- A. De Felice, FL, S. Mukohyama, M. Oliosi; *On the absence of conformally flat slicings of the Kerr spacetime*, Phys. Rev. D **100**, 124044 (2019). arXiv: 1908.03456 [gr-qc]. [151]

All the computations presented in this dissertation have been implemented with the *Mathematica* software [358]. The results of chapters VI and VII have been obtain by the use of the *xAct* package [279]. The post-Newtonian studies of chapters III and IV have been analytically performed by using the *PNComBin* library of the *xAct* package, developed by G. Faye, J. Laidet and S. Marsat, and its d-dimensional extension, *ExtendedPNComBin*, implemented by T. Marchand.

Part A

The relativistic two-body problem

Chapter II

Introduction to the relativistic two-body problem

II.1 From the Newtonian to the relativistic problem

II.1.1 Revisiting the Newtonian two-body problem

As an appetizer, we will show that the Newtonian gravitation and the two-body problem are closely linked. To this purpose, starting from a system composed of two particles and imposing only "natural" requirements on the gravitational interaction, we aim at recovering the Newtonian law.

Let us consider two point-like test particles of masses m_1 and m_2, interacting *via* a gravitational field Φ_{12}. Using for the moment a generic Galilean reference frame, we denote the positions of the masses as y_1^i and y_2^i, their speeds as $v_a^i \equiv \mathrm{d}y_a^i/\mathrm{d}t$ $(a = 1, 2)$, together with $r_{12} = |y_1^i - y_2^i|$ and $n_{12}^i = (y_1^i - y_2^i)/r_{12}$. As for the specific form of Φ_{12}, we only require that it satisfies usual expectations for a gravitational field: fulfilling Newton's first law, being static, attractive, and vanish when the distance between the masses becomes infinite. The two first conditions together imply that Φ_{12} can only depend on r_{12}, the attractivity condition, that Φ_{12} is a monotonically increasing function of r_{12}, and the last one, that $\Phi_{12}(r_{12} \to \infty) = 0$. In addition we will require that the equivalence principle holds, *ie.* that Φ_{12} is linear in each of the masses. All those conditions impose to use $\Phi_{12} = m_1 m_2\, \Phi(r_{12})$, where Φ does not depend on the masses. The test particles thus obey the Lagrangian

$$\mathcal{L} = \frac{m_1}{2}\, v_1^2 + \frac{m_2}{2}\, v_2^2 - m_1 m_2\, \Phi(r_{12})\,. \tag{II.1}$$

Defining the usual variables $X^i \equiv (m_1 y_1^i + m_2 y_2^i)/(m_1 + m_2)$ and $x^i \equiv y_1^i - y_2^i$, together with the mass parameters $M \equiv m_1 + m_2$ and $\mu \equiv m_1 m_2/M$, the Lagrangian is rewritten as

$$\mathcal{L} = \frac{M}{2}\, V^2 + \frac{\mu}{2}\, v^2 - M\mu\, \Phi(r)\,, \tag{II.2}$$

where we have denoted $r = |x^i|$ together with $n^i = x^i/r$ and the new speeds obey $v_1^i = V^i + m_2/M v^i$ and $v_2^i = V^i - m_1/M v^i$. We thus see that the motion of the two particles can be decomposed into the motion of the center-of-mass X^i and the motion of an effective body of mass μ, immersed in the gravitational field created by a mass M. They obey the equations of motion

$$\frac{\mathrm{d}V^i}{\mathrm{d}t} = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}v^i}{\mathrm{d}t} = -M\, \frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}\, n^i\,. \tag{II.3}$$

The center-of-mass, on which no external forces apply, behaves as an isolated body and is at rest, which is nothing but the practical translation of the principle of inertia.[1] As the motion of the center-of-mass is trivial, we focus henceforth on the motion of the effective particle of mass μ. Its equation

[1] As expected this principle is independent of the precise form of the gravitational interaction, Φ, as it embodies a broader framework than just Newtonian gravitational law.

of motion indicates that the motion takes place within a plane, spanned by the initial position and velocity vectors[2] and we set z^i as a constant unit vector, normal to this plane. Defining as usual the λ^i vector so that (n^i, λ^i, z^i) forms an orthonormal basis of the Euclidean space, we associate an angular coordinate θ to the λ^i vector. In this coordinate system, the equation of motion (II.3) takes the form

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \left(\ddot{r} - r\dot{\theta}^2\right) n^i + \frac{\mathrm{d}\, r^2\dot{\theta}}{\mathrm{d}t}\frac{\lambda^i}{r} = -M\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\, n^i\,, \tag{II.4}$$

where a dot denotes time derivative. Integrating the projections onto each direction, two conserved quantities arise, corresponding to the usual energy and angular momentum, explicitly given by

$$E = \frac{\mu}{2}\,\dot{r}^2 + \frac{\mu\, j^2}{2r^2} + M\mu\,\Phi(r) \equiv \mu\,\varepsilon\,, \qquad \text{and} \qquad J = \mu\, r^2\,\dot{\theta} \equiv \mu\, j\,. \tag{II.5}$$

Using a trick due to J. Binet, we introduce $u \equiv 1/r$ and trade the time dependence for an angular one, so that the energy per unit mass is rewritten as

$$\varepsilon = \frac{j^2}{2}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2 + \frac{2\varphi(u)}{j^2}\right]. \tag{II.6}$$

where we used the shorthand $\varphi(u) = M\Phi(1/r)$. The first additional requirement that we impose on the gravitational interaction is that it allows bound orbits to form, as such orbits are to be observed in Nature. From eq. (II.6), we can see that the presence of closed orbits is linked to the well-definition and existence of positive roots of the quantity

$$K(u) \equiv \sqrt{\frac{2\varepsilon}{j^2} - \frac{2\varphi(u)}{j^2} - u^2}\,. \tag{II.7}$$

To be more precise, we need K to be well-defined between two of its strictly positive roots. Together with the initial requirements, this translates into the existence of at least one strictly positive number u_0, such that the gravitational field satisfies the set of conditions

$$\varphi(0) = 0\,, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}u} < 0\,, \qquad \left.\frac{\mathrm{d}\varphi}{\mathrm{d}u}\right|_{u_0} = -j^2 u_0\,, \qquad \text{and} \qquad \left.\frac{\mathrm{d}^2\varphi}{\mathrm{d}u^2}\right|_{u_0} > -j^2\,. \tag{II.8}$$

Taking as an example a simple power-law $\varphi(u) = \varphi_0\, u^\alpha$, those conditions are met as long as $\varphi_0 < 0$ and $0 < \alpha < 2$: there is thus a large class of theories accommodating bound orbits.

Let us require in addition that those closed orbits do not precess. We could impose such non-precession condition in a strong manner, by requiring a full periodicity of the orbit, $u(\theta + 2\pi) = u(\theta)$. We will rather enforce non-precession in a weak way, by fixing only the periastron and apastron, allowing the orbits to "breath" between those two points. This amounts to impose that the apastron, origin and periastron are exactly aligned, thus

$$\int_{u_-}^{u_+} \mathrm{d}\theta(u) = \pi\,, \qquad \text{or equivalently} \qquad \frac{1}{\pi}\int_{u_-}^{u_+}\left(\frac{\mathrm{d}\theta}{\mathrm{d}u}\right)\mathrm{d}u = \frac{1}{\pi}\int_{u_-}^{u_+}\frac{\mathrm{d}u}{K(u)} = 1\,, \tag{II.9}$$

where $u_- < u_+$ are two strictly positive roots of K, corresponding respectively to the apastron and periastron of the orbit. In order to express this integral in a simpler form, we rely on a method exposed in [19], and define a new variable τ such that $2u = u_+ + u_- + (u_+ - u_-)\cos\tau$, which yields

$$\frac{1}{\pi}\int_{u_-}^{u_+}\frac{\mathrm{d}u}{K(u)} = \frac{1}{\pi}\int_0^{\pi}\frac{\mathrm{d}\tau}{\sqrt{1-\Delta(\tau)}}\,, \tag{II.10}$$

[2]Of course, if the initial speed is null or colinear to the initial position, the motion will follow a straight line, which is of little interest and will not be considered in the following.

with

$$\Delta(\tau) = \frac{2}{\sin^2\tau}\frac{u_+ + u_-}{u_+ - u_-}\left\{\cos\tau - \frac{4\,j^{-2}}{u_+^2 - u_-^2}\left[\frac{\varphi\,(u_+)}{2} + \frac{\varphi\,(u_-)}{2} - \varphi\left(\frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2}\cos\tau\right)\right]\right\}. \tag{II.11}$$

Inverting this relation yields

$$\varphi(u) = -\frac{j^2}{2}\left\{u^2 + u_+u_- + \frac{(u_+ - u)(u - u_-)}{A^2(u)}\right\}, \tag{II.12}$$

where $\frac{1}{\pi}\int_0^\pi \mathrm{d}\tau A[u(\tau)] = 1$ in order to satisfy the condition (II.9). Let us note that, injecting the function (II.12) in eq. (II.7) evaluated in u_+, it comes $u_+u_- = -2\varepsilon/j^2$. As u_+ and u_- are positive by definition, this traduces the fact that the energy of a bound system is necessarily negative, as expected by its conservation and its cancellation at infinity. On the other hand, there is one freedom left in the u_+ and u_- coefficients, and we label $u_+ + u_- = 2GM/j^2$, with G a positive constant.

Note also that the two last conditions of (II.8) are automatically satisfied by the function (II.12) as long as it is regular, *ie.* as long as A does not vanish. Moreover, as there is a large freedom in the choice of the function A, it can be taken so that the two first conditions of eq. (II.8) are met. Nevertheless, unless $A = 1$, the potential contains a contribution $-j^2u_+u_-/2 = \varepsilon$, which violates the equivalence principle. Therefore only the solution with $A(u) = 1$, *ie.* $\varphi = -Gu$, satisfies our requirements.

Let us sum up this rather arid discussion. Starting from two gravitationally interacting test bodies, we have only made a few natural assumptions (that the gravitational field obeys Newton's first law and the principle of equivalence, is attractive, static, vanishes at infinity and allows bound orbits to form) and have added a weak condition of non-precession for the bound orbits. With this minimal set of requirements we have obtained a unique solution for the gravitational field, namely

$$\Phi_{12} = -\frac{Gm_1m_2}{r_{12}}, \tag{II.13}$$

with G a positive constant, and we have unsurprisingly rediscovered Newtonian gravitation. Moreover, it is now clear that precession, that plays an important role in the relativistic discussion of sec. II.2, is not a special attribute of GR, but rather a common feature of mostly all non-Newtonian theories of gravitation.

When switching from Newtonian gravitation to General Relativity, three major discrepancies show up, that are to be discussed in the remaining of this section.

II.1.2 The relativistic notion of trajectory

The first and most important discrepancy between Newtonian and Einstein gravitations is of conceptual nature. As discussed in sec. I.2.1, GR is a geometrical description of gravitational phenomena, and it relies on a unification of space and time into a four-dimensional manifold. Therefore the Newtonian conception of motion (*ie.* the time evolution of a particle's position) looses its significance. The relativistic description of motion is encoded in the notion of *world-line*: a one-dimensional, time-like section of the manifold. There is thus no genuine notion of "time evolution" in GR: what we (as observers) consider as all the successive positions of a body are to be considered as a whole.

The second consequence of the geometrical nature of GR is the disappearance of the notion of "gravitational force". Apples do not fall because they are attracted by Earth *via* its gravitational field, but because they follow their "natural" trajectories : freely falling bodies are isolated systems. For test particles, such "natural" trajectories are the geodesics of the space-time, which are nothing but the generalizations of the straight lines in the case of curved manifolds. The gravitational attraction manifests itself *via* a curvature of the space-time, created by a distribution of energy, which affects the form of the geodesics, and thus the trajectories of particles. This directly applies to the two-body system: the bodies are not interacting *via* a gravitational force as in Newtonian gravity, but are simply freely falling in the space-time they create. Nevertheless, the exact geodesics of such space-times are not yet known exactly, and we have to rely on approximation schemes in order to acquire knowledge on the relativistic two-body problem, as will be presented in more details in the next sections.

Taking an affine parameterization of the world-line, with parameter λ, the trajectory is thus a one-dimensional, connected, future-oriented ensemble of points $x^\mu(\lambda)$, and we can define its tangential vector field, dubbed four-velocity, $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\lambda$. For an isolated test particle, the world-line is defined by the fact that the tangent vector is parallel transported along itself, which reads

$$u^\nu \nabla_\nu u^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0\,, \tag{II.14}$$

where we refer to app. A.1 for the definition of the covariant derivative ∇_μ and the Christoffel symbols $\Gamma^\rho_{\mu\nu}$. The whole effects of gravitation are encrypted in $\Gamma^\mu_{\rho\sigma}$, and naturally, beyond the test mass approximation, there is a back reaction of the particle on the geometry. In addition to obeying the geodesic equation (II.14), the four-velocity should have a negative norm $g_{\mu\nu}u^\mu u^\nu < 0$, as we deal with matter particles.

It is nevertheless extremely convenient to recover the usual Newtonian conception of motion by foliating the space-time in three-dimensional, space-like hypersurfaces, *cf.* app. A.3. In this setup, the continuous label of the slice can be used as an obvious time parameter, and one can naturally mimic the time evolution of a particle's position. Another manner to recover a Newtonian-like interpretation of the trajectory is by using an approximation scheme that weakly departs from the Newtonian case. As an example, let us consider the simplest realization of the two-body problem: a spinless test particle in the space-time created by a mass M, sitting at the origin.[3] We naturally neglect the back-reaction of the particle on the geometry, and use a weak-field approximation, *ie.* focus on large enough radial separations r between the two masses, so that our small parameter is given by $\epsilon \sim \frac{GM}{rc^2} \ll 1$. The world-line of the test body $\{x^\mu\} = (ct, r, \theta, \phi\}$ obeys the geodesic equation (II.14) with the Cristoffel symbols given by the geometry created by a point-particle of mass M, namely the Schwarzchild line element [325]. The 0 component of the geodesic equation (II.14) reads

$$c\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} = \mathcal{O}(c\,\epsilon^2)\,, \tag{II.15}$$

so we can identify the affine parameter $\lambda = t$, at leading order, thus recovering the Newtonian interpretation of the coordinate time being a label tracking the evolution of the particles. The angular sector of the geodesic equation reads

$$\ddot{\theta} - 2\frac{\dot{r}\dot{\theta}}{r} - \cos\theta\sin\theta\,\dot{\varphi}^2 = \mathcal{O}(\epsilon^2)\,, \qquad \text{and} \qquad \ddot{\varphi} + 2\left(\frac{\dot{r}}{r} + \frac{\dot{\theta}}{\tan\theta}\right)\dot{\varphi} = \mathcal{O}(\epsilon^2)\,. \tag{II.16}$$

At leading order, it is naturally solved by $\theta = \pi/2$ (*ie.* restricting the motion to the equatorial plane) and $r^2\dot{\varphi} = j$ where j is a constant. This constant is naturally the Newtonian conserved angular momentum per unit mass (II.5) (where θ corresponds to φ). Last, the radial part is then given by

$$\ddot{r} - \frac{j^2}{r^3} + \frac{2GM}{r^2} = \mathcal{O}(\epsilon^2)\,. \tag{II.17}$$

Multiplying by $m\dot{r}$ and integrating, the conservation of energy (II.5) is recovered. Appropriately, the weak-field limit of GR is indeed Newtonian gravitation. The whole point of the approximation schemes that are to be used in this dissertation is to make this analogy persist at higher orders, *ie.* treating the relativistic problem in a Newtonian fashion.

[3] To be accurate, we should rather describe the trajectory of this mass M by a four-velocity $u^\mu = (c, \vec{0})$.

II.1.3 The relativistic gravitational radiation

The second difference between Newtonian and Einstein gravitations has deep observational consequences, as it is the existence of gravitational radiation. This radiation can be defined as small departures from flatness at infinity.

Note however that, strictly speaking, the definition of such radiation does not make sense in GR. Indeed, as discussed just above, nothing can radiate in GR: the manifold structure is given at once and the notions of "time evolution" or "wave propagation" are absolutely meaningless. But when adopting a Newtonian-like approach by singling out a time parameter (for example in the framework of a weak-field approximation scheme), the notion of radiation recovers some sense.

A direct consequence of the existence of such radiation is that energy and angular momentum are carried away from two-body systems. If it is complicated to rigorously define a notion of energy in GR, it is doable when approximation schemes are at play, as will be done in this dissertation. In such case, the equations of motion, given by the relativistic extensions of eqs. (II.15)–(II.17), can be separated in two sectors: a *conservative* one, given by conserved quantities, and a *dissipative* sector (also dubbed *radiative* sector), encoded in fluxes. Both sectors are linked by the so-called *flux-balance* equations,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}\,, \qquad \text{and} \qquad \frac{\mathrm{d}J^i}{\mathrm{d}t} = -\mathcal{G}^i\,, \tag{II.18}$$

where E and J^i are the relativistic extensions of the Newtonian conserved energy and angular momentum, and $\mathcal{F}$ and $\mathcal{G}^i$ are their associated fluxes. The second equation indicates that, in general, the relativistic motion is not planar: when J^i and $\mathcal{G}^i$ are not aligned, the direction of the angular momentum evolves in time. At the lowest order, the energy and angular momentum fluxes $\mathcal{F}$ and $\mathcal{G}^i$ are vanishing, and we recover the usual Newtonian conservation laws. Note that similar flux-balance equations also exist for the relativistic extensions of the six Newtonian conserved quantities associated with the center-of-mass: its position X^i and its linear momentum MV^i.

This separation in conservative and dissipative sectors is vital, as both are to be treated very differently. The conservative sector is associated with the dynamics of the binary system, as *eg.* E is composed of the kinetic energy and gravitational binding of the two bodies. Those quantities make sense in the so-called *near-zone*, *ie.* the surroundings of the binary. On the other hand, the dissipative sector is linked to the emission of gravitational waves, and the fluxes are computed *via* suitable formulas, presented in detail in chapter IV. Such computations are defined in the *wave-* or *radiative-zone*, that is simply defined as the exterior of the binary. Those two zones overlap in the *matching-zone*, where interplays between conservative and dissipative effects occur, as discussed in the next chapters.

II.1.4 The relativistic non-linearities

The last main discrepancy between the two gravitational theories is more technical (but bears great physical significance): when Newtonian gravitation is a linear theory, GR is a (highly) non-linear one. The former is clear from the two-body equations of motion written in Binet's variables: $\mathrm{d}^2u/\mathrm{d}\theta^2 + u = GM/j^2$, which is nothing but a driven linear oscillator. The most straightforward consequence of this linearity is that the two-body problem is fully solvable in Newtonian gravitation. Regarding Einstein gravitation, the theory is highly non-linear, as clearly indicated by the geodesic equation (II.14): the world-line of the particle sources the metric, which enters non-linearly in the Cristoffel symbols (A.4), that affect the world-line of the particle. Those non-linearities are a strong bridle towards the achievement of non-perturbatively solving the two-body problem.

But in addition to that, those non-linearities yield appealing physical implications, as they entail scattering of the gravitational waves. If a gravitational wave scatters off the static monopolar structure, the effect is called a *tail*. Indeed an observer located at infinity will see delayed arrivals of waves, the signal decaying as a power-law (typically as t^{-3}), thus the name. As for the wave-to-wave scatterings, they are usually dubbed *memory* effects, as they encode the whole history of the evolution of the binary. Those non-linear interactions in the radiative sector are treated in chapter IV. Such effects also affect the conservative sector, as the backreaction of gravitational waves changes the dynamics of the binary system. The tail contribution to the conservative sector is the object of chapter III.

II.2 The case of the exoplanet HD 80606b as an illustration

Before outlining the state-of-the-art relativistic methods and results in sec. II.3, we present a simple computation with rewarding observational consequences, that illustrates the strategy of using Newtonian-looking methods to perturbatively solve the two-body problem. Together with L. Blanchet and G. Hébrard, we have studied the binary system formed by the star HD 80606 and the exoplanet HD 80606b, and have shown in [95] that the lowest-order relativistic effects on such peculiar system are detectable and could thus provide a new channel to test GR.

By their conceptual simplicity and abundance in Nature, binary systems have been widely used to test the predictions of GR and alternative theories, in a large variety of regimes, from the Earth-Moon system [108, 346] to the vicinity of the central black hole of our galaxy [12, 21]. Using stellar binary systems was proposed in [207], but it is quite complicated to discriminate the relativistic precession from the Newtonian one due to tidal effects (see *eg.* [357]). To the best of our knowledge, our proposition is the only one to suggest a complete and realistic test in an exoplanetary system,[4] and thus the only one allowing to realize a Solar system-like test of GR at a distant location.

II.2.1 HD 80606b, a remarkable exoplanet

HD 80606b as the perfect candidate

For a stellar binary system of total mass M, the orbit is well described by a Newtonian one, *ie.* an ellipse of semi-major axis a and eccentricity e. The relativistic, or "post-Newtonian", corrections[5] to this orbit are of order $\frac{v^2}{c^2} \sim \frac{GM}{ac^2} \ll 1$, where v is the typical velocity of the system, and both quantities are of the same order of magnitude as we suppose that the bound system is virialized.

In such weak-field slow-motion regime, the dominant relativistic effect on the Newtonian orbit is the precession of the periastron, given by

$$\Delta_{\mathrm{GR}} = \frac{6\pi}{1-e^2}\frac{GM}{ac^2}\,. \tag{II.19}$$

Thus in order to have a substantial effect, one needs either a large compactness[6] $\frac{GM}{ac^2}$ or a large eccentricity. In addition, the relativistic precession should be clearly distinguishable from the Newtonian one, that can be caused by tidal effects, a third body, *etc.* As they are largely stiffer than stars (and so less subject to tidal precession), exoplanets are thus particularly recommended for such detection. In order to detect the effect, one should be able to easily observe the consequences of the precession. If such orbital variation should leave imprints on the radial velocity of the star,[7] its detection would

[4] In the final stage of our study, we found that similar methods have been suggested by [243, 300] and later by [239]. Nevertheless those works assume a central transit and only consider the effect of the periastron shift, when we take the full relativistic effects into account, with a non-vanishing impact parameter. Furthermore they are generic studies while we apply our computations to the remarkable case of HD 80606b and explicitly derive an observable quantity.

[5] As explained in I.3.1, we count perturbative "post-Newtonian" (PN) orders as powers of c^{-2}. Thus the nPN correction consists of the c^{-2n} order.

[6] Obviously if the compactness is too large, *ie.* if it approaches 1, the system departs from the weak-field regime. However for Solar system values, this corresponds to an orbital separation of a few km, which is naturally unreachable.

[7] Due to the presence of one or many exoplanets, the star moves around the center-of-mass of the system and this radial velocity induces an observable periodic Doppler shift in its spectrum. This technique is widely used to detect exoplanets and we let the interested reader refers to the reviews [272, 360].

require tremendous efforts in precisely monitoring the spectrum of the star. It would be much easier to observe a (anti-)transiting exoplanet, as the detection of precession-induced modifications of the transits would be much more perceptible. A similar method, called "transit-timing variation", is already used to detect exoplanets, *via* the precession they induce on a known, transiting planetary companion (see [229] for the proposition and [41] for the first detection using such technique).

Among the more that 4700 exoplanets detected (and confirmed) at the time of writing of this dissertation, 1847 have known eccentricity values, and only four of those exceed 0.9, according to the regularly updated catalog exoplanet.eu [323]. Among those four exoplanets, only HD 80606b has been observed (anti-)transiting, and is the only planet with measured mass, radius and inclination. Its eccentricity and the compactness of the binary system are respectively $e \simeq 0.934$ and $\frac{GM}{ac^2} \simeq 2.2 \cdot 10^{-8}$, to be compared *eg.* to the value for Mercury $e \simeq 0.206$ and $\frac{GM}{ac^2} \simeq 2.6 \cdot 10^{-8}$. This leads to the relativistic precession

$$\Delta_{\rm GR}^{\rm HD\,80606b} \simeq 215 \text{ arcsec/century}\,, \quad \text{to be compared with} \quad \Delta_{\rm GR}^{\rm Mercury} \simeq 43 \text{ arcsec/century}\,. \tag{II.20}$$

In addition, more than 15 years of radial-velocity monitoring did not provide a hint of any additional planetary companion [190, 223]: the exoplanet HD 80606b is thus the perfect candidate for the measure of the relativistic effects that we plan to conduct.

Detection and characterization of HD 80606b

As many other exoplanets, HD 80606b has been discovered through radial velocity measurements, in 2001 [290]. The relatively long orbital period of the planet, $P \simeq 111$ days, together with its high eccentricity implied a low probability of the orbital plan to be aligned with the line of sight to the Earth, and thus less than one chance over a hundred that it would transit. Fortunately, HD 80606b was successively discovered to pass behind its parent star [258] and in front of it [288] (see also [198, 204, 230, 356]), and the first monitoring of a full twelve-hour long transit has been conducted in 2010 [223]. Those observations allowed to derive the planetary radius $R_p \simeq 0.98\,R_{\rm Jup}$, and the inclination of its orbit $i \simeq 89°$. Together with the sky-projected mass (derived with radial velocities), the inclination provides the mass $M_p \simeq 4\,M_{\rm Jup}$. The obliquity of the system could also be measured from the Rossiter-McLaughlin anomaly[8] observed during transits, which revealed that the orbit of HD 80606b is prograde but quite inclined [288, 307, 356]: [223] reported an obliquity of $\lambda \simeq 40°$.

The host star, HD 80606, is a Solar-like star: its radius and mass are similar to the Sun's one and it is of the G5 spectral type. In addition, it is part of a binary system, together with HD 80607, located 1200 AU away. For the precise characterization of the stellar system and the associated accuracy, we let the interested reader refer to [223].

Other remarkable features of HD 80606b

The favored mechanism for the formation of such a peculiar orbit relies on a "Kozai-Lidov migration" [361], due to the influence of the stellar companion, HD 80607. Such mechanism was independently introduced by M. L. Lidov in the case of artificial satellites [266] and Y. Kozai in the case of asteroids [256], and consists in a specific resonance of the three-body problem. In the case of HD80606b, the formation of the orbit presumably happened as follows. First, the planet is formed with a small eccentricity ($e \lesssim 0.1$) and a wide orbit ($a \gtrsim 5$ AU), the angle between its orbital plane and the orbital plane of the stellar companion, α, being slightly less that $\pi/2$. Then Kozai-Lidov cycles take place, meaning that the orbits of the planet and of the stellar companion

[8] Due to the Doppler effect, the spectrum of a rotating star is not homogeneous on its apparent surface. When the planet transits, it screens only a part of this surface, which results in a change of the apparent color of the star. Monitoring this effect, dubbed Rossiter-McLaughlin anomaly, gives access to the angle between the orbital plane of the planet and the spin orientation of the star. See [337] for more details.

enter in resonance, the quantity $J_{\rm KL} = \sqrt{1-e^2}\cos\alpha$, being conserved. During those cycles (that are quite rapid, lasting a few Myrs) eccentricity and relative inclination of the two planes are thus exchanged. In addition, and during those many cycles, the energy of the system is dissipated through tidal effects, and therefore the orbit of the planet shrinks. Finally the Kozai-Lidov mechanism stops when the planet is close enough from its parent star so that the relativistic precession takes over the influence of the stellar companion. The computation realized in [361] shows that the current orbital parameters are reached only 1.2 Gyrs after the birth of the planet, and that the Kozai-Lidov mechanism does not occur anymore today. Note that, in order to reach this state, HD 80606b had to undergo through a much higher eccentricity, $e \simeq 0.99$!

For completeness, let us mention two other, disfavored, formation mechanisms for HD 80606b. This first consists in a planet-planet scattering, assuming that HD 80606b has been formed together with a planetary companion, in an unstable configuration. It would then have acquired its high eccentricity during the ejection process of the companion. Nevertheless, [197] has studied more than a thousand of such unstable systems, and none resulted in eccentricities larger than 0.90: the formation of HD 80606b *via* this channel is thus unlikely. The second discarded possibility relies on dynamical friction between the planet and the gas disks from which it formed. Such interactions have been investigated in [212], and were not efficient enough to produce eccentricities above 0.6.

In addition to those fascinating features, HD 80606b is also remarkable by its atmosphere properties. The transiting nature of the planet allows to study its atmospheric composition and, for example, traces of potassium were found [125]. But the most impressive aspect of this gas giant is that it grazes its host star: the periastron is only 0.03 AU away from the star, which corresponds to less than 8 times the radius of the star ! The surface temperature of the planet thus increases from $\sim$ 800 K to $\sim$ 1500 K within 6 hrs, inducing violent heat shock waves [258]. But this heating is so prompt that, unlike many "hot Jupiters", the mass loss of HD 80606b is absolutely negligible: the relative mass loss was estimated in [257] to be at most $3 \cdot 10^{-4}$ during the whole lifetime of the planet.

II.2.2 Computation of the relativistic effects on the trajectory

Geometry of the orbit and transits

As discussed in sec. II.1.3, the position and impulsion of the center-of-mass are conserved, up to 3.5PN order [77]. This accuracy is far from our current concern and we can assume that they are both perfectly conserved: as in the Newtonian case, we will only focus on the relative motion and thus work in a frame $\{x, y, z\}$ with origin at the center of the star. By convention, the observer is in the direction x, while the projected angular momentum of the star lies along the direction z. Hence the plane $\{y, z\}$ is the plane of the sky. The orbital plane of the planet $\{X, Y\}$ is defined by its inclination I with respect to the plane $\{x, y\}$, and by its longitude Ω with respect to the plane $\{x, z\}$, following the conventions described in fig. II.1a.

As depicted in fig. II.1c, the motion of the planet in the orbital plane is parametrized by polar coordinates (r, φ), where φ denotes the sum of the true anomaly and of ω_0, the latter denoting a constant angle, which can be viewed as defining the initial argument of the periastron at a reference time. Then the coordinates (x, y, z) of the planet in this frame read (see figs. II.1a and II.1c)

$$x = r\Big[-\cos\varphi\cos\Omega + \sin\varphi\cos I\sin\Omega\Big]\,, \tag{II.21a}$$

$$y = r\Big[-\cos\varphi\sin\Omega - \sin\varphi\cos I\cos\Omega\Big]\,, \tag{II.21b}$$

$$z = r\sin\varphi\sin I\,. \tag{II.21c}$$

We define the positions of the planet $\{T_1, T_2, T_3, T_4\}$ to be respectively the entry in the transit, the start of the full transit, and the exits from the full and partial transits. Similarly the points

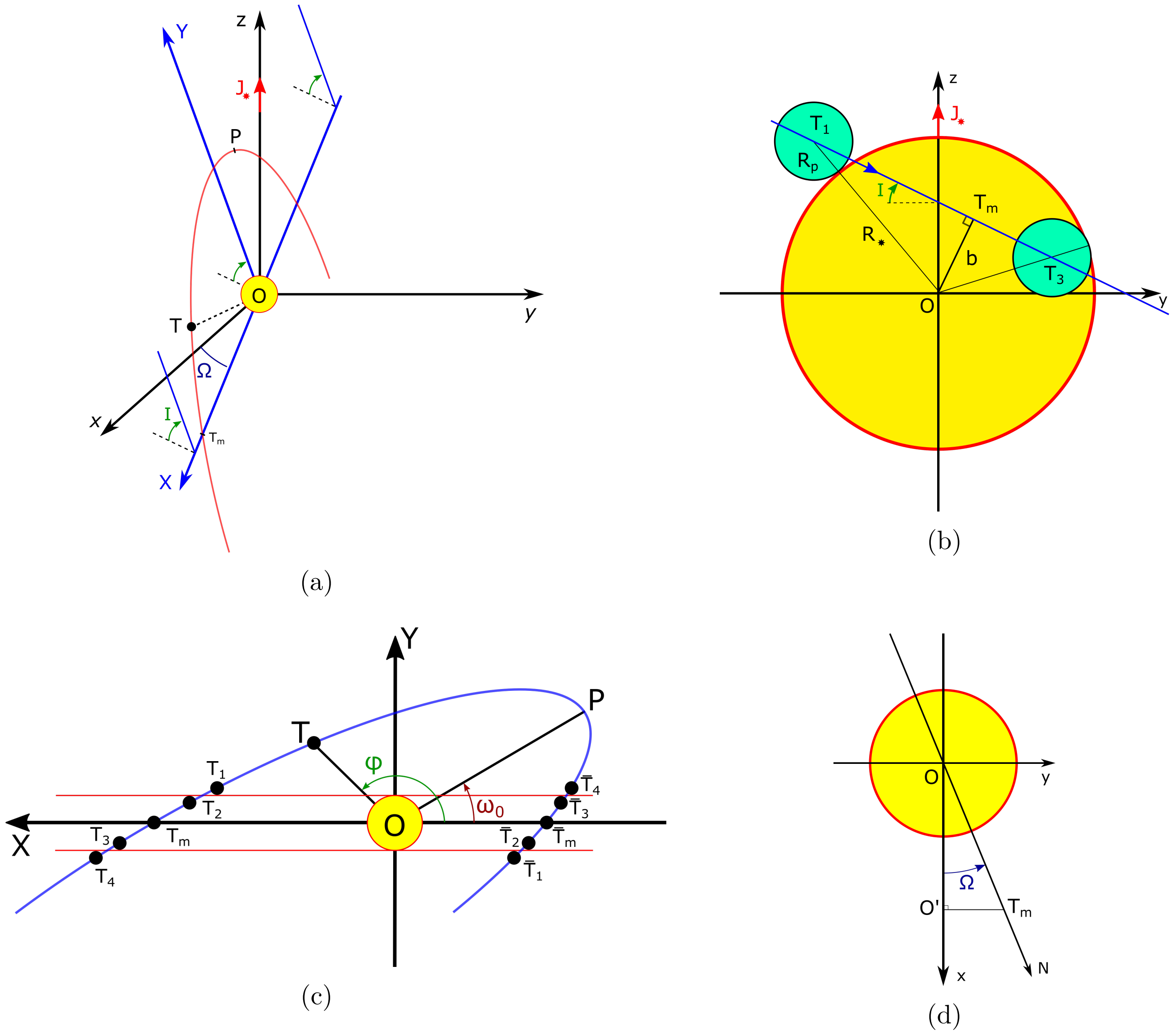


Figure II.1: Geometry of the planetary transit: (a) Motion of the planet with respect to the observer, located at infinity in the x direction. The center of the star is denoted O, P is the periastron at reference time t_P, and $J_\star$ is the angular momentum of the star projected on the plane of the sky $\{y, z\}$. (b) The transit as seen by the observer and the definition of the points $\{T_i\}$; here are only shown T_1 and T_3. (c) The trajectory of the planet and the points $\{T_i\}$ (transit) and $\{\bar{T}_i\}$ (eclipse) as seen in the orbital plane $\{X, Y\}$. Up to an angle Ω, the observer is in the direction of X. (d) View of the transit from above. The direction (ON) is the line of "nodes", corresponding to (OX) in (a).

$\{\bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4\}$ corresponding to the eclipse are shown in fig. II.1c. Since x is the direction of the observer, the condition for the (partial or complete) transit of the planet is that $y^2+z^2 \leqslant (R_\star+R_p)^2$, where $R_\star$ and R_p are the radius of the star and planet, hence

$$\sin^2 I \sin^2 \varphi + \Big(\cos I \cos\Omega \sin\varphi + \sin\Omega\cos\varphi\Big)^2 \leqslant \left(\frac{R_\star + R_p}{r}\right)^2 . \tag{II.22}$$

Since $R_\star + R_p \ll r$ at the transits, it is easy to see that this condition can be satisfied when $\sin I \ll 1$ or $\sin\varphi \ll 1$, and $|\cos I \cos\Omega \sin\varphi + \sin\Omega\cos\varphi| \ll 1$. With the conventions of fig. II.1a we are interested in a transit for which $\varphi \simeq \pi$ and $\Omega \simeq 0$.[9] On the other hand, for an eclipse (or anti-transit), we have $\varphi \simeq 0$ and $\Omega \simeq 0$. In an expansion to the first order in $\frac{R_\star+R_p}{r} \ll 1$, we find that

[9]For HD 80606b, [223] gives $\sin\Omega \simeq 0.019$.

the coordinates of the planet during and around the transit are

$$x \simeq \frac{b}{\sin I}\left[\cot\Omega + \cos I\,\sin\varphi\right], \tag{II.23a}$$

$$y \simeq \frac{b}{\sin I}\left[1 - \frac{\sin\varphi\cos I}{\sin\Omega}\right], \tag{II.23b}$$

$$z \simeq \frac{b\,\sin\varphi}{\sin\Omega}, \tag{II.23c}$$

where the impact parameter b of the planet is the distance between the trajectory of the planet projected on the plane $\{y, z\}$ and the center of the star. Neglecting the bending of the trajectory during the transits and eclipses,[10] the impact parameter is given by (see fig. II.1b)

$$b = \begin{cases} r(\pi)\sin\Omega\sin I & (\text{transits } T_i)\,, \\ r(0)\sin\Omega\sin I & (\text{eclipses } \bar{T}_i)\,, \end{cases} \tag{II.24}$$

where $r(\pi)$ and $r(0)$ are the values of the radial coordinate of the planet at $\varphi = \pi$ and $\varphi = 0$, when the transit and eclipse occur. The transit and eclipse conditions for the polar coordinates (r, φ) at the passage of these points read

$$r\sin\varphi = Y(b)\,. \tag{II.25}$$

Given a model $r(\varphi)$ for the planetary orbit, this determines the true anomaly φ for each of these points, where the vertical coordinate $Y(b)$ is given in terms of the impact parameter b by (see figs. II.1b and II.1c)

$$Y(b) = \begin{cases} \sqrt{(R_\star + R_p)^2 - b^2} + b\cot I & (T_1 \text{ and } \bar{T}_4)\,, \\ \sqrt{(R_\star - R_p)^2 - b^2} + b\cot I & (T_2 \text{ and } \bar{T}_3)\,, \\ -\sqrt{(R_\star - R_p)^2 - b^2} + b\cot I & (T_3 \text{ and } \bar{T}_2)\,, \\ -\sqrt{(R_\star + R_p)^2 - b^2} + b\cot I & (T_4 \text{ and } \bar{T}_1)\,. \end{cases} \tag{II.26}$$

The cases for which $(R_\star \pm R_p)^2 = b^2$ correspond to the planet just grazing tangentially the star, either from the exterior or from the interior of the stellar disc. In addition to the entry and exit of the (anti-)transit, $\{T_n\}$ and $\{\bar{T}_n\}$, we compute also the points corresponding to the minimal approach to the center of the star, T_m and $\bar{T}_m$, whose transit conditions are obviously (see fig. II.1b)

$$Y(b) = b\cot I \qquad (T_m \text{ and } \bar{T}_m)\,. \tag{II.27}$$

Now that we have clearly stated the geometry of the transits, we need to derive the dynamics of the system, in order to solve (II.25), and to deduce the instants t_i and $\bar{t}_i$ of passage at each of these points. Note that, since the eclipse will be seen from behind the star, we have to add to the instants of eclipse a Roemer time delay accounting for the propagation of light at finite velocity c. Thus $\bar{t}_i \longrightarrow \bar{t}_i + \Delta t_\mathrm{R}$ where

$$\Delta t_\mathrm{R} = \frac{r(0) + r(\pi)}{c} \simeq 2.7\ \text{min} \qquad (\text{for HD 80606b})\,. \tag{II.28}$$

[10] The correction induced by taking into account this effect is of the order of $\frac{\delta b}{b} \sim \frac{1}{2}\left(\frac{R_\star}{r}\right)^2$, so $\sim 10^{-4}$ for the transits and $\sim 10^{-2}$ for the anti-transits, in the case of HD 80606b.

Post-Newtonian motion of the planet

We will investigate the first relativistic correction to the dynamics of the two-body problem with a Lagrangian method. In [95], we have also used two other methods to derive it (relying on the 1PN Hamiltonian in Delaunay-Poincaré variables, and using a quasi-Keplerian representation of the motion [131]), and they gave excellent agreement. Note that the Hamiltonian and the hereafter presented Lagrangian computations rely on celestial perturbation computational methods, that were intensively developed during the XIX$^{\rm th}$ century. The "XX$^{\rm th}$ century" piece of the computation consists only in the explicit form of the perturbation function $\mathcal{R}$ defined hereafter.

The Lagrangian of the relative motion of two point masses at the first post-Newtonian order is simply an extension of eq. (II.2) and reads

$$\mathcal{L}_{\rm 1PN} = \frac{M}{2}V^2 + \frac{m}{2}v^2 - \frac{GMm}{r} + m\,\mathcal{R}\left(x^i, v^i\right) . \tag{II.29}$$

The 1PN perturbation function reads (see *eg.* [79])

$$\mathcal{R}\left(x^i, v^i\right) = \frac{1}{2c^2}\left[\frac{1-3\nu}{4}v^4 + \frac{GM}{r}\left(\nu\,\dot{r}^2 + (3+\nu)\,v^2\right) - \frac{G^2M^2}{r^2}\right] , \tag{II.30}$$

where we have defined the "symmetric mass ratio" $\nu \equiv m/M = m_1m_2/(m_1+m_2)^2$ and we recall that $\dot{r} = \mathrm{d}r/\mathrm{d}t$. The post-Newtonian approximation is well suited to the case of HD 80606b, as $|m\mathcal{R}/\mathcal{L}_{\rm 1PN}| \lesssim 10^{-5}$.

As the non-perturbed orbit is Newtonian, thus elliptic, we are going to use an osculating elliptic orbit, meaning that at each instant of time, we approximate the relativistic motion by an ellipse. Thus we use a Newtonian-looking parametrization of the orbit, which orbital elements, given by $\{a, e, \omega, \ell, I, \Omega\}$, are naturally allowed to depend on time. Together with the semi-major axis a, eccentricity e and argument of the periastron ω, we have defined the mean anomaly $\ell = n(t - t_P)$ where $n \equiv \sqrt{GM/a^3}$ and t_P is the (time-dependent) instant of passage at periastron. The two angles defining the orbital plane, I and Ω, are still defined as in fig. II.1a. For each successive ellipse, those orbital elements are linked to the usual $\{r, \varphi\}$ coordinates depicted in fig. II.1c by the Keplerian relations

$$r = a\left(1 - e\cos\psi\right) , \qquad \text{and} \qquad \varphi = \omega + 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\psi\right) , \tag{II.31}$$

where the eccentric anomaly ψ is solution of

$$\psi - e\,\sin\psi = \ell\,. \tag{II.32}$$

We are seeking for secular effects, *ie.* relativistic effects that accumulate over time. Therefore, we can directly apply the usual perturbation equations of celestial mechanics with the perturbation function $\mathcal{R}$ in the Lagrangian (see *eg.* [133]). Those equations are explicitly given by [83, 111]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{an}\frac{\partial\mathcal{R}}{\partial\ell} , \tag{II.33a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{ea^2n}\left[\sqrt{1-e^2}\frac{\partial\mathcal{R}}{\partial\ell} - \frac{\partial\mathcal{R}}{\partial\omega}\right] , \tag{II.33b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{a^2n}\left[\frac{\sqrt{1-e^2}}{e}\frac{\partial\mathcal{R}}{\partial e} - \frac{\cot I}{\sqrt{1-e^2}}\frac{\partial\mathcal{R}}{\partial I}\right] , \tag{II.33c}$$

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = n - \frac{1}{a^2n}\left[2a\frac{\partial\mathcal{R}}{\partial a} + \frac{1-e^2}{e}\frac{\partial\mathcal{R}}{\partial e}\right] , \tag{II.33d}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{a^2n\sqrt{1-e^2}\sin I}\left[\cos I\frac{\partial\mathcal{R}}{\partial\omega} - \frac{\partial\mathcal{R}}{\partial\Omega}\right] , \tag{II.33e}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{a^2 n\sqrt{1-e^2}\sin I}\frac{\partial \mathcal{R}}{\partial I}\,. \tag{II.33f}$$

Note that Kepler's third law does not hold here, as $n \equiv \sqrt{GM/a^3}$ is not equal to the orbital frequency $2\pi/P$. Instead the orbital frequency will be given by averaging eq. (II.33d) for ℓ.

As the perturbing function $\mathcal{R}$ (II.30) only depends on r, $\dot{r}^2$ and v^2, the only orbital elements entering it are a, e and ℓ (the latter entering *via* $\psi(e,\ell)$, solution of eq. (II.32)). As a direct consequence, I and Ω are constant in time: the orbital plane is fixed. This is directly related to the fact that, for a spinless particle, the angular momentum flux $\mathcal{G}^i$ is vanishing at 1PN order, see eq. (II.18).

As for the other quantities, we denote the zeroth order constant orbital elements as a_0, e_0, and similarly define $n_0 = \sqrt{GM/a_0^3}$. To first order the equations for a and e can readily be integrated as

$$a = a_0 + \frac{2}{a_0 n_0^2}\mathcal{R}\,, \qquad \text{and} \qquad e = e_0 + \frac{1-e_0^2}{e_0 a_0^2 n_0^2}\mathcal{R}\,, \tag{II.34}$$

where the perturbation function reduces in this case to

$$\mathcal{R} = \frac{G^2M^2}{2a_0^2c^2}\left[\frac{1-3\nu}{4} + \frac{\nu-4}{\mathcal{X}_0} + \frac{6+\nu}{\mathcal{X}_0^2} - \nu\,\frac{1-e_0^2}{\mathcal{X}_0^3}\right]\,, \tag{II.35}$$

with $\mathcal{X}_0 \equiv 1 - e_0\cos\psi$. Next, the equations for ℓ and ω at first order become

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = n_0 - \frac{3}{a_0^2 n_0}\mathcal{R} - \frac{1}{a_0^2 n_0}\left[2a_0\frac{\partial \mathcal{R}}{\partial a_0} + \frac{1-e_0^2}{e_0}\frac{\partial \mathcal{R}}{\partial e_0}\right]\,, \tag{II.36a}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e_0^2}}{e_0 a_0^2 n_0}\frac{\partial \mathcal{R}}{\partial e_0}\,. \tag{II.36b}$$

As we are mainly interested in the secular evolution of the orbit,[11] we perform the orbital average $\langle f\rangle \equiv \frac{1}{P}\int_0^P \mathrm{d}t\, f(t)$ of those equations before integrating them, with P being the orbital period. It is equivalent to directly average the equations (II.36) or to replace in them the perturbing function $\mathcal{R}$ by its average $\langle\mathcal{R}\rangle$

$$\langle\mathcal{R}\rangle = \frac{GMa_0n_0^2}{8c^2}\left(\frac{24}{\sqrt{1-e_0^2}} - 15 + \nu\right)\,. \tag{II.37}$$

It finally comes

$$\left\langle\frac{\mathrm{d}\ell}{\mathrm{d}t}\right\rangle = n_0 + \frac{GMn_0\,(\nu-15)}{8\,a_0c^2} \equiv n_0\,(1+\zeta)\,, \qquad \text{and} \qquad \left\langle\frac{\mathrm{d}\omega}{\mathrm{d}t}\right\rangle = \frac{3GMn_0}{a_0c^2(1-e_0^2)} \equiv n_0 k\,, \tag{II.38}$$

where we have introduced the relativistic precession parameter k (corresponding to the periastron advance) and the relativistic shift in the orbital period, ζ. Note that for HD 80606b,

$$\frac{k}{\zeta} = \frac{-24}{(15-\nu)(1-e_0^2)} \simeq -12\,, \tag{II.39}$$

the periastron advance is thus dominant over the shift in orbital period.

The relativistic evolution of the impact parameter is simply given by a perturbation of eq. (II.24),

$$\delta b = \mp\frac{b_0\,\sin\omega_0}{1\mp e_0\cos\omega_0}\,\delta\omega\,, \tag{II.40}$$

where b_0 is naturally the constant Newtonian value of the impact parameter, and the minus (resp. plus) sign corresponds to a transit (resp. eclipse). Injecting δb in eq. (II.26), one can solve the perturbed conditions for the (anti-)transit (II.25), and thus extract the relativistic modifications of the time of (anti-)transits.

[11] In [95], the computation has been done also for the periodic quantities, and the final results showed excellent agreement with the averaged method that we use here, the two methods differing only by 2% after 33 periods, *ie.* 10 years.

Seeking for an observable quantity

When dealing with relativistic corrections, it is of prime importance to study an appropriate observable quantity. For example, in the case of HD 80606b, despite 15 years of radial velocity data, [190] reported that the relativistic effects were not detectable. They measured the variation of the longitude of the periastron to be $\dot{\omega} = 9\,720 \pm 11\,160\,\mathrm{arcsec/century}$, thus in a non-significant way. This inconclusive result is explained by the insufficient accuracy reached on ω: $\pm\,540\,\mathrm{arcsec}$ [223].

As briefly explained in the beginning of sec. II.2.1 and clear from the previous computation, our method does not rely on radial velocities but rather on the variation of the time of transits. We thus have to construct an appropriate observable quantity with those times and compare it to a reference one, quite naturally defined at the first complete observation of the transit, realized in January 2010 [223]. For the N^{th} orbit following the reference one, we defined the post-Newtonian time corrections

$$\delta t_i(N) \equiv t_i(N) - t_{0,i}(N)\,, \qquad \text{and} \qquad \delta \bar{t}_i(N) \equiv \bar{t}_i(N) - \bar{t}_{0,i}(N)\,, \tag{II.41}$$

where $i \in (1, 2, m, 3, 4)$ and the index 0 denotes the Newtonian prediction: $t_{0,i}(N) = t_{0,i}(0) + NP_0$ and $\bar{t}_{0,i}(N) = \bar{t}_{0,i}(0) + NP_0$, where P_0 is the Newtonian period of the orbit, evaluated in January 2010. Note that taking the Newtonian predicted times as reference times is merely a convenient trick, as Newtonian orbits do not exist: the "real" motion of the system is obviously the fully relativistic one. But, as P_0 is known, it is quite convenient to use those Newtonian orbits as references.

After 10 years (*ie.* for $N = 33$), the post-Newtonian time corrections for the transit times are of order of 170 s. As the measure of Jan. 2020 had an accuracy of 85 s. [223], this indicates that the effect could be measurable in this way. But such measure relying only on the variation of the entrance time t_1, for instance, would require a extreme precision on the Newtonian period, P_0, that is not yet achieved. In order to have a realistic observable quantity, one should eliminate this uncertainty by comparing durations instead of absolute times. Unfortunately, the relativistic variation of the transit time $t_4 - t_1$ is only of order 2 s. after 10 years, and thus seems hardly detectable (as for the eclipse, it is even worse, of the order of 0.2 s). Instead, we propose to monitor the time interval between the passage at the minimum approach point during an eclipse and the passage at the minimum approach point during the following transit:

$$t_{\text{tr-ec}}(N) - t_{\text{tr-ec}}(0) = t_m(N) - \bar{t}_m(N) - \Big[t_{0,m}(N) - \bar{t}_{0,m}(N)\Big]\,. \tag{II.42}$$

The important point about this result is that the right-hand side of the equation represents the relativistic prediction, that we have previously computed, while the left-hand side is directly measurable. In fact $t_{\mathrm{tr-ec}}(0)$ has been measured [223, 258]

$$t_{\mathrm{tr-ec}}(0) = (5.8491 \pm 0.003)\,\mathrm{days}\,, \tag{II.43}$$

where the precision is 275 seconds. The relativistic effect becomes of the same order than this precision after 49 orbits (corresponding to the transit of December 2024), the time interval between the eclipse and transit being shorter by 271.4 seconds (4.5 minutes) than to the one measured in 2010. Therefore we conclude that, starting from 2025, observational campaigns monitoring the eclipses and transits of HD 80606b should be able to detect the relativistic effects .

II.2.3 Feasibility of the measure

The observational uncertainty on the $t_{\mathrm{tr-ec}}(0)$ measure ($\pm\,271\,\mathrm{sec}$) is a combination of the accuracies of $\pm\,260\,\mathrm{sec}$ [258] and $\pm\,85\,\mathrm{sec}$ [223] that were respectively obtained on the mid-points of eclipse and transit, using the IRAC instruments of the *Spitzer Space Telescope*.

Precision of the computation

As for the accuracy of the relativistic computation, one can estimate the effects of the second post-Newtonian order to be of relative order $5 \cdot 10^{-6}$ [95]. Stopping at the first post-Newtonian order is thus accurate enough.

The main source of error in the relativistic computation comes from the accuracy of the physical parameters of the system. In order to estimate their repercussions, we have saturated one-by-one the uncertainties tabulated in [223], and have re-evaluated our observable quantity after 33 cycles. It appeared clearly that the major uncertainty comes from the measure of the mass of the star: it is known at the 5% level and induces a 5% error on our prediction. Similarly the value of the semi-major axis, known up to 1.7% leads to a 1.7% uncertainty in our result. Note that, as expected, the physical properties of the planet play a marginal role: the mass of HD 80606b is constrained up to 3%, but it impacts our derivation by 10^{-4} only, as it only enters ν. Note also that using the averaged method that we presented in this dissertation yields results that differ by 2% only from the exact methods used in [95], thus largely under the uncertainty due to the mass of the star. Such convenient averaging procedure can thus be used without problems. Similarly, neglecting the bending of the trajectory induces a error under the percent level, thus sub-leading.

Other disturbing effects

As presented in sec. II.2.1, HD 80606b has no known planetary companion. To be more precise, companions with sky-projected masses larger than $0.5\, M_{\mathrm{Jup}}$ and orbital periods shorter than 40 years are excluded [190, 223]. In addition, the stellar companion HD 80607 is currently too far away (1200 AU) to significantly influence the orbital precession of HD 80606b: we do not expect disturbing effects from third bodies. To compare with the usual Solar system case, the periastron advance of Mercury is largely dominated by the Newtonian effect of the other planets, mainly Venus and Jupiter: $\Delta_{\mathrm{planets}} = 532.3\,\mathrm{arcsec/century} \simeq 12.4\,\Delta_{\mathrm{GR}}$. In this regard the HD 80606 system is much cleaner than the Solar system.

Besides the possibility that there might be some disturbing third bodies, we have to consider the precession of the orbit caused by the oblateness of the star, the tidal effects, and the Lense-Thirring effect. The orbital precession rate due to the quadrupolar deformation J_2 of the star is

$$\Delta_{J_2} = \frac{3\pi J_2 R_\star^2}{a^2(1-e^2)^2} \,. \tag{II.44}$$

Assuming the Solar value $J_2 \sim 10^{-7}$ we obtain $\Delta_{J_2}/\Delta_{\mathrm{GR}} \sim 2\cdot 10^{-3}$ for HD 80606b, which is definitively negligible for our purpose. As for the orbital precession induced by the tidal interaction of the planet with its parent star, it produces a supplementary orbital precession at the rate [183, 243]

$$\Delta_{\mathrm{T}} = 30\pi \left(k_p \frac{M_\star R_p^5}{M_p} + k_\star \frac{M_p R_\star^5}{M_\star} \right) \frac{1 + \frac{3}{2}e^2 + \frac{1}{8}e^4}{a^5(1-e^2)^5} \,, \tag{II.45}$$

where k_p denotes the Love number of the planet, which is approximately $k_p \sim 0.25$ for a hot Jupiter, and $k_\star$ that of the star, expected to be approximately $k_\star \sim 0.01$ [243]. We find that the first term, due to the tidal deformation of the planet, gives a rather large contribution of about 32 arcsec/century, while the second term, due to the deformation of the star, is much smaller, approximately 2 arcsec/century. Hence the tidal interactions represent a non-negligible effect, but nevertheless six times smaller than the GR relativistic precession:

$$\Delta_{\mathrm{T}} \sim 34\,\mathrm{arcsec/century} \simeq 0.16\,\Delta_{\mathrm{GR}} \,. \tag{II.46}$$

Hence we conclude that the tidal interactions should be taken into account in the precise data analysis of the transit times aimed at measuring the GR effect. Finally the Lense-Thirring effect, due to the angular momentum $J_\star$ of the star, induces a precession of the line of nodes with a rate of

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{2G\,J_\star}{c^2 a^3 (1-e^2)^{3/2}}\,, \tag{II.47}$$

when $M_p \ll M_\star$. Considering the extreme case of a spin-orbit alignment and assuming a solar value $J_\star \sim 10^{42}\,\mathrm{kg.m^2/s}$, it is negligible as expected: $\Delta_{\mathrm{LT}}/\Delta_{\mathrm{GR}} \sim 3\cdot 10^{-4}$ arcsec/century. To compare with the Solar system, those three effects are negligible for Mercury, for which they are on the order of $\Delta_{J_2} \simeq 0.2$ arcsec/century, $\Delta_{\mathrm{T}} \simeq 2\cdot 10^{-6}$ arcsec/century, and $\Delta_{\mathrm{LT}} \sim 2\cdot 10^{-3}$ arcsec/century.

The last possible effect, the modification of the orbit by the mass loss that can happen in hot Jupiter systems [106, 263], is absolutely marginal here as the mass losses of HD 80606b are negligible [257], as already advertised.

Practical implementation of the observation

Starting from 2025, an instrument with a perfect accuracy could make the clear detection of the relativistic precession in the HD 80606 system, thus (*a priori*) confirming GR in a distant stellar system.

More realistically, a 12 hour-long observation of the full transit is required to make such detection, which implies necessarily a space-borne instrument. If the 2010 observations were conducted with the *Spitzer Space Telescope* and its IRAC instruments, this spacecraft has been decommissioned at the end of 2020. Fortunately, at the time of writing of this dissertation, the *James Webb Space Telescope* (JWST) is planned to be launched in October 2021. Thanks to its NIRCam and MIRI instruments, it would provide an accuracy on $t_{\mathrm{tr-ec}}$ a few times better than *Spitzer*, and thus could detect the effect (if the mission is not delayed eternally). Naturally other current or future telescopes, such as TESS, CHEOPS or PLATO, could also sample long-duration transits and perform the measure. However such instruments are smaller and less sensitive than *Spitzer* or JWST.

Among the telescopes dedicated to the study of exoplanets, the *Atmospheric Remote-sensing Infrared Exoplanet Large-survey* (ARIEL) instrument, to be launched around 2030, is devoted to the study of their atmospheres (see *eg.* [206]). As HD 80606b has remarkable atmospheric properties, it is already listed as a potential target for ARIEL [173]. It would thus be clever to use this opportunity of performing both observations (atmospheric and relativistic effect) at once. The feasibility of the measure with the ARIEL spacecraft is thus currently discussed with J.-P. Beaulieu, the coPI of the mission, and A. Bocchieri, in charge of simulating the noise of the instrument. The first results give good hopes that ARIEL would successfully achieve the detection, and that the relativistic effects will be measured in the HD 80606 system.

II.3 Different approaches to tackle the relativistic problem

II.3.1 Weak-field, slow-motion approaches

The relativistic computation of the previous section relies on a weak-field, slow-motion approximation, *ie.* by expanding the equations of motion in terms of the compactness $\frac{GM}{rc^2}$ and the square of the dimensionless speed $\frac{v^2}{c^2}$, both being of the same order of magnitude for virialized bound systems. In addition we have seen that post-Newtonian (PN) effects begin at the formal c^{-2} order, therefore it is usual to denote the c^{-2n} correction to the Newtonian motion as a "nPN effect". Such an approach has been developed as early as 1917, by H. A. Lorentz and J. Droste, in order to tackle the first relativistic correction to the N-body problem [270], and rediscovered more than twenty years later by A. Einstein, L. Infeld and B. Hoffmann [180]. Since those works, it has been extensively applied to the two-body problem, and has known many developments, to reach extremely high accuracy.

The weak-field, slow-motion regime is particularly adequate for bound, virialized binary systems, and is valid during the inspiralling phase (when the two bodies are well enough separated). This approximation naturally fails during the last cycles of the binary, when the velocities of the bodies become comparable to the speed of light. To describe this extreme regime, one has to rely on other frameworks like numerical relativity. Note that resummation methods have been developed to push forward the validity of the expansions, for example the "effective one-body" framework [115]. In the following, we intend to outline the modern computational methods that rely on weak-field, slow-motion approximations for spinless point-particles.

Traditional post-Newtonian approaches

Concerning the conservative sector, the state-of-the art is the 4PN approximation.[12] This accuracy has first been reached by an ADM Hamiltonian method [134], together with an action-angle-Delaunay averaging [135], both agreeing with previous partial results [136]. But, quoting their own developers [134], those Hamiltonian-based methods are only «incomplete representations of the two-body conservative dynamics». Indeed they involve an *ambiguity parameter*, that has to be fixed by a careful matching with the radial potential in the circular limit, that is known by effective one-body techniques [61]. This ambiguity parameter is related to the near-zone non-linear tail effect: the conservative dynamics becomes non-local in time at 4PN, which is known since long time [74]. The Hamiltonian methods are unable to properly deal with this time non-locality, and therefore they cannot achieve the complete conservative dynamics from first principles.

Using the Fokker Lagrangian integration technique, a first computation of the conserved dynamics was also plagued by an ambiguity coming from the tail effect [54]. If such ambiguity parameter could also be fixed *via* a matching to the test mass limit results [55], its genuine derivation has been achieved [56, 277], yielding the first computation of the conservative sector from first principles. The physical quantities finally agreed with the Hamiltonian results [57]. The fact that, conversely to the Hamiltonian method, this method could deal with the time non-locality is highly non-trivial. It relies on the fact that the radiative sector can be treated by using the same (harmonic) coordinate system. Thus, computing the tail term with the methods developed for the radiative sector, one can integrate it in the conservative computation in a natural way.

This method used to compute the radiative sector is the so-called "post-Newtonian-multipolar-post-Minkowskian" (PN-MPM) scheme, that is currently the only one to reach high accuracy. As advertised by its name, it combines a post-Newtonian expansion in the near-zone and a post-Minkowkian one (PM, *ie.* in powers of Newton's constant G) in the radiative zone. Those two expansions have an overlapping region of validity, dubbed matching zone. In this buffer zone, the two expansions should obviously agree with each other, which is realized if the PN expansion of the multipolar solution agrees with the multipolar expansion of the PN solution. Using overbars to denote PN expansions, and $\mathcal{M}$ for PM ones, this matching requirement can be restated by the elegant equation

$$\mathcal{M}\left(\overline{g_{\mu\nu}}\right) = \overline{\mathcal{M}\left(g_{\mu\nu}\right)}\,. \tag{II.48}$$

Using this PN-MPM approach, the waveform and flux are known up to 3.5PN accuracy, see [73] for a review. The 4PN order is currently under investigation, and will be discussed in great details in chapter IV. Note that the 4.5PN sector of the gravitational flux is already known [276], as it does not rely on the knowledge of the 4PN one, and has been confirmed by the technique of factorized waveforms [281].

[12] Partial 5PN and 6PN results have been obtained [63, 64], but only the 4PN order is fully mastered.

Effective field theoretical approach

The effective field theoretical (EFT) approach of the two-body problem has been initiated in 2004 [210], and relies on an effective field treatment of the matter sector: the Lagrangian is composed of the most general set of operators that are invariant under diffeomorphisms, with unknown "Wilsonian" coefficients that are to be determined by matchings, see *eg.* [193, 308] for reviews. This approach led to the derivation of the 4PN dynamics in either Lagrangian [194, 196] or Hamiltonian [98] formulations. Naturally, both those computations are in perfect agreement with the results derived by traditional PN methods. As for the radiative sector, the EFT techniques reached the 2PN accuracy [264], and efforts are currently made to push forward higher precisions.

The EFT community is currently developing "bound-to-boundary" relations [245, 246], as dictionaries relating observable quantities of unbound systems (like the deflection angle of two scattering black holes) to physical quantities of bound systems (like conserved energy or periastron advance), *via* some well-chosen analytic continuations. (Note that such dictionaries were first introduced in the language of traditional PN methods [68]). Therefore, in addition to direct computations of bound quantities, it is interesting to perform post-Minkowskian expansions for unbound system, *ie.* releasing the small-velocity approximation [247]. In this vein, the conservative dynamics of a binary system has been computed at 3PM from an EFT perspective [249], recovering previous results exposed in the next subsection [48, 49].

In addition, synergies between the traditional PN and EFT communities, such as the one presented in the next chapter, are exciting paths towards novel and efficient computational methods. If not common today, those synergies may play a crucial role in the coming years.

II.3.2 Other approaches

Other approaches to tackle the relativistic two-body problem have naturally been developed in parallel to the weak-field slow motion ones. For instance the non-perturbative, numerical approach has been initiated during the 70's [181, 330], at the crossroads between the first efficient computers and vital theoretical breakthroughs, consisting in recasting Einstein's field equations as a Cauchy problem (see notably the work of Y. Choquet-Bruhat, *eg.* [124]). If the first numerical integrations of the relativistic field equations were not very reliable, this field has known great breakthroughs and became competitive in 2005 [40, 117, 310]. It is yet an extremely active field of research and its results are widely used for the detection of gravitational radiation, see *eg.* [66] for a review.

Another (perturbative) technique to deal with the relativistic two-body problem is the so-called "gravitational self-force" approach. Such method considers a particle of mass m evolving in the space-time created by a mass $M \gg m$, and expands the field equations in terms of the small quantity m/M, see *eg.* [42] for a review. At zeroth-order, the motion is simply given by the geodesics of the Schwarzchild (or Kerr, if spin is included) space-time. The first-order corrections are explicitly known up to extremely high post-Newtonian orders: 21.5PN for analytical computations [251] and 22PN for numerical integration [200]. The first attempts to push the method towards second-order encountered strong but unphysical divergences on the world-line of the small object, that are currently being understood and regularized [339]. Note that the future LISA detector is expected to observe many "extreme mass ratio inspirals" events, *ie.* gravitational waves that are generated by small bodies orbiting gigantic black holes. The gravitational self-force approach is obviously particularly indicated to provide templates for such events.

Inspired by the laws of thermodynamics, the first law of black hole mechanics relates the variations of the mass, spin and surface area of a black hole [43]. In the case of a binary of compact objects, on a circular orbit, a similar law has been found to relate the variations of the total ADM mass, total angular momentum and both individual masses [199, 261]. This "first law of binary point-like particle mechanics" has been then generalized to include spins [92], eccentricity [259], non-linear tail effects [82], and finally finite-size effects [311]. Those relations are widely employed when dealing with the two-body problem, and we will explicitly use them in chapter III.

The last, but definitively not least, present-day method to address the relativistic two-body problem is the use of scattering amplitude techniques. Taking a particle-physicist point of view, the scattering of two black holes is not so different from the scattering of two quarks, as both processes are made by the interactions of tensorial particles (gravitons or gluons), in a highly non-linear theory. The similarity is even more striking, as the vertices of the graviton interactions can be extracted from those of gluonic interactions: this is the famous "double copy" [47]. Therefore, from the quantum amplitudes involving gluons, one can extract "amplitudes" of the scattering process for two black holes, and derive observable quantities, such as the conserved energy or angular momentum. The equivalent of the gluon coupling constant is naturally the strength of the gravitational force, G, and thus those amplitude techniques are formally post-Minkowskian expansions, valid regardless of the value of the speed. The last decades saw tremendous efforts to push forward the comprehension and computations of Quantum Chromodynamics processes, so the computational techniques used to derive gluonic amplitudes can be mapped to the relativistic problem. This brilliant idea led to the computation of the dynamics at the third post-Minkowskian order [48, 49], and partial results (the potential contributions) have been achieved at 4PM accuracy [51]. Those computations are naturally valid for unbound, scattering black holes, but they can lead to results for bound orbits when a dictionary such as the one described in the previous section is employed.

In addition to be extremely interesting *per se*, those methods are vital for the post-Newtonian computations, as they derive the same observables (conserved energy, gravitational phase,...) but for different regimes of validity. Agreement of physical results in the overlapping regions is thus a strong validation of our computations. For instance, we always derive the $m_1/m_2 \to 0$ limit of a post-Newtonian observable to compare it with the corresponding gravitational self-force result. Furthermore, those approaches are usually very powerful in different regions. Combining computations with different approaches, depending of their strengths, allows to derive high-accuracy quantities in a drastically simpler manner than with a single approach. Such "tutti frutti" method has been used to derive partial 5 and 6PN results in the conservative sector [63, 64].

II.3.3 Beyond the point-particle approximation

Just as in the case of the study of HD 80606b, the work done during this thesis focuses on spinless, point-particle modelizations of compact objects. Nevertheless, all the above presented frameworks are naturally generalizable to include spins and finite-size effects.

We will not review the tremendous efforts that have been put in including spins and finite-size effects in the two-body problem, but only sample some of the latest results, and let the interested reader refer to references therein. For the traditional PN computations, the results with spins effects are gathered in [73], see references therein. A double copy framework including spins has been derived, and applied up to 2PM to the spin-spin coupling [50]. As for the EFT method, the conservative and radiative spin-spin effects at next-to-leading order have also been recently computed [123], and were found to perfectly agree with previous results.

Regarding finite-size effects, which enter at 5PN, they are known up to 2.5PN [225, 226, 227] and 2PM [122, 248] beyond the leading order, those results perfectly agreeing in the overlapping regions, as proven in [227].

Chapter III

Conservative sector

The main contributions of this thesis to the post-Newtonian field belong to the radiative sector, and are exposed in the next chapter. As for the contribution to the conservative sector, it consisted in the derivation of the logarithmic "tail" effects, *ie.* the impact of non-linear radiative processes on the conservative dynamics. Those computations were pushed to high accuracy, using new synergies between traditional PN and EFT methods. The technical parts of this chapter, presented in sec. III.2 and III.3, are thus extensively based on the work done in collaboration with L. Blanchet, S. Foffa and R. Sturani, published in [96].

III.1 The tail effects in General Relativity

As advertised in sec. II.1.4, the non-linear nature of GR yields interesting "tail" effects, that are known since a long time [103, 335] and were formalized in the multipolar post-Minkowskian context[1] in [74, 75].

From a phenomenological point of view, the tails can be understood as scatterings of gravitational waves onto the static curvature of the space-time. Those scatterings induce effects in both conservative and radiative sectors. In the conservative sector, back-reactions of the waves into the binary modify its motion, and thus naturally change its dynamics. In the radiative sector, waves that are scattered back into the observers line of sight will induce a typical power-law decay of the signal, thus the name of "tails". The tail effects for an instantaneous burst of gravitational radiation is schematically represented in fig. III.1.

In the multipolar post-Minkowskian language of app. B, the three-dimensional[2] linearized metric can be expressed as

$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathrm{M}_L\right)\,, \tag{III.1a}$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{M}^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{ijk}\partial_{jL-1}\left(\frac{1}{r}\mathrm{S}_{kL-1}\right)\right\}\,, \tag{III.1b}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}\mathrm{M}^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{kL-2}\left(\frac{1}{r}\varepsilon_{kl(i}\mathrm{S}^{(1)}_{j)L-2}\right)\right\}\,, \tag{III.1c}$$

with M_L (S_L) the mass (current) type canonical multipole moments, numbers in parenthesis denote time derivatives and the indices L are shorthands for ℓ indices, *eg.* $\partial_L \equiv \partial_{i_1 i_2 \dots i_\ell}$, see app. A.1.3. The

[1] As tail effects are radiative by nature, we will make extensive use of the radiative machinery in this chapter. We thus strongly invite the unfamiliar reader to use the post-Minkowskian toolkit of app. B.

[2] Here and henceforth, the split between space and time has been performed, and we speak about spatial dimensions. In the following, we will use d-dimensional quantities (d being arbitrary) for regularization purposes.

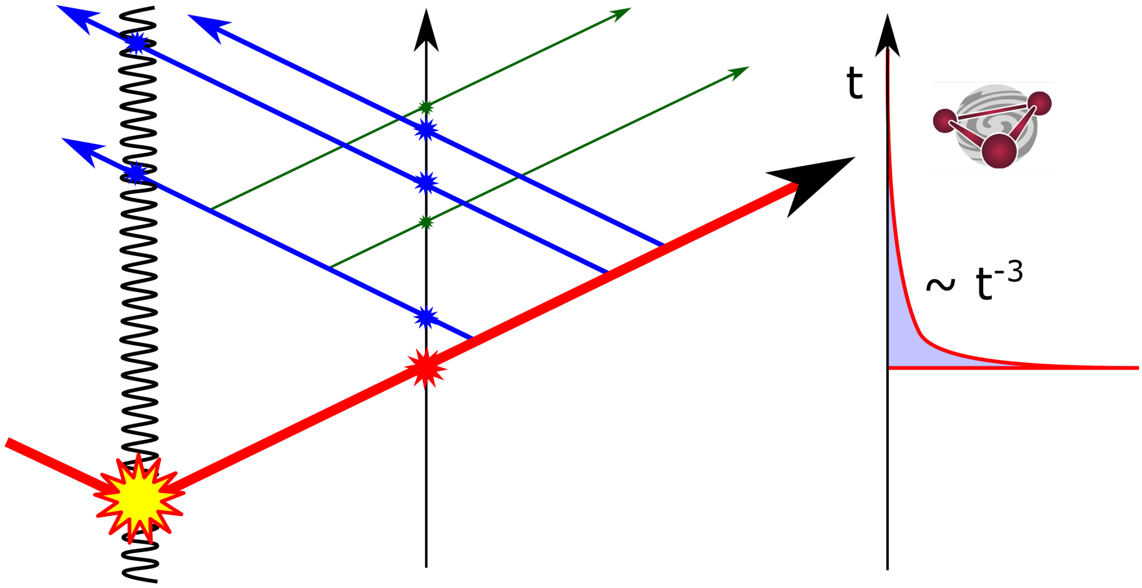


Figure III.1: Schematic effects of the scattering of gravitational waves onto the static curvature. In this space-time diagram, time flows from bottom to top, whereas the transverse direction is the radial dimension. In red is the primary propagation of a flash of radiation; some of the tail effects are represented in blue; in green are the tail-of-tail ones (higher scattering processes are omitted). The black straight line represents the world-line of an observer located at infinity, *eg.* the LISA instrument. Both conservative effects (*via* the back-reactions of the waves into the dynamics of the binary) and radiative effects (*via* the delayed arrivals of gravitational radiation) are to be observed.

$\ell = 0$ mass moment, M, is simply identified with the conserved ADM mass M. The tails can thus be understood as the interplay of multipolar moments in the quadratic (and higher) iterations, that arise when injecting this linear metric in the non-linear source $\Lambda^{\mu\nu}$, as explained in app. B.2.3. The interaction $\mathrm{M} \times \mathrm{M}$ is nothing but the PM expansion of the Schwarzschild geometry, and the dipole moment M_i is naturally vanishing in the center-of-mass. Thus the leading interesting interaction is $\mathrm{M} \times \mathrm{M}_{ij}$, *ie.* the interaction of the quadrupolar radiation and the static mass, which is nothing but the simple tail process. Similarly, the octupolar tail is the interaction $\mathrm{M} \times \mathrm{M}_{ijk}$, the quadrupolar tail-of-tail is given by $\mathrm{M} \times \mathrm{M} \times \mathrm{M}_{ij}$, *etc*, and this is trivially generalized to current type tails. Such formalism has been developed in the radiative zone, and thus directly applies to the radiative sector. In the waveform, the simple tails appear at 1.5PN order and each iteration is 1.5PN higher than the precedent. They have been computed (using three-dimensional techniques) up to 4.5PN [276], *ie.* including the tail-of-tail and tail-of-tail-of-tail contributions.

In the conservative sector, the mathematical emergence and proper treatment of tails is more subtle, as they arise *via* the crucial matching of the near zone and far zone solutions. This matching has to be dealt with great care by a well chosen and properly regularized homogeneous solution of the PN-expanded d'Alembertian operator. Understanding how to treat this term was crucial to fix the last ambiguity parameter in the 4PN conserved energy from first principles, see [56, 277]. Those simple tails appear thus in the conserved energy starting at the (state-of-the-art) 4PN order and each iteration is again 1.5PN higher than the precedent. The 1PN corrections to the simple tail contributions have been derived [91, 132, 261], as well as the 2PN corrections, but only in the gravitational self-force limit [62]. The leading tail-of-tail effects (thus entering at 5.5PN in the energy) are also known in the gravitational self-force limit [62].

An extremely interesting feature of those tails is that they are non-local in time. Indeed, at a given instant, the binary receives scattered waves from all of its previous emissions, and thus its dynamics is influenced by its whole history. In the simple case of quasi-circular orbits, this non-locality in time implies that the conserved energy bears logarithmic dependencies in the orbital frequency. In the following of this chapter, we will focus on those tail-induced logarithmic contributions in the energy. As the subtleties linked with the 4PN ambiguity came from the instantaneous sector of the tail effects [56, 277], we can safely neglect them in the following of the chapter. It thus follows that we can directly do our computation in a three-dimensional space, as there is no need for d-dimensional regularization schemes.

III.2 The logarithmic simple tail contributions

III.2.1 An effective action for the simple tail terms

The starting point of this computation is an effective action, that was proposed in [195], for all the non-local simple tails, involving all multipolar contributions $\ell \geqslant 2$. This action only considers non-local terms and can be recognized as being part of the effective action of EFT or equivalently, the Fokker action of traditional PN methods (see *eg.* eq. (3.18) of [277]). Originally written in the frequency domain, it reads

$$S^{\rm nl}_{\rm tail} = -\sum_{\ell=2}^{+\infty} \frac{2G^2M}{c^{2\ell+4}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k_0}{2\pi} \ln\Big(|k_0|\ell_0\Big)\, k_0^{2\ell+2} \left[a_\ell\, |\tilde{I}_L(k_0)|^2 + \frac{b_\ell}{c^2}\, |\tilde{J}_L(k_0)|^2\right], \tag{III.2}$$

where M is the ADM mass, $I_L(t)$ and $J_L(t)$ denote the mass and current type source multipole moments[3] with Fourier integrals

$$\tilde{I}_L(k_0) = \int_{-\infty}^{+\infty} \mathrm{d}t\, I_L(t)\, e^{\mathrm{i}k_0 t} \quad \text{and} \quad \tilde{J}_L(k_0) = \int_{-\infty}^{+\infty} \mathrm{d}t\, J_L(t)\, e^{\mathrm{i}k_0 t}\,. \tag{III.3}$$

In traditional PN methods the mass and current multipoles are defined in harmonic coordinates by the metric (III.1). In (III.2) ℓ_0 is an arbitrary energy scale coming from dimensional regularization, defined by the ratio of Newton constants in d and three dimensions, see eq. (A.7). Finally the coefficients in (III.2) are exactly those which appear in the multipole expansion of the gravitational wave energy flux [335], namely

$$a_\ell = \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}\,, \qquad \text{and} \qquad b_\ell = \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}\,. \tag{III.4}$$

In the time domain, the non-local action (III.2) becomes

$$S^{\rm nl}_{\rm tail} = \sum_{\ell=2}^{+\infty} \frac{G^2M}{c^{2\ell+4}} \int_{-\infty}^{+\infty} \mathrm{d}t \left[a_\ell\, I_L^{(\ell+1)}(t)\, \mathcal{I}_L^{(\ell+1)}(t) + \frac{b_\ell}{c^2} J_L^{(\ell+1)}(t)\, \mathcal{J}_L^{(\ell+1)}(t)\right], \tag{III.5}$$

where we have defined the non-local pieces

$$\mathcal{I}_L(t) \equiv \int_0^{+\infty} \mathrm{d}\tau\, \ln\left(\frac{\tau}{\tau_0}\right) \left[I_L^{(1)}(t-\tau) - I_L^{(1)}(t+\tau)\right], \tag{III.6}$$

together with the same definition for the functional $\mathcal{J}_L[J_L]$. Here the scale τ_0 is related to ℓ_0 by $c\tau_0 \equiv \ell_0\, e^{-\gamma_{\rm E}}$, with $\gamma_{\rm E}$ being the Euler constant.[4] The expression (III.6) is equivalent to the following form, involving the Hadamard finite part prescription in terms of the scale τ_0,

$$\mathcal{I}_L(t) = \mathop{\rm Pf}_{\tau_0} \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|}\, I_L(t')\,, \tag{III.7}$$

[3]We recall that they differ from the mass and current type canonical multipole moments M_L and S_L by 2.5PN corrections [187].

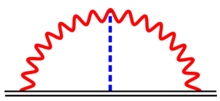
[4]For a conservative dynamics, the source moments are time symmetric, $I_L(-t) = I_L(t)$, and we can check that the definition (III.6) is also time symmetric, *ie.* $\mathcal{I}_L(-t) = \mathcal{I}_L(t)$.

in terms of which the time-domain action (III.5) takes the elegant form

$$S_{\rm tail}^{\rm nl} = \sum_{\ell=2}^{+\infty} \frac{G^2 M}{c^{2\ell+4}} \mathop{\rm Pf}_{\tau_0} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} \left[a_\ell \, I_L^{(\ell+1)}(t) I_L^{(\ell+1)}(t') + \frac{b_\ell}{c^2} J_L^{(\ell+1)}(t) J_L^{(\ell+1)}(t') \right] . \tag{III.8}$$

In the derivation of the 4PN equations of motion, either by Hamiltonian, Lagrangian or EFT methods, it was proved that the unphysical scale ℓ_0 (or equivalently τ_0) originally present in the tail action (III.2) disappears from the final total action, and that the distance between the two bodies plays the role of the cut-off scale in the logarithmic term. Therefore we fix the unphysical scale τ_0 in eq. (III.8) to be $\tau_0 = r/c$, where r is the distance in harmonic coordinates. For a more detailed derivation of the substitution $\tau_0 \to r/c$, due to the interplay between near and far zone logarithms, we refer the interested reader to Sec. IV of [194].

A proof that the effective action (III.2) indeed describes the dynamics of the non-local tail effects has been provided in the EFT language in [96] (see also [195]), and relies on the relation sketched in fig. III.2. This relation states that the logarithmic, non-local part of the tail contributions to the self-energy Feynman diagram can be directly extracted from to the imaginary part of the "raw" self-energy diagram. A proof relying on traditional PN methods has been sketched in the App. A of [96], and achieved at the 1PN, 1PM level.



$$\ni -\frac{GM\,k_0}{c^2}\left[\frac{1}{d-3} + 2\ln\Big(|k_0|\ell_0\Big) - i\,\pi\right]\times \mathrm{Im}:$$

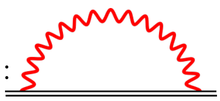


Figure III.2: Schematic representation of the relation between the tail self-energy amplitude (which includes also a real finite, local term) and the purely imaginary self-energy diagram. The black lines represent the two compact objects; the red curly lines, the graviton propagators; and the blue dashed line, the potential interaction sourced by the mass M. The logarithmic term fully determines the non-local tail action (III.2); the imaginary part relates the tail contribution to the energy flux, which is proportional *via* a factor $GMk_0\pi$ to the non-tail radiation flux; finally the UV pole $1/(d-3)$ cancels the IR pole coming from near-zone conservative contributions. See [96] and reference therein.

So it is clear that the tail terms yield logarithmic contributions to the energy, but are they the only ones to contribute ? From the EFT point of view, such logarithmic contributions are associated with UV divergent diagrams in the far zone, their divergence being exactly compensated by an opposite IR divergence from the near zone, and the logarithmic terms from near and far zones combine to give a $\ln(k_0 r)$ term [194]. As explicitly shown in [195], divergent terms do not appear in leading order memory diagrams (*eg.* $\mathrm{M}_L \times \mathrm{M}_L$ interactions), which give instantaneous contributions to the self energy. Simultaneously, the "failed" tail diagrams (involving an angular momentum instead of the mass M at the insertion of the blue dashed line onto the source in fig. III.2) yield only instantaneous contributions. However additional sub-leading logarithmic terms are expected from mixed tail-memory processes, *eg.* from $\mathrm{M} \times \mathrm{M}_L \times \mathrm{M}_L$ interactions, but those appear beyond the accuracy we intend to reach.

Similarly, in traditional PN methods, all the logarithms are generated by tails propagating in the far zone, as well as iterated tail-of-tails and sub-leading tail-memory couplings. In turn, the logarithms in the far zone give rise to logarithms in the near zone, through the matching procedure. Notice that in intermediate steps of the calculation there are other logarithms which appear, but these are pure gauge and cancel out in gauge invariant quantities. This is the case of the logarithms at the 3PN order in the equations of motion in harmonic coordinates, which disappear from the invariant circular energy and angular momentum [73]. In the following we assume (rather than explicitly check) that these gauge logarithms properly cancel up to the required PN order.

III.2.2 Contributions to the dynamics

In the action (III.8), the multipole moments I_L and J_L (and also the mass M) are functionals of the particles' positions, velocities, accelerations, *etc.* We know that the action can be order reduced by means of the equations of motion, and that the variables can be expressed in the center-of-mass frame, so that the action (III.8) is an ordinary action depending on the relative separation $x^i = y_1^i - y_2^i$ and velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$ of the two particles. And, as we said, the tail term is obtained with $\tau_0 = r/c$ where $r = |x^i|$ is the separation between particles. We denote by $m = m_1+m_2$ the total mass, $\nu = m_1 m_2/m^2$ the symmetric mass ratio and will omit to write explicitly the contributions due to current type multipoles, as they are easily restored by $(I_L, \mathcal{I}_L, a_\ell) \to (J_L, \mathcal{J}_L, c^{-2}b_\ell)$.

We vary the action (III.8) with respect to the particles' relative variables, taking into account the non-local structure of the action (see [54] for more details). Considering the multipole moments as functionals of the independent variables x^i and v^i we get the direct contribution of tails in the acceleration $a^i = \mathrm{d}v^i/\mathrm{d}t$ as[5]

$$\begin{aligned}\Delta a^i_{\text{tail}} = \frac{1}{m\nu}\sum_{\ell=2}^{+\infty}\frac{G^2 a_\ell}{c^{2\ell+4}}\biggl\{&\frac{\partial M}{\partial x^i} I_L^{(\ell+1)}\,\mathcal{I}_L^{(\ell+1)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial M}{\partial v^i} I_L^{(\ell+1)}\,\mathcal{I}_L^{(\ell+1)}\right) \\ &-2M(-)^\ell\left[\frac{\partial I_L}{\partial x^i}\,\mathcal{I}_L^{(2\ell+2)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial I_L}{\partial v^i}\,\mathcal{I}_L^{(2\ell+2)}\right)\right]\biggr\}\,.\end{aligned} \tag{III.9}$$

Furthermore, at high PN orders, there will be also other tail contributions (not detailed here) coming from the replacements of accelerations in lower order terms of the final equations of motion, as well as coming later from the reduction to quasi circular orbits. From the tail contribution (III.9) in the acceleration we obtain the corresponding tail contribution in the conserved energy, *via* the usual multiplication by $m\nu\, v^i$ and subsequent integration (see [55] for details)

$$\begin{aligned}\Delta E_{\text{tail}} = \sum_{\ell=2}^{+\infty}\frac{G^2 a_\ell}{c^{2\ell+4}}\biggl\{&v^i\frac{\partial M}{\partial v^i} I_L^{(\ell+1)}\mathcal{I}_L^{(\ell+1)} + 2M\sum_{p=1}^{\ell}(-)^p I_L^{(\ell+1-p)}\mathcal{I}_L^{(\ell+1+p)} \\ &+ M I_L^{(\ell+1)}\mathcal{I}_L^{(\ell+1)} - 2(-)^\ell M v^i\frac{\partial I_L}{\partial v^i}\,\mathcal{I}_L^{(2\ell+2)} + M\,\delta H_\ell\biggr\}\,.\end{aligned} \tag{III.10}$$

Those terms are easily derived, except the last one δH_ℓ which represents a non trivial correction to be added in the case of the non-local dynamics, for which the Noetherian conserved energy E actually differs from the value of the non-local Hamiltonian H computed on shell [55]. This term satisfies

$$\frac{\mathrm{d}\delta H_\ell}{\mathrm{d}t} = I_L^{(\ell+1)}\mathcal{I}_L^{(\ell+2)} - I_L^{(\ell+2)}\mathcal{I}_L^{(\ell+1)}\,, \tag{III.11}$$

which is the generalization of eq. (3.8) of [55] in the case of generic multipole moments. In analogy to similar relations discussed in [69, 195, 209], note that the time average over one period of this term is related to the energy flux,

$$\langle \delta H\rangle = -\frac{2GM}{c^3}\,\mathcal{F}^{\text{source}}_{\text{GW}}\,, \tag{III.12}$$

where we have naturally denoted $\delta H = \sum \frac{G^2 M a_\ell}{c^{2\ell+4}}\delta H_\ell +$ identical contribution for the current moments, and where the total (averaged) energy flux associated with the source moments I_L and J_L reads

$$\mathcal{F}^{\text{source}}_{\text{GW}} = \sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\left[a_\ell\, I_L^{(\ell+1)}\, I_L^{(\ell+1)} + \frac{b_\ell}{c^2} J_L^{(\ell+1)}\, J_L^{(\ell+1)}\right]\,. \tag{III.13}$$

[5] We ignore the variation of the scale $\tau_0 = r/c$ since this gives an instantaneous term, without logarithms.

This result seems to confirm the soundness of the general action (III.8) with the general coefficients (III.4) and source moments I_L and J_L. Nevertheless, and as detailed in the discussion in sec. III.3.3, it appears that we should rectify this formalism at high PN orders.

As presented hereafter, and contrary to the other contributions in (III.10), the term δH_ℓ will not contribute to the logarithmic part of the conserved energy when dealing with quasi-circular orbits.

Derivation of the extra contribution in the non-local energy

Let us focus on the extra contribution satisfying eq. (III.11). By recasting the functional (III.7) as

$$\mathcal{I}_L(t) = \underset{\tau_0}{\mathrm{Pf}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{|\tau|}\, I_L(t+\tau)\,, \tag{III.14}$$

one can perform a formal Taylor expansion when $\tau \to 0$. As the kernel $1/|\tau|$ is even, it only suffices to select even powers of τ in the expansion of $I_L(t+\tau)$. This procedure will naturally lead to divergent integrals (due to the behaviour at $\tau \to \pm\infty$). To cure those divergences, let us modify the kernel by introducing a regulator $e^{-\epsilon|\tau|}$, with an arbitrary parameter $\epsilon > 0$, that will be put to 0 at the end of the computation. Thanks to this regulator, the Hadamard partie finie is no longer needed (the integrals are now convergent at $\tau = 0$). With this modified kernel, and after some integrations by part, it comes as a generalization of eq. (3.11) of [55]:

$$\delta H_\ell = \sum_{n=1}^{+\infty}\left[I_L^{(\ell+1)} I_L^{(\ell+2n+1)} - 2\sum_{s=0}^{n-2}(-)^s I_L^{(\ell+s+2)} I_L^{(\ell+2n-s)} + (-)^n \left(I_L^{(\ell+n+1)}\right)^2\right] \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau\, e^{-\epsilon|\tau|}}{|\tau|}\frac{\tau^{2n}}{(2n)!}\,. \tag{III.15}$$

To resum the Taylor series, we introduce the Fourier decomposition of the multipoles, following [35],

$$I_L(t) = \sum_{p=-\infty}^{+\infty}\sum_{m=-\ell}^{+\ell} \underset{(p,m)}{\tilde{I}}{}_L\, e^{\mathrm{i}\,(p+m\,\mathrm{k})\,\mathcal{M}}\,, \quad \text{with} \quad \underset{(p,m)}{\tilde{I}}{}_L = \int_0^{2\pi}\frac{\mathrm{d}\mathcal{M}}{2\pi}\, I_L\, e^{-\mathrm{i}\,(p+m\,\mathrm{k})\,\mathcal{M}}\,, \tag{III.16}$$

where $\mathcal{M} \equiv \omega(t-t_0)$ is the mean anomaly of the binary motion, with $\omega = 2\pi/P$ being the orbital frequency, and t_0 an instant of reference. The index p corresponds to the usual orbital motion, and the "magnetic-type" index m, to the relativistic precession (k being defined by the precession of the periastron *per* period, $\Delta_{\mathrm{GR}} = 2\pi\mathrm{k}$). The discrete Fourier coefficients ${}_{(p,m)}\tilde{I}_L$ naturally satisfy ${}_{(-p,-m)}\tilde{I}_L = {}_{(p,m)}\tilde{I}_L^\star$, with the star denoting the complex conjugation. In the following, we will denote $\tilde{p} \equiv p + m\,\mathrm{k}$ and thus ${}_{\tilde{p}}\tilde{I}_L \equiv {}_{(p,m)}\tilde{I}_L$. Separating the constant part $\tilde{p}+\tilde{q} = 0$ from the oscillating one, and resuming with respect to n, the equation (III.15) reads

$$\begin{aligned}\delta H_\ell = &-\sum_{\tilde{p}} (\tilde{p}\omega)^{2\ell+2}\, |\underset{\tilde{p}}{\tilde{I}}{}_L|^2 \int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau\, e^{-\epsilon|\tau|}}{|\tau|}\tilde{p}\omega\tau\, \sin(\tilde{p}\omega\tau) \\ &- \frac{(-)^\ell}{2}\sum_{\tilde{p}+\tilde{q}\neq 0}\left(\tilde{p}\tilde{q}\omega^2\right)^{\ell+1} \underset{\tilde{p}}{\tilde{I}}{}_L\, \underset{\tilde{q}}{\tilde{I}}{}_L\, \frac{\tilde{p}-\tilde{q}}{\tilde{p}+\tilde{q}}\, e^{\mathrm{i}(\tilde{p}+\tilde{q})\mathcal{M}} \int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau\, e^{-\epsilon|\tau|}}{|\tau|}\Big[\cos(\tilde{p}\omega\tau) - \cos(\tilde{q}\omega\tau)\Big]\,.\end{aligned} \tag{III.17}$$

Using the integration formulas

$$\int \frac{\mathrm{d}\tau\, e^{-\epsilon|\tau|}}{|\tau|}\tilde{p}\omega\tau\, \sin(\tilde{p}\omega\tau) = 2 + \mathcal{O}(\epsilon)\,, \quad \text{and} \quad \int \frac{\mathrm{d}\tau\, e^{-\epsilon|\tau|}}{|\tau|}\Big[\cos(\tilde{p}\omega\tau) - \cos(\tilde{q}\omega\tau)\Big] = -2\ln\left|\frac{\tilde{p}}{\tilde{q}}\right| + \mathcal{O}(\epsilon)\,, \tag{III.18}$$

it finally comes, by setting the regulator ϵ to zero,

$$\delta H_\ell = -2\omega^{2\ell+2}\left[\sum_{\tilde{p}} |\underset{\tilde{p}}{\tilde{I}}{}_L|^2 \tilde{p}^{2\ell+2} - \frac{(-)^\ell}{2}\sum_{\tilde{p}+\tilde{q}\neq 0} \underset{\tilde{p}}{\tilde{I}}{}_L\, \underset{\tilde{q}}{\tilde{I}}{}_L\, \frac{\tilde{p}^{\ell+1}\tilde{q}^{\ell+1}(\tilde{p}-\tilde{q})}{\tilde{p}+\tilde{q}} \ln\left|\frac{\tilde{p}}{\tilde{q}}\right| e^{\mathrm{i}(\tilde{p}+\tilde{q})\mathcal{M}}\right]. \tag{III.19}$$

The total contribution in the conserved energy that is due to those extra terms reads

$$\delta H = \sum_{\ell=2}^{\infty} \frac{G^2 M\, a_\ell}{c^{2\ell+4}}\, \delta H_\ell + (\text{current moments contribution})\,. \tag{III.20}$$

Averaging over one period, only the first term in (III.19) survives, and yields

$$\langle \delta H \rangle = -\frac{2GM}{c^3} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[a_\ell\, I_L^{(\ell+1)}\, I_L^{(\ell+1)} + \frac{b_\ell}{c^2} J_L^{(\ell+1)}\, J_L^{(\ell+1)} \right]. \tag{III.21}$$

This confirms the previously stated relation (III.12).

III.2.3 The case of quasi-circular orbits

In order to explicitly compute the tail effects for a binary, we need to specify its whole history, as the tail integral (III.7) involves an integration on its remote past. From now on, we model the evolution of the binary as a series of shrinking circular orbits in the adiabatic approximation, the so-called "quasi-circular orbits". Taking circular orbits simplifies drastically the computation, as non-circular parametrizations usually rely on small-eccentricity developments, and the tail sector becomes quite involving (see *eg.* [135]). The adiabatic approximation implies that the timescale of the decrease in radius of those orbits (cause by the energy loss through gravitational radiation) is very large compared to the orbital period. Therefore in practical computations, one can consider only one timescale: the orbital period $2\pi/\omega$.

When specialized to bound orbits, the tail integrals $\mathcal{I}_L$ and $\mathcal{J}_L$ become local (as the time integration can be explicitly performed) and, in the case of quasi circular orbits, they become (see *eg.* [54]):

$$\mathcal{I}_L = -2 I_L \left[\ln\left(\frac{r\,\omega}{c}\right) + \gamma_\text{E} \right], \qquad \text{and} \qquad \mathcal{J}_L = -2 J_L \left[\ln\left(\frac{r\,\omega}{c}\right) + \gamma_\text{E} \right], \tag{III.22}$$

where γ_E denotes the Euler constant and we recall that $\tau_0 = r/c$. As we focus on logarithmic contributions, we can naturally ignore the Euler constant. Hence we obtain the purely logarithmic contributions in the tail acceleration (III.9) for circular orbits as

$$\begin{aligned}\Delta a^i_\text{tail} = &- \frac{2}{m\nu} \sum_{\ell=2}^{+\infty} \frac{G^2 a_\ell}{c^{2\ell+4}} \left[\frac{\partial M}{\partial x^i} \left(I_L^{(\ell+1)}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial M}{\partial v^i} \left(I_L^{(\ell+1)}\right)^2 \right) \right] \ln\left(\frac{r\,\omega}{c}\right) \\ &- \frac{4}{m\nu} \sum_{\ell=2}^{+\infty} \frac{G^2 M a_\ell (-)^{\ell+1}}{c^{2\ell+4}} \left[\frac{\partial I_L}{\partial x^i}\, I_L^{(2\ell+2)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial I_L}{\partial v^i}\, I_L^{(2\ell+2)} \right) \right] \ln\left(\frac{r\,\omega}{c}\right) \\ &+ \text{identical contribution for the current moments}\,.\end{aligned} \tag{III.23}$$

As for the logarithmic contributions in the conserved energy (III.10), we find for circular orbits

$$\begin{aligned}\Delta E_\text{tail} = &- 2 \sum_{\ell=2}^{+\infty} \frac{G^2 a_\ell}{c^{2\ell+4}} v^i \frac{\partial M}{\partial v^i} \left(I_L^{(\ell+1)}\right)^2 \ln\left(\frac{r\,\omega}{c}\right) \\ &- 4 \sum_{\ell=2}^{+\infty} \frac{G^2 M a_\ell (-)^{\ell+1}}{c^{2\ell+4}} \left[\sum_{p=1}^{\ell} (-)^p I_L^{(p)} I_L^{(2\ell+2-p)} - \frac{(-)^\ell}{2} \left(I_L^{(\ell+1)}\right)^2 + v^i \frac{\partial I_L}{\partial v^i}\, I_L^{(2\ell+2)} \right] \ln\left(\frac{r\,\omega}{c}\right) \\ &+ \text{identical contribution for the current moments}\,.\end{aligned} \tag{III.24}$$

We recall that the extra contribution δH_ℓ in the energy is explicitly given by eq. (III.19) and does not contain logarithmic terms for circular orbits.

III.3 Logarithmic contributions in the conserved energy

III.3.1 Simple tail contributions

As an application, we compute logarithmic contributions to the circular energy, as a function of the orbital frequency, or the equivalent 1PN gauge invariant quantity $x \equiv (Gm\omega/c^3)^{2/3}$, with $m \equiv m_1 + m_2$. Up to now, such contributions have been computed up to 5PN order [91, 132, 261], while at 6PN order only the leading term in the symmetric mass ratio $\nu \equiv \frac{m_1 m_2}{m^2}$ is known [62]. Using the material displayed in the previous section, the only limitation to compute all the logarithmic terms coming from simple tails is the knowledge of the multipole moments. Given that the state-of-the-art is 3PN for the dominant $\ell = 2$ mass quadrupole moment, and that the effect starts at 4PN precision, we are presently able to provide such terms up to 7PN order.[6]

As advertised in the previous chapter, the multipole moments have not been computed at that order by means of EFT methods, so we rely on the source moments computed by PN-MPM techniques. In order to apply the formula of the previous section at 7PN, we have to order-reduce the derivatives of the multipole moments by means of the equations of motion obtained from the non-local formulation, so that the acceleration (III.23) and energy (III.24) are expressed only in terms of r and ω, or equivalently, in terms of x and $\gamma \equiv Gm/(rc^2)$. We then use the expression of the acceleration in the adiabatic approximation, $a^i = -r\omega^2 n^i$, and express it in terms of the two PN variables, as $a^i = -x^3c^4/(Gm\gamma)n^i$. By plugging the values of the multipolar moments, we express a^i (which is the sum of the usual acceleration at 3PN order and of the tail contribution (III.23)) in terms of γ and x. This yields the relation $\gamma(x)$ at the required accuracy. The energy (which is similarly the sum of the usual energy at the 3PN order and of the tail contribution (III.24)) is also expressed in terms of γ and x and, by plugging the previously found $\gamma(x)$ and selecting only the logarithmic contributions, it comes

$$\begin{aligned}E^{\ln}_{\text{simple-tail}} = -\frac{m\nu^2}{2}x^5 \ln x \bigg[&\frac{448}{15} + \left(-\frac{4988}{35} - \frac{656}{5}\nu\right)x + \left(-\frac{1967284}{8505} + \frac{914782}{945}\nu + \frac{32384}{135}\nu^2\right)x^2 \\ &+ \bigg(\frac{16785520373}{2338875} - \frac{1424384}{1575}\ln\left(\frac{r}{r_0}\right) + \left(\frac{2132}{45}\pi^2 - \frac{41161601}{51030}\right)\nu \\ &-\frac{13476541}{5670}\nu^2 - \frac{289666}{1215}\nu^3\bigg)x^3 + \mathcal{O}\left(x^4\right)\bigg] ,\end{aligned} \tag{III.25}$$

where r_0 is the unphysical UV regulator which appears in the expression of the 3PN mass quadrupole moment, see *eg.* [73]. Such constant is nevertheless just an artefact and as such should disappear from final results. As discussed in the following, this is realized by taking into account the tail-of-tail-of-tail contributions.

The 4PN and 5PN coefficients, as well as the leading order 6PN coefficient in the test mass limit $\nu \to 0$, confirm previous findings in the literature [62, 91, 132, 261], while all the remaining ones are derived here for the first time.

III.3.2 The relative 3PN logarithmic contributions

As the iterated tails appear at relative 1.5PN orders, the result (III.25) does not encompass the whole logarithmic sector at 7PN: the tail-of-tails and tail-of-tail-of-tails may also contribute.

The tail-of-tail contributions

The simplest iterated tails are the so-called tail-of-tails (or $(\text{tail})^2$) and correspond to the cubic interactions $\mathrm{M}\times\mathrm{M}\times\mathrm{M}_L$ and $\mathrm{M}\times\mathrm{M}\times\mathrm{S}_L$. They arise at 5.5PN in the conserved energy (corresponding

[6] Note that the source and canonical moments are similar at this accuracy. Indeed the source and canonical mass quadrupole moments differ by a 3.5PN term, and other moments, by 2.5PN terms, see [187].

to 3PN in the asymptotic waveform [70, 209]), which is possible because of the non-locality involved[7] (half-integer PN orders are usually associated with dissipation effects). By contrast to the simple tails, or to any $(\text{tail})^n$ with n odd, the $(\text{tail})^2$ will not bring any logarithmic dependence in the conserved energy, at least up to 7.5PN order (see [93, 94] for discussions). The effects of the $(\text{tail})^2$ in the redshift variable have been computed using traditional PN methods at the leading order when $\nu \to 0$, *ie.* in the gravitational self-force (GSF) limit [168]. From this quantity, it is possible to extract the contribution of the $(\text{tail})^2$ to the energy function, by using the first law of binary point-like particle mechanics. At the best of our knowledge, this has not been done yet, and only the leading PN order is known [62], thus we present the derivation in the present section.

In the non-spinning case, and for circular orbits, the first law of binary point-like particle mechanics [199, 261] relates the variation of the total ADM mass M and the total angular momentum J to the variation of the individual masses m_1 and m_2 as

$$\delta M - \omega\, \delta J = z_1\, \delta m_1 + z_2\, \delta m_2\,, \tag{III.26}$$

where ω is the circular orbital frequency and z_1 and z_2 are the gravitational redshift variables.[8] In the GSF limit, the expressions for those variables is known analytically from usual PN methods up to 3PN order [90] (the logarithmic coefficients at 4PN and 5PN being added in [91, 261]), from analytical GSF methods up to 21.5PN [251] and from numerical ones up to 22PN [200]. Expressed in terms of the 1PN variable x, it reads

$$z_1 = \sqrt{1-3x} + \nu\, z_{\text{SF}}(x) + \mathcal{O}(\nu^2)\,, \tag{III.27}$$

where $\sqrt{1-3x}$ is the Schwarzschild redshift in the test mass limit. The tail-of-tails in the GSF part of the redshift variable are known analytically up to the 2PN relative order, and read [93, 94][9]

$$z_{\text{SF}}^{(\text{tail})^2} = \pi\, x^{13/2} \left[\frac{13696}{525} - \frac{368693}{3675}\, x - \frac{361209292}{3274425}\, x^2 + \mathcal{O}\left(x^3\right)\right]. \tag{III.28}$$

Note that these terms represent the full contributions in the redshift variable to these orders, and they are in agreement with modern analytic SF computations of the redshift up to high PN order [251].

Integrating the first law (III.26), it is possible to express the conserved energy of the test particle in terms of its redshift variable as [260]

$$\frac{E}{m\nu} = \frac{1-2x}{\sqrt{1-3x}} - 1 + \nu\, \mathcal{E}_{\text{SF}}(x) + \mathcal{O}\left(\nu^2\right)\,, \tag{III.29}$$

where the interesting contribution reads

$$\mathcal{E}_{\text{SF}}(x) = \frac{1}{2}\, z_{\text{SF}}(x) - \frac{x}{3}\, z'_{\text{SF}}(x) + \sqrt{1-3x} - 1 + \frac{x}{6}\frac{7-24x}{(1-3x)^{3/2}}\,. \tag{III.30}$$

Plugging the redshift (III.28) in this expression, the energy contribution of the tail-of-tails reads

$$E_{(\text{tail})^2}(x) = -\frac{m\nu\, x}{2}\left\{\nu\, \pi\, x^{11/2}\left[\frac{27392}{315} - \frac{1474772}{3675}\, x - \frac{722418584}{1403325}\, x^2 + \mathcal{O}\left(x^3\right)\right] + \mathcal{O}\left(\nu^2\right)\right\}. \tag{III.31}$$

Starting at the 6.5PN order we also expect couplings between the simple tail and memory effects, *eg.* $\mathrm{M} \times \mathrm{M}_L \times \mathrm{M}_L$, to contribute to the conservative dynamics. However such effects do not enter the formulas (III.28) and (III.31), as they only affect terms of higher-order in ν.

The leading 5.5PN term of (III.31) was already computed in [62], and shows perfect agreement. Both the 6.5 and 7.5PN sub-leading terms are new contributions.

[7] The fact that the conservative dynamics contains half-integer PN orders starting at 5.5PN has been discovered in high-precison numerical gravitational self-force calculations [328].

[8] Defined as the redshift experienced by a photon emitted by one of the masses and observed at infinity in the direction of the angular momentum.

[9] Note that the quantity used in those references is $u_1^T = 1/z_1$ and is expressed in terms of $y = x(1+m_1/m_2)^{-2/3}$, thus the different numerical coefficients. Here $m_1 \ll m_2$ is the smaller mass orbiting the larger one m_2.

The tail-of-tail-of-tail contributions

The simple tail contributions displayed in eq. (III.25) are the only logarithmic terms contributing to the observable $E(x)$ up to 6PN order, while at 7PN one should account also for the leading order tail-of-tail-of-tail or $(\text{tail})^3$ terms, which are expected to cancel out the residual UV regulator r_0.

The leading order $(\text{tail})^3$ contribution to the energy, which contains both ln and $(\ln)^2$ terms, is purely quadratic in the mass ratio ν (because the quadrupole is linear in ν at Newtonian order). This means that while the ν-dependent 7PN terms in the square bracket of eq. (III.25) account for the total logarithmic contributions, this is not true for ν-independent ones, so that we can write the 7PN logarithmic sector of the total energy as

$$E^{\ln}_{7\text{PN}}(x) = -\frac{m\nu^2}{2}x^8 \ln x \left[c^{\ln^2}_{7\text{PN}} \ln x + c^{\ln}_{7\text{PN}} + \left(\frac{2132}{45}\pi^2 - \frac{41161601}{51030}\right)\nu - \frac{13476541}{5670}\nu^2 - \frac{289666}{1215}\nu^3\right]. \tag{III.32}$$

The coefficient $c^{\ln^2}_{7PN}$ will be computed from first principles in sec. III.3.4. For the moment we just notice that both $c^{\ln}_{7PN}$ and $c^{\ln^2}_{7PN}$ can be derived from GSF redshift results by proceeding along the same lines as in the $(\text{tail})^2$ case. Using the results of [235], we find

$$c^{\ln}_{7\text{PN}} = \frac{85229654387}{16372125} - \frac{1424384}{1575}\left(\gamma_{\text{E}} + \ln 4\right), \qquad \text{and} \qquad c^{\ln^2}_{7\text{PN}} = -\frac{356096}{1575}. \tag{III.33}$$

This result allows us to predict the leading order $(\text{tail})^3$ contribution as the difference between eq. (III.32) and the 7PN part of eq. (III.25), namely

$$E^{\ln}_{(\text{tail})^3}(x) = -\frac{m\nu^2}{2}x^8 \ln x \left\{-\frac{356096}{1575}\ln x - \frac{108649792}{55125} + \frac{1424384}{1575}\left[\ln\left(\frac{r}{r_0}\right) - \gamma_{\text{E}} - \ln 4\right] + \mathcal{O}(x)\right\}. \tag{III.34}$$

Note that the coefficient of γ_{E} in (III.33) is exactly matching the one of $\ln(r/r_0)$ in eq. (III.25), thus giving an indication that the cancellation of the UV scale by the tail-of-tail-of-tails will be straightforward, when performed from first principles.

The total logarithmic contributions at the relative 3PN order

In a nutshell, starting from an effective action for the simple tails (III.2) derived by EFT methods, and employing source multipoles moments computed by traditional PN-MPM means, we were able to derive the logarithmic contributions of simple tails in the conserved energy, up to 7PN order. At this order, the full logarithmic structure receives contributions also from the tail-of-tail-of-tails, whose direct computation has been reserved for future work. Nevertheless we were able to extract this contribution from gravitational self-force results, and to derive the whole logarithmic sector of the conserved energy at 7PN, that can be written as

$$\begin{aligned} E^{\ln} = -\frac{m\nu^2}{2}x^5 \ln x \Bigg\{ & \frac{448}{15} + \left(-\frac{4988}{35} - \frac{656}{5}\nu\right)x + \left(-\frac{1967284}{8505} + \frac{914782}{945}\nu + \frac{32384}{135}\nu^2\right)x^2 \\ & + \left[\frac{85229654387}{16372125} - \frac{1424384}{1575}\left(\gamma_{\text{E}} + \ln 4\right) + \left(\frac{2132}{45}\pi^2 - \frac{41161601}{51030}\right)\nu \right. \\ & \left. - \frac{13476541}{5670}\nu^2 - \frac{289666}{1215}\nu^3 - \frac{356096}{1575}\ln x\right]x^3 + \mathcal{O}\left(x^4\right)\Bigg\}. \end{aligned} \tag{III.35}$$

Although it does not contain logarithmic terms, we have also derived the contribution of the tail-of-tails at leading order in the mass ratio, up to 7.5PN order, as presented in eq. (III.31).

III.3.3 Discussion on the reliability of the formalism

The fact that our result (III.35) agrees with all the previous computations, that the coefficient of γ_E is exactly matching the one of $\ln(r/r_0)$ in eq. (III.25) and the structure of eq. (III.12) are strong indications that the framework developed in this section should be accurate. However, after consequent discussions with L. Blanchet, S. Foffa, Q. Henry and R. Sturani, and long after the publication of our work, I came to the conclusion that we applied it in an erroneous manner, but that the result (III.35) is nevertheless correct. In a nutshell, I am convinced that the formalism should be applied with some kind of *radiative* moments instead of *source* ones.[10] Note that this would not spoil agreements with the previous computations reported in the literature, as the difference strikes only at 3PN beyond the leading order, *ie.* 7PN in the conserved energy, and that our work was the first one to reach this accuracy.

When deriving the source quadrupole moment at the 4PN accuracy, we have found out that its expression was dependent of the IR regularization scheme employed, as presented in details in the next chapter. To summarize the discussion of sec. IV.3.3, the transition from the traditional Hadamard regularization to dimensional regularization (imposed by the resolution of the ambiguity) leads to the emergence of a divergence $\propto 1/(d-3)$ starting at 3PN order. This pole is exactly compensated by a UV pole coming from the instantaneous part of the d-dimensional tail-of-tails, and thus disappears in the radiative quadrupole, as noticed as early as 2009 in the EFT language [209]. In this section, we have used IR-Hadamard regularized source moments, and thus did not care about this pole (at this time, the computation of the IR dimensional regularization of the mass quadrupole was not even initiated). But this dependence on regularization schemes points out that the source moments are not the good quantities to use.

In addition, the vertices of fig. III.2 (that is at the root of the proof of the effective action that we used) are to be dressed. From a diagrammatic point of view, it means that all processes with two matter and one graviton legs are to be included in the vertices, as schematically represented in fig. III.3. Those additional contributions are simply given by loops involving source moments, *ie.* tail, memory and other non-linear interplays. From a traditional PN point of view, those non-linearly dressed vertices are simply given by radiative moments.

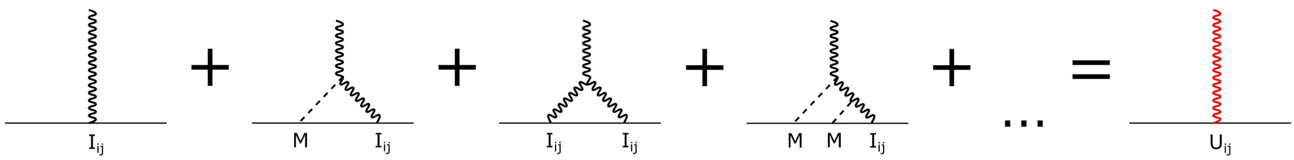


Figure III.3: Schematic dressing of the vertices of fig. III.2, for the quadrupolar moment. The ellipsis encompasses all higher non-linear interactions, building up to give the radiative moment U_{ij}.

A last indication that the multipoles entering the effective action (III.2) should not be the source ones is given by the structure of eq. (III.12). The flux (III.13) is the one associated with source moments, but the physical gravitational flux has to be expressed in terms of mass and current radiative moments U_L and V_L as

$$\mathcal{F}_\mathrm{GW} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[a_\ell \, U_L^{(1)} \, U_L^{(1)} + \frac{b_\ell}{c^2} V_L^{(1)} \, V_L^{(1)} \right] . \tag{III.36}$$

Once those three points are taken in consideration, it becomes less clear that the moments entering the action (III.8) are the source (or canonical) ones. Instead, my conjecture is that, in order to

[10] The confusion has been caused by a misunderstanding between the traditional PN and EFT sides of the collaboration, that definitively use different languages when coming to the radiative sector. This points out the dramatic necessity of building bridges between the different communities, as synergies can be extremely powerful, but require a mutual understanding.

properly treat the logarithmic contributions of the simple tails, we should have used the effective action

$$\tilde{S}^{\text{nl}}_{\text{tail}} = \sum_{\ell=2}^{+\infty} \frac{G^2 M}{c^{2\ell+4}} \mathop{\text{Pf}}_{\tau_0} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} \left[a_\ell \, \tilde{U}_L^{(1)}(t)\tilde{U}_L^{(1)}(t') + \frac{b_\ell}{c^2} \tilde{V}_L^{(1)}(t)\tilde{V}_L^{(1)}(t') \right] , \tag{III.37}$$

where $\tilde{U}_L$ and $\tilde{V}_L$ the instantaneous parts of the radiative moments. Note that $\tilde{U}_L$ and $\tilde{V}_L$ are coordinate dependent, just as I_L and J_L were, but are independent of the regularization scheme at 3PN (and *a priori* up to at least 4PN), as discussed in the next chapter.

Even if we have not performed the computation with the effective action (III.37), we can guarantee that using it instead of eq. (III.8) would not change our final result (III.35). Indeed at 3PN accuracy, the differences between I_L and $\tilde{U}_L$ are the instantaneous sectors of the 1.5PN tail (together with 2.5PN corrections) and 3PN tail-of-tail contributions. Those will induce a 7PN discrepancy in the energy, involving two ADM masses and two Newtonian quadrupole moments. Therefore the disagreement will be purely quadratic in the symmetric mass ratio, and will only affect the $c^{\ln}_{7\text{PN}}$ coefficient of eq. (III.32). As this coefficient has been computed by means of GSF results, and thus its value (III.33) is independent of the effective action used, we can conclude that using the effective action (III.37) would still yield the final result (III.35). Note that using (III.37) would also yield odd contributions in the energy, starting at the 5.5PN order. Those have also been computed by means of GSF results (III.31), and thus the same reasoning applies.

Let us conclude this discussion by some words on the possibility of including the non-local in time parts of U_L and V_L in the effective action (III.37). This would require to drastically refine the machinery that led to the generic eqs. (III.9) and (III.10), in order to include infinitely nested non-localities. Nevertheless, such task seems doable when truncated at some accuracy (*eg.* the relative 3PN order), and one can naively expect that those nested non-localities are linked to iterated tail effects and contribute as higher powers of logarithms in the energy. It would thus be extremely interesting to see if one can recover the whole logarithmic sector (III.35) from first principles by using the 3PN radiative quadrupole instead of the source ones.

III.3.4 Leading logarithm contributions in the conserved energy

When computing the logarithmic tail contributions in the energy up to 7PN accuracy, it appeared that we could also straightforwardly derive the dominant $\log^n$ contributions, that enter at $(3n+1)$PN orders. The dominant log and $\log^2$ contributions are presented in (III.35) and were respectively derived by means of the effective action (III.2) and GSF results. As for the higher dominant $\log^n$ contributions, we have used EFT techniques, deriving from the renormalization group (RG) equations [211].

The renormalization group equations

Setting henceforth $c=1$, we start from the RG equation for the total mass-energy M, see eq. (19) of [211] that we copy here:

$$\frac{\mathrm{d}\ln M(\mu)}{\mathrm{d}\ln\mu} = -\frac{2G^2}{5}\left[2I_{ij}^{(1)}I_{ij}^{(5)} - 2I_{ij}^{(2)}I_{ij}^{(4)} + I_{ij}^{(3)}I_{ij}^{(3)}\right] , \tag{III.38}$$

where μ is the renormalization scale, and both the (Bondi[11]) mass M and quadrupole moment I_{ij} are defined at the scale μ. This equation traduces the leading radiative corrections that are to be included into the definition of the mass. It agrees with the traditional PN eq. (4.6) of [57], with

[11] At the required order, and restricting to the conservative dynamics, the Bondi mass can however be traded for the usual ADM mass. We thus use a similar notation, M, for both.

the obvious replacement $r_{12} \to \mu$ since we are only interested in the running with the scale μ. The angular momentum J^i undergoes a similar renormalization, that reads

$$\frac{\mathrm{d}J^i(\mu)}{\mathrm{d}\ln\mu} = -\frac{8G^2M}{5}\varepsilon^{ijk}\left[I_{jl}I_{kl}^{(5)} - I_{jl}^{(1)}I_{kl}^{(4)} + I_{jl}^{(2)}I_{kl}^{(3)}\right], \tag{III.39}$$

and is the consequence of eq. (4.15a) of [57]. The quadrupole moment itself undergoes a logarithmic renormalization under the RG flow, which is computable from the singularities in the tail-of-tail effects in the EFT language. The relation between the quadrupole moment I_{ij} at the scale μ and the same quantity at the scale μ_0, is reported by eq. (21) of [211] and reads in the Fourier domain (with Ω the Fourier frequency)

$$\tilde{I}_{ij}(\Omega,\mu) = \bar{\mu}^{\beta_I(GM\Omega)^2}\tilde{I}_{ij}(\Omega,\mu_0)\,, \tag{III.40}$$

where $\bar{\mu} \equiv \mu/\mu_0$ and $\beta_I = -\frac{214}{105}$ is the coefficient associated with the logarithmic renormalization of the mass quadrupole moment [209]. The latter relation can be recast in the time domain as

$$I_{ij}(t,\mu) = \sum_{n=0}^{+\infty}\frac{1}{n!}\Big(-\beta_I G^2M^2\ln\bar{\mu}\Big)^n I_{ij}^{(2n)}(t,\mu_0)\,. \tag{III.41}$$

Let us emphasize that this renormalization of the mass quadrupole is associated with the 3PN IR pole that cancels the UV pole coming from the radiative tail-of-tail term, as presented in the next chapter, where the constant β_I plays an important role.

Short-circuiting eqs. (III.38) and (III.41), one can derive the mass renormalization group flow equation, which reads

$$\frac{\mathrm{d}\ln M(\mu)}{\mathrm{d}\ln\bar{\mu}} = -\frac{2G^2}{5}\sum_{k,p\geqslant 0}\frac{\Big(-\beta_I G^2M^2\ln\bar{\mu}\Big)^{k+p}}{k!p!}\left[2I_{ij}^{(2k+1)}I_{ij}^{(2p+5)} - 2I_{ij}^{(2k+2)}I_{ij}^{(2p+4)} + I_{ij}^{(2k+3)}I_{ij}^{(2p+3)}\right](t,\mu_0)\,, \tag{III.42}$$

and by integrating it, we obtain

$$\ln\frac{M(\mu)}{M(\mu_0)} = -\frac{2G^2}{5}\sum_{k,p\geqslant 0}\frac{\Big(-\beta_I G^2M^2\Big)^{k+p}(\ln\bar{\mu})^{k+p+1}}{k!p!(k+p+1)}\left[2I_{ij}^{(2k+1)}I_{ij}^{(2p+5)} - 2I_{ij}^{(2k+2)}I_{ij}^{(2p+4)} + I_{ij}^{(2k+3)}I_{ij}^{(2p+3)}\right], \tag{III.43}$$

in which the quadrupole moment I_{ij} in the right-hand side is defined at the scale μ_0. Finally, and for general orbits, we can average the previous result over an orbital period and approximate $\ln\frac{M(\mu)}{M(\mu_0)} \simeq \frac{M(\mu)}{M(\mu_0)} - 1$ (*ie.* discarding all terms higher than quadratic in the quadrupole moment), to obtain [211]

$$\left\langle\frac{M(\mu)}{M}\right\rangle = 1 - G^2\sum_{n=1}^{+\infty}\frac{(2\ln\bar{\mu})^n}{n!}\Big(\beta_I G^2M^2\Big)^{n-1}\langle I_{ij}^{(n+2)}I_{ij}^{(n+2)}\rangle\,, \tag{III.44}$$

where the brackets denote the time-average and $M = M(\mu_0)$. Exactly the same procedure applied to the angular momentum, *ie.* starting from eq. (III.39) gives similarly

$$\langle J^i(\mu)\rangle = \langle J^i(\mu_0)\rangle - \frac{12G^2M}{5}\varepsilon^{ijk}\sum_{n=1}^{+\infty}\frac{(2\ln\bar{\mu})^n}{n!}\Big(\beta_I G^2M^2\Big)^{n-1}\langle I_{jl}^{(n+1)}I_{kl}^{(n+2)}\rangle\,. \tag{III.45}$$

Explicit leading $\ln^n$ contributions

Let us work out the consequences of the two results (III.44) and (III.45) in the case of quasi-circular orbits. In this case, there is no need of averaging and we no longer mention the time-average process. In the case of circular orbits, the combinations appearing in eqs. (III.44) and (III.45) are given at leading order by

$$I_{ij}^{(n+2)} I_{ij}^{(n+2)} = 2^{2n+3} m^2 \nu^2 \frac{(Gm)^{n+2}}{r^{3n+2}}\,, \quad \text{and} \quad \varepsilon^{ijk} I_{jl}^{(n+1)} I_{kl}^{(n+2)} = 2^{2n+2} m^2 \nu^2 \frac{(Gm)^{n+1}}{r^{3n-1}}\, \omega\, \hat{J}^i\,, \tag{III.46}$$

with $\hat{J}^i$ the constant unit vector normal to the orbital plane such that $\varepsilon^{ijk} x^j v^k = r^2 \omega\, \hat{J}^i$, where we recall that $j = r^2\omega$ is the Newtonian value of the angular momentum per unit mass (II.5). At leading order, the relations (III.44) and (III.45) yield

$$E \equiv M - m = \frac{1}{2} m\nu\, r^2 \omega^2 - \frac{G m^2 \nu}{r} - 8 m \nu^2 \frac{\gamma^2}{\beta_I} \sum_{n=1}^{+\infty} \frac{1}{n!} \Big(8\beta_I \gamma^3 \ln v\Big)^n\,, \tag{III.47a}$$

$$J \equiv |J^i| = m\nu\, r^2 \omega - \frac{48}{5} G^2 m^3 \nu^2 \frac{\omega}{\beta_I \gamma} \sum_{n=1}^{+\infty} \frac{1}{n!} \Big(8\beta_I \gamma^3 \ln v\Big)^n\,. \tag{III.47b}$$

In eq. (III.47) we have fixed the scale ratio $\bar{\mu}$ to be the relevant one for our purpose, that is the ratio between the radiation zone scale $\mu \simeq \lambda_{\rm GW}^{-1}$, where the observer is located, and the orbital scale $\mu_0 \simeq r^{-1}$ at which the eqs. (III.46) hold. Hence we can take $\bar{\mu} \simeq r\omega = v$, where v is the orbital velocity.

Furthermore, we also know that for circular orbits the two invariants $E(\omega)$ and $J(\omega)$ are not independent but are linked by the "thermodynamic" relation

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \omega \frac{\mathrm{d}J}{\mathrm{d}\omega}\,, \tag{III.48}$$

which is the consequence of the first law of binary black hole mechanics (III.26) when the individual masses do not vary. The three relations (III.47) and (III.48) then permit to determine the orbital separation r or equivalently γ, which is defined here in harmonic coordinates, as a function of the orbital frequency:[12]

$$\gamma(x) = x \left[1 + \frac{32\nu}{15} \sum_{n=1}^{+\infty} \frac{3n-7}{n!} (4\beta_I)^{n-1} x^{3n+1} (\ln x)^n\right]\,, \tag{III.49}$$

together with the two invariants $E(x)$ and $J(x)$, that are related to each other by eq. (III.48):

$$E(x) = -\frac{m\nu\, x}{2} \left[1 + \frac{64\nu}{15} \sum_{n=1}^{+\infty} \frac{6n+1}{n!} (4\beta_I)^{n-1} x^{3n+1} (\ln x)^n\right]\,, \tag{III.50a}$$

$$J(x) = \frac{m^2 \nu}{\sqrt{x}} \left[1 - \frac{64\nu}{15} \sum_{n=1}^{+\infty} \frac{3n+2}{n!} (4\beta_I)^{n-1} x^{3n+1} (\ln x)^n\right]\,. \tag{III.50b}$$

Summarizing, we have obtained the leading powers of the logarithms $(\ln)^n$ in the conserved energy for circular orbits, which can quite remarkably be explicitly resumed as

$$E_{\text{leading-}(\ln)^n} = -\frac{8 m \nu^2 x^2}{15 \beta_I} \left[\Big(1 + 24\beta_I x^3 \ln x\Big) x^{4\beta_I x^3} - 1\right]\,, \tag{III.51}$$

where we recall that $\beta_I = -\frac{214}{105}$. Similarly, the leading powers of the logarithms $(\ln)^n$ in the angular momentum are

$$J_{\text{leading-}(\ln)^n} = -\frac{32 m^2 \nu^2 \sqrt{x}}{15 \beta_I} \left[\Big(1 + 6\beta_I x^3 \ln x\Big) x^{4\beta_I x^3} - 1\right]\,. \tag{III.52}$$

[12]This crucial relation $\gamma(x)$, as well as the angular momentum RG equation, are not discussed in [211]. We find that working simply with the RG equation (III.44) for the mass-energy does not allow to get the correct result for the invariant circular energy $E(x)$, eq. (III.50a), for every n.

For the linear and quadratic terms, one recovers the coefficients $c^{\ln}_{4\mathrm{PN}} = \frac{448}{15}$ and $c^{\ln^2}_{7\mathrm{PN}} = -\frac{356096}{1575}$ displayed in eqs. (III.25) and (III.33). Further comparisons with [251] (see App. B and the electronic archive Ref. [19] there) show perfect agreement, *via* the first law of binary dynamics, between the first terms of the infinite series (III.50a) and the high accurate self-force results, that is up to $n \leqslant 7$, *ie.* up to the 21PN accuracy. This stresses the great consistency between EFT methods predicting the RG equations (III.44)–(III.45) [209, 211], the traditional PN approach deriving the first law of binary mechanics [261], and the state-of-the-art 21.5PN accurate SF calculations [251].

Chapter IV

Radiative sector

We conclude the part devoted to the study of the two-body system by presenting the contribution of this thesis to the developments of the radiative sector. As advertised in sec. II.3, only the PN-MPM techniques are currently able to deal with high PN accuracy, and we are relying on such approach to compute the gravitational waveform up to the 4^{th}PN accuracy. To reach such precision, one crucially needs the knowledge of the properly regularized mass-type radiative quadrupole at the 4PN order, which is the subject of this chapter.

If the procedure and results of sec. IV.2 have been published in [278] (up to a small correction to the final result, implemented in this dissertation), the computations of secs. IV.3 and IV.4 are presented here for the first time.

IV.1 Towards the gravitational phase at the 4PN accuracy

The gravitational waveform at infinity, as felt by observers, is given by the radiative metric (B.29).

$$H_{ij}^{\text{TT}} = \frac{4G}{c^2 R}\mathcal{P}_{ijab}^{\text{TT}} \sum_{\ell\geq 2} \frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{abL-2}\left(T-\frac{R}{c}\right) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\varepsilon_{cd(a}V_{b)dL-2}\left(T-\frac{R}{c}\right)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{IV.1}$$

where $\mathcal{P}_{ijab}^{\text{TT}}$ is the transverse and traceless projector. Due to the interferometric technology they rely on, the current and planned gravitational wave detectors are more sensible to the phase than to the amplitude of the waves. If the latter can be directly extracted from the radiative moments $\{U_L, V_L\}$, the former requires the knowledge of their explicit time dependence, which directly follows from the motion of the two bodies. It is thus linked to the conservative sector and cannot be accessed from the radiative sector only.

Instead, one has to rely on the flux-balance equations (II.18) to derive the gravitational phase. We define the orbital phase ϕ as usual by $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega$, where ω is the orbital pulsation[1] satisfying *eg.* $|v^i| = r\omega$. As presented in the previous chapter, the energy for circular orbits is usually expressed in terms of the 1PN parameter $x \equiv (Gm\omega/c^3)^{2/3}$, with $m \equiv m_1 + m_2$. Similarly, the flux emitted by binaries on circular orbits is also to be expressed in terms of x. Therefore the flux-balance equation can be recast as $\frac{\mathrm{d}E(x)}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -\mathcal{F}(x)$. Taking the reference time t_0 as the merger time (or more realistically, as the time at which the PN approximation breaks down), it comes

$$\phi - \phi_0 = \int_{t_0}^{t} \mathrm{d}t\,\omega = -\frac{c^3}{Gm}\int_{x_0}^{x}\mathrm{d}x\,\frac{x^{3/2}}{\mathcal{F}(x)}\frac{\mathrm{d}E(x)}{\mathrm{d}x}\,. \tag{IV.2}$$

[1] Note that, due to their tensorial nature, gravitational waves have a fundamental pulsation $\omega_{\text{GW}} = 2\omega$.

In the case of Newtonian circular orbits, the energy and flux read [73] $E = -\frac{m\nu c^2 x}{2}$ and $\mathcal{F} = \frac{32c^5\nu^2 x^5}{5G}$, with $\nu \equiv \frac{m_1 m_2}{m^2}$. Applying the former integral, it comes $\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu}$. This parametrization in terms of x (or equivalently, in terms of ω) is quite convenient for the data analysis of the gravitational observations, that monitor the time evolution of the frequency of the signal. At the Newtonian level, the flux-balance equation can be expressed in terms of the fundamental frequency of the wave $f = \frac{\omega_{\rm GW}}{2\pi} = \frac{\omega}{\pi} = \frac{\pi c^3}{Gm}\, x^{3/2}$ as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{96}{5}\frac{\pi^{8/3}(Gm)^{5/3}}{c^5}\,\nu\, f^{11/3} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/3} f^{11/3}\,, \tag{IV.3}$$

where we have defined the "chirp mass" $\mathcal{M}_c = m\nu^{3/5}$. By using the lower order approximation for the gravitational radiation, only this special combination of the masses is thus measurable. In order to access individual masses, and naturally to refine the measure of the parameters of the binary system, one needs to break the degeneracy by including higher orders in eq. (IV.3). The state-of-the-art is the 3.5PN accuracy [73], and thus the aim is to push the computation of ϕ up to the 4PN order. Obviously in order to reach such precision, we need the expressions of the energy and flux up to 4PN. The former is known and has been confirmed by independent methods (*cf.* sec. II.3.1), so we need to compute the latter. In terms of radiative moments, the flux is expressed as [335]

$$\mathcal{F} = \sum_{\ell\geq 2}\frac{G}{c^{2\ell+1}}\left[a_\ell\, U_L^{(1)}\, U_L^{(1)} + \frac{b_\ell}{c^2}\, V_L^{(1)}\, V_L^{(1)}\right]\,, \tag{IV.4}$$

where the coefficients are the same as in eq. (III.4), namely

$$a_\ell = \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}\,, \qquad \text{and} \qquad b_\ell = \left(\frac{2\ell}{\ell+1}\right)^2 a_\ell\,. \tag{IV.5}$$

At this point let us emphasize a direct consequence of this formula for high PN orders: the practical computation of the flux is highly dependent on the equations of motion. Indeed the radiative multipoles involve accelerations and derivatives of the acceleration (either in the expressions of the source (B.12) and gauge moments (B.20), or in the relation between the radiative and canonical moments, as in eq. (B.30)). Therefore, one should lead the investigation of the radiative sector in a manner that is consistent with the conservative one. This can be realized notably by using the same coordinate system, as done when computing the flux in the PN-MPM procedure: the radiative coordinates are the same than those used in the Fokker Lagrangian method. This is why this particular method was able to deal with the ambiguities, as both radiative and conservative sectors can be treated within a common formalism.

Thus the regularization scheme used in the conservative sector should also be used for the derivation of the flux. If the historical regularization scheme was the Hadamard one, UV divergences appearing at 3PN were too strong to be cured this way. Those are related to the point-particle approximation, and thus a UV dimensional regularization scheme had to be implemented to yield the consistent dynamics at 3PN order [87]. The final results contained a UV pole $\propto 1/(d-3)$ that has been removed by an unphysical coordinate shift. When computing the flux at 3PN, the used acceleration was the d-dimensional one, and similarly the residual pole was removed by the exact same coordinate shift [88]. This was a strong confirmation of the accuracy of the PN-MPM framework.

At the 4PN order, an additional IR dimensional regularization had to be implemented in the equations of motion, to deal with divergences coming from the PN expansion formally extending to infinity. This IR regularization was part of the resolution of the ambiguities, and similarly a residual IR pole was removed by means of a coordinate shift. Therefore, we need to perform this IR dimensional regularization when computing the flux, and applying the subsequent coordinate shift. If this succeeds in compensating the remaining poles, it will be a powerful consistency check of the method used.

The dominant contribution to the flux (IV.4) is given by the radiative mass quadrupole U_{ij}. Therefore, in order to compute the flux at 4PN, one needs U_{ij} at the same order, and thus one has to implement UV and IR regularization schemes when computing it. In practice, this computation (that is currently reaching its completion) has been split in three major steps, corresponding to the three next sections. First, the source mass quadrupole I_{ij} is computed with the historical Hadamard regularization and a UV regularization scheme is implemented. Second, the IR dimensional regularization scheme is applied on I_{ij}. Finally the non-linear interactions are computed and properly regularized, to yield U_{ij}. Starting by computing I_{ij} with a Hadamard regularization scheme rather than directly in d dimensions may seem a loss of time. It is in fact a powerful gain of energy. Indeed the Hadamard computations are largely simpler to perform than the d-dimensional ones, and the sectors in which both regularization schemes differ are well identified. Therefore, we rarely compute quantities in d dimensions, but rather derive and apply formulas to compute the difference between the two regularization schemes, that we add to the Hadamard regularized quantities.

As for the other radiative moments, the difficult computation of the current quadrupole V_{ij} at 3PN order, and its proper UV dimensional regularization[2] were tackled by Q. Henry, L. Blanchet and G. Faye [228]. The (UV regularized) 3PN mass octupole U_{ijk} has been derived in [188], and checked (*ie.* recomputed) by Q. Henry. The other moments are at most 2PN, and thus do not require dimensional regularization. They are all known at the required PN order, see *eg.* [73, 89].

IV.2 The Hadamard regularized mass quadrupole

The first step towards the computation of the 4PN radiative mass quadrupole is the derivation of the source mass quadrupole, using a UV regularization scheme on top of the Hadamard one. Such project was initiated during the PhD thesis of T. Marchand (under the direction of L. Blanchet and in collaboration with G. Faye), and was nearly finished at the end of the thesis. Nevertheless, such computation is extremely involving and errors are quite difficult to track down, so it is customary to perform a "double-blind" computation: two different persons compute independently the same quantity, and compare their final results. As T. Marchand's computation of I_{ij} lacked a "double-blind" confirmation, Q. Henry and myself recomputed it at the beginning of our PhD thesis. While Q. Henry carried the major part of this unglamorous task, I recomputed the "quadratic potentials", as presented in sec. IV.2.2, and reapplied the UV shift, as described in sec. IV.2.4. But my main contribution to this project was to derive the surface terms (see sec. IV.2.3), in a "double-blind" configuration, together with S. Marsat. If the next section presents an overview of the various methods used for the 4PN computation of I_{ij}, we will only detail the computations in which I effectively took part, and let the interested reader refer to [278] for a more detailed outline of the whole derivation. Note that we have recently noticed that the Hadamard regularized published mass quadrupole misses a small piece. The expressions displayed in eq. (IV.65) are thus more accurate than the ones published in [278].

IV.2.1 The mass quadrupole in d dimensions

As we intend to apply a UV dimensional regularization scheme, a d-dimensional generalization of eq. (B.12a) is required, and reads [88]

$$I_L(t) = \frac{d-1}{2(d-2)}\underset{B}{\mathrm{PF}}\int\frac{\mathrm{d}^d\mathbf{x}}{\ell_0^\varepsilon}\left(\frac{r}{r_0}\right)^B\left\{\hat{x}_L\,\overline{\Sigma}_{[\ell]}-\frac{\alpha_\ell^{(d)}}{c^2}\,\hat{x}_{iL}\,\overline{\Sigma}^{(1)}_{i[\ell+1]}+\frac{\beta_\ell^{(d)}}{c^4}\,\hat{x}_{ijL}\,\overline{\Sigma}^{(2)}_{ij[\ell+2]}-\frac{(d-3)B\,\alpha_\ell^{(d)}}{(d-1)c^2r^2}\,\hat{x}_{iL}x_j\,\overline{\Sigma}_{ij[\ell+1]}\right\}, \tag{IV.6}$$

[2]Interestingly the Levi-Civita tensors used to define J_L in three dimensions, see eq. (B.12b), do not exist in generic d dimensions. They had to rely on Young tableaux to perform the UV dimensional regularization of J_{ij}.

with ℓ_0 the d-dimensional length scale (defined by the ratio of gravitational strengths, see (A.7)) and the parenthesis in exponent indicate time derivatives. The coefficients are given by

$$\alpha_\ell^{(d)} = \frac{4(d+2\ell-2)}{(d+\ell-2)(d+2\ell)}\,, \qquad \text{and} \qquad \beta_\ell^{(d)} = \frac{2(d+2\ell-2)}{(d+\ell-1)(d+\ell-2)(d+2\ell+2)}\,. \tag{IV.7}$$

Note the appearance of a fourth term, which is proportional to $(d-3)B$ and thus plays no role in this section. Indeed in the UV regime (say for $r < \mathcal{R}$ with $\mathcal{R}$ some constant scale) the integral is convergent and thus does not develop poles in $1/B$: the finite part kills this contribution. In the IR regime (for $r > \mathcal{R}$), the regularization scheme is three-dimensional, and thus no pole $1/(d-3)$ can appear: this contribution vanishes by virtue of the $(d-3)$ factor.

The source terms are defined by a formal PN expansion of the Landau-Lifschitz pseudo tensor (B.5) as They explicitly read

$$\overline{\Sigma} = \frac{2}{d-1}\frac{(d-2)\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2}\,, \qquad \overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c}\,, \qquad \text{and} \qquad \overline{\Sigma}_{ij} = \overline{\tau}^{ij}\,, \tag{IV.8}$$

and thus the overall d-dependent factor in front of I_L (IV.6) is such that the source quadrupole reduces to the usual Newtonian-looking expression $I_L = \int \mathrm{d}^3\mathbf{x}\,\rho\,\hat{x}_L + \mathcal{O}(c^{-2})$. From those source densities, we define

$$\overline{\Sigma}_\ell(\mathbf{x},t) = \int_{-1}^{1} \mathrm{d}z\, \delta_\ell^{(\varepsilon)}(z)\, \overline{\Sigma}(\mathbf{x}, t+zr/c)\,, \tag{IV.9}$$

where $\delta_\ell^{(\varepsilon)}(z)$ is the d-dimensional generalization of $\delta_\ell(z)$ and reads

$$\delta_\ell^{(\varepsilon)}(z) \equiv \frac{\Gamma\left(\ell+\frac{3}{2}+\frac{\varepsilon}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\ell+1+\frac{\varepsilon}{2}\right)}\,(1-z^2)^{\ell+\frac{\varepsilon}{2}}, \qquad \text{such that} \qquad \int_{-1}^{1}\mathrm{d}z\,\delta_\ell^{(\varepsilon)}(z) = 1\,. \tag{IV.10}$$

This allows to use the formal PN expansion

$$\overline{\Sigma}_{[\ell]}(\mathbf{x},t) = \sum_{k\geq 0} \frac{1}{2^{2k}k!}\frac{\Gamma(\ell+\frac{d}{2})}{\Gamma(\ell+\frac{d}{2}+k)}\left(\frac{r}{c}\frac{\partial}{\partial t}\right)^{2k}\overline{\Sigma}(\mathbf{x},t)\,, \tag{IV.11}$$

which is the one entering the expression of the moments (IV.6).

Those source moments involve the stress-energy tensor of the matter distribution $T^{\mu\nu}$ through the Landau-Lifschitz pseudo tensor (B.5), which we now have to specify in the case of a binary system of compact objects. For celestial bodies composed of matter (neutron star, white dwarf, *etc*), we neglect finite-size effects that arise at 5PN and thus it is quite natural to use a point-particle approximation, *ie.* modeling the objects by a Dirac distribution with a constant mass m_a. As for black holes, the modeling is more subtle as those celestial bodies are vacuum solutions by definition (the "mass" of a black hole space-time is very different conceptually from the mass of a star). Nevertheless, as we will not directly probe the vicinity of those objects (in which the PN approximation breaks down), one can model them as point-"particle" of "mass" given by the Schwarzschild parameter. Therefore we will not discriminate between the different possible natures of the compact objects, but model them all by Dirac distributions of mass m_a localized in $\mathbf{y}_a$, *ie.* an coordinate energy density[3] $\rho = m_a\delta(\mathbf{x}-\mathbf{y}_a)$. This non-discrimination can be related to the fact that the space-time generated by a spherical non-rotating star is given by the Schwarzschild metric [65, 242]: far from it, one cannot distinguish a static black hole from a perfectly spherical and spinless matter distribution *via* gravitational interaction only. The stress-energy tensor associated with two of such objects is given by

$$T^{\mu\nu} = \mu_1\, v_1^\mu v_1^\nu\, \delta^{(d)}(\mathbf{x}-\mathbf{y}_1) + \mu_2\, v_2^\mu v_2^\nu\, \delta^{(d)}(\mathbf{x}-\mathbf{y}_2)\,, \tag{IV.12}$$

[3]This quantity represents the energy contained in a box of coordinates, and is thus not covariantly conserved.

where $v_1^\mu = \mathrm{d}y_1^\mu/\mathrm{d}t = (c, v_1^i)$ is the coordinate velocity of the particle 1, and $\delta^{(d)}$ is the d-dimensional Dirac distribution, see fig. A.1 for our geometrical conventions. In order to account for all PN corrections, the effective masses entering $T^{\mu\nu}$ are given by

$$\mu_1(t) = \frac{1}{\sqrt{-(g)_1}} \frac{m_1}{\sqrt{-(g_{\mu\nu})_1 \frac{v_1^\mu v_1^\nu}{c^2}}}\,, \tag{IV.13}$$

where the index 1 indicates that the metric has to be evaluated at the location $\mathbf{x} = \mathbf{y}_1$. The self-gravitating divergences are removed by means of Hadamard regularization up to 2PN, and by dimensional regularization from 3PN order on. It is very useful to define matter currents in a similar guise than the source terms (IV.8)

$$\sigma \equiv \frac{2}{d-1}\frac{(d-2)T^{00} + T^{ii}}{c^2}\,, \qquad \sigma_i \equiv \frac{T^{0i}}{c}\,, \qquad \text{and} \qquad \sigma_{ij} \equiv T^{ij}\,. \tag{IV.14}$$

In the case of compact binary systems they read

$$\sigma = \tilde{\mu}_1\, \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2\,, \tag{IV.15a}$$

$$\sigma_i = \mu_1 v_1^i\, \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2\,, \tag{IV.15b}$$

$$\sigma_{ij} = \mu_1 v_1^i v_1^j\, \delta^{(d)}(\mathbf{x} - \mathbf{y}_1) + 1 \leftrightarrow 2\,, \tag{IV.15c}$$

with the novel effective parameter

$$\tilde{\mu}_1 = \frac{2}{d-1}\left(d - 2 + \frac{v_1^2}{c^2}\right)\mu_1\,. \tag{IV.16}$$

Metric parametrization in terms of potentials

As the source moments $\{I_L, J_L\}$ encode the whole physical information of the binary system, the metric to inject in (IV.6) is the PN-expanded metric $\bar{h}^{\mu\nu}$, valid in the near-zone. This metric has been naturally computed in order to solve the conservative sector, but we cannot use directly those results. Indeed in the Fokker Lagrangian method, the computation of the nPN equations of motion requires the knowledge of the metric at $\mathcal{O}(c^{-n-2})$, this is the so-called "n+2" method [54]. The metric was therefore computed up to the 3PN order. Nevertheless no equivalent of such "n+2" method exists for the computation of the source moments, and thus we need to compute the metric at 4PN order (more precisely we require the knowledge of $\bar{h}^{00}$, $\bar{h}^{0i}$ and $\bar{h}^{ij}$ at respectively 4PN, 3.5PN and 4PN accuracies).

It is convenient to parametrize the metric in terms of potentials [69]. As $\bar{h}^{00}$, $\bar{h}^{0i}$ and $\bar{h}^{ij}$ start respectively at 1PN, 1.5PN and 2PN orders, we need a relative 3PN knowledge for $\bar{h}^{00}$ and relative 2PN knowledges for $\bar{h}^{0i}$ and $\bar{h}^{ij}$. Parametrizing each order with a new potential, we thus need four scalar $\{V, K, \hat{X}, \hat{T}\}$, three vector $\{V_i, \hat{R}_i, \hat{Y}_i\}$ and three tensor $\{\hat{W}_{ij}, \hat{Z}_{ij}, \hat{M}_{ij}\}$ potentials.[4] The structure of the metric at the required order is then given by

$$\bar{h}^{00} = -\frac{2}{c^2}\frac{(d-1)V}{d-2} + \frac{4}{c^4}\left[\frac{(d-3)(d-1)K}{(d-2)^2} + \dots\right] - \frac{8}{c^6}\left[\frac{(d-1)\hat{X}}{d-2} + \dots\right] - \frac{32}{c^8}\left[\frac{(d-1)\hat{T}}{d-2} + \dots\right], \tag{IV.17a}$$

$$\bar{h}^{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\left[\hat{R}_i + \dots\right] - \frac{16}{c^7}\left[\hat{Y}_i + \dots\right], \tag{IV.17b}$$

$$\bar{h}^{ij} = -\frac{4}{c^4}\left[\hat{W}_{ij} - \frac{\hat{W}}{2}\delta_{ij}\right] - \frac{16}{c^6}\left[\hat{Z}_{ij} - \frac{\hat{Z}}{2}\delta_{ij}\right] - \frac{32}{c^8}\left[\hat{M}_{ij} + \dots\right], \tag{IV.17c}$$

[4] The hats were introduced to distinguish those potentials from older definitions, that were not involving traces in the $\bar{h}^{ij}$ component of the metric (IV.17c). For historical reasons, we will retain this notation with hats.

where ellipses denote combinations of other potentials, and we have used shorthands for the trace, *eg.* $\hat{W} \equiv \hat{W}_{ii}$. Note that the potential K is a pure feature of the d-dimensional space and does not enter the three-dimensional metric (but it plays a crucial role in regularization processes). The full metric at the required order is presented in app. C.1. The potentials are defined by Einstein's field equations (B.4) in the near zone, and thus obey flat wave equations with sources given by compact support terms coming from the matter sector, and products of other potentials, coming from the $\Lambda^{\mu\nu}$ source term. The simplest potentials have compact support sources (*ie.* proportional to Dirac distributions)

$$\Box V = -4\pi G\sigma\,, \qquad \Box V_i = -4\pi G\sigma_i\,, \qquad \text{and} \qquad \Box K = -4\pi G V\sigma\,. \tag{IV.18}$$

The other potentials involve non-linear terms, such as the $\hat{W}_{ij}$ potential obeying

$$\Box \hat{W}_{ij} = -4\pi G\left(\sigma_{ij} - \frac{\sigma_{kk}}{d-2}\,\delta_{ij}\right) - \frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_i V \partial_j V\,. \tag{IV.19}$$

The wave equations governing the ten potentials that we need are presented in app. C.2.

The different types of terms involved in the Hadamard computation

Injecting the metric (IV.17) in the expression (IV.6) for $\ell = 2$, and truncating at 4PN order, one gets the expression of the source quadrupole in terms of potentials and matter densities (IV.14). In theory, a simple integration and regularization process would then allow to get its explicit expression in terms of the masses, positions and speeds of the two compact objects.

But in practice we do not known the expression of the complicated potentials (such as $\hat{M}_{ij}$), as solving the associated wave equations is, to the best of our computational techniques, not feasible. In order to avoid using them, we employ integrations by parts. For example, let us consider a term $\int \mathrm{d}^d\mathbf{x} r^B \hat{x}_L \phi P$ involving a complicated potential P (with source S, so that $\Box P = S$) and a simple one, ϕ (where we have set $r_0 = 1$ as it will play no role here). Defining the *superpotentials* associated to ϕ as obeying

$$\Delta \Psi_L^\phi = \hat{x}_L\,\phi\,, \tag{IV.20}$$

(note that superpotentials are defined with a Laplacian and not a d'Alembertian), one can integrate the complicated term by parts as

$$\begin{aligned}\int \mathrm{d}^d\mathbf{x}\, r^B\, \hat{x}_L\phi\, P &= \int \mathrm{d}^d\mathbf{x}\, r^B \left\{\Psi_L^\phi \Delta P + \partial_i\left(P\partial_i\Psi_L^\phi - \Psi_L^\phi\partial_i P\right)\right\} \\ &= \int \mathrm{d}^d\mathbf{x}\, r^B \left\{\Psi_L^\phi S + \frac{1}{c^2}\Psi_L^\phi \partial_t^2 P\right\} + \int \mathrm{d}^3\mathbf{x}\, r^B \partial_i\left(P\partial_i\Psi_L^\phi - \Psi_L^\phi\partial_i P\right)\,,\end{aligned} \tag{IV.21}$$

where we have substituted $\Box P$ by S in the last line. This integration by parts pushes the unknown potential to a higher PN order, which would usually contribute beyond the aimed accuracy. The potential P also enters *surface terms* (the second integral in the last line of (IV.21)), that only involve an expansion of P at spatial infinity, always computable as detailed in sec. IV.2.3. If such integrations by parts are very convenient to deal with unknown potentials, they involve the sources of those potentials and thus induce extremely long expressions (see *eg.* eq. (C.5a)). Note that all the required superpotentials are straightforward to compute, see [94, 278] for the detailed procedure. The (eleven pages long) integrated by parts source quadrupole, given in terms of potentials, superpotentials and matter densities, is presented in the app. C of [278].

Let us first focus on the Hadamard computation of the source quadrupole, thus taking $d = 3$. This computation involves three different types of terms: compact support, volume and surface ones. The first type appears when a matter density enters, as *eg.* in $\int \mathrm{d}^3\mathbf{x}(r/r_0)^B \hat{x}_L V_i \hat{R}_i \sigma$. Indeed matter densities involve Dirac distributions, as clear from eq. (IV.15), and thus one simply needs to evaluate the associated product of potentials ($V_i\hat{R}_i$ in our example) in $\mathbf{x} = \mathbf{y}_1$ and $\mathbf{y}_2$. The second, and most common, type of terms are volume terms, *ie.* usual Hadamard regularized volume integrals of products of known potentials. The last class of terms, surface integrals, that appear *eg. via* the described integrations by parts are dealt with in sec. IV.2.3.

Performing the accurate dimensional regularization requires five supplementary types of computations. First, one has to compute the *difference* between Hadamard and dimensional regularizations for compact support and volume terms (as the dimensional regularizations is of UV type, it does not affect surface terms). Convenient formulas have been derived and are extensively used. Note that, in practice, the compact support terms are directly computed in d dimensions. Next, as we use distributions when defining the potential, we should use *distributional* derivatives instead of ordinary ones. To remedy this problem, a generalization of the Gel'fand-Shilov formula [205] in d dimensions is applied. Note that this problem is not specific to the d-dimensional case, and a similar procedure should be applied in the pure three-dimensional case. Nevertheless, it is easier to apply it directly in d dimensions, which is why we have not implemented it in three dimensions. The fourth type of contributions comes from the time derivatives entering eq. (IV.6) and eq. (IV.11): indeed the non-shifted 3PN acceleration bears a pole $1/\varepsilon$ that combines with the $\mathcal{O}(\varepsilon)$ order of 1PN terms, to give a non-vanishing 4PN contribution in the $d \to 3$ limit. Finally, the UV shift coming from the equations of motion has to be applied, and should (and did) remove the remaining poles.

IV.2.2 Computation of the potentials

It appears clearly that the core of the computation of the source moment is the knowledge of the potentials parametrizing the metric (and some superpotentials). In full generality, those potentials are derived at the required PN order by applying the Hadamard regularized retarded propagator $\widetilde{\Box_\mathrm{R}^{-1}}$. But the odd PN terms are already known and begin at 2.5PN order [86]: their first contributions *via* odd-odd couplings are 5PN. Therefore we can focus on even PN terms only, and trade the retarded propagator for the PN-expanded symmetric one

$$\widetilde{\Box^{-1}_{\mathrm{sym}}} = \sum_{n\in\mathbb{N}} \left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2n} \widetilde{\Delta^{-n-1}}\,. \tag{IV.22}$$

The inverse Laplacian operator is given by the usual Hadamard regularized Poisson integral in d dimensions

$$\left(\widetilde{\Delta^{-1}}S\right)(\mathbf{x}) \equiv \mathop{\mathrm{PF}}_{B}\left[-\frac{\tilde{k}}{4\pi}\int \mathrm{d}^d\mathbf{x}'\left(\frac{r'}{r_0}\right)^B \frac{S\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|^{d-2}}\right], \tag{IV.23}$$

which naturally reduces to the canonical three dimensional Poisson integral for $d = 3$. The constant $\tilde{k} = \Gamma(\frac{d-2}{2})/\pi^{\frac{d-2}{2}}$ has been defined in app. A.1.2. Note that this prescription for the propagator is the one compatible with the matching procedure: it automatically selects the accurate homogeneous function to satisfy the matching equation (II.48). When applying other propagators, we will have to correct the procedure by adding an accurate homogeneous solution, so that the matching equation is fulfilled.

Computation of the linear and cubic potentials

The compact support sources are easily integrated, as the Green function of the Laplace operator is simply given by

$$\widetilde{\Delta^{-1}}\delta^{(d)}(\mathbf{x}-\mathbf{y}_1) = -\frac{\tilde{k}}{4\pi}\,r_1^{2-d}\,. \tag{IV.24}$$

The "linear" potentials V, V_i and K are thus easily computed in all space, at any PN order, and automatically fulfill the matching equation.

The only "cubic" potential whose knowledge is required in all space is $\hat{X}$, at Newtonian order. It is cubic in the sense that its source (C.4a) involves a non-linearity between a quadratic and a linear potential, *via* a term $\hat{W}_{ij}\partial_{ij}V$. How to compute and match such difficult interaction in three dimensions has been understood in [84, 85].

Computation of the quadratic potentials

The remaining of this section deals with the computation of the quadratic potentials $\hat{W}_{ij}$ at 2PN, $\hat{R}_i$ and $\hat{Z}_{ij}$ at 1PN, in three dimensions. The previous knowledge of those potentials was respectively 1PN and Newtonian orders. We focus on the most involving case, namely the computation of the non-linear part of $\hat{W}_{ij}$ at 2PN, that involves all the ingredients required to compute the two others potentials. So we seek for the 2PN solution of

$$\Box \hat{W}_{ij}^{(\mathrm{NL})} = -\partial_i V \partial_j V\,, \qquad \text{with} \qquad \Box V = -4\pi G \tilde{\mu}_1 \delta_1 + 1 \leftrightarrow 2\,, \tag{IV.25}$$

where $\delta_1 \equiv \delta^{(3)}(\mathbf{x}-\mathbf{y}_1)$ and we recall that $\tilde{\mu}_1$ is a time-dependent effective mass defined by eq. (IV.16). The iteration of the Poisson integrals straightforwardly yields the 2PN expression for the potential V as

$$V = \frac{G\tilde{\mu}_1}{r_1} + \frac{G}{c^2}\,\partial_t^2\Big(\tilde{\mu}_1 \frac{r_1}{2}\Big) + \frac{G}{c^4}\,\partial_t^4\Big(\tilde{\mu}_1 \frac{r_1^3}{24}\Big) + 1 \leftrightarrow 2 + \mathcal{O}(c^{-6})\,. \tag{IV.26}$$

When plugging this expression into the source term of $\hat{W}_{ij}^{(\mathrm{NL})}$ up to 2PN order, it is convenient to split it into "self" terms, which are essentially proportional to $\tilde{\mu}_1^2$ or $\tilde{\mu}_2^2$, and "interaction" terms proportional to $\tilde{\mu}_1\tilde{\mu}_2$, as

$$\hat{W}_{ij}^{(\mathrm{NL})} = \hat{W}_{ij}^{\mathrm{self}} + \hat{W}_{ij}^{\mathrm{inter}} + \mathcal{O}(c^{-6})\,. \tag{IV.27}$$

Introducing the partial derivatives with respect to the source points ${}_a\partial_i \equiv \partial/\partial y_a^i$, and using the fact that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are just functions of time, the interaction part is expressed in terms of elementary kernel functions $\{g, f, f^{12}, f^{21}, h, h^{12}, h^{21}, k\}$ as

$$\begin{aligned}
\underset{1}{\hat{W}}{}^{\mathrm{inter}}_{ij} = &- G^2\tilde{\mu}_1\tilde{\mu}_2\, \underset{1}{\partial}_{(i}\underset{2}{\partial}_{j)} g - \frac{G^2}{c^2}\Big\{\partial_t^2\Big[\tilde{\mu}_1\tilde{\mu}_2\, \underset{1}{\partial}_{(i}\underset{2}{\partial}_{j)} f\Big] + 2\ddot{\tilde{\mu}}_1\tilde{\mu}_2\, \underset{1}{\partial}_{(i}\underset{2}{\partial}_{j)} f^{12} + 4\dot{\tilde{\mu}}_1\tilde{\mu}_2 v_1^k\, \underset{1}{\partial}_{k(i}\underset{2}{\partial}_{j)} f^{12} \\
&\qquad\qquad + 2\tilde{\mu}_1\tilde{\mu}_2 a_1^k\, \underset{1}{\partial}_{k(i}\underset{2}{\partial}_{j)} f^{12} + 2\tilde{\mu}_1\tilde{\mu}_2 v_1^k v_1^l\, \underset{1}{\partial}_{kl(i}\underset{2}{\partial}_{j)} f^{12}\Big\} \\
&- \frac{G^2}{c^4}\Big\{\partial_t^4\Big[\tilde{\mu}_1\tilde{\mu}_2\, \underset{1}{\partial}_{(i}\underset{2}{\partial}_{j)} h\Big] + \partial_t^2\Big[2\ddot{\tilde{\mu}}_1\tilde{\mu}_2\, \underset{1}{\partial}_{(i}\underset{2}{\partial}_{j)} k^{21} + 4\dot{\tilde{\mu}}_1\tilde{\mu}_2 v_1^k\, \underset{1}{\partial}_{k(i}\underset{2}{\partial}_{j)} k^{21} \\
&\qquad + 2\tilde{\mu}_1\tilde{\mu}_2 a_1^k\, \underset{1}{\partial}_{k(i}\underset{2}{\partial}_{j)} k^{21} + 2\tilde{\mu}_1\tilde{\mu}_2 v_1^k v_1^l\, \underset{1}{\partial}_{kl(i}\underset{2}{\partial}_{j)} k^{21}\Big] \\
&\qquad + \tilde{\mu}_1\tilde{\mu}_2 a_1^m a_2^k\, \underset{1}{\partial}_{m(i}\underset{2}{\partial}_{j)k} k + \tilde{\mu}_1\tilde{\mu}_2 a_1^m v_2^k v_2^l\, \underset{1}{\partial}_{m(i}\underset{2}{\partial}_{j)kl} k + \tilde{\mu}_1\tilde{\mu}_2 v_1^m v_1^n a_2^k\, \underset{1}{\partial}_{mn(i}\underset{2}{\partial}_{j)k} k \\
&\qquad + \tilde{\mu}_1\tilde{\mu}_2 v_1^m v_1^n v_2^k v_2^l\, \underset{1}{\partial}_{mn(i}\underset{2}{\partial}_{j)kl} k + 2\tilde{\mu}_1\tilde{\mu}_2 \ddot{a}_1^k\, \underset{1}{\partial}_{k(i}\underset{2}{\partial}_{j)} h^{12} + 8\tilde{\mu}_1\tilde{\mu}_2 \dot{a}_1^k v_1^l\, \underset{1}{\partial}_{kl(i}\underset{2}{\partial}_{j)} h^{12} \\
&\qquad + 6\tilde{\mu}_1\tilde{\mu}_2 a_1^k a_1^l\, \underset{1}{\partial}_{kl(i}\underset{2}{\partial}_{j)} h^{12} + 12\tilde{\mu}_1\tilde{\mu}_2 a_1^k v_1^l v_1^m\, \underset{1}{\partial}_{klm(i}\underset{2}{\partial}_{j)} h^{12} + 2\tilde{\mu}_1\tilde{\mu}_2 v_1^k v_1^l v_1^m v_1^n\, \underset{1}{\partial}_{klmn(i}\underset{2}{\partial}_{j)} h^{12}\Big\}\,,
\end{aligned} \tag{IV.28}$$

where dots denote time derivatives, $\mathbf{v}_{1,2}$ are the velocities and $\mathbf{a}_{1,2}$ the accelerations. The kernel functions introduced in this term are defined by the Poisson equations they obey:

$$\Delta g = \frac{1}{r_1 r_2}\,, \tag{IV.29a}$$

$$\Delta f = g\,, \qquad \Delta f^{12} = \frac{r_1}{2r_2}\,, \qquad \Delta f^{21} = \frac{r_2}{2r_1}\,, \tag{IV.29b}$$

$$\Delta h = f\,, \qquad \Delta h^{12} = \frac{r_1^3}{24 r_2}\,, \qquad \Delta h^{21} = \frac{r_2^3}{24 r_1}\,, \tag{IV.29c}$$

$$\Delta k = \frac{r_1 r_2}{4}\,, \qquad \Delta k^{12} = f^{21}\,, \qquad \Delta k^{21} = f^{12}\,, \tag{IV.29d}$$

where the numerical coefficients have been introduced for later convenience, and differ *eg.* from those employed in the definition of f, f^{12} and f^{21} in [76].

To the "interaction" term given in eq. (IV.28), one must add the corresponding "self" term, obtained from $\hat{W}_{ij}^{\text{inter}}$ by performing the limit of source points $y_2^i \to y_1^i$ and replacing $\tilde{\mu}_2$ by $\tilde{\mu}_1$. However this term becomes divergent when performing the limit $y_2^i \to y_1^i$ and we have to carefully use the self-field regularization. With this caveat in mind we have

$$\underset{1}{\hat{W}}{}_{ij}^{\text{self}} = \lim_{\substack{\mathbf{y}_2\to\mathbf{y}_1\\ \tilde{\mu}_2\to\tilde{\mu}_1}} \left[\underset{1}{\hat{W}}{}_{ij}^{\text{inter}}\right]. \tag{IV.30}$$

Naturally, we have to add to (IV.28) and (IV.30) the terms corresponding to $1 \leftrightarrow 2$.

The aim is thus to compute the kernels defined by eq. (IV.29). If g is well-known as the Fock function [192]

$$g = \ln\left(\frac{r_1 + r_2 + r_{12}}{r_c}\right), \tag{IV.31}$$

where $r_{12} \equiv |\mathbf{y}_1 - \mathbf{y}_2|$, and r_c is an arbitrary constant, the others are not as famous. As presented hereafter, we will select the particular solution of the kernels by adapting previous findings in the literature. Such procedure will yield potentials that do not satisfy the matching condition, and thus we have to add the accurate homogeneous solution by a matching to the far-zone. A subtle point is that it is equivalent to match the potentials, or directly the kernels. Choosing the second way of proceeding, we will have to select the accurate homogeneous functions so that the matching equation is fulfilled (in the case of g, this is equivalent to choosing the accurate constant r_c).

Computing the particular solutions

Particular solutions for the Poisson equations (IV.29) have been already used within the Hamiltonian approach of PN, see *eg.* the app. A of [241]. The general structure of these solutions is constituted of two parts: a homogeneous regular solution of the Laplace or iterated Laplace operator, multiplied by the Fock function, and a specific polynomial of r_1, r_2 and r_{12}. As we seek for particular solutions, we are free to add a global homogeneous solution, for example by adding a numerical constant to the Fock function. This will not affect the true final solution as the proper homogeneous function will be selected later by the matching procedure. Thus we have chosen to start with the following particular solutions, that differ from those in app. A of [241] by homogeneous solutions, and will be denoted with a hat:

$$\hat{g} = \ln S + \frac{197}{810}\,, \tag{IV.32a}$$

$$\hat{f} = \frac{1}{12}\left[\left(r_1^2 + r_2^2 - r_{12}^2\right)\left(\ln S - \frac{73}{810}\right) + r_1 r_{12} + r_2 r_{12} - r_1 r_2\right], \tag{IV.32b}$$

$$\begin{aligned}\hat{h} = \frac{1}{320}\Bigg[&\left(r_1^4 + r_2^4 - r_{12}^4 - 2r_{12}^2(r_1^2 + r_2^2) + \frac{2}{3}r_1^2 r_2^2\right)\left(\ln S - \frac{37}{81}\right)\\ &+ r_1 r_2(r_{12}^2 - r_1^2 - r_2^2) + r_{12}(r_1 + r_2)(r_1 r_2 - r_{12}) + \frac{4}{9}r_1^2 r_2^2 + \frac{5}{3}r_{12}(r_1^3 + r_2^3)\Bigg],\end{aligned} \tag{IV.32c}$$

$$\begin{aligned}\hat{k} = \frac{1}{120}\Bigg[&\left(r_{12}^4 - 3r_1^4 - 3r_2^4 + 6r_1^2 r_2^2 + 2r_{12}^2(r_1^2 + r_2^2)\right)\ln S\\ &+ \frac{21}{10}(r_1^4 + r_2^4) - \frac{r_{12}^4}{30} + 3r_{12}(r_1^3 + r_2^3) + (r_1^2 + r_2^2)\left(3r_1 r_2 - \frac{31}{15}r_{12}^2\right)\\ &+ r_1 r_2 r_{12}^2 - \frac{21}{5}r_1^2 r_2^2 - r_{12}(r_1 + r_2)\left(r_{12}^2 - 3r_1 r_2\right)\Bigg],\end{aligned} \tag{IV.32d}$$

with the shorthand $S \equiv r_1 + r_2 + r_{12}$. The functions $\hat{f}^{12}$, $\hat{h}^{12}$ and $\hat{k}^{12}$ are naturally obtained by exchanging the field point $\mathbf{x}$ with the source point $\mathbf{y_1}$:

$$\hat{f}^{12} = \hat{f}\Big|_{\mathbf{x}\longleftrightarrow\mathbf{y_1}}\,, \qquad \hat{h}^{12} = \hat{h}\Big|_{\mathbf{x}\longleftrightarrow\mathbf{y_1}}\,, \qquad \hat{k}^{12} = \hat{k}\Big|_{\mathbf{x}\longleftrightarrow\mathbf{y_1}}\,, \tag{IV.33}$$

and similarly $\hat{f}^{21}$, $\hat{h}^{21}$ and $\hat{k}^{21}$ are obtained by exchanging $\mathbf{x}$ and $\mathbf{y_2}$.

It is straightforward to check that the kernel functions (IV.32) and (IV.33) satisfy the constitutive relations (IV.29) and also, in addition, the relations

$$\Delta_1 \hat{f}^{12} = \hat{g}\,, \qquad \Delta_1 \hat{h}^{12} = \hat{f}^{12}\,, \qquad \Delta_1 \hat{k} = \hat{f}^{21}\,, \qquad \text{and} \qquad \Delta_1 \hat{k}^{21} = \hat{f}\,, \tag{IV.34}$$

together with the relations obtained from $1 \leftrightarrow 2$, where $\Delta_1 = \underset{1}{\partial_i}\underset{1}{\partial_i}$. Those extra relations are specifically true for the homogeneous solutions chosen in eq. (IV.32). All those relations suggest that there is an underlying algebra relating those particular kernel functions to higher orders, and thus that we should be able to compute a particular solution to the general Poisson equation $\Delta\varphi_{nm} = r_1^{2n-1} r_2^{2m-1}$, with $(n,m) \in \mathbb{N}^2$. This would lead to the knowledge at all even PN orders of the quadratic potentials $\hat{W}_{ij}$, $\hat{Z}_{ij}$ and $\hat{R}_i$, up to the possible odd-odd couplings.

Matching procedure

As the particular solutions (IV.32) and (IV.33) have not been derived by using the accurate propagator (IV.23), they have to be matched to the far-zone by adding the accurate homogeneous solution. To this purpose, it is more convenient to work with d'Alembertian rather than Poisson equations, hence we resum the kernels into the functions

$$\mathcal{G} \equiv g + \frac{1}{c^2}\,\partial_t^2 f + \frac{1}{c^4}\,\partial_t^4 h + \mathcal{O}(c^{-6})\,, \tag{IV.35a}$$

$$\mathcal{F}^{12} \equiv f^{12} + \frac{1}{c^2}\,\partial_t^2 k^{21} + \mathcal{O}(c^{-4})\,, \tag{IV.35b}$$

$$\mathcal{H}^{12} \equiv h^{12} + \mathcal{O}(c^{-2})\,, \tag{IV.35c}$$

$$\mathcal{K} \equiv k + \mathcal{O}(c^{-2})\,, \tag{IV.35d}$$

that obey

$$\Box\mathcal{G} = \frac{1}{r_1 r_2}\,, \qquad \Box\mathcal{F}^{12} = \frac{r_1}{2r_2}\,, \qquad \Box\mathcal{K} = \frac{r_1 r_2}{4}\,, \qquad \text{and} \qquad \Box\mathcal{H}^{12} = \frac{r_1^3}{24 r_2}\,. \tag{IV.36}$$

Let $\Box\Psi = S(\mathbf{x}, t)$ be one of those wave equations, with some non-compact support source S. The matching equation (II.48) can be worked out to state that the multipolar expansion of the solution Ψ, denoted $\mathcal{M}(\Psi)$, should satisfy [73]

$$\mathcal{M}(\Psi) = \widetilde{\Box_{\mathrm{R}}^{-1}}\mathcal{M}(S) - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,\mathcal{S}_L\left(t - \frac{r}{c}\right)\right]\,, \tag{IV.37}$$

where the first term is a solution of the multipole expanded wave equation $\Box\mathcal{M}(\Psi) = \mathcal{M}(S)$, defined by means of the Hadamard regularized retarded propagator, and the second term is a homogeneous solution constructed out of the multipole moments:

$$\mathcal{S}_L(u) = \underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B x_L\, S(\mathbf{x}, u)\,, \tag{IV.38}$$

which themselves integrate over the source S. Suppose now that we know a particular solution of the wave equation, say $\hat{\Psi}$ such that $\Box\hat{\Psi} = S$. We look for a homogeneous solution Ψ^{hom} such that

$\Psi = \hat{\Psi} + \Psi^{\rm hom}$ satisfies eqs. (IV.37)–(IV.38). Since the homogeneous solution is directly in the form of a multipole expansion, we obtain the following relation:

$$\Psi^{\rm hom} = \mathcal{M}(\Psi^{\rm hom}) = \widetilde{\Box_R^{-1}}\mathcal{M}\,(S) - \mathcal{M}\big(\hat{\Psi}\big) - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,\mathcal{S}_L\left(t-\frac{r}{c}\right)\right]. \tag{IV.39}$$

The previous recipe (IV.39) completely determines the homogeneous solution, since all the terms in the right-hand side are computable. In practical computations, we trade the Hadamard regularized retarded propagators $\widetilde{\Box_R^{-1}}$ for symmetric ones (IV.22), as only even PN terms are relevant to our purpose.

We have applied this method to determine all the relevant homogeneous solutions in the kernel functions g, f, *etc.* For example, expanding the last term of eq. (IV.39), and identifying the relevant PN orders, it comes

$$g^{\rm hom} = \widetilde{\Delta^{-1}}\mathcal{M}\Big(\frac{1}{r_1 r_2}\Big) - \mathcal{M}\big(\hat{g}\big)\,, \tag{IV.40a}$$

$$f^{\rm hom} = \widetilde{\Delta^{-2}}\mathcal{M}\Big(\frac{1}{r_1 r_2}\Big) - \mathcal{M}\big(\hat{f}\big) + \frac{1}{4}\Big(r\,Y - n^i Y_i\Big)\,, \tag{IV.40b}$$

where we have denoted (notice the STF multipole factor $\hat{x}_L$)

$$Y_L \equiv -\frac{1}{2\pi}\mathop{\rm FP}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\frac{\hat{x}_L}{r_1 r_2} = \frac{r_{12}}{\ell+1}\sum_{m=0}^{\ell} y_1^{\langle M}y_2^{L-M\rangle}\,. \tag{IV.41}$$

We emphasize that, although we introduced the “Poisson” kernels g, f, f^{12}, *etc.* for convenience, it is better to consider the “d'Alembertian” kernels ($\mathcal{G}$, $\mathcal{F}^{12}$, *etc.*) as the fundamental quantities. Indeed, when working with the Poisson kernels, we have to take into account the fact that the Hadamard regularization and the inverse Laplacian do not commute, thus for instance

$$\widetilde{\Delta^{-1}}\mathcal{M}\,(f) \neq \widetilde{\Delta^{-3}}\mathcal{M}\Big(\frac{1}{r_1 r_2}\Big)\,, \tag{IV.42}$$

and the matching procedure in more complicated. Nonetheless, after adding some correction terms accounting for this non-commutativity, the previous results are recovered. Note however that the non-commutativity emerges at 2PN order, and only affects the computation of h.

Having matched the particular solutions, one has selected the good prescription for the elementary kernels. The results for g and f are for example

$$g = \ln\left(\frac{r_1+r_2+r_{12}}{2r_0}\right) - 1\,, \tag{IV.43a}$$

$$f = \frac{r_1 r_2\,\mathbf{n}_1\cdot\mathbf{n}_2}{6}\left[\ln\left(\frac{r_1+r_2+r_{12}}{2r_0}\right)+\frac{1}{6}\right] + \frac{r_{12}r_1 + r_{12}r_2 - r_1 r_2 + 2r\,\mathbf{n}\cdot(\mathbf{y_1}+\mathbf{y_2}) - 3r^2}{12}\,, \tag{IV.43b}$$

where r_0 is the usual scale associated with Hadamard regularization. We do not display the other kernels since their homogeneous solutions are not very enlightening, and we have described the general procedure to obtain them.

In the end, by means of this technique, we have obtained in the whole space (and 3 dimensions) the potential $\hat{W}_{ij}$ at 2PN order, as well as the potentials $\hat{Z}_{ij}$ and $\hat{R}_i$ at 1PN order. Those computations had two main applications. First it gave the correct value for the 2PN trace value of $\hat{W}_{ij}$ evaluated in $\mathbf{y}_1$ (such quantity was previously computed by another method, plagued by a subtle issue of non-commutativity of the d'Alembertian and Hadamard regularized retarded propagator). Second, it allowed to avoid the lengthy matching procedure at infinity required for surface terms, as exposed in the next section.

IV.2.3 Computation of the surface terms

The surface terms entering the mass quadrupole are classified in two types: the *divergence* ones, coming from the integrations by parts (IV.21) and the *Laplacian* ones. The latter arise from the double gradient of a linear potential ϕ (*ie.* V, V_i or K) and an involving one, P (with source S), as

$$\begin{aligned}\int \mathrm{d}^d\mathbf{x}\, r^B\, \hat{x}_L \partial_i\phi\, \partial_i P &= \frac{1}{2}\int \mathrm{d}^d\mathbf{x}\, r^B \Big\{\Delta(\phi P) - \phi\Delta P - P\Delta\phi\Big\} \\ &= -\frac{1}{2}\int \mathrm{d}^d\mathbf{x}\, r^B \Big\{\phi S + P\Delta\phi\Big\} + \int \mathrm{d}^3\mathbf{x}\, r^B \Delta\left(\phi P\right) + \mathcal{O}\left(\frac{1}{c^2}\right) .\end{aligned} \tag{IV.44}$$

The first term is easy to evaluate as $\Delta\phi$ has a compact support, and thus only the knowledge of P at $\mathbf{y}_1$ is required. The so-created surface term involves a Laplacian operator, thus the name. Obviously the UV regularization scheme does not apply to such terms, and as such we can restrict ourselves to a three-dimensional computation. The list and expressions of surface terms we had to deal with are presented in app. C.3.

Integration of the divergence terms

Let us consider a generic divergence term

$$\mathfrak{D} = \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \partial_i \mathcal{A}_i\,, \tag{IV.45}$$

where $\mathcal{A}_i$ is typically a product of potentials, superpotentials and their derivatives. Performing an integration by parts, it comes

$$\mathfrak{D} = \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x}\, \partial_i \left[\left(\frac{r}{r_0}\right)^B \mathcal{A}_i\right] - \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x}\, B \left(\frac{r}{r_0}\right)^B \frac{n_i}{r}\mathcal{A}_i\,. \tag{IV.46}$$

Using Ostrogradski theorem, the first term reduces to the angular integral of $(r/r_0)^B \mathcal{A}_i n_i$, evaluated at infinity. Such term vanishes by virtue of Hadamard regularization: B can be chosen so that $(r/r_0)^B \mathcal{A}_i n_i$ is asymptotically null, and by analytic continuation in B, the integral vanishes. As for the second integral, let us split it in two pieces, separated by an arbitrary constant scale $\mathcal{R}$. For $r < \mathcal{R}$, the smoothness of the metric in the near zone implies that the integrand is regular and thus the B prefactor kills the integral. On the other hand, for $r > \mathcal{R}$, we multipolarly expand $\mathcal{A}_i$ as

$$\mathcal{M}\left(\mathcal{A}_i\right) = \sum_{p\geq p_0} \frac{1}{r^p}\left[\alpha_{i,p}\left(\mathbf{n}\right) + \alpha^{\log}_{i,p}\left(\mathbf{n}\right)\ln\left(\frac{r}{r_0}\right)\right]\,, \tag{IV.47}$$

where p_0 is a (possibly negative) integer, and we have not included higher powers of logarithms as we have not encountered them in practical computations, nor the time dependence as it plays no role. Then

$$\begin{aligned}\mathfrak{D} &= -\underset{B}{\mathrm{PF}} \int_{\mathcal{R}}^{\infty} \mathrm{d}^3\mathbf{x}\, B \left(\frac{r}{r_0}\right)^B \frac{n_i}{r}\mathcal{M}\left(\mathcal{A}_i\right) \\ &= -\underset{B}{\mathrm{PF}} \sum_{p\geq p_0} B \int \mathrm{d}\Omega_2 \int_{\mathcal{R}}^{\infty} \mathrm{d}r \frac{r^{B+1-p}}{r_0^B} n_i \left[\alpha_{i,p}\left(\mathbf{n}\right) + \alpha^{\log}_{i,p}\left(\mathbf{n}\right)\ln\left(\frac{r}{r_0}\right)\right] \\ &= \underset{B}{\mathrm{PF}} \sum_{p\geq p_0} \frac{B}{B+2-p}\frac{\mathcal{R}^{B+2-p}}{r_0^B} \int \mathrm{d}\Omega_2\, n_i \left[\alpha_{i,p}\left(\mathbf{n}\right) + \alpha^{\log}_{i,p}\left(\mathbf{n}\right)\ln\left(\frac{\mathcal{R}}{r_0}\right) - \frac{\alpha^{\log}_{i,p}\left(\mathbf{n}\right)}{B+2-p}\right]\,,\end{aligned} \tag{IV.48}$$

where the upper boundary does not contribute by analytic continuation in B and $\mathrm{d}\Omega_2$ is the volume element of the two-sphere. It is clear that only the $p = 2$ terms survive the Hadamard regularization:

$$\mathfrak{D} = \int \mathrm{d}\Omega_2\, n_i\, \alpha_{i,2}(\mathbf{n})\ . \tag{IV.49}$$

This divergence term thus reduces to a simple angular integral, that does not depend on $\mathcal{R}$ nor on r_0. In order to practically compute it, one has to extract the r^{-2} coefficient of the product of potentials and superpotentials $\mathcal{A}_i$.

Integration of the Laplacian terms

The Laplacian terms are treated in a similar fashion. Let us consider a generic Laplacian term

$$\mathfrak{L}_L = \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \hat{x}_L \, \Delta\mathcal{B} \,, \tag{IV.50}$$

where $\mathcal{B}$ is typically a product of potentials and their derivatives, and ℓ is either 2 or 3 in practical implementations. Performing two integrations by parts, it comes

$$\mathfrak{L}_L = \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \left\{ \left(\frac{r}{r_0}\right)^B \hat{x}_L \, \partial_i\mathcal{B} - \mathcal{B}\,\partial_i \left[\left(\frac{r}{r_0}\right)^B \hat{x}_L \right] \right\} + \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \, B(B+2\ell+1) \left(\frac{r}{r_0}\right)^B \frac{\hat{x}_L}{r^2} \mathcal{B} \,, \tag{IV.51}$$

where we have notably used $\Delta\hat{x}_L = 0$. Using the same argument than for the divergence terms, the first integral is vanishing, and the second one can be restricted to $r > \mathcal{R}$. Expanding $\mathcal{B}$ as

$$\mathcal{M}\left(\mathcal{B}\right) = \sum_{p\geq p_0} \frac{1}{r^p} \left[\beta_p\left(\mathbf{n}\right) + \beta_p^{\log}\left(\mathbf{n}\right) \ln\left(\frac{r}{r_0}\right) \right] \,, \tag{IV.52}$$

where we have dropped the time dependence, it comes

$$\begin{aligned}
\mathfrak{L}_L &= \underset{B}{\mathrm{PF}} \int_{\mathcal{R}}^\infty \mathrm{d}^3\mathbf{x} \, B(B+2\ell+1) \left(\frac{r}{r_0}\right)^B \frac{\hat{x}_L}{r^2} \mathcal{M}\left(\mathcal{B}\right) \\
&= \underset{B}{\mathrm{PF}} \sum_{p\geq p_0} B(B+2\ell+1) \int \mathrm{d}\Omega_2 \int_{\mathcal{R}}^\infty \mathrm{d}r \frac{r^{B+\ell-p}}{r_0^B} \hat{n}_L \left[\beta_p\left(\mathbf{n}\right) + \beta_p^{\log}\left(\mathbf{n}\right) \ln\left(\frac{r}{r_0}\right) \right] \\
&= -\underset{B}{\mathrm{PF}} \sum_{p\geq p_0} \frac{B(B+2\ell+1)}{B+1+\ell-p} \frac{\mathcal{R}^{B+1+\ell-p}}{r_0^B} \int \mathrm{d}\Omega_2 \, \hat{n}_L \left[\beta_p\left(\mathbf{n}\right) + \beta_p^{\log}\left(\mathbf{n}\right) \ln\left(\frac{\mathcal{R}}{r_0}\right) - \frac{\beta_p^{\log}\left(\mathbf{n}\right)}{B+1+\ell-p} \right] \,,
\end{aligned} \tag{IV.53}$$

where the upper boundary still does not contribute by analytic continuation in B. Only the $p = \ell+1$ terms contribute, to yield

$$\mathfrak{L}_L = \int \mathrm{d}\Omega_2 \, \hat{n}_L \left[-(2\ell+1)\beta_{\ell+1}\left(\mathbf{n}\right) + \beta_{\ell+1}^{\log}\left(\mathbf{n}\right) \right] \,. \tag{IV.54}$$

As for the divergence terms, the Laplacian ones reduce to simple angular integrals, that do not depend on $\mathcal{R}$ nor on r_0, and require the knowledge of the $r^{-\ell-1}$ coefficients of the product of potentials $\mathcal{B}$.

Development of the potentials at spatial infinity

Once the functions $\alpha_{i,2}(\mathbf{n})$, $\beta_{\ell+1}(\mathbf{n})$ and $\beta_{\ell+1}^{\log}(\mathbf{n})$ are known, performing the angular integrals of eqs. (IV.49) and (IV.54) is direct (all the needed formulas are encoded in the *PNComBin* library of *xAct*). Therefore, the crucial part of the computation of the surface terms is the knowledge of the asymptotic value of the potentials (and superpotentials). Most of those are already known in all space at the required PN order (notably thanks to the machinery developed in sec. IV.2.2), namely V at 3PN, V_i, $\hat{W}_{ij}$ and $\Psi_{ij}^{\partial_{ab}V}$ at 2PN, and $\hat{Z}_{ij}$, $\hat{R}_i$, $\Psi_{ij}^{\partial_t\partial_a V}$, $\Psi_{ij}^{\partial_a V_b}$ and $\Psi_{ijk}^{\partial_a V}$ at 1PN. We thus straightforwardly[5] expand them and their required derivatives up to the proper order in $1/r$.

[5]Note that, if this expansion is conceptually simple, it may become cumbersome in practice. For instance, the expansion of $\hat{W}_{ij}$ at 2PN at the required $\mathcal{O}(r^{-2})$ order involves $\sim 34\,000$ terms, and its first derivative, $\sim 100\,000$.

But the main interest of creating surface terms is to deal with unknown potentials, in our case $\hat{X}$ at 1PN and $\hat{T}$, $\hat{Y}_i$ and $\hat{M}_{ij}$ at Newtonian order. Obviously the compact support parts of their sources can be integrated in whole space, and expanded subsequently. Similarly, the contributions of the distributional derivatives are to be treated in the same way, as those yield Dirac distributions. For example, the source of the $\hat{X}$ potential (C.4a) contains a contribution of the distributional derivative arising in $\partial_{ij}V$ as

$$\hat{W}_{ij}\partial_{ij}V = \frac{1}{2}\hat{W}_{ij}\partial^{\text{ord}}_{ij}V - \frac{4\pi G\tilde{\mu}_1}{3}\hat{W}\,\delta_1 - \frac{4\pi G\tilde{\mu}_1}{15c^2}\left(2v_1^i v_1^j + v_1^2\delta_{ij}\right)\hat{W}_{ij}\,\delta_1 + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^2}\right), \quad \text{(IV.55)}$$

where ∂^{ord}_i is the ordinary derivation in the sense of functions.

The unknown pieces come from the non-linear source terms, and we construct their accurate expansions at infinity by iterating the propagator on the expanded non-compact sources. Let us consider a potential P obeying $\Box P = S$, where the source term S is known in all space.[6] The asymptotic value of the source $\mathcal{M}(S)$ is easily derived, and we seek for the multipolar expansion $\mathcal{M}(P)$. This is exactly the situation that we have encountered when dealing with the matching of the kernels in the previous section. Indeed, eq. (IV.37) tells us that the multipolar expansion of P is given by

$$\mathcal{M}(P) = \widetilde{\Box^{-1}_R}\mathcal{M}(S) - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,\mathcal{S}_L\left(t-\frac{r}{c}\right)\right], \quad \text{(IV.56)}$$

where we recall that the particular solution is iterated with the Hadamard regularized retarded propagator $\widetilde{\Box^{-1}_R}$, and that the homogeneous solution is constructed out of the quantity

$$\mathcal{S}_L\left(u\right) \equiv \underset{B}{\text{PF}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B x_L\, S(\mathbf{x},u)\,. \quad \text{(IV.57)}$$

For the potentials required at the Newtonian order, eq. (IV.56) reduces to

$$\mathcal{M}(P) = \widetilde{\Delta^{-1}}\mathcal{M}(S) - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\mathcal{S}_L(t)\,\partial_L\left(\frac{1}{r}\right), \quad \text{(IV.58)}$$

and the first term is computed by use of the so-called *Matthieu* formulas[7] (see app. C of [87])

$$\Delta^{-1}\left[r^\alpha\hat{n}_L\right] = \frac{r^{\alpha+2}\hat{n}_L}{(\alpha+3+\ell)(\alpha+2-\ell)}\,, \qquad \text{for } \alpha\in\mathbb{C}\setminus\left\{\ell-2,-\ell-3\right\}, \quad \text{(IV.59a)}$$

$$\Delta^{-1}\left[r^{\ell-2}\hat{n}_L\right] = \frac{1}{2\ell+1}\left[\ln\left(\frac{r}{r_0}\right) - \frac{1}{2\ell+1}\right] r^\ell\hat{n}_L\,, \quad \text{(IV.59b)}$$

$$\Delta^{-1}\left[\frac{\hat{n}_L}{r^{\ell+3}}\right] = -\frac{1}{2\ell+1}\left[\ln\left(\frac{r}{r_0}\right) + \frac{1}{2\ell+1}\right]\frac{\hat{n}_L}{r^{\ell+1}}\,. \quad \text{(IV.59c)}$$

For the particular solution of $\hat{X}$ at 1PN, we naturally truncate $\widetilde{\Box^{-1}_R} = \widetilde{\Delta^{-1}} + \frac{1}{c^2}\partial_t^2\widetilde{\Delta^{-2}}$. As the source term involves also some logarithms $\ln(r/r_0)$, we use (in the generic case $\alpha\in\mathbb{C}\setminus\{\ell-2,-\ell-3\}$),

$$\widetilde{\Delta^{-1}}\left[\ln\left(\frac{r}{r_0}\right)r^\alpha\hat{n}_L\right] = \frac{r^{\alpha+2}\hat{n}_L}{(\alpha+3+\ell)(\alpha+2-\ell)}\left[\ln\left(\frac{r}{r_0}\right) - \frac{2\alpha+5}{(\alpha+3+\ell)(\alpha+2-\ell)}\right]. \quad \text{(IV.60)}$$

Thanks to the formulas encoded in the *PNComBin* library, the integration of the three-dimensional source terms (IV.57) develops no conceptual problems.

The surface terms presented in app. C.3 have consequently been computed in a “double-blind” fashion, together with S. Marsat, and added to the source quadrupole.

[6] As the Newtonian value of $\hat{X}$ is known, this is always the case in our computations.

[7] Note that the name of those useful formulas do not come from a bad spelling of the name of the physicist É. Mathieu, but from the evangelist [1]: the $(\alpha+3+\ell)(\alpha+2-\ell)$ structure directly echos the verse «for to everyone who has will be given more, and he will have more than enough; but from the man who has not, even what he has will be taken away.» [280].

IV.2.4 Application of the UV shift and final sum

Application of the shifts

In fact not one, but two UV shifts were applied in the equations of motions: the first was constructed to remove the remaining UV poles [55, 56] and the second, merely for convenience [57]. The complete expressions of those UV shifts are given in app. C.4.1.

It is of prime importance for the computation of the source quadrupole to use the same coordinate system as for the equations of motion. Therefore those two shifts have to be applied to the raw source quadrupole and should (and do) remove the remaining poles. Therefore, starting from the computed quadrupole, we have to perform the coordinate change

$$y_1^i \to y_1^i + \delta y_1^i\,, \qquad \text{and} \qquad y_2^i \to y_2^i + \delta y_2^i\,, \tag{IV.61}$$

where δy_1^i is the sum of the two shifts, and can be decomposed as

$$\delta y_1^i = \frac{1}{c^6}\left(\frac{1}{\varepsilon}\,\Psi^i_{(3,-1)} + \Psi^i_{(3,0)}\right) + \frac{1}{c^8}\left(\frac{1}{\varepsilon}\,\Psi^i_{(4,-1)} + \Psi^i_{(4,0)}\right)\,. \tag{IV.62}$$

As δy_1^i starts at 3PN, is has to be applied to the d-dimensional 1PN source quadrupole, that reads

$$\begin{aligned} I_{ij} = m_1 \hat{y}_1^{ij} + \frac{m_1}{c^2}\Bigg\{ & \frac{d}{2(d-2)}\,v_1^2\,\hat{y}_1^{ij} - \frac{2(d-2)}{d-1}\frac{\tilde{k}\,G\,m_2}{r_{12}^{d-2}}\hat{y}_1^{ij} - \frac{2(d-1)(d+2)}{d(d+4)(d-2)}\frac{\mathrm{d}}{\mathrm{d}t}\left[v_1^a \hat{y}_1^{aij}\right] \\ & + \frac{1}{2(d+4)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\hat{y}_1^{ij}\right]\Bigg\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^2}\right)\,. \end{aligned} \tag{IV.63}$$

Note that in this expression the accelerations can be replaced by their on-shell values and thus I_{ij} only depends on positions and speeds. Obviously we shift the speeds as $v_1^i \to v_1^i + \frac{\mathrm{d}\delta y_1^i}{\mathrm{d}t}$ and similarly for v_2^i.

Enforcing the UV shifts given in app. C.4.1, the residual UV poles disappear, which is a strong confirmation of the soundness of the PN-MPM framework.

The Hadamard regularized mass quadrupole at 4PN

Adding all the different contributions yields an expression for the mass quadrupole that is far too long to be presented in this dissertation (well simplified, it involves roughly 50 000 terms). Fortunately, reducing I_{ij} to the center-of-mass and expressing it on circular orbits only require the knowledge of the center-of-mass coordinates at 3PN, and the 3PN equations of motion for circular orbits. Those 3PN quantities are naturally insensitive to the 4PN IR dimensional regularization scheme and thus, even if we have not yet performed the accurate IR regularization, we can present this temporary result for circular orbits in the center-of-mass frame. It takes the compact form

$$I_{ij}^{\text{Had}} = m\nu\left(A\,x_{\langle i}x_{j\rangle} + B\,\frac{r^2}{c^2}v_{\langle i}v_{j\rangle} + \frac{G^2m^2\nu}{c^5 r}\,C\,x_{\langle i}v_{j\rangle}\right) + \mathcal{O}\left(\frac{1}{c^9}\right)\,. \tag{IV.64}$$

For completeness, we have added the time-odd 2.5PN and 3.5PN contributions computed in [187]. We have denoted here $r = |\mathbf{y}_1 - \mathbf{y}_2|$ the radial separation in harmonic coordinates, and similarly, $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ the relative distance and $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity. The coefficients entering this expression are explicitly given by

$$A = 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right)$$

$$
\begin{aligned}
&+\gamma^3\left(\frac{395899}{13200}-\frac{428}{105}\ln\left(\frac{r}{r_0}\right)+\left[\frac{3304319}{166320}-\frac{44}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu+\frac{162539}{16632}\nu^2+\frac{2351}{33264}\nu^3\right)\\
&+\gamma^4\left(-\frac{1023844001989}{12713500800}+\frac{31886}{2205}\ln\left(\frac{r}{r_0}\right)+\left[-\frac{18862022737}{470870400}-\frac{2783}{1792}\pi^2\right.\right.\\
&\quad\left.-\frac{24326}{735}\ln\left(\frac{r}{r_0}\right)+\frac{8495}{63}\ln\left(\frac{r}{r_0'}\right)\right]\nu+\left[\frac{171906563}{4484480}+\frac{44909}{2688}\pi^2-\frac{4897}{21}\ln\left(\frac{r}{r_0'}\right)\right]\nu^2\\
&\quad\left.-\frac{22063949}{5189184}\nu^3+\frac{71131}{314496}\nu^4\right),
\end{aligned}
\tag{IV.65a}
$$

$$
\begin{aligned}
B=&\frac{11}{21}-\frac{11}{7}\nu+\gamma\left(\frac{1607}{378}-\frac{1681}{378}\nu+\frac{229}{378}\nu^2\right)\\
&+\gamma^2\left(-\frac{357761}{19800}+\frac{428}{105}\ln\left(\frac{r}{r_0}\right)-\frac{92339}{5544}\nu+\frac{35759}{924}\nu^2+\frac{457}{5544}\nu^3\right)\\
&+\gamma^3\left(\frac{17607264287}{1589187600}-\frac{4922}{2205}\ln\left(\frac{r}{r_0}\right)+\left[\frac{5456382809}{529729200}+\frac{143}{192}\pi^2-\frac{1714}{49}\ln\left(\frac{r}{r_0}\right)-\frac{968}{63}\ln\left(\frac{r}{r_0'}\right)\right]\nu\right.\\
&\quad\left.+\left[\frac{351838141}{5045040}-\frac{41}{24}\pi^2+\frac{968}{21}\ln\left(\frac{r}{r_0'}\right)\right]\nu^2-\frac{1774615}{81081}\nu^3-\frac{3053}{432432}\nu^4\right),
\end{aligned}
\tag{IV.65b}
$$

$$
C=\frac{48}{7}+\gamma\left(-\frac{4096}{315}-\frac{24512}{945}\nu\right), \tag{IV.65c}
$$

where we recall that the 1PN parameter γ is defined as

$$
\gamma=\frac{Gm}{rc^2}. \tag{IV.66}
$$

The two constants r_0 and r_0' entering the coefficients (IV.65) are associated respectively with Hadamard and dimensional UV regularization schemes. As they are unphysical scales, they are expected to be exactly compensated in the radiative moment, by virtue of non-linear effects and application of the time derivatives. Such compensation is perfectly mastered at the 3PN level [73].

The intrepid reader would have noticed that the presented coefficients are slightly different from those published in [278]. Indeed they read

$$
A=A^{\text{publ}}-\frac{4}{63}\gamma^4\nu^2, \qquad \text{and} \qquad B=B^{\text{publ}}+\frac{4}{63}\gamma^3\nu^2. \tag{IV.67}
$$

This 4PN discrepancy comes from an error in the d-dimensional computation of the value of $\hat{R}_i$ at 1PN order, when evaluated in $\mathbf{y}_1$. If this mistake has been corrected in the value of $\tilde{\mu}_1$, we forgot to include it in the computation of the compact support terms.

Note that this preliminary result perfectly agrees with the 3.5PN mass quadrupole [187]. Therefore we can naively expect the contributions of the IR dimensional regularization to enter at the 4PN order only.

IV.3 IR dimensional regularization of the mass quadrupole

When deriving the equations of motion at the 4PN order by means of the Fokker Lagrangian, it was crucial to perform an IR dimensional regularization. This is due to the fact that the Hadamard regularization is not strong enough to deal with the divergences related to the extension at infinity of the formal PN expansion. Indeed, in the expression of the source densities (IV.11), higher PN orders come with higher powers of r, that naturally yield strongly divergent integrals, when plugged into (IV.6). If the Hadamard regularization scheme was sufficient to cure them up to 3PN, it cannot handle those appearing at the 4PN level.

In order to be consistent, and to use the accurate coordinate system, an IR dimensional regularization of the source quadrupole is thus required. In practice, we will only compute the differences between the dimensionally regularized I_{ij}^{DR} and Hadamard regularized I_{ij}^{Had} source quadrupoles, and add it to our previous result for I_{ij}^{Had}. This difference of regularization schemes can be decomposed in five different contributions

$$\mathcal{D}I_{ij} \equiv I_{ij}^{\mathrm{DR}} - I_{ij}^{\mathrm{Had}} = \mathcal{D}I_{ij}^{\mathrm{Vol}} + \mathcal{D}I_{ij}^{\mathrm{Comp}} + \mathcal{D}I_{ij}^{\mathrm{Surf}} + I_{ij}^{\mathrm{extra}} + \delta_\chi I_{ij}\,, \tag{IV.68}$$

where $\mathcal{D}I_{ij}^{\mathrm{Vol}}$, $\mathcal{D}I_{ij}^{\mathrm{Comp}}$ and $\mathcal{D}I_{ij}^{\mathrm{Surf}}$ are the differences in the volume, compact support[8] and surface terms, respectively. I_{ij}^{extra} denotes the value of the last piece of eq. (IV.6) (indeed the previous argument relying on the $(d-3)B$ factor to discard such contribution does not apply anymore). Finally, $\delta_\chi I_{ij}$ is the contribution of the IR shift χ^i that has been applied in the equations of motion.

As the Hadamard regularization scheme is crucial to the matching procedure, we do not use a pure dimensional regularization scheme, but rather a "mixed" Hadamard-dimensional regularization scheme. We defined thus the "dimensional regularization" of the integral $\int \mathrm{d}^d\mathbf{x}\, A(\mathbf{x},t)$ to be

$$\mathrm{DR}\left[\int \mathrm{d}^d\mathbf{x}\, A(\mathbf{x},t)\right] \equiv \lim_{d\to 3}\left\{\underset{\eta}{\mathrm{PF}}\left[\int \mathrm{d}^d\mathbf{x}\left(\frac{r}{r_0}\right)^\eta A(\mathbf{x},t)\right]\right\}\,, \tag{IV.69}$$

where r_0 is the same IR scale as in the three-dimensional Hadamard regularization, but we have changed the name of the regulator from B to η to emphasize that the procedure in now taking place in d dimensions. The difference between the two regularization schemes is thus formally given by the commutator of the finite part and $d\to 3$ limit as

$$\begin{aligned}\mathcal{D}\left\{\int \mathrm{d}^d\mathbf{x} A(\mathbf{x},t)\right\} &\equiv \left[\lim_{d\to 3}, \underset{\eta}{\mathrm{PF}}\right]\left(\int \mathrm{d}^d\mathbf{x} A(\mathbf{x},t)\right) \\ &= \lim_{d\to 3}\left\{\underset{\eta}{\mathrm{PF}}\left[\int \mathrm{d}^d\mathbf{x}\left(\frac{r}{r_0}\right)^\eta A(\mathbf{x},t)\right]\right\} - \underset{\eta}{\mathrm{PF}}\left[\lim_{d\to 3}\left\{\int \mathrm{d}^d\mathbf{x}\left(\frac{r}{r_0}\right)^\eta A(\mathbf{x},t)\right\}\right]\,.\end{aligned} \tag{IV.70}$$

IV.3.1 Computation of the different contributions

Regularization of the volume terms

Let us consider a volume term of the type

$$\mathcal{V} = \int \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^\varepsilon}\left(\frac{r}{r_0}\right)^\eta F(\mathbf{x},t)\,. \tag{IV.71}$$

The function $F(\mathbf{x},t)$ is a product of (super)-potentials and $\hat{x}_L$, and we will drop the time dependence as it will play no role. As we investigate IR regularization schemes, we can restrict the integral to $r > \mathcal{R}$, where $\mathcal{R}$ is an arbitrary constant scale, significantly larger than r_1 and r_2. We will thus expand $F(\mathbf{x})$ in multipolar series

$$F(\mathbf{x}) = \sum_{p\geq p_0}\sum_{q=q_0}^{q_1}\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\,\varphi_{p,q}(\mathbf{n}) = \sum_{p\geq p_0}\sum_{q=q_0}^{q_1}\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\left[\frac{\hat{\varphi}_{p,q}(\mathbf{n})}{\varepsilon} + \varphi_{p,q}^{(\varepsilon)}(\mathbf{n})\right]\,, \tag{IV.72}$$

where p_0, q_0 and q_1 are (possibly negative) integers and we will drop the angular dependencies for convenience. We have allowed the function F to bear a simple pole $1/\varepsilon$ (as we have checked that no double poles appear in volume terms), and thus $\hat{\varphi}_{p,q}$ does not depend on ε. All higher order in ε are thus contained in $\varphi_{p,q}^{(\varepsilon)}$. The three-dimensional limit of F is given by

$$F^{\mathrm{3D}}(\mathbf{x}) = \sum_{p\geq p_0} r^{-p}\left[\varphi_p^{\mathrm{3D}} + \varphi_p^{\log}\ln\left(\frac{r}{\ell_0}\right)\right]\,, \tag{IV.73}$$

[8]Note that, due to the Dirac distribution they involve, we include the contributions of distributional derivatives in the class of compact support terms.

where the coefficients are related to those entering eq. (IV.72) by

$$\varphi_p^{\rm 3D} \equiv \left(\sum_{q=q_0}^{q_1} \varphi_{p,q}^{(\varepsilon)}\right)_{\varepsilon=0}\,, \qquad \varphi_p^{\rm log} \equiv -\sum_{q=q_0}^{q_1} q\,\hat\varphi_{p,q}\,, \qquad \text{and} \qquad \sum_{q=q_0}^{q_1} \hat\varphi_{p,q} \equiv 0\,. \tag{IV.74}$$

The last equality can be understood as a condition on F so that its three-dimensional limit is non-pathological.

With $\mathrm{d}\Omega_{d-1}$ the volume element of the $d-1$ sphere, the dimensional regularization of the volume intergral (IV.71) reads

$$\begin{aligned}
\mathcal{V}^{\rm DR} &= \lim_{d\to 3}\left\{\mathop{\rm PF}_{\eta}\int_{r>\mathcal{R}}\mathrm{d}r\,\frac{r^{2+\varepsilon+\eta}}{\ell_0^\varepsilon r_0^\eta}\sum_{p,q}\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\int\mathrm{d}\Omega_{d-1}\,\varphi_{p,q}\right\}\\
&= -\lim_{d\to 3}\left\{\mathop{\rm PF}_{\eta}\sum_{p,q}\frac{\mathcal{R}^{3-p+\eta+(1-q)\varepsilon}}{3-p+\eta+(1-q)\varepsilon}\frac{\ell_0^{(q-1)\varepsilon}}{r_0^\eta}\int\mathrm{d}\Omega_{d-1}\,\varphi_{p,q}\right\}\\
&= -\lim_{d\to 3}\left\{\sum_p\sum_{q\neq 1}\frac{\mathcal{R}^{3-p+(1-q)\varepsilon}}{3-p+(1-q)\varepsilon}\,\ell_0^{(q-1)\varepsilon}\int\mathrm{d}\Omega_{d-1}\,\varphi_{p,q}+\sum_{p\neq 3}\frac{\mathcal{R}^{3-p}}{3-p}\int\mathrm{d}\Omega_{d-1}\,\varphi_{p,1}+\ln\left(\frac{\mathcal{R}}{r_0}\right)\int\mathrm{d}\Omega_{d-1}\,\varphi_{3,1}\right\}.
\end{aligned}\tag{IV.75}$$

Performing the $\varepsilon\to 0$ limit and using the relations (IV.74), it comes

$$\begin{aligned}
\mathcal{V}^{\rm DR} = &-\sum_{p\neq 3}\frac{\mathcal{R}^{3-p}}{3-p}\int\mathrm{d}\Omega_2\left[\varphi_p^{\rm 3D}+\varphi_p^{\rm log}\ln\left(\frac{\mathcal{R}}{\ell_0}\right)-\frac{\varphi_p^{\rm log}}{3-p}\right]-\ln\left(\frac{\mathcal{R}}{r_0}\right)\int\mathrm{d}\Omega_2\varphi_3^{\rm 3D}\\
&-\frac{1}{2}\ln^2\left(\frac{\mathcal{R}}{\ell_0}\right)\int\mathrm{d}\Omega_2\varphi_3^{\rm log}+\sum_{q\neq 1}\left[\frac{1}{(q-1)\varepsilon}-\ln\left(\frac{r_0}{\ell_0}\right)\right]\int\mathrm{d}\Omega_{d-1}\,\varphi_{3,q}\,.
\end{aligned}\tag{IV.76}$$

On the other hand, the three-dimensional Hadamard regularized volume integral (IV.71) reads

$$\begin{aligned}
\mathcal{V}^{\rm Had} &= \mathop{\rm PF}_{B}\int_{r>\mathcal{R}}\mathrm{d}r\,\frac{r^{2+B}}{r_0^B}\sum_p\frac{1}{r^p}\int\mathrm{d}\Omega_2\left[\varphi_p^{\rm 3D}+\varphi_p^{\rm log}\ln\left(\frac{r}{\ell_0}\right)\right]\\
&= -\mathop{\rm PF}_{B}\sum_p\frac{\mathcal{R}^{3-p+B}}{(3-p+B)r_0^B}\int\mathrm{d}\Omega_2\left[\varphi_p^{\rm 3D}+\varphi_p^{\rm log}\ln\left(\frac{\mathcal{R}}{\ell_0}\right)-\frac{\varphi_p^{\rm log}}{3-p+B}\right]\\
&= -\sum_{p\neq 3}\frac{\mathcal{R}^{3-p}}{3-p}\int\mathrm{d}\Omega_2\left[\varphi_p^{\rm 3D}+\varphi_p^{\rm log}\ln\left(\frac{\mathcal{R}}{\ell_0}\right)-\frac{\varphi_p^{\rm log}}{3-p}\right]-\ln\left(\frac{\mathcal{R}}{r_0}\right)\int\mathrm{d}\Omega_2\left[\varphi_3^{\rm 3D}+\frac{\varphi_3^{\rm log}}{2}\ln\left(\frac{\mathcal{R}r_0}{\ell_0^2}\right)\right].
\end{aligned}\tag{IV.77}$$

Using the relations (IV.74), the difference between the two regularization schemes is then

$$\mathcal{DV}\equiv\mathcal{V}^{\rm DR}-\mathcal{V}^{\rm Had}=\sum_{q\neq 1}\left[\frac{1}{(q-1)\varepsilon}-\ln\left(\frac{r_0}{\ell_0}\right)+\frac{(q-1)\varepsilon}{2}\ln^2\left(\frac{r_0}{\ell_0}\right)\right]\int\mathrm{d}\Omega_{d-1}\left(\frac{\hat\varphi_{3,q}}{\varepsilon}+\varphi_{3,q}^{(\varepsilon)}\right)\,, \tag{IV.78}$$

where the $\varepsilon\to 0$ limit is implicitly understood. There are a few comments to be made on this formula. Firstly, the scale $\mathcal{R}$ has disappeared, which was expected as the difference of IR regularization schemes should not depend on the integration scale. Secondly, the value of the difference is sensitive to very peculiar multipolar orders: only the $\mathcal{O}(r^{-3-q\varepsilon})$ orders, with $q\neq 1$, contribute. Note that this $q\neq 1$ condition is not imposed by hand to make the formula well-defined, but is really selected by the computation: when they exist, terms with $q=1$ do not contribute. Then, this difference between regularization schemes naturally yields poles $1/\varepsilon$, and can even yield double poles $1/\varepsilon^2$. Fortunately, those double poles do not show up in the computations as they all are vanishing in the three-dimensional limit. Finally, let us note that this formula for the difference of volume terms (IV.78) can be compacted in the elegant formulation

$$\mathcal{DV}=\sum_{q\neq 1}\frac{1}{(q-1)\varepsilon}\left(\frac{\ell_0}{r_0}\right)^{(q-1)\varepsilon}\int\mathrm{d}\Omega_{d-1}\,\varphi_{3,q}+\mathcal{O}(\varepsilon)\,. \tag{IV.79}$$

Regularization of the compact support terms

An interesting and non-trivial feature of the IR dimensional regularization scheme is that it affects the evaluation of the potentials in the location of the particles. This feature is due to the fact that most of the potentials have a non-compact source: the non-linear terms extend towards infinity and thus the value of the potential in say $\mathbf{y}_1$ is sensitive to the IR regularization process. Naturally this effect does not affect the linear potentials, whose sources are proportional to Dirac distributions, and thus sensitive to UV regularization processes only. The IR regularization scheme could therefore affect the compact support terms, the contributions of distributional derivatives and the values of μ_1 (IV.13) and $\tilde{\mu}_1$ (IV.16), that all involve some potentials evaluated in $\mathbf{y}_1$ or $\mathbf{y}_2$.

Let us consider a potential P with d-dimensional source S. As in the previous section, we are interested in the IR behaviour of S and hence we will expand it in a multipolar manner. In practice, none of the sources we are interested in develop poles (or equivalently, logarithms in three dimensions), so we can safely take

$$S(\mathbf{x}) = \sum_{p\geq p_0} \sum_{q=q_0}^{q_1} \frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}} \varphi_{p,q}(\mathbf{n})\,, \tag{IV.80}$$

where $\varphi_{p,q}$ has no pole, and we will drop the angular dependencies. The source has a (non-pathological) three-dimensional limit

$$S^{\text{3D}}(\mathbf{x}) = \sum_{p\geq p_0} \frac{\varphi_p^{\text{3D}}}{r^p}\,, \qquad \text{with} \qquad \varphi_p^{\text{3D}} \equiv \left(\sum_{q=q_0}^{q_1} \varphi_{p,q}\right)_{\varepsilon=0}\,. \tag{IV.81}$$

Let us first deal with the Newtonian case, *ie.* compute the difference induced by the change of IR regularization scheme in the Poisson integral (IV.23) evaluated in $\mathbf{y}_1$

$$P_1^{\text{Newt}} = \left(\widetilde{\Delta^{-1}S}\right)(\mathbf{y}_1) \equiv \underset{\eta}{\text{PF}}\left[-\frac{\tilde{k}}{4\pi}\int_{r'>\mathcal{R}} \mathrm{d}^d\mathbf{x}' \left(\frac{r'}{r_0}\right)^{\eta} \frac{S\left(\mathbf{x}'\right)}{|\mathbf{y}_1 - \mathbf{x}'|^{d-2}}\right]\,, \tag{IV.82}$$

where we have safely replaced $\mathbf{x}$ by $\mathbf{y}_1$ in the kernel, as we integrate for $r' > \mathcal{R} > r_1$, and thus are free from UV divergences. Taylor-expanding the kernel

$$\frac{1}{|\mathbf{y}_1 - \mathbf{x}'|^{d-2}} = \sum_{\ell\in\mathbb{N}} \frac{2^\ell}{\ell!} \frac{\Gamma\left(\frac{d-2}{2}+\ell\right)}{\Gamma\left(\frac{d-2}{2}\right)} \frac{y_1^L \hat{n}_L}{r^{d-2+\ell}}\,, \tag{IV.83}$$

the Poisson integral becomes

$$P_1^{\text{Newt}} = \underset{\eta}{\text{PF}}\left[\frac{\tilde{k}}{4\pi} \sum_{p,q,\ell} \frac{2^\ell}{\ell!} \frac{\Gamma\left(\frac{d-2}{2}+\ell\right)}{\Gamma\left(\frac{d-2}{2}\right)} \frac{\mathcal{R}^{2-p-\ell+\eta-q\varepsilon}}{2-p-\ell+\eta-q\varepsilon} \frac{\ell_0^{q\varepsilon}}{r_0^\eta} y_1^L \int \mathrm{d}\Omega_{d-1} \hat{n}_L \varphi_{p,q}\right]\,. \tag{IV.84}$$

Using the machinery developed for the treatment of volume terms, the difference between both regularizations turns out to be

$$\mathcal{D}P_1^{\text{Newt}} = -\sum_{\ell\in\mathbb{N}} \frac{(2\ell-1)!!}{4\pi\ell!} y_1^L \sum_{q\neq 0}\left[\frac{1}{q\varepsilon} - \ln\left(\frac{r_0}{\ell_0}\right) - \frac{\ln\left(4\pi e^{\gamma_E}\right)}{2q} + \frac{1}{q}\sum_{k=1}^{\ell} \frac{1}{2k-1}\right] \int \mathrm{d}\Omega_{d-1} \hat{n}_L\, \varphi_{2-\ell,q}\,, \tag{IV.85}$$

where γ_E is the Euler constant, and naturally $\mathcal{D}P_2^{\text{Newt}}$ is obtained under the replacement $y_1^L \to y_2^L$. Once again, the $q \neq 0$ criterion is not an artificial requirement to ensure that the formula is well-defined, but it comes naturally out of the derivation. The main difference with the formula for the volume terms (IV.79) is the sum over ℓ. Nevertheless, this sum is finite by virtue of the structure of the source (IV.80): ℓ is bounded by $2 - p_0$.

But if the potentials $\hat{T}$, $\hat{Y}_i$ or $\hat{M}_{ij}$ enter at Newtonian order in the source quadrupole, all the other potentials enter at higher orders. For those, the accurate propagator is the one in d dimensions (IV.125), that will be investigated in great details in sec. IV.4. For our purpose, we can restrict ourselves to an integration region $r' > \mathcal{R} > r_1$ and thus safely replace $\mathbf{x}$ by $\mathbf{y}_1$, so that

$$P_1 = \left(\widetilde{\Box_{\mathrm{R}}^{-1}} S\right)(\mathbf{y}_1) \equiv \underset{\eta}{\mathrm{PF}}\left[-\frac{\tilde{k}}{4\pi}\int_{r'>\mathcal{R}}\frac{\mathrm{d}^d\mathbf{x}'}{|\mathbf{y}_1-\mathbf{x}'|^{d-2}}\left(\frac{r'}{r_0}\right)^{\eta}\int_1^{+\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,S\left(\mathbf{x}',t-z\frac{|\mathbf{y}_1-\mathbf{x}'|}{c}\right)\right], \tag{IV.86}$$

where the time-dependence of the source is now crucial, and the z-integration translates the fact that this propagator is not localized on the light cone, and involves the kernel

$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma\left(-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1+\varepsilon}{2}\right)}\left(z^2-1\right)^{-1-\frac{\varepsilon}{2}}. \tag{IV.87}$$

PN-expanding this propagator in the same fashion than what is presented in sec. IV.4.2, it comes

$$P_1 = \left(\widetilde{\Delta^{-1}}\bar{S}^{\text{even}}\right)(\mathbf{y}_1) + \left(\widetilde{\Delta^{-1}}\bar{S}^{\text{odd}}\right)(\mathbf{y}_1), \tag{IV.88}$$

where $\left(\widetilde{\Delta^{-1}}A\right)(\mathbf{y}_1)$ is the Poisson integral (IV.82) and the PN expansion of the source term reads

$$\bar{S}^{\text{even}} = \sum_{j\in\mathbb{N}}\frac{\sqrt{\pi}}{(2j)!}\frac{\Gamma\left(\frac{1+\varepsilon}{2}-j\right)}{\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1}{2}-j\right)}\left(\frac{r_1}{c}\right)^{2j}S^{(2j)}(\mathbf{x},t), \tag{IV.89a}$$

$$\bar{S}^{\text{odd}} = \sum_{j\in\mathbb{N}}\frac{(-)^j}{j!}\frac{2\sqrt{\pi}\,\Gamma(\varepsilon)}{\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(-j-\frac{\varepsilon}{2}\right)\Gamma(2j+2+\varepsilon)}\left(\frac{r_1}{c}\right)^{1+2j+\varepsilon}\int_0^{+\infty}\mathrm{d}\tau\,\tau^{-\varepsilon}S^{(2j+2)}(\mathbf{x},t-\tau). \tag{IV.89b}$$

As expected, the $j=0$ term of $\bar{S}^{\text{even}}$ is exactly given by S. Note that the odd piece, which we will not consider, appears to be non-local.

Hence for potentials entering at post-Newtonian orders, we can simply apply the Newtonian formula (IV.85), but using the even PN-expanded source (IV.89a), truncated at the accurate order. It turns out that all the non-vanishing corrections are strongly connected. Indeed they obey the relations (naturally valid up to the $\mathcal{O}(\varepsilon^0)$ order only, as the $\mathcal{O}(\varepsilon)$ corrections do not play any role in the IR dimensional regularization of the compact support terms)

$$\mathcal{D}\hat{W}_1^{\text{2PN}} = \frac{4}{c^2}\,\mathcal{D}\hat{Z}_1^{\text{1PN}} = -\frac{2\varepsilon}{c^2}\,\mathcal{D}\hat{X}_1^{\text{1PN}} = \frac{8}{3c^4}\,\mathcal{D}\hat{T}_1^{\text{Newt}} = -\frac{16\varepsilon}{3c^4}\,\mathcal{D}\hat{M}_1^{\text{Newt}}. \tag{IV.90}$$

Note that only the trace of the Newtonian $\hat{M}_{ij}$ and the 1PN $\hat{X}$ develop poles. Note also that only the "scalar" sector of the potentials is affected: neither $\hat{R}_i$ nor $\hat{Y}_i$ receive corrections, and regarding the tensor potentials $\hat{Z}_{ij}$ and $\hat{M}_{ij}$, only their traces are impacted. In addition to those nice relations, the difference itself can be compactly written in terms of the moment of inertia $I \equiv m_1 y_1^2 + m_2 y_2^2$, as

$$\mathcal{D}\hat{M}_1^{\text{0PN}} = \left(\frac{3}{8}-\varepsilon\right)c^2\,\mathcal{D}\hat{X}_1^{\text{1PN}} = -\frac{G^2(m_1+m_2)}{8}\left[\frac{1}{\varepsilon}-1-2\ln\left(\frac{r_0\sqrt{\bar{q}}}{\ell_0}\right)\right]I^{(4)}, \tag{IV.91}$$

where we have defined $\bar{q} \equiv 4\pi e^{\gamma_E}$.

The regularization induced differences (IV.90) yield corrections in the $\tilde{\mu}_1$ and in a few compact terms, that all enter at 4PN in the source quadrupole, but not in the contributions of the distributional derivatives.

Regularization of the surface terms

Turning now to the surface terms, we will not compute the difference between the two regularization schemes, but rather directly compute them in d dimensions and show that they vanish. Therefore their contributions to the difference in regularization schemes will simply be given by

$$\mathcal{D}I_{ij}^{\mathrm{Surf}} = -I_{ij}^{\mathrm{Surf,Had}}\,. \tag{IV.92}$$

Considering a surface term of the divergence type

$$\mathfrak{D} = \mathop{\mathrm{PF}}_{\eta} \int \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{\varepsilon}} \left(\frac{r}{r_0}\right)^{\eta} \partial_i \mathcal{A}_i\,, \tag{IV.93}$$

we naturally perform the multipolar expansion

$$\mathcal{A}_i(\mathbf{x},t) = \sum_{p\geq p_0} \sum_{q=q_0}^{q_1} \frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\, \alpha^i_{p,q}(\mathbf{n},t)\,, \tag{IV.94}$$

where the $\alpha^i_{p,q}$ can develop poles, and we drop the time and angular dependencies. Integrating (IV.93) by parts and dropping the surface integrals that are vanishing by analytic continuation in η, we have

$$\mathfrak{D} = \mathop{\mathrm{PF}}_{\eta} \sum_{p,q} \frac{\eta}{2-p+\eta+(1-q)\varepsilon} \frac{\mathcal{R}^{2-p+\eta+(1-q)\varepsilon}}{r_0^{\eta}\,\ell_0^{(1-q)\varepsilon}} \int \mathrm{d}\Omega_{d-1} n_i \alpha^i_{p,q} = \int \mathrm{d}\Omega_{d-1} n_i\, \alpha^i_{2,1}\,. \tag{IV.95}$$

Similarly, let us consider a surface term of the Laplacian type

$$\mathfrak{L}_L = \mathop{\mathrm{PF}}_{\eta} \int \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{\varepsilon}} \left(\frac{r}{r_0}\right)^{\eta} \hat{x}_L\, \Delta\mathcal{B}\,. \tag{IV.96}$$

Once again, we perform the multipolar expansion

$$\mathcal{B}(\mathbf{x},t) = \sum_{p\geq p_0} \sum_{q=q_0}^{q_1} \frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\, \beta^i_{p,q}(\mathbf{n},t)\,, \tag{IV.97}$$

allowing the $\beta_{p,q}$ to develop poles. Integrating (IV.96) by parts and dropping the surface integrals that are vanishing by analytic continuation in η, we have

$$\begin{aligned}\mathfrak{L}_L &= -\mathop{\mathrm{PF}}_{\eta} \sum_{p,q} \frac{\eta(\eta+2\ell+1+\varepsilon)}{\ell+1-p+\eta+(1-q)\varepsilon} \frac{\mathcal{R}^{\ell+1-p+\eta+(1-q)\varepsilon}}{r_0^{\eta}\,\ell_0^{(1-q)\varepsilon}} \int \mathrm{d}\Omega_{d-1} \hat{n}_L\, \beta_{p,q} \\ &= -(2\ell+1+\varepsilon) \int \mathrm{d}\Omega_{d-1} \hat{n}_L\, \beta_{\ell+1,1}\,.\end{aligned} \tag{IV.98}$$

So the values of the d-dimensional surface terms are extremely simple and only involve terms with $q=1$. But all of them have $q \geq 2$. Indeed, as can be intuited from their three-dimensional expressions (C.7) and (C.8) and confirmed by their d-dimensional expressions displayed in app. C of [278], all surface terms are products of potentials, superpotentials and their derivatives. It is easy to understand from the Green function (IV.24) that compact support potentials have $q=1$. As non-compact support potentials are made of products of compact support ones, and as the iteration of Poisson integral does not produces $q<1$,[9] all potentials have $q \geq 1$. A similar argument applies to the case of superpotentials, and thus the surface terms, composed of products of potentials, superpotentials and their derivatives, cannot contributes to the d-dimensional source quadrupole.

[9] This can be understood from the *Matthieu* formula (IV.110), the homogeneous solution to add bearing $q=1$, as explicit by (IV.108).

Contribution of the extra term

The argument developed in sec. IV.2.1 to discard the contribution of the extra piece of I_{ij} is obsolete in d-dimensional IR regularization: we have to compute it. Nevertheless, the argument preventing contributions for $r < \mathcal{R}$ still holds, and thus we can restrict ourselves to an integration zone $r > \mathcal{R}$, and recast

$$I_{ij}^{\text{extra}} = -\frac{2}{c^2}\,\underset{\eta}{\mathrm{PF}} \sum_{k\in\mathbb{N}} \gamma_k^{(d)} \left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2k} \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^\varepsilon} \left(\frac{r}{r_0}\right)^\eta \eta\, \frac{r^{2k}\,\hat{x}_{aij}x_b}{r^2}\,\overline{\Sigma}_{ab}\,, \tag{IV.99}$$

where we have shortened

$$\gamma_k^{(d)} \equiv \frac{1}{2^k\,k!}\frac{\Gamma\left(\frac{d}{2}+3\right)}{\Gamma\left(\frac{d}{2}+3+k\right)}\frac{(d-3)(d+2)}{d(d-2)(d+4)}\,. \tag{IV.100}$$

As usual now, we expand the source density $\overline{\Sigma}_{ab}$ as

$$\overline{\Sigma}_{ab}(\mathbf{x},t) = \sum_{p\geq p_0}\sum_{q=q_0}^{q_1} \frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\,\overline{\sigma}_{p,q}^{ab}(\mathbf{n},t)\,, \tag{IV.101}$$

where the $\overline{\sigma}_{p,q}^{ab}$ contain many orders in c and can develop poles. We will drop the angular and time dependencies, and perform the integration

$$\begin{aligned} I_{ij}^{\text{extra}} &= \frac{2}{c^2}\,\underset{\eta}{\mathrm{PF}} \sum_{k,p,q} \gamma_k^{(d)} \left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2k} \left[\eta\,\frac{\mathcal{R}^{5+2k-p+\eta+(1-q)\varepsilon}}{5+2k-p+\eta+(1-q)\varepsilon}\,\frac{\ell_0^{(q-1)\varepsilon}}{r_0^\eta}\int \mathrm{d}\Omega_{d-1}\,\hat{n}_{aij}n_b\,\overline{\sigma}_{p,q}^{ab}\right] \\ &= \frac{2}{c^2}\sum_k \gamma_k^{(d)}\left(\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2k}\left[\int \mathrm{d}\Omega_{d-1}\,\hat{n}_{aij}n_b\,\overline{\sigma}_{5+2k,1}^{ab}\right]\,, \end{aligned} \tag{IV.102}$$

where the fact that the $\gamma_k^{(d)}$ do not depend on η was crucial. Exactly as in the case of the surface terms, we see that the finite part procedure selects only the terms with $q = 1$. But, recalling that $\overline{\Sigma}_{ij} = \bar{\tau}_{ij}$ and the definition of the Landau-Lifschitz pseudo tensor (B.5), one can see that the $\overline{\sigma}_{p,q}^{ab}$ are made of two pieces. The first one, coming from the stress-energy tensor $T^{\mu\nu}$, involves Dirac distributions and thus does not enter our computation as we are restricted to $r > \mathcal{R}$. The second entails the non-linear source term $\Lambda^{\mu\nu}$ and thus is composed of products of potentials. As already discussed, it can not involve $q = 1$ terms. Therefore, we can conclude that, even in the IR dimensional regularization scheme, the extra piece does not contribute

$$I_{ij}^{\text{extra}} = 0\,. \tag{IV.103}$$

Contribution of the IR shift

The IR shift $\chi_{1,2}^i$ used to compensate the remaining poles in the equations of motion [56] is given in app. C.4.2.[10] It starts at the 4PN order, and thus its contribution to the source quadrupole is simply given by

$$\delta_\chi I_{ij} = 2m_1\,y_1^{\langle i}\chi_1^{j\rangle} + 2m_2\,y_2^{\langle i}\chi_2^{j\rangle}\,. \tag{IV.104}$$

[10] We thank L. Bernard for providing us the file containing this shift.

IV.3.2 Computation of the required potentials in d dimensions

Just as in the case of the three-dimensional Hadamard regularized surface terms, the difference between Hadamard and dimensional regularizations only involves specific multipolar orders, see eqs. (IV.79) and (IV.85). Therefore the asymptotic expansions of the d-dimensional potentials are required.

The linear potentials V, V_i and K and the superpotentials are known in the whole d-dimensional space, thus one has just to expand them in $r \to \infty$.

But the volume terms and the source of the potential involved in compact support terms also include the quadratic potentials $\hat{W}_{ij}$ at 1PN, $\hat{R}_i$ and $\hat{Z}_{ij}$ at Newtonian order, and the cubic potential $\hat{X}$ at Newtonian order. If the compact support parts of their sources (and contributions of the distributional derivatives in the $\hat{X}$) can be treated easily,[11] none of those potentials are known in all d-dimensional space. We thus have to compute their asymptotic behaviors by iterating the propagator at infinity, and we will construct such multipolar expansions by adding the accurate homogeneous solution to a particular one. For a potential P with source S, the generalization of the matching procedure (IV.56) in the case of a d-dimensional potential reads [56]

$$\mathcal{M}(P) = \widetilde{\Box^{-1}_{\mathrm{R}}}\mathcal{M}(S) - \frac{1}{4\pi}\sum_{\ell\in\mathbb{N}} \frac{(-)^\ell}{\ell!}\overline{\partial_L S^L_\star}\,, \tag{IV.105}$$

where we will trade $\widetilde{\Box^{-1}_{\mathrm{R}}}$ for the d-dimensional symmetric propagator, as we are only interested in even orders. The homogeneous solution is constructed out of the PN expansion of

$$S^L_\star(t,r) \equiv \frac{\tilde{k}}{r^{d-2}}\int_1^{+\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\left[\mathcal{S}_L\left(t-\frac{zr}{c}\right)+\mathcal{S}_L\left(t+\frac{zr}{c}\right)\right]\,, \tag{IV.106}$$

with

$$\mathcal{S}_L(u) \equiv \mathop{\mathrm{PF}}_{\eta}\int \mathrm{d}^d\mathbf{x}\left(\frac{r}{r_0}\right)^\eta x_L\,S(\mathbf{x},u)\,, \tag{IV.107}$$

and the $\gamma_{\frac{1-d}{2}}$ kernel is defined by eq. (IV.87). Note that $\mathcal{S}_L(u)$ is the PN-even d-dimensional generalization of eq. (IV.57), but with the main difference of the presence of the z integration, due to the fact that the propagator does not lie anymore on the light-cone. Performing explicitly the PN expansion, eq. (IV.105) becomes

$$\mathcal{M}(P) = \sum_{k\in\mathbb{N}}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2k}\widetilde{\Delta^{-k-1}}\mathcal{M}(S) - \frac{1}{4\pi^{\frac{d-1}{2}}}\sum_{k,\ell}\frac{(-)^\ell}{\ell!(2k)!}\frac{\Gamma\left(\frac{d}{2}-1-k\right)}{\Gamma\left(\frac{1}{2}-k\right)}\frac{1}{c^{2k}}\,\mathcal{S}^{(2k)}_L(t)\,\partial_L r^{2k-1-\varepsilon}\,. \tag{IV.108}$$

Computing the particular solutions

There is no particular issues with the multipolar development of the sources,[12] so computing the particular solutions

$$\mathcal{M}(P)_{\mathrm{part}} \equiv \sum_{k\in\mathbb{N}}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2k}\widetilde{\Delta^{-k-1}}\mathcal{M}(S) = \sum_{k\in\mathbb{N}}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^{2k}\mathop{\mathrm{PF}}_{\eta}\left[\Delta^{-k-1}\left(\frac{r}{r_0}\right)^\eta\mathcal{M}(S)\right]\,, \tag{IV.109}$$

is easily done by means of iterations of the d-dimensional *Matthieu* formula [87]

$$\Delta^{-1}\left[r^{\alpha+\eta}\,\hat{n}_L\right] = \frac{r^{\alpha+2+\eta}\,\hat{n}_L}{(\alpha+d+\eta+\ell)(\alpha+2+\eta-\ell)}\,. \tag{IV.110}$$

[11] To the only difficult point that we need to know the value of the Newtonian $\hat{W}$ evaluated at $\mathbf{y}_1$, required to compute $\hat{X}$, see eq. (IV.55).

[12] The source of the $\hat{X}$ potential involves $\hat{W}_{ij}$, which has thus to be computed and matched before deriving $\hat{X}$.

Homogeneous solutions for the quadratic potentials

Computing the homogeneous solutions is more involving. Indeed, it relies on the source integral $\mathcal{S}_L(u)$ (IV.107), and thus on the integration of the source in the whole space. For the quadratic potentials $\hat{W}_{ij}$, $\hat{R}_i$ and $\hat{Z}_{ij}$, the source is a product of (derivatives) of the linear potentials and thus known in all space. Therefore, in full generality, we need to integrate products like

$$\int \mathrm{d}^d\mathbf{x}\, x_L r_1^\alpha r_2^\beta r_{12}^\gamma\, n_1^K n_2^P n_{12}^Q\,. \tag{IV.111}$$

First, by substituting $x^i = y_1^i + r_1 n_1^i$ and $n_2^i = (r_{12}n_{12}^i + r_1 n_1^i)/r_2$ (*cf.* fig. A.1), and noticing that neither y_1^i, r_{12} nor n_{12}^i depend on $\mathbf{x}$, we only need to compute integrals such as

$$\int \mathrm{d}^d\mathbf{x}\, r_1^\alpha r_2^\beta\, n_1^L\,. \tag{IV.112}$$

There are two approaches to compute such integrals. The first one is to express $r_1^\alpha n_1^L$ in terms of derivatives with respect to y_1^i, for instance

$$r_1^\alpha n_1^i = -\frac{1}{\alpha+1}\,\underset{1}{\partial_i}\, r_1^{\alpha+1}\,, \qquad \text{or} \qquad r_1^\alpha n_1^{ij} = \frac{1}{\alpha(\alpha+2)}\,\underset{1}{\partial_{ij}}\, r_1^{\alpha+2} - \frac{r_1^\alpha}{\alpha}\delta_{ij}\,. \tag{IV.113}$$

Those derivatives can be brought outside of the integral and it only remains to use the d-dimensional Riesz formula [87, 315]

$$\int \mathrm{d}^d\mathbf{x}\, r_1^\alpha r_2^\beta = \pi^{d/2}\frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(\frac{-\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\, r_{12}^{\alpha+\beta+d}\,. \tag{IV.114}$$

If this procedure is very convenient for small values of ℓ, it becomes extremely cumbersome when ℓ is larger than 3, as practically implementing the generalizations of eq. (IV.113) is then quite laborious. In addition, this method yields spurious divergences, *eg.* for $\alpha = -d$, that have to be dealt with by the introduction of a new regulator. Therefore, we would like to directly compute integrals as (IV.112), with a "generalized Riesz" procedure. Such computation has been performed by using a method exposed in app. A.6 of [220], which relies on a projection of $\mathbf{n}_1$ onto $\mathbf{n}_{12}$, given by

$$n_1^i = \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}}\, n_{12}^i + n_\perp^i\,, \tag{IV.115}$$

where $n_\perp^i$ is the component of $\mathbf{n}_1$ orthogonal to $\mathbf{n}_{12}$. The interest of such projection is that, by using prolate spheroidal coordinates, one can single the integration of products of $n_\perp^i$ out from the total integration. As the averages of products of $n_\perp^i$ are directly expressed in terms of r_1, r_2, r_{12}, $\mathbf{n}_{12}$ and δ^{ij}, the complicated integral (IV.112) reduces to a sum of Riesz ones (IV.114), and thus can be computed. By means of this technique, we were able to match all the required quadratic potentials.[13]

Homogeneous solution of the $\hat{X}$ cubic potential

The non-linear part of the source of the cubic $\hat{X}$ potential (C.4a) can be decomposed into quadratic and cubic sectors as

$$\Box\hat{X}^{(\mathrm{NL})} = S^{\mathrm{q}} + S^{\mathrm{c}}\,, \tag{IV.116}$$

with

$$S^{\mathrm{q}} = 2V_i\,\partial_t\partial_i V + \frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V + \frac{d(d-1)}{4(d-2)^2}\left(\partial_t V\right)^2 - 2\partial_i V_j\,\partial_j V_i \tag{IV.117a}$$

[13] G. Faye provided an explicit formula to express integrals such as (IV.111). Unfortunately, it was after the practical computations were achieved, so we have not used it.

$$S^{\mathrm{c}} = \hat{W}_{ij}\,\partial_{ij}V\,. \tag{IV.117b}$$

The integration of the quadratic part is naturally performed by using the "generalized Riesz" procedure described previously. The major difficulty comes from the cubic part, as it involves the potential $\hat{W}_{ij}$ in all space in d dimensions. As we need it only at Newtonian level, we could rely on a d-dimensional generalization of the Fock function (IV.43a). The latter has been derived, but only in an integral form [87], which is of little practical use. Instead, we will employ the method of superpotentials in order to replace $\hat{W}_{ij}$ by the expression of its source. As superpotentials are defined in a STF guise, *cf.* eq. (IV.20), we will compute

$$\hat{\mathcal{S}}^{\mathrm{c}}_L \equiv \underset{\eta}{\mathrm{PF}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^\eta \hat{x}_L\,\hat{W}_{ij}\partial_{ij}V\,, \tag{IV.118}$$

instead of eq. (IV.107), and un-STFize it at the end of the computation (as it is clear that eq. (IV.107) is symmetrical under its ℓ indices, this simply accounts to putting the traces back). Two integrations by parts and the use of eq. (C.3a) give

$$\hat{\mathcal{S}}^{\mathrm{c}}_L = -4\pi G \underset{\eta}{\mathrm{PF}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^\eta \left(\sigma_{ij} - \frac{\delta_{ij}\,\sigma_{kk}}{d-2}\right)\Psi_L^{\partial_{ij}V} - \frac{1}{2}\frac{d-1}{d-2}\underset{\eta}{\mathrm{PF}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^\eta \Psi_L^{\partial_{ij}V}\partial_i V\partial_j V\,, \tag{IV.119}$$

where we have introduced the superpotentials $\Psi_L^{\partial_{ij}V}$ obeying

$$\Delta\Psi_L^{\partial_{ij}V} = \hat{x}_L\,\partial_{ij}V\,. \tag{IV.120}$$

We have discarded the surface terms coming from the integration by parts, as they bear $q > 1$ and thus are vanishing. The first integral of (IV.119) has a compact support and is straightforwardly computed as we know the value of $\Psi_L^{\partial_{ij}V}$ in $\mathbf{y}_1$. The second integral can be evaluated by means of the "generalized Riesz" procedure, and thus the matching of the $\hat{X}$ is doable, and has been done.

Consistency checks performed on the potentials

With all those techniques, we can compute and match all the required potentials, and extract their asymptotic behaviors. Nevertheless, these computations are heavy[14] and some verifications are thus required. The first and most stringent one is that those computations were performed in a "double-blind" fashion, together with Q. Henry. In addition to that, we investigated the three-dimensional limits of our potentials, and confirmed that they agree with the matched asymptotic expressions that were used to compute the three-dimensional surface terms. We have also verified that the harmonicity relations $\partial_\mu\bar{h}^{\mu\nu} = 0$ hold, up to 1PN and to the achievable order in $1/r$. At 1PN order, those conditions read [278]

$$\begin{aligned}
\partial_\mu\bar{h}^{\mu 0} =& \frac{d-1}{2(d-2)}\partial_t V + \partial_i V_i \\
&+ \frac{1}{c^2}\left[\partial_t\left(-\frac{(d-1)(d-3)}{(d-2)^2}\,\hat{K} + \frac{\hat{W}}{2} + \frac{(d-1)^2}{2(d-2)^2}\,V^2\right) + \partial_i\left(2\hat{R}_i + \frac{d-1}{d-2}\,VV_i\right)\right]\,,
\end{aligned} \tag{IV.121a}$$

$$\partial_\mu\bar{h}^{\mu i} = \partial_t V_i + \partial_j\left(\hat{W}_{ij} - \frac{\hat{W}}{2}\,\delta_{ij}\right) + \frac{1}{c^2}\left[\partial_t\left(2\hat{R}_i + \frac{d-1}{d-2}\,VV_i\right) + 4\,\partial_j\left(\hat{Z}_{ij} - \frac{\hat{Z}}{2}\,\delta_{ij}\right)\right]. \tag{IV.121b}$$

The cancellation of $\partial_\mu\bar{h}^{\mu 0}$ has been checked exactly at Newtonian order, and up to $\mathcal{O}(r^{-6})$ at 1PN order. The fact that $\partial_\mu\bar{h}^{\mu i}$ vanishes, up to $\mathcal{O}(r^{-7})$ at Newtonian order, and up to $\mathcal{O}(r^{-5})$ at 1PN order.

[14] For instance, it took more than 160 CPU hours to match the 1PN $\hat{W}_{ij}$ up to $\mathcal{O}(r^{-4})$. At this order, it has more than 10 000 terms, of which $\sim 8\,000$ come from the homogeneous part.

IV.3.3 The IR regularized mass quadrupole

Once the asymptotic expansions of the potentials are derived, the computation of the difference of regularization schemes for volume and compact support terms can be implemented by means of eqs. (IV.79) and (IV.85). This has been done in a "double-blind" fashion, together with Q. Henry. Adding the contribution of the IR shift (IV.104) and removing the Hadamard regularized surface terms, the difference of regularization schemes for the source quadrupole reads

$$\begin{aligned}\mathcal{D}I_{ij} &= \mathcal{D}I_{ij}^{\text{Vol}} + \mathcal{D}I_{ij}^{\text{Comp}} - I_{ij}^{\text{Surf,Had}} + \delta_\chi I_{ij} \\ &= -\beta_I \frac{G^2 M^2 Q_{ij}^{(2)}}{2c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{r_0\sqrt{\bar{q}}}{\ell_0}\right) - \frac{246\,299}{22\,470}\right] + \beta_I \frac{G^2 M P_{\langle i}P_{j\rangle}}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{r_0\sqrt{\bar{q}}}{\ell_0}\right) - \frac{252\,599}{22\,470}\right] \\ &\quad + \frac{1}{c^8}\left[\frac{1}{\varepsilon}\mathcal{D}I_{ij}^{\text{4PN,p}} + \mathcal{D}I_{ij}^{\text{4PN}}\right],\end{aligned} \tag{IV.122}$$

where $Q_{ij} = m_1 \hat{y}_1^{ij} + m_2 \hat{y}_2^{ij}$ is the Newtonian mass quadrupole, $P_i = m_1 v_1^i + m_2 v_2^i$ is the Newtonian linear momentum (that vanishes in the center-of-mass) and $\beta_I = -\frac{214}{105}$ is the coefficient associated with the logarithmic renormalization of the mass quadrupole moment [209].

Recalling that the Hadamard regularized mass quadrupole (IV.64) agrees with previous results at the 3PN accuracy [187], this 3PN contribution due to dimensional regularization indicates that there is an effect that we have not taken into account. Note that this 3PN pole cannot result from a failure of the application of the IR shift, as χ^i is a 4PN quantity, and thus cannot generate nor remove poles entering at 3PN.

The number associated with this 3PN pole, β_I, indicates that the missing effect is given by the tail-of-tails. Indeed those enter at 3PN, and it is known by EFT methods that they bear a UV pole that should cancel the IR pole appearing in the mass quadrupole [209]. Therefore the physical quantity to regularize is the radiative moment U_{ij} rather than the source one I_{ij}. In addition, it means that the IR shift χ^i should remove the remaining poles in U_{ij} and not in I_{ij}.

IV.4 Dimensional regularization of the radiative quadrupole

In order to check if our computation of the source quadrupole I_{ij} is correct, and notably if the crucial IR shift compensates accurately the remaining poles, we should thus compute and regularize the radiative quadrupole U_{ij}. This dimensional regularization of the radiative quadrupole is currently, and for the first time, under investigation.

IV.4.1 Dimensional regularization of the non-linear interactions

We first focus on the transition between the canonical and radiative moments. Most of the pieces required to perform this transition are already known in three dimensions. For instance, the iterated tail contributions have been computed up to 4.5PN order [276]. Therefore, we do not intend to perform a full computation in d dimensions, but rather only to derive and apply formulas that give the difference between Hadamard and dimensional regularizations, leaving the computation of the three-dimensional missing pieces for future works. Let us also note that we only need to iterate twice the propagator, as the dominant contribution of the third iteration, the tail-of-tail-of-tail effect, enters at 4.5PN order when we aim to 4PN accuracy. Therefore we only seek for cubic interactions.

Last but not least, the effect to be compensated is local, as clear from eq. (IV.122), whereas the non-linear interactions reveal non-local features by nature. Fortunately, and as explicitly shown hereafter, those non-local effects will play no role in the change of regularization scheme. From an EFT point of view, this can be related to the relation displayed in fig. III.2: we seek for the local pole that is associated to the non-local piece of the tails (represented by the logarithmic dependence).

The computation of the dimensional regularization of those non-linear interactions, and the difference with the usual Hadamard regularized ones, is one of the main achievements of this thesis. The detailed derivation is thus presented in app. D, we only sketch it rapidly hereafter.

Dimensional regularization of the post-Minkowskian metric

The aim is to study the d-dimensional solution of the vacuum Einstein equations

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu}(\mathbf{x}, t)\,. \tag{IV.123}$$

As they will play no role, we will henceforth drop space-time indices, and focus on multipolar ones. Therefore our object of study is the wave equation

$$\Box h_L = N_L(\mathbf{x}, t)\,, \tag{IV.124}$$

where N_L is some source function that will be specified later on. The accurate prescription for the d-dimensional propagator is

$$h_L(\mathbf{x}, t) = \frac{-\tilde{k}}{4\pi}\int \mathrm{d}^d\mathbf{x}'\left(\frac{r'}{r_0}\right)^\eta \int_1^{+\infty}\mathrm{d}z\gamma_{\frac{1-d}{2}}(z)\,\frac{N_L\left(\mathbf{x}', t - z|\mathbf{x}'-\mathbf{x}|/\right)}{|\mathbf{x}'-\mathbf{x}|^{d-2}}\,, \tag{IV.125}$$

where the kernel of the z-integration is given by the function

$$\gamma_{\frac{1-d}{2}-\ell}(z) = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{d-2}{2}+\ell\right)\Gamma\left(\frac{3-d}{2}-\ell\right)}\left(z^2-1\right)^{\frac{1-d}{2}-\ell}\,, \tag{IV.126}$$

which is properly normalized for any $\ell \in \mathbb{C}$ by analytic continuation as

$$\int_1^{+\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}-\ell}(z) = 1\,. \tag{IV.127}$$

The presence of this kernel accounts for the fact that the propagator is not localized on the light cone. In fact such a localization only happens in even-dimensional space-times, as can be seen by the distributional limit of $\gamma_{\frac{1-d}{2}-\ell}(z)$, see app. C of [56],

$$\gamma_{-1-\ell}(z) = \sum_{k=0}^{\ell}\frac{2^{k-\ell}}{(2\ell-1)!!}\frac{(2\ell-k)!}{(\ell-k)!}\,\delta^{(k)}(z-1)\,, \qquad \text{for } \ell\in\mathbb{N}\,, \tag{IV.128}$$

where $\delta^{(k)}$ is the k^{th} derivative of the Dirac distribution. Note also the similarities between this kernel $\gamma_{\frac{1-d}{2}}$ and the $\delta_\ell^{(\varepsilon)}$ functions entering the source terms (IV.10).

As we will see in the next section, the source can be recast as

$$N_L(\mathbf{x}, t) = \sum_{p\geq p_0}\sum_{q=q_0}^{q_1}\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\left[\frac{\tilde{\psi}_{p,q}(t)}{\varepsilon} + \psi_{p,q}(t)\right]\hat{n}_L\,, \tag{IV.129}$$

where the $\tilde{\psi}_{p,q}$ coefficients do not depend on ε. As shown in app. D, the only contributions to the difference of regularization schemes come from the terms with $p-\ell-3\in 2\mathbb{N}$ and read

$$\mathcal{D}h_L = -4\frac{(-)^\ell}{\ell!}\hat{\partial}_L\left\{\frac{\tilde{k}}{r^{d-2}}\int_1^\infty \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\mathcal{D}H\left(t-\frac{zr}{c}\right)\right\}\,, \tag{IV.130}$$

with

$$\mathcal{D}H(t) = -\sum_{q\neq 1}\frac{2^{\ell-2}}{q-1}\frac{\ell!\,(\ell+j)!}{j!\,(p+\ell-2)!}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\ell_0^{q-1}\sqrt{\bar{q}}}{r_0^{q-1}}\right)-\sum_{k=0}^{\ell+j}\frac{1}{2k+1}\right]\frac{\psi_{p,q}^{(2j)}(t)}{c^{2j}}\,, \tag{IV.131}$$

where we have denoted $j\equiv(p-\ell-3)/2\in\mathbb{N}$, and we recall our definition $\bar{q}=4\pi e^{\gamma_E}$. Note that this contribution is local, as expected. Conversely, contributions coming from the pole terms $\tilde{\psi}_{p,q}$ reveal non-local pieces, as displayed in eq. (D.35). Fortunately, none of the poles entering in the sources we deal with satisfies the condition $p-\ell-3\in 2\mathbb{N}$ and as such, contributes to the difference of regularization schemes. Therefore this effect is purely local.

From the corrections in the metric to the corrections in the moments

Once the contribution of the dimensional regularization scheme in the metric has been established, one should translate it into a contribution in the radiative quadrupole. This is to be done by identifying it as a non-linear correction to the linear canonical moment, just as the canonical moments were identified as non-linear completions of the source ones. In d dimensions, the mass canonical moments enter the linear metric as

$$k^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\partial_L\,\tilde{\mathcal{M}}_L\,,\quad k^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\,\partial_{L-1}\,\tilde{\mathcal{M}}^{(1)}_{iL-1}\,,\quad k^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell}{\ell!}\,\partial_{L-2}\,\tilde{\mathcal{M}}^{(2)}_{ijL-2}\,, \tag{IV.132}$$

where we have defined the "tilded" versions of the moments as

$$\tilde{\mathcal{M}}_L \equiv \frac{\tilde{k}}{r^{d-2}}\int_1^\infty \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,\mathrm{M}_L\left(t-\frac{zr}{c}\right)\,, \tag{IV.133}$$

which reduce to the usual moments in the $d\to 3$ limit.

The interest of writing the dimensional contribution under the form (IV.130) is now manifest: it mimics the linearized metric (IV.132), and thus is directly formatted in the appropriate fashion to be recognized as a non-linear completion of the source moments.

But before any recognition, one has first to remove the divergence, as $\mathcal{D}h_L$ is divergenceful. This is done following a very similar procedure as the three-dimensional one, presented in app. B.2.3 (by roughly replacing potentials by their "tilded" versions). Once the metric obeys the de Donder gauge condition, two techniques are possible to recognize the difference in regularization schemes as corrections to the moments. The brute force one consists in recasting the metric into source and gauge moments, as in the d-dimensional generalization of eq. (B.18)

$$\begin{aligned}
h^{00}_{(1)} &= -\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\partial_L\left\{\tilde{\mathcal{I}}_L + \frac{\tilde{\mathcal{W}}^{(1)}_L}{c^2} + \frac{\tilde{\mathcal{X}}^{(2)}_L}{c^4} + \frac{\tilde{\mathcal{Y}}_L}{c^2}\right\}\,,\\
h^{0i}_{(1)} &= \frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\tilde{\mathcal{I}}^{(1)}_{iL-1} + \frac{\tilde{\mathcal{Y}}^{(1)}_{iL-1}}{c^2}\right) + \partial_{iL}\left(\tilde{\mathcal{W}}_L + \frac{\tilde{\mathcal{X}}^{(1)}_L}{c^2}\right)\right\}\,,\\
h^{ij}_{(1)} &= -\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\tilde{\mathcal{I}}^{(2)}_{ijL-2} + 2\partial_{ijL}\tilde{\mathcal{X}}_L + 2\partial_{L-1(i}\tilde{\mathcal{Y}}_{j)L-1} + \delta_{ij}\,\partial_L\left(\tilde{\mathcal{W}}^{(1)}_L - \frac{\tilde{\mathcal{X}}^{(2)}_L}{c^2} - \tilde{\mathcal{Y}}_L\right)\right\}\,,
\end{aligned} \tag{IV.134}$$

where, for the sake of clarity, we have not included the current-type moments, nor the third type of moments that appears in d-dimensional space-times (see [228]). Identifying the contribution that behaves as $\tilde{\mathcal{I}}_{ij}$ yields the correction to the source quadrupole.

The second possible way to proceed is to compute the $0i0j$ components of the linearized Riemann tensor (A.3)

$$\mathcal{R}_{0i0j} = \frac{G}{2}\left(\partial_{ij}h^{00} + \frac{2}{c}\,\partial_t\partial^{(i}h^{j)0} + \frac{1}{c^2}\,\partial_t^2 h^{ij} + \frac{\partial_{ij}h}{d-1} - \frac{1}{c^2}\frac{\delta_{ij}}{d-1}\,\partial_t^2 h\right) + \mathcal{O}\left(h^2\right)\,, \tag{IV.135}$$

with $h = \eta_{\mu\nu}h^{\mu\nu}$ the four-dimensional trace of the metric perturbation. Indeed, those components are gauge independent, *ie.* plugging the metric (IV.134) in eq. (IV.135), the gauge moments disappear

$$\mathcal{R}_{0i0j} = -\frac{2G}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\left(\frac{d-2}{d-1}\,\partial_{ijL}\tilde{\mathcal{I}}_L - \frac{2}{c^2}\,\partial_{L-1(i}\tilde{\mathcal{I}}^{(2)}_{j)L-1} + \frac{1}{c^4}\,\partial_{L-2}\tilde{\mathcal{I}}^{(4)}_{ijL-2} + \frac{1}{c^2}\frac{\delta_{ij}}{d-1}\,\partial_L\tilde{\mathcal{I}}^{(2)}_L\right)\,, \tag{IV.136}$$

and one can more easily identify the contributions to the $\tilde{\mathcal{I}}_L$ moments.

IV.4.2 Computation of the d-dimensional quadratic interactions

As we have advertised, the 3PN and 4PN corrections are due to cubic interactions, or 3PM corrections. Therefore one needs to known the 3PM source of (IV.123), *ie.* compute the d-dimensional 2PM effect, or quadratic interactions. We will present hereafter the computation of the tail effect $\mathrm{M}\times\mathrm{M}_{ij}$, as it is the most representative quadratic interaction.

The source of the d-dimensional tails is given by the interplay of the $\ell=0$ and $\ell=2$ modes of the canonical metric (IV.132). As $\tilde{\mathcal{M}}=\tilde{k}Mr^{2-d}$, we have to solve equations of the generic form

$$\Box h_L = N_L(\mathbf{x},t) = \hat{n}_L \int_1^\infty \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,z^k\,\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\,F\left(t-\frac{zr}{c}\right)\,. \tag{IV.137}$$

If we know the exact prescription to apply, namely the propagator (IV.125), we do not need to use it here. Instead, following [56],[15] we can rely on a formal PN expansion as

$$\bar{h}_L = \widetilde{\Box_{\mathrm{R}}^{-1}}\bar{N}_L - \underset{\eta}{\mathrm{PF}}\sum_{p\in\mathbb{N}}\frac{(-)^p}{p!}\sum_{j\in\mathbb{N}}\Delta^{-j}\hat{x}_P\int\mathrm{d}^d\mathbf{x}'\left(\frac{r'}{r_0}\right)^\eta\hat{\partial}'_P\left[\frac{\tilde{\mathcal{N}}_L^{(2j)}(\mathbf{y},t-zr'/c)}{r'^{d-2}}\right]_{\mathbf{y}=\mathbf{x}'}\,, \tag{IV.138}$$

with

$$\Delta^{-j}\hat{x}_L \equiv \frac{\Gamma\left(\frac{d}{2}+\ell\right)}{\Gamma\left(\frac{d}{2}+\ell+j\right)}\frac{r^{2j}\hat{x}_L}{2^{2j}j!}\,. \tag{IV.139}$$

As now usual, we can recognize a particular and a homogeneous solution, the latter being constructed so that the metric satisfies the matching condition (II.48).

The particular solution involves the PN expansion of the source $\bar{N}_L$. Introducing $\tau=zr/c$ and using the asymptotic development of the $\gamma_{\frac{1-d}{2}}$ kernel

$$\gamma_{\frac{1-d}{2}}(z) \underset{z\to\infty}{=} \sum_{j\in\mathbb{N}}\frac{(-)^j}{j!}\frac{2\sqrt{\pi}\,z^{-2-2j-\varepsilon}}{\Gamma\left(-j-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1+\varepsilon}{2}\right)}\,, \tag{IV.140}$$

we can decompose $\bar{N}_L$ into "even" and "odd" sectors, generalizing the results of [56] in the case of a non-vanishing k, as

$$\bar{N}_L^{\mathrm{even}} = \sum_{j\in\mathbb{N}} A_j^k\,\frac{r^{2j+\beta_k-p-q\varepsilon}}{c^{2j+\beta_k}}\,F^{(2j+\beta_k)}(t)\,\hat{n}_L\,, \tag{IV.141a}$$

$$\bar{N}_L^{\mathrm{odd}} = \sum_{j\in\mathbb{N}} B_j^k\,\frac{r^{1+2j-k-p+(1-q)\varepsilon}}{c^{1+2j-k+\varepsilon}}\int_0^\infty\mathrm{d}\tau^{-\varepsilon}\,F^{(2j+2-k)}(t-\tau)\,\hat{n}_L\,, \tag{IV.141b}$$

where $\beta_k=0$ if k is even and 1 if k is odd. The coefficients that enter those expressions are

$$A_j^k \equiv \frac{(-)^j}{2^{2j+\beta_k}j!}\frac{\Gamma\left(\frac{1}{2}-j-\beta_k\right)}{\Gamma\left(\frac{1+\varepsilon}{2}\right)}\frac{\Gamma\left(\frac{1-\beta_k-k+\varepsilon}{2}-j\right)}{\Gamma\left(\frac{1-\beta_k-k}{2}-j\right)}\,, \tag{IV.142a}$$

$$B_j^k \equiv \frac{(-)^k}{2^{2j}j!}\frac{2\sqrt{\pi}\,\Gamma\left(2-k+\varepsilon\right)}{\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(1-\varepsilon\right)\Gamma\left(\frac{-\varepsilon}{2}\right)}\frac{\Gamma\left(1+j+\frac{\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}\frac{\Gamma\left(1-\frac{k}{2}+\frac{\varepsilon}{2}\right)}{\Gamma\left(1+j-\frac{k}{2}+\frac{\varepsilon}{2}\right)}\frac{\Gamma\left(\frac{3-k+\varepsilon}{2}\right)}{\Gamma\left(j+\frac{3-k+\varepsilon}{2}\right)}\,. \tag{IV.142b}$$

The expressions (IV.141), once integrated with the PN-expanded propagator, can (and will) yield poles. Fortunately, those poles disappear in the three-dimensional limit by giving logarithmic terms, and thus this limit is non-pathological. Moreover, such poles will not contribute to the difference in regularization schemes in the iterated cubic solution, as already advertised.

[15] Note that the following computations are generalizations of the results exposed in [56].

As for the homogeneous sector, *ie.* the second part of eq. (IV.138), once well massaged, it comes

$$\bar{h}_L^{\mathrm{hom}} = \frac{-1}{d+2\ell-2}\frac{\Gamma\left(q\varepsilon-\eta\right)}{\Gamma\left(p+\ell-1+q\varepsilon-\eta\right)}\,C_\ell^{k,p,q}\sum_{j\in\mathbb{N}}\Delta^{-j}\hat{x}_L\int_0^{\infty}\mathrm{d}\tau\tau^{\eta-q\varepsilon}F^{(2j+\ell+p-1)}(t-\tau)\,, \quad \text{(IV.143)}$$

with the coefficients

$$C_\ell^{k,p,q} \equiv \int_1^\infty \mathrm{d}y\,\gamma_{\frac{1-d}{2}-\ell}(y)\int_1^\infty \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,z^k(y+z)^{\ell-2+p+q\varepsilon-\eta}\,, \quad \text{(IV.144)}$$

which are generalizations of the ones introduced in [56] for $q \neq 2$, and are derivable by similar methods to those developed in the app. D of this work.[16]

Therefore we have access to the knowledge of the PN-expanded d-dimensional quadratic interactions (IV.138), which consist in series in powers of $1/r$. As we know that only certain contributions (namely those satisfying $p-\ell-3 \in 2\mathbb{N}$) will contribute to the cubic result (IV.130), we can thus restrict our computations to the necessary orders only. Injecting those in the cubic source (IV.124), we see in addition that we were allowed to decompose it as in eq. (IV.129).

The only quadratic interaction that does not fall within the presented computation is the memory $\mathrm{M}_{ij}\times\mathrm{M}_{ij}$ that is needed for 4PN corrections. For this interaction, the source term (IV.137) has to be generalized into

$$\Box h_L = N_L(\mathbf{x},t) = \hat{n}_L\int_1^\infty \mathrm{d}y\,\gamma_{\frac{1-d}{2}}(y)\,y^s\int_1^\infty \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,z^k\,\frac{\ell_0^{q\varepsilon}}{r^{p+q\varepsilon}}\,F\left(t-\frac{yr}{c}\right)\,G\left(t-\frac{zr}{c}\right)\,. \quad \text{(IV.145)}$$

In this case the PN expansions of the sources are naturally given by four terms: "even-even", "even-odd", *etc.* and the homogeneous solution keeps the same structure.

Note that in the case of the sought cubic interactions, performing such PN expansion instead of the machinery developed in app. D would not have helped. Indeed, putting the result in the crucial linear metric-looking form (IV.130) would have been definitively more involving. Moreover we would have had a potentially extremely large number of terms to study, coming from the iterated PN-expanded propagator acting on $\bar{N}_L$.

IV.4.3 The 3PN dimensional corrections to the radiative quadrupole

A dimensional analysis indicates that the only possible interactions entering at 3PN in the radiative quadrupole are the tail-of-tail $\mathrm{M}\times\mathrm{M}\times\mathrm{M}_{ij}$ and tail-of-"linear memory" $\mathrm{M}\times\mathrm{M}_i\times\mathrm{M}_i$. The difference between dimensional and Hadamard regularizations of those interactions were thus computed by using the machinery developed in this section.

The extraction of the correction to the mass quadrupole from the metric has been performed in both direct recognition (IV.134) and through the gauge independent components of the linearized Riemann tensor (IV.136), and they agreed to yield

$$\delta\mathrm{M}_{ij} = \beta_I\frac{G^2M^2\mathrm{M}_{ij}^{(2)}}{2c^6}\left[\frac{1}{\varepsilon}-2\ln\left(\frac{r_0\sqrt{\bar{q}}}{\ell_0}\right)-\frac{246\,299}{22\,470}\right]-\beta_I\frac{G^2M\mathrm{M}_{<i}^{(1)}\mathrm{M}_{j>}^{(1)}}{c^6}\left[\frac{1}{\varepsilon}-2\ln\left(\frac{r_0\sqrt{\bar{q}}}{\ell_0}\right)-\frac{252\,599}{22\,470}\right]. \quad \text{(IV.146)}$$

At the Newtonian order, we naturally have $\mathrm{M}_{ij} = Q_{ij}+\mathcal{O}(c^{-2})$ and $\mathrm{M}_i^{(1)} = P_i+\mathcal{O}(c^{-2})$, so that this contribution exactly compensates the 3PN sector of $\mathcal{D}I_{ij}$ (IV.122). This result, and the fact that we recover the tail-of-tail EFT results of [209], are strong confirmations of the soundness of the d-dimensional computations we conducted. But it is also a confirmation of the previously led 3PN computation, using only an IR Hadamard regularization [88] . Indeed, if the two regularization schemes yield different results for the source quadrupole, they converge when dealing with the physical radiative quadrupole. Therefore, both regularization schemes are suitable at 3PN order, as long as they are consistently implemented.

[16]Note that our notations differ from those of [56] by the exchange $p \leftrightarrow k$.

IV.4.4 Towards the 4PN dimensional corrections

At the 4PN order, using dimensional arguments, the only possible interactions that can yield corrections to the radiative quadrupole are the tail-of-memory effects $\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}$, $\mathrm{M}\times\mathrm{M}_{i}\times\mathrm{M}_{ijk}$, $\mathrm{M}\times\mathrm{S}_{i}\times\mathrm{M}_{ij}$ and $\mathrm{M}\times\mathrm{M}_{i}\times\mathrm{S}_{ij}$ together with the "memory-of-memory" $\mathrm{M}_{i}\times\mathrm{M}_{i}\times\mathrm{M}_{ij}$. Note that the only interactions to exist in the center-of-mass are $\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}$ and $\mathrm{M}\times\mathrm{S}_{i}\times\mathrm{M}_{ij}$. The dimensional regularizations of those contributions are currently under investigation, by using the whole machinery presented above. One of the main differences with the 3PN case is that the 3PN tail-of-tail $\mathrm{M}\times\mathrm{M}\times\mathrm{M}_{ij}$ and tail-of-"linear memory" $\mathrm{M}\times\mathrm{M}_{i}\times\mathrm{M}_{i}$ could only contribute as corrections to the mass quadrupole. The identification was thus quite easy to perform. On the contrary, in the 4PN case, the tail-of-memory $\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}$ can *eg.* contribute to the monopole, quadrupole and $\ell=4$ moments, depending on contractions of its indices. In addition, the interactions involving S_i and S_{ij} can naturally also contribute as corrections to the current moments. The identification of the accurate dimensional regularization effects in the mass moment will thus be more involving.

But in addition to those tail and memory contributions to the radiative quadrupole, the canonical quadrupole can also receive corrections from dimensional regularization. Indeed at 4PN, the following interactions are dimensionally authorized to enter M_{ij}: $M\times M\times G_{ij}$, $M\times I_{ij}\times G$, $M\times I_i\times G_i$ and $I_i\times I_i\times G$, where G_L stands for one of both gauge moments W_L or Y_L. Note that only the first two types of interactions remain when expressed in the center-of-mass, and W is vanishing at leading order for circular orbits [187]. If such corrections are too high orders to affect the computation of the tail and memory interactions, they may play a crucial role in the final radiative moments. Fortunately, the machinery developed to deal with the transition from M_{ij} to U_{ij} is directly adaptable to the transition from I_{ij} to M_{ij}, so we won't have to develop a novel procedure to deal with such gauge interactions in d dimensions.

Nevertheless, it is to note that, once expressed in the center-of-mass, the pole in the 4PN contribution of $\mathcal{D}I_{ij}$ (IV.122) can be expressed in terms of tail-of-memory effects only, as

$$\mathcal{D}I_{ij}^{\text{4PN,p,CoM}} = \frac{G^2 M}{c^8}\left(\frac{10}{7}\,Q_{a\langle i}Q^{(4)}_{j\rangle a} - \frac{12}{7}\,Q^{(1)}_{a\langle i}Q^{(3)}_{j\rangle a} + \frac{6}{7}\,Q^{(2)}_{a\langle i}Q^{(2)}_{j\rangle a} - \frac{2}{3}\,\varepsilon_{ab\langle i}Q^{(3)}_{j\rangle a}\mathrm{J}_b\right)\,, \tag{IV.147}$$

with $Q_{ij} = m_1\hat{y}_1^{ij} + m_2\hat{y}_2^{ij}$ the Newtonian mass quadrupole and $\mathrm{J}_i = \epsilon_{ijk}(m_1 y_1^j v_1^k + m_2 y_2^j v_2^k)$ the conserved Newtonian angular momentum. This feature seems to indicate that the pole can be compensated by implementing dimensional regularization only in the transition from canonical to radiative moment, and thus that the gauge moments may not play a significant role. But this is just an intuition and obviously, concrete computations are to be done, and will be performed in a near future.

Part B

Alternative theories of gravitation

Chapter V

Non-canonical domain walls

To begin with the second part of this dissertation, devoted to the study of alternatives to the canonical gravitational theory, we first discuss a novel proposition for the construction of stable solitonic defects.

The discussion on the stability of those solitonic defects usually invokes a *topological* argument, reviewed in sec. V.1.4, based on the vacuum structure of the potential energy of the field. Together with C. Deffayet, we have shown that viable alternatives exist beyond this usual construction, which are to be presented in this chapter, extensively based on [165]. Note that we do not seek for exhaustivity, nor for a classification of those theories: after presenting the general framework we focus on a few examples of such non-canonical defects.

V.1 The canonical domain walls

V.1.1 Solitonic defects

Solitonic defects are static and self-sustained non-trivial configurations of a given field, with localized energy (*ie.* their energy must decay "fast enough" in at least one spatial direction), see *eg.* the textbooks [275, 329, 340, 349].

The first of such defects that was studied is of hydrodynamical origin, and results from the propagation of self-sustained front waves in rectangular narrow channels. This effect was scientifically described as soon as 1845 by J. Scott-Russell [326], and an accurate theoretical modeling was derived in 1871 by J. Boussinesq [109]. By solving the hydrodynamical equations, he recovered the relation between the speed of the front wave, v, the depth of water ahead of the soliton, H, and the height of the wave, h:

$$v^2 = g\left(H + \frac{3\,h}{2}\right) , \tag{V.1}$$

that had been observed by J. Scott-Russell and M. Bazin (in this formula $g \simeq 9.81\,\mathrm{m.s^{-2}}$ is the standard gravity). Note that similar (but widely more impressive) effects can happen at river mouths: the tidal bores (*mascarets* in French). Such front waves are extremely powerful, as the Seine tidal bores could easily reach Rouen, for instance.[1]

Beyond those hydrodynamical examples, solitons exist and are used in a large variety of domains, ranging from liquid crystals to communications. Focusing on the scope of this thesis, *ie.* gravitational theories, such defects are present in cosmological contexts. Indeed, implementing GR (or an alternative theory of gravitation) with non-linear matter fields can lead to self-sustained non-trivial configurations. The simplest examples of such scenarios lie within the scalar-tensor framework, briefly reviewed in sec. I.2.2, and thus we will restrict ourselves to this class of theories.

[1] As such front waves can be extremely devastating, the rives were adapted to break them. Those events are thus extremely rare nowadays.

An interesting formation mechanism for those cosmological defects is through phase transitions in the early stages of the evolution of our Universe [345]. Let us consider a field whose vacuum configuration is unique at high energy, but degenerate at low energy (such as the Higgs field). In the very early times such field is thus perfectly homogeneous but, when the temperature of the Universe has decreased enough, it has to settle down to one of the possible vacuum configurations. During this phase transition, causally disconnected patches of the Universe have no reason to settle down to the same configuration. Therefore, localized and self-sustained non-vacuum configurations of the field have to appear at some point: cosmological defects are hence formed.

From an observational point of view, those cosmological solitons are localized sources of energy, and thus should leave detectable imprints on the cosmic microwave background. Those effects have been investigated by the Planck collaboration, that was able to put bounds on the parameter space [13]. Moreover, string-shaped cosmological defects would induce polarizations in the cosmic microwave background [59, 306, 327]. Current experiments (such as the BICEP3 instrument [250]), or planned ones (*eg.* the LiteBIRD satellite [222]) are conducted to detect this polarization. In addition to those cosmological impacts, defects also emit gravitational radiation. Considering a cosmological network of solitons, the individual contributions add up to be observed as an unresolved stochastic background of gravitational radiation [36, 60, 191, 232], which is currently searched for, either by using the pulsar timing array techniques [97, 265] or directly by the LIGO-Virgo-KAGRA collaboration [11].

V.1.2 Some generalities about canonical domain walls

The simplest realizations of solitonic defects are the *domain walls*, that are invariant under translations in two directions: the energy is localized on a two-dimensional (thick) brane, thus the name. For the sake of simplicity, we will restrict ourselves to those domain wall geometry, and, apart in the last section, turn off the gravitational interaction, working on a Minkowskian space-time. The generalization to gravitating solutions, and to *string* or *monopole* configurations, is left for future studies.

Before investigating alternatives, let us review the usual domain walls in flat space-time. The canonical Lagrangian density for a scalar field ϕ is simply given by

$$\mathcal{L}_{\text{can}} = X - V(\phi)\,, \tag{V.2}$$

where the Lorentz-invariant kinetic term reads

$$X \equiv -\frac{1}{2}\,\eta^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,, \tag{V.3}$$

with $\eta^{\mu\nu}$ the inverse Minkowskian metric with mostly plus signature.

As we seek for domain wall configurations, that are invariant under translations in two spatial directions, we will restrict ourselves to the study of one-dimensional defects, usually called *kinks*. Using a 1+1 dimensional space-time instead of a 3+1 dimensional one does not spoil any of our results (as long as we work in flat space-time). We denote z the spatial coordinate that remains, write derivatives with respect to z with primes, and time derivatives with dots.

The specific form of the potential $V(\phi)$ that appears in the Lagrangian (V.2) determines the existence and shape of possible domain wall configurations. In the canonical case, V must have degenerate minima, *ie.* two or more distinct values ϕ^k_{min} for which V is minimal. In topological terms, the vacuum structure of V has to be non-connected. A kink configuration is then a static solution that interpolates between one of those minima at $z = -\infty$, and a distinct one at $z = +\infty$. As the field tends towards a state of minimal energy at both infinities, the energy of the kink is well localized. Moreover the decay of such configuration is forbidden as the two extremities of the kink are "trapped" in their vacuum states: exiting one of those states would require an infinite amount of energy (this rather vague assertion will be formalized in sec. V.1.4).

For a profile $\phi(z)$, the Euler-Lagrange equation for the theory (V.2) can be recast in the form of a total derivative, yielding the first integral[2]

$$\frac{\phi'^2}{2} - V(\phi) \approx \mathcal{J}_0 \,, \tag{V.4}$$

where $\mathcal{J}_0$ is a constant. This leads to the very simple differential equation

$$\phi' \approx \pm\sqrt{2\left(V(\phi) + \mathcal{J}_0\right)} \,. \tag{V.5}$$

As we seek for kinks, we require that ϕ settles down to some ϕ^k_{min} at spatial infinities, and that $\phi'(z = \pm\infty) = 0$. Thus evaluating eq. (V.4) at $z = +\infty$, we see that $\mathcal{J}_0$ is nothing but minus the minimal value of V (which renders eq. (V.5) well-defined for all ϕ). As we have the freedom of shifting the potential, we will only consider potentials that have vanishing minima, *ie.* we take $\mathcal{J}_0 = 0$ in the following.

Using a trick originally due to Bogomolny [99], let us write the energy carried by the kink as

$$\mathcal{H}_{\text{kin}} = \int \mathrm{d}z \left[\frac{\phi'^2}{2} + V(\phi)\right] = \int \mathrm{d}z \left[\frac{1}{2}\left(\phi' \mp \sqrt{2V}\right)^2 \pm \phi'\sqrt{2V}\right] . \tag{V.6}$$

Choosing the sign inside the parenthesis so that the perfect square vanishes by virtue of (V.5), it only remains

$$\mathcal{H}_{\text{kink}} = \pm \int \mathrm{d}z \phi' \sqrt{2V} = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}\phi \sqrt{2V} \,. \tag{V.7}$$

As the energy density vanishes at infinity, it is natural to require that the total energy (V.7) is finite.

Two of the usual potentials that accommodate domain walls are the "mexican hat" and "sine-Gordon" ones, given respectively by[3]

$$V_{\text{mh}} = \frac{(1-\phi^2)^2}{2} \,, \qquad \text{and} \qquad V_{\text{sG}} = 1 - \cos\phi \,. \tag{V.8}$$

Those two potentials are quite different: the mexican hat potential posses only two vacuum configurations, at $\phi = \pm 1$, when the sine-Gordon one has an infinite number of minima, for $\phi = 2\pi k$, $k \in \mathbb{Z}$. Integrating (V.5) with $\mathcal{J}_0 = 0$, the resulting configurations are given by

$$\phi_{\text{mh}}(z) = \pm \tanh z \,, \qquad \text{and} \qquad \phi_{\text{sG}}(z) = 2\pi k \pm 4 \arctan e^z \,. \tag{V.9}$$

Those configurations have finite energies, explicitly given by

$$\mathcal{H}_{\text{mh}} = \frac{4}{3} \,, \qquad \text{and} \qquad \mathcal{H}_{\text{sG}} = 8 \,. \tag{V.10}$$

V.1.3 An interesting change of variables

Eq. (V.5) points out a very convenient change of variable: defining ψ as

$$\mathrm{d}\psi \equiv \frac{\mathrm{d}\phi}{\sqrt{2V(\phi)}} \,, \tag{V.11}$$

[2] In this chapter, we denote with curly symbols "$\approx$" relations that hold on-shell, *ie.* when the field and its derivatives are replaced by the configuration that satisfies the equation of motion.

[3] As we do not seek for exhaustivity, we only consider the most "natural" form of those potentials. Parameters could naturally be added, *eg.* by $V_{\text{mh}} = V_0 - \mu^2\phi^2 + \lambda\phi^4$, but taking such parameters into account changes the discussion in a quantitative rather than a qualitative manner. Similarly, we consider that our domain walls "sit" in the origin (which amounts to integrating out the translationnal zero mode), and we will treat the $+$ and $-$ solutions of (V.5) as being the same configuration.

the Lagrangian density is factorized as

$$\mathcal{L}_{\rm can} = 2\left(X_\psi - \frac{1}{2}\right) V\left[\phi(\psi)\right] , \tag{V.12}$$

with X_ψ being naturally given by replacing ϕ by ψ in eq. (V.3). This form is interesting as it is easy to see that $\psi = \pm z$ generates solutions, whatever the shape of the potential V is.

More noteworthy, the quite different mexican hat and sine-Gordon theories (V.8) appear very similar when written in the ψ variable

$$\mathcal{L}_{mh} = \left(X_\psi - \frac{1}{2}\right) \cosh^{-4}\psi\,, \qquad \text{and} \qquad \mathcal{L}_{sG} = 4\left(X_\psi - \frac{1}{2}\right) \cosh^{-2}\psi\,. \tag{V.13}$$

Note that those two Lagrangians appear singular at the $\psi = \pm\infty$ points, which is to be expected as the transformation (V.11) is ill-defined when V vanishes. Note also that the inverse change of variable $\phi[\psi]$ induced by this transformation maps the real line (domain of variation of ψ) to a finite interval (bounded by the two vacuum states between which the kink interpolates) which does not represent the full range of variation of the ϕ field of the original model, *eg.* it does not cover the large values of ϕ in the mexican hat potential.

Given the similarity between the Lagrangians (V.13), we can easily generalize these canonical models to a larger family

$$\mathcal{L}_{\rm k,can} = \frac{\mathcal{K}}{\cosh^{2k}(\psi)}\left(X_\psi - \frac{1}{2}\right) , \tag{V.14}$$

where $\mathcal{K}$ is some positive constant and k a half integer. As discussed, is easy to see that $\psi' = \pm 1$ provides a solution of the field equations of the kink type. The energy of such a solution is finite and given by

$$\mathcal{H}_{\rm k,can} = \mathcal{K}\int_{-\infty}^{+\infty} \frac{{\rm d}z}{\cosh^{2k} z} = \mathcal{K}\mathcal{I}_k\,, \tag{V.15}$$

where we have defined $\mathcal{I}_k$ as

$$\mathcal{I}_k \equiv \int_{-\infty}^{+\infty} \frac{{\rm d}z}{\cosh^{2k}(z)} = \frac{\sqrt{\pi}\,\Gamma(k)}{\Gamma(k+1/2)} \qquad \text{for} \quad k \geq \frac{1}{2}\,. \tag{V.16}$$

In order to express those theories in the original ϕ variable, let us perform a change of variable of the form

$$\phi[\psi] = \sqrt{\mathcal{K}}\int_0^{\psi} \frac{{\rm d}u}{\cosh^k u}\,. \tag{V.17}$$

When ψ varies over the whole real line, the interval of variation of ϕ is just given by $\left]-\frac{\sqrt{\mathcal{K}}}{2}\mathcal{I}_{\frac{k}{2}}, \frac{\sqrt{\mathcal{K}}}{2}\mathcal{I}_{\frac{k}{2}}\right[$ and because the hyperbolic cosine is a positive function, we see that the above defined $\phi[\psi]$ is invertible into a $\psi[\phi]$ on this interval. This change of variable puts the Lagrangian (V.14) in the standard form (V.2) with the specific potential

$$V(\phi) = \frac{\mathcal{K}}{2}\cosh^{-2k}\left(\psi[\phi]\right) , \tag{V.18}$$

where, at this stage, V is defined for $\phi \in \left]-\frac{\sqrt{\mathcal{K}}}{2}\mathcal{I}_{\frac{k}{2}}, \frac{\sqrt{\mathcal{K}}}{2}\mathcal{I}_{\frac{k}{2}}\right[$. However, it is easy to see that ${\rm d}V/{\rm d}\phi$ vanishes at the ends of this interval (where ψ diverges) allowing to extend the domain of variation of ϕ to the entire real line, either by making V periodic (which is always possible, with period then given by $\sqrt{\mathcal{K}}\mathcal{I}_{\frac{k}{2}}$) or by using an analytic extension, possibly non periodic. This later possibility arises for instance in the case of the canonical mexican hat model, which corresponds to $k=2$. The $k=6$ or $k=10$ also yield analytical expressions for $\psi[\phi]$ (however not very enlightening) which in turn result in potentials having similar shapes to the mexican hat one. In turn, the sine-Gordon ($k=1$) and the $k=1/2$ cases have potentials which are periodic by analytic extension. We show these potentials in the fig. V.1.

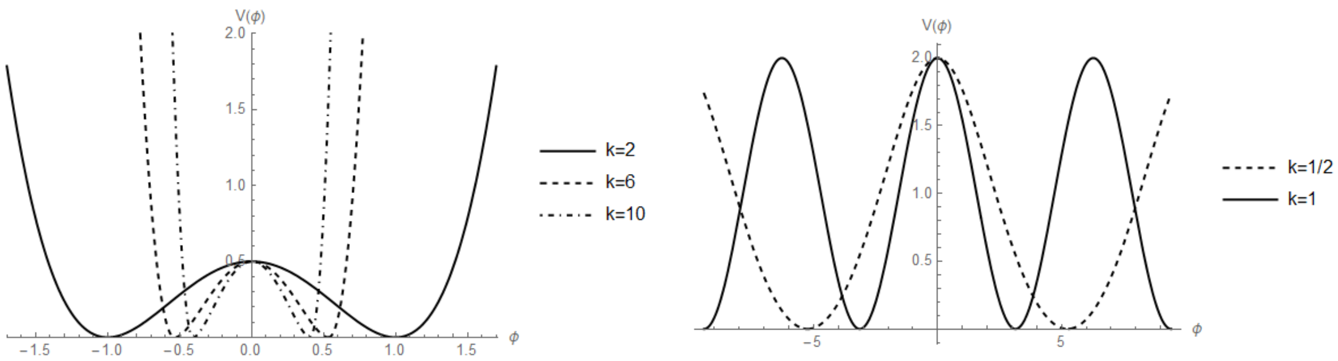


Figure V.1: Analytic extensions of the potential $V(\phi)$ (V.18). The left panel displays the cases $k=2$, $k=6$ and $k=10$, yielding a mexican hat-like profile; the right one, $k=1$ and $k=1/2$, yielding a periodic profile.

V.1.4 The usual argument sustaining stability

The Hamiltonian of a generic configuration $\phi(t,z)$ is easily derived from the Lagrangian density (V.2) and, using the Bogomolny trick, it reads

$$\mathcal{H}(t) = \int \mathrm{d}z \left[\frac{\dot{\phi}^2}{2} + \frac{\phi'^2}{2} + V(\phi)\right] = \int \mathrm{d}z \left[\frac{\dot{\phi}^2}{2} + \frac{1}{2}\left(\phi' \mp \sqrt{2V}\right)^2 \pm \phi'\sqrt{2V}\right]. \tag{V.19}$$

This energy can be decomposed as $\mathcal{H}(t) = \mathcal{H}_{\text{kin}}(t) + \mathcal{H}_{\text{topo}}(t)$ with

$$\mathcal{H}_{\text{kin}}(t) \equiv \frac{1}{2}\int \mathrm{d}z \left[\dot{\phi}^2 + \left(\phi' \mp \sqrt{2V}\right)^2\right], \tag{V.20a}$$

$$\mathcal{H}_{\text{topo}}(t) \equiv \pm \int \mathrm{d}z \phi' \sqrt{2V} = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}\phi \sqrt{2V}. \tag{V.20b}$$

The first term is obviously always positive, thus the energy is always greater than $\mathcal{H}_{\text{topo}}$. The last term, which is nothing but $\mathcal{H}_{\text{kink}}$, depends only on the value of ϕ at spatial boundaries, thus it has been dubbed *topological*. Once again, for a kink configuration, one can choose the sign within the parenthesis accordingly to (V.5) so that $\mathcal{H}_{\text{kin}} = 0$ (we recall that such a solution is static). Thus we recover the result (V.7), namely that all the energy of a kink is contained in a topological contribution. In addition, this provides an elegant proof of the perturbative stability of the kinks: the addition of a localized perturbation to a kink configuration will not change the topological term, but only add a contribution $\delta\mathcal{H}_{\text{kin}}$. The energy of the perturbed configuration is thus inevitably greater than the energy of the background one: the kink is stable against perturbations.

The usual argument invoked to prove the non-perturbative stability of the kink relies on the existence of a "topological charge". Using the fully antisymmetric Levi-Civita tensor $\varepsilon^{\mu\nu}$ and a generic smooth function $f(\phi)$, one can construct a "conserved" current $\varepsilon^{\mu\nu}\partial_\nu f(\phi)$ as, by symmetry considerations, $\partial_\mu\left(\varepsilon^{\mu\nu}\partial_\nu f(\phi)\right)$ is trivially vanishing. A convenient choice of f leads to the current

$$J^\mu \equiv \varepsilon^{\mu\nu}\partial_\nu\left(\int_{\phi(-\infty)}^{\phi} \mathrm{d}\varphi \sqrt{2V(\varphi)}\right), \tag{V.21}$$

with associated "charge"

$$Q = \int \mathrm{d}z J^0 = \pm\mathcal{H}_{\text{topo}}, \tag{V.22}$$

where the sign directly corresponds to the sign of ϕ'. One can then argue that among the class of solutions sharing the same value of the "topological charge" Q, the kink one minimizes the energy. Thus, as long as the charge is conserved, the kink is stable in a non-perturbative way.

In order to change the value of its charge, a kink has to be deformed to a configuration with different boundaries. If one considers that this deformation has to be smooth, there exist intermediate configurations in which the field at infinities is not in a vacuum state, as the vacuum structure of the potential is not connected, and thus the total energy of those configurations diverges. Therefore the usual argument supporting the conservation of the charge Q is that it requires an infinite amount of energy to smoothly deform a configuration with a given charge into a configuration with another charge.

But one can be reticent to the idea of considering Q as a physical charge, and thus its conservation as a fundamental principle. Indeed the notion of a conserved current usually implies an underlying symmetry, as discussed in the work of E. Noether [296]. If the present system enjoys a symmetry under time redefinition, the associated conserved quantity is already given by the first integral (V.4). Thus the conservation of the current (V.21) is nothing more that a mathematical artifact, and the "conservation of the charge Q" must be dealt with great care.

V.2 Domain walls in potential-free scalar theories

V.2.1 Bypassing the usual argument

The ambiguity of this concept of "charge", on which the usual argument to prove the stability of kinks relies, is a sufficient motivation to seek for alternatives. As this concept is based on the vacuum structure of the potential, a straightforward way to go beyond the argument is to seek for theories without potentials. But a canonical Lagrangian with no potential simply describes a free field $\mathcal{L} = X$, and is of no interest as it cannot sustain kink configurations. We thus have to use non-canonical kinetic terms, *ie.* seeking for theories belonging to the class of Horndeski models [164, 236]. For simplicity, we will only consider Lagrangian containing at most first derivatives

$$\mathcal{L} = P(\phi, X)\,, \tag{V.23}$$

leaving the extension to second and higher derivatives for future works. We define a potential-free theory by requiring that the Lagrangian density is trivially vanishing when the kinetic term is set to zero

$$P\,(\phi, 0) \equiv 0\,. \tag{V.24}$$

Solitonic solutions in theories involving non-canonical kinetic terms have been already studied and are usually dubbed k-defects (see *eg.* [37, 38, 322, 364] and other references displayed in [165]). But to our knowledge, none of those works investigate theories that satisfy the condition (V.24): they either involve an explicit potential (as in the original paper [37]), or a "hidden" one, as for example in the DBI-like Lagrangian of [20], $\mathcal{L} = -[1 + U(\phi)]\sqrt{1 - 2X}$, that does not vanish when X is set to vanish.

V.2.2 General stability requirements

For a static profile $\phi(z)$, the theory (V.23) admits the first integral[4]

$$\mathcal{J} \equiv 2XP_X - P \approx -\mathcal{J}_0\,, \tag{V.25}$$

which is the generalization of eq. (V.4). The conservation of this quantity is equivalent to the cancellation of the equation of motion $\mathcal{E} \equiv \delta\mathcal{L}/\delta\phi$, as

$$\mathcal{J}' = -\mathcal{E}\,\phi'\,. \tag{V.26}$$

[4] As from now, we denote partial derivatives with a subscript, *eg.* $P_X \equiv \partial P(\phi, X)/\partial X$.

Similarly to what was done in the canonical case, we will only study the solutions with $\mathcal{J}_0 = 0$ for simplicity. It comes as a natural requirement to impose that the energy of the background is finite and strictly positive. For a static configuration, the energy density being simply minus the Lagrangian one, we require that

$$\mathcal{H} = -\int \mathrm{d}z P = -2\int \mathrm{d}z X P_X \in]0, +\infty[\,, \tag{V.27}$$

where we have used eq. (V.25) with $\mathcal{J}_0 = 0$.

Assuming that a background profile has been found, we study its stability by adding a small perturbation $\varphi(t,z)$ on top of it. The quadratic Lagrangian for this perturbation reads

$$\delta^{(2)}\mathcal{L} = -\frac{1}{2}\left[\mathcal{Z}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{M}^2\varphi^2\right] , \tag{V.28}$$

where the kinetic matrix and mass terms read

$$\mathcal{Z}^{00} = -P_X \,, \tag{V.29a}$$
$$\mathcal{Z}^{zz} = \mathcal{J}_X = 2XP_{XX} + P_X \,, \tag{V.29b}$$
$$\mathcal{M}^2 = -\mathcal{E}_\phi = \mathcal{J}_{\phi\phi} - \mathcal{J}_{\phi X}\phi'' \,. \tag{V.29c}$$

In the canonical case (V.2), those quantities are given by $\mathcal{Z}^{\mu\nu} = \eta^{\mu\nu}$ and $\mathcal{M}^2 = V_{\phi\phi}$. In order to avoid tachyonic instabilities, we require that the determinant of the kinetic matrix is strictly negative everywhere[5]

$$-\,\mathcal{Z}^{00}\,\mathcal{Z}^{zz} > 0 \,. \tag{V.30}$$

As the background is time-independent, one can perform a Fourier decomposition of the perturbation $\varphi(t,z) = \sum_k \varphi_k(z)e^{i\omega_k t}$, where the index k can be discrete or continuous. Each mode then obeys

$$(\mathcal{Z}^{zz}\varphi_k')' - \left(\mathcal{Z}^{00}\omega_k^2 + \mathcal{M}^2\right)\varphi_k = 0 \,. \tag{V.31}$$

This equation is in a Sturm-Liouville form, the modes obey orthogonality relations with the measure $-\mathcal{Z}^{00}\mathrm{d}z$ [129]

$$\int \mathrm{d}z \left(-\mathcal{Z}^{00}\right)\varphi_k\varphi_\ell = 0 \qquad \text{when} \quad k \neq \ell \,. \tag{V.32}$$

Note that there is always a zero-mode, given by $\varphi_0 \propto \phi'$, which explicitly appears when taking the derivative of the equation of motion

$$\mathcal{E}' = (\mathcal{J}_X\phi'')' - (\mathcal{J}_{\phi X}\phi'' - \mathcal{J}_{\phi\phi})\,\phi' = (\mathcal{Z}^{zz}\phi'')' - \mathcal{M}^2\,\phi' \approx 0 \,. \tag{V.33}$$

The condition (V.27) can now be understood also as a normalizability condition for this zero-mode, under the measure $-\mathcal{Z}^{00}\mathrm{d}z$, as we recall that for a kink, $X = -\phi'^2/2 \propto -\varphi_0^2$. Naturally we have to require that this massless mode is the lowest lying one, otherwise instabilities would appear. Note that when the background configuration is proportional to ϕ_{mh} or to ϕ_{sG} (V.9), this zero mode has no node, and thus is indeed the lowest lying one.

Moreover, by combining the conditions (V.27) and (V.30) we recover the usual conditions for the theory to be consistent for arbitrary configurations (see *eg.* [32, 39, 113] and other references listed in [165]), namely

$$P_X > 0 \qquad \text{and} \qquad 2XP_{XX} + P_X > 0 \,. \tag{V.34}$$

The first condition is necessary to have a bounded by below Hamiltonian, and, together with the second, leads to a hyperbolic equation of motion for ϕ.

[5]We could have weakened this requirement by allowing the determinant to vanish, as long as $\mathcal{Z}^{00} < 0$, but for simplicity we will not consider such case.

In a nutshell, given a particular theory accommodating a kink configuration (*ie.* satisfying $\mathcal{J} \approx 0$), we require that

$$
- \mathcal{Z}^{00}\, \mathcal{Z}^{zz} > 0\,, \tag{V.35a}
$$

$$
0 < 2 \int \mathrm{d}z \mathcal{Z}^{00} X < +\infty\,, \tag{V.35b}
$$

$$
\varphi_0 \propto \phi' \quad \text{is the lowest lying mode.} \tag{V.35c}
$$

Note that in the case of the canonical theory (V.2), the kinetic matrix reads $\mathcal{Z}^{\mu\nu} = \eta^{\mu\nu}$, and thus the condition (V.35a) is satisfied.

V.2.3 Explicit stability requirements

In order to construct viable alternatives to the usual framework, we will follow a reverse procedure: we start by choosing a convenient kink configuration ϕ, and aim at constructing a theory admitting it as a solution. The advantages of this procedure are that we are trivially confident that our final theory admits a kink solution, and that we can enforce the zero-mode condition (V.35c) from the beginning, by selecting a kink profile which derivative has no node.

Assuming a well-behaving kink profile ϕ, we denote as $f(\phi)$ the *on-shell* value of $|\phi'|$, so that $X \approx -f^2(\phi)/2$. The relation $|\phi'| \approx f(\phi)$ is in some sense the generalization of eq. (V.5). In addition, we restrict ourselves to Lagrangians that are analytical in $\sqrt{-X}$, *ie.* theories that can be power-expanded as

$$
P(\phi, X) = \sum_{n\geq 2} \alpha_n(\phi)(-2X)^{n/2}\,, \tag{V.36}
$$

where we have set $\alpha_0(\phi) = 0$ to fulfill the no-potential condition (V.24) and $\alpha_1(\phi) = 0$ as such term would not contribute in the equation of motion for static profiles.

No domain walls for $P(X)$ theories

Beginning with the simplest $P(X)$ case, the first integral (V.25) vanishes off-shell and is easily integrated as

$$
P(X) = P_0\sqrt{-2X} \qquad \text{with} \qquad P_0 \in \mathbb{R}\,, \tag{V.37}
$$

which does not fall in the class (V.36), and thus will not be considered hereafter.

Separable theories

We then focus on separable theories, *i.e.* consider

$$
P(\phi, X) = \alpha(\phi) \sum_{n\geq 2} \beta_n \left(-2X\right)^{n/2}\,, \tag{V.38}
$$

where $\{\beta_n\}_{n\geq 2}$ is a collection of constant coefficients. In this case, it simply comes

$$
\mathcal{J} \approx \alpha(\phi) \sum_{n\geq 2} \beta_n (n-1) f^n \approx 0\,. \tag{V.39}
$$

Leaving aside the case of a vanishing α which would make the theory trivial, f must be a constant root of the polynomial equation (V.39), say f_0. The conditions (V.35a) and (V.35b) become respectively

$$
\left(\sum_{n\geq 2} n\beta_n f_0^n\right) \left(\sum_{m\geq 2} m(m-1)\beta_m f_0^m\right) > 0\,, \tag{V.40a}
$$

$$
0 < -\left(\sum_{n\geq 2} n\beta_n f_0^n\right) \int \mathrm{d}z\, \alpha[\phi(z)] < +\infty\,, \tag{V.40b}
$$

imposing that α is regular everywhere and has a finite integral.

Non-separable theories

Let finally focus on the most general case (V.36) and define $n_0 \geq 2$ as the smallest integer n for which α_n is non-vanishing. We can extract α_{n_0} from the first integral as

$$\alpha_{n_0} = -\sum_{n>n_0} \frac{n-1}{n_0-1}\alpha_n\, f^{n-n_0}\,. \tag{V.41}$$

Plugging this value in the conditions (V.35a) and (V.35b), they read

$$\left(-\sum_{n>n_0} \frac{n-n_0}{n_0-1}\,\alpha_n f^n\right)\left(\sum_{m>n_0}(m-n_0)(m-1)\,\alpha_m f^m\right) > 0\,. \tag{V.42a}$$

$$0 < \int \mathrm{d}z \left(\sum_{n>n_0} \frac{n-n_0}{n_0-1}\alpha_n f^n\right) < +\infty\,. \tag{V.42b}$$

V.3 Mimicking canonical domain wall profile

In this section we apply the general conditions (V.40) and (V.42) to derive theories accommodating kinks that mimic the canonical ones (V.9).

V.3.1 The case of mexican hat-like profiles

Let us start by seeking a mexican hat-like profile $\phi_{\rm mh} = \pm\tanh z$. As previously discussed, the zero mode $\varphi_0 \propto \phi' = \cosh^{-2} z$ has no node, and thus the condition (V.35c) is satisfied. The absolute value of ϕ' is non-constant, and we can express $f(\phi) = 1-\phi^2$. Therefore no separable theory (when written in terms of the ϕ variable) can sustain such profiles and we have to seek a non separable theory satisfying the conditions (V.42). When decomposing P as in (V.36), we can choose to restrict ourselves to even functions α_n

$$\alpha_n = \sum_{p\in\mathbb{Z}} \frac{\beta_{n,p}}{2(n-1)}\left(1-\phi^2\right)^p\,, \tag{V.43}$$

where $\beta_{n,p}$ are some constants (and the factor $2(n-1)$ is introduced to simplify the computations). We could have added an odd part as well, however it would drop out of the crucial normalization condition (V.42b) and thus we do not need it. Let us further simplify the setting by only considering a deviation from the canonical Lagrangian, *ie.* by imposing $P = X + \mathcal{O}(X^{3/2})$, which amounts to take $n_0 = 2$ and $\alpha_2 = -1/2$. Considering only two non-vanishing coefficients $\beta_{n,p}$, our ansatz reads

$$P(\phi,X) = X + \frac{\beta_{n,p}}{2(n-1)}\left(1-\phi^2\right)^p(-2X)^{n/2} + \frac{\beta_{m,q}}{2(m-1)}\left(1-\phi^2\right)^q(-2X)^{m/2}\,. \tag{V.44}$$

In order to get a finite energy, we must have $n+p>0$ and $m+p>0$ so that the integrals $\int \mathrm{d}z f^{n+p}$ and $\int \mathrm{d}z f^{m+p}$ converge in equation (V.42b). Next, eq. (V.41) with $\alpha_2 = -1/2$ imposes

$$\beta_{n,p} = f^{2-n-p} - \beta_{m,q}\, f^{m+q-n-p}\,. \tag{V.45}$$

As f is not a constant, we get

$$p = 2-n \qquad q = 2-m \qquad \text{and} \qquad \beta_{n,2-n} = 1-\beta_{m,2-m}\,, \tag{V.46}$$

so that we are left with a family of theories parametrized by a single parameter $\kappa \equiv \beta_{m,2-m}$, with Lagrangians

$$P_{n,m}(\phi,X) = X + \frac{1-\kappa}{2(n-1)}\frac{(-2X)^{n/2}}{(1-\phi^2)^{n-2}} + \frac{\kappa}{2(m-1)}\frac{(-2X)^{m/2}}{(1-\phi^2)^{m-2}}\,. \tag{V.47}$$

The total energy of a kink in such a theory is

$$\mathcal{H} = \frac{2}{3}\left(\frac{n-2}{n-1} + \frac{(m-n)\kappa}{(n-1)(m-1)}\right) . \tag{V.48}$$

Hence we get a strictly positive energy provided that

$$\frac{n-m}{n-2}\kappa < m-1 \, . \tag{V.49}$$

Let us finally check the constraint (V.42a). The coefficients of the kinetic matrix are independent of z and given by

$$\mathcal{Z}^{00} = -\frac{1}{2}\left(\frac{n-2}{n-1} + \frac{(m-n)\kappa}{(n-1)(m-1)}\right) = -\frac{3}{4}\mathcal{H} \, , \qquad \mathcal{Z}^{zz} = \frac{2-n+(n-m)\kappa}{2} \, . \tag{V.50}$$

As in the canonical case, the mass term depends on z, and reads

$$\mathcal{M}^2(z) = (2-n+(n-m)\kappa)\left(3\phi(z)-1\right) . \tag{V.51}$$

The positivity of the energy implies that $\mathcal{Z}^{00}$ is strictly negative, so we just have to check that $\mathcal{Z}^{zz}$ is strictly positive, which amounts to

$$1 < \frac{n-m}{n-2}\kappa \, . \tag{V.52}$$

Hence, at this point, we have shown that the family of Lagrangians (V.47) does accommodate a hyperbolic tangent configuration $\phi = \pm\tanh(z)$ with stable perturbations as long as n and m are two distinct integers and together with κ verify the bounds

$$n > 2 \, , \qquad m > 2 \, , \qquad \text{and} \qquad 1 < \frac{n-m}{n-2}\kappa < m-1 \, . \tag{V.53}$$

Note in particular that these bounds cannot be satisfied if $\kappa = 0$, hence we need at least two non trivial terms of the form $(1-\phi^2)^{2-n}(-2X)^{n/2}$ in the Lagrangian $P(\phi, X)$. However, more terms are allowed and we could have considered a larger family with Lagrangians of the form

$$P\left(\phi, X\right) = \sum_{n\geq 2} \kappa_n (1-\phi^2)^{2-n}(-2X)^{n/2} \, , \tag{V.54}$$

where $\{\kappa_n\}$ are more than two non vanishing and properly chosen constants (we will later derive the conditions they must obey). If this family is quite simple, it is not the only one to exhibit such features, and another one, inspired by the DBI action, is presented in the App. A of [165].

In the following of this section, we will study in more details the family (V.47), before discussing the generalization of the method to mimic other canonical profiles in sec. V.3.6.

V.3.2 Apparent singularities

A possible concern with the class of theories (V.47) is its singularity in $\phi = \pm 1$. Let us first note that this is not a problem for the kink configuration as $\phi = \tanh z$ takes values on $]-1;1[$ and the fact that X decays as $(1-\phi^2)^2$ allows the energy density and the quadratic Lagrangian for perturbations (V.28) to remain bounded. Nevertheless if one wants to consider such theories for more generic applications, this singularity may become a real problem.

A simple way to circumvent this singularity is to push it towards infinity, for instance by a change of variables, using the variable ψ introduced in eq. (V.11) where V is taken to be the mexican hat potential $(1-\phi^2)^2/2$. Concretely we set $\psi = \tanh^{-1}\phi$ so that the wall solution reads $\psi \approx z$ and the Lagrangian (V.47) becomes

$$P_{n,m}\left(\psi, X_\psi\right) = \frac{1}{\cosh^4\psi}\left(X_\psi + \frac{1-\kappa}{2(n-1)}\left(-2X_\psi\right)^{n/2} + \frac{\kappa}{2(m-1)}\left(-2X_\psi\right)^{m/2}\right), \tag{V.55}$$

Comparing this form with eq. (V.13) we see that the above family of theories and the canonical scalar with a mexican hat potential belong to the same family of theories, with Lagrangians of the form

$$\mathcal{L} = \left(\sum_{n\in\mathbb{N}} \kappa_n(-2X_\psi)^{n/2}\right)\cosh^{-4}\psi, \tag{V.56}$$

where κ_n are constants. Note that this family also includes the more general class (V.54) when written in terms of ψ variable. One difference between our theories (V.54) and the canonical ones is of course the presence in eq. (V.12) of a pure potential encoded in a non vanishing κ_0 above.

But those Lagrangians are still singular, at infinity. If one wishes to extend elegantly the theory (V.47) "beyond" the $\phi = \pm 1$ singularity with an everywhere regular theory, a possible change of variable to consider is given by

$$\mathrm{d}\xi \equiv \frac{\mathrm{d}\phi}{(1-\phi^2)^{1-\frac{2}{m}}}, \tag{V.57}$$

where we have considered that $m > n$ (if the converse holds, the argument is similar, under a trivial change of coefficients). This can be explicitly integrated to yield

$$\xi(\phi) = {}_2F_1\left[\frac{1}{2}, 1-\frac{2}{m}; \frac{3}{2}; \phi^2\right]\phi, \tag{V.58}$$

where ${}_2F_1(a,b;c;u)$ is the Gauss hypergeometric function (which is well defined on the unit interval for its fourth argument u, and whenever $c > a+b$, see e.g. [214]). The minima $\phi = \pm 1$ of the mexican hat potential are mapped to finite values of the ξ field

$$\xi^{\pm} \equiv \xi\left(\phi = \pm 1\right) = \pm\frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{m}\right)}{2\,\Gamma\left(\frac{1}{2}+\frac{2}{m}\right)}. \tag{V.59}$$

As the mapping (V.57) is (monotonic and hence) one to one between $\phi \in [-1,1]$ and $\xi \in [\xi^-, \xi^+]$, and that $\mathrm{d}\xi/\mathrm{d}\phi$ diverges in $\phi = \pm 1$, the inverse mapping $\phi(\xi)$ can be naturally extended to a periodic everywhere smooth, non singular function, defined on the entire real line and of period $4\xi^+$. In general, this inverse mapping, even though it exists, does not correspond to a simple function, however, this is not true for $m = 4$ and $m = 8$, for which we have respectively

$$\phi = \sin\xi \qquad \text{and} \qquad \phi = \sin\left[2\mathrm{am}(\xi/2)\right], \tag{V.60}$$

where am is the so-called *amplitude* of the elliptic integral of the first kind[6]. Obviously the period of the first function above is $2\pi = 2\sqrt{\pi}\Gamma(1/2)/\Gamma(1)$, while the period of the second function is $2\sqrt{\pi}\Gamma(1/4)/\Gamma(3/4) \sim 10.5$. Of course this is not the only possibility to extend the inverse function beyond the points $\xi^\pm$, however, choosing this way offers an elegant extension of the family of models (which strictly speaking differ from (V.47) where the function $(1-\phi^2)^2$ is not periodic). It would be interesting to investigate if this "periodic" extension would allow to find solutions with a non trivial time dependence interpolating between non adjacent minima, similarly to what is known to exist in the sine-Gordon model (as discussed briefly at the end of sec. V.3.6).

[6] We use here the definition of [214] for the elliptic integral of the first kind, *ie.* $F(\varphi,k) = \int_0^\varphi \frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin\alpha}}$.

The interest of the change of variable (V.58) appears when writing the Lagrangians (V.47) in terms of ξ and X_ξ (defined by replacing ϕ by ξ in eq. (V.3))

$$P(\xi, X_\xi) = \frac{\kappa\,(-2X_\xi)^{m/2}}{2(m-1)} + \frac{1-\kappa}{2(n-1)} f[\phi(\xi)]^{2\left(1-\frac{n}{m}\right)}\,(-2X_\xi)^{n/2} + f[\phi(\xi)]^{2\left(1-\frac{2}{m}\right)}\,X_\xi\,, \tag{V.61}$$

where we recall that $f(\phi) = 1-\phi^2$. In this form the Lagrangian is no longer singular at the finite values $\xi^\pm$ (corresponding to $\phi = \pm 1$), even though the purely "kinetic" term of ξ has the non standard form $\propto (-X_\xi)^{m/2}$. The domain wall profile is found to be

$$\xi(z) = {}_2F_1\left[\frac{1}{2}, 1-\frac{2}{m}, \frac{3}{2}, \tanh^2(z)\right]\,\tanh(z)\,. \tag{V.62}$$

Those profiles are shown in fig. V.2 for the cases $m = 2$ (the usual hyperbolic tangent), $m = 6$ and $m = 8$.

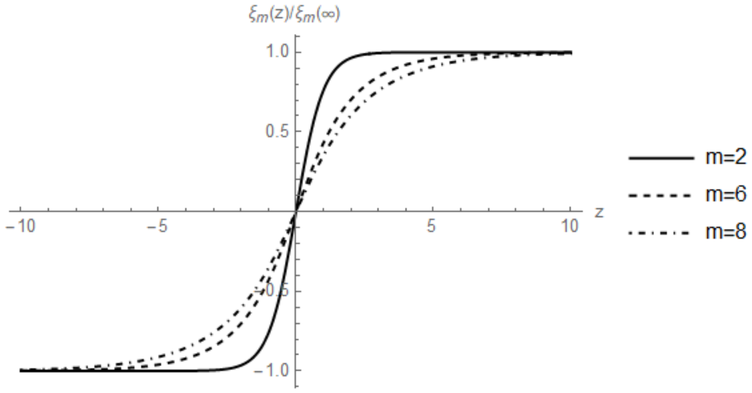


Figure V.2: Behavior of the domain wall profile $\xi(z)$ (V.62) normalized to its value at infinity for different values of m ($m = 2$ corresponds to the usual hyperbolic tangent).

V.3.3 Non-perturbative stability considerations

In this section, we will see what happens to the arguments developed in sec. V.1.4, when applied to our non-canonical theories. To that end we consider the theory written in the ψ variable, and start with the general form (V.56) which also encompasses the canonical mexican hat model by allowing $n = 0$, and restrict it to integer powers of X_ψ for simplicity. The energy density $\mathcal{H}(t, z)$ of a given (arbitrary) field configuration is easily found to be

$$\mathcal{H}(t, z) = -\left(\sum_{n\in\mathbb{N}} \kappa_{2n}(\psi'^2 - \dot{\psi}^2)^n + 2n\,\kappa_{2n}\dot{\psi}^2(\psi'^2 - \dot{\psi}^2)^{n-1}\right)\cosh^{-4}\psi\,. \tag{V.63}$$

In the case of the canonical mexican hat model (recalling that we just have then $\kappa_0 = \kappa_2 = -1/2$) we find an energy density $\mathcal{H}(t, z) = (1 + \psi'^2 + \dot{\psi}^2)/(2\cosh^4\psi)$. Using the shorthands

$$x \equiv \psi'\,, \qquad \text{and} \qquad y \equiv \dot{\psi}\,, \tag{V.64}$$

we see that the Bogomolny decomposition (V.19) amounts here to just rewrite the energy density $\mathcal{H} = (1+x^2+y^2)/(2\cosh^4\psi) = \left[y^2 + (x \mp 1)^2 \pm 2x\right]/(2\cosh^4\psi)$ where the two first terms on the right hand side yield the kinetic energy (V.20a) and the last one gives the equivalent of the "topological" charge (V.20b). A nice feature of our non-canonical theories is that a similar decomposition exists in general. Indeed, the Hamiltonian density (V.63) is just $\Pi_0(x, y)/(2\cosh^4\psi)$, with the polynomial

$$\Pi_0(x, y) \equiv -2\left(\sum_{n\in\mathbb{N}} \kappa_{2n}(x^2 - y^2)^n + 2n\,\kappa_{2n}y^2(x^2 - y^2)^{n-1}\right)\,. \tag{V.65}$$

In order to have a domain wall with profile $\psi = \pm z$, the coefficient κ_n must obey the algebraic relation

$$\Sigma_{\kappa,0} = 2\Sigma_{\kappa,1}\,, \qquad \text{with} \quad \Sigma_{\kappa,k} \equiv \sum_{n\in\mathbb{N}} \kappa_{2n} n^k\,, \tag{V.66}$$

where we imply in particular that $\Sigma_{\kappa,0} = \sum_{n\in\mathbb{N}} \kappa_{2n}$ (using the convention that $0^0 = 1$). At this stage, considering the form of Π_0, we can notice that the Hamiltonian cannot be bounded by below if the largest integer n for which κ_{2n} does not vanish, call it $n_{\max}$, is even. In contrast, if $n_{\max}$ is odd we see that at large x and y the dominant terms in $\Pi_0(x,y)$ read $-2\kappa_{2n_{\max}}(x^2+(2n_{\max}-1)y^2)(x^2-y^2)^{n_{\max}-1}$ which shows that the Hamiltonian is bounded by below for negative $\kappa_{2n_{\max}}$ (and finite ψ). In fact it can further be shown (see below) that it is possible to find, for specific odd $n_{\max}$ and κ_{2n}, an everywhere positive Hamiltonian (the Hamitonian vanishing only at $(x=0, y=0)$). Let us now expand Π_0 around $x=\pm 1$ and $y=0$ corresponding to the domain wall solution. We find after some simple manipulations, and taking eq. (V.66) into consideration

$$\Pi_0(x,y) = \mp 2x\Sigma_{\kappa,0} + \Pi\left((x\mp 1), y^2\right), \tag{V.67}$$

where $\Pi(a,b)$ is a polynomial in a and b which vanishes in $(a=0, b=0)$ and starts only at order a^2 and b around this point. Thus, and exactly as in the case of the canonical kink, we can decompose $\mathcal{H}(t) = \mathcal{H}_{\text{kin}}(t) + \mathcal{H}_{\text{topo}}(t)$ with

$$\mathcal{H}_{\text{kin}}(t) = \int \mathrm{d}z \frac{\Pi\left((\psi'\mp 1), \dot{\psi}^2\right)}{2\cosh^4\psi}\,, \tag{V.68a}$$

$$\mathcal{H}_{\text{topo}}(t) = \mp\Sigma_{\kappa,0}\int \mathrm{d}z \frac{\psi'}{\cosh^4\psi}\,, \tag{V.68b}$$

where one sees that the last term is a “topological conserved charge” just identical (up to a constant factor) to the one of the canonical model (V.22). The above decomposition generalizes the one of Bogomolny in our context, and one can check that with the choice of non vanishing κ_{2n} given by $\kappa_0 = \kappa_2 = -1/2$ we find back exactly the form V.20. As Π vanishes for the kink configuration, we see that the whole energy of the kink is stored in the topological term only, just as for the canonical case. Note however that in contrast to the canonical mexican hat domain wall, no non-canonical kink (*ie.* whenever κ_0 vanishes) can be a global minimum of the energy within the class of configurations with the same $\mathcal{H}_{\text{topo}}$. Indeed a look at the the behavior of Π_0 around $(\pm 1, 0)$, taking into account the constraint (V.66), gives

$$\Pi_0(x,y) = \mp 2x\Sigma_{\kappa,0} - (x\mp 1)^2\left(4\Sigma_{\kappa,2} - \Sigma_{\kappa,0}\right) - y^2\Sigma_{\kappa,0} + \dots \tag{V.69}$$

where the ellipsis encapsulates terms that are at least cubic in $(x\mp 1)$ and y. Thus we see that $\Pi = \Pi_0 \pm 2x\Sigma_{\kappa,0}$, but Π_0 vanishes in $(x=0, y=0)$ with first corrections quadratic in x and y. This means that Π has to change sign across $(x=0, y=0)$ and must be somewhere negative, preventing the local minimum of Π at $x=\pm 1$, $y=0$ (where Π vanishes) to be a global minimum. The existence of such a global minimum and of its possible meaning for field configurations is of course an interesting, but difficult, question. In fact the total energy (say in field configurations with fixed $\mathcal{H}_{\text{topo}}$) does not only depends on x and y through Π but also on the value of the field ψ via the denominator $\cosh^{-4}\psi$. Hence, even if in some cases a global minimum of Π can be found (*eg.* for some models in the class to be discussed later where the Hamiltonian is bounded by below), the discussion above shows that Π is strictly negative there. It is not enough to conclude for what concerns the energy, after the taking into account of the $\cosh\psi^{-4}$ factor (in contrast, the local minimum of Π found for the wall configuration is a local minimum of the energy because Π vanishes there). Those interesting issues, which are of course related to the non perturbative decay of our walls, are left for a future work.

Nevertheless, the expansion (V.69) shows that the domain wall solutions represent a local minimum of the energy in the class of all field configurations having the same $\mathcal{H}_{\rm topo}$, provided that the quantities $\Sigma_{\kappa,0}$ and $\Sigma_{\kappa,2}$ verify

$$4\Sigma_{\kappa,2} < \Sigma_{\kappa,0} < 0\,. \tag{V.70}$$

Setting κ_2, κ_n and κ_m as in (V.55) for some specific even n and m, we can check that above conditions are equivalent to the conditions (V.53). Similarly the conditions that the $\{\kappa_n\}$ coefficients of the family (V.54) must obey are nothing but (V.66) and (V.70).

To discuss a more explicit case, let us consider the simplest model in the class (V.55), taking $n = 4$ and $m = 6$. Explicitly the Lagrangian of this model reads

$$P_{4,6}\left(\psi, X_\psi\right) = \frac{1}{\cosh^4\psi}\left(X_\psi + \frac{1-\kappa}{6}\left(-2X_\psi\right)^2 + \frac{\kappa}{10}\left(-2X_\psi\right)^3\right). \tag{V.71}$$

The bounds (V.53) are satisfied provided that $-5 < \kappa < -1$, which corresponds also to the allowed range for κ given in eq. (V.70). Moreover, in the line of the previous discussion on non-perturbative stability, one can show that restricting further κ to be larger than $-(17+3\sqrt{21})/10 \sim -3.07$ we get an everywhere positive Hamiltonian density. As further expected, we find in that case that Π vanishes at $(x = \pm 1, y = 0)$ which is a local minimum of Π, but Π is negative somewhere on the $y = 0$ line in the (x, y) plane and hence $(x = \pm 1, y = 0)$ is not a global minimum of Π.

V.3.4 The case of mimickers

Let us note that the kinetic matrix (V.50) and mass term (V.51) of the perturbations of our kinks are very similar to the ones of the canonical mexican hat configuration. More precisely, we see that, as $\mathcal{Z}^{zz}$ is constant and $\mathcal{M}^2 = \mathcal{Z}^{zz}\,\mathcal{M}^2_{\rm can}$ (the subscript "can" denoting naturally a quantity related to the perturbations of the canonical mexican hat kink), a perturbation of our wall obeys the same equation as a perturbation of the canonical wall

$$\left(\mathcal{Z}^{zz}_{\rm can}\,\varphi_k'\right)' - \left(\mathcal{Z}^{00}_{\rm can}\,\tilde{\omega}_k^2 + \mathcal{M}^2_{\rm can}\right)\varphi_k = 0\,, \tag{V.72}$$

where the frequency of each mode is simply given by the canonical one, multiplied by a constant factor

$$\tilde{\omega}_k^2 \equiv \frac{(n-2)(m-1)+(m-n)\kappa}{\left(2-n+(n-m)\kappa\right)(n-1)(m-1)}\,\omega^2_{k,\rm can}\,. \tag{V.73}$$

Thus, by setting

$$\kappa = \frac{n(m-1)(n-2)}{(n-m)(2-n+m(n-1))}\,, \tag{V.74}$$

which lies within the bounds (V.53), we get $\tilde{\omega}_k^2 = \omega^2_{k,\rm can}$ and so the theory has exactly the same spectrum as the canonical one. This corresponds explicitly to the family of Lagrangians

$$P_{\rm mim}(\phi, X) = X + \frac{1}{2(m-n)(2-n+m(n-1)}\left[\frac{m(m-2)\left(-2X\right)^{n/2}}{(1-\phi^2)^{n-2}} - \frac{n(n-2)\left(-2X\right)^{m/2}}{(1-\phi^2)^{m-2}}\right], \tag{V.75}$$

which have stable domain walls with profiles identical to the one of the canonical mexican hat, energy densities and kinetic matrices just rescaled by a common factor given by

$$\frac{(n-2)(m-2)}{2(2-n+m(n-1))} = 1 - \frac{mn}{2(mn-(m+n)+2)}\,. \tag{V.76}$$

In this class of models, that we will call here and henceforth mimickers, the simplest ones are obtained by choosing $(n, m) = (4, 6)$ and $\kappa = -5/4$ yielding the Lagrangian

$$\mathcal{L} = X + \frac{3\,X^2}{2\left(1-\phi^2\right)^2} + \frac{X^3}{\left(1-\phi^2\right)^4}\,, \tag{V.77}$$

which has a domain wall solution $\phi = \pm\tanh(z)$, a Hamiltonian everywhere positive (as seen in the previous section), and energy density and kinetic matrix just rescaled by a global factor $1/4$ with respect to the canonical ones.

If we can perfectly mimic the phenomenology of the canonical kinks at the quadratic order, can we push such imitation at higher orders ? The answer is necessarily negative as, within the class of field configurations with fixed boundary conditions, the non-canonical walls are local minima of the energy, whereas the canonical ones are global minima. As a consequence, one should be able to distinguish the two by looking at higher order perturbations, as we now show. Up to surface terms, for a generic theory (V.23), the third-order perturbed Lagrangian reads

$$\delta^{(3)}\mathcal{L} = -\frac{1}{3!}\left[\mathcal{Y}^{\mu\nu\rho}\,\partial_\mu\varphi\,\partial_\nu\varphi\,\partial_\rho\varphi - 3\mathcal{Y}^{\mu\nu}\varphi\,\partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{Y}\,\varphi^3\right], \tag{V.78}$$

where the different coefficients appearing above are given by

$$\mathcal{Y}^{\mu\nu\rho} = P_{XXX}\,\partial^\mu\phi\,\partial^\nu\phi\,\partial^\rho\phi - 3P_{XX}\,\eta^{\mu\nu}\,\partial^\rho\phi\,, \tag{V.79a}$$
$$\mathcal{Y}^{\mu\nu} = P_{XX\phi}\,\partial^\mu\phi\,\partial^\nu\phi - P_{X\phi}\,\eta^{\mu\nu}\,, \tag{V.79b}$$
$$\mathcal{Y} = -P_{\phi\phi\phi} - \partial_\mu\left(P_{X\phi\phi}\,\partial^\mu\phi\right)\,. \tag{V.79c}$$

For the canonical model (V.2), only $\mathcal{Y} = V_{\phi\phi\phi} = 12\phi$ is non-vanishing. For the $P_{n,m}$ models, as well as their subset mimickers, we present some relevant coefficients in the table V.1, with background given as usual by $\phi = \pm\tanh(z)$. One can notice that for all $P_{n,m}$ models, $\mathcal{Y}^{00} = -2\phi/3\,\mathcal{Y}^{00z}$

	Generic $P_{n,m}$	Mimicker $P_{n,m}$	Mimicker $P_{4,6}$
$\mathcal{Y}^{00z}$	$-\frac{3}{2}\left(\frac{n(n-2)(\kappa-1)}{n-1} - \frac{m(m-2)\kappa}{m-1}\right)\frac{1}{1-\phi^2}$	0	0
$\mathcal{Y}^{zzz}$	$\frac{n(n-2)(\kappa-1)-m(m-2)\kappa}{2}\frac{1}{1-\phi^2}$	$\frac{mn(n-2)(m-2)}{2(2-n+m(n-1))}\frac{1}{1-\phi^2}$	$\frac{6}{1-\phi^2}$
$\mathcal{Y}^{00}$	$\left(\frac{n(n-2)(\kappa-1)}{n-1} - \frac{m(m-2)\kappa}{m-1}\right)\frac{\phi}{1-\phi^2}$	0	0
$\mathcal{Y}^{zz}$	$-\left(n(n-2)(\kappa-1) - m(m-2)\kappa\right)\frac{\phi}{1-\phi^2}$	$-\frac{mn(n-2)(m-2)}{(2-n+m(n-1))}\frac{\phi}{1-\phi^2}$	$-\frac{12\phi}{1-\phi^2}$

Table V.1: Some relevant coefficients of the cubic vertices for the generic $P_{n,m}$ Lagrangians of eq. (V.47), for the subset of mimicker models (V.75), and for the specific choice $(n,m) = (4,6)$.

and $\mathcal{Y}^{zz} = -2\phi\,\mathcal{Y}^{zzz}$. Moreover, for the mimickers, all contributions containing time derivatives of the perturbations vanish at cubic order. However, the cubic interactions are found to diverge at large z, for which, for the domain wall profile, $1/(1-\phi^2)$ as well as $\phi/(1-\phi^2)$ diverge. Hence the perturbation theory in the ϕ variable diverges at large z off the wall. Note however, that as we have shown that the wall is a local minimum of the energy in the class of field configurations with fixed boundary conditions, one expects that there is a range of localized perturbations of the wall which are absolutely stable. To conclude, we also notice that one cannot mimic our models with a Lagrangian of the form $f(X) - V(\phi)$ as $\mathcal{Y}^{\mu\nu}$ would be vanishing. Note also that the generic properties of the perturbations found in this section using the ϕ variable (sound quadratic perturbations, off-the-wall strong coupling at cubic order) persist when written in terms of the variable ξ (once the quadratic perturbations are properly normalized).

V.3.5 An extended family of kinks

In addition to the hyperbolic tangential kink $\phi = \pm\tanh z$ (or $\psi = \pm z$) that has been discussed until now, our non-canonical theories (V.56) allow for other kink configurations. Indeed a generic theory described by (V.56) falls in the class of separable theories discussed and the expression of it first integral is given by (V.39)

$$\mathcal{J} = \left(\sum_{n\in\mathbb{N}} \kappa_n(n-1)(-2X_\psi)^{n/2}\right)\cosh^{-4}\psi\,. \tag{V.80}$$

Hence we see that we can get a domain wall solution $\psi = \lambda z$ provided that $\mathcal{J}$ vanishes and that λ verifies (using $-2X_\psi \approx \lambda^2$)

$$\sum_{k\in\mathbb{N}} \kappa_k(k-1)|\lambda|^k = 0\,. \tag{V.81}$$

For the canonical theory which has $\kappa_0 = \kappa_2 = -1/2$, the only roots of such polynomial are given by $\lambda = \pm 1$, and thus the usual kink is the only possible solution. However, considering non-canonical theories, it is possible that eq. (V.81) has other roots than ± 1. Such solutions are extremely appealing as, if they can coexist with the usual $\lambda = \pm 1$ one, and have a different total energy, non-perturbative decay can happen. For instance, taking the theory (V.71), a second set of roots are given by $\lambda = \pm 1/\sqrt{-\kappa}$. Nevertheless, for the case of the Lagrangians (V.55) with generic (n, m), it has been shown in the App. B of [165] that the two sets of solutions cannot be stable simultaneously: the solutions with $\lambda \neq \pm 1$ can be made stable only at the price of violating the bounds (V.53) on κ (which are in turn necessary for the stability of the solutions with $\lambda = \pm 1$). However, having more than three terms in the Lagrangian leads to the possibility to have more roots of the equation (V.81) and hence possibly more than one stable kink configuration.

Another possibility to extend the solutions discussed above is to let the walls move. In particular, using the ψ variable and considering for simplicity the models (V.55), it is easy to see that the part of the field equations that do not contain any second derivatives is in full generality proportional (and as a consequence of Lorentz invariance) to

$$X_\psi + \frac{1-\kappa}{2}(-2X_\psi)^{n/2} + \frac{\kappa}{2}(-2X_\psi)^{m/2}\,, \tag{V.82}$$

which for a static wall is in turn proportional to the expression of the first integral $\mathcal{J}$. This means that any static wall profile extends (including the case $\lambda \neq \pm 1$) to a boosted solution of the form

$$\phi_\beta(t,x) = \pm\tanh\left(\lambda\frac{z \pm \beta t}{\sqrt{1-\beta^2}}\right)\,, \tag{V.83}$$

where $\beta < 1$ is the dimensionless speed, and where one has $-2X_\psi = \psi'^2 - \dot{\psi}^2 \approx \lambda^2$.

V.3.6 Mimicking other canonical domain wall profiles

The above discussion and construction can easily be extended to other kinds of kink profiles such as the one of sine-Gordon or more generally the family of models (V.14). Indeed, considering Lagrangians of the form

$$\mathcal{L} = \left(\sum_{n\in\mathbb{N}} \kappa_n(-2X_\psi)^{n/2}\right)\cosh^{-2k}\psi. \tag{V.84}$$

$\psi = \lambda z$ is a solution as long as λ obeys (V.81) and moreover, the solution with $\lambda = \pm 1$ is stable provided conditions (V.66) and (V.70) hold. Turning back to the original ϕ variable, the corresponding Lagrangians are simply given by eq. (V.54), where the powers of $(1-\phi^2)$ are replaced by powers of $|\phi'|$, considered and expressed in terms of ϕ. It is then easy to get the corresponding domain wall profiles for the ϕ variable. In particular, for $k = 1$, the Lagrangians

$$\mathcal{L} = \sum_{n\geq 2} \kappa_{2n}\sin(\phi/2 - p\pi)^{2-2n}(-2X)^n \qquad \text{with} \qquad p \in \mathbb{Z}\,. \tag{V.85}$$

sustain stable domain wall profiles identical to the one of sine-Gordon model $\phi = 2p\pi \pm 4 \arctan e^z$, as long as the $\{\kappa_{2n}\}$ obey accurate relations. One feature of the sine-Gordon model is its integrability leading in particular to non trivial solutions such as *breathers* or kink-antikink (see *eg.* [329]). It would be interesting to investigate if remnants of such solutions still exist in the kind of models considered here.

V.4 Towards gravitating non-canonical domain walls

Before closing this chapter on non-canonical domain walls, we wish to discuss their possible gravitating configurations. In this section we will show that unfortunately, one cannot just "turn on" the gravitational interaction in the models considered previously. Further studies are thus required to construct gravitating domain walls that are not supported by potentials.

Note that we will not rely on approximations (such as the usual Israel's thin wall formalism [240] that treats the wall as an infinitesimally thin veil, characterized only by junction conditions at its surface, or the expansion in curvature performed in [120]), but rather analytically solve the exact set of equations of motion for the scalar field and metric.

V.4.1 Generalities about gravitating domain walls

By "turning on the gravitational interaction", we mean promoting the Minkowski metric $\eta_{\mu\nu}$ to a dynamical one, $g_{\mu\nu}$, with compatible covariant derivative ∇_μ. As two-dimensional metrics are all conformally flat, and thus of no interest for our purpose, we are forced to restore the transverse spatial directions (x, y). We thus consider the four-dimensional Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left\{\frac{M_{\rm Pl}^2}{2}\,\mathcal{R}\left[g\right] + P\left(\phi, X\right)\right\}, \qquad \text{with} \qquad X \equiv -\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\,. \tag{V.86}$$

By a trivial dilatation of P, we set $M_{\rm Pl} = 1$ in the following.

A well-known fact is that, for generic configurations, the scalar field can be treated as a perfect fluid with stress-energy tensor

$$T^{\mu\nu} \equiv \frac{-1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\,\partial^\nu\phi - \mathcal{L}\,g^{\mu\nu}\right) = P_X\,\partial^\mu\phi\,\partial^\nu\phi + P\,g^{\mu\nu}\,. \tag{V.87}$$

We will notably use such ansatz when dealing with the cosmology of the minimal theory of bigravity in sec. VI.3.2. The equations of motion for the metric and scalar field take the usual form of Einstein's and continuity equations

$$\mathcal{E}_{\mu\nu} = \mathcal{R}_{\mu\nu} - T_{\mu\nu} + \frac{g_{\mu\nu}}{2}\,g^{\alpha\beta}\,T_{\alpha\beta} \approx 0\,, \qquad \text{and} \qquad \mathcal{E}_\phi = \nabla_\mu T^{\mu\nu} \approx 0\,. \tag{V.88}$$

Specifying to the case of one-dimensional configurations for the scalar field $\phi(z)$, we follow [102] to implement an accurate ansatz for the metric. As usual the scalar field settles down to constant values at spatial infinities $z \to \pm\infty$, so, as long as g^{zz} and P_X are regular at spatial infinities, $\mathcal{R}_{\mu\nu} = -P(\phi_{\pm\infty}, 0)\,g_{\mu\nu}$ at the boundaries. Note that we cannot arbitrarily set $P(\phi_{\pm\infty}, 0) = 0$, as we have done in the previous sections, as the introduction of a dynamical metric has spoiled the freedom of shifting P. Therefore the metric has in general an asymptotical (anti)-de Sitter behavior, which induces the requirement of time dependencies in $g_{\mu\nu}$. As for the invariance of ϕ under translations along x and y, they are encoded in the metric by a planar (x, y) symmetry. As all configurations that we have considered in the previous section have energy densities that are invariant under $z \to -z$, it seems natural to impose the reflexivity of the metric along the z axis, thus killing g_{tz}. Finally, choosing z to be the proper distance from the wall, we can take

$$\mathrm{d}s^2 = -A^2(z)\mathrm{d}t^2 + B^2(t, z)\left(\mathrm{d}^2x + \mathrm{d}^2y\right) + \mathrm{d}z^2\,, \tag{V.89}$$

where A and B are yet arbitrary functions (the would-be time dependence of A can be reabsorbed in a redefinition $t \to t'(t)$ that does not spoil any other property of the metric). Note that by taking z to be the proper distance, we recover $X = -\phi'^2/2$.

But the metric (V.89) can be further simplified by considering combinations of the Einstein's equations that do not involve the scalar field. Injecting it in the cross term tz of Einstein's equations (V.88), it comes

$$\mathcal{E}_{tz} = 2\left(\frac{A'}{A}\frac{\dot{B}}{B} - \frac{\dot{B}'}{B}\right) = 0\,. \tag{V.90}$$

As A only depends on z, this imposes $B(t,z) = A(z)b(t)$, with b an arbitrary function of time. Next, the combination

$$\frac{B^2}{A^2}\,\mathcal{E}_{tt} + \mathcal{E}_{zz} = \dot{b}^2 - b\,\ddot{b} = 0\,, \tag{V.91}$$

leads to $b = b_0^2\,e^{\omega t}$, where b_0 and ω are constant. By rescaling $(x, y) \to b_0^{-1}(x, y)$, the constant b_0 can be set to unity. This gives the final form of the metric

$$\mathrm{d}s^2 = A^2(z)\left[-\mathrm{d}t^2 + e^{\omega t}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\right] + \mathrm{d}z^2\,. \tag{V.92}$$

V.4.2 Practical implementation for the models previously investigated

Injecting the metric (V.92) in the Einstein's and continuity equations (V.88), they can be combined to yield only two independent combinations: an evolution equation and a constraint one, that respectively read

$$\mathcal{E}_{\text{evol}} \equiv \frac{A''}{A} - \frac{\rho + 3P}{6} \approx 0\,, \tag{V.93a}$$

$$\mathcal{E}_{\text{constr}} \equiv \left(\frac{A'}{A}\right)^2 - \frac{\omega^2}{A^2} + \rho \approx 0\,, \tag{V.93b}$$

where we have introduced the equivalent of the energy density $\rho \equiv 2XP_X - P$, which is nothing but the covariant version of the first integral (V.25). As the kinetic term simply reduces to $X = -\phi'^2/2$, we even have the equality $\rho = \mathcal{J}$.

All the models investigated in the previous sections were constructed to satisfy $\mathcal{J} \approx 0$. Plugging $\rho = 0$ in eqs. (V.93b) simply yields $A(z) = A_0 \pm \omega z$, with A_0 a constant, and thus eq. (V.93a) implies $P \approx 0$, *ie.* the metric has a Minkowskian asymptotical behavior. In order to be a solution of the gravitating equations of motion, the domain wall has thus to violate the condition (V.35b). Therefore none of the models considered in the flat space-time case can be implemented as a gravitating domain wall configuration.

Chapter VI

Minimalism as a guideline to construct alternative theories of gravitation

As briefly presented in sec. I.2.2, the landscape of alternative theories of gravitation is quite vast, and the possibilities of constructing frameworks "beyond" General Relativity are multiple. Classifying all possible alternatives would be a tremendously difficult (if not impossible) task. Therefore when aiming at constructing new theories, it may help to impose supplementary theoretical requirements, in addition to the breaking of one of Lovelock's hypothesis.

This chapter introduces one of those additional "aesthetic" guidelines, namely the framework of *minimalism*, and presents the construction of one of its explicit realizations, the "minimal theory of bigravity", which is one of the main achievements of this thesis. The next chapter discusses the strong-field regimes of two other minimal theories, by constructing and testing black hole solutions.

VI.1 The principle of minimalism

As discussed in sec. I.2.2, the most explored path to break Lovelock's hypothesis is by implementing additional degrees of freedom (*dofs*). Even when starting by breaking another hypothesis, additional *dofs* can enter the game (*eg.* in the case of $f(\mathcal{R})$ theories). Those additional modes usually have appealing phenomenological implications (*eg.* a natural explanation for the late-time acceleration of the expansion of the Universe, in the case of some scalar-tensor theories and massive gravities). Nevertheless, the gravitational radiation that should betray their presence is still undetected (*cf.* sec. I.2.3). Such additional *dofs* are therefore *convenient* ways to address some of our current problems, but they are not *necessary* from an observational point of view.

The basis of the *minimalistic* approach is thus to seek theories that retain the interesting phenomenological implications of those additional *dofs*, but propagate only the minimal number of *dofs* that are required by observations. Such approach can be related to the philosophical notion of Occam's razor [298], stating[1] that «entities should not be multiplied without necessity».

This approach has been initiated in 2015 by the work of A. De Felice and S. Mukohyama [137], who constructed the "minimal theory of massive gravity", by "minimalizing" the canonical ghost-free massive gravity [160]. By adding appropriate constraints at the Hamiltonian level, they killed the scalar and vector modes, but retained the interesting cosmological implication of massive gravity, namely a natural explanation of the accelerated expansion of our Universe [137, 138]. Nevertheless, in order to contain a viable cosmology, this theory needs an *ad hoc* and non-trivial time dependence of the fiducial metric. In order to bypass this unaesthetic and non-dynamical time evolution, the "minimal theory of massive gravity" has been extended to a "minimal theory of quasidilaton massive gravity" [146, 147, 152], where the time dependence is realized by an additional scalar field.

[1] In its ontological version, *Pluralitas non est ponenda sine necessitate.*

In addition to this original procedure of "minimalization" of previously existing theories, minimal theories can also be generated *per se*, by imposing the sole requirement that they propagate two healthy tensorial *dofs*, and are not GR. This approach was initiated in [269] and extended in [119, 267, 268, 289], and it became clear that those theories are separable in two classes. The *type-I* minimal theories are those that posses an Einstein frame, meaning that they are (at least locally) equivalent to GR in vaccum. Thus a type-I minimal theory only differs from GR by a non canonical matter coupling, and all of them can be generated by following the prescription of [23]. On the other hand, *type-II* theories are those that lack Einstein frames, and no systematic construction method is yet known, but some particular examples have been built [25, 140, 153, 289, 363]. Note that the original minimal theory of massive gravity belongs to this second type. In the the following of this dissertation, we will only study type-II theories.

If all those theories share the interesting feature of a natural explanation for the accelerated expansion of our Universe (and some can even mimic the dark matter phenomenology [140]), they cannot be different from GR while propagating only two tensorial *dofs*, without breaking at least one of the hypothesis of Lovelock's theorem (*cf.* sec. I.2.2). In fact all minimal theories implement a breaking of the invariance under full diffeomorphisms, down to a restricted subset of them. Note however that this breaking only happens in the gravitational sector, the induced Lorentz violations in the matter sector are exclusively due to graviton loops, thus heavily suppressed. Moreover, and for many of the minimal theories (notably all those that are presented in this dissertation), this breaking can be made extremely weak and only effective at cosmological scales: such theories pass all current local experimental constraints.

The phenomenological tests and successes of the minimal theories are reviewed in the beginning of the next chapter.

VI.2 Construction of a minimal theory of bigravity

In this section we will present the "minimalization" of Hassan-Rosen bigravity (HRBG) [221], and seek for a "minimal theory of bigravity" (MTBG) propagating only four tensorial *dofs* (*ie.* two for each metric). This task has been achieved in collaboration with A. De Felice, S. Mukohyama and M. Oliosi, and published in [155]. The advantages of minimalizing HRBG are twofold: firstly we will seek to retain its extremely appealing phenomenological features that are to be discussed hereafter (namely a natural dark energy paradigm and oscillations of the gravitons); secondly, by promoting the fiducial metric to be dynamical, we avoid the *ad hoc* and unaesthetical time evolution of the fiducial metric, that was necessary in the original minimal theory of massive gravity.

VI.2.1 A brief review of Hassan-Rosen bigravity

Construction of the theory

Hassan-Rosen bigravity [221] is a theory of two interacting dynamical metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$. The kinetic sector of each metric is given by an usual Einstein-Hilbert term, thus the theory in vacuum reads

$$\mathcal{S}_{\mathrm{HRBG}} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\mathcal{R}\left[g\right] + \frac{\alpha^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-f}\mathcal{R}\left[f\right] - \frac{m^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\mathcal{V}\left[g,f\right]\,, \tag{VI.1}$$

where α is the ratio of Planck masses in both sectors and m is a coupling constant. If not carefully crafted, the interaction term $\mathcal{V}$ will give rise to a ghost scalar *dof*. In fact, as presented in app. A.3, the Einstein-Hilbert term contains four first-class constraints, with associated Lagrange multipliers given by lapse and shifts. Adding a generic potential $\mathcal{V}[g,f]$ does not change the number of dynamical variables (that are still given by the spatial metrics and their conjugate momenta) but removes the constraint structure of the Hamiltonian, as there is no reason that the lapses and shifts still enter linearly. Therefore the metric $g_{\mu\nu}$ propagates in general six *dofs*, among which one has a wrong sign kinetic term[2] thus being a ghost, usually called the "Boulware-Deser" (BD) ghost [107]. In order to kill this sick *dof*, one has to choose $\mathcal{V}$ so that the Hamiltonian contains either one first-class constraint, or two second-class constraints.

Seeking for a healthy theory of massive gravity,[3] C. de Rham, G. Gabadadze and A. Tolley have derived the accurate structure of the interaction term [160]. This potential only depends on five coupling constants $\{\beta_n\}_{n=0..4}$, and the two metrics enter it *via* the so-called *square-root* matrix $\mathbb{X}^{\mu}{}_{\nu}$ defined by

$$\mathbb{X}^{\mu}{}_{\rho}\mathbb{X}^{\rho}{}_{\nu} \equiv g^{\mu\lambda} f_{\lambda\nu} \,. \tag{VI.2}$$

Introducing the elementary symmetric polynomials[4] e_n (where the brackets denote the trace operation)

$$e_0\left(\mathbb{X}\right) = 1\,, \tag{VI.3a}$$
$$e_1\left(\mathbb{X}\right) = \left[\mathbb{X}\right]\,, \tag{VI.3b}$$
$$e_2\left(\mathbb{X}\right) = \frac{1}{2}\left(\left[\mathbb{X}\right]^2 - \left[\mathbb{X}^2\right]\right)\,, \tag{VI.3c}$$
$$e_3\left(\mathbb{X}\right) = \frac{1}{3!}\left(\left[\mathbb{X}\right]^3 - 3\left[\mathbb{X}\right]\left[\mathbb{X}^2\right] + 2\left[\mathbb{X}^3\right]\right)\,, \tag{VI.3d}$$
$$e_4\left(\mathbb{X}\right) = \frac{1}{4!}\left(\left[\mathbb{X}\right]^4 - 6\left[\mathbb{X}^2\right]\left[\mathbb{X}\right]^2 + 3\left[\mathbb{X}^2\right]^2 + 8\left[\mathbb{X}^3\right]\left[\mathbb{X}\right] - 6\left[\mathbb{X}^4\right]\right)\,, \tag{VI.3e}$$

the ghost-free, or "dRGT", interaction term[5] reads

$$\mathcal{V}_{\text{dRGT}}\left[g,f\right] = \sqrt{-g}\sum_{n=0}^{4}\beta_n\, e_n\left(\mathbb{X}\right)\,. \tag{VI.4}$$

Using the relation $\sqrt{-f}\, e_n\left(\mathbb{X}^{-1}\right) = \sqrt{-g}\, e_{4-n}\left(\mathbb{X}\right)$, we see that $\mathcal{V}_{\text{dRGT}}$ is symmetrical under the exchange $\{g \leftrightarrow f, \beta_n \leftrightarrow \beta_{4-n}\}$, and thus the whole theory (VI.1) is symmetrical under the exchange of both metrics together with a relabeling of the constants: the theory has no "preferred metric", as expected for a bigravity theory in vacuum. Naturally, one of the coupling constants is redundant as it can be reabsorbed in the mass parameter m appearing in (VI.1). For clarity, and to make the $f \leftrightarrow g$ symmetry explicit, we will keep it as such (a similar redundancy appears in our minimal theory and, for the same reason, we will not fix it). The specificity of this interaction term is to contain a particular combination of lapses and shifts that enters linearly in the Hamiltonian, thus giving birth to a second-class constraint. The time conservation of this constraint induces a secondary constraint and together, they kill the BD ghost.

This bimetric theory contains seven *dofs*, distributed into a massless graviton (two *dofs*) and a massive one (five *dofs*). Note that the mass eigenstates and the kinetic ones are different, leading to an oscillating behavior of the graviton, similar to the usual oscillation of neutrinos [202].

When adding matter to HRBG, the story becomes more involving. The simplest and most natural implementation of two matter species, each one being minimally coupled to a metric, is healthy [221]. But when coupling a matter field to both metrics, the BD ghost usually reappears, excepted for some

[2]This can be related to the usual result of field theory stating that a massive spin-2 field has only five polarizations.

[3]In a theory of "massive gravity" only one metric propagates. But in order to build up a mass term, as $g_{\mu\nu}g^{\mu\nu} = 4$ cannot yield any mass effects, a second, frozen, metric is required. The action of massive gravity is thus given by eq. (VI.1) with $\alpha = 0$, see [158] for more details.

[4]For a diagonizable matrix A, $e_n(A)$ is simply given by the sum of n-by-n products of the eigenvalues of A, and they obey $\det\left[\text{Id} + \lambda A\right] = \sum_n \lambda^n e_n\left(A\right)$.

[5]Here and henceforth, we dub "ghost-free" a theory that is free of the BD ghost, but can contain other types of instabilities.

very specific forms of the couplings [362]. Nevertheless, for a large class of couplings to a composite metric (that is, a combination of both metrics), the BD ghost appears only out of the range of validity of the theory (*ie.* beyond the strong-coupling scale) [145, 161, 162, 184, 218], and thus such couplings can be considered as relatively healthy.

Note that, once matter is considered, the bimetric theory (VI.1) implemented with the ghost-free potential (VI.4) has preferred frame effects. Indeed observers couple to only one metric (or to a combination of both metrics), and so the value of the other metric (or the orthogonal combination of the metrics) plays the role of a "reference metric", inducing preferred frame effects. This is deeply related to the fact that HRBG is not invariant under the two copies of full diffeomorphisms, but only under joint diffeomorphisms (*ie.* diffeormorphisms that affect both metrics in the exact same way). In order to restore this invariance, the usual trick is to introduce a set of four scalars, σ^a, called Stückelberg fields [333], and to replace one of the metrics, say $f_{\mu\nu}$, by

$$f^{ab} \to f^{\mu\nu}\, \partial_\mu \sigma^a\, \partial_\nu \sigma^b \,. \tag{VI.5}$$

Under a change of coordinates, the Stückelberg fields behave in such a way that they absorb the variation of the metric $f_{\mu\nu}$. Therefore the theory becomes invariant under the two copies of diffeomorphisms, at the cost of introducing new fields.

The cosmology of HRBG

Contrarily to the dRGT massive gravity that lacks stable homogeneous and isotropic cosmologies [130, 142], HRBG contains the usual cosmological configuration. Taking two Friedmann-Lemaître-Robertson-Walker (FLRW) line elements, with respective lapses N and M and scale factors a and b,

$$\begin{aligned} \mathrm{d}s_g^2 &= g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2(t)\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j \,, \\ \mathrm{d}s_f^2 &= f_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = -M^2(t)\mathrm{d}t^2 + b^2(t)\gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j \,, \end{aligned} \tag{VI.6}$$

the Bianchi identity is factorizable as

$$\left(\frac{\partial_t a}{N} - \frac{\partial_t b}{M}\right)\mathcal{B} = 0 \,, \tag{VI.7}$$

where $\mathcal{B}$ is a complicated function of the scale factors and coefficients β_n. This factorization indicates the presence of two branches of solution [17, 126, 144, 273]. The first one, corresponding to $\mathcal{B} = 0$, gives rise to a natural cosmological constant, proportional to m^2. Nevertheless, it is the equivalent of the cosmological solution of the ghost-free massive gravity and, as such, is unstable. As for the second branch, defined by $\partial_t a/N = \partial_t b/M$, it contains two "dark fluids", that come as natural explanations for the late-time acceleration of the expansion of the Universe. By dark fluid, we mean that the contribution of the mass sector can be put in the form of a canonical stress-energy tensor for a perfect fluid. It is thus an unobservable cosmological contribution mimicking a fluid, hence the name. Such dark fluid can naturally source an accelerated expansion of the Universe. But this branch also suffers from a gradient-type instability [127, 255].

A possible way to cure the gradient instabilities is by adequately tune the ratio of Planck masses in such a way that they occur at unobservably early times [17]. Similarly, under a careful choice of the coefficients β_n, the instability of the normal branch can be pushed away from the range of validity of the theory (beyond the strong coupling scale) [143]. Those methods are nevertheless quite harsh, and we would prefer to have a theory that contains stable cosmological solutions without too much tuning of the coupling constants.

Possible extensions of HRBG

A another way to cure those instabilities is by adding a *chameleonic* scalar field χ [149, 150]. This approach is inspired by the Vainhstein screening mechanism, and considers matter fields minimally coupled to the conformal metric $A^2(\chi)g_{\mu\nu}$, instead of $g_{\mu\nu}$, in addition to promoting the β_n to functions of χ. The function $A(\chi)$ is chosen so that the effective mass of the graviton scales as the energy density of the medium, and so the behavior of the field depends on its environment, hence the name [253].

Another possible extension of HRBG is given by the "Dipolar Dark Matter" framework, that attempts to unify the descriptions of dark matter and dark energy phenomenologies within a single theory [52, 53, 78]. As expected, the modeling of the late-time acceleration of the expansion of the Universe relies on the cosmology of HRBG, as discussed above (the instability being pushed beyond the decoupling limit [78]). The dark matter sector is implemented *via* the MONDian dielectric analogy [72, 81]. For a non-relativistic body of mass m_0 experiencing an acceleration $\vec{a}$, the MONDian phenomenological law states that Newton's law $\vec{F} = m_0\,\vec{a}$ should be modified to

$$\vec{F} = m_0\,\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a}\,, \qquad \text{where} \qquad \mu(x \gg 1) = 1 \quad \text{and} \quad \mu(x \ll 1) = x\,. \tag{VI.8}$$

The a_0 constant is observationally fixed to $\sim 10^{-10}\text{m.s}^{-2}$ Note that the interpolating function μ has the same behavior as the permittivity of a dielectric medium, and thus Newtonian gravity is recovered for accelerations greater than a_0. Therefore, if one can mimic the gravitational interaction as being immersed in a dielectric-like medium, the MONDian phenomenology can be inherently recovered. The point of using a bimetric theory is that it naturally provides a polarization-like mechanism: identifying the particles living in the g-sector as "positive" charges and those coupled to f-sector as "negative" ones,[6] and linking them *via* a "graviphoton",[7] we have a consistent dielectric analogy. This Dipolar Dark Matter framework thus offers an elegant extension of HRBG that implements also dark matter phenomenology. But, if this theory and its achievements seem really appealing, a typo in the field equations of its most recent version [78] spoils the whole polarization mechanism [80].

VI.2.2 The precursor theory

The procedure to "minimalize" a given theory proceeds by two steps: firstly, one builds a *precursor* theory that explicitly breaks the full diffeomorphisms invariance. Next, one adds appropriate constraints to the precursor theory to kill the adequate *dofs*.

We will build MTBG in vacuum, and discuss the issues related to inclusion of matter at the very end of sec. VI.2.3.

Building blocks

In order to explicitly break the diffeomorphisms invariance, we write the two metrics in their ADM forms, *cf.* app. A.3

$$\begin{aligned} \mathrm{d}s_g^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)\,, \\ \mathrm{d}s_f^2 &= f_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = -M^2\mathrm{d}t^2 + \phi_{ij}\left(\mathrm{d}x^i + M^i\mathrm{d}t\right)\left(\mathrm{d}x^j + M^j\mathrm{d}t\right)\,. \end{aligned} \tag{VI.9}$$

Note that we could have covariantized our theory by the introduction of four Stückelberg fields σ^a in the f-sector. This would be done by promoting the corresponding lapse, shifts and spatial metric to

$$\mathbf{M} = \frac{1}{\sqrt{-f^{\mu\nu}\partial_\mu\sigma^0\partial_\nu\sigma^0}}\,, \qquad \mathbf{M}^i = \mathbf{M}^2\,f^{\mu\nu}\partial_\mu\sigma^0\partial_\nu\sigma^i\,, \quad \text{and} \quad \boldsymbol{\phi}^{ij} = f^{\mu\nu}\partial_\mu\sigma^i\partial_\nu\sigma^j + \frac{\mathbf{M}^i\mathbf{M}^j}{\mathbf{M}^2}\,, \tag{VI.10}$$

[6] Note that we do not introduce negative masses, but rather a new charge.

[7] *Ie.* a massless vector field that couples adequately to the two metrics so that the instability lies beyond the decoupling limit [78].

so that eq. (VI.5) holds. Nevertheless, for the sake of simplicity, we will work in the so-called "unitary gauge", *ie.* setting the Stückelberg fields as $\sigma^a \equiv x^a$, which is sufficient for our purpose: constructing the theory and deriving its cosmological phenomenology.

Let us introduce the spatial covariant derivatives compatible with γ_{ij} and ϕ_{ij} as respectively $\mathcal{D}_i$ and $\mathfrak{D}_i$, and the two extrinsic curvatures

$$K_{ij} \equiv \frac{1}{2N}\left(\partial_t \gamma_{ij} - 2\mathcal{D}_{(i}N_{j)}\right) , \qquad \text{and} \qquad \Phi_{ij} \equiv \frac{1}{2M}\left(\partial_t \phi_{ij} - 2\mathfrak{D}_{(i}M_{j)}\right) . \tag{VI.11}$$

Here and henceforth, we operate the g-related quantities (N^i, K_{ij},...) with γ_{ij}, and the f-related ones with ϕ_{ij} without more precision, the two sectors being well enough separated so that no confusion can arise.

We will use two usual Einstein-Hilbert terms for the kinetic sector of our theory. As for the interaction term, we mimic the construction that led to HRBG by defining two square-root matrices, $\mathfrak{K}^p{}_q$ and $\mathcal{K}^p{}_q$, inverse to each other, as

$$\mathfrak{K}^p{}_s\mathfrak{K}^s{}_q = \gamma^{pr}\phi_{rq} , \qquad \mathcal{K}_q{}^s\mathcal{K}_s{}^p = \gamma_{qr}\phi^{rp} , \qquad \text{and} \qquad \mathfrak{K}^p{}_s\mathcal{K}_q{}^s = \mathfrak{K}_q{}^s\mathcal{K}_s{}^p = \delta^p_q . \tag{VI.12}$$

The non-vanishing elementary symmetric polynomials of the square-root matrix $\mathfrak{K}^p{}_q$ are given by

$$e_0(\mathfrak{K}) = 1 , \qquad e_1(\mathfrak{K}) = [\mathfrak{K}] , \qquad e_2(\mathfrak{K}) = \frac{1}{2}\left([\mathfrak{K}]^2 - \left[\mathfrak{K}^2\right]\right) , \qquad e_3(\mathfrak{K}) = \det[\mathfrak{K}] , \tag{VI.13}$$

and similarly for $e_n(\mathcal{K})$. From those building blocks, we define the action of the precursor theory

$$\mathcal{S}_{\text{pre}} = \frac{M_{\text{Pl}}^2}{2}\int \mathrm{d}^4x \left\{\sqrt{-g}\mathcal{R}[g] + \alpha^2\sqrt{-f}\mathcal{R}[f] - m^2\sum_{n=0}^{3}\left[N\sqrt{\gamma}\, c_{4-n}e_n(\mathfrak{K}) + M\sqrt{\phi}\, c_n e_n(\mathcal{K})\right]\right\} , \tag{VI.14}$$

where α is still the ratio between the Planck masses of the two sectors, m and the set $\{c_n\}_{n=0..4}$ are coupling constants. Naturally there is a redundancy as one of the c_n can be absorbed in m^2, but we will keep it as it is for simplicity.

This theory is naturally symmetrical under the exchange of both metrics, together with a relabeling of the constants.

Another construction of the precursor theory

If the construction of the precursor theory (VI.14) by mimicking HRBG could seem somehow arbitrary, there exists another path leading to it. Let us introduce two sets of four-dimensional vielbeins $e^A_{\ \mu}$ and $E^A_{\ \mu}$ ($A = \{0..3\}$) so that

$$g_{\mu\nu} = \eta_{AB}\, e^A{}_\mu e^B{}_\nu , \qquad \text{and} \qquad f_{\mu\nu} = \eta_{AB}\, E^A{}_\mu E^B{}_\nu , \tag{VI.15}$$

together with their dual basis $e_A^{\ \mu}$ and $E_A^{\ \mu}$ obeying

$$e_A{}^\mu e^B{}_\mu = \delta^B_A , \qquad e_A{}^\mu e^A{}_\nu = \delta^\mu_\nu , \qquad E_A{}^\mu E^B{}_\mu = \delta^B_A , \qquad \text{and} \qquad E_A{}^\mu E^A{}_\nu = \delta^\mu_\nu . \tag{VI.16}$$

In such language, the ghost-free interaction term (VI.4) reads [158]

$$\begin{aligned}\mathcal{V}_{\text{dRGT}}[e,E] = \varepsilon_{ABCD}\,\varepsilon^{\mu\nu\rho\sigma}\big(&\bar\beta_0\, e^A{}_\mu e^B{}_\nu e^C{}_\rho e^D{}_\sigma + \bar\beta_1\, e^A{}_\mu e^B{}_\nu e^C{}_\rho E^D{}_\sigma + \bar\beta_2\, e^A{}_\mu e^B{}_\nu E^C{}_\rho E^D{}_\sigma \\ &+\bar\beta_3\, e^A{}_\mu E^B{}_\nu E^C{}_\rho E^D{}_\sigma + \bar\beta_4\, E^A{}_\mu E^B{}_\nu E^C{}_\rho E^D{}_\sigma\big) ,\end{aligned} \tag{VI.17}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the four-dimentional totally antisymetric Levi-Civita tensor and the $\{\bar\beta_n\}$ coefficients are linked to the $\{\beta_n\}$ of eq. (VI.4) by simple rescalings. Imposing the so-called ADM gauge for the vielbein [137]

$$e^A{}_\mu = \begin{pmatrix} N & 0_j \\ e^I{}_i N^i & e^I{}_j \end{pmatrix} , \qquad \text{and} \qquad E^A{}_\mu = \begin{pmatrix} M & 0_j \\ E^I{}_i M^i & E^I{}_j \end{pmatrix} , \tag{VI.18}$$

with the two sets of spatial vielbeins ${e^I}_i$ and ${E^I}_i$ ($I = \{1..3\}$), the two metrics take the form (VI.9). The potential (VI.17) becomes

$$\begin{aligned}\mathcal{V}_{\rm dRGT}\left[e,E\right] = &-2N \det\left[{e^I}_j\right]\left(12\bar\beta_0 + 3\bar\beta_1 {X_I}^I + \bar\beta_2\left({X_I}^I{X_J}^J - {X_I}^J{X_J}^I\right) + 3\bar\beta_3 \det\left[{X_I}^J\right]\right)\\ &-2M \det\left[{E^I}_j\right]\left(12\bar\beta_4 + 3\bar\beta_3 {Y_I}^I + \bar\beta_2\left({Y_I}^I{Y_J}^J - {Y_I}^J{Y_J}^I\right) + 3\bar\beta_1 \det\left[{Y_I}^J\right]\right)\,,\end{aligned} \tag{VI.19}$$

where we have introduced the quantities

$${X_I}^J = {e_I}^i {E^J}_i\,, \qquad \text{and} \qquad {Y_I}^J = {E_I}^i {e^J}_i\,. \tag{VI.20}$$

The symmetry of the extrinsic curvature K_{ij} induces the relation $\delta^{AC}{X_C}^B = \delta^{BC}{X_C}^A$, as shown in [138], and naturally a similar relation holds for ${Y_B}^A$. Thus the spatial square-root matrices (VI.12) are expressed in terms of the spatial vielbeins as

$${\mathfrak{K}^p}_q = {e_I}^p {E^I}_q\,, \qquad \text{and} \qquad {\mathcal{K}^p}_q = {E_I}^p {e^I}_q\,. \tag{VI.21}$$

It is then easy to see that the matrices ${X_I}^J$ and ${Y_I}^J$ play a similar role as the square-root matrices, namely that $e_n(X) = e_n(\mathfrak{K})$ and similarly $e_n(Y) = e_n(\mathcal{K})$. Defining $\binom{4}{n}c_n = -24\bar\beta_{4-n}$, we recover the precusor Lagrangian (VI.14). Therefore the precusor theory is nothing but the usual HRBG gauged in a specific manner.

Counting the number of *dofs* propagating in the precursor theory

Once the precursor theory (VI.14) is constructed, one has to investigate how many *dofs* it propagates in order to add the accurate number of constraints. To perform such a counting, we will follow the usual Dirac procedure [170] (presented for GR in app. A.3).

As clear from eq. (VI.14), the lapses enter linearly the mass term, and the shifts do not enter it: the Hamiltonian is thus linear in lapses and shifts. Therefore they can be simply treated as Lagrange multipliers, and we will not associate conjugate momenta to them. We thus start with a twenty-four dimensional phase space spanned by the spatial metrics and their conjugate momenta, defined as usual:

$$\pi^{ij} \equiv \frac{\delta \mathcal{S}_{\rm pre}}{\delta \partial_t \gamma_{ij}} = \frac{M_{\rm Pl}^2}{2}\sqrt{\gamma}\left(K^{ij} - K\gamma^{ij}\right)\,, \quad \text{and} \quad \sigma^{ij} \equiv \frac{\delta \mathcal{S}_{\rm pre}}{\delta \partial_t \phi_{ij}} = \frac{\alpha^2 M_{\rm Pl}^2}{2}\sqrt{\phi}\left(\Phi^{ij} - \Phi\phi^{ij}\right)\,. \tag{VI.22}$$

A Legendre transformation yields the primary Hamiltonian of the precursor theory

$$H^{(1)}_{\rm pre} = -\int {\rm d}^3x\left(N\mathcal{R}_0 + N^i\mathcal{R}_i + M\tilde{\mathcal{R}}_0 + M^i\tilde{\mathcal{R}}_i\right)\,. \tag{VI.23}$$

Note the striking similarity with the Hamiltonian of GR (A.22): both are linear combinations of constraints. In the case of the precursor theory, those constraints read

$$\mathcal{R}_0 = \mathcal{R}_0^{\rm GR} - \frac{m^2 M_{\rm Pl}^2}{2}\sqrt{\gamma}\,\mathcal{H}_0\,, \qquad \tilde{\mathcal{R}}_0 = \tilde{\mathcal{R}}_0^{\rm GR} - \frac{m^2 M_{\rm Pl}^2}{2}\sqrt{\phi}\,\tilde{\mathcal{H}}_0\,, \tag{VI.24a}$$

$$\mathcal{R}_i = 2\sqrt{\gamma}\gamma_{ij}\mathcal{D}_k\left(\frac{\pi^{jk}}{\sqrt{\gamma}}\right)\,, \qquad \tilde{\mathcal{R}}_i = 2\sqrt{\phi}\phi_{ij}\mathfrak{D}_k\left(\frac{\sigma^{jk}}{\sqrt{\phi}}\right)\,, \tag{VI.24b}$$

$$\mathcal{H}_0 = \sum_{n=0}^{3} c_{4-n} e_n\left(\mathfrak{K}\right)\,, \qquad \tilde{\mathcal{H}}_0 = \sum_{n=0}^{3} c_n e_n\left(\mathcal{K}\right)\,, \tag{VI.24c}$$

where $\mathcal{R}_0^{\rm GR}$ is directly given by eq. (A.23a) and $\tilde{\mathcal{R}}_0^{\rm GR}$ is given by eq. (A.23a) under the expected replacements $\{\gamma_{ij}, \pi^{ij}, M_{\rm Pl}\} \to \{\phi_{ij}, \sigma^{ij}, \alpha M_{\rm Pl}\}$. Note that the momentum constraints $\mathcal{R}_i$ and $\tilde{\mathcal{R}}_i$ are exactly the same as in GR, eq. (A.23b).

At this stage, we need to discriminate first-class constraints from second-class ones, by computing the algebra of Poisson brackets (as explained in app. A.3, we use distributional forms for the constraints). In order to greatly simplify the algebra, we eliminate $\tilde{\mathcal{R}}_i$ in favor of the three combinations $\mathrm{R}_i \equiv \mathcal{R}_i + \tilde{\mathcal{R}}_i$, that are first-class constraints, as proven in the app. B of [155] (as the detailed computation is quite arid and not really enlightening, we will not detail it in this dissertation). These three first-class constraints are generators of the joint spatial diffeomorphisms, under which both the precursor theory and MTBG are invariant. This point is worth emphasizing, as understanding that those constraints are first-class in the precursor theory, and remain first-class when the precursor theory is promoted to MTBG (as presented in the next section), was a crucial step towards an accurate construction of MTBG. As it is not very enlightening, we will not present the detailed computation of the Poisson brackets in this dissertation, but let the interested reader refer to the app. A of [155]. The only non-vanishing Poisson brackets are

$$\{\mathcal{R}_0, \mathcal{R}_i\} \approx \mathcal{A}_i \,, \qquad \left\{\mathcal{R}_0, \tilde{\mathcal{R}}_0\right\} \approx \mathcal{B}_0 \,, \qquad \left\{\tilde{\mathcal{R}}_0, \mathcal{R}_i\right\} \approx \mathcal{B}_i \,, \tag{VI.25}$$

where we have introduced (with the shorthands $\pi \equiv \gamma_{ij}\pi^{ij}$ and $\sigma \equiv \phi_{ij}\sigma^{ij}$)

$$\mathcal{A}_i\left[\xi^i\right] = \frac{m^2 M_{\rm Pl}^2}{4}\int \mathrm{d}^3y\sqrt{\gamma}\sum_{n=1}^{3} c_{5-n}\left(U^i_{(n)\,j} + \gamma^{iq}\gamma_{jp}\,U^p_{(n)\,q}\right)\mathcal{D}_i\xi^j - \frac{m^2 M_{\rm Pl}^2}{2}\int \mathrm{d}^3y\sqrt{\gamma}\mathcal{D}_i\left(\mathcal{H}_0\xi^i\right) \,, \tag{VI.26a}$$

$$\mathcal{B}_0 = -m^2\sum_{n=1}^{4}\left[c_{4-n}\left(\pi^i_j - \frac{\pi}{2}\delta^i_j\right)U^j_{(n)\,i} - \frac{c_n}{\alpha^2}\left(\sigma^i_j - \frac{\sigma}{2}\delta^i_j\right)V^j_{(n)\,i}\right] \,, \tag{VI.26b}$$

$$\mathcal{B}_i\left[\xi^i\right] = \frac{m^2 M_{\rm Pl}^2}{4}\int \mathrm{d}^3y\sqrt{\gamma}\sum_{n=1}^{3} c_{4-n}\left(U^i_{(n)\,j} + \gamma^{iq}\gamma_{jp}\,U^p_{(n)\,q}\right)\mathcal{D}_i\xi^j \,, \tag{VI.26c}$$

where the derivatives of the elementary symmetric polynomials are given by

$$U^p_{(n)\,q} \equiv \frac{\partial e_n(\mathfrak{K})}{\partial \mathfrak{K}^q{}_p} = \sum_{\ell=0}^{n-1}(-)^\ell e_{n-1-\ell}(\mathfrak{K})\,\left(\mathfrak{K}^\ell\right)^p{}_q \,, \tag{VI.27}$$

and similarly for $V^p_{(n)\,q} \equiv \frac{\partial e_n(\mathcal{K})}{\partial \mathcal{K}^q{}_p}$. The time evolution of the constraints is thus given by

$$\dot{\mathcal{R}}_0 \equiv \left\{\mathcal{R}_0, H^{(1)}_{\rm pre}\right\} \approx -M\mathcal{B}_0 - \left(N^i - M^i\right)\mathcal{A}_i \,, \tag{VI.28a}$$

$$\dot{\tilde{\mathcal{R}}}_0 \equiv \left\{\tilde{\mathcal{R}}_0, H^{(1)}_{\rm pre}\right\} \approx N\mathcal{B}_0 - \left(N^i - M^i\right)\mathcal{B}_i \,, \tag{VI.28b}$$

$$\dot{\mathcal{R}}_i \equiv \left\{\mathcal{R}_i, H^{(1)}_{\rm pre}\right\} \approx N\mathcal{A}_i + M\mathcal{B}_i \,, \tag{VI.28c}$$

$$\dot{\mathrm{R}}_i \equiv \left\{\mathrm{R}_i, H^{(1)}_{\rm pre}\right\} \approx 0 \,. \tag{VI.28d}$$

The conservation in time of the constraints imposes that each of the above equations has to vanish. It is easy to see that this system of eight equations is of rank four, as the non-trivial combination $N\dot{\mathcal{R}}_0 + M\dot{\tilde{\mathcal{R}}}_0 + (N^i - M^i)\,\dot{\mathcal{R}}_i$ trivially vanishes, thus the presence of a fourth first-class constraint, that allows to discard one of the $\dot{\mathcal{R}}_i \approx 0$ constraints. As $\dot{\mathcal{R}}_0 \approx 0$ and $\dot{\tilde{\mathcal{R}}}_0 \approx 0$ can be used to fix M and N in terms of $N^i - M^i$, the remaining constraints are $\dot{\mathcal{R}}_i \propto (N^j - M^j)(\mathcal{A}_i\mathcal{B}_j - \mathcal{A}_j\mathcal{B}_i)$. This gives thus rise to two secondary constraints, that we will denote $\mathcal{C}_{\tau=1,2}$. Note that, as neither $\mathcal{A}_i$ nor $\mathcal{B}_i$ depend on the conjugate momenta, the secondary constraints $\mathcal{C}_\tau$ do not involve π^{ij} nor σ^{ij}. The secondary Hamiltonian of the precursor theory is thus given by

$$H^{(2)}_{\rm pre} = -\int \mathrm{d}^3x\left[N\mathcal{R}_0 + M\tilde{\mathcal{R}}_0 + \left(N^i - M^i\right)\mathcal{R}_i + M^i\mathrm{R}_i + \lambda^\tau\mathcal{C}_\tau\right] \,, \tag{VI.29}$$

and its constraint algebra is displayed in table VI.1. As it is straightforward to prove by use of Jacobi identities that the R_i weakly commute with $\{\mathcal{R}_0, \mathcal{R}_i\}$ and $\{\tilde{\mathcal{R}}_0, \mathcal{R}_i\}$, the Poisson brackets $\{\mathrm{R}_i, \mathcal{C}_\tau\}$ are also vanishing (meaning that the R_i remain first-class). In addition, as $\mathcal{C}_\tau$ do not contain conjugate momenta, the Poisson brackets $\{\mathcal{C}_\tau, \mathcal{C}_{\tau'}\}$ strongly vanish. The time evolution of the secondary set of constraints is thus given by

$$\dot{\mathcal{R}}_0 \equiv \left\{\mathcal{R}_0, H^{(2)}_{\text{pre}}\right\} \approx -M\mathcal{B}_0 - \left(N^i - M^i\right)\mathcal{A}_i - \lambda^\tau \left\{\mathcal{R}_0, \mathcal{C}_\tau\right\} , \tag{VI.30a}$$

$$\dot{\tilde{\mathcal{R}}}_0 \equiv \left\{\tilde{\mathcal{R}}_0, H^{(2)}_{\text{pre}}\right\} \approx N\mathcal{B}_0 - \left(N^i - M^i\right)\mathcal{B}_i - \lambda^\tau \left\{\tilde{\mathcal{R}}_0, \mathcal{C}_\tau\right\} , \tag{VI.30b}$$

$$\dot{\mathcal{R}}_i \equiv \left\{\mathcal{R}_i, H^{(2)}_{\text{pre}}\right\} \approx N\mathcal{A}_i + M\mathcal{B}_i - \lambda^\tau \left\{\mathcal{R}_i, \mathcal{C}_\tau\right\} , \tag{VI.30c}$$

$$\dot{\mathrm{R}}_i \equiv \left\{\mathrm{R}_i, H^{(2)}_{\text{pre}}\right\} \approx 0 , \tag{VI.30d}$$

$$\dot{\mathcal{C}}_\tau \equiv \left\{\mathcal{C}_\tau, H^{(2)}_{\text{pre}}\right\} \approx N\left\{\mathcal{R}_0, \mathcal{C}_\tau\right\} + M\left\{\tilde{\mathcal{R}}_0, \mathcal{C}_\tau\right\} + \left(N^i - M^i\right)\left\{\mathcal{R}_i, \mathcal{C}_\tau\right\} , \tag{VI.30e}$$

where we do not give explicit expressions for the complicated Poisson brackets involving $\mathcal{C}_\tau$. Again, all those equations shall weakly vanish, and the combination $N\dot{\mathcal{R}}_0 + M\dot{\tilde{\mathcal{R}}}_0 + \left(N^i - M^i\right)\dot{\mathcal{R}}_i + \lambda^\tau \dot{\mathcal{C}}_\tau \approx 0$ indicates the presence of an additional first-class constraint, which allows to discard $\dot{\tilde{\mathcal{R}}}_0$. The three conditions $\dot{\mathcal{R}}_i \approx 0$ can be solved for N and the two λ^τ. Then $\dot{\mathcal{R}}_0 \approx 0$ can be solved for M, and the remaining two $\dot{\mathcal{C}}_\tau \approx 0$, for two of the N^i. Thus the system can be solved by itself, there is no need for tertiary constraints. Note that the freedom in the choice of the remaining N^i and the three M^i is naturally related to the four first-class constraints.

To sum up, we have four first-class and six second-class constraints, removing a total of fourteen phase space *dofs* at each point. As the secondary Hamiltonian contains twenty-six variables (twenty-four from the metrics and their conjugate momenta, and the two Lagrange multipliers λ^τ), it contains twelve phase space *dofs* left at each point. In order to built a minimal theory containing only eight phase space *dofs*, we still have to eliminate four of them, by introducing appropriate constraints.

$\{\downarrow, \rightarrow\}$	$\mathcal{R}_0$	$\tilde{\mathcal{R}}_0$	$\mathcal{R}_i$	R_i	$\mathcal{C}_\tau$
$\mathcal{R}_0$	≈ 0	$\mathcal{B}_0$	$\mathcal{A}_i$	≈ 0	$\not\approx 0$
$\tilde{\mathcal{R}}_0$		≈ 0	$\mathcal{B}_i$	≈ 0	$\not\approx 0$
$\mathcal{R}_j$			≈ 0	≈ 0	$\not\approx 0$
R_j				≈ 0	≈ 0
$\mathcal{C}_\tau$					0

Table VI.1: Final constraint algebra of the precursor theory, including the secondary constraints. The omitted entries are due to the antisymmetric nature of the Poisson brackets.

VI.2.3 A minimal theory of bigravity

Hamiltonian construction

As advertised, we construct our minimal theory of bigravity at the Hamiltonian level, adding constraints to the precursor Hamiltonian (VI.23). As the precursor theory propagates twelve phase space *dofs*, a natural guess would be that we need to add four constraints to recover eight phase space propagating *dofs*

In order to choose the appropriate form for the constraints, we require in addition that MTBG has the same background equations of motion than HRBG on generic FLRW manifolds. In this case, MTBG naturally becomes a stable (its stability will be proven later on) nonlinear completion of HRBG. It turns out that this is achieved by adding a constraint $\mathcal{C}_0 - \tilde{\mathcal{C}}_0$, where the two pieces are defined as

$$\mathcal{C}_0\left[\zeta\right] = \left\{\mathcal{R}_0^{\mathrm{GR}}\left[\zeta\right], -\frac{m^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^3x\,\sqrt{\phi}\tilde{\mathcal{H}}_0\right\}\,, \qquad \tilde{\mathcal{C}}_0\left[\zeta\right] = \left\{\tilde{\mathcal{R}}_0^{\mathrm{GR}}\left[\zeta\right], -\frac{m^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^3x\,\sqrt{\gamma}\mathcal{H}_0\right\}\,, \tag{VI.31}$$

and read explicitly

$$\mathcal{C}_0 = -m^2\left(\pi^{pr}\gamma_{rq} - \frac{\pi}{2}\,\delta^p_q\right)\mathcal{U}^q{}_p\,, \qquad \text{and} \qquad \tilde{\mathcal{C}}_0 = -\frac{m^2}{\alpha^2}\left(\sigma^{pr}\phi_{rq} - \frac{\sigma}{2}\,\delta^p_q\right)\tilde{\mathcal{U}}^q{}_p\,, \tag{VI.32}$$

with the shorthands

$$\mathcal{U}^p{}_q \equiv \frac{1}{2}\sum_{n=1}^{3} c_{4-n}\left(U^p_{(n)\,q} + \gamma^{pr}\gamma_{qs}U^s_{(n)\,r}\right)\,, \quad \text{and} \qquad \tilde{\mathcal{U}}^p{}_q \equiv \frac{1}{2}\sum_{n=1}^{3} c_n\left(V^p_{(n)\,q} + \phi^{pr}\phi_{qs}V^s_{(n)\,r}\right)\,. \tag{VI.33}$$

Having now stable FLRW backgrounds makes the cosmology of the HRBG-backgrounds appealing once again. In order to prevent a spoiling of such nice property, any additional constraint shall vanish on homogeneous and isotropic backgrounds.

As we have *a priori* three more constraints to add, and as the three-dimensional invariance is not broken in any way in MTBG, we can mimic the pieces (VI.31) and define

$$\mathcal{C}_i\left[\xi^i\right] = \left\{\mathcal{R}_i^{\mathrm{GR}}\left[\xi^i\right], -\frac{m^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^3x\,\sqrt{\phi}\tilde{\mathcal{H}}_0\right\}\,, \qquad \tilde{\mathcal{C}}_i\left[\xi^i\right] = \left\{\tilde{\mathcal{R}}_i^{\mathrm{GR}}\left[\xi^i\right], -\frac{m^2 M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^3x\,\sqrt{\gamma}\mathcal{H}_0\right\}\,. \tag{VI.34}$$

Those quantities read

$$\mathcal{C}_i = \frac{m^2 M_{\mathrm{Pl}}^2}{2}\sqrt{\gamma}\,\mathcal{D}_j\mathcal{U}^j{}_i\,, \qquad \text{and} \qquad \tilde{\mathcal{C}}_i = \frac{m^2 M_{\mathrm{Pl}}^2}{2}\sqrt{\phi}\,\mathfrak{D}_j\tilde{\mathcal{U}}^j{}_i\,. \tag{VI.35}$$

The associated constraint is taken as $\propto \mathcal{C}_i - \beta\tilde{\mathcal{C}}_i$, which vanishes on cosmological background by isotropy. We have made an attempt to treat a rather general case, by introducing the parameter β as an extra possible free parameter of the theory. However, this β-constant will play no role in the cosmological study of MTBG, and thus its numerical value is not relevant for the present study. Although we were tempted to fix it to unity by symmetry with the $\mathcal{C}_0 - \tilde{\mathcal{C}}_0$ constraint, we have chosen to let it be free, as it may play a role (and so be constrained) in the study of strong gravity regime (*eg.* for black-hole type solutions). Note also that, just by fixing to three the number of these vector-like constraints, several other possibilities could have been taken into account, but we had to make a choice.

At this point we could hope that we have added the accurate number of constraints. Nevertheless, a Hamiltonian analysis shows that the fourth first-class constraint of the precursor theory is degraded to second-class in this theory, removing only one phase space *dof*. Thus we need a fifth additional constraint, that vanishes on cosmological background. By symmetry with the $\mathcal{C}_0 - \tilde{\mathcal{C}}_0$ constraint, we want to add a constraint looking as $\mathcal{C}_0 + \tilde{\mathcal{C}}_0$. In order to preserve the cosmological background, we plug spatial derivatives on this quantity, finally achieving the Hamiltonian construction of a minimal theory of bigravity

$$\begin{aligned} H = -\int \mathrm{d}^3x \bigg\{ & N\mathcal{R}_0 + N^i\mathcal{R}_i + M\tilde{\mathcal{R}}_0 + M^i\tilde{\mathcal{R}}_i + \lambda\left(\mathcal{C}_0 - \tilde{\mathcal{C}}_0\right) + \lambda^i\left(\mathcal{C}_i - \beta\,\tilde{\mathcal{C}}_i\right) \\ & + \bar{\lambda}\left[\sqrt{\gamma}\gamma^{ij}\mathcal{D}_{ij}\left(\frac{\mathcal{C}_0}{\sqrt{\gamma}}\right) + \sqrt{\phi}\phi^{ij}\mathfrak{D}_{ij}\left(\frac{\tilde{\mathcal{C}}_0}{\sqrt{\phi}}\right)\right]\bigg\}\,, \end{aligned} \tag{VI.36}$$

where $\{\lambda, \lambda^i, \bar{\lambda}\}$ is a set of five Lagrange multipliers, β is a free constant and the eight first constraints $\{\mathcal{R}_0, \mathcal{R}_i, \tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_i\}$ are the same as in eq. (VI.24). Note that the fifth additional constraint is the one associated with $\bar{\lambda}$.

As discussed in the previous section, it has been proven in app. B of [155] that the combinations

$$\mathrm{R}_i \equiv \mathcal{R}_i + \tilde{\mathcal{R}}_i \,, \tag{VI.37}$$

are still first-class constraints, and this statement holds independently of the value of the constant β. This is similar to what is happening in usual bigravity [362]. Therefore, starting from twenty-four phase space variables, the three first-class and the ten second-class constraints remove sixteen phase space variables: our theory propagates at most four physical degrees of freedom. For readers that are more familiar with formulations in terms of vielbeins, app. C of [155] presents the construction of MTBG in this language.

The fact that MTBG only contains the three first-class constraints (VI.37) indicates that it breaks both temporal copies of the diffeomorphisms, as well as one copy of the spatial ones, retaining only the invariance under joint spatial diffeomorphisms. Introducing Stückelberg fields as in (VI.10) would restore one copy of the full diffeomorphisms. As such reintroduction was of no interest for our purpose, we have not explicitly performed it.

In order to demonstrate that the theory (VI.36) propagates exactly four physical *dofs* at each point, we should in principle properly check that the algebra of the Poisson brackets is closed, *ie.* that no secondary constraints are generated by the time conservation of the primary ones. As this seems quite involving, we do a little detour and investigate the cosmology of MTBG. Indeed, by studying the linear perturbations that propagate on FLRW backgrounds, we will prove that the theory propagates at least four *dofs*, and thus we will conclude that it propagates exactly four *dofs*.

Lagrangian formulation

From the Hamiltonian (VI.36), it follows that

$$\partial_t\gamma_{ij} = \frac{\partial H}{\partial \pi^{ij}} = \frac{4N}{M_{\rm Pl}^2\sqrt{\gamma}}\left(\pi_{ij} - \frac{\pi}{2}\,\gamma_{ij}\right) + 2\gamma_{k(i}\mathcal{D}_{j)}N^k + m^2\left(\mathcal{U}^p{}_{(i}\gamma_{j)p} - \frac{1}{2}\mathcal{U}^p{}_p\gamma_{ij}\right)\left(\lambda + \gamma^{kl}\mathcal{D}_{kl}\bar{\lambda}\right)\,, \tag{VI.38a}$$

$$\partial_t\phi_{ij} = \frac{\partial H}{\partial \sigma^{ij}} = \frac{4M}{\alpha^2 M_{\rm Pl}^2\sqrt{\phi}}\left(\sigma_{ij} - \frac{\sigma}{2}\,\phi_{ij}\right) + 2\phi_{k(i}\mathfrak{D}_{j)}M^k - \frac{m^2}{\alpha^2}\left(\tilde{\mathcal{U}}^p{}_{(i}\phi_{j)p} - \frac{1}{2}\tilde{\mathcal{U}}^p{}_p\phi_{ij}\right)\left(\lambda - \phi^{kl}\mathfrak{D}_{kl}\bar{\lambda}\right)\,, \tag{VI.38b}$$

which can be inverted to give

$$\pi^{ij} = \frac{M_{\rm Pl}^2}{2}\sqrt{\gamma}\left\{K^{ij} - K\,\gamma^{ij} - \frac{m^2\left(\lambda + \gamma^{kl}\mathcal{D}_{kl}\bar{\lambda}\right)}{2N}\,\mathcal{U}^{(i}_{\ p}\gamma^{j)p}\right\}\,, \tag{VI.39a}$$

$$\sigma^{ij} = \frac{\alpha^2 M_{\rm Pl}^2}{2}\sqrt{\phi}\left\{\Phi^{ij} - \Phi\,\phi^{ij} + \frac{m^2\left(\lambda - \phi^{kl}\mathfrak{D}_{kl}\bar{\lambda}\right)}{2M\alpha^2}\,\tilde{\mathcal{U}}^{(i}_{\ p}\phi^{j)p}\right\}\,. \tag{VI.39b}$$

Legendre-transforming the Hamiltonian, the Lagrangian density of MTBG finally reads

$$\begin{aligned}\mathcal{L} = {}& \frac{M_{\rm Pl}^2}{2}\sqrt{-g}\,\mathcal{R}\left[g\right] + \frac{\alpha^2 M_{\rm Pl}^2}{2}\sqrt{-f}\,\mathcal{R}\left[f\right] - \frac{m^2 M_{\rm Pl}^2}{2}\left(N\sqrt{\gamma}\,\mathcal{H}_0 + M\sqrt{\phi}\,\tilde{\mathcal{H}}_0\right)\\ & - \frac{m^2 M_{\rm Pl}^2}{2}\Bigg\{\sqrt{\gamma}\,\mathcal{U}^p{}_q\mathcal{D}_p\lambda^q - \beta\sqrt{\phi}\,\tilde{\mathcal{U}}^p{}_q\mathfrak{D}_p\lambda^q + \left(\lambda + \gamma^{ij}\mathcal{D}_{ij}\bar{\lambda}\right)\sqrt{\gamma}\,\mathcal{U}^p{}_qK^p{}_q - \left(\lambda - \phi^{ij}\mathfrak{D}_{ij}\bar{\lambda}\right)\sqrt{\phi}\,\tilde{\mathcal{U}}^p{}_q\Phi^p{}_q\\ & + \frac{m^2\left(\lambda + \gamma^{ij}\mathcal{D}_{ij}\bar{\lambda}\right)^2}{4N}\sqrt{\gamma}\left(\mathcal{U}^p{}_q - \frac{\mathcal{U}^k{}_k}{2}\,\delta^p_q\right)\mathcal{U}^q{}_p + \frac{m^2\left(\lambda - \gamma^{ij}\mathfrak{D}_{ij}\bar{\lambda}\right)^2}{4M\alpha^2}\sqrt{\phi}\left(\tilde{\mathcal{U}}^p{}_q - \frac{\tilde{\mathcal{U}}^k{}_k}{2}\,\delta^p_q\right)\tilde{\mathcal{U}}^q{}_p\Bigg\}\,.\end{aligned} \tag{VI.40}$$

We find important at this point to recall that the principle of minimalism is a guideline towards *phenomenological* simplicity, which can be very different from *computational* simplicity, as obviously proven by our theory (VI.40).

Some remarks on MTBG

Before discussing the cosmology of MTBG, let us conclude this section by three remarks.

Firstly, the choice of the additional constraints in (VI.36) is totally arbitrary, and naturally other possibilities are allowed: we do not claim to have derived *the* minimal theory of bigravity, but rather *one* minimal theory of bigravity. For example, if the $\bar{\lambda}$ and λ^i constraints have no influence on the FLRW background dynamics, they will affect in general the dynamics of other background solutions and, at least, the linear perturbations around FLRW space-times. Choices had to be done, and we have chosen the constraints so that our theory has the same cosmological background as HRBG, and stable linear cosmological perturbations. Needless to say, each inequivalent choice of constraints would lead to a different minimal theory.

Secondly, as clear from the Lagrangian density (VI.40), the breaking of the invariance under diffeomorphisms is only due to the interaction term, and thus of order m^2. This parameter is naturally related to the acceleration of the expansion of the Universe (see sec. VI.3.1) and to the mass of the graviton (see sec. VI.3.2), so it can be taken of order of the current Hubble parameter. In such a natural setup, the breaking of the temporal diffeomorphism is extremely weak and only occurs at cosmological scales, as advertised at the end of sec. VI.1.

Last but not least, the construction has been performed in vacuum. As discussed in sec. VI.2.1, coupling matter to a bimetric theory can be dangerous if not carefully done, as it can reintroduce the BD ghost. We will not tackle in detail this subtle issue here, but let us emphasize two important points. The first one is that in order to study cosmology, it is enough to implement the simplest setup consisting of two perfect fluids (or dust components), each one minimally coupled to one metric. This setup is viable in HRBG and, as proven in the next section, also in MTMG. The second point is that, in HRBG, dubious matter couplings are not linear in the specific combination of lapses and shifts that removes the BD ghost: the redeeming constraint is no longer present and the BD ghost is thus reintroduced. In our theory, the situation is quite different as we do not rely on this specific combination, but we introduce constraints by hand. Therefore we expect that, unless being very exotic, no matter coupling can spoil any of our constraints, the only danger being a dubious coupling that degrades our first-class constraints R_i to second-class. As R_i is associated with the three-dimensional rotational invariance, this should not happen as long as the matter coupling is covariant. Therefore we have good clues to guess that no covariant matter couplings can introduce ghosts in MTBG, whatever the combination of metrics enters it. Naturally this assertion should be carefully probed by a detailed analysis.

If verified, a direct consequence of this claim is that the Dipolar Dark Matter framework (discussed at the end of sec. VI.2.1) could revive if implemented in MTBG rather than in HRBG. Firstly, the "graviphoton" that couples to the two metrics should be perfectly healthy. Secondly, the failure of the polarization mechanism in the current version [78] is due to the so-called van Dam–Veltman–Zakharov discontinuity [341] (roughly stated, the phenomenology of GR is not recovered in the $m \to 0$ limit of the phenomenology of HRBG). This discontinuity is due to a coupling between the healthy scalar mode and the trace of the stress-energy tensor of matter, coupling that survives in the $m \to 0$ limit. As there is no scalar modes in MTBG, the trace of the stress-energy tensor of matter cannot couple to any dynamical *dof*, and thus we expect to be free from van Dam–Veltman–Zakharov-like discontinuities in MTBG. Therefore a Dipolar Dark Matter framework may work if implemented in MTBG, which is currently under investigation.

VI.3 Cosmology of our minimal theory of bigravity

In order to check that only four *dofs* propagate in MTBG, but also to derive its interesting phenomenological implications, let us study the cosmology of the theory (VI.40), at the background and linear perturbation levels. In this section, we assume that we live in the g-sector and thus, we will favor g-related observable quantities in the final expressions.

VI.3.1 Cosmological background

Let us first consider homogeneous and isotropic background solutions, *ie.* let us begin with two flat Friedman-Lemaître-Robertson-Walker (FLRW) line elements

$$\begin{aligned} \mathrm{d}s_g^2 &= -N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,, \\ \mathrm{d}s_f^2 &= -M^2(t)\,\mathrm{d}t^2 + b^2(t)\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,, \end{aligned} \tag{VI.41}$$

with δ_{ij} the flat Euclidean three-dimensional metric. We define the two Hubble-Lemaître expansion rates $H = \dot{a}/Na$ and $L = \dot{b}/Mb$, where a dot denotes time derivative. We will also introduce the ratios of the scale factors and the ratio of speeds of light in each sector as

$$\mathcal{X} \equiv \frac{b}{a}\,, \qquad \text{and} \qquad r \equiv \frac{M}{N\mathcal{X}} = \frac{c_f}{c_g}\,. \tag{VI.42}$$

By virtue of homogeneity of the background, we will consider only time-dependent auxiliary fields $\lambda(t)$, $\bar{\lambda}(t)$ and $\lambda^i(t)$, so that $\bar{\lambda}$ and λ^i disappear from the action.[8] As for the matter, we will consider two general perfect fluids with barotropic equations of state $p(\rho)$ and $\tilde{p}\,(\tilde{\rho})$, minimally coupled respectively to $g_{\mu\nu}$ and $f_{\mu\nu}$. For later convenience, we introduce the combination

$$\Gamma_1 = c_1\mathcal{X}^2 + 2c_2\mathcal{X} + c_3\,. \tag{VI.43}$$

The equations of motion $\mathcal{E}_\star = 0$ for the components of the metric can be put in the usual form

$$\mathcal{E}_N = 3H^2 - \frac{\rho + \rho_m + \rho_{\rm aux}}{M_{\rm Pl}^2}\,, \tag{VI.44a}$$

$$\mathcal{E}_M = 3L^2 - \frac{\tilde{\rho} + \tilde{\rho}_m + \tilde{\rho}_{\rm aux}}{\alpha^2 M_{\rm Pl}^2}\,, \tag{VI.44b}$$

$$\mathcal{E}_a = \frac{2\dot{H}}{N} + 3H^2 + \frac{p + p_m + p_{\rm aux}}{M_{\rm Pl}^2}\,, \tag{VI.44c}$$

$$\mathcal{E}_b = \frac{2\dot{L}}{M} + 3L^2 + \frac{\tilde{p} + \tilde{p}_m + \tilde{p}_{\rm aux}}{\alpha^2 M_{\rm Pl}^2}\,, \tag{VI.44d}$$

where we have introduced the usual matter energy densities and pressures ρ, $\tilde{\rho}$, p and $\tilde{p}$, together with the effective energy densities

$$\rho_m = \frac{M_{\rm Pl}^2 m^2}{2}\left(c_1\mathcal{X}^3 + 3c_2\mathcal{X}^2 + 3c_3\mathcal{X} + c_4\right)\,, \quad \tilde{\rho}_m = \frac{M_{\rm Pl}^2 m^2}{2\mathcal{X}^3}\left(c_0\mathcal{X}^3 + 3c_1\mathcal{X}^2 + 3c_2\mathcal{X} + c_3\right)\,, \tag{VI.45a}$$

$$\rho_{\rm aux} = -\frac{3m^2 M_{\rm Pl}^2\lambda}{2N}\left(H + \frac{m^2\Gamma_1\lambda}{8N}\right)\Gamma_1\,, \qquad \tilde{\rho}_{\rm aux} = \frac{3m^2 M_{\rm Pl}^2\lambda}{2M\mathcal{X}^2}\left(L - \frac{m^2\Gamma_1\lambda}{8M\alpha^2\mathcal{X}^2}\right)\Gamma_1\,, \tag{VI.45b}$$

and pressures (which detailed expressions are given in eq. (25) of [155]). The two perfect fluids obey their respective continuity equations

$$\mathcal{E}_c^g = \dot{\rho} + 3NH\,(\rho + p)\,, \qquad \text{and} \qquad \mathcal{E}_c^f = \dot{\tilde{\rho}} + 3ML\,(\tilde{\rho} + \tilde{p})\,. \tag{VI.46}$$

[8] The fact that λ^i disappears from the action is also naturally enforced by the joint requirement of isotropy.

All those equations take the same form as in the usual HRBG [143]. For latter convenience, we construct two "Bianchi" identities, one for each sector, as

$$\mathcal{E}_B^g = \dot{\mathcal{E}}_N + 3NH\left(\mathcal{E}_N - \mathcal{E}_a\right) + \frac{\mathcal{E}_c^g}{M_{\rm Pl}^2}\,, \qquad \text{and} \qquad \mathcal{E}_B^f = \dot{\mathcal{E}}_M + 3ML\left(\mathcal{E}_M - \mathcal{E}_b\right) + \frac{\mathcal{E}_c^f}{\alpha^2 M_{\rm Pl}^2}\,. \tag{VI.47}$$

The equation of motion of the only non-vanishing auxiliary field reads

$$\mathcal{E}_\lambda = \left(\frac{\Gamma_1\lambda}{N} + \frac{\Gamma_1\lambda}{\alpha^2 M\mathcal{X}} + \frac{4(H-\mathcal{X}L)}{m^2}\right)\Gamma_1\,. \tag{VI.48}$$

Note the similar structure as the Bianchi identity of HRBG (VI.7), revealing that there exist also two branches of cosmological solutions in MTBG. As we will prove hereafter, those two branches are the same as the two branches of HRBG cosmology. The "self-accelerating" branch, defined by $\Gamma_1 = 0$ (and thus a constant $\mathcal{X}$) is the analogue of the strongly-coupled branch of HRBG (defined by $\mathcal{B} = 0$ in eq. (VI.7)), up to the only important difference that it is now perfectly healthy. The second, "normal", branch corresponds to the healthy[9] branch of HRBG, defined by $H = \mathcal{X}L$.

Self-accelerating branch

In the self-accelerating branch, $\Gamma_1 = 0$ and thus $\mathcal{X}$ is constant, which implies in turn $ML = NH$. Closing the set of equations of motion (VI.44) and (VI.46) is equivalent to verify that the two identities (VI.47) hold. They are proportional to each other and give the condition

$$3m^2H^2\left(c_1\mathcal{X} + c_2\right)\frac{1-\mathcal{X}^2 r}{\mathcal{X}\,r}\,\lambda = 0\,. \tag{VI.49}$$

As we will see later, in addition to be a severe tuning, strong coupling issues appear when the quantity $c_1\mathcal{X} + c_2$ is vanishing, so we will keep it non-zero. On the other hand, imposing $\mathcal{X}^2 r = 1$ would yield $H = L/\mathcal{X}$ which, as $\mathcal{X}$ is constant, totally fixes the dynamics of the g-sector with respect to the f-sector. This would naturally yield an ill-defined cosmology, thus we discard this possibility. Therefore the only possible way to satisfy the Bianchi's identities is to fix $\lambda = 0$.

Defining the two constants

$$\Lambda_0 = \frac{m^2}{2}\left(c_2\mathcal{X}^2 + 2c_3\mathcal{X} + c_4\right) \qquad \text{and} \qquad \tilde{\Lambda}_0 = \frac{m^2}{2\alpha^2}\frac{c_0\mathcal{X}^2 + 2c_1\mathcal{X} + c_2}{\mathcal{X}^2}\,, \tag{VI.50}$$

the energies and pressures coming from the mass sector are then

$$\rho_m = M_{\rm Pl}^2\Lambda_0\,, \qquad p_m = -\rho_m\,, \qquad \tilde{\rho}_m = \alpha^2 M_{\rm Pl}^2\tilde{\Lambda}_0\,, \qquad \tilde{p}_m = -\tilde{\rho}_m\,, \tag{VI.51}$$

so that the mass sector yields a set of two effective cosmological constants, thus the name "self-accelerating" branch. As we will show later on, since MTBG propagates only tensor modes, this branch is not strongly coupled any longer, and one can then study its phenomenology.

Normal branch

In the normal branch, recalling that by definition $\Gamma_1 \neq 0$, eq. (VI.48) is solved by

$$\lambda = \frac{4\alpha^2 N\mathcal{X}^2 r\,(\mathcal{X}L - H)}{m^2\Gamma_1\left(1 + \alpha^2\mathcal{X}^2 r\right)}\,. \tag{VI.52}$$

[9] The self-accelerating and normal branches have also been called "algebraic" and "dynamical", respectively (for example in [17]).

Injecting this result in the identities (VI.47) and forming the linear combination $\mathcal{E}_B^g + \mathcal{X}^4\,\mathcal{E}_B^f$ yields

$$\mathcal{E}_B^g + \mathcal{X}^4\,\mathcal{E}_B^f = (H - \mathcal{X}L)\,\mathcal{Q} = 0\,, \tag{VI.53}$$

where $\mathcal{Q}$ is a complicated quantity that has no reason to vanish (or it will be equivalent to add more constraints). This relation thus imposes that $H - \mathcal{X}L$, and thus λ, vanish. So in the normal branch, we have

$$\lambda = 0 \qquad \text{and} \qquad H = \mathcal{X}L\,. \tag{VI.54}$$

Note here that the previous argument against proportionality between H and L does not hold here as neither $\mathcal{X}$ nor r are constant. The normal sector is thus filled with two effective dark fluids (ρ_m, p_m) and ($\tilde{\rho}_m$, $\tilde{p}_m$) that satisfy their respective conservation equations. The equations of state of those effective dark fluids are given by

$$w_m \equiv \frac{p_m}{\rho_m} = -1 + \frac{c_1\mathcal{X}^3 + 2c_2\mathcal{X}^2 + c_3\mathcal{X}}{c_1\mathcal{X}^3 + 3c_2\mathcal{X}^2 + 3c_3\mathcal{X} + c_4}\,(1-r)\,, \tag{VI.55a}$$

$$\tilde{w}_m \equiv \frac{\tilde{p}_m}{\tilde{\rho}_m} = -1 + \frac{c_1\mathcal{X}^3 + 2c_2\mathcal{X}^2 + c_3\mathcal{X}}{c_0\mathcal{X}^3 + 3c_1\mathcal{X}^2 + 3c_2\mathcal{X} + c_3}\,\frac{r-1}{r}\,. \tag{VI.55b}$$

This branch is analogous to the healthy branch of HRBG, defined by $H = \mathcal{X}L$ [143]. Moreover, as in HRBG, the evolution of the ratios $\mathcal{X}$ and r (and thus of the dark fluids) is totally fixed by the matter content, as

$$\frac{\dot{\mathcal{X}}}{N\mathcal{X}} = (r-1)\,H\,, \qquad \text{and} \qquad 2\left(\frac{m^2\Gamma_1}{4\mathcal{X}}\frac{1+\alpha^2\mathcal{X}^2}{\alpha^2} - H^2\right)(r-1) = \frac{\rho+p}{M_{\rm Pl}^2} - \frac{\tilde{\rho}+\tilde{p}}{\alpha^2 M_{\rm Pl}^2}\,\mathcal{X}^2\,r\,. \tag{VI.56}$$

VI.3.2 Cosmological perturbations

Let us now study linear cosmological perturbations, on both branches. This study has two aims: on the one hand, by showing that four gravitational *dofs* propagate at the linear level, we can conclude that the theory contains exactly 4 gravitational *dofs* since the Hamiltonian analysis has already shown that the number of propagating *dofs* is at most four at the fully nonlinear level. On the other hand, it allows us to investigate the cosmological phenomenology, and possible deviations from canonical GR results.

Let us perturb the gravitational sector as

$$N = N(t)\,(1+\eta)\,, \qquad M = M(t)\,(1+\omega)\,, \tag{VI.57a}$$

$$N_i = N(t)\,a^2(t)\,(\beta_i + \partial_i\beta)\,, \qquad M_i = M(t)\,b^2(t)\,(\theta_i + \partial_i\theta)\,, \tag{VI.57b}$$

$$\gamma_{ij} = a^2(t)\left[(1+2\zeta)\,\delta_{ij} + 2\partial_{ij}\chi + 2\partial_{(i}\chi_{j)} + \pi_{ij}\right], \qquad \phi_{ij} = b^2(t)\left[(1+2\xi)\,\delta_{ij} + 2\partial_{ij}\tau + 2\partial_{(i}\tau_{j)} + \psi_{ij}\right], \tag{VI.57c}$$

$$\lambda = \delta\lambda\,, \qquad \bar{\lambda} = \delta\bar{\lambda}\,, \tag{VI.57d}$$

$$\lambda^i = \delta\ell^i + \frac{\delta^{ij}}{a^2(t)}\,\partial_j\delta\ell\,, \tag{VI.57e}$$

where all the perturbation variables are of the same order in perturbation and depend on space and time, the vector quantities $\{\beta_i, \theta_i, \chi_i, \tau_i, \delta\ell^i\}$ are divergenceless and the tensor quantities $\{\pi_{ij}, \psi_{ij}\}$ are divergenceless, symmetric and trace-free. We operate indices in both sectors with the flat δ_{ij} metric.

As for the matter, we will present the computation with two pure dust components, *à la* Schutz-Sorkin [112, 324], one in each sector

$$\mathcal{S}_{\rm dust} = -\int \mathrm{d}^4x \left\{\sqrt{-g}\,\left[\rho(n) + J^\alpha\partial_\alpha\varphi\right] + \sqrt{-f}\,\left[\tilde{\rho}(\tilde{n}) + \tilde{J}^\alpha\partial_\alpha\tilde{\varphi}\right]\right\}, \tag{VI.58}$$

where J^α and $\tilde{J}^\alpha$ are two four-vectors. The dusts are made of particles of individual masses μ_0 and $\tilde{\mu}_0$ so that the coordinate densities read

$$\rho = \mu_0\, n\,, \qquad\qquad n = \sqrt{-g_{\alpha\beta}J^\alpha J^\beta}\,, \tag{VI.59a}$$

$$\tilde{\rho} = \tilde{\mu}_0\, \tilde{n}\,, \qquad\qquad \tilde{n} = \sqrt{-f_{\alpha\beta}\tilde{J}^\alpha \tilde{J}^\beta}\,, \tag{VI.59b}$$

and the four-velocities $u^\alpha = J^\alpha/n$ and $\tilde{u}^\alpha = \tilde{J}^\alpha/\tilde{n}$ are properly normalized $u^\alpha u_\alpha = \tilde{u}^\alpha \tilde{u}_\alpha = -1$. Varying $\mathcal{S}_{\rm dust}$ with respect to J^α and $\tilde{J}^\alpha$, it follows that

$$u_\alpha = \frac{\partial_\alpha \varphi}{\mu_0}\,, \qquad \text{and} \qquad \tilde{u}_\alpha = \frac{\partial_\alpha \tilde{\varphi}}{\tilde{\mu}_0}\,. \tag{VI.60}$$

The dust quantities are perturbed as

$$J^0 = \frac{1}{Na^3}\left(\mathcal{N}_0 + j_0\right)\,, \qquad\qquad \tilde{J}^0 = \frac{1}{Mb^3}\left(\tilde{\mathcal{N}}_0 + \tilde{j}_0\right)\,, \tag{VI.61a}$$

$$J^i = \frac{1}{a^3(t)}\left(j^i + \frac{\delta^{ik}}{a^2(t)}\,\partial_k j\right)\,, \qquad\qquad \tilde{J}^i = \frac{1}{b^3(t)}\left(\tilde{j}^i + \frac{\delta^{ik}}{b^2(t)}\,\partial_k \tilde{j}\right)\,, \tag{VI.61b}$$

$$\varphi = -\mu_0 \int^t \mathrm{d}\tau N(\tau) - \mu_0 v_m\,, \qquad\qquad \tilde{\varphi} = -\tilde{\mu}_0 \int^t \mathrm{d}\tau M(\tau) - \tilde{\mu}_0 \tilde{v}_m\,, \tag{VI.61c}$$

where $\mathcal{N}_0$ and $\tilde{\mathcal{N}}_0$ are constants of integration of the background equations for φ and $\tilde{\varphi}$, that represent the comoving number density of particles, as they satisfy $n = \mathcal{N}_0/a^3$ and $\tilde{n} = \tilde{\mathcal{N}}_0/b^3$. We will also introduce the gauge invariant density contrasts

$$\delta_m \equiv \frac{\delta\rho}{\rho} + 3Hv_m\,, \qquad \text{and} \qquad \tilde{\delta}_m \equiv \frac{\delta\tilde{\rho}}{\tilde{\rho}} + 3L\tilde{v}_m\,. \tag{VI.62}$$

Using (VI.59) and (VI.61a), it follows that

$$\delta_m \equiv \frac{j_0}{\mathcal{N}_0} + \eta + 3Hv_m\,, \qquad \text{and} \qquad \tilde{\delta}_m \equiv \frac{\tilde{j}_0}{\tilde{\mathcal{N}}_0} + \omega + 3L\tilde{v}_m\,. \tag{VI.63}$$

For a cross-check and as a complementary calculation, we have also made the computation with two minimally coupled k-essences

$$\mathcal{S}_{k-\rm ess} = \int \mathrm{d}^4x \left\{\sqrt{-g}\,P(X) + \sqrt{-f}\,Q(Y)\right\}\,, \tag{VI.64}$$

where

$$X \equiv -\frac{g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi}{2}\,, \qquad \text{and} \qquad Y \equiv -\frac{f^{\mu\nu}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi}}{2}\,. \tag{VI.65}$$

For this system of matter, the background pressures and energy densities are $p = P(X)$, $\tilde{p} = Q(Y)$, $\rho = 2XP_X - P$ and $\tilde{\rho} = 2YQ_Y - Q$. As for the perturbations, the GR no-ghost condition and squared sound speed are given in the g-sector by

$$2XP_{XX} + P_X > 0\,, \qquad \text{and} \qquad c_s^2 = \frac{P_X}{2XP_{XX} + P_X}\,, \tag{VI.66}$$

with similar relations holding in the f-sector.

Tensor sector

In both branches, the quadratic action of the tensor sector reads

$$\delta^{(2)}\,\mathcal{S}^T = \frac{M_{\rm Pl}^2}{8}\int \mathrm{d}t\mathrm{d}^3x N a^3 \left\{ \left(\frac{\dot\pi_{ij}}{N}\right)^2 - \left(\frac{\partial_k \pi_{ij}}{a}\right)^2 + \frac{\alpha^2\mathcal{X}^3 M}{N}\left[\left(\frac{\dot\psi_{ij}}{M}\right)^2 - \left(\frac{\partial_k\psi_{ij}}{\mathcal{X}a}\right)^2\right] - \mu_T^2\,(\pi_{ij}-\psi_{ij})^2 \right\}\,, \tag{VI.67}$$

the only difference being the expression of the effective mass

$$\mu_T^2 = \frac{m^2\mathcal{X}^2}{2}\,(c_1\mathcal{X}+c_2)\,(r-1) \qquad \text{in the self-accelerating branch}\,, \tag{VI.68a}$$

$$\mu_T^2 = \frac{m^2\mathcal{X}^2}{2}\left[(c_1\mathcal{X}+c_2)\;r + \frac{c_2\mathcal{X}+c_3}{\mathcal{X}}\right] \qquad \text{in the normal branch}\,. \tag{VI.68b}$$

As the kinetic terms are the canonical ones, we are automatically free from ghost and gradient instabilities. In order to avoid tachyonic instabilities, one needs to ensure the positivity of the squared mass, which does not drastically restrict the parameter space of MTBG. It is clear that the tensor sector propagates four *dofs* at the linear level. On the phenomenological point of view, we recover the usual bigravity features: the mass eigenstates are different from either g-perturbation or f-perturbation, with the massive eigenstate being $h_{ij} \propto \pi_{ij} - \psi_{ij}$. This leads to oscillations of the gravitons during their propagation, and would naturally have observable consequences [144].

Vector sector

In the vector sector, defining $D_i = \chi_i - \tau_i$, $B_i = \beta_i - \dot\chi_i/N$ and similarly $\tilde B_i = \theta_i - \dot\tau_i/M$, the quadratic action can be represented in both branches as

$$\delta^{(2)}\,\mathcal{S}^V = \frac{M_{\rm Pl}^2}{4}\int \mathrm{d}t\mathrm{d}^3x N a^3 \left\{ (\partial_i B_j)^2 + \frac{\alpha^2\mathcal{X}^3 M}{N}\left(\partial_i \tilde B_j\right)^2 + \frac{2a^2\rho}{M_{\rm Pl}^2}\left(\beta_i + \frac{j_i}{\mathcal{N}_0}\right)^2 + \frac{2a^2\mathcal{X}^2\tilde\rho}{M_{\rm Pl}^2\, r}\left(\theta_i + \frac{\tilde j_i}{\tilde{\mathcal{N}}_0}\right)^2 \right.$$
$$\left. -\mu_T^2\,(\partial_i D_j)^2 - m^2\frac{(c_1\mathcal{X}+c_2)\,(1-\beta\mathcal{X}) - \beta\Gamma_1}{N}\partial_i D_j\,\partial^i\delta\ell^j \right\}\,, \tag{VI.69}$$

where μ_T^2 is the squared tensor mass (VI.68). The auxiliary field imposes the constraint $D_i = 0$, which eliminates the whole massive sector, so the action simply reduces to two linked copies of the usual GR vector sector. The phenomenology of the vector sector of MTBG is thus the same as the one of GR, separately in each metric. Moreover this clearly shows that no vector gravitational degree of freedom is dynamical, as in GR.

Scalar sector in the self-accelerating branch

In the scalar sector of the self-accelerating branch, all auxiliary fields enter linearly in the quadratic action. As variations with respect to $\delta\lambda$ and $\delta\bar\lambda$ yield redundant constraints, one has only the two relations

$$\zeta = \xi\,, \qquad \text{and} \qquad \chi = \tau\,. \tag{VI.70}$$

Those two conditions suffice to eliminate the whole mass sector, leaving only two copies of the usual GR scalar sector.

In the following, we eliminate the quantities j_0 and $\tilde j_0$ in favor of the gauge invariant density contrasts (VI.63). Moreover, we still have one freedom to fully fix the gauge, and thus we will take $\zeta = 0$. One can then integrate out j and $\tilde j$ as

$$j = -\mathcal{N}_0\left(v_m + a^2\beta\right)\,, \qquad \text{and} \qquad \tilde j = -\tilde{\mathcal{N}}_0\left(\tilde v_m + \mathcal{X}^2 a^2\theta\right)\,, \tag{VI.71}$$

so that the shift perturbations β and θ enter linearly and allow to express $\eta = \rho v_m/(2M_{\rm Pl}^2 H)$ and $\omega = \tilde{\rho}\tilde{v}_m/(2\alpha^2 M_{\rm Pl}^2 L)$. At this point χ vanishes by virtues of its own equation of motion and v_m and $\tilde{v}_m$ can be integrated out so to give

$$\delta^{(2)}\mathcal{S}^S_{\rm sa} = \int {\rm d}t{\rm d}^3x \left\{ \frac{Na^5\rho}{2k^2}\left(\frac{\dot{\delta}_m^2}{N^2} + 4\pi G_N\,\rho\,\delta_m^2\right) + \frac{Mb^5\tilde{\rho}}{2k^2}\left(\frac{\dot{\tilde{\delta}}_m^2}{M^2} + 4\pi\,\tilde{G}_N\,\tilde{\rho}\,\tilde{\delta}_m^2\right)\right\}\,, \tag{VI.72}$$

where we have naturally defined $G_N = (8\pi M_{\rm Pl}^2)^{-1}$ and $\tilde{G}_N = (8\pi\alpha^2 M_{\rm Pl}^2)^{-1}$ the Newton constants in both sectors. So the no-ghost conditions ($\rho > 0$ and $\tilde{\rho} > 0$) and the equations of motion for the density contrasts

$$\frac{1}{N}\frac{\rm d}{{\rm d}t}\left(\frac{\dot{\delta}_m}{N}\right) + 2H\,\frac{\dot{\delta}_m}{N} - 4\pi G_N\rho\,\delta_m = 0 \qquad \text{and} \qquad \frac{1}{M}\frac{\rm d}{{\rm d}t}\left(\frac{\dot{\tilde{\delta}}_m}{M}\right) + 2L\,\frac{\dot{\tilde{\delta}}_m}{M} - 4\pi\tilde{G}_N\tilde{\rho}\,\tilde{\delta}_m = 0\,, \tag{VI.73}$$

are strictly the same as in GR. It is thus clear that only two *dofs* propagate in the scalar sector, and that they can be identified as two matter *dofs*. Each of the equations (VI.73) reproduces exactly the phenomenology of a dust component in GR.

When using the k-essence ansatz (VI.64), we also recover exactly two copies of the usual GR quadratic action, notably the no-ghost conditions and squared sound speeds are given exactly by (VI.66). Therefore we can conclude that the phenomenology of the scalar sector of the self-accelerating branch is exactly given by two unrelated copies of GR. Note that, contrary to the fate of the self-accelerating branch of HRBG, the self-accelerating branch of MTBG is perfectly healthy, at least at the level of linear perturbations.

Scalar sector in the normal branch

In the scalar sector of the normal branch, the auxiliary field $\delta\ell$ enters linearly in the quadratic action, leading to the constraint

$$\zeta = \xi\,. \tag{VI.74}$$

As in the self-accelerating branch, we eliminate j_0 and $\tilde{j}_0$ in favor of the gauge invariant density contrasts (VI.63), and still have a remaining gauge freedom, that we use to fix $\chi = 0$. One can then integrate out j and $\tilde{j}$ *via* the same relations as in the self-accelerating branch (VI.71), so that the shift perturbations β and θ still enter linearly and allow to express $\eta(v_m, \dot{\zeta}, \delta\lambda, \delta\bar{\lambda})$ and $\omega(\tilde{v}_m, \dot{\zeta}, \delta\lambda, \delta\bar{\lambda})$. At this point $\delta\lambda$ and $\delta\bar{\lambda}$ enter linearly and can be used to express $\zeta(\delta_m, \tilde{\delta}_m, v_m, \tilde{v}_m)$ and $\tau(\delta_m, \tilde{\delta}_m, v_m, \tilde{v}_m)$. Finally, the fields v_m and $\tilde{v}_m$ are not dynamical and can be integrated out to give a complicated action for δ_m and $\tilde{\delta}_m$, that are both dynamical. Thus we can conclude that, also in the normal branch, the scalar sector of the theory propagates only two *dofs*, which can be identified as two matter *dofs*.

From a phenomenological point of view, let us take the sub-horizon (large k) limit. At leading k order, the action reads

$$\delta^{(2)}\mathcal{S}^S_{\rm norm} = \int {\rm d}t{\rm d}^3x \left\{ \frac{Na^5\rho}{2k^2}\left(\frac{\dot{\delta}_m^2}{N^2} + 4\pi\mathcal{G}_{11}\,\rho\,\delta_m^2\right) + \frac{Mb^5\tilde{\rho}}{2k^2}\left(\frac{\dot{\tilde{\delta}}_m^2}{M^2} + 4\pi\,\mathcal{G}_{22}\,\tilde{\rho}\,\tilde{\delta}_m^2\right) + \frac{Na^3b^2\rho\,\tilde{\rho}}{k^2}\,\pi\,\mathcal{G}_{12}\,\delta_m\,\tilde{\delta}_m\right\}\,, \tag{VI.75}$$

as the kinetic coupling term $\propto \dot{\delta}_m\,\dot{\tilde{\delta}}_m$ and the friction term $\propto \dot{\delta}_m\,\tilde{\delta}_m - \delta_m\,\dot{\tilde{\delta}}_m$ appear at subleading $\mathcal{O}(k^{-4})$ order. The no-ghost conditions ($\rho > 0$ and $\tilde{\rho} > 0$) are the same as in GR. Nevertheless, the phenomenology of the scalar sector is different from the GR one, due to the non-standard mass terms. Indeed, the components of the $\mathcal{G}$ matrix are given by

$$\frac{\mathcal{G}_{11}}{G_N} = 1 + \frac{8m^2\,(c_1\mathcal{X} + c_2)\,\mathcal{X}^4H^2\,(1-r) + 4m^2\mathcal{X}^3H^2\Gamma_1 + m^4\,(\mathcal{X}^2+\alpha^{-2})\,(r-2)\,\mathcal{X}^2\Gamma_1^2 - 6m^2H^2\mathcal{X}^3\Gamma_1\,\Omega_m}{\left[4H^2\mathcal{X} - m^2\,(\mathcal{X}^2+\alpha^{-2})\,\Gamma_1\right]^2}\,, \tag{VI.76a}$$

$$\frac{\mathcal{G}_{12}}{G_N} = \frac{16m^2H^2\mathcal{X}^3\left(c_1\mathcal{X}^2 - c_3\right)r - 32m^2H^2\mathcal{X}^4\left(c_1\mathcal{X} + c_2\right) + 4m^4\mathcal{X}^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1^2 + 24m^2H^2\mathcal{X}^3\Gamma_1\,\Omega_m}{\alpha^2\left[4H^2\mathcal{X} - m^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1\right]^2}\,, \tag{VI.76b}$$

$$\frac{\mathcal{G}_{22}}{\tilde{G}_N} = 1 + \frac{8m^2H^2\mathcal{X}\left(c_2\mathcal{X} + c_3\right)r + 4m^2H^2\mathcal{X}\left(c_1\mathcal{X}^2 - c_3\right) - m^4\left(\mathcal{X}^2 + \alpha^{-2}\right)r\,\Gamma_1^2 - 6m^2H^2\mathcal{X}\Gamma_1\,\Omega_m}{\alpha^2 r\left[4H^2\mathcal{X} - m^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1\right]^2}\,, \tag{VI.76c}$$

with the g-sector matter density $\Omega_m \equiv \rho/(3M_{\rm Pl}^2H^2)$. The relations (VI.56) were used to eliminate $\tilde{\rho}$ in favor of the g-observable Ω_m.

When using the k-essence ansatz (VI.64), we also recover, at the leading sub-horizon approximation, the usual no-ghost conditions and squared sound speeds (VI.66), and a non-standard, non-diagonal mass matrix. It should be noticed that if the dynamics of the background leads to $4H^2\mathcal{X} \to m^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1$, then linear perturbation theory would break down, and in this case the system needs to be further studied. This possibility was present also in MTMG [138].

When the Hubble constant is notably larger than the mass of the massive graviton, as it is the case in the early Universe, one can expand the previous relations as

$$\frac{\mathcal{G}_{11}}{G_N} = 1 + \frac{m^2\mathcal{X}}{2H^2}\left[\frac{3c_1\mathcal{X}^2 + 4c_2\mathcal{X} + c_3}{2} - \left(c_1\mathcal{X}^2 + c_2\mathcal{X}\right)r - \frac{3\Gamma_1}{4}\,\Omega_m\right] + \mathcal{O}\left(\frac{m^4}{H^4}\right)\,, \tag{VI.77a}$$

$$\frac{\mathcal{G}_{12}}{G_N} = \frac{m^2\mathcal{X}}{H^2\alpha^2}\left[\left(c_1\mathcal{X}^2 - c_3\right)r - 2\,\mathcal{X}\left(c_1\mathcal{X} + c_2\right) + \frac{3\Gamma_1}{2}\,\Omega_m\right] + \mathcal{O}\left(\frac{m^4}{H^4}\right)\,, \tag{VI.77b}$$

$$\frac{\mathcal{G}_{22}}{\tilde{G}_N} = 1 + \frac{m^2}{2H^2\alpha^2 r\mathcal{X}}\left[\frac{c_1\mathcal{X}^2 - c_3}{2} + \left(c_2\mathcal{X} + c_3\right)r - \frac{3\Gamma_1}{4}\,\Omega_m\right] + \mathcal{O}\left(\frac{m^4}{H^4}\right)\,. \tag{VI.77c}$$

Deviations from the usual phenomenology are thus strongly suppressed. Nevertheless, if one seeks to mimic the late-time acceleration of the Universe by means of the effective dark fluid (VI.55a), it requires $m \sim H_0$, so the deviations can be large at the present epoch.

Considering only one dust component, minimally coupled in the g-sector, the action reads

$$\delta^{(2)}\mathcal{S}^S_{\rm norm} = \int {\rm d}t{\rm d}^3x\frac{Na^5\rho}{2k^2}\left(\frac{\dot{\delta}_m^2}{N^2} + 4\pi G_{\rm eff}\,\rho\,\delta_m^2\right) + \mathcal{O}\left(k^{-4}\right)\,. \tag{VI.78}$$

Once again, the no-ghost condition ($\rho > 0$) is the same as in GR, and the dust experiences the effective Newton constant

$$\begin{aligned}\frac{G_{\rm eff}}{G_N} &= 1 + \frac{m^2\Gamma_1\mathcal{X}^2}{4H^2\mathcal{X} - m^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1} + \frac{24m^2H^4\mathcal{X}^4\left(c_1\mathcal{X}^2 - c_3\right)\,\Omega_m}{\left[4H^2\mathcal{X} - m^2\left(\mathcal{X}^2 + \alpha^{-2}\right)\Gamma_1\right]^3} \\ &= 1 + \frac{m^2}{4H^2}\left[\Gamma_1\mathcal{X} + \frac{3\left(c_1\mathcal{X}^2 - c_3\right)\mathcal{X}}{2}\,\Omega_m\right] + \mathcal{O}\left(\frac{m^4}{H^4}\right)\,.\end{aligned} \tag{VI.79}$$

Therefore we find that $G_{\rm eff}/G_N = 1 + \mathcal{O}(1) \times m^2M_{\rm Pl}^2/\rho_m$. The situation is similar to what happens in MTMG [138]. The only gravity modes introduced by the theory are tensorial, no extra scalar *dof* is present, and m is related to the mass of the massive graviton. Then, whenever the mass-energy of the environment, ρ_m, becomes much larger than $m^2M_{\rm Pl}^2$, we would naturally expect the graviton (sourced by matter fields) to be ultra-relativistic, that $G_{\rm eff}/G_N \to 1$, and that dust matter fields in the dark sector coupled to the f-metric would decouple from the dark matter/baryonic components belonging to our physical sector coupled to the g-metric, at least, as long as linear perturbation theory is concerned.

Cosmological perturbations in a nutshell

Let us sum up the cosmological phenomenology of MTBG (VI.40). At the background level, we found two branches of solutions that are equivalent to the ones present in HRBG, the only difference being that, in MTBG, they are both perfectly healthy and do not suffer from either strong-coupling issues or the Higuchi ghost.[10] In the first (self-accelerating) branch, the mass term produces two pure cosmological constants, one for each metric, and the phenomenology of the scalar and vector perturbations is the same as in GR. The tensor perturbations acquire an effective mass and oscillate, as usual in bigravity theories. It is thus interesting to note that this branch provides the simplest testing ground for those oscillations [144], massive graviton dark matter [22] as well as the enhancement mechanism of stochastic gravitational waves [201]. Indeed it perfectly mimics the ΛCDM model at the background, scalar and vector perturbative levels, but has a non-canonical tensor sector.

In the background of the second (normal) branch, the mass term behaves as two dark fluids, with equations of state given by (VI.55). While the vector sector is the same as in GR, the tensor perturbations acquire an effective mass and oscillate as in the self-accelerating branch. More interestingly, the scalar perturbations have a non-trivial phenomenology, with an effective Newton constant depending on the graviton mass, the background energy density and the scale of perturbations. It would be interesting to investigate this branch more deeply, to see how well this modified gravitational strength can accommodate the most recent cosmological data.

VI.3.3 Gravitational Cherenkov radiation in MTBG

We finish this chapter by discussing the possible gravitational Cherenkov effect in MTBG. In fact, as it is clear from the action of tensor perturbations (VI.67), gravitational waves propagate along light cones, despite the (weak) breaking of temporal diffeomorphism invariance. Therefore, and if r is less than unity, the f-sector gravitational perturbations travel slower than the g-sector electromagnetic waves. This would yield a gravitational Cherenkov effect, meaning that ultra-high-energy cosmic rays in the g-sector propagating at speeds greater than c_f would loose their energy by emission of coherent f-gravitons. As ultra-high-energy cosmic rays are observed on Earth, this mechanism could be used to put a lower bound on r [287]. However, such an effect has been investigated in detail in HRBG and the bounds on (r, m^2) were found to be extremely weak [254]. An essential reason for this is that in HRBG, cosmic rays in the g-sector do not couple directly to the f-metric. Although we have not performed the detailed computation, we expect that a similar (if not the same) conclusion should hold in MTBG. Indeed the enforcement of the ADM vielbeins and the additional constraints that we use are not tensorial in nature, so it is expected that they would not significantly impact the computation of the Cherenkov effect. Moreover the quadratic action of the tensor perturbations (VI.67) takes exactly the same form as the one in [254], and the matter couplings are the same in both works. Therefore we expect that the gravitational Cherenkov radiation would not put stringent bounds on the parameter space of MTBG.

[10] This Higuchi ghost usually appears in de Sitter space-times when the mass of the graviton is positive and smaller than $\sqrt{2}H$. In such regimes (than are easily reached in the early Universe), the scalar mode acquires a negative norm, and thus becomes a spurious degree of freedom [231]. As we have no dynamical scalar *dof*, we are automatically free from this instability.

Chapter VII

Testing the strong field regimes of minimal theories

The cosmological phenomenology of minimal theories is usually derived when they are constructed, as we did in the previous chapter when constructing a minimal theory of bigravity (MTBG). Indeed, deriving the linear cosmological perturbations is an essentially part of the proof that those theories propagate the accurate number of *dofs*. In addition, more specific cosmological predictions have been investigated for the original minimal theory of massive gravity (MTMG), such as redshift space distortions [139], integrated Sachs-Wolfe effects [100] or non-linear dynamics *via* N-body simulations [219]. Those computations gave noteworthy results as MTMG can achieve better fits with the current cosmological data than the canonical ΛCDM framework. In addition, the cosmological phenomenology of type-I models has been investigated and constrained in [23], and some other type-II theories have been confronted against the Planck data: they also gave better results than the canonical ΛCDM model [24, 140, 156]. The dispersion relation of gravitational radiation as well as the dynamics of Bianchi-I types of Universe were also investigated in a specific model [26]. In addition, an inflationary scenario has been implemented in a type-II model, with interesting deviations from the usual paradigm [321].

If very interesting *per se*, we have to point out that all those results belong to the same type of tests, namely cosmological ones. In order to have better phenomenological knowledge and observational constraints on minimal theories, it is thus necessary to exit the cosmological realm and to study them in others regimes. In this spirit, this chapter is devoted to the short-distance, strong-field regimes of two minimal theories: the original MTMG [137] and a type-II theory dubbed VCDM [153]. This will be realized by constructing black hole and star space-times in both those specific theories. We emphasize that the presented tests are currently the only ones that have been performed out of the cosmological realm for minimal theories.

VII.1 Strong field regime of the minimal theory of massive gravity

VII.1.1 The minimal theory of massive gravity

As discussed in the previous chapter, the minimal theory of massive gravity is the first minimal theory that has been constructed [137, 138]. The method that has been used to construct it is very similar to what we have presented for MTBG in sec. VI.2, but was first achieved in the vielbein language. This theory is the minimalization of the ghost-free massive gravity, that is given by the action (VI.1) with $\alpha = 0$ and the potential (VI.17), and thus contains a dynamical metric $g_{\mu\nu}$ and a *fiducial* (*ie.* non-dynamical) one, $f_{\mu\nu}$. In contrast to MTBG, the metric $f_{\mu\nu}$ is here part of the definition of theory and, as such, frozen. Imposing the ADM gauge (VI.18) for the two sets of vielbein gives a precursor theory, propagating three *dofs* instead of the five present in the original theory. The unwanted scalar *dof* is then removed by the addition of appropriate constraints. As expected by the use of ADM decomposition, the resulting theory breaks the temporal sector of the diffeomorphisms invariance. Note that, contrary to MTBG, MTMG bears a uniqueness feature: it is the unique minimal theory of massive gravity that is invariant under three-dimensional diffeomorphisms.

The building blocks of MTMG are thus two metrics, written in their usual ADM decompositions (VI.9) and a set of four Lagrange multipliers $\{\lambda, \lambda^i\}$. In this section, we use the notations introduced in sec. VI.2, to the only exception that the lapse of the fiducial metric is denoted as N_f, to avoid confusion with the symbol M that will be used as a mass. The fiducial metric $f_{\mu\nu}$ only enters the theory *via* its lapse N_f, the square-root matrix $\mathfrak{K}^p{}_q$ and the matrix $\zeta^p{}_q$ that obey

$$\mathfrak{K}^p{}_r\mathfrak{K}^r{}_q \equiv \gamma^{ps}\phi_{sq}\,, \qquad \text{and} \qquad \zeta^p{}_q \equiv \frac{1}{2N_f}\,\phi^{ps}\,\partial_t\phi_{sq}\,. \tag{VII.1}$$

The action of MTMG is then given by

$$\mathcal{S}_{\mathrm{MTMG}} = \mathcal{S}_{\mathrm{EH}}\left[g_{\mu\nu}\right] + \mathcal{S}_{\mathrm{mat}}\left[\psi_{\mathrm{mat}};g_{\mu\nu}\right] - \frac{M_{\mathrm{Pl}}^2 m^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{W}\,, \tag{VII.2}$$

where $\mathcal{S}_{\mathrm{EH}}$ and $\mathcal{S}_{\mathrm{mat}}$ are respectively the canonical Einstein-Hilbert kinetic term and the usual action for matter fields minimally coupled to the metric $g_{\mu\nu}$. The potential term $\mathcal{W}$ is the diffeomorphisms breaking part of the action[1] and explicitly reads

$$\mathcal{W} \equiv \frac{N_f}{N}\,\mathcal{E} + \tilde{\mathcal{E}} + \frac{N_f\lambda}{N}\left(\hat{\mathcal{F}}^i{}_j\zeta^j{}_i - \tilde{\mathcal{E}}\zeta^p{}_p + \tilde{\mathcal{F}}^i{}_jK^{jk}\gamma_{ik}\right) + \frac{N_f}{N}\tilde{\mathcal{F}}^i{}_j\,\mathcal{D}_j\lambda^i - \frac{m^2N_f^2\lambda^2}{4N^2}\left(\tilde{\mathcal{F}}^i{}_j - \frac{\tilde{\mathcal{F}}^p{}_p}{2}\delta^i_j\right)\tilde{\mathcal{F}}^j{}_i\,, \tag{VII.3}$$

where we recall that K^{ij} is the extrinsic curvature (VI.11) and $\mathcal{D}_i$ the covariant derivative compatible with γ_{ij}. The quantities entering $\mathcal{W}$ are given by

$$\mathcal{E} \equiv \sum_{n=0}^{3} c_n e_{3-n}\left(\mathfrak{K}\right)\,, \qquad \tilde{\mathcal{E}} \equiv \sum_{n=1}^{4} c_n e_{4-n}\left(\mathfrak{K}\right)\,, \qquad \hat{\mathcal{E}} \equiv \sum_{n=2}^{4} c_n e_{5-n}\left(\mathfrak{K}\right)\,, \tag{VII.4a}$$

$$\tilde{\mathcal{F}}^p{}_q \equiv \frac{\delta\tilde{\mathcal{E}}}{\delta\mathfrak{K}^q{}_p}\,, \qquad \hat{\mathcal{F}}^p{}_q \equiv \frac{\delta\hat{\mathcal{E}}}{\delta\mathfrak{K}^q{}_p}\,, \tag{VII.4b}$$

the elementary polynomials being expressed in eq. (VI.13). Note here the striking similarities with the action of MTBG (VI.40).

Cosmological phenomenology of MTMG

Just as in MTBG, the cosmology of MTMG contains two branches. Taking two FLRW line elements (VI.6) and homogeneous Lagrange multipliers, the equation of motion (*eom*) for λ reads

$$\left(c_3 + 2c_2\mathcal{X} + c_1\mathcal{X}^2\right)\tilde{\mathcal{Q}} = 0\,, \tag{VII.5}$$

where $\mathcal{X}$ is again the ratio of the two scale factors, and $\tilde{\mathcal{Q}}$ is some complicated expression (the interested reader will refer to eq. (83) of [138]). Note that the combination $c_3 + 2c_2\mathcal{X} + c_1\mathcal{X}^2$ is simply the quantity Γ_1 introduced in eq. (VI.43). The "self-accelerating" branch is still given by $\Gamma_1 = 0$ and has a phenomenology similar to the one of MTBG (excepted for the graviton oscillations of course): it contains a natural cosmological constant, a tensor sector with a non-vanishing mass and the scalar and vector sectors are the same as in GR. On the other hand, in the "normal" branch (given by $\tilde{\mathcal{Q}} = 0$), the mass terms give rise to a dark fluid, the tensor sector is massive and the scalar sector has a different phenomenology than GR, leading to a modified growth of perturbations.

[1] Thus, if the mass parameter m is chosen to provide a late-time acceleration of the expansion of the Universe, this breaking is weak and only occurs at cosmological scales, just as in the case of MTBG.

A valuable lemma

Leaving the phenomenology of the strong field regime in the normal branch for future studies, we focus on the self-accelerating branch only. In this branch the search is heavily simplified by the use of the following lemma, that we derived and exploited together with A. De Felice, S. Mukohyama and M. Oliosi in [148].

Lemma : Any GR solution that can be written with flat constant-time surfaces is a solution of the self-accelerating branch of MTMG, with the additional feature of a bare cosmological constant.

The proof of this lemma is given as follows. A metric with flat constant-time surfaces is written as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a^2(t)\,\delta^F_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)\,, \tag{VII.6}$$

where the laspe and shifts are free functions of the four-dimensional coordinates x^μ, $a(t)$ is a function of t corresponding to a scale factor and δ^F_{ij} is a flat three-dimensional metric. Similarly, we use the fiducial metric

$$f_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N_f^2(t)\mathrm{d}t^2 + b^2(t)\,\delta^F_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{VII.7}$$

and the Lagrange multipliers are kept in their most general form $\lambda(x^\mu)$ and $\lambda^i(x^\mu)$.

The *eom* for λ reveals the same splitting in two branches as in cosmology, and we will work in the self-accelerating one, where $\mathcal{X} \equiv b(t)/a(t)$ is constant. In this branch, the *eom* for λ^i are automatically satisfied, so one can safely choose the Lagrange multipliers to be equal to their cosmological values, $\lambda = \lambda^i = 0$. With this choice, the Einstein equations for the metric (VII.6) are

$$M^2_{\rm Pl}\left[G_{\mu\nu} + \frac{m^2}{2}\left(c_4 + 2c_3\mathcal{X} + c_2\mathcal{X}^2\right)g_{\mu\nu}\right] = T_{\mu\nu}\,, \tag{VII.8}$$

where $T_{\mu\nu}$ is the usual stress-energy tensor of matter. Thus the Einstein equations are indeed the same as in GR with an effective cosmological constant

$$\Lambda_{\rm eff} \equiv \frac{m^2}{2}\left(c_4 + 2c_3\mathcal{X} + c_2\mathcal{X}^2\right)\,. \tag{VII.9}$$

QED

This lemma uncovers a large class of solutions of MTMG, given by all solutions of GR (with a cosmological constant) of the form (VII.6). Deriving and classifying all those metrics appears as an unmanageable task, therefore we will adopt a less ambitious approach and, starting from a chosen interesting space-time, we will seek for the appropriate diffeomorphism that puts it in the form (VII.6). Note that while the chosen examples are all spherically symmetric systems, this assumption was not made in the lemma.

VII.1.2 Non-rotating black holes and stars

Let us first apply this approach to non-rotating black holes, by seeking for the diffeomorphism that foliates GR black hole space-times in flat constant-time surfaces.

The metric of a generic spherically symmetric system can be written as

$$\mathrm{d}s^2 = -f(t,r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{2\mu(t,r)}{r}} + r^2\mathrm{d}^2\Omega\,, \tag{VII.10}$$

where $\mathrm{d}^2\Omega$ is the usual line element on the two-sphere. By a simple change of the time coordinate $t \to \tau + T(\tau, r)$ satisfying

$$\left(\frac{\partial T(\tau,r)}{\partial r}\right)^2 = \frac{1}{f(\tau,r)}\frac{2\mu(\tau,r)}{r-2\mu(\tau,r)}\,, \tag{VII.11}$$

the metric (VII.10) can be put in a spatially flat form

$$\mathrm{d}s^2 = -N^2\mathrm{d}\tau^2 + \left[(\mathrm{d}r + \beta\,\mathrm{d}\tau)^2 + r^2\mathrm{d}^2\Omega\right] , \tag{VII.12}$$

with

$$N^2 = (1+\partial_\tau T)^2 f\left(1+(\partial_r T)^2 f\right) , \qquad \text{and} \qquad \beta = -\partial_r T\ (1+\partial_\tau T)\, f\,. \tag{VII.13}$$

Simplest non-rotating black holes

The simplest representation of a non-rotating black hole in GR is given by the Schwarzschild-de Sitter solution in Schwarzschild coordinates, *ie.* by (VII.10) with $\mu(t,r) = M - \Lambda r^2/6$ and $f(t,r) = 1 - 2\mu(r)/r$, with both M and Λ constant. Applying the transformation as in eq. (VII.11) and (VII.13), it simply comes

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left(\mathrm{d}r \pm \sqrt{\frac{2M}{r} - \frac{\Lambda r^2}{3}}\mathrm{d}\tau\right)^2 + r^2\mathrm{d}^2\Omega\,. \tag{VII.14}$$

Going to the pure Schwarzschild solution (*ie.* taking $\Lambda = 0$), this particular form of the metric has been long known and was firstly independently proposed by P. Painlevé [299] and A. Gullstrand [216].

By the lemma, the metric (VII.14) is a solution of MTMG for a flat fiducial metric $\phi_{ij} = \mathcal{X}^2\delta^F_{ij}$ with $\mathcal{X}$ constant and $\lambda = \lambda^i = 0$, provided that $\Lambda = \Lambda_{\mathrm{eff}}$. This demonstrates the existence of static black holes in MTMG, identical to those in GR.[2]

Inclusion of matter

Let us now turn to the study of time-dependent non-rotating stars in MTMG, filled with a fluid of density ρ obeying a general barotropic equation of state $P(\rho)$. If such objects are absolutely not likely to exist, their construction is a first step towards realistic star solutions. More importantly, such solutions allow to test if the addition of matter is harmless or if it introduces instabilities. As exposed previously, in MTMG two solutions that differ from a temporal diffeomorphism are physically distinct, and thus including matter may lead to a coordinate singularity at the center of solutions. As a condition for the regularity of solutions with matter, we require that in the $r \to 0$ limit, the extrinsic curvature remains regular and becomes isotropic. This translates into requesting that the anisotropic part of the extrinsic curvature, $K_{rr} - K/3$ (where K is the trace of K_{ij}), vanishes at the center. For the metric ansatz (VII.12) it is sufficient to show that

$$\lim_{r\to 0}\left(\partial_r\beta - \frac{\beta}{r}\right) = 0\,. \tag{VII.15}$$

For this purpose we expand all quantities around $r = 0$, as

$$N(\tau,r) = \sum_{n=0}^{\infty} N_n(\tau) r^n\,, \qquad \beta(\tau,r) = \sum_{n=0}^{\infty}\beta_n(\tau) r^n\,, \qquad \text{and} \qquad \rho(\tau,r) = \sum_{n=0}^{\infty}\rho_n(\tau) r^n\,. \tag{VII.16}$$

In the lowest order in r, the $\tau\tau$ component of the Einstein equations becomes $M_{\mathrm{Pl}}^2\beta_0^2/r^2 = 0$, imposing $\beta_0 = 0$. Then, taking the next relevant order in r, one finds that the $\tau\tau$ and rr components of the

[2] More generally, *ie.* beyond the lemma, $\lambda(r) = \lambda_0$ and $\lambda^i\partial_i = \pm\lambda_0\sqrt{2M/r - \Lambda r^2/3}\,\partial_r$ are also allowed, where λ_0 is a constant. However, if we require a matching with the cosmological boundary conditions, one has to impose $\lambda_0 = 0$ so that $\lambda = \lambda^i = 0$.

Einstein equations are $N_0^2\rho_0 = 3M_{\rm Pl}^2\beta_1^2/2$ and $N_1/N_0 = 0$, leaving $N_0 \neq 0$ and $N_1 = 0$. Iterating this procedure yields $\beta_2 = 0$ and thus

$$\beta(\tau, r) = \pm \frac{N_0\, r}{M_{\rm Pl}} \sqrt{\frac{2}{3}\rho_0} + \mathcal{O}(r^3)\,, \tag{VII.17}$$

in which we have omitted the negligible contribution from the effective cosmological constant term. It is thus obvious that the condition (VII.15) is satisfied, *ie.* that the inclusion of matter is harmless and that we can have healthy time-dependent non-rotating stars in MTMG. Note that our argument does not depend on the effective equation of state, and applies to both dynamical and static configurations.

Regular solutions found here include static solutions with matter as a special case. As a concrete and simple example, let us consider the interior Schwarzschild solution, as described for instance in [332]. The solution, after matching to the Schwarzschild solution of mass M at the stellar radius r_0, is described by (VII.10) with

$$\mu = M \left(\frac{r}{r_0}\right)^3\,, \qquad \text{and} \qquad f = \left[\frac{3}{2}\sqrt{1 - \frac{2M}{r_0}} - \frac{1}{2}\sqrt{1 - \frac{2\mu(r)}{r}}\right]^2\,, \tag{VII.18}$$

where we have once again omitted the contribution of the effective cosmological constant as physically negligible at scales of order of the stellar radius. After a transformation of the form of eqs. (VII.11) and (VII.13), we recover a spatially flat space-time with

$$N^2 = \frac{rf}{r - 2\mu}\,, \qquad \text{and} \qquad \beta = \pm\sqrt{\frac{2\mu f}{r - 2\mu}}\,. \tag{VII.19}$$

Thanks to the lemma, we can claim that this is a solution of MTMG with $\lambda = \lambda^i = 0$.

As for the outer part of the non-rotating stars, just as in GR, they are described by the black hole solutions (VII.14). This important result allows us to extract the PPN parameters $\beta^{\rm PPN}$ and $\gamma^{\rm PPN}$, that are unity, as in GR. Therefore the self-accelerating branch of MTMG passes the most usual Solar-system tests as successfully as GR does. Nevertheless, a more detailed study would be necessary to obtain all other parameters, and thus to put constraints using the whole range of Solar-system tests.

Matching black holes to cosmology

The presence of time-dependent solutions with matter allows to construct non-homogeneous cosmological solutions, as discussed in [130] for dRGT massive gravity. Those solutions are good approximations of the canonical FLRW Universe for patches larger than the Vainshtein radius. Fortunately, we can go beyond this approximation in MTMG and embed black holes into homogeneous and isotropic cosmological setups, as presented in the following.

Indeed the first result (VII.14) can be extended by noticing that the special form of the metric used in the lemma (VII.6) allows for a non-trivial scale factor $a(t)$. To include non-trivial $a(t)$, one has also to implement a non-trivial scale factor in the fiducial sector as $\phi_{ij} = b^2(t)\delta^F_{ij}$, so that the ratio $\mathcal{X} \equiv b(t)/a(t)$ remains constant for an expanding universe ($\partial_t a > 0$), as required in the self-accelerating branch. Note that in MTMG this class of solutions is distinct from the generic solution (VII.12). Indeed, in MTMG in the unitary gauge, due to the breaking of the invariance under diffeormorphisms, two solutions related by a temporal diffeomorphism are physically different and thus the Schwarzschild-de Sitter black hole solution of the form (VII.14) is not equivalent to a black hole embedded in a de Sitter Universe.

Starting in GR with the Schwarzschild-de Sitter black hole written in Schwarzschild coordinates, the aim is thus to write it in "generalized Painlevé-Gullstrand" coordinates

$$\mathrm{d}s^2 = -N(t,r)^2\mathrm{d}t^2 + a^2(t)\left[(\mathrm{d}r + \beta(t,r)\,\mathrm{d}t)^2 + r^2\mathrm{d}^2\Omega\right]\,. \tag{VII.20}$$

This can be done with a time redefinition of the form of eq. (VII.11) and (VII.13) together with $r \to a(t) r$ (the function $a(t)$ being unspecified), to have

$$N(t,r) = 1, \qquad \text{and} \qquad \beta(t,r) = \frac{\partial_t a}{a} r \pm \sqrt{\frac{2M}{a^3 r} - \frac{\Lambda r^2}{3}}\,. \tag{VII.21}$$

Turning to MTMG, we inject the metric (VII.20) in the *eoms*, taking spherically symmetric ansätze for the auxiliary fields as $\lambda = \lambda(t,r)$ and $\lambda^i \partial_i = \lambda^r(t,r)\,\partial_r$. This yields

$$N(t,r) = 1, \qquad \text{and} \qquad \beta(t,r) = \frac{\partial_t a}{a} r \pm \sqrt{\frac{2\tilde{M}(t)}{a^3 r} - \frac{\Lambda_{\rm eff}\, r^2}{3}}\,, \tag{VII.22}$$

with the time-dependent modified mass function

$$\tilde{M}(t) = M_0 \pm \frac{m^2\,(c_1 \mathcal{X} + c_2)\,\mathcal{X}}{2\sqrt{3}} \int_{-\infty}^{t} \mathrm{d}\tau\, a^3(\tau) N_f(\tau) \tilde{\lambda}(\tau)\,, \tag{VII.23}$$

where M_0 is a constant and $\tilde{\lambda}(t)$ an arbitrary function of t. The Lagrange multipliers λ and λ^r are

$$\lambda(t,r) = 0, \qquad \text{and} \qquad \lambda^r(t,r) = \frac{\tilde{\lambda}(t)}{\sqrt{r\left(\sqrt{3}\tilde{M}(t) - \Lambda_{\rm eff}\, r^3\right)}}\,. \tag{VII.24}$$

Considering a BH or an exterior solution of a star formed from smooth and asymptotically FLRW initial data with matter, both λ and λ^r should vanish at spatial infinity to recover their cosmological values. Also, $\lambda^r(t,r)$ should vanish at $r=0$ for regularity, at least until a physical singularity forms there. Thus, considering black holes formed by gravitational collapse, it comes $\tilde{\lambda}(t) = 0$ due to regular initial data, meaning that $\tilde{M}(t) = M_0$. In this case the solution (VII.22) in MTMG recovers the solution (VII.20) in GR, with $M_0 = M$ and $\Lambda_{\rm eff} = \Lambda$.

Contrary to the solutions found in [130] and [217] for dRGT massive gravity, this solution (with $\tilde{\lambda} = 0$) in the $M_0 \to 0$ limit is strictly homogeneous and isotropic and is free from strong coupling issues, which seems to be a unique and significant feature of MTMG among massive gravity theories. Indeed, it has been shown that in the canonical massive gravity, no static solution is healthy, although there exist time-dependent non-GR solutions [316].

VII.1.3 Towards rotating solutions

Until now, we have presented a collection of corollaries to the lemma: (i) spherically symmetric static solutions in vacuum, (ii) spherically symmetric solutions with matter which are either time dependent or time independent, and (iii) a Schwarzschild-de Sitter solution matched with a de Sitter background in the FLRW form. As it appears that most of the detected black holes do spin, the next step would naturally be to construct a rotating vacuum solution of MTMG. The most straightforward way to proceed in MTMG would be by repeating the preceding strategy and finding a change of coordinates that puts the Kerr-de Sitter metric in the form (VII.6). Using the lemma we could thus claim that we have obtained a rotating black hole, solution of MTMG.

Unfortunately a study performed by A. Garat and R. H. Price [203][3] seems to indicate that such change of coordinates does not exist. Starting from a Kerr space-time with spin parameter a, written in Boyer-Lindquist coordinates (t, r, θ, ϕ), they performed a slicing $t = F(r, \theta, a)$, required conformally flatness for the induced spatial metric, and demonstrated that such requirement was not possible to achieve. As our target metric (VII.6) is only a specific case of metrics that have conformally flat spatial sections, it seems that this is a no-go theorem for the simplest implementation

[3] Kindly brought to our attention by E. Gourgoulhon.

of rotating black holes in MTMG. However their work cannot be directly applied to our case for two reasons. First they started from the Kerr metric, with vanishing cosmological constant, whereas the lemma would in general involve a Kerr-de Sitter-like geometry. More importantly they required the spinless limit of the foliation to be given by a constant, thus the foliated Kerr space-time boils down to the Schwarzschild metric, written in Schwarzschild coordinates. This restriction reduces drastically the number of possibilities, and notably forbids to consider spinning completions of the Painlevé-Gullstrand coordinates. As we know that we should seek for a space-time that reduces to the metric (VII.14) in the spinless limit, we have to extend the work done in [203] to apply it to MTMG.

In a collaboration with A. De Felice, S. Mukohyama and M. Oliosi, we have thus extended [203] for a Kerr geometry, taking the spinless limit $F(r,\theta,0) = f_0(r)$ [151]. We had to expand the foliation up to the fifth power in a to prove that conformally flat slicings were indeed not achievable. Nevertheless, due to the fact that we worked with a Kerr geometry, this result does not strictly forbid to use the lemma for constructing Kerr-de Sitter-like solutions of MTMG.

In the following, we will reproduce the computation performed in app. A of [151] that implements a general coordinate change in the Kerr-de Sitter metric, thus directly answering to the question of the usefulness of the lemma for rotating solutions. This method is less subtle than the slicing $t = F(r,\theta,a)$, but it has the merit of encompassing all possible options. As we will find out that no coordinate change can satisfy our requirements, we will finally conclude that the lemma can definitively not be applied to construct rotating vacuum solutions of MTMG.

The Kerr-de Sitter line element written in Boyer-Lindquist-like coordinates $x^\mu = (t,r,\theta,\phi)$ reads

$$\begin{aligned}\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = &-\frac{\tilde{\Delta} - a^2\zeta\sin^2\theta}{\Xi}\,\mathrm{d}t^2 + \frac{\Sigma}{\tilde{\Delta}}\,\mathrm{d}r^2 + \frac{\Sigma}{\zeta}\,\mathrm{d}\theta^2 + \frac{\left(r^2+a^2\right)^2\zeta - a^2\tilde{\Delta}\sin^2\theta}{\Xi}\,\sin^2\theta\mathrm{d}\phi^2 \\ &- \frac{2a\left[6Mr - \Lambda(r^2+a^2)\Sigma\right]}{3\,\Xi}\,\mathrm{d}t\mathrm{d}\phi\,,\end{aligned} \tag{VII.25}$$

where Λ is a cosmological constant and

$$\begin{aligned}\tilde{\Delta} &\equiv \left(r^2+a^2\right)\left(1+\frac{\Lambda r^2}{3}\right) - 2Mr\,, &\qquad \Sigma &\equiv r^2 + a^2\cos^2\theta\,, \\ \Xi &\equiv \Sigma\left(1-\frac{\Lambda a^2}{3}\right)^2\,, &\qquad \zeta &\equiv 1 - \frac{\Lambda a^2\,\cos^2\theta}{3}\,.\end{aligned} \tag{VII.26}$$

As expected, in the $\Lambda \to 0$ limit, eq. (VII.25) reduces to the usual Kerr metric in Boyer-Lindquist coordinates.

As in the case of non-rotating black holes, we will first aim to achieve purely flat spatial sections.[4] This is not contradictory with the Kerr-de Sitter geometry, as the metric (VII.25) is static. We will perform a general coordinate change $x^\mu \to \chi^\mu(a, x^\nu) = (\tau,\rho,\vartheta,\varphi)$ and require that the spatially induced metric is flat

$$\gamma_{ij} \equiv \bar{g}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\chi^i}\frac{\mathrm{d}x^\nu}{\mathrm{d}\chi^j} = \hat{\delta}_{ij}\,, \tag{VII.27}$$

where $\hat{\delta}_{ij}$ is the usual three-dimensional Euclidean metric in spherical coordinates

$$\hat{\delta}_{ij}\mathrm{d}\chi^i\mathrm{d}\chi^j = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\vartheta^2 + \rho^2\sin^2\vartheta\,\mathrm{d}\varphi^2\,. \tag{VII.28}$$

We also require the change of coordinates to be invertible, *ie.* that the Jacobian of the transformation is non-vanishing

$$\mathcal{J} \equiv \det\left[\frac{\partial\chi^\alpha}{\partial x^\mu}\right] \neq 0\,. \tag{VII.29}$$

[4] Note that the "rotating" Minkowskian metric, *ie.* written in Born coordinates, has purely flat spatial sections.

In the spinless limit $a \to 0$, we naturally already know the result: the Boyer-Lindquist-like coordinates reduce to the Schwarzschild-de Sitter ones and thus the transformation to apply is the extended Painlevé-Gullstrand one that was applied in the previous section to obtain the metric (VII.14)

$$t = \tau + \int^{\rho} \mathrm{d}u \, \frac{\sqrt{2u\,\mu(u)}}{u - 2\,\mu(u)}, \qquad x^i = \chi^i \,, \tag{VII.30}$$

where the effective mass is given by $\mu(r) = M - \Lambda r^3/6$. Starting from this zeroth-order solution, we expand the coordinate change in a as

$$t = \tau + \int^{\rho} \mathrm{d}u \, \frac{\sqrt{2u\,\mu}}{u - 2\mu} + \sum_{n\geq 1} a^n \, T^{(n)}(\rho, \vartheta, \varphi) \,, \tag{VII.31a}$$

$$r = \rho + \sum_{n\geq 1} a^n \, R^{(n)}(\rho, \vartheta, \varphi) \,, \tag{VII.31b}$$

$$\theta = \vartheta + \sum_{n\geq 1} a^n \, \Theta^{(n)}(\rho, \vartheta, \varphi) \,, \tag{VII.31c}$$

$$\phi = \varphi + \sum_{n\geq 1} a^n \, \Phi^{(n)}(\rho, \vartheta, \varphi) \,. \tag{VII.31d}$$

Note the absence of temporal dependencies in the functions $(T^{(n)}, R^{(n)}, \Theta^{(n)}, \Phi^{(n)})$, as we start from a static metric and aim for flat constant-time surfaces. We will then solve order by order the equations

$$\mathcal{E}_{ij} \equiv \gamma_{ij} - \hat{\delta}_{ij} = \sum_{n\geq 1} a^n \mathcal{E}_{ij}^{(n)} = 0 \,. \tag{VII.32}$$

Denoting $F_\mu^{(n)}$ the collection $\{T^{(n)}, R^{(n)}, \Theta^{(n)}, \Phi^{(n)}\}$, one can decompose $\mathcal{E}_{ij}^{(n)} = \mathcal{O}_{ij}\left[F_\mu^{(n)}\right] + \mathcal{S}_{ij}^{(n)}$, where $\mathcal{S}_{ij}^{(n)}$ is a source term (depending only on $F_\mu^{(m)}$ with $1 \leq m \leq n-1$) and the linear operator $\mathcal{O}_{ij}$ is given by

$$\mathcal{O}_{ij}\left[F_\mu^{(n)}\right] = \left(g_{\mu\nu}^{\mathrm{SdS}} \frac{\partial F_\nu^{(0)}}{\partial \chi^i} \frac{\partial}{\partial \chi^j} + g_{\mu\nu}^{\mathrm{SdS}} \frac{\partial F_\nu^{(0)}}{\partial \chi^j} \frac{\partial}{\partial \chi^i} + \frac{\partial g_{\nu\lambda}^{\mathrm{SdS}}}{\partial x^\mu} \frac{\partial F_\nu^{(0)}}{\partial \chi^i} \frac{\partial F_\lambda^{(0)}}{\partial \chi^j} \right) F_\mu^{(n)} \,, \tag{VII.33}$$

where $g_{\mu\nu}^{\mathrm{SdS}}$ is the usual Schwarzschild-de Sitter metric. Explicitly, it reads

$$\mathcal{O}_{\rho\rho} = 2\left(\partial_\rho A + \frac{3M + \Lambda\rho^3}{6M - \Lambda\rho^3} \, \frac{A - R}{\rho} \right) \,, \tag{VII.34a}$$

$$\mathcal{O}_{\rho\vartheta} = \partial_\vartheta A + \rho^2 \, \partial_\rho \Theta \,, \tag{VII.34b}$$

$$\mathcal{O}_{\rho\varphi} = \partial_\varphi A + \rho^2 \, \sin^2\vartheta \, \partial_\rho \Phi \,, \tag{VII.34c}$$

$$\mathcal{O}_{\vartheta\vartheta} = 2\rho^2 \left(\partial_\vartheta \Theta + \frac{R}{\rho} \right) \,, \tag{VII.34d}$$

$$\mathcal{O}_{\vartheta\varphi} = \rho^2 \left(\partial_\varphi \Theta + \sin^2\vartheta \, \partial_\vartheta \Phi \right) \,, \tag{VII.34e}$$

$$\mathcal{O}_{\varphi\varphi} = 2\rho^2 \left(\partial_\varphi \Phi + \frac{R}{\rho} + \Theta \cot\vartheta \right) \sin^2\vartheta \,, \tag{VII.34f}$$

where we have introduced the ancillary function $A(\rho, \vartheta, \varphi) \equiv -\sqrt{\frac{2\mu(\rho)}{\rho}} \, T(\rho, \vartheta, \varphi) + \frac{\rho}{\rho - 2\mu(\rho)} R(\rho, \vartheta, \varphi)$. At a given order, we have thus to solve a system of coupled linear differential equations: the most general solution will be given by the sum of a homogeneous solution of the operator $\mathcal{O}_{ij}$ and a particular solution.

Homogeneous solution

We first seek a general solution of the operator $\mathcal{O}_{ij}$, defined in (VII.34). Eliminating all but Θ-dependencies in the angular sector gives

$$\frac{1}{2\rho^2}\partial_\vartheta\left(\mathcal{O}_{\vartheta\vartheta}-\frac{\mathcal{O}_{\varphi\varphi}}{\sin^2\vartheta}\right)+\partial_\varphi\left(\frac{\mathcal{O}_{\vartheta\varphi}}{\rho^2\sin^2\vartheta}\right)=\partial_\vartheta\left[\sin\vartheta\,\partial_\vartheta\left(\frac{\Theta}{\sin\vartheta}\right)\right]+\frac{1}{\sin^2\vartheta}\,\partial^2_\varphi\Theta=0\,. \tag{VII.35}$$

Imposing periodicity in φ and regularity in ϑ, the solution reads

$$\Theta=\Theta_s(\rho)\sin\vartheta+\Theta_c(\rho)\cos\vartheta\sin\left[\varphi-\varphi_0(\rho)\right]\,, \tag{VII.36}$$

where Θ_s, Θ_c and φ_0 are constants (in angular variables) of integration. Plugging this solution back in eqs. (VII.34d), (VII.34e) and (VII.34f) and solving them, it comes

$$R=-\rho\,\Theta_s(\rho)\cos\vartheta+\rho\,\Theta_c(\rho)\sin\vartheta\sin\left[\varphi-\varphi_0(\rho)\right]\,, \tag{VII.37a}$$

$$\Phi=\Phi_0(\rho)+\Theta_c(\rho)\,\frac{\cos\left[\varphi-\varphi_0(\rho)\right]}{\sin\vartheta}\,, \tag{VII.37b}$$

with Φ_0 a constant of integration. Then $\partial_\vartheta\mathcal{O}_{\rho\varphi}-\partial_\varphi\mathcal{O}_{\rho\vartheta}=\rho^2\sin(2\vartheta)\partial_\rho\Phi_0=0$ forces Φ_0 to be constant. Injecting those solutions in eqs. (VII.34b) and (VII.34c) yields

$$A=A_0(\rho)+\rho^2\cos\vartheta\,\partial_\rho\Big(\Theta_s-\Theta_c\sin\left[\varphi-\varphi_0(\rho)\right]\tan\vartheta\Big)\,. \tag{VII.38}$$

with A_0 a last constant of integration. Finally eq. (VII.34a) gives

$$A_0'+\aleph\,\frac{A_0}{\rho}+\Big(\rho^2\Theta_s''+(2+\aleph)\rho\Theta_s'+\aleph\Theta_s\Big)\cos\vartheta-\Big(\rho^2\tilde{\Theta}_c''+(2+\aleph)\rho\tilde{\Theta}_c'+\aleph\,\tilde{\Theta}_c\Big)\sin\vartheta=0\,, \tag{VII.39}$$

where we have shortened $\aleph=(3M+\Lambda\rho^3)/(6M-\Lambda\rho^3)$ and $\tilde{\Theta}_c=\Theta_c(\rho)\sin\left[\varphi-\varphi_0(\rho)\right]$ and denoted with primes derivatives with respect to ρ. This imposes $A_0=-T_0\sqrt{2\mu/\rho}$, $\Theta_s=\left(\kappa_1+\kappa_2\int\mathrm{d}u\sqrt{2\mu/u}\right)/\rho$ and $\Theta_c=0$, where T_0, κ_1 and κ_2 are constants of integration. Let us point out that in the $\Lambda=0$ limit, it simply comes $\Theta_s=\kappa_1/\rho+2\kappa_2\sqrt{2M/\rho}$. Turning back to the original variables (*ie.* expressing T in terms of A and R), it finally comes

$$T_h=T_0+\left\{\kappa_1\,\frac{\sqrt{2\rho\mu}}{\rho-2\mu}+\kappa_2\,\left(\rho+\frac{\sqrt{2\rho\mu}}{\rho-2\mu}\int\!\mathrm{d}u\sqrt{\frac{2\mu}{u}}\right)\right\}\cos\vartheta\,, \tag{VII.40a}$$

$$R_h=\left\{\kappa_1+\kappa_2\,\int\!\mathrm{d}u\sqrt{\frac{2\mu}{u}}\right\}\cos\vartheta\,, \tag{VII.40b}$$

$$\Theta_h=-\frac{1}{\rho}\left\{\kappa_1+\kappa_2\,\int\!\mathrm{d}u\sqrt{\frac{2\mu}{u}}\right\}\sin\vartheta\,, \tag{VII.40c}$$

$$\Phi_h=\Phi_0\,. \tag{VII.40d}$$

We can easily recognize that T_0 and Φ_0 are respectively linked to the staticity and axisymmetry of the Schwarzschild-de Sitter metric. The two other constants are certainly associated with the two remaining generators of the group of isometries of the Schwarzschild-de Sitter space-time.

Particular solution at linear order

At the linear order in a, the only non-vanishing source term is $\mathcal{S}^{(1)}_{\rho\varphi}$, leading to

$$A^{(1)}_\varphi+\left[\rho^2\,\Phi^{(1)}_\rho-\left(\frac{2\mu}{\rho}\right)^{3/2}\frac{\rho}{\rho-2\mu}\right]\sin^2\vartheta=0\,, \tag{VII.41}$$

which is easily solved by imposing $\Phi^{(1)}_p = \int^\rho \frac{\mathrm{d}u}{u}\left(\frac{2\mu}{u}\right)^{3/2}\frac{1}{u-2\mu}$. Together with the previously found homogeneous solution (VII.40), it comes

$$t = \tau + aT_0 + \int^\rho \mathrm{d}u\,\frac{\sqrt{2Mu}}{u-2M} + a\left\{\kappa_1\,\frac{\sqrt{2\rho\mu}}{\rho-2\mu} + \kappa_2\,\left(\rho + \frac{\sqrt{2\rho\mu}}{\rho-2\mu}\int \mathrm{d}u\sqrt{\frac{2\mu}{u}}\right)\right\}\cos\vartheta + \mathcal{O}(a^2)\,, \tag{VII.42a}$$

$$r = \rho + a\left\{\kappa_1 + \kappa_2\,\int \mathrm{d}u\sqrt{\frac{2\mu}{u}}\right\}\cos\vartheta + \mathcal{O}(a^2)\,, \tag{VII.42b}$$

$$\theta = \vartheta - \frac{a}{\rho}\left\{\kappa_1 + \kappa_2\,\int \mathrm{d}u\sqrt{\frac{2\mu}{u}}\right\}\sin\vartheta + \mathcal{O}(a^2)\,, \tag{VII.42c}$$

$$\phi = \varphi + a\varphi_0 + a\int^\rho \frac{\mathrm{d}u}{u}\left(\frac{2\mu}{u}\right)^{3/2}\frac{1}{u-2\mu} + \mathcal{O}(a^2)\,. \tag{VII.42d}$$

Note that the Jacobian $\mathcal{J} = 1 + a\left(\kappa_1 - \kappa_2\sqrt{2\mu\rho} + \kappa_2\int \mathrm{d}u\sqrt{\frac{2\mu}{u}}\right)\frac{\cos\vartheta}{\rho} + \mathcal{O}(a^2)$ cannot vanish but in localized points.

Particular solution at quadratic order

At the second order in a, introducing $\mathcal{K} = \kappa_1 + \kappa_2\int^\rho \mathrm{d}u\sqrt{\frac{2\mu}{u}}$, the source term is slightly more involving

$$\begin{aligned}\mathcal{S}^{(2)}_{\rho\rho} &= \frac{(\rho+2\mu)\cos(2\vartheta)}{2\rho^3} - \frac{\rho^2-20\mu^2}{2\rho^3\,(\rho-2\mu)} - \frac{2M}{\rho^3}\frac{\rho^2+6\rho\mu-8\mu^2}{(\rho-2\mu)^2} \\ &\quad - \left[\left(\frac{(3M-2\mu)^2\,(\rho^2-12\rho\mu-12\mu^2)}{2\mu\rho\,(\rho-2\mu)} + \rho - 6\mu\right)\frac{\rho\,\cos^2\vartheta}{(\rho+2\mu)^2} - 1\right]\frac{\mathcal{K}^2}{\rho^2} \\ &\quad + 2\frac{\sqrt{2\mu}}{\rho^{3/2}}\left(\frac{4\mu^2\,(\rho+2\mu) + 3M\rho\,(\rho-6\mu)}{2\mu\,(\rho-2\mu)^2}\cos^2\vartheta - 1\right)\kappa_2\mathcal{K} + \frac{2\mu - (\rho-2\mu)\cos^2\vartheta}{\rho}\,\kappa_2^2\,,\end{aligned} \tag{VII.43a}$$

$$\mathcal{S}^{(2)}_{\rho\vartheta} = \left[\frac{\rho\,(\rho-6\mu) + 3M\,(\rho+2\mu)}{\rho\,(\rho-2\mu)^2}\,\mathcal{K}^2 - \frac{3M}{\sqrt{2\rho\mu}}\,\kappa_2\mathcal{K} + \kappa_2^2\rho\right]\cos\vartheta\sin\vartheta\,, \tag{VII.43b}$$

$$\mathcal{S}^{(2)}_{\rho\varphi} = 2\mu\,\left[\frac{2\mu\,(\rho+2\mu) - 3M\,(3\rho-2\mu)}{\sqrt{2\mu}\,\rho^{3/2}}\,\mathcal{K} + (\rho-2\mu)\,\kappa_2\right]\cos\vartheta\,\sin^2\vartheta\,, \tag{VII.43c}$$

$$\mathcal{S}^{(2)}_{\vartheta\vartheta} = \left[1 + \frac{\Lambda\rho^2}{3} - 2\mathcal{K}^2\right]\cos^2\vartheta + \left[\left(\kappa_1 + \kappa_2\,\int\mathrm{d}u\sqrt{\frac{2\mu}{u}} - \kappa_2\sqrt{2\rho\mu}\right)^2 - \kappa_2^2\,\rho^2\right]\sin^2\vartheta\,, \tag{VII.43d}$$

$$\mathcal{S}^{(2)}_{\vartheta\varphi} = \frac{2\mu}{\rho-2\mu}\,\left[\sqrt{\frac{2\mu}{\rho}}\,\mathcal{K} + (\rho+2\mu)\,\kappa_2\right]\sin^3\vartheta\,, \tag{VII.43e}$$

$$\mathcal{S}^{(2)}_{\varphi\varphi} = \left[1 + \frac{\Lambda\rho^2}{3} + \frac{2M}{\rho}\sin^2\vartheta - \left(1+\cos^2\vartheta\right)\mathcal{K}^2\right]\sin^2\vartheta\,. \tag{VII.43f}$$

We first focus on the angular part. The combination $\partial_\vartheta\left(\mathcal{E}^{(2)}_{\vartheta\vartheta} - \frac{\mathcal{E}^{(2)}_{\varphi\varphi}}{\sin^2\vartheta}\right) + 2\,\partial_\varphi\left(\frac{\mathcal{E}^{(2)}_{\vartheta\varphi}}{\sin^2\vartheta}\right)$ yields

$$\partial_\vartheta\left[\sin\vartheta\,\partial_\vartheta\left(\frac{\Theta^{(2)}_p}{\sin\vartheta}\right)\right] + \frac{\partial^2_\varphi\Theta^{(2)}_p}{\sin^2\vartheta} = \left[1 + \frac{2M}{\rho} + \frac{\Lambda\rho^2}{3} - 2\mathcal{K}^2 + 2\sqrt{2\rho\mu}\,\kappa_2\mathcal{K} + \kappa_2^2\,\rho\,(\rho-2\mu)\right]\frac{\sin(2\vartheta)}{2\rho^2}\,. \tag{VII.44}$$

This is notably solved by

$$\Theta_p^{(2)} = -\left[1 + \frac{2M}{\rho} + \frac{\Lambda\rho^2}{3} - 2\mathcal{K}^2 + 2\sqrt{2\rho\mu}\,\kappa_2\mathcal{K} + \kappa_2^2\,\rho\,(\rho - 2\mu)\right]\frac{\sin(2\vartheta)}{4\rho^2}\,, \tag{VII.45}$$

and thus, when injected back in $\mathcal{E}^{(2)}_{\vartheta\vartheta} - \frac{\mathcal{E}^{(2)}_{\varphi\varphi}}{\sin^2\vartheta}$ and $\mathcal{E}^{(2)}_{\vartheta\varphi}$, one obtains

$$\Phi_p^{(2)} = \left[\kappa_2 + \sqrt{\frac{2\mu}{\rho}}\frac{\mathcal{K}}{\rho - 2\mu}\right]\frac{2\mu\cos\vartheta}{\rho^2}\,. \tag{VII.46}$$

Then $\mathcal{E}^{(2)}_{\vartheta\vartheta}$ gives

$$R_p^{(2)} = \left(\frac{M}{\rho^2} + \frac{\rho - 2\mu}{2}\,\kappa_2^2 + \sqrt{\frac{2\mu}{\rho}}\,\kappa_2\mathcal{K}\right)\cos^2\vartheta + \left(\frac{\mu - 2M}{\rho^2} + \frac{\mathcal{K}^2 - 1}{2\rho}\right)\sin^2\vartheta\,. \tag{VII.47}$$

Turning to the radial part of the system, $\mathcal{E}^{(2)}_{\rho\varphi}$ gives

$$A_p^{(2)} = \frac{6M\,\kappa_2\cos\vartheta\sin^2\vartheta}{2\rho}\,\varphi + f_2(\rho,\vartheta)\,, \tag{VII.48}$$

where f_2 is the yet unconstrained φ-independent part of $A_p^{(2)}$. Plugged into $\partial_\varphi\mathcal{E}^{(2)}_{\rho\vartheta}$, it imposes that $\kappa_2 = 0$. $\mathcal{E}^{(2)}_{\rho\vartheta}$ then yields

$$f_2 = \left[\frac{3M + \rho}{\rho} - \frac{\rho^2 - 2\rho\mu + 8\mu^2 - 3M\,(\rho + 2\mu)}{(\rho + 2\mu)^2}\right]\frac{\cos(2\vartheta)}{4\rho}\,. \tag{VII.49}$$

But then

$$\partial_\vartheta\mathcal{E}^{(2)}_{\rho\rho} = \frac{9M^2\sin(2\vartheta)}{2\rho^3\mu}, \tag{VII.50}$$

cannot vanish. Thus we conclude that it is impossible to achieve a spatially flat slicing of the Kerr-de Sitter space-time beyond linear order in a.

Other attempts to construct rotating black holes in MTMG

If the previous result prevents us to use the lemma for constructing rotating black hole solutions in MTMG, it naturally does not preclude the existence of such solutions.

A usual trick to generate rotating solutions from static ones is the Newman–Janis algorithm [291]. This method relies on a complexification of the non-rotating metric (and hence a doubling of the number of coordinates), and then a careful slicing of this higher-dimensional line element to recover a rotating metric. But this algorithm is usually performed in theories that are invariant under the full set of diffeomorphisms (*eg.* such as $f(\mathcal{R})$ [118]), and we have failed to implement it in MTMG.

Unfortunately, all other attempts that have been undertaken until now have also miserably failed,[5] either by trying to build up such a rotating solution from scratch (imposing only that it satisfies the vacuum equations of MTMG and bears a non-vanishing Komar momentum [213]), or by attempting to find some extension of Painlevé-Gullstrand coordinates (allowing for a small departure from spatial flatness) in which the Kerr-de Sitter metric becomes solution of MTMG.

[5] To the only exception of taking a ficudical metric whose spatial sector is given by the Kerr-de Sitter one, up to a proportionality constant. The ugliness of such a construction is definitively too repellent for us to consider it as a serious candidate.

VII.2 Strong field regime of the VCDM model

VII.2.1 The model under consideration

Let us now turn to another minimal theory, belonging to the type-II (*ie.* it does not admit an Einstein frame), dubbed VCDM for reasons that will be clear in the following. This theory was proposed in [153], and albeit it formally contains a metric field $g_{\mu\nu}$, an ancillary scalar field ϕ, a Lagrange multiplier λ and a gauge-fixing field λ^i, it only propagates two tensor *dofs*, as expected. ADM-decomposing the metric $g_{\mu\nu}$ as usual and introducing the extrinsic curvature K_{ij}, the action that will be considered is

$$\mathcal{S}_{V\mathrm{CDM}} = \mathcal{S}_{\mathrm{EH}}\left[g_{\mu\nu}\right] - M_{\mathrm{Pl}}^2 \int \mathrm{d}^4x N\sqrt{\gamma} \left\{ \lambda\left(K+\phi\right) + \frac{3\lambda^2}{4} + \frac{\lambda^i \partial_i \phi}{N} + V(\phi) \right\} + \mathcal{S}_{\mathrm{mat}}\left[\psi_{\mathrm{mat}}, g_{\mu\nu}\right] \,, \tag{VII.51}$$

with $V(\phi)$ a free function and $\mathcal{S}_{\mathrm{mat}}$ the usual action for matter fields minimally coupled to $g_{\mu\nu}$. The not so obvious fact that this theory has no Einstein frame was recently proven [26]. In addition it was shown in the same study that this theory is equivalent to the cuscuton model, introduced in [14]. It is clear that the diffeomorphisms invariance is only broken by the second term. In fact the theory happens to be invariant under the so-called foliation-preserving diffeomorphisms

$$t \to \tau(t)\,, \qquad x^i \to \chi^i(t, x^i)\,. \tag{VII.52}$$

In the next sections, following the work done in collaboration with A. De Felice, A. Doll and S. Mukohyama [154], we will study the strong-field regime of the theory (VII.51), in non-rotating[6] configurations. The resulting spherical symmetry allows us to use the ansatz

$$N = N(t,r)\,, \qquad N_i \mathrm{d}x^i = B(t,r)F(t,r)\mathrm{d}r\,, \qquad \text{and} \qquad \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = F(t,r)^2\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega\,, \tag{VII.53}$$

so that B is the tetrad component of the shift vector and the four-dimensional metric reads

$$\mathrm{d}s_g^2 = -N(t,r)^2\mathrm{d}t^2 + \left[F(t,r)\mathrm{d}r + B(t,r)\mathrm{d}t\right]^2 + r^2\mathrm{d}^2\Omega\,, \tag{VII.54}$$

As for the auxiliary fields, spherical symmetry fixes the ansatz

$$\phi = \phi(t,r)\,, \qquad \lambda = \lambda(t,r)\,, \qquad \text{and} \qquad \lambda^i\,\partial_i = \lambda^r(t,r)\,\frac{\partial_r}{F(t,r)}\,. \tag{VII.55}$$

Sketch of the construction of the theory

For completeness, we will sketch the procedure that led to the theory (VII.51), as it is quite elegant. Starting from GR in vacuum, a cosmological constant is added in a new frame; the theory is then pulled back to the original frame, where matter is finally included. For a detailed description of the construction, we let the reader refer to [153].

In practice, and as usual when dealing with minimal theories, the construction is performed at the Hamiltonian level. The starting point is the Hamiltonian of GR in vacuum (A.22)

$$\mathcal{H}_{\mathrm{EH}} = -\int \mathrm{d}t\mathrm{d}^3x\left(N\mathcal{R}_0 + N^i\mathcal{R}_i + \kappa\,\pi + \kappa^i\,\pi_i\right)\,, \tag{VII.56}$$

where the four constraints $\{\mathcal{R}_0, \mathcal{R}_i\}$ are given in eq. (A.23), and we have included the momenta $\{\pi, \pi_i\}$ that are conjugated to $\{N, N^i\}$, toghether with associated Lagrange multipliers $\{\kappa, \kappa^i\}$. The

[6] Contrarily to the case of MTMG, we have not yet tried to implement rotation in VCDM.

change of frame is realized by a canonical transformation of the variables $\{N, N^i, \gamma_{ij}\}$ to $\{\mathfrak{N}, \mathfrak{N}^i, \Gamma_{ij}\}$ *via* the generating functional (where f is an arbitrary function)

$$\mathcal{F} = -\int \mathrm{d}^3x \left[M_{\mathrm{Pl}}^2 \sqrt{\Gamma}\, f\left(\frac{\pi^{ij}\Gamma_{ij}}{M_{\mathrm{Pl}}^2\sqrt{\Gamma}}, \frac{\pi\, \mathfrak{N}}{M_{\mathrm{Pl}}^2\sqrt{\Gamma}} \right) + \mathfrak{N}^i \pi_i \right], \tag{VII.57}$$

ie. the new momenta $\{\Pi, \Pi_i, \Pi^{ij}\}$ and old variables obey

$$\Pi = -\frac{\delta \mathcal{F}}{\delta \mathfrak{N}}, \qquad \Pi_i = -\frac{\delta \mathcal{F}}{\delta \mathfrak{N}^i} = \pi_i, \qquad \Pi^{ij} = -\frac{\delta \mathcal{F}}{\delta \Gamma_{ij}}, \tag{VII.58a}$$

$$N = -\frac{\delta \mathcal{F}}{\delta \pi}, \qquad N^i = -\frac{\delta \mathcal{F}}{\delta \pi_i} = \mathfrak{N}^i, \qquad \gamma_{ij} = -\frac{\delta \mathcal{F}}{\delta \pi^{ij}}. \tag{VII.58b}$$

Only the combination $\phi \equiv \pi^{ij}\Gamma_{ij}/(M_{\mathrm{Pl}}^2\sqrt{\Gamma})$ will play a role at the end of the process. In this new frame, a cosmological constant term $M_{\mathrm{Pl}}^2 \mathfrak{N}\sqrt{\Gamma}\,\Lambda$ as well as a gauge-fixing term $\sqrt{\Gamma}\lambda_{\mathrm{gf}}^i \partial_i\phi$ are added. The former is vital to have a theory that differs from GR, and the latter to ensure that the final theory propagates only two *dofs*. Reversing the transformation, relabeling the Lagrange multipliers and performing a Legendre transformation, it comes

$$\mathcal{S}_{V\mathrm{CDM,vac}} = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4x N\sqrt{\gamma} \left\{ \mathcal{R}\left[\gamma\right] + K_{ij}K^{ij} - K^2 - 2\lambda\left(K + \phi\right) - \frac{3\lambda^2}{2} - 2\frac{\lambda^i\partial_i\phi}{N} - 2V(\phi) \right\}, \tag{VII.59}$$

where we have defined

$$V(\phi) \equiv \frac{\Lambda}{f_1\, f_0^{3/2}}, \qquad \text{with} \qquad f_0(\phi) \equiv \frac{\partial f(\phi, 0)}{\partial \phi}, \quad \text{and} \quad f_1(\phi) \equiv \left.\frac{\partial f(\phi, y)}{\partial y}\right|_{y=0}. \tag{VII.60}$$

Introducing minimally coupled matter in this old frame, we obtain the theory (VII.51).

Cosmological phenomenology

As usual when constructing minimal theories, the cosmological phenomenology of VCDM has been investigated as soon as it was constructed [153]. On a flat FLRW background with Hubble factor H, the Lagrange multipliers obey

$$\lambda = -\frac{2}{3}\phi - 2H, \qquad \text{and} \qquad \lambda^i = 0. \tag{VII.61}$$

As for the auxiliary field ϕ, it gives rise to a dark fluid entering the usual Friedman equations. The equation of state of this fluid is determined by the shape of V as

$$\rho_\phi \equiv M_{\mathrm{Pl}}^2 \left\{ V(\phi) - \phi\frac{\mathrm{d}V}{\mathrm{d}\phi} + \frac{3}{4}\left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \right\}, \qquad \text{and} \qquad P_\phi \equiv -\frac{3}{2}\left(\rho + P\right)\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2} - \rho_\phi, \tag{VII.62}$$

where ρ and P are the density and pressure of the usual matter sector. As expected, ϕ is not dynamical but is fixed by the relation

$$\frac{3}{2}\frac{\mathrm{d}V}{\mathrm{d}\phi} - \phi = 3H. \tag{VII.63}$$

Instead of determining the evolution of the Universe from the behavior of $V(\phi)$, one can reverse the problem and, given a background dynamic for the Universe, reconstruct the shape of $V(\phi)$ that sources it. The only assumptions that are to be made are that the total stress energy tensor obeys the null energy condition $\rho + P > 0$, and that the Universe is expanding, *ie.* $H > 0$. This ensures that ϕ is a steadily increasing function of the number of e-folds $\mathcal{N} \propto \ln a$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\mathcal{N}} = \frac{3}{2}\frac{\rho + P}{M_{\rm Pl}^2 H} \,. \tag{VII.64}$$

In such a case, $\phi(\mathcal{N})$ is invertible, and the metric equations allow to express ρ in term of ϕ. So, given a specific background dynamic for the Universe, one can reconstruct the potential $V(\phi)$ as

$$V(\phi) = \frac{\phi^2}{3} - \frac{\rho\left[\mathcal{N}(\phi)\right]}{M_{\rm Pl}^2} \,. \tag{VII.65}$$

This feature has naturally been applied to the evolution of our Universe, using Planck data, and the reconstructed $V(\phi)$ gave excellent results [156]. In conclusion, adding a cold dark matter component in the matter sector, the theory (VII.51) could challenge the current ΛCDM paradigm, thus the name "VCDM".

A last nice feature of VCDM at cosmological scales is that the gravitational waves propagate at the speed of light, and obey the canonical dispersion relations [26], thus the theory is in perfect agreement with the most up-to-date tests on the propagation of gravitational radiation.

VII.2.2 Static solutions

We first focus on static configurations in vacuum, *ie.* implementing the ansatz (VII.53)-(VII.55) with time-independent quantities. Here and henceforth we denote by $\mathcal{E}_\star$ the equation of motion of the variable $\star$, and, in this section, we do not write explicitly the radial dependencies, as all quantities can depend only on r. With this ansatz, only one gauge-fixing constraint remains, namely

$$\mathcal{E}_{\lambda^r} = -M_{\rm Pl}^2 r^2 F\,\partial_r\phi \,, \tag{VII.66}$$

thus $\phi = \phi_0$ is constant. The non-dynamical fields λ and ϕ obey the equations of motion

$$\mathcal{E}_\lambda = -\frac{3M_{\rm Pl}^2 r^2 NF}{2}\left[\lambda + \frac{2\phi_0}{3} - \frac{2}{3r^2NF}\,\partial_r\left(r^2B\right)\right] \,, \tag{VII.67a}$$

$$\mathcal{E}_\phi = M_{\rm Pl}^2\left[\partial_r\left(r^2\lambda^r - \frac{2r^2B}{3}\right) + r^2NF\left(\frac{2\phi_0}{3} - V_0'\right)\right] \,, \tag{VII.67b}$$

where we naturally shortened $V_0' = (\mathrm{d}V(\phi)/\mathrm{d}\phi)_{\phi=\phi_0}$. Those equations allow to write λ and λ^r in terms of N, B and F. The equation of motion for the shift B is of the form

$$\mathcal{E}_B = -\frac{2M_{\rm Pl}^2}{3rF}\,\partial_r\left[\frac{r^4}{NF}\partial_r\left(\frac{B}{r}\right)\right] \,, \tag{VII.68}$$

so we can express $B = 3\kappa_0 r\int_{r_0}^r \mathrm{d}u NFu^{-4}$, where r_0 and κ_0 are constants of integration. Finally the equations of motion for N and F are solved by

$$F(r) = \frac{N_0}{N(r)} = \left(1 - \frac{2\mu_0}{r} - \frac{\Lambda_0 r^2}{3} + \frac{\kappa_0^2}{r^4}\right)^{-1/2} \,, \tag{VII.69}$$

where μ_0 is a constant of integration and we have defined $\Lambda_0 = V_0 - \phi_0^2/3$. The constant N_0 can be set to any positive value by the space-independent time reparametrization, which is a part of the foliation-preserving diffeomorphisms (VII.52), and it is convenient to set it to unity.

So finally, the static and spherically symmetric solutions of our theory are parameterized by five constants $\{\mu_0, b_0, \kappa_0, \phi_0, \ell_0\}$ as

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{F^2(r)} + \left[F(r)\,\mathrm{d}r + \left(b_0 r - \frac{\kappa_0}{r^2}\right)\mathrm{d}t\right]^2 + r^2\mathrm{d}^2\Omega\,, \tag{VII.70a}$$

$$\phi = \phi_0\,, \qquad \lambda = 2b_0 - \frac{2\phi_0}{3}\,, \qquad \lambda^i\,\partial_i = \left[\frac{\ell_0}{r^2} + \left(V_0' + 2b_0 - \frac{2\phi_0}{3}\right)\frac{r}{3}\right]\frac{\partial_r}{F(r)}\,, \tag{VII.70b}$$

where F is defined in (VII.69). We have also defined $b_0 = \kappa_0/r_0^3$ and ℓ_0 is the constant of integration associated with the equation of motion for ϕ (VII.67b). When setting b_0 and κ_0 to zero, this family of metrics reduces to the usual Schwarzschild-de Sitter ones with mass μ_0 and cosmological constant Λ_0.

The function F (VII.69) differs from the usual Schwarzschild solution and is obviously divergent. It is thus vital to seek if those apparent divergences are only coordinate ones (as in GR), or point out physical singularities. In other words, we have to investigate the regularity of the family of metrics (VII.70a). Scalars made of the intrinsic and extrinsic curvatures for those metrics are given by

$$\mathcal{R}_{ijkl}\mathcal{R}^{ijkl} = \frac{4\Lambda_0^2}{3} + \frac{24\mu_0^2}{r^6} + \frac{8\kappa_0^2}{r^6}\left(\Lambda_0 - \frac{6\mu_0}{r^3} + \frac{9\kappa_0^2}{2r^6}\right)\,, \tag{VII.71a}$$

$$\mathcal{R}_{ij}\mathcal{R}^{ij} = \frac{4\Lambda_0^2}{3} + \frac{6\mu_0^2}{r^6} + \frac{8\kappa_0^2}{r^6}\left(\Lambda_0 - \frac{3\mu_0}{2r^3} + \frac{9\kappa_0^2}{4r^6}\right)\,, \tag{VII.71b}$$

$$\mathcal{R} = 2\Lambda_0 + \frac{6\kappa_0^2}{r^6}\,, \tag{VII.71c}$$

$$K_{ij}K^{ij} = 3b_0^2 + \frac{6\kappa_0^2}{r^6} \qquad \text{and} \qquad K = -3b_0\,. \tag{VII.71d}$$

They are well-behaved in all space but the origin $r = 0$: there is no geometrical singularity at the horizons.

Black hole mass and effective cosmological constant

The family of metrics (VII.70a) describes a set of Einstein space-times, as their associated four-dimensional Ricci tensors read

$$\mathcal{R}_{\mu\nu} = \left(\Lambda_0 + 3b_0^2\right)g_{\mu\nu}\,. \tag{VII.72}$$

This indicates that those space-times bear an effective cosmological constant $\Lambda_{\text{eff}} = \Lambda_0 + 3b_0^2$. Moreover, Birkhoff theorem [65, 242] implies that the four-dimensional metric is locally Schwarzschild-de Sitter ($\Lambda_{\text{eff}} > 0$), Schwarzschild ($\Lambda_{\text{eff}} = 0$), or Schwarzschild-AdS ($\Lambda_{\text{eff}} < 0$). In order to determine the mass of the black hole, let us compute the (generalized) Misner-Sharp mass [116, 286]. For a block-diagonal, spherically symmetric metric of the form

$$\mathrm{d}s^2 = h_{ab}\mathrm{d}x^a\mathrm{d}x^b + r^2\mathrm{d}^2\Omega\,, \tag{VII.73}$$

where $h_{ab}\mathrm{d}x^a\mathrm{d}x^b$ is the metric of a two-dimensional space-time spanned by the coordinates $\{x^c\}_{c=0..1}$, often called the orbit space, and r is a non-negative function of the coordinates of the orbit space, the (generalized) Misner-Sharp mass M is defined by

$$h^{ab}\partial_a r\partial_b r = 1 - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2\,. \tag{VII.74}$$

Here, h^{ab} is the inverse of h_{ab}. For the family (VII.70a), a straightforward calculation results in

$$M = \mu_0 - \kappa_0 b_0\,. \tag{VII.75}$$

Therefore, the mass and cosmological constant of our Schwarzschild-de Sitter black holes are different from their naive expectation values μ_0 and Λ_0, respectively.

The fact that κ_0 does not enter $\Lambda_{\rm eff}$ can be understood from the fact that this parameter is linked to a coordinate change. Starting with an usual Schwarzschild-de Sitter space-time with mass μ_0 and cosmological constant Λ_0, and performing the time redefinition $t \to t - \kappa_0 \int \frac{F^3(r) r^2\,\mathrm{d}r}{\kappa_0^2 F^2(r) - r^4}$, the line element reads

$$\mathrm{d}s^2_{\rm SdS} = -\left(1 - \frac{2\mu_0}{r} - \frac{\Lambda_0 r^2}{3}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2\mu_0}{r} - \frac{\Lambda_0 r^2}{3} + \frac{\kappa_0^2}{r^4}} - \frac{2\kappa_0}{r^2}\frac{\mathrm{d}t\,\mathrm{d}r}{\sqrt{1 - \frac{2\mu_0}{r} - \frac{\Lambda_0 r^2}{3} + \frac{\kappa_0^2}{r^4}}} + r^2\mathrm{d}^2\Omega\,, \tag{VII.76}$$

which is nothing but the line element (VII.70a) with $b_0 = 0$. As κ_0 is only associated to a boost, it is natural that it does not enter the effective cosmological constant. We nevertheless recall that while such a parameter would be unphysical in GR, this is not the case in the framework of this study: due to the breaking of temporal diffeomorphisms, two solutions with different κ_0 are physically distinct, as the applied time redefinition is space-dependent, and as such does not enter the class (VII.52).

The b_0 parameter represents the leftover freedom in space-time slicing. Indeed, as shown in (VII.71), it does not enter R_{ijkl}, but only the extrinsic curvature, when κ_0 only enters R_{ijkl}.

Matching to cosmology

In the empty cosmological setup, in general a de Sitter solution in the flat FLRW form is present, as $H^2 = H_0^2 = \rho_\phi(\phi_0)/(3M_{\rm Pl}^2) \equiv \Lambda_{\rm eff}/3 = b_0^2 + \Lambda_0/3$, which sets the value of b_0^2 [153]. The four-dimensional metric of the de Sitter solution in the flat slicing is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2H_0 t}(\mathrm{d}\rho^2 + \rho^2\mathrm{d}^2\Omega)\,. \tag{VII.77}$$

Since the theory enjoys the spatial diffeomorphisms invariance, the metric is physically equivalent to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\mathrm{d}r - H_0 r\,\mathrm{d}t)^2 + r^2\mathrm{d}^2\Omega\,, \tag{VII.78}$$

which is spatially flat and manifestly static. Here, $r = e^{H_0 t}\rho$.

If one requires that the Schwarzschild-de Sitter black holes asymptotically match with this de Sitter solution, then the intrinsic curvature should vanish at infinity. By means of (VII.71c) it implies that $\Lambda_0 = 0$, and thus $V(\phi_0) = \phi_0^2/3$. In this case, it is convenient to set $\lim_{r\to\infty} N = 1$ (*ie.* setting $N_0 = 1$ in eq. (VII.69), as $F \to 1$) by the space-independent time reparametrization, which is a part of the foliation-preserving diffeomorphisms (VII.52), so that one can match the metric (VII.70a) to the metric (VII.78) at infinity not only up to the foliation-preserving diffeomorphisms but also explicitly. The matching between the two metrics then requires $b_0 = -H_0$ as well. In this same case, the auxiliary fields ϕ and λ are also well matching their cosmological values: ϕ is constant, which agrees with its homogeneous behavior in cosmology and $\lambda = -2(\phi + K)/3$ (as expected from (VII.61)). Last, but not least, the fields λ^i should vanish at spatial infinity. As clear from eq. (VII.70b), this is realized if $2b_0 + V_0' - 2\phi_0/3 = 0$, which agrees with (VII.63).

In the above we have imposed the exact matching to the de Sitter solution with the flat FLRW slicing. In a more realistic situation, the boundary condition within the spherically symmetric ansatz should be modified one way or another to the extent that the would-be mismatch between the cosmological solution and the black hole solution can be absorbed by physical effects in the intermediate region (such as deviations from spherical symmetry, the existence of interstellar matter, etc.) or by non-trivial cosmology (*eg.* the existence of cold dark matter, baryons, radiation, etc.). Nonetheless, the properties of the black hole solution at astrophysical scales are expected to be insensitive to such modifications of the boundary condition as far as there is a large enough hierarchy between the size of the black hole and the scales associated with the physical effects in the intermediate and cosmological scales. For this reason we have considered the idealized situation where the spherically symmetric solution asymptotically matches with the de Sitter (or Minkowski) space-time in the spatially homogeneous and isotropic slicing.

Let us finally note that if one takes the limit $b_0 \to 0$ (after matching to the flat FLRW de Sitter solution), then the Schwarzschild-de Sitter solution reduces to the asymptotically flat Schwarzschild solution.

Summary of static black hole solutions

In a nutshell, the static and spherically symmetric solutions of our theory are given by Schwarzschild-de Sitter space-times (VII.70a)–(VII.69), parameterized by four parameters: the mass M, the effective cosmological constant Λ_{eff}, and the two slicing parameters b_0 and κ_0, linked to the way the space-time is foliated by constant time hypersurfaces. The two other parameters entering (VII.70a), Λ_0 and μ_0, are not independent of the parameters mentioned above, and are defined by

$$\mu_0 = M + \kappa_0 b_0 \,, \qquad \text{and} \qquad \Lambda_0 = \Lambda_{\text{eff}} - 3b_0^2 \,. \tag{VII.79}$$

Let us emphasize that the effective cosmological constant depends not only on the gravity action but also on the way the space-time is foliated by constant-time hypersurfaces, which is a natural consequence of the breaking of temporal diffeomorphism. As for the auxiliary sector (VII.70b), it is parameterized by two additional constants: the value of the auxiliary field ϕ, and an integration constant entering the gauge-fixing field.

Those solutions are perfectly asymptotically matching the de Sitter solutions found in [153] as long as $V(\phi_0) = \phi_0^2/3$ (*ie.* $\Lambda_0 = 0$) and $b_0 = -H_0 = \phi_0/3 - V_0'/2$. This completely eliminates the mismatch between the effective cosmological constant deduced from the solution and that from the action: the effective cosmological constant agrees with the cosmological value that is determined by the gravitational action.

Let us conclude the static case with a word on the parameter κ_0. Since it is independent from Λ_{eff} and M, it does not affect the covariant properties of the four-dimensional metric. Therefore, as long as the matter action is covariant, no astrophysical processes involving matter can probe the value of κ_0. The only way to probe it is through gravity, *eg.* gravitational radiation.

VII.2.3 Time dependent, non-rotating solutions

Let us consider now vacuum time-dependent solutions of VCDM. The gauge-fixing constraint imposes that $\phi(t,r) = \phi(t)$. One can then solve $\mathcal{E}_N$ for B, and then $\mathcal{E}_B$ for F to find

$$F = \left(1 - \frac{2\mu(t)}{r} - \frac{1}{3}\,\Lambda(t)\,r^2 + \frac{\kappa^2(t)}{r^4}\right)^{-1/2}, \tag{VII.80}$$

where we have fixed $F > 0$. Here $\mu(t)$ and $\kappa(t)$ are two constants (in r) of integration and we have defined $\Lambda(t) = V[\phi(t)] - \phi^2(t)/3$. Note the similarity with the static case (VII.69). Then on fixing F to such a solution, one finds that, on reconsidering $\mathcal{E}_N$, one can solve it for N, finding the following equation[7]

$$N = \frac{r^2\,(rB_{,r} - B - rF_{,t})}{3\kappa F} \,. \tag{VII.81}$$

This solution for N is valid unless $\kappa = 0$. The case for which $\kappa = 0$ turns out to reproduce the static case with $\kappa_0 = 0$, as demonstrated in the appendix of [154], and thus will be discarded here. Appropriate boundary conditions need to be set for the solution so that, for instance, N remains

[7] In order to avoid confusion with other underscripts, we denote henceforth partial or total derivatives with a comma, as $B_{,r} \equiv \partial B/\partial r$ or $\Lambda_{,\phi} \equiv \mathrm{d}\Lambda/\mathrm{d}\phi$. A dot denotes a total derivative with respect to time.

finite in the wanted coordinate patch (in particular at infinity). These same boundary conditions will take care of the case $0 < |\kappa| \ll 1$. At this point, both the equations $\mathcal{E}_N$ and $\mathcal{E}_B$ are satisfied, and $\mathcal{E}_{rr}$ can be used to find

$$B = \frac{b_1(t)}{r^2} + r\, b_2(t) + \frac{r}{6}\int^r \mathrm{d}u\,\frac{\Omega(t,u)}{u^5} - \frac{1}{6r^2}\int^r \mathrm{d}u\,\frac{\Omega(t,u)}{u^2}\,, \tag{VII.82a}$$

$$\Omega = -2\kappa\dot{\phi}F^3r^3 + F^3\left(r^6\Lambda_{,\phi}\dot{\phi} + 12\kappa\dot{\kappa}\right) + 3F^2F_{,t}\left(r^6\Lambda - r^4 + 3\kappa^2\right) + 7r^4F_{,t}\,, \tag{VII.82b}$$

where the functions $b_{1,2}$ are free functions which come as constants (in r) of integration for the two integrals defining the solution for B.

We will now proceed to understand the properties of such a solution. First of all, we get the following relations

$$\mathcal{R} = \frac{2}{r^2} - \frac{2}{r^2F^2} + \frac{4F_{,r}}{rF^3} = 2\Lambda(t) + \frac{6\kappa^2}{r^6}\,, \tag{VII.83a}$$

$$K = -\frac{3}{r}\left(\frac{B}{NF} + \frac{\kappa}{r^2}\right)\,, \tag{VII.83b}$$

$$K^{ij}K_{ij} = \frac{3}{r^2}\left(\frac{B}{NF} + \frac{\kappa}{r^2}\right)^2 + \frac{6\kappa^2}{r^6} \tag{VII.83c}$$

$$\lambda = \frac{2}{r}\left(\frac{B}{NF} + \frac{\kappa}{r^2}\right) - \frac{2}{3}\phi\,, \tag{VII.83d}$$

$$\lambda^r = \frac{\ell(t)}{r^2} + \frac{1}{r^2}\int^r \mathrm{d}u NFu^2\left[\Lambda_{,\phi} + \frac{2}{u}\left(\frac{B}{NF} + \frac{\kappa}{u^2}\right)\right]\,. \tag{VII.83e}$$

Here $\ell(t)$ is a constant (in r) of integration. As $\kappa \neq 0$, the 3D Ricci scalar only blows up at the origin $r = 0$. In this case we see that a singularity is present at the origin in general, and thus we need a horizon to make sure that it is not naked.

Coordinate singularity

We first investigate the apparent singularity at $r = r_0(t)$, defined by the property

$$\frac{1}{F(r_0)^2} = 0\,, \tag{VII.84}$$

which can be formally solved for $\mu(t)$ as

$$\mu(t) = \frac{r_0}{2} + \frac{\kappa^2}{2r_0^3} - \frac{1}{6}\Lambda r_0^3\,. \tag{VII.85}$$

Then if we expand F around $r_0(t)$, by taking $r = r_0(t) + \rho$ with $0 < \rho \ll r_0$, it comes

$$\frac{1}{F^2} = \frac{r_0^4 - 3\kappa^2 - r_0^6\Lambda}{r_0^5}\rho + \mathcal{O}(\rho^2)\,, \tag{VII.86}$$

which leads to the condition $r_0^4 - 3\kappa^2 - r_0^6\Lambda \geq 0$. It is easy to show that in this case the proper distance, $\int \mathrm{d}r\, F$, remains finite around this point. On calculating the limit of several expressions around the critical point $r = r_0(t)$, we find that all the scalars remain finite. We can thus conclude that this point corresponds only to a coordinate singularity and focus henceforth on the presence of horizons for this metric. Note the difference with the usual Schwarzschild-de Sitter case in GR, in which the critical points that cancel F^{-2} correspond to the horizons.

Trapping horizons and null surfaces

In order to study the presence of horizons for an effective metric defined as in eq. (VII.54)

$$\mathrm{d}s^2 = -(N^2 - B^2)\,\mathrm{d}t^2 + 2BF\,\mathrm{d}t\mathrm{d}r + F^2\mathrm{d}r^2 + r^2\mathrm{d}^2\Omega\,, \tag{VII.87}$$

it is convenient to introduce two future-oriented four-vectors

$$l^\mu = \frac{1}{F}\,(F, N - B, 0, 0)\,, \qquad \text{and} \qquad n^\mu = \frac{1}{F}\,(F, -N - B, 0, 0)\,, \tag{VII.88}$$

l^μ being "outgoing" and n^μ, "ingoing." Those vectors are null and conveniently normalized, namely

$$n^\mu n_\mu = 0\,, \qquad l^\mu l_\mu = 0\,, \qquad \text{and} \qquad n^\mu l_\mu = -2N^2\,. \tag{VII.89}$$

Out of these two light-like vectors, on following *eg.* [186], we can then build up a projector

$$\pi^{\mu\nu} = g^{\mu\nu} + \frac{l^\mu n^\nu + l^\nu n^\mu}{(-n^\alpha l_\alpha)}\,, \tag{VII.90}$$

and we can introduce the corresponding expansion scalars, that roughly express the dilatation undergone by a box made of photon, in the directions of respectively n^μ and l^μ,

$$\theta_{\rm in} = \pi^{\mu\nu}\,\nabla_\mu n_\nu = -\frac{2}{rF}\,(N + B)\;, \qquad \text{and} \qquad \theta_{\rm out} = \pi^{\mu\nu}\,\nabla_\mu l_\nu = \frac{2}{rF}\,(N - B)\;. \tag{VII.91}$$

In terms of the expansion scalars, g^{rr} reads

$$g^{rr} = \frac{N^2 - B^2}{N^2 F^2} = -\frac{\theta_{\rm in}\,\theta_{\rm out}}{4N^2}\,. \tag{VII.92}$$

We now define $r_H(t)$ by

$$B\Big(t, r = r_H(t)\Big) = N\Big(t, r = r_H(t)\Big)\,, \tag{VII.93}$$

and S_H by

$$S_H \equiv r - r_H(t) = 0\,. \tag{VII.94}$$

Then the surface $S_H = 0$ with $B > 0$ and $0 < F < \infty$ represents a Marginally Outgoing Trapped Surface (MOTS), as $\theta_{\rm out} = 0$ and $\theta_{\rm in} < 0$ there, *ie.* null geodesics pointing towards the center converge and those pointing outward are at the edge of converging: light rays are marginally trapped. This surface represents an apparent horizon as (VII.92) implies $g^{rr} = 0$ on the MOTS.

Instead the surface representing the coordinate singularity $S_0 \equiv r - r_0(t) = 0$, with r_0 satisfying $F\Big(r = r_0(t)\Big) \to \infty$, is then trapped (since both $\theta_{\rm in}$ and $\theta_{\rm out}$ are negative). In fact, for this S_0 surface, we find that both N and B/F get the following values

$$\lim_{r\to r_0^+} N = \frac{2r_0^3}{3}\,\frac{r_0^3\,\dot\phi - 3\dot\kappa}{r_0^4 - r_0^6\Lambda - 3\kappa^2}\,, \tag{VII.95a}$$

$$\lim_{r\to r_0^+} \frac{B}{F} = -\frac{r_0}{3}\,\frac{r_0^6\,\Lambda_{,\phi}\dot\phi + 6r_0^3\,\dot\mu - 6\kappa\,\dot\kappa}{r_0^4 - r_0^6\Lambda - 3\kappa^2}\,, \tag{VII.95b}$$

which sets $r_0^3\dot\phi - 3\dot\kappa \geq 0$, as the denominator is positive by virtue of (VII.86).

The two surfaces $S_0 = 0$ and $S_H = 0$ coincide only if $B = N$ and $1/F \to 0$. The conditions (VII.95b) then impose that $r_0^6\,\Lambda_{,\phi}\dot\phi + 6r_0^3\,\dot\mu - 6\kappa\,\dot\kappa = 0$, which in turn imposes that r_0 (and thus r_H) is constant in time, as can be seen with the help of eq. (VII.85). In this case the expansion scalars are vanishing at the surface, thus the MOTS condition degrades to a marginally trapped surface (MTS) one. Note that this property is strongly linked to the static case, where $N = F^{-1}$, thus $\theta_{\rm in}$ and $\theta_{\rm out}$ vanish on S_0, which is then a MTS.

Otherwise, when the two surfaces are distinct, the continuity of $\theta_{\rm out}$ implies that $r_0(t) < r_H(t)$, *ie.* that the coordinate singularity lies inside the MOTS and is thus hidden for observers living at infinity.

Type of horizons

Let us consider the MOTS, $S_H = 0$, and study whether it is spacelike or not. Since g^{rr} vanishes on this surface, we find

$$g^{\mu\nu}\partial_\mu S_H \partial_\nu S_H = -\frac{\dot{r}_H^2}{N^2} - \frac{2\dot{r}_H}{NF}\,, \tag{VII.96}$$

so that this surface is spacelike if $0 < -\dot{r}_H < 2N/F[t, r_H(t)]$. Taking the time derivative of eq. (VII.93), we can express $\dot{r}_H$, and find that S_H is spacelike if

$$0 < \left.\frac{B_{,t} - N_{,t}}{B_{,r} - N_{,r}}\right|_{r=r_H} < \left.\frac{2N}{F}\right|_{r=r_H}\,, \tag{VII.97}$$

and lightlike if one of the inequalities is saturated.

As for the coordinate singularity surface $S_0 = r - r_0(t) = 0$ (with $r_0(t) > 0$ and $F[r = r_0(t)] \to \infty$), we have

$$g^{\mu\nu}\partial_\mu S_0 \partial_\nu S_0 = -\frac{1}{N^2}\left(\dot{r}_0 + \frac{B}{F}\right)^2 + \frac{1}{F^2} = 0\,. \tag{VII.98}$$

To obtain the last equality, we need to take the time derivative of eq. (VII.85), and use eq. (VII.95b), which leads to $\dot{r}_0 = -\lim_{r\to r_0} B/F$, so that $S_0 = 0$ is always a null surface.

A possible violation of the null convergence condition

In GR, the null energy condition for the stress energy tensor of matter and the null convergence condition for the geometry are equivalent. The latter states that for any future directed null vector ℓ^μ, $\mathcal{R}_{\mu\nu}\ell^\mu\ell^\nu \geq 0$. In non-GR gravitational theories, the null energy condition does not necessarily implies the null convergence condition. In our theory, we will see below that the null convergence condition can in principle be violated, even in the absence of matter. For simplicity we study the possible violation of the null convergence condition on the MOTS ($S_H = 0$) and on the $S_0 = 0$ surface.

In our theory, using the "outgoing" null vector (VII.88), we have

$$\mathcal{R}_{\mu\nu}l^\mu l^\nu = \frac{2\,(N-B)^2}{F^2 r}\left(\frac{F_{,r}}{F} + \frac{N_{,r}}{N}\right) + \frac{4\,(N-B)\,F_{,t}}{F^2 r} + \frac{2B}{Fr}\left(\frac{B_{,t}}{B} - \frac{N_{,t}}{N}\right)\,. \tag{VII.99}$$

On the MOTS, on imposing the condition $B = N$, we simply find

$$\mathcal{R}_{\mu\nu}l^\mu l^\nu = \frac{2}{r_H F}\left(B_{,t} - N_{,t}\right)|_{r=r_H} = \frac{2}{F}\frac{\dot{r}_H}{r_H}\left(N_{,r} - B_{,r}\right)|_{r=r_H}\,. \tag{VII.100}$$

Solving the eq. (VII.93) in terms of $b_2(t)$ and inserting the solution in eq. (VII.100), we find a complicated relation which does not need to be non-negative, in principle. Thus the null convergence condition can be violated in our theory in the absence of matter.

Matching to cosmology

From what we have said in the case of static solution, we match the vacuum time-dependent solutions with a vacuum FLRW cosmology solution at infinity. In other words, we impose

$$\lim_{r\to\infty}\mathcal{R} = 0\,, \quad \lim_{r\to\infty} K = 3H_0\,, \quad \lim_{r\to\infty} K^{ij}K_{ij} = 3H_0^2\,, \quad \lim_{r\to\infty}\lambda^r = 0\,, \quad \dot{\phi} = 0\,, \tag{VII.101}$$

where H_0 is a positive constant. The last condition comes from the fact that, in vacuum FLRW cosmology, ρ_ϕ has to be constant, see eq. (VII.62). We also demand that the mass of the compact object be finite. A well-known definition of the mass in an asymptotically de Sitter space-time is the

Abbott-Deser mass [8], which is known to be conserved. In spherically symmetric configurations, the Abbott-Deser mass reduces to the $r \to \infty$ limit of the generalized Misner-Sharp mass M defined in eq. (VII.74) with $\Lambda_{\text{eff}} = 3H_0^2$. Therefore, we require that

$$h^{ab}\partial_a r\, \partial_b r = 1 - H_0^2 r^2 + \mathcal{O}(r^{-1})\,. \tag{VII.102}$$

On demanding the first conditions of (VII.101), we need to impose $\Lambda = 0$ (*cf.* eq. (VII.83a)) so that we can set

$$F = \left(1 - \frac{2\mu(t)}{r} + \frac{\kappa^2(t)}{r^4}\right)^{-1/2}\,. \tag{VII.103}$$

We then demand the second condition of (VII.101) to find $b_2 = H_0(5\mu^2\dot{\mu} + 2b_1)/(2\kappa)$, which ensures the third condition as well. So far the free function b_1 is left unspecified. In order to simplify the expressions, we can trade it for $n(t)$, as

$$b_1 \equiv -n(t)\kappa - \frac{5}{2}\,\dot{\mu}\,\mu^2\,, \qquad \text{so that} \qquad \lim_{r\to\infty} N = n(t) > 0\,, \tag{VII.104}$$

to match a yet undetermined and free behavior at infinity[8]. In this case $b_2 = -H_0\, n(t)$. The forth condition then leads to

$$(2H_0 - \Lambda_{,\phi})n = 0\,. \tag{VII.105}$$

As n cannot vanish, this reduces to the same condition we have found in the static case, namely $2H_0 - V_{,\phi} + 2\phi/3 = 0$. Finally, demanding the finite mass condition (VII.102), one obtains

$$\dot{\mu} + H_0\dot{\kappa} = 0\,. \tag{VII.106}$$

After imposing all the above mentioned boundary conditions, the four-dimensional Ricci tensor is shown to be

$$\mathcal{R}_{\mu\nu} = 3H_0^2\, g_{\mu\nu}\,. \tag{VII.107}$$

Therefore, from a GR point of view, the effective four-dimensional metric is locally Schwarzschild ($H_0 = 0$) or Schwarzschild-AdS ($H_0 \neq 0$), again thanks to the Birkhoff theorem. By using the definition (VII.74), we find that the generalized Misner-Sharp mass is

$$M = \mu + H_0\kappa\,, \tag{VII.108}$$

which is constant due to the condition (VII.106). Note again the similarity with the static case (VII.75), where $H_0 = -b_0$. This is consistent with the fact that the four-dimensional metric is either the Schwarzschild ($H_0 = 0$) or Schwarzschild-AdS ($H_0 \neq 0$) metric. Therefore, the Abbott-Deser mass $\lim_{r\to\infty} M$ is constant and finite.

Summary of time-dependent black hole solutions

In the time-dependent case, despite the absence of extra local physical degrees of freedom, the theory admits non-trivial solutions that are not Einstein spaces in general and that possess regular trapping and event horizons. Those generic solutions are given by eqs. (VII.80)–(VII.82) and are parametrized by several functions of time. They are in general different from Schwarzschild or Schwarzschild-(A)dS solutions even locally, and can break the null convergence condition despite the fact that they are written in vacuum.

Those exotic solutions become simpler if we require that they should asymptotically approach the corresponding cosmological solution, (a de Sitter (or Minkowski) space-time in the spatially

[8] We can *eg.* choose $n = 1$ by means of the freedom of the space-independent time reparametrization, which is a part of the foliation-preserving diffeomorphisms (VII.52).

homogeneous and isotropic slicing), but in general they still deviate from those in GR, *ie.* they are not Einstein spaces in general. Thus, despite the absence of extra physical degrees of freedom, the Birkhoff theorem does not hold in the model under consideration and a spherically symmetric vacuum solution depends not only on the cosmological constant and the mass but also on several free functions of the globally defined time coordinate. Those free functions are specified by suitable asymptotic conditions at the spatial infinity or boundary conditions at a finite distance.

However, for those non-GR solutions, the Abbott-Deser mass diverges and thus they do not represent compact objects. Such solutions may be interesting as possible large-scale structures in the Universe but our main focus in the present study is compact objects at astrophysical scales. Requiring that the Abbott-Deser mass be finite eliminates the non-GR time-dependent solutions and thus only possible solutions in the setup turned out to be Schwarzschild or Schwarzschild-(A)dS ones even in the time-dependent case.

VII.2.4 Inclusion of matter

As what was done in MTMG, let us have a word on the inclusion of matter, *ie.* study non-rotating stars. So far, we have dealt with solutions which represent black holes in vacuum and we were able to fix some of the free parameters (or functions) of the solutions by their asymptotic behavior at infinity, but some of those remain unconstrained. For example, while the combination (VII.108) is constant, each of $\mu(t)$ and $\kappa(t)$ may depend on time and one of them can be thought as being a free function of the solution, still unrestricted. Nonetheless, the theory is free from extra local degrees of freedom and the functions $\mu(t)$ and $\kappa(t)$ shall be unambiguously determined by matching the boundary condition and the initial condition of dynamical degrees of freedom. Obviously, if for a black hole solution one cannot impose a boundary condition at $r = 0$, the situation is different for stars, and we can expect that by building star solutions and matching them with exterior configurations, we will be able to fix more parameters (or functions).

To model such a non-rotating star, we include a perfect fluid with density ρ and a barotropic equation of state $P(\rho)$.

Exterior solution of static stars

Let us first focus on the static case with $B = 0$. In this case, the interior metric radial function F takes the form

$$F(r) = \left(1 + \frac{\Phi_0}{r} - \frac{\int^r dR\, R^2\, \rho(R)}{r M_{\rm Pl}^2} - \frac{\Lambda_0\, r^2}{3}\right)^{-1/2}, \tag{VII.109}$$

and we need to set the constant $\Phi_0 = 0$ for regularity at the origin. It should be noted that in this solution no term proportional to r^{-4} is present,[9] so that matter does not source r^{-4} terms in the function $F(r)$. Then for this interior solution, we can set at its border, *ie.* at the radius of the star, $r = r_\star$ defined as $\rho(r_\star) = 0$, the matching condition with the exterior vacuum metric. On doing this, we find that also in the exterior metric, we need to set $B = 0$, and in particular $\kappa_0 = 0$. On the other hand, the exterior solution fixes $\Lambda_0 = 0$ also for the interior solution. Since $B = 0$ also for the exterior solution, this solution represents a static asymptotically flat solution. In other words, we reach the conclusion that, on choosing $B = 0$ for the interior matter static solution, the four-dimensional metric solution surrounding a spherically symmetric star reduces to the standard Schwarzschild metric of GR.

For the general case $B \neq 0$, on the other hand, we were unable to derive an analytic interior solution in general. Therefore we cannot use the matching conditions between the interior and

[9] This is not highly surprising as, in the vacuum case, the r^{-4} term is proportional to κ_0, that arises as an integration constant of the equation of motion of B (VII.68). So if one imposes $B = 0$ from the beginning, there will be no r^{-4} term in vacuum also.

exterior solutions to determine the value of κ_0 analytically. Instead, one should set the value of κ_0 by looking for the numerical interior solution of the star for a given equation of state, after setting appropriate boundary conditions at the origin, and then matching the interior solution to the exterior solution. On the other hand, if the solution is to represent a static black hole formed by gravitational collapse then κ_0 should be obtained by the dynamics of gravitational collapse starting from a regular initial condition.

The case of time-dependent stars

In the time dependent case, we completely failed to derive an analytic general interior solution, and we will have to rely on a numerical integration of the equations of motion. When doing so, we will need to impose the boundary conditions which are compatible with the absence of a singularity at the origin. Through the matching conditions at the border of the compact object, these boundary conditions will in turn affect the exterior solution. On the other hand, the boundary conditions imposed by cosmology will also affect the interior solution by fixing the free functions which cannot be set by the boundary conditions at the origin. For example, around the origin, we expect to have in general $B = \zeta_1(t)\, r + \mathcal{O}(r^2)$, where $\zeta_1(t)$ will be determined by the interplay between the dynamics of matter and the matching conditions with the exterior solution.

Conclusion

This PhD thesis was devoted to the study of the gravitational interaction. As exposed in chapter I, our current cosmological model, the ΛCDM framework relying on General Relativity, faces two observational challenges. Those challenges, the missing mass problem and the current acceleration of the expansion of our Universe, could be explained by non-gravitational physics (*eg.* a cold dark matter particle and a dark cosmological fluid), but also by purely gravitational effects. In order to discriminate between those possible origins, the path to undertake is threefold: we should acquire a better observational knowledge of our Universe; we should deepen our theoretical comprehension of the current gravitational theory and of its phenomenological implications; and we should construct and investigate viable alternatives to the canonical gravitational theory. This thesis aimed at following the two last directions.

In the first part, we have focused on the two-body problem within General Relativity, and the emission of gravitational waves. The study of this problem can yield rewarding observational consequences, such as providing the first realistic test of GR in a distant exoplanetary system, as presented in chapter II. Nevertheless, the main interest of such study is to push forward our theoretical knowledge of GR, deep within the realm of high precision computations. In this vein, using synergies between the traditional post-Newtonian methods, Effective Field Theoretical approaches and other computational techniques, we have derived in chapter III high accuracy predictions for the logarithmic contributions of "tail" effects in the conserved energy. Those tail effects are genuine consequences of the non-linear nature of GR, and their proper treatment is always a subtle issue. The ansatz we have used is questionable, and its validity for high precision computations is not obvious to me. Nevertheless, this shortcoming appears as an exciting opportunity, as it merely indicates that there is a great deal of discussion and mutual understanding between the two communities, which will certainly lead to promising breakthroughs. The main contribution of this thesis to the two-body problem is the IR regularizations of the source mass quadrupole and of the non-linear interactions, at the fourth post-Newtonian order, presented in chapter IV. The investigation of the iterated non-linearities in a d-dimensional space is a premiere, and the preliminary results at the third post-Newtonian approximation, a strong confirmation of the soundness of the methods we employed. Moreover, by recovering EFT results and translating them in the traditional post-Newtonian language, we have opened a new space for discussions between the two communities.

The second part of this dissertation has been dedicated to the construction and study of theories beyond the canonical ones. First, a new construction of domain wall configurations has been presented in chapter V. By breaking the hypothesis on which usual arguments sustaining the stability of such defects rely, we have built stable kink configurations, that can mimic the canonical ones, at background and first order levels. Moreover those configurations are distinguishable from the canonical ones (and thus testable) at second order. Turning back to gravitational theories, we have then presented the idea of "minimalism", that can be used as a guideline when constructing alternatives to GR. This concept relies on the observational fact that only tensorial modes are observationally necessary, and thus proposes to build gravitational theories that contain only those tensorial modes. This idea has been applied by constructing a minimal theory of bigravity in chapter VI, which is one of the main results of this thesis. The study of its cosmological phenomenology revealed that it is a viable completion of the canonical Hasasn-Rosen bigravity theory at cosmological scales. Finally we have investigated the strong-field regimes of two of those minimal theories in chapter VII, by constructing non-rotating black hole solutions. We have found out interesting black hole space-times with non-trivial time dependences and possible violations of the null convergence condition. This second part thus offered an overview of the different steps that are to be followed when investigating alternative frameworks: studying the canonical hypothesis and breaking them, constructing novel theories, and then deriving and testing their phenomenological implications, notably in the cosmological and strong-field regimes.

Perspectives

If this PhD thesis left many exciting doors wide open, three major directions of future studies are worth being emphasized.

- The most natural one is inevitably to finish the computation of the radiative sector at the fourth post-Newtonian approximation. All the necessary ingredients are available, apart from the radiative quadrupole moment, which is currently under investigation. The proper treatment of the dimensional regularization of the tail and memory effects is now fully understood, and will be implemented in a very near future. The last remaining conceptual difficulty is the derivation of the second-order non-linear interplay between gauge and source moments, that will be treated for the first time (in either three of d dimensions).

- The second stimulating direction is to deepen synergies with the EFT community, notably by fully understanding and completing the proper treatment of the high orders tail effects in the conservative sector. This would naturally open many other doors, as current and future challenges of the two-body problem, notably in the radiative sector, would crucially benefit from a common understanding.

- A third exciting open door is the derivation of the phenomenological implications of our minimal theory of bigravity, and the possible deviations from the canonical ones. If we have seen that the cosmological phenomenology is appealing, and calls for a deeper analysis, it would also be extremely interesting to build rotating black hole and star solutions. The exploration of this strong-field regime by the construction and investigation of (yet non-rotating) black hole space-times is currently undertaken, and will be pursued in a near future.

Appendices

Appendix A

Conventions and some technical aspects of General Relativity

A.1 Conventions

In this thesis, we use natural units (where $\hbar = c = 1$), unless otherwise specified and define the Planck mass as

$$M_{\rm Pl} \equiv \sqrt{\frac{\hbar c}{8\pi G}}\,, \tag{A.1}$$

where $G \simeq 6.67 \cdot 10^{-11}\,\mathrm{m}^3.\mathrm{k}^{-1}.\mathrm{s}^{-2}$ is the Newton constant. Greek letters denote space-time indices, latin ones denote spatial indices and we naturally use the Einstein summation convention. We denote with "$\equiv$" a definition relation and with curly symbols (*eg.* "$\approx$" or "$\lesssim$") relations that hold only *on-shell*.

A.1.1 Geometrical conventions

We choose a mostly plus signature for the metric, defining the Minkowski metric as

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \equiv -c^2\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{A.2}$$

with δ_{ij} the Euclidean metric. Subsequently, the Riemann tensor is defined as

$$\mathcal{R}_{\mu\nu\rho}{}^{\sigma} = \partial_\nu\Gamma^\sigma_{\mu\rho} - \partial_\mu\Gamma^\sigma_{\nu\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}\,, \tag{A.3}$$

with $\partial_\mu \equiv \partial/\partial x^\mu$ the partial derivatives and the Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{g^{\sigma\lambda}}{2}\Big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big)\,. \tag{A.4}$$

The covariant derivative associated with $g_{\mu\nu}$ is denoted as ∇_μ, and we recall that for a vector V^μ and a one-form W_μ,

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta}V^\beta\,, \qquad \nabla_\mu W_\alpha = \partial_\mu W_\alpha - \Gamma^\beta_{\mu\alpha}W_\beta\,. \tag{A.5}$$

Symmetrizors and antisymmetrizors are weighted by the usual factorial terms. For example,

$$\begin{aligned} A_{(\mu\nu)} &\equiv \frac{1}{2}\Big(A_{\mu\nu} + A_{\nu\mu}\Big)\,, \qquad A_{[\mu\nu]} \equiv \frac{1}{2}\Big(A_{\mu\nu} - A_{\nu\mu}\Big)\,, \\ B_{(\mu\nu\rho)} &\equiv \frac{1}{3!}\Big(B_{\mu\nu\rho} + B_{\mu\rho\nu} + B_{\nu\mu\rho} + B_{\nu\rho\mu} + B_{\rho\mu\nu} + B_{\rho\nu\mu}\Big)\,. \end{aligned} \tag{A.6}$$

A.1.2 Dimensional conventions

Most of this dissertation deals with four-dimensional space-times. Nevertheless, in chapter IV we will also work in a $d+1$ dimensional manifold, *ie.* bearing a single time dimension and d spatial ones, d being an arbitrary complex number. For simplicity, we will focus on spatial dimensions, and thus dub "d-dimensional" this $d+1$-dimensional analysis (and similarly "three-dimensional" the usual four-dimensional space-time). In this d-dimensional copy of General Relativity, the gravitational strength G is related to the usual Newton constant G_N by a new length scale ℓ_0 as

$$G = \ell_0^{d-3}\, G_N\,. \tag{A.7}$$

We will also extensively use the notations $\varepsilon \equiv d-3$ and

$$\tilde{k} \equiv \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} = \frac{1}{d-2}\,\frac{4\pi}{\Omega_{d-1}}\,, \tag{A.8}$$

with Γ the usual Euler function and Ω_{d-1} the volume of the $d-1$ Euclidean sphere. The definition of $\tilde{k}$ was chosen so that it goes to unity in the three-dimensional limit:

$$\tilde{k} = 1 - \frac{\ln(4\pi)+\gamma_E}{2}\,\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\,, \tag{A.9}$$

with γ_E the Euler constant.

A.1.3 Conventions specific to post-Newtonian computations

In the post-Newtonian and post-Minkowskian analysis led in chapters chapter III and IV, we focus on two point-particles of mass m_1 and m_2, located in $\mathbf{y}_1$ and $\mathbf{y}_2$, respectively. Taking a point $\mathbf{x} \equiv r\,\mathbf{n}$ in the d-dimensional Euclidean space, it is useful to define the vectors $\mathbf{r}_1 \equiv \mathbf{x} - \mathbf{y}_1 = r_1\,\mathbf{n}_1$ and $\mathbf{r}_2 \equiv \mathbf{x} - \mathbf{y}_2 = r_2\,\mathbf{n}_2$ together with the relative separation of the bodies $\mathbf{r}_{12} \equiv \mathbf{y}_1 - \mathbf{y}_2 = r_{12}\,\mathbf{n}_{12}$, as presented in fig. A.1. Naturally the $\mathbf{n}$ vectors are unit vectors.

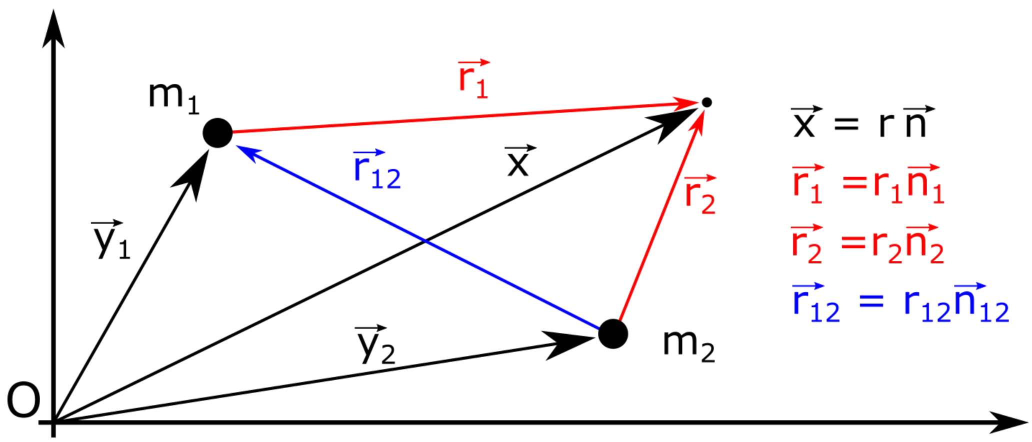


Figure A.1: Geometric conventions used in the analysis led in chapter III and chapter IV.

In addition to those geometric conventions, we will use a particular set of shorthands, presented hereafter. As we employ a flat spatial metric δ_{ij}, we do not discriminate between covariant and contravariant spatial indices. Similarly, repeated indices denote trace operation, irrespectively of their position : $A^{ii} = A_{ii} = A^i_i \equiv \delta^{ij} A_{ij}$.

We write multi-index with uppercase shorthands: $L \equiv i_1 i_2 \dots i_\ell$. This allows to stack repeated vectors or derivatives as for example $n_K \equiv n_{i_1} n_{i_2} \dots n_{i_k}$ or $\partial_L \equiv \partial_{i_1}\partial_{i_2}\dots\partial_{i_\ell}$. This notation is straightforwardly extended in many situations, as *eg.* $\partial_{aL-2} \equiv \partial_a \partial_{i_1} \partial_{i_2} \dots \partial_{i_{\ell-2}}$ and accounts also for implicit summation conventions as in $\partial_L A_L$.

We make extensive use of symmetric and trace-free (STF) projections, denoted either explicitly by the shorthands

$$\underset{L}{\mathrm{STF}}\,[A_L] \equiv A_{\langle L\rangle} \equiv \hat{A}_L\,, \tag{A.10}$$

or implicitly by directly defining tensors in a STF fashion (*eg.* all the multipoles of app. B). As stated by the name, a STF quantity is totally symmetric under the exchange of any two of its indices, and vanishing under any contraction of its indices. For example in d-dimensional space-times, the STF projections of quantities with two and three indices are easily worked out

$$A_{\langle ij\rangle} = A_{(ij)} - A_{aa}\frac{\delta_{ij}}{d}\,, \qquad \text{and} \qquad A_{\langle ijk\rangle} = A_{(ijk)} - \frac{A_{aai}\,\delta_{jk}}{d+2} - \frac{A_{aaj}\,\delta_{ik}}{d+2} - \frac{A_{aak}\,\delta_{ij}}{d+2}\,. \tag{A.11}$$

A.2 General Relativity: Lagrangian formulation

From the Riemann tensor (A.3), one can construct the Ricci tensor and scalar by contracting appropriate indices

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\lambda\nu}{}^{\lambda}\,, \qquad \mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}\,. \tag{A.12}$$

The Einstein-Hilbert action is then simply given by [176]

$$\mathcal{S}_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\mathcal{R}\,, \tag{A.13}$$

the square-root of (minus) the determinant of the metric, $\sqrt{-g}$, being the canonical measure for generic manifolds. In addition to this Einstein-Hilbert sector, GR can also contain a cosmological constant, denoted as Λ, and a matter sector, described by a Lagrangian density $\mathcal{L}_{\mathrm{mat}}\,[g_{\mu\nu};\psi_{\mathrm{mat}}]$, where ψ_{mat} denotes the collection of matter fields (*ie.* all non-gravitational forms of energy) living in the manifold. The action for GR finally reads

$$\mathcal{S}_{\mathrm{GR}} = \frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\,(\mathcal{R}-2\Lambda) + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{mat}}\,[g_{\mu\nu};\psi_{\mathrm{mat}}]\;. \tag{A.14}$$

The equations of motion for the metric and matter fields result from the usual least-action principle

$$\mathcal{E}_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left[\frac{M_{\mathrm{Pl}}^2}{2}\sqrt{-g}\,g^{\alpha\beta}\mathcal{R}_{\alpha\beta} - M_{\mathrm{Pl}}^2\Lambda\sqrt{-g} + \sqrt{-g}\mathcal{L}_{\mathrm{mat}}\right]\,, \qquad \mathcal{E}_{\mathrm{mat}} \equiv \frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta\psi_{\mathrm{mat}}}\,. \tag{A.15}$$

The variation of the measure is given by $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}$, and the variation of the Ricci tensor is obtained by using the Palatini formula

$$\delta\Gamma^{\rho}_{\mu\nu} = -\frac{g^{\rho\lambda}}{2}\bigg(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\bigg)\,, \qquad g^{\alpha\beta}\delta\mathcal{R}_{\alpha\beta} = \nabla_{\mu\nu}\left[\delta g^{\mu\nu} - g^{\mu\nu}g_{\rho\lambda}\,\delta g^{\rho\lambda}\right]\,, \tag{A.16}$$

with the perturbation $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\,\delta g^{\alpha\beta}$. Injecting it in the action, we see that it gives rise to a surface term, that will not contribute to the equations of motion. Defining the *stress-energy* tensor as $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{mat}})}{\delta g^{\mu\nu}}$, we finally obtain the famous Einstein's field equations

$$\mathcal{E}_{\mu\nu} = M_{\mathrm{Pl}}^2\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}\,g_{\mu\nu} + \Lambda\,g_{\mu\nu}\right) - T_{\mu\nu} \approx 0\,. \tag{A.17}$$

A.3 Arnowitt-Deser-Misner formalism and Hamiltonian analysis of General Relativity

Any metric $g_{\mu\nu}$ can be reshaped in the so-called Arnowitt-Deser-Misner (ADM) form by introducing a lapse function N, a shift vector N^i and a spatial metric γ_{ij} as [34]

$$g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2 \mathrm{d}t^2 + \gamma_{ij} \left(N^i \mathrm{d}t + \mathrm{d}x^i \right) \left(N^j \mathrm{d}t + \mathrm{d}x^j \right) . \tag{A.18}$$

If this decomposition explicitly breaks the temporal diffeomorphism, it retains the spatial ones. Indeed the lapse is a spatial scalar, the shifts constitute a spatial vector, and the spatial metric is naturally a spatial tensor.

The physical meaning of this "$3+1$" decomposition can be easily understood when the manifold is foliated by a (infinite) family of space-like hypersurfaces (which is always possible for space-times that are considered within this thesis). In this framework, the spatial metric is simply the induced metric on the hypersurface, the shift vector is the projection of the coordinate time vector on the hypersurface and the lapse is the normal component of the coordinate time vector. For a comprehensive review on this decomposition (and its powerful applications in numerical relativity), see [213].

This ADM decomposition is convenient as it singles out the time sector of the metric, which is required[1] when performing a Hamiltonian analysis. Denoting $\mathcal{D}_i$ the covariant derivative compatible with γ_{ij}, and introducing the *extrinsic* curvature

$$K_{ij} \equiv \frac{1}{2N} \Big(\partial_t \gamma_{ij} - 2\mathcal{D}_{(i} N_{j)} \Big) , \tag{A.19}$$

and its trace $K \equiv \gamma^{ij} K_{ij}$, the Gauss-Codazzi relations [213] allow to express the Einstein-Hilbert action as

$$\mathcal{S}_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}t \mathrm{d}^3x N \sqrt{\gamma} \left(\mathcal{R}\left[\gamma\right] + K_{ij}K^{ij} - K^2 \right) = \int \mathrm{d}t \mathrm{d}^3x\, L_{\mathrm{EH}} . \tag{A.20}$$

Where $\mathcal{R}\left[\gamma\right]$ is obviously the Ricci scalar constructed out of the spatial metric. The lapse and shifts enter non-dynamically, thus we will not associate conjugate momenta to them, but treat them as ancillary variables. As for the spatial metric, its time derivative only enters the extrinsic curvature, thus its conjugate momentum is easily computed as

$$\pi^{ij} \equiv \frac{\delta L_{\mathrm{EH}}}{\delta(\partial_t \gamma_{ij})} = \frac{M_{\mathrm{Pl}}^2}{2} \sqrt{\gamma} \left(K^{ij} - K\, \gamma^{ij} \right) . \tag{A.21}$$

At this point, we have twelve dynamical phase-space degrees of freedom $\{\gamma_{ij}, \pi^{\mathrm{ij}}\}$. Constructing the primary Hamiltonian in the usual way, and discarding surface terms, it comes

$$\int \mathrm{d}t \mathrm{d}^3x\, H_{\mathrm{EH}}^{(1)} \equiv \int \mathrm{d}t \mathrm{d}^3x \left(\pi^{ij} \partial_t \gamma_{ij} - L_{\mathrm{EH}} \right) = - \int \mathrm{d}t \mathrm{d}^3x \left(N \mathcal{R}_0 + N^i \mathcal{R}_i \right) . \tag{A.22}$$

The primary Hamiltonian is thus a pure constraint, with Lagrange multipliers given by the lapse and shifts. Denoting $\pi \equiv \gamma_{ij} \pi^{ij}$, the so-called *Hamiltonian* and *momentum* constraints are given by

$$\mathcal{R}_0 = \frac{M_{\mathrm{Pl}}^2}{2} \sqrt{\gamma}\, \mathcal{R}\left[\gamma\right] - \frac{2}{M_{\mathrm{Pl}}^2 \sqrt{\gamma}} \left(\pi_{ij} \pi^{ij} - \frac{\pi^2}{2} \right) , \tag{A.23a}$$

$$\mathcal{R}_i = 2\sqrt{\gamma}\, \gamma_{ij} \mathcal{D}_k \left(\frac{\pi^{jk}}{\sqrt{\gamma}} \right) . \tag{A.23b}$$

[1]To be exact, one needs to single out one of the coordinates to perform the Legendre transformation, and we could have chosen one of the spatial coordinates. But retaining the invariance under spatial diffeomorphisms is extremely useful, and thus we have chosen to single out time.

Following the usual Dirac procedure [170], we investigate the nature of those constraints. For simplicity, we use "distributional forms" , *ie.* using well-behaved test functions ζ and ξ^i that vanish at spatial boundaries, we define

$$\mathcal{R}_0\left[\zeta\right] \equiv \int \mathrm{d}^3 z\, \mathcal{R}_0\, \zeta(z)\,, \qquad \mathcal{R}_i\left[\xi^i\right] \equiv \int \mathrm{d}^3 z\, \mathcal{R}_i\, \xi^i(z)\,. \tag{A.24}$$

Using Hamilton's equations of motion, one can show that the time evolution of a constraint $\mathcal{C}$, that does not explicitly depend on time, is given by its Poisson bracket with the Hamiltonian, as

$$\begin{aligned}\frac{\mathrm{d}\mathcal{C}\left[\zeta\right]}{\mathrm{d}t} &= \frac{\partial\mathcal{C}\left[\zeta\right]}{\partial\gamma_{ij}}\frac{\mathrm{d}\gamma_{ij}}{\mathrm{d}t} + \frac{\partial\mathcal{C}\left[\zeta\right]}{\partial\pi^{ij}}\frac{\mathrm{d}\pi^{ij}}{\mathrm{d}t} = \int \mathrm{d}^3 y \left(\frac{\partial\mathcal{C}\left[\zeta\right]}{\partial\gamma_{ij}(y)}\frac{\partial H^{(1)}_{\mathrm{EH}}\left[\chi\right]}{\partial\pi^{ij}(y)} - \frac{\partial\mathcal{C}\left[\zeta\right]}{\partial\pi^{ij}(y)}\frac{\partial H^{(1)}_{\mathrm{EH}}\left[\chi\right]}{\partial\gamma_{ij}(y)}\right) \\ &= \left\{\mathcal{C}\left[\zeta\right], H^{(1)}_{\mathrm{EH}}\left[\chi\right]\right\}\,.\end{aligned} \tag{A.25}$$

Therefore the constraints can be split in two classes: the *first-class* constraints have vanishing Poisson brackets with all other constraints (and thus are automatically conserved in time), and the *second-class* constraints have non-vanishing Poisson brackets with at least one other constraint.

A *first-class* constraint removes two phase-space degrees of freedom. Firstly, the algebraic relation $\mathcal{C} = 0$ is to be solved for one dynamical variable. Secondly, as its time conservation is automatically ensured, its associated Lagrange multiplier is not determined, but remains free. As such non-dynamical variable enters the equations of motion of the dynamical variables, it can be used to fix one of them (this behavior is very similar to a gauge choice, and indeed, primary *first-class* constraints are associated with gauge freedom [170]). Regarding the *second-class* constraints, the algebraic relation $\mathcal{C} = 0$ is still to be solved for one dynamical variable. But enforcing the conservation in time of such constraint $\mathrm{d}\mathcal{C}/\mathrm{d}t \equiv 0$ can either be solved by fixing one Lagrange multiplier, or yield a secondary constraint. Thus *second-class* constraints remove only one phase-space degree of freedom each.

For the Hamiltonian (A.22), the Dirac algebra is given by

$$\left\{\mathcal{R}_0\left[\zeta\right], \mathcal{R}_0\left[\chi\right]\right\} = \int \mathrm{d}^3 z\, \mathcal{R}_i \left[\zeta\partial^i\chi - \chi\partial^i\zeta\right]\,, \tag{A.26a}$$

$$\left\{\mathcal{R}_0\left[\zeta\right], \mathcal{R}_i\left[\xi^i\right]\right\} = -\int \mathrm{d}^3 z \sqrt{\gamma}\, \mathcal{D}_i \left[\mathcal{R}_0\zeta\xi^i\right]\,, \tag{A.26b}$$

$$\left\{\mathcal{R}_i\left[\xi^i\right], \mathcal{R}_j\left[\chi^j\right]\right\} = \int \mathrm{d}^3 z\, \mathcal{R}_i \left[\chi^j\mathcal{D}_j\xi^i - \xi^j\mathcal{D}_j\chi^i\right]\,, \tag{A.26c}$$

and thus vanishes *on-shell*, which means that all constraints are *first-class*. Therefore there is no need of secondary constraints: the primary Hamiltonian is also the final one. As each *first-class* constraint fixes two phase-space variables, the number of independent phase-space variables is $12 - 2 \times 4 = 4$, thus two physical degrees of freedom propagates, corresponding to the usual two gravitational wave polarizations.

As advertised, primary *first-class* constraints are usually associated with gauge symmetries. In the case of the Einstein-Hilbert Hamiltonian (A.22), the Hamiltonian constraint is associated with the invariance under boosts, and the momentum constraints with the spatial rotational invariance.

Appendix B

Toolkit for post-Minkowskian expansions

B.1 Recasting Einstein's field equations

We ultimately seek a metric $g_{\mu\nu}$ solution of the GR field equations (A.17) with vanishing cosmological constant, and matter sector given by $\mathcal{L}_{\rm mat}$. Instead of studying directly $g_{\mu\nu}$, we define the perturbation of the so-called "gothic" metric

$$h^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}\,, \tag{B.1}$$

where $\eta^{\mu\nu}$ is the inverse Minkowskian metric with mostly plus signature, and the results of this section are valid also in d-dimensional space. As advertised in the previous appendix, there remains some gauge freedom, so we can impose the *de Donder* conditions (sometimes also called *Henry* gauge)

$$\partial_\mu h^{\mu\nu} \equiv 0\,. \tag{B.2}$$

The associated coordinate system is usually dubbed "harmonic". Indeed, considering the coordinates $\{x^\mu\}$ as a set of *scalar* functions, it comes

$$\partial_\nu h^{\mu\nu} = \partial_\nu \left(\sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}\right) = -\sqrt{-g} g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = -\sqrt{-g} g^{\rho\sigma}\Gamma^\nu_{\rho\sigma}\partial_\nu x^\mu = \sqrt{-g}\Box_g x^\mu\,, \tag{B.3}$$

where $\Box_g \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the curved space d'Alembertian operator. Therefore in this gauge $\Box_g x^\mu = 0$: the coordinates enjoy being homogeneous solutions of the d'Alembertian operator, thus the name "harmonic" coordinate system. The interest of the particular definition (B.1), together with this coordinate system, is that the metric field equations are simply recast as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\,\tau^{\mu\nu}\,, \tag{B.4}$$

ie. a wave equation with propagator given by the usual flat space d'Alembertian operator $\Box \equiv \eta^{\mu\nu}\partial_{\mu\nu}$ and source term involving the Landau-Lifschitz pseudo tensor

$$\tau^{\mu\nu} \equiv |g|\, T^{\mu\nu} + \frac{c^4}{16\pi G}\,\Lambda^{\mu\nu}\,, \tag{B.5}$$

where $T^{\mu\nu}$ is the usual stress-energy tensor for matter. The second piece, given by $\Lambda^{\mu\nu}$, is constructed out of the metric only and reads in d dimensions

$$\begin{aligned}\Lambda^{\mu\nu} =& - h^{\alpha\beta}\partial_{\alpha\beta}h^{\mu\nu} + \partial_\alpha h^{\mu\beta}\partial_\beta h^{\nu\alpha} + \frac{1}{2}\, g^{\mu\nu} g_{\alpha\beta}\partial_\rho h^{\alpha\sigma}\partial_\sigma h^{\beta\rho} - 2 g^{\alpha(\mu} g_{\rho\beta}\partial_\sigma h^{\nu)\beta}\partial_\alpha h^{\rho\sigma} \\ &+ g^{\alpha\beta} g_{\rho\sigma}\partial_\alpha h^{\mu\rho}\partial_\beta h^{\nu\sigma} + \frac{1}{4}\left(2 g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta}\right)\left(g_{\rho\sigma} g_{\lambda\tau} - \frac{1}{d-1}\, g_{\rho\lambda} g_{\sigma\tau}\right)\partial_\alpha h^{\rho\lambda}\partial_\beta h^{\sigma\tau}\,.\end{aligned} \tag{B.6}$$

Note that $\Lambda^{\mu\nu}$ is highly non-linear, as expected for GR, but begins at the quadratic order in $h^{\mu\nu}$ only.

B.2 The multipolar-post-Minkowskian iteration scheme

As advertised by the name, the spirit of post-Minkowskian (PM) computations is to expand quantities around their Minkowskian vacuum configurations, outside of the distribution of matter. Naturally this expansion performed in the radiation zone has to be carefully matched with the post-Newtonian expansion performed in the near zone, inside the matter distribution.[1] We formally expand the perturbation in powers of Newton constant as

$$h^{\mu\nu} \equiv \sum_{n\geq 1} G^n \, h^{\mu\nu}_{(n)} \,. \tag{B.7}$$

In addition to this PM expansion, we also decompose each quantity in a multipolar way, *ie.* we project them on spherical harmonics. It can be shown (see *eg.* the formulas displayed in [187]) that spin-2 weighted spherical harmonics and symmetric trace-free (STF) products of unit vectors $\hat{n}_L$ are equivalent. Therefore we will work for example with

$$h^{00}_{(n)} = \sum_{\ell\in\mathbb{N}} A^L_{(n)}(r,t)\, \hat{n}_L \,, \qquad \text{or} \qquad h^{0i}_{(n)} = \sum_{\ell\in\mathbb{N}} \left\{ B^{iL}_{(n)}(r,t)\, \hat{n}_L + C^L_{(n)}(r,t)\, \hat{n}_{iL} \right\} . \tag{B.8}$$

This MPM double formal expansion is to be injected in the vacuum field equations (*ie.* eq. (B.4) with $T^{\mu\nu} = 0$), that reduce to

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu} \,. \tag{B.9}$$

The aim of MPM computations is thus to solve such equations, order by order in G and for each mulitpolarity ℓ, together with the gauge-fixing condition (B.2).

In the rest of this section, we briefly outline the most striking features of this method in three-dimensional spaces, and the generalization to d dimensions is to be worked out in chapter IV. We let interested readers refer to [73] for a more comprehensive presentation of the MPM framework.

B.2.1 At linear order

Injecting the formal expansion (B.7) in the non-linear source term (B.6), it straightforwardly appears that $\Lambda^{\mu\nu}$ begins at order G^2. Therefore the linear piece of the metric, $h^{\mu\nu}_{(1)}$, is simply given by a homogeneous solution of the flat d'Alembertian operator.

Source moments

A homogeneous solution of the flat d'Alembertian operator can be parametrized by a series of STF multipolar advanced and retarded waves

$$\mathcal{H}^{\mu\nu} = \sum_{\ell\in\mathbb{N}} \partial_L \left(\frac{\mathcal{A}^{\mu\nu}_L(t+r/c) + \mathcal{R}^{\mu\nu}_L(t-r/c)}{r} \right) , \tag{B.10}$$

as easily verified by applying the d'Alembertian operator (that obviously commutes with the partial derivatives) on $\mathcal{H}^{\mu\nu}$. Note that the multipolarity is here encoded in the ℓ derivatives (as long as A_L is STF, $\partial_L A_L(t\pm r/c) \propto A_L^{(\ell)}(t\pm r/c)\, \hat{n}_L$). If both advanced and retarded waves are mathematically allowed, the advanced ones physically correspond to radiation coming from infinity towards the matter system. Therefore it is usual to impose in addition a "no-incoming radiation" condition that, as stated by its name, accounts for discarding the moments $\mathcal{A}^{\mu\nu}_L$.

[1] We will naturally apply this formalism to the two-body problem, but note that the formulas presented in this appendix hold for any compact support matter distribution.

Requiring in addition that the homogeneous solution $\mathcal{H}^{\mu\nu}$ obeys the gauge condition (B.2), the ten components of $\mathcal{R}_L^{\mu\nu}$ can be naturally parametrized by two sets of STF moments $\{I_L, J_L\}$, as

$$\begin{aligned}
k^{00}_{(1)} &= -\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,I_L\left(t-\frac{r}{c}\right)\right]\,,\\
k^{0i}_{(1)} &= \frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}\,I^{(1)}_{iL}\left(t-\frac{r}{c}\right)\right]+\frac{\ell}{\ell+1}\,\partial_{aL-1}\left[\frac{1}{r}\,\varepsilon_{abi}J_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\}\,,\\
k^{ij}_{(1)} &= -\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[\frac{1}{r}\,I^{(2)}_{ijL}\left(t-\frac{r}{c}\right)\right]+\frac{2\ell}{\ell+1}\,\partial_{aL-2}\left[\frac{1}{r}\,\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]\right\}\,,
\end{aligned}\tag{B.11}$$

where ε_{ijk} is the flat Levi-Civita symbol and parenthesis in exponent denote time derivation. Due to their apparent parity, I_L are called *mass* moments, and J_L are called *current* moments (they are respectively analogous to the electric and magnetic moments of electromagnetism).

This expansion has to be matched to the post-Newtonian expansion performed within the source. This matching is the most delicate part of the process and formally reads [71]

$$I_L(u) = \underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\left\{\delta_\ell(z)\,\hat{x}_L\,\overline{\Sigma}-\frac{4(2\ell+1)\delta_{\ell+1}(z)}{c^2(\ell+1)(2\ell+3)}\,\hat{x}_{iL}\overline{\Sigma}^{(1)}_i+\frac{2(2\ell+1)\delta_{\ell+2}(z)\,\hat{x}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\overline{\Sigma}^{(2)}_{ij}\right\}\,,\tag{B.12a}$$

$$J_L(u) = \underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\,\varepsilon_{ab\langle i_\ell}\left\{\delta_\ell(z)\,\hat{x}_{L-1\rangle a}\,\overline{\Sigma}_b-\frac{(2\ell+1)\delta_{\ell+1}(z)}{c^2(\ell+2)(2\ell+3)}\,\hat{x}_{L-1\rangle ac}\overline{\Sigma}^{(1)}_{bc}\right\}\,,\tag{B.12b}$$

where the source densities are evaluated in $(\mathbf{x}, u+zr/c)$ and defined from the PN-expanded components of the Landau-Lifschitz pseudo tensor (B.5) (an overbar denoting a formal PN expansion)

$$\overline{\Sigma}\equiv\frac{\overline{\tau}^{00}+\delta_{ij}\overline{\tau}^{ij}}{c^2}\,,\qquad\overline{\Sigma}_i\equiv\frac{\overline{\tau}^{0i}}{c}\,,\qquad\text{and}\qquad\overline{\Sigma}_{ij}\equiv\overline{\tau}^{ij}\,.\tag{B.13}$$

Note the particularity of GR, that is encoded in the definition of those source densities: the moments depend naturally on the matter content of the source, but also on the energy contained in the metric field, *via* the $\Lambda^{\mu\nu}$ piece of the Landau-Lifschitz pseudo tensor. The z-integration involving the kernels

$$\delta_\ell(z)\equiv\frac{(2\ell+1)!!}{2^{\ell+1}\,\ell!}\left(1-z^2\right)^\ell\,,\tag{B.14}$$

allows to formally define the PN expansion of the moments, as

$$\int_{-1}^1\mathrm{d}z\,\delta_\ell(z)\,A\left(\mathbf{x},u+\frac{zr}{c}\right)=\sum_{k\in\mathbb{N}}\frac{(2\ell+1)!!}{2^k\,k!(2\ell+2k+1)!!}\left(\frac{r}{c}\frac{\partial}{\partial u}\right)^{2k}A\left(\mathbf{x},u\right)\,.\tag{B.15}$$

Note that this expansion is even in c. Last, but definitively not least, the definitions of the multipoles (B.12) involve the crucial *Hadamard partie finie* regularization scheme. This is due to the fact, that, plugging the formal expansions (B.15) in eq. (B.12), the integrals are fiercely IR divergent. Those integrals are thus to be computed with a regulator $(r/r_0)^B$ and evaluated by analytic continuation in $B\in\mathbb{C}$. At the end of the computation, only the zeroth order of the Laurent series on B is to be considered (this notably implies that all the poles $1/B$ are to be discarded). The final expression of the moments will then inevitably depend on the IR scale r_0, but such scale is not physical, and in fact r_0 has to disappear in the observables (flux, gravitational phase,...).

If the expressions (B.12) are quite massive, they fortunately give back the usual source moments in the Newtonian limit. In such limit, $T^{00}=\rho\,c^2+\mathcal{O}(1)$, $T^{0i}=\rho\,c\,v^i+\mathcal{O}(c^{-1})$ and $T^{ij}\sim\mathcal{O}(1)$, with ρ the matter density, and v^i the speed of the system. Therefore, only $\overline{\Sigma}=\rho+\mathcal{O}(c^{-2})$ will contribute to the dominant order in I_L, and $\overline{\Sigma}_i=\rho v^i+\mathcal{O}(c^{-2})$, to J_L. Injecting those expressions in (B.12) , it comes

$$I_L(u)=\int\mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x},u)\hat{x}_L+\mathcal{O}(c^{-2})\,,\qquad\text{and}\qquad J_L(u)=\int\mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x},u)\,\varepsilon_{ab(i}\hat{x}_{L-1)a}v_b+\mathcal{O}(c^{-2})\,,\tag{B.16}$$

which are the usual Newtonian source moments for an arbitrary distribution of matter.

Gauge moments

If the linear perturbation (B.11) is indeed a homogeneous solution of the d'Alembertian operator, satisfying the gauge condition (B.2), it is not the most general one. In fact, there remains a freedom in the coordinate choice, which can be linked to the fact that two physical degrees of freedom propagate in GR, whereas eq. (B.2) only fixes four of the ten components of $h^{\mu\nu}$. Performing the coordinate change $x^\mu \to \chi^\mu = x^\mu + G\xi^\mu$, and perturbing eq. (B.1) at 1PM order, it comes

$$h^{\mu\nu} \to \tilde{h}^{\mu\nu} = h^{\mu\nu} + 2\partial^{(\mu}\xi^{\nu)} - \eta^{\mu\nu}\,\partial_\alpha\xi^\alpha\,, \tag{B.17}$$

where we naturally raise indices with $\eta^{\mu\nu}$. In order not to spoil the de Donder gauge condition, we have to impose in addition that $\Box\xi^\mu = 0$. Let us point out that the coordinate transformation ξ^μ fixes four additional components of $h^{\mu\nu}$, which is now totally gauged: there is no residual freedom. Therefore the most general solution at linear order is given by

$$h^{\mu\nu}_{(1)} = k^{\mu\nu}_{(1)} + \partial^\mu\varphi^\nu_{(1)} + \partial^\nu\varphi^\mu_{(1)} - \eta^{\mu\nu}\,\partial_\alpha\varphi^\alpha_{(1)}\,, \tag{B.18}$$

where the tensorial piece, $k^{\mu\nu}_{(1)}$, is naturally given by eq. (B.11), and the linear gauge vector is parametrized by four sets of STF moments $\{W_L, X_L, Y_L, Z_L\}$ as

$$\varphi^0_{(1)} = \frac{4}{c^3}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,W_L\left(t-\frac{r}{c}\right)\right]\,, \tag{B.19a}$$

$$\begin{aligned}\varphi^i_{(1)} = &-\frac{4}{c^4}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\partial_{iL}\left[\frac{1}{r}\,X_L\left(t-\frac{r}{c}\right)\right]\\ &-\frac{4}{c^4}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}\,Y_{iL-1}\left(t-\frac{r}{c}\right)\right] + \frac{\ell}{\ell+1}\,\partial_{aL-1}\left[\frac{1}{r}\,\varepsilon_{abi}Z_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\}\,,\end{aligned} \tag{B.19b}$$

where the no-incoming radiation condition accounts for selecting the minus sign in the argument of the moments. Those moments are naturally dubbed *gauge* moments, as they only account for a gauge transformation. They have also to be carefully matched to the PN expansion of the near zone, and formally read [73]

$$W_L(u) = \underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\,\frac{2\ell+1}{(\ell+1)(2\ell+3)}\left\{\delta_{\ell+1}(z)\,\hat{x}_{iL}\,\overline{\Sigma}_i - \frac{(2\ell+3)\,\delta_{\ell+2}(z)}{2c^2(\ell+2)(2\ell+5)}\,\hat{x}_{ijL}\overline{\Sigma}^{(1)}_{ij}\right\}\,, \tag{B.20a}$$

$$X_L(u) = \underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\,\frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}(z)\,\hat{x}_{ijL}\,\overline{\Sigma}_{ij}\,, \tag{B.20b}$$

$$\begin{aligned}Y_L(u) = \underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\,\Bigg\{&-\delta_\ell(z)\,\hat{x}_L\,\overline{\Sigma}_{ii} + \frac{3(2\ell+1)\,\delta_{\ell+1}(z)}{(\ell+1)(2\ell+3)}\,\hat{x}_{iL}\overline{\Sigma}^{(1)}_i\\ &-\frac{2(2\ell+1)\,\delta_{\ell+2}(z)}{c^2(\ell+1)(\ell+2)(2\ell+5)}\,\hat{x}_{ijL}\overline{\Sigma}^{(2)}_{ij}\Bigg\}\,,\end{aligned} \tag{B.20c}$$

$$Z_L(u) = -\underset{B}{\mathrm{PF}}\int\mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\int_{-1}^1\mathrm{d}z\,\varepsilon_{ab\langle i}\hat{x}_{L-1\rangle bc}\frac{2\ell+1}{(\ell+2)(2\ell+3)}\,\delta_{\ell+1}(z)\,\overline{\Sigma}_{ac}\,, \tag{B.20d}$$

where $\delta_\ell(z)$ is still given by (B.14); the source densities, by (B.13) and are evaluated in $(\mathbf{x}, u+zr/c)$.

B.2.2 Canonical moments

The linear post-Minkowskian solution is thus parametrized by six sets of moments, given by (B.12) and (B.20). All those moments naturally enter the source term of eq. (B.9), and thus contribute to higher order iterations, which induces involving computations. Fortunately, it has been shown that, to any order in G, the non-linear metric parametrized by those six sets of moments is equivalent up to a coordinate change to a simpler metric, parametrized only by two sets of moments [73].

At linear order, this simpler metric is given by $k^{\mu\nu}_{(1)}$ under the replacement $\{I_L, J_L\} \to \{\mathrm{M}_L, \mathrm{S}_L\}$, where those new moments are called mass and current *canonical* moments. As the four gauge moments parametrize an unphysical linear coordinate transformation, they are absolutely irrelevant at linear order. Therefore the discrepancy between $\{\mathrm{M}_L, \mathrm{S}_L\}$ and $\{I_L, J_L\}$ starts only at relative 1PM (and 2.5PN) order. For example, it comes [187]

$$\mathrm{M}_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{G}{c^7}; \frac{G^2}{c^8}\right), \tag{B.21a}$$

$$\mathrm{M}_{ijk} = I_{ijk} + \frac{4G}{c^5}\left[W^{(2)} I_{ijk} - W^{(1)} I^{(1)}_{ijk} + 3 I_{\langle ij} Y^{(1)}_{k\rangle}\right] + \mathcal{O}\left(\frac{G}{c^7}; \frac{G^2}{c^8}\right), \tag{B.21b}$$

$$\mathrm{S}_{ij} = J_{ij} + \frac{2G}{c^5}\left[3 J_{\langle i} Y^{(1)}_{j\rangle} - 2 J^{(1)}_{ij} W^{(1)} + \varepsilon_{ab\langle i}\left(I^{(1)}_{j\rangle b} Y^{(1)}_a - I^{(3)}_{j\rangle b} W_a - 2 I_{j\rangle b} Y^{(2)}_a\right)\right] + \mathcal{O}\left(\frac{G}{c^7}; \frac{G^2}{c^8}\right). \tag{B.21c}$$

Let us emphasize that those sets of canonical moments are merely mathematical conveniences, and contain exactly the same amount of physical information than the six sets of moments entering the linear metric (B.18). It is thus physically equivalent to iterate the linear metric (B.18) or the linear canonical metric

$$\begin{aligned}
h^{00}_{(1),\mathrm{can}} &= -\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\partial_L\left[\frac{1}{r}\,\mathrm{M}_L\left(t-\frac{r}{c}\right)\right],\\
h^{0i}_{(1),\mathrm{can}} &= \frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}\,\mathrm{M}^{(1)}_{iL}\left(t-\frac{r}{c}\right)\right] + \frac{\ell}{\ell+1}\,\partial_{aL-1}\left[\frac{1}{r}\,\varepsilon_{abi}\mathrm{S}_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\},\\
h^{ij}_{(1),\mathrm{can}} &= -\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[\frac{1}{r}\,\mathrm{M}^{(2)}_{ijL}\left(t-\frac{r}{c}\right)\right] + \frac{2\ell}{\ell+1}\,\partial_{aL-2}\left[\frac{1}{r}\,\varepsilon_{ab(i}\mathrm{S}^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]\right\},
\end{aligned} \tag{B.22}$$

where the canonical moments $\{\mathrm{M}_L, \mathrm{S}_L\}$ are non-linear completions of the source moments $\{I_L, J_L\}$.

B.2.3 Iteration scheme

From the linear solution (B.22), the higher PM orders of the metric (B.7) are derived by iteratively solving eq. (B.9). Indeed it is easy to see that the source term $\Lambda^{\mu\nu}$ (B.6) expanded up to $(n>1)$PM order only involves the metric perturbations of order $m<n$, and therefore eq. (B.9) reduces to a flat wave equation with a non-trivial source term.

As the linear order has been expanded in multipolarities, those higher orders can be split between the different interactions they traduce. To illustrate such representation, let us consider only the mass monopole M and quadrupole M_{ij}, and so unambiguously separate the linear perturbation as $h^{\mu\nu}_{(1)} \equiv h^{\mu\nu}_{\mathrm{M}} + h^{\mu\nu}_{\mathrm{M}_{ij}}$. When plugged into the source term (B.6), it is clear that the 2PM order can be separated in three contributions: a static one, realized by the quadratic interaction $h^{\mu\nu}_{\mathrm{M}} \times h^{\mu\nu}_{\mathrm{M}}$, a "tail" one, given by the coupling $h^{\mu\nu}_{\mathrm{M}} \times h^{\mu\nu}_{\mathrm{M}_{ij}}$ and finally a "memory" one, $h^{\mu\nu}_{\mathrm{M}_{ij}} \times h^{\mu\nu}_{\mathrm{M}_{ij}}$. Naturally, the 2PM solution will then be easily split into the corresponding static $h^{\mu\nu}_{\mathrm{M}\times\mathrm{M}}$, "tail" $h^{\mu\nu}_{\mathrm{M}\times\mathrm{M}_{ij}}$ and "memory" $h^{\mu\nu}_{\mathrm{M}_{ij}\times\mathrm{M}_{ij}}$ interactions. In a similar way, any PM order can be arranged into the different interactions it represents. For instance the so-called "tail-of-memory" $h^{\mu\nu}_{\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}}$ will be sourced by the quadratic interaction of the monopole and memory sectors $h^{\mu\nu}_{\mathrm{M}} \times h^{\mu\nu}_{\mathrm{M}_{ij}\times\mathrm{M}_{ij}}$, the quadratic interplay between quadrupole and tail $h^{\mu\nu}_{\mathrm{M}_{ij}} \times h^{\mu\nu}_{\mathrm{M}\times\mathrm{M}_{ij}}$ and the cubic interaction of a monopole and two quadrupoles $h^{\mu\nu}_{\mathrm{M}} \times h^{\mu\nu}_{\mathrm{M}_{ij}} \times h^{\mu\nu}_{\mathrm{M}_{ij}}$. From a particle-physicist point of view, those interplays are simply represented by scatterings of gravitons with different multipolarities, as sketched in chapter III.

Let us consider that we have enough knowledge of the metric perturbations to build up an explicit expression of the n^{th} PM order source term $\Lambda^{\mu\nu}_{(n)}(\mathbf{x},t)$ for a given interaction. The solution $h^{\mu\nu}_{(n)}$ is thus composed of two pieces: a particular solution of (B.9) and a homogeneous solution of the flat d'Alembertian operator. The particular solution is given by the three-dimensional Hadamard regularized *retarded* propagator

$$u^{\mu\nu}_{(n)}(\mathbf{x},t) \equiv \underset{B}{\mathrm{PF}}\,\Box^{-1}_{\text{ret}}\left[\left(\frac{r}{r_0}\right)^B \Lambda^{\mu\nu}_{(n)}\right] \equiv \underset{B}{\mathrm{PF}}\left[\frac{-1}{4\pi}\int\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{r'}{r_0}\right)^B \Lambda^{\mu\nu}_{(n)}\left(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]. \tag{B.23}$$

But, if this solution satisfies (B.9) by construction, does it obey the gauge relation (B.2)? If the source is divergenceless $\partial_\mu\Lambda^{\mu\nu}_{(n)}=0$ (as can be explicitely checked by applying the divergence operator on its expression (B.6)), the presence of the regulator $(r/r_0)^B$ contributes to the divergence of $u^{\mu\nu}_{(n)}$ as

$$w^{\mu}_{(n)} \equiv \partial_\nu u^{\mu\nu}_{(n)} = \underset{B}{\mathrm{PF}}\,\Box^{-1}_{\text{ret}}\left[B\left(\frac{r}{r_0}\right)^B \frac{n_i}{r}\Lambda^{i\mu}_{(n)}\right]. \tag{B.24}$$

This term is vanishing only when $r^{B-1}n_i\Lambda^{i\mu}_{(n)}$ does not develop poles $\propto 1/B$. In general such poles appear, and thus the particular solution $u^{\mu\nu}_{(n)}$ is not the accurate solution to our problem: we need to add a homogeneous solution to compensate the divergence. Nevertheless, it is easy to see that the divergence satisfies $\Box w^{\mu}_{(n)}=0$ and, as it is a vector, it can be recast[2] in terms of four sets of STF moments

$$w^0_{(n)} = \sum_{\ell\geq 0}\partial_L\left[\frac{1}{r}N_L\left(t-\frac{r}{c}\right)\right], \tag{B.25a}$$

$$w^i_{(n)} = \sum_{\ell\geq 0}\partial_{iL}\left[\frac{1}{r}P_L\left(t-\frac{r}{c}\right)\right] + \sum_{\ell\geq 1}\left\{\partial_{L-1}\left[\frac{1}{r}Q_{iL-1}\left(t-\frac{r}{c}\right)\right] + \partial_{aL-1}\left[\frac{1}{r}\varepsilon_{abi}R_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\}. \tag{B.25b}$$

Note the expected similarity with the gauge vector $\varphi^{\mu}_{(1)}$ (B.19). So, in order to complete $u^{\mu\nu}_{(n)}$ into an accurate solution $h^{\mu\nu}_{(n)}$, one should find a quantity $v^{\mu\nu}_{(n)}$ such that $\Box v^{\mu\nu}_{(n)}=0$ and $\partial_\nu v^{\mu\nu}_{(n)} = -w^{\mu}_{(n)}$. Of course, such quantity will not be unique, as it remains the freedom of an unphysical non-linear coordinate change, and a convenient choice is

$$v^{00}_{(n)} = -\frac{N^{(-1)}}{r} - \partial_i\left(\frac{N_i^{(-1)} - Q_i^{(-2)} + 3P_i}{r}\right), \tag{B.26a}$$

$$v^{0i}_{(n)} = \frac{3P_i^{(1)} - Q_i^{(-1)}}{r} - \partial_a\left(\varepsilon_{abi}\frac{R_b^{(-1)}}{r}\right) - \sum_{\ell\geq 2}\partial_{L-1}\left(\frac{N_{iL-1}}{r}\right), \tag{B.26b}$$

$$\begin{aligned} v^{ij}_{(n)} = &-\delta_{ij}\frac{P}{r} + \sum_{\ell\geq 2}\left\{2\delta_{ij}\,\partial_{L-1}\left(\frac{P_{L-1}}{r}\right) - 6\partial_{L-2(i}\left(\frac{P_{j)L-2}}{r}\right) + \partial_{L-2}\left(\frac{N^{(1)}_{ijL-2} + 3P^{(2)}_{ijL-2} - Q_{ijL-2}}{r}\right)\right.\\ &\left. -2\partial_{aL-2}\left(\frac{\varepsilon_{ab(i}R_{j)bL-2}}{r}\right)\right\}, \end{aligned} \tag{B.26c}$$

where all the moments are naturally evaluated in $t-r/c$ and we have denoted anti-derivatives with minus signs: $A^{(-1)}(u) \equiv \int_{-\infty}^{u}\mathrm{d}\tau A(\tau)$, which are well-defined as we consider sources that are stationary in the remote past. The PM solution satisfying the vacuum Einstein equations (B.9) at the n^{th} PM order while obeying the de Donder gauge condition (B.2) is thus given by

$$h^{\mu\nu}_{(n)} \equiv u^{\mu\nu}_{(n)} + v^{\mu\nu}_{(n)}. \tag{B.27}$$

[2] Naturally, in practical computation this divergence is not computed by means of eq. (B.24), but rather by directly applying partial derivatives on $u^{\mu\nu}_{(n)}$.

B.3 Radiative moments

The whole iteration procedure relies on the crucial Hadamard regularization, that inherently yields logarithmic dependencies in $h^{\mu\nu}$. Indeed such logarithms appear as soon as the quantity to be regularized develops poles in B, as $\underset{B}{\mathrm{PF}}\left[r^{n+B}/B^q\right] = r^n(\ln r)^q/q!$. Nevertheless, such logarithms are only artifacts of the harmonic coordinates, as it is known that there exists a coordinate system in which the metric is written only in terms of inverse powers of the radial distance, without logarithmic dependencies. Such *radiative* coordinates are deeply linked with the study of the structure at future null infinity, performed in the Bondi-Sachs-Penrose approach [101, 303, 304, 319].

It has been shown in [67] that the MPM procedure can be performed in such radiative coordinates instead of the harmonic ones, by defining the coordinate change iteratively. For example the linear piece of this coordinate change is given by eq. (B.17) with

$$\xi^0 = -\frac{2M}{c^2}\ln\left(\frac{r}{r_h}\right), \qquad \text{and} \qquad \xi^i = 0\,, \tag{B.28}$$

where r_h is a new (unphysical) scale associated with this change of coordinates. If the harmonic coordinates are extremely useful to deal with the processes that generate gravitational radiation, those radiative coordinates are particularly indicated for dealing with the observation of this radiation, as they are defined to be well-behaved at infinity, where observers are located. We will denote those radiative coordinates as $(T, \mathbf{X})$, and naturally define the radial distance $R \equiv |\mathbf{X}|$ together with the unit vector $\mathbf{N} = \mathbf{X}/R$. For the current and planned configurations of the detectors, it is particularly convenient to use the traceless and transverse (TT) gauge. The corresponding projector is given by $\mathcal{P}^{\mathrm{TT}}_{ijab} \equiv \mathcal{P}_{ia}\mathcal{P}_{jb} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{ab}$ with $\mathcal{P}_{ab} = \delta_{ab} - N_aN_b$ is the projector onto the plane orthogonal to N^i. In the radiative coordinate systems, one can formally resum the post-Minkowskian expansion and thus define the mass and current *radiative* moments U_L and V_L as parametrizing the leading $1/R$ order of the TT metric [335]

$$\begin{aligned} H^{\mathrm{TT}}_{ij} = &\frac{4G}{c^2R}\mathcal{P}^{\mathrm{TT}}_{ijab}\sum_{\ell\geq 2}\frac{1}{c^\ell \ell!}\left\{N_{L-2}U_{abL-2}\left(T-\frac{R}{c}\right) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\varepsilon_{cd(a}V_{b)dL-2}\left(T-\frac{R}{c}\right)\right\} \\ &+ \mathcal{O}\left(\frac{1}{R^2}\right), \end{aligned} \tag{B.29}$$

As H^{TT}_{ij} directly produces the observed gravitational wave signal, the two sets of *radiative* moments are thus the only physically relevant ones. The relation between the radiative moments and the source (or canonical) ones is obtained by performing the iterative coordinate change that led to H^{TT}_{ij}. For instance, at the relative 1PM (and 1.5PN) order, it comes

$$U_L(T) = \mathrm{M}^{(\ell)}_L(T) + \frac{2GM}{c^3}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(\ell)}_L(T-\tau)\left[\ln\left(\frac{c\tau}{2r_h}\right) + \frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)} + \mathcal{H}_{\ell-2}\right] + \mathcal{O}\left(\frac{G}{c^5}, \frac{G^2}{c^6}\right), \tag{B.30a}$$

$$V_L(T) = \mathrm{S}^{(\ell)}_L(T) + \frac{2GM}{c^3}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{S}^{(\ell)}_L(T-\tau)\left[\ln\left(\frac{c\tau}{2r_h}\right) + \frac{\ell-1}{\ell(\ell+1)} + \mathcal{H}_{\ell-1}\right] + \mathcal{O}\left(\frac{G}{c^5}, \frac{G^2}{c^6}\right), \tag{B.30b}$$

where $\mathcal{H}_p \equiv \sum_{k=1}^p \frac{1}{k}$ is the harmonic number. Those 1PM corrections are "tail" contributions, that can be separated between a non-local in time sector (the logarithmic one), and an instantaneous one. Similarly the "memory" contributions, starting at 2.5PN, are split into hereditary and instantaneous sectors. Note the presence of the scale associated with the change of coordinates (B.28), r_h. This unphysical dependency is naturally compensated when expressing the radiative moments in terms of harmonic coordinates, in order to derive the gravitational phase.

Appendix C

Lengthy post-Newtonian expressions

C.1 The 4PN metric in the near zone

As explained in sec. IV.2.1 the knowledge of the near zone, PN-expanded metric $\overline{h}^{\mu\nu}$ is required at the 4PN order to compute the source mass quadrupole. As $\overline{h}^{00}$, $\overline{h}^{0i}$ and $\overline{h}^{ij}$ start respectively at 1PN, 1.5PN and 2PN orders, we need a relative 3PN knowledge for $\overline{h}^{00}$ and relative 2PN knowledge for $\overline{h}^{0i}$ and $\overline{h}^{ij}$. We reproduce here the expression of the near zone metric in terms of potentials, that has been used [278]

$$
\begin{aligned}
\overline{h}^{00} = & -\frac{1}{c^2}\frac{2(d-1)V}{d-2} + \frac{1}{c^4}\left[\frac{4(d-3)(d-1)K}{(d-2)^2} - \frac{2(d-1)^2V^2}{(d-2)^2} - 2\hat{W}\right] \\
& + \frac{1}{c^6}\left[-\frac{4(d-1)^3V^3}{3(d-2)^3} + \frac{8(d-3)V_aV_a}{d-2} + \frac{8(d-3)(d-1)^2VK}{(d-2)^3} - \frac{4(d-1)V\hat{W}}{d-2} - \frac{8(d-1)\hat{X}}{d-2} - 8\hat{Z}\right] \\
& + \frac{1}{c^8}\left[\frac{2d\hat{W}_{ab}\hat{W}_{ab}}{d-2} - \frac{2(d-3)^2(d-1)(-4+3d)K^2}{(d-2)^4} + \frac{32\hat{M}}{d-2} - \frac{32(d-1)\hat{T}}{d-2} - \frac{2(d-1)^4V^4}{3(d-2)^4} - 2\hat{W}^2\right. \\
& \qquad + \frac{16(d-4)\hat{R}_aV_a}{d-2} + \frac{8(d-3)(d-1)K\hat{W}}{(d-2)^2} + \frac{8(d-3)(d-1)^3KV^2}{(d-2)^4} - \frac{4(d-1)^2\hat{W}V^2}{(d-2)^2} \\
& \qquad \left. + \frac{16(d-3)(d-1)VV_aV_a}{(d-2)^2} - \frac{4(d-1)(-4+3d)\hat{X}V}{(d-2)^2} - \frac{16(d-1)\hat{Z}V}{d-2}\right] + \mathcal{O}\left(\frac{1}{c^{10}}\right) ,
\end{aligned} \tag{C.1a}
$$

$$
\begin{aligned}
\overline{h}^{0i} = & -\frac{4}{c^3}V_i + \frac{1}{c^5}\left[-8\hat{R}_i - \frac{4(d-1)V_iV}{d-2}\right] \\
& + \frac{1}{c^7}\left[-16\hat{Y}_i + \frac{8(d-3)(d-1)V_iK}{(d-2)^2} - \frac{8(d-1)\hat{R}_iV}{d-2} - \frac{4(d-1)^2V_iV^2}{(d-2)^2} + 8\hat{W}_{ia}V_a - 8V_i\hat{W}\right] + \mathcal{O}\left(\frac{1}{c^9}\right) ,
\end{aligned} \tag{C.1b}
$$

$$
\begin{aligned}
\overline{h}^{ij} = & \frac{1}{c^4}\left[-4\hat{W}_{ij} + 2\delta_{ij}\hat{W}\right] + \frac{1}{c^6}\left[-16\hat{Z}_{ij} + 8\delta_{ij}\hat{Z}\right] \\
& + \frac{1}{c^8}\left[-32\hat{M}_{ij} - 16V_i\hat{R}_j - 16\hat{R}_iV_j - 8\hat{W}_{ij}\hat{W} + 8\hat{W}_{ia}\hat{W}_{ja}\right. \\
& \qquad \left. + \delta_{ij}\left\{-\frac{2(d-3)^2(d-1)K^2}{(d-2)^3} + 16\hat{R}_aV_a + 2\hat{W}^2 - 2\hat{W}_{ab}\hat{W}_{ab} - \frac{4(d-1)V\hat{X}}{d-2}\right\}\right] + \mathcal{O}\left(\frac{1}{c^{10}}\right) ,
\end{aligned} \tag{C.1c}
$$

where we have used unambiguous shorthands for the trace, *eg.* $\hat{W} \equiv \hat{W}_{ii}$.

C.2 The potentials entering the 4PN metric

The ten potentials entering the near zone metric are governed by flat wave equations, that can be classified depending on the degree of non-linearity they involve. The three linear potentials (*ie.* with compact support sources) are defined as

$$\Box V = -4\pi G\,\sigma\,, \tag{C.2a}$$
$$\Box V_i = -4\pi G\,\sigma_i\,, \tag{C.2b}$$
$$\Box K = -4\pi G\,V\sigma\,, \tag{C.2c}$$

where the matter currents are given in eq. (IV.15). The "quadratic" potentials are

$$\Box\hat{W}_{ij} = -\,4\pi G\left(\sigma_{ij}-\delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right)-\frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_i V\partial_j V\,, \tag{C.3a}$$

$$\Box\hat{R}_i = -\,\frac{4\pi G}{d-2}\left(\frac{5-d}{2}\,V\sigma_i-\frac{d-1}{2}\,V_i\,\sigma\right)-\frac{d-1}{d-2}\,\partial_k V\partial_i V_k-\frac{d(d-1)}{4(d-2)^2}\,\partial_t V\partial_i V\,, \tag{C.3b}$$

$$\begin{aligned}\Box\hat{Z}_{ij} = &-\,\frac{4\pi G}{d-2}\,V\left(\sigma_{ij}-\delta_{ij}\,\frac{\sigma_{kk}}{d-2}\right)-\frac{d-1}{d-2}\,\partial_t V_{(i}\,\partial_{j)}V+\partial_i V_k\,\partial_j V_k+\partial_k V_i\,\partial_k V_j-2\partial_k V_{(i}\,\partial_{j)}V_k\\&-\frac{\delta_{ij}}{d-2}\,\partial_k V_m\left(\partial_k V_m-\partial_m V_k\right)-\frac{d(d-1)}{8(d-2)^3}\,\delta_{ij}\left(\partial_t V\right)^2+\frac{(d-1)(d-3)}{2(d-2)^2}\,\partial_{(i}V\partial_{j)}K\,.\end{aligned} \tag{C.3c}$$

The "cubic" potentials read

$$\begin{aligned}\Box\hat{X} = &-\,4\pi G\left[\frac{V\sigma_{ii}}{d-2}+2\left(\frac{d-3}{d-1}\right)\sigma_i V_i+\left(\frac{d-3}{d-2}\right)^2\sigma\left(\frac{V^2}{2}+K\right)\right]\\&+\hat{W}_{ij}\,\partial_{ij}V+2V_i\,\partial_t\partial_i V+\frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V+\frac{d(d-1)}{4(d-2)^2}\left(\partial_t V\right)^2-2\partial_i V_j\,\partial_j V_i\,,\end{aligned} \tag{C.4a}$$

$$\begin{aligned}\Box\hat{Y}_i = &-\,4\pi G\Bigg[-\frac{1}{2}\left(\frac{d-1}{d-2}\right)\sigma\hat{R}_i-\frac{(5-d)(d-1)}{4(d-2)^2}\,\sigma V V_i+\frac{1}{2}\,\sigma_k\hat{W}_{ik}+\frac{1}{2}\sigma_{ik}V_k+\frac{1}{2(d-2)}\,\sigma_{kk}V_i\\&\qquad-\frac{d-3}{(d-2)^2}\,\sigma_i\left(V^2+\frac{5-d}{2}\,K\right)\Bigg]\\&+\hat{W}_{kl}\,\partial_{kl}V_i-\frac{1}{2}\left(\frac{d-1}{d-2}\right)\partial_t\hat{W}_{ik}\,\partial_k V+\partial_i\hat{W}_{kl}\,\partial_k V_l-\partial_k\hat{W}_{il}\,\partial_l V_k-\frac{d-1}{d-2}\,\partial_k V\partial_i\hat{R}_k\\&-\frac{d(d-1)}{4(d-2)^2}\,V_k\,\partial_i V\partial_k V-\frac{d(d-1)^2}{8(d-2)^3}\,V\partial_t V\partial_i V-\frac{1}{2}\left(\frac{d-1}{d-2}\right)^2 V\partial_k V\partial_k V_i+\frac{1}{2}\left(\frac{d-1}{d-2}\right)V\partial_t^2 V_i\\&+2V_k\,\partial_k\partial_t V_i+\frac{(d-1)(d-3)}{(d-2)^2}\,\partial_k K\partial_i V_k+\frac{d(d-1)(d-3)}{4(d-2)^3}\left(\partial_t V\partial_i K+\partial_i V\partial_t K\right)\,,\end{aligned} \tag{C.4b}$$

$$\begin{aligned}\Box\hat{T} = &-\,4\pi G\Bigg[\frac{1}{2(d-1)}\,\sigma_{ij}\hat{W}_{ij}+\frac{5-d}{4(d-2)^2}\,V^2\sigma_{ii}+\frac{1}{d-2}\,\sigma V_i V_i-\frac{1}{2}\left(\frac{d-3}{d-2}\right)\sigma\hat{X}-\frac{1}{12}\left(\frac{d-3}{d-2}\right)^3\sigma V^3\\&\qquad-\frac{1}{2}\left(\frac{d-3}{d-2}\right)^3\sigma V K+\frac{(5-d)(d-3)}{2(d-1)(d-2)}\,\sigma_i V_i V+\frac{d-3}{d-1}\,\sigma_i\hat{R}_i-\frac{d-3}{2(d-2)^2}\,\sigma_{ii}K\Bigg]\\&+\hat{Z}_{ij}\,\partial_{ij}V+\hat{R}_i\,\partial_t\partial_i V-2\partial_i V_j\,\partial_j\hat{R}_i-\partial_i V_j\,\partial_t\hat{W}_{ij}+\frac{1}{2}\left(\frac{d-1}{d-2}\right)V V_i\,\partial_t\partial_i V+\frac{d-1}{d-2}\,V_i\,\partial_j V_i\,\partial_j V\\&+\frac{d(d-1)}{4(d-2)^2}\,V_i\,\partial_t V\,\partial_i V+\frac{1}{8}\left(\frac{d-1}{d-2}\right)^2 V^2\partial_t^2 V+\frac{d(d-1)^2}{8(d-2)^3}\,V\left(\partial_t V\right)^2-\frac{1}{2}\left(\partial_t V_i\right)^2\\&-\frac{(d-1)(d-3)}{4(d-2)^2}\,V\partial_t^2 K-\frac{d(d-1)(d-3)}{4(d-2)^3}\,\partial_t V\,\partial_t K-\frac{(d-1)(d-3)}{4(d-2)^2}\,K\partial_t^2 V\end{aligned}$$

$$
-\frac{d-3}{d-2}V_i\,\partial_t\partial_i K-\frac{1}{2}\left(\frac{d-3}{d-2}\right)\hat{W}_{ij}\,\partial_{ij}K\,. \tag{C.4c}
$$

Finally the only "quartic" potential required is governed by

$$
\begin{aligned}
\Box\hat{M}_{ij}=4\pi G\Bigg[&\delta^{ij}\left[-\frac{1}{2}\hat{R}^a+\frac{(d-4)VV^a}{2(d-2)}\right]\sigma_a+\left[\hat{R}^{(i}-\frac{(d-5)VV^{(i}}{2(d-2)}\right]\sigma^{j)}+\frac{1}{8}\delta^{ij}\hat{W}^{ab}\sigma_{ab}\\
&+\left(-\frac{(d-1)V^iV^j}{2(d-2)}+\delta^{ij}\left[\frac{(d-3)^2(d-1)KV}{4(d-2)^3}+\frac{(d-3)^2(d-1)V^3}{16(d-2)^3}+\frac{(d-1)V_aV^a}{4(d-2)}+\frac{(d-1)\hat{X}}{8(d-2)}\right]\right)\sigma\\
&+\frac{(d-1)\delta^{ij}V^2\sigma^a{}_a}{8(d-2)^2}-\frac{\delta^{ij}\hat{W}^a{}_a\sigma^b{}_b}{8(d-2)}+\left[\frac{(d-3)K}{(d-2)^2}-\frac{V^2}{(d-2)^2}\right]\sigma^{ij}-\frac{1}{2}\hat{W}^{a(i}\sigma^{j)}{}_a\Bigg]\\
&+\frac{1}{2}\hat{W}^{ab}\partial_b\partial_a\hat{W}^{ij}-\frac{1}{2}\partial_a\hat{W}^i{}_b\partial^b\hat{W}^{ja}-\frac{(d-3)^2(d-1)\partial^iK\partial^jK}{4(d-2)^3}\\
&+\partial_tV^{(i}\left[-\partial_tV^{j)}+\frac{(d-3)(d-1)\partial^{j)}K}{(d-2)^2}\right]-2\partial_aV^{(i}\partial^{j)}\hat{R}^a+2\partial^{(i}V_a\partial^{j)}\hat{R}^a-\frac{(d-1)\partial_t\hat{R}^{(i}\partial^{j)}V}{d-2}\\
&+\frac{(d-1)\hat{W}^{(i}{}_a\partial^{j)}V\partial^aV}{4(d-2)}-\frac{(d-1)\partial^{(i}\hat{X}\partial^{j)}V}{2(d-2)}+V^{(i}\left[\frac{(d-1)\partial_aV^{j)}\partial^aV}{d-2}+\frac{(d-1)\partial_tV\partial^{j)}V}{4(d-2)}\right]\\
&+V\left[\frac{(d-1)\partial_t^2\hat{W}^{ij}}{4(d-2)}-\frac{(d-1)^2\partial_tV^{(i}\partial^{j)}V}{2(d-2)^2}\right]+V^a\left[\partial_t\partial_a\hat{W}^{ij}-\frac{(d-1)\partial_aV^{(i}\partial^{j)}V}{d-2}\right]\\
&-2\partial^a\hat{R}^{(i}\partial^{j)}V_a+\partial_t\hat{W}^{(i}{}_a\left[-\partial^aV^{j)}+\partial^{j)}V^a\right]+\partial^b\hat{W}^{a(i}\partial^{j)}\hat{W}_{ab}-\frac{1}{4}\partial^i\hat{W}^{ab}\partial^j\hat{W}_{ab}\\
&+\delta^{ij}\Bigg(-\frac{(d-3)(d-1)d\partial_tK\partial_tV}{8(d-2)^3}-\frac{(d-1)^2V^2\partial_t^2V}{16(d-2)^2}+V^a\left[-\frac{1}{2}\partial_t\partial_a\hat{W}^b_b-\frac{(d-1)\partial_tV\partial_aV}{8(d-2)}\right]\\
&\qquad-\frac{(d-3)(d-1)\partial_tV_a\partial^aK}{2(d-2)^2}+\frac{(d-1)\partial_t\hat{R}_a\partial^aV}{2(d-2)}-\frac{1}{4}\hat{W}^{ab}\partial_b\partial_a\hat{W}^k{}_k+\partial_aV_b\partial^b\hat{R}^a\\
&\qquad-\frac{(d-1)\hat{W}_{ab}\partial^aV\partial^bV}{16(d-2)}-\frac{1}{2}\partial_t\hat{W}_{ab}\partial^bV^a-\frac{1}{4}\partial_b\hat{W}_{ak}\partial^k\hat{W}^{ab}\\
&\qquad+V\Bigg[\frac{(d-1)^2d(\partial_tV)^2}{32(d-2)^3}-\frac{(d-1)V^a\partial_t\partial_aV}{4(d-2)}-\frac{(d-1)\partial_t^2\hat{W}^a{}_a}{8(d-2)}\\
&\qquad\quad+\frac{(d-1)^2\partial_tV_a\partial^aV}{4(d-2)^2}-\frac{(d-1)\hat{W}^{ab}\partial_b\partial_aV}{8(d-2)}+\frac{(d-1)\partial_aV_b\partial^bV^a}{4(d-2)}\Bigg]\Bigg)\,.
\end{aligned}\tag{C.5a}
$$

C.3 The surface terms entering the source quadrupole

The mass quadrupole, once written in terms of potentials and superpotentials as in App. C of [278], contains the following surface terms

$$
\begin{aligned}
I_{ij} \ni \underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B & \left\{\partial_a \left(\frac{\mathfrak{D}^{2,0}_{aij}}{c^4} + \frac{\mathfrak{D}^{3,0}_{aij}}{c^6} + \frac{\mathfrak{D}^{4,0}_{aij}}{c^8}\right) + \hat{x}_{ij}\Delta\left(\frac{\mathfrak{L}^{1,0}}{c^2} + \frac{\mathfrak{L}^{2,0}}{c^4} + \frac{\mathfrak{L}^{3,0}}{c^6} + \frac{\mathfrak{L}^{4,0}}{c^8}\right)\right\} \\
+ \frac{\mathrm{d}}{\mathrm{d}t} & \left[\underset{B}{\mathrm{PF}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \left\{\partial_a \left(\frac{\mathfrak{D}^{3,1}_{aij}}{c^6} + \frac{\mathfrak{D}^{4,1}_{aij}}{c^8}\right) + \hat{x}_{aij}\Delta\left(\frac{\mathfrak{L}^{2,1}_a}{c^4} + \frac{\mathfrak{L}^{3,1}_a}{c^6} + \frac{\mathfrak{L}^{4,1}_a}{c^8}\right)\right\}\right] ,
\end{aligned} \tag{C.6}
$$

with the divergence terms

$$
\mathfrak{D}^{2,0}_{aij} = \frac{1}{\pi G}\left(\Psi^{\partial_{bk}V}_{ai}\partial_j \hat{W}_{bk} - \hat{W}_{bk}\partial_a \Psi^{\partial_{bk}V}_{ij}\right) , \tag{C.7a}
$$

$$
\begin{aligned}
\mathfrak{D}^{3,0}_{aij} = & \frac{4}{\pi G}\left(\Psi^{\partial_{tb}V}_{ai}\partial_j \hat{R}_b - \hat{R}_b\partial_a \Psi^{\partial_{tb}V}_{ij}\right) + \frac{4}{\pi G}\left(\Psi^{\partial_{bk}V}_{ai}\partial_j \hat{Z}_{bk} - \hat{Z}_{bk}\partial_a \Psi^{\partial_{bk}V}_{ij}\right) \\
& - \frac{8}{\pi G}\left(\Psi^{\partial_b V_k}_{ai}\partial_{kj}\hat{R}_b - \partial_a \Psi^{\partial_b V_k}_{ij}\partial_k \hat{R}_b\right) ,
\end{aligned} \tag{C.7b}
$$

$$
\begin{aligned}
\mathfrak{D}^{4,0}_{aij} = & \frac{8}{\pi G}\left(\Psi^{\partial_{tb}V}_{ai}\partial_j \hat{Y}_b - \hat{Y}_b\partial_a \Psi^{\partial_{tb}V}_{ij}\right) + \frac{8}{\pi G}\left(\Psi^{\partial_{bk}V}_{ai}\partial_j \hat{M}_{bk} - \hat{M}_{bk}\partial_a \Psi^{\partial_{bk}V}_{ij}\right) \\
& - \frac{16}{\pi G}\left(\Psi^{\partial_b V_k}_{ai}\partial_{kj}\hat{Y}_b - \partial_a \Psi^{\partial_b V_k}_{ij}\partial_k \hat{Y}_b\right) ,
\end{aligned} \tag{C.7c}
$$

$$
\mathfrak{D}^{3,1}_{aij} = \frac{40}{21\pi G}\left(\Psi^{\partial_k V}_{aib}\partial_{kj}\hat{R}_b - \partial_a \Psi^{\partial_k V}_{ijb}\partial_k \hat{R}_b\right) , \tag{C.7d}
$$

$$
\mathfrak{D}^{4,1}_{aij} = \frac{80}{21\pi G}\left(\Psi^{\partial_k V}_{aib}\partial_{kj}\hat{Y}_b - \partial_a \Psi^{\partial_k V}_{ijb}\partial_k \hat{Y}_b\right) , \tag{C.7e}
$$

and the Laplacian ones

$$
\mathfrak{L}^{1,0} = -\frac{1}{2\pi G}V^2 , \tag{C.8a}
$$

$$
\mathfrak{L}^{2,0} = -\frac{2}{3\pi G}V^3 - \frac{1}{2\pi G}V\hat{W} , \tag{C.8b}
$$

$$
\mathfrak{L}^{3,0} = -\frac{2}{3\pi G}V^4 - \frac{1}{\pi G}V^2\hat{W} - \frac{1}{4\pi G}\hat{W}^2 + \frac{1}{2\pi G}\hat{W}_{ab}\hat{W}_{ab} - \frac{4}{\pi G}V\hat{X} - \frac{2}{\pi G}V\hat{Z} , \tag{C.8c}
$$

$$
\begin{aligned}
\mathfrak{L}^{4,0} = & -\frac{8}{15\pi G}V^5 - \frac{1}{2\pi G}V\hat{W}^2 - \frac{4}{3\pi G}V^3\hat{W} - \frac{2}{\pi G}\hat{W}\hat{X} - \frac{2}{\pi G,}\hat{W}\hat{Z} + \frac{4}{\pi G}\hat{W}_{ab}\hat{Z}_{ab} \\
& - \frac{16}{\pi G}\hat{T}V + \frac{8}{\pi G}\hat{M}V ,
\end{aligned} \tag{C.8d}
$$

$$
\mathfrak{L}^{2,1}_a = \frac{10}{21\pi G}VV_a , \tag{C.8e}
$$

$$
\mathfrak{L}^{3,1}_a = \frac{10}{21\pi G}V^2V_a + \frac{10}{21\pi G}V_a\hat{W} - \frac{10}{21\pi G}V_b\hat{W}_{ab} + \frac{20}{21\pi G}\hat{R}_aV , \tag{C.8f}
$$

$$
\begin{aligned}
\mathfrak{L}^{4,1}_a = & \frac{10}{7\pi G}V_aV_bV_b + \frac{20}{21\pi G}VV_a\hat{W} - \frac{20}{21\pi G}VV_b\hat{W}_{ab} + \frac{40}{21\pi G}\hat{R}_aV^2 + \frac{20}{21\pi G}\hat{R}_a\hat{W} \\
& - \frac{20}{21\pi G}\hat{R}_b\hat{W}_{ab} + \frac{40}{21\pi G}V_a\hat{X} + \frac{40}{21\pi G}V_a\hat{Z} - \frac{40}{21\pi G}V_b\hat{Z}_{ab} + \frac{40}{21\pi G}V\hat{Y}_a .
\end{aligned} \tag{C.8g}
$$

C.4 The coordinate shifts applied in the equations of motion

C.4.1 The UV shifts

We present here the two UV shifts that have to be applied in the quadrupole. The first, denoted $\xi^i_{1,2}$, was constructed to remove the UV poles in the equations of motion [54]. The second, $\eta^i_{1,2}$, was mainly used for convenience [57]. The shift ξ^i_1 is composed of 3PN and 4PN contributions and is given by

$$\xi^i_1 = \frac{11}{3}\frac{G^2\,m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{\overline{q}^{1/2}r'_0}{\ell_0}\right) - \frac{327}{1540}\right]a^i_1 + \frac{1}{c^8}\xi^i_{1,\,4\mathrm{PN}}\,. \tag{C.9}$$

The first term represents the 3PN contribution determined in [87]. Here $\overline{q} = 4\pi e^{\gamma_\mathrm{E}}$ where γ_E is the Euler constant, r'_0 is the (unphysical) length scale associated with the UV dimensional regularization and $\mathbf{a}_1$ is the Newtonian acceleration of the particle 1 in d dimensions. The 4PN term in the shift is conveniently decomposed as

$$\xi^i_{1,\,4\mathrm{PN}} = \frac{1}{\varepsilon}\xi^{i\,(-1)}_{1,\,4\mathrm{PN}} + \xi^{(0,y_1)}_{1,\,4\mathrm{PN}}y^i_1 + \xi^{(0,n_{12})}_{1,\,4\mathrm{PN}}n^i_{12} + \xi^{(0,v_1)}_{1,\,4\mathrm{PN}}v^i_1 + \xi^{(0,v_{12})}_{1,\,4\mathrm{PN}}v^i_{12}\,, \tag{C.10}$$

with $1/\varepsilon$ being the UV pole and $v^i_{12} = v^i_1 - v^i_2$. We have

$$\begin{aligned}
\xi^{i\,(-1)}_{1,\,4\mathrm{PN}} &= \frac{G^3 m_1^2 m_2 v^i_{12}}{r_{12}^2}\Big(11(n_{12}v_{12}) + \frac{11}{3}(n_{12}v_1)\Big) + n^i_{12}\left[\frac{G^4}{r_{12}^3}\Big(\frac{55}{3}m_1^3 m_2 + \frac{22}{3}m_1^2 m_2^2 + 4 m_1 m_2^3\Big)\right.\\
&\quad\left. + \frac{G^3 m_1^2 m_2}{r_{12}^2}\Big(\frac{11}{2}(n_{12}v_{12})^2 - 11(n_{12}v_{12})(n_{12}v_1) + \frac{11}{2}(n_{12}v_1)^2 - \frac{22}{3}v_{12}^2\Big)\right],
\end{aligned} \tag{C.11a}$$

$$\begin{aligned}
\xi^{(0,y_1)}_{1,\,4\mathrm{PN}} &= G^3(m_1+m_2)\frac{m_1 m_2}{15 r_{12}^3}\left(\left[-67 + 48\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})^2 + \left[17 - 16\ln\Big(\frac{r_{12}}{r_0}\Big)\right]{v_{12}}^2\right)\\
&\quad + G^4(m_1+m_2)^2\frac{m_1 m_2}{15 r_{12}^4}\left[-17 + 16\ln\Big(\frac{r_{12}}{r_0}\Big)\right],
\end{aligned} \tag{C.11b}$$

$$\begin{aligned}
\xi^{(0,n_{12})}_{1,\,4\mathrm{PN}} &= G^3 m_1 m_2^2\left[\frac{1}{r_{12}^2}\left(\left[-\frac{1753}{80} + \frac{72}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})^2 + \left[\frac{1753}{40} + \frac{96}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})(n_{12}v_1)\right.\right.\\
&\qquad\left. + \left[-\frac{1753}{60} - \frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](v_{12}v_1) + \left[\frac{1753}{120} - \frac{32}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right]v_{12}^2\right)\\
&\quad + \frac{1}{r_{12}^3}\left(\left[-\frac{431}{10} + 48\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})^2(n_{12}y_1) + \left[\frac{53}{2} - \frac{144}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})(v_{12}y_1)\right.\\
&\qquad\left.\left. + \left[\frac{67}{10} - \frac{48}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}y_1)v_{12}^2\right)\right]\\
&\quad + G^3 m_1^2 m_2\left[\frac{1}{r_{12}^2}\left(\left[-\frac{2761}{168} - \frac{33}{2}\ln\Big(\frac{\overline{q}^{1/2}r'_0}{\ell_0}\Big) - \frac{11}{2}\ln\Big(\frac{r_{12}}{r'_0}\Big)\right](n_{12}v_{12})^2\right.\right.\\
&\qquad + \left[\frac{4367}{168} + 33\ln\Big(\frac{\overline{q}^{1/2}r'_0}{\ell_0}\Big) + \frac{96}{5}\ln\Big(\frac{r_{12}}{r_0}\Big) + 11\ln\Big(\frac{r_{12}}{r'_0}\Big)\right](n_{12}v_{12})(n_{12}v_1)\\
&\qquad + \left[\frac{7489}{840} - \frac{33}{2}\ln\Big(\frac{\overline{q}^{1/2}r'_0}{\ell_0}\Big) - \frac{11}{2}\ln\Big(\frac{r_{12}}{r'_0}\Big)\right](n_{12}v_1)^2 + \left[-\frac{1753}{60} - \frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](v_{12}v_1)\\
&\qquad\left. + \left[\frac{1753}{120} + 22\ln\Big(\frac{\overline{q}^{1/2}r'_0}{\ell_0}\Big) + \frac{16}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right]v_{12}^2\right)\\
&\quad + \frac{1}{r_{12}^3}\left(\left[-\frac{431}{10} + 48\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})^2(n_{12}y_1) + \left[\frac{53}{2} - \frac{144}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}v_{12})(v_{12}y_1)\right.
\end{aligned}$$

$$+\left[\frac{67}{10}-\frac{48}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\right](n_{12}y_1)v_{12}^2\Big)\Bigg]$$
$$+G^4m_1^2m_2^2\bigg(\frac{1}{r_{12}^3}\bigg[-\frac{88}{3}\ln\Big(\frac{\bar{q}^{1/2}r'_0}{\ell_0}\Big)-\frac{16}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg]+\frac{1}{r_{12}^4}\bigg[\frac{67}{15}-\frac{32}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}y_1)\bigg)$$
$$+G^4m_1^3m_2\bigg(\frac{1}{r_{12}^3}\bigg[-\frac{220}{3}\ln\Big(\frac{\bar{q}^{1/2}r'_0}{\ell_0}\Big)-\frac{32}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg]+\frac{1}{r_{12}^4}\bigg[\frac{67}{30}-\frac{16}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}y_1)\bigg)$$
$$+G^4m_1m_2^3\bigg(\frac{1}{r_{12}^3}\bigg[-16\ln\Big(\frac{\bar{q}^{1/2}r'_0}{\ell_0}\Big)+\frac{16}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg]+\frac{1}{r_{12}^4}\bigg[\frac{67}{30}-\frac{16}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}y_1)\bigg)\,, \tag{C.11c}$$

$$\xi_{1,\,4\mathrm{PN}}^{(0,v_1)}=G^3\bigg(\frac{m_1^2m_2}{r_{12}^2}\bigg[-\frac{269}{120}+\frac{32}{15}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_{12})+\frac{m_1m_2^2}{r_{12}^2}\bigg[-\frac{269}{120}+\frac{32}{15}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_{12})\bigg)\,, \tag{C.11d}$$

$$\begin{aligned}\xi_{1,\,4\mathrm{PN}}^{(0,v_{12})}=&\;G^3m_1m_2^2\bigg[\frac{1}{r_{12}^2}\bigg(\bigg[\frac{755}{48}-8\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_{12})+\bigg[-\frac{1753}{60}-\frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_1)\bigg)\\&+\frac{1}{r_{12}^3}\bigg(\bigg[\frac{53}{2}-\frac{144}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_{12})(n_{12}y_1)+\bigg[-\frac{134}{15}+\frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](v_{12}y_1)\bigg)\bigg]\\&+G^3m_1^2m_2\bigg[\frac{1}{r_{12}^2}\bigg(\bigg[\frac{23113}{1680}-33\ln\Big(\frac{\bar{q}^{1/2}r'_0}{\ell_0}\Big)+\frac{11}{3}\ln\Big(\frac{r_{12}}{r'_0}\Big)\bigg](n_{12}v_{12})\\&+\bigg[-\frac{2572}{105}-11\ln\Big(\frac{\bar{q}^{1/2}r'_0}{\ell_0}\Big)-\frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)-\frac{11}{3}\ln\Big(\frac{r_{12}}{r'_0}\Big)\bigg](n_{12}v_1)\bigg)\\&+\frac{1}{r_{12}^3}\bigg(\bigg[\frac{53}{2}-\frac{144}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](n_{12}v_{12})(n_{12}y_1)+\bigg[-\frac{134}{15}+\frac{64}{5}\ln\Big(\frac{r_{12}}{r_0}\Big)\bigg](v_{12}y_1)\bigg)\bigg]\,.\end{aligned} \tag{C.11e}$$

The shift η_1^i is more compact. It starts at the 4PN order and is made only of G^3 and G^4 terms,

$$\eta_1=\frac{G^3}{c^8}\eta_{1,\,4\mathrm{PN}}^{(3)}+\frac{G^4}{c^8}\eta_{1,\,4\mathrm{PN}}^{(4)}\,, \tag{C.12}$$

where

$$\begin{aligned}\eta_{1,\,4\mathrm{PN}}^{(3)}=&\;\frac{\mathbf{v}_{12}}{r_{12}^2}\Big(\frac{769}{24}m_1^2m_2(n_{12}v_{12})+\frac{561}{35}m_1m_2^2(n_{12}v_{12})\Big)\\&+\frac{\mathbf{n}_{12}}{r_{12}^2}\bigg[m_1m_2^2\Big(\frac{21719}{1400}(n_{12}v_{12})^2-\frac{2096}{175}v_{12}^2\Big)+m_1^2m_2\Big(-\frac{2119}{50}(n_{12}v_{12})^2+\frac{58769}{2100}v_{12}^2\Big)\bigg]\,,\end{aligned} \tag{C.13a}$$

$$\eta_{1,\,4\mathrm{PN}}^{(4)}=\Big(\frac{8861}{2100}m_1^3m_2+\frac{613}{350}m_1^2m_2^2-\frac{5183}{2100}m_1m_2^3\Big)\frac{\mathbf{n}_{12}}{r_{12}^3}\,. \tag{C.13b}$$

Obviously, the shifts ξ_2^i and η_2^i are obtained from ξ_1^i and η_1^i by the exchange of the two particles.

C.4.2 The IR shift

The shift used to remove the remaining IR poles in the equations of motion [56] was provided to us by L. Bernard. It can be decomposed as

$$\chi_1^i = \frac{1}{c^8}\left(\frac{1}{\varepsilon}\,\chi_1^{i\,(-1)} + \chi_1^{(0,y_1)}\,y_1^i + \chi_1^{(0,n_{12})}\,n_{12}^i + \chi_1^{(0,v_1)}\,v_1^i + \chi_1^{(0,v_{12})}\,v_{12}^i\right)\,, \tag{C.14}$$

where

$$\begin{aligned}
\chi_1^{i\,(-1)} =& \frac{8G^4 m_1 m_2 (m_1+m_2)}{5{r_{12}}^4}\left[(m_1+m_2)(n_{12}y_1)n_{12}^i + r_{12}(2m_1-m_2)n_{12}^i - \frac{m_1+m_2}{3}\,y_1^i\right] \\
&- \frac{4G^3 m_1 m_2\, n_{12}^i}{5r_{12}^3}\bigg[2v_{12}^2\left(r_{12}(m_1-2m_2) - 3\left(n_{12}y_1\right)\left(m_1+m_2\right)\right) - 8r_{12}\left(v_1v_{12}\right) \\
&\qquad + \left(n_{12}v_{12}\right)^2\left(30\left(n_{12}y_1\right)\left(m_1+m_2\right) + 9m_2r_{12}\right) \\
&\qquad + 6\left(n_{12}v_{12}\right)\left(m_1+m_2\right)\left(2r_{12}\left(n_{12}v_1\right) - 3\left(y_1v_{12}\right)\right)\left(m_1+m_2\right)\bigg] \\
&+ \frac{4G^3 m_1 m_2\, v_{12}^i}{5{r_{12}}^3}\left[8(m_1+m_2)\left(r_{12}\left(n_{12}v_1\right) - \left(y_1v_{12}\right)\right) + \left(n_{12}v_{12}\right)\left(18\left(n_{12}y_1\right)\left(m_1+m_2\right) + 5m_2r_{12}\right)\right] \\
&- \frac{8G^3 m_1 m_2 (m_1+m_2)\,y_1^i}{15r_{12}^3}\left[3\left(n_{12}v_{12}\right)^2 - v_{12}^2\right] - \frac{16G^3 m_1 m_2\left(n_{12}v_{12}\right)\left(m_1+m_2\right)}{15r_{12}^2}\,v_1^i\,,
\end{aligned} \tag{C.15a}$$

$$\begin{aligned}
\chi_1^{(0,y_1)} =& \frac{G^4 m_1 m_2 (m_1+m_2)^2}{75r_{12}^4}\left[80\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 80\ln\left(\frac{r_{12}}{r_0}\right) + 73\right] \\
&+ \frac{G^3 m_1 m_2 (m_1+m_2)}{75r_{12}^3}\bigg[\left(n_{12}v_{12}\right)^2\left(180\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 120\ln\left(\frac{r_{12}}{r_0}\right) + 359\right) \\
&\qquad - v_{12}^2\left(60\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 40\ln\left(\frac{r_{12}}{r_0}\right) + 133\right)\bigg]\,,
\end{aligned} \tag{C.15b}$$

$$\begin{aligned}
\chi_1^{(0,n_{12})} =& -\frac{G^4 m_1 m_2 (m_1+m_2)}{2100r_{12}^4}\bigg[r_{12}\left(6720(2m_1-m_2)\left(\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + \ln\left(\frac{r_{12}}{r_0}\right)\right) + 8861m_1 - 5183m_2\right) \\
&\qquad + 42\left(n_{12}y_1\right)\left(m_1+m_2\right)\left(160\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 160\ln\left(\frac{r_{12}}{r_0}\right) + 61\right)\bigg] \\
&+ \frac{G^3 m_1 m_2}{4200r_{12}^3}\bigg[2r_{12}v_{12}^2\left((m_1-2m_2)\left(5040\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 3360\ln\left(\frac{r_{12}}{r_0}\right)\right) - 121916m_1 - 37995m_2\right) \\
&\qquad - 84v_{12}^2\left(n_{12}y_1\right)\left(m_1+m_2\right)\left(360\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 240\ln\left(\frac{r_{12}}{r_0}\right) + 223\right) \\
&\qquad + 2100\left(n_{12}v_{12}\right)^2\left(n_{12}y_1\right)\left(m_1+m_2\right)\left(72\ln\left(\bar{q}{r_0}^2\right) + 48\ln\left(\frac{r_{12}}{r_0}\right) + 35\right) \\
&\qquad + 3r_{12}\left(n_{12}v_{12}\right)^2\left(15120m_2\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 10080m_2\ln\left(\frac{r_{12}}{r_0}\right) + 126959m_1 + 45908m_2\right) \\
&\qquad + 21r_{12}\left(n_{12}v_1\right)\left(n_{12}v_{12}\right)\left(m_1+m_2\right)\left(2880\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 1920\ln\left(\frac{r_{12}}{r_0}\right) - 9661\right) \\
&\qquad - 252\left(n_{12}v_{12}\right)\left(y_1v_{12}\right)\left(m_1+m_2\right)\left(360\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 240\ln\left(\frac{r_{12}}{r_0}\right) + 223\right) \\
&\qquad - 42r_{12}\left(v_1v_{12}\right)\left(m_1+m_2\right)\left(960\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right) + 640\ln\left(\frac{r_{12}}{r_0}\right) - 3007\right)\bigg]\,,
\end{aligned} \tag{C.15c}$$

$$\chi_1^{(0,v_1)} = \frac{G^3 m_1 m_2 \left(n_{12}v_{12}\right)\left(m_1+m_2\right)}{600 r_{12}^2}\left[960\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right)+640\ln\left(\frac{r_{12}}{r_0}\right)+2113\right], \tag{C.15d}$$

$$\begin{aligned}\chi_1^{(0,v_{12})} =& -\frac{G^3 m_1 m_2}{4200 r_{12}^3}\left[5r_{12}\left(n_{12}v_{12}\right)\left(5040 m_2\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right)+3360 m_2\ln\left(\frac{r_{12}}{r_0}\right)+55132 m_1+41681 m_2\right)\right.\\ &+42 r_{12}\left(n_{12}v_1\right)\left(m_1+m_2\right)\left(960\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right)+640\ln\left(\frac{r_{12}}{r_0}\right)-3007\right)\\ &+252\left(n_{12}v_{12}\right)\left(n_{12}y_1\right)\left(m_1+m_2\right)\left(360\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right)+240\ln\left(\frac{r_{12}}{r_0}\right)+223\right)\\ &\left.-366\left(y_1v_{12}\right)\left(m_1+m_2\right)\left(120\ln\left(\frac{r_0^2\bar{q}}{\ell_0^2}\right)+80\ln\left(\frac{r_{12}}{r_0}\right)+101\right)\right]. \end{aligned} \tag{C.15e}$$

Obviously, the shift χ_2^i is obtained from χ_1^i by the exchange of the two particles.

Appendix D

Explicit dimensional regularization of the radiative quadrupole

In this appendix, we present the derivation of the dimensional regularization for the non-linear interactions in U_{ij}. In order to lighten the expressions that will be displayed, we set $c = r_0 = \ell_0 = 1$.

D.1 Iteration of the d-dimensional propagator

We aim at solving the d-dimensional wave equation

$$\Box h_L = N_L(\mathbf{x}, t) = N(r, t)\,\hat{n}_L\,. \tag{D.1}$$

We recall that the accurate prescription for the d-dimensional propagator is given by eq. (IV.125)

$$h_L(\mathbf{x}, t) = \frac{-\tilde{k}}{4\pi}\int \mathrm{d}^d\mathbf{x}'\,r'^{\eta}\int_1^{+\infty}\mathrm{d}z\gamma_{\frac{1-d}{2}}(z)\,\frac{N_L\left(\mathbf{x}', t - z|\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|^{d-2}}\,, \tag{D.2}$$

with

$$\gamma_{\frac{1-d}{2}-\ell}(z) = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{d-2}{2}+\ell\right)\Gamma\left(\frac{3-d}{2}-\ell\right)}\left(z^2 - 1\right)^{\frac{1-d}{2}-\ell}\,. \tag{D.3}$$

The strategy to compute the integral (D.2) relies on a split into two integration zones, depending on whether r' is larger than r or not. In the first integration zone, for $r' > r$, we expand the integrand as

$$\frac{N_L\left(\mathbf{x}', t - z|\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|^{d-2}} = \sum_{p\in\mathbb{N}}\frac{x^P}{p!}\partial_P\left[\frac{N_L\left(\mathbf{x}', t - z|\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|^{d-2}}\right]_{\mathbf{x}=0} = \sum_{p\in\mathbb{N}}\frac{(-)^p}{p!}x^P\partial'_P\left[\frac{N_L\left(\mathbf{y}, t - zr'\right)}{r'^{d-2}}\right]_{\mathbf{y}=\mathbf{x}'}\,, \tag{D.4}$$

and in the second integration zone, defined by $r' < r$, we expand it as

$$\frac{N_L\left(\mathbf{x}', t - z|\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|^{d-2}} = \sum_{p\in\mathbb{N}}\frac{x'^P}{p!}\partial'_P\left[\frac{N_L\left(\mathbf{y}, t - z|\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|^{d-2}}\right]_{\mathbf{x}'=0,\mathbf{y}=\mathbf{x}'} = \sum_{p\in\mathbb{N}}\frac{(-)^p}{p!}x'^P\partial_P\left[\frac{N_L\left(\mathbf{x}', t - zr\right)}{r^{d-2}}\right]\,. \tag{D.5}$$

Next, injecting those expansions in eq. (D.2), and massaging it notably by using the relations

$$\sum_{p\in\mathbb{N}}\frac{(-)^p}{p!}x^P\partial'_P = \sum_{(j,q)\in\mathbb{N}^2}\frac{(-)^q\,r^{2j+q}}{2^{2j}\,j!q!}\frac{\Gamma\left(\frac{d}{2}+q\right)}{\Gamma\left(\frac{d}{2}+q+j\right)}\,\hat{n}_Q\,\hat{\partial}'_Q\Delta'^j\,, \qquad \hat{\partial}_Q A(r) = r^q\hat{n}_Q\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^q A(r)\,, \tag{D.6a}$$

$$\int_1^{+\infty}\mathrm{d}z\gamma_{\frac{1-d}{2}}(z)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\ell}\left[\frac{F(t-zr)}{r^{d-2}}\right]=(-2)^{\ell}\frac{\Gamma\left(\frac{d-2}{2}+\ell\right)}{\Gamma\left(\frac{1-d}{2}\right)}r^{2-2d-2\ell}\int_1^{+\infty}\mathrm{d}z\gamma_{\frac{d-2}{2}-\ell}(z)F(t-zr)\,, \tag{D.6b}$$

it comes out

$$h_L=-\frac{1}{2}\sum_{j\in\mathbb{N}}\frac{\hat{n}_L}{2^{2j}j!}\frac{\Gamma\left(\frac{d-2}{2}+\ell\right)}{\Gamma\left(\frac{d}{2}+\ell+j\right)}\int_1^{+\infty}\mathrm{d}z\gamma_{\frac{1-d}{2}-\ell}(z)\left\{r^{2-d-\ell}\int_0^r\mathrm{d}\lambda\lambda^{d-1+2j+\ell+\eta}\,N^{(2j)}(\lambda,t-zr)\right.$$
$$\left.+r^{2j+\ell}\int_r^{+\infty}\mathrm{d}\lambda\lambda^{1-\ell+\eta}\,N^{(2j)}(\lambda,t-z\lambda)\right\}\,, \tag{D.7}$$

where the separation into two integration zones appears clearly. If already complicated, this expression is not yet satisfactory. Indeed its brute force three-dimensional limit does not directly reduce to the usual Hadamard non-linear interactions. Therefore we need some more massaging of the integrals before studying the differences between regularization schemes.

Using the formal asymptotic expansion

$$\gamma_{\frac{1-d}{2}-\ell}(z)\underset{z\to\infty}{=}\sum_{j\in\mathbb{N}}\frac{(-)^j}{j!}\frac{2\sqrt{\pi}\,z^{-2-2j-2\ell-\varepsilon}}{\Gamma\left(-\ell-j-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1+\varepsilon}{2}+\ell\right)}\,, \tag{D.8}$$

one obtains the expression for the “retarded minus advanced” propagator

$$\int_1^{\infty}\mathrm{d}z\,\gamma_{\frac{1-d}{2}-\ell}(z)\Big[A(t-zr)-A(t+zr)\Big]=\frac{2\sqrt{\pi}\Gamma\left(\frac{d}{2}+\ell\right)\Gamma\left(2-2\ell-d\right)}{\Gamma\left(\frac{d-2}{2}+\ell\right)\Gamma\left(1-\varepsilon\right)\Gamma\left(-\frac{\varepsilon}{2}-\ell\right)}\sum_{j\in\mathbb{N}}\frac{1}{2^{2j}j!}\frac{r^{2j+2\ell+1+\varepsilon}}{\Gamma\left(\frac{d}{2}+\ell+j\right)}$$
$$\times\int_0^{\infty}\mathrm{d}\tau\tau^{-\varepsilon}\left[A^{(2+2\ell+2j)}(t-\tau)-A^{(2+2\ell+2j)}(t+\tau)\right]\,, \tag{D.9}$$

where r is here a dummy variable. We have used this “retarded minus advanced” form in order to discard the even sector, irrelevant for our purpose. But we now have to recognize the right-hand side of eq. (D.9) within the integral (D.7). It is straightforward (in Fourier space) to show that

$$\mathcal{A}(t)=\int_0^{\infty}\mathrm{d}\tau\tau^{-\varepsilon}\left[A(t-\tau)-A(t+\tau)\right]\quad\Rightarrow\quad A(t)=\frac{1}{\pi}\tan\left(\frac{\pi\varepsilon}{2}\right)\int_0^{\infty}\mathrm{d}\tau\tau^{\varepsilon}\left[\mathcal{A}^{(2)}(t-\tau)-\mathcal{A}^{(2)}(t+\tau)\right]\,. \tag{D.10}$$

Let us now concentrate on only one given multipolar order, *ie.* write $N(r,t)=r^{-p-q\varepsilon}F(t)$. Plugging it, together with the relations (D.9) and (D.10), in the integral (D.7), it finally comes

$$h_L=\frac{\Gamma_\ell^\varepsilon\,\hat{n}_L}{r^{\ell+1+\varepsilon}}\int_1^{\infty}\mathrm{d}y\,\gamma_{-1-\frac{\varepsilon}{2}-\ell}(y)\int_1^{\infty}\mathrm{d}z\,\gamma_{-1-\frac{\varepsilon}{2}-\ell}(z)\int_{-\infty}^{+\infty}\mathrm{d}\tau\,\varepsilon\,|\tau|^{\varepsilon-1}$$
$$\times\left\{\int_0^r\mathrm{d}\lambda\lambda^{1-\ell-p-q\varepsilon+\eta}\left[F^{(-2\ell-1)}\left(t-yr-z\lambda+\tau\right)-F^{(-2\ell-1)}\left(t-yr+z\lambda+\tau\right)\right]\right.$$
$$\left.+\int_r^{\infty}\mathrm{d}\lambda\lambda^{1-\ell-p-q\varepsilon+\eta}\left[F^{(-2\ell-1)}\left(t-yr-z\lambda+\tau\right)-F^{(-2\ell-1)}\left(t+yr-z\lambda+\tau\right)\right]\right\}, \tag{D.11}$$

where we have compacted the prefactor

$$\Gamma_\ell^\varepsilon\equiv(-)^\ell\frac{2^{2(\ell-1)}}{\pi^{3/2}}\frac{\Gamma\left(\frac{1-\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}\left[\Gamma\left(\ell+\frac{1+\varepsilon}{2}\right)\right]^2\,, \tag{D.12}$$

whose three-dimensional limit is simply given by $\Gamma_\ell^0=(-)^\ell\left[(2\ell-1)!!\right]^2/4$. Even if not obvious *per se*, the three-dimensional limit of the integral (D.11) is naturally defined, and agrees with the usual three-dimensional propagator. Therefore it is the formulation we will use henceforth.

D.2 Dimensional regularization of the metric

We know that the pole compensating the IR pole of the source moment is of UV nature, and thus should come from the 0 boundary of the integral (D.11). Therefore, the difference between the two regularization schemes comes only from the sector

$$\begin{aligned}\tilde{h}_L \equiv& \frac{\Gamma^\varepsilon_\ell\, \hat{n}_L}{r^{\ell+1+\varepsilon}} \int_1^\infty \mathrm{d}y\, \gamma_{-1-\frac{\varepsilon}{2}-\ell}(y) \int_1^\infty \mathrm{d}z\, \gamma_{-1-\frac{\varepsilon}{2}-\ell}(z) \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \varepsilon\, |\tau|^{\varepsilon-1} \\ &\times \int_0^\Lambda \mathrm{d}\lambda \lambda^{1-\ell-p-q\varepsilon+\eta} \left[F^{(-2\ell-1)}\left(t-yr-z\lambda+\tau\right) - F^{(-2\ell-1)}\left(t-yr+z\lambda+\tau\right)\right] ,\end{aligned} \tag{D.13}$$

where Λ is an irrelevant cut-off scale that can be safely set to $+\infty$. The aim is then to put this integral in a nicer looking form. This is realized by setting

$$\begin{aligned}\varphi(t) &\equiv \frac{\Gamma^\varepsilon_\ell\, \hat{n}_L}{r^{\ell+1+\varepsilon}} \int_1^\infty \mathrm{d}y \gamma_{-1-\frac{\varepsilon}{2}-\ell}(y) \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \varepsilon |\tau|^{\varepsilon-1} F^{(-2\ell-1)}(t-yr+\tau) \\ &= \frac{(-)^\ell}{2^\ell}\, \Gamma^\varepsilon_\ell\, \frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)}{\Gamma\left(\ell+\frac{1+\varepsilon}{2}\right)} \int_1^\infty \mathrm{d}y \gamma_{-1-\frac{\varepsilon}{2}}(y) \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \varepsilon |\tau|^{\varepsilon-1} \hat{\partial}_L \left[\frac{F^{(-2\ell-1)}(t-yr+\tau)}{r^{d-2}}\right] ,\end{aligned} \tag{D.14}$$

where we have used the relation (D.6b). Defining $\rho \equiv \lambda z$, it comes

$$\begin{aligned}\tilde{h}_L &= \left(\int_1^\infty \mathrm{d}z\, z^{\ell-2+p+q\varepsilon-\eta}\, \gamma_{-1-\frac{\varepsilon}{2}-\ell}(z)\right) \int_0^\infty \mathrm{d}\rho\, \rho^{1-\ell-p-q\varepsilon+\eta} \left[\varphi(t-\rho) - \varphi(t+\rho)\right] \\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{3+\ell-p-q\varepsilon+\varepsilon+\eta}{2}\right)}{\Gamma\left(\ell+\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{3-\ell-p-q\varepsilon+\eta}{2}\right)} \frac{(-)^{\ell+p+1}\Gamma(q\varepsilon-\eta)}{\Gamma(p+\ell-1+q\varepsilon-\eta)} \\ &\qquad\qquad \times \int_0^\infty \mathrm{d}\rho\, \rho^{\eta-q\varepsilon} \left[\varphi^{(\ell+p-1)}(t-\rho) + (-)^{\ell+p}\varphi^{(\ell+p-1)}(t+\rho)\right] .\end{aligned} \tag{D.15}$$

The interest of this definition is that the metric perturbation is now written so that it looks like the linear d-dimensional metric, namely

$$\tilde{h}_L = -4\,\frac{(-)^\ell}{\ell!}\,\hat{\partial}_L \left\{\frac{\tilde{k}}{r^{d-2}} \int_1^\infty \mathrm{d}z \gamma_{-1-\frac{\varepsilon}{2}}(z)\, H\left(t-zr\right)\right\} , \tag{D.16}$$

where the role of the multipolar moments is played by H, that reads

$$\begin{aligned}H(t) \equiv& \frac{(-)^p\,\ell!}{2^{4-\ell}\pi^{\frac{1-\varepsilon}{2}}} \frac{\Gamma\left(\frac{1-\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)} \frac{\Gamma\left(\frac{3+\ell-p-q\varepsilon+\varepsilon+\eta}{2}\right)}{\Gamma\left(\frac{3-\ell-p-q\varepsilon+\eta}{2}\right)} \frac{\Gamma\left(q\varepsilon-\eta\right)}{\Gamma\left(p+\ell-1+q\varepsilon-\eta\right)} \\ &\times \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \varepsilon |\tau|^{\varepsilon-1} \int_0^{+\infty} \mathrm{d}\rho\, \rho^{\eta-q\varepsilon} \left[F^{(p-\ell-2)}(t+\tau-\rho) + (-)^{\ell+p}F^{(p-\ell-2)}(t+\tau+\rho)\right] .\end{aligned} \tag{D.17}$$

We are thus interested in the difference of regularization schemes of H more than of $\tilde{h}_L$. Therefore we aim at computing

$$\mathcal{D}H \equiv \lim_{\varepsilon\to 0}\left[\underset{\eta}{\mathrm{PF}}\, H\right] - \underset{\eta}{\mathrm{PF}}\left[\lim_{\varepsilon\to 0} H\right] . \tag{D.18}$$

For simplicity, the previous formulas have been obtained by considering only one multipolar order in (p,q), but naturally the source consists in a summation over many of them. Formally, we can decompose

$$N(r,t) = \sum_{p,q} r^{-p-q\varepsilon}\, \varphi_{p,q}(t) = \sum_{p,q} r^{-p-q\varepsilon}\left[\frac{\tilde{\psi}_{p,q}(t)}{\varepsilon} + \psi_{p,q}(t)\right] , \tag{D.19}$$

where the $\tilde{\psi}_{p,q}$ coefficients do not depend on ε. The $F(t)$ function corresponds to one particular $\varphi_{p,q}(t)$. As we have explicitly checked in practical computations, all the considered sources have a non-pathological three-dimensional limit, meaning that

$$N^{3\mathrm{D}}(r,t) = \lim_{\varepsilon\to 0} N(r,t) = \sum_p r^{-p}\left[\varphi_p(t) + \tilde{\varphi}_p(t)\,\ln r\right] = \mathop{\mathrm{PF}}_{\epsilon} \sum_p r^{-p}\left[\varphi_p(t) + \tilde{\varphi}_p(t)\,\frac{r^{\epsilon}}{\epsilon}\right], \tag{D.20}$$

with

$$\sum_q \tilde{\psi}_{p,q} = 0\,, \quad \sum_q q\,\tilde{\psi}_{p,q} = -\tilde{\varphi}_p \quad \text{and} \quad \sum_q \psi_{p,q} = \varphi_p + \mathcal{O}(\varepsilon)\,. \tag{D.21}$$

Note that the rewriting of the logarithms by means of a finite part has been made for practical computational purposes.

D.2.1 Three-dimensional computation

In order to perform the three-dimensional computation with a natural implementation of logarithms, let us identify ϵ with ε in (D.20). The three-dimensional limit thus becomes a finite part operation on ε, where only the ε associated with a q are to be considered, and one takes $q \equiv -1$. Therefore the three-dimensional limit is formally given by

$$\begin{aligned}\lim_{\varepsilon\to 0} H = \frac{(-)^p\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p+\eta}{2}\right)}{\Gamma\left(\frac{3-\ell-p+\eta}{2}\right)}\frac{\Gamma\left(-\eta\right)}{\Gamma\left(p+\ell-1-\eta\right)}\int_0^{+\infty}\mathrm{d}\tau\,\tau^{\eta}\Bigg\{&\varphi_p^{(p-\ell-2)}(t-\tau) + (-)^{\ell+p}\varphi_p^{(p-\ell-2)}(t+\tau)\\ + \Bigg[\ln\tau + \psi\left(p+\ell-1-\eta\right) - \psi(-\eta) + \frac{\psi\left(\frac{3+\ell-p+\eta}{2}\right) - \psi\left(\frac{3-\ell-p+\eta}{2}\right)}{2}\Bigg]&\\ \times\left[\tilde{\varphi}_p^{(p-\ell-2)}(t-\tau) + (-)^{\ell+p}\tilde{\varphi}_p^{(p-\ell-2)}(t+\tau)\right]\Bigg\}&\,,\end{aligned} \tag{D.22}$$

with ψ the digamma function.

To accurately perform the finite part operation on η, we have to distinguish cases following the values of p and ℓ. First, for $p+\ell\leq 1$, it comes

$$\begin{aligned}\mathop{\mathrm{Pf}}_{\eta}\left(\lim_{\varepsilon\to 0} H\right) =& \frac{(-)^{\ell+1}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)}{\Gamma\left(\frac{3-\ell-p}{2}\right)}\Gamma\left(2-p-\ell\right)\\ &\times\Bigg\{\left[1-(-)^{\ell+p}\right]\left[\varphi_p^{(p-\ell-3)}(t) + \left(\mathcal{H}_{1-p-\ell} + \frac{\psi\left(\frac{3+\ell-p}{2}\right) - \psi\left(\frac{3-\ell-p}{2}\right)}{2}\right)\tilde{\varphi}_p^{(p-\ell-3)}(t)\right]\\ &+\int_0^{\infty}\mathrm{d}\tau\ln\tau\left[\tilde{\varphi}_p^{(p-\ell-2)}(t-\tau) + (-)^{\ell+p}\tilde{\varphi}_p^{(p-\ell-2)}(t+\tau)\right]\Bigg\}\,.\end{aligned} \tag{D.23}$$

Then in the case $p+\ell-2\in 2\mathbb{N}$, it reads

$$\begin{aligned}\mathop{\mathrm{Pf}}_{\eta}\left(\lim_{\varepsilon\to 0} H\right) = \frac{(-)^{p+1}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)}{\Gamma\left(\frac{3-\ell-p}{2}\right)}\frac{1}{\Gamma\left(p+\ell-1\right)}\int_0^{\infty}\mathrm{d}\tau\ln\tau\Bigg\{&\varphi_p^{(p-\ell-2)}(t-\tau) + \varphi_p^{(p-\ell-2)}(t+\tau)\\ + \left(\mathcal{H}_{p+\ell-2} + \frac{\psi\left(\frac{3+\ell-p}{2}\right) - \psi\left(\frac{3-\ell-p}{2}\right)}{2} + \frac{\ln\tau}{2}\right)\left[\tilde{\varphi}_p^{(p-\ell-2)}(t-\tau) + \tilde{\varphi}_p^{(p-\ell-2)}(t+\tau)\right]\Bigg\}&\,.\end{aligned} \tag{D.24}$$

In the $p+\ell=2j+3$ case, with $j<\ell$, the finite part reduces to

$$\underset{\eta}{\mathrm{Pf}}\left(\lim_{\varepsilon\to 0} H\right) = \frac{(-)^{\frac{1+p-\ell}{2}}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)\Gamma\left(\frac{\ell+p-1}{2}\right)}{\Gamma\left(p+\ell-1\right)}\Bigg\{\varphi_p^{(p-\ell-3)}(t)+\left(\mathcal{H}_{p+\ell-2}+\frac{\mathcal{H}_{\frac{1-p+\ell}{2}}}{2}-\frac{\mathcal{H}_{\frac{p+\ell-3}{2}}}{2}\right)\tilde{\varphi}_p^{(p-\ell-3)}(t)$$
$$+\int_0^\infty \mathrm{d}\tau\,\ln\tau\,\frac{\tilde{\varphi}_p^{(p-\ell-2)}(t-\tau)-\tilde{\varphi}_p^{(p-\ell-2)}(t+\tau)}{2}\Bigg\}\,. \tag{D.25}$$

Finally, for $p-\ell-3=2j$ with $j\in\mathbb{N}$, it comes

$$\underset{\eta}{\mathrm{Pf}}\left(\lim_{\varepsilon\to 0} H\right) = \frac{\ell!}{2^{3-\ell}}\frac{\Gamma\left(\ell+j+1\right)}{\Gamma\left(j+1\right)\Gamma\left(2j+2\ell+2\right)}\Bigg\{2\tilde{\mathcal{H}}\varphi_p^{(2j)}(t)+\int_0^\infty \mathrm{d}\tau\,\ln\tau\left[\varphi_p^{(2j+1)}(t-\tau)-\varphi_p^{(2j+1)}(t+\tau)\right]$$
$$+\left(\tilde{\mathcal{H}}^2+\frac{\psi_1(j+\ell+1)-\psi_1(j+1)-4\psi_1(2j+2\ell+2)}{4}+\frac{\pi^2}{6}\right)\tilde{\varphi}_p^{(2j)}(t)$$
$$+\int_0^\infty \mathrm{d}\tau\left(\tilde{\mathcal{H}}\ln\tau+\frac{\ln^2\tau}{2}\right)\left[\tilde{\varphi}_p^{(2j+1)}(t-\tau)-\tilde{\varphi}_p^{(2j+1)}(t+\tau)\right]\Bigg\}\,, \tag{D.26}$$

where we have shortened the combination of harmonic numbers $\tilde{\mathcal{H}}\equiv\mathcal{H}_{2j+2\ell+1}+\frac{\mathcal{H}_j}{2}-\frac{\mathcal{H}_{j+\ell}}{2}$, and ψ_1 denotes the trigamma function.

D.2.2 d-dimensional computation

For clarity purposes, we will omit here the summation over the values of q. In this vein, we note *eg.* $F=\sum_q(\tilde{\psi}_{p,q}/\varepsilon+\psi_{p,q})$ or $qF=\sum_q q(\tilde{\psi}_{p,q}/\varepsilon+\psi_{p,q})$. It was verified that no source terms with $q=0$ appear in our computation (this fact can be related to the fact that all considered d-dimensional quantities have at least $q=1$). Therefore, as long as $p-\ell-3\notin\mathbb{N}$, we are protected by a ε and thus the finite part on η is trivially realized by taking $\eta=0$. Discriminating again on the values of p and ℓ, for $p+\ell\leq 1$, it comes

$$\lim_{\varepsilon\to 0}\left(\underset{\eta}{\mathrm{Pf}}\,H\right)=\frac{(-)^{\ell+1}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)}{\Gamma\left(\frac{3-\ell-p}{2}\right)}\Gamma\left(2-p-\ell\right)$$
$$\times\Bigg\{\sigma\left[1+\varepsilon\left(\frac{\ln\bar{q}+\gamma_E}{2}-q\mathcal{H}_{1-p-\ell}+\frac{(1-q)\psi\left(\frac{3+\ell-p}{2}\right)+q\psi\left(\frac{3-\ell-p}{2}\right)}{2}\right)\right]F^{(p-\ell-3)}(t)$$
$$+\varepsilon\int_0^\infty \mathrm{d}\tau\,\ln\tau\left[\left(\frac{\sigma}{2}-q\right)F^{(p-\ell-2)}(t-\tau)-\left(\frac{\sigma}{2}+(-)^{\ell+p}q\right)F^{(p-\ell-2)}(t+\tau)\right]\Bigg\}\,, \tag{D.27}$$

where we have shortened $\sigma\equiv 1-(-)^{\ell+p}$. By virtue of the relations (D.21), eq. (D.23) is recovered: the two regularization schemes agree.

For $p+\ell-2\in 2\mathbb{N}$, the d-dimensional computation yields

$$\lim_{\varepsilon\to 0}\left(\underset{\eta}{\mathrm{Pf}}\,H\right)=\frac{(-)^{p+1}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)}{\Gamma\left(\frac{3-\ell-p}{2}\right)}\frac{1}{\Gamma\left(p+\ell-1\right)}$$
$$\times\int_0^\infty \mathrm{d}\tau\,\ln\tau\Bigg\{\left[1+\varepsilon\left(\frac{\ln\bar{q}+\gamma_E}{2}-q\mathcal{H}_{p+\ell-2}+\frac{(1-q)\psi\left(\frac{3+\ell-p}{2}\right)+q\psi\left(\frac{3-\ell-p}{2}\right)}{2}-\frac{q\ln\tau}{2}\right)\right]$$
$$\times\left[F^{(p-\ell-2)}(t-\tau)+F^{(p-\ell-2)}(t+\tau)\right]$$
$$+\frac{\varepsilon}{2}\int_0^\infty \mathrm{d}\lambda\,\ln\lambda\left[F^{(p-\ell-1)}(t-\tau-\lambda)-F^{(p-\ell-1)}(t-\tau-\lambda)\right]\Bigg\}\,, \tag{D.28}$$

so that the three-dimensional result (D.24) is recovered.

As for $p+\ell = 2j+3$, with $j<\ell$, it reads

$$\lim_{\varepsilon\to 0}\left(\underset{\eta}{\mathrm{Pf}}\, H\right) = \frac{(-)^{\frac{1+p-\ell}{2}}\,\ell!}{2^{3-\ell}}\frac{\Gamma\left(\frac{3+\ell-p}{2}\right)\Gamma\left(\frac{\ell+p-1}{2}\right)}{\Gamma\left(p+\ell-1\right)}\Bigg\{ F^{(p-\ell-3)}(t) + \varepsilon\left(\frac{\ln\bar{q}}{2} - q\mathcal{H}_{p+\ell-2} + \frac{(1-q)\mathcal{H}_{\frac{1-p+\ell}{2}} + q\mathcal{H}_{\frac{p+\ell-3}{2}}}{2}\right) F^{(p-\ell-3)}(t) + \frac{\varepsilon(1-q)}{2}\int_0^\infty \mathrm{d}\tau \ln\tau \left[F^{(p-\ell-2)}(t-\tau) - F^{(p-\ell-2)}(t+\tau)\right]\Bigg\}\,. \tag{D.29}$$

Once again, the d-dimensional and three-dimensional (D.25) computations yield the same result.

Finally for $p-\ell-3 = 2j$ with $j\in\mathbb{N}$, we separate the d-dimensional result into two pieces: the first one sums over all q

$$\begin{aligned}\underset{\eta}{\mathrm{Pf}}\left(\lim_{\varepsilon\to 0} H\right)_1 = \frac{\ell!}{2^{3-\ell}}\frac{\Gamma\left(\ell+j+1\right)}{\Gamma\left(j+1\right)\Gamma\left(2j+2\ell+2\right)}\Bigg\{ & 2\tilde{\mathcal{H}}F^{(2j)}(t) + \int_0^\infty \mathrm{d}\tau\ln\tau\left[F^{(2j+1)}(t-\tau) - F^{(2j+1)}(t+\tau)\right] \\ & + 2\varepsilon\left[\frac{\mathcal{H}_j+\ln\bar{q}}{2}\tilde{\mathcal{H}} - \frac{q+1}{2}\left(\tilde{\mathcal{H}}^2 - \psi_1(2j+2\ell+2) + \frac{\psi_1(1+\ell+j)}{4}\right)\right. \\ & \left. + \frac{q-1}{8}\psi_1(1+j) - \frac{q}{12}\pi^2\right] F^{(2j)}(t) \\ & - \varepsilon\int_0^\infty \mathrm{d}\tau\ln\tau\left(q\tilde{\mathcal{H}} - \frac{\mathcal{H}_j+\ln\bar{q}}{2} + \frac{q+1}{2}\ln\tau\right)\left[F^{(2j+1)}(t-\tau) - F^{(2j+1)}(t+\tau)\right] \\ & + \frac{\varepsilon}{2}\iint_0^\infty \mathrm{d}\tau\mathrm{d}\lambda\ln\tau\ln\lambda\left[F^{(2j+2)}(t-\tau-\lambda) - 2F^{(2j+2)}(t-\tau+\lambda) + F^{(2j+2)}(t+\tau+\lambda)\right]\Bigg\}\,,\end{aligned} \tag{D.30}$$

and, thanks to the conditions (D.21), exactly agrees with the three-dimensional limit (D.26). Therefore the difference in regularization schemes is exactly given by the second contribution, that sums over $q\neq 1$, namely

$$\begin{aligned}\underset{\eta}{\mathrm{Pf}}\left(\lim_{\varepsilon\to 0} H\right)_2 = \frac{\ell!}{2^{3-\ell}}\frac{\Gamma\left(\ell+j+1\right)}{\Gamma\left(j+1\right)\Gamma\left(2j+2\ell+2\right)}\sum_{q\neq 1}\frac{1}{q-1}\Bigg\{ & 2\left(\frac{-1}{\varepsilon} + \mathcal{H}_{2j+2\ell+1} - \frac{\mathcal{H}_{j+\ell}}{2} - \frac{\ln\bar{q}}{2}\right)F^{(2j)}(t) \\ & - \varepsilon\left[\tilde{\mathcal{H}}^2 + \left(\tilde{\mathcal{H}}^2 - \frac{\mathcal{H}_j+\ln\bar{q}}{2}\right)^2 - 2\psi_1(2j+2\ell+2) + \frac{\psi_1(1+\ell+j)}{2} + \frac{\pi^2}{6}\right] F^{(2j)}(t) \\ & + \frac{\varepsilon}{2}\iint_0^\infty \mathrm{d}\tau\mathrm{d}\lambda\ln\tau\ln\lambda\left[F^{(2j+2)}(t-\tau-\lambda) - 2F^{(2j+2)}(t-\tau+\lambda) + F^{(2j+2)}(t+\tau+\lambda)\right]\Bigg\}\,.\end{aligned} \tag{D.31}$$

D.2.3 Difference in regularization schemes for the non-linear interactions

The main result of this technical appendix is that for a source

$$N(r,t) = \sum_{p,q} r^{-p-q\varepsilon}\left[\frac{\tilde{\psi}_{p,q}(t)}{\varepsilon} + \psi_{p,q}(t)\right]\,, \tag{D.32}$$

the difference in regularization schemes only comes from the terms that have $p=\ell+3+2j$, for $j\in\mathbb{N}$. One can separate the contribution coming from the pole from the one coming from the regular sector as

$$\mathcal{D}H = (\mathcal{D}H)_{\mathrm{reg}} + (\mathcal{D}H)_{\mathrm{p}}\,, \tag{D.33}$$

with

$$(\mathcal{D}H)_{\text{reg}} = -\sum_{q\neq 1} \frac{2^{\ell-2}}{q-1} \frac{\ell!\,(\ell+j)!}{j!\,(p+\ell-2)!} \left[\frac{1}{\varepsilon} + \ln\sqrt{\bar{q}} - \sum_{k=0}^{\ell+j} \frac{1}{2k+1}\right] \psi_{p,q}^{(2j)}(t)\,, \tag{D.34}$$

and

$$\begin{aligned}(\mathcal{D}H)_{\text{p}} = -\sum_{q\neq 1} \frac{2^{\ell-2}}{q-1} \frac{\ell!\,(\ell+j)!}{j!\,(p+\ell-2)!} \Bigg\{ &\left[\frac{1}{\varepsilon^2} + \frac{\ln\sqrt{\bar{q}}}{\varepsilon} - \frac{1}{\varepsilon}\sum_{k=0}^{\ell+j} \frac{1}{2k+1} - \mathcal{A}_{j,\ell}\right] \tilde{\psi}_{p,q}^{(2j)}(t) \\ &- \iint_0^\infty \mathrm{d}\tau \mathrm{d}\lambda \ln\tau \ln\lambda \frac{\tilde{\psi}_{p,q}^{(2j+2)}(t-\tau-\lambda) - 2\tilde{\psi}_{p,q}^{(2j+2)}(t-\tau+\lambda) + \tilde{\psi}_{p,q}^{(2j+2)}(t+\tau+\lambda)}{4}\Bigg\}\,,\end{aligned} \tag{D.35}$$

where

$$\mathcal{A}_{j,\ell} \equiv \frac{\tilde{\mathcal{H}}^2}{2} + \frac{1}{2}\left(\tilde{\mathcal{H}}^2 - \frac{\mathcal{H}_j + \ln\bar{q}}{2}\right)^2 - \psi_1(2j+2\ell+2) + \frac{\psi_1(1+\ell+j)}{4} + \frac{\pi^2}{12}\,. \tag{D.36}$$

ψ_1 is the trigamma function and we recall the shorthand $\tilde{\mathcal{H}} \equiv \mathcal{H}_{2j+2\ell+1} + \frac{\mathcal{H}_j}{2} - \frac{\mathcal{H}_{j+\ell}}{2}$.

It is to note that the regular contribution is local, whereas the one coming from the pole bears a non-local part. Fortunately, in our practical case, there is no poles when $p-\ell-3\in 2\mathbb{N}$!

Appendix E

Résumé en français

La théorie actuellement la plus à même de rendre compte des phénomènes gravitationnels, la Relativité Générale (RG), a été testée dans une variété de régimes sans précédent : depuis les accélérations très faibles ressenties par les sondes Pioneer 10 et 11 aux confins du système Solaire, aux voisinages des trous noirs, en passant naturellement par notre mouvement autour du Soleil. Ces tests ont tous été concluants, et ont confirmé la fermeté des assises de la RG. Cependant, il reste aujourd'hui au moins deux phénomènes qui n'ont pas d'explication convaincante : la phénoménologie de la matière noire et l'accélération de la dilatation de notre univers. La phénoménologie de la matière noire a été postulée pour rendre compte d'anomalies dans le comportement des étoiles en périphérie des galaxies, et se retrouve aujourd'hui à toutes les échelles : depuis les expériences sondant la structure fondamentale de la matière au CERN jusqu'aux observations des reliques du début de l'univers. Cette "matière noire" pourrait être composée d'une (ou de plusieurs) nouvelle particule, qui résiste encore et toujours à l'observateur, ou alors elle pourrait être dûe à une insuffisance de la RG pour décrire la phénoménologie des accélérations extrêmement ténues. De même, l'accélération de la dilatation de notre univers peut être décrite par une simple "constante cosmologique" ajoutée aux équations de la RG, par une "énergie sombre" de nature matérielle, ou encore par une complétion de la RG dans le régime des énergies très faibles.

Pour avancer vers une explication convaincante de ces phénomènes, trois voies semblent primordiales à suivre. La première, que nous ne suivrons pas, consiste naturellement à tendre vers une meilleure connaissance empirique de notre univers, en raffinant les observations actuelles et en inventant de nouveaux modes d'observation. Un effort conséquent est actuellement porté en ce sens et de nombreux instruments sont développés et exploités, sur terre, sous terre, dans les profondeurs marines et dans l'espace. Les deux autres voies, plus théoriques, sont celles que nous avons choisies d'explorer lors de cette thèse. Tout d'abord, et en parallèle de cette nécessité d'affiner notre connaissance empirique de l'univers, il y a un vrai besoin d'affuter nos prédictions théoriques pour la RG. Ce n'est qu'en comparant les observations à des prédictions robustes que nous serons capables de discerner de possibles déviations à la RG, qui pourront indiquer dans quelle direction celle-ci a besoin d'être dépassée. La troisième et dernière voie consiste à construire et tester des scénarios alternatifs à la RG. Même si les théories exotiques développées ne sont pas forcément de meilleures descriptions de la nature que la RG, il est toujours intéressant de les explorer pour au moins deux raisons. La première tient à ce que de telles constructions nous renseignent sur la nature même de la RG : en la déconstruisant pour construire une autre théorie, on en expose au grand jour les mécaniques internes. La seconde raison est d'ordre plus pratique: en sachant quelles alternatives sont viables, on peut réaliser des tests plus affutés en cherchant à isoler un effet particulier, inexistant en RG.

Naturellement ces deux dernières voies (une meilleure connaissance théorique de la RG, et la recherche d'alternatives viables) se doivent d'être suivies en parallèle, et d'être mises en regard l'une de l'autre. C'est dans cet esprit que nous avons abordé cette thèse, dédiée d'une part à l'étude du problème à deux corps, et de l'autre, à la construction et au test de théories alternatives.

Étude du problème à deux corps

Le problème à deux corps est un cadre idéal pour l'étude de la RG. Conceptuellement c'est l'un des systèmes les plus simples que l'on puisse imaginer, puisqu'il n'est composé que de deux corps, liés uniquement par l'interaction gravitationnelle. Empiriquement, on trouve de telles configurations en abondance dans l'univers, de la pomme qui tombe aux binaires de trous noirs, en passant par le système Terre-Lune, ce qui permet de multiplier les régimes d'observation. Mais d'un point de vue théorique, si le problème à deux corps Newtonien est très simple car linéaire, le problème relativiste exhibe toutes les non-linéarités de la RG, et est donc bien plus ardu. Bien que centenaire, c'est ainsi un champ toujours très actif et dynamique, et de nombreux pans restent à défricher.

Pour simplifier la configuration étudiée, nous allons considérer les deux corps comme des particules ponctuelles, négligeant les effets de rotation et de taille finie. Traiter le problème relativiste à deux corps tel quel semble une tache impossible avec les techniques actuelles de calcul, nous allons donc devoir recourir à des méthodes d'approximation, dites "post-Newtonienne" (PN, qui consiste à considérer des corps de faible vitesse) et "post-Minkowskienne" (PM, qui traite des champs faibles). La première approximation traite des caractéristiques des corps (en l'occurrence, leur vitesse), elle est donc valable en zone proche, c'est-à-dire à proximité des-dits corps. Elle sera donc utilisée pour déduire la structure de l'espace-temps dans cette zone uniquement. La seconde approximation considère des déviations à un espace vide, elle est donc particulièrement adaptée à l'étude de la radiation gravitationnelle, et est valide à l'extérieur de la distribution de matière. Ces deux approches engendrent deux solutions, chacune dans une zone différente : nous allons donc devoir les associer pour obtenir une unique solution, valable dans tout l'espace-temps. Cette association se fait dans la zone de recouvrement où les deux approximations sont valides, au moyen d'une puissante *équation de raccord*. Ce formalisme "post-Newtonien-post-Minkowskien-multipolaire" (PN-MPM, de l'anglais), qui consiste à raccorder ces deux approches, est celui que nous utiliserons dans cette thèse. Comme la première correction à la dynamique Newtonienne est quadratique en l'inverse de la vitesse de la lumière, il est d'usage de nommer nPN l'ordre c^{-2n}. En revanche le $n^{\text{ième}}$ ordre PM est traditionnellement donné par la correction en G^n.

Une première contribution de cette thèse au domaine du problème à deux corps fut le calcul des contributions logarithmiques dues aux effets de *sillage* (*tail effects* en anglais) dans l'énergie. Ces effets sont dûs à la diffusion d'ondes gravitationnelles sur la courbure statique de l'espace-temps créée par la binaire, et sont donc des phénomènes non-linéaires par essence. En "ricochant" sur la structure statique du fond, les ondes vont se disperser et certaines vont revenir interagir avec la binaire, ce qui va naturellement affecter son mouvement. Ces effets sont par essence non-instantanés puisqu'à un instant donné, les ondes qui interagissent avec la binaire sont celles qui ont été émises dans le passé de la source. Cette non-localité en temps se traduit par l'apparition de logarithmes en vitesse dans l'énergie d'une binaire évoluant sur une orbite circulaire. En exploitant de nouvelles synergies entre les méthodes PN-MPM, les techniques provenant des théories effectives des champs et celle dite "première loi de la dynamique des binaires de trous noirs", et en collaboration avec L. Blanchet, S. Foffa et R. Sturani, nous avons calculé cette dépendance logarithmique à une précision qui n'avait pas encore été atteinte (3PN en sus de l'ordre dominant, soit 7PN dans l'énergie). En outre, nous avons tiré profit de l'équation de renormalisation pour extraire une formule analytique donnant, pour chaque puissance du logarithme, la valeur du coefficient dominant. Nous avons naturellement vérifié que, pour les ordres les plus bas (*ie.* jusqu'à 21PN), notre formule concorde bien avec les résultats préexistants.

Si cette première étude fut importante car révélatrice du potentiel encore largement inexploré des synergies entre ces différentes approches du problème à deux corps, la contribution principale de cette thèse réside dans le travail relatif à la génération d'onde. Les générations actuelles et futures de détecteurs sont bien plus sensibles à l'évolution de la fréquence des ondes qu'à celle de leur amplitude. Il est donc essentiel de connaitre la phase gravitationnelle (*ie.* la dépendance temporelle de ces ondes) à de très grandes précisions, notamment pour le futur détecteur LISA (*Laser Interferometer Space Antenna*). Si l'état de l'art est la précision 3.5PN, le programme visant à déterminer le comportement à 4PN a été initié il y a plusieurs années et touche à sa fin. Il est important de souligner qu'actuellement seule la méthode PN-MPM est capable d'atteindre de telles précisions, c'est donc celle que nous implémenterons. Parmi les multiples étapes qui conduisent à l'établissement de la phase à 4PN, nous nous sommes attachés lors de cette thèse à la détermination du moment quadrupolaire de masse. Ce calcul peut se décomposer en trois étapes : le quadrupole source est d'abord évalué avec une régularisation dimensionnelle[1] dans l'UV et une régularisation de Hadamard dans l'IR; ensuite la régularisation dimensionnelle est appliquée dans l'IR; enfin le quadrupole radiatif, c'est-à-dire observable, est déterminé à partir du quadrupole source et de la prise en compte des nombreuses interactions non-linéaires, qui doivent être évaluées en dimension arbitraire. L'établissement du quadrupole source Hadamard a été achevé par T. Marchand avant le début de cette thèse, mais nous en avons recalculé quelques secteurs avec L. Blanchet et S. Marsat dans l'optique de réaliser des vérifications en double aveugle, rendues nécessaires par la longueur et la technicité des calculs. Néanmoins l'apport majeur de cette thèse fut l'implémentation de la régularisation dimensionnelle dans l'IR pour les quadrupoles source et radiatif, réalisée en collaboration avec Q. Henry, L. Blanchet et G. Faye. Ce procédé de régularisation est vital pour obtenir un quadrupole radiatif physique, c'est-à-dire sans divergences fallacieuses provenant de l'approximation PN étendue hors de son régime de validité. La régularisation dimensionnelle du quadrupole source a été achevée et satisfait tous les critères de validité, nous sommes actuellement en train de mettre au point celle du quadrupole radiatif, en calculant les effets de *sillage*, de *mémoire* et leurs itérations.

Un troisième projet fut entrepris lors de cette thèse, et concerne l'établissement d'un test de la RG dans un système exoplanétaire bien particulier, HD 80606b. En effet cette exoplanète a une excentricité très élevée et a été observée lors de transits et d'éclipses. Avec L. Blanchet et G. Hébrard, nous avons calculé la prédiction relativiste de la variation du temps écoulé entre une éclipse et le transit suivant, et avons montré qu'elle pourrait être détectable à partir de 2025. Les lancements prévus des télescopes spatiaux *James–Webb* et *Ariel*, à même d'observer précisément les éclipses et transits d'HD 80606b, nous donnent bon espoir que le test soit effectué (des discussions en ce sens avec J.-P. Beaulieu, coPI d'*Ariel* sont en cours). À notre connaissance, il s'agit là du premier test réaliste de la RG dans un système stellaire qui ne soit pas le système Solaire.

Étude de théories alternatives

En parallèle de l'étude du problème à deux corps en RG, nous avons aussi considéré des théories alternatives. Avec C. Deffayet nous nous sommes intéressés aux défauts topologiques (ou *solitons*), c'est-à-dire des configurations d'énergie localisées et auto-entretenues, les exemples les plus sensationnels étant les mascarets, remontant l'embouchure des fleuves. Nous nous sommes concentrés sur de tels défauts réalisés par des champs scalaires, une configuration qui a notamment d'importantes implications cosmologiques. L'argument habituel justifiant leur stabilité est topologique, et repose sur la dégénérescence de la structure du vide d'un potentiel. Pour dépasser un tel argument et évaluer sa nécessité, nous avons cherché à construire des défauts non-topologiques, sans potentiel mais néanmoins stables, et nous nous sommes restreints à des géométries planaires non-gravitantes par simplicité. Après avoir obtenu des formules générales permettant de construire de tels solitons, nous nous sommes concentrés sur une réalisation particulière et avons démontré qu'un certain choix de paramètres permettait d'imiter exactement les solitons canoniques, aussi bien au niveau du profil de la solution, qu'au niveau de ses perturbations linéaires. Malheureusement, de tels défauts non-topologiques ne peuvent pas être directement solutions de théories cosmologiques car l'implémentation de la gravitation altère trop gravement leurs conditions d'existence.

[1] Un tel procédé consiste à effectuer les calculs en dimension arbitraire, puis à prendre la limite tridimensionnelle.

Conjointement à l'étude de ces défauts non-topologiques, cette thèse s'est attachée à l'étude de théories alternatives de gravitation dites *minimales*. La philosophie du *minimalisme* repose sur le constat que si l'ajout de degrés de liberté est un moyen pratique de rendre compte de phénomènes tels que l'accélération de la dilatation de notre univers, de tels supplétifs ne sont pas nécessaires d'un point de vue empirique. De fait, ni le suivi de binaires de pulsars, ni la détection directe de radiation gravitationnelle ne laisse transparaître de traces de degrés de liberté autres que les deux polarisations tensorielles présentes en RG. Une théorie minimale est donc une théorie qui ne propage que des polarisations tensorielles, tout en offrant une phénoménologie sensiblement plus riche que celle de la RG. Naturellement une telle construction a un coût, et les théories minimales brisent faiblement l'invariance sous les difféomorphismes temporels, présente en RG. Une telle brisure est néanmoins peu contraignante car elle n'est effective qu'aux échelles cosmologiques et dans le secteur purement gravitationnel : ces théories satisfont donc tous les tests habituels, réalisés à des échelles terrestres. La première de ces théories, construite en 2015 par A. De Felice et S. Mukohyama est une *minimalisation* de la gravité massive canonique, qui offre une explication naturelle à l'accélération de la dilatation de notre univers. Mais pour pouvoir donner lieu à une cosmologie attrayante, l'un des champs non-dynamiques de cette théorie minimale de gravité massive doit être muni d'une artificielle dépendance en temps.

Un tel artefact étant peu convaincant et encore moins esthétique, nous nous sommes efforcés de le dépasser, en collaboration avec A. De Felice, S. Mukohyama et M. Oliosi. Cela nous a conduit à chercher une théorie minimale de bigravité, c'est-à-dire une théorie faisant apparaître deux métriques dynamiques, mais seulement quatre polarisations (contrairement aux sept habituellement présentes dans de telles théories de bigravité). En jaugeant la théorie canonique de bigravité, due à S. Hassan et R. Rosen, et en introduisant des contraintes supplémentaires au niveau du Hamiltonien, nous avons réalisé une telle théorie. Une étude de sa phénoménologie cosmologique a révélé qu'elle donne lieu à deux branches de solutions, toutes deux stables (contrairement à la bigravité canonique, dont la cosmologie est instable) et conduisant à des mécanismes naturels d'accélération de la dilatation de l'univers. En outre cette théorie conduit à des oscillations et une relation de dispersion non triviale pour les gravitons et, dans l'une des deux branches, la croissance des structures à grande échelle est différente de celle de la RG. Toutes ces disparités avec la cosmologie orthodoxe sont naturellement observables, et pourraient permettre de tester notre théorie.

Outre de tels tests cosmologiques, il est important de sonder la phénoménologie à petite échelle, notamment en explorant les régimes forts par la construction de solutions de trous noirs. C'est ce que nous avons réalisé, en collaboration avec A. De Felice, A. Doll, S. Mukohyama et M. Oliosi, dans deux théories minimales, la théorie initiale de gravité massive et le modèle dit VCDM. Dans la première de ces théories, nous avons démontré un lemme qui relie les solutions de RG ayant des sections spatiales plates à des solutions de la théorie alternative. Ce lemme technique est en fait extrêmement puissant puisque les espaces-temps de la RG représentant des étoiles et des trous noirs statiques admettent de telles sections spatiales plates, et sont donc directement solutions de la théorie minimale. Le corollaire direct de ce résultat est que cette théorie alternative satisfait la majorité des tests réalisés dans le système Solaire aussi bien que la RG, c'est-à-dire avec brio. Nous avons essayé d'implémenter des trous noirs en rotation par un processus similaire, mais avons montré que les trous noirs en rotation de la RG n'admettent pas de sections spatiales plates, et nous n'avons donc pas pu appliquer notre lemme. L'étude du régime de champ fort du modèle VCDM s'est révélée bien différente. En effet nous avons pu construire des espaces-temps de trou noirs distincts de ceux de la RG, notamment par des phénomènes de non-staticité et une possible violation de la condition de convergence nulle. De telles disparités avec les trous noirs usuels sont évidemment observables, que ce soit lors des effondrements gravitationnels dont ils émergent, par le suivi du mouvement des corps dans leur voisinage ou par l'émission de radiation gravitationnelle.

Bibliography

[1] Discussions with L. Blanchet.

[2] G. Aad *et al.*; "Search for new phenomena in events with two opposite-charge leptons, jets and missing transverse momentum in pp collisions at $\sqrt{s} = 13$ TeV with the ATLAS detector"; (2021)arXiv: 2102.01444.

[3] B. P. Abbott *et al.*; "Observation of Gravitational Waves from a Binary Black Hole Merger"; Phys. Rev. Lett. **116**, p. 061102 (2016). arXiv: 1602.03837.

[4] B. P. Abbott *et al.*; "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral"; Phys. Rev. Lett. **119**, p. 161101 (2017). arXiv: 1710.05832.

[5] B. P. Abbott *et al.*; "Multi-messenger Observations of a Binary Neutron Star Merger"; Astrophys. J. Lett. **848**, p. L12 (2017). arXiv: 1710.05833.

[6] B. P. Abbott *et al.*; "Tests of General Relativity with GW170817"; Phys. Rev. Lett. **123**, p. 011102 (2019). arXiv: 1811.00364.

[7] B. P. Abbott *et al.*; "GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs"; Phys. Rev. X **9**, p. 031040 (2019). arXiv: 1811.12907.

[8] L. F. Abbott & S. Deser; "Stability of gravity with a cosmological constant"; Nuclear Physics B **195**, pp. 76–96 (1982).

[9] R. Abbott *et al.*; "GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run"; (2020)arXiv: 2010.14527.

[10] R. Abbott *et al.*; "Tests of General Relativity with Binary Black Holes from the second LIGO-Virgo Gravitational-Wave Transient Catalog"; (2020)arXiv: 2010.14529.

[11] R. Abbott *et al.*; "Constraints on cosmic strings using data from the third Advanced LIGO-Virgo observing run"; (2021)arXiv: 2101.12248.

[12] R. Abuter *et al.*; "Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole"; Astron. Astrophys. **636**, p. L5 (2020). arXiv: 2004.07187.

[13] P. A. R. Ade *et al.*; "Planck 2013 results. XXV. Searches for cosmic strings and other topological defects"; Astron. Astrophys. **571**, p. A25 (2014). arXiv: 1303.5085.

[14] N. Afshordi, D. J. H. Chung & G. Geshnizjani; "Cuscuton: A Causal Field Theory with an Infinite Speed of Sound"; Phys. Rev. D **75**, p. 083513 (2007). arXiv: hep-th/0609150.

[15] N. Aghanim *et al.*; "Planck 2018 results. VI. Cosmological parameters"; Astron. Astrophys. **641**, p. A6 (2020). arXiv: 1807.06209.

[16] K. Akiyama *et al.*; "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole"; Astrophys. J. **875**, p. L1 (2019). arXiv: 1906.11238.

[17] Y. Akrami, S. F. Hassan, F. Könnig, A. Schmidt-May & A. R. Solomon; "Bimetric gravity is cosmologically viable"; Phys. Lett. B **748**, pp. 37–44 (2015). arXiv: 1503.07521.

[18] A. Albert *et al.*; "Combined search for neutrinos from dark matter self-annihilation in the Galactic Center with ANTARES and IceCube"; Phys. Rev. D **102**, p. 082002 (2020). arXiv: 2003.06614.

[19] P. Amore & F. M. Fernández; "Exact and approximate expressions for the period of anharmonic oscillators"; European Journal of Physics **26**, pp. 589–601 (2005). arXiv: math-ph/0409034.

[20] M. Andrews, M. Lewandowski, M. Trodden & D. Wesley; "Distinguishing k-defects from their canonical twins"; Phys. Rev. D **82**, p. 105006 (2010). arXiv: 1007.3438.

[21] T. Anson, E. Babichev & C. Charmousis; "Deformed black hole in Sagittarius A"; (2021)arXiv: 2103.05490.

[22] K. Aoki & S. Mukohyama; "Massive gravitons as dark matter and gravitational waves"; Phys. Rev. D **94**, p. 024001 (2016). arXiv: 1604.06704.

[23] K. Aoki, A. D. Felice, C. Lin, S. Mukohyama & M. Oliosi; "Phenomenology in type-I minimally modified gravity"; Journal of Cosmology and Astroparticle Physics **2019**, p. 017 (2019). arXiv: 1810.01047.

[24] K. Aoki, A. De Felice, S. Mukohyama, K. Noui, M. Oliosi & M. C. Pookkillath; "Minimally modified gravity fitting Planck data better than Λ-CDM"; The European Physical Journal C **80** (2020). arXiv: 2005.13972.

[25] K. Aoki, M. A. Gorji & S. Mukohyama; "A consistent theory of $D \to 4$ Einstein-Gauss-Bonnet gravity"; Phys. Lett. B **810**, p. 135843 (2020). arXiv: 2005.03859.

[26] K. Aoki, F. Di Filippo & S. Mukohyama; "Non-uniqueness of massless transverse-traceless graviton"; (2021)arXiv: 2103.15044.

[27] E. Aprile *et al.*; "Excess electronic recoil events in XENON1T"; Phys. Rev. D **102**, p. 072004 (2020). arXiv: 2006.09721.

[28] Aristote; *La Physique, Livre IV* (Flammarion) (1999); préf. by P. Pellegrin.

[29] Aristote; *La Physique, Livre VII* (Flammarion) (1999); préf. by P. Pellegrin.

[30] Aristote; *Traité du Ciel, Livre I* (Flammarion) (2004).

[31] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty & S. Mukohyama; "Ghost condensation and a consistent infrared modification of gravity"; JHEP **05**, p. 074 (2004). arXiv: hep-th/0312099.

[32] C. Armendariz-Picon, T. Damour & V. F. Mukhanov; "k - inflation"; Phys. Lett. B **458**, pp. 209–218 (1999). arXiv: hep-th/9904075.

[33] C. Armendariz-Picon, V. F. Mukhanov & P. J. Steinhardt; "A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration"; Phys. Rev. Lett. **85**, pp. 4438–4441 (2000). arXiv: astro-ph/0004134.

[34] R. L. Arnowitt, S. Deser & C. W. Misner; "The Dynamics of general relativity"; Gen. Rel. Grav. **40**, pp. 1997–2027 (2008). arXiv: gr-qc/0405109.

[35] K. G. Arun, L. Blanchet, B. R. Iyer & M. S. S. Qusailah; "Tail effects in the 3PN gravitational wave energy flux of compact binaries in quasi-elliptical orbits"; Phys. Rev. D **77**, p. 064034 (2008). arXiv: 0711.0250.

[36] P. Auclair, D. A. Steer & T. Vachaspati; "Particle emission and gravitational radiation from cosmic strings: observational constraints"; Phys. Rev. D **101**, p. 083511 (2020). arXiv: 1911.12066.

[37] E. Babichev; "Global topological k-defects"; Physical Review D **74** (2006). arXiv: hep-th/0608071.

[38] E. Babichev; "Gauge k-vortices"; Phys. Rev. D **77**, p. 065021 (2008). arXiv: 0711.0376.

[39] E. Babichev, C. Charmousis, G. Esposito-Farèse & A. Lehébel; "Hamiltonian unboundedness vs stability with an application to Horndeski theory"; Phys. Rev. D **98**, p. 104050 (2018). arXiv: 1803.11444.

[40] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz & J. van Meter; "Gravitational wave extraction from an inspiraling configuration of merging black holes"; Phys. Rev. Lett. **96**, p. 111102 (2006). arXiv: gr-qc/0511103.

[41] S. Ballard *et al.*; "The Kepler-19 System: A Transiting 2.2 $R_\oplus$ Planet and a Second Planet Detected via Transit Timing Variations"; Astrophys. J. **743**, p. 200 (2011). arXiv: 1109.1561.

[42] L. Barack; "Gravitational self force in extreme mass-ratio inspirals"; Class. Quant. Grav. **26**, p. 213001 (2009). arXiv: 0908.1664.

[43] J. M. Bardeen, B. Carter & S. W. Hawking; "The Four laws of black hole mechanics"; Commun. Math. Phys. **31**, pp. 161–170 (1973).

[44] S. Baumgartner, M. Bernardini, J. R. Canivete Cuissa, H. de Laroussilhe, A. M. W. Mitchell, B. A. Neuenschwander, P. Saha, T. Schaeffer, D. Soyuer & L. Zwick; "Towards a polarization prediction for LISA via intensity interferometry"; Monthly Notices of the Royal Astronomical Society **498**, pp. 4577–4589 (2020). arXiv: 2008.11538.

[45] J. D. Bekenstein; "Relativistic gravitation theory for the MOND paradigm"; Phys. Rev. D **70**, p. 083509 (2004). [Erratum: Phys.Rev.D 71, 069901 (2005)]; arXiv: astro-ph/0403694.

[46] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui & G. Tasinato; "Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order"; JHEP **12**, p. 100 (2016). arXiv: 1608.08135.

[47] Z. Bern, J. J. M. Carrasco & H. Johansson; "New Relations for Gauge-Theory Amplitudes"; Phys. Rev. D **78**, p. 085011 (2008). arXiv: 0805.3993.

[48] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon & M. Zeng; "Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order"; Phys. Rev. Lett. **122**, p. 201603 (2019). arXiv: 1901.04424.

[49] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon & M. Zeng; "Black Hole Binary Dynamics from the Double Copy and Effective Theory"; JHEP **10**, p. 206 (2019). arXiv: 1908.01493.

[50] Z. Bern, A. Luna, R. Roiban, C.-H. Shen & M. Zeng; "Spinning Black Hole Binary Dynamics, Scattering Amplitudes and Effective Field Theory"; (2020)arXiv: 2005.03071.

[51] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon & M. Zeng; "Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$"; (2021)arXiv: 2101.07254.

[52] L. Bernard & L. Blanchet; "Phenomenology of Dark Matter via a Bimetric Extension of General Relativity"; Phys. Rev. D **91**, p. 103536 (2015). arXiv: 1410.7708.

[53] L. Bernard, L. Blanchet & L. Heisenberg; "Bimetric gravity and dark matter"; in "50th Rencontres de Moriond on Gravitation: 100 years after GR," (2015); arXiv: 1507.02802.

[54] L. Bernard, L. Blanchet, A. Bohé, G. Faye & S. Marsat; "Fokker action of nonspinning compact binaries at the fourth post-Newtonian approximation"; Phys. Rev. D **93**, p. 084037 (2016). arXiv: 1512.02876.

[55] L. Bernard, L. Blanchet, A. Bohé, G. Faye & S. Marsat; "Energy and periastron advance of compact binaries on circular orbits at the fourth post-Newtonian order"; Phys. Rev. D **95**, p. 044026 (2017). arXiv: 1610.07934.

[56] L. Bernard, L. Blanchet, A. Bohé, G. Faye & S. Marsat; "Dimensional regularization of the IR divergences in the Fokker action of point-particle binaries at the fourth post-Newtonian order"; Phys. Rev. D **96**, p. 104043 (2017). arXiv: 1706.08480.

[57] L. Bernard, L. Blanchet, G. Faye & T. Marchand; "Center-of-Mass Equations of Motion and Conserved Integrals of Compact Binary Systems at the Fourth Post-Newtonian Order"; Phys. Rev. D **97**, p. 044037 (2018). arXiv: 1711.00283.

[58] G. Bertone, D. Hooper & J. Silk; "Particle dark matter: Evidence, candidates and constraints"; Phys. Rept. **405**, pp. 279–390 (2005). arXiv: hep-ph/0404175.

[59] N. Bevis, M. Hindmarsh, M. Kunz & J. Urrestilla; "CMB polarization power spectra contributions from a network of cosmic strings"; Phys. Rev. D **76**, p. 043005 (2007). arXiv: 0704.3800.

[60] P. Binetruy, A. Bohe, T. Hertog & D. A. Steer; "Gravitational wave signatures from kink proliferation on cosmic (super-) strings"; Phys. Rev. D **82**, p. 126007 (2010). arXiv: 1009.2484.

[61] D. Bini & T. Damour; "Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation"; Phys. Rev. D **87**, p. 121501 (2013). arXiv: 1305.4884.

[62] D. Bini & T. Damour; "High-order post-Newtonian contributions to the two-body gravitational interaction potential from analytical gravitational self-force calculations"; Phys. Rev. D **89**, p. 064063 (2014). arXiv: 1312.2503.

[63] D. Bini, T. Damour & A. Geralico; "Novel approach to binary dynamics: application to the fifth post-Newtonian level"; Phys. Rev. Lett. **123**, p. 231104 (2019). arXiv: 1909.02375.

[64] D. Bini, T. Damour, A. Geralico, S. Laporta & P. Mastrolia; "Gravitational scattering at the seventh order in G: nonlocal contribution at the sixth post-Newtonian accuracy"; Phys. Rev. D **103**, p. 044038 (2021). arXiv: 2012.12918.

[65] G. D. Birkhoff; *Relativity and Modern Physics* (Harvard University Press) (1923).

[66] N. T. Bishop & L. Rezzolla; "Extraction of Gravitational Waves in Numerical Relativity"; Living Rev. Rel. **19**, p. 2 (2016). arXiv: 1606.02532.

[67] L. Blanchet; "Radiative Gravitation Fields in General Relativity II. Asymptotic Behaviour at Future Null Infinity"; Proc. of the Roy. Soc. of London Series A **409**, pp. 383–399 (1987).

[68] L. Blanchet & G. Schäfer; "Higher order gravitational radiation losses in binary systems"; Mon. Not. Roy. Astron. Soc. **239**, p. 845 (1989).

[69] L. Blanchet; "Second post-Newtonian generation of gravitational radiation"; Phys. Rev. D **51**, pp. 2559–2583 (1995). arXiv: gr-qc/9501030.

[70] L. Blanchet; "Gravitational wave tails of tails"; Class. Quant. Grav. **15**, pp. 113–141 (1998). [Erratum: Class.Quant.Grav. 22, 3381 (2005)]; arXiv: gr-qc/9710038.

[71] L. Blanchet; "On the multipole expansion of the gravitational field"; Class. Quant. Grav. **15**, pp. 1971–1999 (1998). arXiv: gr-qc/9801101.

[72] L. Blanchet; "Gravitational polarization and the phenomenology of MOND"; Class. Quant. Grav. **24**, pp. 3529–3540 (2007). arXiv: astro-ph/0605637.

[73] L. Blanchet; "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries"; Living Rev. Rel. **17**, p. 2 (2014). arXiv: 1310.1528.

[74] L. Blanchet & T. Damour; "Tail-transported temporal correlations in the dynamics of a gravitating system"; Phys. Rev. D **37**, pp. 1410–1435 (1988).

[75] L. Blanchet & T. Damour; "Hereditary effects in gravitational radiation"; Phys. Rev. D **46**, pp. 4304–4319 (1992).

[76] L. Blanchet & G. Faye; "General relativistic dynamics of compact binaries at the third post-Newtonian order"; Phys. Rev. D **63**, p. 062005 (2001). arXiv: gr-qc/0007051.

[77] L. Blanchet & G. Faye; "Flux-balance equations for linear momentum and center-of-mass position of self-gravitating post-Newtonian systems"; Class. Quant. Grav. **36**, p. 085003 (2019). arXiv: 1811.08966.

[78] L. Blanchet & L. Heisenberg; "Dipolar Dark Matter as an Effective Field Theory"; Phys. Rev. D **96**, p. 083512 (2017). arXiv: 1701.07747.

[79] L. Blanchet & B. R. Iyer; "Third post-Newtonian dynamics of compact binaries: Equations of motion in the center-of-mass frame"; Class. Quant. Grav. **20**, p. 755 (2003). arXiv: gr-qc/0209089.

[80] L. Blanchet & F. Larrouturou; unpublished computations.

[81] L. Blanchet & A. Le Tiec; "Model of Dark Matter and Dark Energy Based on Gravitational Polarization"; Phys. Rev. D **78**, p. 024031 (2008). arXiv: 0804.3518.

[82] L. Blanchet & A. Le Tiec; "First Law of Compact Binary Mechanics with Gravitational-Wave Tails"; Class. Quant. Grav. **34**, p. 164001 (2017). arXiv: 1702.06839.

[83] L. Blanchet & J. Novak; "External field effect of modified Newtonian dynamics in the Solar system"; Mon. Not. Roy. Astron. Soc. **412**, pp. 2530–2542 (2011). arXiv: 1010.1349.

[84] L. Blanchet, T. Damour & B. R. Iyer; "Gravitational waves from inspiralling compact binaries: Energy loss and wave form to second post-Newtonian order"; Phys. Rev. D **51**, p. 5360 (1995). [Erratum: Phys.Rev.D 54, 1860 (1996)]; arXiv: gr-qc/9501029.

[85] L. Blanchet, G. Faye & B. Ponsot; "Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order"; Phys. Rev. D **58**, p. 124002 (1998). arXiv: gr-qc/9804079.

[86] L. Blanchet, B. R. Iyer & B. Joguet; "Gravitational waves from inspiralling compact binaries: Energy flux to third post-Newtonian order"; Phys. Rev. D **65**, p. 064005 (2002). [Erratum: Phys.Rev.D 71, 129903 (2005)]; arXiv: gr-qc/0105098.

[87] L. Blanchet, T. Damour & G. Esposito-Farese; "Dimensional regularization of the third post-Newtonian dynamics of point particles in harmonic coordinates"; Phys. Rev. D **69**, p. 124007 (2004). arXiv: gr-qc/0311052.

[88] L. Blanchet, T. Damour, G. Esposito-Farese & B. R. Iyer; "Gravitational radiation from inspiralling compact binaries completed at the third post-Newtonian order"; Phys. Rev. Lett. **93**, p. 091101 (2004). arXiv: gr-qc/0406012.

[89] L. Blanchet, G. Faye, B. R. Iyer & S. Sinha; "The Third post-Newtonian gravitational wave polarisations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits"; Class. Quant. Grav. **25**, p. 165003 (2008). [Erratum: Class.Quant.Grav. 29, 239501 (2012)]; arXiv: 0802.1249.

[90] L. Blanchet, S. L. Detweiler, A. Le Tiec & B. F. Whiting; "Post-Newtonian and Numerical Calculations of the Gravitational Self-Force for Circular Orbits in the Schwarzschild Geometry"; Phys. Rev. D **81**, p. 064004 (2010). arXiv: 0910.0207.

[91] L. Blanchet, S. L. Detweiler, A. Le Tiec & B. F. Whiting; "High-Order Post-Newtonian Fit of the Gravitational Self-Force for Circular Orbits in the Schwarzschild Geometry"; Phys. Rev. D **81**, p. 084033 (2010). arXiv: 1002.0726.

[92] L. Blanchet, A. Buonanno & A. Le Tiec; "First law of mechanics for black hole binaries with spins"; Phys. Rev. D **87**, p. 024030 (2013). arXiv: 1211.1060.

[93] L. Blanchet, G. Faye & B. F. Whiting; "Half-integral conservative post-Newtonian approximations in the redshift factor of black hole binaries"; Phys. Rev. D **89**, p. 064026 (2014). arXiv: 1312.2975.

[94] L. Blanchet, G. Faye & B. F. Whiting; "High-order half-integral conservative post-Newtonian coefficients in the redshift factor of black hole binaries"; Phys. Rev. D **90**, p. 044017 (2014). arXiv: 1405.5151.

[95] L. Blanchet, G. Hébrard & F. Larrouturou; "Detecting the General Relativistic Orbital Precession of the Exoplanet HD 80606b"; Astron. Astrophys. **628**, p. A80 (2019). arXiv: 1905.06630.

[96] L. Blanchet, S. Foffa, F. Larrouturou & R. Sturani; "Logarithmic tail contributions to the energy function of circular compact binaries"; Phys. Rev. D **101**, p. 084045 (2020). arXiv: 1912.12359.

[97] J. J. Blanco-Pillado, K. D. Olum & J. M. Wachter; "Comparison of cosmic string and superstring models to NANOGrav 12.5-year results"; (2021)arXiv: 2102.08194.

[98] J. Blümlein, A. Maier, P. Marquard & G. Schäfer; "Fourth post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach"; Nucl. Phys. B **955**, p. 115041 (2020). arXiv: 2003.01692.

[99] E. Bogomolny; "Stability of Classical Solutions"; Sov. J. Nucl. Phys. **24**, p. 449 (1976).

[100] N. Bolis, A. De Felice & S. Mukohyama; "Integrated Sachs-Wolfe-galaxy cross-correlation bounds on the two branches of the minimal theory of massive gravity"; Phys. Rev. D **98** (2018). arXiv: 1804.01790.

[101] H. Bondi, M. G. J. van der Burg & A. W. K. Metzner; "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems"; Proc. of the Roy. Soc. of London Series A **269**, pp. 21–52 (1962).

[102] F. Bonjour, C. Charmousis & R. Gregory; "Thick domain wall universes"; Class. Quant. Grav. **16**, pp. 2427–2445 (1999). arXiv: gr-qc/9902081.

[103] W. B. Bonnor & M. A. Rotenberg; "Gravitational waves from isolated sources"; Proc. R. Soc. Lond. A **289**, pp. 247–274 (1966).

[104] Bordas, sous la direction de R. Caratini; *Encyclopédie – 2 – Astronomie* (Éditions Bordas) (1968).

[105] A. Bosma; "21-cm line studies of spiral galaxies. II. The distribution and kinematics of neutral hydrogen in spiral galaxies of various morphological types." Astron. J. **86**, pp. 1825–1846 (1981).

[106] G. Boué, P. Figueira, A. C. M. Correia & N. C. Santos; "Orbital migration induced by anisotropic evaporation. Can hot Jupiters form hot Neptunes?" Astron. Astrophys. **537**, p. L3 (2012). arXiv: 1109.2805.

[107] D. G. Boulware & S. Deser; "Can Gravitation Have a Finite Range?" Phys. Rev. D **6**, pp. 3368–3382 (1972).

[108] A. Bourgoin, A. Hees, S. Bouquillon, C. Le Poncin-Lafitte, G. Francou & M. C. Angonin; "Testing Lorentz symmetry with Lunar Laser Ranging"; Phys. Rev. Lett. **117**, p. 241301 (2016). arXiv: 1607.00294.

[109] J. Boussinesq; "Théorie de l'intumescence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire"; C. R. Acad. Sci. (Paris) **72**, pp. 755–759 (1871)`https://gallica.bnf.fr/ark:/12148/bpt6k3029x/f759`.

[110] C. Brans & R. H. Dicke; "Mach's Principle and a Relativistic Theory of Gravitation"; Phys. Rev. **124**, pp. 925–935 (1961).

[111] D. Brouwer & G. Clemence; *Methods of celestial mechanics* (Academic Press (New York)) (1961).

[112] J. D. Brown; "Action functionals for relativistic perfect fluids"; Class. Quant. Grav. **10**, pp. 1579–1606 (1993). arXiv: gr-qc/9304026.

[113] J.-P. Bruneton & G. Esposito-Farese; "Field-theoretical formulations of MOND-like gravity"; Phys. Rev. D **76**, p. 124012 (2007). [Erratum: Phys.Rev.D 76, 129902 (2007)]; arXiv: 0705.4043.

[114] H. A. Buchdahl; "Non-Linear Lagrangians and Cosmological Theory"; Monthly Notices of the Royal Astronomical Society **150**, pp. 1–8 (1970).

[115] A. Buonanno & T. Damour; "Effective one-body approach to general relativistic two-body dynamics"; Phys. Rev. D **59**, p. 084006 (1999). arXiv: gr-qc/9811091.

[116] M. E. Cahill & G. C. McVittie; "Spherical Symmetry and Mass-Energy in General Relativity"; Journal of Mathematical Physics **11**, p. 1392 (1970).

[117] M. Campanelli, C. O. Lousto, P. Marronetti & Y. Zlochower; "Accurate evolutions of orbiting black-hole binaries without excision"; Phys. Rev. Lett. **96**, p. 111101 (2006). arXiv: gr-qc/0511048.

[118] S. Capozziello, M. De laurentis & A. Stabile; “Axially symmetric solutions in f(R)-gravity”; Class. Quant. Grav. **27**, p. 165008 (2010). arXiv: 0912.5286.

[119] R. Carballo-Rubio, F. D. Filippo & S. Liberati; “Minimally modified theories of gravity: a playground for testing the uniqueness of general relativity”; Journal of Cosmology and Astroparticle Physics **2018**, p. 026 (2018). arXiv: 1802.02537.

[120] B. Carter & R. Gregory; “Curvature corrections to dynamics of domain walls”; Phys. Rev. D **51**, pp. 5839–5846 (1995). arXiv: hep-th/9410095.

[121] C. Charmousis, E. J. Copeland, A. Padilla & P. M. Saffin; “General second order scalar-tensor theory, self tuning, and the Fab Four”; Phys. Rev. Lett. **108**, p. 051101 (2012). arXiv: 1106.2000.

[122] C. Cheung & M. P. Solon; “Tidal Effects in the Post-Minkowskian Expansion”; Phys. Rev. Lett. **125**, p. 191601 (2020). arXiv: 2006.06665.

[123] G. Cho, B. Pardo & R. A. Porto; “Gravitational radiation from inspiralling compact objects: Spin-spin effects completed at the next-to-leading post-Newtonian order”; (2021)arXiv: 2103.14612.

[124] Y. Choquet-Bruhat & R. Geroch; “Global aspects of the Cauchy problem in general relativity”; Comm. in Math. Phys. **14**, pp. 329–335 (1969).

[125] K. D. Colón, E. B. Ford, S. Redfield, J. J. Fortney, M. Shabram, H. J. Deeg & S. Mahadevan; “Probing potassium in the atmosphere of HD 80606b with tunable filter transit spectrophotometry from the Gran Telescopio Canarias”; Mon. Not. Roy. Astr. Soc. **419**, pp. 2233–2250 (2012). arXiv: 1008.4800.

[126] D. Comelli, M. Crisostomi, F. Nesti & L. Pilo; “FRW Cosmology in Ghost Free Massive Gravity”; JHEP **03**, p. 067 (2012). [Erratum: JHEP 06, 020 (2012)]; arXiv: 1111.1983.

[127] D. Comelli, M. Crisostomi & L. Pilo; “Perturbations in Massive Gravity Cosmology”; JHEP **06**, p. 085 (2012). arXiv: 1202.1986.

[128] N. Copernicus; *De revolutionibus orbium cœlestium* (J. Petreius ed. (Nürnberg)) (1530).

[129] R. Courant & D. Hilbert; *Methods of Mathematical Physics* (Interscience (New York)) (1953).

[130] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava & A. J. Tolley; “Massive Cosmologies”; Phys. Rev. D **84**, p. 124046 (2011). arXiv: 1108.5231.

[131] T. Damour & N. Deruelle; “General relativistic celestial mechanics of binary systems. I. The post-Newtonian motion”; Annales Inst. H. Poincaré Phys. Théor. **43**, pp. 107–132 (1985)`http://www.numdam.org/article/AIHPA_1985__43_1_107_0.pdf`.

[132] T. Damour; “Gravitational Self Force in a Schwarzschild Background and the Effective One Body Formalism”; Phys. Rev. D **81**, p. 024017 (2010). arXiv: 0910.5533.

[133] T. Damour & G. Esposito-Farèse; “Testing for preferred frame effects in gravity with artificial earth satellites”; Phys. Rev. D **49**, pp. 1693–1706 (1994). arXiv: gr-qc/9311034.

[134] T. Damour, P. Jaranowski & G. Schäfer; “Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems”; Phys. Rev. D **89**, p. 064058 (2014). arXiv: 1401.4548.

[135] T. Damour, P. Jaranowski & G. Schäfer; "Fourth post-Newtonian effective one-body dynamics"; Phys. Rev. D **91**, p. 084024 (2015). arXiv: 1502.07245.

[136] T. Damour, P. Jaranowski & G. Schäfer; "Conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity"; Phys. Rev. D **93**, p. 084014 (2016). arXiv: 1601.01283.

[137] A. De Felice & S. Mukohyama; "Minimal theory of massive gravity"; Physics Letters B **752**, p. 302 (2016). arXiv: 1506.01594.

[138] A. De Felice & S. Mukohyama; "Phenomenology in minimal theory of massive gravity"; JCAP **04**, p. 028 (2016). arXiv: 1512.04008.

[139] A. De Felice & S. Mukohyama; "Graviton mass might reduce tension between early and late time cosmological data"; Phys. Rev. Lett. **118**, p. 091104 (2017). arXiv: 1607.03368.

[140] A. De Felice & S. Mukohyama; "Weakening gravity for dark matter in a type-II minimally modified gravity"; (2020); arXiv: 2011.04188.

[141] A. De Felice & S. Tsujikawa; "f(R) theories"; Living Rev. Rel. **13**, p. 3 (2010). arXiv: 1002.4928.

[142] A. De Felice, A. E. Gümrükçüoğlu & S. Mukohyama; "Massive gravity: nonlinear instability of the homogeneous and isotropic universe"; Phys. Rev. Lett. **109**, p. 171101 (2012). arXiv: 1206.2080.

[143] A. De Felice, A. E. Gümrükçüoğlu, S. Mukohyama, N. Tanahashi & T. Tanaka; "Viable cosmology in bimetric theory"; JCAP **06**, p. 037 (2014). arXiv: 1404.0008.

[144] A. De Felice, T. Nakamura & T. Tanaka; "Possible existence of viable models of bi-gravity with detectable graviton oscillations by gravitational wave detectors"; PTEP **2014**, p. 043E01 (2014). arXiv: 1304.3920.

[145] A. De Felice, A. E. Gümrükçüoğlu, L. Heisenberg & S. Mukohyama; "Matter coupling in partially constrained vielbein formulation of massive gravity"; JCAP **01**, p. 003 (2016). arXiv: 1509.05978.

[146] A. De Felice, S. Mukohyama & M. Oliosi; "Minimal theory of quasidilaton massive gravity"; Phys. Rev. D **96** (2017). arXiv: 1701.01581.

[147] A. De Felice, S. Mukohyama & M. Oliosi; "Horndeski extension of the minimal theory of quasidilaton massive gravity"; Phys. Rev. D **96** (2017). arXiv: 1709.03108.

[148] A. De Felice, F. Larrouturou, S. Mukohyama & M. Oliosi; "Black holes and stars in the minimal theory of massive gravity"; Phys. Rev. D **98**, p. 104031 (2018). arXiv: 1808.01403.

[149] A. De Felice, S. Mukohyama, M. Oliosi & Y. Watanabe; "Stable cosmology in chameleon bigravity"; Phys. Rev. D **97**, p. 024050 (2018). arXiv: 1711.04655.

[150] A. De Felice, S. Mukohyama & J.-P. Uzan; "Extending applicability of bimetric theory: chameleon bigravity"; Gen. Rel. Grav. **50**, p. 21 (2018). arXiv: 1702.04490.

[151] A. De Felice, F. Larrouturou, S. Mukohyama & M. Oliosi; "On the absence of conformally flat slicings of the Kerr spacetime"; Phys. Rev. D **100**, p. 124044 (2019). arXiv: 1908.03456.

[152] A. De Felice, S. Mukohyama & M. Oliosi; "Phenomenology of minimal theory of quasidilaton massive gravity"; Physical Review D **99** (2019). arXiv: 1806.00602.

[153] A. De Felice, A. Doll & S. Mukohyama; "A theory of type-II minimally modified gravity"; JCAP **09**, p. 034 (2020). arXiv: 2004.12549.

[154] A. De Felice, A. Doll, F. Larrouturou & S. Mukohyama; "Black holes in a type-II minimally modified gravity"; JCAP **03**, p. 004 (2021). arXiv: 2010.13067.

[155] A. De Felice, F. Larrouturou, S. Mukohyama & M. Oliosi; "Minimal Theory of Bigravity: construction and cosmology"; JCAP **04**, p. 015 (2021). arXiv: 2012.01073.

[156] A. De Felice, S. Mukohyama & M. C. Pookkillath; "Addressing H_0 tension by means of VCDM"; Phys. Lett. B **816**, p. 136201 (2021). arXiv: 2009.08718.

[157] P.-S. de Laplace; *Exposition du système du monde, vol. 2* (Imprimerie du Cercle-Social) (An IV de la République française).

[158] C. de Rham; "Massive Gravity"; Living Rev. Rel. **17**, p. 7 (2014). arXiv: 1401.4173.

[159] C. de Rham & V. Pozsgay; "New class of Proca interactions"; Phys. Rev. D **102**, p. 083508 (2020). arXiv: 2003.13773.

[160] C. de Rham, G. Gabadadze & A. J. Tolley; "Resummation of Massive Gravity"; Phys. Rev. Lett. **106**, p. 231101 (2011). arXiv: 1011.1232.

[161] C. de Rham, L. Heisenberg & R. H. Ribeiro; "Ghosts and matter couplings in massive gravity, bigravity and multigravity"; Phys. Rev. D **90**, p. 124042 (2014). arXiv: 1409.3834.

[162] C. de Rham, L. Heisenberg & R. H. Ribeiro; "On couplings to matter in massive (bi-)gravity"; Class. Quant. Grav. **32**, p. 035022 (2015). arXiv: 1408.1678.

[163] C. Deffayet, G. Esposito-Farese & A. Vikman; "Covariant Galileon"; Phys. Rev. D **79**, p. 084003 (2009). arXiv: 0901.1314.

[164] C. Deffayet, X. Gao, D. A. Steer & G. Zahariade; "From k-essence to generalised Galileons"; Phys. Rev. D **84**, p. 064039 (2011). arXiv: 1103.3260.

[165] C. Deffayet & F. Larrouturou; "Domain walls without a potential"; Phys. Rev. D **103**, p. 036010 (2021). arXiv: 2009.00404.

[166] C. Deffayet & D. A. Steer; "A formal introduction to Horndeski and Galileon theories and their generalizations"; Class. Quant. Grav. **30**, p. 214006 (2013). arXiv: 1307.2450.

[167] C. Deffayet, G. Esposito-Farese & R. P. Woodard; "Field equations and cosmology for a class of nonlocal metric models of MOND"; Phys. Rev. D **90**, p. 064038 (2014). [Addendum: Phys.Rev.D 90, 089901 (2014)]; arXiv: 1405.0393.

[168] S. L. Detweiler; "A Consequence of the gravitational self-force for circular orbits of the Schwarzschild geometry"; Phys. Rev. D **77**, p. 124026 (2008). arXiv: 0804.3529.

[169] H. Ding, A. T. Deller, P. Freire, D. L. Kaplan, T. J. W. Lazio, R. Shannon & B. Stappers; "Very long baseline astrometry of PSR J1012+5307 and its implications on alternative theories of gravity"; Astrophys. J. **896**, p. 85 (2020). [Erratum: Astrophys.J. 900, 89 (2020)]; arXiv: 2004.14668.

[170] P. A. M. Dirac; *Lectures on Quantum Mechanics*; Belfer Graduate School of Science, monograph series (Dover Publications) (2001); ISBN 9780486417134.

[171] G. R. Dvali, G. Gabadadze & M. Porrati; "4-D gravity on a brane in 5-D Minkowski space"; Phys. Lett. B **485**, pp. 208–214 (2000). arXiv: hep-th/0005016.

[172] F. W. Dyson, A. S. Eddington & C. Davidson; "A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919"; Philosophical Transactions of the Royal Society of London Series A **220**, pp. 291–333 (1920).

[173] B. Edwards, L. Mugnai, G. Tinetti, E. Pascale & S. Sarkar; "An Updated Study of Potential Targets for Ariel"; Astronom. J. **157**, p. 242 (2019). arXiv: 1905.04959.

[174] A. Einstein; "Erklärung der Perihelbewegung des Merkur"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 831–839 (1915)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[175] A. Einstein; "Die Feldgleichungen der Gravitation"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 844–847 (1915)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[176] A. Einstein; "Anhang. Darstellung der Theorie ausgehend von einem Variationsprinzip"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 340–345 (1916)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[177] A. Einstein; "Näherungsweise Integration der Feldgleichungen der Gravitation"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 688–696 (1916)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[178] A. Einstein; "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 142–152 (1917)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[179] A. Einstein; "Über Gravitationswellen"; Sitz. der Kön. Preu. Ak. der Wiss. pp. 154–167 (1918)`https://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/sitzungsberichte`.

[180] A. Einstein, L. Infeld & B. Hoffmann; "The Gravitational Equations and the Problem of Motion"; Annals of Mathematics **39**, pp. 65–100 (1938).

[181] K. Eppley; *The numerical evolution of the collision of two black holes*; Ph.D. thesis; Princeton University (1975).

[182] G. Esposito-Farese; "Comparing solar system, binary pulsar, and gravitational wave tests of gravity"; in "34th Rencontres de Moriond: Gravitational Waves and Experimental Gravity," pp. 475–480 (1999); arXiv: gr-qc/9903058.

[183] D. Fabrycky; "Non-Keplerian dynamics of exoplanets"; Exoplanets, ed. S. Seager p. 217 (2010).

[184] A. Falkowski & G. Isabella; "Matter coupling in massive gravity"; JHEP **04**, p. 014 (2020). arXiv: 2001.06800.

[185] B. Famaey & S. McGaugh; "Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions"; Living Rev. Rel. **15**, p. 10 (2012). arXiv: 1112.3960.

[186] V. Faraoni; "Evolving black hole horizons in General Relativity and alternative gravity"; Galaxies **1**, pp. 114–179 (2013). arXiv: 1309.4915.

[187] G. Faye, S. Marsat, L. Blanchet & B. R. Iyer; "The third and a half post-Newtonian gravitational wave quadrupole mode for quasi-circular inspiralling compact binaries"; Class. Quant. Grav. **29**, p. 175004 (2012). arXiv: 1204.1043.

[188] G. Faye, L. Blanchet & B. R. Iyer; "Non-linear multipole interactions and gravitational-wave octupole modes for inspiralling compact binaries to third-and-a-half post-Newtonian order"; Class. Quant. Grav. **32**, p. 045016 (2015). arXiv: 1409.3546.

[189] M. Fierz & W. Pauli; "On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field"; Proceedings of the Royal Society of London Series A **173**, pp. 211–232 (1939).

[190] P. Figueira *et al.*; "Is the activity level of HD 80606 influenced by its eccentric planet?" Astron. Astrophys. **592**, p. A143 (2016). arXiv: 1606.05549.

[191] D. G. Figueroa, M. Hindmarsh, J. Lizarraga & J. Urrestilla; "Irreducible background of gravitational waves from a cosmic defect network: update and comparison of numerical techniques"; Phys. Rev. D **102**, p. 103516 (2020). arXiv: 2007.03337.

[192] V. Fock; *Theory of space, time and gravitation* (Pergamon (London)) (1959).

[193] S. Foffa & R. Sturani; "Effective field theory methods to model compact binaries"; Class. Quant. Grav. **31**, p. 043001 (2014). arXiv: 1309.3474.

[194] S. Foffa & R. Sturani; "Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian"; Phys. Rev. D **100**, p. 024047 (2019). arXiv: 1903.05113.

[195] S. Foffa & R. Sturani; "Hereditary terms at next-to-leading order in two-body gravitational dynamics"; Phys. Rev. D **101**, p. 064033 (2020). arXiv: 1907.02869.

[196] S. Foffa, R. A. Porto, I. Rothstein & R. Sturani; "Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian"; Phys. Rev. D **100**, p. 024048 (2019). arXiv: 1903.05118.

[197] E. B. Ford, M. Havlickova & F. A. Rasio; "Dynamical instabilities in extrasolar planetary systems containing two giant planets"; Icarus **150**, pp. 303–313 (2001). arXiv: astro-ph/0010178.

[198] S. J. Fossey, I. P. Waldmann & D. M. Kipping; "Detection of a transit by the planetary companion of HD 80606"; Mon. Not. Roy. Astr. Soc. **396**, p. L16 (2009). arXiv: 0902.4616.

[199] J. L. Friedman, K. Uryu & M. Shibata; "Thermodynamics of binary black holes and neutron stars"; Phys. Rev. D **65**, p. 064035 (2002). [Erratum: Phys.Rev.D 70, 129904 (2004)]; arXiv: gr-qc/0108070.

[200] R. Fujita; "Gravitational Waves from a Particle in Circular Orbits around a Schwarzschild Black Hole to the 22nd Post-Newtonian Order"; Prog. Theor. Phys. **128**, pp. 971–992 (2012). arXiv: 1211.5535.

[201] T. Fujita, S. Kuroyanagi, S. Mizuno & S. Mukohyama; "Blue-tilted Primordial Gravitational Waves from Massive Gravity"; Phys. Lett. B **789**, pp. 215–219 (2019). arXiv: 1808.02381.

[202] Y. Fukuda *et al.*; "Evidence for oscillation of atmospheric neutrinos"; Phys. Rev. Lett. **81**, pp. 1562–1567 (1998). arXiv: hep-ex/9807003.

[203] A. Garat & R. H. Price; "Nonexistence of conformally flat slices of the Kerr space-time"; Phys. Rev. D **61**, p. 124011 (2000). arXiv: gr-qc/0002013.

[204] E. Garcia-Melendo & P. R. McCullough; "Photometric Detection of a Transit of HD 80606b"; Astrophys. J. **698**, p. 558 (2009). arXiv: 0902.4493.

[205] I. M. Gel'fand & G. E. Shilov; *Generalized functions* (Academic Press (New York)) (1964).

[206] P. Giacobbe *et al.*; "Five carbon- and nitrogen-bearing species in a hot giant planet's atmosphere"; Nature **592**, pp. 205–208 (2021).

[207] A. Gimenez; "General-relativistic periastron advances in eclipsing binary systems"; Astrophys. J. **405**, p. 167 (1985)`http://adsabs.harvard.edu/pdf/1985ApJ...297..405G`.

[208] J. Gleyzes, D. Langlois, F. Piazza & F. Vernizzi; "Healthy theories beyond Horndeski"; Phys. Rev. Lett. **114**, p. 211101 (2015). arXiv: 1404.6495.

[209] W. D. Goldberger & A. Ross; "Gravitational radiative corrections from effective field theory"; Phys. Rev. D **81**, p. 124015 (2010). arXiv: 0912.4254.

[210] W. D. Goldberger & I. Z. Rothstein; "An Effective field theory of gravity for extended objects"; Phys. Rev. D **73**, p. 104029 (2006). arXiv: hep-th/0409156.

[211] W. D. Goldberger, A. Ross & I. Z. Rothstein; "Black hole mass dynamics and renormalization group evolution"; Phys. Rev. D **89**, p. 124033 (2014). arXiv: 1211.6095.

[212] P. Goldreich & R. Sari; "Eccentricity evolution for planets in gaseous disks"; Astrophys. J. **585**, pp. 1024–1037 (2003). arXiv: astro-ph/0202462.

[213] E. Gourgoulhon; "3+1 formalism and bases of numerical relativity"; (2007)arXiv: gr-qc/0703035.

[214] I. S. Gradshteyn & I. M. Ryzhik; *Table of Integrals, Series, and Products* (Academic Press) (2000); 6th edition.

[215] A. M. Green & B. J. Kavanagh; "Primordial Black Holes as a dark matter candidate"; J. Phys. G **48**, p. 043001 (2021). arXiv: 2007.10722.

[216] A. Gullstrand; "Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie"; Ark. Mat. Astron. Fys. **16**, p. 8 (1922).

[217] A. E. Gümrükçüoğlu, C. Lin & S. Mukohyama; "Open FRW universes and self-acceleration from nonlinear massive gravity"; JCAP **11**, p. 030 (2011). arXiv: 1109.3845.

[218] A. E. Gümrükçüoğlu, L. Heisenberg, S. Mukohyama & N. Tanahashi; "Cosmology in bimetric theory with an effective composite coupling to matter"; JCAP **04**, p. 008 (2015). arXiv: 1501.02790.

[219] R. Hagala, A. De Felice, D. F. Mota & S. Mukohyama; "Nonlinear Dynamics of the Minimal Theory of Massive Gravity"; (2020); arXiv: 2011.14697.

[220] J. Hartung, J. Steinhoff & G. Schafer; "Next-to-next-to-leading order post-Newtonian linear-in-spin binary Hamiltonians"; Annalen Phys. **525**, pp. 359–394 (2013). arXiv: 1302.6723.

[221] S. F. HASSAN & R. A. ROSEN; "Bimetric Gravity from Ghost-free Massive Gravity"; JHEP **02**, p. 126 (2012). arXiv: 1109.3515.

[222] M. HAZUMI *et al.*; "LiteBIRD: JAXA's new strategic L-class mission for all-sky surveys of cosmic microwave background polarization"; Proc. SPIE Int. Soc. Opt. Eng. **11443**, p. 114432F (2020). arXiv: 2101.12449.

[223] G. HÉBRARD *et al.*; "Observation of the full 12-hour-long transit of the exoplanet HD80606b. Warm-Spitzer photometry and SOPHIE spectroscopy"; Astron. Astrophys. **516**, p. A95 (2011). arXiv: 1004.0790.

[224] R. W. HELLINGS & K. NORDTVEDT; "Vector-Metric Theory of Gravity"; Phys. Rev. D **7**, pp. 3593–3602 (1973).

[225] Q. HENRY, G. FAYE & L. BLANCHET; "Tidal effects in the equations of motion of compact binary systems to next-to-next-to-leading post-Newtonian order"; Phys. Rev. D **101**, p. 064047 (2020). arXiv: 1912.01920.

[226] Q. HENRY, G. FAYE & L. BLANCHET; "Tidal effects in the gravitational-wave phase evolution of compact binary systems to next-to-next-to-leading post-Newtonian order"; Phys. Rev. D **102**, p. 044033 (2020). arXiv: 2005.13367.

[227] Q. HENRY, G. FAYE & L. BLANCHET; "Hamiltonian for tidal interactions in compact binary systems to next-to-next-to-leading post-Newtonian order"; Phys. Rev. D **102**, p. 124074 (2020). arXiv: 2009.12332.

[228] Q. HENRY, L. BLANCHET & G. FAYE; "The current-type quadrupole moment and gravitational-wave mode $(\ell, m) = (2, 1)$ of compact binary systems at the third post-Newtonian order"; (2021)arXiv: 2105.10876.

[229] J. S. HEYL & B. J. GLADMAN; "Using long-term transit timing to detect terrestrial planets"; Mon. Not. Roy. Astr. Soc. **377**, pp. 1511–1519 (2007). arXiv: astro-ph/0610267.

[230] M. G. HIDAS *et al.*; "An ingress and a complete transit of HD80606 b"; Mon. Not. Roy. Astr. Soc. **406**, p. 1146 (2010). arXiv: 1002.1052.

[231] A. HIGUCHI; "Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time"; Nucl. Phys. B **282**, pp. 397–436 (1987).

[232] M. B. HINDMARSH, M. LÜBEN, J. LUMMA & M. PAULY; "Phase transitions in the early universe"; (2020). arXiv: 2008.09136.

[233] G. HINSHAW *et al.*; "Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results"; A. P. J. S. **208**, 19 (2013). arXiv: 1212.5226.

[234] HOMÈRE; *L'Odyssée, Chant II* (La Découverte) (1982); trad. by P. Jaccottet.

[235] S. HOPPER, C. KAVANAGH & A. C. OTTEWILL; "Analytic self-force calculations in the post-Newtonian regime: eccentric orbits on a Schwarzschild background"; Phys. Rev. D **93**, p. 044010 (2016). arXiv: 1512.01556.

[236] G. W. HORNDESKI; "Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space"; International Journal of Theoretical Physics **10**, pp. 363–384 (1974).

[237] P. HOŘAVA; "Quantum Gravity at a Lifshitz Point"; Phys. Rev. D **79**, p. 084008 (2009). arXiv: 0901.3775.

[238] R. A. Hulse & J. H. Taylor; "Discovery of a pulsar in a binary system." Astrophys. J. **195**, pp. L51–L53 (1975).

[239] L. Iorio; "Classical and relativistic long-term time variations of some observables for transiting exoplanets"; Mon. Not. Roy. Astr. Soc. **411**, p. 167 (2011). arXiv: 1007.2780.

[240] W. Israel; "Thin shells in general relativity"; Nuovo Cimento **44B**, p. 1 (1966).

[241] P. Jaranowski & G. Schaefer; "Third post-Newtonian higher order ADM Hamilton dynamics for two-body point mass systems"; Phys. Rev. D **57**, pp. 7274–7291 (1998). [Erratum: Phys.Rev.D 63, 029902 (2001)]; arXiv: gr-qc/9712075.

[242] J. T. Jebsen; "Über die allgemeinen Kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum"; Arkiv for Mat. Astr. och Fys. **15**, p. 18 (1921).

[243] A. Jordàn & G. Bakos; "Observability of the general relativistic precession of periastra in exoplanets"; Astrophys. J. **685**, p. 543 (2008). arXiv: 0806.0630.

[244] P. Jordan; *Schwerkraft und Weltall: Grundlagen der theoretischen Kosmologie.* (Braunschweig) (1952).

[245] G. Kälin & R. A. Porto; "From Boundary Data to Bound States"; JHEP **01**, p. 072 (2020). arXiv: 1910.03008.

[246] G. Kälin & R. A. Porto; "From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)"; JHEP **02**, p. 120 (2020). arXiv: 1911.09130.

[247] G. Kälin & R. A. Porto; "Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics"; JHEP **11**, p. 106 (2020). arXiv: 2006.01184.

[248] G. Kälin, Z. Liu & R. A. Porto; "Conservative Tidal Effects in Compact Binary Systems to Next-to-Leading Post-Minkowskian Order"; Phys. Rev. D **102**, p. 124025 (2020). arXiv: 2008.06047.

[249] G. Kälin, Z. Liu & R. A. Porto; "Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach"; Phys. Rev. Lett. **125**, p. 261103 (2020). arXiv: 2007.04977.

[250] J. H. Kang *et al.*; "2017 upgrade and performance of BICEP3: a 95GHz refracting telescope for degree-scale CMB polarization"; Proc. SPIE Int. Soc. Opt. Eng. **10708**, p. 107082N (2018). arXiv: 1808.00567.

[251] C. Kavanagh, A. C. Ottewill & B. Wardell; "Analytical high-order post-Newtonian expansions for extreme mass ratio binaries"; Phys. Rev. D **92**, p. 084025 (2015). arXiv: 1503.02334.

[252] R. P. Kerr; "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics"; Phys. Rev. Lett. **11**, pp. 237–238 (1963).

[253] J. Khoury & A. Weltman; "Chameleon fields: Awaiting surprises for tests of gravity in space"; Phys. Rev. Lett. **93**, p. 171104 (2004). arXiv: astro-ph/0309300.

[254] R. Kimura, T. Tanaka, K. Yamamoto & Y. Yamashita; "Constraint on ghost-free bigravity from gravitational Cherenkov radiation"; Phys. Rev. D **94**, p. 064059 (2016). arXiv: 1605.03405.

[255] F. Könnig & L. Amendola; "Instability in a minimal bimetric gravity model"; Phys. Rev. D **90**, p. 044030 (2014). arXiv: 1402.1988.

[256] Y. Kozai; "Secular perturbations of asteroids with high inclination and eccentricity"; Astronom. J. **67**, pp. 591–598 (1962).

[257] H. Lammer *et al.*; "Determining the mass loss limit for close-in exoplanets: what can we learn from transit observations?" Astron. Astrophys. **506**, pp. 399–410 (2009).

[258] G. Laughlin, D. Deming, J. Langton, D. Kasen, S. Vogt, P. Butler, E. Rivera & S. Meschiari; "Rapid heating of the atmosphere of an extrasolar planet"; Nature **457**, p. 562 (2009).

[259] A. Le Tiec; "First Law of Mechanics for Compact Binaries on Eccentric Orbits"; Phys. Rev. D **92**, p. 084021 (2015). arXiv: 1506.05648.

[260] A. Le Tiec, E. Barausse & A. Buonanno; "Gravitational Self-Force Correction to the Binding Energy of Compact Binary Systems"; Phys. Rev. Lett. **108**, p. 131103 (2012). arXiv: 1111.5609.

[261] A. Le Tiec, L. Blanchet & B. F. Whiting; "The First Law of Binary Black Hole Mechanics in General Relativity and Post-Newtonian Theory"; Phys. Rev. D **85**, p. 064039 (2012). arXiv: 1111.5378.

[262] U. Le Verrier; *Recherches sur les mouvements de la planète Herschel* (Bachelier (Paris)) (1846). `http://hdl.handle.net/1908/3096`.

[263] A. Lecavelier des Étangs; "A diagram for the evaporation status of extrasolar planets"; Astron. Astrophys. **461**, p. 1185 (2007). arXiv: astro-ph/0609744.

[264] A. K. Leibovich, N. T. Maia, I. Z. Rothstein & Z. Yang; "Second post-Newtonian order radiative dynamics of inspiralling compact binaries in the Effective Field Theory approach"; Phys. Rev. D **101**, p. 084058 (2020). arXiv: 1912.12546.

[265] S.-L. Li, L. Shao, P. Wu & H. Yu; "NANOGrav Signal from First-Order Confinement/Deconfinement Phase Transition in Different QCD Matters"; (2021)arXiv: 2101.08012.

[266] M. L. Lidov; "The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies"; Planetary and Space Science **9**, pp. 719–759 (1962).

[267] C. Lin; "The self-consistent matter coupling of a class of minimally modified gravity theories"; Journal of Cosmology and Astroparticle Physics **2019**, p. 037 (2019). arXiv: 1811.02467.

[268] C. Lin & Z. Lalak; "Novel matter coupling in Einstein gravity"; (2019)arXiv: 1911.12026.

[269] C. Lin & S. Mukohyama; "A class of minimally modified gravity theories"; Journal of Cosmology and Astroparticle Physics **2017**, p. 033 (2017). arXiv: 1708.03757.

[270] H. A. Lorentz & J. Droste; "The Motion of a System of Bodies under the Influence of their Mutual Attraction, According to Einstein's Theory"; Collected Papers: Volume V (Springer Netherlands) pp. 330–355 (1917). Trad. of Versl. K. Akad. Wet. Amsterdam. **26**, 392, (1917).

[271] D. Lovelock; "The Einstein Tensor and Its Generalizations"; Journal of Mathematical Physics **12**, pp. 498–501 (1971).

[272] C. Lovis & D. Fischer; "Radial Velocity Techniques for Exoplanets"; Exoplanets, ed. S. Seager pp. 27–53 (2010).

[273] K.-i. Maeda & M. S. Volkov; "Anisotropic universes in the ghost-free bigravity"; Phys. Rev. D **87**, p. 104009 (2013). arXiv: 1302.6198.

[274] M. Maggiore & M. Mancarella; "Nonlocal gravity and dark energy"; Phys. Rev. D **90**, p. 023005 (2014). arXiv: 1402.0448.

[275] N. Manton & P. Sutcliffe; *Topological solitons*; Cambridge Monographs on Mathematical Physics (Cambridge University Press) (2004); ISBN 978-0-511-20783-9.

[276] T. Marchand, L. Blanchet & G. Faye; "Gravitational-wave tail effects to quartic non-linear order"; Class. Quant. Grav. **33**, p. 244003 (2016). arXiv: 1607.07601.

[277] T. Marchand, L. Bernard, L. Blanchet & G. Faye; "Ambiguity-Free Completion of the Equations of Motion of Compact Binary Systems at the Fourth Post-Newtonian Order"; Phys. Rev. D **97**, p. 044023 (2018). arXiv: 1707.09289.

[278] T. Marchand, Q. Henry, F. Larrouturou, S. Marsat, G. Faye & L. Blanchet; "The mass quadrupole moment of compact binary systems at the fourth post-Newtonian order"; Class. Quant. Grav. **37**, p. 215006 (2020). arXiv: 2003.13672.

[279] J. M. Martín-García, A. García-Parrado, A. Stecchina, B. Wardell, C. Pitrou, D. Brizuela, D. Yllanes, G. Faye, L. Stein, R. Portugal & T. Bäckdahl; "xAct: Efficient tensor computer algebra for Mathematica"; (GPL 2002–2012); http://www.xact.es/.

[280] Matthew; *Gospel, 25:29* (ed. Jerusalem).

[281] F. Messina & A. Nagar; "Parametrized-4.5PN TaylorF2 approximants and tail effects to quartic nonlinear order from the effective one body formalism"; Phys. Rev. D **95**, p. 124001 (2017). [Erratum: Phys.Rev.D 96, 049907 (2017)]; arXiv: 1703.08107.

[282] J. Michell; "Letter to Henry Cavendish"; Phil. Trans. of the Roy. Soc. of London **74**, pp. 35–57 (1784).

[283] A. Michelson; "Some of the Objects and Methods of Physical Science"; University of Chicago Quaterly Calendar p. 15 (1894).

[284] M. Milgrom; "A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis." A. P. J. **270**, pp. 365–370 (1983).

[285] C. W. Misner, K. S. Thorne & J. A. Wheeler; *Gravitation* (W. H. Freeman and Company) (1973).

[286] C. W. Misner & D. H. Sharp; "Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse"; Phys. Rev. **136**, pp. B571–B576 (1964).

[287] G. D. Moore & A. E. Nelson; "Lower bound on the propagation speed of gravity from gravitational Cherenkov radiation"; JHEP **09**, p. 023 (2001). arXiv: hep-ph/0106220.

[288] C. Moutou *et al.*; "Photometric and spectroscopic detection of the primary transit of the 111-day-period planet HD 80 606 b"; Astron. Astrophys. **498**, p. L5 (2009). arXiv: 0902.4457.

[289] S. Mukohyama & K. Noui; "Minimally modified gravity: a Hamiltonian construction"; Journal of Cosmology and Astroparticle Physics **2019**, p. 049 (2019). arXiv: 1905.02000.

[290] D. Naef *et al.*; "HD 80606b, a planet on an extremely elongated orbit"; Astron. Astrophys. **375**, p. L27 (2001). arXiv: astro-ph/0106256.

[291] E. T. Newman & A. I. Janis; "Note on the Kerr Spinning-Particle Metric"; Journal of Mathematical Physics **6**, pp. 915–917 (1965).

[292] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash & R. Torrence; "Metric of a Rotating, Charged Mass"; Journal of Mathematical Physics **6**, pp. 918–919 (1965).

[293] I. Newton; *Philosophiæ naturalis principia mathematica* (1756); trad. by E. du Châtelet.

[294] A. Nicolis, R. Rattazzi & E. Trincherini; "The Galileon as a local modification of gravity"; Phys. Rev. D **79**, p. 064036 (2009). arXiv: 0811.2197.

[295] H. Niikura *et al.*; "Microlensing constraints on primordial black holes with Subaru/HSC Andromeda observations"; Nature Astron. **3**, pp. 524–534 (2019). arXiv: 1701.02151.

[296] E. Noether; "Invariante Variationsprobleme"; Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen **1918**, pp. 235–257 (1918)`http://eudml.org/doc/59024`.

[297] G. Nordström; "On the Energy of the Gravitation field in Einstein's Theory"; Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences **20**, pp. 1238–1245 (1918).

[298] W. Occam; *Quatuor libros Sententiarum cum centilogio theologico* (1319).

[299] P. Painlevé; "La mécanique classique et la théorie de la relativité"; C. R. Acad. Sci. (Paris) **173**, pp. 677–680 (1921).

[300] A. Pàl & B. Kocsis; "Periastron precession measurements in transiting extrasolar planetary systems at the level of general relativity"; Mon. Not. Roy. Astr. Soc. **389**, p. 191 (2008). arXiv: 0806.0629.

[301] P. J. E. Peebles & B. Ratra; "The Cosmological Constant and Dark Energy"; Rev. Mod. Phys. **75**, pp. 559–606 (2003). arXiv: astro-ph/0207347.

[302] P. Pellegrin; *Le vocabuaire d'Aristote* (Ellipses) (2009).

[303] R. Penrose; "Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour"; Proc. of the Roy. Soc. of London Series A **284**, pp. 159–203 (1965).

[304] R. Penrose; "Asymptotic Properties of Fields and Space-Times"; Phys. Rev. Lett. **10**, pp. 66–68 (1963).

[305] S. Perlmutter *et al.*; "Measurements of Ω and Λ from 42 high redshift supernovae"; Astrophys. J. **517**, pp. 565–586 (1999). arXiv: astro-ph/9812133.

[306] L. Pogosian & M. Wyman; "B-modes from cosmic strings"; Phys. Rev. D **77**, p. 083509 (2008). arXiv: 0711.0747.

[307] F. Pont *et al.*; "Spin-orbit misalignment in the HD 80606 planetary system"; Astron. Astrophys. **502**, p. 695 (2009). arXiv: 0906.5605.

[308] R. A. Porto; "The effective field theorist's approach to gravitational dynamics"; Phys. Rept. **633**, pp. 1–104 (2016). arXiv: 1601.04914.

[309] R. V. Pound & G. A. Rebka; “Gravitational Red-Shift in Nuclear Resonance”; PRL **3**, pp. 439–441 (1959).

[310] F. Pretorius; “Evolution of binary black hole spacetimes”; Phys. Rev. Lett. **95**, p. 121101 (2005). arXiv: gr-qc/0507014.

[311] P. Ramond & A. Le Tiec; “Multipolar Particles in Helically Symmetric Spacetimes”; (2020)arXiv: 2005.00602.

[312] B. Ratra & P. J. E. Peebles; “Cosmological consequences of a rolling homogeneous scalar field”; Phys. Rev. D **37**, pp. 3406–3427 (1988).

[313] H. Reissner; “Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie”; Annalen der Physik **355**, pp. 106–120 (1916).

[314] A. G. Riess *et al.*; “Observational evidence from supernovae for an accelerating universe and a cosmological constant”; Astron. J. **116**, pp. 1009–1038 (1998). arXiv: astro-ph/9805201.

[315] M. Riesz; “L’intégrale de Riemann-Liouville et le problème de Cauchy”; Acta Mathematica **81**, pp. 1 – 222 (1949).

[316] R. A. Rosen; “Black Hole Mechanics for Massive Gravitons”; Phys. Rev. D **98**, p. 104008 (2018). arXiv: 1805.12135.

[317] V. C. Rubin, J. Ford, W. K., N. Thonnard & D. Burstein; “Rotational properties of 23Sb galaxies.” Astrophys. J. **261**, pp. 439–456 (1982).

[318] A. I. Sabra; “The Andalusian revolt against Ptolemaic astronomy: Averroes and al-Bitrûjî”; Transformation and Tradition in the Sciences: Essays in honor of I. Bernard Cohen pp. 133–153 (1984).

[319] R. K. Sachs; “Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time”; Proc. of the Roy. Soc. of London Series A **270**, pp. 103–126 (1962).

[320] A. D. Sakharov; “Vacuum Quantum Fluctuations In Curved Space And The Theory Of Gravitation”; Sov. Phys. Dokl. **12**, p. 104 (1968).

[321] J. Sangtawee & K. Karwan; “Inflationary model in minimally modified gravity theories”; (2021)arXiv: 2103.11463.

[322] S. Sarangi; “DBI global strings”; JHEP **07**, p. 018 (2008). arXiv: 0710.0421.

[323] J. Schneider, C. Dedieu, P. Le Sidaner, R. Savalle & I. Zolotukhin; “Defining and cataloging exoplanets: the exoplanet.eu database”; Astron. Astrophys. **532**, p. A79 (2011). arXiv: 1106.0586.

[324] B. F. Schutz & R. Sorkin; “Variational aspects of relativistic field theories, with application to perfect fluids”; Annals Phys. **107**, pp. 1–43 (1977).

[325] K. Schwarzschild; “Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie”; Sitz. der Kön. Preu. Ak. der Wiss. pp. 189–196 (1916)`https://archive.org/details/sitzungsberichte1916deutsch/page/188/mode/2up?view=theater`.

[326] J. Scott-Russell; “Report on Waves: Made to the Meetings of the British Association”; (1845). `https://archive.org/details/reportonwavesma00russgoog`.

[327] U. Seljak, U.-L. Pen & N. Turok; "Polarization of the microwave background in defect models"; Phys. Rev. Lett. **79**, pp. 1615–1618 (1997). arXiv: astro-ph/9704231.

[328] A. G. Shah, J. L. Friedman & B. F. Whiting; "Finding high-order analytic post-Newtonian parameters from a high-precision numerical self-force calculation"; Phys. Rev. D **89**, p. 064042 (2014). arXiv: 1312.1952.

[329] Y. M. Shnir; *Topological and Non-Topological Solitons in Scalar Field Theories* (Cambridge University Press) (2018); ISBN 978-1-108-63625-4.

[330] L. Smarr; "Space-time generated by computes: black holes with gravitational radiation"; Annals of the New York Academy of Sciences **302**, pp. 569–604 (1977).

[331] A. A. Starobinsky; "A new type of isotropic cosmological models without singularity"; Physics Letters B **91**, pp. 99–102 (1980).

[332] H. Stephani; *Relativity: An introduction to special and general relativity* (Cambridge University Press) (2004).

[333] E. C. G. Stückelberg; "Die Wechselwirkungs Kräfte in der Elektrodynamik und in der Feldtheorie der Kernkräfte"; Helv. Phys. Acta **11**, pp. 225–244 (1938).

[334] W. Thomson; "Nineteenth-Century Clouds over the Dynamical Theory of Heat and Light"; Philosophical Magazine and Journal of Science pp. 1–40 (1901)`https://www.equipes.lps.u-psud.fr/Montambaux/histoire-physique/Kelvin-1900.pdf`.

[335] K. S. Thorne; "Multipole expansions of gravitational radiation"; Rev. Mod. Phys. **52**, pp. 299–339 (1980).

[336] P. Touboul *et al.*; "Space test of the Equivalence Principle: first results of the MICROSCOPE mission"; Class. Quant. Grav. **36**, p. 225006 (2019). arXiv: 1909.10598.

[337] A. Triaud; "The Rossiter-McLaughlin Effect in Exoplanet Research"; Handbook of Exoplanets, ed. H. J. Deeg and J. A. Belmonte (2018). arXiv: 1709.06376.

[338] R. B. Tully, O. de Marseille & J. R. Fisher; "A New Method of Determining Distances to Galaxies"; in "Bulletin of the American Astronomical Society," , vol. 7p. 426 (1975).

[339] S. D. Upton & A. Pound; "Second-order gravitational self-force in a highly regular gauge"; (2021)arXiv: 2101.11409.

[340] T. Vachaspati; *Kinks and domain walls: An introduction to classical and quantum solitons* (Cambridge University Press) (2010); ISBN 978-0-511-24290-8.

[341] H. van Dam & M. Veltmann; "On the mass of the graviton"; Gen. Rel. Grav. **3**, pp. 215–220 (1972).

[342] E. P. Verlinde; "On the Origin of Gravity and the Laws of Newton"; JHEP **04**, p. 029 (2011). arXiv: 1001.0785.

[343] J.-P. Vernant; "Géométrie et astronomie sphérique dans la première cosmologie grecque"; La Pensée **109**, pp. 82–92 (1963).

[344] J.-P. Vernant; "Structures géométriques et notions politiques das la cosmologie d'Anaximandre"; Eirene. Studia Græca et Latina **VII**, pp. 5–23 (1968).

[345] A. Vilenkin & E. S. Shellard; *Cosmic Strings and Other Topological Defects* (Cambridge University Press) (2000); ISBN 978-0-521-65476-0.

[346] V. Viswanathan *et al.*; "Extending Science from Lunar Laser Ranging"; (2020)arXiv: 2008.09584.

[347] T. A. Wagner, S. Schlamminger, J. H. Gundlach & E. G. Adelberger; "Torsion-balance tests of the weak equivalence principle"; Class. Quant. Grav. **29**, p. 184002 (2012). arXiv: 1207.2442.

[348] R. M. Wald; *General Relativity* (University of Chicago Press) (1984).

[349] E. J. Weinberg; *Classical solutions in quantum field theory: Solitons and Instantons in High Energy Physics*; Cambridge Monographs on Mathematical Physics (Cambridge University Press) (2012); ISBN 978-1-107-43805-7.

[350] S. Weinberg; "The cosmological constant problem"; Rev. Mod. Phys. **61**, pp. 1–23 (1989).

[351] J. M. Weisberg & J. H. Taylor; "Relativistic binary pulsar B1913+16: Thirty years of observations and analysis"; ASP Conf. Ser. **328**, p. 25 (2005)arXiv: astro-ph/0407149.

[352] C. Wetterich; "Effective nonlocal Euclidean gravity"; Gen. Rel. Grav. **30**, pp. 159–172 (1998). arXiv: gr-qc/9704052.

[353] C. M. Will; *Theory and Experiment in Gravitational Physics* (Cambridge University Press) (1993).

[354] C. M. Will; "The Confrontation between General Relativity and Experiment"; Living Rev. Rel. **17**, p. 4 (2014). arXiv: 1403.7377.

[355] C. M. Will & K. J. Nordtvedt; "Conservation Laws and Preferred Frames in Relativistic Gravity. I. Preferred-Frame Theories and an Extended PPN Formalism"; APJ **177**, p. 757 (1972).

[356] J. N. Winn *et al.*; "The Transit Ingress and the Tilted Orbit of the Extraordinarily Eccentric Exoplanet HD 80606b"; Astrophys. J. **703**, p. 2091 (2009). arXiv: 0907.5205.

[357] M. Wolf *et al.*; "Relativistic apsidal motion in eccentric eclipsing binaries"; Astron. Astrophys. **509**, p. A18 (2010).

[358] Wolfram Research, Inc.; "Mathematica, Version 12.1"; `https://www.wolfram.com/mathematica`; Champaign, IL, 2020.

[359] R. P. Woodard; "Nonlocal Models of Cosmic Acceleration"; Found. Phys. **44**, pp. 213–233 (2014). arXiv: 1401.0254.

[360] J. T. Wright; "Radial Velocities as an Exoplanet Discovery Method"; Handbook of Exoplanets, ed. H. J. Deeg and J. A. Belmonte (2018). arXiv: 1707.07983.

[361] Y. Wu & N. Murray; "Planet migration and binary companions: The case of HD 80606b"; Astrophys. J. **589**, pp. 605–614 (2003). arXiv: astro-ph/0303010.

[362] Y. Yamashita, A. De Felice & T. Tanaka; "Appearance of Boulware–Deser ghost in bigravity with doubly coupled matter"; Int. J. Mod. Phys. D **23**, p. 1443003 (2014). arXiv: 1408.0487.

[363] Z.-B. Yao, M. Oliosi, X. Gao & S. Mukohyama; "Minimally modified gravity with an auxiliary constraint: a Hamiltonian construction"; (2020); arXiv: 2011.00805.

[364] Y. Zhong, R.-Z. Guo, C.-E. Fu & Y.-X. Liu; "Kinks in higher derivative scalar field theory"; Physics Letters B **782**, pp. 346–352 (2018). arXiv: 1804.02611.

François LARROUTUROU **18 juin 2021**

Méthodes analytiques pour l'étude du problème à deux corps, et des théories alternatives de gravitation

Résumé : Le travail effectué durant cette thèse visait à parfaire notre connaissance des phénomènes gravitationnels, en poursuivant deux buts: (i) améliorer notre compréhension du problème relativiste à deux corps, (ii) construire et tester des modèles alternatifs.
Dans un premier temps, nous avons utilisé une approximation post-Newtonienne (champs et vitesses faibles) pour chercher une expression analytique et très précise de la phase des ondes gravitationnelles. Pour ce faire, nous avons dû implémenter une régularisation dimensionnelle pour le quadrupole de masse, et pour les effets non-linéaires dits de "sillage" et de "mémoire". En exploitant des synergies communes avec d'autres méthodes, nous avons aussi calculé les contributions logarithmiques qui entrent dans l'énergie conservée, jusqu'à une haute précision. Enfin, nous avons proposé le premier test réaliste de la RG dans un système exoplanétaire.
Quant au second but, nous avons étudié des murs de domaine originaux, en proposant une nouvelle classe de défauts non-topologiques. Nous avons prouvé leur stabilité, et montré qu'ils peuvent imiter les défauts canoniques. De plus, nous avons construit et étudié la cosmologie d'une "théorie minimale de bigravité", basée sur le principe du rasoir d'Occam. Pour finir, nous avons étudié le régime de champs forts de deux théories "minimales" en construisant leurs solutions de trous noirs.

Mots clés : Problème à deux corps, défauts non-topologiques, théories alternatives

Analytical methods for the study of the two-body problem, and alternative theories of gravitation

Abstract: The work completed during this thesis aimed at pushing forward our knowledge of gravitational phenomena, by following two directions: (i) deepening our comprehension of the relativistic two-body problem, (ii) building and testing alternative models.
First, we used a post-Newtonian (weak-field and slow motion) approximation to seek analytic expressions for the phase of gravitational waves at high accuracy. Two of the main results of this thesis are thus the proper dimensional regularization of the mass quadrupole, and of the non-linear "tail" and "memory" effects, appearing in the radiative quadrupole. Using synergies with other approaches, we have also derived the logarithmic tail contributions in the conserved energy at high accuracies. Finally, and as a side result, we have also proposed the first realistic test of GR in an exoplanetary system.
As for the second direction, we have investigated non-canonical domain walls by building a new class of non-topological kinks. We have shown that those were stable, and could mimic the canonical ones. A "minimal theory of bigravity" was also constructed by requiring to avoid observationally unnecessary gravitational polarizations, and its cosmology was proven stable. Finally, we have investigated the strong-field regime of two of such minimal theories by deriving their (non-trivial) black hole solutions.

Keywords : Two-body problem, non-topological defects, alternatives theories