# Uniform stability for some systems arising from fluid mechanics and plasma physics

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## Stabilité uniforme pour certains systèmes issus de la mécanique des fluides et de la physique des plasmas

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Thèse de doctorat


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# Chapter 1

# Introduction-Francais

## 1.1 Introduction générale

Dans de nombreux systèmes physiques concrets, il existe des nombres physiques fondamentaux (nombre de Reynolds, nombre de Mach, nombre de Rossby, nombre de Froude, etc.) qui sont très petits ou très grands selon les différents contextes. Par exemple, le nombre de Reynolds, qui est un nombre physique sans dimension mesurant le rapport de la force d'inertie et de la force visqueuse, est très grand pour les fluides autour d'un grand bateau de croisière. Le nombre de Mach, qui est défini par le rapport d'une vitesse caractéristique de l'écoulement à la vitesse du son, est généralement très petit pour les écoulements hautement subsoniques (par exemple le flux d'air près d'un plan en mouvement rapide). Ces paramètres conduisent généralement aux termes singuliers dans les systèmes fluides qui donnent lieu à des difficultés pour l'analyse des systèmes. Une approche courante utilisée en physique consiste à essayer de considérer un système approché. Par exemple, on peut utiliser le système non visqueux pour décrire le mouvement d'un système légèrement visqueux avec un nombre de Reynolds grand et on peut approximer les écoulements compressibles avec un faible nombre de Mach par le système incompressible correspondant. Dans cette thèse, nous nous concentrerons sur les problèmes de limite non-visqueuse (grand nombre de Reynolds) et incompressible (faible nombre de Mach) pour une solution suffisamment régulière. Afin de justifier ces limites, il faut prouver des estimations uniformes (par rapport à ces quantités physiques) qui sont les principales préoccupations et le cœur de la cette thèse.

La première partie de la thèse (chapitre 3-5) est consacrée au problème de stabilité uniforme (par rapport au nombre de Reynolds ou à faible viscosité) pour certains modèles visqueux compressibles (on prendra système de Navier-Stokes-Poisson par example). D'une part, les solutions globales régulières pour ces modèles visqueux proches des équilibres sont construites dans [85, 126, 26, 64] en supposant que les données initiales sont proches des équilibres dans un espace approprié avec la distance proportionnelle à l'inverse du nombre de Reynolds (noté $\varepsilon$). Ce n'est pas si attrayant pour le problème de limite non visqueux puisque ces solutions a la limite disparaissent. D'autre part, la version non visqueuse de ces modèles admet des propriétés dispersives favorables et une existence globale proche des équilibres constants a été montrée [55, 56, 58, 69, 84, 10] pour les flux irrotationnels. L'hypothèse qui suppose que les données initiales sont irrotationnelles semble optimale, du moins pour certains de ces modèles, car des singularités peuvent se former en temps fini quelle que soit la taille des données initiales si l'on se débarrasse de cette hypothèse. Par conséquent, pour le système visqueux avec un grand nombre de Reynolds, on ne peut s'attendre à l'existence globale qu'en permettant à la partie rotationnelle de la vitesse initiale d'être petite proportionnelle à $\varepsilon$. Mais la chose importante est que toutes les autres parties de la perturbation initiale peuvent être prises petites indépendamment de $\varepsilon$. Nous dirons que le système satisfait la propriété de stabilité uniforme si la solution globale existe sous des perturbations (presque) uniformes autour de l'équilibre constant comme mentionné ci-dessus.

Pour obtenir la stabilité uniforme, il y a plusieurs difficultés à surmonter. Par exemple, la condition d'irrotationalité n'est pas préservée pour les systèmes visqueux compressibles, ce qui est une propriété significative utilisée pour établir l'existence globale du système non visqueux correspondant. De plus, le semi-groupe linéaire de la partie compressible du système étant très différent du cas où $\varepsilon = 0$, on est obligé de tronquer en fréquence afin d'isoler les effets dispersifs et les effets de dissipation. Au chapitre 3, nous prouvons un résultat de stabilité uniforme pour le système tridimensionnel de Navier-Stokes-Poisson (NSP). Dans le chapitre 4, nous montrons un résultat similaire pour un système NSP bidimensionnel avec une viscosité spécifique subtil dépendante de la densité ($\mu(\rho) = \rho, \lambda(\rho) = 0$) ce qui est plus impliqué en raison de la dispersion plus faible. De plus, nous présentons un résultat de régularité en temps long pour le 2d NSP général. Les preuves reposent sur une étude minutieuse des phénomènes dispersives et des phénomènes de diffusion pour les EDP dispersifs avec de petites perturbations visqueuses et elles combinent de techniques à partir de Pdes dispersives telles que la méthode de "résonance espace-temps" et les estimations d'énergie parabolique classiques. Au chapitre 5, nous montrerons un sous-produit de la méthode employée aux chapitres 2 et 3, pour prouver des résultats d'existence en temps long pour certains modèles non visqueux (Euler-Maxwell, Euler-Korteweg) avec un tourbillon non trivial.

La deuxième partie (chapitre 6-7) de la thèse porte sur le problème de limite du nombre de Mach bas pour certains fluides visqueux dans des domaines avec des frontières fixes ou libres. Du point de vue physique, lorsque la vitesse du fluide est bien inférieure à celle de la vitesse du son (ce qui signifie que le nombre de Mach est très petit), l'écoulement compressible doit se comporter comme l'écoulement incompressible. Du point de vue mathématique, au moins pour les systèmes à fluide isentropiques, en régime à faible nombre de Mach, la pression (et donc la densité) varie peu de sorte que le système compressible tend formellement à l'incompressible. Nous nous intéressons à la vérification mathématique de ce processus limite pour solutions fortes aux systèmes compressibles des Navier-Stokes. En raison de l'apparition des termes singuliers dans les équations et également de la présence des frontières, il faut traiter simultanément les oscillations temporelles et les effets de la couche limite et de nouvelles idées doivent être trouvées.

Au chapitre 6, nous considérons le estimations de régularité uniforme pour les équations de Navier-Stokes compressibles isentropiques dans des domaines fixes lisses avec condition de glissement de Navier au bord dans le cas général de données initiales mal préparées. Pour tenir compte des effets de couche limite dus aux oscillations rapides et de hypothèse de données initiale mal préparée, nous prouvons des estimations uniformes dans un cadre fonctionnel anisotrope avec une seule dérivée normale proche de la frontière. Ceci permet de prouver l'existence locale d'une solution forte sur un intervalle de temps indépendant du nombre de Mach et de justifier la limite incompressible par un simple argument de compacité. Pour présenter les stratégies générales, nous présentons la preuve dans la section 8 de ce chapitre lorsque le domaine fluide est un demi-espace $\mathbb{R}^3_+$ où l'estimation légèrement meilleure est disponible en raison de la géométrie triviale de la frontière.

Le chapitre 7 est consacré au problème de la limite du nombre de Mach bas pour les équations de Navier-Stokes compressibles à surface libre. Par rapport au cas du domaine fixe, ce genre de problème est plus complexe car une attention particulière doit être portée à la régularité de la surface. Il y a en effet eu plusieurs investigations concernant le problème limite incompressible pour un système fluide avec des frontières libres [37, 88, 92] qui se concentrent sur les systèmes *non visqueuse*. Contrairement aux fluides visqueux, les couches limites n'apparaissent pas dans les études des fluides non visqueux, en raison de l'absence du terme de dissipation. Le résultat présenté dans ce chapitre sert de première exploration pour les fluides *visqueux* à surface libre.
Nous détaillerons ces problèmes, énoncerons les principaux résultats et expliquerons brièvement les difficultés et les stratégies. dans les deux sections suivantes.

## 1.2 Stabilité uniforme et régularité en temps long pour les modèles de fluides visqueux en physique des plasmas.

### 1.2.1 Système de Navier-Stokes-Poisson.

L'une des principales préoccupations de la physique des plasmas est d'étudier la dynamique des particules chargées (électrons et ions). Cependant, très couramment, le système cible que l'on étudie est composé d'un grand nombre de particules, il n'est donc parfois pas possible et pratique de suivre les mouvements de chaque particule. Le modèle fluide consiste à considérer ces particules comme fluides et à utiliser les équations de la mécanique des fluides pour étudier leur dynamique. Dans la première partie de la thèse, nous nous intéressons au système dit de Navier-Stokes-Poisson qui est un modèle un-fluide décrivant la dynamique du plasma. En supposant que le champ magnétique est négligeable et que les ions sont fixes, le mouvement des électrons peut être illustré par le système de Navier-Stokes-Poisson (ENSP) (électron) suivant:

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon \mathrm{div}\mathcal{L}u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon \nabla \phi^\varepsilon = 0, \\ \Delta \phi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{1.2.1}$$

Ici, les inconnues $\rho^\varepsilon(t,x) \in \mathbb{R}_+$, $u^\varepsilon \in \mathbb{R}^3$, $\nabla\phi^\varepsilon \in \mathbb{R}^3$ sont respectivement la densité électronique, la vitesse électronique et le champ électrique auto-cohérent. La pression thermique des électrons $P(\rho^\varepsilon) > 0$ est généralement supposée être une fonction régulière de la densité tandis que le terme visqueux est sous la forme

$$\mathrm{div}\mathcal{L}u^\varepsilon = \mu \Delta u^\varepsilon + (\mu + \lambda)\nabla \mathrm{div} u^\varepsilon \tag{1.2.2}$$

où les coefficients de Lamé $\mu, \lambda$ sont supposé être des constantes qui satisfont à la condition:

$$\mu > 0, \qquad 2\mu + \lambda > 0.$$

Notons que nous considérons une version mise à l'échelle du système avec le coefficient $\varepsilon$ devant les termes de diffusion qui est l'inverse du nombre de Reynolds et qui est supposé petit dans cette thèse.

En revanche, lorsque les électrons sont considérés en équilibre thermodynamique, un système simplifié pour la dynamique des ions est le système (ion) de Navier-Stokes-Poisson (INSP) suivant:

$$\begin{cases} \partial_t \rho_+^\varepsilon + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon) = 0, \\ \partial_t(\rho_+^\varepsilon u_+^\varepsilon) + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon \otimes u_+^\varepsilon) - \varepsilon \mathrm{div}\mathcal{L}u_+^\varepsilon + \nabla P(\rho_+^\varepsilon) - \rho_+^\varepsilon \nabla \phi_+^\varepsilon = 0, \\ \Delta \phi_+^\varepsilon = \rho_+^\varepsilon - e^{-\phi_+^\varepsilon} \\ u_+|_{t=0} = u_{+0}^\varepsilon, \rho_+|_{t=0} = \rho_{+0}^\varepsilon. \end{cases} \tag{1.2.3}$$

Ces deux modèles admettent une solution stationnaire $(\rho^\varepsilon, u^\varepsilon, \nabla\phi^\varepsilon) = (1,0,0)$ qui s'appelle l'équilibre. Nous nous intéressons à la stabilité de ces systèmes autour de l'équilibre, à savoir résoudre l'équation globalement sous une petite perturbation initiale de l'équilibre.

Il existe une abondante littérature traitant de la stabilité sous des perturbations suffisamment petites et lisses de l'équilibre constant de (ENSP) lorsque $\varepsilon = 1$. On se réfère par exemple à [85] où l'existence globale dans $H^l$ pour $l \geq 4$ est prouvée sous l'hypothèse que la perturbation initiale est petite dans $H^l$ et $L^1$. Plus récemment, dans [126], l'existence globale dans $H^N$ ($N \geq 3$) de (ENSP) est obtenue en utilisant uniquement des estimations d'énergie sous l'hypothèse que la perturbation initiale appartient à $H^N$ et est petite dans $H^3$. Ces résultats utilisent fortement le fait que l'équation de la vitesse est une équation parabolique et que le couplage entre les deux équations d'évolution de (ENSP) produit une décroissance de la densité. Dans [85], l'existence globale en dimension $d$ est obtenue dans les espaces hybrides de Besov lorsque la perturbation initiale est proche de l'équilibre dans une norme critique $L^2$ en utilisant des estimations d'énergie et en considérant les fréquences basses et hautes différemment. Ce résultat a ensuite été généralisé à un cadre $L^p$ critique [135, 26].

Tous ces travaux portent sur un système non mis à l'échelle, c'est-à-dire (ENSP) avec $\varepsilon = 1$. Nous pouvons facilement vérifier que pour le système dépendant de $\varepsilon$, ces travaux donnent des solutions globales fluides si la perturbation initiale est suffisamment petite par rapport à $\varepsilon$ et que les taux de décroissance obtenus sont valables en termes de la variable de temps $\varepsilon t$ (par exemple [126] donnerait cela dans $L^\infty$, $(\rho^\varepsilon - 1)$ est borné par $\varepsilon(1+\varepsilon t)^{-\frac{3}{2}}$). En effet, l'existence globale est obtenue par des arguments bootstrap et des estimations a priori, Il existe deux façons d'obtenir les estimations a priori. Une façon est, comme dans [85, 126], d'utiliser des estimations d'énergie et d'obtenir une dissipation pour $u^\varepsilon$ en utilisant le terme de diffusion $\varepsilon\Delta u^\varepsilon$ et la dissipation pour $\rho^\varepsilon - 1$ en utilisant une "estimation d'énergie croisée". Les termes non linéaires peuvent être absorbés si une certaine quantité est petite par rapport à $\varepsilon$. L'autre façon est, comme dans [64, 135, 26], lorsque l'on considère l'existence globale dans les espaces critiques de

Besov consiste à utiliser l'effet de lissage maximal du noyau de la chaleur $e^{\varepsilon t\Delta}$ ce qui donne par exemple pour l'équation de la chaleur mise à l'échelle

$$\|e^{\varepsilon t\Delta} f\|_{L^1(\mathbb{R}_+,\dot{B}^{s+2}_{p,1})} \lesssim \varepsilon^{-1}\|f\|_{\dot{B}^s_{p,1}}.$$

Par conséquent, pour contrôler les termes non linéaires, cela conduit également à supposer que la taille de la perturbation initiale doit être petite par rapport à $\varepsilon$.

Néanmoins, lorsque $\varepsilon = 0$, le système (1.2.1) se réduit au système dit électronique Euler-Poisson (EEP). Pour le système (EEP), l'existence globale de solutions fluides proche de l'équilibre constant $(1,0)$ a d'abord été obtenu par Guo [55] sous une petite perturbation neutre, irrotationnelle de l'équilibre de référence $(\rho^0, u^0) = (1,0)$. L'hypothèse neutre $(\int(\rho_0^0 - 1)\mathrm{d}x = 0)$ a ensuite été supprimée dans [48]. La propriété importante qui a été utilisée dans ces travaux est que le système (EEP) a de meilleures propriétés dispersives que les équations d'Euler pour les fluides compressibles en raison de la présence du champ électrique. Néanmoins, en dimension 3, la seule utilisation des estimations énergétiques et de la décroissance dispersive ne suffisent pas pour obtenir des solutions globales régulière en présence de non-linéarités quadratiques. Un ingrédient supplémentaire est donc nécessaire, à savoir soit les estimations d'énergie utilisant les méthodes de champ de vecteurs, soit la méthode de forme normale. Pour le système d'Euler-Poisson, la méthode de forme normale de Shatah [112] ou plus généralement, la philosophie des "résonances espace-temps" peut être utilisée pour contrôler les termes non linéaires. On se réfère à [47, 48, 82, 112], pour plus d'informations sur la méthode de la forme normale et l'approche de la "résonance espace-temps". Ce type d'approche a été récemment utilisé avec succès pour gérer le système (EEP) en dimension deux [69, 84] et une [56].

Puisque dans les écoulements physiques concrets, le nombre de Reynolds est généralement très grand (donc $\varepsilon$ très petit), il est naturel de demander des résultats de stabilité qui soient uniformes par rapport à $\varepsilon$ pour (ENSP). Bien que les méthodes utilisées dans les deux types de résultats que nous venons de présenter soient complètement différentes, il est plutôt naturel de s'attendre à obtenir des solutions globales régulières pour les perturbations de l'équilibre constant $(1,0)$ avec une hypothèse de petitesse sur la perturbation indépendante de $\varepsilon$ sauf pour la partie curl de la vitesse (rappelez-vous que pour $\varepsilon = 0$ nous avons des solutions globales régulières uniquement pour les données irrotationnelles). C'est le résultat que nous avons obtenu dans [108].

**Theorem 1.2.1** (Rousset-Sun, [108]). *Notons $\mathcal{P}$ le projecteur de Leray sur les champs de vecteurs à divergence nulle de sorte que $\mathcal{P}^\perp = Id - \mathcal{P} = \nabla\Delta^{-1}\mathrm{div}$ et notons $\nabla\phi_0^\varepsilon = -\nabla(-\Delta)^{-1}(\rho_0^\varepsilon - 1)$. Il existe $\delta_0 > 0$ tel que pour toute famille de données initiales qui satisfont pour chaque $\varepsilon \in (0,1]$ les estimations :*

$$\|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,1}} + \|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{H^N} \leq \delta_0$$
$$\|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \delta_0\varepsilon$$

*avec $\sigma \geq 5$ et $N \geq \sigma + 7$, alors, pour tous $\varepsilon \in (0,1]$, il existe une solution globale unique du système (ENSP) (1.2.1) dans $C([0,+\infty), H^3)$. En plus, si nous supposons que $\sup_{\varepsilon\in(0,1]}\|\mathcal{P}u_0^\varepsilon\|_{\dot{H}^{-s}} < +\infty$ pour quelque $0 < s < \frac{1}{2}$, alors on a*

$$\|(\rho^\varepsilon - 1, \nabla\phi^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}} \leq C\big(\min\{\varepsilon, (1+t)^{-\frac{s}{2+s}}\} + (1+t)^{-(\frac{11}{8}+)}\big), \quad \forall t \geq 0.$$

*où $a^+$ représentetout nombre strictement plus grand mais arbitrairement proche de $a$.*

Notons que l'hypothèse que l'on fait sur la taille de la partie "curl" des données initiales, c'est-à-dire l'hypothèse sur $\mathcal{P}u_0^\varepsilon$, semble être le naturel. En effet, même si nous supposons que $\mathcal{P}u_0^\varepsilon = 0$, cette propriété n'est pas propagée par le système (ENSP), l'équation de convection-diffusion pour la partie rotationnelle de la vitesse est forcée par un terme source de taille $\varepsilon$, donc une partie rotationnelle de taille $\varepsilon$ soit instantanément créé.

#### 1.2.1.1 Stratégies.

Notre stratégie pour prouver la Théorème 1.2.1 est de séparer le système en deux systèmes visqueux, avec les données initiales $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ et $(0, 0, \mathcal{P}u_0^\varepsilon)$ respectivement. Le premier aura des solutions globales sous des hypothèses indépendantes de $\varepsilon$ sur les données initiales $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ et les solutions bénéficieront des mêmes estimations décroissance que le système (EEP). L'autre n'est que la perturbation du système original (1.2.1) par la solution du premier, les points importants sont que pour ce système, les données initiales et le terme source sont petits par rapport à $\varepsilon$ et que le terme source a une décroissance intégrable dans $L^2$. Nous pouvons ainsi utiliser les estimations d'énergie et les bonnes propriétés décroissance des solutions du système précédent pour prouver l'existence et la décroissance globales.

Pour la simplicité de la présentation, nous supposerons que $\mu = 1$, $\lambda = 0$ et que $P(\rho^\varepsilon) = (\rho^\varepsilon)^2/2$. Néanmoins, il n'y a pas d'annulation spéciale résultant de ce choix (le cas le plus simple pour l'analyse serait le choix $\mu(\rho) = \rho$, $\lambda = -\mu$, car dans ce cas il y a des solutions irrotationnelles solutions de (1.2.1)). Les résultats sont valables pour la pression générale et pour la densité générale dépendant de $\mu, \lambda$ tant que $\mu(1) > 0$, $2\mu(1) + \lambda(1) > 0$ (voir Section 6 du chapitre 2). Nous écrivons la solution $(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon)$ de (ENSP) comme

$$(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon) = (\rho, \nabla\phi, u) + (n, \nabla\psi, v),$$

où $(\rho, \nabla\phi, u)$ et $(n, \nabla\psi, v)$ sont les soloutions des systèmes suivants:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u - \varepsilon\mathrm{div}\mathcal{L}u + \nabla\rho - \nabla\phi = 0, \\ \Delta\phi = \rho - 1, \\ u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \tag{1.2.4}$$

$$\begin{cases} \partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u\cdot\nabla v + v\cdot(\nabla u + \nabla v) - \varepsilon\mathrm{div}\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho+n} - 1)(\mathcal{L}v + \mathcal{L}u), \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0} = 0. \end{cases} \tag{1.2.5}$$

Notez que pour ces deux systèmes, nous omettons la dépendance $\varepsilon$ des solutions dans notre notation.

Nous pouvons définir $\varrho = \rho - 1$, pour changer le système (1.2.4) en:

$$\begin{cases} \partial_t\varrho + \mathrm{div}u = -\mathrm{div}(\varrho u), \\ \partial_t u + u\cdot\nabla u - \varepsilon\mathrm{div}\mathcal{L}u + \nabla\varrho - \nabla\phi = 0, \\ \Delta\phi = \varrho, \\ u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \varrho|_{t=0} = \varrho_0 = \rho_0^\varepsilon - 1. \end{cases} \tag{1.2.6}$$

Notez que la donnée initiale du dernier système est telle que $\mathrm{curl}(\mathcal{P}^\perp u_0^\varepsilon) = 0$, et cette propriété irrotationnelle sera propagée, ce qui signifie qu'une solution régulière de ce système restera irrotationnelle. Ce système est donc une très bonne approximation visqueuse du système Euler-Poisson. Comme nous le verrons au Chapitre 2, la partie linéaire de ce système a les mêmes propriétés de décroissance pour les basses fréquences que le système (EEP), c'est-à-dire pour les données initiales localisées, la norme $L^p$ de $(\varrho, \nabla\phi, u)$ décroît comme $(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}$ uniformément pour $\varepsilon \in (0, 1]$.

### 1.2.2 Stabilité uniforme pour 3d INSP.

On peut également considérer le système INSP qui est un modèle simplifié pour le mouvement des ions:

$$\begin{cases} \partial_t\rho_+^\varepsilon + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon) = 0, \\ \partial_t(\rho_+^\varepsilon u_+^\varepsilon) + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon \otimes u_+^\varepsilon) - \varepsilon\mathrm{div}\mathcal{L}u_+^\varepsilon + \nabla P(\rho_+^\varepsilon) - \rho_+^\varepsilon\nabla\phi_+^\varepsilon = 0, \\ \Delta\phi_+^\varepsilon - \phi_+^\varepsilon = \rho_+^\varepsilon - 1 \\ u_+|_{t=0} = u_{+0}^\varepsilon, \rho_+|_{t=0} = \rho_{+0}^\varepsilon. \end{cases} \tag{1.2.7}$$

Notez que nous avons utilisé l'approximation dite linéarisée puisque dans le système (INSP), nous avons remplacé l'équation de Poisson $\Delta\phi_+^\varepsilon = \rho_+^\varepsilon - e^{-\phi_+^\varepsilon}$ par une version linéarisée. Ce n'est pas une hypothèse gênante puisque nous traitons de petites perturbations de l'equilibre constant $(1, 0)$. Pour le système d'Euler-Poisson décrivant la dynamique des ions (IEP) (c'est-à-dire $\varepsilon = 0$ in (1.2.3)), des solutions globales d'irrotation lisses ont été construites par Guo et Pausader [58]. L'idée est à nouveau de trouver des estimations dispersives pour le système linéarisé (qui se révèle être plus faible que celle des équations linéaires de Klein-Gordon) et d'utiliser la méthode de forme normale. Néanmoins, l'analyse de ce modèle est beaucoup plus complexe. En effet, la relation de dispersion est plus proche de celle de l'équation d'onde qui conduit à l'apparition de "résonances temporelles". Par exemple, l'ensemble de "résonances temporelles" de la fonction de phase $\Phi_{++} = -p(\xi) + p(\xi - \eta) + p(\eta)$,$(p(\xi) = |\xi|\sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}})$ est $\{\eta = 0\} \cup \{\xi - \eta = 0\}$. Après intégration en temps, les opérateurs multilinéaires ont désormais un noyau singulier et pour les contrôler, il faut utiliser les normes $\dot{H}^{-1}$.

Nous énonçons maintenant la contrepartie du Théorème 1.2.1 qui est présenté dans [108].

**Theorem 1.2.2** (Rousset-Sun, [108]). *Fixons un nombre absolu $\kappa > 0$ assez petit. Il existe $\delta_2 > 0$ tel que pour toute famille de conditions initiales qui satisfont pour tout $\varepsilon \in (0, 1]$ les estimations*

$$\||(\rho^\varepsilon_{+0} - 1, \mathcal{P}^\perp u^\varepsilon_{+0})\|_{W^{\sigma+3,8'_\kappa}} + \||\nabla|^{-1}(\rho^\varepsilon_{+0} - 1, \mathcal{P}^\perp u^\varepsilon_{+0})\|_{H^N} \leq \delta_2,$$
$$\|\mathcal{P}u^\varepsilon_{+0}\|_{H^3} \leq \delta_2\varepsilon$$

*avec $8_\kappa = \frac{8}{1-3\kappa}$, $8'_\kappa = \frac{8_\kappa}{8_\kappa-1}$, $\sigma \geq 6$, $N \geq 2\sigma + 1$, alors on a pour tout $\varepsilon \in (0, 1]$, il existe une solution globale unique pour le système (1.2.7) dans $C([0, +\infty), H^3)$. En outre, si $\sup_{\varepsilon\in(0,1]} \|\mathcal{P}u^\varepsilon_{+0}\|_{\dot{H}^{-s}} < +\infty$ avec $s < \frac{3}{8}$, alors on a les estimations de décroissance temporelle suivantes: il eixste $C > 0$ tel que pour tous $\varepsilon \in (0, 1]$,*

$$\|(\rho^\varepsilon_+ - 1, u^\varepsilon_+)\|_{W^{1,\infty}} \leq C\big(\min\{\varepsilon, (1 + t)^{-\min\{\frac{s}{2+s}, \frac{\kappa}{2}\}}\} + (1 + t)^{-(1+\kappa)}\big), \quad \forall t \geq 0.$$

### 1.2.3 Régularité en temps long pour l'ENSP en dimension 2.

Comme cela est bien connu, les effets de dispersion sont plus faibles lorsque la dimension est plus faible. Il est donc intéressant d'étudier le problème de stabilité uniforme en 2-d. Nous nous concentrerons dans cette sous-section sur le système 2d ENSP:

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon\mathrm{div}\mathcal{L}u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon\nabla\phi^\varepsilon = 0, \\ \Delta\phi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u^\varepsilon_0, \rho|_{t=0} = \rho^\varepsilon_0. \end{cases} \tag{1.2.8}$$

Ici le terme visqueux $\mathrm{div}\mathcal{L}u^\varepsilon$ est défini dans (1.2.2). Si on prend $\mu(\rho) = \rho, \lambda(\rho) = -\rho$ dans (1.2.2), le système (1.2.8) peut être réduit à

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon\Delta u^\varepsilon + \nabla\rho^\varepsilon - \nabla\phi^\varepsilon = 0, \\ \Delta\phi^\varepsilon = \rho^\varepsilon - 1, \\ u^\varepsilon|_{t=0} = u^\varepsilon_0, \rho^\varepsilon|_{t=0} = \rho^\varepsilon_0. \end{cases} \tag{1.2.9}$$

Comme expliqué dans la dernière sous-section, on s'attend à ce que ce système soit une bonne approximation visqueuse du système d'Euler-Poisson et nous attendons que la petite diffusion visqueuse n'influence pas autant la dynamique de la partie compressible des solutions. En d'autres termes, les effets dispersifs dominent les effets de dissipation pour les flux irrotationnels. Puisque l'existence globale du système 2d d'Euler-Poisson pour la perturbation initiale irrotationnelle a été montrée dans [69, 84], nous cherchons d'abord à prouver les résultats globaux correspondants pour (1.2.9) en supposant que l'initiale est irrotationnelle et petite mais indépendante de $\varepsilon$. Définir l'espace

$$\begin{aligned} \|(\rho_0 - 1, u_0, \nabla\varphi_0)\|_{Y^\sigma} \triangleq\ & \|(\rho_0 - 1, u_0, \nabla\varphi_0)^L\|_{W^{\sigma+4,1}} + \|x(\rho_0 - 1, u_0, \nabla\varphi_0)^L\|_{H^{\sigma+4+\delta}} \\ & + \|x(\rho_0 - 1, u_0, \nabla\varphi_0)^h\|_{L^2} + \|(\rho_0 - 1, u_0, \nabla\varphi_0)\|_{H^{11+2\sigma}} \end{aligned} \tag{1.2.10}$$

où $\sigma \geq 0$ est un paramètre positif et $\delta = \frac{1}{1000}$. Le théorème suivant est tiré de [120].

**Theorem 1.2.3** (Sun, [120]). *Soit $\sigma \geq 0$. Il existe deux constantes $C_1 > 0$, $\vartheta_1 > 0$ tel que pour tout $\varepsilon \in (0, 1]$, tous $\bar{\vartheta} \in (0, \vartheta_1]$ si*

$$\|(u^\varepsilon_0, \rho^\varepsilon_0 - 1, \nabla\varphi^\varepsilon_0)\|_{Y^\sigma} \leq \bar{\vartheta},$$

*alors le système* (1.2.9) *admet une solution globale* $(u, \varrho, \nabla\varphi)$ *dans* $C([0,\infty), H^{\sigma+7})$, *qui bénéficie l'estimation décroissance uniforme: pour tout* $t > 0$,

$$(1+t)\|(\rho^\varepsilon - 1, \nabla u^\varepsilon, \nabla\varphi^\varepsilon)(t)\|_{W^{\sigma,\infty}} + \|(\rho^\varepsilon - 1, \nabla u^\varepsilon, \nabla\varphi^\varepsilon)(t)\|_{H^{\sigma+7}} \leq C_1\bar{\vartheta}.$$

Les stratégies pour prouver ce théorème seront présentées dans la section suivante. Néanmoins, une fois que nous considérons le système (1.2.8), la condition irrotationnel n'est pas propagée par le flot. Même si nous supposons que les données initiales sont sans curl, une partie curl de taille $\varepsilon$ est instantanément créée. De plus, du fait de la plus faible dispersion en 2-d, la partie rotationnelle de la vitesse est pilotée par un terme source dont la norme $L^2_x$ ne bénéficie au mieux que de la décroissance temporelle critique $(1+t)^{-1}$. Par conséquent, la partie rotationnelle de la vitesse est susceptible d'avoir une croissance logarithmique qui est un grand obstacle pour prouver l'existence globale. Néanmoins, nous pouvons prouver les estimations suivantes pour la durée de vie d'une solution régulière:

**Theorem 1.2.4** (Sun, [120])**.** *Il existe deux constantes* $\vartheta_0, C$, *tel que pour tout* $\varepsilon \in (0,1], \vartheta \in (0, \vartheta_0]$, *si l'hypothèse suivante est vraie:*

$$\|(\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla\varphi_0^\varepsilon)\|_{Y^4} \leq \frac{1}{C}\vartheta, \qquad \|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \vartheta\varepsilon,$$

*où la norme* $Y^4$ *est définie dans* (1.2.10), *alors le système* (1.2.8) *possède une solution dans* $C([0,T), H^3)$ *avec* $T > \varepsilon^{-(1-\vartheta)}$.

Semblable au cas 3d, la preuve de ce théorème est basée sur Théorème 1.2.3 et des estimations d'énergie faciles pour le système perturbé.

### 1.2.4 Un modèle jouet: perturbation visqueuse de l'équation de demi-Klein-Gordon avec des non-linéarités quadratiques

Cette sous-section est consacrée à l'existence globale et à la décroissance rapide du système (1.2.4) en 3d et 2d, qui peuvent être utilisées pour prouver le Théorème 1.2.1 et 1.2.4. Nous expliquons maintenant les idées principales pour la preuve. En utilisant la condition irrotationnelle, nous considérons la nouvelle inconnue $V = (\frac{\langle\nabla\rangle}{|\nabla|}\varrho, \frac{\mathrm{div}}{|\nabla|}u)$. Le système linéarisé pour $V$ est

$$\partial_t V + AV = 0, \quad A = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & -2\varepsilon\Delta \end{pmatrix}.$$

où nous utilisons $\langle\nabla\rangle = \sqrt{1-\Delta}$ le multiplicateur de Fourier avec le symbole $\sqrt{1+|\xi|^2}$. Les valeurs propres de ce système sont

$$\lambda_\pm = -\varepsilon|\xi|^2 \pm i\sqrt{1+|\xi|^2 - \varepsilon^2|\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm ib(\xi).$$

Un modèle jouet pour présenter les idées est donc

$$\begin{cases} \partial_t\beta - \lambda_-(D)\beta = \beta^2, \\ \beta|_{t=0} = \beta_0. \end{cases} \tag{1.2.11}$$

Les principales observations sont, d'une part, lorsque nous nous concentrons sur les basses fréquences, (disons $\varepsilon|\xi|^2 \leq 2\kappa_0$ avec $\kappa_0$ à choisir petit mais indépendant de $\varepsilon$) alors $b(\xi)$ est très proche de $\langle\xi\rangle$, ce indique que la partie imaginaire $e^{itb(D)}$ devrait nous donner une estimation de décroissance $L^p$ $(p > 2)$ qui est uniforme pour $\varepsilon \in (0,1]$. Par contre, quand on traite de hautes fréquences (au sens où $\varepsilon|\xi|^2 \geq \kappa_0$), les calculs directs montrent qu'il existe une constante positive $c = c(\kappa_0)$ tel que $\mathrm{Re}\,(\lambda_\pm) \leq -c(\kappa_0)$ pour tout $\varepsilon \in (0,1]$, nous nous donc attendre que la partie haute fréquence de la solution a une bonne décroissance même en normes $L^2$ norme. On peut se référer à l'introduction en anglais pour plus de détails sur l'estimation a priori pour ce modèle jouet.

### 1.2.5 Une application de la méthode de séparation.

L'idée de séparation employée dans la preuve du Théorème 1.2.1 and 1.2.2 peut également être utilisé pour prouver l'estimation de la durée de vie de solutions pour certains systèmes fluides à tourbillon non-nul pour lesquels l'existence globale avec tourbillon nul est connue. Nous nous concentrerons sur deux exemples, le système Euler-Korteweg (EK) et le système Euler-Maxwell bifluide. Nous présentons ici le résultat pour le système Euler-Korteweg qui s'écrit:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla P(\rho) - \rho \nabla \left( K(\rho)\Delta\rho + \frac{1}{2} K'(\rho)|\nabla\rho|^2 \right) = 0, \\ u|_{t=0} = u_0, \rho|_{t=0} = \rho_0 \end{cases} \tag{1.2.12}$$

où $\rho, u$ sont la densité et la vitesse du fluide, $P(\rho)$ la pression et est supposé être une fonction lisse de la densité. $K(\rho)$ est le tenseur de Korteweg, qui prend en compte les effets capillaires. Le théorème suivant est tiré de [119].

**Theorem 1.2.5** (Sun, [119])**.** *Soit $\bar\rho > 0$. Supposons que le tenseur de Korteweg $K(\rho)$ est régulier et satisfait $K(\rho) \geq K_0 > 0$ pour $\bar\rho/2 \leq \rho \leq 3\bar\rho/2$. Supposons également que la pression satisfait: $P'(\rho)/\rho > 0$ pour $\bar\rho/2 \leq \rho \leq 3\bar\rho/2$. Alors il existe trois constantes $\delta_1, \epsilon_1 > 0$ petites, et $N$ grand telles que si les données initiales $(\rho_0 - \bar\rho, u_0)$ satisfont:*

$$\begin{gathered} \|\mathcal{P}u_0\|_{H^s} < \epsilon_1, \\ \|\mathcal{P}^\perp u_0\|_{H^N} + \|\rho_0 - \bar\rho\|_{H^{N+1}} + \|x(\rho_0 - \bar\rho, \mathcal{P}^\perp u_0)\|_{L^2} \\ + \|u_0\|_{W^{6,1}} + \|\rho_0 - \bar\rho\|_{W^{7,1}} \leq \delta_1, \end{gathered}$$

*où $5/2 < s \leq 3$. Alors il existe $T_{\epsilon_1} \gtrsim \epsilon_1^{-1}$ tel que les équations (1.2.12) admettent une solution unique et*

$$(\rho - \bar\rho, u) \in C([0, T_{\epsilon_1}], H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)).$$

*De plus, si $\mathcal{P}u_0 \in H^N$, alors la solution*

$$(\rho - \bar\rho, u) \in C([0, T_{\epsilon_1}], H^{N+1}(\mathbb{R}^3) \times H^N(\mathbb{R}^3))$$

*vérifie pour tout $0 \leq t \leq T_{\epsilon_1}$,*

$$\|(\rho - \bar\rho, u)(t)\|_{H^{N+1}\times H^N} \lesssim e^{ct} \|(\rho - \bar\rho, u)(0)\|_{H^{N+1}\times H^N}.$$

Notre idée pour prouver le Théorème 1.2.5 est de diviser (1.2.12) en deux systèmes, le premier garde la même forme que (1.2.12) mais avec les données initiales irrotationnelles, pour laquelle l'existence globale et la décroissance temporelle sont connues [10]. La tâche pour nous est simplement de prouver l'existence du système perturbé. Bien sûr, en raison de la présence d'ordre supérieur, nous devons utiliser une 'technique de jauge' pour éviter de perdre des dérivés dans les estimations d'énergie. On se référer au Chapitre 4 pour plus de détails.

Notez qu'un résultat similaire est également obtenu par Audiard [9]. Cependant, la méthode que nous proposons repose uniquement sur les estimations d'énergie et les estimations de décroissance rapide du système avec un tourbillon trivial, et est donc flexible. De plus, nous n'exigeons pas l'hypothèse de localisation sur la partie rotationnelle de la vitesse initiale $\mathcal{P}u_0$.

## 1.3 Limite incompressible.

### 1.3.1 Contexte.

Nous considérons les équations de Navier-Stokes compressibles suivantes qui exprimer la conservation de la quantité de mouvement et la conservation de la masse pour les fluides newtoniens:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\tilde{\mathcal{L}}u + \dfrac{\nabla P(\rho)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0, \rho|_{t=0} = \rho_0, \end{cases} \tag{1.3.1}$$

où $\Omega \subset \mathbb{R}^3$ est un domaine régulier, $\rho(t,x)$ et $u(t,x)$ sont la densité et la vitesse du fluide respectivement, $P(\rho)$ est la pression qui est une fonction régulier donnée de la densité satisfaisant $\frac{\mathrm{d}P}{\mathrm{d}\rho} > 0$, pour $\rho > 0$. Le tenseur des contraintes visqueuses prend la forme:

$$\tilde{\mathcal{L}}u = 2\tilde{\mu}\mathbb{S}u + \tilde{\lambda}\mathrm{div}u\mathrm{Id}, \quad \mathbb{S}u = \frac{1}{2}(\nabla u + \nabla^t u).$$

Ici, $\tilde{\mu}, \tilde{\lambda}$ sont paramètres de viscosité supposés constants et satisfaisant à la condition: $\tilde{\mu} > 0, 2\tilde{\mu} + 3\tilde{\lambda} > 0$. En effectuant la mise à l'échelle suivante:

$$\rho^\varepsilon(\varepsilon t, x) = \rho(t,x), \quad \varepsilon u^\varepsilon(\varepsilon t, x) = u(t,x), \quad \varepsilon\mu = \tilde{\mu}, \quad \varepsilon\lambda = \tilde{\lambda},$$

alors $(\rho^\varepsilon, u^\varepsilon)$ satisfaire $(CNS)_\varepsilon$

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mathrm{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \tag{1.3.2}$$

avec $\mathcal{L}u = 2\mu\mathbb{S}u + \lambda\mathrm{div}u\,\mathrm{Id}$. Mesurant la compressibilité des fluides, le paramètre $\varepsilon$ est le nombre de Mach et est défini par le rapport entre la vitesse du fluide et la vitesse du son. Pour illustrer le lien entre les fluides compressibles et les fluides incompressibles, nous supposerons qu'il est petit, c'est-à-dire $\varepsilon \in (0,1]$. Formellement, en raison du terme singulièr $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, la pression (et donc la densité $\rho^\varepsilon$) devrait tendre vers un état constant. On s'attend donc à obtenir dans la limite une solution au système Navier-Stokes incompressible suivant:

$$\begin{cases} \bar{\rho}(\partial_t u^0 + \mathrm{div}(u^0 \otimes u^0)) - \Delta u^0 + \nabla \pi^0 = 0, \\ \mathrm{div}u^0 = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^0|_{t=0} = u_0^0, \end{cases} \tag{1.3.3}$$

Ce processus limite est donc fréquemment appelé limite incompressible.

La justification rigoureuse de ce processus limite a été largement étudiée dans des contextes différents selon la généralité du système (isentropique ou non isentropique), le type de système (Navier-Stokes ou Euler), le type de solutions (solutions fortes ou solutions faibles), les propriétés du domaine (tout l'espace, tore ou domaine borné avec diverses conditions aux limites), ainsi que le type de données initiales considérées (bien préparées ou mal préparées). En gros, dans le cas du système compressible d'Euler, on prouve d'abord que la solution forte existe sur un intervalle de temps indépendant du nombre de Mach, puis des arguments de compacité sont développés pour passer à la limite. Dans le cas du système compressible Navier-Stokes, on peut soit essayer d'utiliser la même approche que pour le cas non visqueux (prouver l'existence d'une solution forte sur un intervalle de temps indépendant du nombre de Mach et puis essayer de passer à la limite) ou essayer de passer à la limite directement à partir de solutions faibles globales. Les deux approches ont été utilisées dans des domaines sans frontières (espace entier ou tore), néanmoins, lorsqu'une frontière est présente, la question de la régularité élevée uniforme pour les données générales est plus difficile et n'a pas été est traité.

La justification mathématique de la limite incompressible a été initiée par Ebin [39], Klainerman-Majda [79, 80] pour des *solutions locales fortes* de fluides compressibles (Navier-Stokes ou Euler) dans $\mathbb{R}^3$ avec des données bien préparées ($\mathrm{div}u_0^\varepsilon = o(\varepsilon), \nabla P_0^\varepsilon = o(\varepsilon^2)$) et ensuite par Ukai [125] pour données mal préparées ($\mathrm{div}u_0^\varepsilon = o(1), \nabla P_0^\varepsilon = o(\varepsilon)$). Dans ce dernier cas, il existe des ondes acoustiques d'amplitude 1 et de fréquence $\varepsilon^{-1}$ dans le système. Ces travaux ont été prolongés par plusieurs auteurs dans différents contextes. Par exemple, on peut se référer à [2, 20, 100, 101] pour le système non isentropique et des données initiales mal préparées chaque fois que le domaine est $\mathbb{R}^3$ ou le tore, et aussi [74, 109] pour les domaines bornés avec des données initiales bien préparées. Les estimations de régularité uniformes (en nombre de Mach) pour les équations d'Euler non isentropiques dans le domaine borné sont établies dans [1]. La limite incompressible de *solutions faibles* pour le système des fluides visqueux a été étudié par Lions et Masmoudi [89], [90] qui ont établi la convergence des solutions faibles globales du système isentropique de Navier-Stokes vers une solution du système incompressible. En général, pour des données mal préparées, on ne peut obtenir qu'une convergence faible en temps, néanmoins, en utilisant la dispersion des ondes acoustiques dans tout l'espace, Desjardins et Grenier [34] peuvent obtenir une convergence locale forte. Il existe également de nombreux autres travaux connexes, on peut voir par exemple [14, 30, 32, 41, 45, 54, 71]. Pour des informations plus exhaustives, on peut se référer par exemple aux survols bien rédigés par Alazard [3], Danchin [31], Feireisl [43], Gallagher [46], Jiang-Masmoudi [73], Schochet [110].

### 1.3.2 Limite incompressible pour système visqueux dans un domaine avec bords fixés.

Puisque nous traitons les systèmes dans un domaine avec des bords, nous devons compléter le système (1.3.2) avec des conditions aux bords. Il existe typiquement deux types de conditions aux limites, à savoir la condition aux limites de Dirichlet (sans glissement) et condition aux limites de Navier-glissement.

- Condition aux limites de Dirichlet:

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1.3.4}$$

- Condition aux limites de Navier-glissement:

$$u^\varepsilon \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^\varepsilon \mathbf{n}) + a\Pi u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1.3.5}$$

où $a$ est une constante liée à une longueur de glissement (notre analyse est également valable si $a$ est une fonction régulière). On utilise la notation $\Pi f$ pour la partie tangentielle d'un vecteur $f$, $\Pi f^\varepsilon = f^\varepsilon - (f^\varepsilon \cdot \mathbf{n}) \cdot \mathbf{n}$. La condition aux limites de Dirichlet (1.3.4) indique que la particule de fluide ne peut pas glisser sur la frontière tandis que la condition de Navier-slip, qui est proposée par Navier [103], indique que la vitesse tangentielle du fluide au bord est reliée au tenseur de contraintes comme dans (1.3.5).

Notre premier objectif est d'obtenir des estimations de régularité d'ordre élevé uniformes (par rapport à $\varepsilon$) pour $(CNS)_\varepsilon$ dans des domaines bornés avec des données initiales mal préparées et des conditions aux limites de Navier-glissement afin d'obtenir l'existence d'une solution forte locale sur un intervalle de temps indépendant de $\varepsilon$. Il n'existe que quelques articles traitant de cette question. Dans [105], les auteurs établissent des estimations globales uniformes (pour des petites données) $H^2$ sous une hypothèse de données initiale (très) *bien préparées*, à savoir la deuxième dérivée temporelle de la vitesse doit être borné initialement. Pour les données initiales mal préparées, la situation est plus subtile et une estimation uniforme dans $H^2$, même localement en temps, ne peut pas être obtenue. En effet, au premier ordre, après linéarisation et symétrisation, le système (1.3.2) devient:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon} L U^\varepsilon - \begin{pmatrix} 0 \\ \mathrm{div}\mathcal{L}u^\varepsilon. \end{pmatrix} = 0, \qquad L = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \quad U = (\sigma^\varepsilon, u^\varepsilon) \in \mathbb{R} \times \mathbb{R}^3. \tag{1.3.6}$$

En raison de la présence du terme de diffusion ainsi que du terme linéaire singulier, une correction de couche limites apparaît pour les ondes acoustique de sorte que les dérivées normales (l'order $\geq 2$) de la vitesse ne peuvent pas être bornées. Notez qu'ici, on ne part pas d'un problème de petit viscosité, néanmoins, à l'échelle $\tau = t/\varepsilon$, le système (1.3.6) se comporte comme une approximation du système acoustique par une petite diffusion. Par exemple, dans le cas le plus simple où le bord est plat (par exemple $\Omega = \mathbb{R}^3_+$), nous attendons le développement suivant des solutions à (1.3.6) impliquant des couches limites

$$\begin{cases} \sigma^\varepsilon(t,x) = \sigma_0^I(\frac{t}{\varepsilon}, t, x) + \varepsilon^{\frac{3}{2}} \sigma^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots, \\ u^\varepsilon(t,x) = u_0^I(\frac{t}{\varepsilon}, t, x) + \sqrt{\varepsilon} \begin{pmatrix} u_{1,\tau}^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) \\ 0 \end{pmatrix} + \varepsilon u_2^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots \end{cases} \tag{1.3.7}$$

où $x = (y, z)$, $z > 0$, ce qui suggère que $\|u_\tau\|_{L_t^2 H^1}, \|u_3^\varepsilon\|_{L_t^2 H^2}, \|\sigma^\varepsilon\|_{L_t^2 H^3}$ peuvent être contrôlés uniformément tandis que $\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_{t,x}}$ et $\|\partial_z^2 u_\tau^\varepsilon\|_{L^2_{t,x}}$ exploseront lorsque $\varepsilon$ tend vers 0.

Pour continuer, nous introduisons la nouvelle inconnue

$$\sigma^\varepsilon = \frac{P(\rho^\varepsilon) - P(\bar{\rho})}{\varepsilon}$$

où $\overline{\rho}$ est un état constant positif, nous pouvons réécrire le système (1.3.2) sous la forme suivante, ce qui est plus pratique pour effectuer des estimations d'énergie:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t\sigma^\varepsilon + u^\varepsilon\cdot\nabla\sigma^\varepsilon\big) + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon\big)(\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon\big) - \mathrm{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \sigma|_{t=0} = \sigma_0^\varepsilon. \end{cases} \tag{1.3.8}$$

où les fonctions scalaires $g_1, g_2$ sont définies par

$$g_2(s) = \rho^\varepsilon = P^{-1}(\bar{P}+s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar{P} = -P(\bar{\rho}). \tag{1.3.9}$$

Nous allons maintenant expliquer les difficultés pour obtenir des estimations uniformes. Comme déjà mentionné, la principale caractéristique de notre problème est la présence d'oscillations temporelles rapides et une couche limite en espace. Ces deux aspects sont bien compris lorsqu'ils se produisent séparément, mais afin de les gérer simultanément, de nouvelles idées seront nécessaires. D'une part, concernant le problème de la limite non visqueuse, on contrôle [93, 106, 127] les dérivées tangentielles d'ordre élevé par des estimations d'énergie, puis on utilise le tourbillon pour contrôler les dérivées normales. Néanmoins, pour le système à faible nombre de Mach, même les estimations de dérivées tangentielles ne sont pas faciles à obtenir, car les dérivées tangentielles spatiales ne commutent pas avec $\nabla, \mathrm{div}$, défini avec les dérivations standard dans $\mathbb{R}^3$ , sauf si la frontière est plate, et donc créent des commutateurs singuliers. Sans cette connaissance a priori sur les dérivées tangentielles, l'estimation du tourbillon ne peut pas être effectuée en raison du manque d'information sur sa trace sur la frontière. D'autre part, pour le système Euler compressible à faible nombre de Mach, des estimations uniformes de haut régularité sont établies par exemple dans [1]. On peut obtenir des estimations $H^s(s>5/2)$ uniformes en utilisant d'abord les dérivés $\varepsilon\partial_t$, puis récupérer des dérivés spatiales en utilisant les équations pour estimer la divergence de la vitesse et le gradient de la pression et les estimations d'énergie directe pour le tourbillon qui résout une équation de transport avec un champ de vecteur caractéristique. Ici, dans le cas des fluides visqueux, on est confronté au fait que les estimations du tourbillon sont difficiles en raison du manque d'information sur sa trace au bord à ce stade.

#### 1.3.2.1 Estimations uniformes pour le domaine borné fixe.

Comme indiqué dans (1.3.7), il semble impossible que les solutions à (1.3.2) restent bornées dans un espace de Sobolev d'ordre élevé, nous devons donc utiliser un cadre fonctionnel basé sur des espaces de 'Sobolev conormaux' qui minimisent l'utilisation de dérivés normales proches de la frontière inspiré par [93, 94]. En introduisant des poids près des bords limites, les 'espaces conormaux' sont cohérents avec les effets des couches limites. Quand $\Omega = \mathbb{R}^3_+$, il est défini par:

$$L^p_t H^m_{co}\big(\mathbb{R}^3_+\big) = \{Z^\alpha f \in L^p([0,t]; L^2(\mathbb{R}^3_+)), |\alpha|\le m\},$$

$$L^p_t H^m_{tan}\big(\mathbb{R}^3_+\big) = \{Z^\alpha f \in L^p([0,t]; L^2(\mathbb{R}^3_+)), |\alpha|\le m, \alpha_3 = 0\},$$

où $\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$, $Z^\alpha = (\varepsilon\partial_t)^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$, et les champs de vecteurs tangentiels au bord sont défini par (supposons $x = (y,z)$):

$$Z_1 = \partial_{y_1} = \partial_1, \quad Z_2 = \partial_{y_2} = \partial_2, \quad Z_3 = \phi(z)\partial_z = \frac{z}{1+z}\partial_z.$$

Pour capturer l'interaction des oscillation temporelles rapides et des effets de couche limite, on peut se référer à la version anglaise de l'introduction pour les estimations uniformes lorsque $\Omega = \mathbb{R}^3_+$. Cependant, lorsque $\Omega$ est un domaine fixe avec des bords non plats, il y aurait des difficultés supplémentaires pour obtenir des estimations uniformes. En effet, même les dérivées tangentielles $(\mathbf{n}\times\nabla)_1, (\mathbf{n}\times\nabla)_2$, ($\mathbf{n}$ représente le vecteur normal extérieur) ne peuvent pas être contrôlés directement car ils ne commutent pas avec $\nabla, \mathrm{div}$ défini avec les dérivations standard dans $\mathbb{R}^3$ et donc les estimations directes de l'énergie créeraient des commutateurs. Sans ce contrôle a priori des dérivées tangentielles, on ne peut pas réduire le contrôle du gradient de vitesse à celui de $\mathrm{div} u^\varepsilon$ et $\omega^\varepsilon\times\mathbf{n}$. Afin d'obtenir les informations manquantes, nous utiliserons d'abord la projection de Leray (voir la définition précise (6.3.2)) pour séparer la vitesse en une partie compressible et une partie incompressible: $u^\varepsilon = \nabla\Psi^\varepsilon + v^\varepsilon$. D'une part, la partie compressible $\nabla\Psi^\varepsilon$ de la vitesse peut être contrôlée par $\mathrm{div} u^\varepsilon$ grâce à la théorie elliptique standard et donc en utilisant les équations et les estimations pour les dérivées $\varepsilon\partial_t$. Par contre, la partie incompressible $v^\varepsilon$ résout une équation de convection-diffusion sans oscillations, et donc on peut utiliser des estimations d'énergie directe pour obtenir un contrôle de $\|v^\varepsilon\|_{L_t^\infty H_{co}^{m-1}}$ et $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. Notez que nous ne pouvons pas estimer le nombre maximal de dérivées $m$ en raison du manque de structure des termes de couplage impliquant la partie compressible dans les estimations d'énergie. Le point clé ici est que la diffusion (qui en revanche crée de nouvelles difficultés dans le contrôle du tourbillon) permet d'obtenir l'estimation de $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. La tâche restante est de contrôler la norme $L_t^\infty H_{co}^{m-2}$ de $\nabla v^\varepsilon$, qui peut être dérivée des estimations de $\omega^\varepsilon \times \mathbf{n}$. Pour établir les estimations uniformes, nous utilisons les quantités

$$\mathcal{N}_{m,T}^\varepsilon \approx \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty \mathcal{H}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^{m-2} \cap L_T^2 H_{co}^{m-1} \cap L_T^\infty L^\infty}$$

où $L_t^\infty \mathcal{H}^m$ n'implique que des dérivés temporelles à poids $\varepsilon\partial_t$ à l'ordre $m$. Nous nous référons à la section (6.1.1) pour la définition des espaces conormaux et (6.1.17) pour la définition précise de $\mathcal{N}_{m,T}^\varepsilon$.

Ce qui suit est notre résultat principal qui est le mélange de Théorèmes 6.1.1 et 6.1.6 dans [95]:

**Theorem 1.3.1** (Masmoudi-Rousset-Sun, [95]). *Donné un entier $m \geq 6$ et un domaine borné $\Omega$ qui est $C^{m+2}$ régulier. Considérons une famille de données initiales $(\sigma_0^\varepsilon, u_0^\varepsilon)$ satisfaisant à certaines conditions de compatibilité raisonnables (voir* (2.3.11)*) et*

$$\sup_{\varepsilon \in (0,1]} \mathcal{N}_{m,0}^\varepsilon < +\infty,$$

$$-\bar{c}\bar{P} \leq \varepsilon\sigma_0^\varepsilon(x) \leq \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0,1],$$

*où $0 < \bar{c} < 1/4$ est une constante fixe, $\bar{P} = P(\bar{\rho})$. Il existe $\varepsilon_0 \in (0,1]$ et $T_0 > 0$, tel que, pour tout $0 < \varepsilon \leq \varepsilon_0$, le système* (1.3.2) *avec condition aux limites* (1.3.5) *admets une solution $(\sigma^\varepsilon, u^\varepsilon)$ unique qui satisfait:*

$$-2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 2\bar{P}/\bar{c}, \quad \forall (t,x) \in [0,T_0] \times \Omega,$$

*and*

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \mathcal{N}_{m,T_0}^\varepsilon < +\infty.$$

*De plus, $\rho^\varepsilon = g_2(\varepsilon\sigma^\varepsilon)$ converge vers $\bar{\rho}$ dans $C([0,T_0], L^2)$, $u^\varepsilon$ converge dans $L_w^2([0,T_0], L^2(\Omega))$ (convergence faible en temps) vers $u^0$ qui est la (unique dans la classe avec une régularité supplémentaire) solution faible de l'équation de Navier-Stokes incompressible* (1.3.3) *avec condition aux limites de Navier.*

Nous terminerons cette sous-section en esquissant les estimations a priori pour $\mathcal{N}_{m,T}^\varepsilon$ ce qui conduit au Théorème 1.3.1. Ceci sera réalisé dans les étapes suivantes (nous omettons la dépendance $\varepsilon$ dans les notations pour simplifier).

**Étape 1: Contrôle uniforme des dérivées temporelles pondérées et $\varepsilon$-dépendant contrôle des dérivés conormales** Dans cette étape, nous visons à prouver deux types d'estimations. À savoir, des estimations uniformes pour les dérivés $\varepsilon\partial_t$, d'ordre élevé $\|(\sigma,u)\|_{L_t^\infty \mathcal{H}^m}, \|(\nabla\sigma, \mathrm{div} u)\|_{L_t^\infty \mathcal{H}^{m-1}}$, et $\varepsilon-$dépendant estimations: $\varepsilon\|(\sigma,u)\|_{L_t^\infty H_{co}^m}$. Comme les champs de vecteurs conormaux spatiaux ne commutent pas avec $\nabla, \mathrm{div}$, la partie singulière du système, nous devons à ce stade ajouter ce poids $\varepsilon$ supplémentaire pour contrôler le commutateur.

**Étape 2: Estimations uniformes pour la partie incompressible de la vitesse.** Notons $v = \mathbb{P}u$, et $\nabla\Psi = \mathbb{Q}u$ le partie incompressible et compressible de vitesse respectivement, où $\mathbb{P}, \mathbb{Q}$ sont définis dans (6.3.2). En appliquant la projection $\mathbb{P}$ sur l'équation de la vitesse et en développant les des conditions aux limites, nous trouvons que $v$ résout:

$$\begin{cases} \bar{\rho}\partial_t v - \mu\Delta v + \nabla q + \frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u = 0 \quad \text{dans} \quad \Omega \\ v \cdot \mathbf{n} = 0, \quad \Pi(\partial_{\mathbf{n}} v) = \Pi(-2au + D\mathbf{n} \cdot \nabla\Psi + D\mathbf{n} \cdot u) \quad \text{sur} \quad \partial\Omega \end{cases}$$

où

$$\nabla q = -\mathbb{Q}\Big(\frac{g_2 - 1}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u - \mu\Delta v\Big).$$

La partie incompressible $v$ interagit avec la partie compressible $\nabla\Psi$ via le terme source et la condition aux limites. En raison de l'absence de termes singulier, on peut obtenir les estimations uniformes pour $v$ ($\|v\|_{L_t^\infty H_{co}^{m-1}}$ et $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$) par des estimations d'énergie directe.

**Étape 3: Estimations uniformes pour la partie compressible du système.** Dans cette étape, nous visons à contrôler $\|(\nabla\sigma, \mathrm{div}u)\|_{L_t^\infty H_{co}^{m-2}\cap L_t^2 H_{co}^{m-1}}$. Cela peut être fait en utilisant les équations et les arguments de récurrence.

**Étape 4**: Le contrôle de $\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}$. La difficulté est l'estimation proche du bord. Nous pouvons travailler dans une carte locale $\Omega_i$. Au vu des identités:

$$\partial_{\mathbf{n}} v \cdot \mathbf{n} = -(\Pi\partial_{y_1} v)^1 - (\Pi\partial_{y_2} v)^2, \quad \Pi(\partial_{\mathbf{n}} v) = \Pi(\omega \times \mathbf{n}) - \Pi[(D\mathbf{n})v],$$

où $\mathbf{n}$ est une extension de la normale unité et $\Pi$ projette sur $(\mathbf{n})^\perp$, il suffit de contrôler $\|\omega \times \mathbf{n}\|_{L_t^\infty H_{co}^{m-2}}$. On remarque que l'avantage de travailler sur $\omega \times \mathbf{n}$ plutôt que $\omega$ est que la condition aux limites pour $\omega \times \mathbf{n}$ (voir (6.3.33)) n'implique que des termes d'ordre inférieur sur la frontière. Pour estimer $\omega \times \mathbf{n}$, une manière naturelle, utilisé dans [93], consiste à effectuer des estimations d'énergie sur l'équation du 'tourbillon modifié' $w = \omega \times \mathbf{n} + 2\Pi(au - (D\mathbf{n})u)$ et pour profiter du fait que $w$ s'annule à la frontière. Cependant, les équations pour $w$ impliquent toujours un terme raide $\frac{1}{\varepsilon}\nabla^\perp\sigma$, ce qui est évidemment un obstacle à l'obtention d'estimations uniformes. Nous utiliserons donc la technique de 'splitting' pour obtenir l'estimation de $\omega \times \mathbf{n}$.

**Étape 5: Estimations des normes de type $L_{t,x}^\infty$ .** Le contrôle de la $L_{t,x}^\infty$ normes provient principalement de l'injection de Sobolev et du principe du maximum pour le système résolu par le tourbillon.

### 1.3.3 Limite incompressible pour les équations de Navier-Stokes en surface libre

Dans cette sous-section, nous considérons un écoulement compressible dans un domaine à frontières libres, où la frontière se déplace selon le mouvement du fluides. Plus précisément, nous nous intéressons aux système suivant:

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon w^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon w^\varepsilon) + \mathrm{div}(\rho^\varepsilon w^\varepsilon \otimes w^\varepsilon) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \Omega_t^\varepsilon \tag{1.3.10}$$

où $\rho^\varepsilon, w^\varepsilon$ sont la densité et la vitesse des fluides, $P(\rho^\varepsilon)$, une fonction régulière de $\rho^\varepsilon$, représente la pression. Le tenseur des contraintes $\mathcal{L}u^\varepsilon$ est

$$\mathcal{L}w^\varepsilon = \mu(\nabla w^\varepsilon + \nabla^t w^\varepsilon) + \lambda \mathrm{div} w^\varepsilon \mathrm{Id} =: 2\mu S w^\varepsilon + \lambda \mathrm{div} w^\varepsilon \mathrm{Id}.$$

Nous nous concentrons sur le cas où le domaine fluide est donné par:

$$\Omega_t^\varepsilon = \{x = (y,z) | y \in \mathbb{R}^2, z < h^\varepsilon(t,y)\}.$$

Ici, $h(t,y)$, la surface du domaine fluide est inconnue et doit être résolue avec $(\rho^\varepsilon, w^\varepsilon)$. Comme les particules de fluide ne traversent pas la surface, elle est régie par l'équation suivante:

$$\partial_t h^\varepsilon - w^\varepsilon(t,y,h^\varepsilon(t,y)) \cdot \mathbf{N}^\varepsilon = 0, \quad y \in \mathbb{R}^2 \tag{1.3.11}$$

où $\mathbf{N}^\varepsilon = (-\partial_1 h^\varepsilon, -\partial_2 h^\varepsilon, 1)^t$ désigne le vecteur normal extérieur à la surface $\Sigma_t^\varepsilon = \{x = (y,z), z = h^\varepsilon(t,y)\}$. Nous complétons le système (1.3.10) et (1.3.11) avec les conditions aux limites suivantes:

$$\mathcal{L}u^\varepsilon \mathbf{N}^\varepsilon = \frac{1}{\varepsilon}\big(P(\rho^\varepsilon) - P(\bar\rho)\big)\mathbf{N}^\varepsilon \quad \text{sur} \quad \Sigma_t^\varepsilon \tag{1.3.12}$$

où $\bar\rho > 0$ est la densité constante de référence. Nous remarquons que (1.3.12) exprime la continuité du tenseur des contraintes sur la surface.

Concernant le problème de limite incompressible pour les systèmes fluidiques dans un domaine en mouvement, il n'y a que quelques travaux sur les systèmes non visqueux compressibles à surface libre. Dans [88], Lindblad-Luo prouve d'abord les estimations a priori uniformes pour les équations d'Euler compressibles à surface libre dans un domaine de référence borné. Plus récemment, ce résultat est prolongé par Luo [92] pour domaine de référence illimité et par Disconzi-Luo [37] pour domaine de référence borné mais avec tension superficielle. Tous ces résultats reposent sur l'hypothèse que la donnée initiale est suffisamment bien préparée en ce sens que les dérivées temporelles jusqu'à au moins d'ordre deux sont bornées initialement. Néanmoins, à notre connaissance, il n'y a pas de travaux connexes pour le système visqueux. En raison de l'absence d'une théorie appropriée des solutions faibles, nous traiterons des solutions suffisamment fortes.

La première étape de l'étude du l'équation à surface libre est d'introduire une certaine transformation pour réduire le problème au cas du domaine de correction. Nous utiliserions le changement de variable suivant donné par l'extension harmonique:

$$\begin{aligned} \Phi_t^\varepsilon : \mathcal{S} = \mathbb{R}^2 \times [-\infty, 0] &\to \Omega_t^\varepsilon \\ (y, z) &\to \Phi^\varepsilon(t, y, z) = (y, \varphi^\varepsilon(t, y, z))^t \end{aligned}$$

où

$$\varphi^\varepsilon(t, y, z) = Az + \eta^\varepsilon(t, x) \tag{1.3.13}$$

Ici $\eta^\varepsilon$ est donné par

$$(\mathcal{F}\eta^\varepsilon)(t, \xi, z) = e^{-(1+|\xi|^2)z^2}(\mathcal{F}h^\varepsilon)(t, \xi), \tag{1.3.14}$$

où $\mathcal{F}$ représente la transformée de Fourier par rapport à la variable horizontale, $A > 0$ est une constante à choisir plus tard suffisamment grande tel que, dans un intervalle de temps d'existence $[0, T]$,

$$\sup_{0\le t\le T} \|\partial_z \eta^\varepsilon(t)\|_{L^\infty(\mathcal{S})} < A/2. \tag{1.3.15}$$

Notez que $\Phi_t^\varepsilon$ est un difféomorphisme quand (1.3.15) est satisfait car $D\Phi^\varepsilon(t, x) = \partial_z\varphi^\varepsilon = A + \partial_z\eta^\varepsilon > 0$. En définissant

$$\varrho^\varepsilon(t, x) = \rho^\varepsilon(t, \Phi_t^\varepsilon(x)), u^\varepsilon(t, x) = w^\varepsilon(t, \Phi_t^\varepsilon(x))$$

et en introduisant une nouvelle inconnue

$$\sigma^\varepsilon = (P(\varrho^\varepsilon) - P(\bar\rho))/\varepsilon,$$

nous reformulons le problème pour étudier le système suivant:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t^{\varphi^\varepsilon}\sigma^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\sigma^\varepsilon\big) + \dfrac{\operatorname{div}^{\varphi^\varepsilon} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon\big)\big(\partial_t^{\varphi^\varepsilon} u^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon} u^\varepsilon\big) - \operatorname{div}^{\varphi^\varepsilon}\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon + \dfrac{\nabla^{\varphi^\varepsilon}\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega_t^\varepsilon \\ u^\varepsilon|_{t=0} = w_0^\varepsilon(\Phi_0^\varepsilon(x)) := u_0^\varepsilon, \quad \sigma^\varepsilon|_{t=0} = \varrho_0^\varepsilon(\Phi_0^\varepsilon(x)) := \sigma_0^\varepsilon. \end{cases} \tag{1.3.16}$$

$$\partial_t h^\varepsilon - u^\varepsilon(t, y, h(t, y))\cdot\mathbf{N}^\varepsilon = 0. \tag{1.3.17}$$

$$\mathcal{L}^\varphi u^\varepsilon \mathbf{N}^\varepsilon = \frac{\sigma^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon \quad \text{sur} \quad \{z = 0\}. \tag{1.3.18}$$

où

$$\partial_i^{\varphi^\varepsilon} = \partial_i - \frac{\partial_i\varphi^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z, \quad i = 0, 1, 2, \quad \partial_z^{\varphi^\varepsilon} = \frac{1}{\partial_z\varphi^\varepsilon}\partial_z. \tag{1.3.19}$$

Nous utiliserons les champs de vecteurs tangentiels suivants: $Z_0 = \varepsilon\partial_t, Z_1 = \partial_{y_1}, Z_2 = \partial_{y_2}, Z_3 = \frac{z}{1-z}\partial_z$, et les normes conormales:

$$\|f\|_{L_t^p\mathcal{H}^m} = \sum_{k\le m}\|(\varepsilon\partial_t)^k f\|_{L^p([0,t],L^2(\mathcal{S})}, \quad \|f\|_{L_t^p H_{co}^m} = \sum_{|\alpha|\le m}\|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))},$$

$$|g|_{L_t^p\tilde{H}^s} = \sum_{k=0}^{[s]} |(\varepsilon\partial_t)^k h|_{L^p([0,t],H^{s-k}(\mathbb{R}^2))}$$

où $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$. Il est utile d'introduire la quantité suivante composée de normes conormales et définie précisément en (7.1.30):

$$\begin{aligned}\mathcal{N}^{\varepsilon}_{m,T} \approx\, & \varepsilon^{-\frac{1}{2}}\|(\nabla^{\varphi^\varepsilon}\sigma^\varepsilon, \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon)\|_{L^\infty_T H^{m-2}_{co}\cap L^2_t H^{m-1}_{co}\cap L^\infty_T L^\infty} + \|\nabla u^\varepsilon\|_{L^\infty_T H^{m-4}_{co}\cap L^2_T H^{m-1}_{co}\cap L^\infty_T L^\infty} \\ & + \varepsilon^{\frac{1}{2}}\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_T \mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}}\|\partial_t \nabla u^\varepsilon\|_{L^2_T\mathcal{H}^{m-1}\cap L^2_T H^{m-2}_{co}\cap L^\infty_T H^{m-4}_{co}} \\ & + \varepsilon^{\frac{1}{2}}\|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_T H^m_{co}} + \varepsilon^{\frac{1}{2}}\|\nabla u^\varepsilon\|_{L^\infty_T H^{m-1}_{co}\cap L^2_T H^m_{co}} + |h|_{L^\infty_t \tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|h|_{L^\infty_t \tilde{H}^{m+\frac{1}{2}}}.\end{aligned}$$

Nous sommes maintenant prêts à présenter notre résultat principal:

**Theorem 1.3.2** (Masmoudi-Rousset-Sun, [96])**.** *Soit $m \geq 7$ un nombre entier. Supposons que la donnée initiale satisfait certaines conditions de compatibilité (voir* (7.1.27)*) et*

$$\sup_{\varepsilon\in(0,1]} \mathcal{N}^\varepsilon_{m,0} < +\infty,$$

$$-c_1\bar{P} \leq \varepsilon\sigma^\varepsilon_0(x) \leq \bar{P}/c_1, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1]$$

*où $0 < c_1 < \frac{1}{4}$ est une constante fixe, $\bar{P} = P(\bar{\rho})$. Alors il existe $T_0 \in (0,1], \varepsilon_0 \in (0,1]$, tel que $\forall \varepsilon, 0 < \varepsilon \leq \varepsilon_0$ le système* (1.3.16)-(1.3.18) *admet une solution unique qui satisfait:*

$$\mathcal{N}^\varepsilon_{m,T_0} < +\infty,$$

$$-2c_1\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 2\bar{P}/c_1, \forall (t,x) \in [0,T_0]\times\mathcal{S}, \varepsilon \in (0,\varepsilon_0].$$

*De plus, quand $\varepsilon$ tends vers 0, $(\bar{P} + \varepsilon\sigma^\varepsilon, u^\varepsilon, h^\varepsilon)$ converge dans $C^\gamma([0,T_0]\times\mathcal{S}) \times C([0,T_0], L^2_{loc}(\mathcal{S})) \times C([0,T_0], H^s_{loc}(\mathbb{R}^2))$ $(0 \leq \gamma < \frac{1}{2}, 0 \leq s < m - \frac{1}{2})$ vers $(0, u^0, h^0)$ qui est la solution de l'équations de Navier-Stokes incompressibles à surface libre (voir* (7.1.33)*).*

**Remark 1.3.3.** *Nous avons supposé que la restriction de la dérivée première de la solution à $t = 0$ est d'ordre $\varepsilon^{-\frac{1}{2}}$, ce qui est légèrement meilleur que les données bien préparées case (où $\partial_t(\sigma^\varepsilon, u^\varepsilon)|_{t=0}$ est supposé être l'ordre* 1*). Cette hypothèse doit être faite en raison d'une possible perte de régularité sur la surface.*

**Remark 1.3.4.** *On peut aussi prouver les estimations uniformes en imposant une hypothèse alternative sur la taille des ondes acoustiques, c'est-à-dire qu'elles sont de taille d'ordre $\varepsilon$ avec une régularité assez faible $H^1_{co}$ mais d'ordre* 1 *dans la régularité supérieure $H^m_{co}$.*

Les estimations uniformes sont obtenues grâce à des modifications soigneuses des stratégies employées dans le cas du domaine fixe, et une attention supplémentaire doit être accordée à la régularité de la surface. Comme précédemment, puisque les champs de vecteurs tangentiels spatiaux ne commutent pas avec $\nabla^{\varphi^\varepsilon}, \mathrm{div}^{\varphi^\varepsilon}$, nous n'avons pas la préconnaissance des dérivées tangentielles d'ordre supérieur. Par conséquent, nous devons introduire le projecteur de Leray approprié et diviser le système en une partie compressible et une partie incompressible. Nous déduisons les estimations pour la partie compressible de celles de $\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon$ par des estimations elliptiques appropriées et pour la partie incompressible en effectuant des estimations d'énergie directe. Veuillez vous référer à la version anglaise de l'introduction pour les principales étapes de la preuve.

# Chapter 2

# Introduction-English

## 2.1 General introduction.

In many concrete physical systems, there are some fundamental physical quantities (Reynolds number, Mach number, Rossby number, Froude number, etc) which are very small or very large depending on different contexts. For example, the Reynolds number, which is a dimensionless physical number measuring the ratio of the inertial force and viscous force, is very large for the nearby fluids passing by a large cruise ship. The Mach number, which is defined by the ratio of a characteristic velocity in the flow to the sound speed, is usually quite small for highly subsonic flows (for instance the airflow near a fast-moving plane). These parameters usually lead to singular terms in the fluid systems which raise difficulties for the analysis of the systems. A common approach used in physics is trying to consider approximated systems. For example, one can use the inviscid system to describe the motion of a slightly viscous system with a high Reynolds number and can approximate the compressible flows with low Mach number by the corresponding incompressible system. We are thus motivated to establish the rigorous mathematical justification of these approximations. In this thesis, we will focus on the inviscid (large Reynolds number) limit and incompressible (low Mach number) limit problems for a smooth enough solution. In order to verify these limits, one has to prove some uniform estimates (with these physical quantities) which are the main concerns and the heart of the current thesis.

The first part of the thesis (Chapter 3-5) is devoted to the uniform (with respect to the Reynolds number or small viscosity) stability problem for some compressible viscous models (we take the Navier-Stokes-Poisson system for example). On one hand, the global smooth solutions for these viscous models near the equilibria are constructed in [85, 126, 26, 64] under the assumption that initial data are close to the equilibria in some suitable space with the distance proportional to the inverse of the Reynolds number (denoted as $\varepsilon$). This is not that appealing for the inviscid limit problem since the limit solutions vanish. On the other hand, the inviscid versions of these models admit amenable dispersive properties and global existence near the constant equilibria has been shown [55, 56, 58, 69, 84, 10] for the irrotational flows. The hypothesis which assumes that the initial data is irrotational seems optimal, at least for some of these models, because singularities can form in finite time regardless of the magnitude of initial data if we get rid of this hypothesis. Nevertheless, concerning the viscous fluids, even though the solution is assumed to be irrotational initially, a rotational part of size $\varepsilon$ is simultaneously created. Therefore, for the viscous system with a high Reynolds number, one naturally expects a global existence result by allowing the rotational part of the initial velocity to be small proportional to $\varepsilon$. But the important feature is that all the other parts of the initial perturbation can be taken small independent of $\varepsilon$. We shall say that the system satisfies the uniform stability property if the global solution exists under (almost) uniform perturbations around the constant equilibrium as mentioned above.

To get uniform stability, there are several difficulties to overcome. For example, the curl-free condition is not preserved for the compressible viscous systems which is a significant property used to establish global existence for the corresponding inviscid system. Moreover, since the linear semigroup of the compressible part of the system differs greatly from the case when $\varepsilon = 0$, one is obliged to split the frequency to isolate the dispersive effects and the dissipation effects. In Chapter 3, we prove a uniform stability result for the three-dimensional Navier-Stokes-Poisson (NSP) system. In Chapter 4, we show a similar result for the two-dimensional NSP with a specific density-dependent viscosity ($\mu(\rho) = \rho, \lambda(\rho) = 0$) which is more involved due to the weaker dispersion. We present also a long-term regularity result for the general 2d NSP. The proofs rely on a careful study of the dispersive phenomena and the diffusion phenomena for dispersive PDEs with small viscous perturbation and they serve as successful combinations of techniques from dispersive PDEs such as the 'space-time resonance' method and the classical parabolic energy estimates. In Chapter 5, we will show a by-product of the method employed in Chapter 2 and 3, to prove a long time existence result for some inviscid models (Euler-Maxwell, Euler-Korteweg) with nontrivial vorticity.

The second part (Chapter 6-7) of the thesis is concerned with the low Mach number limit problem for some *viscous* fluids in domains with fixed or free boundaries. From the physical point of view, when the fluid speed is much lower than that of the sound speed (which means that the Mach number is very small), the compressible flows will behave like incompressible ones. From the mathematical point of view, at least for isentropic fluid systems, when lying in the low Mach number regime, the pressure (and so the density) is expected to enjoy only small variations proportional to Mach number and the compressible system tends formally to the incompressible one. We are interested in the mathematical verification of this limit process for the strong solution to the compressible Navier-Stokes systems in domains with boundaries. Due to the appearance of the singular terms in the equations and also the presence of the boundaries, one needs to deal with fast oscillations and boundary layer effects simultaneously and some new ideas are needed.

In Chapter 6, we consider the uniform (with respect to Mach number) regularity estimates for isentropic compressible Navier-Stokes equations in fixed smooth domains with Navier-slip condition on the boundary in the general case of ill-prepared initial data. To match the boundary layer effects due to the fast oscillations and the ill-prepared initial data assumption, we prove uniform estimates in an anisotropic functional framework with only one normal derivative close to the boundary. This allows to prove the local existence of a strong solution on a time interval independent of the Mach number and to justify the incompressible limit through a simple compactness argument. To showcase the general strategies and the main features, we would like to present the proof in Section 8 of this chapter when the fluid domain is the half space $\mathbb{R}^3_+$ where slightly better estimates are available due to the trivial geometry of the boundary.

Chapter 7 is devoted to the low Mach number limit problem for a slightly compressible *viscous* fluid in the presence of free boundaries. Compared to the fixed domain case, this kind of problem is more involved since some extra attention needs to be paid to the regularity of the surface. There have indeed been several investigations concerning the incompressible limit problem for a fluid system with free boundaries [37, 88, 92], which focus on *inviscid* systems. In contrast with viscous fluids, the boundary layers does not appear in the studies of the inviscid fluids, due to the lack of the dissipation term. The result presented in this chapter serves as the first exploration for multidimensional free surface *viscous* fluids.

We shall give more details for these problems, state the main results and explain the difficulties and strategies briefly in the following two sections.

## 2.2 Uniform stability and long term regularity for viscous fluid models in plasma physics.

### 2.2.1 Navier-Stokes-Poisson system

One of the main concerns in plasma physics is to study the dynamics of charged particles (electrons and ions). However, very commonly, the target system one studies is composed by huge number of particles, so sometimes it is not possible and practical to keep track of the motions of each particle. The fluid model is to consider these particles as fluid and to use the equations from fluid mechanics to study their dynamics. In the first part of the thesis, we are interested in the so-called Navier-Stokes-Poisson system which is a one-fluid model describing the dynamics of the plasma. Assuming that the magnetic field is negligible and ions are fixed, the motion of electrons can be illustrated by the following electronic Navier-Stokes-Poisson system (ENSP):

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon \mathrm{div}\mathcal{L}u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon \nabla \phi^\varepsilon = 0, \\ \Delta \phi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{2.2.1}$$

Here the unkowns $\rho^\varepsilon(t,x) \in \mathbb{R}_+$, $u^\varepsilon \in \mathbb{R}^3$, $\nabla\phi^\varepsilon \in \mathbb{R}^3$ are the electron density, the electron velocity and the self-consistent electric field respectively. The thermal pressure of electrons $P(\rho^\varepsilon) > 0$ is usually assumed to be a smooth function of the density while the viscous term is under the form

$$\mathrm{div}\mathcal{L}u^\varepsilon = \mu\Delta u^\varepsilon + (\mu + \lambda)\nabla \mathrm{div} u^\varepsilon \tag{2.2.2}$$

where the Lamé coefficients $\mu, \lambda$ are supposed to be constants that satisfy the condition:

$$\mu > 0, \qquad 2\mu + \lambda > 0.$$

Note that we consider a scaled version of the system with the coefficient $\varepsilon$ in front of the diffusion terms which is the inverse of the Reynolds number and is assumed small in this thesis.

When the electrons are considered in thermodynamical equilibrium, a simplified system for the dynamics of ions is the following ionic Navier-Stokes-Poisson (INSP) system:

$$\begin{cases} \partial_t \rho_+^\varepsilon + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon) = 0, \\ \partial_t(\rho_+^\varepsilon u_+^\varepsilon) + \mathrm{div}(\rho_+^\varepsilon u_+^\varepsilon \otimes u_+^\varepsilon) - \varepsilon \mathrm{div}\mathcal{L}u_+^\varepsilon + \nabla P(\rho_+^\varepsilon) - \rho_+^\varepsilon \nabla \phi_+^\varepsilon = 0, \\ \Delta \phi_+^\varepsilon = \rho_+^\varepsilon - e^{-\phi_+^\varepsilon} \\ u_+|_{t=0} = u_{+0}^\varepsilon, \rho_+|_{t=0} = \rho_{+0}^\varepsilon. \end{cases} \tag{2.2.3}$$

Both of these models have a stationary solution $(\rho^\varepsilon, u^\varepsilon, \nabla\phi^\varepsilon) = (1, 0, 0)$ which is a constant equilibrium. We are interested in the stability of these systems around the equilibrium, namely to solve the equation globally under a small initial perturbation to the equilibrium.

### 2.2.2 Uniform stability for 3-dimensional ENSP with high Reynolds number.

There is a large body of literature dealing with the stability under small and smooth enough perturbations of the constant equilibrium of (ENSP) when $\varepsilon = 1$. We refer for example to [85] where global existence in $H^l$ for $l \geq 4$ is proven under the assumption that the initial perturbation is small in $H^l$ and $L^1$. More recently, in [126] global existence in $H^N (N \geq 3)$ of (ENSP) is obtained by using only energy estimates under the assumption that the initial perturbation belongs to $H^N$ and is small in $H^3$. Moreover, as in works on the compressible Navier-Stokes system [61], by assuming that the initial data is in a negative Sobolev space $\dot{H}^{-s}$ $(0 < s < \frac{3}{2})$, explicit decay rates can be obtained by using interpolation inequalities and energy estimates. These results use heavily the fact that the equation for the velocity is a parabolic equation and that the coupling between the two evolution equations of (ENSP) yields decay of the density. In [85], global existence in dimension $d$ is obtained in hybrid Besov spaces when the initial perturbation is close to equilibrium in a $L^2$ critical norm by using energy estimates and by considering low and high frequencies differently. This result was then generalized to a $L^p$ critical frameworks [135], [26].

All these works deal with an unscaled system, that is to say (ENSP) with $\varepsilon = 1$. We can easily check that for the $\varepsilon$ dependent system, these works give global smooth solutions if the initial perturbation is small enough compared to $\varepsilon$ and that the obtained decay rates hold in terms of the slow time variable $\varepsilon t$ (for example [126] would give that in $L^\infty$, $(\rho^\varepsilon - 1)$ is bounded by $\varepsilon(1 + \varepsilon t)^{-\frac{3}{2}}$). Indeed, global existence is obtained by bootstrap arguments and a priori estimates. There are roughly two ways to get the a priori estimates. One way is, as in [85], [126], to use energy estimates and to get dissipation for $u^\varepsilon$ by using the diffusion term $\varepsilon\Delta u^\varepsilon$ and dissipation for $\rho^\varepsilon - 1$ by using a "cross energy estimate". The nonlinear terms can be absorbed if some quantity is small compared to $\varepsilon$. The other way is, as in [64], [135], [26] when considering global existence in critical Besov spaces is to use the maximal smoothing effect of the heat kernel $e^{\varepsilon t\Delta}$, which gives for example for the scaled heat equation

$$\|e^{\varepsilon t\Delta} f\|_{L^1(\mathbb{R}_+,\dot{B}^{s+2}_{p,1})} \lesssim \varepsilon^{-1}\|f\|_{\dot{B}^s_{p,1}}.$$

Therefore, to control the nonlinear terms, this also leads to the assumption that the size of the initial perturbation has to be small compared to $\varepsilon$.

Nevertheless, when $\varepsilon = 0$, the system (2.2.1) reduces to the so-called electron Euler-Poisson (EEP) system. For the (EEP) system, the global existence of smooth solutions close to the constant equilibrium $(1,0)$ was first obtained by Guo [55] under neutral, irrotational, small perturbations to the reference equilibrium $(\rho^0, u^0) = (1,0)$. The neutral assumption ($\int(\rho_0^0 - 1)\mathrm{d}x = 0$) was then removed in [48]. The important property which was used in these works is that the (EEP) system has better dispersive properties than the Euler equations for compressible fluids due to the presence of the electric field. For example, when restricted to irrotational solutions, the linearized (EEP) system can be rewritten as a Klein-Gordon equation which verifies in space dimension $d$ the decay estimate

$$\|e^{it\langle\nabla\rangle} f\|_{L^\infty} \lesssim (1+t)^{-\frac{d}{2}}\|f\|_{W^{d,1}}$$

which is better than the one of the wave equation. Nevertheless, in dimension 3, the only use of energy estimates and of the above dispersive decay (or its $L^p \to L^{p'}$ counterpart) is not enough to get global smooth solutions in the presence of quadratic nonlinearities. Some additional ingredient is thus needed namely either energy estimates using the vector fields method or the normal form method. For the Euler-Poisson system the normal form method of Shatah [112] or more generally, the 'space-time resonances' philosophy can be used to control the nonlinear terms. We refer to [47, 48, 82, 112] for more information about the normal form method and the 'space-time resonance' approach. This type of approach was recently successfully used to handle the (EEP) system in dimension two [69, 84] and one [56].

Since in concrete physical flows the Reynolds number is usually very high (thus $\varepsilon$ very small), it is natural to ask for stability results that hold uniformly with respect to $\varepsilon$ for (ENSP). Though the methods used in the two lines of results that we just presented are completely different, it is rather natural to expect to get global smooth solutions for perturbations of the constant equilibrium $(1,0)$ with a smallness assumption on the perturbation that is independent of $\varepsilon$ except for the curl part of the velocity (remember that for $\varepsilon = 0$ we have global smooth solutions only for irrotational data). This is the result that we obtained in [108].

**Theorem 2.2.1** (Rousset-Sun, [108])**.** *Denote $\mathcal{P}$ the Leray projector on divergence free vector fields so that $\mathcal{P}^\perp = Id - \mathcal{P} = \nabla\Delta^{-1}\mathrm{div}$ and set $\nabla\phi_0^\varepsilon = -\nabla(-\Delta)^{-1}(\rho_0^\varepsilon - 1)$. There exists $\delta_0 > 0$ such that for every family of initial data that satisfy for every $\varepsilon \in (0,1]$ the estimates :*

$$\|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,1}} + \|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{H^N} \leq \delta_0,$$
$$\|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \delta_0\varepsilon$$

*with $\sigma \geq 5$ and $N \geq \sigma + 7$, then, for every $\varepsilon \in (0,1]$, there exists a unique global solution of the (ENSP) system (2.2.1) in $C([0,+\infty), H^3)$. If in addition, we assume that $\sup_{\varepsilon\in(0,1]}\|\mathcal{P}u_0^\varepsilon\|_{\dot{H}^{-s}} < +\infty$ for some $0 < s < \frac{1}{2}$, then we have the following time decay estimates that are uniform in $\varepsilon$. There exists $C > 0$ such that for every $\varepsilon \in (0,1]$, we have*

$$\|(\rho^\varepsilon - 1, \nabla\phi^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}} \leq C\big(\min\{\varepsilon, (1+t)^{-\frac{s}{2+s}}\} + (1+t)^{-(\frac{11}{8}+)}\big), \quad \forall t \geq 0.$$

*where $a^+$ stands for any number strictly larger arbitrarily close to $a$.*

Note that the assumption that we make on the size of the "curl" part of the initial data, that is to say, the assumption on $\mathcal{P}u_0^\varepsilon$, seems to be the natural one. Indeed, even if we assume that $\mathcal{P}u_0^\varepsilon = 0$, this property is not propagated by the system (ENSP), the convection-diffusion equation for the rotational part of the velocity is forced by a source term of size $\varepsilon$ so that a curl part of size $\varepsilon$ is instantaneously created.

#### 2.2.2.1 Some attempts for the proof.

The main difficulty to get Theorem 2.2.1 lies in the interaction between the dynamics of the potential part and the incompressible part of the solution. Indeed, the natural attempt is to consider the equations satisfied by the compressible part and the incompressible part respectively. One expects to use the dispersive property from the coupled equations for density and potential part of velocity to prove the time decay of $\rho^\varepsilon - 1$ and $\mathcal{P}^\perp u^\varepsilon$, and to prove the global existence for the rotational part $\mathcal{P}u$ by using the smoothing effect of small diffusion $\varepsilon\Delta$. For the potential part, we could expect a $L^\infty$ decay given by the linear inviscid dispersive estimates of the order $(1+t)^{-\frac{3}{2}}$. For the incompressible part, we expect that this component will remain of order $\varepsilon$ in $H^s$ but its decay is driven by the heat equation with diffusivity $\varepsilon$, in terms of uniform in $\varepsilon$ estimate this can only yield at best a rather slow decay rate of order $(1+t)^{-1}$ which is in turn not enough to close the decay estimate for the potential part. Moreover, owing to the presence of the diffusion term, the eigenvalues $\lambda_\pm = \varepsilon\Delta \pm i\sqrt{\langle\nabla\rangle^2 - (\varepsilon\Delta)^2}$ of the linearized matrix for the dispersive part of the system are far from $\pm i\langle\nabla\rangle$ − the eigenvalues for (EP). It seems necessary to cut the frequency to isolate the dispersive effects and dissipation effects, which forces us to control the interactions between different frequencies. We prefer not to take this way since it is more sophisticated to treat the potential-rotational interactions and low-high frequencies interactions at the same time.

Another possible way to get such a result would be to write the solution of (ENSP) as the global solution of (EEP) plus a remainder and to try to control the remainder. Since the source term in the equation for the perturbation is of order $\varepsilon$, one could hope to use the parabolic methods described above to control the remainder. Nevertheless, such a naive approach cannot work. Indeed, even in dimension 3, the source term in the equation for the remainder has a non-integrable decay in the energy norm so that there is no hope to be able to control the remainder globally in time. We thus really need to develop a method that allows us to use the type of ideas introduced in the study of dispersive PDE when there is a small dissipative term in addition.

#### 2.2.2.2 Strategies.

Our strategy to prove Theorem 2.2.1 is to split the system into two viscous systems, with initial data $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and $(0, 0, \mathcal{P}u_0^\varepsilon)$ respectively. The first one will have global solutions under $\varepsilon$-independent assumptions on the initial data $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and the solutions will enjoy the same decay estimates as the (EEP) system. The other is just the perturbation of the original system (2.2.1) by the solution to the former one, the important points are that for this system the initial data and the source term are small compared to $\varepsilon$ and that the source term has integrable decay in $L^2$. We can thus use energy estimates and the good decay properties of the solutions to the former system to prove global existence and decay.

For the simplicity of the presentation, we shall assume that $\mu = 1$, $\lambda = 0$ and that $P(\rho^\varepsilon) = (\rho^\varepsilon)^2/2$. Nevertheless, there is no special cancellation arising from this choice (the easiest case for the analysis would be the choice $\mu(\rho) = \rho$, $\lambda = -\mu$, since in this case there are curl free solutions of (2.2.1)). The results hold for general pressure (with $P'(\bar{1}) > 0$) and for general density dependent $\mu, \lambda$ as long as $\mu(1) > 0$, $2\mu(1) + \lambda(1) > 0$ (see Section 6 of Chapter 2). We write the solution $(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon)$ of (ENSP) as

$$(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon) = (\rho, \nabla\phi, u) + (n, \nabla\psi, v),$$

where $(\rho, \nabla\phi, u)$ and $(n, \nabla\psi, v)$ are the solutions of the following systems:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u - \varepsilon \mathrm{div}\mathcal{L}u + \nabla\rho - \nabla\phi = 0, \\ \Delta\phi = \rho - 1, \\ u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{2.2.4}$$

$$\begin{cases} \partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u \cdot \nabla v + v \cdot (\nabla u + \nabla v) - \varepsilon \mathrm{div}\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho + n} - 1)(\mathcal{L}v + \mathcal{L}u), \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0} = 0. \end{cases} \tag{2.2.5}$$

Note that for these two systems we skip the $\varepsilon$ dependence of the solutions in our notation.

We can set $\varrho = \rho - 1$, to change system (2.2.4) into:

$$\begin{cases} \partial_t \varrho + \mathrm{div} u = -\mathrm{div}(\varrho u), \\ \partial_t u + u \cdot \nabla u - \varepsilon \mathrm{div} \mathcal{L} u + \nabla \varrho - \nabla \phi = 0, \\ \Delta \phi = \varrho, \\ u|_{t=0} = \mathcal{P}^{\perp} u_0^{\varepsilon},\ \varrho|_{t=0} = \varrho_0 = \rho_0^{\varepsilon} - 1. \end{cases} \tag{2.2.6}$$

Note that the initial datum for the last system is such that $\mathrm{curl}(\mathcal{P}^{\perp} u_0^{\varepsilon}) = 0$, and this irrotational property will be propagated which means that a smooth solution of this system will remain irrotational. This system is thus a really good viscous approximation of the Euler-Poisson system. As we shall see in Chapter 2, the linear part of this system has the same decay properties for low frequencies as the (EEP) system, that is for localized initial data, the $L^p$ norm of $(\varrho, \nabla \phi, u)$ decay like $(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}$ uniformly for $\varepsilon \in (0, 1]$.

### 2.2.3 Uniform stability for 3d INSP.

We can also consider the INSP which is a simplified model for the motion of ions:

$$\begin{cases} \partial_t \rho_+^{\varepsilon} + \mathrm{div}(\rho_+^{\varepsilon} u_+^{\varepsilon}) = 0, \\ \partial_t(\rho_+^{\varepsilon} u_+^{\varepsilon}) + \mathrm{div}(\rho_+^{\varepsilon} u_+^{\varepsilon} \otimes u_+^{\varepsilon}) - \varepsilon \mathrm{div} \mathcal{L} u_+^{\varepsilon} + \nabla P(\rho_+^{\varepsilon}) - \rho_+^{\varepsilon} \nabla \phi_+^{\varepsilon} = 0, \\ \Delta \phi_+^{\varepsilon} - \phi_+^{\varepsilon} = \rho_+^{\varepsilon} - 1 \\ u_+|_{t=0} = u_{+0}^{\varepsilon}, \rho_+|_{t=0} = \rho_{+0}^{\varepsilon}. \end{cases} \tag{2.2.7}$$

Note that we have used the so-called linearized approximation since in the (INSP) system, we have replaced the Poisson equation $\Delta \phi_+^{\varepsilon} = \rho_+^{\varepsilon} - e^{-\phi_+^{\varepsilon}}$, by a linearized version. This is not a stringent assumption since we are again dealing with small perturbations of the constant equilibrium $(1, 0)$.

For the Euler-Poisson system describing ions' dynamics (IEP) (that is $\varepsilon = 0$ in (2.2.3)), global smooth irrotational solutions with small amplitude have been constructed by Guo and Pausader [58]. The idea is again to find dispersive estimates for the linearized system (which turn out to be weaker than the one of the linear Klein-Gordon equations) and to use the normal form method. Nevertheless, the analysis for this model is much more involved. Indeed, the dispersion relation is closer to the one of the wave equation which leads to the appearance of "time resonances". For example, the 'time resonances' of the phase function $\Phi_{++} = -p(\xi) + p(\xi - \eta) + p(\eta), (p(\xi) = |\xi| \sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}})$ is $\{\eta = 0\} \cup \{\xi - \eta = 0\}$. After integration in time, the multilinear operators now have a singular kernel and to control them the use of $\dot{H}^{-1}$ norms is needed.

We now state the counterpart of Theorem 2.2.1 which is obtained in [108].

**Theorem 2.2.2** (Rousset-Sun, [108]). *Let us fix some absolute number $\kappa > 0$ small enough. There exists $\delta_2 > 0$ such that for any family of initial datum that satisfy for every $\varepsilon \in (0, 1]$ the estimates*

$$\begin{aligned} \| |(\rho_{+0}^{\varepsilon} - 1, \mathcal{P}^{\perp} u_{+0}^{\varepsilon}) \|_{W^{\sigma+3, 8'_\kappa}} + \| |\nabla|^{-1} (\rho_{+0}^{\varepsilon} - 1, \mathcal{P}^{\perp} u_{+0}^{\varepsilon}) \|_{H^N} \leq \delta_2, \\ \|\mathcal{P} u_{+0}^{\varepsilon}\|_{H^3} \leq \delta_2 \varepsilon \end{aligned}$$

*with $8_\kappa = \frac{8}{1-3\kappa}$, $8'_\kappa = \frac{8_\kappa}{8_\kappa - 1}$, $\sigma \geq 6$, $N \geq 2\sigma + 1$, then we have that for every $\varepsilon \in (0, 1]$ there exists a unique global solution for system (2.2.7) in $C([0, +\infty), H^3)$. Besides, if $\sup_{\varepsilon \in (0,1]} \|\mathcal{P} u_{+0}^{\varepsilon}\|_{\dot{H}^{-s}} < +\infty$ with $s < \frac{3}{8}$, then we have the following time decay estimates: there exists $C > 0$ such that for every $\varepsilon \in (0, 1]$ :*

$$\|(\rho_+^{\varepsilon} - 1, u_+^{\varepsilon})\|_{W^{1,\infty}} \leq C \big( \min\{\varepsilon, (1+t)^{-\min\{\frac{s}{2+s}, \frac{\kappa}{2}\}}\} + (1+t)^{-(1+\kappa)} \big), \quad \forall t \geq 0.$$

### 2.2.4 Long-term regularity for two-dimensional ENSP.

As is well-known, the dispersion effects are weaker when the dimension is lower. It is thus interesting to investigate the uniform stability problem in 2-d. We will focus in this subsection on 2d (ENSP) system:

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) - \varepsilon\mathrm{div}\mathcal{L}u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon\nabla\phi^\varepsilon = 0, \\ \Delta\phi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \tag{2.2.8}$$

here $\mathrm{div}\mathcal{L}u^\varepsilon$ is defined in (2.2.2). If one takes $\mu(\rho) = \rho, \lambda(\rho) = -\rho$ in (2.2.2), the 2d (ENSP) can be reduced to the system (dividing by $\rho^\varepsilon$ in the equations of the velocity)

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \varepsilon\Delta u^\varepsilon + \nabla\rho^\varepsilon - \nabla\phi^\varepsilon = 0, \\ \Delta\phi^\varepsilon = \rho^\varepsilon - 1, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{2.2.9}$$

As is explained in last subsection, this system is expected to be a good viscous approximation of the 2-dimensional Euler-Poisson system and we expect that the small viscous diffusion has no remarkable influence on the dynamics of the the compressible part of solutions. In other words, the dispersive effects dominate the dissipation effects for irrotational flows. Since the global existence of the 2d Euler-Poisson system for irrotational initial perturbation has been shown in [69, 84], we first prove the corresponding global result for (2.2.9) by assuming the irrotational initial perturbation is small but independent of $\varepsilon$. Define the norm:

$$\begin{aligned} \|(\rho_0-1, u_0, \nabla\varphi_0)\|_{Y^\sigma} \triangleq\ & \|(\rho_0-1, u_0, \nabla\varphi_0)^L\|_{W^{\sigma+4,1}} + \|x(\rho_0-1, u_0, \nabla\varphi_0)^L\|_{H^{\sigma+4+\delta}} \\ & + \|x(\rho_0-1, u_0, \nabla\varphi_0)^h\|_{L^2} + \|(\rho_0-1, u_0, \nabla\varphi_0)\|_{H^{11+2\sigma}} \end{aligned} \tag{2.2.10}$$

where $\sigma \geq 0$ is a positive parameter and $\delta = \frac{1}{1000}$, where $f^L, f^h$ are the projections of $f$ on the low and high frequencies.

The following theorem is taken from [120].

**Theorem 2.2.3** (Sun, [120])**.** *Let $\sigma \geq 0$. There exist two constants $C_1 > 0$, $\vartheta_1 > 0$ such that for any $\varepsilon \in (0,1]$, any $\bar\vartheta \in (0,\vartheta_1]$ if*

$$\|(u_0^\varepsilon, \rho_0^\varepsilon - 1, \nabla\varphi_0^\varepsilon)\|_{Y^\sigma} \leq \bar\vartheta,$$

*then the system (2.2.9) admits a global solution $(u, \varrho, \nabla\varphi)$ in $C([0,\infty), H^{\sigma+7})$, which enjoys the uniform (in $\varepsilon$) time decay: for any $t > 0$,*

$$(1+t)\|(\rho^\varepsilon - 1, \nabla u^\varepsilon, \nabla\varphi^\varepsilon)(t)\|_{W^{\sigma,\infty}} + \|(\rho^\varepsilon - 1, \nabla u^\varepsilon, \nabla\varphi^\varepsilon)(t)\|_{H^{\sigma+7}} \leq C_1\bar\vartheta.$$

The strategies to prove this theorem will be presented in the next section.

Nevertheless, once we are considering system (2.2.8), the curl-free condition is not be propagated by the flow. Even though we assume the initial data to be curl-free, a curl part of size $\varepsilon$ is instantaneously created. Moreover, due to the weaker dispersion in 2-d, the rotational part of the velocity is driven by a source term whose $L_x^2$ norm enjoys only at best the critical time decay $(1+t)^{-1}$. Consequently, the rotational part of the velocity is likely to have a logarithmic growth which is a big obstacle to prove the global existence. Nevertheless, we can prove the following estimates for the lifespan of a smooth solution:

**Theorem 2.2.4** (Sun, [120])**.** *There exist two constants $\vartheta_0, C$, such that for any $\varepsilon \in (0,1], \vartheta \in (0,\vartheta_0]$, if the following assumption holds:*

$$\|(\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla\varphi_0^\varepsilon)\|_{Y^4} \leq \frac{1}{C}\vartheta, \qquad \|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \vartheta\varepsilon,$$

*where the $Y^4$ norm is defined in (2.2.10), then the system (2.2.8) admits a solution in $C([0,T), H^3)$ with $T > \varepsilon^{-(1-\vartheta)}$.*

Similar to the 3d case, the proof of this theorem is based on Theorem 2.2.3 and easy energy estimates for the perturbed system.

### 2.2.5 A toy model: small viscosity approximation of the half Klein-Gordon equation with quadratic nonlinearities.

This subsection is devoted to the global well-posedness and long time decay of system (2.2.4) in 3d and 2d, which can be used to prove Theorem 2.2.1 and Theorem 2.2.4. Let us now explain the main ideas for the proof. Using the 'curl-free' condition, we consider the new unkown $V = (\frac{\langle\nabla\rangle}{|\nabla|}\varrho, \frac{\mathrm{div}}{|\nabla|}u)$. The linearized system for $V$ is

$$\partial_t V + AV = 0, \quad A = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & -2\varepsilon\Delta \end{pmatrix}.$$

where we use $\langle\nabla\rangle = \sqrt{1-\Delta}$ the Fourier multiplier with symbol $\sqrt{1+|\xi|^2}$. The eigenvalues for this system are

$$\lambda_\pm = -\varepsilon|\xi|^2 \pm i\sqrt{1+|\xi|^2-\varepsilon^2|\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm ib(\xi)$$

A toy model to present the ideas is thus

$$\begin{cases} \partial_t\beta - \lambda_-(D)\beta = \beta^2 \\ \beta|_{t=0} = \beta_0 \end{cases} \tag{2.2.11}$$

The key observations are, on the one hand, when we focus on low frequencies, ($\varepsilon|\xi|^2 \leq 2\kappa_0$ with $\kappa_0$ to be chosen small but independent of $\varepsilon$) then $b(\xi)$ is very close to $\langle\xi\rangle$, this indicates that the imaginary part $e^{itb(D)}$ should give us an $L^p$ decay estimate ($p > 2$) which is uniform for $\varepsilon \in (0,1]$. On the other hand, when we deal with high frequencies (in the sense that $\varepsilon|\xi|^2 \geq \kappa_0$), direct computations show that there exists a positive constant $c = c(\kappa_0)$ such that $\mathrm{Re}\,(\lambda_\pm) \leq -c(\kappa_0)$ for any $\varepsilon \in (0,1]$, so we can expect that the high frequency part of the solution has good decay even in $L^2$ norm.

#### 2.2.5.1 Case $d = 3$.

We first consider the global well-posedness for (2.2.11) for physical dimension $d = 3$ which is relatively easy due to the stronger dispersive property. Define $\beta = P^L\beta + P^H\beta = \beta^L + \beta^H$ where $P^L, P^H$ are the Fourier multipliers that project on low ($\varepsilon|\xi|^2 \leq 2\kappa_0$) and high ($\varepsilon|\xi|^2 \geq \kappa_0$) frequencies respectively. We then define the norm

$$\|\beta\|_{X_T} = \sup_{t\in[0,T)} \Big(\|\beta(t)\|_{H^{10}} + \langle t\rangle^{\frac{3}{2}}\|\beta^H(t)\|_{H^{10}} + \langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|\beta^L(t)\|_{W^{3,p}}\Big). \tag{2.2.12}$$

where we use the notation $\langle t\rangle = \sqrt{1+t^2}$. The first Sobolev norm can be estimated by standard energy estimates. The other two terms involve time decay estimates. The high frequencies piece is easier because we have a uniform (with respect to $\varepsilon$) upper bound for $\mathrm{Re}\,(\lambda_\pm)$ and thus an $L^2 \to L^2$ type estimate with exponential decay uniformly in $\varepsilon$ for the semi-group. The low frequency piece is more difficult to get. We first check that $e^{itb(D)}$ enjoys the same dispersive estimates as $e^{it\langle\nabla\rangle}$ uniformly for $\varepsilon \in (0,1]$. As for the (EEP) systems the linear dispersive estimates are not enough to control the quadratic nonlinearity, we thus have to use normal form transformation to close the low frequencies decay estimate. In this step, we have to carefully track the contribution of the viscous term that creates new error terms. More precisely, let us write $\alpha = e^{-itb(D)}P^L\beta$ then, $\alpha$ satisfies the equation

$$\partial_t\alpha - \varepsilon\Delta\alpha = e^{-itb(D)}(\beta^2)^L.$$

By Duhamel's formula, we have:

$$\beta = e^{itb(D)}\alpha = e^{itb(D)}(e^{\varepsilon t\Delta}\beta_0^L + \int_0^t e^{\varepsilon(t-s)\Delta}e^{-isb(D)}\chi^L(D)((\beta^L)^2 + \beta\beta^H + \beta^H\beta^L)(s)\mathrm{d}s).$$

We focus only on the first term in the above integral, the decay for the other terms is easy to obtain because of the $L^2$ decay of the high frequency part. We can see the first term as

$$e^{itb(D)}\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^3} e^{-\varepsilon(t-s)|\xi|^2}e^{is\varphi}\hat{\alpha}(s,\xi-\eta)\hat{\alpha}(s,\eta)\mathrm{d}\eta\mathrm{d}s \tag{2.2.13}$$

where $\varphi = -b(\xi) + b(\eta) + b(\xi - \eta) > 0$ for $\kappa_0$ small enough, which means that when localized on the low frequency, the phase function does not involve any time resonance.

Following the 'space-time resonance' method, by using the identity $e^{is\varphi} = \frac{1}{i\varphi}\partial_s e^{is\varphi}$, we can integrate by parts in time so that (2.2.13) becomes:

$$i\int_0^t e^{i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}\big(\varepsilon\Delta T_{\frac{1}{\varphi}}(\beta^L,\beta^L) + T_{\frac{1}{\varphi}}(\varepsilon\Delta\beta^L + (\beta^2)^L,\beta^L)\big)\mathrm{d}s$$

plus boundary terms and symmetric term which are similar to handle. Here the bilinear operator $T_{\frac{1}{\varphi}}$ is defined as:

$$T_{\frac{1}{\varphi}}(f,g) \triangleq \mathcal{F}^{-1}\Big(\int \frac{1}{\varphi}(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\mathrm{d}\eta\Big)$$

The last term is cubic and thus can be estimated as in the study of the (EEP) system (we shall check that for $\kappa_0$ sufficiently small the operator $T_{\frac{1}{\varphi}}$ has the similar continuity properties as in the inviscid case). The first two terms are still quadratic but are $\varepsilon$ small, we can thus get additional decay by using the decay provided by the heat equation: for example, we expect that the $L^2$ norm of $\varepsilon\Delta\beta^L$ has decay like $(1+t)^{-1}$. This is enough to close the a-priori estimates when $6 < p \le 12$. However, if one wants to propagate the estimate for larger $p$ which involves a faster rate of decay, another step of integration by parts in time needs to be performed in order to close the estimate.

This idea works also for the case where $b(\xi)$ is changed into $b(\xi) = \sqrt{|\xi|\frac{2+|\xi|}{1+|\xi|} - \varepsilon^2|\xi|^4}$ which is the dispersive relation for the linearized system of ionic Navier-Stokes-Poisson system, and thus can be used to prove Theorem 2.2.2. The only difference is that the phase functions are not necessarily non-vanishing unless away from the planes $\{\xi = 0\}$ and $\{\xi - \eta = 0\}$ so that the bilinear estimate are much harder to obtain and the norm $\dot{H}^{-1}$ need to be involved in order to offset some singularity due to the integration by parts in time. Nevertheless, once localizing on the low frequency, the properties of phase functions, bear much resemblance with the case when $\varepsilon = 0$, we thus could use them to prove the similar bilinear estimates shown in [58]. Therefore, we can focus on the viscous terms performing normal form transformation. See [Sec 7, Chap 2].

#### 2.2.5.2 Case d=2.

Now we explain how to establish global well-posedness for (2.2.11) when the physical dimension is 2-d. In 2-d, the dispersion effects only give critical decay in $L^\infty$ norm so that the (highest) energy norm is forced to grow as time evolves and the $L^\infty$ norm with the critical decay is necessary to be involved in the a-priori estimates. Even with the aid of normal form transformation, it is still not easy to close the estimates. In this case, the weighted norm needs to be involved and some 'space resonances' of phase function needs to be detected. When $\varepsilon = 0$, the global well-posedness has been proved in [69, 84] (this would also be sketched in Chapter 3 for this specific quadratic nonlinearity). Nevertheless, when one considers a small viscous perturbation of half Klein-Gordon, as we have seen above, it is necessary to cut the frequency to isolate the dispersive phenomena and dissipation phenomena and therefore some interactions between different frequencies need to be dealt with. In dimension three, these interactions are easily treated. But it will be harder for the two-dimension case due to the slower time decay of low frequencies (we will explain more in below). Therefore, some new ideas concerning the way of cutting frequencies need to be found.

In practice, we first try to introduce some norms which indicate the decay properties for both low and high frequency. We define the norm (the reason for evolving the weighted norm will be explained later):

$$\|\beta\|_{X_T} = \sup_{t\in[0,T)} \langle t\rangle\|\beta^L(t)\|_{W^{1,\infty}} + \|xe^{itb(D)}\beta^L\|_{H^4} + (1+t)\|\beta^H(t)\|_{H^9} + \langle t\rangle^{-\delta}\|\beta(t)\|_{H^{10}} + \|\beta(t)\|_{H^8}.$$

where $\beta^L = \chi^L(D)\beta, \beta^H = \chi^H(D)\beta$. Note that in the definition of the norm, we have time decay of rate $(1+t)^{-1}$ rather than $e^{-ct}$ for high frequency due to the weak decay property for low frequency. To get the a priori estimate, we need to consider several interactions between different frequencies. However, due to the slow decay provided by the dispersive estimate for low frequency, the low frequency output of the interactions between the low frequency and the high frequency is difficult to close.

More precisely, in order to estimate the low frequency, by rewriting $\beta^2 = (\beta^L)^2 + 2\beta^L\beta^H + (\beta^H)^2$, we need to estimate the term $\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)\mathrm{d}s$ which can be controlled by dispersive estimate for $e^{itb(D)}\chi^L$ (see Lemma 4.4.1) as:

$$\begin{aligned}&\|\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)\mathrm{d}s\|_{L^\infty} \lesssim \int_0^t (1+t-s)^{-1}\|\beta^H\beta^L\|_{W^{2,1}}\mathrm{d}s\\ &\lesssim \int_0^t (1+t-s)^{-1}(1+s)^{-1}\mathrm{d}s\|\beta\|^2_{X_T} \lesssim (1+t)^{-\iota}\|U\|^2_{X_T}\end{aligned} \tag{2.2.14}$$

where $0 < \iota < 1$. Unfortunately, the desired case $\iota = 1$ cannot be obtained in this way. To overcome this difficulty, we need more accurate splitting of frequencies. We observe that one can indeed split the frequency into three parts, namely, lowest frequency: $\{\varepsilon|\xi|^2 \le \kappa_0\}$, intermediate frequency $\{\frac{\kappa_0}{2} \le \varepsilon|\xi|^2 \le 3\kappa_0\}$ and highest frequency $\{\varepsilon|\xi|^2 \ge \frac{5}{2}\kappa_0\}$. In this way, due to the lack of interaction lowest$\times$ lowest $\to$ highest, we could expect the highest frequency enjoys faster decay. The intermediate frequency part has also good decay since in this region, we have $e^{t\lambda_\pm(\xi)} \le e^{-ct}$ for some $c > 0$ independent of $\varepsilon$. The lowest frequency is now manageable since for the lowest$\times$intermediate$\to$ lowest interaction we can use normal form transformation by noticing that the intermediate frequency still lies in the region that dispersive property holds. To summarize, after some crude analysis, we expect that the lowest frequency part enjoys the $L^\infty_x$ decay of $(1+t)^{-1}$, the intermediate part enjoy the $L^p_x$ $(2 \le p \le 4)$ decay like $(1+t)^{-(2-\frac{2}{p})}$, and the high frequency parts has $L^2_x$ decay like $(1+t)^{-2}$. We explain for instance the high frequency case. By choosing three smooth radial functions $\chi^l, \chi^m, \chi^h$ which satisfy $\chi^l(\xi-\eta)\chi^l(\eta)\chi^h(\xi) = 0$ and defining $\chi^L = \chi^l + \chi^m, \beta^L = \beta^l + \beta^m$, one can write $(\beta^2)^h = \big(2\beta^L\beta^h + (\beta^h)^2 + (\beta^m)^2 + 2\beta^l\beta^m\big)^h$. We can expect the worst part $\beta^l\beta^m$ enjoys $L^2$ decay at $(1+t)^{-2}$.

We thus need to modify our norm to be (with $N > 10$ to be chosen) :

$$\begin{aligned}\|\beta\|_{X_T} \triangleq \sup_{t\in[0,T)} &\langle t\rangle\|\beta^L\|_{W^{1,\infty}} + \langle t\rangle\|\beta^m\|_{H^{N-1}} + \langle t\rangle^2\|\beta^h\|_{H^{N-1}}\\ &+ \|xe^{itb(D)}\beta^L\|_{H^4} + \langle t\rangle^{-\delta}\|\beta\|_{H^N} + \|\beta\|_{H^{N-2}}\end{aligned} \tag{2.2.15}$$

The Low$\times$Low$\to$ Low estimate is again controlled by using the 'space-time resonance' techniques. By following the procedures explained in the 3d case, after integrating by parts in time, we need to estimate the typical term:

$$\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^L\beta^L)^L\beta^L(s)\mathrm{d}s.$$

Nevertheless, the same problem like (4.1.6) shows up, since we could estimate $\|(\beta^L\beta^L)^L\beta^L\|_{W^{2,1}}$ by $\|\beta^L\|^2_{H^2}\|\beta^L\|_{L^\infty}$ which has only the decay $(1+s)^{-1}$. Following [69, 84], the 'vector field-like' norm $e^{-itb(D)}xe^{itb(D)}\beta^L$ needs to be involved to detect some space resonance information of the phase function.

### 2.2.6 An application of the "splitting" idea.

The 'splitting' idea employed in the proof of Theorem 2.2.1 and 2.2.2 can also be used to prove lower bound of the lifespan of solutions to some fluid systems with nontrivial vorticity for which the global existence with zero vorticity is known. We shall focus on two examples, namely the Euler-Korteweg system and the two-fluid Euler-Maxwell system. We present here the result for the Euler-Korteweg system which reads:

$$\begin{cases}\partial_t\rho + \mathrm{div}(\rho u) = 0,\\ \rho\partial_t u + \rho u\cdot\nabla u + \nabla P(\rho) - \rho\nabla\left(K(\rho)\Delta\rho + \dfrac{1}{2}K'(\rho)|\nabla\rho|^2\right) = 0,\\ u|_{t=0} = u_0, \rho|_{t=0} = \rho_0.\end{cases} \tag{2.2.16}$$

where $\rho, u$ are the density and velocity of the fluid, $P(\rho)$ the pressure and is assumed to be a smooth function of density. $K(\rho)$ is the Korteweg tensor which takes the capillary effects into account and is a smooth function of the density $\rho$. The following theorem is taken from [119].

**Theorem 2.2.5** (Sun, [119]). *Let $\bar\rho > 0$. Suppose that the Korteweg tensor $K(\rho)$ is smooth and satisfies: $K(\rho) \geq K_0 > 0$ for $\bar\rho/2 \leq \rho \leq 3\bar\rho/2$. Assume also that the pressure satisfies: $P'(\rho)/\rho > 0$ for $\bar\rho/2 \leq \rho \leq 3\bar\rho/2$. Then there exist three constants $\delta_1, \epsilon_1 > 0$ small, and $N$ large such that if the initial datum $(\rho_0 - \bar\rho, u_0)$ satisfies the following:*

$$\|\mathcal{P}u_0\|_{H^s} < \epsilon_1,$$

$$\|\mathcal{P}^\perp u_0\|_{H^N} + \|\rho_0 - \bar\rho\|_{H^{N+1}} + \|x(\rho_0 - \bar\rho, \mathcal{P}^\perp u_0)\|_{L^2} \\ + \|u_0\|_{W^{6,1}} + \|\rho_0 - \bar\rho\|_{W^{7,1}} \leq \delta_1,$$

*where $5/2 < s \leq 3$. Then there exists $T_{\epsilon_1} \gtrsim \epsilon_1^{-1}$ such that the Euler-Korteweg equation* (2.2.16) *has a unique solution and*

$$(\rho - \bar\rho, u) \in C([0, T_{\epsilon_1}], H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)).$$

*In addition, if $\mathcal{P}u_0 \in H^N$, then the solution*

$$(\rho - \bar\rho, u) \in C([0, T_{\epsilon_1}], H^{N+1}(\mathbb{R}^3) \times H^N(\mathbb{R}^3)).$$

*with exponential growth: for any $0 \leq t \leq T_{\epsilon_1}$, there exists a constant $c$ independent of $\epsilon_1$, such that*

$$\|(\rho - \bar\rho, u)(t)\|_{H^{N+1}\times H^N} \lesssim e^{ct}\|(\rho - \bar\rho, u)(0)\|_{H^{N+1}\times H^N}.$$

Note that a similar result is also obtained by Audiard [9]. However, the method we propose relies only on the energy estimates and the fast decay estimates of the system with trivial vorticity, and thus is flexible. Moreover, we do not require the localization assumption on the rotational part of the initial velocity $\mathcal{P}u_0$.

To explain the main idea, we prefer to restrict ourselves to a rather abstract setting. Consider a system:

$$\begin{cases} \partial_t U + JLU = U' \cdot \nabla U \\ U|_{t=0} = U_0 \end{cases} \tag{2.2.17}$$

where $U(t,x) = (U_1, U') : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^4$, is a four-elements vector function, $J$ is a skew symmetric differential matrix, $L$ is self-adjoint and positive in some suitable space in the sense that

$$(Lu, v)_{L^2} = (u, Lv)_{L^2},\ (LU, U)_{L^2} \geq \|U\|_{L^2}^2$$

. For example, we can take

$$J = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} Id - \Delta & 0 \\ 0 & Id_{3\times3} \end{pmatrix},$$

for Euler-Korteweg type equations (the simplified case that the term $\rho\mathrm{div}u$ is dropped in $(2.2.16)_1$ and $P(\rho) = \frac{1}{2}\rho^2, K(\rho) = 1$ in $(2.2.16)_2$ is assumed) while

$$J = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} Id + (-\Delta)^{-1} & 0 \\ 0 & Id_{3\times3} \end{pmatrix},$$

for one fluid Euler-Poisson type equations.

We suppose firstly that for the curl-free ($\mathrm{curl}\, U_0' = 0$) smooth initial datum, there exists global solutions in some Sobolev space $H^N$ (where $N$ is large enough) which decays fast enough to 0 as the time goes to infinity. More precisely, we suppose that $\|U\|_{W^{4,\infty}} \lesssim (1+t)^{-\alpha}$ with $\alpha > 1$. Now, we want to analyze the large time existence of system (2.2.17) with general (not necessarily curl-free) smooth initial data. Our strategy is to split the system (2.2.17) into two systems, with initial data $(U_1, \mathcal{P}^\perp U_0')$ and $(0, \mathcal{P}U_0')$. To be more concrete, we write $U = W + V$, where $W$ solves (2.2.17) with initial data $(0, \mathcal{P}^\perp U_0')$, and $V$ satisfies the equation:

$$\begin{cases} \partial_t V + JLV = (V' + W') \cdot \nabla V + V' \cdot \nabla W =: F(V, W) \\ W|_{t=0} = (0, \mathcal{P}U_0') \end{cases} \tag{2.2.18}$$

To study the long time existence of (2.2.18), it suffices for us to get appropriate a priori energy estimates. Let us define energy functional

$$E_s(t) = \int_{\mathbb{R}^3} \Lambda^s V \cdot L\Lambda^s V(t)\,\mathrm{d}x$$

where $\Lambda = \sqrt{1-\Delta}$ and $s > 5/2$. Taking $\Lambda^s$ on system (2.2.18), and testing $L\Lambda^s V$, we then get

$$\partial_t E_s \leq \int_{\mathbb{R}^3} \Lambda^s F(V, W) \cdot L\Lambda^s V \mathrm{d}x$$

which yields by commutator estimates, if $L$ is zero order differential operator (for example Euler-Poisson system),

$$\partial_t E_s \lesssim E_s^{\frac{3}{2}} + \|W\|_{W^{s+1,\infty}} E_s \lesssim E_s^{\frac{3}{2}} + (1+t)^{-\alpha} E_s$$

from which, one deduce by the Grönwall inequality and continuation arguments, that, there exists solutions for system (2.2.18) in $C([0,T), H^s)$ for $T \gtrsim 1/\|\mathcal{P}U_0'\|_{H^s}$. However, when $L$ is a differential operator of positive order (for example Euler-Korteweg type), direct energy estimate will inevitably lose derivatives. In this case, the 'gauge' technique used in [17] need to be employed.

## 2.3 Low Mach number limit problem.

### 2.3.1 Background

We consider the following compressible Navier-Stokes equations which express conservation of momentum and conservation of mass for Newtonian fluids:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\tilde{\mathcal{L}}u + \nabla P(\rho) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0, \rho|_{t=0} = \rho_0, \end{cases} \tag{2.3.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain, $\rho(t,x)$ and $u(t,x)$ are the density and the velocity of the fluid respectively, $P(\rho)$ is the pressure which is a given smooth function of the density satisfying $\frac{\mathrm{d}P}{\mathrm{d}\rho} > 0$, for $\rho > 0$. The viscous stress tensor takes the form:

$$\tilde{\mathcal{L}}u = 2\tilde{\mu}\mathbb{S}u + \tilde{\lambda}\operatorname{div}u\,\mathrm{Id}, \quad \mathbb{S}u = \frac{1}{2}(\nabla u + \nabla^t u).$$

Here, $\tilde{\mu}, \tilde{\lambda}$ are viscosity parameters that are assumed to be constant and to satisfy the condition: $\tilde{\mu} > 0, 2\tilde{\mu} + 3\tilde{\lambda} > 0$. Performing the following scaling:

$$\rho^\varepsilon(\varepsilon t, x) = \rho(t,x), \quad \varepsilon u^\varepsilon(\varepsilon t, x) = u(t,x), \quad \varepsilon\mu = \tilde{\mu}, \quad \varepsilon\lambda = \tilde{\lambda},$$

then $(\rho^\varepsilon, u^\varepsilon)$ satisfies the scaled compressible Navier-Stokes system $(CNS)_\varepsilon$ :

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \operatorname{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \tag{2.3.2}$$

with $\mathcal{L}u = 2\mu\mathbb{S}u + \lambda\operatorname{div}u\,\mathrm{Id}$. Measuring the compressibility of fluids, the parameter $\varepsilon$ is the Mach number and is defined by the ratio between the fluid speed and sound speed. To reveal the link between compressible fluids and incompressible fluids, we will assume it to be small, that is $\varepsilon \in (0,1]$.

Formally, due to the stiff term $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, the pressure (and hence the density $\rho^\varepsilon$) is expected to tend to a constant state. One thus expects to obtain in the limit a solution to the following incompressible Navier-Stokes system:

$$\begin{cases} \bar{\rho}(\partial_t u^0 + \operatorname{div}(u^0 \otimes u^0)) - \Delta u^0 + \nabla \pi^0 = 0, \\ \operatorname{div}u^0 = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^0|_{t=0} = u_0^0, \end{cases} \tag{2.3.3}$$

This limit process is therefore frequently referred to as the incompressible limit.

The rigorous justification of this limit process has been studied extensively in different contexts depending on the generality of the system (isentropic or non-isentropic), the type of the system (Navier-Stokes or Euler), the type of solutions (strong solutions or weak solutions), the properties of the domain (whole space, torus or bounded domain with various boundary conditions), as well as the type of the initial data considered (well-prepared or ill-prepared). Roughly speaking, in the case of the compressible Euler system, one proves first that the local strong solution exists on a time interval independent of the Mach number, and then compactness arguments are developed to pass to the limit. In the case of the compressible Navier-Stokes system, one can either try to use the same approach as for the inviscid case (prove the existence of a strong solution on a time interval independent of the Mach number and then try to pass to the limit) or try to pass to the limit directly from global weak solutions. Both approaches have been used in domains without boundaries (whole space or torus), nevertheless when a boundary is present the question of uniform high regularity for general data is more subtle and has not been addressed.

The mathematical justification of the low Mach number limit was initiated by Ebin [39, 40], Klainerman-Majda [79, 80] for *local strong solutions* of compressible fluids (Navier-Stokes or Euler), in the whole space with well-prepared data ($\text{div} u_0^\varepsilon = \mathcal{O}(\varepsilon), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon^2)$) and later, by Ukai [125] for ill-prepared data ($\text{div} u_0^\varepsilon = \mathcal{O}(1), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon)$). In the latter case, there are acoustic waves of amplitude 1 and frequency $\varepsilon^{-1}$ in the system. These works were extended by several authors in different settings. For instance, one can refer to [2, 20, 100, 101] for the non-isentropic system and ill-prepared initial data whenever the domain is the whole space or the torus, and also [74, 109] for bounded domains with well-prepared initial data. Uniform (in Mach number) regularity estimates for the non-isentropic Euler equations in a bounded domain are established in [1]. The low Mach limit of *weak solutions* for the viscous fluid system (6.1.1) was studied by Lions and Masmoudi [89], [90] who established convergence of the global weak solutions of the isentropic Navier-Stokes system towards a solution of the incompressible system. Their result holds for ill-prepared initial data and several different domains (whole space, torus and bounded domain with suitable boundary conditions). In general, for ill-prepared data, one can only obtain weak convergence in time, nevertheless, by using the dispersion of acoustic waves in the whole space, Desjardin and Grenier [34] could get local strong convergence. There are also many other related works, one can see for example [14, 30, 32, 36, 41, 45, 54, 71]. For more exhaustive information, one can refer for example to the well-written survey papers by Alazard [3], Danchin [31], Feireisl [43], Gallagher [46], Jiang-Masmoudi [73], Schochet [110].

### 2.3.2 Low Mach number limit for viscous system in domains with fixed boundaries and slip boundary condition.

Since we are dealing with the system in domain with boundaries, some boundary conditions need to be supplemented for system (2.3.2). There are two kinds of typical boundary conditions, namely the Dirichlet boundary condition (no-slip) and Navier-slip boundary condition.

- Dirichlet boundary condition:

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{2.3.4}$$

- Navier-slip boundary condition:

$$u^\varepsilon \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^\varepsilon \mathbf{n}) + a\Pi u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{2.3.5}$$

where $a$ is a constant related to a slip length (our analysis also holds if $a$ is a smooth function). We use the notation $\Pi f$ for the tangential part of a vector $f$, $\Pi f^\varepsilon = f^\varepsilon - (f^\varepsilon \cdot \mathbf{n}) \cdot \mathbf{n}$. The Dirichlet boundary condition (2.3.4) indicates that the fluid particle cannot slip on the boundary while the Navier-slip condition, which is proposed by Navier [103], indicates that the tangential velocity of the fluid at the boundary is connected to the stress tensor as in (2.3.5).

We focus now more specifically on the study of the low Mach limit of the isentropic compressible Navier-Stokes $(CNS)_\varepsilon$ system in domains with boundaries with *ill-prepared* initial data, which is more related to the interest of this thesis. As mentioned above, Lions and Masmoudi [89] studied the convergence of *weak solutions* to $(CNS)_\varepsilon$ in bounded domains with Navier-slip boundary conditions. Later on, for low Mach limit in bounded domains with Dirichlet boundary condition, the authors in [35] noticed that, under some geometric assumption on the domain, the acoustic waves are damped in a boundary layer so that local in time strong convergence ($L^2_{t,x}$) holds. Recently, this result is extended by Feireisl et al [44] and Xiong [131] to the case of Navier-slip boundary conditions with $a$ of the order $\varepsilon^{-\frac{1}{2}}$. In this case, the boundary layer effect is comparable to the one in the Dirichlet case. One can also refer to [38, 41, 42] for the justification of convergence in unbounded domains by using the local energy decay for the acoustic system. Without one of the above properties of the domain, strong convergence does not hold for ill-prepared data.

Our first aim is to obtain uniform (with respect to $\varepsilon$) high order regularity estimates for $(CNS)_\varepsilon$ in bounded domains with ill-prepared initial data and Navier-slip boundary conditions, in order to get the existence of a local strong solution on a time interval independent of $\varepsilon$. There are only few papers addressing this issue. In [105], the authors establish uniform global (for small data) $H^2$ estimates under a (very) *well-prepared* initial data assumption, namely the second time derivative of the velocity needs to be uniformly bounded initially. For ill-prepared initial data, the situation is more subtle and a uniform $H^2$ estimate, even locally in time, cannot be expected. Indeed, at leading order, after linearization and symmetrization, the system (2.3.2) becomes:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon} L U^\varepsilon - \begin{pmatrix} 0 \\ \mathrm{div}\mathcal{L}u^\varepsilon \end{pmatrix} = 0, \qquad L = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \quad U = (\sigma^\varepsilon, u^\varepsilon) \in \mathbb{R}\times\mathbb{R}^3. \tag{2.3.6}$$

Due to the presence of the diffusion term as well as the singular linear term, a boundary layer correction to the highly oscillating acoustic waves appear and create unbounded high order normal derivatives of the velocity. Note that here, we do not start from a small viscosity problem, nevertheless, at the scale $\tau = t/\varepsilon$ of the acoustic waves the system (2.3.6) behaves like a small viscosity perturbation of the acoustic system. For example, in the easiest case where the boundary is flat (for instance $\Omega = \mathbb{R}^3_+$), we expect the following expansion of the solutions to (2.3.6) involving boundary layers

$$\begin{cases} \sigma^\varepsilon(t,x) = \sigma_0^I(\frac{t}{\varepsilon}, t, x) + \varepsilon^{\frac{3}{2}}\sigma^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots, \\ u^\varepsilon(t,x) = u_0^I(\frac{t}{\varepsilon}, t, x) + \sqrt{\varepsilon}\begin{pmatrix} u^B_{1,\tau}(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) \\ 0 \end{pmatrix} + \varepsilon u_2^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots \end{cases} \tag{2.3.7}$$

where $x = (y, z)$, $z > 0$, which suggests that $\|u_\tau\|_{L^2_t H^1}, \|u_3^\varepsilon\|_{L^2_t H^2}, \|\sigma^\varepsilon\|_{L^2_t H^3}$ can be uniformly bounded whereas $\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_{t,x}}$ and $\|\partial_z^2 u_\tau^\varepsilon\|_{L^2_{t,x}}$ will blow up as $\varepsilon$ tends to 0.

To continue, let us introduce the new unknown

$$\sigma^\varepsilon = \frac{P(\rho^\varepsilon) - P(\bar\rho)}{\varepsilon},$$

where $\bar\rho$ is a positive constant state, we can rewrite the system (2.3.2) into the following form which is more convenient to perform energy estimates:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t\sigma^\varepsilon + u^\varepsilon\cdot\nabla\sigma^\varepsilon\big) + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon)(\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon\big) - \mathrm{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon,\ \sigma|_{t=0} = \sigma_0^\varepsilon. \end{cases} \tag{2.3.8}$$

where the scalar functions $g_1$, $g_2$ are defined by

$$g_2(s) = \rho^\varepsilon = P^{-1}(\bar P + s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar P = -P(\bar\rho). \tag{2.3.9}$$

We shall now explain the main difficulties and the main strategies in order to obtain uniform estimates. As already mentioned the main feature of our problem is the presence of both fast time oscillations and a boundary layer in space. These two aspects are well-understood when they occur separately, but in order to handle them simultaneously, some new ideas will be needed. On the one hand, concerning the inviscid limit problem, one controls [93, 106, 127] the high order tangential derivatives by direct energy estimates, and then uses the vorticity to control the normal derivatives. Nevertheless, for the system with a low Mach number, even the tangential derivative estimates are not easy to get, since the spatial tangential derivatives do not commute with $\nabla, \mathrm{div}$, defined with the standard derivations in $\mathbb{R}^3$, unless the boundary is flat, and thus create singular commutators. Without this a priori knowledge on the tangential derivatives, the estimate of the vorticity cannot be performed because of the consequent lack of information on its trace on the boundary. On the other hand, for the compressible Euler system with a low Mach number, uniform high regularity estimates are established for example in [1]. One can get uniform $H^s(s > 5/2)$ estimates by using first $\varepsilon\partial_t$ derivatives and then recover space derivatives by using the equations to estimate the divergence of the velocity and the gradient of the pressure and direct energy estimates for the vorticity which solves a transport equation with a characteristic vector field. Here, in the case of viscous fluids, we face again the fact that the estimates of the vorticity are challenging due to the lack of information on its trace on the boundary at this stage.

#### 2.3.2.1 Uniform (with respect to the Mach number) estimates when $\Omega = \mathbb{R}^3_+$.

As indicated in (2.3.7), the solutions to (2.3.2) are unlikely to stay bounded in common high order Sobolev space, we thus need to use a functional framework based on conormal Sobolev spaces that minimize the use of normal derivatives close to the boundary in the spirit of [93, 94]. By introducing some weights near the boundaries, the cornormal space are coherent with the boundary layers effects. To capture the interplay of fast time oscillation and the boundary layer effects, we shall firstly neglect the difficulties arising from the nontrivial boundary geometry and explain the ideas to get uniform estimates when $\Omega = \mathbb{R}^3_+$.

Let us define the vector fields that are tangential to the boundary: (suppose $x = (y, z)$)

$$Z_1 = \partial_{y_1} = \partial_1, \quad Z_2 = \partial_{y_2} = \partial_2, \quad Z_3 = \phi(z)\partial_z = \frac{z}{1+z}\partial_z,$$

and also the conormal Sobolev spaces:

$$L^p_t H^m_{co}(\mathbb{R}^3_+) = \{Z^\alpha f \in L^p([0,t]; L^2(\mathbb{R}^3_+)), |\alpha| \le m\},$$

$$L^p_t H^m_{tan}(\mathbb{R}^3_+) = \{Z^\alpha f \in L^p([0,t]; L^2(\mathbb{R}^3_+)), |\alpha| \le m, \alpha_3 = 0\},$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4$, $Z^\alpha = (\varepsilon\partial_t)^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$. We shall use the following quantities (one can refer to section 5.8 for the precise definition)

$$\mathcal{N}^\varepsilon_{m,T} \approx \|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_T H^m_{tan}} + \|(\nabla\sigma^\varepsilon, \mathrm{div} u^\varepsilon)\|_{L^\infty_T H^{m-1}_{tan} \cap L^\infty_T H^{m-2}_{co}} + \|\omega^\varepsilon_\tau\|_{L^\infty_T H^{m-1}_{co}} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_T H^{m-1}_{co} \cap L^\infty_T L^\infty}.$$

where $\omega^\varepsilon_\tau = (\omega^\varepsilon_1, \omega^\varepsilon_2, 0)^t$ denotes the horizontal part of the vorticity $\omega^\varepsilon = \mathrm{curl}\, u^\varepsilon$.

We aim to show the following type of a-priori energy estimates which is sufficient to find an existence time interval that is uniform in $\varepsilon$ :

$$\mathcal{N}^\varepsilon_{m,T} \lesssim \mathcal{N}^\varepsilon_{m,0} + (T+\varepsilon)^{\frac{1}{4}} P(\mathcal{N}^\varepsilon_{m,T}) \tag{2.3.10}$$

where $P$ is a continuous function independent of $\varepsilon$. Let us first give some remarks concerning the selections of the norms:

**Remark 2.3.1.** *Due to the ill-prepared data assumption, we can only have the control of the weighted time derivative $\varepsilon\partial_t$, which prevents us from bounding the $L^\infty_{t,x}$ norm for $(\sigma^\varepsilon, u^\varepsilon)$ by Sobolev embedding in time as is done in the inviscid limit problem [106, 127]. Therefore, it seems necessary for us to control the gradients in a $L^\infty$ in time norm, which eventually can be derived from the control of spatial tangential derivatives,* $\mathrm{div} u^\varepsilon$ *and* $\omega^\varepsilon_\tau$.

**Remark 2.3.2.** *The norm $L^\infty_t H^{m-1}_{co}$ prescribed for $\omega^\varepsilon_\tau$ seems to be optimal. Due to the ill-prepared data assumption (and also as indicated in* (2.3.7)*), one cannot expect any normal derivatives of $\omega_\tau$ to be bounded uniformly. Indeed, by parabolic regularity theory, any control of higher order spatial norms requires the boundedness of (fractional) time derivative. As $\omega^\varepsilon_\tau$ interacts with the compressible part through the boundary, there seems no hope to get a uniform bound for $\partial_t\omega^\varepsilon_\tau$.*

**Remark 2.3.3.** *We can control* $\mathrm{div} u^\varepsilon, \nabla\sigma^\varepsilon$ *up to order* $m-2$ *and control* $\omega_\tau^\varepsilon$ *up to highest order* $m-1$ *in* $L_t^\infty L^2$. *This is however not strange. Since we recover the higher order weighted normal regularity for* $(\nabla\sigma^\varepsilon, \mathrm{div} u^\varepsilon)$ *by using the equation, we shall lose one derivative due to the appearance of the (second order) viscous term. On the other hand, as one can control the spatial tangential derivatives in advance and* $Z_3$ *vanishes on the boundary, we can indeed control the vorticity up to the highest order by the ideas explained below. Nevertheless, this mismatch is mainly due to the trivial boundary geometry. As we will see in the next subsection, for a general smooth bounded domain, we can only control* $\omega^\varepsilon \times \boldsymbol{n}$ *up to order* $m-2$.

We explain now the main ideas to prove the uniform energy estimates.

Our strategy is illustrated in the following four steps:

• **Step 1. Control of the tangential derivatives** $(\varepsilon\partial_t, \partial_1, \partial_2)$ **by energy estimates and control the highest conormal derivatives with a weight** $\varepsilon$. On the one hand, since the tangential derivatives $\varepsilon\partial_t, \partial_1, \partial_2$ commute with the spatial derivatives, we can get uniform estimates for high order tangential derivatives by standard energy estimates. Note that we use $\varepsilon\partial_t$ instead of $\partial_t$ since we are dealing with ill-prepared data. On the other hand, as the conormal vector field $Z_3$ does not commute with $\nabla, \mathrm{div}$, the singular part of the system, we need at this stage to add this additional $\varepsilon$ weight to control the commutator.

• **Step 2. Recover the weighted normal derivatives by the equations and induction arguments.** This can be done by using the equations and induction arguments. Indeed, we can rewrite the system (2.3.2) as

$$\begin{aligned} -\mathrm{div} u^\varepsilon &= g_1 \varepsilon\partial_t \sigma^\varepsilon + \varepsilon g_1 u^\varepsilon \cdot \nabla\sigma^\varepsilon, \\ -\nabla\sigma^\varepsilon &= g_2 \varepsilon\partial_t u^\varepsilon + \varepsilon(g_2 u^\varepsilon \cdot \nabla u^\varepsilon - \mathrm{div}\mathcal{L}u^\varepsilon). \end{aligned}$$

In view of the above equations, one can 'trade' one spatial derivative by one (small scale) time derivative $\varepsilon\partial_t$. We can thus recover the high order spatial (conormal) derivatives by using iteratively this observation.

• **Step 3. Control of the vorticity.** Since the vorticity solves by a transport-diffusion system without any singular terms, we expect to control the high-order conormal estimates by energy estimates. However, it is not applicable due to the lack of information on the trace of $\omega$. Our strategy is to use a lifting of the boundary conditions by using Green's function for the solution of the heat equation with non-homogenous boundary conditions and estimate the remainder by energy estimates. More precisely, we split the system into two systems, one of which satisfies a heat equation with nontrivial Dirichlet boundary conditions which admits explicit formulae whereas the other one carries on all the nonlinear terms and initial data but with homogeneous Dirichlet boundary condition so that the direct energy estimates can be performed.

• **Step 4.** $L_{t,x}^\infty$ **estimate**: The $L_{t,x}^\infty$ norm of $(\sigma, u)$ (and its tangential derivatives) can be controlled by Sobolev embedding and the norms controlled in the previous steps. We then use the maximum principle for transport-diffusion equation satisfied by $\omega_\tau$ and for the damped transport equation solved by $\partial_z\sigma$ (which is obtained by applying $\partial_z$ on the equation of $\sigma$ and substituting $\partial_z \mathrm{div} u$ by using the equation for $u_3$) to get the $L_{t,x}^\infty$ estimates of $\partial_z u_\tau$ and $\partial_z\sigma$ respectively.

#### 2.3.2.2 Uniform estimates for a fixed bounded domain.

When $\Omega$ is a generic fixed domain with non-flat boundaries, there will be some extra difficulties in order to obtain uniform estimates. Indeed, once the boundaries are not flat, even the tangential derivatives $(\mathbf{n}\times\nabla)_1, (\mathbf{n}\times\nabla)_2$, ($\mathbf{n}$ stands for the outward normal vector) do not commute with the singular part of the system, and thus can not be controlled directly. Without foreknowledge of the tangential derivatives, one cannot follow what we did in the former subsection, to reduce the control of the gradient of velocity to that of $\mathrm{div} u^\varepsilon$ and $\omega^\varepsilon \times \mathbf{n}$.

In order to get the missing information, we shall first use the Leray projection (the precise definition (6.3.2) is in Section 5.3) to split the velocity into a compressible part and an incompressible part: $u^\varepsilon = \nabla\Psi^\varepsilon + v^\varepsilon$. On the one hand, the compressible part $\nabla\Psi^\varepsilon$ of the velocity can be controlled by

$\mathrm{div} u^\varepsilon$ thanks to the standard elliptic theory and hence by using the equations and the estimates for $\varepsilon\partial_t$ derivatives. On the other hand, the incompressible part $v^\varepsilon$ solves, up to the control of non-local commutators, a convection-diffusion equation without oscillations, and thus one can use direct energy estimates to get a control of $\|v^\varepsilon\|_{L_t^\infty H_{co}^{m-1}}$ and $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. Note that we cannot estimate the maximal number of derivatives $m$ due to the lack of structure of the coupling terms involving the compressible part in the energy estimates. The key point here is that the diffusion (which on the other hand creates new difficulties in the control of the vorticity) allows to get the estimate of $\|\nabla v^\varepsilon\|_{L_t^2 H_{co}^{m-1}}$. The remaining task is to control the $L_t^\infty H_{co}^{m-2}$ norm of $\nabla v^\varepsilon$, which can be derived from the estimates for $\omega^\varepsilon \times \mathbf{n}$.

We modify the quantities used to establish the uniform estimates:

$$\mathcal{N}_{m,T}^\varepsilon \approx \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty \mathcal{H}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^{m-2} \cap L_T^2 H_{co}^{m-1} \cap L_T^\infty L^\infty}$$

where $L_t^\infty \mathcal{H}^m$ involves only weighted time derivatives $\varepsilon\partial_t$ up to order $m$. We refer to Section (6.1.1) for the definition of conormal spaces and to (6.1.17) for the precise definition of $\mathcal{N}_{m,T}^\varepsilon$. Before stating the main result, we define the compatibility conditions:

**Definition 1** (Compatibility conditions)**.** *We say that $(\sigma_0^\varepsilon, u_0^\varepsilon)$ satisfy the compatibility conditions up to order $m$ if:*

$$(\varepsilon\partial_t)^j u^\varepsilon\big|_{t=0} \cdot \boldsymbol{n} = 0, \qquad \Pi\big[\mathbb{S}\big((\varepsilon\partial_t)^j u^\varepsilon|_{t=0}\big)\boldsymbol{n}\big] = -a\Pi\big[(\varepsilon\partial_t)^j u|_{t=0}\big] \quad on \quad \partial\Omega, j = 0, 1 \cdots m-1. \tag{2.3.11}$$

Note that the restriction of the time derivatives of the solution at the initial time can be expressed inductively by using the equations. For example, we have

$$(\varepsilon\partial_t u^\varepsilon)(0) = \frac{1}{\rho_0^\varepsilon}(-\varepsilon u_0^\varepsilon \cdot \nabla u_0^\varepsilon + \varepsilon \mathrm{div} \mathcal{L} u_0^\varepsilon - \nabla\sigma_0^\varepsilon).$$

The following is our main result which is a restatement of Theorems 6.1.1 and 6.1.6 in [95]:

**Theorem 2.3.4** (Masmoudi-Rousset-Sun, [95])**.** *Given an integer $m \geq 6$ and a $C^{m+2}$ smooth bounded domain $\Omega$. Consider a family of initial data such that $(\sigma_0^\varepsilon, u_0^\varepsilon)$ satisfy compatibility conditions up to order $m$ and*

$$\sup_{\varepsilon \in (0,1]} \mathcal{N}_{m,0}^\varepsilon < +\infty,$$

$$-\bar{c}\bar{P} \leq \varepsilon\sigma_0^\varepsilon(x) \leq \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0,1],$$

*where $0 < \bar{c} < 1/4$ is a fixed constant, $\bar{P} = P(\bar{\rho})$. There exist $\varepsilon_0 \in (0,1]$ and $T_0 > 0$, such that, for any $0 < \varepsilon \leq \varepsilon_0$, the system (2.3.2), (2.3.5) has a unique solution $(\sigma^\varepsilon, u^\varepsilon)$ which satisfies:*

$$-2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 2\bar{P}/\bar{c}, \quad \forall (t,x) \in [0,T_0] \times \Omega,$$

*and*

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \mathcal{N}_{m,T_0}^\varepsilon < +\infty.$$

*Moreover, $\rho^\varepsilon = g_2(\varepsilon\sigma^\varepsilon)$ converges to $\bar{\rho}$ in $C([0,T_0], L^2)$, and $u^\varepsilon$ converges in $L_w^2([0,T_0], L^2(\Omega))$ (weak convergence in time) to $u^0$ which is the (unique in the class with additional regularity (6.1.23)) weak solution to the incompressible Navier-Stokes system (2.3.3) with Navier boundary condition.*

We finish this subsection by sketching the a-priori estimates for $\mathcal{N}_{m,T}^\varepsilon$ which further leads to Theorem 2.3.4. This is achieved in the following steps which are slightly different from the half-space case: (we shall skip the $\varepsilon$ dependence in the notations for the sake of simplicity).

**Step 1: Uniform high-order $\varepsilon\partial_t$ derivatives and $\varepsilon-$dependent high-order conormal derivatives.** In this step, we aim to prove two kinds of estimates. Namely, uniform estimates for high order $\varepsilon\partial_t$ derivatives, $\|(\sigma, u)\|_{L_t^\infty \mathcal{H}^m}$, $\|(\nabla\sigma, \mathrm{div} u)\|_{L_t^\infty \mathcal{H}^{m-1}}$, and $\varepsilon-$dependent estimates: $\varepsilon\|(\sigma, u)\|_{L_t^\infty H_{co}^m}$. Again, as the spatial conormal vector fields do not commute with $\nabla, \mathrm{div}-$ the singular part of the system, we need at this stage to add this additional $\varepsilon$ weight to control the commutator.

**Step 2: Uniform estimates for the incompressible part of the velocity.** Let us denote by $v = \mathbb{P}u$, and $\nabla\Psi = \mathbb{Q}u$ the incompressible and compressible part of the velocity respectively, where $\mathbb{P}, \mathbb{Q}$ are defined in (6.3.2). By applying the projection $\mathbb{P}$ on the equation for the velocity and expanding the boundary conditions, we find that $v$ solves:

$$\begin{cases} \bar{\rho}\partial_t v - \mu\Delta v + \nabla q + \frac{g_2-\bar{\rho}}{\varepsilon}\varepsilon\partial_t u + g_2 u\cdot\nabla u = 0 \quad \text{in} \quad \Omega \\ v\cdot\mathbf{n} = 0, \quad \Pi(\partial_{\mathbf{n}} v) = \Pi(-2au + D\mathbf{n}\cdot\nabla\Psi + D\mathbf{n}\cdot u) \quad \text{on} \quad \partial\Omega \end{cases}$$

where

$$\nabla q = -\mathbb{Q}(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t u + g_2 u\cdot\nabla u - \mu\Delta v).$$

The incompressible part $v$ interacts with the compressible part $\nabla\Psi$ through the source term and the boundary condition. D Due to the absence of singular terms, one can get the uniform estimates for $v$ (namely $\|v\|_{L_t^\infty H_{co}^{m-1}}$ and $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$) by direct energy estimates. Nevertheless, for latter use in the proof, we need to track in the energy estimates the counts of time and spatial conormal derivatives.

**Step 3: Uniform estimates for the compressible part of the system.** In this step, we aim to get the control of $\|(\nabla\sigma, \mathrm{div}u)\|_{L_t^\infty H_{co}^{m-2}\cap L_t^2 H_{co}^{m-1}}$. This can be done by using the equations and induction arguments as employed in the half-space case.

**Step 4**: Control of $L_t^\infty H_{co}^{m-2}$ norm of $\nabla u$. The difficulty is the estimate close to the boundary. We can work in a local chart $\Omega_i$. In light of the identities

$$\partial_{\mathbf{n}} v\cdot\mathbf{n} = -(\Pi\partial_{y_1}v)^1 - (\Pi\partial_{y_2}v)^2, \quad \Pi(\partial_{\mathbf{n}}v) = \Pi(\omega\times\mathbf{n}) - \Pi[(D\mathbf{n})v],$$

where $\mathbf{n}$ is an extension of the unit normal and $\Pi$ projects on $(\mathbf{n})^\perp$, it suffices to control $\|\omega\times\mathbf{n}\|_{L_t^\infty H_{co}^{m-2}}$. We remark that the advantage of working on $\omega\times\mathbf{n}$ rather than $\omega$ is that the boundary condition for $\omega\times\mathbf{n}$ (see (6.3.33)) involves only lower order terms on the boundary. To estimate $\omega\times\mathbf{n}$, a natural attempt, used in [93], is to perform energy estimates on the equation for the 'modified vorticity' $w = \omega\times\mathbf{n} + 2\Pi(au - (D\mathbf{n})u)$ and to take advantage of the fact that $w$ vanishes on the boundary. However, the equations for $w$ still involve a stiff term $\frac{1}{\varepsilon}\nabla^\perp\sigma$, which is obviously an obstacle to obtain uniform energy estimates. We shall thus use the similar 'splitting' technique explained in half-space case to obtain the estimate of $\omega\times\mathbf{n}$.

**Step 5: $L_{t,x}^\infty$ estimates.** The control of the $L_{t,x}^\infty$ norms mainly stems from the Sobolev embedding and the maximum principle for the system solved by the vorticity and $\partial_{\mathbf{n}}\sigma$.

### 2.3.3 Low Mach number limit for free surface Navier-Stokes equations

In this subsection, we consider a slightly compressible viscous flow occupied in a domain with free boundaries. More precisely, we are interested in the following system:

$$\begin{cases} \partial_t\rho^\varepsilon + \mathrm{div}(\rho^\varepsilon w^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon w^\varepsilon) + \mathrm{div}(\rho^\varepsilon w^\varepsilon\otimes w^\varepsilon) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \end{cases} \qquad (t,x)\in\mathbb{R}_+\times\Omega_t^\varepsilon \tag{2.3.12}$$

where $\rho^\varepsilon, w^\varepsilon$ are the density and the velocity of the fluids, $P(\rho^\varepsilon)$, a smooth function of $\rho^\varepsilon$, stands for the pressure. The stress tensor $\mathcal{L}u^\varepsilon$ takes the form:

$$\mathcal{L}u^\varepsilon = 2\mu\mathbb{S}u^\varepsilon + \lambda\mathrm{div}u^\varepsilon\,\mathrm{Id}, \quad \mathbb{S}u^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + \nabla^t u^\varepsilon).$$

Here, $\mu, \lambda$ are viscosity parameters that are assumed to be constant and to satisfy the condition: $\mu > 0, 2\mu + 3\lambda > 0$. The parameter $\varepsilon$ is the scaled Mach number which is assumed small, that is $\varepsilon\in(0,1]$. We focus on the case where the fluid domain is given by:

$$\Omega_t^\varepsilon = \{x = (y,z) |\ y\in\mathbb{R}^2,\ z < h^\varepsilon(t,y)\},$$

where $h^\varepsilon(t,y)$, the surface of the fluid domain, is unknown and needs to be solved together with $(\rho^\varepsilon, w^\varepsilon)$. Since the fluid particles do not cross the surface, $h^\varepsilon$ solves the following equation:

$$\partial_t h^\varepsilon - w^\varepsilon(t,y,h^\varepsilon(t,y))\cdot\mathbf{N}^\varepsilon = 0, \quad y\in\mathbb{R}^2 \tag{2.3.13}$$

where $\mathbf{N}^\varepsilon = (-\partial_1 h^\varepsilon, -\partial_2 h^\varepsilon, 1)^t$ denotes the outward normal vector supported on the surface $\Sigma_t^\varepsilon = \{x = (y,z), z = h^\varepsilon(t,y)\}$. We supplement the system (2.3.12) and (2.3.13) with the following physical condition, which expresses the continuity of the stress tensor on the surface:

$$\mathcal{L}u^\varepsilon \mathbf{N}^\varepsilon = \frac{1}{\varepsilon^2}\big(P(\rho^\varepsilon) - P(\bar{\rho})\big)\mathbf{N}^\varepsilon \quad \text{on} \quad \Sigma_t^\varepsilon \tag{2.3.14}$$

where $\bar{\rho} > 0$ is a reference constant density.

The mathematical study of free boundary problems has received much attention since the last four decades. For the compressible viscous systems, local well-posedness was established by Secchi-Valli [111], Solonnikov-Tani [117], Zajaczkowski [132], Tanaka-Tani [121] with or without surface tension. As for the incompressible free surface Navier-Stokes equations, we refer to Solonnikov [116], Beale [13], Tani [123] Guo-Tice [60] for the local well-posedness and Guo-Tice [59] for global well-posedness. For the well-posedness of free surface inviscid incompressible system, the first local existence result was due to Wu [128, 129] for a layer with infinite depth and irrotational data. Later on, similar results are obtained by Lannes [81] for a layer with finite depth and by Zhang-Zhang [133] for rotational data, see also the works [4, 5]. The global well-posedness of 3d gravity water waves system was established by Germain-Masmoudi-Shatah [49] and Wu [130] independently, and is then generalized to the case with surface tension [33] and 2d gravity water waves system [6, 70]. For the compressible free-surface Euler equations, one can refer to [87, 28, 124] for local well-posedness.

Concerning the low Mach number in the presence of free boundaries, there are only a few works on inviscid compressible systems. In [88], Lindblad-Luo prove uniform a-priori estimates for the free boundary compressible Euler equations in the case of a bounded reference domain. More recently, this result is extended by Luo [92] for unbounded reference domains and by Disconzi-Luo [37] for a bounded reference domain but with surface tension. All these results are based on the assumption that the initial datum is sufficiently well-prepared in the sense that the time derivatives up to at least order two are bounded initially. Nevertheless, within our knowledge, there is no related work for the viscous system with free boundaries. Therefore, motivated by the work on the incompressible limit problem for viscous fluids in a fixed domain done in the last subsection, we find it interesting to investigate the similar problem in the presence of free boundaries. In the case of the fixed boundary for viscous fluids, one can start from a weak solution, nevertheless, due to the absence of a suitable theory of weak solutions in the case of the free boundary, we have to deal with strong enough solutions.

In the last subsection, we establish uniform estimates for $(CNS)_\varepsilon$ in a domain with fixed boundaries and slip boundary condition in a general setting of ill-prepared initial data. In there, we get around difficulties arising from the fast oscillations and the boundary layer effects. Nevertheless, it cannot be generalized naively to the case of moving domain since extra difficulties due to the appearance of the free surface show up. Indeed, because of the occurrence of the singular terms, the compressible part of the system behaves at time scale $\tau = t/\varepsilon$ like the small viscosity approximation of the Navier-Stokes equation, we thus cannot obtain extra regularity for the surface from the diffusion term. Therefore, we are forced to allow that the initial data to be slightly well-prepared in the sense that the first time derivative of the solution is uniformly bounded in a rather low regularity space, see also Remark 2.3.6. We shall explain more below after the reformulation of the system and statement of the main result.

We first choose some appropriate change of coordinates to reduce the free-surface domain to a fixed one. One natural possibility is to use Lagrangian coordinates, nevertheless, since we shall consider the problem in the conormal Sobolev setting, the Lagrangian transformation would be only bounded in the conormal setting. Therefore, we shall instead use the following smoothing diffeomorphism [83], where the map will be in the usual Sobolev spaces. Denote $\mathcal{S} = \mathbb{R}^2 \times (-\infty, 0]$, and consider the map:

$$\begin{aligned} \Phi_t^\varepsilon : \mathcal{S} = \mathbb{R}^2 \times (-\infty, 0] &\to \Omega_t^\varepsilon \\ (y,z) &\to \Phi^\varepsilon(t,y,z) = (y, \varphi^\varepsilon(t,y,z))^t \end{aligned}$$

where

$$\varphi^\varepsilon(t,y,z) = Az + \eta^\varepsilon(t,y,z) \tag{2.3.15}$$

Here $\eta^\varepsilon$ is given by

$$(\mathcal{F}\eta^\varepsilon)(t,\xi,z) = e^{-(1+|\xi|^2)z^2}(\mathcal{F}h^\varepsilon)(t,\xi) \tag{2.3.16}$$

where $\mathcal{F}$ stands for the Fourier transform with respect to the horizontal variable, $A>0$ is a constant to be chosen sufficiently large later such that, within a existence time interval $[0,T]$,

$$\sup_{0\le t\le T}\|\partial_z\eta^\varepsilon(t)\|_{L^\infty(\mathcal{S})}<A/2. \tag{2.3.17}$$

Note that $\Phi_t^\varepsilon$ is a diffeomorphism when (2.3.17) is satisfied since

$$\det(D\Phi^\varepsilon)(t,x)=\partial_z\varphi^\varepsilon=A+\partial_z\eta^\varepsilon>0.$$

Set

$$\varrho^\varepsilon(t,x)=\rho^\varepsilon(t,\Phi_t^\varepsilon(x)),\ u^\varepsilon(t,x)=w^\varepsilon(t,\Phi_t^\varepsilon(x))$$

and introduce the new unknown

$$\sigma^\varepsilon=(P(\varrho^\varepsilon)-P(\bar\rho))/\varepsilon,$$

we find that it suffices to study of the following system:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t^{\varphi^\varepsilon}\sigma^\varepsilon+u^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\sigma^\varepsilon\big)+\dfrac{\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon}{\varepsilon}=0,\\ g_2\big(\varepsilon\sigma^\varepsilon\big)\big(\partial_t^{\varphi^\varepsilon}u^\varepsilon+u^\varepsilon\cdot\nabla^{\varphi^\varepsilon}u^\varepsilon\big)-\mathrm{div}^{\varphi^\varepsilon}\mathcal{L}^{\varphi^\varepsilon}u^\varepsilon+\dfrac{\nabla^{\varphi^\varepsilon}\sigma^\varepsilon}{\varepsilon}=0, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_-\\ u^\varepsilon|_{t=0}=w_0^\varepsilon(\Phi_0^\varepsilon(x)):=u_0^\varepsilon,\quad \sigma^\varepsilon|_{t=0}=\varrho_0^\varepsilon(\Phi_0^\varepsilon(x)):=\sigma_0^\varepsilon,\end{cases} \tag{2.3.18}$$

with boundary conditions:

$$\partial_t h^\varepsilon-u^\varepsilon(t,y,h(t,y))\cdot\mathbf{N}^\varepsilon=0. \tag{2.3.19}$$

$$\mathcal{L}^\varphi u^\varepsilon\mathbf{N}^\varepsilon=\frac{\sigma^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon\quad\text{on}\quad\{z=0\}. \tag{2.3.20}$$

where

$$\partial_i^{\varphi^\varepsilon}=\partial_i-\frac{\partial_i\varphi^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z,\quad i=0,1,2,\quad \partial_z^{\varphi^\varepsilon}=\frac{1}{\partial_z\varphi^\varepsilon}\partial_z,\quad \partial_0=\partial_t, \tag{2.3.21}$$

We shall use the following tangential vector fields: $Z_0=\varepsilon\partial_t, Z_1=\partial_{y_1}, Z_2=\partial_{y_2}, Z_3=\frac{z}{1-z}\partial_z$, and conormal norms:

$$\|f\|_{L_t^p\mathcal{H}^m}=\sum_{k\le m}\|(\varepsilon\partial_t)^kf\|_{L^p([0,t],L^2(\mathcal{S})}, \quad \|f\|_{L_t^pH_{co}^m}=\sum_{|\alpha|\le m}\|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))},$$

$$|g|_{L_t^p\tilde H^s}=\sum_{k=0}^{[s]}|(\varepsilon\partial_t)^kh|_{L^p([0,t],H^{s-k}(\mathbb{R}^2))}$$

where $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$.

It is useful to introduce the following quantity composed by conormal norms and defined precisely in (7.1.30):

$$\begin{aligned}\mathcal{N}_{m,T}^\varepsilon\approx{}&\varepsilon^{-\frac12}\|(\nabla^{\varphi^\varepsilon}\sigma^\varepsilon,\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon)\|_{L_T^\infty H_{co}^{m-2}\cap L_t^2H_{co}^{m-1}\cap L_T^\infty L^\infty}+\|\nabla u^\varepsilon\|_{L_T^\infty H_{co}^{m-4}\cap L_T^2H_{co}^{m-1}\cap L_T^\infty L^\infty}\\ &+\varepsilon^{\frac12}\|\partial_t(\sigma^\varepsilon,u^\varepsilon)\|_{L_T^\infty\mathcal{H}^{m-1}}+\varepsilon^{\frac12}\|\partial_t\nabla u^\varepsilon\|_{L_T^2\mathcal{H}^{m-1}\cap L_T^2H_{co}^{m-2}\cap L_T^\infty H_{co}^{m-4}}\\ &+\varepsilon^{\frac12}\|(\sigma^\varepsilon,u^\varepsilon)\|_{L_T^\infty H_{co}^m}+\varepsilon^{\frac12}\|\nabla u^\varepsilon\|_{L_T^\infty H_{co}^{m-1}\cap L_T^2H_{co}^m}+|h|_{L_t^\infty\tilde H^{m-\frac12}}+\varepsilon^{\frac12}|h|_{L_t^\infty\tilde H^{m+\frac12}}.\end{aligned}$$

Again, before stating the main result, we introduce the definition of the compatibility conditions:

**Definition 2** (Compatibility conditions). *We say that $(\sigma_0^\varepsilon,u_0^\varepsilon)$ satisfy the compatibility condition up to order $m$ if for $j=0,1\cdots m-1$,*

$$(\varepsilon\partial_t)^j\big(\mathcal{L}^{\varphi^\varepsilon}u^\varepsilon\boldsymbol{n}^\varepsilon\big)|_{t=0}=(\varepsilon\partial_t)^j(\sigma^\varepsilon/\varepsilon)\big|_{t=0},\quad \varepsilon^j\partial_t^{j+1}h^\varepsilon\big|_{t=0}=(\varepsilon\partial_t)^j(u^\varepsilon\cdot\boldsymbol{N}^\varepsilon)\big|_{t=0}\quad \textit{on } \{z=0\}.$$

We are now ready to give a brief statement for the main result which is taken from [96].

**Theorem 2.3.5** (Masmoudi-Rousset-Sun, [96]). *Given $m \geq 7$ an integer. Assume that the initial datum satisfies compatibility conditions up to order $m$ and*

$$\sup_{\varepsilon \in (0,1]} \mathcal{N}^{\varepsilon}_{m,0} < +\infty,$$

$$-c_1 \bar{P} \leq \varepsilon \sigma^{\varepsilon}_0(x) \leq \bar{P}/c_1, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1]$$

*where $0 < c_1 < \frac{1}{4}$ is a fixed constant, $\bar{P} = P(\bar{\rho})$. Then there exist $0 < T_0 \leq 1, 0 < \varepsilon_0 \leq 1$, such that for any $0 < \varepsilon \leq \varepsilon_0$, the system (2.3.18)-(2.3.20) has a unique solution which satisfies:*

$$\mathcal{N}^{\varepsilon}_{m,T_0} < +\infty,$$

$$-2c_1 \bar{P} \leq \varepsilon \sigma^{\varepsilon}(t,x) \leq 2\bar{P}/c_1, \forall (t,x) \in [0,T_0] \times \mathcal{S}, \varepsilon \in (0,\varepsilon_0].$$

*Moreover, as $\varepsilon$ tends to 0, $(\bar{P}+\varepsilon\sigma^{\varepsilon}, u^{\varepsilon}, h^{\varepsilon})$ converge in $C^{\gamma}([0,T_0]\times\mathcal{S})\times C([0,T_0], L^2_{loc}(\mathcal{S}))\times C([0,T_0], H^s_{loc}(\mathbb{R}^2))$ to $(0, u^0, h^0)$ $0 \leq \gamma < \frac{1}{2}, 0 \leq s < m - 1/2$. Moreover, $u^0$ is the solution to the incompressible free-surface Navier-Stokes equations (see (7.1.33)).*

**Remark 2.3.6.** *In view of the definition of $Y^{\varepsilon}_m$, we have assumed that the restriction of the first time derivative of the solution at $t = 0$ is of order $\varepsilon^{-\frac{1}{2}}$, which is slightly better than the well-prepared data case (where $\partial_t(\sigma^{\varepsilon}, u^{\varepsilon})|_{t=0}$ is assumed to be order 1). This assumption has to be made due to some possible loss of regularity on the surface.*

**Remark 2.3.7.** *We can also prove the uniform estimates by imposing alternative assumption on the size of the acoustic waves, that is assuming them to be of size of order $\varepsilon$ in a rather low regularity $H^1_{co}$ but order 1 in the higher regularity $H^m_{co}$.*

The uniform estimates are achieved through careful modifications of the strategies employed in the fix domain case, and some extra attention needs to be paid to the regularity of the surface. As before, since the spatial tangential vector fields do not commute with $\nabla^{\varphi^{\varepsilon}}, \mathrm{div}^{\varphi^{\varepsilon}}$, we cannot have the foreknowledge of the higher order tangential derivatives. Therefore, we need to use the appropriate Leray projector and split the system into the compressible part and incompressible part. We derive estimates for the compressible part from those of $\mathrm{div}^{\varphi^{\varepsilon}} u^{\varepsilon}$ by suitable elliptic estimates and for the incompressible part by performing direct energy estimates.

The usual Leray-Helmholtz projection is defined by solving the compressible part by a elliptic equation with Neumann boundary condition. That is, by supposing $u^{\varepsilon} = \nabla^{\varphi^{\varepsilon}} \Psi^{\varepsilon} + v^{\varepsilon}$, and the compressible part $\nabla^{\varphi^{\varepsilon}} \Psi^{\varepsilon}$ satisfies

$$\Delta^{\varphi^{\varepsilon}} \Psi^{\varepsilon} = \mathrm{div}^{\varphi^{\varepsilon}} u^{\varepsilon} \quad \text{in } \mathcal{S}, \qquad \nabla^{\varphi^{\varepsilon}} \Psi^{\varepsilon} \cdot \mathbf{N}^{\varepsilon} = u^{\varepsilon} \cdot \mathbf{N}^{\varepsilon} \text{ on } \{z = 0\}.$$

However, this means that the incompressible part satisfies $v^{\varepsilon} \cdot \mathbf{N}^{\varepsilon}|_{z=0} = 0$, which contradicts with our expectation that $\partial_t h^0 + v^0 \cdot \mathbf{N}^0 = 0$ where $h^0, v^0, \mathbf{N}^0$ are the corresponding limits of sequences $h^{\varepsilon}, v^{\varepsilon}, \mathbf{N}^{\varepsilon}$. We thus need to modify the definition of the projection. It seems that the natural one is to supplement the trivial Neumann boundary condition. Nevertheless, since the projection (defined in this way) does not commute with time derivation $\partial_t^{\varphi^{\varepsilon}}$, the commutators between them are hard to handle. We thus eventually choose the homogeneous Dirichlet boundary condition so that the commutators between the projection and time derivation would be easier to control. Note that this kind of projection has been used in [13].

We shall establish the uniform estimates in the following steps (we skip the $\varepsilon-$dependence of the solutions):

**Step 1: $\varepsilon-$dependent high order energy estimates and $\varepsilon-$independent high order time estimates.** This is similar to the fixed domain case but the computations are more involved mainly due to the appearance of surface equation. In addition, since $h$ solves the transport equation, we have the (non-uniform) control of surface (ie. $\varepsilon^{\frac{1}{2}}|h|_{L^{\infty}_t H^{m+\frac{1}{2}}}$) thanks to the dissipation effects.

**Step 2. Uniform lower order energy estimates.** In this step, we show the boundedness of $\|\varepsilon^{\frac{1}{2}} \partial_t(\sigma, u)\|_{L^{\infty}_t L^2}$.

**Step 3. Recovering high order spatial derivatives of $(\nabla^\varphi\sigma, \nabla^\varphi\nabla^\varphi\Psi)$ by induction.** Denote $\nabla^\varphi\Psi$ the compressible part of the velocity which is defined by the unique solution to the following elliptic problem:

$$\begin{cases} -\mathrm{div}^\varphi\nabla^\varphi\Psi = -\mathrm{div}^\varphi u, \\ \Psi|_{z=0} = 0. \end{cases} \tag{2.3.22}$$

In this step, we aim to control the $L_t^2H_{co}^{m-1}$ norm of $\nabla^\varphi(\sigma, \nabla^\varphi\Psi)$ which can be deduced from the estimate of $\varepsilon^{-\frac{1}{2}}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{m-1}}$. We will use the equation and induction arguments to recover the high order spatial derivatives of $(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)$. Let us rewrite the system (2.3.18) as follows:

$$\begin{cases} -\mathrm{div}^\varphi u = g_1\varepsilon\partial_t\sigma + \varepsilon g_1\underline{u}\cdot\nabla\sigma, \\ -\mu\varepsilon\,\mathrm{curl}^\varphi\,\omega - \nabla^\varphi\big(\sigma - (2\mu+\lambda)\varepsilon\mathrm{div}^\varphi u\big) = g_2\varepsilon\partial_t u + \varepsilon g_2\underline{u}\cdot\nabla u. \end{cases} \tag{2.3.23}$$

where $\underline{u} = (u_1, u_2, u_z) =: (u_1, u_2, \frac{u\cdot\mathbf{N}-\partial_t\varphi}{\partial_z\varphi})$.

$$\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j,l}} \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,l}} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \lesssim \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j+1,l-1}} + \mathcal{O}(\varepsilon^{\frac{1}{2}}),$$

$$\begin{aligned} \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\sigma\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi\Psi\|_{L_t^2\mathcal{H}^{j+1,l}} + \mathcal{X}_{m,t} + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \\ &\lesssim \varepsilon^{-\frac{1}{2}}\|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{j+1,l-1}} + \mathcal{X}_{m,t} + \mathcal{O}\big((T+\varepsilon)^{\frac{1}{2}}\big) \end{aligned}$$

where

$$\mathcal{X}_{m,t} = (\varepsilon^{\frac{1}{2}}\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\|\nabla^\varphi u\|_{L_t^2H_{co}^m})\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)$$

which has been controlled in the first step. These two inequalities in hand, we can conclude by induction arguments. These two inequalities in hand, we can conclude by the induction arguments.

Now that $\mathrm{div}^\varphi u$ has been bounded, we can control the compressible part of the velocity $\nabla^\varphi\nabla^\varphi\Psi$ by again elliptic estimates. Nevertheless, there shall be loss of one derivatives on the surface. Indeed, as $\nabla^\varphi\Psi$ solves equation (2.3.22), we have by the elliptic estimates that

$$\|\nabla^\varphi\nabla^\varphi\Psi\|_{L_t^2\mathcal{H}^{0,m-1}} \lesssim (|h|_{L_t^2H^{m+\frac{1}{2}}} + \|\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{0,m-1}})\Lambda(\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{tan}^1} + |h|_{2,\infty,t}),$$

which requires one more regularity of the surface than that we can expect (note that we have only the control of $|h|_{L_t^2H^{m-\frac{1}{2}}}$ ). Nevertheless, when performing the variational arguments to get elliptic estimates, the main problematic term is indeed $\nabla\Psi Z^\alpha\nabla\mathbf{N}$ ($|\alpha| = m-1, \alpha_0 = 0$), whose $L_t^2L^2(\mathcal{S})$ norm can be bounded by

$$\|\nabla\Psi\|_{L_{t,x}^\infty}|h|_{L_t^2H^{m+\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{tan}^1}|h|_{L_t^2H^{m+\frac{1}{2}}}.$$

The right hand side can be controlled if $\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{tan}^1} = \mathcal{O}(\varepsilon)$ (remember that we have the control of $\varepsilon^{\frac{1}{2}}|h|_{L_t^2H^{m+\frac{1}{2}}}$ in the first step). Hopefully, as is shown in Step 2, once assuming $\varepsilon^{\frac{1}{2}}(\partial_t\sigma, \partial_t u)(0)$ to be bounded uniformly in $H_{co}^1(\mathcal{S})$, we have that $\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^1} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$. This is one reason why we need the initial data to be slightly well-prepared in our assumption.

**Step 4. Uniform energy estimate of the incompressible part of the velocity** Denote $v = u - \nabla^\varphi\Psi$ the incompressible part of the velocity. By a straightforward calculation, we find that $v$ is governed by the following system:

$$\begin{cases} \bar{\rho}\partial_t^\varphi v - \mu\Delta^\varphi v + \nabla^\varphi\pi = -(f + \nabla^\varphi q + \bar{\rho}[\mathbb{P}_t, \partial_t^\varphi]u), \\ \mathrm{div}^\varphi v = 0, \\ (2\mu S^\varphi v - \pi\mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^\varphi u\mathrm{Id} - \nabla^\varphi\nabla^\varphi\Psi)\mathbf{N}|_{z=0} \end{cases} \tag{2.3.24}$$

where

$$f = (g_2 u\cdot\nabla^\varphi u + \frac{g_2-\bar{\rho}}{\varepsilon}\partial_t^\varphi u),\ \nabla^\varphi q = -\mathbb{Q}_t(f - \mu\Delta^\varphi v),\ \nabla^\varphi\pi = \mathbb{P}_t[\nabla^\varphi(\frac{\sigma}{\varepsilon} - (2\mu+\lambda)\mathrm{div}^\varphi u)]$$

and $\mathbb{Q}_t, \mathbb{P}_t$ are time-dependent projectors projecting vector functions to its compressible part and incompressible part (see (7.5.2) (7.5.3) for the exact definition). Note that $\nabla^\varphi\pi$ does not vanish identically since $\mathbb{Q}\nabla^\varphi \neq \nabla^\varphi$ (which is essentially due to the very definition of $\mathbb{Q}_t$ which lying on solving the elliptic problem with *homogeneous Dirichlet* boundary condition). Nevertheless, although including singular term $\sigma/\varepsilon$, it can indeed be estimated uniformly since it solves a Laplace equation with amenable boundary conditions. In view of (7.1.46), we can get the a priori control of $\|v\|_{L_t^\infty H_{co}^{m-1}}, \|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^\infty H_{co}^{m-2}}$ and $\|\nabla^\varphi v\|_{L_t^2 H_{co}^{m-1}}, \|\varepsilon^{\frac{1}{2}}\partial_t \nabla v\|_{L_t^\infty H_{co}^{m-2}}$ by performing direct energy estimates. Due to the interactions with the compressible part through the boundary condition, its control relies also on the information of compressible part $\nabla^\varphi\Psi$ and therefore we cannot estimate the maximal number of derivatives $m$ for $v$. As will be seen in Chapter 7, even to control the second highest $(m-1)$ derivatives of $v$, we need also to use the 'slightly well-prepared' initial data assumption again to avoid losing regularity on the surface.

**Step 5. Control of the normal derivative of the velocity.** We have obtained the estimates of $\|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-1}}$ in Step 3 and Step 4. It remains to control $\varepsilon^{-\frac{1}{2}}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-2}}$ and $\|\nabla v\|_{L_t^\infty H_{co}^{m-3}}$, which is useful to control the $L_{t,x}^\infty$ norm of the solution. The former quantity can be obtained by again induction arguments while the latter quantity can be deduced from that of $\omega\times\mathbf{n}$. we have roughly the estimate:

$$\|(\nabla v, \varepsilon^{\frac{1}{2}}\partial_t\nabla v)\|_{L_t^\infty H_{co}^{m-4}} \lesssim \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\omega\times\mathbf{n}\|_{L_t^\infty H_{co}^{m-4}} + \|(v, \varepsilon^{\frac{1}{2}}\partial_t v)\|_{L_t^\infty H_{co}^{m-3}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde{H}^{m-2}}.$$

Let us explain the estimate of $\|\omega\times\mathbf{n}\|_{L_t^\infty H_{co}^{m-3}}$. Direct computations show that:

$$\omega\times\mathbf{n} = -2\Pi(\partial_1 u\cdot\mathbf{n}, \partial_2 u\cdot\mathbf{n}, 0)^t \quad \text{on } \{z=0\}. \tag{2.3.25}$$

where $\Pi = \mathrm{Id}_{3\times3} - \mathbf{n}\otimes\mathbf{n}$. We define the modified vorticity $\omega_{\mathbf{n}} = \omega\times\mathbf{n} + 2\Pi(\partial_1 v\cdot\mathbf{n}, \partial_2 v\cdot\mathbf{n}, 0)$, so that:

$$\omega_{\mathbf{n}}|_{\partial\mathcal{S}} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n}, \partial_2\nabla^\varphi\Psi\cdot\mathbf{n}, 0)^t.$$

The advantage of working on $\omega_{\mathbf{n}}$ rather than $\omega\times\mathbf{n}$ is that the former one only involves the compressible part of velocity on the boundary, whose estimates has been established in Step 3. To estimate $\omega_{\mathbf{n}}$, we use again a lifting of the boundary conditions by using Green's function for the solution of the heat equation with non-homogenous boundary conditions and estimate the remainder by energy estimates. More precisely, let $\omega^h$ solves the free heat equation with boundary condition $\omega^{b,1}|_{z=0} = \omega_{\mathbf{n}}|_{z=0}$, we use the Green function to get that:

$$\begin{aligned}\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\omega_{\mathbf{n}}^h\|_{L_t^\infty H_{co}^{m-4}} &\lesssim T^{\frac{1}{4}}\big(|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\nabla\Psi|_{L_t^\infty\tilde{H}^{m-3}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)h|_{L_t^\infty\tilde{H}^{m-3}}\big)\\ &\lesssim T^{\frac{1}{4}}(\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^{m-3}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)h|_{L_t^\infty\tilde{H}^{m-3}}).\end{aligned}$$

The remainder $\omega^n - \omega_n^h$ can then be controlled by direct energy estimates.

**Step 6. $L_{t,x}^\infty$ estimate.** Their estimates basically stem from the Sobolev embedding, the maximum principle of (damped) transport equations satisfied by $\nabla\sigma$ and also the explicit formulae for the heat equation satisfied by $\omega$.

## The arrangement of the thesis:

Chapter 3 is devoted to the uniform stability of 3-d Navier-Stokes-Poisson system, it is based on a joint work [108] with Prof. Rousset which has been published in Ann. Inst. H. Poincaré Anal. Non Linéaire.

In Chapter 4, we study the long time regularity of 2-d Navier-Stokes-Poisson system which is taken from [120]-a manuscript accepted for publication by SIAM Journal on Mathematical Analysis.

In Chapter 5, we use the 'splitting' technique proposed in [108, 120] to investigate large time existence of the Euler-Korteweg system and two-fluid Euler-Maxwell system with nontrivial vorticity. This is based on the work [119] published in Nonlinear Analysis.

Chapter 6 and Chapter 7 are dedicated to the low Mach number limit problem in a domain with fixed and free boundaries respectively. They are excerpted from [95] (submitted) and [96] (in preparation) which are joint works with Professors F. Rousset and N. Masmoudi.

# Part I: Uniform (in Reynolds number) stability for viscous fluid models in plasma physics

# Chapter 3

# Uniform stability for 3-d Navier-Stokes-Poisson system

Taken from [108] which has been published in Ann. Inst. H. Poincaré Anal. Non Linéaire, this chapter contains a joint work with professor Frédéric Rousset on the uniform stability result of Navier-Stokes-Poisson system.

**Abstract** We prove a stability result of constant equilibra for the three dimensional Navier-Stokes-Poisson system uniform in the inviscid limit. We allow the initial density to be close to a constant and the potential part of the initial velocity to be small independently of the rescaled viscosity parameter $\varepsilon$ while the incompressible part of the initial velocity is assumed to be small compared to $\varepsilon$. We then get a unique global smooth solution. We also prove a uniform in $\varepsilon$ time decay rate for these solutions. Our approach allows to combine the parabolic energy estimates that are efficient for the viscous equation at $\varepsilon$ fixed and the dispersive techniques (dispersive estimates and normal forms) that are useful for the inviscid irrotational system.

## 3.1 Introduction

The Navier-Stokes-Poisson system is a hydrodynamical model of plasma which describes the dynamics of electrons and ions that interact with its self-consistent electric field. If we neglect the motion of ions, then the dynamics of electrons can be described by the following electron Navier-Stokes-Poisson system (ENSP)

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon \mathcal{L} u^\varepsilon + \nabla p(\rho^\varepsilon) - \rho^\varepsilon \nabla \phi^\varepsilon = 0, \\ \Delta \phi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{3.1.1}$$

We shall always consider in this paper that the spatial domain is the whole space, $x \in \mathbb{R}^3$. Here the unkowns $\rho^\varepsilon(t,x) \in \mathbb{R}_+$ , $u^\varepsilon \in \mathbb{R}^3$, $\nabla\phi^\varepsilon \in \mathbb{R}^3$ are the electron density, the electron velocity and the self-consistent electric field respectively. The thermal pressure of electrons $p(\rho^\varepsilon)$ is usually assumed to follow a polytropic $\gamma$-law: $p(\rho^\varepsilon) = C(\rho^\varepsilon)^\gamma$, $\gamma > 1$ while the viscous term is under the form

$$\mathcal{L}u^\varepsilon = \mu \Delta u^\varepsilon + (\mu + \lambda)\nabla \mathrm{div} u^\varepsilon$$

where the Lamé coefficients $\mu, \lambda$ are supposed to be constants which satisfy the condition:

$$\mu > 0 \qquad 2\mu + \lambda > 0$$

Note that we consider a scaled version of the system with the coefficient $\varepsilon$ in front of the diffusion terms which is the inverse of the Reynolds number and which will be assumed small in this paper. For the simplicity of the presentation, we shall assume in this paper that $\mu = 1$, $\lambda = 0$ and that $p(\rho^\varepsilon) = (\rho^\varepsilon)^2/2$.

Nevertheless, there is no special cancellation arising from this choice (the easiest case for the analysis in this paper would be the choice $\mu(\rho) = \rho$, $\lambda = -\mu$, since in this case there are curl free solutions of (3.1.1)). The results of this paper can thus be easily extended to general pressure (with $P'(1) > 0$) and to general density dependent $\mu, \lambda$ as long as $\mu(1) > 0$, $2\mu(1) + \lambda(1) > 0$. We shall also handle in this paper a simplified system for the dynamics of ions, the electrons being considered in thermodynamical equilibrium which reads

$$\begin{cases} \partial_t \rho_+^\varepsilon + \operatorname{div}(\rho_+^\varepsilon u_+^\varepsilon) = 0, \\ \partial_t(\rho_+^\varepsilon u_+^\varepsilon) + \operatorname{div}(\rho_+^\varepsilon u_+^\varepsilon \otimes u_+^\varepsilon) - \varepsilon \mathcal{L} u_+^\varepsilon + \nabla p(\rho_+^\varepsilon) - \rho_+^\varepsilon \nabla \phi_+^\varepsilon = 0, \\ \Delta \phi_+^\varepsilon - \phi_+^\varepsilon = \rho_+^\varepsilon - 1, \\ u_+|_{t=0} = u_{+0}^\varepsilon, \rho_+|_{t=0} = \rho_{+0}^\varepsilon. \end{cases} \tag{3.1.2}$$

There is a large body of literature dealing with the stability under small and smooth enough perturbations of the constant equilibrium (say $(\rho^\varepsilon, u^\varepsilon) = (1, 0)$) of (ENSP) when $\varepsilon = 1$. Here stability means global existence (in a suitable Sobolev or Besov space) and decay for small perturbations. We refer for example to [85] where global existence in $H^l$ for $l \geq 4$ is proven under the assumption that the initial perturbation is small in $H^l$ and $L^1$. An explicit time decay rate for the perturbation is obtained by a careful analysis of the Green function of the linearized system (we also refer to [66]). More recently, in [126] global existence in $H^N (N \geq 3)$ of (ENSP) is obtained by using only energy estimates under the assumption that the initial perturbation belongs to $H^N$ and is small in $H^3$. Moreover, as in works on the compressible Navier-Stokes system [61], by assuming that the initial data is in a negative Sobolev space $\dot{H}^{-s}$ $(0 < s < \frac{3}{2})$, explicit decay rates can be obtained by using interpolation inequalities and energy estimates. These results use heavily the fact that the equation for the velocity is a parabolic equation and that the coupling between the two evolution equations of (ENSP) yields decay of the density. In [85], global existence in dimension $d$ is obtained in hybrid Besov spaces when the initial perturbation is close to equilibrium in a $L^2$ critical norm by using energy estimates and by considering low and high frequencies differently. This result was then generalized to a $L^p$ critical frameworks [135], [26].

All these works deal with an unscaled system, that is to say (ENSP) with $\varepsilon = 1$. We can easily check that for the $\varepsilon$ dependent system, these works give global smooth solutions if the initial perturbation is small enough compared to $\varepsilon$ and that the obtained decay rates hold in terms of the slow time variable $\varepsilon t$ (for example [126] would give that in $L^\infty$, $(\rho^\varepsilon - 1)$ is bounded by $\varepsilon(1 + \varepsilon t)^{-\frac{3}{2}}$). Indeed, global existence is obtained by bootstrap arguments and a priori estimates. There are roughly two ways to get the a priori estimates. One way is, as in [85], [126], to use energy estimates and to get dissipation for $u^\varepsilon$ by using the diffusion term $\varepsilon \Delta u^\varepsilon$ and dissipation for $\rho^\varepsilon - 1$ by using a "cross energy estimate". The nonlinear terms can be absorbed if some quantity is small compared to $\varepsilon$. The other way is, as in [64], [135], [26] when considering global existence in critical Besov spaces is to use the maximal smoothing effect of the heat kernel $e^{\varepsilon t \Delta}$, which gives for example for the scaled heat equation

$$\|e^{\varepsilon t \Delta} f\|_{L^1(\mathbb{R}_+, \dot{B}^{s+2}_{p,1})} \lesssim \varepsilon^{-1} \|f\|_{\dot{B}^s_{p,1}}.$$

Therefore, to control the nonlinear terms, this also leads to the assumption that the size of the initial perturbation has to be small compared to $\varepsilon$.

Nevertheless, when $\varepsilon = 0$, the system (3.1.1) reduces to the so-called electron Euler-Poisson (EEP) system. For the (EEP) system, the global existence of smooth solutions close to the constant equilibrium $(1, 0)$ was first obtained by Guo [55] under neutral, irrotational, small perturbation to the reference equilibrium $(\rho^0, u^0) = (1, 0)$. The neutral assumption ($\int (\rho_0^0 - 1) \mathrm{d}x = 0$) was then removed in [48]. The important property which was used in these works is that the (EEP) system has better dispersive properties than the Euler equations for compressible fluids due to the presence of the electric field. For example, when restricted to irrotational solutions, the linearized (EEP) system can be rewritten as a Klein-Gordon equation which verifies in space dimension $d$ the decay estimate

$$\|e^{it\langle \nabla \rangle} f\|_{L^\infty} \lesssim (1 + t)^{-\frac{d}{2}} \|f\|_{W^{d,1}}$$

which is better than the one of the wave equation. Nevertheless, in dimension 3, the only use of energy estimates and of the above dispersive decay (or its $L^p \to L^{p'}$ counterpart) is not enough to get global smooth solutions in the presence of quadratic nonlinearities. Some additional ingredient is thus needed namely either energy estimates using the vector fields methods or the normal form method. For the Euler-Poisson system the normal form method of Shatah [112] or more generally, the 'space-time resonances' philosophy can be used to control the nonlinear terms. We refer to [112] and [47, 48], for more information about normal form method and the 'space-time resonance' approach. This type of approach was recently successfully used to handle the (EEP) system in dimension two [84, 69] and one [56].

Since in concrete physical flows the Reynolds number is usually very high (thus $\varepsilon$ very small), it is natural to ask for stability results that hold uniformly with respect to $\varepsilon$ for (ENSP). Though the methods used in the two lines of results that we just presented are completely different, it is rather natural to expect to get global smooth solutions for perturbations of the constant equilibrium $(1,0)$ with a smallness assumption on the perturbation that is independent of $\varepsilon$ except for the curl part of the velocity (remember that for $\varepsilon = 0$ we have global smooth solutions only for irrotational data). This is the result that we shall obtain in this paper. A first attempt to get such a result would be to write the solution of (ENSP) as the global solution of (EEP) plus a remainder and to try to control the remainder. Since the source term in the equation for the perturbation is of order $\varepsilon$, one could hope to use the parabolic methods described above to control the remainder. Nevertheless, such a naive approach cannot work. Indeed, even in dimension 3, the source term in the equation for the remainder has a non integrable decay in the energy norm so that there is no hope to be able to control the remainder globally in time. We thus really need to develop a method that allows to use the type of ideas introduced in the study of dispersive PDE when there is a small dissipative term in addition. This is the main aim of this paper. As far as we know, there are few works addressing this type of question, in [21] it is the extension of the vector field method that is developed. The situation that we are dealing with here for (ENSP) occurs for many other systems of mathematical physics. Indeed, there are many other systems for which we have for the viscous version of the physical model, global existence for small, viscosity dependent data and for the inviscid version (which is often a dispersive perturbation of a compressible type Euler equation) global existence for small irrotational data. We can think about MHD, water-waves...We thus hope that the approach developed in this paper can be useful to handle other systems. As an illustration, we shall also handle the Navier-Stokes-Poisson system for ions, the results are described in the end of the introduction.

We shall denote by $\mathcal{P}$ the Leray projector on divergence free vector fields so that $\mathcal{P}^\perp = Id - \mathcal{P} = \nabla\Delta^{-1}\mathrm{div}$. The following is our main result for the (ENSP) system:

**Theorem 3.1.1.** *Let us set $\nabla\phi_0^\varepsilon = -\nabla(-\Delta)^{-1}(\rho_0^\varepsilon - 1)$. There exists $\delta_0 > 0$ such that for every family of initial data that satisfy for every $\varepsilon \in (0,1]$ the estimates :*

$$\|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,1}} + \|(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{H^N} \leq \delta_0$$
$$\|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \delta_0\varepsilon$$

*with $\sigma \geq 5$ and $N \geq \sigma + 7$, then, for every $\varepsilon \in (0,1]$, there exists a unique global solution of the (ENSP) system* (3.1.1) *in $C([0,+\infty), H^3)$. If in addition, we assume that $\sup_{\varepsilon\in(0,1]} \|\mathcal{P}u_0^\varepsilon\|_{\dot{H}^{-s}} < +\infty$ for some $0 < s < \frac{1}{2}$, then we have the following time decay estimates that are uniform in $\varepsilon$. There exists $C > 0$ such that for every $\varepsilon \in (0,1]$, we have*

$$\|(\rho^\varepsilon - 1, \nabla\phi^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}} \leq C\big(\min\{\varepsilon, (1+t)^{-\frac{s}{2+s}}\} + (1+t)^{-(\frac{11}{8}+)}\big), \quad \forall t \geq 0.$$

*where $a^+$ stands for any number strictly larger arbitrarily close to $a$.*

**Remark 3.1.2.** *If in addition, $\mathcal{P}u_0^\varepsilon$ is in $H^M$ (say $\sup_{\varepsilon\in(0,1]} \|\mathcal{P}u_0^\varepsilon\|_{\dot{H}^M} < +\infty$) and $\sigma \geq M + 2 > 5$, then the solution constructed in Theorem 3.1.1 also belongs to $C([0,\infty), H^M)$.*

Note that the assumption that we make on the size of the "curl" part of the initial data, that is to say the assumption on $\mathcal{P}u_0^\varepsilon$, seems to be the natural one. Indeed, even if we assume that $\mathcal{P}u_0^\varepsilon = 0$, this property is not propagated by the system (ENSP), the convection diffusion equation for the rotational part of the velocity is forced by a source term of size $\varepsilon$ so that a curl part of size $\varepsilon$ is instantaneously created.

The main difficulty in order to get Theorem 3.1.1 lies in the interaction between the dynamics of the potential part and the incompressible part of the solution. For the potential part we could expect a $L^\infty$ decay given by the linear inviscid dispersive estimates of the order $(1+t)^{-\frac{3}{2}}$. For the incompressible part, we expect that this component will remain of order $\varepsilon$ in $H^s$ but its decay is driven by the heat equation with diffusivity $\varepsilon$, in terms of uniform in $\varepsilon$ estimate this can only yield at best a rather slow decay rate of order $(1+t)^{-1}$ which could be difficult to handle especially in the control of the interaction with the potential part. Our strategy to prove Theorem 3.1.1 is to split the system into two viscous systems, with initial data $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and $(0, 0, \mathcal{P}u_0^\varepsilon)$ respectively. The first one will have global solutions under $\varepsilon$-independent assumptions on the inital data $(\rho_0^\varepsilon - 1, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and the solutions will enjoy the same decay estimates as the (EEP) system. The other is just the perturbation of the original system (3.1.1) by the solution to the former one, the important points are that for this system the initial data and the source term are small compared to $\varepsilon$ and that the source term has integrable decay in $L^2$. We can thus use energy estimates and the good decay properties of the solutions to the former system to prove global existence and decay. More precisely, we write the solution $(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon)$ of (ENSP) as

$$(\rho^\varepsilon, \nabla\phi^\varepsilon, u^\varepsilon) = (\rho, \nabla\phi, u) + (n, \nabla\psi, v),$$

where $(\rho, \nabla\phi, u)$ and $(n, \nabla\psi, v)$ are the solutions of the following systems:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u - \varepsilon\mathcal{L}u + \nabla\rho - \nabla\phi = 0, \\ \Delta\phi = \rho - 1, \\ u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{3.1.3}$$

$$\begin{cases} \partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u\cdot\nabla v + v\cdot(\nabla u + \nabla v) - \varepsilon\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho+n} - 1)(\mathcal{L}v + \mathcal{L}u), \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0} = 0. \end{cases} \tag{3.1.4}$$

Note that for these two systems we skip the $\varepsilon$ dependence of the solutions in our notation.

We can set $\varrho = \rho - 1$, to change system (3.1.3) into:

$$\begin{cases} \partial_t\varrho + \mathrm{div}u = -\mathrm{div}(\varrho u), \\ \partial_t u + u\cdot\nabla u - \varepsilon\mathcal{L}u + \nabla\varrho - \nabla\phi = 0, \\ \Delta\phi = \varrho, \\ u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \varrho|_{t=0} = \varrho_0 = \rho_0^\varepsilon - 1. \end{cases} \tag{3.1.5}$$

Note that the initial datum for the last system is such that $\mathrm{curl}(\mathcal{P}^\perp u_0^\varepsilon) = 0$, and this irrotational property will be propagated which means that a smooth solution of this system will remain irrotational. This system is thus a really good viscous approximation of the Euler-Poisson system. As we shall see below, the linear part of this system has the same decay properties for low frequencies as the (EEP) system, that is for localized initial data, the $L^p$ norm of $(\varrho, \nabla\phi, u)$ decay like $(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}$ uniformly for $\varepsilon \in (0,1]$.

The following is the main result for the system (3.1.5).

**Theorem 3.1.3.** *For any $6 < p < +\infty$, there exists $\delta_0 > 0$ such that for any family of initial data satisfying*

$$\sup_{\varepsilon\in(0,1]} \left( \|(\varrho_0^\varepsilon, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,1}} + \|(\varrho_0^\varepsilon, \nabla\phi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{H^N} \right) \le \delta_0$$

*with $\sigma \ge 3, N \ge \sigma + 7$, then for every $\varepsilon \in (0,1]$, there exist a unique solution for system (3.1.5) in $C([0,\infty), H^N)$. Moreover, we have the following time decay estimates that are uniform for $\varepsilon \in (0,1]$. There exists a constant $C$ such that for every $\varepsilon \in (0,1]$, we have*

$$\|(\varrho, \nabla\phi, u)(t)\|_{W^{\sigma,p}} \le C\delta_0(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}, \quad \forall t \ge 0. \tag{3.1.6}$$

Let us now explain the main ideas for the proof. Using the 'curl-free' condition, we consider the new unkown $V = (\frac{\langle\nabla\rangle}{|\nabla|}\varrho, \frac{\mathrm{div}}{|\nabla|}u)$. The linearized system for $V$ is

$$\partial_t V + AV = 0, \quad A = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & -2\varepsilon\Delta \end{pmatrix}.$$

where we use $\langle\nabla\rangle = \sqrt{1-\Delta}$ the Fourier multiplier with symbol $\sqrt{1+|\xi|^2}$. The eigenvalues for this system are

$$\lambda_\pm = -\varepsilon|\xi|^2 \pm i\sqrt{1+|\xi|^2-\varepsilon^2|\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm ib(\xi)$$

A toy model to present the ideas is thus

$$\begin{cases} \partial_t\beta - \lambda_-(D)\beta = \beta^2 \\ \beta|_{t=0} = \beta_0 \end{cases}$$

The key observations are, on the one hand, when we focus on low frequencies, (say $\varepsilon|\xi|^2 \le 2\kappa_0$ with $\kappa_0$ to be chosen small but independent of $\varepsilon$) then $b(\xi)$ is very close to $\langle\xi\rangle$, this indicates that the imaginary part $e^{itb(D)}$ should give us an $L^p$ decay estimate ($p>2$) which is uniform for $\varepsilon\in(0,1]$. On the other hand, when we deal with high frequencies (in the sense that $\varepsilon|\xi|^2 \ge \kappa_0$), direct computations show that there exists a positive constant $c=c(\kappa_0)$ such that $\mathrm{Re}\,(\lambda_\pm) \le -c(\kappa_0)$ for any $\varepsilon\in(0,1]$, so we can expect that the high frequency part of the solution has good decay even in $L^2$ norm.

Define $\beta = \chi_{\varepsilon,\kappa_0}(D)\beta + (1-\chi_{\varepsilon,\kappa_0})(D)\beta =: \beta^L + \beta^H$ where $\chi_{\varepsilon,\kappa_0}(D), (1-\chi_{\varepsilon,\kappa_0})(D)$ are the Fourier multipliers that project on low and high frequencies in the above sense respectively (cf. Section 3.2 for the precise definition) . We then define the norm

$$\|\beta\|_{X_T} = \sup_{t\in[0,T)} \left( \|\beta(t)\|_{H^{10}} + \langle t\rangle^{\frac{3}{2}}\|\beta^H(t)\|_{H^{10}} + \langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|\beta^L(t)\|_{W^{3,p}} \right). \tag{3.1.7}$$

where we use the notation $\langle t\rangle = \sqrt{1+t^2}$. The first Sobolev norm can be estimated by standard energy estimates. The other two terms involve time decay estimates. The high frequencies piece is easier because we have uniform (with respect to $\varepsilon$) upper bounds for $\mathrm{Re}\,(\lambda_\pm)$ and thus an $L^2\to L^2$ type estimate with exponential decay uniformly in $\varepsilon$ for the semi-group. The low frequency piece is more difficult to get. We first check that $e^{itb(D)}$ enjoys the same dispersive estimates as $e^{it\langle\nabla\rangle}$ uniformly for $\varepsilon\in(0,1]$. As for the (EEP) systems the linear dispersive estimates are not enough to control the quadratic nonlinearity, we thus have to use normal form transformation to close the low frequencies decay estimate. In this step, we have to carefully track the contribution of the viscous term that creates new error terms. More precisely, let us write $\alpha = e^{-itb(D)}P^L\beta$ then, $\alpha$ satisfies the equation

$$\partial_t\alpha - \varepsilon\Delta\alpha = e^{-itb(D)}(\beta^2)^L.$$

By Duhamel's formula, we have:

$$\beta = e^{itb(D)}\alpha = e^{itb(D)}(e^{\varepsilon t\Delta}\beta_0^L + \int_0^t e^{\varepsilon(t-s)\Delta}e^{-isb(D)}\chi_{\varepsilon,\kappa_0}(D)((\beta^L)^2 + \beta\beta^H + \beta^H\beta^L)(s)\mathrm{d}s).$$

We focus only on the first term in the above integral, the decay for the other terms is easy to obtain because of the $L^2$ decay of the high frequency part. We can see the first term as

$$e^{itb(D)}\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^3} e^{-\varepsilon(t-s)|\xi|^2}e^{is\varphi}\hat{\alpha}(s,\xi-\eta)\hat{\alpha}(s,\eta)\mathrm{d}\eta\mathrm{d}s \tag{3.1.8}$$

where $\varphi = -b(\xi)+b(\eta)+b(\xi-\eta) > 0$ for $\kappa_0$ small enough. Following the 'space-time resonance' method, by using the identity $e^{is\varphi} = \frac{1}{i\varphi}\partial_s e^{is\varphi}$, we can integrate by parts in time so that (3.1.8) becomes:

$$i\int_0^t e^{i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}\big(\varepsilon\Delta T_{\frac{1}{\varphi}}(\beta^L,\beta^L) + T_{\frac{1}{\varphi}}(\varepsilon\Delta\beta^L + (\beta^2)^L,\beta^L)\big)\mathrm{d}s$$

plus boundary terms and symmetric term which are similar to handle (we refer to Section 3.2 for the definition of the bilinear operator $T_{\frac{1}{\varphi}}$). The last term is cubic and thus can be estimated as in the study of the (EEP) system (we shall check that for $\kappa_0$ sufficiently small the operator $T_{\frac{1}{\varphi}}$ has the same continuity properties as in the inviscid case). The first two terms are still quadratic but are $\varepsilon$ small, we can thus get additional decay by using the decay provided by the heat equation: for example, we expect that the $L^2$ norm of $\varepsilon\Delta\beta^L$ has decay like $(1+t)^{-1}$. This is enough to get Theorem 3.1.3 for $6<p\le 12$. To propagate the estimate for larger p which involves a faster rate of decay, the previous $(1+t)^{-1}$ gain is not enough and we shall therefore perform another step of integration by parts in time in order to close the estimate.

Let us now consider the system (3.1.4). We shall see the system (3.1.4) as a perturbation of (3.1.1) by $(\rho, \nabla\phi, u)$. Thanks to the good decay estimates for $(\rho, \nabla\phi, u)$(in the sense that the time decay of the $L^\infty$ norm is integrable in time), we can still get global existence by energy estimates for this system. We will prove the following result.

**Theorem 3.1.4.** *We fix the number $p \geq 24$ in Theorem 3.1.3. Consider $(\varrho, u, \nabla\phi)$ and $\delta_0$ given by Theorem 3.1.3. If $\delta_0$ is small enough and $\|\mathcal{P}^\perp u_0^\varepsilon\|_{H^3} \leq \delta_0\varepsilon$, then the system (3.1.4) has a solution in $C([0, +\infty), H^3)$ and*

$$\sup_{0 \leq t < +\infty} \|(n, \nabla\psi, v)(t)\|_{H^3} \leq 8\delta_0\varepsilon.$$

*Moreover, if we assume in addition that for some $s$, $0 < s < \frac{1}{2}$, $\sup_{\varepsilon\in(0,1)} \|\mathcal{P}u_0^\varepsilon\|_{\dot{H}^{-s}} < +\infty$, then we have the following uniform in $\varepsilon$ time decay estimates for $(n, \nabla\psi, v)$. There exists $C > 0$ which does not depend on $\varepsilon$, such that*

$$\|\nabla^l(n, \nabla\psi, v)(t)\|_{H^{3-l}} \leq C \min\{\varepsilon, (1+t)^{-\min\{\frac{l+s}{2+l+s}, \frac{1}{3}-\}}\}$$

*where $l = 0, 1, 2$ and $a^-$ stands for a real number smaller but arbitrarily close to $a$.*

**Remark 3.1.5.** *If in addition, $\mathcal{P}u_0^\varepsilon$ is in $H^M$, where $3 \leq M \leq \sigma - 2$, then the solution to (3.1.4) constructed above belongs to $C([0, +\infty), H^M)$. Besides, as we do not assume that $\|\mathcal{P}u_0^\varepsilon\|_{H^M}$ is small, we have some time decay estimate in terms of the slow variable $'\varepsilon t'$:*

$$\|\nabla^k(n, \nabla\psi, v)(t)\|_{H^{M-k}} \leq C(1+\varepsilon t)^{-\min\{\frac{k+s}{2}, \frac{1}{3}-\}}$$

*where $k = 0, 1, 2 \cdots M-1$.*

Inspired by [61] [126] , we use merely energy estimates to prove global existence. By using a modified energy functional $\tilde{\mathcal{E}}_M$ that roughly controls the same Sobolev norms as the usual energy functional

$$\mathcal{E}_M(n, v, \nabla\psi) = \sum_{|\alpha|\leq M} \frac{1}{2} \int \rho|\partial^\alpha v|^2 + |\partial^\alpha n|^2 + |\partial^\alpha \nabla\psi|^2 \mathrm{d}x,$$

for $M \geq 3$, we shall get that if $\mathcal{E}_3 \leq \delta\varepsilon^2$, and $\delta$ small enough, we have a positive constant $c$ such that the inequality

$$\partial_t \tilde{\mathcal{E}}_M + c\varepsilon(\|n\|_{H^M} + \|\nabla u\|^2_{H^M}) \lesssim \delta^3\varepsilon^2(1+t)^{-\frac{5}{3}} \tag{3.1.9}$$

holds. Note that the interest of this modified functional is that it detects also damping of the $n$ component. The global existence then follows from continuation arguments.

For the decay estimate, we first prove that the solution remains bounded in $\dot{H}^{-s}$ if the initial data is in $\dot{H}^{-s}$. Then by using an interpolation inequality and (3.1.9), we can obtain the energy inequality: $\partial_t \tilde{\mathcal{E}}_M + c\varepsilon(\tilde{\mathcal{E}}_M)^{1+\frac{1}{s}} \lesssim \varepsilon^2(1+t)^{-\frac{5}{3}}$ from which we get the desired decay estimate.

Once we have proven Theorem 3.1.3 and Theorem 3.1.4, Theorem 3.1.1 is an easy consequence of them.

In the last part of the paper, we shall explain how we can also handle the Navier-Stokes-Poisson system for the ions dynamics (INSP) introduced in (3.1.2) by using the same approach. Note that we have used the so-called linearized approximation since in the (INSP) system, we have replaced the Poisson equation $\Delta\phi_+^\varepsilon = \rho_+^\varepsilon - e^{-\phi_+^\varepsilon}$, by a linearized version. This is not a stringent assumption since we are again dealing with small perturbations of the constant equilibrium $(1, 0)$. For the Euler-Poisson system describing ions dynamics (IEP) (that is $\varepsilon = 0$ in (3.1.2)), global smooth irrotational solutions with small amplitude have been constructed by Guo and Pausader [58]. The idea is again to find dispersive estimates for the linearized system (which turn out to be weaker than the one of the linear Klein-Gordon equations) and to use the normal form method. Nevertheless, the analysis for this model is much more involved. Indeed, the dispersion relation is closer to the one of the wave equation which leads to the appearance of "time resonances". For example, the 'time resonances' of the phase function $\Phi_{++} = -p(\xi) + p(\xi - \eta) + p(\eta)$,$(p(\xi) = |\xi|\sqrt{\frac{2+|\xi|^2}{1+|\xi|^2}})$ is $\{\eta = 0\} \cup \{\xi - \eta = 0\}$. After integration in time, the multilinear operators now have a singular kernel and to control them the use of $\dot{H}^{-1}$ norms is needed.

We now state the counterpart of Theorem 3.1.1.

**Theorem 3.1.6.** *Let us fix some absolute number $\kappa > 0$ small enough. There exists $\delta_2 > 0$ such that for any family of initial conditions that satisfy for every $\varepsilon \in (0,1]$ the estimates*

$$\||(\rho_{+0}^\varepsilon - 1, \mathcal{P}^\perp u_{+0}^\varepsilon)\|_{W^{\sigma+3,8'_\kappa}} + \||\nabla|^{-1}(\rho_{+0}^\varepsilon - 1, \mathcal{P}^\perp u_{+0}^\varepsilon)\|_{H^N} \le \delta_2,$$
$$\|\mathcal{P}u_{+0}^\varepsilon\|_{H^3} \le \delta_2 \varepsilon$$

*with $8_\kappa = \frac{8}{1-3\kappa}$, $8'_\kappa = \frac{8_\kappa}{8_\kappa - 1}$, $\sigma \ge 6$, $N \ge 2\sigma + 1$, then we have that for every $\varepsilon \in (0,1]$ there exists a unique global solution for system (3.1.2) in $C([0,+\infty), H^3)$. Besides, if $\sup_{\varepsilon\in(0,1]} \|\mathcal{P}u_{+0}^\varepsilon\|_{\dot H^s} < +\infty$ with $s < \frac{3}{8}$, then we have the following time decay estimates. There exists $C > 0$ such that for every $\varepsilon \in (0,1]$, we have the estimate*

$$\|(\rho_+^\varepsilon - 1, u_+^\varepsilon)\|_{W^{1,\infty}} \le C\big(\min\{\varepsilon, (1+t)^{-\min\{\frac{s}{2+s}, \frac{\kappa}{2}\}}\} + (1+t)^{-(1+\kappa)}\big), \quad \forall t \ge 0.$$

**Organization of this chapter.** In the second section, we introduce some notations. In Section 3, we establish some useful preliminary estimates (in particular linear decay estimates) in order to prove Theorem 3.1.3. Then, we prove Theorem 3.1.3 and Theorem 3.1.4 in Section 4, and Section 5 respectively. In Section 6, we will explain briefly the modifications needed to extend the results to general pressure laws and density-dependent viscosities. In Section 7, we shall explain how to deal with the ions system. Finally, we recall some classical inequalities in the appendix.

## 3.2 Some Notations

- We define $\varphi_0(\xi), \chi(\xi)$ as two radial symmetric $C_c^\infty$ functions, which are both supported on $\{\xi \big| |\xi| \le 2\}$ and equal to 1 when $\{\xi \big| |\xi| \le 1\}$, and $\tilde\chi \in C_c^\infty$ equal to 1 on $\{\xi \big| |\xi| \le 3\}$ and vanish on $\{\xi \big| |\xi| \ge 4\}$.
- We shall also use the truncation function $\chi_{\varepsilon,\kappa_0}(\xi) = \chi\big(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi\big)$ in the proof of Theorem 3.1.3.
- We denote by $m(D)$ the Fourier multiplier defined by $m(D)f = \mathcal{F}^{-1}(m(\cdot)\mathcal{F}f(\cdot))$.
- We also introduce the classical Littlewood-Paley decomposition: define $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and $\varphi_j = \varphi(\frac{\xi}{2^j})$, $j \in \mathbb{N}^*$, $\Delta_j f = \mathcal{F}^{-1}(\varphi_j(\xi)\mathcal{F}f(\xi))$, $j \in \mathbb{N}$. The norm in the inhomogeneous Besov space $B^s_{p,r}(p, r \ge 1, s \in \mathbb{R})$ is defined by $\|f\|_{B^s_{p,r}} = (\sum_{j=0}^\infty \|\Delta_j f\|^r_{L^p} 2^{jsr})^{\frac{1}{r}}$.
- For a given function $m(\zeta,\eta)$, we define the bilinear operator $T_m(f,g)$ as:

$$\begin{aligned} T_m(f,g) &\triangleq \mathcal{F}^{-1}\Big(\int m(\xi-\eta,\eta)\hat f(\xi-\eta)\hat g(\eta)\mathrm{d}\eta\Big) \\ &= \frac{1}{(2\pi)^3}\int m(\zeta,\eta)\hat f(\zeta)\hat g(\eta)e^{ix(\zeta+\eta)}\mathrm{d}\zeta\mathrm{d}\eta \end{aligned} \tag{3.2.1}$$

- We use $\langle\cdot\rangle$ for $\sqrt{1+|\cdot|^2}$.
- We denote $a^+$ a constant which is larger and arbitrarily close to $a$.
- We shall always use the notation $\lesssim$ for $\le C$ for $C > 0$ a harmless number that can be chosen independent of $\varepsilon \in (0,1]$ and $t > 0$.

## 3.3 Preliminary estimates

In this section, we analyze the system (3.1.5). At first, we observe that as long as a smooth solution exists on an interval $[0,T]$, then $\omega(t) \triangleq \operatorname{curl} u(t) = 0$ on this interval. Indeed, by taking the curl in the second equation of system (3.1.5), we get the equation for $\omega$

$$\begin{cases} \partial_t\omega - \varepsilon\Delta\omega + \omega\mathrm{div}u + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0 \\ \omega|_{t=0} = 0 \end{cases}$$

By the standard energy estimate and Grönwall's inequality, we have

$$\|\omega(t)\|_{L^2}^2 \le e^{c\int_0^T \|\nabla u(s)\|_{L^\infty}\mathrm{d}s}\|\omega_0\|_{L^2}^2 = 0$$

A direct consequence is that $u = \mathcal{P}^\perp u = \nabla\Delta^{-1}\mathrm{div}u$. Thus by using the identity $\mathrm{curl}\,\mathrm{curl}\, u = -\Delta u + \nabla\mathrm{div}u$, the second equation of system (3.1.5) turns out to be:

$$\partial_t u + u\cdot\nabla u - 2\varepsilon\Delta u + \nabla\varrho - \nabla\phi = 0.$$

Based on the above facts, let us set

$$h = \frac{\langle\nabla\rangle}{|\nabla|}\varrho, \quad c = \frac{\mathrm{div}}{|\nabla|}u, \qquad V = (h,c)^\top,$$

we then obtain that $(h,c)$ satisfies the system:

$$\begin{cases} \partial_t h + \langle\nabla\rangle c = -\langle\nabla\rangle\dfrac{\mathrm{div}}{|\nabla|}\big(\dfrac{|\nabla|}{\langle\nabla\rangle}h\cdot\mathcal{R}c\big) = \langle\nabla\rangle\mathcal{R}^*\big(\dfrac{|\nabla|}{\langle\nabla\rangle}h\cdot\mathcal{R}c\big), \\ \partial_t c - \langle\nabla\rangle h - 2\varepsilon\Delta c = -\dfrac{1}{2}\dfrac{\mathrm{div}}{|\nabla|}\nabla|\mathcal{R}c|^2 = \dfrac{1}{2}|\nabla||\mathcal{R}c|^2, \\ h|_{t=0} = \dfrac{\langle\nabla\rangle}{|\nabla|}\varrho_0, c|_{t=0} = \dfrac{\mathrm{div}}{|\nabla|}u_0. \end{cases} \tag{3.3.1}$$

which we shall rewrite as:

$$\partial_t V + A(D)V = \begin{pmatrix} \langle\nabla\rangle\mathcal{R}^*\big(\frac{|\nabla|}{\langle\nabla\rangle}h\cdot\mathcal{R}c\big) \\ \mathcal{R}^*\nabla|\mathcal{R}c|^2 \end{pmatrix} \triangleq B(V,V), \quad A(D) = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & -2\varepsilon\Delta \end{pmatrix}. \tag{3.3.2}$$

In the above systems, $\mathcal{R}$ is the vectorial Riesz transform: $\mathcal{R} = \frac{\nabla}{|\nabla|}$ and $\mathcal{R}^* = -\frac{\mathrm{div}}{|\nabla|}$ is its adjoint for the $L^2$ scalar product.
By elementary computations, we get that the eigenvalues of $-A(\xi)$ are:

$$\lambda_\pm = -\varepsilon|\xi|^2 \pm i\sqrt{1+|\xi|^2-\varepsilon^2|\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm ib(\xi) \tag{3.3.3}$$

where we cut the lower half imaginary axis to define the square root of a complex number. Note that $b$ is in fact dependent on $\varepsilon$, but we do not write it explicitly for simplicity. One can easily check that the Green matrix is

$$e^{-tA(\xi)} = \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t} & (e^{\lambda_- t} - e^{\lambda_+ t})\langle\xi\rangle \\ (e^{\lambda_+ t} - e^{\lambda_- t})\langle\xi\rangle & \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \end{pmatrix} \triangleq \begin{pmatrix} \mathcal{G}_1(t,\xi) & -\mathcal{G}_2(t,\xi) \\ \mathcal{G}_2(t,\xi) & \mathcal{G}_3(t,\xi) \end{pmatrix}.$$

Note that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are actually well defined everywhere since there is no singularity when $\lambda_+ = \lambda_-$ (see the proof of Lemma 3.3.5).

Let us observe that for low frequencies, ie, when $\varepsilon|\xi|^2 \le 2\kappa_0 << 1$ (since the eigenvalues do not cross), we can smoothly diagonalize $A$ under the form:

$$\begin{aligned} A(D) &= \begin{pmatrix} 1 & 1 \\ -\frac{\lambda_-(D)}{\langle\nabla\rangle} & -\frac{\lambda_+(D)}{\langle\nabla\rangle} \end{pmatrix}\begin{pmatrix} -\lambda_- & 0 \\ 0 & -\lambda_+ \end{pmatrix}\begin{pmatrix} \lambda_+ & \langle\nabla\rangle \\ -\lambda_- & -\langle\nabla\rangle \end{pmatrix}\frac{1}{2ib} \\ &\triangleq Q\begin{pmatrix} -\lambda_- & 0 \\ 0 & -\lambda_+ \end{pmatrix}Q^{-1}, \quad Q^{-1} = \begin{pmatrix} \lambda_+ & \langle\nabla\rangle \\ -\lambda_- & -\langle\nabla\rangle \end{pmatrix}\frac{1}{2ib}. \end{aligned} \tag{3.3.4}$$

Since by Duhamel principle, we can rewrite (3.3.2) as

$$V = e^{-tA}V_0 + \int_0^t e^{-(t-s)A}B(V,V)(s)\mathrm{d}s, \tag{3.3.5}$$

we shall first study the main properties of $e^{-tA}$ and $B(V,V)$ in the following two subsections.

### 3.3.1 Linear estimates

This subsection is devoted to the study of $e^{-tA}$. We shall carry out the analysis in any space dimension $\mathbb{R}^d$, $d \geq 2$ although in this paper, we only use it for dimension 3. The behavior will be different for low frequencies $\varepsilon|\xi|^2 \lesssim 1$ where uniform in $\varepsilon$ decay estimates will come from the dispersive behavior and for high frequencies $\varepsilon|\xi|^2 \gtrsim 1$ where dissipative damping dominates.

#### 3.3.1.1 Linear estimates for low frequencies: $\varepsilon|\xi|^2 \leq 2\kappa_0$

For low frequencies, we can get decay estimates that are similar to the ones of the linear Klein-Gordon equation by using dispersive properties. Let us recall that we use the notation $\chi_{\varepsilon,\kappa_0}(\xi) = \chi\big(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi\big)$ (see Section 3.2). We will fix the threshold $\kappa_0$ in the proof of Lemma 3.3.1.

**Lemma 3.3.1.** *There exists $\kappa_0 > 0$, small enough such that uniformly for $\varepsilon \in (0,1]$, and for every $f \in B^d_{1,2}$, we have the estimate*

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)f\|_{B^0_{\infty,2}} \lesssim_{\kappa_0} (1+|t|)^{-\frac{d}{2}}\|f\|_{B^d_{1,2}}. \quad \forall t \in \mathbb{R}$$

*Proof.* Note that on the support of $\chi_{\varepsilon,\kappa_0}$, $b(\xi)$ behaves like $\langle\xi\rangle$, thus, to prove this lemma, we can follow the proof of the dispersive estimate for the linear Klein-Gordon equation by keeping track of the perturbation. The key point is that this dispersive estimate is uniform with respect to $\varepsilon$.

The proof will thus follow from the following two lemmas.

**Lemma 3.3.2.** *For every $\kappa_0$ small enough, we have uniformly for $\varepsilon \in (0,1]$, the estimate*

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)\varphi_0(D)f\|_{L^\infty} \lesssim (1+|t|)^{-\frac{d}{2}}\|f\|_{L^1}. \quad \forall t \in \mathbb{R}$$

*Proof.* By using the Fourier transform, we only need to show that

$$\|\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi)\mathrm{d}\xi\|_{L^\infty} \lesssim_{\kappa_0} (1+|t|)^{-\frac{d}{2}}.$$

At first, note that:

$$\|\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi)\mathrm{d}\xi\|_{L^\infty} \lesssim \|\varphi_0\|_{L^1}.$$

Thus in the following, we only prove that:

$$\|\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi)\mathrm{d}\xi\|_{L^\infty} \lesssim_{\kappa_0} |t|^{-\frac{d}{2}}.$$

Let us write

$$\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi)\mathrm{d}\xi = \int e^{it\Phi(\xi)}\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi)\mathrm{d}\xi, \quad \Phi(\xi) = b(\xi) + \frac{x}{t}\cdot\xi.$$

By direct computations, the first and second derivative of $\Phi(\xi)$ are given by the following expressions:

$$\nabla_\xi\Phi(\xi) = \nabla_\xi b + \frac{x}{t} = \frac{(1-2\varepsilon^2|\xi|^2)}{b(\xi)}\xi + \frac{x}{t}$$

$$\partial_{\xi_i}\partial_{\xi_j}\Phi(\xi) = \frac{1-2\varepsilon^2|\xi|^2}{b(\xi)}(\delta_{ij} - \frac{(1+4\varepsilon^2)\xi_i\xi_j}{b^2(\xi)(1-2\varepsilon^2|\xi|^2)}).$$

We then obtain that on the support of $\chi_{\varepsilon,\kappa_0}(\xi)\varphi_0(\xi) \subset \{\xi\big||\xi| \leq 2, \varepsilon|\xi|^2 \leq 2\kappa_0\}$, we have

$$\begin{aligned}\det(\nabla^2\Phi(\xi)) = (\frac{1-2\varepsilon^2|\xi|^2}{b(\xi)})^d\frac{1-6\varepsilon^2|\xi|^2-3\varepsilon^2|\xi|^4+2\varepsilon^4|\xi|^6}{b^2(\xi)(1-2\varepsilon^2|\xi|^2)} \quad &\geq \quad \frac{(1-4\varepsilon\kappa_0)^d(1-12\varepsilon\kappa_0-12\kappa_0^2)}{b^{d+2}(\xi)} \\ &\geq \quad \frac{1}{2^{d+1}b^{d+2}(\xi)} \geq \frac{1}{2^{d+1}\cdot 5^{\frac{d+1}{2}}}\end{aligned}$$

for $\varepsilon \in (0,1]$ as long as $\kappa_0$ is small enough.

By using the classical stationary phase lemma (we refer to [102],[136] for example), we arrive at the desired result. $\square$

**Lemma 3.3.3.** *Suppose $d \geq 2$. For every $\kappa_0 > 0$ small enough and for every $\lambda \geq 1$, we have uniformly for $\varepsilon \in (0,1]$,*

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)\varphi(\frac{D}{\lambda})f\|_{L^\infty} \lesssim_{\kappa_0} |t|^{-\frac{d}{2}}\lambda^{\frac{d+2}{2}}\|f\|_{L^1}, \qquad \forall t \in \mathbb{R}$$

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)\varphi(\frac{D}{\lambda})f\|_{L^\infty} \lesssim_{\kappa_0} \lambda^d\|f\|_{L^1}.$$

*Proof.* It suffices to prove:

$$\|\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi(\frac{\xi}{\lambda})\mathrm{d}\xi\|_{L^\infty} \lesssim_{\kappa_0} |t|^{-\frac{d}{2}}\lambda^{\frac{d+2}{2}}, \qquad \forall t \in \mathbb{R}$$

$$\|\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi(\frac{\xi}{\lambda})\mathrm{d}\xi\|_{L^\infty} \lesssim_{\kappa_0} \lambda^d.$$

The second estimate just comes from a change of variable, we thus only need to prove the first one. We will also restrict ourselves to the case $t > 0$ as the other case is similar. As $\chi_{\varepsilon,\kappa_0}, \phi, b$ are all radially symmetric, we actually have:

$$\begin{aligned}\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi(\frac{\xi}{\lambda})\mathrm{d}\xi &= \int_0^{+\infty} e^{itb(r)}\chi_{\varepsilon,\kappa_0}(r)\phi(\frac{r}{\lambda})\mathcal{F}(\sigma_{\mathbb{S}^{d-1}})(|x|r)r^{d-1}\mathrm{d}r \\ &= \lambda^d\int_{\frac{1}{2}}^2 e^{itb(\lambda r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)\mathcal{F}(\sigma_{\mathbb{S}^{d-1}})(\lambda|x|r)r^{d-1}\mathrm{d}r\end{aligned}$$

where we have used the fact that the Fourier transform of the Lebesgue measure on the sphere $\mathcal{F}(\sigma_{\mathbb{S}^{d-1}})(x)$ is (see [50], Appendix B)

$$\mathcal{F}(\sigma_{\mathbb{S}^{d-1}})(x) = |x|^{-\frac{d-2}{2}}J_{\frac{d-2}{2}}(|x|) = e^{i|x|}Z(|x|) - e^{-i|x|}\bar{Z}(|x|)$$

where $J_{\frac{d-2}{2}}(s)$ is the Bessel function and $Z(s)$ satisfies [50] for all integer $k \geq 0$ and all $s > 0$,

$$|\partial^k Z(s)| \lesssim_{k,d} (1+s)^{-\frac{d-1}{2}-k}. \tag{3.3.6}$$

Therefore, we can write:

$$\int e^{itb(\xi)}e^{ix\cdot\xi}\chi_{\varepsilon,\kappa_0}(\xi)\varphi(\frac{\xi}{\lambda})\mathrm{d}\xi = \sum_{\pm}\lambda^d\int_{\frac{1}{2}}^2 e^{it\Phi_\lambda^{\pm}(r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)Z_{\pm}(\lambda|x|r)\mathrm{d}r$$

where $Z_\pm = Z, \bar{Z}$ and $\Phi_\lambda^{\pm}(r) = b(\lambda r) \pm \lambda r\frac{|x|}{t}$.

For the $'+'$ case, we can easily check that

$$\partial_r\Phi_\lambda^+ = \lambda b'(\lambda r) + \frac{\lambda|x|}{t} = \lambda\frac{\lambda r(1-2\varepsilon^2\lambda^2r^2)}{b(\lambda r)} + \frac{\lambda|x|}{t} \geq \frac{\lambda r(1-2\varepsilon^2\lambda^2r^2)}{b(r)} \gtrsim_{\kappa_0} \lambda$$

as long as $\kappa_0$ is small enough. Moreover, for $k \geq 2$, we have

$$|\partial_r^k\Phi_\lambda^+| = |\lambda^k\partial_r^k b(\lambda r)| \lesssim_{\kappa_0} \lambda.$$

This yields by direct induction, that

$$|\partial_r^k\frac{1}{\partial_r\Phi_\lambda^+}| \lesssim_{\kappa_0,k} \lambda^{-1}. \tag{3.3.7}$$

In addition, by (3.3.6), we have on the support of $\phi$, that

$$\partial_r^k\big(Z_\pm(\lambda|x|r)\big) \lesssim (\lambda|x|)^k(1+\lambda|x|r)^{-\frac{d-1}{2}-k} \lesssim (1+\lambda|x|r)^{-\frac{d-1}{2}} \leq 1. \tag{3.3.8}$$

Consequently, by using the classical (non-)stationary phase lemma and (3.3.7), (3.3.8), we have that for any integer $N \geq 0$

$$\Big|\int_{\frac{1}{2}}^{2} e^{it\Phi_\lambda^+(r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)Z_+\mathrm{d}r\Big| \lesssim_{\kappa_0} (\lambda t)^{-N}.$$

To conclude, we choose $N = \frac{d}{2}$ if $d$ is even, and we choose $N = \frac{d-1}{2}$ and $N = \frac{d+1}{2}$ if $d$ is odd to get:

$$\Big|\int_{\frac{1}{2}}^{2} e^{it\Phi_\lambda^+(r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)Z_+\mathrm{d}r\Big| \lesssim_{\kappa_0} (\lambda t)^{-\frac{d}{2}} \lesssim \lambda^{-\frac{d-2}{2}}t^{-\frac{d}{2}}$$

which is the desired result for '+' case.

For the $'-'$ case, the first derivative of $\Phi_\lambda^-(r)$ can vanish. Indeed, we have

$$\partial_r\Phi_\lambda^- = \lambda b'(\lambda r) - \frac{\lambda|x|}{t} = \lambda\frac{\lambda r(1-2\varepsilon^2\lambda^2r^2)}{b(\lambda r)} - \frac{\lambda|x|}{t}.$$

At first, if $|x| \leq \frac{t}{100}$ or $t \leq \frac{|x|}{100}$ , then we have

$$|\partial_r\Phi_\lambda^-| \gtrsim_{\kappa_0} \frac{\lambda}{t}(|x|+t)$$

and for $k \geq 2$,

$$|\partial_r^k\Phi_\lambda^-| = |\partial_r^k\Phi_\lambda^+| \lesssim_{\kappa_0} \lambda \leq \frac{\lambda}{t}(|x|+t).$$

As before, this yields by induction, for any $l \geq 0$,

$$\Big|\partial_r^l\frac{1}{\partial_r\Phi_\lambda^-}\Big| \lesssim_{\kappa_0} \big(\lambda(|x|+t)\big)^{-1}t.$$

Consequently, by using again the (non-)stationary phase method, we get:

$$\Big|\int_{\frac{1}{2}}^{2} e^{it\Phi_\lambda^-(r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)\bar{Z}(\lambda r)\mathrm{d}r\Big| \lesssim_{\kappa_0} \langle\lambda(t+|x|)\rangle^{-N}.$$

If $|x| \approx t$, ie $\frac{1}{100} \leq \frac{|x|}{t} \leq 100$, we first notice that if $\kappa_0$ is sufficient small, then on the support of $\chi_{\varepsilon,\kappa_0}(\lambda r)$, one has $\partial_r^2\Phi_\lambda^- = \lambda^2\frac{1-6\varepsilon^2(\lambda r)^2-3\varepsilon^2(\lambda r)^4+2\varepsilon^4(\lambda r)^6}{b^3(\lambda r)} \gtrsim_{\kappa_0} \lambda^{-1}$. Combining this fact with the behavior of $Z$ (see (4.1.10)), we then apply Van der Corput Lemma (see for example [118]) to get

$$\begin{aligned}
&\Big|\int_{\frac{1}{2}}^{2} e^{it\Phi_\lambda^-(r)}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)\bar{Z}(\lambda|x|r)\mathrm{d}r\\
\lesssim_{\kappa_0}\ & (\lambda^{-1}t)^{-\frac{1}{2}}\Big(\frac{1}{2^{d-1}}\chi(\frac{1}{2}(\frac{\varepsilon}{\kappa_0})^{\frac{1}{2}}\lambda)\phi(\frac{1}{2})\bar{Z}(\frac{1}{2}\lambda|x|) + \sup_r \partial_r(r^{d-1}\chi_{\varepsilon,\kappa_0}(\lambda r)\phi(r)\bar{Z}(\lambda|x|r))\Big)\\
\lesssim_{\kappa_0}\ & (\lambda^{-1}t)^{-\frac{1}{2}}(1+\lambda|x|)^{-\frac{d-1}{2}} \lesssim_{\kappa_0} \lambda^{-\frac{d}{2}+1}t^{-\frac{d}{2}}.
\end{aligned}$$

This ends the proof. ☐

Once we have the above two lemmas, we can sum the frequencies over the dyadic decomposition to get Lemma 3.3.1. ☐

From now on, we fix $\kappa_0$ sufficiently small independent of $\varepsilon$ such that the statement of Lemma 3.3.1 and proposition 3.8.3 in Appendix holds.

**Corollary 3.3.4.** *For $j = 1, 2, 3$ and $f \in B_{1,2}^d$, we have uniformly in $\varepsilon \in (0,1]$ the estimate*

$$\|\mathcal{G}_j(t,D)\chi_{\varepsilon,\kappa_0}(D)f\|_{B^0_{\infty,2}} \lesssim (1+t)^{-\frac{d}{2}}\|f\|_{B^d_{1,2}}, \quad \forall t > 0.$$

*Proof.* We focus on the proof for $\mathcal{G}_1$, the other two terms can be handled with similar arguments. Simple computations show that:

$$\mathcal{G}_1(t,D)=\frac{1}{2}e^{\varepsilon t\Delta}\big((e^{ib(D)t}+e^{-ib(D)t})+i\frac{\varepsilon\Delta}{b(D)}(e^{ib(D)t}-e^{-ib(D)t})\big).$$

By Lemma 3.3.1 and the continuous property of the operator $e^{\varepsilon t\Delta}$ on $L^p$, $1\le p\le+\infty$, it suffices for us to show the same result as in Lemma 3.3.1 when $e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)$ is changed into $e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)\frac{\varepsilon\Delta}{b(D)}$. The proof is similar to that of Lemma 3.3.1 once we notice that on the support of $\chi_{\varepsilon,\kappa_0}$, $\partial_r^k\big(\frac{\varepsilon r^2}{b(r)}\big)\lesssim C$, we thus omit the details.

□

#### 3.3.1.2 Linear estimates for high frequencies: $\varepsilon|\xi|^2\ge\kappa_0$

**Lemma 3.3.5.** *There exists $c_0>0$ such that, for $j=1,2,3$ and for every $\varepsilon\in(0,1]$, we have the estimate*

$$|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_j(t,\xi)|\lesssim e^{-c_0t},\quad\forall t\ge0.$$

*Proof.* There are two cases:
Case 1: $1+|\xi|^2\ge\varepsilon^2|\xi|^4$. We first observe that

$$\mathcal{G}_1(t,\xi)=\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}=e^{-\varepsilon|\xi|^2t}\big(\cos(bt)+\varepsilon\frac{\sin(bt)}{b}|\xi|^2\big),$$

$$\mathcal{G}_3(t,\xi)=\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}=e^{-\varepsilon|\xi|^2t}((\cos(bt)-\varepsilon\frac{\sin(bt)}{b}|\xi|^2)).$$

Therefore, for $k=1,3$, we have:

$$|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_k|\le|(1-\chi_{\varepsilon,\kappa_0})|e^{-\varepsilon|\xi|^2t}(1+\varepsilon|\xi|^2t)\lesssim|(1-\chi_{\varepsilon,\kappa_0})|e^{-\frac{1}{2}\varepsilon|\xi|^2t}\lesssim e^{-\frac{1}{2}\kappa_0t}.$$

For $\mathcal{G}_2$, if $b(\xi)\ge\frac{\langle\xi\rangle}{2}$, then we have

$$|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_2|=|(1-\chi_{\varepsilon,\kappa_0})|e^{-\varepsilon|\xi|^2t}\Big|\frac{\sin(bt)}{b}\langle\xi\rangle\Big|\lesssim e^{-\frac{1}{2}\kappa_0t}.$$

If $b(\xi)\le\frac{\langle\xi\rangle}{2}$, we have $\langle\xi\rangle\le\frac{2}{\sqrt{3}}\varepsilon|\xi|^2$, thus

$$|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_2|\le e^{-\varepsilon|\xi|^2t}2\varepsilon|\xi|^2tI_{\{\varepsilon|\xi|^2\ge\kappa_0\}}\lesssim e^{-\frac{1}{2}\kappa_0t}.$$

Case 2. $1+|\xi|^2\le\varepsilon^2|\xi|^4$. Let us introduce $\tilde{b}=\sqrt{\varepsilon^2|\xi|^4-(1+|\xi|^2)}$ then $\lambda_\pm=-\varepsilon|\xi|^2\mp\tilde{b}(\xi)$.

Firstly, we have

$$\begin{aligned}|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_1| &= |(1-\chi_{\varepsilon,\kappa_0})|e^{\lambda_-t}(1+\frac{1-e^{-2\tilde{b}t}}{2\tilde{b}}(-\lambda_-))|\\ &\le e^{\lambda_-t}(1+(-\lambda_-)t)\lesssim e^{\frac{1}{2}\lambda_-t}\lesssim e^{-\frac{1}{4\varepsilon}t}\lesssim e^{-\frac{1}{4}t}\end{aligned}$$

Here we have used the fact $\lambda_+>-2\varepsilon|\xi|^2, \lambda_-=\frac{1+|\xi|^2}{\lambda_+}\le-\frac{1}{2\varepsilon}$.

Secondly, we also have

$$|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_2|\quad\le\quad\Big|e^{\lambda_-t}(\frac{1-e^{-2\tilde{b}t}}{2\tilde{b}})\Big|\lesssim e^{\lambda_-t}t\lesssim2\varepsilon e^{-\frac{1}{4\varepsilon}t}\lesssim e^{-\frac{1}{4\varepsilon}t}\lesssim e^{-\frac{1}{4}t}.$$

Finally, for $\mathcal{G}_3=e^{\lambda_-t}[1+\lambda_+\frac{1-e^{-2\tilde{b}t}}{2\tilde{b}}]$, we write

- if $\tilde{b} > \frac{\varepsilon|\xi|^2}{2}$, then $|(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_3| \leq e^{\lambda_- t}(1+\frac{-\lambda_+}{\tilde{b}}) \lesssim 5e^{-\lambda_- t} \lesssim e^{-\frac{1}{2}t}$;
- if $0 \leq \tilde{b} \leq \frac{\varepsilon|\xi|^2}{2}$, then $\lambda_- \leq -\frac{1}{2}\varepsilon|\xi|^2$, and therefore,

$$\begin{aligned} |(1-\chi_{\varepsilon,\kappa_0})\mathcal{G}_3| \leq e^{\lambda_- t}(1+(-\lambda_+)t)I_{\{\varepsilon|\xi|^2\geq\kappa_0\}} &\lesssim e^{\lambda_- t}(1+2\varepsilon|\xi|^2 t)I_{\{\varepsilon|\xi|^2\geq\kappa_0\}} \\ &\lesssim e^{-\frac{1}{4}\varepsilon|\xi|^2 t}I_{\{\varepsilon|\xi|^2\geq\kappa_0\}} \lesssim e^{-\frac{1}{4}\kappa_0 t}. \end{aligned}$$

This ends the proof. □

#### 3.3.1.3 Additional estimates of $e^{-tA}$

**Lemma 3.3.6.** *For j=1,2,3, for every $s \geq 0$ and uniformly for $\varepsilon \in (0,1]$, we have the estimate*

$$\|\mathcal{G}_j(t,D)f\|_{H^s} \lesssim \|f\|_{H^s}.$$

*Proof.* We only need to show that $|\mathcal{G}_j(t,\xi)| \leq C$. Note that we have proven in the last lemma that if $\varepsilon|\xi|^2 \geq \kappa_0$, then we have $|\mathcal{G}_j(t,\xi)| \leq e^{-c_0 t}$. In the remaining region $\varepsilon|\xi|^2 \leq 2\kappa_0$, we have,

$$|\mathcal{G}_1| = |e^{-\varepsilon|\xi|^2 t}(\cos(bt)+\varepsilon|\xi|^2\frac{\sin(bt)}{b})| \leq e^{-\varepsilon|\xi|^2 t}(1+\varepsilon|\xi|^2 t) \leq C,$$

$$|\mathcal{G}_2| = |e^{-\varepsilon|\xi|^2 t}\frac{\sin(bt)}{b}\langle\xi\rangle| \leq e^{-\varepsilon|\xi|^2 t}\frac{\langle\xi\rangle}{\sqrt{1-4\kappa_0^2+|\xi|^2}} \leq C.$$

The estimate of $\mathcal{G}_3$ is similar to that of $\mathcal{G}_1$. This ends the proof. □

By combining Corollary 3.3.4 and Lemma 3.3.6, we also obtain:

**Corollary 3.3.7.** *For $p \geq 2$, we have uniformly for $\varepsilon \in (0,1]$ the estimates*

$$\begin{aligned} \|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)f\|_{L^p} &\lesssim (1+|t|)^{-\frac{d}{2}(1-\frac{2}{p})}\|f\|_{W^{(1-\frac{2}{p})d,p'}}, \\ \|\mathcal{G}_j(t,D)\chi_{\varepsilon,\kappa_0}(D)f\|_{L^p} &\lesssim (1+t)^{-\frac{d}{2}(1-\frac{2}{p})}\|f\|_{W^{(1-\frac{2}{p})d,p'}}. \end{aligned}$$

**Corollary 3.3.8.** *For j=1,2,3, we have uniformly for $\varepsilon \in (0,1]$ the estimate*

$$\|\mathcal{G}_j(t,D)f\|_{L^\infty} \lesssim (1+t)^{-\frac{d}{2}}\|f\|_{W^{d,1}} + e^{-c_0 t}\|f\|_{H^{\frac{d+1}{2}}}.$$

*Proof of Corollaries 3.3.7, 3.3.8.* For Corollary 3.3.7, we can interpolate in a classical way between the estimates of Corollary 3.3.4 and Lemma 3.3.6 and use the embeddings $B^s_{p,2} \hookrightarrow W^{s,p}$, $W^{s,p'} \hookrightarrow B^s_{p',2}$ with $p \geq 2, s \geq 0$. One can refer for instance to the books [11] [51] for the relations between Besov spaces and Sobolev spaces.

For Corollary 3.3.8, we write $\mathcal{G}_j(t,D)f = \mathcal{G}_j(t,D)\chi_{\varepsilon,\kappa_0}(D)f + \mathcal{G}_j(t,D)(1-\chi_{\varepsilon,\kappa_0})(D)f$, and the result follows from Corollary 3.3.7, Lemma 3.3.5 and the inequality: $\|\hat{f}\|_{L^1} \lesssim \|f\|_{H^{\frac{d+1}{2}}}$. □

**Lemma 3.3.9.** *Let us define the operators*

$$n_1(D) = |\nabla| \quad or \quad \varepsilon\Delta\tilde{\chi}_{\varepsilon,\kappa_0}(D)\frac{\mathrm{div}}{|\nabla|} \quad or \quad ib(D)\tilde{\chi}_{\varepsilon,\kappa_0}(D)\frac{\mathrm{div}}{|\nabla|},$$

$$n_2 = \frac{\varepsilon\Delta + b(D)}{\langle\nabla\rangle}\tilde{\chi}_{\varepsilon,\kappa_0}(D) \quad or \quad \mathcal{R} \quad or \quad \frac{|\nabla|}{\langle\nabla\rangle}.$$

*Then, for any $p \in (1,\infty)$, we have the estimate:*

$$\|n_1(D)f\|_{L^p} \lesssim \|f\|_{W^{1,p}} \qquad \|n_2(D)f\|_{L^p} \lesssim \|f\|_{L^p}$$

*Proof.* We can apply the Hörmander-Mikhlin Theorem (we refer for instance to Theorem 5.2.7 in [50]). One can easily check that $n_1(\xi)$, $n_2(\xi)$ satisfy homogeneous 0 type conditions uniformly in $\varepsilon \in (0,1]$. $\square$

From the definition of $Q(D)$, $Q^{-1}(D)$ (see (3.3.4)), we also have the following property for $Q(D)$, $Q^{-1}(D)$:

**Corollary 3.3.10.** *For any $1 < p < +\infty$, $\chi_{\varepsilon,\kappa_0}(D)Q(D), \chi_{\varepsilon,\kappa_0}(D)Q^{-1}(D)$ are both continous in $L^p$ uniformly in $\varepsilon \in (0,1]$:*

$$\|\chi_{\varepsilon,\kappa_0}(D)Q(D)F\|_{L^p} \lesssim \|F\|_{L^p}, \qquad \|\chi_{\varepsilon,\kappa_0}(D)Q^{-1}(D)F\|_{L^p} \lesssim \|F\|_{L^p}.$$

We will also need to use some elementary parabolic estimates.

**Lemma 3.3.11.** *For any integer $k \in \mathbb{N}^*$ and $1 < q < +\infty$, we have:*

$$\|e^{\varepsilon t\Delta}(\varepsilon\Delta)^k \chi_{\varepsilon,\kappa_0}(D) f\|_{L^q} \lesssim (1+t)^{-k}\|f\|_{L^q}.$$

*Proof.* On the one hand, by Young's inequality, we have

$$\|e^{\varepsilon t\Delta}(\varepsilon\Delta)^k \chi_{\varepsilon,\kappa_0}(D) f\|_{L^q} \lesssim t^{-k}\|\chi_{\varepsilon,\kappa_0}(D) f\|_{L^q} \lesssim t^{-k}\|f\|_{L^q}.$$

On the other hand, as $(\varepsilon\Delta)^k \chi_{\varepsilon,\kappa_0}(D)$ is a $L^q$ multiplier, we also have:

$$\|e^{\varepsilon t\Delta}(\varepsilon\Delta)^k \chi_{\varepsilon,\kappa_0}(D) f\|_{L^q} \lesssim \|(\varepsilon\Delta)^k \chi_{\varepsilon,\kappa_0}(D) f\|_{L^q} \lesssim \|f\|_{L^q}.$$

$\square$

### 3.3.2 Nonlinear and bilinear estimates

**Lemma 3.3.12.** *For every $1 < p < +\infty$, $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$ $1 < r_1, q_1 < +\infty$, $1 < r_2, q_2 < +\infty$, we have the estimate*

$$\|B(V,V)\|_{W^{s,p}} \lesssim \|V\|_{W^{s+1,q_1}}\|V\|_{L^{r_1}} + \|V\|_{L^{r_2}}\|V\|_{W^{s+1,q_2}}. \tag{3.3.9}$$

*Proof.* By the definition of B, the boundedness of the Riesz transform in $L^q(1 < q < +\infty)$ and the Kato-Ponce inequality (recalled in Lemma 3.8.1), we have

$$\begin{aligned}
\|B(V,V)_1\|_{W^{s,p}} &\lesssim \|(|\nabla|\langle\nabla\rangle^{-1}h)\mathcal{R}c\|_{W^{s+1,p}} \\
&\lesssim \||\nabla|\langle\nabla\rangle^{-1}h\|_{W^{s+1,q_1}}\|\mathcal{R}c\|_{L^{r_1}} + \||\nabla|\langle\nabla\rangle^{-1}h\|_{L^{r_2}}\|\mathcal{R}c\|_{W^{s+1,q_2}} \\
&\lesssim \|h\|_{W^{s+1,q_1}}\|c\|_{L^{r_1}} + \|h\|_{L^{r_2}}\|c\|_{W^{s+1,q_2}}.
\end{aligned}$$

The estimates of the other components follow from the same arguments, we omit the proof. $\square$

We finally state the bilinear estimate that will be heavily used in Section 4. We will give the proof in the appendix.

**Lemma 3.3.13.** *Let us assume that d=3, and let us define*

$$\phi_{j,k}(\xi,\eta) = (-1)^{j+1}b(\xi) + (-1)^{k+1}b(\eta) - b(\xi+\eta), \quad j,k = 1,2$$

*and*

$$m(\xi,\eta) = \tilde{\chi}_{\varepsilon,\kappa_0}(\xi)\tilde{\chi}_{\varepsilon,\kappa_0}(\eta)\tilde{\chi}_{\varepsilon,\kappa_0}(\xi+\eta)\frac{\langle\xi+\eta\rangle}{2ib(\xi+\eta)}.$$

*Then, we have the following estimates that are uniform for $\varepsilon \in (0,1]$:*

$$\|T_{\frac{m}{\phi_{jk}}}(f,g)\|_{W^{\sigma,p}} \lesssim \|f\|_{W^{\sigma+2_+,q_1}}\|g\|_{W^{2,r_1}} + \|f\|_{W^{2,r_2}}\|g\|_{W^{\sigma+2_+,q_2}},$$

$$\|T_{\frac{m}{\phi_{jk}^2}}(f,g)\|_{W^{\sigma,p}} \lesssim \|f\|_{W^{\sigma+2_+,q_1}}\|g\|_{W^{3,r_1}} + \|f\|_{W^{3,r_2}}\|g\|_{W^{\sigma+2_+,q_2}}.$$

*where $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$, $1 < r_1, r_2 \le +\infty, 1 \le q_1, q_2 < +\infty$ and $T_{\frac{m}{\phi_{jk}}}$ is the bilinear operator defined in* (3.2.1).

## 3.4 Proof of Theorem 2.13

This section is devoted to the proof of Theorem 3.1.3. Let us observe that from a standard iteration argument, (similar to the one for the compressible Navier-Stokes system as in [97]), one can show that the system 3.1.3 admits a unique solution in $C([0,T_\varepsilon),H^3)$ for some $T_\varepsilon>0$ and that if the initial data are in $H^l$, $l\geq 3$, then this additional regularity also propagates on $[0,T_\varepsilon)$ . We thus only focus on the proof of a priori estimates that are uniform in time and in $\varepsilon$.

We shall use the norms :

$$\begin{aligned}
\|X\|_Y &\triangleq \|X\|_{W^{\sigma+3(1-\frac{2}{p}),p'}}+\|X\|_{H^N},\\
\|U\|_{X_T} &\triangleq \sup_{t\in[0,T)}\Big(\langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|U(t)\|_{W^{\sigma,p}}+\langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|\chi^H(D)U(t)\|_{H^{N-1}}+\|U(t)\|_{H^N}\Big).
\end{aligned}$$

where $U=(\varrho,\nabla\phi,u)$, $\chi^H(D)=(1-\chi_{\varepsilon,\kappa_0})(D)$, $6<p<\infty,\sigma\geq 3,N\geq\sigma+7$.

By standard bootstrap argument, it suffices to prove that there exists $\tilde{\delta}_1>0$, and $C>0$ that are independent of $T$ such that for every $\delta_1\in(0,\tilde{\delta}_1]$, if $\|U\|_{X_T}\leq\delta_1$, then we have uniformly for $\varepsilon\in(0,1]$ an estimate under the form

$$\|U\|_{X_T}\leq C\Big(\|U(0)\|_Y+\|U\|_{X_T}^{\frac{3}{2}}+\|U\|_{X_T}^2+\|U\|_{X_T}^3\Big). \tag{3.4.1}$$

Indeed, let us set

$$T_*=\sup\{T\in[0,T_\varepsilon),\ \|U\|_{X_T}\leq\delta_1\}.$$

Then, we can deduce from (3.4.1) that $T_*=T_\varepsilon=+\infty$ by choosing $\delta$ (which is such that $\|U(0)\|_Y\leq\delta$) and $\delta_1$ small enough such that by (3.4.1), $\|U\|_{X_T}\leq C(\delta+3\delta_1^{\frac{3}{2}})<\delta_1$ for any $T<T_*$. The result follows by time continuity and a local well-posedness result.

The a priori estimate (3.4.1) will follow from the following two propositions.

**Proposition 3.4.1** (Energy estimates). *We define the energy functional*

$$E_N=\sum_{|\alpha|\leq N}E_\alpha=\sum_{|\alpha|\leq N}\int\frac{|\partial^\alpha\varrho|^2}{2}+\frac{|\partial^\alpha\nabla\phi|^2}{2}+\rho\frac{|\partial^\alpha u|^2}{2}\mathrm{d}x.$$

*Assuming that $\|\varrho\|_{H^2}\leq\delta_1$ and that $\delta_1$ is small enough so that $\|\varrho\|_{L^\infty}\leq\frac{1}{6},\|\nabla\varrho\|_{L^3}<\frac{1}{2\tilde{c}}$ where $\tilde{c}$ is the biggest one among the Sobolev constants coming from the embedding $H^2\hookrightarrow L^\infty,\dot{H}^{\frac{1}{2}}\hookrightarrow L^3,\dot{H}^1\hookrightarrow L^6$, then there exists a constant $C>0$ which depends only on $\tilde{c}$, such that*

$$\sup_{0\leq t<T}E_N(t)\leq E_N(0)+C\|U\|_{X_T}^3. \tag{3.4.2}$$

*Proof.* By taking the time derivative of the energy functional and by using the equations, we get:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha &= -\int\mathrm{div}(\rho u)\frac{|\partial^\alpha u|^2}{2}+\rho\partial^\alpha u\cdot\partial^\alpha\big[(u\cdot\nabla u)+\nabla\varrho-\nabla\phi-2\varepsilon\Delta u\big]\\
&\qquad +\partial^\alpha\varrho\partial^\alpha\mathrm{div}(\rho u)-\partial^\alpha\nabla\phi\cdot\partial^\alpha\nabla\partial_t\phi\mathrm{d}x\\
&= \int\rho\partial^\alpha u\cdot\big[u,\partial^\alpha\big]\nabla u+\partial^\alpha\rho\mathrm{div}\big([\rho,\partial^\alpha]u\big)\\
&\quad +\big[\partial^\alpha\nabla\phi\cdot\partial^\alpha\partial_t\nabla\phi+\rho\partial^\alpha u\cdot\partial^\alpha\nabla\phi\big]+2\varepsilon\rho\partial^\alpha u\cdot\partial^\alpha\Delta u\mathrm{d}x\\
&\triangleq J_1+J_2+J_3+J_4.
\end{aligned}$$

We now estimate these four terms. For $J_1$, by using Lemma 3.8.1 in the appendix, we have,

$$J_1=-\int\rho\partial^\alpha u\cdot\big[\partial^\alpha,u\big]\nabla u\mathrm{d}x\ \leq\ \|\rho\|_{L^\infty}\|\partial^\alpha u\|_{L^2}\|\big[\partial^\alpha,u\big]\nabla u\|_{L^2}\leq 2\|u\|_{\dot{H}^{|\alpha|}}^2\|\nabla u\|_{L^\infty}.$$

For $J_2$, which is non-zero only if $|\alpha| \geq 1$, by using Lemma 3.8.1 again, we have

$$\begin{aligned}
J_2 &= \int \partial^\alpha \varrho\big(\mathrm{div}(\rho\partial^\alpha u) - \partial^\alpha \mathrm{div}(\rho u)\big) \\
&= \int |\partial^\alpha \varrho|^2 \mathrm{div} u - \partial^\alpha \varrho([\partial^\alpha, u]\nabla\varrho + [\partial^\alpha, \rho]\mathrm{div} u - \nabla\varrho\partial^\alpha u)\mathrm{d}x \\
&\lesssim \|(u,\varrho)\|^2_{\dot{H}^{|\alpha|}}(\|(\nabla u, \nabla\varrho)\|_{L^\infty}.
\end{aligned}$$

In a similar way, we estimate $J_3$ and $J_4$ as follows:

$$\begin{aligned}
J_3 &= \int \rho\partial^\alpha u \partial^\alpha \nabla\phi + \partial^\alpha\phi\partial^\alpha \mathrm{div}(\rho u)\mathrm{d}x = -\int \partial^\alpha\nabla\phi[\partial^\alpha, \rho]u\mathrm{d}x \\
&\lesssim \|\nabla\phi\|_{\dot{H}^{|\alpha|}}(\|u\|_{\dot{H}^{|\alpha|-1}}\|\nabla\varrho\|_{L^\infty} + \|u\|_{L^\infty}\|\varrho\|_{\dot{H}^{|\alpha|}}),
\end{aligned}$$

$$J_4 = 2\varepsilon\int \rho\partial^\alpha u\partial^\alpha \Delta u\mathrm{d}x = -2\varepsilon\int \rho|\partial^\alpha\nabla u|^2 + \nabla\varrho\partial^\alpha u\cdot\partial^\alpha\nabla u\mathrm{d}x.$$

We estimate the second term in the above equality by $\tilde{c}\varepsilon\|\nabla\varrho\|_{L^3}\|\nabla u\|^2_{\dot{H}^{|\alpha|}}$ where $\tilde{c}$ is the Sobolev constant associated to Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$. We finally get:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_N + \varepsilon\sum_{|\alpha|\leq N}\int \rho|\partial^\alpha\nabla u|^2\mathrm{d}x \lesssim (\|u\|_{W^{1,\infty}} + \|\varrho\|_{W^{1,\infty}})\|U\|^2_{H^N}. \tag{3.4.3}$$

By integrating in time and by using the definition of $\|U\|_{X_T}$, we get the inequality (3.4.2).

□

**Remark 3.4.2.** *Note that by the assumptions of Proposition 3.4.1, we have* $\frac{5}{6} \leq \rho \leq \frac{7}{6}$ *so that* $E_N \approx \|U\|^2_{H^N}$ *which combine with* (3.4.2) *gives:* $\sup_{0\leq t<T}\|U(t)\|_{H^N} \lesssim \|U(0)\|_{H^N} + \|U\|^{\frac{3}{2}}_{X_T}$.

Now we begin to deal with the other two terms in the definition of the $X_T$ norm. By the definition of $V$ and the boundedness of the Riesz transform in $L^q$, $1 < q < +\infty$, we have $\|U\|_{X_T} \sim \|V\|_{X_T}$ which leads us to prove the corresponding estimate for $V$ :

**Proposition 3.4.3.** *For any* $6 < p < \infty$, *we have the decay estimate:*

$$\sup_{t\in[0,T)}\Big(\langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|V\|_{W^{\sigma,p}} + \langle t\rangle^{\frac{3}{2}(1-\frac{2}{p})}\|\chi^H(D)V\|_{H^{N-1}}\Big) \lesssim \|V(0)\|_Y + \|V(0)\|^2_Y + \|V\|^2_{X_T} + \|V\|^3_{X_T}.$$

*Proof.* For notational convenience, we denote $V^L = \chi^L(D)V = \chi_{\varepsilon,\kappa_0}(D)V, V^H = \chi^H(D)V = (1-\chi_{\varepsilon,\kappa_0})(D)V$. By using (3.3.5), Lemma 3.3.5, Lemma 3.3.12 and Sobolev embedding, we get:

$$\begin{aligned}
\|V^H(t)\|_{H^{N-1}} &\lesssim \|e^{-tA}\chi^H(D)V(0)\|_{H^{N-1}} + \int_0^t \|e^{-(t-s)A}\chi^H(D)B(V,V)(s)\|_{H^{N-1}}\mathrm{d}s \\
&\lesssim e^{-c_0t}\|V(0)\|_{H^{N-1}} + \int_0^t e^{-c_0(t-s)}\|B(V,V)(s)\|_{H^{N-1}}\mathrm{d}s \\
&\lesssim e^{-c_0t}\|V(0)\|_{H^{N-1}} + \int_0^t e^{-c_0(t-s)}\|V(s)\|_{H^N}\|V(s)\|_{W^{\sigma,p}}\mathrm{d}s \\
&\lesssim e^{-c_0t}\|V(0)\|_{H^{N-1}} + (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|^2_{X_T}.
\end{aligned}$$

For the $W^{\sigma,p}$ estimate, we just use Sobolev embedding and the above estimate,

$$\|V^H(t)\|_{W^{\sigma,p}} \lesssim \|V^H(t)\|_{H^{\sigma+3(\frac{p-2}{2p})}} \lesssim e^{-c_0t}\|V(0)\|_{H^{N-1}} + (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|^2_{X_T}.$$

We shall now prove the decay estimate for low frequencies. By applying $Q^{-1}\chi^L$ to the system for $V$ (see (3.3.2)) and by setting $R = (r_1, r_2)^\top = Q^{-1}\chi^L(D)V$, we find that $R$ solves the system:

$$\partial_t R + \begin{pmatrix} -\lambda_-(D) & 0 \\ 0 & -\lambda_+(D)\end{pmatrix} R = Q^{-1}\chi^L(D)B(V,V) \tag{3.4.4}$$

with initial data $R(0)=Q^{-1}\chi^L(D)V(0)$. We thus obtain from the Duhamel formula that

$$\begin{aligned} R &= \begin{pmatrix} e^{\lambda_-(D)t} & 0 \\ 0 & e^{\lambda_+(D)t} \end{pmatrix} R_0 + \int_0^t \begin{pmatrix} e^{\lambda_-(D)(t-s)} & 0 \\ 0 & e^{\lambda_+(D)(t-s)} \end{pmatrix} Q^{-1}\chi^L(D)B(V,V)\mathrm{d}s \\ &\triangleq J_1+J_2+J_3+J_4 \end{aligned} \tag{3.4.5}$$

where:

$$\begin{aligned} J_1 &= \begin{pmatrix} e^{\lambda_-(D)t} & 0 \\ 0 & e^{\lambda_+(D)t} \end{pmatrix} R_0, \\ J_2 &= \int_0^t \begin{pmatrix} e^{\lambda_-(D)(t-s)} & 0 \\ 0 & e^{\lambda_+(D)(t-s)} \end{pmatrix} Q^{-1}\chi^L(D)B\big(V^H,V\big)\mathrm{d}s, \\ J_3 &= \int_0^t \begin{pmatrix} e^{\lambda_-(D)(t-s)} & 0 \\ 0 & e^{\lambda_+(D)(t-s)} \end{pmatrix} Q^{-1}\chi^L(D)B\big(V^L,V^H\big)\mathrm{d}s, \\ J_4 &= \int_0^t \begin{pmatrix} e^{\lambda_-(D)(t-s)} & 0 \\ 0 & e^{\lambda_+(D)(t-s)} \end{pmatrix} Q^{-1}\chi^L(D)B(V^L,V^L)\mathrm{d}s. \end{aligned}$$

For the term $J_1$, note that $R_0=Q^{-1}\chi^L(D)V(0)=\tilde{\chi}_{\varepsilon,\kappa_0}(D)Q^{-1}\chi^L(D)V(0)$, thus by Corollary 3.3.7, 3.3.10 we have:

$$\|J_1\|_{W^{\sigma,p}} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|Q^{-1}\chi^L(D)V(0)\|_{W^{\sigma+3(1-\frac{2}{p}),p'}} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V(0)\|_{W^{\sigma+3(1-\frac{2}{p}),p'}}.$$

For the term $J_2$, we use Corollaries 3.3.7, 3.3.10 and Lemma 3.3.12 to get:

$$\begin{aligned} \|J_2\|_{W^{\sigma,p}} &\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}\|Q^{-1}\chi^L(D)B\big(V^H,V\big)\|_{W^{\sigma+3(1-\frac{2}{p}),p'}}\mathrm{d}s \\ &\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}(\|V^H\|_{H^{\sigma+3(1-\frac{2}{p})+1}}\|V\|_{L^{\frac{2p}{p-2}}}+\|V^H\|_{L^2}\|V\|_{W^{\sigma+3(1-\frac{2}{p})+1,\frac{2p}{p-2}}})\mathrm{d}s \\ &\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}(1+s)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|_{X_T}^2\mathrm{d}s \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|_{X_T}^2. \end{aligned}$$

The estimate for $J_3$ is similar to the one for $J_2$, we thus skip it.

It remains to estimate $J_4$ which is the most difficult one. To get over the difficulty of the quadratic nonlinearity, we need to use the normal form method. By the definition of $Q,Q^{-1}$ in (3.3.4) and $R=Q^{-1}\chi^L(D)V$, we have that

$$\begin{aligned} B(V^L,V^L)=B(QR,QR) &= \begin{pmatrix} \langle\nabla\rangle\mathcal{R}^*\{\frac{|\nabla|}{\langle\nabla\rangle}(r_1+r_2)\mathcal{R}[i\frac{b(D)}{\langle\nabla\rangle}(r_2-r_1)+\frac{\varepsilon\Delta}{\langle\nabla\rangle}(r_1+r_2)]\} \\ |\nabla|\big|\mathcal{R}[i\frac{b(D)}{\langle\nabla\rangle}(r_2-r_1)+\frac{\varepsilon\Delta}{\langle\nabla\rangle}(r_1+r_2)]\big|^2 \end{pmatrix} \\ &\triangleq \begin{pmatrix} B_1(R,R) \\ B_2(R,R) \end{pmatrix}. \end{aligned}$$

Define:

$$A(R,R) = Q^{-1}B(QR,QR) = \begin{pmatrix} \frac{1}{2ib}(\lambda_+B_1+\langle\nabla\rangle B_2) \\ \frac{-1}{2ib}(\lambda_-B_1-\langle\nabla\rangle B_2) \end{pmatrix} \triangleq \begin{pmatrix} A_1(R,R) \\ A_2(R,R) \end{pmatrix}.$$

We shall only study the first component, ie.

$$J_{41}=\int_0^t e^{(t-s)\lambda_-(D)}\chi_{\varepsilon,\kappa_0}(D)A_1(R,R)\mathrm{d}s,$$

the other can be handled in a similar way. For notational convenience (although with a little abuse of notation), we write $A_1(R,R)=\frac{\langle\nabla\rangle}{2ib}\sum n_1(D)(n_2(D)r_1\cdot n_2(D)r_2)$, here the summation runs over all the possibilities in the definition of $n_1(D),n_2(D)$ defined in Lemma 3.3.9 from the definition of $\lambda_\pm$.

Set $\tilde{R}=n_2(D)R$, then by recalling $\tilde{\chi}_{\varepsilon,\kappa_0}\chi_{\varepsilon,\kappa_0}=\chi_{\varepsilon,\kappa_0}$, $J_{41}$ is the sum of the following term:

$$G_{jk}=e^{-ib(D)t}\mathcal{F}^{-1}\Big(\int_0^t\int_{\mathbb{R}^3}e^{-\varepsilon|\xi|^2(t-s)}e^{ib(\xi)s}m(\xi-\eta,\eta)n_1(\xi)\chi_{\varepsilon,\kappa_0}(\xi)\hat{\tilde{r}}_j(s,\xi-\eta)\hat{\tilde{r}}_k(s,\eta)\mathrm{d}\eta\mathrm{d}s\Big).$$

where $m(\xi-\eta,\eta)=\tilde{\chi}_{\varepsilon,\kappa_0}(\xi-\eta)\tilde{\chi}_{\varepsilon,\kappa_0}(\eta)\tilde{\chi}_{\varepsilon,\kappa_0}(\xi)\frac{\langle\xi\rangle}{2ib(\xi)}$.

Set $W=(W_1,W_2)^\top=\begin{pmatrix}e^{ib(D)t} & 0\\ 0 & e^{-ib(D)t}\end{pmatrix}R$, then from (3.4.4), $W$ satisfies:

$$\partial_t W=\begin{pmatrix}e^{ib(D)t} & 0\\ 0 & e^{-ib(D)t}\end{pmatrix}\big[\varepsilon\Delta R+Q^{-1}\chi^L(D)B(V,V)\big].$$

By the definition of $W$, we have $w_j=e^{i(-1)^{j+1}b(D)}r_j$, so by defining $\tilde{w}_j=e^{i(-1)^{j+1}b(D)}\tilde{r}_j$ we get:

$$G_{jk}=e^{-ib(D)t}\mathcal{F}^{-1}\Big(\int_0^t\int_{\mathbb{R}^3}e^{-\varepsilon|\xi|^2(t-s)}e^{-is\phi_{jk}}m(\xi-\eta,\eta)n_1(\xi)\chi_{\varepsilon,\kappa_0}(\xi)\widehat{\tilde{w}}_j(s,\xi-\eta)\widehat{\tilde{w}}_k(s,\eta)\mathrm{d}\eta\mathrm{d}s\Big)$$

thus, by using that $\phi_{jk}$ does not vanish in the support of $\chi_{\varepsilon,\kappa_0}(D)$, we can integrate by parts in time:

$$\begin{aligned}e^{ib(D)t}G_{jk}&=\mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^3}e^{-\varepsilon|\xi|^2(t-s)}\frac{\partial_s e^{-is\phi_{jk}}}{-i\phi_{jk}}m(\xi-\eta,\eta)n_1(\xi)\chi_{\varepsilon,\kappa_0}(\xi)\widehat{\tilde{w}}_j(s,\xi-\eta)\widehat{\tilde{w}}_k(s,\eta)\mathrm{d}\eta\mathrm{d}s\\&=e^{ib(D)t}\big(\sum_{j=1}^7 I_j\big)\end{aligned}\tag{3.4.6}$$

where

$$\begin{aligned}I_1&=i\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(t),\tilde{r}_k(t)),\\
I_2&=-ie^{\varepsilon t\Delta}e^{-itb(D)}\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(0),\tilde{r}_k(0)),\\
I_3&=i\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}(\varepsilon\Delta)\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(s),\tilde{r}_k(s))\mathrm{d}s,\\
I_4&=-i\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde{r}_j(s),\tilde{r}_k(s))\mathrm{d}s,\\
I_5&=\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{B}_j(s),\tilde{r}_k(s))\mathrm{d}s,\\
I_6&=\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(s),\varepsilon\Delta\tilde{r}_k(s))\mathrm{d}s,\\
I_7&=\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(s),\tilde{B}_k(s))\mathrm{d}s\end{aligned}$$

and $\tilde{B}=n_2(D)Q^{-1}\chi^L B(V,V)$, we recall that $B(V,V),Q^{-1}$ are defined in (3.3.2) and (3.3.4). We now estimate $I_1$ to $I_7$. In the following, we shall use the estimates for the bilinear operator $T_{\frac{m}{\phi_{jk}}}$ in Lemma 3.3.13 with the choice $(k)_+=\frac{3}{p}$.

By Lemma 3.3.13 and Sobolev embedding, we can estimate $I_1$ as follows:

$$\begin{aligned}\|I_1\|_{W^{\sigma,p}}\lesssim\|T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(t),\tilde{r}_k(t))\|_{W^{\sigma+1,p}}&\lesssim\|\tilde{r}_j(t)\|_{W^{\sigma+3_+,p}}\|\tilde{r}_k(t)\|_{W^{2,\infty}}+\|\tilde{r}_j(t)\|_{W^{2,\infty}}\|\tilde{r}_k(t)\|_{W^{\sigma+3_+,p}}\\&\lesssim\|R(t)\|_{H^{\sigma+\frac{3}{2}(1-\frac{2}{p})+3_+}}\|R(t)\|_{W^{3,p}}\lesssim(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|_{X_T}^2.\end{aligned}$$

By Corollaries 3.3.7, 3.3.10, Lemma 3.3.13 and the Sobolev embedding, we have for $I_2$:

$$\begin{aligned}\|I_2\|_{W^{\sigma,p}}&\lesssim(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|T_{\frac{m}{\phi_{jk}}}(\tilde{r}_j(0),\tilde{r}_k(0))\|_{W^{\sigma+3(1-\frac{2}{p})+1,p'}}\\&\lesssim(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}(\|r_j(0)\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}\|r_k(0)\|_{W^{2,\frac{2p}{p-2}}}+\|r_j(0)\|_{W^{2,\frac{2p}{p-2}}}\|r_k(0)\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}})\\&\lesssim(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|R(0)\|_{H^N}^2\lesssim(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V(0)\|_{H^N}^2.\end{aligned}$$

For the term $I_5$, we have:

$$\begin{aligned}
\|I_5\|_{W^{\sigma,p}} &\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})} \|T_{\frac{m}{\phi_{jk}}}(\tilde{B}_j, \tilde{r}_k)\|_{W^{\sigma+3(1-\frac{2}{p})+1,p'}} \mathrm{d}s \\
&\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})} (\|B_j\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}} \|r_k\|_{W^{2,\frac{2p}{p-2}}} + \|B_j\|_{W^{2,\frac{2p}{p-2}}} \|r_k\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}) \mathrm{d}s \\
&\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})} (\|B_j\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}} \|R\|_{H^3} + \|B_j\|_{H^3} \|R\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}) \mathrm{d}s \\
&\lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})} (1+s)^{-\frac{3}{2}(1-\frac{2}{p})} \|V\|_{X_T}^3 \mathrm{d}s \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})} \|V\|_{X_T}^3.
\end{aligned}$$

Here, we have used Corollaries 3.3.7, 3.3.10, Lemma 3.3.12 3.3.13, and Sobolev embedding. The estimate for $I_7$ is very similar to that of $I_5$, we omit the details.

The terms $I_3, I_4$ correspond to the error terms created by $\varepsilon\Delta$. As explained in the introduction, since one can only expect that $\|\varepsilon\Delta u\|_{H^{N-1}} \lesssim (1+t)^{-1}$ which is not a fast enough decay, to control them, we need to perform normal form transformation again.

By integrating by parts again, we get

$$\begin{aligned}
I_4 &= i\int_0^t e^{\varepsilon(t-s)\Delta} e^{-i(t-s)b(D)} n_1(D) T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde{r}_j, \tilde{r}_k) \mathrm{d}s \\
&= -\chi_{\varepsilon,\kappa_0}(D) n_1(D) T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j(t), \tilde{r}_k(t)) + e^{\varepsilon t\Delta} e^{-itb(D)} \chi_{\varepsilon,\kappa_0}(D) n_1(D) T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j(0), \tilde{r}_k(0)) \\
&\quad - \int_0^t e^{\varepsilon(t-s)\Delta} e^{-i(t-s)b(D)} (\varepsilon\Delta) \chi_{\varepsilon,\kappa_0}(D) n_1(D) T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j, \tilde{r}_k) \mathrm{d}s \\
&+ \int_0^t e^{\varepsilon(t-s)\Delta} e^{-i(t-s)b(D)} \chi_{\varepsilon,\kappa_0}(D) n_1(D) \big[T_{\frac{m}{\phi_{jk}^2}}((\varepsilon\Delta)^2\tilde{r}_j + \varepsilon\Delta\tilde{B}_j, \tilde{r}_k) + T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j, \varepsilon\Delta\tilde{r}_k + \tilde{B}_k)\big] \mathrm{d}s \\
&\triangleq I_{41} + \cdots + I_{47}.
\end{aligned} \tag{3.4.7}$$

The terms $I_{41}, I_{42}$ are similar to $I_1, I_2$. For instance, by using the fact that $\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)$ is a bounded multiplier in $L^p$, $1 < p < \infty$, we get

$$\begin{aligned}
\|I_{41}\|_{W^{\sigma,p}} &\lesssim \|T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j(t), \tilde{r}_k(t))\|_{W^{\sigma+1,p}} \\
&\lesssim \|\varepsilon\Delta r_j(t)\|_{W^{\sigma+3_+,\infty}} \|r_k(t)\|_{W^{3,p}} + \|\varepsilon\Delta r_j(t)\|_{W^{3,p}} \|r_k(t)\|_{W^{\sigma+3_+,\infty}} \\
&\lesssim \|R\|_{W^{\sigma+3_+,\infty}} \|R\|_{W^{\sigma,p}} \lesssim \|R\|_{H^{\sigma+5}} \|R\|_{W^{\sigma,p}} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})} \|V\|_{X_T}^2.
\end{aligned}$$

Up to now, we have only used the dispersive estimates, but not yet the viscous dissipation, we shall use it in the estimate for $I_{43}$. We write

$$I_{43} = \int_0^{t-1} + \int_{t-1}^t e^{\varepsilon(t-s)\Delta} e^{-i(t-s)b(D)} (\varepsilon\Delta) \chi_{\varepsilon,\kappa_0}(D) n_1(D) T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j, \tilde{r}_k) \mathrm{d}s \triangleq I_{431} + I_{432}$$

By Young's inequality, Lemma 3.3.13, Sobolev embedding, we get:

$$\begin{aligned}
\|I_{431}\|_{W^{\sigma,p}} &\lesssim \int_0^{t-1} \|\mathcal{F}^{-1}(\varepsilon|\xi|^2 e^{-\varepsilon(t-s)|\xi|^2})\|_{L^{\frac{2p}{p+2}}} \|e^{-i(t-s)b(D)} T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j, \tilde{r}_k)\|_{H^{\sigma+1}} \mathrm{d}s \\
&\lesssim \int_0^{t-1} \varepsilon^{\frac{1}{4}+\frac{3}{2p}} (t-s)^{-(\frac{7}{4}-\frac{3}{2p})} \|T_{\frac{m}{\phi_{jk}^2}}(\Delta\tilde{r}_j, \tilde{r}_k)\|_{H^{\sigma+1}} \mathrm{d}s \\
&\lesssim \int_0^{t-1} (t-s)^{-(\frac{7}{4}-\frac{3}{2p})} \|R(s)\|_{H^{\sigma+\frac{3}{p}+5_+}} \|R(s)\|_{W^{3,p}} \mathrm{d}s \\
&\lesssim \int_0^{t-1} (t-s)^{-(\frac{7}{4}-\frac{3}{2p})} (1+s)^{-\frac{3}{2}(1-\frac{2}{p})} \|V\|_{X_T}^2 \mathrm{d}s \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})} \|V\|_{X_T}^2.
\end{aligned}$$

Here, we have used that: $\varepsilon^{\frac{1}{4}+\frac{3}{2p}} \leq \varepsilon^{\frac{1}{4}} \leq 1$, $\frac{7}{4} - \frac{3}{2p} > \frac{3}{2}(1-\frac{2}{p})$ and $\frac{3}{p} < \frac{1}{2}$ if $6 < p < +\infty$. By using again the fact that $\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)$ is a bounded multiplier in $L^p$ $(1 < p < \infty)$, Lemma 3.3.13 and Sobolev embedding, we have:

$$\begin{aligned}
\|I_{432}\|_{W^{\sigma,p}} &\lesssim \int_{t-1}^{t} \|T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j,\tilde{r}_k)\|_{H^{\sigma+1+3(\frac{p-2}{2p})}}\mathrm{d}s\\
&\lesssim \int_{t-1}^{t} \|\varepsilon\Delta r_j\|_{H^{\sigma+(\frac{9}{2})+}}\|r_k\|_{W^{3,p}}+\|\varepsilon\Delta r_j\|_{W^{3,p}}\|r_k\|_{H^{\sigma+(\frac{9}{2})+}}\mathrm{d}s\\
&\lesssim \int_{t-1}^{t} \|R\|_{H^{\sigma+5}}\|R\|_{W^{\sigma,p}}\mathrm{d}s \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|_{X_T}^2.
\end{aligned}$$

For $I_{44}$, we need to use the following lemma.

**Lemma 3.4.4.** *For $k\le N-1$, we have the following uniform for $\varepsilon\in(0,1]$ estimates:*

$$\begin{aligned}
\|\varepsilon\Delta R\|_{H^k} &\lesssim (1+t)^{-1}(\|V(0)\|_Y+\|V\|_{X_T}^2),\\
\|(\varepsilon\Delta)^2 R\|_{H^k} &\lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}(\|V(0)\|_Y+\|V\|_{X_T}^2).
\end{aligned}$$

*Proof.* By using the equation (3.4.5) for $R$ , we obtain from Lemma 3.3.11, tame estimate and Sobolev embedding,

$$\begin{aligned}
\|\varepsilon\Delta R\|_{H^k} &\le \|e^{\varepsilon t\Delta}\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)Q^{-1}V(0)\|_{H^k}+\int_0^t \|e^{\varepsilon(t-s)\Delta}\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)Q^{-1}B(V,V)\|_{H^k}\mathrm{d}s\\
&\lesssim (1+t)^{-1}\|V(0)\|_{H^k}+\int_0^t (1+t-s)^{-1}\|\tilde{\chi}_{\varepsilon,\kappa_0}(D)Q^{-1}B(V,V)\|_{H^k}\mathrm{d}s\\
&\lesssim (1+t)^{-1}\|V(0)\|_{H^k}+\int_0^t (1+t-s)^{-1}(1+s)^{-\frac{3}{2}(1-\frac{2}{p})}\|V\|_{X_T}^2\mathrm{d}s\\
&\lesssim (1+t)^{-1}(\|V(0)\|_{H^k}+\|V\|_{X_T}^2).
\end{aligned}$$

where we have used that for $6<p<+\infty$, $1<\frac{3}{2}(1-\frac{2}{p})<\frac{3}{2}$. The other inequality follows from the same arguments by noticing that $\frac{3}{2}(1-\frac{2}{p})<2$. □

We go back to the estimate of $I_{44}$ in (3.4.7). By using the last lemma, we get:

$$\begin{aligned}
&\|I_{44}\|_{W^{\sigma,p}} \lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}\|T_{\frac{m}{\phi_{jk}^2}}((\varepsilon\Delta)^2\tilde{r}_j,\tilde{r}_k)\|_{W^{\sigma+3(1-\frac{2}{p})+1,p'}}\mathrm{d}s\\
\lesssim\ &\int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}\big(\|(\varepsilon\Delta)^2 r_j\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}\|r_k\|_{W^{3,\frac{2p}{p-2}}}+\|(\varepsilon\Delta)^2 r_j\|_{W^{3,\frac{2p}{p-2}}}\|r_k\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}\big)\mathrm{d}s\\
\lesssim\ &\int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}(1+s)^{-\frac{3}{2}(1-\frac{2}{p})}\big(\|V\|_{X_T}^3+\|V\|_{X_T}^2+\|V(0)\|_Y^2\big)\mathrm{d}s\\
\lesssim\ &(1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\big(\|V\|_{X_T}^3+\|V\|_{X_T}^2+\|V(0)\|_Y^2\big).
\end{aligned}$$

The term $I_{46}$ can be estimated in the same way as $I_{44}$:

$$\begin{aligned}
&\|I_{46}\|_{W^{\sigma,p}} \lesssim \int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}\|T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde{r}_j,\varepsilon\Delta\tilde{r}_k)\|_{W^{\sigma+3(1-\frac{2}{p})+1,p'}}\mathrm{d}s\\
\lesssim\ &\int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}\big(\|\varepsilon\Delta r_j\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}\|\varepsilon\Delta r_k\|_{W^{3,\frac{2p}{p-2}}}+\|\varepsilon\Delta r_k\|_{H^{\sigma+3(1-\frac{2}{p})+3_+}}\|\varepsilon\Delta r_j\|_{W^{3,\frac{2p}{p-2}}}\big)\mathrm{d}s\\
\lesssim\ &\int_0^t (1+t-s)^{-\frac{3}{2}(1-\frac{2}{p})}(1+s)^{-2}\big(\|V\|_{X_T}^2+\|V(0)\|_Y\big)^2\mathrm{d}s \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})}\big(\|V\|_{X_T}^4+\|V(0)\|_Y^2\big).
\end{aligned}$$

The terms $I_{45}, I_{47}$ are similar to $I_5, I_7$, we thus skip them.

It remains to estimate

$$I_3 = i\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)T_{\frac{m}{\phi_{jk}}}(r_j(s),r_k(s))\mathrm{d}s.$$

Integrating by parts in time again, we get

$$\begin{aligned}
I_3 &= i\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}(\varepsilon\Delta)\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}}}(\tilde r_j(s),\tilde r_k(s))\mathrm{d}s\\
&= -\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j(t),\tilde r_k(t)) + e^{\varepsilon t\Delta}e^{-itb(D)}\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j(0),\tilde r_k(0))\\
&\quad -\int_0^t e^{\varepsilon(t-s)\Delta}e^{-i(t-s)b(D)}(\varepsilon\Delta)^2\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j,\tilde r_k)(s)\mathrm{d}s\\
&\quad +\int_0^t e^{\varepsilon(t-s)\Delta}e^{-i(t-s)b(D)}\varepsilon\Delta\chi_{\varepsilon,\kappa_0}(D)n_1(D)[T_{\frac{m}{\phi_{jk}^2}}(\varepsilon\Delta\tilde r_j+\tilde B_j,r_k)+T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j,\varepsilon\Delta\tilde r_k+\tilde B_k)]\mathrm{d}s\\
&\triangleq I_{31}+\cdots+I_{37}.
\end{aligned}$$

Note that $I_{34}=I_{43}$, and that the estimates for $I_{31}, I_{32}, I_{35}, I_{37}$ are similar to the ones for $I_1, I_2, I_{45}, I_{47}$, we thus skip them.

For $I_{33}$, we use Lemma 3.3.13 and Lemma 3.3.11 to get:

$$\begin{aligned}
\|I_{33}\|_{W^{\sigma,p}} &\lesssim \int_0^t \|e^{\varepsilon(t-s)\Delta}(\varepsilon\Delta)^2\chi_{\varepsilon,\kappa_0}(D)n_1(D)T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j,\tilde r_k)(s)\|_{H^{\sigma+3(\frac12-\frac1p)}}\mathrm{d}s\\
&\lesssim \int_0^t (1+t-s)^{-2}\|T_{\frac{m}{\phi_{jk}^2}}(\tilde r_j,\tilde r_k)(s)\|_{H^{\sigma+3(\frac12-\frac1p)+1}}\mathrm{d}s\\
&\lesssim \int_0^t (1+t-s)^{-2}(1+s)^{-\frac32(1-\frac2p)}\|V\|_{X_T}^2\mathrm{d}s \lesssim (1+t)^{-\frac32(1-\frac2p)}\|V\|_{X_T}^2.
\end{aligned}$$

This ends the proof of Proposition 3.4.3. □

We thus have (3.4.1) by combining Proposition 3.4.1 and 3.4.3. Theorem 3.1.3 then follows from the interpolation inequality: for any $1<p'<2$,

$$\|(\varrho_0^\varepsilon,\nabla\phi_0^\varepsilon,\mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,p'}} \lesssim \|(\varrho_0^\varepsilon,\nabla\phi_0^\varepsilon,\mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma+3,1}}^\theta \|(\varrho_0^\varepsilon,\nabla\phi_0^\varepsilon,\mathcal{P}^\perp u_0^\varepsilon)\|_{H^{\sigma+3}}^{1-\theta}.$$

**Remark 3.4.5.** *If we only prove the Theorem 3.1.3 for $6<p\le 12$, the decay estimate for $I_3, I_4$ will be easier, that is, we do not need to integrate by parts in time again. Indeed, for example, when $p=12$, we could estimate $I_3$ as follows:*

$$\begin{aligned}
\|I_3\|_{W^{\sigma,12}} \lesssim \|I_3\|_{W^{\sigma+1,\frac{12}{5}}} &\lesssim \int_0^t (1+t-s)^{-\frac54}\|T_{\frac{m}{\phi_{jk}}}(\tilde r,\tilde r)\|_{W^{\sigma+\frac32,\frac{12}{7}}}\mathrm{d}s\\
&\lesssim \int_0^t (1+t-s)^{-\frac54}\|\tilde r\|_{W^{2,12}}\|\tilde r\|_{H^{\sigma+(\frac72)+}}\mathrm{d}s\\
&\lesssim \int_0^t (1+t-s)^{-\frac54}(1+s)^{-\frac54}\mathrm{d}s\|U\|_{X_T}^2 \lesssim (1+t)^{-\frac54}\|U\|_{X_T}^3.
\end{aligned}$$

*For the estimate of $I_4$, we can use the identity*

$$T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde r,\tilde r) = \varepsilon\Delta T_{\frac{m}{\phi_{jk}}}(\tilde r,\tilde r) - 2\sum_{l=1}^3 T_{\frac{m}{\phi_{jk}}}(\varepsilon^{\frac12}\partial_l\tilde r,\varepsilon^{\frac12}\partial_l\tilde r).$$

*and the a priori estimates:*

$$\|\varepsilon^{\frac12}\nabla\tilde r\|_{H^{N-1}} \lesssim (1+s)^{-\frac12}\|U\|_X,\quad \|\varepsilon^{\frac12}\nabla\tilde r\|_{W^{N-2,\frac{12}{5}}} \lesssim (1+s)^{-\frac34}\|U\|_X.$$

*Nevertheless, we are interested also in $12<p<\infty$, and in this case, it is necessary to use a normal form transformation again because $p'$ is too small to allows us to conclude the estimate directly.*

**Remark 3.4.6.** *We now choose $24\le p<+\infty$. By interpolation, for any $2\le q\le p$, we have the decay estimate:*

$$\|(\varrho,u)\|_{W^{\sigma,q}} \lesssim (1+t)^{-\frac32(1-\frac2q)}\|(\varrho,u)\|_X,$$

$$\|(\varrho,u)(t)\|_{W^{\sigma,\infty}} \lesssim (1+t)^{-\frac{4}{3}}\|(\varrho,u)\|_X. \tag{3.4.8}$$

*Indeed, we only need to prove (3.4.8) for $\nabla^\sigma(\varrho,u)$ as the other is almost obvious. By the Gagliardo-Nirenberg inequality, we have:*

$$\|\nabla^\sigma(\varrho,u)\|_{L^\infty} \lesssim \|(\varrho,u)\|^{\theta}_{\dot W^{\sigma,p}}\|(\varrho,u)\|^{1-\theta}_{\dot H^{\sigma+l}} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2}{p})\theta}\|(\varrho,u)\|_X,$$

*where $\theta = 1-\frac{1}{(\frac{l}{3}-\frac{1}{2})p+1}$ and $l=7$. When $p\geq 24$, we have: $\frac{3}{2}(1-\frac{2}{p})\theta > \frac{4}{3}$.*

## 3.5 Proof of Theorem 3.1.4

Now our aim is to prove Theorem 3.1.4, that is to say, to get global existence for system (3.1.4) under the assumption that the incompressible part of the initial velocity is small compared to $\varepsilon$. We adapt the energy estimate showed in [126] where the original (ENSP) system was treated. However, we need to focus more on the perturbation term where we should make the most use of the integrability of time decay of $(\varrho,u)$ in some Sobolev spaces. Global existence for $(n,u,\nabla\psi)$ follows from the energy estimate (see lemma 3.5.1 and lemma 3.5.2) and classical bootstrap arguments. To prove the decay estimate, again, inspired by [126] we use an interpolation argument between the energy estimate and an $\dot H^{-s}$ estimate which is true if the initial data is in this space. This yields a good energy inequality (see (3.5.28)), which finally gives the decay estimate.

For the reader's convenience, we recall that we are talking about the system (3.1.4) which takes the form:

$$\begin{cases} \partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u\cdot\nabla v + v\cdot(\nabla u+\nabla v) - \varepsilon\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho+n}-1)(\mathcal{L}v+\mathcal{L}u), \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0}=0. \end{cases} \tag{3.5.1}$$

We define the energy functional:

$$\mathcal{E}_M(n,u,\nabla\psi) = \sum_{|\alpha|\leq M}\mathscr{E}_\alpha = \sum_{|\alpha|\leq M}\frac{1}{2}\int \rho|\partial^\alpha v|^2 + |\partial^\alpha n|^2 + |\partial^\alpha\nabla\psi|^2\mathrm{d}x. \tag{3.5.2}$$

Denote also $\mathscr{E}_k = \sum_{|\alpha|=k}\mathscr{E}_\alpha$. We carry out energy estimates in the following two lemmas.

**Lemma 3.5.1.** *Assuming that ($\rho = \varrho+1, \nabla\phi, u$) are given by Theorem 3.1.3, so that in particular $\|(\varrho,\nabla\phi,u)\|_{H^3}\leq C\delta$, and that $\mathcal{E}_3\leq C\delta\varepsilon$, with $C$ an absolute constant and $\delta$ small enough, such that $\|\varrho,n\|_{L^\infty}\leq C\|\varrho,n\|_{H^2}\leq C\delta\leq\frac{1}{6}$. Then the following energy inequality holds: for any $k=1,2\cdots M$ we have:*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_k + \frac{1}{2}\varepsilon\|\nabla v\|^2_{\dot H^k} \lesssim{}& \|(u,\varrho)\|^{\frac{3}{4}}_{W^{M+2,\infty}}\mathscr{E}_k + (\mathcal{E}_3^{\frac{1}{2}}+\varepsilon\|\varrho\|_{H^M})\|(\nabla v,n)\|^2_{\dot H^k} \\ &+\varepsilon^2\|u\|^{\frac{5}{4}}_{W^{k+2,\infty}}(\|\varrho\|^2_{H^{|\alpha|}}+\mathcal{E}_3) + \mathcal{E}_3\|\varrho\|^{\frac{5}{4}}_{W^{k+1,\infty}} + \varepsilon\mathcal{E}_3^{\frac{1}{2}}\|\varrho\|^2_{\dot W^{k,6}}. \end{aligned} \tag{3.5.3}$$

*where $3\leq M\leq\sigma-2$.*

*Proof.* By local existence, we have enough regularity to do energy estimates. We take the time derivative of the energy functional $\mathscr{E}_\alpha, |\alpha|=k$, and we make use of the equation (3.5.1) to get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_\alpha + \varepsilon\int\rho\big(|\partial^\alpha\nabla v|^2 + |\partial^\alpha\mathrm{div}v|^2\big)\mathrm{d}x \triangleq \sum_{j=1}^{10}F_j \tag{3.5.4}$$

where

$$
\begin{aligned}
F_1 &= \int \partial^\alpha n\big(\mathrm{div}(\rho\partial^\alpha v) - \partial^\alpha \mathrm{div}(\rho v)\big)\mathrm{d}x, \qquad F_2 = \int \partial^\alpha \nabla\psi\cdot\rho\partial^\alpha v - \partial^\alpha\nabla\psi\cdot\partial^\alpha(\rho v)\mathrm{d}x,\\
F_3 &= \int \rho\partial^\alpha v\big(u\cdot\partial^\alpha\nabla v - \partial^\alpha(u\cdot\nabla v)\big)\mathrm{d}x, \qquad F_4 = -\int \rho\partial^\alpha v\partial^\alpha(v\cdot\nabla u)\mathrm{d}x,\\
F_5 &= -\int \partial^\alpha n\partial^\alpha \mathrm{div}(nu)\mathrm{d}x, \qquad F_6 = -\int \partial^\alpha\nabla\psi\cdot\partial^\alpha(nu)\mathrm{d}x,\\
F_7 &= -\varepsilon\int \nabla\varrho\cdot\partial^\alpha v\partial^\alpha \mathrm{div}v + (\nabla\varrho\otimes\partial^\alpha v):\partial^\alpha\nabla v\mathrm{d}x,\\
F_8 &= \varepsilon\int \rho\partial^\alpha v\partial^\alpha\big((\frac{1}{\rho+n}-1)\mathcal{L}v\big)\mathrm{d}x, \qquad F_9 = \varepsilon\int \rho\partial^\alpha v\partial^\alpha\big((\frac{1}{\rho+n}-1)\mathcal{L}u\big)\mathrm{d}x,\\
F_{10} &= -\int \rho\partial^\alpha v\partial^\alpha(v\cdot\nabla v) + \partial^\alpha n\partial^\alpha\mathrm{div}(nv) + \partial^\alpha\nabla\psi\partial^\alpha(nv)\mathrm{d}x.
\end{aligned}
$$

One first notices that $F_1, F_2, F_3$ equal to 0 when $|\alpha| = k = 0$. When $k \geq 1$, using product estimate and Young's inequality, we have for $F_1$

$$
\begin{aligned}
F_1 &= -\int \partial^\alpha n\big([\partial^\alpha,\rho]\mathrm{div}v + \partial^\alpha(\nabla\varrho\cdot v) - \nabla\varrho\cdot\partial^\alpha v\big)\mathrm{d}x\\
&\lesssim \|n\|_{\dot H^{|\alpha|}}\|\nabla\varrho\|_{W^{|\alpha|,\infty}}(\|v\|_{\dot H^{|\alpha|}} + \|v\|_{H^1})\\
&\lesssim \|\nabla\varrho\|_{W^{|\alpha|,\infty}}\|(n,v)\|^2_{\dot H^{|\alpha|}} + \|\nabla\varrho\|^{\frac34}_{W^{|\alpha|,\infty}}\|n\|^2_{\dot H^{|\alpha|}} + \|\nabla\varrho\|^{\frac54}_{W^{|\alpha|,\infty}}\|v\|^2_{H^1}\\
&\lesssim (\|\nabla\varrho\|_{W^{|\alpha|,\infty}} + \|\nabla\varrho\|^{\frac34}_{W^{|\alpha|,\infty}})\|(n,v)\|^2_{\dot H^{|\alpha|}} + \mathcal{E}_3\|\nabla\varrho\|^{\frac54}_{W^{|\alpha|,\infty}}\\
&\lesssim \|\nabla\varrho\|^{\frac34}_{W^{|\alpha|,\infty}}\mathscr{E}_k + \mathcal{E}_3\|\nabla\varrho\|^{\frac54}_{W^{k,\infty}}. \qquad (3.5.5)
\end{aligned}
$$

Here, in the last inequality we have used that $\|\nabla\varrho\|_{W^{M,\infty}}$ is small. We point out that we use the power $\frac54$ in the above mainly to get more time integrability for the 'perturbation term', that is we want $b$ larger than $\frac32$ in (3.5.24) and (3.5.25) which will lead to the better decay estimate for $(n,\nabla\psi,v)$.

The estimates for $F_2$ and $F_3$. For $F_2$, we write

$$
\begin{aligned}
F_2 &= \int \partial^\alpha\nabla\psi\cdot(\varrho\partial^\alpha v - \partial^\alpha(\varrho v))\mathrm{d}x \leq \|\partial^\alpha\nabla\psi\|_{L^2}\|\varrho\|_{W^{|\alpha|,\infty}}(\|v\|_{L^2} + \|v\|_{\dot H^{|\alpha|}})\\
&\lesssim (\|\varrho\|_{W^{|\alpha|,\infty}} + \|\varrho\|^{\frac34}_{W^{|\alpha|,\infty}})(\|\nabla\psi\|^2_{\dot H^{|\alpha|}} + \|v\|^2_{\dot H^{|\alpha|}}) + \mathcal{E}_3\|\nabla\varrho\|^{\frac54}_{W^{|\alpha|-1,\infty}}\\
&\lesssim \|\varrho\|^{\frac34}_{W^{k,\infty}}\mathscr{E}_k + \mathcal{E}_3\|\varrho\|^{\frac54}_{W^{k,\infty}}, \qquad (3.5.6)
\end{aligned}
$$

and we can get in the same way

$$
F_3 \lesssim \|\nabla u\|^{\frac34}_{W^{k-1,\infty}}\mathscr{E}_k + \mathcal{E}_3\|u\|^{\frac54}_{W^{k,\infty}}. \qquad (3.5.7)
$$

We now estimate $F_4 - F_7$ with $|\alpha| = k \geq 0$. By product estimates and Young's inequality again, we have for $F_4$,

$$
\begin{aligned}
F_4 &\lesssim \|\partial^\alpha v\|_{L^2}\big(\|\nabla u\|_{W^{|\alpha|,\infty}}(\|v\|_{L^2} + \|v\|_{\dot H^{|\alpha|}})\big) \lesssim (\|\nabla u\|_{W^{|\alpha|,\infty}} + \|\nabla u\|^{\frac34}_{W^{|\alpha|,\infty}})\|v\|^2_{\dot H^{|\alpha|}} + \mathcal{E}_3\|\nabla u\|^{\frac54}_{W^{|\alpha|,\infty}}\\
&\lesssim \|\nabla u\|^{\frac34}_{W^{k,\infty}}\mathscr{E}_k + \mathcal{E}_3\|\nabla u\|^{\frac54}_{W^{k,\infty}}. \qquad (3.5.8)
\end{aligned}
$$

For $F_5$, we integrate by parts for the first term and use Hölder's inequality for the other two terms to get

$$
\begin{aligned}
F_5 &= -\int \partial^\alpha n\big(\partial^\alpha\nabla n\cdot u + [\partial^\alpha, u]\nabla n + \partial^\alpha(n\mathrm{div}u)\big)\mathrm{d}x\\
&\lesssim \|\partial^\alpha n\|^2_{L^2}\|\nabla u\|_{L^\infty} + \|\partial^\alpha n\|_{L^2}\|\nabla u\|_{W^{|\alpha|,\infty}}(\|n\|_{\dot H^{|\alpha|}} + \|n\|_{H^1})\\
&\lesssim (\|\nabla u\|_{W^{|\alpha|,\infty}} + \|\nabla u\|^{\frac34}_{W^{|\alpha|,\infty}})\|n\|^2_{\dot H^{|\alpha|}} + \mathcal{E}_3\|u\|^{\frac54}_{W^{|\alpha|+1,\infty}} \lesssim \|\nabla u\|^{\frac34}_{W^{k,\infty}}\mathscr{E}_k + \mathcal{E}_3\|\nabla u\|^{\frac54}_{W^{k,\infty}}. \qquad (3.5.9)
\end{aligned}
$$

In a similar way, we have

$$F_6 \lesssim \|u\|_{W^{|\alpha|,\infty}}^{\frac{3}{4}}\|(n,\nabla\psi)\|_{\dot H^{|\alpha|}}^2+\mathcal{E}_3\|u\|_{W^{|\alpha|,\infty}}^{\frac{5}{4}} \lesssim \|u\|_{W^{k+1,\infty}}^{\frac{3}{4}}\mathscr{E}_k+\mathcal{E}_3\|u\|_{W^{k+1,\infty}}^{\frac{5}{4}}. \tag{3.5.10}$$

as well as

$$F_7 \lesssim \varepsilon\|\nabla\varrho\|_{L^\infty}(\|\nabla v\|_{\dot H^k}^2+\|v\|_{\dot H^k}^2). \tag{3.5.11}$$

For $F_8$, we only handle $k=|\alpha|>0$ since the case $k=|\alpha|=0$ is easier. Integrating by parts, and denoting $\partial^\alpha=\partial_j\partial^{\tilde\alpha}$ and using Lemma 3.8.2 we get that:

$$\begin{aligned}
F_8 &= -\varepsilon\int(\rho\partial_j\partial^\alpha v+\partial_j\varrho\partial^\alpha v)\partial^{\tilde\alpha}((\frac{1}{\rho+n}-1)\mathcal{L}v)\\
&\lesssim \varepsilon(\|\rho\|_{L^\infty}+\|\nabla\varrho\|_{L^3})\|\nabla v\|_{\dot H^{|\alpha|}}\|(\frac{1}{\rho+n}-1)\mathcal{L}v\|_{\dot H^{|\alpha|-1}}\\
&\lesssim \varepsilon\|\nabla v\|_{\dot H^{|\alpha|}}(\|\varrho+n\|_{L^\infty}\|\nabla^2 v\|_{\dot H^{|\alpha|-1}}+\|\frac{1}{\rho+n}-1\|_{\dot W^{|\alpha|-1,6}}\|\nabla^2 v\|_{L^3})\\
&\lesssim \varepsilon\|\nabla v\|_{\dot H^{|\alpha|}}(\|\varrho+n\|_{L^\infty}\|\nabla^2 v\|_{\dot H^{|\alpha|-1}}+\|\varrho+n\|_{\dot W^{|\alpha|-1,6}}\|\nabla^2 v\|_{L^3})\\
&\lesssim \varepsilon(\mathcal{E}_3^{\frac12}+\|\varrho\|_{H^M})\|(n,\nabla v)\|_{\dot H^k}^2+\varepsilon\mathcal{E}_3^{\frac12}\|\varrho\|_{W^{k-1,6}}^2,
\end{aligned} \tag{3.5.12}$$

where we have used the fact that $\|\varrho\|_{H^2}$ is bounded in the second inequality.
We now deal with $F_9$ in the same fashion:

$$\begin{aligned}
F_9 &\lesssim \varepsilon\|\partial^\alpha v\|_{L^2}\|\partial^\alpha\big((\frac{1}{\rho+n}-1)\mathcal{L}u\big)\|_{L^2}\\
&\lesssim \varepsilon\|\partial^\alpha v\|_{L^2}\big(\|\nabla^2 u\|_{W^{|\alpha|,\infty}}(\|\varrho+n\|_{L^2}+\|\varrho+n\|_{\dot H^{|\alpha|}})\big)\\
&\lesssim \|\nabla^2 u\|_{W^{|\alpha|,\infty}}^{\frac34}\|v\|_{\dot H^{|\alpha|}}^2+\varepsilon^2\|\nabla^2 u\|_{W^{|\alpha|,\infty}}^{\frac54}(\|\varrho\|_{H^{|\alpha|}}^2+\mathcal{E}_3)\\
&\quad+\varepsilon\|\nabla^2 u\|_{W^{|\alpha|,\infty}}(\|v\|_{\dot H^{|\alpha|}}^2+\|n\|_{\dot H^{|\alpha|}}^2)\\
&\lesssim \|\nabla^2 u\|_{W^{k,\infty}}^{\frac34}\mathscr{E}_k+\varepsilon^2\|\nabla^2 u\|_{W^{k,\infty}}^{\frac54}(\|\varrho\|_{H^k}^2+\mathcal{E}_3).
\end{aligned} \tag{3.5.13}$$

Finally, for $F_{10}$, inspired by [126], we actually have:

$$F_{10} \lesssim \mathcal{E}_3^{\frac12}(\|\nabla v\|_{\dot H^k}^2+\|n\|_{\dot H^k}^2). \tag{3.5.14}$$

We just give details for the third term of $F_{10}$, the first two terms are similar and easier. Integrating by parts and using the Poisson equation, we have:

$$\begin{aligned}
&\int\partial^\alpha\nabla\psi\cdot\partial^\alpha(nv)\mathrm{d}x=\int\partial^\alpha\nabla\psi\cdot\partial^\alpha v\Delta\psi)\mathrm{d}x=-\int\partial^\alpha\nabla^2\psi:\partial^\alpha(\nabla\psi\otimes v)+\partial^\alpha\nabla\psi\cdot\partial^\alpha\big((\nabla\psi\cdot\nabla)v\big)\mathrm{d}x\\
&\triangleq (3.5.15)_1+(3.5.15)_2.
\end{aligned} \tag{3.5.15}$$

For the estimate of $(3.5.15)_1$, we use Kato-Ponce inequality (see Lemma 3.8.1) again to get:

$$|(3.5.15)_1| \lesssim \|\nabla^2\psi\|_{\dot H^{|\alpha|}}(\|\nabla\psi\|_{\dot W^{|\alpha|,6}}\|v\|_{L^3}+\|v\|_{\dot W^{|\alpha|,6}}\|\nabla\psi\|_{L^3}) \lesssim \mathcal{E}_3^{\frac12}(\|n\|_{\dot H^k}^2+\|\nabla v\|_{\dot H^k}^2).$$

For $(3.5.15)_2$, by using Kato-Ponce inequality and Gagliardo-Nirenberg inequality, we have:

$$\begin{aligned}
|(3.5.15)_2| &\lesssim \|\nabla\psi\|_{\dot W^{|\alpha|,6}}\|(\nabla\psi\nabla v)\|_{\dot W^{|\alpha|,\frac65}}\\
&\lesssim \|n\|_{\dot H^{|\alpha|}}(\|\nabla v\|_{\dot H^{|\alpha|}}\|\nabla\psi\|_{L^3}+\|\nabla\psi\|_{\dot W^{|\alpha|,3}}\|\nabla v\|_{L^2})\\
&\lesssim \|n\|_{\dot H^{|\alpha|}}(\|\nabla v\|_{\dot H^{|\alpha|}}\|\nabla\psi\|_{L^3}+\|\nabla\psi\|_{\dot H^{|\alpha|+1}}^\theta\|\nabla\psi\|_{\dot H^{\frac12}}^{1-\theta}\|v\|_{\dot H^{\frac12}}^\theta\|v\|_{\dot H^{|\alpha|+1}}^{1-\theta})\\
&\lesssim \mathcal{E}_3^{\frac12}\|(n,\nabla\psi)\|_{\dot H^k}^2,
\end{aligned}$$

where in the above $\theta=\frac{|\alpha|}{|\alpha|+\frac12}$.
Using (3.5.5-3.5.14) , and summing up for any $|\alpha|=k$ we get the Lemma 3.5.1. □

As indicated in [126], to close the energy estimate, we must get some damping for $n$, this can be achieved by doing the 'cross' energy estimate.

**Lemma 3.5.2.** *Under the assumption of Lemma 3.5.1, we have for any $k = 0, 1 \cdots M-1$,*

$$\begin{aligned}
&\sum_{|\alpha|=k} \frac{d}{dt}\int \partial^\alpha \nabla n \cdot \partial^\alpha v \mathrm{d}x + \frac{1}{2}(\|n\|^2_{\dot{H}^k} + \|n\|^2_{\dot{H}^{k+1}}) \\
\lesssim\ & (\|v\|^2_{\dot{H}^{k+1}} + \|v\|^2_{\dot{H}^{k+2}}) + \|(\varrho, u)\|^{\frac{3}{4}}_{W^{k+2,\infty}}(\mathscr{E}_k + \mathscr{E}_{k+1}) \\
&\qquad + \varepsilon \mathcal{E}_3 (\|(\varrho, u)\|^{\frac{5}{4}}_{W^{k+2,\infty}} + \|(\varrho, u)\|^2_{\dot{W}^{k,6}}).
\end{aligned} \tag{3.5.16}$$

*Proof.* Taking $\partial^\alpha \nabla$ (respectively $\partial^\alpha$) on the first (respectively second) equation in system (3.5.1), multiplying by $\partial^\alpha v$(respectively $\partial^\alpha \nabla n$), integrating in space and adding together, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\int \partial^\alpha \nabla n \cdot \partial^\alpha v \mathrm{d}x + \int |\partial^\alpha \nabla n|^2 + |\partial^\alpha n|^2 \mathrm{d}x = G_1 + G_2 + G_3 + G_4 + G_5 \\
&\triangleq -\int \partial^\alpha \nabla \mathrm{div}(\rho v + nu)\cdot \partial^\alpha v \mathrm{d}x - \int \partial^\alpha \nabla n \cdot \partial^\alpha (u\cdot \nabla v + v \cdot \nabla u)\mathrm{d}x - \varepsilon \int \partial^\alpha \nabla n \cdot \partial^\alpha (\frac{1}{\rho + n}\mathcal{L}v)\mathrm{d}x \\
&\qquad - \varepsilon \int \partial^\alpha \nabla n \cdot \partial^\alpha \big((\frac{1}{\rho+n} - 1)\mathcal{L}u\big)\mathrm{d}x - \int \partial^\alpha \nabla \mathrm{div}(nv)\cdot \partial^\alpha v + \partial^\alpha \nabla n \cdot \partial^\alpha (v\cdot \nabla v)\mathrm{d}x.
\end{aligned} \tag{3.5.17}$$

We handle the estimates for $|\alpha| = k \geq 1$, $k = 0$ being easier.

Similar to the estimate in Lemma 3.5.1, by Hölder's and Young's inequality, we have that

$$\begin{aligned}
G_1 &= \int \partial^\alpha \mathrm{div} v \cdot \partial^\alpha \mathrm{div}(v + \varrho v + nu)\mathrm{d}x \leq \|\partial^\alpha \mathrm{div} v\|^2_{L^2} + \|\partial^\alpha \mathrm{div} v\|_{L^2}\|\varrho v + nu\|_{\dot{H}^{|\alpha|+1}} \\
&\lesssim \|\partial^\alpha \mathrm{div} v\|^2_{L^2} + \|\partial^\alpha \mathrm{div} v\|_{L^2}\|(\varrho, u)\|_{W^{|\alpha|+1,\infty}}(\|(n,v)\|_{L^2} + \|(n,v)\|_{\dot{H}^{|\alpha|+1}}) \\
&\lesssim \|\nabla v\|^2_{\dot{H}^{|\alpha|}} + (\|(\varrho, u)\|_{W^{|\alpha|+1,\infty}} + \|(\varrho, u)\|^{\frac{3}{4}}_{W^{|\alpha|+1,\infty}})\|(n,v)\|^2_{\dot{H}^{|\alpha|+1}} + \mathcal{E}_3\|(\varrho, u)\|^{\frac{5}{4}}_{W^{|\alpha|+1,\infty}} \\
&\lesssim \|\nabla v\|^2_{\dot{H}^k} + \|(\varrho, u)\|^{\frac{3}{4}}_{W^{k+1,\infty}}\mathscr{E}_{k+1} + \mathcal{E}_3\|(\varrho, u)\|^{\frac{5}{4}}_{W^{k+1,\infty}},
\end{aligned} \tag{3.5.18}$$

as well as

$$\begin{aligned}
G_2 &= -\int \partial^\alpha \nabla n \cdot \partial^\alpha (u \cdot \nabla v + v \cdot \nabla u)\mathrm{d}x \\
&\lesssim \|\partial^\alpha \nabla n\|_{L^2}(\|u\cdot \nabla v\|_{\dot{H}^{|\alpha|}} + \|v \cdot \nabla u\|_{\dot{H}^{|\alpha|}}) \\
&\lesssim \|\partial^\alpha \nabla n\|_{L^2}\|u\|_{W^{|\alpha|+1,\infty}}(\|v\|_{H^1} + \|\nabla v\|_{\dot{H}^{|\alpha|}} + \|v\|_{\dot{H}^{|\alpha|}}) \\
&\lesssim (\|u\|_{W^{k+1,\infty}} + \|u\|^{\frac{3}{4}}_{W^{k+1,\infty}})(\mathscr{E}_k + \mathscr{E}_{k+1}) + \mathcal{E}_3\|u\|^{\frac{5}{4}}_{W^{k+1,\infty}}.
\end{aligned} \tag{3.5.19}$$

By Lemma 3.8.1, 3.8.2 in the appendix, we estimate $G_3$ as follows:

$$\begin{aligned}
G_3 &= -\varepsilon \int \partial^\alpha \nabla n \cdot \partial^\alpha (\frac{1}{\rho+n}\mathcal{L}v)\mathrm{d}x \\
&\lesssim \varepsilon \|\partial^\alpha \nabla n\|_{L^2}(\|\frac{1}{\rho+n} - 1\|_{\dot{W}^{|\alpha|,6}}\|\nabla^2 v\|_{L^3} + \|\frac{1}{\rho+n}\|_{L^\infty}\|\nabla^2 v\|_{\dot{H}^{|\alpha|}}) \\
&\lesssim \varepsilon \|\partial^\alpha \nabla n\|_{L^2}\big(\|\nabla^2 v\|_{L^3}\|(\varrho, n)\|_{\dot{W}^{|\alpha|,6}} + \|\nabla^2 v\|_{\dot{H}^{|\alpha|}}\big) \\
&\leq (\frac{1}{8} + c\mathcal{E}_3^{\frac{1}{2}})\|n\|^2_{\dot{H}^{|\alpha|+1}} + c\|\nabla^2 v\|^2_{\dot{H}^{|\alpha|}} + c\varepsilon^2 \mathcal{E}_3 \|\varrho\|^2_{W^{|\alpha|,6}} \\
&\leq \frac{1}{4}\|\partial^\alpha \nabla n\|^2_{L^2} + c\|\nabla^2 v\|^2_{\dot{H}^k} + c\varepsilon^2 \mathcal{E}_3\|\varrho\|^2_{W^{k,6}}
\end{aligned} \tag{3.5.20}$$

using that $c\mathcal{E}_3^{\frac{1}{2}} \leq c\delta\varepsilon \leq \frac{1}{8}$ where $\delta$ is small enough. Note that the first term in (3.5.20) could be absorbed by the left hand side of (3.5.17). Next, $G_4$ can be estimated exactly as $F_9$. For $|\alpha| = k \geq 1$, we have:

$$\begin{aligned}
G_4 &= -\varepsilon \int \partial^\alpha \nabla n \cdot \partial^\alpha \big( (\frac{1}{\rho+n} - 1)\mathcal{L}u \big) \mathrm{d}x \\
&\lesssim \varepsilon \|\partial^\alpha \nabla n\|_{L^2} \|\nabla^2 u\|_{W^{|\alpha|,\infty}} (\|\varrho + n\|_{L^2} + \|\varrho + n\|_{\dot{H}^{|\alpha|}}) \\
&\lesssim (\|\nabla^2 u\|_{W^{|\alpha|,\infty}} + \|\nabla^2 u\|_{W^{|\alpha|,\infty}}^{\frac{3}{4}}) \|(\nabla n, n)\|_{\dot{H}^{|\alpha|}}^2 + \varepsilon \mathcal{E}_3 \|\nabla^2 u\|_{W^{|\alpha|,\infty}}^{\frac{5}{4}} + \varepsilon^2 \|\varrho\|_{H^{|\alpha|}}^2 \|\nabla^2 u\|_{W^{|\alpha|,\infty}}^2 \\
&\lesssim \|\nabla^2 u\|_{W^{k,\infty}}^{\frac{3}{4}} (\mathscr{E}_k + \mathscr{E}_{k+1}) + \varepsilon \mathcal{E}_3 \|\nabla^2 u\|_{W^{k,\infty}}^{\frac{5}{4}} + \varepsilon^2 \|\varrho\|_{H^k}^2 \|\nabla^2 u\|_{W^{k,\infty}}^2 .
\end{aligned} \tag{3.5.21}$$

For $G_5$, as in [126], one can show that if $\mathcal{E}_3^{\frac{1}{2}} \leq \delta\varepsilon$ with $\delta$ small enough, we have:

$$G_5 \leq \frac{1}{8}(\|n\|_{\dot{H}^k}^2 + \|\nabla n\|_{\dot{H}^k}^2) + c(\|\nabla u\|_{\dot{H}^k}^2 + \|\nabla^2 u\|_{\dot{H}^k}^2). \tag{3.5.22}$$

Summing up from (3.5.18) to (3.5.22), we get Lemma 3.5.2. □

*Proof of Theorem 3.1.4.* We first prove global existence. Summing up from $k = 0$ to $k = M$, we can conclude from Lemma 3.5.1 and Lemma 3.5.2 , Remark 3.4.6 that if $\mathcal{E}_3 \leq \delta^2\varepsilon^2$ and $\delta$ is small enough, then there are some constants which depend only on $M$, such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_M + C\varepsilon \|\nabla v\|_{H^M}^2 \\
\leq& C_1(1+t)^{-a} \|(\varrho, u)\|_X^{\frac{3}{4}} \mathcal{E}_M + C_2\delta\varepsilon \|(\nabla v, n)\|_{H^M}^2 + C_3\varepsilon^2(1+t)^{-b}(\delta\|(\varrho,u)\|_X^2 + \|(\varrho,u)\|_X^3) \\
\leq& C_1\delta^{\frac{3}{4}}(1+t)^{-a}\mathcal{E}_M + C_2\delta\varepsilon\|(\nabla v, n)\|_{H^M}^2 + C_3\delta^3\varepsilon^2(1+t)^{-b},
\end{aligned} \tag{3.5.23}$$

and

$$\begin{aligned}
&\sum_{|\alpha|\leq M-1} \frac{d}{dt} \int \partial^\alpha \nabla n \cdot \partial^\alpha v \mathrm{d}x + \frac{1}{2}\|n\|_{H^M}^2 \\
\leq& C_4\|\nabla v\|_{H^M}^2 + C_5(1+t)^{-a}\|(\varrho,u)\|_X^{\frac{3}{4}}\mathcal{E}_M + C_6\mathcal{E}_3(1+t)^{-b}\|(\varrho,u)\|_X \\
\leq& C_4\|\nabla v\|_{H^M}^2 + C_5\delta^{\frac{3}{4}}\mathcal{E}_M + C_6\delta^3\varepsilon^2(1+t)^{-b},
\end{aligned} \tag{3.5.24}$$

where $a > 1, b > \frac{5}{3}$ (here we use $\|(\varrho, u)\|_{W^{\sigma,\infty}} \lesssim (1+t)^{-\frac{4}{3}}$).

Multiplying (3.5.24) by $8C_2\delta\varepsilon$ and add it to (3.5.23), if $\mathcal{E}_3^{\frac{1}{2}} \leq \delta\varepsilon$ and $\delta$ is small enough, (say, $32C_2C_4\delta \leq C$) we get that there exist constant $C_7, C_8, C_9$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_M + 8C_2\delta\varepsilon \sum_{|\alpha|\leq M-1} \int \partial^\alpha\nabla n \cdot \partial^\alpha v \mathrm{d}x) + C_7\varepsilon\|(n, \nabla v)\|_{H^M}^2 \leq C_8(1+t)^{-a}\delta^{\frac{3}{4}}\mathcal{E}_M + C_9\delta^3\varepsilon^2(1+t)^{-b}. \tag{3.5.25}$$

Define $\tilde{\mathcal{E}}_M = \mathcal{E}_M + 8C_2\delta\varepsilon\sum_{|\alpha|\leq M-1}\int \partial^\alpha\nabla n\cdot\partial^\alpha v\mathrm{d}x$, we see that $\tilde{\mathcal{E}}_M \approx \mathcal{E}_M$(say $\frac{1}{2}\mathcal{E}_M \leq \tilde{\mathcal{E}}_M \leq 2\mathcal{E}_M$) if $\delta$ is small enough.

From inequality (3.5.25), Grönwall's inequality and the fact $\|(\varrho, u)\|_X \leq \delta$, we achieve that:

$$\begin{aligned}
\mathcal{E}_M(t) + C_7\varepsilon \int_0^t \|(n,\nabla v)\|_{H^M}^2 \mathrm{d}s &\leq e^{C_8\delta^{\frac{3}{4}}\int_0^t (1+s)^{-a}\mathrm{d}x}(4\mathcal{E}_M(0) + 2C_9\delta^3\varepsilon^2 \int_0^t (1+s)^{-b}\mathrm{d}s) \\
&\leq C_{10}\mathcal{E}_M(0) + C_{11}\delta^3\varepsilon^2.
\end{aligned} \tag{3.5.26}$$

Global existence of $(n, \nabla\psi, v)$ in $C([0,+\infty), H^3)$ then is direct by bootstrap arguments. Moreover, we have $\mathcal{E}_3(t) \leq \delta^2\varepsilon^2$ if $\mathcal{E}_3(0) \leq \frac{1}{16}\delta^2\varepsilon^2$ and $\delta$ is small enough (Note that $C_{10} \leq 8$ if $\delta$ is small enough.) Finally, as can be seen easily from (3.5.26), if in addition, $\mathcal{E}_M(0) < +\infty$, then the solution constructed also belongs to $C([0,\infty), H^M)$.

**Remark 3.5.3.** *If we define $\mathcal{E}_k^M = \sum_{l=k}^M \mathscr{E}_l$, then by adding* (3.5.3) *from $k$ to $M$,* (3.5.16) *from $k$ to $M-1$ and the same arguments for proving* (3.5.25), *we can have(with another constant $C_7$):*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_k^M + 8C_2\delta\varepsilon \sum_{k\le|\alpha|\le M-1} \int \partial^\alpha\nabla n\cdot\partial^\alpha v\mathrm{d}x) + C_7\varepsilon(\|\nabla^k n\|^2_{H^{M-k}} + \|\nabla^{k+1}v\|^2_{H^{M-k}}) \\ &\le C_8(1+t)^{-a}\delta^{\frac{3}{4}}\mathcal{E}_k^M + C_9\varepsilon^2\delta^3(1+t)^{-b}. \end{aligned} \tag{3.5.27}$$

Motivated by [126], we can prove that if the initial data belongs to some negative Sobolev spaces, the solution for system (3.1.4) will propagate in this space. This will allow us to obtain some time decay for $(n, \nabla\psi, v)$.

**Lemma 3.5.4.** *For $0 < s \le \frac{1}{2}$, we have:*

$$\begin{aligned} &\frac{d}{\mathrm{d}t}\int |\Lambda^{-s}n|^2 + |\Lambda^{-s}\nabla\psi|^2 + |\Lambda^{-s}v|^2\mathrm{d}x + \int |\Lambda^{-s}\nabla v|^2 + |\Lambda^{-s}\mathrm{div}v|^2\mathrm{d}x \\ &\lesssim \|\Lambda^{-s}(n,\nabla\psi,v)\|_{L^2}\big(\|n\|^2_{H^2} + \|\nabla v\|^2_{H^1} + \|(\varrho,u)\|_{W^{2,3/s}}(\|(n,v)\|_{H^1} + \|(\varrho,u)\|_{H^1})\big). \end{aligned}$$

*Proof.* Applying $\Lambda^{-s}$ to the equations (3.1.4) and multiplying by $\Lambda^{-s}n, \Lambda^{-s}v$ respectively, we get, after using the Poisson equation:

$$\begin{aligned} &\frac{1}{2}\frac{d}{\mathrm{d}t}\int |\Lambda^{-s}n|^2 + |\Lambda^{-s}\nabla\psi|^2 + |\Lambda^{-s}v|^2\mathrm{d}x + \int |\Lambda^{-s}\nabla v|^2 + |\Lambda^{-s}\mathrm{div}v|^2\mathrm{d}x \\ = \; & -\int \Lambda^{-s}v\Lambda^{-s}(u\cdot\nabla v + v\cdot\nabla u + v\cdot\nabla v) - \Lambda^{-s}v\Lambda^{-s}(\frac{1}{\rho+n} - 1)(\mathcal{L}u + \mathcal{L}v))\mathrm{d}x \\ & -\int \Lambda^{-s}n\Lambda^{-s}\mathrm{div}(\varrho v + nu + nv) + \Lambda^{-s}\nabla\psi\Lambda^{-s}(\varrho v + nu + nv)\mathrm{d}x \\ \triangleq \; & H_1 + H_2 + H_3 + H_4. \end{aligned}$$

We only estimate $H_1, H_2$, since the other two terms can be handled by similar arguments. Using Hölder's inequality and the Hardy-Littlewood-Sobolev inequality:

$$\|\Lambda^{-s}f\|_{L^2} \lesssim \|f\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \quad 0 \le s < \frac{3}{2},$$

we get:

$$\begin{aligned} H_1 &= -\int \Lambda^{-s}v\Lambda^{-s}(u\cdot\nabla v + v\cdot\nabla u + v\cdot\nabla v)\mathrm{d}x \\ &\lesssim \|\Lambda^{-s}v\|_{L^2}\big(\|\nabla v\|_{L^2}\|u\|_{L^{3/s}} + \|\nabla u\|_{L^{3/s}}\|v\|_{L^2} + \|\nabla v\|_{L^2}\|v\|_{L^{3/s}}\big) \\ &\lesssim \|\Lambda^{-s}v\|_{L^2}\big(\|\nabla v\|^2_{H^1} + \|u\|_{W^{1,s/3}}\|v\|_{H^1}\big). \end{aligned}$$

$$\begin{aligned} H_2 &= \int \Lambda^{-s}v\Lambda^{-s}(\frac{1}{\rho+n} - 1)(\mathcal{L}u + \mathcal{L}v))\mathrm{d}x \lesssim \|\Lambda^{-s}v\|_{L^2}\big(\|\nabla^2 u\|_{L^{\frac{3}{s}}}\|(\varrho,n)\|_{L^2} + \|\nabla^2 v\|_{L^2}\|(\varrho,n)\|_{L^{3/s}}\big) \\ &\lesssim \|\Lambda^{-s}v\|_{L^2}\big(\|\nabla^2 u\|_{L^{\frac{3}{s}}}\|(\varrho,n)\|_{L^2} + \|\nabla^2 v\|^2_{L^2} + \|n\|^2_{H^2} + \|\varrho\|_{L^{3/s}}\|\nabla^2 v\|_{L^2}\big). \end{aligned}$$

This ends the proof. $\square$

Now we can prove the decay estimate for $(n, v, \nabla\psi)$ which is stated in Theorem 3.1.4. Here we follow the arguments in [126] with a few considerations on perturbation terms.

Step 1:
Prove $(n, \nabla\psi, v)$ propagate in the negative Sobolev space $\dot{H}^{-s}$. We should make use of the damping property of $(n, \nabla v)$ and decay estimate in time of $(\varrho, u)$ .

Define

$$\mathcal{E}_{-s} = \|(n,\nabla\psi,v)\|^2_{\dot{H}^{-s}}.$$

By Lemma 3.5.4, the decay estimate of $(\varrho, u)$: $\|(\varrho,u)\|_{W^{2,\frac{3}{s}}} \lesssim (1+t)^{-\frac{3}{2}(1-\frac{2s}{3})}$ (note $\frac{3}{2}(1-\frac{2s}{3}) > 1$ if $0 < s < \frac{1}{2}$) and the damping property of $(n, \nabla v)$ (see (3.5.26)), we have:

$$
\begin{aligned}
&\sup_{0\le\tau\le t} \mathcal{E}_{-s}(\tau)\\
\le\;& \mathcal{E}_{-s}(0) + C\int_0^t \|n\|_{H^2}^2 + \|\nabla v\|_{H^1}^2 + \|(\varrho,u)\|_{W^{2,3/s}}(\|(n,v)\|_{H^1} + \|(\varrho,u)\|_{H^1})\mathrm{d}\tau \sup_{0\le\tau\le t} \mathcal{E}_{-s}^{\frac{1}{2}}\\
\le\;& \mathcal{E}_{-s}(0) + (\sup_{0\le\tau\le t} \mathcal{E}_{-s})^{\frac{1}{2}},
\end{aligned}
$$

which yields the boundedness of $\|(n,\nabla\psi,v)\|_{H^{-s}}$ if we suppose $\mathcal{E}_{-s}(0) < +\infty$.

**Remark 3.5.5.** *The case $s = \frac{1}{2}$ is critical in the sense that the source term $\varepsilon(\varrho\mathcal{L}u)$ (which comes from $\varepsilon[(\frac{1}{1+\varrho+n} - 1)\mathcal{L}u)]$) has critical decay $(1+t)^{-1}$ in $\dot{H}^{\frac{1}{2}}$.*

Step 2: Using interpolation and energy estimate to get new energy inequality, and then get the time decay estimate.
By interpolation, we have:

$$\|u\|_{L^2} \le \|u\|_{\dot{H}^{-s}}^{\frac{1}{1+s}} \|u\|_{\dot{H}^1}^{\frac{s}{1+s}},$$

which is equivalent to

$$\|u\|_{\dot{H}^1} \ge \|u\|_{L^2}^{\frac{1+s}{s}} \|u\|_{\dot{H}^{-s}}^{-\frac{1}{s}}.$$

Combined with (3.5.25), we get that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_M + C_{12}\varepsilon(\|n\|_{H^M}^2 + \|v\|_{H^M}^2)^{1+\frac{1}{s}} \le C_{13}(1+t)^{-b}\varepsilon^2 + C_{14}(1+t)^{-a}\mathcal{E}_M. \tag{3.5.28}$$

We now prove the time decay estimate when $M = 3$. We recall that we assume $\mathcal{E}_3(0)$ is small respect to $\varepsilon$. Defining firstly $\beta_s = \frac{2}{s} + 1$, $f = \exp(-\frac{C_{14}}{a-1}(1+t)^{-(a-1)})\mathcal{E}_3$, then multiplying (3.5.28) by $(1+\varepsilon^{\beta_s}t)^\gamma, (s < \gamma < b-1)$ and integrating in time, we have by Young's inequality:

$$
\begin{aligned}
&(1+\varepsilon^{\beta_s}t)^\gamma f + C_{12}\varepsilon\int_0^t (1+\varepsilon^{\beta_s}\tau)^\gamma f^{1+\frac{1}{s}}(\tau)\mathrm{d}\tau \le f(0) + \gamma\varepsilon^{\beta_s}\int_0^t (1+\varepsilon^{\beta_s}\tau)^{\gamma-1} f\mathrm{d}\tau + C_{15}\varepsilon^2\\
\le\;& (f(0) + C_{15}\varepsilon^2) + \frac{C_{12}}{2}\varepsilon\int_0^t (1+\varepsilon^{\beta_s}\tau)^\gamma f^{1+\frac{1}{s}}(\tau)\mathrm{d}\tau + C_{16}\varepsilon^2(1+\varepsilon^{\beta_s}t)^{\gamma-s}
\end{aligned}
$$

which yields:

$$f \;\lesssim\; \varepsilon^2(1+\varepsilon^{\beta_s}t)^{-\gamma} + \varepsilon^2(1+\varepsilon^{\beta_s}t)^{-s} \lesssim \varepsilon^2(1+\varepsilon^{\beta_s}t)^{-s}.$$

We thus get that:

$$\|(n,\nabla\psi,v)(t)\|_{H^3} \lesssim \varepsilon(1+\varepsilon^{\beta_s}t)^{-\frac{s}{2}}.$$

which, by considering $\varepsilon^{\beta_s}t \lesssim 1$ and $\varepsilon^{\beta_s}t \gtrsim 1$ respectively, yields

$$\|(n,\nabla\psi,v)(t)\|_{H^3} \lesssim \min\{\varepsilon, (1+t)^{-\frac{s}{2+s}}\}. \tag{3.5.29}$$

This ends the proof of Theorem 3.1.4. $\square$

**Remark 3.5.6.** *For $M > 3$, as we do not assume the initial data $\|\mathcal{P}u_0^\varepsilon\|_{H^M}$ is small proportional to $\varepsilon$, we do not expect that $\|(n,\nabla\psi,u)\|_{H^M}$has decay like* (3.5.29) *which is independent of $\varepsilon$. However, we could still get some decay in the slow variable "$\varepsilon t$". Defining $g = \exp(-C_{14}\int_0^t(1+\tau)^{-a}\mathrm{d}\tau)\mathcal{E}_M = \exp(-\frac{C_{14}}{a-1}(1+t)^{-(a-1)})\mathcal{E}_M$. We choose again a constant $\gamma$ with condition $s < \gamma < b-1$ multiply* (3.5.28) *by $(1+\varepsilon t)^\gamma$, and integrate then in time, we achieve that:*

$$
\begin{aligned}
&(1+\varepsilon t)^\gamma g + C_{12}\varepsilon\int_0^t (1+\varepsilon\tau)^\gamma g^{1+\frac{1}{s}}(\tau)\mathrm{d}\tau \le g(0) + C_{15}\varepsilon^2 + \gamma\varepsilon\int_0^t (1+\varepsilon\tau)^{\gamma-1}g(\tau)\mathrm{d}\tau\\
&\le g(0) + C_{15}\varepsilon^2 + \frac{C_{12}}{2}\varepsilon\int_0^t (1+\varepsilon\tau)^\gamma g^{1+\frac{1}{s}}(\tau)\mathrm{d}\tau + C_{16}(1+\varepsilon t)^{\gamma-s}
\end{aligned}
$$

*which yields $g \lesssim (1+\varepsilon t)^{-s}$.*

**Remark 3.5.7.** *By* (3.5.27) *and interpolation* $\|v\|_{\dot H^{k+1}} \geq \|v\|_{\dot H^k}^{1+\frac{1}{k+s}} \|u\|_{\dot H^{-s}}^{-\frac{1}{k+s}}$, *we can also have:*

$$\|\nabla^k(n,\nabla\psi,v)\|_{H^{M-k}} \lesssim (1+\varepsilon t)^{-\min\{\frac{k+s}{2},\frac{1}{3}-\}},$$

$$\|\nabla^l(n,\nabla\psi,v)\|_{H^{3-l}} \lesssim \varepsilon(1+\varepsilon^{\beta_{s,l}}t)^{-\min\{\frac{l+s}{2},\frac{1}{3}-\}} \lesssim \min\{\varepsilon,(1+t)^{-\min\{\frac{l+s}{2+l+s},\frac{1}{3}-\}}\}.$$

*where* $k=0,1,2\cdots M-1$, $l=0,1,2$ *and* $\beta_{s,l}=1+\frac{2}{l+s}$.

## 3.6 Remarks on more general pressure laws and viscosity coefficients

In this short section, we will explain briefly how our results can be easily extended to more general pressure laws and viscosity coefficients. Here, we suppose that the pressure $p(\rho^\varepsilon)=\frac{1}{\gamma}(\rho^\varepsilon)^\gamma, \gamma>1$, and $\mu=\mu(\rho^\varepsilon), \lambda=\lambda(\rho^\varepsilon)$ are both density dependent. We assume that $\mu,\lambda$ are smooth functions in the vicinity of 1 and that $\mu(1)>0$, $2\mu(1)+\lambda(1)>0$. As previously , we write $(\rho^\varepsilon,\nabla\phi^\varepsilon,u^\varepsilon)=(1+\varrho,\nabla\phi,u)+(n,\nabla\psi,v)$ where $(1+\varrho,\nabla\phi,u)$ and $(n,\nabla\psi,v)$ satisfy the following two systems respectively:

$$\begin{cases} \partial_t\varrho+\mathrm{div}u=-\mathrm{div}(\varrho u),\\ \partial_t u+u\cdot\nabla u-\varepsilon\mathcal{L}_1 u+\nabla\varrho-\nabla\phi=-\nabla\big(\dfrac{(1+\varrho)^{\gamma-1}}{\gamma-1}-\varrho\big),\\ \Delta\phi=\varrho,\\ u|_{t=0}=\mathcal{P}^\perp u_0^\varepsilon, \varrho|_{t=0}=\varrho_0=\rho_0^\varepsilon-1. \end{cases} \tag{3.6.1}$$

$$\begin{cases} \partial_t n+\mathrm{div}v+\mathrm{div}(\varrho v+nu+nv)=0,\\ \partial_t v+u\cdot\nabla v+v\cdot(\nabla u+\nabla v)-\varepsilon\mathcal{L}_1 v+p'(1+\varrho+n)\nabla n-\nabla\psi=\varepsilon(\dfrac{1}{1+\varrho+n}-1)(\mathcal{L}_{\rho^\varepsilon}v+\mathcal{L}_{\rho^\varepsilon}u)\\ \qquad\qquad +\varepsilon(\mathcal{L}_{\rho^\varepsilon}-\mathcal{L}_1)(u+v)+(p'(1+\varrho+n)-p'(1+\varrho))\nabla\varrho,\\ \Delta\psi=n,\\ v|_{t=0}=\mathcal{P}u_0^\varepsilon, n|_{t=0}=0. \end{cases} \tag{3.6.2}$$

where we denote

$$\mathcal{L}_1(u)=\mu(1)\Delta u+(\mu(1)+\lambda(1))\nabla\mathrm{div}u, \qquad \mathcal{L}_{\rho^\varepsilon}(u)=\mathrm{div}\big(\mu(\rho^\varepsilon)\nabla u\big)+\nabla\big((\mu+\lambda)\big(\rho^\varepsilon)\mathrm{div}u\big)\big).$$

For the system (3.6.1), only minor modifications need to be taken into account due to the extra term $((1+\varrho)^{\gamma-2}-1)\nabla\varrho$. Indeed, on the one hand, the solution is still irrotational as long as it exists. On the other hand, the term $((1+\varrho)^{\gamma-2}-1)\nabla\varrho$ is essentially quadratic term, since we can expand it as $\big((\gamma-2)\varrho+g(\varrho)\big)\nabla\varrho$ where $g(x)$ is smooth when $x>-1$ and $g(0)=g'(0)=0$, therefore we can consider the term $g(\varrho)\nabla\varrho$ as cubic term. In the process of decay estimate for the solutions to (3.1.5), we just have to perform an additional normal form transformation for the quadratic term $(\gamma-2)\varrho\nabla\varrho$, since the term $g(\varrho)\nabla\varrho$ is already cubic. The energy estimates can be obtained in a classical way for general pressure laws.

As for the system (3.6.2), one can still perform energy estimates. The last two extra terms also can be controlled easily. In fact, we can write:

$$\begin{aligned} &\varepsilon(\mathcal{L}_{\rho^\varepsilon}-\mathcal{L}_1)(u+v)\\ =\ &\mu'(1)\Delta(u+v)+(\mu+\lambda)(1)\nabla\mathrm{div}(u+v)+\mathrm{div}\big(h_1(\varrho+n)\nabla(u+v)\big)+\nabla\big(h_2(\varrho+n)\mathrm{div}(u+v)\big),\\ &(p'(1+\varrho+n)-p'(1+\varrho))\nabla\varrho=p''(1)n\nabla\varrho+(h_3(\rho)+h_4(\rho+n))\nabla\varrho \end{aligned}$$

where $h_j(j=1,2,3,4)$ are smooth functions and satisfy: $h(0)=h'(0)=0$. All the terms in the above two identities can be controlled similarly as in Section 5, indeed the new higher order terms have faster decay.

## 3.7 Navier-Stokes-Poisson system for ion dynamics

In this section, we consider the ion dynamic Navier-Stokes-Poisson system (3.1.2). We shall give a sketch of the proof of Theorem 3.1.6.

### 3.7.1 A viscous perturbation of ion Euler-Poisson

Following the global scheme of the proof for the electrons case, we shall first study the following intermediate system which has the property of propagating curl free solutions.

$$\begin{cases} \partial_t \varrho + \mathrm{div} u = -\mathrm{div}(\varrho u), \\ \partial_t u + (u \cdot \nabla) u - 2\varepsilon \Delta u + \nabla \varrho - \nabla \phi = 0, \\ (\Delta - 1)\phi = \varrho, \\ u|_{t=0} = \mathcal{P} u_0^\varepsilon, \varrho|_{t=0} = \varrho_0 = \rho_0^\varepsilon - 1. \end{cases} \tag{3.7.1}$$

We first prove the following result:

**Proposition 3.7.1.** *There exists $\delta_3 > 0$ such that for any family of initial data satisfying*

$$\sup_{\varepsilon \in (0,1]} \Big( \|(\varrho_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{W^{\sigma + \frac{9}{4}(1+\kappa), 8'_\kappa}} + \||\nabla|^{-1}(\varrho_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)\|_{H^N} \Big) \le \delta_3$$

*with $\sigma \ge 5, N \ge 2\sigma + 1$, $8_\kappa = \frac{8}{1-3\kappa}, \kappa = \frac{1}{200}$. Then for every $\varepsilon \in (0,1]$, there exist a unique solution for system* (3.7.1) *in $C([0,\infty), H^N)$. Moreover, there exists a constant $C > 0$ such that for every $\varepsilon \in (0,1]$, we have the estimate*

$$\|(\varrho, \nabla \phi, u)(t)\|_{W^{\sigma, 8_\kappa}} \le C\delta_3 (1+t)^{-(1+\kappa)}, \quad \forall t \ge 0.$$

**Remark 3.7.2.** *The choice of the $L^p$ type exponent $8_\kappa$ in the above result comes from a constraint in order to get continuous properties of the bilinear operators used in the normal form transformation and the slow decay of viscous term. More explanations will be given after Proposition 3.7.6.*

Let $h = \sqrt{1 + (1-\Delta)^{-1}}\varrho$, $c = \frac{\mathrm{div}}{|\nabla|} u$, then we get as a counterpart of (3.3.1),

$$\begin{cases} \partial_t h + p(|\nabla|) c = q(|\nabla|) \mathrm{div}\big( \dfrac{h}{\sqrt{1 + (1-\Delta)^{-1}}} \cdot \mathcal{R} c \big), \\ \partial_t c - p(|\nabla|) h - 2\varepsilon \Delta c = |\nabla| |\mathcal{R} c|^2, \\ h|_{t=0} = \dfrac{1}{\sqrt{1 + (1-\Delta)^{-1}}} \varrho_0, c|_{t=0} = \mathcal{R}^* u_0. \end{cases}$$

where $p(|\nabla|) = |\nabla|\sqrt{1 + (1-\Delta)^{-1}}$. We note that we still have (3.3.2) – (3.3.3) by replacing $\langle \nabla \rangle$ with $p(|\nabla|)$. We will analyze the high and low frequency separately as before. As for high frequency, similar arguments as in Lemma 3.3.5 show that the smoothing effect of $\chi^H e^{-tA}$ is still true. We now focus on the low frequency. To start, we need to analysis $b(r) = \sqrt{p(r)^2 - (\varepsilon r^2)^2}$ precisely.

**Lemma 3.7.3.** *Suppose $0 < \varepsilon \le 1$, $\kappa_0$ small enough, then on the region $\{\varepsilon r^2 \le 2\kappa_0\}$, $b(r)$ satisfies the following property:*
*1. $b'(r) \ge c_1(\kappa_0) > 0$,*
*2. $b''(r)$ only have one zero point $r_0^{\varepsilon,\kappa_0}$ and $1 \le r_0^{\varepsilon,\kappa_0} \le 10$,*
*3. There exists a small interval $[r_0^{\varepsilon,\kappa_0} - \iota, r_0^{\varepsilon,\kappa_0} + \iota]$ st. $b'''(r) \ge c_2 > 0$, where $c_2$ is a small constant independent of $\varepsilon$.*

*Proof.*

$$b'(r) = \sqrt{\frac{2+r^2}{1+r^2} - \varepsilon^2 r^2} - r \frac{\frac{r}{(1+r^2)^2} + \varepsilon^2 r}{\sqrt{\frac{2+r^2}{1+r^2} - \varepsilon^2 r^2}} = \frac{1 + \frac{1}{(1+r^2)^2} - 2\varepsilon^2 r^2}{\sqrt{1 + \frac{1}{1+r^2} - \varepsilon^2 r^2}} \ge \frac{1}{2\sqrt{2}}$$

on the support of $\chi_{\varepsilon,\kappa_0}(\xi)$ if $4\varepsilon\kappa_0 \le 4\kappa_0 \le \frac{1}{2}$.

2. After direct computations, we have that:

$$b''(r) = (1+r^2)^{-4}(\frac{2+r^2}{1+r^2} - \varepsilon^2 r^2)^{-\frac{3}{2}} r\{[1-(5-8\varepsilon^2)\varepsilon^2 r^4]r^4 \\ -[2-2\varepsilon^4 r^8 + (22-12\varepsilon^2)\varepsilon^2 r^4]r^2 - [6+(31-8\varepsilon^2)\varepsilon^2 r^4 + (20\varepsilon - 2\varepsilon^3)\varepsilon r^2 + 6\varepsilon^2]\}.$$

Note that if $\varepsilon \leq 1$ and $\kappa_0$ is small enough, then on the region $\{\varepsilon r^2 \leq 2\kappa_0\}$, we have that for $\kappa_0$ sufficiently small, the polynomial in the bracket is a small perturbation of $r^4 - 2r^2 - 6(1+\varepsilon^2)$ which has only one real simple positive root that is uniformly in $[2, 9]$. Therefore, for $\kappa_0$ small enough $b''$ has only one nonnegative zero which is uniformly for $\varepsilon \in (0, 1]$ in $[1, 10]$.

3. For simplicity, we write $r_0 = r_0^{\varepsilon,\kappa_0}$. One can check that:

$$\begin{aligned} b'''(r_0) &= (1+r_0^2)^{-4}(\frac{2+r_0^2}{1+r_0^2} - \varepsilon^2 r_0^2)^{-\frac{3}{2}}\{[4-8(5-8\varepsilon^2)\varepsilon^2 r_0^4]r_0^4 \\ &\qquad -[4-20\varepsilon^4 r_0^8 + 6(22-12\varepsilon^2)\varepsilon^2 r_0^4]r_0^2 - [4(31-8\varepsilon^2)\varepsilon^2 r_0^2 - 2\varepsilon r_0^2]\} \\ &\triangleq (1+r_0^2)^{-4}(\frac{2+r_0^2}{1+r_0^2} - \varepsilon^2 r_0^2)^{-\frac{3}{2}}(a_1 r_0^4 + a_2 r_0^2 - a_3) \end{aligned}$$

Notice that when $\kappa_0$ is very close to 0, $a_1$ and $a_2$ are very close to 4 while $a_3$ is very close to 0. Therefore, as long as $\kappa_0$ is small enough, there exist constants $\iota, c_2$ which are independent of $\varepsilon$, st. $b'''(r) \geq c_2 > 0$ on the interval $[r_0 - \iota, r_0 + \iota]$. □

This lemma in hand, we could keep track of the proof of Lemma 3.1-3.3 of [58] to get:

**Lemma 3.7.4.** *Suppose $\kappa_0$ satisfy the assumptions of the Lemma 3.7.3, then*

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)f\|_{L^\infty} \lesssim_{\kappa_0} (1+|t|)^{-\frac{4}{3}}\|f\|_{W^{3,1}}, \quad \forall t \in \mathbb{R}$$

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)f\|_{L^p} \lesssim_{\kappa_0} (1+|t|)^{-\frac{4}{3}(1-\frac{2}{p})}\|f\|_{W^{3(1-\frac{2}{p}),p'}}, \quad \forall t \in \mathbb{R}.$$

We omit the proof, since thanks to the above properties of $b$, it follows exactly the same lines as in [58] in the same way as the proof of Lemma 3.3.1 was following the proof for the classical Klein-Gordon equation. Note again that the above estimates are independent of $\varepsilon$.

To treat low frequencies, we need also to get some continous property of $T_{m/\phi_{j,k}}$ on $L^p$.

**Lemma 3.7.5.** *Bilinear estimate: Define $\phi_{j,k}(\xi,\eta) = (-1)^{j+1}b(\xi) + (-1)^{k+1}b(\eta) - b(\xi+\eta)$, $j, k = 1, 2$*

$$m(\xi,\eta) = \tilde{\chi}_{\varepsilon,\kappa_0}(\xi)\tilde{\chi}_{\varepsilon,\kappa_0}(\eta)\tilde{\chi}_{\varepsilon,\kappa_0}(\xi-\eta)|\xi|n_1(\xi)n_2(\xi-\eta)n_3(\eta).$$

*where $n_1, n_2, n_3$ are homogeneous-0 functions whose corresponding multiplier is bounded in $L^p(1 < p < +\infty)$. By choosing $\kappa_0$ smaller if necessary, we have similar results as in Proposition 6.1 in [58]: ie.*

$$\|T_{\frac{m}{\phi_{j,k}}}(f,g)\|_{W^{\sigma,p'}} \lesssim_{\kappa_0} \||\nabla|^{-1}f\|_{H^{\sigma+\lambda}}\||\nabla|^{-1}g\|_{W^{\lambda,r}} + \||\nabla|^{-1}f\|_{W^{\lambda,r}}\||\nabla|^{-1}g\|_{H^{\sigma+\lambda}}. \tag{3.7.2}$$

*where $\lambda \geq \frac{9}{4} + \kappa$, and $\frac{1}{r} + \frac{1}{p} = 1 - \frac{\frac{5}{4}-\kappa}{3}$, $2 \leq p, r \leq \frac{12}{1+4\kappa}$, $\kappa$ can be chosen very small.*

*Proof.* This Lemma is a consequence of the next proposition along with Theorem 6.1 of [58] which states that if $\mathcal{M}(\xi,\eta)$ satisfies $\|\mathcal{M}\|_{L^\infty_\xi \dot{H}^s_\eta} + \|\mathcal{M}\|_{L^\infty_\eta \dot{H}^s_\xi} < \infty$, then

$$\|T_{\mathcal{M}}(f,g)\|_{L^{p'}} \lesssim \|g\|_{L^2}\|f\|_{L^r},$$

where $\frac{1}{r} + \frac{1}{p} = 1 - \frac{s}{3}, 2 \leq p, r \leq \frac{6}{3-2s}$. □

**Proposition 3.7.6.** *Define $\mathcal{M}_{jk}(\xi,\eta) = \frac{\langle\xi\rangle^\sigma|\xi||\eta||\xi-\eta|}{\phi_{jk}\langle\xi-\eta\rangle^{\lambda+\sigma}\langle\eta\rangle^\lambda}\Phi(\frac{|\eta|}{|\xi-\eta|})\tilde{\chi}_{\varepsilon,\kappa_0}(\xi)\tilde{\chi}_{\varepsilon,\kappa_0}(\eta)\tilde{\chi}_{\varepsilon,\kappa_0}(\xi-\eta)$ where $\Phi \in C_c^\infty(\mathbb{R})$ is supported in $B_2(\mathbb{R})$, then for any $\kappa > 0$, if $\lambda > \frac{9}{4} + \kappa$, then the following estimate holds:*

$$\|\mathcal{M}_{jk}\|_{L^\infty_\xi \dot{H}^{\frac{5}{4}-\kappa}_\eta} + \|\mathcal{M}_{jk}\|_{L^\infty_\eta \dot{H}^{\frac{5}{4}-\kappa}_\xi} \lesssim_\kappa 1.$$

*Proof.* For the proof of this proposition, we can adapt the proof of Proposition 6.1 in [58]. We only explain for the case $\phi_{11}$ as other cases could be obtained by symmetry. We split $\mathbb{R}^3$ into three regions $\{|\eta| < \frac{1}{3}|\xi|\}$, $\{|\xi| < \frac{1}{3}|\eta|\}$ and $\{\frac{1}{4} \leq \frac{|\xi|}{|\eta|} \leq 4\}$. For example, on the region $\{|\eta| < \frac{1}{3}|\xi|\}$, to estimate $\|\mathcal{M}_{11}\|_{L^\infty_\eta \dot{H}^s_\xi}$, one first fix $\eta$ and compute the $\|\varphi_l(\xi)\mathcal{M}_{11}\|_{\dot{H}^s_\xi}$ norm by interpolation between $\|\varphi_l(\xi)\mathcal{M}_{11}\|_{L^2}$ and $\|\varphi_l(\xi)\mathcal{M}_{11}\|_{\dot{H}^2_\xi}$ for any $l$ (recall $\phi_l$ is $l$-th dyadic function), and find the optimal number $s$ (which finally turns out to be $\frac{5}{4}-$) such that it is summable for $l$ uniformly for $\eta$. In light of this strategy, one sees that the main ingredients are the elementary estimates for $\phi_{11}$. We list briefly the properties needed for $\phi_{11}$ which are essentially the same as Lemma 6.3 and Lemma 6.4 of [58].

1. Lower boundedness of $\phi_{11}$.

If $|\xi| \leq \min\{|\xi-\eta|, |\eta|\}$, then $|\phi_{11}(\xi,\eta)| = |b(\xi-\eta)+b(\eta)-b(\xi)| \geq_{\kappa_0} \max\{|\xi-\eta|, |\eta|\}$;

if $|\xi|$ is not smallest, for example, $|\eta| \leq \min\{|\xi-\eta|, |\xi|\}$, then $|\phi_{11}(\xi,\eta)| \gtrsim_{\kappa_0} \frac{|\xi||\eta||\xi-\eta|}{\langle\eta\rangle^2\langle\xi-\eta\rangle\langle\xi\rangle} + |\eta|(1-\cos\beta+1-\cos\theta)$. where $\beta, \theta$ are the angle between $\eta$ and $\xi-\eta$, $\eta$ and $\xi$ respectively.

2. The first and second derivative for $\phi_{11}$ can be estimated as

$$|\partial_\xi\phi_{11}| \lesssim_{\kappa_0} \frac{|\eta|}{\langle\max\{|\xi-\eta|, |\xi|\}\rangle\langle\min\{|\xi-\eta|, |\xi|\}\rangle^2} + 2|\sin\frac{\gamma}{2}|,$$
$$|\partial_\eta\phi_{11}| \lesssim_{\kappa_0} \frac{|\xi|}{\langle\max\{|\xi-\eta|, |\xi|\}\rangle\langle\min\{|\xi-\eta|, |\xi|\}\rangle^2} + 2|\sin\frac{\beta}{2}|,$$
$$|\Delta_\xi\phi_{11}(\xi,\eta)| \lesssim_{\kappa_0} \frac{|\eta|}{|\xi-\eta||\xi|}, \qquad |\Delta_\eta\phi_{11}(\xi,\eta)| \lesssim_{\kappa_0} \frac{1}{\min\{|\xi-\eta|, |\eta|\}}.$$

where $\gamma$ denotes the angle between $\xi$ and $\xi-\eta$.

Nevertheless, as in [58], all the information needed for $b(r)$ and $q(r) = \frac{1}{r}b(r) = \sqrt{\frac{2+r^2}{1+r^2} - \varepsilon^2 r^2}$ to prove the above two properties are the following facts which are consistent with the case $\varepsilon = 0$.

(1) $b''(r) \lesssim_{\kappa_0} 1$,

(2) if $\kappa_0$ is sufficient small, one still has that there exists two constants $K_1, K_2$ which are independent of $\varepsilon \in (0,1]$, st.

$$-q'(r) \approx \frac{1}{r}, \quad when \quad r \leq K_1,$$
$$-q'(r) \approx \frac{1}{r^3}, b''(r) \approx \frac{1}{r^3}, b'''(r) \approx \frac{1}{r^4} \quad when \quad r \geq K_2.$$

Since the above two facts are easy to see, we omit the proof. □

From now on, we fix $\kappa_0$ such that Lemma 4.8.6(1-3), Lemma 3.7.4 and Lemma 3.7.5 holds.

In view of Lemma 3.7.4, 3.7.5, we can only expect to get decay estimates in some $L^p$ framework with $8 < p < 12$ (due to the appearance of 'time resonances', we can only perform the normal form transformation one time). To overcome the difficulty that $\|\frac{\varepsilon\Delta}{|\nabla|}\chi_{\varepsilon,\kappa_0}(D)R\|_{L^2}$ decays only like $(1+t)^{-1}$, we need to use a 'slow' decay estimate for $\||\nabla|^{-1}R\|_{L^r}$ where $r$ is larger but close to 2. By Lemma 3.7.5, if we choose $p$ larger, we need to estimate $\||\nabla|^{-1}R\|_{L^r}$ for a smaller $r$ which obviously has slower decay. Therefore, to close our decay estimate, we need to choose $p$ small, this is why we choose $p = 8_\kappa$, where $\frac{1}{8_\kappa} = \frac{1}{8} - \frac{3\kappa}{8}$. By this choice, we have that:

$$\|e^{itb(D)}\chi_{\varepsilon,\kappa_0}(D)f\|_{L^{8_\kappa}} \lesssim (1+t)^{-(1+\kappa)}\|f\|_{W^{\frac{9}{4}(1+\kappa),8'_\kappa}}.$$

*Proof of Proposition 3.7.1.* We shall use the norm:

$$\|V\|_{X_T} \triangleq (1+t)^{-(1+\kappa)}\|V\|_{W^{\sigma,8_\kappa}} + (1+t)^{-(1+\kappa)}\|(1-\chi_{\varepsilon,\kappa_0})(D)V\|_{H^{N-2}} + \||\nabla|^{-1}V\|_{H^N},$$

$$\|f\|_Y \triangleq \|f\|_{W^{\sigma+\frac{9(1+\kappa)}{4},8'_\kappa}} + \||\nabla|^{-1}f\|_{H^N}.$$

where $\sigma \geq 5, N \geq 2\sigma + 1$.
Global existence for $(\rho, u)$ follows if we prove the a priori estimate:

$$\|V\|_{X_T} \lesssim \|V_0\|_Y + \|V\|_{X_T}^{\frac{3}{2}} + \|V\|_{X_T}^2 + \|V\|_{X_T}^3. \tag{3.7.3}$$

Sketch of the proof of (3.7.3):
1. The bound for $H^N$ norm. We can perform energy estimates in the same way as in Proposition 3.4.1. One only needs to change the norm a little bit by

$$E_N = \sum_{|\alpha| \leq N} E_\alpha = \sum_{|\alpha| \leq N} \int \frac{|\partial^\alpha \varrho|^2}{2} + \frac{|\partial^\alpha \langle \nabla \rangle \phi|^2}{2} + \rho \frac{|\partial^\alpha u|^2}{2} \mathrm{d}x.$$

2. The bound for $H^{-1}$ norm.
It can easily be seen that the nonlinear terms are under the form $B_l(V, V) = \sum |\nabla| n_1(D) \big(n_2(D) V n_3(D) V\big)$ where $n_1(D), n_2(D)$, $n_3(D)$ are $L^p (1 < p < +\infty)$ multipliers. So by Duhamel's principle, tame estimates and Sobolev embedding, we have:

$$\begin{aligned}\||\nabla|^{-1} V\|_{L^2} &\lesssim \||\nabla|^{-1} V_0\|_{L^2} + \int_0^t \||\nabla|^{-1} B(V, V)\|_{L^2} \mathrm{d}s \lesssim \||\nabla|^{-1} V_0\|_{L^2} + \int_0^t \|V\|_{L^2} \|V\|_{W^{\sigma, 8_\kappa}} \mathrm{d}s \\ &\lesssim \||\nabla|^{-1} V_0\|_{L^2} + \|V\|_{X_T}^2.\end{aligned}$$

3. Estimates of $\|\chi^H V\|_{H^{N-2}}$ and $\|\chi^H V\|_{W^{\sigma, 8_\kappa}}$ can be performed in the same fashion as in the electron case, we thus skip them.
4. Estimate of $\|\chi_{\varepsilon, \kappa_0}(D) V\|_{W^{\sigma, 8_\kappa}}$. For clarity, we will use the same notation as in the electron case. More precisely, we set $R = Q^{-1} \chi_{\varepsilon, \kappa_0}(D) V = \sum_{k=1}^4 J_k$ where $J_1 - J_4$ are defined in (3.4.5). Nevertheless, $J_1, J_2, J_3$ can be easily estimated using the Kato-Ponce inequality (Lemma 3.8.1), we thus omit their estimate.

Now, it remains to estimate the typical term of $J_4$: $G_{jk} = \sum_{j=1}^7 I_j$, which is defined in the same way as in the electron case (with slightly adaptation of multiplier m and $n_j$), see (3.4.6). We need to prove that

$$\|G_{j,k}\|_{W^{\sigma, 8_\kappa}} \lesssim_{\kappa_0} (1 + t)^{-(1+\kappa)} \|V\|_{X_T}.$$

In the above decomposition, $I_1, I_2, I_5, I_7$ correspond to boundary terms and cubic terms, which have essentially been treated in [58] where the authors proved global existence for the ions Euler-Poisson system. Note that in [58], the authors proved $L^{10}$ decay estimate. Nevertheless, it is the same to prove decay in $L^{8_\kappa}$ framework, we leave the details. We will only detail the estimate of the 'viscous term' $I_4$, since $I_6$ is 'symmetric' term and the estimate for $I_3$ can be reduced to that for $I_4$.

To start, we prove the following two claims:
Claim 1:

$$\||\nabla|^{-1} R(t)\|_{W^{\lambda, r}} \lesssim (1 + t)^{-\kappa} \|V\|_{X_T},$$

where $\frac{1}{r} = \frac{11+17\kappa}{24}$, $\lambda = \frac{9}{4} + \kappa$.
Claim 2:

$$\begin{aligned}\|\frac{\varepsilon \Delta}{|\nabla|} R\|_{H^{N-1}} &\lesssim (1 + t)^{-1} (\||\nabla|^{-1} V_0\|_{H^{N-1}} + \|V\|_{X_T}^2), \\ \|\frac{\varepsilon \Delta}{|\nabla|} R\|_{W^{\lambda, r}} &\lesssim (1 + t)^{-(1+\kappa)} (\|V_0\|_Y + \|V\|_{X_T}^2).\end{aligned}$$

**Proof of Claim 1:** By interpolation, we have

$$\||\nabla|^{-1} R\|_{W^{\lambda, 8_\kappa}} \lesssim \|R\|_{W^{\lambda, \frac{24}{11-9\kappa}}} \lesssim \|R\|_{W^{\lambda, 8_\kappa}}^\theta \|R\|_{H^\lambda}^{1-\theta} \lesssim (1 + t)^{-\frac{1+9\kappa}{9}} \|V\|_{X_T}.$$

where $\theta = \frac{1+9\kappa}{9(1+\kappa)}$. Claim 1 follows from another interpolation, that is:

$$\||\nabla|^{-1} R(t)\|_{W^{\lambda, r}} \lesssim \||\nabla|^{-1} R(t)\|_{W^{\lambda, 8_\kappa}}^\vartheta \||\nabla|^{-1} R(t)\|_{H^\lambda}^{1-\vartheta} \lesssim (1 + t)^{-\frac{1}{100}} \|V\|_{X_T}.$$

where $\vartheta = \frac{1-17\kappa}{9(1+\kappa)}$ and $\frac{1-17\kappa}{9(1+\kappa)}\frac{1+9\kappa}{9} \geq \frac{1}{100}$ if we choose $\kappa$ small enough, say $\kappa \leq \frac{1}{200}$.
**Proof of Claim 2:** The first inequality can be proved like Lemma 3.4.4, we thus do not detail it. For the second inequality, we have by the decay estimate (3.7.4) and the Kato-Ponce inequality (Lemma 3.8.1) that

$$\begin{aligned}
\|\frac{\varepsilon\Delta}{|\nabla|}R\|_{W^{\lambda,r}} &\lesssim \|\begin{pmatrix} e^{\lambda_-(D)t} & 0 \\ 0 & e^{\lambda_+(D)t}\end{pmatrix}\frac{\varepsilon\Delta}{|\nabla|}R_0\|_{W^{\lambda,r}} \\
&\quad + \int_0^t \|\begin{pmatrix} e^{\lambda_-(D)(t-s)} & 0 \\ 0 & e^{\lambda_+(D)(t-s)}\end{pmatrix}\frac{\varepsilon\Delta}{|\nabla|}Q^{-1}\chi_{\varepsilon,\kappa_0}(D)B(V,V)\|_{W^{\lambda,r}}\mathrm{d}s \\
&\lesssim (1+t)^{-\frac{10-17\kappa}{9}}\||\nabla|^{-1}V_0\|_{W^{\lambda+3(1-\frac{2}{r}),r'}} + \int_0^t (1+t-s)^{-\frac{10-17\kappa}{9}}\||\nabla|^{-1}B(V,V)\|_{W^{\lambda+3(1-\frac{2}{r}),r'}}\mathrm{d}s \\
&\lesssim (1+t)^{-\frac{10-17\kappa}{9}}\||\nabla|^{-1}V_0\|_{W^{\sigma,r'}} + \int_0^t (1+t-s)^{-\frac{10-17\kappa}{9}}\|V\|_{H^\sigma}\|V\|_{W^{\lambda+3(1-\frac{2}{r}),\frac{24}{1-17\kappa}}}\mathrm{d}s \\
&\lesssim (1+t)^{-(1+\kappa)}(\||\nabla|^{-1}V_0\|_{W^{\sigma,r'}} + \|V\|^2_{X_T}) \lesssim (1+t)^{-(1+\kappa)}\|V_0\|_Y + \|V\|^2_{X_T}),
\end{aligned}$$

where $r' = \frac{24}{13-17\kappa}$. Note that $\lambda + 3(1-\frac{2}{r}) \leq \sigma - 1$. In the last inequality, we used the fact $\frac{1}{r'} + \frac{1}{3} < \frac{1}{8'_\kappa}$ and interpolation to get:

$$\||\nabla|^{-1}V_0\|_{W^{\sigma,r'}} \lesssim \|V_0\|_{W^{\sigma,\frac{24}{21-17\kappa}}} \lesssim \|V_0\|_Y.$$

These two claims, combine with the bilinear estimate (3.7.2), we can estimate

$$I_4 = -i\int_0^t e^{\varepsilon(t-s)\Delta}e^{i(t-s)b(D)}\chi_{\varepsilon,\kappa_0}(D)T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde{r}_j, \tilde{r}_k)$$

as follows:

$$\begin{aligned}
\|I_4\|_{W^{\sigma,8_\kappa}} &\lesssim \int_0^t (1+t-s)^{-(1+\kappa)}\|T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde{r}_j,\tilde{r}_k)\|_{W^{\sigma+\frac{9(1+\kappa)}{4},8'_\kappa}}\mathrm{d}s \\
&\lesssim \int_0^t (1+t-s)^{-(1+\kappa)}(\|\frac{\varepsilon\Delta}{|\nabla|}\tilde{r}_j\|_{H^{\sigma+\lambda+\frac{9(1+\kappa)}{4}}}\||\nabla|^{-1}\tilde{r}_k\|_{W^{\lambda,r}} + \|\frac{\varepsilon\Delta}{|\nabla|}\tilde{r}_j\|_{W^{\lambda,r}}\||\nabla|^{-1}\tilde{r}_k\|_{H^{\sigma+\lambda+\frac{9(1+\kappa)}{4}}})\mathrm{d}s \\
&\lesssim \int_0^t (1+t-s)^{-(1+\kappa)}(1+s)^{-(1+\kappa)}\|V\|^2_{X_T}\mathrm{d}s \lesssim (1+t)^{-(1+\kappa)}\|V\|^2_{X_T}.
\end{aligned}$$

For the estimate of $I_3$, we use the identity

$$\varepsilon\Delta T_{\frac{m}{\phi_{jk}}}(\tilde{r},\tilde{r}) = T_{\frac{m}{\phi_{jk}}}(\varepsilon\Delta\tilde{r},\tilde{r}) + 2\sum_{l=1}^{3}T_{\frac{m}{\phi_{jk}}}(\varepsilon^{\frac{1}{2}}\partial_l\tilde{r},\varepsilon^{\frac{1}{2}}\partial_l\tilde{r})$$

and the following inequalities whose proofs are similar to that of Claim 2.

$$\begin{aligned}
&\|\frac{\varepsilon^{\frac{1}{2}}\nabla}{|\nabla|}R\|_{H^{N-1}} \lesssim (1+t)^{-\frac{1}{2}}(\|V_0\|_Y + \|V\|^2_{X_T}), \\
&\|\frac{\varepsilon^{\frac{1}{2}}\nabla}{|\nabla|}R\|_{w^{\lambda,r}} \lesssim (1+t)^{-(\frac{11-34\kappa}{18})}(\|V_0\|_Y + \|V\|^2_{X_T}).
\end{aligned}$$

This ends the proof of a priori estimate 3.7.3. □

### 3.7.2 Perturbing the ion Navier-Stokes-Poisson by the solution of (3.7.1)

As before, we consider now the following system:

$$\begin{cases}
\partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\
\partial_t v + u\cdot\nabla v + v\cdot(\nabla u + \nabla v) - \varepsilon\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\frac{1}{\rho+n} - 1)(\mathcal{L}v + \mathcal{L}u), \\
\Delta\psi - \psi = n \\
v|_{t=0} = \mathcal{P}u^\varepsilon_{+0}, n|_{t=0} = 0.
\end{cases} \tag{3.7.4}$$

then $(\rho_+^\varepsilon, u_+^\varepsilon, \phi_+^\varepsilon) = (n, \psi, v) + (\rho, u, \phi)$.

We define the energy functional similar to (3.5.2):

$$\mathcal{E}_M(n, u, \nabla\psi) = \sum_{|\alpha|\le M} \mathscr{E}_\alpha = \sum_{|\alpha|\le M} \frac{1}{2}\int \rho|\partial^\alpha v|^2 + |\partial^\alpha n|^2 + |\partial^\alpha \langle\nabla\rangle\psi|^2 \mathrm{d}x.$$

We can derive similar energy estimates as in the electron case by using almost the same computations as in Lemma 3.5.1 and Lemma 3.5.2. In fact, one can check that (3.5.4) in Lemma 3.5.1 do not change, while (3.5.17) in Lemma 3.5.2 is changed by replacing $\int |\partial^\alpha n|^2 \mathrm{d}x$ by $\int |\partial^\alpha \psi|^2 + |\partial^\alpha \Delta\psi|^2 \mathrm{d}x$. We finally get the following a priori estimate: if we have $\|(u, \varrho)\|_X \lesssim \delta, \mathcal{E}_3 \lesssim \delta\varepsilon$ for some $\delta$ sufficiently small independent of $\varepsilon$, then we have uniformly in $\varepsilon$:

$$\mathcal{E}_3(t) \lesssim \mathcal{E}_3(0) + \int_0^t (1+s)^{-(1+\kappa)}(\delta\mathcal{E}_3(s) + \varepsilon^2\delta^3)\mathrm{d}s.$$

Global existence for system (3.7.4) follows again by Grönwall's inequality and bootstrap arguments. The decay estimate follows in the similar way as that in electron case, the only difference is now that it is the $L^8$ norm rather than the $L^6$ has the critical decay $(1+t)^{-1}$.

## 3.8 Appendix

We first recall two classical estimates:

**Lemma 3.8.1** (Kato-Ponce inequality)**.** *Given real number $s > 0$,two functions $f, g$, we have:*

$$\|fg\|_{W^{s,q}} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{r_1}} + \|f\|_{L^{r_2}}\|g\|_{W^{s,p_2}}, \tag{3.8.1}$$

$$\|fg\|_{\dot{W}^{s,q}} \lesssim \|f\|_{\dot{W}^{s,p_1}}\|g\|_{L^{r_1}} + \|f\|_{L^{r_2}}\|g\|_{\dot{W}^{s,p_2}} \tag{3.8.2}$$

*where $\frac{1}{p_j} + \frac{1}{r_j} = \frac{1}{q}$, $q \le p_j < +\infty$, $q < r_j \le +\infty$.*

**Lemma 3.8.2.** *Suppose $F$: $\mathbb{R} \to \mathbb{R}$ is a smooth function with the condition $F(0) = 0$. Then for any function $u$ that belongs to $L^\infty \cap W^{k,p}$ ($k \ge 0$ is an integer and $1 \le p \le +\infty$), we have:*

$$\|F(u)\|_{\dot{W}^{k,p}} \lesssim C(\|u\|_{L^\infty})\|u\|_{\dot{W}^{k,p}}. \tag{3.8.3}$$

*Proof.* For $k = 0$, we Taylor expand $F$ at first order. For $k > 0$, we use the Gagliardo-Nirenberg interpolation inequality. Indeed, for any $|\alpha| = k > 0$, we have:

$$\partial^\alpha F(u) = \sum F^{(l)}(u)\partial^{\alpha_1}u\partial^{\alpha_2}u\cdots\partial^{\alpha_l}u.$$

where $\alpha_1 + \alpha_2 + \cdots \alpha_l = \alpha$ and by using

$$\|u\|_{\dot{W}^{|\alpha_j|,p_j}} \lesssim \|u\|_{L^\infty}^{1-\frac{|\alpha_j|}{k}}\|u\|_{\dot{W}^{k,p}}^{\frac{|\alpha_j|}{k}}.$$

where $p_j|\alpha_j| = kp$. The result follows from the Hölder inequality. □

At last, we present the proof of the bilinear estimate stated in Lemma 3.3.13. We first give a proposition which shows that $\frac{m}{\phi_{jk}}$ has the same properties as the Klein-Gordon phase $\frac{1}{\pm\langle\xi\rangle\pm\langle\eta\rangle-\langle\xi+\eta\rangle}$ ([69][84]) as long as the threshold $\kappa_0$ is small enough.

**Proposition 3.8.3.** *Let $m$ and $\phi_{jk}$ defined in 3.3.13, if $\kappa_0$ is small enough then for any multi-index $\alpha, \beta \in \mathbb{N}^3$, we have the following estimate uniformly in $\varepsilon \in (0, 1]$:*

$$|\partial_\xi^\alpha\partial_\eta^\beta \frac{m}{\phi_{jk}}(\xi-\eta, \eta)| \lesssim_{\alpha,\beta,\kappa_0} \min\{\langle\xi\rangle, \langle\eta\rangle, \langle\xi-\eta\rangle\},$$

$$|\partial_\xi^\alpha\partial_\eta^\beta \frac{m}{\phi_{jk}^2}(\xi-\eta, \eta)| \lesssim_{\alpha,\beta,\kappa_0} \min\{\langle\xi\rangle^2, \langle\eta\rangle^2, \langle\xi-\eta\rangle^2\}.$$

We postpone the proof of this proposition and prove firstly Lemma 3.3.13.

*Proof of Lemma 3.3.13.* We choose two smooth functions $\psi_1, \psi_2 \in C_b^\infty(\mathbb{R}^6)$ which satisfy the following conditions:

$$\begin{cases} \psi_1 + \psi_2 = 1 \qquad \forall(\xi,\eta), \\ \operatorname{Supp}\psi_1 \subset \{(\xi,\eta) \big| \langle \xi-\eta\rangle \geq \dfrac{\langle\eta\rangle}{2}\}, \\ \operatorname{Supp}\psi_2 \subset \{(\xi,\eta)\big|\langle\eta\rangle > \langle\xi-\eta\rangle\}. \end{cases}$$

And write

$$\begin{aligned} \langle\xi\rangle^\sigma \frac{m}{\phi_{jk}}(\xi-\eta,\eta) &= \frac{m\psi_1(\xi-\eta,\eta)\langle\xi\rangle^\sigma}{\phi_{jk}\langle\xi-\eta\rangle^{\sigma+2+}\langle\eta\rangle^2}\langle\xi-\eta\rangle^{2+}\langle\eta\rangle^2 + \frac{m\psi_2(\xi-\eta,\eta)\langle\xi\rangle^\sigma}{\phi_{jk}\langle\eta\rangle^{\sigma+2+}\langle\xi-\eta\rangle^2}\langle\eta\rangle^{2+}\langle\xi-\eta\rangle^2 \\ &\triangleq M_1(\xi-\eta,\eta)\langle\xi-\eta\rangle^{2+}\langle\eta\rangle^2 + M_2(\xi-\eta,\eta)\langle\eta\rangle^{2+}\langle\xi-\eta\rangle^2. \end{aligned}$$

By Proposition 3.8.3, we have for any $\alpha,\beta\in\mathbb{N}^3$ with $|\alpha|+|\beta|\leq 4$,

$$|\partial_\xi^\alpha\partial_\eta^\beta M_1| \leq 1_{\langle\xi-\eta\rangle\geq\frac{\langle\eta\rangle}{2}}\langle\xi-\eta\rangle^{-2+}\langle\eta\rangle^{-1}.$$

In particular, we have proved that: $M_1, \partial_\xi^4 M_1, \partial_\eta^4 M_1 \in L^2(\mathbb{R}^6)$. So we get that $\mathcal{F}^{-1}(M_1)(x,y)\in L^1_{x,y}$, as

$$\|\mathcal{F}^{-1}(M_1)(x,y)\|_{L^1_{x,y}} \lesssim \|(1+|x|^4+|y|^4)^{-1}\|_{L^2_{x,y}}(\|M_1\|_{L^2}+\|\partial_\xi^4 M_1\|_{L^2}+\|\partial_\eta^4 M_1\|_{L^2}).$$

By using the definition of the bilinear operator $T_m$ (3.2.1) and properties of the Fourier transform, we can write:

$$T_{M_1\langle\xi\rangle^{\sigma+2+}\langle\eta\rangle^2}(f,g) = \int(\mathcal{F}^{-1}M_1)(x',y')(\langle D_x\rangle^{\sigma+2+}f)(x-x')\langle D_x\rangle^2 g(x-y')\mathrm{d}x'\mathrm{d}y',$$

thus by the Minkowski's inequality, we have:

$$\begin{aligned} \|T_{M_1\langle\xi\rangle^{\sigma+2+}\langle\eta\rangle^2}(f,g)\|_{L^p} &\leq \int\|\langle D_x\rangle^2 g\|_{L^{r_1}}\|\int(\mathcal{F}^{-1}M_1)(x',y')(\langle D_x\rangle^{\sigma+2+}f)(x-x')\mathrm{d}x'\|_{L^{p_1}}\mathrm{d}y' \\ &\leq \|\mathcal{F}^{-1}M_1\|_{L^1_{x,y}}\|f\|_{W^{\sigma+2+,p_1}}\|g\|_{W^{2,r_1}}. \end{aligned}$$

The similar result for $M_2$ can be derived in the same fashion. □

*Proof of Proposition 3.8.3.* We only prove the estimate of $\frac{m}{\phi_{11}}$, the ones of $\frac{m}{\phi_{12}}, \frac{m}{\phi_{21}}$ can be obtained by symmetry, $\frac{m}{\phi_{22}}$ is easier. At first, we have

$$\frac{1}{\phi_{11}}(\xi,\eta) = \frac{b(\xi)+b(\eta)+b(\xi+\eta)}{(b(\xi)+b(\eta))^2-b^2(\xi+\eta)} \triangleq \frac{b(\xi)+b(\eta)+b(\xi+\eta)}{A}.$$

In the following, we will assume $\kappa_0\leq\frac{1}{200}$, which ensure that: on the support $m(\xi,\eta)=\tilde{\chi}^L(\xi)\tilde{\chi}^L(\eta)\tilde{\chi}^L(\xi+\eta)\frac{\langle\xi+\eta\rangle}{2ib(\xi+\eta)}$, we have that: $\varepsilon^2|\xi|^4\leq 16\kappa_0^2\leq\frac{1}{2500}$ and $\frac{99}{100}\langle\xi\rangle\leq b(\xi)\leq\langle\xi\rangle$. Under this assumption, $A$ has the lower bound:

$$\begin{aligned} A &= 1+2b(\xi)b(\eta)-2\xi\cdot\eta-\varepsilon^2(|\xi|^4+|\eta|^4-|\xi+\eta|^4)\geq 1-32\kappa_0^2+2b(\xi)b(\eta)-2\xi\cdot\eta \\ &\geq \frac{(1-32\kappa_0^2+2b(\xi)b(\eta))^2-4|\xi\cdot\eta|^2}{1-32\kappa_0^2+2b(\xi)b(\eta)+2\xi\cdot\eta}\gtrsim\frac{(b(\xi)+b(\eta))^2}{b(\xi)b(\eta)}\gtrsim\frac{(\langle\xi\rangle+\langle\eta\rangle)^2}{\langle\xi\rangle\langle\eta\rangle}\gtrsim 1. \end{aligned} \tag{3.8.4}$$

Inspired by [84][69], we will prove that on the support of $m(\xi,\eta)$, for any multi-index $\alpha,\beta\in\mathbb{N}^3$, the following property holds:

$$|\partial_\xi^\alpha\partial_\eta^\beta\frac{1}{A}|\lesssim_{\alpha,\beta,\kappa_0}\frac{1}{A}. \tag{3.8.5}$$

This is an easy consequence of Leibniz's rule and the estimate

$$|\partial_\xi^\alpha\partial_\eta^\beta A|\lesssim_{\alpha,\beta,\kappa_0} A, \quad \forall\alpha,\beta\in\mathbb{N}^3. \tag{3.8.6}$$

In the following, we will thus prove (3.8.6). At first, we prove that

$$|\partial_{\xi,\eta} A| \lesssim_{\kappa_0} A. \tag{3.8.7}$$

We will focus on $\partial_\xi A \lesssim_{\kappa_0} A$. One first notices that on the support of $\tilde{\chi}^L(\xi)\tilde{\chi}^L(\eta)\tilde{\chi}^L(\xi+\eta)$

$$\begin{aligned}
|\partial_\xi A| &= \big|2\partial_\xi b(\xi) b(\eta) - 2\eta - \varepsilon^2(4|\xi|^2\xi - 4|\xi+\eta|^2(\xi+\eta))\big| \\
&= \big|2\frac{b(\eta)}{b(\xi)}(1-2\varepsilon^2|\xi|^2)\xi - 2\eta - \varepsilon^2(4|\xi|^2\xi - 4|\xi+\eta|^2(\xi+\eta))\big| \\
&\leq 2\big|\frac{b(\eta)}{b(\xi)}(1-2\varepsilon^2|\xi|^2)\xi - \eta\big| + 64\varepsilon^{\frac{1}{2}}\kappa_0^{\frac{3}{2}} \leq 2\big|\frac{b(\eta)}{b(\xi)}(1-2\varepsilon^2|\xi|^2)\xi - \eta\big| + \frac{4}{125},
\end{aligned}$$

if $\varepsilon \leq 1$, $\kappa_0 \leq \frac{1}{200}$. Thus, by noticing again that $A \geq 1 - 32\kappa_0^2 + 2b(\xi)b(\eta) - 2\xi\cdot\eta \geq \frac{199}{200} + +2b(\xi)b(\eta) - 2\xi\cdot\eta$, we only need to show that

$$\big|\frac{b(\eta)}{b(\xi)}(1-2\varepsilon^2|\xi|^2)\xi - \eta\big| \lesssim_{\kappa_0} 1 + b(\xi)b(\eta) - \xi\cdot\eta.$$

Besides, we also observe that on the support of $\tilde{\chi}^L(\xi)\tilde{\chi}^L(\eta)\tilde{\chi}^L(\xi+\eta)$, if $|\eta| \lesssim |\xi|$, we have $\frac{b(\eta)}{b(\xi)}\varepsilon^2|\xi|^3 \leq 8\varepsilon^{\frac{1}{2}}\kappa_0^{\frac{3}{2}} \leq \frac{1}{250}$ and if $|\xi| \leq |\eta|$, we have $\frac{b(\eta)|\xi|}{b(\xi)|\eta|}\varepsilon^2|\xi|^2|\eta| \leq \varepsilon^2|\xi|^2|\eta| \leq 8\varepsilon^{\frac{1}{2}}\kappa_0^{\frac{3}{2}} \leq \frac{1}{250}$. It thus suffices for us to prove that

$$\big|\frac{b(\eta)}{b(\xi)}\xi - \eta\big| \lesssim_{\kappa_0} 1 + b(\xi)b(\eta) - \xi\cdot\eta. \tag{3.8.8}$$

Let $\theta = \frac{\xi\cdot\eta}{|\xi||\eta|}$, to prove (3.8.8), we only need to prove that there exists a constant $C$, $4 < C < \infty$, such that:

$$\frac{b^2(\eta)}{b^2(\xi)}|\xi|^2 + |\eta|^2 - 2\frac{b(\eta)}{b(\xi)}|\xi||\eta|\theta \leq C[1 + b^2(\xi)b^2(\eta) + |\xi|^2|\eta|^2\theta^2 - 2b(\xi)b(\eta)|\xi||\eta|\theta]$$

Define $F(\theta) = (|\xi|^2|\eta|^2)\theta^2 - 2b(\eta)|\xi||\eta|(b(\xi) - \frac{1}{Cb(\xi)})\theta$. The critical point of $F(\theta)$ is $\theta_0 = \frac{b(\eta)(b(\xi) - \frac{1}{Cb(\xi)})}{|\xi||\eta|}$ and

$$\begin{aligned}
&\theta_0 \geq 1 \iff |\xi|^2|\eta|^2 b^2(\xi) \leq b^2(\eta)(b^2(\xi) - \frac{1}{C})^2 \\
\iff\ & |\xi|^2|\eta|^2(1+|\xi|^2 - \varepsilon^2|\xi|^4) \leq \big((1+|\xi|^2-\varepsilon^2|\xi|^4)^2 + \frac{1}{C^2} - \frac{2}{C}(1+|\xi|^2-\varepsilon^2|\xi|^4)\big)(1+|\eta|^2-\varepsilon^2|\eta|^4)
\end{aligned}$$

Nevertheless, by the assumption $\kappa_0 \leq \frac{1}{200}$, and $C > 4$, we have that: $32\kappa_0^2 + \frac{2}{C} < 1, (1-16\kappa_0^2)^2 + \frac{1}{C^2} - \frac{2}{C} > 0$, which leads to the lower bound of right hand side of the last inequality: $\big((1-16\kappa_0^2)^2 + |\xi|^4 + 2(1-16\kappa_0^2)|\xi|^2 + \frac{1}{C^2} - \frac{2}{C}(1+|\xi|^2)\big)(1-16\kappa_0^2+|\eta|^2) \geq (1+|\xi|^2)|\xi|^2|\eta|^2$. We thus have $\theta_0 \geq 1$ and only need to prove (3.8.8) for $\theta = 1$. However,

$$\begin{aligned}
\big|\frac{b(\eta)|\xi|}{b(\xi)} - |\eta|\big| &\leq (b(\eta) - |\eta|)\frac{|\xi|}{b(\xi)} + (1 - \frac{|\xi|}{b(\xi)})|\eta| \\
&\leq 1 + \frac{|\eta|(b(\xi) - |\xi|)}{b(\xi)} \leq 1 + b(\xi)b(\eta) - |\xi||\eta|,
\end{aligned}$$

this proves (3.8.8) for $\theta = 1$ which finish the proof of (3.8.7).

We now prove $\partial_\xi^\alpha \partial_\eta^\beta A \lesssim_{\kappa_0} A$ for $|\alpha| + |\beta| \geq 2$. Indeed, it is direct to show

$$\partial_\xi^\alpha \partial_\eta^\beta A \lesssim_{\kappa_0} \frac{\langle\eta\rangle}{\langle\xi\rangle} + \frac{\langle\xi\rangle}{\langle\eta\rangle}, |\alpha| + |\beta| \geq 2.$$

which, combined with (3.8.4), yields $\partial_\xi^\alpha \partial_\eta^\beta A \lesssim_{\kappa_0} A$. This ends the proof of (3.8.6) and thus of (3.8.5).

Next, we have

$$\frac{1}{b(\xi) + b(\eta) - b(\xi+\eta)} = \frac{b(\xi) + b(\eta) + b(\xi+\eta)}{A} \lesssim_{\kappa_0} \min\{b(\xi), b(\eta), b(\xi+\eta)\}. \tag{3.8.9}$$

In fact, if $b(\xi+\eta)$ is not the biggest, we have

$$b(\xi)+b(\eta)-b(\xi+\eta)\geq \min\{b(\xi),b(\eta)\}\geq 1.$$

Otherwise, by the lower bound for $A$ (3.8.4),

$$\frac{b(\xi)+b(\eta)+b(\xi+\eta)}{A}\lesssim_{\kappa_0}(b(\xi)+b(\eta)+b(\xi+\eta))\frac{b(\xi)b(\eta)}{(b(\xi)+b(\eta))^2}\lesssim_{\kappa_0}\min\{b(\xi),b(\eta)\}.$$

Finally, by inequality (3.8.5), we have:

$$\begin{aligned}\left|\partial_\xi^\alpha\partial_\eta^\beta\frac{m(\xi,\eta)}{\phi_{11}}\right| &= \left|\sum c_{\alpha_1\alpha_2\beta_1\beta_2}\partial_\xi^{\alpha_1}\partial_\eta^{\beta_1}m(\xi,\eta))\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\frac{b(\xi)+b(\eta)+b(\xi+\eta)}{A}\right|\\ &\lesssim_{\kappa_0} (b(\xi)+b(\eta)+b(\xi+\eta))\frac{1}{A}\lesssim_{\kappa_0}\min\{\langle\xi\rangle,\langle\eta\rangle,\langle\xi-\eta\rangle\}.\end{aligned}$$

Similarly, one has, by choosing $\kappa_0$ small if necessary,

$$|\partial_\xi^\alpha\partial_\eta^\beta\frac{m(\xi,\eta)}{\phi_{11}^2}| \lesssim_{\kappa_0} \frac{1}{\phi_{11}^2}\lesssim_{\kappa_0}\min\{\langle\xi\rangle^2,\langle\eta\rangle^2,\langle\xi+\eta\rangle^2\}.$$

□

# Chapter 4

# Long-term regularity of the two dimensional Navier-Stokes-Poisson equations

This chapter is excerpted from [120] which is accepted for publication in SIAM Journal on Mathematical analysis.

**Abstract** This article is devoted to the long-term regularity of the 2-d Navier-Stokes-Poisson system. We allow the initial density to be close to a constant and the potential part of the initial velocity to be small independently of the rescaled viscosity parameter $\varepsilon$ while the rotational part of the initial velocity is assumed to be small compared to $\varepsilon$. We then show that the lifespan of the system $T^\varepsilon$ satisfies $T^\varepsilon > \varepsilon^{-(1-\vartheta)}$, where the small constant $\vartheta$ is the size of the initial perturbation in some suitable space.

## 4.1 Introduction

In the present chapter, we are concerned with the large time regularity of the scaled two dimensional Navier-Stokes-Poisson system (NSP):

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon \mathcal{L} u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon \nabla \varphi^\varepsilon = 0, \\ \Delta \varphi^\varepsilon = \rho^\varepsilon - 1, \\ u|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon. \end{cases} \tag{4.1.1}$$

Here $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$, the unkowns $\rho^\varepsilon(t,x) \in \mathbb{R}_+, u^\varepsilon \in \mathbb{R}^2, \nabla\varphi^\varepsilon \in \mathbb{R}^2$ are the electron density, the electron velocity and the self-consistent electric field respectively. The thermal pressure $P(\rho^\varepsilon)$ is assumed to follow a polytropic $\gamma$-law: $P(\rho^\varepsilon) = \frac{(\rho^\varepsilon)^\gamma}{\gamma}, \gamma > 1$, while the viscous term is under the form $\mathcal{L}u^\varepsilon = \mu \Delta u^\varepsilon + (\mu + \lambda)\nabla \mathrm{div} u^\varepsilon$, where the Lamé coefficients $\mu, \lambda$ are supposed to be constants which satisfy the condition: $\mu > 0, 2\mu + \lambda > 0$. For the conciseness of the presentation, we shall assume $\mu = 1, \lambda = 0$ and $\gamma = 2$, since there are no specific cancellations arising from this choice. Note also that the scaled parameter $\varepsilon$ in front of the diffusion term is the inverse of the Reynolds number and is assumed to be small in this paper, that is $\varepsilon \in (0, 1]$.

As mentioned in the last chapter, there has been extensive studies concerning the global well-posedness of (NSP) under small and smooth perturbations of the constant equilibrium ($(\rho^\varepsilon, u^\varepsilon) = (1, 0)$) when the scaled parameter $\varepsilon = 1$ and the spatial dimension $d = 3$. We refer for example to [85] [126], for the global existence in Sobolev space $H^N$ for $N \geq 4$ and [26, 85, 113, 135], in hybrid Besov spaces. Although these works are done in spatial dimension $d = 3$, global existence in $H^N$ ($N \geq 3$) for $d = 2$ could still be proved by the same arguments as in [126]. That is, by using the dissipation for $u$ provided by the diffusion term $\varepsilon \Delta u^\varepsilon$ and the damping for $\rho^\varepsilon - 1$ resulting from the coupling structure, one could control the nonlinear term as long as some lower order Sobolev norm of $(\rho^\varepsilon - 1, u^\varepsilon, \nabla\varphi^\varepsilon)$ is small. Nevertheless, if we consider the scaled equations, that is $\varepsilon \in (0,1]$, the above strategy requires the initial perturbation $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \nabla\varphi_0^\varepsilon)$ to be small proportional to $\varepsilon$ in some suitable Sobolev spaces.

On the other hand, when $\varepsilon = 0$, the system (4.1.1) reduces to the so-called Euler-Poisson (EP) equation. Regarding (EP), Guo [55] constructs in dimension d=3 the global smooth solutions close to the reference equilibrium $(1,0)$ under neutral $(\int_{\mathbb{R}^2}(\rho_0^\varepsilon - 1)\mathrm{d}x = 0)$, irrotational, small perturbation to the equilibrium. The good dispersive properties due to the presence of the electric field and the normal form transformation technique developed by Shatah [112] are the main two ingredients of his proof. More recently, similar results was obtained in dimension $d = 2$ by Ionescu-Pausader [69] and Li-Wu [84] independently and $d = 1$ by Guo-Han-Zhang [56]. See also the result about the large time regularity of 2-d (EP) on the torus [134].

Nevertheless, since in practical physics, the Reynolds number is usually very high (that is $\varepsilon$ very small), it is natural to ask global existence results that hold uniformly in $\varepsilon$. In last chapter, we successfully combine the parabolic energy estimate which works for (NSP) and normal form transformation used in the works for (EP) to prove a uniform stability result for 3-d (NSP) system. That is, we construct the global smooth solutions around the constant equilibria $(1,0)$ with a smallness assumption on the perturbation which is independent of $\varepsilon$ except for the curl part of the velocity (recall that for $\varepsilon = 0$ we have global smooth solutions only for irrotational data). In this paper, we aim to prove the analogous results in 2-d. However, due to the weaker dispersion in 2-d, the rotational part of the velocity is driven by a source term whose $L_x^2$ norm enjoys only at best the critical time decay $(1+t)^{-1}$. Consequently, the rotational part of the velocity has a logarithmic growth which prevents one from establishing the global existence. Therefore, in this chapter, we only devote ourselves to proving a large time existence result of 2-d (NSP) system.

We shall denote by $\mathcal{P} = Id - \nabla\Delta^{-1}\mathrm{div}$ the Leray projector which projects a vector to its divergence-free (or rotational) part. Denote also $\mathcal{P}^\perp = Id - \mathcal{P}$, which projects a vector field to its curl-free (or potential) part.

**Theorem 4.1.1.** *There exist two constants $\vartheta_0, C$, such that for any $\varepsilon \in (0,1], \vartheta \in (0,\vartheta_0]$, if the following assumption holds:*

$$\|(\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla\varphi_0^\varepsilon)\|_{Y^4} \le \frac{1}{C}\vartheta, \qquad \|\mathcal{P}u_0^\varepsilon\|_{H^3} \le \vartheta\varepsilon,$$

*where the $Y^4$ norm is defined in (4.1.4), then the system (4.1.1) admits a solution in $C([0,T), H^3)$ with $T > \varepsilon^{-(1-\vartheta)}$.*

**Remark 4.1.2.** *We remark that the assumption we add on the rotational part of the initial velocity, that is, $\mathcal{P}u_0^\varepsilon$ small proportional to $\varepsilon$ is rather natural. Indeed, as the rotational part $\mathcal{P}u^\varepsilon$ is driven by a source term of size $\varepsilon$, even if we assume it to vanish initially , a rotational part of size $\varepsilon$ is instantaneously created.*

**Remark 4.1.3.** *As explained before, the rotational part of the velocity satisfies at the leading order a heat equation with a source term of size $\varepsilon$ and critical time decay in $L_x^2$, so that we can not even extend its existence time to $\varepsilon^{-1}$.*

A natural attempt to prove Theorem 4.1.1 is to consider the highly coupled equations for the potential part and rotational part of the velocity respectively. One expects to use the dispersive property from the coupled equations (see (4.4.3) below) for density and potential part of velocity to prove the time decay of $\rho^\varepsilon - 1$ and $\mathcal{P}^\perp u^\varepsilon$, and to use the smoothing effect to prove the large time existence of of the rotational part $\mathcal{P}u$. Nevertheless, there will be some difficulties stemming from the interactions between the rotational part and the potential part. To be more precise, on one hand, by the dispersive estimate, the $L_x^\infty$ norm of the solution to (4.4.3) decays at best at rate $(1+t)^{-1}$. On the other hand, since $\mathcal{P}u^\varepsilon$ is governed by the heat equation with a source term $\varepsilon\mathcal{P}\big[(\frac{1}{\rho^\varepsilon} - 1)\Delta\mathcal{P}^\perp u^\varepsilon\big]$, whose $L_x^2$ norm has at best critical time decay $(1+t)^{-1}$, it is far from being bounded in $L_x^2$, which in turn prevents us from proving the time decay of $(\rho^\varepsilon - 1, \mathcal{P}^\perp u^\varepsilon)$ in $L_x^\infty$. Moreover, owing to the presence of the diffusion term, the eigenvalues $\lambda_\pm = \varepsilon\Delta \pm i\sqrt{\langle\nabla\rangle^2 - (\varepsilon\Delta)^2}$ of the linearized matrix for the dispersive part of the system are far from $\pm i\langle\nabla\rangle$ − the eigenvalues for (EP). It seems necessary to cut the frequency to isolate the dispersive effects and dissipation effects, which forces us to control the interactions between different frequencies. We prefer not to take this way since it is more sophisticated to treat the potential-rotational interactions and low-high frequencies interactions in the same time.

Another attempt is to write the solution of NSP $(\rho^\varepsilon - 1, \nabla\varphi^\varepsilon, u^\varepsilon)$ by that of EP $(\rho^0, \nabla\varphi^0, u^0)$ plus a remainder, and try to control the remainder by the dissipation term. However in that case the equation satisfied by the remainder has source term $\varepsilon\Delta u^0$ which has size $\varepsilon$ but without any decay, this forces the remainder to grow linearly (in $L^2_x$) and leads to the time existence only at order $O(1)$.

We shall thus adopt the same strategy employed in the last chapter where the uniform stability problem for 3d NSP is investigated. More precisely, we split the (NSP) into two viscous systems, with initial data $(\rho_0^\varepsilon - 1, \nabla\varphi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and $(0, 0, \mathcal{P}u_0^\varepsilon)$ respectively. The first one will have global solutions under $\varepsilon$-independent assumptions on the initial data $(\rho_0^\varepsilon - 1, \nabla\varphi_0^\varepsilon, \mathcal{P}^\perp u_0^\varepsilon)$ and the solutions will enjoy the same time decay as the 2-d (EP) system. The other is the perturbation of the original system (4.1.1) by the former one. The source term $\varepsilon(\rho-1)\Delta u$ in this system is small compared to $\varepsilon$ and has critical decay in $L^2_x$. We can thus get the desired lifespan by merely energy estimates. More precisely, we write the solution $(\rho^\varepsilon, u^\varepsilon, \nabla\varphi^\varepsilon)$ of (NSP) as

$$(\rho^\varepsilon, u^\varepsilon, \nabla\varphi^\varepsilon) = (\rho, u, \nabla\varphi) + (n, v, \nabla\psi),$$

where $(\rho, \nabla\phi, u)$ and $(n, \nabla\psi, v)$ are the solutions of the following two systems

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u - \varepsilon\mathcal{L}u + \nabla\rho - \nabla\varphi = 0, \\ \Delta\varphi = \rho - 1, \\ u|_{t=0} = u_0 = \mathcal{P}^\perp u_0^\varepsilon, \rho|_{t=0} = \rho_0 = \rho_0^\varepsilon, \end{cases} \tag{4.1.2}$$

and

$$\begin{cases} \partial_t n + \operatorname{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u\cdot\nabla v + v\cdot(\nabla u + \nabla v) - \varepsilon\mathcal{L}v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho+n} - 1)(\mathcal{L}v + \mathcal{L}u), \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0} = 0. \end{cases} \tag{4.1.3}$$

Note that we skip the $\varepsilon$ dependence of the solutions in our notation for the last two systems. We also point out that we choose this kind of splitting mainly to ensure that the smooth solutions of the first system remain irrotational which is crucial to establish the global existence. As we shall see below, system (3.1.3) is a good viscous approximation of the Euler-Poisson system, in the sense that the linear part of this system has the same decay properties for low frequencies as the (EP) system, that is for localized initial data, the $L^p_x$ norm of $\nabla(\rho-1, \nabla\phi, u)$ decay like $(1+t)^{-(1-\frac{2}{p})}$ uniformly for $\varepsilon \in (0, 1]$.

To prove the global existence of (3.1.3), we shall use the following norm for the initial data:

$$\begin{aligned} \|(\rho_0 - 1, u_0, \nabla\varphi_0)\|_{Y^\sigma} \triangleq \|(\rho_0 - 1, u_0, \nabla\varphi_0)^L\|_{W^{\sigma+4,1}} + \|x(\rho_0 - 1, u_0, \nabla\varphi_0)^L\|_{H^{\sigma+4+\delta}} \\ + \|x(\rho_0 - 1, u_0, \nabla\varphi_0)^h\|_{L^2} + \|(\rho_0 - 1, u_0, \nabla\varphi_0)\|_{H^{11+2\sigma}} \end{aligned} \tag{4.1.4}$$

where $\sigma \geq 0$ is a positive parameter and $\delta = \frac{1}{1000}$.

We now state our results for system (4.1.2) and (4.1.3):

**Theorem 4.1.4.** *Let $\sigma \geq 0$. There exist two constants $C_1 > 0$, $\vartheta_1 > 0$ such that for any $\varepsilon \in (0, 1]$, any $\bar\vartheta \in (0, \vartheta_1]$ if*

$$\|(u_0, \rho_0 - 1, \nabla\varphi_0)\|_{Y^\sigma} \leq \bar\vartheta,$$

*then the system (3.1.3) admits a global solution $(u, \varrho, \nabla\varphi)$ in $C([0,\infty), H^{\sigma+7})$, which enjoys the uniform (in $\varepsilon$) time decay: for any $t > 0$,*

$$(1+t)\|(\rho - 1, \nabla u, \nabla\varphi)(t)\|_{W^{\sigma,\infty}} + \|(\rho - 1, \nabla u, \nabla\varphi)(t)\|_{H^{\sigma+7}} \leq C_1\bar\vartheta.$$

Once the above theorem proved, Theorem 4.1.1 is an easy consequence of the following one:

**Theorem 4.1.5.** *Let $(\rho, u, \nabla\phi)$ be the solutions constructed in the Theorem 4.1.4 with $\sigma = 4$. There exists $C > 1, \vartheta_0 \in (0, \vartheta_1]$, such that for any $\varepsilon \in (0, 1], \vartheta \in (0, \vartheta_0]$, if the following assumption holds:*

$$\|(\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla\varphi_0^\varepsilon)\|_{Y^4} \leq \frac{\vartheta}{C}, \quad \|\mathcal{P}u_0^\varepsilon\|_{H^3} \leq \vartheta\varepsilon,$$

*then the system* (4.1.3) *admits a solution in $C([0, T), H^3)$ with $T > \varepsilon^{-(1-\vartheta)}$.*

Since Theorem 4.1.5 is quite easy to obtain by merely energy estimates, we shall only explain the difficulties and strategies for proving Theorem 4.1.4.

We introduce the new unknown $U = (\frac{\langle\nabla\rangle}{|\nabla|}(\rho - 1), \frac{\mathrm{div}}{|\nabla|}u)$. Using the curl-free condition, it suffices for us to consider the system:

$$\partial_t U + AU = F(U, U), \quad A = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & -2\varepsilon\Delta \end{pmatrix}. \tag{4.1.5}$$

where $F$ is a quadratic form defined in (4.4.3). Simple computations show that the eigenvalues of $\hat{A}(\xi)$ are $\lambda_\pm = -\varepsilon|\xi|^2 \pm \sqrt{1 + |\xi|^2 - \varepsilon^2|\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm b(\xi)$. To present our ideas about decay estimate, we thus consider the toy model (we change the nonlinear term for clarity since there will be no loss of derivative in energy estimate):

$$\begin{cases} \partial_t\beta - \varepsilon\Delta\beta + ib(D)\beta = \beta^2 \\ \beta|_{t=0} = \beta_0 \end{cases}$$

where $\beta \in \mathbb{C}^2$. As indicated in the 3d case [108], we need to consider different frequencies to isolate the dispersive effects and dissipation effects. On one hand, when we focus on low frequency (say $\varepsilon|\xi|^2 \lesssim \kappa_0$ where $\kappa_0$ is very small but independent of $\varepsilon$), $e^{\varepsilon t\Delta}$ is useless since we want to get estimate uniformly for $\varepsilon$, but we can expect that $e^{itb(D)}$ behaves very like $e^{it\langle\nabla\rangle}$, which shall provide us with the dispersive estimate uniformly in $\varepsilon \in (0, 1]$. On the other hand, when we focus on the high frequency (say $\varepsilon|\xi|^2 \geq \kappa_0$), we have that $\mathrm{Re}\,\lambda_\pm \leq -c(\kappa_0)$ where $c(\kappa_0)$ is independent of $\varepsilon \in (0, 1]$, the operator $e^{t\lambda_\pm(D)}$ can provide exponential decay, we thus expect the solution to have good decay even in $L_x^2$ norm.

In practice, we first try to introduce some norms which indicate the decay properties for both low and high frequency. In order to do this, let us choose a compactly supported function $\chi$, which equals to 1 on the unit ball $B_1(\mathbb{R}^2)$ and vanishes outside $B_2(\mathbb{R}^3)$. Denote then $\chi^L(\xi) = \chi(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$ , $\chi^H = 1 - \chi^L$ where $\kappa_0$ is a threshold to be chosen later. We define the norm (the reason for evolving the weighted norm will be explained later):

$$\|\beta\|_{X_T} = \sup_{t\in[0,T)} \langle t\rangle\|\beta^L(t)\|_{W^{1,\infty}} + \|xe^{itb(D)}\beta^L\|_{H^4} + (1+t)\|\beta^H(t)\|_{H^9} + \langle t\rangle^{-\delta}\|\beta(t)\|_{H^{10}} + \|\beta(t)\|_{H^8}.$$

where $\beta^L = \chi^L(D)\beta, \beta^H = \chi^H(D)\beta$. Note that in the definition of the norm, we have time decay of rate $(1+t)^{-1}$ rather than $e^{-ct}$ for high frequency due to the weak decay property for low frequency. To get the a priori estimate, we need to consider several interactions between different frequencies. However, due to the slow decay provided by dispersive estimate for low frequency, the low frequency output of the interactions between the low frequency and the high frequency is difficult to close.

More precisely, in order to estimate the low frequency, by rewriting $\beta^2 = (\beta^L)^2 + 2\beta^L\beta^H + (\beta^H)^2$, we need to estimate the term $\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)\mathrm{d}s$ which can be estimated as, by dispersive estimate for $e^{itb(D)}\chi^L$ (see Lemma 4.4.1)

$$\begin{aligned} &\|\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)\mathrm{d}s\|_{L^\infty} \lesssim \int_0^t (1+t-s)^{-1}\|\beta^H\beta^L\|_{W^{2,1}}\mathrm{d}s \\ &\lesssim \int_0^t (1+t-s)^{-1}(1+s)^{-1}\mathrm{d}s\|\beta\|_X^2 \lesssim (1+t)^{-\iota}\|U\|_X^2 \end{aligned} \tag{4.1.6}$$

where $0 < \iota < 1$. Unfortunately, the desired case $\iota = 1$ is not true. To overcome this difficulty, we need more accurate splitting of frequencies. We observe that one can indeed split the frequency into three parts, namely, lowest frequency: $\{\varepsilon|\xi|^2 \leq \kappa_0\}$, intermediate frequency $\{\frac{\kappa_0}{2} \leq \varepsilon|\xi|^2 \leq 3\kappa_0\}$ and highest frequency $\{\varepsilon|\xi|^2 \geq \frac{5}{2}\kappa_0\}$. In this way, due to the lack of interaction lowest$\times$ lowest $\to$ highest, we could expect the highest frequency enjoys faster decay. What is more, the intermediate frequency part has also good decay since on this region, we have $e^{t\lambda_\pm(\xi)} \leq e^{-ct}$ for some $c > 0$ independent of $\varepsilon$. The lowest frequency is now manageable since for the lowest$\times$intermediate$\to$ lowest interaction we could use normal form transformation (see details below) by noticing that the intermediate frequency still lies in the region that dispersive property holds. To summarize, after some crude analysis, we expect the lowest frequency part enjoys the $L_x^\infty$ decay of $(1+t)^{-1}$, the intermediate part enjoy the $L_x^p$ $(2 \leq p \leq 4)$ decay like $(1+t)^{-(2-\frac{2}{p})}$, and the high frequency parts has $L_x^2$ decay like $(1+t)^{-2}$. We explain for instance the high frequency case. By choosing three smooth radial function $\chi^l, \chi^m, \chi^h$ which satisfies $\chi^l(\xi-\eta)\chi^l(\eta)\chi^h(\xi) = 0$ (see the definition in Section 2) and defining $\chi^L = \chi^l + \chi^m, \beta^L = \beta^l + \beta^m$, one can write $(\beta^2)^h = \big(2\beta^L\beta^h + (\beta^h)^2 + (\beta^m)^2 + 2\beta^l\beta^m\big)^h$. We could expect the worst part $\beta^l\beta^m$ enjoys $L^2$ decay of $(1+t)^{-2}$.

We thus need to modify our norm to be (with $N > 10$ to be chosen) :

$$\begin{aligned} \|\beta\|_{X_T} \triangleq \sup_{t\in[0,T)} \big(&\langle t\rangle\|\beta^L\|_{W^{1,\infty}} + \langle t\rangle\|\beta^m\|_{H^{N-1}} + \langle t\rangle^2\|\beta^h\|_{H^{N-1}} \\ &+ \|xe^{itb(D)}\beta^L\|_{H^4} + \langle t\rangle^{-\delta}\|\beta\|_{H^N} + \|\beta\|_{H^{N-2}}\big) \end{aligned} \tag{4.1.7}$$

Now we explain the Low$\times$Low$\to$ Low estimate where only dispersive estimate is available. To overcome the difficulty of quadratic nonlinearity, the normal form transformation (or more generally 'space-time resonance' philosophy [47]) need be enforced. To be more precise, we set $\alpha = e^{itb(D)}\beta^L$ and write

$$\int_0^t e^{-i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}\chi^L(D)(\beta^L)^2\mathrm{d}s = \mathcal{F}^{-1}(e^{-itb(\xi)}\int_0^t e^{is\phi_1(\xi,\eta)}e^{\varepsilon(t-s)|\xi|^2}\chi^L(\xi)\hat{\alpha}(\xi-\eta)\hat{\alpha}(\eta)\mathrm{d}\eta\mathrm{d}s) \tag{4.1.8}$$

where $\phi_1 = b(\xi) - b(\xi-\eta) - b(\eta) < 0$ on the support of $\chi^L(\xi)\chi^L(\xi-\eta)\chi^L(\eta)$. Following the 'space-time resonance' philosophy, by identity $e^{is\phi_1} = \frac{1}{i\phi_1}\partial_s e^{is\phi_1}$, one integrates by parts in time so that (4.1.8) becomes

$$-\int_0^t e^{-i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}\chi^L(D)\big(\varepsilon\Delta T_{\frac{1}{i\phi_1}}(\beta^L,\beta^L) + T_{\frac{1}{i\phi_1}}(\varepsilon\Delta\beta^L + (\beta^2)^L, \beta^L)\big)\mathrm{d}s \tag{4.1.9}$$

plus boundary terms and symmetric terms which can be handled similarly. Here, $T_{\frac{1}{i\phi_1}}$ is the bilinear operator defined by (4.2.1). Note that we have also used the equation satisfied by $\alpha$:

$$\partial_t\alpha = \varepsilon\Delta\alpha + e^{itb(D)}(\beta^2)^L.$$

In view of (4.1.9), Besides the viscous terms, we need to estimate the typical term:

$$\int_0^t e^{\lambda_-(D)(t-s)}\chi^L(D)(\beta^L\beta^L)^L\beta^L(s)\mathrm{d}s.$$

Nevertheless, the same problem like (4.1.6) emerges, since we could estimate $\|(\beta^L\beta^L)^L\beta^L\|_{W^{2,1}}$ by $\|\beta^L\|_{H^2}^2\|\beta^L\|_{L^\infty}$ which has only the decay $(1+s)^{-1}$. Following [69, 84], the 'vector field-like' norm $e^{-itb(D)}xe^{itb(D)}\beta^L$ needs to be involved to detect some space resonance information of the phase function.

We now explain the extra difficulty due to the dissipation term $\varepsilon\Delta\beta$. By noticing that $e^{\varepsilon(t-s)\Delta}\chi^L\varepsilon\Delta$ is a multiplier in $L_x^2$ with norm $(1+t-s)^{-1}$, we expect that $\varepsilon\Delta\beta^L$ has $L^2$ decay like $(1+t)^{-1}$. However, we shall still encounter the difficulty that $\|(\beta^L)^2\|_{L^2}$ have decay like $(1+t)^{-1}$, which forces us to use normal form transformation (or integrate by parts in time) again. This will increase the complexity of computations. The trick to simplify the arguments is that in the process of performing normal form transformations, we could introduce $\tilde{\alpha} = e^{-\varepsilon t\Delta}e^{itb(D)}\beta^L$ as the intermediate profile. By defining the complex phase function $\phi = i\phi_1 + \varepsilon(|\xi|^2 - |\xi-\eta|^2 - |\eta|^2)$ which does not vanish on the support of $\chi^L(\xi)\chi^L(\xi-\eta)\chi^L(\eta)$, we could integrate by parts in time as before to get:

$$\begin{aligned} &\int_0^t e^{-i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}\chi^L(D)(\beta^L)^2\mathrm{d}s = \mathcal{F}^{-1}(e^{-itb(\xi)}\int_0^t e^{is\phi(\xi,\eta)}\chi^L(\xi)\hat{\alpha}(\xi-\eta)\hat{\alpha}(\eta)\mathrm{d}\eta\mathrm{d}s) \\ = &\ \textit{boundary terms} + \int_0^t e^{-i(t-s)b(D)}e^{\varepsilon(t-s)\Delta}T_{\frac{1}{\phi}}((\beta^2)^L,\beta^L)\mathrm{d}s + \textit{symmetric term}. \end{aligned} \tag{4.1.10}$$

which allows us not to care about $\varepsilon\Delta\beta^L$. Note that there is no singularity on $\frac{1}{\phi}$ since $i\phi_1$ never vanishes. Moreover, computations (see Section 3) show that the bilinear operator $T_{\frac{1}{\phi}}$ enjoys the same good quasi-product estimates as $T_{\frac{1}{\phi_1}}$. The strategy for dealing with this term shall then have similarities with [69], [84] where the global existence for 2-d (EP) is proved.

**Organization of this chapter**: We first introduce some notations in Section 2. To prove Theorem 4.1.4, some reformulations and useful lemmas (linear estimates, bilinear estimates) are presented in Section 3. The local existence in weighted space for system (3.1.3) shall be shown in Section 4. Section 5 to Section 8 are dedicated to establish several a priori estimates. The conclusion for Theorem 4.1.4 are then made in Section 9. Theorem 4.1.5 shall be proved in Section 10. Finally, in appendix, we sketch the proofs for part of low frequency estimates.

## 4.2 Notations

- We denote $a_+(resp. a_-)$ for a constant larger (resp.smaller) but arbitrarily closed to $a$.
- We choose three radial smooth functions $\chi_1, \chi_2, \chi_3 : \mathbb{R}^2 \to \mathbb{R}$ st.$\chi_1 + \chi_2 + \chi_3 = 1$ and $\mathrm{Supp}\,\chi_1(\xi) \subset \{\xi \big| |\xi| \leq 1\}$, $\mathrm{Supp}\,\chi_2(\xi) \subset \{\xi \big| \frac{1}{2} \leq |\xi| \leq 3\}$, $\mathrm{Supp}\,\chi_3(\xi) \subset \{\xi \big| |\xi| \geq \frac{5}{2}\}$. Denote $\chi^l = \chi_1(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$, $\chi^m = \chi_2(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$, $\chi^h = \chi_3(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$, $\chi^L = \chi^l + \chi^m$, $\chi^H = \chi^m + \chi^h$. We also write: $f^L = \mathcal{F}^{-1}(\chi^L(\xi)\mathcal{F}f(\xi))$, $f^l = \mathcal{F}^{-1}(\chi^l(\xi)\mathcal{F}f(\xi))$, $f^m = \mathcal{F}^{-1}(\chi^m(\xi)\mathcal{F}f(\xi))$, $f^H = \mathcal{F}^{-1}(\chi^H(\xi)\mathcal{F}f(\xi))$.
- We define the bilinear operator $T_m(f,g)$ and trilinear operator $T_{\tilde{m}}(f,g,h)$

$$T_m(f,g) \triangleq \mathcal{F}^{-1}(\int m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\mathrm{d}\eta) \tag{4.2.1}$$

$$T_{\tilde{m}}(f,g,h) \triangleq \mathcal{F}^{-1}(\int m(\xi,\eta,\sigma)\hat{f}(\xi-\eta)\hat{g}(\eta-\sigma)\hat{h}(\eta)\mathrm{d}\eta) \tag{4.2.2}$$

- We recall the classical Littlewood-Paley decomposition: choose a cut-off function $\Psi, 0 \leq \Psi \leq 1, \Psi \equiv 1$ on $B_{3/2}$ and vanishes on $B^c_{5/3}$. We set

$$\Phi_j(x) = \Phi(\frac{x}{2^j}), \quad \text{where} \quad \Phi(x) = \Psi(x) - \Psi(2x). \tag{4.2.3}$$

Note that $\Phi(x)$ supported on the annulus $\{\frac{3}{4} \leq |x| \leq \frac{5}{3}\}$ and $1 = \Psi(x) + \sum_{j\in\mathbb{N}^*}\Phi_j(x)$. Recall the homogeneous dyadic block: $\dot{\Delta}_k f \triangleq \mathcal{F}^{-1}(\Phi_k(\xi)\hat{f}(\xi))$ $(k \in \mathbb{Z})$, inhomogeneous dyadic block: $\Delta_{-1}f \triangleq \mathcal{F}^{-1}(\Psi(\xi)\hat{f}(\xi)), \Delta_l f \triangleq \mathcal{F}^{-1}(\Phi_k(\xi)\hat{f}(\xi)), (l \in \mathbb{N})$, and $S_k = \sum_{-1\leq j\leq k-1}\Delta_j$.

## 4.3 Preliminaries I: The global existence of 2-d half Klein-Gordon equation with quadratic nonlinearity

In this subsection, we establish global well-posedness for quadratic Klein-Gordon equation in $2-d$ to show the idea of 'space-time resonance' method, the proof is essentially adapted from [69, 84]. We consider the following equation

$$\partial_t u - i\langle\nabla\rangle u = u^2, \quad u|_{t=0} = u_0 \tag{4.3.1}$$

where $u \in \mathbb{C}^2$. Local existence is easy, we only need to add a viscous perturbation $\varepsilon\Delta$ and prove there are solutions $u^\varepsilon \in C([0,T), H^N)$ where $T$ is uniform in $\varepsilon$, and $N \geq 8$. Letting $\varepsilon$ tends to 0, we get the solution of 4.3.1 in $C([0,T), H^N)$.

We now prove the global existence. Define the norm

$$\|u\|_X \triangleq \langle t\rangle\|u\|_{L^\infty} + \langle t\rangle^{-\delta}\|u\|_{H^{12}} + \|u\|_{H^8} + \|xe^{-it\langle\nabla\rangle}u\|_{L^2} \tag{4.3.2}$$

$$\|u_0\|_Y \triangleq \|u_0\|_{W^{2,1}} + \|xu_0\|_{L^2} + \|u_0\|_{H^{12}} \tag{4.3.3}$$

Combining the local existence, we only prove the a priori estimate

$$\|u\|_X \lesssim \|u_0\|_Y + \|u\|_X^2 \tag{4.3.4}$$

Define the profile of $u$: $f = e^{it\langle\nabla\rangle}u$, then $f$ satisfy $\partial_t f = e^{-it\langle\nabla\rangle}u^2$

we thus have

$$\begin{aligned}
f &= f_0 + \int_0^t e^{-is\langle\nabla\rangle}u^2\mathrm{d}s = f_0 + \mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^2} e^{is\phi}\hat{f}(s,\xi-\eta)\hat{f}(s,\eta)\mathrm{d}\eta\mathrm{d}s \\
&= f_0 + \mathcal{F}^{-1}\int_0^t\int_{\mathbb{R}^2}\frac{\partial_s e^{is\phi}}{i\phi}\hat{f}(s,\xi-\eta)\hat{f}(s,\eta)\mathrm{d}\eta\mathrm{d}s \\
&= f_0 - ie^{-it\langle\nabla\rangle}T_{\frac{1}{\phi}}(u(t),u(t)) + iT_{\frac{1}{\phi}}(u_0,u_0) \\
&\quad + i\int_0^t e^{-is\langle\nabla\rangle}T_{\frac{1}{\phi}}(u^2,u)\mathrm{d}s + symmetric \quad term
\end{aligned}$$

where $\phi = \langle\xi-\eta\rangle + \langle\eta\rangle - \langle\xi\rangle > 0$ and we have used $e^{is\langle\nabla\rangle}\partial_s f = u^2(s)$. Therefore, we have $u = e^{it\langle\nabla\rangle}f \triangleq I_1 + I_2 + I_3 + I_4$.

• Energy estimate. By standard energy estimate, we get:

$$\partial_t\|u\|_{H^8}^2 \lesssim \|u\|_{H^8}^2\|u\|_{L^\infty} \tag{4.3.5}$$

• Immediate Sobolev norm. We first recall bilinear estimate: for any $\sigma \geq 0, 1 \leq p \leq \infty, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

$$\|T_{\frac{1}{\phi}}(f,g)\|_{W^{\sigma,p}} \lesssim \|f\|_{W^{\sigma+4,p_1}}\|g\|_{L^{q_2}} + \|f\|_{L^{q_1}}\|g\|_{W^{\sigma+4,q_2}}$$

which easily leads to the estimate:

$$\begin{aligned}
\|I_2\|_{H^8} &\lesssim \|u(t)\|_{H^{12}}\|u\|_{L^\infty} \lesssim \langle t\rangle^{-(1-\delta)}\|u\|_X^2 \\
\|I_3\|_{H^8} &\lesssim \|u_0\|_{H^{12}}\|u_0\|_{L^\infty} \lesssim \|u_0\|_Y^2 \\
\|I_4\|_{H^8} &\lesssim \int_0^t(\|u^2\|_{H^{12}}\|u\|_{L^\infty} + \|u\|_{H^{12}\|u^2\|_{L^\infty}})\mathrm{d}s \\
&\lesssim \int_0^t\|u\|_{L^\infty}^2\|u\|_{H^{12}}\mathrm{d}s \lesssim \int_0^t\langle s\rangle^{-2+\delta}\mathrm{d}s \lesssim \|u\|_X^2
\end{aligned}$$

• Decay estimate

Firstly, by dispersive estimate: (cf. [102])

$$\|e^{it\langle\nabla\rangle}f\|_{L^\infty} \lesssim (1+t)^{-1}\|f\|_{W^{2,1}}$$

we control $L^\infty$ norm of $I_1$ as:

$$\|I_1\|_{L^\infty} = \|e^{it\langle\nabla\rangle}u_0\|_{L^\infty} \lesssim (1+t)^{-1}\|u_0\|_{W^{2,1}}$$

By bilinear estimate, we have:

$$\|I_2\|_{L^\infty} \lesssim \|u\|_{L^\infty}\|u\|_{W^{3+,\infty}} \lesssim (1+t)^{-1}\|u\|_X^2$$

By dispersive and bilinear estimate, we have:

$$\begin{aligned}
\|I_3\|_{L^\infty} = \|e^{it\langle\nabla\rangle}T_{\frac{1}{\phi}}(u_0,u_0)\|_{L^\infty} &\lesssim (1+t)^{-1}\|T_{\frac{1}{\phi}}(u_0,u_0)\|_{W^{2,1}} \\
&\lesssim (1+t)^{-1}\|u_0\|_{H^5}\|u_0\|_{H^1}
\end{aligned}$$

For $I_4$, a direct way to estimate is :

$$\begin{aligned}
\|I_4\|_{L^\infty} &\lesssim \int_0^t(1+t-s)^{-1}\|T_{\frac{1}{\phi}}(u^2,u)\|_{W^{2,1}}\mathrm{d}s \\
&\lesssim \int_0^t(1+t-s)^{-1}\|u\|_{H^5}\|u\|_{H^1}\|u\|_{L^\infty}\mathrm{d}s \\
&\lesssim \int_0^t(1+t-s)^{-1}(1+s)^{-1}\mathrm{d}s\|u\|_{X_T}^2 \lesssim (1+t)^{-(1-\kappa)}\|u\|_{X_T}^2
\end{aligned}$$

for any $0 < \kappa < 1$. However, it is not right for $\kappa = 0$. This is why we need involve weighted Sobolev norm for profile in the a priori norm.

We write $I_4 = e^{it\langle\nabla\rangle} I_4'$ by dispersive estimate:

$$\|I_4\|_{L^\infty} \lesssim (1+t)^{-1}\|I_4'\|_{W^{2,1}} \lesssim_\delta (1+t)^{-1}(\|I_4'\|_{H^2} + \|xI_4'\|_{W^{2,2-\delta}})$$

where $\delta$ is chosen to be very small. The estimate of $\|I_4'\|_{H^2}$ is easy, we skip it. We also postpone the estimate for $\|xI_4'\|_{W^{2,2-\delta}}$ to the end.

- Weighted Sobolev estimate for profile.

$$\|xe^{it\langle\nabla\rangle} T_{\frac{1}{\phi}}(u(t), u(t))\|_{H^1}$$

To avoid losing derivative of weighted norm, we need to distinguish the low and high frequency.

Write $\frac{1}{\phi}(\xi-\eta,\eta) = \frac{1}{\phi}(\xi-\eta,\eta)(\chi_1(\frac{\langle\xi-\eta\rangle}{\langle\eta\rangle}) + 1 - \chi_1(\frac{\langle\xi-\eta\rangle}{\langle\eta\rangle})) = m_1 + m_2$, we thus need to estimate $\|xe^{it\langle\nabla\rangle} T_{m_j}(u(t), u(t))\|_{L^\infty}$ $j = 1, 2$. We only prove for $j = 1$, as the other case can be treated by integrating by parts in $\eta$. Direct computation gives

$$\begin{aligned}
& xe^{it\langle\nabla\rangle} T_{m_1}(u(t), u(t)) \\
= \;& it\frac{\nabla}{\langle\nabla\rangle} e^{it\langle\nabla\rangle} T_{\frac{1}{\phi}}(u(t), u(t)) + e^{it\langle\nabla\rangle} T_{\partial_\xi m_1}(u(t), u(t)) + e^{it\langle\nabla\rangle} T_{m_1}(xu, u) \\
\triangleq \;& I_{21} + I_{22} + I_{23}
\end{aligned}$$

For $I_{21}$, using the bilinear estimate (see Corollary 4.4.18), we easily have:

$$\|I_{21}\|_{L^2} \lesssim t\|u(t)\|_{H^4}\|u(t)\|_{L^\infty} \lesssim \|u\|_X^2$$

$I_{22}$ is similar as $\partial_\xi m_1$ has the same property as $\frac{1}{\phi}$ and by the cut-off of the frequency in $m_1$, we also have:

$$\|I_{23}\|_{L^2} \lesssim \|u(t)\|_{H^4}\|xu(t)\|_{L^2} \lesssim \|u\|_X^2$$

Finally, by Sobolev embedding

$$\|xI_4'\|_{L^2} \lesssim \|xI_4'\|_{W^{1,2-\delta}}$$

so it remains for us to estimate $\|xI_4'\|_{W^{2,2-\delta}}$. However, we have:

$$\|xI_4'\|_{W^{2,2-\delta}} \lesssim \|I_4'\|_{W^{2,2-\delta}} + \|x\langle\nabla\rangle^2 I_4\|_{L^{2-\delta}}$$

so we only prove the later, as the other case is much easier.

$$\begin{aligned}
& \mathcal{F}(xI_4') = \partial_\xi(\langle\xi\rangle^2 I_4') = \partial_\xi(i\int_0^t e^{is\phi}\frac{\langle\xi\rangle^2}{\phi}\partial_s\partial_s\hat{f}(\eta)\hat{f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s) \\
= \;& i\int_0^t e^{is\phi}\partial_\xi(\frac{\langle\xi\rangle^2}{\phi})\partial_s\hat{f}(\eta)\hat{f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s - \int_0^t se^{is\phi}\frac{\langle\xi\rangle^2\partial_\xi\phi}{\phi}\partial_s\hat{f}(\eta)\hat{f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s \\
& +i\int_0^t e^{is\phi}\frac{\langle\xi\rangle^2}{\phi}\partial_s\hat{f}(\eta)\partial_\xi\hat{f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s \\
\triangleq \;& I_{41} + I_{42} + I_{43}
\end{aligned}$$

The estimate of $I_{41}$ is actually similar to that of $I_{31}$:

$$\begin{aligned}
\|I_{41}\|_{L^{2-\delta}} \;\lesssim\; & \int_0^t \|e^{is\langle\nabla\rangle} T_{\partial_\xi(\frac{\langle\xi\rangle^2}{\phi})}(u^2, u)\|_{2-\delta}\mathrm{d}s \lesssim \int_0^t \langle s\rangle^\delta \|T_{\partial_\xi(\frac{\langle\xi\rangle^2}{\phi})}(u^2, u)\|_{W^{\delta,2-\delta}}\mathrm{d}s \\
\lesssim\; & \int_0^t \langle s\rangle^\delta (\|u\|_{W^{6+2\delta,(\frac{1}{2-\delta}-2\delta)^{-1}}}\|u^2\|_{L^{\frac{1}{2\delta}}} + \|u\|_{L^{\frac{1}{\delta}}}\|u^2\|_{W^{6+2\delta,(\frac{1}{2-\delta}-\delta)^{-1}}})\mathrm{d}s \\
\lesssim\; & \int_0^t \langle s\rangle^\delta \|u\|_{L^{\frac{1}{\delta}}}^2\|u\|_{H^7}\mathrm{d}s \lesssim \int_0^t \langle s\rangle^\delta\langle s\rangle^{2(1-2\delta)}\langle s\rangle^\delta\mathrm{d}s\|u\|_X^3 \lesssim \|u\|_X^3
\end{aligned}$$

Next, we treat $I_{42}$

$$I_{42} = -\int_0^t se^{is\tilde{\phi}} \frac{\langle\xi\rangle^2\partial_\xi\phi}{\phi} \hat{f}(\xi-\eta)\hat{f}(\eta-\sigma)\hat{f}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s$$

where $\tilde{\phi} = -\langle\xi\rangle + \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle + \langle\sigma\rangle \gtrsim \frac{1}{\langle\xi\rangle}$, we thus can integrate by parts in $s$ again.

$$\begin{aligned}
I_{42} &= ite^{-it\langle\nabla\rangle} T_{\frac{\langle\xi\rangle^2\partial_\xi\phi}{\tilde{\phi}\phi}}(u(t),u(t),u(t)) - i\int_0^t e^{-is\langle\nabla\rangle} T_{\frac{\langle\xi\rangle^2\partial_\xi\phi}{\tilde{\phi}\phi}}(u,u,u)\mathrm{d}s \\
&\quad -i\int_0^t se^{-is\langle\nabla\rangle} T_{\frac{\langle\xi\rangle^2\partial_\xi\phi}{\tilde{\phi}\phi}}(u,u,u^2)\mathrm{d}s \\
&\triangleq I_{421} + I_{422} + I_{423}
\end{aligned}$$

By recalling the trilinear estimate: for $\frac{1}{p} = \sum_{j=1}^3 \frac{1}{p_j} = \sum_{j=1}^3 \frac{1}{q_j} = \sum_{j=1}^3 \frac{1}{r_j}$

$$\|T_{\frac{\langle\xi\rangle^2\partial_\xi\phi}{\tilde{\phi}\phi}}\|_{L^p} \lesssim \|f\|_{W^{11+\delta,p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}} + \|f\|_{L^{q_1}}\|g\|_{W^{11+\delta,q_2}}\|h\|_{L^{q_3}} + \|f\|_{L^{r_1}}\|g\|_{L^{r_2}}\|h\|_{W^{11+\delta,r_3}}$$

$$\begin{aligned}
\|I_{421}\|_{L^{2-\delta}} &\leq \langle t\rangle^{1+\delta}\|T_{\frac{\langle\xi\rangle^2\partial_\xi\phi}{\tilde{\phi}\phi}}(u(t),u(t),u(t))\|_{W^{\delta,2-\delta}} \\
&\lesssim \langle t\rangle^{1+\delta}\|u(t)\|^2_{L^{\frac{1}{\delta}}}\|u\|_{W^{11+2\delta,(\frac{1}{2-\delta}-2\delta)^{-1}}} \\
&\lesssim \langle t\rangle^{1+\delta}\langle t\rangle^{-2(1-2\delta)}\langle t\rangle^{\delta}\|u\|_X^3 \lesssim \|u\|_X^3\langle t\rangle^{-1+6\delta}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|I_{421}\|_{L^{2-\delta}} &\lesssim \int_0^t \langle s\rangle^{-2+6\delta}\|f\|_X^3\mathrm{d}s \lesssim \|u\|_X^3 \\
\|I_{433}\|_{L^{2-\delta}} &\lesssim \int_0^t \langle s\rangle^{1+\delta}\|u\|^3_{L^{\frac{1}{\delta}}}\|u\|_{W^{11+2\delta,(\frac{1}{2-\delta}-3\delta)^{-1}}}\mathrm{d}s \lesssim \int_0^s \langle t\rangle^{1+2\delta-2(1-2\delta)}\mathrm{d}s\|u\|_X^3
\end{aligned}$$

We now begin the estimate of $I_{43}$. Write

$$I_{43} = i\int_0^t e^{-is\langle\nabla\rangle}\big(T_{\langle\xi\rangle^2 m_1}\big(u^2, e^{is\langle\nabla\rangle}xf\big) + T_{\langle\xi\rangle^2 m_2}(u^2, e^{is\langle\nabla\rangle}xf)\big)\mathrm{d}s \triangleq I_{431} + I_{432}$$

$$\begin{aligned}
\|I_{431}\|_{L^{2-\delta}} &\lesssim \int_0^t \langle s\rangle^{\delta}\|u^2\|_{W^{5+\delta,(\frac{1}{2-\delta}-\frac{1}{2})^{-1}}}\|xf\|_{L^2}\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{\delta}\|u\|_{L^\infty}\|u\|_{W^{5+\delta,(\frac{1}{2-\delta}-\frac{1}{2})^{-1}}}\|xf\|_{L^2}\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{\delta}\|u\|_{L^\infty}\|u\|_{W^{6,3}}\|xf\|_{L^2}\mathrm{d}s \lesssim \int_0^t \langle s\rangle^{-1-\frac{1}{3}+\delta}\mathrm{d}s\|u\|_X^3 \lesssim \|u\|_X^3
\end{aligned}$$

where in the above, we have used Sobolev embedding in the third inequality (note $\delta < \frac{1}{3}$) and the fact $\|u(s)\|_{W^{6,3}} \lesssim (1+s)^{-\frac{1}{3}}\|u\|_X$ in the last inequality. To estimate $I_{432}$ we write $m_2(\xi,\eta,\sigma) = m_2(\chi_1(\frac{\langle\eta-\sigma\rangle}{\langle\sigma\rangle}) + 1 - \chi_1(\frac{\langle\eta-\sigma\rangle}{\langle\sigma\rangle})) = m_{21} + m_{22}$ then

$$\begin{aligned}
I_{432} &= i\int_0^t\int\int\int e^{is\tilde{\phi}}\langle\xi\rangle^2(m_{21}+m_{22})(\xi-\eta,\eta-\sigma,\sigma)\partial_\xi\hat{f}(\xi-\eta)\hat{f}(\eta-\sigma)\hat{f}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s \\
&\triangleq I_{4321} + I_{4322}
\end{aligned}$$

We only explain the estimate for $I_{4311}$, the other case is similar once we change variable by $\sigma \to \eta - \sigma$. Using $\partial_\xi\hat{f}(\xi-\eta) = -\partial_\eta\hat{f}(\xi-\eta)$, and integrating by parts in $\eta$, we get:

$$\begin{aligned}
I_{4311} &= -\int_0^t\int\int e^{is\tilde{\phi}}\langle\xi\rangle^2(s\partial_\eta\tilde{\phi}m_2 + \partial_\eta m_2)\hat{f}(\xi-\eta)\hat{f}(\eta-\sigma)\hat{f}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s \\
&\quad + \int_0^t e^{is\tilde{\phi}}\langle\xi\rangle^2 m_2\hat{f}(\xi-\eta)\partial_\eta\hat{f}(\eta-\sigma)\hat{f}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s
\end{aligned}$$

The second term can be estimated like that $I_{431}$, the first term can be estimated in the same fashion as $I_{42}$.

## 4.4 Preliminaries II

Set $\varrho = \rho - 1$, system (4.1.2) is equivalent to the following system:

$$\begin{cases} \partial_t \varrho + \mathrm{div} u + \mathrm{div}(\varrho u) = 0, \\ \partial_t u + u \cdot \nabla u - \varepsilon \mathcal{L} u + \nabla \varrho - \nabla \varphi = 0, \\ \Delta \varphi = \varrho \\ u|_{t=0} = \mathcal{P}^{\perp} u_0^{\varepsilon}, \varrho|_{t=0} = \rho_0^{\varepsilon} - 1 \end{cases} \tag{4.4.1}$$

We first remark that since $\mathrm{curl}(\mathcal{P}^{\perp} u_0^{\varepsilon}) = 0$, standard energy estimates indicate that this curl-free property will propagate as long as smooth solution exists. Note also that by identity $\Delta u = -\,\mathrm{curl}\,\mathrm{curl}\, u + \nabla \mathrm{div} u$, we have $\mathcal{L} u = \Delta u + \nabla \mathrm{div} u = 2\Delta u$.
As in the 3d case, we first introduce the new unkowns to symmetrize the system,

$$\mathrm{a} = \frac{\langle \nabla \rangle}{|\nabla|} \varrho, \quad \mathrm{c} = \frac{\mathrm{div}}{|\nabla|} u, \qquad U = (\mathrm{a}, \mathrm{c})^{\top}$$

It is direct to see that $(\mathrm{a}, \mathrm{c})$ satisfies the system:

$$\begin{cases} \partial_t \mathrm{a} + \langle \nabla \rangle \mathrm{c} = \langle \nabla \rangle \dfrac{\mathrm{div}}{|\nabla|} \Big( (\dfrac{|\nabla|}{\langle \nabla \rangle} \mathrm{a}) \mathcal{R} \mathrm{c} \Big) = \langle \nabla \rangle \mathcal{R} \cdot \Big( (\dfrac{|\nabla|}{\langle \nabla \rangle} \mathrm{a}) \mathcal{R} \mathrm{c} \Big) \\ \partial_t \mathrm{c} - \langle \nabla \rangle \mathrm{a} - 2\varepsilon \Delta \mathrm{c} = \dfrac{1}{2} \dfrac{\mathrm{div}}{|\nabla|} \nabla |\mathcal{R} \mathrm{c}|^2 = -\dfrac{1}{2} |\nabla| |\mathcal{R} \mathrm{c}|^2 \\ \mathrm{a}|_{t=0} = \dfrac{\langle \nabla \rangle}{|\nabla|} \varrho_0, \mathrm{c}|_{t=0} = \dfrac{\mathrm{div}}{|\nabla|} u_0 \end{cases} \tag{4.4.2}$$

ie.

$$\partial_t U + \begin{pmatrix} 0 & \langle \nabla \rangle \\ -\langle \nabla \rangle & -2\varepsilon \Delta \end{pmatrix} U = \begin{pmatrix} \langle \nabla \rangle \mathcal{R} \cdot \big( (\frac{|\nabla|}{\langle \nabla \rangle} \mathrm{a}) \cdot \mathcal{R} \mathrm{c} \big) \\ -\frac{1}{2} |\nabla| |\mathcal{R} \mathrm{c}|^2 \end{pmatrix} \triangleq \begin{pmatrix} F_1(\mathrm{a}, \mathrm{c}) \\ F_2(\mathrm{a}, \mathrm{c}) \end{pmatrix} = F(\mathrm{a}, \mathrm{c}) \tag{4.4.3}$$

where we denote $\mathcal{R} = \frac{\nabla}{|\nabla|}$ the Riesz potential. Note also that we have used the fact that $u = \mathcal{R}\mathrm{c}$ which is a consequence of $\mathrm{curl}\, u = 0$.
Define

$$A = \begin{pmatrix} 0 & \langle \nabla \rangle \\ -\langle \nabla \rangle & -2\varepsilon \Delta \end{pmatrix} \tag{4.4.4}$$

By elementary computation, we get that the eigenvalues of $-\hat{A}(\xi)$ are the same presented in (3.3.3), ie.

$$\lambda_{\pm} = -\varepsilon |\xi|^2 \pm i \sqrt{1 + |\xi|^2 - \varepsilon^2 |\xi|^4} \triangleq -\varepsilon |\xi|^2 \pm i b(\xi)$$

What is more, one can easily check that the Green matrix is

$$\begin{aligned} e^{-t\hat{A}(\xi)} &= \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t} & (e^{\lambda_- t} - e^{\lambda_+ t}) \langle \xi \rangle \\ (e^{\lambda_+ t} - e^{\lambda_- t}) \langle \xi \rangle & \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \end{pmatrix} \\ &\triangleq \begin{pmatrix} \mathcal{G}_1 & -\mathcal{G}_2 \\ \mathcal{G}_2 & \mathcal{G}_3 \end{pmatrix} \end{aligned} \tag{4.4.5}$$

Note $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are well defined everywhere since there is no singularity when $\lambda_+ = \lambda_-$. When we focus on the Low frequency, ie, when $\varepsilon |\xi|^2 \leq 3\kappa_0 << 1$, we can diagonalize $A$ as:

$$\begin{aligned} A(D) &= \begin{pmatrix} 1 & 1 \\ -\frac{\lambda_-(D)}{\langle \nabla \rangle} & -\frac{\lambda_+(D)}{\langle \nabla \rangle} \end{pmatrix} \begin{pmatrix} -\lambda_- & 0 \\ 0 & -\lambda_+ \end{pmatrix} \begin{pmatrix} \lambda_+ & \langle \nabla \rangle \\ -\lambda_- & -\langle \nabla \rangle \end{pmatrix} \frac{1}{2ib} \\ &\triangleq Q \begin{pmatrix} -\lambda_- & 0 \\ 0 & -\lambda_+ \end{pmatrix} Q^{-1}, \quad Q^{-1} = \begin{pmatrix} \lambda_+ & \langle \nabla \rangle \\ -\lambda_- & -\langle \nabla \rangle \end{pmatrix} \frac{1}{2ib}. \end{aligned} \tag{4.4.6}$$

We denote then $W = Q^{-1}\chi^L U \triangleq (w, \bar{w})$ for which the first component satisfies the equation:

$$\begin{aligned}
\partial_t w - \lambda_-(D)w &= \frac{\lambda_+}{2ib}\chi^L(D)F_1(\mathrm{a}, \mathrm{c}) + \frac{\langle\nabla\rangle}{2ib}\chi^L(D)F_2(\mathrm{a}, \mathrm{c}) \\
&= \frac{\lambda_+}{2ib}\chi^L(D)F_1(\mathrm{a}^L, \mathrm{c}^L)+\frac{\langle\nabla\rangle}{2ib}\chi^L(D)F_2(\mathrm{a}^L, \mathrm{c}^L) \\
&\qquad +[Q^{-1}\chi^L(D)(F(\mathrm{a}^h, \mathrm{c}^L) + F(\mathrm{a}^L, \mathrm{c}^h) + F(\mathrm{a}^h, \mathrm{c}^h))]_1 \\
&\triangleq \mathcal{R}\big(B(w, w) + \langle\nabla\rangle\chi^L H\big). \qquad (4.4.7)
\end{aligned}$$

where, $H = \mathcal{R}\mathrm{a}^L\mathcal{R}\mathrm{c}^h + \mathcal{R}\mathrm{c}^L\mathcal{R}\mathrm{a}^h + \mathcal{R}\mathrm{a}^h + \mathcal{R}\mathrm{c}^h \approx \mathcal{R}U^L\mathcal{R}U^h + \mathcal{R}U^h\mathcal{R}U^h$ and by relation $\mathrm{a}^L = w + \bar{w}, \mathrm{c}^L = -(\frac{\lambda_-}{\langle\nabla\rangle}w + \frac{\lambda_+}{\langle\nabla\rangle}\bar{w})$, $B(w, w)$ is defined by

$$\mathcal{F}B(w, w) = \sum_{\mu,\nu\in\{+,-\}} \int m_{\mu\nu}(\xi, \eta)\widehat{\mathcal{R}w^\mu}(\xi - \eta)\widehat{\mathcal{R}w^\nu}(\eta)\mathrm{d}\eta \qquad (4.4.8)$$

with $m_{\mu,\nu}(\xi, \eta) = \langle\xi\rangle n_\mu(\xi - \eta)n_\nu(\eta)\chi^L(\xi)\chi^L(\xi - \eta)\chi^L(\eta)$, $n_+ \in \{-\frac{\lambda_-}{\langle\nabla\rangle}, 1\}, n_- \in \{-\frac{\lambda_+}{\langle\nabla\rangle}, 1\}$ and exponent $\{\pm\} = \{Id, conjugate\}$.

Note that in the above (and hereafter), for notational convenience, we do not make difference between 'real' Riesz potential and general zero order Fourier multipliers whose symbol satisfies zero homogeneous condition and is smooth away from the origin, since they have similar properties. For example, they are both bounded operators in $L^p(1 < p < \infty)$. Moreover, we do not distinguish the scalar Riesz potential $\frac{\nabla_j}{|\nabla|}(j = 1, 2)$ and vector one $\frac{\nabla}{|\nabla|}$. One easily checks that in the above, $\mathcal{R}$ can represent anyone of the set $\{\frac{\mathrm{div}}{|\nabla|}\frac{\lambda_+}{2ib(D)}\chi^L(D), \frac{|\nabla|}{4ib}\chi^L(D), \frac{|\nabla|}{\langle\nabla\rangle}, \frac{\nabla}{|\nabla|}\}$.

After recalling the definition: $U^L = \chi^L(D)U = \chi^l(D)U + \chi^m(D)U = U^l + U^m, U^m = \chi^m(D)U, U^h = \chi^h(D)U$, we define the following norm:

$$\begin{aligned}
\|U\|_{X_T^\sigma} \triangleq \sup_{t\in[0,T)} &\langle t\rangle\||\nabla|^{\frac{1}{2}}\langle\nabla\rangle Q^{-1}U^L(t)\|_{W^{\sigma,\infty}} + \|xe^{itb(D)}w(t)\|_{W^{\sigma+4,\frac{2}{1-\delta}}} + \|U^L(t)\|_{H^{\sigma+N'}} \\
&+ \langle t\rangle^{1-3\delta}\|U^m(t)\|_{H^{2\sigma+N-1}} + \langle t\rangle^{\frac{3}{2}}\|U^m(t)\|_{W^{1,4}} + \langle t\rangle^\alpha\|U^h(t)\|_{H^{2\sigma+N-2}} + \langle t\rangle^{-\delta}\|U(t)\|_{H^{2\sigma+N}},
\end{aligned} \qquad (4.4.9)$$

where $N' = 7, N = N' + 4$, $\alpha = 2 - 5\delta$ and $\delta$ is chosen to be very small (say $\delta = \frac{1}{1000}$). We comment that the norm defined above is slightly different from (4.1.7) mainly due to the presence of Riesz potential in the nonlinear term (see (4.4.7)). We shall prove the global existence of system (4.4.3) in the Banach space $X_T^\sigma$ defined by the norm $\|\cdot\|_{X_T^\sigma}$. In the sequel, for notational clarity, we shall assume that $\sigma = 0$ (and denote $X_T^0 = X_T$), since the case $\sigma > 0$ can be easily generalized. We first remark that by dispersive estimate (4.4.10) and Hölder's inequality, for any $0 \le t < T$

$$\|U^L(t)\|_{W^{2,\frac{1}{\delta}}} \lesssim \|w(t)\|_{W^{2,\frac{1}{\delta}}} \lesssim \langle t\rangle^{-(1-2\delta)}\|e^{itb(D)}w(t)\|_{W^{4(1-\delta),\frac{1}{1-\delta}}} \lesssim \langle t\rangle^{-(1-2\delta)}\|U\|_{X_T}.$$

Moreover, we have:

$$\begin{aligned}
\|\nabla u(t)\|_{L^\infty} = \|\nabla\mathcal{R}U(t)\|_{L^\infty} &\le \|\nabla\mathcal{R}U^L(t)\|_{L^\infty} + \|\nabla\mathcal{R}U^h(t)\|_{L^\infty} \\
&\lesssim \sum_k 2^{\frac{1}{2}k}\langle 2^k\rangle^{-1}\|\dot{\Delta}_k|\nabla|^{\frac{1}{2}}\langle\nabla\rangle w(t)\|_{L^\infty} + \langle t\rangle^{-\alpha}\|U\|_{X_T} \\
&\lesssim \||\nabla|^{\frac{1}{2}}\langle\nabla\rangle w(t)\|_{L^\infty} + \langle t\rangle^{-\alpha}\|U\|_X \lesssim \langle t\rangle^{-1}\|U\|_{X_T}.
\end{aligned}$$

In the following of this section, we will give some preliminary lemmas which will be used later.

### 4.4.1 Linear estimates

We present in this subsection the linear estimates for Low (lowest and intermediate) and highest frequency.

#### 4.4.1.1 Linear estimates for Low frequency

**Lemma 4.4.1.** *Dispersive estimate for $e^{itb(D)}\chi^L$.*
*For every $\kappa_0$ is small enough (say $\kappa_0 \leq \frac{1}{200}$) and for any $2 \leq p \leq \infty$, we have the following dispersive estimate:*

$$\|e^{itb(D)}\chi^L(D)f\|_{L^p} \lesssim_{\kappa_0} (1+|t|)^{-(1-\frac{2}{p})}\|f\|_{W^{2(1-\frac{2}{p}),p'}}, \quad \forall t \in \mathbb{R}. \tag{4.4.10}$$

*Proof.* Indeed, (4.4.10) holds when $e^{itb(D)}\chi^L(D)$ is replaced by $e^{it\langle\nabla\rangle}$, which follows from the classical stationary phase arguments. Nevertheless, when $\kappa_0$ is chosen small enough, $e^{itb(D)}\chi^L(D)$ enjoys the similar algebraic properties as $e^{it\langle\nabla\rangle}$. One can refer to Corollary 2.3.7 for the details. □

**Lemma 4.4.2.** *$L^p \to L^p$ boundedness for $e^{itb(D)}\chi^L$.*

*Suppose $\kappa_0 \leq \frac{1}{200}$. For any $1 < p < \infty$, we have the following estimate:*

$$\|\Delta_k e^{itb(D)}\chi^L(D)u\|_{L^p} \lesssim_{\kappa_0} \langle t\rangle^{|1-\frac{2}{p}|}\langle 2^k\rangle^{|1-\frac{2}{p}|}\|\Delta_k u\|_{L^p}, (k \geq -1) \tag{4.4.11}$$

$$\|e^{itb(D)}\chi^L(D)u\|_{L^p} \lesssim_{\kappa_0} \langle t\rangle^{|1-\frac{2}{p}|}\|u\|_{W^{s,p}}. \tag{4.4.12}$$

*where $s > |1-\frac{2}{p}|$.*

*Proof.* This lemma has essentially been proved in Lemma 2.2 of [84] where $e^{it\langle\nabla\rangle}$ rather than $e^{itb(D)}\chi^L(D)$ is considered. We will sketch the proof of (4.4.11) for $p = 1, 2, \infty$, the other case for (4.4.11) and (4.4.12) follows from interpolation and summation respectively. By Young's inequality, it suffices for us to show:

$$\|\mathcal{F}^{-1}\big(\Phi_k(\xi)e^{itb(D)}\chi^L(\xi)\big)\|_{L^1} \lesssim_{\kappa_0} \langle t\rangle\langle 2^k\rangle, k \geq 0; \qquad \|\mathcal{F}^{-1}\big(\Psi(\xi)e^{itb(D)}\chi^L(\xi)\big)\|_{L^1} \lesssim_{\kappa_0} \langle t\rangle \tag{4.4.13}$$

where $\Phi_k, \Psi_k$ is defined in (4.2.3). To prove (4.4.13), one uses the inequality: $\|f\|_{L^1} \lesssim \|f\|_{L^2}^{\frac{1}{2}}\|x^2 f\|_{L^2}^{\frac{1}{2}}$ and elementary estimate:

$$\|\Phi_k(\xi)e^{itb(D)}\chi^L(\xi)\|_{L^2} \lesssim 2^k, \qquad \|\partial_\xi^2\big(\Phi_k(\xi)e^{itb(D)}\chi^L(\xi)\big)\|_{L^2} \lesssim_{\kappa_0} 2^{-k}\langle 2^k t\rangle^2,$$

$$\|\Psi(\xi)e^{itb(D)}\chi^L(\xi)\|_{L^2} \lesssim 1, \qquad \|\partial_\xi^2\big(\Phi_k(\xi)e^{itb(D)}\chi^L(\xi)\big)\|_{L^2} \lesssim_{\kappa_0} \langle t\rangle^2.$$

□

#### 4.4.1.2 Linear estimate for high frequency

**Lemma 4.4.3.** *Linear estimate for $e^{-tA}\chi^h$.*

*There exists a constant $c = c(\kappa_0)$, such that, for any real number $s$, we have:*

$$\|e^{-tA}\chi^h U\|_{H^s} \lesssim_{\kappa_0} e^{-ct}\|U\|_{H^s}.$$

*Proof.* One needs to study carefully the Green matrix (4.4.5) localized on high frequency, since the algebraic computations does not depend on the dimension, one can refer to Lemma 3.5 of [108] where the similar property is shown in dimension 3. □

#### 4.4.1.3 Additional estimate for intermediate frequency

For the intermediate frequency, we could use the spectral localization to get the boundedness of $e^{-tA}\chi^m$ from $W^{|1-\frac{2}{p}|+,p}$ to $L^p(1 < p < \infty)$.

**Lemma 4.4.4.** *Recall $\chi^m(\xi) = \chi_2(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$ where $\chi_2$ is smooth function supported on $\{\xi | \frac{1}{2} \leq |\xi| \leq 3\}$. We have for any $1 < p < \infty$,*

$$\|e^{-tA}\chi^m(D)u\|_{L^p} \lesssim_{\kappa_0} e^{-\frac{\kappa_0}{5}t}\|u\|_{W^{|1-\frac{2}{p}|+,p}}.$$

*Proof.* We first prove
$$\|e^{\varepsilon t\Delta}\chi^m(D)f\|_{L^p}\lesssim_{\kappa_0} e^{-\frac{1}{4}\kappa_0 t}\|u\|_{L^p}.$$
which follows from the Young's inequality and the fact: $\|f\|_{L^1}\lesssim\|f\|_{L^2}^{\frac{1}{2}}\|x^2f\|_{L^2}^{\frac{1}{2}}$. Indeed, one has:
$$\begin{aligned}\|\mathcal{F}^{-1}\big(e^{-\varepsilon|\xi|^2t}\chi^m(\xi)\big)\|_{L^1} &= \|\mathcal{F}^{-1}(e^{-\kappa_0\xi|^2t}\chi_2(\xi))\|_{L^1}\\ &\lesssim \|\mathcal{F}^{-1}\big(e^{-\kappa_0|\xi|^2t}\chi_2(\xi)\big)\|_{L^2}^{\frac{1}{2}}\|x^2\mathcal{F}^{-1}\big(e^{-\kappa_0|\xi|^2t}\chi_2(\xi)\big)\|_{L^2}^{\frac{1}{2}}\\ &\lesssim e^{-\frac{1}{2}\kappa_0 t}\langle\kappa_0 t\rangle\lesssim e^{-\frac{1}{4}\kappa_0 t}.\end{aligned}$$
By the definition of the Green matrix,
$$e^{-t\hat{A}(\xi)} = \frac{1}{\lambda_+-\lambda_-}\begin{pmatrix}\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t} & (e^{\lambda_-t}-e^{\lambda_+t})\langle\xi\rangle\\ (e^{\lambda_+t}-e^{\lambda_-t})\langle\xi\rangle & \lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}\end{pmatrix}.$$
and eigenvalue $\lambda_\pm=\varepsilon\Delta\pm ib(D)$ , we see that $e^{-tA}$ is indeed the combination of terms like $e^{\lambda_\pm(D)}q(D)$ where $q(D)\in\{\frac{\varepsilon\Delta}{b(D)},\frac{\langle\nabla\rangle}{b(D)},Id\}$. Therefore, by Lemma 4.4.2 and the definition of $\chi^m$,
$$\begin{aligned}\|e^{itb(D)}e^{\varepsilon t\Delta}\chi^m q(D)(u)\|_{L^p} &\lesssim \|e^{itb(D)}\tilde{\chi}^m e^{\varepsilon t\Delta}\chi^m q(D)\|_{L^p}\\ &\lesssim_{\kappa_0} \langle t\rangle^{|1-\frac{2}{p}|}\|e^{\varepsilon t\Delta}\chi^m q(D)u\|_{W^{|1-\frac{2}{p}|_+,p}}\\ &\lesssim_{\kappa_0} \langle t\rangle^{|1-\frac{2}{p}|}e^{-\frac{1}{4}\kappa_0 t}\|\chi^m q(D)u\|_{W^{|1-\frac{2}{p}|_+,p}}\\ &\lesssim_{\kappa_0} e^{-\frac{1}{5}\kappa_0 t}\|u\|_{W^{|1-\frac{2}{p}|_+,p}}.\end{aligned}$$
□

### 4.4.2 Bilinear estimates

As we shall use the normal form transformation, it is necessary for us get some continuous properties for bilinear operators defined by (4.2.1). To start, we present some elementary properties of bilinear multipliers which is useful to derive the bilinear estimates.

**Proposition 4.4.5.** *Define the phase function*
$$\phi_{\mu,\nu}(\xi+\eta,\eta)=i(b(\xi+\eta)-\mu b(\xi)-\nu b(\eta))+Z(\xi,\eta),$$
*and multiplier function*
$$m_{\mu,\nu}(\xi+\eta,\eta)=\langle\xi+\eta\rangle\chi^L(\xi)\chi^L(\eta)\chi^L(\xi+\eta)n_\mu(\xi)n_\nu(\eta)$$
*where* $b(\xi)=\sqrt{1+|\xi|^2-\varepsilon^2|\xi|^4}, Z(\xi,\eta)=\varepsilon(|\xi+\eta|^2-|\xi|^2-|\eta|^2)$ *and* $\mu,\nu\in\{+,-\}$. $n_+\in\{-\frac{\lambda_-}{\langle\nabla\rangle},1\}$, $n_-\in\{-\frac{\lambda_+}{\langle\nabla\rangle},1\}$. *Suppose* $\kappa_0\le\frac{1}{200}$, *then for any multi-index* $\alpha,\beta\in\mathbb{N}^2$, *the following estimate hlods uniformly in* $\varepsilon\in(0,1]$*:*
$$|\partial_\xi^\alpha\partial_\eta^\beta\frac{m_{\mu,\nu}}{\phi_{\mu,\nu}}(\xi+\eta,\eta)|\lesssim_{\alpha,\beta,\kappa_0}\langle\xi+\eta\rangle\min\{b(\xi),b(\eta),b(\xi+\eta)\}$$

*Proof.* We only present the proof for $\mu=\nu='+'$, since the others are easier or can be obtained by symmetry. At first, we have
$$\begin{aligned}\frac{1}{\phi_{++}} &= i\frac{b(\xi)+b(\eta)+b(\xi+\eta)+iZ(\xi,\eta)}{(b(\xi)+b(\eta)+iZ(\xi,\eta))^2-b^2(\xi+\eta)}\\ &\triangleq i\frac{b(\xi)+b(\eta)+b(\xi+\eta)+iZ(\xi,\eta)}{B}.\end{aligned}$$
where
$$\begin{aligned}B &= \big(b(\xi)+b(\eta)\big)^2-b^2(\xi+\eta)-Z^2(\xi,\eta)+2iZ(\xi,\eta)\big(b(\xi)+b(\eta)\big)\\ &\triangleq A-Z^2(\xi,\eta)+2iZ(\xi,\eta)\big(b(\xi)+b(\eta)\big)\end{aligned}$$

Note that $A$ has the lower bound:

$$\begin{aligned} A &= 1+2b(\xi)b(\eta)-2\xi\cdot\eta+\varepsilon^2(|\xi|^4+|\eta|^4-|\xi+\eta|^4) \\ &\geq 1-27\kappa_0^2+2b(\xi)b(\eta)-2\xi\cdot\eta \\ &= \frac{(1-27\kappa_0^2+2b(\xi)b(\eta))^2-4|\xi\cdot\eta|^2}{1-27\kappa_0^2+2b(\xi)b(\eta)+2\xi\cdot\eta} \gtrsim \frac{(b(\xi)+b(\eta))^2}{b(\xi)b(\eta)} \gtrsim 1. \end{aligned} \tag{4.4.14}$$

We will prove that on the support of $\chi^L(\xi)\chi^L(\eta)\chi^L(\xi+\eta)$, for any multi-index $\alpha,\beta\in\mathbb{N}^2$, the following property holds:

$$|\partial_\xi^\alpha\partial_\eta^\beta\frac{1}{B}|\lesssim_{\alpha,\beta,\kappa_0}\frac{1}{|B|} \tag{4.4.15}$$

which is an easy consequence of Leibniz's rule and

$$|\partial_\xi^\alpha\partial_\eta^\beta B|\lesssim_{\alpha,\beta,\kappa_0}|B|,\quad \forall\alpha,\beta\in\mathbb{N}^2. \tag{4.4.16}$$

However, (4.4.16) can be derived once we have the estimate for $A$:

$$|\partial_\xi^\alpha\partial_\eta^\beta A|\lesssim_{\alpha,\beta,\kappa_0}A,\quad \forall\alpha,\beta\in\mathbb{N}^2. \tag{4.4.17}$$

Indeed, since

$$|\partial_\xi^\alpha\partial_\eta^\beta Z^2(\xi,\eta)|+|\partial_\xi^\alpha\partial_\eta^\beta Z(\xi,\eta)(b(\xi)+b(\eta))|\leq P(\alpha,\beta,\kappa_0),\qquad \text{for } (\xi,\eta)\in\operatorname{Supp} m(\xi,\eta)$$

where $P(\alpha,\beta,\kappa_0)$ is a polynomial with respect to $\kappa_0$ which can be bounded by a constant $C(\alpha,\beta)$ if we choose $\kappa_0$ small (say $\kappa_0\leq\frac{1}{200}$), we can use (4.4.17), (4.4.14) to get that:

$$|\partial_\xi^\alpha\partial_\eta^\beta B|\leq|\partial_\xi^\alpha\partial_\eta^\beta A|+C(\alpha,\beta)\lesssim_{\alpha,\beta,\kappa_0}A+C(\alpha,\beta)\lesssim_{\alpha,\beta,\kappa_0}A\lesssim_{\alpha,\beta,\kappa_0}|B|.$$

Nevertheless, we note that the estimate of (4.4.17) has been proved in the appendix of last chapter. Therefore, inequality (4.4.15) holds, which leads to the following computation:

$$\begin{aligned} |\partial_\xi^\alpha\partial_\eta^\beta\frac{m_{++}(\xi,\eta)}{\phi_{++}}| &= |\sum c_{\alpha_1\alpha_2\beta_1\beta_2}\partial_\xi^{\alpha_1}\partial_\eta^{\beta_1}(m_{++}(\xi,\eta))\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\frac{b(\xi)+b(\eta)+b(\xi+\eta)+iZ(\xi,\eta)}{B}| \\ &\lesssim_{\kappa_0} \langle\xi+\eta\rangle(b(\xi)+b(\eta)+b(\xi+\eta))\frac{1}{A} \\ &\lesssim_{\kappa_0} \langle\xi+\eta\rangle\min\{b(\xi),b(\eta),b(\xi+\eta)\}. \end{aligned}$$

$\square$

This proposition in hand, we then show the following bilinear estimate:

**Lemma 4.4.6.** *Let $m_{\mu\nu},\phi_{\mu\nu}$ being defined as the last proposition, one has bilinear estimate:*

$$\|T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}(f,g)\|_{L^p}\lesssim_{\kappa_0}\|f\|_{W^{2+,q_1}}\|g\|_{W^{2,r_1}}+\|f\|_{W^{2,r_2}}\|g\|_{W^{2+,q_2}} \tag{4.4.18}$$

*where $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{r_1}=\frac{1}{q_2}+\frac{1}{r_2}$, $1<r_1,r_2\leq+\infty, 1\leq q_1,q_2<+\infty$, $T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}$ is the bilinear operator defined in* (3.2.1) *and $k_+$ is a real number slightly larger than $k$.*

*Proof.* As before, we only treat the case $T_{\frac{m_{++}}{\phi_{++}}}$. Let $\psi_1,\psi_2\in C_b^\infty(\mathbb{R}^4)$ which satisfy the following conditions:

$$\begin{cases} \psi_1+\psi_2=1 & \forall(\xi,\eta), \\ \operatorname{Supp}\psi_1\subset\{(\xi,\eta)\big|\langle\xi-\eta\rangle\geq\frac{\langle\eta\rangle}{2}\}, \\ \operatorname{Supp}\psi_2\subset\{(\xi,\eta)\big|\langle\eta\rangle>\langle\xi-\eta\rangle\}. \end{cases}$$

We write

$$\begin{aligned} \frac{m_{++}}{\phi_{++}}(\xi,\eta) &= \frac{m_{++}\psi_1(\xi,\eta)}{\phi_{++}\langle\xi-\eta\rangle^{2+}\langle\eta\rangle^2}\langle\xi-\eta\rangle^{2+}\langle\eta\rangle^2+\frac{m_{++}\psi_2(\xi,\eta)}{\phi_{++}\langle\eta\rangle^{2+}\langle\xi-\eta\rangle^2}\langle\eta\rangle^{2+}\langle\xi-\eta\rangle^2 \\ &\triangleq M_1(\xi,\eta)\langle\xi-\eta\rangle^{2+}\langle\eta\rangle^2+M_2(\xi,\eta)\langle\eta\rangle^{2+}\langle\xi-\eta\rangle^2. \end{aligned}$$

By Proposition 4.4.5, we have for any $\alpha, \beta$ with $|\alpha| + |\beta| \le 3$,

$$|\partial_\xi^\alpha \partial_\eta^\beta M_1| \le I_{\langle \xi-\eta\rangle \ge \frac{\langle \eta\rangle}{2}} \langle \xi - \eta\rangle^{-1+} \langle \eta\rangle^{-1}.$$

Therefore, $M_1, \partial_\xi^3 M_1, \partial_\eta^3 M_1 \in L^2(\mathbb{R}^4)$, which leads to the fact: $\mathcal{F}^{-1}(M_1)(x,y) \in L^1_{x,y}$. Indeed,

$$\|\mathcal{F}^{-1}(M_1)(x,y)\|_{L^1_{x,y}} \lesssim \|(1+|x|^3+|y|^3)^{-1}\|_{L^2_{x,y}} (\|M_1\|_{L^2} + \|\partial_\xi^3 M_1\|_{L^2} + \|\partial_\eta^3 M_1\|_{L^2}).$$

By the definition of bilinear operator $T_m$ (3.2.1) and Fourier transform:

$$T_{M_1\langle\xi\rangle^{2+}\langle\eta\rangle^2}(f,g) = \int (\mathcal{F}^{-1}M_1)(x', y'-x')\langle D_x\rangle^{2+} f(x-x')\langle D_x^2\rangle g(x-y')\mathrm{d}x'\mathrm{d}y'.$$

Therefore, By Minkowski's inequality,

$$\begin{aligned} \|T_{M_1\langle\xi\rangle^{2+}\langle\eta\rangle^2}(f,g)\|_{L^p} &\le \int \|\langle D_x^2\rangle g\|_{L^{r_1}} \| \int (\mathcal{F}^{-1}M_1)(x', y'-x')\langle D_x\rangle^{2+} f(x-x')\mathrm{d}x'\|_{L^{q_1}}\mathrm{d}y' \\ &\le \|\mathcal{F}^{-1}M_1\|_{L^1_{x,y}} \|f\|_{W^{2+,p_1}} \|g\|_{W^{2,r_1}}. \end{aligned}$$

The similar result for $M_2$ can be derived in the same fashion. □

**Remark 4.4.7.** *It is easy to adapt the proof of the above lemma to get that:*

$$\|T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}(f,g)\|_{L^p} \lesssim_{\kappa_0} \|f\|_{W^{3+,q_1}}\|g\|_{W^{1,r_1}} + \|f\|_{W^{1,r_2}}\|g\|_{W^{3+,q_2}}. \tag{4.4.19}$$

**Remark 4.4.8.** *Of course the norm of $T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}$ in* (4.4.18) *and* (4.4.19) *is dependent on $2_+ - 2$ or $3_+ - 3$, but when we use this lemma, we fixed $2_+$ and $3_+$.*

**Remark 4.4.9.** *From now on, we will fix $\kappa_0 = \frac{1}{200}$.*

**Corollary 4.4.10.** *Recall $B(w,w) \approx \mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2$ is defined in* (4.4.8)*, the following trilinear estimate holds*

$$\|T_{\frac{m}{\phi}}(\mathcal{R}B(w,w), \mathcal{R}w)\|_{W^{\sigma,p}} \lesssim \|w\|_{W^{2,p_1}}\|w\|_{W^{2,p_2}}\|w\|_{W^{\sigma+3_+,p_3}}$$

*where $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.*

### 4.4.3 Useful Lemmas for local existence.

In this subsection, we give some preliminary lemmas which will be used in the proof of local existence.

**Lemma 4.4.11.** *Let $\zeta$ be a compactly support smooth function and $\theta \in \dot{W}^{1,\infty}$, and denote $\zeta_R = \zeta(\frac{x}{R})$. then*

$$\|[\zeta_R(\cdot), \Theta(D)]f\|_{L^2} \lesssim \frac{1}{R}\|f\|_{L^2}$$

*Proof.*

$$\begin{aligned} \mathcal{F}([\zeta_R, \Theta(D)]f) &= \int \widehat{\zeta_R}(\eta)\hat{f}(\xi-\eta)[\Theta(\xi-\eta) - \Theta(\xi)]\mathrm{d}\eta \\ &= -\int \widehat{\zeta_R}(\eta)\hat{f}(\xi-\eta)\int_0^1 \eta\cdot\nabla\Theta(\xi-\tau\eta)\mathrm{d}\tau\mathrm{d}\eta \\ &= i\int \hat{f}(\xi-\eta)\widehat{\nabla\zeta_R}(\eta)\cdot\int_0^1 \nabla\Theta(\xi-\tau\eta))\mathrm{d}\tau\mathrm{d}\eta. \end{aligned} \tag{4.4.20}$$

Therefore, by Parseval's inequality and Young's inequality,

$$\begin{aligned} \|[\zeta_R, \Theta(D)]f\|_{L^2} &\lesssim \|\mathcal{F}([\zeta_R, \Theta(D)]f)\|_{L^2} \\ &\lesssim \|\widehat{\nabla\zeta_R}\|_{L^1}\|\hat{f}\|_{L^2}\|\nabla_\xi\Theta\|_{L^\infty} \lesssim \frac{1}{R}\|f\|_{L^2}. \end{aligned}$$

Note that in the above, we have used the fact: $\|\widehat{\nabla\zeta_R}\|_{L^1} \lesssim \frac{1}{R}$. □

**Corollary 4.4.12.** *Denote $\Phi_j$ the j-th dyadic function coming from Littlewood-Paley theory, and $\chi^L(\xi) = (\chi_1+\chi_2)(\sqrt{\frac{\varepsilon}{\kappa_0}}\xi)$ (See the definitions in Section 2). Then, for any $t\in(0,1]$, $2\le p<\infty$,*

$$\|[\Phi_j, e^{it\lambda_-(D)}\langle\nabla\rangle^2\chi^L(D)]f\|_{L^p} \lesssim_{\kappa_0,\varepsilon,p} 2^{-j}\|f\|_{L^2}. \tag{4.4.21}$$

*Proof.* By virtue of Hausdorff's inequality, identity (4.4.20) and Young's inequality, we have:

$$\begin{aligned}
\|[\Phi_j, e^{it\lambda_-(D)}]\langle\nabla\rangle^2\chi^L(D)f\|_{L^p} &\lesssim \|\mathcal{F}([\Phi_j, e^{it\lambda_-(D)}\langle\nabla\rangle^2\chi^L(D)]f)\|_{L^{p'}}\\
&\lesssim \|\widehat{\nabla\Phi_j}\|_{L^1}\|\hat f\|_{L^2}\|\nabla_\xi(e^{it\lambda_-(\xi)}\langle\xi\rangle^2\chi^L)\|_{L^{\frac{2p}{p-2}}}\\
&\lesssim 2^{-j}(\frac{\varepsilon}{\kappa_0})^{-(\frac12-\frac1p)}(1+\frac{\kappa_0}{\varepsilon})\|f\|_{L^2}.
\end{aligned}$$

□

We will need to estimate 'weighted product term' like $xfg$, the following lemma allows us not to lose derivative on weighted term.

**Lemma 4.4.13.** *For $s\ge 0$, $\iota>0$, the following weighted product estimate holds :*

$$\|xfg\|_{H^s} \lesssim \min\{\|xf\|_{L^2}\|g\|_{H^{s+1+\iota}}, \|xf\|_{L^\infty}\|g\|_{H^{s+\iota}}\} + \|xg\|_{L^2}\|f\|_{H^{s+1+\iota}} + \|f\|_{H^{s+\iota}}\|g\|_{H^{s+\iota}}.$$

*Proof.* Write $fg = \sum_{j\ge1} S_{j-1}f\Delta_j g + \Delta_j f S_j g$. Thanks to the Bernstein inequality and Young's inequality the first term can be estimated as:

$$\begin{aligned}
\|x\sum_{j\ge1}S_{j-1}f\Delta_j g\|_{H^s} &= \sum_{j\ge1}2^{js}(\|[x,S_{j-1}]f\Delta_j g\|_{L^2} + \|S_{j-1}(xf)\Delta_j g\|_{L^2})\\
&\le \sum_{j\ge1}2^{js}(\|f\|_{L^2}\|\Delta_j g\|_{L^2} + \min\{2^j\|xf\|_{L^2}, \|xf\|_{L^\infty}\}\|g\|_{L^2})\\
&\le \min\{\|xf\|_{L^2}\|g\|_{H^{s+1+\iota}}, \|xf\|_{L^\infty}\|g\|_{H^{s+\iota}}\} + \|xf\|_{L^2}\|g\|_{H^{s+\iota}}.
\end{aligned}$$

The second term can be controlled similarly, we omit the details. □

## 4.5 Local existence and time continuity of weighted norm

By classical iteration technique, one could construct solution for system (4.4.1) in $C([0,T_\varepsilon];H^N)$, $(N\ge 3)$ for some $T_\varepsilon>0$ (please refer to [97] for example), which leads to the local existence for system (4.4.3). We thus focus on the local boundedness of weighted norm $\|xe^{itb(D)}w(t)\|_{W^{4,\frac{2}{1-\delta}}}$ and its continuity in time. We start with the weighted $L^2_x$ estimate for high frequency which shall be useful later.

**Lemma 4.5.1.** *There exists a constant $M_0>0$, such that for small but fixed time $T_0<1$, the following a-priori estimate holds*

$$\sup_{t\in[0,T_0]}\|x(u^h,\varrho^h)\|_{L^2} \lesssim M_0 e^{M_0T_0}T_0(1+\|x(u^L,\varrho^L)\|_{L^\infty([0,T_0],L^\infty_x)}). \tag{4.5.1}$$

*Proof.* We consider the system satisfied by the high frequency:

$$\begin{cases}
\partial_t\varrho^h + \mathrm{div}u^h + \mathrm{div}(\varrho u)^h = 0,\\
\partial_t u^h + (u\cdot\nabla u)^h - 2\varepsilon\Delta u^h + \nabla\varrho^h - \nabla\varphi^h = 0,\\
\Delta\varphi^h = \varrho^h,\\
u|_{t=0} = \mathcal{P}^\perp u_0^\varepsilon, \varrho|_{t=0} = \varrho_0^\varepsilon.
\end{cases} \tag{4.5.2}$$

Set $\psi_R(x) = x\Psi(\frac{x}{R})$, where the compactly supported function $\Psi \equiv 1$ on $B_{\frac{3}{2}}$ and vanish on $B^c_{\frac{5}{3}}$. Multiplying the system (4.5.2) by $\psi_R(x)$, and test $\psi_R(\varrho^h, u^h)$, one gets the energy equality:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_R(\varrho^h, u^h)\|^2_{L^2} \\
= \;& -\int \psi_R^2(\varrho^h \mathrm{div} u^h - \nabla\varrho^h u^h)\mathrm{d}x + \int \psi_R(\nabla\varphi)^h\psi_R u^h \mathrm{d}x + 2\varepsilon\int \psi_R\Delta u^h \psi_R u^h \mathrm{d}x \\
& -\int \psi_R \mathrm{div}(\varrho u)^h \psi_R\varrho^h \mathrm{d}x - \int \psi_R(u\cdot\nabla u)^h\psi_R u^h \mathrm{d}x \\
\triangleq \;& \mathcal{T}_1 + \mathcal{T}_2 + \cdots \mathcal{T}_5.
\end{aligned}$$

We now estimate $\mathcal{T}_1, \cdots \mathcal{T}_5$. For $\mathcal{T}_1$, integration by parts and Höder inequality yield:

$$\mathcal{T}_1 = 2\int \psi_R\varrho^h\nabla\psi_R u^h \mathrm{d}x \le \|\psi_R\varrho^h\|_{L^2}\|\nabla\psi_R u^h\|_{L^2} \lesssim \|\psi_R\varrho^h\|_{L^2}\|u^h\|_{L^2}.$$

Note that $\nabla\psi_R$ is pointwise bounded uniformly in $R$. For $\mathcal{T}_2$, by Höder inequality,

$$\mathcal{T}_2 \lesssim \|\psi_R u^h\|_{L^2}\|\psi_R\nabla\varphi^h\|_{L^2} \lesssim \|\psi_R u^h\|_{L^2}(\|\psi_R\varrho^h\|_{L^2} + \|\varrho^h\|_{L^2})$$

where we have used

$$\psi_R\nabla\varphi^h = \psi_R\nabla(\Delta)^{-1}\tilde{\chi}^h\varrho^h = [\psi_R, \nabla(\Delta)^{-1}\tilde{\chi}^h]\varrho^h + \nabla(\Delta)^{-1}\tilde{\chi}^h(D)(\psi_R\varrho^h).$$

Notice that by the Lemma 4.4.11 and the spectral localization of $\tilde{\chi}^h$, one has that:

$$\begin{aligned}
\|[\psi_R, \nabla(\Delta)^{-1}\tilde{\chi}^h]\varrho^h\|_{L^2} &\lesssim \|\mathcal{F}^{-1}(\nabla\psi_R)\|_{L^1}\|\partial_\xi(\frac{\xi}{|\xi|^2}\tilde{\chi}^h)\|_{L^\infty_\xi}\|\varrho^h\|_{L^2} \lesssim \frac{\varepsilon}{\kappa_0}\|\varrho^h\|_{L^2}, \\
\|\nabla(\Delta)^{-1}\tilde{\chi}^h(D)(\psi_R\varrho^h)\|_{L^2} &\lesssim \sqrt{\frac{\varepsilon}{\kappa_0}}\|\psi_R\varrho^h\|_{L^2}.
\end{aligned}$$

For $K_3$, it is direct to see

$$\begin{aligned}
\mathcal{T}_3 + 2\varepsilon\int |\psi_R\nabla u^h|^2 \mathrm{d}x &= -4\varepsilon\int \psi_R u^h\nabla u^h\nabla\psi_R \mathrm{d}x \\
&\lesssim \|\psi_R u^h\|_{L^2}\|\nabla u^h\|_{L^2}.
\end{aligned}$$

For $\mathcal{T}_4$, using again $[\psi_R, \chi^h(D)]$ belongs to $\mathcal{L}(L^2(\mathbb{R}^2))$ whose norm is independent of $R$, we get:

$$\begin{aligned}
\mathcal{T}_4 &= -\int [\psi_R, \chi^h(D)](u\cdot\nabla u)\psi_R u^h + \chi^h(D)(\psi_R u\cdot\nabla u)\psi_R u^h \mathrm{d}x \\
&\lesssim \|\psi_R u^h\|_{L^2}\big(\|u\cdot\nabla u\|_{L^2} + \|\psi_R u^h\|_{L^2}\|\nabla u\|_{L^\infty} + \|\psi_R u^L\|_{L^\infty}\|\nabla u\|_{L^2}\big).
\end{aligned}$$

Similarly, for $\mathcal{T}_5$, we have:

$$\begin{aligned}
\mathcal{T}_5 &= -\int [\psi_R, \chi^h(D)]\mathrm{div}(\varrho u)\psi_R\varrho^h + \chi^h(\nabla\varrho\cdot\psi_R u + \psi_R\varrho \mathrm{div} u)\psi_R\varrho^h \mathrm{d}x \\
&\lesssim \|\psi_R\varrho^h\|_{L^2}\big(\|\mathrm{div}(\varrho u)\|_{L^2} + \|\psi_R(u^h, \varrho^h)\|_{L^2}\|(u, \varrho)\|_{W^{1,\infty}} + \|\psi_R(u^L, \varrho^L)\|_{L^\infty}\|(\varrho, u)\|_{H^1}\big).
\end{aligned}$$

Summing up the above estimates, we finally get:

$$\begin{aligned}
\partial_t\|\psi_R(u^h, \varrho^h)\|_{L^2} \lesssim \;& \|\psi_R(u^h, \varrho^h)\|_{L^2}(1 + \|(u, \varrho)\|_{W^{1,\infty}}) \\
& + (1 + \|(u, \varrho)\|_{L^\infty})\|(u, \varrho)\|_{H^1} + \|\psi_R(u^L, \varrho^L)\|_{L^\infty}\|(\varrho, u)\|_{H^1}.
\end{aligned}$$

Grönwall's inequality then gives us for any $t < T_0$, there exists constant $M_0$ which depends on $\|(u, \varrho)\|_{L^\infty([0,T_0], H^1\cap W^{1,\infty})}$,

$$\|\psi_R(u^h, \varrho^h)(t)\|_{L^2} \lesssim M_0 e^{M_0 T_0} T_0(1 + \|\psi_R(u^L, \varrho^L)\|_{L^\infty([0,T_0], L^\infty_x)}).$$

Letting $R$ tends to $+\infty$, we obtain (4.5.1) $\square$

.

One can easily adapt the above proof to derive the following Corollary:

**Corollary 4.5.2.** *There exists another constant $M_0'$ which is dependent on $\|(u,\varrho)\|_{L^\infty([0,T_0],H^1\cap W^{1,\infty}\cap W^{1,4})}$, such that for any $j\geq 0$,*

$$\sup_{t\in[0,T_0]}\|\Phi_j(\cdot)(u^h,\varrho^h)(\cdot)\|_{L^2}\lesssim M_0'e^{M_0'T_0}T_0(2^{-j}+\|\Phi_j(u^L,\varrho^L)\|_{L^\infty([0,T_0],L^4_x)}).$$

*where $\Phi_j$ is the dyadic function defined in* (4.2.3).

### 4.5.1 Weighted norm for low frequency: $x(u^L,\varrho^L)$

By the Duhamel formula,

$$w=e^{t\lambda_-(D)}w_0+\int_0^t e^{(t-s)\lambda_-(D)}\chi^L(D)\big[\frac{\lambda_+(D)}{2ib}\langle\nabla\rangle\frac{\mathrm{div}}{|\nabla|}(\varrho u)+\frac{\langle\nabla\rangle}{2ib}|\nabla||u|^2\big]\mathrm{d}s. \tag{4.5.3}$$

Since $\varrho^L=\frac{|\nabla|}{\langle\nabla\rangle}(w+\bar{w})$, $u^L=-\mathcal{R}\big(\frac{\lambda_-(D)}{\langle\nabla\rangle}w+\overline{\frac{\lambda_-(D)}{\langle\nabla\rangle}w}\big)$, we define the linear and nonlinear flow for $\varrho^L,u^L$:

$$\mathcal{L}_{u^L}=\mathcal{R}\frac{\lambda_-(D)}{\langle\nabla\rangle}e^{it\lambda_-(D)}w_0\qquad \mathcal{L}_{\varrho^L}=\frac{|\nabla|}{\langle\nabla\rangle}e^{it\lambda_-(D)}w_0 \tag{4.5.4}$$

$$\begin{aligned}\mathcal{N}_{u^L}&=\int_0^t e^{i(t-s)\lambda_-(D)}\chi^L(D)\big[\mathcal{R}\frac{\lambda_-\lambda_+(D)}{2ib}\frac{\mathrm{div}}{|\nabla|}(\varrho u)+\frac{\lambda_-(D)}{2ib}\nabla|u|^2\big]\mathrm{d}s\\&=\int_0^t e^{i(t-s)\lambda_-(D)}\chi^L(D)\big[\mathcal{R}\frac{\langle\nabla\rangle^2}{2ib}\frac{\mathrm{div}}{|\nabla|}(\varrho u)+\frac{\lambda_-(D)}{2ib}\nabla|u|^2\big]\mathrm{d}s\\ \mathcal{N}_{\varrho^L}&=\int_0^t e^{i(t-s)\lambda_-(D)}\chi^L(D)\big[\frac{\lambda_+(D)}{2ib}\mathrm{div}(\varrho u)-\frac{\mathrm{div}}{2ib}\nabla|u|^2\big]\mathrm{d}s\end{aligned}$$

Inspired by [84], we need to prove the following claim:
**Claim**: For $j\geq 0$, one has:

$$\sup_{t\in[0,T_0]}\|\Phi_j(\cdot)(\langle\nabla\rangle u^L,\langle\nabla\rangle\varrho^L)(\cdot)\|_{L^4_x}\lesssim 2^{-j} \tag{4.5.5}$$

We will postpone the proof of (7.11.30) and first show the local boundedness of $\|xe^{itb(D)}w\|_{W^{4,\frac{2}{1-\delta}}}$ and its continuity in time. By (7.11.30), we have for any $j\in\mathbb{N}$,

$$\sup_{t\in[0,T_0]}\|\Phi_j(u^L,\varrho^L)\|_{W^{1,4}}\lesssim 2^{-j}.$$

which leads to, by Sobolev embedding,

$$\sup_{t\in[0,T_0]}\|x(u^L,\varrho^L)\|_{L^\infty_x}\lesssim 1.$$

Note that by (4.5.1), we have also $\sup_{t\in[0,T_0]}\|x(u^h,\varrho^h)(t)\|_{L^2_x}<+\infty$.

• Local boundedness of $\|xe^{itb(D)}w\|_{W^{4,\frac{2}{1-\delta}}}$. The boundedness of the linear term $xw_0$ stems from the assumption imposed upon the initial data. We thus focus on the boundedness of the nonlinear term. In light of (4.5.3), it suffices for us to consider the typical term :

$$x\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\mathcal{R}\chi^L(\varrho u)\mathrm{d}s.$$

Nevertheless, since for arbitrary function $g$,

$$\|x\mathcal{R}g\|_{W^{4,\frac{2}{1-\delta}}}\lesssim\||\nabla|^{-1}g\|_{W^{4,\frac{2}{1-\delta}}}+\|xg\|_{W^{4,\frac{2}{1-\delta}}}\lesssim\|g\|_{H^{4+\delta}}+\|xg\|_{H^{4+\delta}}, \tag{4.5.6}$$

it remains for us to control $\|x\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\chi^L(\varrho u)\mathrm{d}s\|_{H^{4+\delta}}$.

To begin with, we write:

$$\begin{aligned}
& x\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\chi^L(\varrho u)\mathrm{d}s \\
= & \int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}(sb'(D)+2\varepsilon(t-s)\nabla)\chi^L\langle\nabla\rangle(\varrho u)\mathrm{d}s \\
& +\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}(\chi^L\langle\cdot\rangle)'(D)(\varrho u)\mathrm{d}s+\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\chi^L\langle\nabla\rangle(x\varrho u)\mathrm{d}s \\
\triangleq & \ I+II+III.
\end{aligned}$$

On one hand, the first two terms can be controlled directly by:

$$\|I+II\|_{H^{4+\delta}}\lesssim\int_0^t\|\varrho u\|_{H^{5+\delta}}\mathrm{d}s\lesssim T_0\sup_{t\in[0,T_0]}\|\varrho\|_{H^{5+\delta}}\|u\|_{H^{5+\delta}}. \tag{4.5.7}$$

On the other hand, by Lemma 4.4.13:

$$\begin{aligned}
\|III\|_{H^{4+\delta}} & \lesssim \int_0^t\|x\varrho u\|_{H^{5+\delta}}\mathrm{d}s \\
& \lesssim T_0\sup_{t\in[0,T_0]}(\|x(u^L,\varrho^L)\|_{L^\infty}\|(u,\varrho)\|_{H^{5+2\delta}} \\
& \qquad +\|x(u^h,\varrho^h)\|_{L^2}\|(u,\varrho)\|_{H^{6+2\delta}}+\|(u,\varrho)\|^2_{H^{5+2\delta}}).
\end{aligned} \tag{4.5.8}$$

• Continuity in time of the weighted norm. By (4.5.6) again, it suffices for us to prove the continuity in time of the quantity

$$\|xe^{\varepsilon t\Delta}\int_0^t e^{isb(D)}e^{-\varepsilon s\Delta}\chi^L\langle\nabla\rangle(\varrho u)\mathrm{d}s\|_{H^{4+\delta}}.$$

Denote $z(t)=\int_0^t e^{isb(D)}e^{-\varepsilon s\Delta}\chi^L\langle\nabla\rangle(\varrho u)(s)\mathrm{d}s$. Notice that $xe^{\varepsilon t\Delta}z(t)=\varepsilon t\nabla e^{\varepsilon t\Delta}z(t)+e^{\varepsilon t\Delta}(xz(t))$. Since $e^{\varepsilon t\Delta}$ is a continuous operator in $H^{4+\delta}$, we reduce the problem to the continuity of $\|xz(t)\|_{H^{4+\delta}}$, which is the consequence of the following:

$$\sup_{t\in[0,T_0]}\|xe^{itb(D)}e^{-\varepsilon s\Delta}\chi^L\langle\nabla\rangle(\varrho u)(t)\|_{H^{2+\delta}}<+\infty.$$

However, it has essentially been included in the proof of (4.5.7),(4.5.8). One notes here that $e^{-\varepsilon s\Delta}\chi^L$ is a $L^2$ multiplier whose norm is less than $e^{\kappa_0 t}\le e^{\kappa_0 T_0}$.

### 4.5.2 Proof of the claim

We are now in position to prove (7.11.30).

• Linear flow estimate. In light of (4.5.4) and the crude approximation:

$$w_0=\frac{\lambda_+}{2ib(D)}\frac{\langle\nabla\rangle}{|\nabla|}\varrho_0^L+\frac{\langle\nabla\rangle}{2ib}\mathcal{R}\cdot u_0^L=\mathcal{R}(\frac{\langle\nabla\rangle}{|\nabla|}\varrho_0^L+u_0^L)=\mathcal{R}(\langle\nabla\rangle\mathcal{R}\cdot\nabla\varphi_0^L+u_0^L),$$

it suffices for us to show that for $\forall j\ge 0$,

$$\|\Phi_j\mathcal{R}n_1(D)e^{it\lambda_-(D)}g\|_{L^4}\lesssim 2^{-j}\|\langle x\rangle g\|_{H^{\frac{3}{2}}} \tag{4.5.9}$$

where we denote $n_1(D)=\lambda_-(D)$ or $|\nabla|$, $g=u_0^L$ or $\langle\nabla\rangle\nabla\varphi^L$. Nevertheless, we have by Sobolev embedding, Hausdorff-Young inequality that,

$$\begin{aligned}
& \|x\mathcal{R}n_1(D)e^{it\lambda_-(D)}\tilde{\chi}^L(D)g\|_{L^4} \\
\lesssim & \|n_1(D)e^{it\lambda_-(D)}\tilde{\chi}^L(D)g\|_{L^2}+\|xn_1(D)e^{it\lambda_-(D)}\tilde{\chi}^L(D)g\|_{H^{\frac{1}{2}}} \\
\lesssim & \|g\|_{H^1}+\|n_1(D)e^{it\lambda_-(D)}\tilde{\chi}^L(D)xg\|_{H^{\frac{1}{2}}}+\|[x,n_1(D)e^{it\lambda_-(D)}\tilde{\chi}^L(D)]g\|_{H^{\frac{1}{2}}} \\
\lesssim & \|g\|_{H^1}+\|xg\|_{H^{\frac{3}{2}}}+(1+t)\|g\|_{H^{\frac{1}{2}}}\lesssim\langle t\rangle\|\langle x\rangle g\|_{H^{\frac{3}{2}}}.
\end{aligned}$$

• Nonlinear flow estimate.

For notational brevity, we replace again $\chi^L\frac{\lambda_\pm}{2ib},\chi^L\frac{\langle\nabla\rangle}{2ib},\chi^L\frac{\langle\nabla\rangle}{2ib},\frac{|\nabla|}{\langle\nabla\rangle}$ by $\mathcal{R}$ since they have the similar properties. Therefore, it remains for us to show: for any $j\in\mathbb{Z}$

$$\|\Phi_j(x)\int_0^t e^{i(t-s)\lambda_-(D)}\chi^L(D)\mathcal{R}\langle\nabla\rangle^2(\varrho u+|u|^2)\mathrm{d}s\|_{L^4}\lesssim_{\varepsilon,\kappa_0,T_0}2^{-j} \tag{4.5.10}$$

By Corollary 4.4.12:

$$\begin{aligned}
&\|\Phi_j(x)\int_0^t e^{i(t-s)\lambda_-(D)}\chi^L(D)\mathcal{R}\langle\nabla\rangle^2(\varrho u+|u|^2)\mathrm{d}s\|_{L^4}\\
\lesssim_{\varepsilon,\kappa_0,T_0}\quad&2^{-j}+\|\int_0^t e^{i(t-s)\lambda_-(D)}\langle\nabla\rangle^2\chi^L(D)\big(\Phi_j(x)\mathcal{R}(\varrho u+|u|^2)\big)\mathrm{d}s\|_{L^4}\\
\lesssim_{\varepsilon,\kappa_0,T_0}\quad&2^{-j}+\|\int_0^t e^{i(t-s)\lambda_-(D)}\langle\nabla\rangle^{\frac52}\chi^L(D)\big(\Phi_j(x)\mathcal{R}(\varrho u+|u|^2)\big)\mathrm{d}s\|_{L^2}\\
\lesssim_{\varepsilon,\kappa_0,T_0}\quad&2^{-j}+T_0(1+\frac{\kappa_0}{\varepsilon})^{\frac54}\sup_{t\in[0,T_0]}\|\Phi_j(x)\mathcal{R}(\varrho u+|u|^2)(t)\|_{L^2}
\end{aligned}\tag{4.5.11}$$

To prove (4.5.10), we are left to establish that:

$$\sup_{t\in[0,T_0]}\|\Phi_j(x)\mathcal{R}(\Phi_{\geq j+2}+\Phi_{\leq j-2})(\varrho u+|u|^2)\|_{L^2}\lesssim 2^{-j} \tag{4.5.12}$$

We show for example the case of $\Phi_{\geq j+2}$ as the other is similar. Denote $\dot{\Delta}_k\mathcal{R}\tilde{\chi}^L g=G_k\star g$, where

$$G_k(x)=\int e^{ix\cdot\xi}\Phi_k(\xi)\frac{\xi}{|\xi|}\mathrm{d}\xi=2^{2k}\int e^{i2^kx\cdot\xi}\Phi(\xi)\frac{\xi}{|\xi|}\mathrm{d}\xi$$

By virtue of the identity $e^{i2^kx\cdot\xi}=\frac{1-i2^kx\cdot\nabla_\xi}{\langle 2^kx\rangle^2}e^{i2^kx\cdot\xi}$, one can integrate by parts to get that, for any non-negative integer $l$,

$$|G_k(x)|\lesssim 2^{2k}\langle 2^k|x|\rangle^{-l}.$$

Note that there is no singularity when the derivative hits on $\frac{\xi}{|\xi|}$ since $\Phi(\xi)$ is supported on annulus. Therefore, we have that

$$\begin{aligned}
&\|\Phi_j(x)\mathcal{R}\Phi_{\geq j+2}(\varrho u+|u|^2)\|_{L^2}\lesssim\sum_{k\in\mathbb{Z}}\|\Phi_j(x)\dot{\Delta}_k\mathcal{R}\Phi_{\geq j+2}(\varrho u+|u|^2)\|_{L^2}\\
\lesssim\quad&\sum_{k\in\mathbb{Z}}\|G_kI_{|\cdot|\gtrsim 2^j}\star\Phi_{\geq j+2}(\varrho u+|u|^2)\|_{L^2}\lesssim\sum_{k\in\mathbb{Z}}\|G_kI_{|\cdot|\gtrsim 2^j}\|_{L^2}\|\varrho u+|u|^2\|_{L^1}\\
\lesssim\quad&\sum_{k\in\mathbb{Z}}\langle 2^k2^j\rangle^{-3}2^k\|(\varrho,u)\|_{L^2}^2\lesssim 2^{-j}\|(\varrho,u)\|_{L^2}^2.
\end{aligned}$$

where the following two inequalities has been used:

$$\|G_kI_{|\cdot|\gtrsim 2^j}\|_{L^2}\lesssim 2^{2k}(\int_{|x|\gtrsim 2^j}\langle 2^k|x|\rangle^{-8}\mathrm{d}x)^{\frac12}\lesssim\langle 2^k2^j\rangle^{-3}2^k,$$

$$\sum_{k\in\mathbb{Z}}\langle 2^k2^j\rangle^{-3}2^k\lesssim(\sum_{k\leq -j}+\sum_{k\geq -j})\langle 2^k2^j\rangle^{-3}2^k\lesssim 2^{-j}.$$

Finally, denote $\tilde{\Phi}_j=\Phi_{j-1}+\Phi_j+\Phi_{j+1}$, Corollary 4.5.2 implies :

$$\begin{aligned}
\|\Phi_j\mathcal{R}\tilde{\Phi}_j(\varrho u+|u|^2)\|_{L^2}\quad&\lesssim\quad\|\tilde{\Phi}_j(\varrho u+|u|^2)\|_{L^2}\\
&\lesssim\quad\|\tilde{\Phi}_j(u^L,\varrho^L)\|_{L^4}\|(u,\varrho)\|_{L^4}+\|\tilde{\Phi}_j(u^h,\varrho^h)\|_{L^2}\|(u,\varrho)\|_{L^\infty}\\
&\lesssim_{\varepsilon,\kappa_0}\ (1+T_0)(2^{-j}+\|\tilde{\Phi}_j(u^L,\varrho^L)\|_{L^4}).
\end{aligned}\tag{4.5.13}$$

Define $y_j=\|\Phi_j(x)(\langle\nabla\rangle u^L,\langle\nabla\rangle\varrho^L)\|_{L^4}$, by (4.5.9), (4.5.11)-(4.5.13), one gets that, as long as $T_0$ is chosen small enough,

$$y_j\lesssim 2^{-j}+\frac{1}{16}(y_{j-1}+y_j+y_{j+1}),$$

which, combined with the fact $y_j\lesssim 1$ ($j\geq 0$) and iteration arguments, yields that $y_j\lesssim 2^{-j}$. We thus finish the proof of (7.11.30).

## 4.6 Weighted $L^2$ norm for high frequency: a priori estimate

The goal of this section is to get the weighted $L^2$ estimate for $(\varrho, u, \nabla\varphi)^h$ which shall be used in Section 7.

**Lemma 4.6.1.** *There exists a constant $\vartheta_2 > 0$, for any $\varepsilon \in (0,1]$, if $\|U\|_{X_T} \leq \vartheta_2$, we have the a priori estimate:*

$$\|\langle x\rangle\big(\varrho, u, \nabla\varphi)^h(t)\|_{L^2} \lesssim \langle t\rangle^{2\delta}(\|\langle x\rangle(\varrho_0, u_0, \nabla\varphi_0)^h\| + \|U\|_{X_T}\big), \quad \forall t \in [0,T) \tag{4.6.1}$$

*Proof.* We first claim that the following estimate holds,

$$\|x(\varrho, u, \nabla\varphi)^L(t)\|_{W^{1,4}} \lesssim \langle t\rangle^{\frac{1}{2}+\delta}\|U\|_{X_T}. \tag{4.6.2}$$

Indeed, by virtue of the Hardy-Littlewood-Sobolev inequality, the $L^p$ boundedness of $e^{itb(D)}\tilde{\chi}^L$ (Lemma 4.4.2),

$$\begin{aligned}
\|x(\varrho, u, \nabla\varphi)^L\|_{W^{1,4}} &= \|x\mathcal{R}\tilde{\chi}^L w\|_{W^{1,4}} \approx \||\nabla|^{-1}w\|_{W^{1,4}} + \|xw\|_{W^{1,4}} \lesssim \|\langle x\rangle w\|_{W^{1+\delta,4-\delta}} \\
&\lesssim \|w\|_{H^2} + \|xe^{itb(D)}f\|_{W^{1+\delta,4-\delta}} \lesssim \|U\|_X + t\|w\|_{W^{1+\delta,4-\delta}} + \|e^{itb(D)}\tilde{\chi}^L xf\|_{W^{1+\delta,4-\delta}} \\
&\lesssim \|U\|_{X_T} + t^{1-(1-\frac{2}{4-\delta})}\|U\|_{X_T} + \|e^{itb(D)}\tilde{\chi}^L xf\|_{W^{\frac{3}{2}+\delta,2+\delta}} \lesssim \langle t\rangle^{\frac{1}{2}+\delta}\|U\|_{X_T}.
\end{aligned}$$

Note that $f = e^{-itb(D)}w$. Hereafter, unless specifically emphasized, the spatial Sobolev norm is estimated in each temporary time $t \in [0,T)$.
The system projected onto the high frequency reads:

$$\begin{cases}
\partial_t\varrho^h + \mathrm{div}u^h + \mathrm{div}(\varrho u)^h = 0, \\
\partial_t u^h + (u\cdot\nabla u)^h - 2\varepsilon\Delta u^h + \nabla\varrho^h - \nabla\varphi^h = 0, \\
\Delta\varphi^h = \varrho^h, \\
u|_{t=0} = \mathcal{P}^{\perp}u_0^{\varepsilon}, \varrho|_{t=0} = \varrho_0^{\varepsilon}.
\end{cases} \tag{4.6.3}$$

Multiplying the system (4.6.3) by $x$, and testing it by $x(\varrho^h, u^h)$, we obtain the energy equality:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(\varrho^h, u^h, \nabla\varphi^h)\|_{L^2}^2 + 2\varepsilon\int|x\nabla u^h|^2\mathrm{d}x \\
= &-\int x^2(\mathrm{div}u^h\varrho^h - \nabla\varrho^h u^h)\mathrm{d}x + \int x^2((\nabla\varphi)^h u^h - \nabla\varphi^h\nabla(\Delta)^{-1}\mathrm{div}u^h)\mathrm{d}x - 4\varepsilon\int\nabla u^h x u^h\mathrm{d}x \\
&-\int x\mathrm{div}(\varrho u)^h x\varrho^h\mathrm{d}x - \int x(u\cdot\nabla u)^h x u^h\mathrm{d}x - \int x\nabla\varphi^h x\nabla(\Delta)^{-1}\mathrm{div}(\varrho u)^h\mathrm{d}x \\
\triangleq\ & G_1 + G_2 + \cdots G_6.
\end{aligned}$$

The following task is to estimate $G_1, \cdots G_6$. At first, by integration by parts and Hölder inequality, $G_1$ can be controlled as:

$$G_1 = -2\int x\varrho^h u^h\mathrm{d}x \lesssim \|x\varrho^h\|_{L^2}\|u^h\|_{L^2}.$$

Similarly, by using the fact $\|\varphi^h\|_{L^2} \lesssim \sqrt{\frac{\varepsilon}{\kappa_0}}\|\nabla\varphi^h\|_{L^2}$,

$$\begin{aligned}
G_2 &= 2\int\varphi^h x(\nabla(\Delta)^{-1}\mathrm{div} - 1)u^h\mathrm{d}x \\
&\lesssim \|\varphi^h\|_{L^2}(\|xu^h\|_{L^2} + \|[x, \nabla(\Delta)^{-1}\mathrm{div}\tilde{\chi}^h]u^h\|_{L^2}) \\
&\lesssim \|\varphi^h\|_{L^2}(\|xu^h\|_{L^2} + \|\partial_\xi(\frac{\xi_i\xi_j}{|\xi|^2}\tilde{\chi}^h)\|_{L^\infty_\xi}\|u^h\|_{L^2}) \\
&\lesssim (\sqrt{\frac{\varepsilon}{\kappa_0}} + \frac{\varepsilon}{\kappa_0})\|\nabla\varphi^h\|_{L^2}(\|u^h\|_{L^2} + \|xu^h\|_{L^2}).
\end{aligned}$$

For $G_3$, we just use Hölder inequality: $G_3 \leq 2\varepsilon \|\nabla u^h\|_{L^2} \|xu^h\|_{L^2}$.
By setting $P(f,g) = f^h g + f^L g^h + f^L g^m + f^m g^l$, we estimate $G_4$ as follows:

$$\begin{aligned}
G_4 &= -\int x\mathrm{div}(\varrho u)^h x\varrho^h \mathrm{d}x = -\int ([x,\chi^h]\mathrm{div}(\varrho u) + \chi^h x\big(P(\nabla\varrho, u) + P(\varrho, \mathrm{div}u)\big)) x\varrho^h \mathrm{d}x \\
&\lesssim \|x\varrho^h\|_{L^2}\big(\sqrt{\frac{\varepsilon}{\kappa_0}}\|\mathrm{div}(\varrho u)^h\|_{L^2} + \|x\big(P(\nabla\varrho,u) + P(\varrho,\mathrm{div}u)\big)\|_{L^2}\big) \\
&\lesssim \|x\varrho^h\|_{L^2}\sqrt{\frac{\varepsilon}{\kappa_0}}\|(\varrho,u)\|_{L^2}\|\nabla(\varrho,u)\|_{L^\infty} + \|x\varrho^h\|_{L^2}\big(\langle t\rangle^{-(1-\delta)}\|U\|_{X_T}^2 + \|x(\varrho^h,u^h)\|_{L^2}\langle t\rangle^{-1}\|U\|_{X_T}\big) \\
&\lesssim \langle t\rangle^{-1}\|U\|_{X_T}(\|x\varrho^h\|_{L^2}^2 + \|xu^h\|_{L^2}^2) + \langle t\rangle^{-(1-\delta)}\|U\|_{X_T}^2\|x\varrho^h\|_{L^2}.
\end{aligned}$$

where we have used that:

$$\|xP(\nabla\varrho,u)\|_{L^2} + \|xP(\varrho,\mathrm{div}u)\|_{L^2} \lesssim (1+t)^{-1}\|x(\varrho,u)^h\|_{L^2} + (1+t)^{-(1-\delta)}\|U\|_{X_T}^2.$$

For example, since $\|(u^m,\varrho^m)\|_{W^{1,4}} \lesssim \langle t\rangle^{-\frac{3}{2}}\|U\|_{X_T}, \|u^h\|_{H^M} \lesssim \langle t\rangle^{-\alpha}\|U\|_{X_T}, (\alpha = 2-5\delta)$, one has that

$$\begin{aligned}
\|xP(\nabla\varrho,u)\|_{L^2} &\lesssim \|xu^h\|_{L^2}\|\nabla\varrho^h\|_{L^\infty} + \|xu^L\|_{L^4}\|\nabla\varrho^h\|_{L^4} + \|x\nabla\varrho^L\|_{L^4}\|u^h + u^m\|_{L^4} + \|xu^L\|_{L^4}\|\nabla\varrho^m\|_{L^4} \\
&\lesssim (1+t)^{-1}\|x(\varrho,u)^h\|_{L^2} + (1+t)^{-(1-\delta)}\|U\|_{X_T}^2.
\end{aligned}$$

The estimate of $G_5, G_6$ is similar, we thus omit the details. To summarize, we have obtained:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|x(\varrho,u,\nabla\varphi)^h\|_{L^2}^2) \\
\lesssim\ &(1+t)^{-1}\|U\|_{X_T}\|x(\varrho,u,\nabla\varphi)^h\|_{L^2}^2 + (1+t)^{-(1-\delta)}\|U\|_{X_T}^2\|x(\varrho,u,\nabla\varphi)^h\|_{L^2} \\
&+(1+t)^{-\alpha}\|U\|_{X_T}\|\langle x\rangle(\varrho,u,\nabla\varphi)^h\|_{L^2}.
\end{aligned} \tag{4.6.4}$$

In the same fashion, one can show that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\varrho,u,\nabla\varphi)^h\|_{L^2}^2) \lesssim (1+t)^{-1}\|U\|_{X_T}\|(\varrho,u,\nabla\varphi)^h\|_{L^2}^2 \tag{4.6.5}$$

Finally, we set $g(t) = \|\langle x\rangle(\varrho,u,\nabla\varphi)^h\|_{L^2}$. Summing up the above two estimates (4.6.4)-(4.6.5), we see that there exists three constants $c_1, c_2, c_3$ which are independent of $\varepsilon$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) \leq c_1(1+t)^{-1}\|U\|_{X_T}g(t) + c_2(1+t)^{-(1-\delta)}\|U\|_{X_T}^2 + c_3(1+t)^{-\alpha}\|U\|_{X_T}.$$

Suppose that $\|U\|_{X_T} \leq \vartheta_2 \leq \frac{\delta}{c_1}$, then for any $0 \leq t < T$, the Grönwall inequality leads to:

$$g(t) \lesssim \langle t\rangle^{\delta}g(0) + \langle t\rangle^{2\delta}\|U\|_{X_T}^2 + \|U\|_{X_T} \lesssim \langle t\rangle^{2\delta}(\|\langle x\rangle(\varrho_0,u_0,\nabla\varphi_0)^h\|_{L^2} + \|U\|_{X_T}).$$

□

## 4.7 Estimate of Sobolev norm

In this section, we aim to get the highest Sobolev estimate for $U$: $\|U\|_{H^N}$ and Sobolev estimate for high and intermediate frequencies: $\|U^h\|_{H^{N-2}}$, $\|U^m\|_{H^{N-1}}$ and $\|U^m\|_{W^{1,4}}$.

### 4.7.1 Control of highest Sobolev norms

Define the energy norm

$$E_N(t) = \sum_{|\alpha|\leq N}\frac{1}{2}\int \rho|\partial^\alpha u(t)|^2 + |\partial^\alpha\varrho(t)|^2 + |\partial^\alpha\nabla\phi(t)|^2\mathrm{d}x.$$

In the former chapter where the 3d NSP is considered, it has been shown that if $\|\varrho\|_{L^\infty} \lesssim \frac{1}{6}$, then the following energy inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_N(t) \lesssim (\|u(t)\|_{W^{1,\infty}} + \|\varrho(t)\|_{L^\infty})E_N(t).$$

However, such an inequality is not enough to close the energy estimate in 2d case. Indeed, due to the presence of Riesz potential in the quadratic nonlinearity (see (4.4.3)), one could only expect that $\|\nabla u(t)\|_{L^\infty}$ rather than $\|u(t)\|_{W^{1,\infty}}$ has the critical decay $(1+t)^{-1}$ . Nevertheless, it is not hard to modify the proof in [108] to get that:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_N(t) \lesssim (\|\nabla u(t)\|_{L^\infty} + \|\varrho(t)\|_{L^\infty})E_N(t). \tag{4.7.1}$$

Indeed, denote $E_\alpha = \frac{1}{2}\int \rho|\partial^\alpha u|^2 + |\partial^\alpha \rho|^2 + |\partial^\alpha \nabla\phi|^2 \mathrm{d}x$, we then have by using the equations (4.4.1) that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha &= \int \rho\partial^\alpha u \cdot [\partial^\alpha, u]\nabla u + \int \partial^\alpha \rho \mathrm{div}\big([\partial^\alpha, u]\rho\big) \\
&\quad + \int [\rho\partial^\alpha u \partial^\alpha \nabla\varphi + \partial^\alpha\varphi\partial^\alpha \mathrm{div}(\rho u)] + 2\varepsilon \int \rho\partial^\alpha u \cdot \partial^\alpha \Delta u \\
&\triangleq L_1 + L_2 + L_3 + L_4
\end{aligned}$$

One can estimate all the terms in the same way as that in [108] except the term $L_3$. However, for any $|\alpha| \geq 1$, it can be rewritten as

$$\begin{aligned}
L_3 &= \int \varrho\partial^\alpha u\partial^\alpha\nabla\varphi + \partial^\alpha\varphi\partial^\alpha \mathrm{div}(\varrho u)\mathrm{d}x \\
&= \int \varrho\partial^\alpha u\partial^\alpha\nabla\varphi + \partial^\alpha(\varrho \mathrm{div} u)\partial^\alpha\varphi + [\partial^\alpha, u]\nabla\varrho\partial^\alpha\varphi + \partial^\alpha\nabla\varrho\cdot u\partial^\alpha\nabla\varphi \mathrm{d}x \\
&= \int \partial^\alpha\varphi\big([\partial^\alpha, \varrho]\mathrm{div} u + \nabla\varrho\cdot\partial^\alpha u\big) + [\partial^\alpha, u]\nabla\varrho\partial^\alpha\varphi - \partial^\alpha\varrho \mathrm{div} u\partial^\alpha\varphi - \frac{\partial^\alpha|\nabla\varphi|^2}{2}\mathrm{div} u \\
&\qquad + \partial^\alpha\nabla\varphi\cdot\nabla u\cdot\partial^\alpha\nabla\varphi \mathrm{d}x
\end{aligned}$$

Notice that in the above expressions, there is at least one spatial derivative in front of $u$, we thus conclude by standard commutator estimate that:

$$|L_3| \leq (\|\nabla u\|_{L^\infty} + \|\varrho\|_{W^{1,\infty}})(\|u\|^2_{\dot{H}^{|\alpha|}} + \|\varrho\|^2_{\dot{H}^{|\alpha|}} + \|\nabla\varphi\|^2_{\dot{H}^{|\alpha|-1}\cap\dot{H}^{|\alpha|}}),$$

which ends the proof (4.7.1).

It is easy to see that $\|U\|^2_{H^N} \approx E_N$ by noting the relation $u = \mathcal{R}\mathrm{c}, \varrho = \frac{|\nabla|}{\langle\nabla\rangle}\mathrm{a}$. This, combined with (4.7.1) and the definition of $X_T$ norm (4.4.9), yields:

$$\begin{aligned}
\|U\|^2_{H^N} \lesssim E_N(t) &\lesssim E_N(0) + \int_0^t (\|\nabla u\|_{L^\infty} + \|\varrho\|_{L^\infty})\|U(s)\|^2_{H^N}\mathrm{d}s \\
&\lesssim E_N(0) + \int_0^t (1+s)^{-1}(1+s)^{2\delta}\|U\|^2_{X_T}\mathrm{d}s \lesssim E_N(0) + (1+t)^{2\delta}\|U\|^2_{X_T}.
\end{aligned}$$

### 4.7.2 High and intermediate frequency estimate

We have firstly the following estimate for nonlinear term $F(a, c) = F(U, U)$.

**Lemma 4.7.1.** *For every $t \in [0, T)$, the following estimate holds:*

$$\|\chi^h F(U, U)(t)\|_{H^{N-2}} \lesssim (1+t)^{-(2-5\delta)}\|U\|^2_X, \tag{4.7.2}$$

$$\|F(U, U)(t)\|_{H^{N-1}} \lesssim (1+t)^{-(1-3\delta)}\|U\|^2_{X_T}. \tag{4.7.3}$$

*Proof.* We begin with the proof of (4.7.2). By the definition of truncation functions, one has $\chi^l(\xi-\eta)\chi^l(\eta)\chi^h(\xi)=0$, which leads to the decomposition:

$$\chi^h(D)F(U,U)=\chi^h(D)\big(F(U^L,U^h)+F(U^h,U)+F(U^l,U^m)+F(U^m,U^l)+F(U^m,U^m)\big).$$

We only detail the estimation of $F(U^l,U^m)$, the other terms are much easier. By the definition, one has that $F(U,U)\approx\langle\nabla\rangle\mathcal{R}(\mathcal{R}U\cdot\mathcal{R}U)$. Therefore, owing to the tame estimate, Sobolev embedding and the definition of $X_T$ norm,

$$\begin{aligned}\|F(U^l,U^m)\|_{H^{N-2}}&\lesssim\|\mathcal{R}U^l\cdot\mathcal{R}U^m\|_{H^{N-1}}\lesssim\|\mathcal{R}U^l\|_{L^\infty}\|\mathcal{R}U^m\|_{H^{N-1}}\\&\lesssim\|\mathcal{R}U^l\|_{W^{1,\frac{1}{\delta}}}\|U^m\|_{H^{N-1}}\lesssim(1+t)^{-(2-5\delta)}\|U\|^2_{X_T}.\end{aligned}\tag{4.7.4}$$

We next show (4.7.3), by splitting $F(U,U)$ into:

$$F(U,U)=F(U^L,U^L)+F(U^L,U^h)+F(U^h,U).$$

Similar to (4.7.4), $F(U^L,U^L)$ can be controlled as:

$$\|F(U^L,U^L)\|_{H^{N-1}}\lesssim\|(\mathcal{R}U^L)^2\|_{H^N}\lesssim\|\mathcal{R}U^L\|_{W^{1,\frac{1}{\delta}}}\|\mathcal{R}U^L\|_{H^N}\lesssim(1+t)^{-(1-3\delta)}\|U\|^2_{X_T}.$$

The other two terms are easier, we omit the detail. □

#### 4.7.2.1 High frequency estimate: control of $\|U^h\|_{H^{N-2}}$

By Duhamel's formula:

$$U^h(t)=e^{-tA}U^h_0+\int_0^t e^{-(t-s)A}\chi^hF(U,U)(s)\mathrm{d}s,$$

Lemma 4.4.3 and Lemma 4.7.1 then imply:

$$\begin{aligned}\|U^h\|_{H^{N-2}}&\lesssim e^{-ct}\|U_0\|_{H^{N-2}}+\int_0^t e^{-c(t-s)}\|\chi^h(F(U,U))\|_{H^{N-2}}\mathrm{d}s\\&\lesssim e^{-ct}\|U_0\|_{H^{N-2}}+\int_0^t e^{-c(t-s)}(1+s)^{-\alpha}\|U\|^2_{X_T}\mathrm{d}s\\&\lesssim e^{-ct}\|U_0\|_{H^{N-2}}+(1+t)^{-\alpha}\|U\|^2_{X_T}.\end{aligned}$$

#### 4.7.2.2 Intermediate frequency estimate: control of $\|U^m\|_{H^{N-1}}$ and $\|U^m\|_{W^{1,4}}$

By Duhamel's formula, Lemma 4.7.1, and Lemma 4.4.3, one can control the Sobolev norm of intermediate frequency as follows

$$\begin{aligned}\|U^m\|_{H^{N-1}}&\lesssim e^{-ct}\|U^m_0\|_{H^{N-1}}+\int_0^t e^{-c(t-s)}\|F(U,U)(s)\|_{H^{N-1}}\mathrm{d}s\\&\lesssim e^{-ct}\|U_0\|_{H^{N-1}}+\int_0^t e^{-c(t-s)}(1+s)^{-(1-3\delta)}\|U\|^2_{X_T}\mathrm{d}s\\&\lesssim e^{-ct}\|U_0\|_{H^{N-1}}+(1+t)^{-(1-3\delta)}\|U\|^2_{X_T}.\end{aligned}$$

We can estimate $\|U^m\|_{W^{1,4}}$ in the same fashion. In fact, by Corollary 4.4.4 (we will use $(\frac{1}{2})_+=\frac{3}{4}$), Lemma 4.8.1(we use relation $\frac{11}{4}\le\frac{2}{5}\cdot 7$), the definition of $X_T$ norm, we get:

$$\begin{aligned}\|U^m\|_{W^{1,4}}&\lesssim e^{-ct}\|U^m_0\|_{W^{\frac{7}{4},4}}+\int_0^t e^{-c(t-s)}\|F(U,U)\|_{W^{\frac{7}{4},4}}\mathrm{d}s\\&\lesssim e^{-ct}\|U^m_0\|_{W^{\frac{7}{4},4}}+\int_0^t e^{-c(t-s)}\|U\|_{W^{\frac{11}{4},5}}\|U\|_{L^{20}}\mathrm{d}s\\&\lesssim e^{-ct}\|U^m_0\|_{H^{\frac{9}{4}}}+\int_0^t e^{-c(t-s)}(1+s)^{-\frac{3}{2}}\|U\|^2_{X_T}\mathrm{d}s\\&\lesssim e^{-ct}\|U_0\|_{H^{\frac{9}{4}}}+(1+t)^{-\frac{3}{2}}\|U\|^2_{X_T}.\end{aligned}$$

## 4.8 Low frequency estimate

In this section, we focus on the a priori estimate of Low frequency: $\||\nabla|^{\frac{1}{2}}\langle\nabla\rangle Q^{-1}U^L\|_{L^\infty}$, $\|xe^{itb(D)}w\|_{W^{4,\frac{2}{1-\delta}}}$, $\|U^L\|_{H^{N'}}$. In practice, we shall perform the decay estimate and weighted estimate in the same time.

By equation (4.4.7) and Duhamel principle:

$$\begin{aligned} w &= e^{t\lambda_-(D)}w_0 + \mathcal{R}\int_0^t e^{(t-s)\lambda_-(D)}(B(w,w)+n(D)\chi^L H)(s)\mathrm{d}s \\ &\triangleq K_1 + \mathcal{R}\big(K_2+K_3\big) \end{aligned} \tag{4.8.1}$$

To close the decay estimate for $\mathcal{R}K_2$, the 'space-time resonance' philosophy that change the quadratic nonlinearity to the cubic one needs to be enforced. More specifically, we rewrite (2) in the following fashion. Recall the definition of the phase function $\phi_{\mu,\nu}(\xi,\eta) = i(b(\xi)-\mu b(\xi-\eta)-\nu b(\eta)) + \varepsilon(|\xi|^2-|\xi-\eta|^2-|\eta|^2)$ $(\mu,\nu\in\{+,-\})$ and the bilinear operator $T_m$ in (4.2.1). Denote $\tilde{f} = e^{-t\lambda_-(D)}w$, then $f$ is governed by

$$e^{t\lambda_-(D)}\partial_t\tilde{f} = \mathcal{R}^2(B(w,w)+\langle\nabla\rangle\chi^L H).$$

One thus has by identity $e^{s\phi_{\mu,\nu}} = \frac{1}{\phi_{\mu,\nu}}\partial_s e^{is\phi_{\mu,\nu}}$ and integration by parts in time that:

$$\begin{aligned} &\int_0^t e^{(t-s)\lambda_-(D)}B(w,w)\mathrm{d}s \\ &= e^{t\lambda_-(D)}\sum_{\mu,\nu\in\{+,-\}}\mathcal{F}^{-1}\Big(\int_0^t\int e^{-s\lambda_-(D)}m_{\mu\nu}(\xi,\eta)\widehat{\mathcal{R}w^\mu}(s,\xi-\eta)\widehat{\mathcal{R}w^\nu}(s,\eta)\mathrm{d}\eta\mathrm{d}s\Big) \\ &= e^{t\lambda_-(D)}\sum_{\mu,\nu\in\{+,-\}}\mathcal{F}^{-1}\Big(\int_0^t\int e^{s\phi_{\mu,\nu}}m_{\mu\nu}(\xi,\eta)\widehat{\mathcal{R}\tilde{f}^\mu}(s,\xi-\eta)\widehat{\mathcal{R}\tilde{f}^\nu}(s,\eta)\mathrm{d}\eta\mathrm{d}s\Big) \end{aligned}$$

$$\begin{aligned} =& \sum_{\mu,\nu\in\{+,-\}}\big[T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}(\mathcal{R}w^\mu(t),\mathcal{R}w^\nu(t)) - e^{t\lambda_-(D)}e^{\varepsilon t\Delta}[T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}(\mathcal{R}w_0^\mu,\mathcal{R}w_0^\nu) \\ &-\int e^{-i(t-s)b(D)}e^{(t-s)\varepsilon\Delta}T_{\frac{m_{\mu\nu}}{\phi_{\mu\nu}}}(\mathcal{R}(B(w,w)+\langle\nabla\rangle\chi^L H)^\mu,\mathcal{R}w^\nu)\mathrm{d}s + symmetric\quad terms\big] \\ \triangleq& I_1+\cdots+I_4 + symmetric\quad terms. \end{aligned} \tag{4.8.2}$$

Note that we denote $\mathcal{R}^2 = \mathcal{R}$ as they have the same property (they are both $L^p(1<p<\infty)$ multiplier). It is also worthy to remark that the operator $e^{isb(D)}\hat{w}$ is well defined as $\hat{w}$ is supported on the low frequency region. For notational brevity, we shall not distinguish $m_{\mu,\nu}$ (just write them as $m$) and ignore the summation on $\mu,\nu$. Therefore, in the following, we will write $I_1 - I_4$ as follows:

$$\begin{aligned} I_1 &= T_{\frac{m}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t)),\quad I_2 = -T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0), \\ I_3 &= -\int e^{-i(t-s)b(D)}e^{(t-s)\varepsilon\Delta}T_{\frac{m}{\phi}}(\mathcal{R}B(w,w),\mathcal{R}w)\mathrm{d}s, \\ I_4 &= -\int e^{-i(t-s)b(D)}e^{(t-s)\varepsilon\Delta}T_{\frac{m}{\phi}}(\mathcal{R}\langle\nabla\rangle\chi^L H,\mathcal{R}w)\mathrm{d}s. \end{aligned}$$

### 4.8.1 Decay estimate and weighted estimate

#### 4.8.1.1 Estimate of $K_1$ and boundary terms $I_1, I_2$

We begin with the decay estimate of $K_1$. Since $w_0 = (Q^{-1}\chi^L U_0)_1 \approx \mathcal{R}\frac{\langle\nabla\rangle}{|\nabla|}\varrho_0 + i\mathcal{R}^*u_0$, we have by the dispersive estimate (4.4.10)

$$\||\nabla|^{\frac{1}{2}}e^{-itb(D)}e^{\varepsilon t\Delta}w_0\|_{W^{1,\infty}} \lesssim \||\nabla|^{\frac{1}{2}}w_0\|_{W^{3,1}} \lesssim \|(\varrho_0,u_0,\nabla\varphi_0)\|_{W^{\frac{7}{2}+\delta,1}}$$

Note that the last inequality in the above arises from the fact that $\mathcal{R}\dot{\Delta}_j$ is $L^1$ multiplier for any $j \in \mathbb{Z}$. Next, for the decay estimate for $\mathcal{R}I_2$, we take benefits of the Sobolev embedding, dispersive estimate (4.4.10), bilinear estimate (4.4.18) (use $2_+ = \frac{9}{4} - \delta$) to get:

$$\||\nabla|^{\frac{1}{2}}\mathcal{R}e^{itb(D)}\tilde{\chi}^L e^{\varepsilon t\Delta}T_{\frac{m}{\phi}}(\mathcal{R}w(0),\mathcal{R}w(0))\|_{W^{1,\infty}} \lesssim \langle t\rangle^{-1}\sum_{j\in\mathbb{Z}} 2^{\frac{1}{2}j}\langle 2^j\rangle^3\|T_{\frac{m}{\phi}}(\mathcal{R}w(0),\mathcal{R}w(0))\|_{L^1}$$
$$\lesssim \langle t\rangle^{-1}\|T_{\frac{m}{\phi}}(\mathcal{R}w(0),\mathcal{R}w(0))\|_{W^{\frac{7}{2}+\delta,1}} \lesssim \langle t\rangle^{-1}\|w(0)\|_{H^{\frac{23}{4}}}\|w(0)\|_{H^2}.$$

As for the decay estimate of $\mathcal{R}I_1$, it is helpful to establish the following lemma:

**Lemma 4.8.1.** *For any $2 \le p < \infty$, and $k < \frac{2}{p}N' + (1-\frac{2}{p})\frac{3}{2}$, we have for every $t \in [0,T)$,*

$$\|U^L(t)\|_{W^{k,p}} \le (1+t)^{-(1-\frac{2}{p})}\|U\|_{X_T}. \tag{4.8.3}$$

*similarly, if $k < \frac{2}{p}N + (1-\frac{2}{p})\frac{3}{2}$, then*

$$\|U^L(t)\|_{W^{k,p}} \le (1+t)^{-(1-\frac{2}{p})+\frac{2}{p}\delta}\|U\|_{X_T}. \tag{4.8.4}$$

*Proof.* We only detail the proof of (4.8.3) since (4.8.4) can be treated in the same manner. We shall use decomposition $U^L = \Delta_{-1}U^L + \sum_{j\ge 0}\Delta_j U^L$. On one hand, the low frequency can be dealt with as follows:

$$\|\Delta_{-1}U^L\|_{W^{k,p}} \lesssim \|U^L\|_{L^p} \lesssim (1+t)^{-(1-\frac{2}{p})}\|U\|_{X_T}.$$

On the other hand, the high frequency term can be controlled by interpolation and the definition of $X_T$ norm:

$$\begin{aligned}\sum_{j\ge 0}\|\Delta_j U^L\|_{W^{k,p}} &\lesssim \sum_{j\ge 0} 2^{kj}\|\Delta_j U^L\|_{L^2}^{\frac{2}{p}}\|\Delta_j U^L\|_{L^\infty}^{1-\frac{2}{p}}\\ &\lesssim \sum_{j\ge 0} 2^{j(k-N'\frac{2}{p}-\frac{3}{2}(1-\frac{2}{p}))}(2^{N'j}\|\Delta_j U^L\|_{L^2})^{\frac{2}{p}}\||\nabla|^\delta\langle\nabla\rangle\Delta_j U^L\|_{L^\infty}^{1-\frac{2}{p}}\\ &\lesssim (1+t)^{-(1-\frac{2}{p})}\|U\|_{X_T}.\end{aligned}$$

□

In light of (4.4.18), (4.8.3) and condition $\frac{19}{4} < \frac{2}{3}\cdot 7 + \frac{1}{2}$, one has that

$$\|\mathcal{R}|\nabla|^{\frac{1}{2}}T_{\frac{m}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t))\|_{W^{1,\infty}} \lesssim \|T_{\frac{m}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t))\|_{H^{\frac{5}{2}+\delta}}$$
$$\lesssim \|w(t)\|_{W^{\frac{19}{4},3}}\|\mathcal{R}w\|_{W^{2,6}} \lesssim (1+t)^{-1}\|U\|_{X_T}^2. \tag{4.8.5}$$

We are now committed to the weighted estimate. Let us first detail the estimate of boundary terms: $x\mathcal{R}e^{itb(D)}(I_1,I_2)$. Using (4.5.6) again, it suffices to estimate $\|xe^{itb(D)}(I_1,I_2)\|_{H^{4+\delta}}$. Denote $f = e^{itb(D)}w$ the profile of $w$, one then writes

$$\begin{aligned}&xe^{itb(D)}T_{\frac{m}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t)) = xe^{it\operatorname{Im}\phi}T_{\frac{m}{\phi}}(\mathcal{R}f(t),\mathcal{R}f(t))\\ =\ &-te^{itb(D)}T_{\frac{m\partial_\xi(\operatorname{Im}\phi)}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t)) + ie^{itb(D)}T_{\partial_\xi(\frac{m}{\phi})}(\mathcal{R}w(t),\mathcal{R}w(t))\\ &+e^{itb(D)}T_{\frac{m}{\phi}}(e^{isb(D)}x\mathcal{R}f(t),\mathcal{R}w(t)).\end{aligned}$$

where $\operatorname{Im}\phi = b(\xi)\pm b(\xi-\eta)\pm b(\eta)$. Thanks to (4.4.18), (4.8.3) and relation $\frac{25}{4} \le \frac{16}{17}\cdot 7$, the first term can be controlled as:

$$\|te^{itb(D)}T_{\frac{m}{\phi}}(\mathcal{R}w(t),\mathcal{R}w(t))\|_{H^{4+\delta}} \lesssim t\|\mathcal{R}w\|_{W^{2,34}}\|\mathcal{R}w\|_{W^{\frac{25}{4},\frac{17}{8}}} \lesssim \|U\|_{X_T}^2.$$

The second term is easier, since it does not contain prefactor $t$. Moreover, the quadratic form $T_{\partial_\xi(\frac{m}{\phi})}$ admits the similar bilinear estimates as $T_{\frac{m}{\phi}}$. The third term is much involved since one could not put the loss of derivative on the weighted term. We thus write:

$$
\begin{aligned}
&\mathcal{F}(e^{itb(D)}T_{\frac{m}{\phi}}(e^{isb(D)}x\mathcal{R}f(t),\mathcal{R}w(t)))\\
= &\int e^{itb(\xi)}\frac{m}{\phi}(\xi,\eta)e^{-isb(\xi-\eta)}\widehat{x\mathcal{R}f}(\xi-\eta)\widehat{\mathcal{R}w}(\eta)\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}}\mathrm{d}\eta\\
&+\int e^{it\mathrm{Im}\,\phi(\xi,\eta)}\frac{m}{\phi}(\xi,\eta)\partial_\xi\widehat{\mathcal{R}f}(\xi-\eta)\widehat{\mathcal{R}f}(\eta)\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}}\mathrm{d}\eta \triangleq I_{131}+I_{132},
\end{aligned}
$$

By virtue of (4.4.18), (4.8.3) and condition $7.5\le\frac{2}{3}\cdot 11+\frac{1}{2}$, one gets that:

$$
\begin{aligned}
\|I_{131}\|_{H^{4+\delta}} &\lesssim \|T_{\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}}}(e^{itb(D)}x\mathcal{R}f(t),\mathcal{R}w(t))\|_{H^{4+\delta}}\\
&\lesssim \|e^{itb(D)}x\mathcal{R}f(t)\|_{W^{2,\frac{2}{1-2\delta}}}\|\mathcal{R}w\|_{W^{\frac{25}{4},\frac{1}{\delta}}}\\
&\lesssim \langle t\rangle^{2\delta}\|\langle x\rangle f\|_{W^{3,\frac{2}{1-\delta}}}\|w\|_{W^{7.5,3}}\lesssim \langle t\rangle^{-(\frac{1}{3}-3\delta)}\|U\|_{X_T}^2,
\end{aligned}
$$

For the term $I_{132}$, thanks to identity $\partial_\xi\widehat{\mathcal{R}f}(\xi-\eta)=-\partial_\eta\widehat{\mathcal{R}f}(\xi-\eta)$, one could integrate by parts in $\eta$ to rewrite it as:

$$
\begin{aligned}
I_{132}=T_{\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}}}(\mathcal{R}w(t),e^{itb(D)}x\mathcal{R}f(t))+T_{\partial_\eta(\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}})}(\mathcal{R}w(t),\mathcal{R}w(t))\\
+itT_{\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}}\partial_\eta(Im\phi)}(\mathcal{R}w(t),\mathcal{R}w(t)).
\end{aligned}
$$

Nevertheless, the first term in the above can be estimated exactly as $I_{311}$, the last two terms can be treated in the same manner as that of $\|I_1\|_{H^{2+\delta}}$, see (4.8.5).

We are now in position to show the estimate of $xe^{itb(D)}I_2$. By definition,

$$
\begin{aligned}
xe^{itb(D)}I_2 &= xe^{\varepsilon t\Delta}T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0)\\
&= 2e^{\varepsilon t\Delta}\varepsilon t\nabla T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0)+e^{\varepsilon t\Delta}T_{\frac{m}{\phi}}(x\mathcal{R}w_0,\mathcal{R}w_0)+iT_{\partial_\xi(\frac{m}{\phi})}(\mathcal{R}w_0,\mathcal{R}w_0).
\end{aligned}
$$

Let us focus on the estimate of the first two terms, since the last one is easier. Owing to the bilinear estimate (4.4.18), one has

$$
\begin{aligned}
&\|e^{\varepsilon t\Delta}\varepsilon t\nabla T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0)\|_{H^{4+\delta}}+\|e^{\varepsilon t\Delta}T_{\frac{m}{\phi}(\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}})}(x\mathcal{R}w_0,\mathcal{R}w_0)\|_{H^{4+\delta}}\\
\lesssim\ &\varepsilon^{\frac{1}{2}}t\|\mathcal{F}^{-1}(e^{-\varepsilon t|\xi|^2}\varepsilon^{\frac{1}{2}}\xi)\|_{L^2}\|T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0)\|_{W^{4+\delta,1}}+\|T_{\frac{m}{\phi}(\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}})}(x\mathcal{R}w_0,\mathcal{R}w_0)\|_{H^{4+\delta}}\\
\lesssim\ &\|T_{\frac{m}{\phi}}(\mathcal{R}w_0,\mathcal{R}w_0)\|_{W^{4+\delta,1}}+\|x\mathcal{R}(\varrho_0,\varphi_0,u_0)\|_{W^{2,3}}\|\mathcal{R}(\varrho_0,\varphi_0,u_0)\|_{W^{\frac{25}{4},6}}\\
\lesssim\ &\|\mathcal{R}(\varrho_0,\varphi_0,u_0)\|_{H^2}\|\mathcal{R}(\varrho_0,\varphi_0,u_0)\|_{H^{\frac{25}{4}}}+\|\langle x\rangle(\varrho_0,\varphi_0,u_0)\|_{H^{\frac{7}{3}}}\|(\varrho_0,\varphi_0,u_0)\|_{H^7}\\
\lesssim\ &\|\langle x\rangle(\varrho_0,\varphi_0,u_0)\|_{H^{\frac{7}{3}}}\|(\varrho_0,\varphi_0,u_0)\|_{H^7}.
\end{aligned}
$$

Note that in the above, the following fact has been used:

$$
\begin{aligned}
&\|\mathcal{F}^{-1}(e^{-\varepsilon t|\xi|^2}\varepsilon^{\frac{1}{2}}\xi)\|_{L^2}\lesssim\varepsilon^{-\frac{1}{2}}t^{-1},\\
&\|x\mathcal{R}f\|_{L^3}\lesssim\|xf\|_{L^3}+\||\nabla|^{-1}f\|_{L^3}\lesssim\|xf\|_{H^{\frac{1}{3}}}+\|f\|_{L^{\frac{6}{5}}}\lesssim\|\langle x\rangle f\|_{H^{\frac{1}{3}}}.
\end{aligned}
\tag{4.8.6}
$$

The estimate of $T_{\frac{m}{\phi}(\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}})}(x\mathcal{R}w_0,w_0)$ can be obtained by integrating by parts in $\eta$ as before.

#### 4.8.1.2 Estimate of $K_3$ and $\mathcal{R}I_4$

We begin with the decay estimate of $K_3$ (which is defined in (4.8.1)). By dispersive estimate, $\|\mathcal{R}K_3\|_{L^\infty}$ can be estimated as follows:

$$
\begin{aligned}
& \|\mathcal{R}\int_0^t e^{i(t-s)\lambda_-(D)}|\nabla|^{\frac{1}{2}}\langle\nabla\rangle\langle\nabla\rangle\chi^L H(s)\mathrm{d}s\|_{L^\infty} \\
\lesssim\ & \int_0^t (1+t-s)^{-1}\sum_{j\in\mathbb{Z}} 2^{\frac{1}{2}j}\langle 2^j\rangle^4\|\dot{\Delta}_j H(s)\|_{L^1}\mathrm{d}s \lesssim \int_0^t (1+t-s)^{-1}\|H(s)\|_{B^5_{1,2}}\mathrm{d}s \\
\lesssim\ & \int_0^t (1+t-s)^{-1}\|U^h\|_{H^5}\|U\|_{H^5}\mathrm{d}s \lesssim \int_0^t (1+t-s)^{-1}(1+s)^{-\alpha}\|U\|^2_{X_T}\mathrm{d}s \lesssim (1+t)^{-1}\|U\|^2_{X_T}
\end{aligned}
$$

We now prove the weighted estimate of $K_3$. According to (4.5.6), it suffices to estimate $\|x(3)\|_{H^{4+\delta}}$. Rewrite $H = \mathcal{R}U^h\mathcal{R}U^L + \mathcal{R}U^h\mathcal{R}U^h \triangleq H_1 + H_2$, one has that by the definition of $W : W = Q^{-1}U^L$,

$$
\begin{aligned}
& x\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\chi^L H_1\mathrm{d}s \\
=\ & \mathcal{F}^{-1}\big(\int_0^t\int s(b'(\xi)\pm b'(\xi-\eta)) + 2\varepsilon(t-s)i\xi)e^{isb(\xi)}e^{-\varepsilon(t-s)|\xi|^2}\langle\xi\rangle\widehat{\mathcal{R}U^h}(\eta)\widehat{\mathcal{R}U^L}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s \\
& + \int_0^t e^{isb(\xi)}e^{-\varepsilon(t-s)|\xi|^2}\partial_\xi\big(\langle\xi\rangle\chi^L(\xi)\big)\hat{H}_1(\xi)\mathrm{d}s \\
& + \int_0^t e^{is(b(\xi)\pm b(\xi-\eta))}e^{-\varepsilon(t-s)|\xi|^2}\langle\xi\rangle\chi^L(\xi)\widehat{\mathcal{R}U^h}(\eta)\partial_\xi\widehat{Q\tilde{\chi}^L\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s\big) \\
\triangleq\ & (3)_{11} + (3)_{12} + (3)_{13} \qquad (4.8.7)
\end{aligned}
$$

For $(3)_{11}$, by virtue of (4.8.6), the fact $b'\chi^L(D)$ is $L^p(1<p<\infty)$ multiplier as well as Young's inequality:

$$
\begin{aligned}
\|(3)_{11}\|_{H^{4+\delta}} &\lesssim \int_0^t s\|\mathcal{R}U^h\mathcal{R}U^L\|_{H^{5+\delta}}\mathrm{d}s + \int_0^t \|\mathcal{R}U^h\mathcal{R}U^L\|_{W^{5+\delta,1}}\mathrm{d}s \\
&\lesssim \int_0^t s\|\mathcal{R}U^h\|_{H^{5+\delta}}\|\mathcal{R}U^L\|_{W^{1,4}}\mathrm{d}s + \int \|\mathcal{R}U^h\|_{H^5}\|\mathcal{R}U^L\|_{L^2}\mathrm{d}s \\
&\lesssim \int_0^t s\langle s\rangle^{-(2-5\delta+\frac{1}{2})}\mathrm{d}s\|U\|^2_{X_T} + \int\langle s\rangle^{-(2-5\delta)}\|U\|^2_{X_T} \lesssim \|U\|^2_{X_T}.
\end{aligned}
$$

The estimate of $(3)_{12}$ is similar, we thus skip it. For $(3)_{13}$, by (4.4.18), (4.4.12),

$$
\begin{aligned}
& \|(3)_{13}\|_{H^{4+\delta}} \lesssim \int_0^t \|\mathcal{R}U^h\|_{W^{5+\delta,6}}\|e^{isb(D)}xQ\tilde{\chi}^L\mathcal{R}f\|_{L^3}\mathrm{d}s \\
\lesssim\ & \int_0^t \|\mathcal{R}U^h\|_{H^6}\langle s\rangle^{\frac{1}{3}}\|xQ\tilde{\chi}^L\mathcal{R}f\|_{W^{\frac{1}{2},3}}\mathrm{d}s \lesssim \int_0^t\langle s\rangle^{-(2-5\delta-\frac{1}{3})}\|U\|^2_X\mathrm{d}s \lesssim \|U\|^2_{X_T}.
\end{aligned}
$$

Note that in the above, we have also used the fact $xQ\tilde{\chi}^L\mathcal{R}f \approx |\nabla|^{-1}\mathcal{R}f + \mathcal{R}(xf)$ which gives:

$$
\|xQ\tilde{\chi}^L\mathcal{R}f\|_{W^{\frac{1}{2},3}} \lesssim \|\mathcal{R}f\|_{W^{\frac{1}{2},\frac{6}{5}}} + \|xf\|_{W^{\frac{1}{2},3}} \lesssim \|f\|_{H^{\frac{1}{2}}} + \|xf\|_{W^{\frac{5}{6},\frac{2}{1-\delta}}}.
$$

For the case of $H_2$, since in the original definition $H_2 = \varrho^h u^h$ or $(u^h)^2$, one can split it into three terms:

$$
\begin{aligned}
& x\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\chi^L H_2\mathrm{d}s \\
=\ & \int_0^t\int sb'(D) + 2(t-s)\varepsilon i\nabla)e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\xi\rangle H_2\mathrm{d}s \\
& + \int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\mathcal{F}^{-1}\big(\partial_\xi(\langle\xi\rangle\chi^L)\hat{H}_1(\xi)\big)\mathrm{d}s + \int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}\langle\nabla\rangle\chi^L(x\varrho^h u^h)\mathrm{d}s \\
\triangleq\ & (3)_{21} + (3)_{22} + (3)_{23} \qquad (4.8.8)
\end{aligned}
$$

The estimates of $(3)_{21}, (3)_{22}$ are similar to that of $(3)_{11}, (3)_{12}$ and thus can be omitted. For $(3)_{23}$, one uses Lemma 4.4.13, the estimate (4.6.1) to get:

$$\begin{aligned}
\|\langle\nabla\rangle\chi^L(x\varrho^h u^h)\|_{H^{4+\delta}} &\lesssim \|xu^h\|_{L^2}\|\varrho^h\|_{H^{6+2\delta,\infty}} + \|x\varrho^h\|_{L^2}\|u^h\|_{H^{6+2\delta}} + \|\varrho^h\|_{H^{5+2\delta}}\|u^h\|_{H^{5+2\delta}} \\
&\lesssim \langle s\rangle^{-(2-7\delta)}(\|U\|_{X_T}^2 + \|x(\varrho_0, u_0, \nabla\varphi_0)\|_{L^2}^2).
\end{aligned}$$

#### 4.8.1.3 Estimate of $I_4$

One first observes that

$$\begin{aligned}
\||\nabla|^{\frac{1}{2}}\langle\nabla\rangle\mathcal{R}I_4\|_{L^\infty} &\lesssim \sum_{j\in\mathbb{Z}} 2^{\frac{1}{2}j}\langle 2^j\rangle\|\dot{\Delta}_j I_4\|_{L^\infty} \\
\lesssim\ \langle t\rangle^{-1}\sum_{j\in\mathbb{Z}} 2^{\frac{1}{2}j}\langle 2^j\rangle^3\|\dot{\Delta}_j e^{itb(D)}I_4\|_{L^1} &\lesssim \langle t\rangle^{-1}\|e^{itb(D)}I_4\|_{W^{\frac{7}{2}+\delta,1}}
\end{aligned}$$

Applying Lemma 4.4.2 for $p=1$, we get:

$$\begin{aligned}
\|e^{itb(D)}I_4\|_{W^{\frac{7}{2}+\delta,1}} &\lesssim \int_0^t s\|T_{\frac{m}{\phi}}(\langle\nabla\rangle\mathcal{R}\chi^L H, \mathcal{R}w)\|_{W^{\frac{9}{2}+\delta,1}}\mathrm{d}s \\
\lesssim\ \int_0^t s\|\langle\nabla\rangle\mathcal{R}\chi^L H\|_{H^{\frac{27}{4}}}\|w\|_{H^{\frac{27}{4}}}\mathrm{d}s &\lesssim \int_0^t s\langle s\rangle^{3-7\delta}\mathrm{d}s\|U\|_X^3 \lesssim \|U\|_{X_T}^3.
\end{aligned}$$

where the following crude estimate has been used

$$\|\langle\nabla\rangle\chi^L\mathcal{R}H\|_{H^{N-3}} \lesssim \|H\|_{H^{N-2}} \lesssim \|U\|_{L^\infty}\|U^h\|_{H^{N-2}} \lesssim \langle s\rangle^{-(3-7\delta)}\|U\|_{X_T}^2. \tag{4.8.9}$$

We are now devoted to proving the estimate of $\|xe^{itb(D)}I_4\|_{W^{4,\frac{2}{1-\delta}}}$. As before, let us write

$$\begin{aligned}
xe^{itb(D)}I_4 &= \mathcal{F}^{-1}\Big(\int_0^t s(b'(\xi)\pm b'(\xi-\eta)) + 2\varepsilon(t-s)i\xi e^{is(b(\xi)\pm b(\xi-\eta))}e^{-\varepsilon(t-s)|\xi|^2} \\
&\qquad \frac{m}{\phi}(\xi,\eta)\chi^L(\eta)\langle\eta\rangle\widehat{\mathcal{R}H}(\eta)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s \\
&\quad + \int_0^t e^{is(b(\xi)\pm b(\xi-\eta))}e^{-\varepsilon(t-s)|\xi|^2}\partial_\xi(\frac{m}{\phi})\chi^L(\eta)\langle\eta\rangle\widehat{\mathcal{R}H}(\eta)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s \\
&\quad + \int_0^t e^{is(b(\xi)\pm b(\xi-\eta))}e^{-\varepsilon(t-s)|\xi|^2}\frac{m}{\phi}(\xi,\eta)\chi^L(\eta)\langle\eta\rangle\widehat{\mathcal{R}H}(\eta)\partial_\xi\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s\Big) \\
&\triangleq J_{41} + J_{42} + J_{43}.
\end{aligned}$$

The first term $J_{41}$ can be dealt with similarly as the term $(3)_{11}$. Indeed, by using Lemma 4.4.2, Lemma 4.4.6 and (4.8.9), one obtains

$$\begin{aligned}
\|J_{41}\|_{W^{4,\frac{2}{1-\delta}}} &\lesssim \int_0^t \langle s\rangle^{1+\delta}\|T_{\frac{m}{\phi}}(\langle\nabla\rangle\chi^L\mathcal{R}H, \mathcal{R}w)(s)\|_{W^{4+2\delta,\frac{2}{1-\delta}}}\mathrm{d}s \\
&\quad + \int_0^t \langle s\rangle^{\delta}\|T_{\frac{m}{\phi}}(\langle\nabla\rangle\chi^L\mathcal{R}H, \mathcal{R}w)(s)\|_{W^{4+2\delta,\frac{2}{2-\delta}}}\mathrm{d}s \\
\lesssim\ \int_0^t \langle s\rangle^{1+\delta}\|\mathcal{R}w\|_{W^{\frac{25}{4},\infty}}\|\langle\nabla\rangle\chi^L\mathcal{R}H\|_{W^{\frac{25}{4},\frac{2}{1-\delta}}}\mathrm{d}s &+ \int_0^t \langle s\rangle^{\delta}\|\mathcal{R}w\|_{W^{\frac{25}{4},\frac{2}{1-\delta}}}\|\langle\nabla\rangle\chi^L\mathcal{R}H\|_{H^{\frac{25}{4}}}\mathrm{d}s \\
\lesssim\ \int_0^t \langle s\rangle^{1+\delta}\langle s\rangle^{-(3-7\delta)}\mathrm{d}s\|U\|_X^3 &\lesssim \|U\|_{X_T}^3.
\end{aligned}$$

$J_{42}$ can be estimated in the same manner, we thus do not detail it. For $J_{43}$, one splits it into two terms:

$$J_{43} = \mathcal{F}^{-1}\Big(\int(\chi_{\langle\xi-\eta\rangle\le\langle\eta\rangle} + \chi_{\langle\xi-\eta\rangle\ge\langle\eta\rangle})\cdots\mathrm{d}s\Big) \triangleq J_{431} + J_{432}.$$

The estimate of $J_{431}$ is easy since we can put all the derivatives onto $H$. Indeed, by (4.8.9), Lemma 4.4.2, and the Sobolev embedding, one obtains that

$$\begin{aligned}\|J_{431}\|_{W^{4,\frac{2}{1-\delta}}} &\lesssim \int_0^t \|\langle\nabla\rangle\chi^L H\|_{W^{\frac{25}{4},\frac{6}{1-3\delta}}}\|e^{isb(D)}x\mathcal{R}f\|_{W^{2,3}}\mathrm{d}s\\ &\lesssim \int_0^t \langle s\rangle^{-(3-7\delta)}\langle s\rangle^{\frac{1}{3}}\|\langle x\rangle f\|_{W^{3,\frac{2}{1-\delta}}}\mathrm{d}s\|U\|^2_{X_T}. \lesssim \|U\|^3_{X_T}\end{aligned}$$

For $J_{432}$, we use the identity $\partial_\xi\widehat{\mathcal{R}f}(\xi-\eta) = -\partial_\eta\widehat{\mathcal{R}f}(\xi-\eta)$ to integrate by parts in $\eta$. Eventually, we get the terms like $J_{41}, J_{42}, J_{431}$ as well as the term:

$$\int_0^t e^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{\frac{m\chi_{\{\langle\xi-\eta\rangle\geq\langle\eta\rangle\}}}{\phi}}(x\langle\nabla\rangle\chi^L(D)\mathcal{R}H,\mathcal{R}w)\mathrm{d}s. \tag{4.8.10}$$

Besides,

$$\begin{aligned}x\langle\nabla\rangle\chi^L(D)\mathcal{R}H &= \frac{\nabla}{\langle\nabla\rangle}\chi^L\mathcal{R}H + \langle\nabla\rangle\big((i\chi^L)'(D)\mathcal{R}H + \chi^L(D)x\mathcal{R}H\big)\\ &\approx \frac{\nabla}{\langle\nabla\rangle}\chi^L\mathcal{R}H + \langle\nabla\rangle\big((i\chi^L)'(D)\mathcal{R}H + \chi^L(D)|\nabla|^{-1}H + \chi^L(D)\mathcal{R}xH\big).\end{aligned}$$

To continue, the following estimate for $H$ shall be useful:

**Proposition 4.8.2.**

$$\begin{aligned}\|(\chi^L)'(D)\mathcal{R}H + \chi^L(D)|\nabla|^{-1}H\|_{W^{4,\frac{2}{1-\delta}}} &\lesssim \langle t\rangle^{-(2-5\delta)}\|U\|^2_{X_T},\\ \|\chi^L\mathcal{R}xH\|_{L^2} &\lesssim \langle t\rangle^{-(1-4\delta)}(\|U\|^2_{X_T} + \|\langle x\rangle(\varrho_0,u_0)^h\|^2_{L^2}).\end{aligned}$$

We postpone the proof of this proposition and finish firstly the estimate of term (4.8.10). Indeed, Lemma 4.8.1, Proposition 4.8.2, combined with the Sobolev embedding $W^{\frac{41}{4},\frac{21}{10}} \hookrightarrow W^{\frac{37}{4},\infty}, H^8 \hookrightarrow W^{\frac{25}{4},\infty}$ yield that:

$$\begin{aligned}&\int_0^t \|\mathcal{R}w\|_{W^{\frac{25}{4},\infty}}\|\chi^L\mathcal{R}H + \langle\nabla\rangle\big((\chi^L)'(D)\mathcal{R}H + \chi^L(D)|\nabla|^{-1}H\big)\|_{W^{2,\frac{2}{1-\delta}}}\\ &\qquad+\|\mathcal{R}w\|_{W^{\frac{37}{4},\infty}}\|\chi^L(D)\mathcal{R}xH\|_{L^2}\mathrm{d}s\\ \lesssim &\int_0^t \langle s\rangle^{-(2-5\delta)}\|U\|^3_X + \langle s\rangle^{-(\frac{1}{21}-\delta)}\langle s\rangle^{-(1-4\delta)}(\|U\|^2_X + \|\langle x\rangle(\varrho_0,u_0)^h\|^2_{L^2})\mathrm{d}s\\ \lesssim &\ \|\langle x\rangle(\varrho_0,u_0,\nabla\varphi_0)^h\|^2_{L^2} + \|U\|^2_{X_T} + \|U\|^3_{X_T}.\end{aligned}$$

We are now left to prove Propostion 4.8.2.

*Proof of Proposition 4.8.2.* Firstly, since $(\chi^L)'(D)\mathcal{R}$ is $L^2$ multiplier with norm $\lesssim \sqrt{\frac{\varepsilon}{\kappa_0}}$, one has by Sobolev embedding, the definition $H \approx \mathcal{R}U^h\mathcal{R}U$ that:

$$\|(\chi^L)'(D)\mathcal{R}H\|_{W^{4,\frac{2}{1-\delta}}} \lesssim \sqrt{\frac{\varepsilon}{\kappa_0}}\|H\|_{W^{4,\frac{2}{1-\delta}}} \lesssim \|U^h\|_{H^4}\|U\|_{H^1} \lesssim \langle t\rangle^{-(2-5\delta)}\|U\|^2_{X_T}.$$

Similarly, by Hardy-Littlewood-Sobolev inequality,

$$\||\nabla|^{-1}H\|_{W^{4,\frac{2}{1-\delta}}} \lesssim \|H\|_{W^{4,\frac{2}{2-\delta}}} \lesssim \|\mathcal{R}U^h\|_{H^4}\|\mathcal{R}U\|_{H^1} \lesssim \langle t\rangle^{-(2-5\delta)}\|U\|^2_{X_T}.$$

We are now ready to estimate $\mathcal{R}xH$. Notice that in the original definition $H = \rho^L u^h + \rho^h u$ or $u^L\cdot u^h + u^h\cdot u$. Therefore, due to the Sobolev embedding $W^{1,\frac{1}{\delta}} \hookrightarrow L^\infty$ and weighted Sobolev estimate for high frequency (4.6.1):

$$\begin{aligned}&\|\chi^L\mathcal{R}xH\|_{L^2} \lesssim \|x(\varrho,u)^h\|_{L^2}\|(\varrho,u)\|_{L^\infty}\\ \lesssim &\ \langle t\rangle^{2\delta}(\|U\|_{X_T} + \|\langle x\rangle(\varrho_0,u_0,\nabla\varphi_0)^h\|_{L^2})\langle t\rangle^{-(1-2\delta)}\|U\|_{X_T}\\ \lesssim &\ \langle t\rangle^{-(1-4\delta)}(\|U\|^2_{X_T} + \|\langle x\rangle(\varrho_0,u_0,\nabla\varphi_0)^h\|^2_{L^2}).\end{aligned}$$

□

#### 4.8.1.4 Estimate of $\mathcal{R}I_3$

In view of the definition of $B(w,w)$ (see 4.4.8), we could indeed consider $B(w,w)$ as $\langle\nabla\rangle\mathcal{R}(\mathcal{R}w)^2$ for simplicity. Therefore, by recalling the definition of profile $f(s)=e^{isb(D)}w$, we see that $I_3=e^{-itb(D)}I_3'$ with $I_3'$ under the form

$$I_3'(t)=\int_0^t\int\int e^{is\tilde{\phi}(\xi,\eta,\sigma)}e^{-\varepsilon(t-s)|\xi|^2}\frac{m}{\phi}(\xi,\eta)\langle\eta\rangle\mathcal{R}(\eta)\widehat{\mathcal{R}f}(\xi-\eta)\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s \tag{4.8.11}$$

where $\tilde{\phi}=b(\xi)\pm b(\xi-\eta)\pm b(\eta-\sigma)\pm b(\sigma)$. For the weighted estimate, thanks to Bernstein's inequality and Young's inequality, one has that

$$\begin{aligned}\|x\mathcal{R}I_3'\|_{W^{4,\frac{2}{1-\delta}}}&\le\sum_{k\in\mathbb{Z}}2^{4k_+}\|\dot{\Delta}_kx\mathcal{R}I_3'\|_{L^{\frac{2}{1-\delta}}}\lesssim\sum_{k\in\mathbb{Z}}2^{4k_+}\left(\||\nabla|^{-1}\dot{\Delta}_kI_3'\|_{L^{\frac{2}{1-\delta}}}+\|\dot{\Delta}_kxI_3'\|_{L^{\frac{2}{1-\delta}}}\right)\\&\lesssim\sum_{k\in\mathbb{Z}}2^{4k_+}\left(2^{\delta k}\|\dot{\Delta}_kI_3'\|_{L^1}+2^{\frac{6}{5}\delta k}\|\dot{\Delta}_kxI_3'\|_{L^{2_\delta}}\right)\\&\lesssim\|I_3'\|_{H^5}+\sup_{k\in\mathbb{Z}}2^{-\frac{4}{5}\delta k^-}2^{(4+2\delta)k^+}\left(\|[x,\dot{\Delta}_k]I_3'\|_{L^{2_\delta}}+\|\dot{\Delta}_kxI_3'\|_{L^{2_\delta}}\right)\end{aligned}$$

where we denote $2_\delta=\frac{2}{1+\delta/5}$ and $k^-=\max(-k,0),k^+=\max(k,0)$. Similarly, by dispersive estimate (4.4.10), Hölder's inequality,

$$\||\nabla|^{\frac{1}{2}}I_3'\|_{W^{1,\infty}}\lesssim\langle t\rangle^{-1}\sum_{k\in\mathbb{Z}}2^{\frac{k}{2}}2^{3k_+}\|\dot{\Delta}_kI_3'\|_{L^1}\lesssim\langle t\rangle^{-1}(\|I_3'\|_{H^4}+\sup_{k\in\mathbb{Z}}2^{\delta k_-}2^{4k_+}\|x\dot{\Delta}_kI_3'\|_{L^{2_\delta}})$$

By Lemma 4.4.2, one has that: $\|e^{isb(D)}\chi^L(D)g\|_{L^{2_\delta}}\lesssim\langle s\rangle^{\frac{\delta}{5}}\|g\|_{W^{\delta,2_\delta}}$. Therefore, the commutator term can be bounded by applying Corollary 4.4.10 and Lemma 4.8.1:

$$\begin{aligned}\|[x,\dot{\Delta}_k]I_3'\|_{W^{4+2\delta,2_\delta}}&=\|\int_0^te^{isb(D)}e^{\varepsilon(t-s)\Delta}(\Phi_k)'(D)T_{\frac{m}{\phi}}\left(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,\mathcal{R}w\right)\|_{W^{4+2\delta,2_\delta}}\mathrm{d}s\\\lesssim&\int_0^t\langle s\rangle^{\frac{\delta}{5}}\|T_{\frac{m}{\phi}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,\mathcal{R}w)\|_{W^{4+3\delta,\frac{1}{1-3\delta/10}}}\|\mathcal{F}^{-1}\left(2^{-k}\Phi'(2^{-k}\cdot)\right)\|_{L^{\frac{10}{5+4\delta}}}\mathrm{d}s\\\lesssim&\ 2^{-\frac{4}{5}\delta k}\int_0^t\langle s\rangle^{\frac{\delta}{5}}\|w\|_{H^{N'}}\|w\|_{W^{2,4}}\|w\|_{W^{2,\frac{20}{5-6\delta}}}\mathrm{d}s\lesssim2^{-\frac{4}{5}\delta k}\int_0^t\langle s\rangle^{\frac{\delta}{5}}\langle s\rangle^{-1-\frac{3}{5}\delta}\mathrm{d}s\|U\|_X^3\lesssim2^{-\frac{4}{5}\delta k}\|U\|_{X_T}^3\end{aligned}$$

It now remains for us to estimate $\sup_{k\in\mathbb{Z}}2^{\frac{4}{5}\delta k_-}2^{(4+\delta)k_+}\|\dot{\Delta}_kxI_3'\|_{L^{2_\delta}}$. By the expression of $I_3'$, we have that $xI_3'=\sum_{j=1}^4Z_j$ where

$$\begin{aligned}Z_1&=\int ise^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{\frac{m}{\phi}\partial_\xi\tilde{\phi}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,\mathcal{R}w)\mathrm{d}s,\\Z_2&=\int_0^te^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{\partial_\xi(\frac{m}{\phi})}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,\mathcal{R}w)\mathrm{d}s,\\Z_3&=\int_0^te^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{\frac{m}{\phi}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,e^{-isb(D)}x\mathcal{R}f)\mathrm{d}s,\\Z_4&=\int_0^t\varepsilon(t-s)\nabla e^{\varepsilon(t-s)\Delta}e^{isb(D)}T_{\frac{m}{\phi}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2,\mathcal{R}w)\mathrm{d}s.\end{aligned}$$

We first remark that $\|\dot{\Delta}_kxZ_4\|_{W^{4+2\delta,2_\delta}}$ can be estimated in the same manner as that of $\|[x,\dot{\Delta}_k]I_3'\|$, the only difference is that at this stage we use the fact the $L^{\frac{10}{5+4\delta}}$ norm of the kernel of $\dot{\Delta}_k\varepsilon(t-s)\nabla e^{\varepsilon(t-s)\Delta}$ is less than $2^{-\frac{4}{5}\delta k}$ uniformly in $\varepsilon\in(0,1]$. Indeed, one can think of $\dot{\Delta}_k\varepsilon(t-s)\nabla e^{\varepsilon(t-s)\Delta}$ as $\dot{\Delta}_k|\nabla|^{-1}$, since $\|\mathcal{F}^{-1}\left(\varepsilon(t-s)\xi^2e^{-\varepsilon(t-s)|\xi|^2}\right)\|_{L^1}$ is uniformly bounded. We point out here that we choose $2_\delta=\frac{2}{1+\delta/5}$ mainly to manage to control the commutator term $[x,\dot{\Delta}_k]I_3'$ and $Z_4$, since the situation is better if $2_\delta$ is closer to 2. We also emphasize that the presence of the half derivative when we control the $L_x^\infty$ norm of $I_3$ is necessary in here. Indeed, as we explained in the introduction, due to the weak dispersive estimate, we need to control it by the weighted norm: $\|xI_3'\|_{L^{2-}}$. Nevertheless, when we deal with the $\|Z_4\|_{L^{2-}}$ which corresponds to the frequency derivative hits on $e^{\varepsilon(t-s)\Delta}$, to compensate the growth of $(t-s)$, the best one can use is: $\|\nabla e^{\varepsilon(t-s)\Delta}\|_{L^{2-}} \lesssim (\varepsilon(t-s))^{-1^-}$, which is obviously not enough. Based on this, the extra derivative can help in the sense that we could find some $1^+$, such that $\|\nabla^{1^+} e^{\varepsilon(t-s)\Delta}\|_{L^{2-}} \lesssim (\varepsilon(t-s))^{-1}$.

The estimations for $Z_1$-$Z_3$ are more involved since in these cases we can not use any information of heat kernel. However, since it has been showed that $e^{itb(D)}\chi^L(D)$, $\operatorname{Im}\phi(\xi,\eta) = b(\xi) \pm b(\xi-\eta) \pm b(\eta)$ and $\tilde{\phi}(\xi,\eta,\sigma) = b(\xi) \pm b(\xi-\eta) \pm b(\eta-\sigma) \pm b(\sigma)$ has the same properties as $e^{it\langle\nabla\rangle}$, $\langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta\rangle$ and $\langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta-\sigma\rangle \pm \langle\sigma\rangle$ respectively, they can be achieved by the similar arguments as in [84] where the global existence of 2-d Euler-Poisson is proved. We thus just sketch them in the appendix for completeness and reader's convenience.

### 4.8.2 Estimate of $H^{N'}$

In this short subsection, we deal with the estimate of $\|U^L\|_{H^{N'}}$. By virtue of the definition $U^L = Q^{-1}\chi^L W$ and the fact that $Q^{-1}\chi^L$ is a $L^2$ multiplier, one easily sees that $\|U^L\|_{H^{N'}} \lesssim \|W\|_{H^{N'}}$. By (4.5.3), (4.8.2),

$$w = (1) + (3) + I_1 + I_2 + I_3 + I_4$$

where $(1), (3)$ are defined in (4.8.1) and $I_1 - I_4$ is defined in (4.8.2).

We shall only show the estimation of $I_1 - I_4$. For $I_1, I_2$, by bilinear estimate (4.4.18)(we use $2_+ = \frac{9}{4} - 2\delta$), Sobolev embedding and the definition of $X_T$ norm,

$$\begin{aligned}
\|I_1\|_{H^{N'}} &\lesssim \|T_{\frac{m}{\phi}}(\mathcal{R}w(t)), \mathcal{R}w(t))\|_{H^{N'}} \lesssim \|\mathcal{R}w\|_{H^{N'+\frac{9}{4}}} \|\mathcal{R}w\|_{W^{2,\frac{1}{\delta}}} \lesssim \langle t\rangle^{-(1-3\delta)} \|U\|_{X_T}^2 \\
\|I_2\|_{H^{N'}} &\lesssim \|T_{\frac{m}{\phi}}(\mathcal{R}w_0, \mathcal{R}w_0)\|_{H^{N'}} \lesssim \|w_0\|_{H^{N'+\frac{9}{4}}}^2 \|w_0\|_{W^{2,\infty}} \lesssim \|w_0\|_{H^{N'+\frac{9}{4}}}^2.
\end{aligned}$$

For $I_3$, we use Corollary 4.4.10 with $3_+ = \frac{13}{4} - 2\delta$ and assumption $N' = N - 4$ to get:

$$\begin{aligned}
\|I_3\|_{H^{N'}} &\lesssim \int_0^t \|T_{\frac{m}{\phi}}(\mathcal{R}B(\mathcal{R}w, \mathcal{R}w), \mathcal{R}w(s))\|_{H^{N'}} \mathrm{d}s \\
\lesssim \quad & \int_0^t \|w\|_{H^{N'+\frac{13}{4}}} \|w\|_{W^{2,\frac{1}{\delta}}}^2 \mathrm{d}s \lesssim \int_0^t \langle s\rangle^{-2(1-3\delta)} \|U\|_{X_T}^3 \mathrm{d}s \lesssim \|U\|_{X_T}^3.
\end{aligned}$$

$I_4$ can be estimated in the similar fashion, by bilinear estimate and the definition of $H$,

$$\begin{aligned}
\|I_4\|_{H^{N'}} &\lesssim \int_0^t \|T_{\frac{m}{\phi}}(\mathcal{R}\langle\nabla\rangle\chi^L H, \mathcal{R}w(s))\|_{H^{N'}} \mathrm{d}s \\
&\lesssim \int_0^t \|\langle\nabla\rangle H\|_{W^{N'+\frac{9}{4},(\frac{1}{2}-\delta)^{-1}}} \|w\|_{W^{2,\frac{1}{\delta}}} + \|\mathcal{R}\langle\nabla\rangle H\|_{W^{2,4}} \|w\|_{W^{N'+\frac{9}{4},4}} \mathrm{d}s \\
&\lesssim \int_0^t \|U\|_{L^{\frac{1}{\delta}}} \|U\|_{H^{N'+\frac{13}{4}+2\delta}} \|w\|_{W^{2,\frac{1}{\delta}}} + \|U^h\|_{W^{3,8}} \|U\|_{L^8} \|w\|_{H^{N'+\frac{11}{4}}} \mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{-2(1-3\delta)} \|U\|_{X_T}^3 + \langle s\rangle^{-(2-7\delta)} \|U\|_{X_T}^3 \mathrm{d}s \lesssim \|U\|_{X_T}^3.
\end{aligned}$$

## 4.9 Conclusion of Theorem 4.1.4

By collecting the estimates in Section 6 and Section 7, we find that if $\|U\|_{X_T} \le \vartheta_2$, there exists three constants $d_1, d_2, d_3$ such that for any $\varepsilon \in (0, 1]$, any $T > 0$,

$$\|U\|_{X_T} \le d_1 \|(u_0, \varrho_0, \nabla\varphi_0)\|_Y + d_2 \|U\|_{X_T}^2 + d_3 \|U\|_{X_T}^3. \tag{4.9.1}$$

where

$$\begin{aligned}\|(u_0,\varrho_0,\nabla\varphi_0)\|_Y \quad &\triangleq \quad \|(u_0,\varrho_0,\nabla\varphi_0)^L\|_{W^{4,1}} + \|x(u_0,\varrho_0,\nabla\varphi_0)^L\|_{H^{4+\delta}} \\ &\qquad + \|x(u_0,\varrho_0,\nabla\varphi_0)^h\|_{L^2} + \|(u_0,\varrho_0,\nabla\varphi_0)\|_{H^N}.\end{aligned}$$

Combining with the local existence shown in Section 4, the global existence stems from the standard bootstrap arguments. Indeed, assume $\|(u_0,\varrho_0,\nabla\varphi_0)\|_Y \leq \bar{\vartheta}$, and set

$$T^* = \sup\{T|U \in C([0,T),H^N), \|U\|_{X_T} \leq 2d_1\bar{\vartheta}\}$$

Suppose that $T^* < +\infty$, then by (4.9.1), for any $t < T^*$,

$$\|U\|_{X_t} \leq d_1\bar{\vartheta} + d_2(2d_1\bar{\vartheta})^2 + d_3(2d_1\bar{\vartheta})^3 \leq \frac{3}{2}d_1\bar{\vartheta} \tag{4.9.2}$$

if $\bar{\vartheta}$ is chosen small enough, say $\bar{\vartheta} \leq \vartheta_3$. By the time continuity of $X_t$ norm, one gets that: $\|U\|_{X_{T^*}} \leq \frac{3}{2}d_1\bar{\vartheta}$, which contradicts with the local existence and the definition of $T^*$. We thus finish the proof of Theorem 4.1.4 by setting $\vartheta_1 = \min\{\vartheta_3, \frac{\vartheta_2}{2d_1}\}$.

## 4.10 Proof of Theorem 4.1.5

This section is devoted to the proof of Theorem 4.1.5 concerning the life span of system (4.1.3):

$$\begin{cases} \partial_t n + \mathrm{div}(\rho v + nu + nv) = 0, \\ \partial_t v + u\cdot\nabla v + v\cdot(\nabla u + \nabla v) - \varepsilon\dfrac{1}{\rho+n}\Delta v + \nabla n - \nabla\psi = \varepsilon(\dfrac{1}{\rho+n} - 1)\Delta u, \\ \Delta\psi = n, \\ v|_{t=0} = \mathcal{P}u_0^\varepsilon, n|_{t=0} = 0, \nabla\psi|_{t=0} = 0. \end{cases}$$

*Proof of Theorem 4.1.5.* The local existence of the above system in $C([0,T_\varepsilon,H^3)$ results from the local existence of system (4.1.1) and (3.1.3), it thus suffices for us to extend the existence time to $\varepsilon^{-(1-\vartheta)}$ which follows from the energy estimates. We define the energy functional:

$$\mathscr{E}_3 = \sum_{|\alpha|\leq 3}\mathscr{E}_\alpha = \sum_{|\alpha|\leq 3}\frac{1}{2}\int (1+\varrho+n)|\partial^\alpha v|^2 + |\partial^\alpha n|^2 + |\partial^\alpha\nabla\psi|^2\mathrm{d}x.$$

Taking the time derivative of the above energy functional and using the equations (4.1.3), we get:

$$\partial_t\mathscr{E}_\alpha + \varepsilon\int |\partial^\alpha\nabla v|^2 + |\partial^\alpha\mathrm{div}v|^2\mathrm{d}x = \sum_{j=1}^{9}F_j, \tag{4.10.1}$$

where:

$$\begin{aligned} F_1 \quad &= \quad -\int \rho^\varepsilon\partial^\alpha v\big[\partial^\alpha, u+v\big]\nabla v\mathrm{d}x, \qquad F_2 = -\int \partial^\alpha n\big[\partial^\alpha,\rho^\varepsilon\big]\mathrm{div}v\mathrm{d}x, \\ F_3 \quad &= \quad -\int \partial^\alpha\nabla\psi\big[\partial^\alpha,\rho^\varepsilon\big]v\mathrm{d}x, \qquad F_4 = -\int \partial^\alpha n\partial^\alpha\mathrm{div}(nu)\mathrm{d}x, \\ F_5 \quad &= \quad = \int \partial^\alpha\nabla\psi\partial^\alpha(nu)\mathrm{d}x, \qquad F_6 = -\int \rho^\varepsilon\partial^\alpha v\partial^\alpha(v\cdot\nabla u)\mathrm{d}x, \\ F_7 \quad &= \quad \int \partial^\alpha n\big(\nabla\rho^\varepsilon\partial^\alpha v - \partial^\alpha(\nabla\rho^\varepsilon v)\big)\mathrm{d}x, \quad F_8 = \varepsilon\int \rho^\varepsilon\partial^\alpha v\partial^\alpha\big[(\frac{1}{\rho^\varepsilon} - 1)\Delta u\big], \\ F_9 \quad &= \quad \varepsilon\int \rho^\varepsilon\partial^\alpha v\big[\partial^\alpha, \frac{1}{\rho^\varepsilon}\big](\Delta v + \nabla\mathrm{div}v)\mathrm{d}x. \end{aligned}$$

We recall that $\varrho^\varepsilon = \rho + n = 1 + \varrho + n$. It is easy to see that: $F_1 = F_2 = F_3 = F_9 = 0$ if $|\alpha| = 0$. By standard commutator estimates: (we assume $|\alpha| \geq 1$ in the estimates of $F_1, F_2, F_3, F_9$)

$$
\begin{aligned}
&|F_1| \lesssim \|v\|_{H^{|\alpha|}}^2 (\|\nabla v\|_{L^\infty} + |\nabla u\|_{W^{|\alpha|-1,\infty}}), \quad |F_2 + F_3| \lesssim \|(n, v, \nabla\psi)\|_{H^{|\alpha|}}^2 (\|(v, n)\|_{W^{1,\infty}} + \|\varrho\|_{W^{|\alpha|,\infty}}),\\
&|F_4| \lesssim \|n\|_{H^{|\alpha|}}^2 \|\nabla u\|_{W^{|\alpha|,\infty}}, \qquad |F_5| \lesssim \|(n, \nabla\psi)\|_{H^{|\alpha|}} \|\nabla u\|_{W^{|\alpha|-1,\infty}},\\
&|F_6| \lesssim \|v\|_{H^{|\alpha|}}^2 \|\nabla u\|_{W^{|\alpha|,\infty}}, \qquad |F_7| \lesssim \|(n, v)\|_{H^{|\alpha|}}^2 (\|(\nabla n, \nabla v)\|_{L^\infty} + \|\nabla\varrho\|_{W^{|\alpha|,\infty}}),\\
&|F_8| \lesssim \varepsilon \|v\|_{H^{|\alpha|}} \|\Delta u\|_{W^{|\alpha|,\infty}} \|(n, \varrho)\|_{H^{|\alpha|}}\\
&\quad \lesssim \|(n, v)\|_{H^{|\alpha|}}^2 \|\Delta u\|_{W^{|\alpha|,\infty}} + \varepsilon^2 \|\Delta u\|_{W^{|\alpha|,\infty}} \|\varrho\|_{H^{|\alpha|}}^2\\
&|F_9| \lesssim \varepsilon \|v\|_{\dot{H}^{|\alpha|}} \big(\|\nabla^2 v\|_{\dot{H}^{|\alpha|-1}} \|(\nabla n, \nabla\varrho)\|_{L^\infty} + \|\nabla^2 v\|_{L^\infty} \|(\nabla n, \nabla\varrho)\|_{\dot{H}^{|\alpha|-1}}\big)\\
&\quad \lesssim \varepsilon \|\nabla v\|_{H^3}^2 (\|n\|_{H^3}^2 + \|\varrho\|_{H^3}^2).
\end{aligned}
$$

We only detail the estimate of $F_5$, which seems not direct. Indeed, by the Poisson equation $\Delta\psi = n$, we have:

$$
\begin{aligned}
|F_5| &= \int \partial^\alpha \nabla\psi \cdot u \partial^\alpha n \mathrm{d}x + \int \partial^\alpha \nabla\psi [\partial^\alpha, u] n \mathrm{d}x\\
&= -\int \partial^\alpha \nabla\psi \cdot \nabla(\partial^\alpha \nabla\psi \cdot u) \mathrm{d}x + \int \partial^\alpha \nabla\psi [\partial^\alpha, u] n \mathrm{d}x\\
&= \frac{1}{2}\int |\partial^\alpha \nabla\psi|^2 \mathrm{div} u \mathrm{d}x - \int \partial^\alpha \nabla\psi \cdot (\partial^\alpha \nabla\psi \cdot u) \mathrm{d}x + \int \partial^\alpha \nabla\psi [\partial^\alpha, u] n \mathrm{d}x\\
&\lesssim \|(n, \nabla\psi)\|_{H^{|\alpha|}}^2 \|\nabla u\|_{W^{|\alpha|-1,\infty}}.
\end{aligned}
$$

Define

$$
T_* = \sup_T \left\{ T \Big| \sup_{0 \leq t \leq T} \mathscr{E}_3(t) \leq 4\vartheta^2 \varepsilon^{2-\vartheta} \right\},
$$

where $0 < \vartheta \leq \vartheta_0 \leq \vartheta_1$, $\vartheta_1$ is defined in Theorem (4.1.4) and $\vartheta_0$ is to be chosen.

Summing up the above estimates for any $|\alpha| \leq 3$, we obtain that, there exist three constants $C_2 > 0, C_3 > 0, C_4 > 0$, for any $0 < \vartheta \leq \vartheta_1$, if $\|(\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla\varphi_0^\varepsilon)\|_{Y^4} \leq \frac{\vartheta}{C_3 C_1}$ (which yields $(1 + t)\|(\nabla u, \varrho)\|_{W^{4,\infty}} + (\nabla u, \varrho)\|_{H^7} \leq \frac{\vartheta}{C_3}$ by Theorem 4.1.4, such that the following energy inequality holds:

$$
\partial_t \mathscr{E}_3 + \varepsilon \|\nabla v\|_{H^3}^2 \leq C_2 \mathscr{E}_3^{\frac{3}{2}} + (1 + t)^{-1} \vartheta^3 \varepsilon^2 + (1 + t)^{-1} \vartheta \mathscr{E}_3 + C_4 \varepsilon \|\nabla v\|_{H^3} (\frac{\vartheta}{C_3} + \mathcal{E}_3). \tag{4.10.2}
$$

Now one can choose $\vartheta_0$ small enough, such that for any $0 \leq t < T_*$,

$$
\|(n, \varrho)\|_{L^\infty} \leq \frac{1}{4}, \qquad C_4(\frac{\vartheta}{C_3} + \mathcal{E}_3) \leq \frac{1}{2}
$$

which leads to:

$$
\partial_t \mathscr{E}_3 \leq C_2 \mathscr{E}_3^{\frac{3}{2}} + (1 + t)^{-1} \vartheta^3 \varepsilon^2 + (1 + t)^{-1} \vartheta \mathscr{E}_3 \qquad \forall t \in [0, T_*). \tag{4.10.3}
$$

We are now ready to show $T_* \geq \varepsilon^{-(1-\vartheta)}$. Indeed, for any $t \leq T_0 \triangleq \min\{\varepsilon^{-(1-\vartheta)}, T_*\}$, one has by (4.10.3), Gronwall's inequality and assumptions: $\vartheta \leq \frac{1}{2}$, $16 C_2^{\frac{3}{2}} \vartheta \leq 1$, $\mathscr{E}_3(0) \leq \vartheta^2 \varepsilon^2$ :

$$
\begin{aligned}
\mathscr{E}_3(t) &\leq e^{\int_0^t \vartheta (1+\tau)^{-1} \mathrm{d}\tau} \mathscr{E}_3(0) + \int_0^t e^{\int_s^t \vartheta (1+\tau)^{-1} \mathrm{d}\tau} \big(C_2 \mathscr{E}_3^{\frac{3}{2}} + \vartheta^3 (1 + s)^{-1} \varepsilon^2\big) \mathrm{d}s\\
&\leq (1 + t)^\vartheta \mathscr{E}_3(0) + (1 + t)^\vartheta \int_0^t (1 + s)^{-\vartheta} \big(C_2 (4\vartheta^2 \varepsilon^{2-\vartheta})^{\frac{3}{2}} + \vartheta^3 (1 + s)^{-1} \varepsilon^2\big) \mathrm{d}s\\
&\leq 2^\vartheta \varepsilon^{-\vartheta(1-\vartheta)} \vartheta^2 \varepsilon^2 + \frac{8}{1 - \vartheta} \varepsilon^{-(1-\vartheta)} C_2^{\frac{3}{2}} \vartheta^3 \varepsilon^{3 - \frac{3}{2}\vartheta} + \vartheta^3 \varepsilon^2 \varepsilon^{-\vartheta(1-\vartheta)} \leq \frac{7}{2} \vartheta^2 \varepsilon^{2-\vartheta},
\end{aligned} \tag{4.10.4}
$$

which ensures $T_0 = \varepsilon^{1-\vartheta} < T_*$. Note that since $\frac{1}{2} \leq \rho_0^\varepsilon \leq \frac{3}{2}$, the assumption $\|(n, v, \nabla\psi)\|_{H^3} \leq \vartheta\varepsilon$ leads to $\mathscr{E}_3(0) \leq \vartheta^2 \varepsilon^2$. We thus finish the proof of Theorem 4.1.5 by choosing $C = C_1 C_3$. $\square$

## 4.11 Appendix

We sketch in this appendix the decay and weighted estimates for $\mathcal{R}I_3 \triangleq \mathcal{R}e^{-itb(D)}I_3'$ which reduce to the control of $W^{4+2\delta,2_\delta}$ (recall $2_\delta = \frac{2}{1+\delta/5}$) norm of $Z_1, Z_2, Z_3$. Let us begin with the estimate $Z_2$ which is the easiest. By Lemma 4.4.2, Corollary 4.4.10 and Sobolev embedding,

$$\begin{aligned}\|Z_2\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|T_{\partial_\xi(\frac{m}{\phi})}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2, \mathcal{R}w)\|_{W^{4+2\delta,2_\delta}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|w\|_{H^8}\|w\|^2_{W^{2,12}}\mathrm{d}s \lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}}\langle s\rangle^{\frac{5}{3}}\mathrm{d}s\|U\|^3_X \lesssim \|U\|^3_{X_T}.\end{aligned}$$

For $Z_3$, we split it into two terms:

$$\begin{aligned}Z_3 &= \int_0^t e^{isb(D)}e^{-\varepsilon(t-s)\Delta}T_{\frac{m}{\phi}(\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}}+\chi_{\{\langle\xi-\eta\rangle\ge\langle\eta\rangle\}})}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2, e^{-isb(D)}x\mathcal{R}f)\mathrm{d}s \\ &\triangleq Z_{31} + Z_{32}\end{aligned}$$

The estimate of $Z_{31}$ is similar to that of $Z_2$, as we can put all the loss of derivative on the term $B(\mathcal{R}w, \mathcal{R}w)$. Applying lemma 4.4.2 with $p = 2_\delta = \frac{2}{1+\delta/5}$, bilinear estimate (4.4.18), Lemma 4.8.3 with $7.5 \le \frac{2}{3}N + \frac{1}{2}$, we control $Z_{31}$ as:

$$\begin{aligned}\|Z_{31}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|T_{\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle\le\langle\eta\rangle\}}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2, e^{-isb(D)}x\mathcal{R}f)\|_{W^{4+3\delta,2_\delta}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|\mathcal{R}w\|_{L^\infty}\|\mathcal{R}w\|_{W^{7+4\delta,\frac{10}{1+\delta}}}\|e^{-isb(D)}x\mathcal{R}f\|_{W^{2,\frac{5}{2}}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|\mathcal{R}w\|_{W^{1,\frac{1}{\delta}}}\|\mathcal{R}w\|_{W^{7.5,3}}\|\langle x\rangle f\|_{W^{3,\frac{2}{\delta}}}\langle s\rangle^{\frac{1}{5}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{1+\delta}{5}}\langle s\rangle^{-(1-2\delta)}\langle s\rangle^{-\frac{1}{3}}\mathrm{d}s\|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}.\end{aligned}$$

For $Z_{32}$, one splits it again into two terms:

$$Z_{32} = \mathcal{F}^{-1}\left(\int_0^t \left(\Psi\left(\frac{\xi-\eta}{\langle s\rangle^{\delta_0}}\right) + 1 - \Psi\left(\frac{\xi-\eta}{\langle s\rangle^{\delta_0}}\right)\right)\cdots\mathrm{d}s\right) \triangleq Z_{321} + Z_{322}$$

.

For $Z_{321}$, Corollary 4.4.10 and Sobolev embedding lead to :

$$\begin{aligned}\|Z_{321}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|T_{\frac{m}{\phi}\chi_{\{\langle\xi-\eta\rangle\ge\langle\eta\rangle\}}}(\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2), P_{\le\langle s\rangle^{\delta_0}}e^{-isb(D)}x\mathcal{R}f)\|_{W^{4+3\delta,2_\delta}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|w\|^2_{W^{2,12}}\|P_{\le\langle s\rangle^{\delta_0}}e^{-isb(D)}x\mathcal{R}f)\|_{W^{8,3}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|w\|^2_{W^{2,12}}\langle s\rangle^{5\delta_0}\langle s\rangle^{\frac{1}{3}}\|x\mathcal{R}f\|_{W^{3,3}}\mathrm{d}s \\ &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}}\langle s\rangle^{-\frac{5}{3}}\|U\|_{X_T^2}\langle s\rangle^{\frac{1}{3}+5\delta_0}\|U\|_{X_T}\mathrm{d}s \lesssim \|U\|^3_{X_T}.\end{aligned}$$

Note we can choose $\delta < \delta_0 \le \frac{1}{50}$.

For $Z_{322}$, we define $M_1(\xi,\eta) = \frac{m}{\phi}(1-\Psi(\frac{\xi-\eta}{\langle s\rangle^{\delta_0}}))\chi_{\{\langle\xi-\eta\rangle>\langle\eta\rangle\}}\langle\eta\rangle$ and recall $\tilde{\phi} = b(\xi) \pm b(\xi-\eta) \pm b(\eta -$

$\sigma) \pm b(\sigma)$. Using identity $\partial_\xi \widehat{\mathcal{R}f}(\xi-\eta) = -\partial_\eta \widehat{\mathcal{R}f}(\eta)$ to integrate by parts in $\eta$, one gets:

$$
\begin{aligned}
Z_{322} &= \mathcal{F}^{-1}\Big(\int_0^t \int\int is\partial_\eta\tilde{\phi} e^{is\tilde{\phi}} e^{-\varepsilon(t-s)|\xi|^2} M_1(\xi,\eta)\mathcal{R}(\eta)\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}s \\
&\quad + \int_0^t \int\int e^{is\tilde{\phi}} e^{-\varepsilon(t-s)|\xi|^2} \partial_\eta M_1(\xi,\eta)\mathcal{R}(\eta)\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}s \\
&\quad + \int_0^t \int\int e^{is\tilde{\phi}} e^{-\varepsilon(t-s)|\xi|^2} M_1(\xi,\eta)|\eta|^{-1}\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}s \\
&\quad + \int_0^t \int\int e^{is\tilde{\phi}} e^{-\varepsilon(t-s)|\xi|^2} M_1(\xi,\eta)\mathcal{R}(\eta)\widehat{x\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\widehat{\mathcal{R}f}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\sigma\mathrm{d}s\Big) \\
&\lesssim Z_{3221} + \cdots Z_{3224}
\end{aligned}
$$

The estimation of $Z_{3222} = \int_0^t e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{\partial_\eta M_1}(\mathcal{R}(\mathcal{R}w)^2, \mathcal{R}w)\mathrm{d}s$ is similar to that of $Z_2$, we thus do not detail it. For $G_{3223}$, thanks to Corollary 4.4.18 and Hardy-Littlewood-Sobolev inequality,

$$
\begin{aligned}
\|Z_{3223}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|T_{M_1}(|\nabla|^{-1}(\mathcal{R}w)^2, P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w)\|_{W^{4+3\delta,2_\delta}}\mathrm{d}s \\
&\lesssim \lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w)\|_{H^8} \|(\mathcal{R}w)^2\|_{W^{2,2_\delta}}\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \langle s\rangle^{-(N-8)\delta_0+\delta}\|w\|_{H^N}\langle s\rangle^{-(1-\frac{\delta}{5})}\mathrm{d}s\|U\|_{X_T}^2 \lesssim \|U\|_{X_T}^3.
\end{aligned}
$$

Note $N = 11$, the last inequality holds if we choose $\delta \leq \delta_0$.

The term $Z_{3224}$ can be estimated in the similar manner as that of $Z_{31}$, we omit the detail. The estimate of $Z_{3221}$ is similar to that of $Z_1$, we thus only focus on the estimate of $Z_1$ in the following.

As before, we split it as:

$$
Z_1 = \mathcal{F}^{-1}\Big(\int_0^t (\Psi(\frac{\xi-\eta}{\langle s\rangle^{\delta_0}}) + 1 - \Psi(\frac{\xi-\eta}{\langle s\rangle^{\delta_0}}))\cdots \mathrm{d}s\Big) \triangleq Z_{11} + Z_{12}.
$$

Split $Z_{12}$ further as:

$$
Z_{12} = \int_0^t s e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{M_2}((P_{\geq\langle s\rangle^{\delta_0}} + P_{\leq\langle s\rangle^{\delta_0}})(\mathcal{R}(\mathcal{R}w)^2, P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w)\mathrm{d}s
$$

where $M_2 = \frac{m}{\phi}\langle\eta\rangle(i\partial_\xi\tilde{\phi})$. Therefore, by bilinear estimate lemma 4.4.6 and the Sobolev embedding,

$$
\begin{aligned}
&\|Z_{12}\|_{W^{4+2\delta,2_\delta}} \lesssim \int_0^t s\langle s\rangle^{\frac{\delta}{5}}\big(\|\mathcal{R}(\mathcal{R}w)^2\|^2_{W^{2,\frac{1}{2\delta}}}\|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{H^7} + \|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}(\mathcal{R}w)^2\|_{H^7}\|\mathcal{R}w\|_{W^{2,\frac{1}{\delta}}}\big)\mathrm{d}s \\
\lesssim &\int_0^t s\langle s\rangle^{\frac{\delta}{5}}\|\mathcal{R}w\|^2_{W^{2,\frac{1}{\delta}}}\langle s\rangle^{-(N-7)\delta_0}\|w\|_{H^N}\mathrm{d}s \lesssim \int_0^t s\langle s\rangle^{\frac{\delta}{5}}\langle s\rangle^{-2(1-2\delta)}\langle s\rangle^{-(N-7)\delta_0+\delta}\mathrm{d}s\|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}.
\end{aligned}
$$

if we assume $N = 11$, $\delta_0 \geq 3\delta$.

For $Z_{11}$, one uses decomposition $g^2 = gP_{\geq\langle s\rangle^{\delta_0}}g + P_{\leq\langle s\rangle^{\delta_0}}gP_{\geq\langle s\rangle^{\delta_0}}g + P_{\leq\langle s\rangle^{\delta_0}}gP_{\leq\langle s\rangle^{\delta_0}}g$ to split it into three terms and denote as $Z_{111} + Z_{112} + Z_{113}$. For the term $Z_{111}$, one can use bilinear estimate (4.4.18) Sobolev embedding and the spectral localization of each term to get:

$$
\begin{aligned}
\|Z_{111}\|_{w^{4,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{1+\frac{\delta}{5}}(\|P_{\leq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{W^{7,\frac{1}{\delta}}}\|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w\cdot\mathcal{R}w\|_{W^{2,\frac{10}{5-9\delta}}} \\
&\qquad + \|P_{\leq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{W^{2,\frac{1}{\delta}}}\|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w\cdot\mathcal{R}w\|_{W^{\frac{25}{4}+2\delta,\frac{10}{5-9\delta}}}\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{1+\frac{\delta}{5}}\|P_{\leq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{W^{2,\frac{1}{\delta}}}\|\mathcal{R}w\|_{L^{\frac{1}{\delta}}}\big(\langle s\rangle^{5\delta_0}\|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{H^3} + \|P_{\geq\langle s\rangle^{\delta_0}}\mathcal{R}w\|_{H^8}\big)\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{1+\frac{\delta}{5}}\langle s\rangle^{-2(1-2\delta)}\langle s\rangle^{-(N-8)\delta_0+\delta}\mathrm{d}s\|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}
\end{aligned}
$$

as $N=11$, and $\delta_0 \geq 5\delta$. The estimate of $Z_{112}$ is similar to that of $Z_{111}$, we thus skip it. Now, we focus on the estimation of

$$\begin{aligned} Z_{113} &= \int_0^t s e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{M_2}(\mathcal{R}(P_{\leq \langle s\rangle^{\delta_0}} \mathcal{R}w P_{\leq \langle s\rangle^{\delta_0}} \mathcal{R}w, P_{\leq \langle s\rangle^{\delta_0}} \mathcal{R}w)\mathrm{d}s \\ &= \mathcal{F}^{-1}\Big(\int_0^t s\partial_\xi \tilde{\phi} e^{is\tilde{\phi}} e^{\varepsilon(t-s)\Delta} \frac{m}{\phi}(\xi,\eta)\mathcal{R}(\eta)\langle\eta\rangle \Phi_{\leq\langle s\rangle^{\delta_0}}(\eta-\sigma)\Phi_{\leq\langle s\rangle^{\delta_0}}(\sigma) \\ &\qquad \Phi_{\leq\langle s\rangle^{\delta_0}}(\xi-\eta)\widehat{\mathcal{R}f)}(\eta-\sigma)\widehat{\mathcal{R}f)}(\sigma)\widehat{\mathcal{R}f)}(\xi-\eta)\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}s\Big) \end{aligned}$$

Recall that $\tilde{\phi} = b(\xi) \pm b(\xi-\eta) \pm b(\eta-\sigma) \pm b(\sigma)$. For the case $'+++','++-','+-+','-++','---'$, one could easily prove that: $\frac{1}{\tilde{\phi}} >_{\kappa_0} \min\{\langle\xi\rangle, \langle\xi-\eta\rangle, \langle\eta-\sigma\rangle, \langle\sigma\rangle\}$. Therefore, we shall use the identity $\frac{1}{i\tilde{\phi}}\partial_s e^{is\tilde{\phi}} = e^{is\tilde{\phi}}$ to integrate by parts in time, which gives us essentially the quartic terms that are easy to handle, we do not detail them. The remaining three terms $'--+','-+-','+--'$ are more involved, we will detail the $'+--'$ case for instance. The other two cases are a little bit easier. Firstly, for $\tilde{\phi} = b(\xi)+b(\xi-\eta)-b(\eta-\sigma)-b(\sigma)$, we have $\partial_\xi\tilde{\phi} = \frac{(1-2\varepsilon^2|\xi|^2)\xi}{b(\xi)} + \frac{(1-2\varepsilon^2|\xi-\eta|^2)(\xi-\eta)}{b(\xi-\eta)}$. In this case, by Lemma 4.11.2 below, one can find two matrices $Q_1, Q_2$, st, $\partial_\xi\tilde{\phi}(\xi,\eta,\sigma) = -2Q_1(\xi,\eta)\partial_\eta\tilde{\phi} - Q_2(\xi,\eta,\sigma)\partial_\sigma\tilde{\phi}$, and $Q_j (j=1,2)$ satisfy the condition:

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_\sigma^\gamma Q_j(\xi,\eta,\sigma)| \lesssim_{\alpha,\beta,\gamma,\kappa_0} \langle|\xi|+|\eta|+|\sigma|\rangle^3.$$

We now split $Z_{113}$ again:

$$Z_{113} = \mathcal{F}^{-1}\Big(\int_0^t \int (\Psi_{\geq\langle s\rangle^{-\delta_0}}(\eta) + \Psi_{\leq\langle s\rangle^{-\delta_0}}(\eta)) \cdots \mathrm{d}\eta\mathrm{d}s\Big) \triangleq Y_1 + Y_2.$$

Let us see $Y_1$. In this case, there is no singularity for $\mathcal{R}(\eta)$ as $\eta$ does not vanish. Therefore, we could use the identity:

$$is\partial_\xi\tilde{\phi}e^{is\tilde{\phi}} = -2Q_1(\xi,\eta,\sigma)\partial_\eta(e^{is\tilde{\phi}}) - Q_2(\xi,\eta,\sigma)\partial_\sigma(e^{is\tilde{\phi}})$$

and integrate by parts in $\eta$ and $\sigma$ respectively. We only detail the situation of integration by parts in $\eta$ as the other case is similar. To continue, we denote $m_j(\xi,\eta,\sigma) = Q_j\frac{m}{\phi}(\xi,\eta)\mathcal{R}(\eta)\langle\eta\rangle\Psi_{\leq\langle s\rangle^{\delta_0}}(\xi-\eta)\Psi_{\leq\langle s\rangle^{\delta_0}}(\eta-\sigma)\Psi_{\leq\langle s\rangle^{\delta_0}}(\sigma)\Psi_{\geq\langle s\rangle^{-\delta_0}}(\eta)\mathcal{R}(\eta)\langle\eta\rangle$.

After integrating by parts in $\eta$, there are two terms to be estimated:

$$\begin{aligned} Y_{11} &= \int_0^t e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{\partial_\eta m_1}(\mathcal{R}w, \mathcal{R}w, \mathcal{R}v)\mathrm{d}s \\ Y_{12} &= \int_0^t e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{m_1}(e^{isb(D)}x\mathcal{R}f, \mathcal{R}w, \mathcal{R}w)\mathrm{d}s + \textit{similar} \quad \textit{term}. \end{aligned}$$

However, these two terms can be easily treated once we have the following lemma.

**Lemma 4.11.1.** *$m_j(\xi,\eta,\sigma), j=1,2$ is defined as follows, the following trilinear estimates hold:*

$$\begin{aligned} \|T_{m_j}(f,g,h)\|_{L^p} &\lesssim \langle s\rangle^{12\delta_0}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}, \\ \|T_{\partial_\eta m_j}(f,g,h)\|_{L^p} &\lesssim \langle s\rangle^{13\delta_0}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|h\|_{L^{p_3}}, \end{aligned}$$

*where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$*

*Proof.* It is not hard to check that: $\partial_\xi^\alpha\partial_\eta^\beta\partial_\sigma^\gamma m_1 \lesssim \langle s\rangle^{\delta_0|\beta|}\langle s\rangle^{5\delta_0}$, which yields

$$\begin{aligned} &\|\mathcal{F}^{-1}(m_j(\xi,\eta,\sigma))\|_{L^1} \lesssim \|(1+\partial_\xi^4+\partial_\eta^4+\partial_\sigma^4)m_j\|_{L^2} \\ \lesssim\ & \Big(\int_{|(\xi,\eta,\sigma)|\lesssim\langle s\rangle^{\delta_0}} |(1+\partial_\xi^4+\partial_\eta^4+\partial_\sigma^4)m_j|^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\sigma\Big)^{\frac{1}{2}} \lesssim \langle s\rangle^{9\delta_0}\langle s\rangle^{3\delta_0} \lesssim \langle s\rangle^{12\delta_0} \end{aligned}$$

Similarly, $\|\mathcal{F}^{-1}(\partial_\eta m_j(\xi,\eta,\sigma))\|_{L^1} \lesssim \langle s\rangle^{13\delta_0}$. We thus finish the proof by noticing the explicit formulae of $T(f,g,h)$

$$T_{m_j}(f,g,h) = \int\int\int \mathcal{F}^{-1}(m_j)(y.z-y, w-z)f(x-y)g(x-z)h(x-w)\mathrm{d}y\mathrm{d}z\mathrm{d}w$$

$\square$

We now estimate $Y_{12}$ for example, by the spectral localization and Lemma 4.4.2,

$$\begin{aligned}
\|Y_{12}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}} \langle s\rangle^{(4+3\delta)\delta_0} \|T_{m_1\langle\xi\rangle^\delta}(e^{isb(D)}x\mathcal{R}f, \mathcal{R}w, \mathcal{R}w)\|_{L^{2_\delta}} \mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{17\delta_0} \|e^{isb(D)}x\mathcal{R}f\|_{L^{(\frac{1}{2\delta}-2\delta)^{-1}}} \|\mathcal{R}w\|^2_{L^{\frac{1}{\delta}}} \mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{17\delta_0} \langle s\rangle^{-2(1-2\delta)} \langle s\rangle^{4\delta} \mathrm{d}s \|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}.
\end{aligned}$$

where the folowing fact has been used:

$$\|e^{isb(D)}x\mathcal{R}f\|_{L^{(\frac{1}{2\delta}-2\delta)^{-1}}} \lesssim \langle s\rangle^{4\delta} \|x\mathcal{R}f\|_{W^{\frac{1}{2},(\frac{1}{2\delta}-2\delta)^{-1}}} \lesssim \langle s\rangle^{4\delta} \|xf\|_{W^{1,\frac{2}{1-\delta}}}.$$

We now go back to estimate $Y_2$, as before, we split it into two terms:

$$Y_2 = \mathcal{F}^{-1}\Big(\int_0^t \int \big(\Psi_{\leq 3\langle s\rangle^{-\delta_0/5}}(\xi-\eta) + \Psi_{\geq 3\langle s\rangle^{-\delta_0/5}}(\xi-\eta)\big) \cdots \mathrm{d}\eta \mathrm{d}s\Big) \triangleq Y_{21} + Y_{22}$$

For $Y_{21}$, one can use the specific form of $\partial_\xi \tilde{\phi} = \frac{1-2\varepsilon|\xi|^2}{b(\xi)}\xi + \frac{1-2\varepsilon|\xi-\eta|^2}{b(\xi-\eta)}(\xi-\eta)$. The observation is that, having projected to the low frequency for $\xi-\eta$ and $\eta$, we could make use of $\xi-\eta$ and $\eta$ appearing in $\partial_\xi\tilde{\phi}$. We also recall that $(1-2\varepsilon|\xi|^2)$ here is bounded on the support of $\chi^L(\xi)$. Formally, we could write $Y_{21}$ as

$$\int_0^t ise^{isb(D)} e^{\varepsilon(t-s)\Delta} \nabla T_{\frac{m}{\phi}\frac{1-2\varepsilon|\xi|^2}{b(\xi)}\langle\eta\rangle} (\mathcal{R}P_{\lesssim\langle s\rangle^{-\delta_0}}(P_{\lesssim\langle s\rangle^{\delta_0}}\mathcal{R}w)^2, P_{\lesssim 3\langle s\rangle^{-\delta_0/5}}\mathcal{R}w)\mathrm{d}s \tag{4.11.1}$$

and similar term. One can estimate (4.11.1) as follows:

$$\begin{aligned}
\|4.11.1\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{1+\delta} \langle s\rangle^{-\frac{1}{5}\delta_0} \|\mathcal{R}w\|^2_{W^{2,\frac{1}{\delta}}} \|\mathcal{R}w\|_{W^{8,(\frac{1}{2\delta}-2\delta)^{-1}}} \mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{1+\delta-\frac{1}{5}\delta_0} \langle s\rangle^{-2(1-2\delta)} \mathrm{d}s \|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}
\end{aligned}$$

if we choose $\delta_0 \geq 50\delta$.

For $Y_{22}$, we need to split again into two terms.

$$Y_{22} = \mathcal{F}^{-1}\Big(\int_0^t \int\int \big(\Psi_{\geq 2\langle s\rangle^{-\delta_0}}(\sigma) + \Psi_{\leq 2\langle s\rangle^{-\delta_0}}(\sigma)\big) \cdots \mathrm{d}\sigma \mathrm{d}\eta \mathrm{d}s\Big) \triangleq Y_{221} + Y_{222}$$

Let us see $Y_{221}$, in this case we have $|\sigma-\eta| > |\sigma| - |\eta| > \frac{4}{9}|\sigma|$. Besides, one can find a matrix $Q_3$, such that:

$$\partial_\sigma\tilde{\phi} = -\Big[\frac{1-2\varepsilon|\sigma|^2}{b(\sigma)}\sigma + \frac{1-2\varepsilon|\sigma-\eta|^2}{b(\sigma-\eta)}(\sigma-\eta)\Big] = Q_3(2\sigma-\eta)$$

so we have: $|\partial_\sigma\tilde{\phi}| \gtrsim \|Q_3^{-1}\|^{-1}|2\sigma-\eta| \gtrsim \frac{|\sigma|}{\langle\sigma\rangle\langle\eta-\sigma\rangle} \gtrsim \langle s\rangle^{-3\delta_0}$. We thus could use identity $e^{is\tilde{\phi}} = \frac{\partial_\sigma\tilde{\phi}\cdot\partial_\sigma}{is|\partial_\sigma\tilde{\phi}|^2} \cdot \partial_\sigma e^{is\tilde{\phi}}$ and integrate by parts in $\sigma$, this leads to two terms:

$$\begin{aligned}
Y_{2211} &= \int_0^t e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{m_3}(\mathcal{R}T_{\partial_\sigma\tilde{m}}(\mathcal{R}w)^2, \mathcal{R}w)\mathrm{d}s \\
Y_{2212} &= \int_0^t e^{isb(D)} e^{\varepsilon(t-s)\Delta} T_{m_3}(\mathcal{R}T_{\tilde{m}}(e^{isb(D)}x\mathcal{R}f, \mathcal{R}w), \mathcal{R}w)\mathrm{d}s
\end{aligned}$$

where we denote

$$\begin{aligned}
m_3(\xi,\eta) &= \frac{m}{\phi}\langle\eta\rangle \Psi_{\leq\langle s\rangle^{-\delta_0}}(\eta) \Psi_{\leq\langle s\rangle^{\delta_0}}(\xi-\eta) \Psi_{\geq 3\langle s\rangle^{-\delta_0/5}}(\xi-\eta), \\
\tilde{m}(\eta,\sigma) &= \frac{\partial_\sigma\tilde{\phi}}{|\partial_\sigma\tilde{\phi}|^2} \Psi_{\geq 2\langle s\rangle^{-\delta_0}}(\sigma) \Psi_{\leq\langle s\rangle^{\delta_0}}(\sigma) \Psi_{\leq\langle s\rangle^{\delta_0}}(\eta-\sigma) \tilde{\chi}^L(\eta-\sigma) \tilde{\chi}^L(\sigma).
\end{aligned}$$

Similar to Lemma 4.11.1, one has the following inequality:

$$\|T_{m_3}(f,g)\|_{L^p} \lesssim \langle s\rangle^{6\delta_0}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$$
$$\|T_{\tilde{m}}(u,v)\|_{L^p} \lesssim \langle s\rangle^{14\delta_0}\|u\|_{L^{p_1}}\|v\|_{L^{p_2}} \quad \|T_{\partial_\sigma \tilde{m}}(u,v)\|_{L^p} \lesssim \langle s\rangle^{17\delta_0}\|u\|_{L^{p_1}}\|v\|_{L^{p_2}}$$

for any $1 \le p \le \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. These inequalities in hand, the estimates of $Y_{2211}, Y_{2212}$ are direct. For example:

$$\begin{aligned}
\|Y_{2212}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{\frac{\delta}{5}}\langle s\rangle^{(4+3\delta)\delta_0}\langle s\rangle^{20\delta_0}\|e^{isb(D)}x\mathcal{R}f\|_{L^{(\frac{1}{2\delta}-2\delta)^{-1}}}\|\mathcal{R}w\|^2_{L^{\frac{1}{\delta}}}\mathrm{d}s \\
&\lesssim \int_0^t \langle s\rangle^{25\delta_0}\langle s\rangle^{-2(1-2\delta)}\mathrm{d}s\|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}
\end{aligned}$$

Note we could choose $\delta < \delta_0 \le \frac{1}{50}$.

Finally, it remains for us to estimate $Y_{222}$. In this case, there is no structure for $\partial_\xi\tilde{\phi}$ can be used. Nevertheless, as noted in [84], one can employ kind of 'partial normal form'. We notice that: $b(\eta)+b(\sigma)-2 = \frac{1-\varepsilon^2|\eta-\sigma|^2}{b(\eta-\sigma)+1}|\eta-\sigma|^2 + \frac{1-\varepsilon^2|\sigma|^2}{b(\sigma)+1}|\sigma|^2$, the observation is that we could use $|\eta-\sigma|^2$ and $|\sigma|^2$ appearing in this quantity. On the other hand, $b(\xi)+b(\xi-\eta)-2 = \frac{(1-\varepsilon^2|\xi|^2)}{b(\xi)+1}|\xi|^2 + \frac{(1-\varepsilon^2|\xi-\eta|^2)}{b(\xi-\eta)+1}|\xi-\eta|^2 \ge \frac{(1-\varepsilon^2|\xi-\eta|^2)}{b(\xi-\eta)+1}|\xi-\eta|^2 \ge \langle s\rangle^{-\delta_0/5}$. We thus use identity:

$$\begin{aligned}
e^{is\tilde{\phi}} &= e^{is(b(\xi)+b(\xi-\eta)-2)}e^{-is(b(\eta)+b(\eta-\sigma)-2)} \\
&= \frac{-i}{b(\xi)+b(\xi-\eta)-2}\partial_s(e^{is(b(\xi)+b(\xi-\eta)-2)})e^{-is(b(\eta)+b(\eta-\sigma)-2)}
\end{aligned}$$

to integrate by parts in $s$:

$$\begin{aligned}
Y_{222} &= -te^{itb(D)}T_{m_4}(\mathcal{R}T_{\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w),\mathcal{R}w) + \int_0^t se^{isb(D)}e^{\varepsilon(t-s)\Delta}\varepsilon\Delta T_{m_4}(\mathcal{R}T_{\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w),\mathcal{R}w)\mathrm{d}s \\
&\quad + i\int_0^t se^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{m_4}(\mathcal{R}T_{\tilde{m}_1(b(\eta)+b(\eta-\sigma)-2)}(\mathcal{R}w,\mathcal{R}w),\mathcal{R}w)\mathrm{d}s \\
&\quad - \int_0^t se^{isb(D)}e^{\varepsilon(t-s)\Delta}T_{m_4}(\mathcal{R}T_{\tilde{m}_1}(e^{isb(D)}\partial_s\mathcal{R}f,\mathcal{R}w),\mathcal{R}w)\mathrm{d}s + \quad similar \quad term \\
&\quad + \int_0^t se^{isb(D)}e^{\varepsilon(t-s)\Delta}[T_{\partial_s m_4}(\mathcal{R}T_{\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w),\mathcal{R}w) + T_{m_4}(\mathcal{R}T_{\partial_s\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w),\mathcal{R}w)]\mathrm{d}s \\
&\triangleq Y_{2221} + \cdots Y_{2225}
\end{aligned}$$

where the following notations has been used:

$$\begin{aligned}
m_4(\xi,\eta,s) &= \frac{\partial_\xi\tilde{\phi}}{b(\xi)+b(\xi-\eta)-2}\frac{m}{\phi}\langle\eta\rangle\Psi_{\le\langle s\rangle^{\delta_0}}(\xi-\eta)\Psi_{\ge 3\langle s\rangle^{-\delta_0/5}}(\xi-\eta)\tilde{\Psi}_{\gtrsim\langle s\rangle^{\delta_0}}(\eta), \\
\tilde{m}_1(\eta,\sigma,s) &= \Psi_{\le\langle s\rangle^{-\delta_0}}(\eta)\Psi_{\le\langle s\rangle^{\delta_0}}(\eta-\sigma)\Psi_{\le\langle s\rangle^{\delta_0}}(\sigma)\Psi_{\le 2\langle s\rangle^{-\delta_0}}(\sigma)
\end{aligned}$$

We have again, as in Lemma 4.11.1, for any $1 \le p \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\|T_{m_4}(f,g)\|_{L^p} \lesssim \langle s\rangle^{9\delta_0}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}, \quad \|T_{\tilde{m}_1}(u,v)\|_{L^p} \lesssim \|u\|_{L^{p_1}}\|v\|_{L^{p_2}}.$$

For $Y_{2221}$,

$$\begin{aligned}
\|Y_{2221}\|_{W^{4+2\delta,2_\delta}} &\lesssim t\langle t\rangle^{14\delta_0}\|\mathcal{R}T_{\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w)\|_{L^{\frac{1}{2\delta}}}\|\mathcal{R}w\|_{L^{(\frac{1}{2\delta}-2\delta)^{-1}}} \\
&\lesssim \langle t\rangle^{1+14\delta_0-2(1-2\delta)}\|U\|^3_{X_T} \lesssim \|U\|^3_{X_T}
\end{aligned}$$

if $\delta \le \delta_0 \le \frac{1}{50}$.

For $Y_{2222}$, owing to the above estimate and the fact $e^{\varepsilon(t-s)\Delta}\varepsilon\Delta\chi^L$ is $L^{2-\delta}$ multiplier with norm less than $\langle t-s\rangle^{-1}$, we get that:

$$\|Y_{2222}\|_{W^{4+2\delta,2_\delta}} \lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-(1-18\delta_0)}\mathrm{d}s\|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3$$

For $Y_{2223}$, since on the support of $m_4(\xi,\eta,s)$, $|\xi-\eta|\geq|\eta|$, and $|\xi-\eta|\gtrsim\langle s\rangle^{-\delta_0}$, it is not hard to see that

$$\|\mathcal{F}^{-1}(m_4\langle\xi\rangle^{4+3\delta}\langle\xi-\eta\rangle^{-8}\langle\eta\rangle^{-8})\|_{L^1} \lesssim \|(1+\partial_\xi^3+\partial_\eta^3)(m_4\langle\xi\rangle^{4+3\delta}\langle\xi-\eta\rangle^{-8}\langle\eta\rangle^{-8})\|_{L^2} \lesssim \langle s\rangle^{\frac{4}{5}\delta_0}.$$

Write also $b(\eta)+b(\sigma)-2=\frac{1-\varepsilon^2|\eta-\sigma|^2}{b(\eta-\sigma)+1}|\eta-\sigma|^2+\frac{1-\varepsilon^2|\sigma|^2}{b(\sigma)+1}|\sigma|^2$, we thus could estimate $Y_{2223}$ by

$$\begin{aligned}\|Y_{2223}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{1+\delta}\langle s\rangle^{\frac{4}{5}\delta_0}\|\frac{1-\varepsilon^2\Delta}{b(D)}|\nabla|^2 P_{\lesssim\langle s\rangle^{-\delta_0}}\mathcal{R}w\|_{L^{\frac{1}{\delta}}}\|\mathcal{R}w\|_{L^{\frac{1}{\delta}}}\|w\|_{W^{8,(\frac{1}{2\delta}-2\delta)^{-1}}}\mathrm{d}s\\ &\lesssim \int_0^t \langle s\rangle^{1+2\delta}\langle s\rangle^{\frac{4}{5}\delta_0}\langle s\rangle^{-2\delta_0}\langle s\rangle^{-2(1-2\delta)}\mathrm{d}s\|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3\end{aligned}$$

if $10\delta<\delta_0$. For $Y_{2224}$ it is not tough because it is essentially quartic. Similar to that of $Y_{2221}$, we have

$$\begin{aligned}\|Y_{2224}\|_{W^{4+2\delta,2_\delta}} &\lesssim \int_0^t \langle s\rangle^{1+\delta}\langle s\rangle^{14\delta_0}\|\mathcal{R}w\|_{L^{\frac{1}{\delta}}}^2\|e^{isb(D)}x\mathcal{R}f\|_{L^{(\frac{1}{2\delta}-2\delta)^{-1}}}\mathrm{d}s\\ &\lesssim \int_0^t \langle s\rangle^{1+\delta}\langle s\rangle^{14\delta_0}\langle s\rangle^{-2(1-2\delta)}\langle s\rangle^{-1}\mathrm{d}s\|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3\end{aligned}$$

where in the above, the following identity has been used

$$e^{isb(D)}\partial_s\mathcal{R}f=\mathcal{R}\varepsilon\Delta w+\mathcal{R}\langle\nabla\rangle(\mathcal{R}w)^2+\mathcal{R}H$$

from which one easily gets: $\|e^{isb(D)}\partial_s\mathcal{R}f\|_{H^1}\lesssim\langle s\rangle^{-1}\|U\|_{X_T}^3$.

For the last term $Y_{2225}$, one notices that when we take the time derivative on $m_4(\xi,\eta,s)$ or $\tilde{m}_1(\eta,\sigma,s)$, there will emerge a power $\langle s\rangle^{-1}$ which is enough for us to close the estimate. For instance,

$$\partial_s\Psi(\langle s\rangle^{\delta_0}\eta)=\eta\cdot\nabla\Psi(\langle s\rangle^{\delta_0}\eta)\langle s\rangle^{\delta_0}\frac{s}{\langle s\rangle}\langle s\rangle^{-1}\triangleq\tilde{\Psi}(\langle s\rangle^{\delta_0}\eta)\frac{s}{\langle s\rangle}\langle s\rangle^{-1}$$

where $\tilde{\Psi}$ has the same properties as $\psi$ that we need: compactly supported, smooth.

**Lemma 4.11.2.** *Recall $b(x)=\sqrt{1+|x|^2-\varepsilon^2|x|^4}$, with $x\in\mathbb{R}^2$, $\varepsilon\in(0,1]$. There exists a $2\times2$ matrix $S$ such that the following identity holds:*

$$\frac{(1-2\varepsilon^2|\xi|^2)x}{b(x)}-\frac{(1-2\varepsilon^2|y|^2)y}{b(y)}=S(x,y)(x-y)$$

*Besides, if $\varepsilon|x|^2\leq3\kappa_0,\varepsilon|y|^2\leq3\kappa_0$ with $\kappa_0\leq\frac{1}{200}$, then $S$ is invertible. Moreover, for any $\varepsilon\in(0,1]$ and $\alpha,\beta\in\mathbb{N}^2$, the following uniform (in $\varepsilon$) estimate holds:*

$$\begin{aligned}&|\partial_x^\alpha\partial_y^\beta S(x,y)|\lesssim_{\alpha,\beta,\kappa_0}\frac{1}{\langle y\rangle},\\ &|\partial_x^\alpha\partial_y^\beta S^{-1}(x,y)|\lesssim_{\alpha,\beta,\kappa_0}(\langle x\rangle+\langle y\rangle)^3.\end{aligned}$$

*Proof.*

$$\begin{aligned}&\frac{(1-2\varepsilon^2|\xi|^2)x}{b(x)}-\frac{(1-2\varepsilon^2|y|^2)y}{b(y)}\\ =\;&(1-2\varepsilon^2|x|^2)x(\frac{1}{b(x)}-\frac{1}{b(y)})+\frac{(1-2\varepsilon^2|x|^2)x-(1-2\varepsilon^2|y|^2)y)}{b(y)}\\ =\;&-(1-2\varepsilon^2|x|^2)(|x|^2-|y|^2)x\frac{1-\varepsilon^2(|x|^2+|y|^2)}{b(x)b(y)(b(x)+b(y)}+\frac{(1-2\varepsilon^2|x|^2)(x-y)-2\varepsilon^2(|x|^2-|y|^2)y}{b(y)}\\ =\;&\frac{1-2\varepsilon^2|x|^2}{b(y)}[\mathrm{Id}_{2\times2}-\frac{\varepsilon^2(|x|^2+|y|^2)}{b(x)(b(x)+b(y))}x\otimes(x+y)-2\frac{\varepsilon^2}{1-2\varepsilon^2|x|^2}y\otimes(x+y)](x-y)\\ \triangleq\;&S(x-y).\end{aligned}$$

We now compute $\det S$.

$$\begin{aligned}\det S &= (\frac{1-2\varepsilon^2|x|^2}{b(y)})^2[1-\frac{1-\varepsilon^2(|x|^2+|y|^2)}{b(x)(b(x)+b(y))}x+\frac{2\varepsilon^2}{1-2\varepsilon^2|x|^2}y]\cdot(x+y)\\ &= \frac{(1-2\varepsilon^2|x|^2)^2}{b^2(y)b(x)(b(x)+b(y))}\\ &\quad \Big(1+\varepsilon^2|x|^2|y|^2+b(x)b(y)-\big(1-\varepsilon^2(|x|^2+|y|^2)\big)x\cdot y-\frac{2\varepsilon^2(x+y)\cdot y}{1-2\varepsilon^2|x|^2}b(x)\big(b(x)+b(y)\big)\Big).\end{aligned}$$

We note that if $\varepsilon \le 1, \kappa_0 \le \frac{1}{100}$, $1-2\varepsilon^2|x|^2 \ge \frac{1}{2}$ and

$$4\varepsilon^2 b(x)\big(b(x)+b(y)\big)(x+y)\cdot y \le 9\varepsilon^2(|x|^2+|y|^2)(b^2(x)+b^2(y)) \le 108\kappa_0(\varepsilon+3\kappa_0) \le \frac{2}{3}.$$

We thus have:

$$\det S \ge \frac{1}{4b^2(y)}\frac{\frac{1}{3}+b(x)b(y)-x\cdot y}{b(x)(b(y)+b(x))}.$$

It is thus easy to see that:

$$|\partial_x^\alpha \partial_y^\beta (\frac{1}{\det Q})| \lesssim_{\alpha,\beta,\kappa_0} \langle y\rangle^2\langle x\rangle(\langle x\rangle+\langle y\rangle)^2. \tag{4.11.2}$$

Besides, direct computations shows that:

$$|\partial_x^\alpha \partial_y^\beta S(x,y)| \lesssim_{\alpha,\beta,\kappa_0} \langle y\rangle^{-1},$$

which combined with (4.11.2), yields:

$$|\partial_x^\alpha \partial_y^\beta S^{-1}(x,y)| \lesssim_{\alpha,\beta,\kappa_0} (\langle x\rangle+\langle y\rangle)^3.$$

$\square$

# Chapter 5

# Large time existence of Euler-Korteweg equations and two-fluid Euler-Maxwell equations with vorticity.

The result of this chapter is taken from [119] which has been published in Journal Nonlinear analysis.

**Abstract**. The aim of this chapter is to study the influence of the vorticity on the existence time in fluid systems for which global smoothness and decay is known in the case of small irrotational data. We focus on two examples: the Euler-Korteweg system and the two-fluid Euler Maxwell system. We prove that the lower bound of the lifespan of these systems is no less than the inverse of the $H^s$ $(s > 5/2)$ norm of the rotational part of the initial velocity. Our approach is based on energy estimates and the fast time decay results of global solutions to these systems with irrotational initial data.

## 5.1 Introduction

In this paper, we are concerned with the well-posedness of 3-d compressible Euler-Korteweg system which reads

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u \cdot \nabla u + \nabla P(\rho) - \rho\nabla\Big(K(\rho)\Delta\rho + \dfrac{1}{2}K'(\rho)|\nabla\rho|^2\Big) = 0, \\ u|_{t=0} = u_0, \rho|_{t=0} = \rho_0, \end{cases} \tag{5.1.1}$$

where $\rho, u$ are the density and velocity of the fluid, $P(\rho)$ the pressure and is assumed to be a smooth function of density. $K(\rho)$ is the Korteweg tensor, which takes the capillary effects into account.

As the modifications of compressible Euler equation through the adjunction the Korteweg stress tensor, Euler-Korteweg system is a mathematical model arising from hydrodynamics and quantum hydrodynamics. In fact, in hydrodynamics, it can be used to describe at interface the flow of capillary flows, for example, a liquid-vapor mixture. Moreover, when $K(\rho) = \frac{c}{\rho}, P(\rho) = \frac{1}{2}\rho^2$, the Euler-Korteweg system can be transformed formally by the so-called Madelung transform[22] $\psi = \sqrt{\rho}e^{i\phi}, u = \nabla\phi$ to the Gross-Pitaevskii equation which is a very important equation in geometric optics and quantum mechanics.

In the last two decades, some interesting results concerning the well-posedness of compressible Euler-Korteweg have been obtained. For the cauchy problem of Euler-Korteweg in one dimension, local well-posedness for smooth perturbations of travelling profiles was established by Benzoni-Gavage, Danchin and Descombes [16] by using Lagrangian coordinate. Later on, in [17], the same authors extend the result to multi-dimension by reformulating the system into a nonlinear degenerate Schrödinger equation incorporated with the 'gauge' technique (by introducing some 'gauge' function to recover some cancellations in order to avoid losing derivatives). More precisely, for $d(d \geq 1)$ dimensional Euler-Korteweg system, they proved the local existence under $H^{s+1} \times H^s$ $(s > d/2+1)$ perturbations to the stationary solution $(\bar{\rho}, 0)(\bar{\rho} > 0)$. Recently, Bezoni-Gavage and Chiron [15] establish some uniform estimate in several different wave regimes and justify the asymptotic limit. Concerning the global well-posedness, it is shown by Audiard and Haspot [10] that 3-D Euler-Korteweg system admits global small *irrotational* solutions, by enforcing the so-called 'space-time resonance' method which turns out to be the efficient tools to get the global existence of some models that admits dispersive properties with critical nonlinearities. One can refer to [47, 49, 63]. Regarding to the large time existence with nontrivial vorticity, Audiard proved in [9] that the lifespan $T_\star$ of 3-D Euler-Korteweg system is no less than the inverse of the size of the rotational part in some suitable weighted space. His strategy is to study rigorously the highly coupled system composed by the equations of the 'rotational' and 'irrotational part' of the velocity. Nevertheless, in this process, one needs to deal with the complicated interactions between 'rotational' parts and 'irrotational' part. The first aim of this paper is to give an alternative approach for the lifespan estimate of Euler-Korteweg system with vorticity, where merely energy estimates are used. We remark also that our proof does not need the localization assumption on the 'rotational' part of initial velocity.

We denote $\mathcal{P}$ the Leray projector that maps a vector in $L^2(\mathbb{R}^3)^3$ to its divergence free part and $\mathcal{P}^\perp = Id - \mathcal{P}$ the 'curl-free' projector.

The following is the main results:

**Theorem 5.1.1.** *Let $\bar{\rho} > 0$. Suppose that the Korteweg tensor $K(\rho)$ is smooth and satisfies: $K(\rho) \geq K_0 > 0$ for $\bar{\rho}/2 \leq \rho \leq 3\bar{\rho}/2$. Assume also that the pressure satisfies: $P'(\rho)/\rho > 0$ for $\bar{\rho}/2 \leq \rho \leq 3\bar{\rho}/2$. Then there exist three constants $\delta_1, \epsilon_1 > 0$ small, and $N$ large. If the initial datum $(\rho_0 - \bar{\rho}, u_0)$ satisfies the following:*

$$\|\mathcal{P}u_0\|_{H^s} < \epsilon_1,$$

$$\|\mathcal{P}^\perp u_0\|_{H^N} + \|\rho_0 - \bar{\rho}\|_{H^{N+1}} + \|x(\rho_0 - \bar{\rho}, \mathcal{P}^\perp u_0)\|_{L^2} + \|u_0\|_{W^{6,1}} + \|\rho_0 - \bar{\rho}\|_{W^{7,1}} \leq \delta_1,$$

*where $5/2 < s \leq 3$. Then there exists $T_{\epsilon_1} \gtrsim \epsilon_1^{-1}$ such that the Euler-Korteweg equation (5.1.1) has a unique solution and*

$$(\rho - \bar{\rho}, u) \in C([0, T_{\epsilon_1}], H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)).$$

*In addition, if $\mathcal{P}u_0 \in H^N$, then the solution*

$$(\rho - \bar{\rho}, u) \in C([0, T_{\epsilon_1}], H^{N+1}(\mathbb{R}^3) \times H^N(\mathbb{R}^3)).$$

*with exponential growth: for any $0 \leq t \leq T_{\epsilon_1}$,*

$$\|(\rho - \bar{\rho}, u)(t)\|_{H^{N+1} \times H^N} \lesssim e^{ct} \|(\rho - \bar{\rho}, u)(0)\|_{H^{N+1} \times H^N}.$$

To prove Theorem 5.1.1, a natural attempt is, as in [9], to study the highly coupled system by considering the 'rotational' parts $\mathcal{P}u$ and 'irrotational' parts $\mathcal{P}^\perp u = u - \mathcal{P}u$. However, when one tries to extend the lifespan of $\mathcal{P}u$ to $1/\|\mathcal{P}^\perp u_0\|_{H^s}$ $(s > 5/2)$, it is necessary to prove that the irrotational part of velocity $\mathcal{P}^\perp u$ enjoys the integrable time decay. However, when using 'space-time resonance' method to perform decay estimate for $\mathcal{P}^\perp u$, one needs to study the 'dispersive$\times$ vorticity' interactions which bring a lot of extra work. On the other hand, since one needs that $\mathcal{P}u$ is in some weighted space in order to prove its decay property, this 'dispersive$\times$ vorticity' interactions will force us to assume that $\mathcal{P}^\perp u$ also belongs to some weighted space. In the following, we propose a shorter approach that does not require the rotational part $\mathcal{P}^\perp u$ to lie in any weighted space.

To explain the main ideas, we will restrict ourselves to the more abstract setting. Consider a system:

$$\begin{cases} \partial_t U + JLU = U' \cdot \nabla U \\ U|_{t=0} = U_0 \end{cases} \tag{5.1.2}$$

where $U(t,x) = (U_1, U') : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^4$, is a four-elements vector function, $J$ is a skew symmetric differential matrix, $L$ is self-adjoint and positive in some suitable space in the sense that $(Lu, v)_{L^2} = (u, Lv)_{L^2}, (LU, U)_{L^2} \geq \|U\|_{L^2}^2$. For example

$$J = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} Id - \Delta & 0 \\ 0 & Id_{3\times 3} \end{pmatrix},$$

is the Euler-Korteweg type equations (the simplified case that the term $\rho\mathrm{div}u$ is dropped in $(5.1.1)_1$ and $P(\rho) = \frac{1}{2}\rho^2, K(\rho) = 1$ in $(5.1.1)_2$ is assumed) while

$$J = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} Id + (-\Delta)^{-1} & 0 \\ 0 & Id_{3\times 3} \end{pmatrix},$$

is the one fluid Euler-Poisson type equations.

We suppose firstly that for the curl-free ($\mathrm{curl}U_0' = 0$) smooth initial datum, there exists global solutions in some Sobolev space $H^N$ (where $N$ is large enough) which decays fast enough to 0 as the time goes to infinity. More precisely, we suppose that $\|U\|_{W^{4,\infty}} \lesssim (1+t)^{-\alpha}$ with $\alpha > 1$. Now, we want to analyze the large time existence of system (5.1.2) with general (not necessarily curl-free) smooth initial data. Our strategy is to split the system (5.1.2) into two systems, with initial data $(U_1, \mathcal{P}^\perp U_0')$ and $(0, \mathcal{P}U_0')$. To be more concrete, we write $U = W + V$, where $W$ solves (5.1.2) with initial data $(0, \mathcal{P}^\perp U_0')$, and $V$ satisfies the equation:

$$\begin{cases} \partial_t V + JLV = (V' + W')\cdot\nabla V + V'\cdot\nabla W =: F(V,W) \\ W|_{t=0} = (0, \mathcal{P}U_0'), \end{cases} \tag{5.1.3}$$

where $V = (V_1, V'), W = (W_1, W')$. To study the long time existence of (5.1.3), it suffices for us to get appropriate a priori energy estimates. Let us define energy functional

$$E_s(t) = \int_{\mathbb{R}^3} \Lambda^s V \cdot L\Lambda^s V(t)\mathrm{d}x$$

where $\Lambda = \sqrt{1-\Delta}$ and $s > 5/2$. Taking $\Lambda^s$ on system (5.1.3), and testing $L\Lambda^s V$, we then get

$$\partial_t E_s \leq \int_{\mathbb{R}^3} \Lambda^s F(V,W)\cdot L\Lambda^s V\mathrm{d}x$$

which yields by commutator estimate, if $L$ is lower order operator (for example Euler-Poisson system), we could get that:

$$\partial_t E_s \lesssim E_s^{\frac{3}{2}} + \|W\|_{W^{s+1,\infty}} E_s \lesssim E_s^{\frac{3}{2}} + (1+t)^{-\alpha} E_s, \quad \alpha > 1$$

from which, one deduce by the Grönwall inequality and continuation arguments, that, there exists solutions for system (5.1.3) in $C([0,T), H^s)$ for $T \gtrsim 1/\|\mathcal{P}U_0'\|_{H^s}$. However, when $L$ is higher order (for example Euler-Korteweg type), direct energy estimate will inevitably lose derivatives. In this case, the 'gauge' technique used in [17] need to be employed.

We would like to mention that this strategy is inspired by the former work of the author with Rousset [108], [120] where the uniform stability for Navier-Stokes-Poisson system in the inviscid limit is established. It turns out that this approach is flexible for many models that admit global solutions with integrable time decay under the irrotational initial perturbation to equilibria, one could consider Euler-Poisson, Euler-Maxwell (one fluid, two-fluid)...Moreover, the method proposed in this paper will simplify the proof to large extend when the 'space-time resonance' of phase function is difficult to analyze, since one do not need to take care of the new resonances arising from the 'dispersive$\times$vorticity' interactions. We will prove the similar results to Theorem 5.1.1 for 'two-fluid' Euler-Maxwell equation (5.3.1) which is new. Note in [68], Ionescu and Lie prove the long time existence for one-fluid Euler-Maxwell equations, although their method is likely to be adapted to prove the similar results for 'two-fluid' case, the proof will be much sophisticated since the 'space-time resonance' is harder to analyze than the 'one-fluid' case.

**Organisation:** We will use the strategy stated above to prove Theorem 5.1.1 in Section 2. We then prove the similar result for 'two-fluid' Euler-Maxwell equations in Section 3. Finally, we recall some useful lemmas in appendix.

## 5.2 Proof of Theorem 5.1.1

As explained in the Introduction, we split the original system (5.1.1) into two systems by letting

$$\rho = (\bar{\rho} + \varrho) + n =: \tilde{\rho} + n,$$
$$u = w + v,$$

such that the unknowns $(\varrho, w)$ satisfy the system

$$\begin{cases} \partial_t \tilde{\rho} + \operatorname{div}(\tilde{\rho} w) = 0, \\ \partial_t w + w \cdot \nabla w + g(\tilde{\rho}) \nabla \tilde{\rho} = \nabla (K(\tilde{\rho}) \Delta \tilde{\rho} + \frac{1}{2} K'(\tilde{\rho}) |\tilde{\rho}|^2), \\ w|_{t=0} = \mathcal{P}^{\perp} u_0, \tilde{\rho}|_{t=0} = \rho_0, \end{cases} \tag{5.2.1}$$

and the unknowns $(n, v)$ satisfy the system

$$\begin{cases} \partial_t n + \operatorname{div}(\rho v + n w) = 0, \\ \partial_t v + (w + v) \cdot \nabla v + v \cdot \nabla w + g(\bar{\rho}) \nabla n \\ \qquad\qquad = \nabla \big( K(\rho) \Delta n + \dfrac{1}{2} K'(\rho) \nabla n \cdot \nabla (n + 2\tilde{\rho}) \big) + F, \\ v|_{t=0} = \mathcal{P} u_0, \ n|_{t=0} = 0. \end{cases} \tag{5.2.2}$$

where we denote $g(\rho) = P'(\rho)/\rho$ and

$$\begin{aligned} F = & -\big(g(\rho) - g(\bar{\rho})\big) \nabla n - \big(g(\rho) - g(\tilde{\rho})\big) \nabla \tilde{\rho} \\ & + \nabla \big[ \big(K(\rho) - K(\tilde{\rho})\big) \Delta \tilde{\rho} + \frac{1}{2} \big(K'(\rho) - K'(\tilde{\rho})\big) \nabla |\tilde{\rho}|^2 \big]. \end{aligned} \tag{5.2.3}$$

For the system (5.2.1), we recall the global existence result established in [10],

**Theorem 5.2.1.** *Theorem 2.2 ,[10] :*

*Assume that $g'(\bar{\rho}) > 0$. There exists a small number $\delta > 0$ and a large integer $N \geq 6$, such that if the initial data satisfy*

$$\begin{aligned} & \|\mathcal{P}^{\perp} u_0\|_{H^N} + \|\rho_0 - \bar{\rho}\|_{H^{N+1}} + \|x(\rho_0 - \bar{\rho}, \mathcal{P}^{\perp} u_0)\|_{L^2} \\ & \quad + \|u_0\|_{W^{6,1}} + \|\rho_0 - \bar{\rho}\|_{W^{7,1}} \leq \delta_1, \end{aligned}$$

*then the Cauchy problem* (5.2.1) *admits a global solution $(\varrho, u)$ in $C\left([0,\infty), H^{N+1} \times H^N\right)$ satisfying for any $t \geq 0$,*

$$|\varrho(t)| \leq \frac{1}{4} \min\{\bar{\rho}, 1\}, \tag{5.2.4}$$

*Moreover, there exist $\alpha > 1$ and $C > 1$ such that*

$$\sup_{t \geq 0} \big( \|u(t)\|_{H^N} + \|\varrho\|_{H^{N+1}} + (1+t)^{\alpha} \|(\varrho, u)\|_{W^{6,\infty}} \big) \leq C \delta_1. \tag{5.2.5}$$

To prove the first part of Theorem 5.1.1, it suffices to show the following:

**Theorem 5.2.2.** *Suppose $(\varrho, w)$ are global solutions to the system* (5.2.1) *given by Theorem 5.2.1. There exists $\epsilon_1$ small enough, if $\|\mathcal{P} u_0\|_{H^s} \leq \epsilon_1$ ($\frac{5}{2} < s \leq 3$), then one can find some $T_{\epsilon_1} \gtrsim \epsilon_1^{-1}$ such that the system* (5.2.2) *has a unique solution and $(n, v) \in C([0, T_{\epsilon_1}], H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3))$.*

**Proof of Theorem 5.2.2** The Cauchy problem (5.2.2) is well-posed in $C([0,T), H^s), s > 5/2$ for some positive $T > 0$ (e.g. [17]). We are thus left to show the lifespan of (5.2.2) has the order of $\mathcal{O}(\epsilon_1^{-1})$ by developing an a priori estimate.

The direct energy estimate will cause the loss of derivatives due to the lack of dissipation and the presence of the high order term $\nabla(K(\rho)\Delta n)$ in $(5.2.2)_2$. To get round this difficulty, one needs to introduce certain weight function (which is called 'gauge' function [17]) coherent to the energy functional to eliminate this kind of terms.

Denote $\Lambda$ the Fourier multiplier with symbol $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. We will work on the following energy functionals:

$$\mathcal{E}_s(t) = \frac{1}{2} \int \phi_s(\rho) \big( g(\bar{\rho}) |\Lambda^s n|^2 + K(\rho) |\Lambda^s \nabla n|^2 + \rho |\Lambda^s v|^2 \big) \, \mathrm{d}x,$$

where the 'gauge' function $\phi_s(\rho) = (\rho K(\rho))^{\frac{s}{2}}$. The role of this gauge function is to avoid the loss of derivative when $\mathcal{P} v = 0$. We shall comment that $\phi_s(\rho)$ depend on both $n$ and $\varrho$ (recall that $\rho = \bar{\rho} + \varrho + n$).

Taking the time derivative on functional $\mathcal{E}_s$ and using the equations (5.2.2), one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(t) =& \int g(\bar\rho)\partial_t\phi_s|\Lambda^s n|^2 + \partial_t(K\phi_s(\rho))|\Lambda^s\nabla n|^2 + \partial_t(\rho\phi_s(\rho))|\Lambda^s v|^2 \,\mathrm{d}x\\ &- \int g(\bar\rho)\phi_s(\rho)\Lambda^s n\Lambda^s\mathrm{div}(nw) - \rho\phi_s(\rho)\Lambda^s v\cdot\Lambda^s F\mathrm{d}x\\ &- g(\bar\rho)\int \phi_s(\rho)\big(\Lambda^s n\Lambda^s\mathrm{div}(\rho v) + \rho\Lambda^s v\Lambda^s\nabla n\big)\mathrm{d}x\\ &- \int \rho\phi_s(\rho)\Lambda^s v\Lambda^s\big((w+v)\cdot\nabla v + v\cdot\nabla w\big) + K\phi_s(\rho)\Lambda^s\nabla n\Lambda^s\nabla\mathrm{div}(nw)\mathrm{d}x\\ &- \int K\phi_s(\rho)\Lambda^s\nabla n\Lambda^s\nabla\mathrm{div}(\rho v)\\ &\qquad - \rho\phi_s(\rho)\Lambda^s v\cdot\Lambda^s\nabla\big[K(\rho)\Delta n + \frac{1}{2}K'(\rho)\nabla n\cdot\nabla(n+2\tilde\rho)\big]\,\mathrm{d}x\\ =:& \, I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{5.2.6}$$

Among the five terms $I_i, 0\le i\le 5$, the term $I_5$ is the most difficult one since it involves the loss of derivatives, we will postpone handling it later on and first deal with the other four easier terms.

For $I_1$, by rewriting $(5.1.1)_1$ as $\partial_t\rho = -\mathrm{div}\big(\rho(w+v)\big)$, one may estimate

$$|I_1| \lesssim \|\rho\|_{W^{1,\infty}}(\|w\|_{W^{1,\infty}} + \|v\|_{W^{1,\infty}})\|(\nabla n, v)\|_{H^s}^2. \tag{5.2.7}$$

Note that we have used the fact that $K(\rho)$ and $\phi_s(\rho)$ are smooth functions and the a priori assumption $\bar\rho/2\le\rho\le 3\bar\rho/2$.

For $I_2$, by Hölder inequality, product estimate (5.4.5) and Corollary 5.4.6,

$$\begin{aligned}|I_2| &\lesssim (\|n\|_{W^{1,\infty}} + \|w\|_{W^{s+1,\infty}})\|(n,\nabla n)\|_{H^s}^2 + \|v\|_{H^s}\|F\|_{H^s}\\ &\lesssim (\|(n,\nabla n)\|_{H^s} + \|(w,\varrho)\|_{W^{s+3,\infty}})\|(n,\nabla n, v)\|_{H^s}^2.\end{aligned} \tag{5.2.8}$$

We remind the reader that in order to make use of the fast decay property of $\|(\varrho,w)(t)\|_{L_x^\infty}$, we shall always attribute $L_x^\infty$ norm on $(\varrho,w)$ when we estimate the product terms that could be considered roughly as $\varrho n, wn$.

For $I_3$, integrating by parts, applying product estimates (5.4.5) and commutator estimates (5.4.7), (5.4.8), we get:

$$\begin{aligned}I_3 &= g(\bar\rho)\int \nabla(\phi_s(\rho))\Lambda^s n\Lambda^s(\rho v)\,\mathrm{d}x + g(\bar\rho)\int\phi_s(\rho)\Lambda^s\nabla n[\Lambda^s,\rho]v\,\mathrm{d}x\\ &\lesssim \big(\|(n,v)\|_{W^{1,\infty}} + \|(\varrho,w)\|_{W^{s,\infty}}\big)\|(n,\nabla n,v)\|_{H^s}^2.\end{aligned} \tag{5.2.9}$$

More precisely, writing $[\Lambda^s,\rho]v = [\Lambda^s,\varrho]v + [\Lambda^s,n]v$, we use (5.4.7) to get

$$\|[\Lambda^s,\varrho]v\|_{L^2} \lesssim \|\varrho\|_{W^{s,\infty}}\|v\|_{H^{s-1}},$$

and use (5.4.8) to obtain

$$\|[\Lambda^s,n]v\|_{L^2} \lesssim \|\nabla n\|_{L^\infty}\|v\|_{H^{s-1}} + \|v\|_{L^\infty}\|n\|_{H^s}.$$

In the same fashion as $I_3$, the term $I_4$ can be handled by

$$\begin{aligned}I_4 &= \frac{1}{2}\int \mathrm{div}\big(\rho\phi_s(w+v)\big)\big|\Lambda^s v\big|^2 + \mathrm{div}\big(K\phi_s w\big)\big|\Lambda^s\nabla n\big|^2\mathrm{d}x\\ &\quad - \int\rho\phi_s\Lambda^s v\cdot\big(\big[\Lambda^s, w+v\big]\nabla v + \Lambda^s(v\cdot\nabla w)\big)\,\mathrm{d}x\\ &\quad - \int K\phi_s(\rho)\Lambda^s\nabla n\big(\Lambda^s\nabla(n\mathrm{div}w) + [\Lambda^s\nabla, w]\nabla n\big)\,\mathrm{d}x\\ &\lesssim \big(\|(n,v)\|_{W^{1,\infty}} + \|(\varrho,w)\|_{W^{s+2,\infty}}\big)\|(n,\nabla n,v)\|_{H^s}^2.\end{aligned} \tag{5.2.10}$$

To estimate $I_5$, it will be helpful to extract the principle term of $\Lambda^s\nabla\big[K(\rho)\Delta n+\frac{1}{2}K'(\rho)\nabla n\cdot\nabla(n+2\tilde{\rho})\big]$ that may lose derivatives. At first,

$$\begin{aligned}&\Lambda^s\nabla\big(\frac{1}{2}K'(\rho)\nabla n\cdot\nabla(n+2\tilde{\rho})\big)\\&=\frac{1}{2}[\Lambda^s\nabla,K'(\rho)](\nabla n\cdot\nabla(n+2\tilde{\rho}))+\frac{1}{2}K'(\rho)\Lambda^s\nabla(\nabla n\cdot\nabla(n+2\tilde{\rho}))\\&=l.o.t+K'(\rho)\nabla\rho\cdot\Lambda^s\nabla\nabla n\end{aligned}\tag{5.2.11}$$

where the notation $l.o.t$ stands for terms which can be controlled by

$$\|l.o.t\|_{L^2}\lesssim(\|(n,v)\|_{H^s}+\|\varrho\|_{W^{s+2,\infty}})\|(\nabla n,v)\|_{H^s}.$$

Similarly, we have that:

$$\begin{aligned}\Lambda^s\nabla(K(\rho)\Delta n)&=[\Lambda^s,\nabla K]\Delta n+K\Lambda^s\nabla\Delta n+\nabla K\Lambda^s\Delta n+[\Lambda^s,K]\nabla\Delta n\\&=l.o.t+K\Lambda^s\nabla\Delta n+K'\Lambda^s\Delta n\nabla\rho+sK'\nabla\rho\cdot\Lambda^s\nabla\nabla n\end{aligned}\tag{5.2.12}$$

Note that we have used the commutator estimate (5.4.9)-(5.4.10) to get that:

$$\begin{aligned}&[\Lambda^s,K]\nabla\Delta n-sK'\nabla\rho\Lambda^s\nabla\nabla n\\&=[\Lambda^s,K]\nabla\Delta n-\frac{1}{\mathrm{i}}\{\langle\xi\rangle^s,K\}(D)\nabla\Delta n-sK'\nabla\rho\Lambda^{s-2}\nabla\nabla n\\&=l.o.t\end{aligned}\tag{5.2.13}$$

where we used Poisson bracket: $\{a,b\}=\partial_\xi a\cdot\partial_x b-\partial_\xi b\cdot\partial_x a$. We are now in position to estimate $I_5$. In view of (5.2.11) and (5.2.12), we have by integrating by parts that:

$$\begin{aligned}I_5=&\int\rho\phi_s\Lambda^s v\big(K\Lambda^s\nabla\Delta n+K'\Lambda^s\Delta n\nabla\rho+(s+1)K'\nabla\rho\cdot\Lambda^s\nabla\nabla n\big)\\&\qquad-K\phi_s\Lambda^s\nabla n\Lambda^s\nabla\mathrm{div}(\rho v)\,\mathrm{d}x\\=:&\sum_{j=1}^{3}\int\Lambda^s\partial_j n\cdot G_s^j\,\mathrm{d}x+R,\end{aligned}\tag{5.2.14}$$

where $R$ stands for the terms that can be controlled by

$$R\lesssim(\|(n,\nabla n,v)\|_{H^s}+\|(\varrho,w)\|_{W^{s+2,\infty}})\|(n,\nabla n,v)\|_{H^s}^2.$$

and

$$G_s^j=-\rho\phi_sK'\nabla\rho\cdot\big((s+1)\nabla\Lambda^s v_j+\partial_j\Lambda^s v\big)+\partial_j\mathrm{div}(\rho K\phi_s\Lambda^s v)-K\phi_s\partial_j\mathrm{div}\Lambda^s(\rho v)$$

To find the cancellation, we extract the lower order terms as:

$$\begin{aligned}G_s^j=&-\rho\phi_sK'\nabla\rho\cdot\big((s+1)\nabla\Lambda^s v_j+\partial_j\Lambda^s v\big)+\rho\partial_j(K\phi_s)\Lambda^s\mathrm{div}v\\&+\rho(K\phi_s)'\nabla\rho\cdot\partial_j\Lambda^s v-K\phi_s[\Lambda^s,\rho]\partial_j\mathrm{div}v+K\phi_s v\cdot\nabla\Lambda^s\partial_j n+l.o.t\end{aligned}$$

Note again that by Lemma 5.4.4,

$$\begin{aligned}&[\Lambda^s,\rho]\mathrm{div}\partial_j v-s\nabla\rho\cdot\Lambda^s\partial_j\mathcal{P}^\perp v\\&=[\Lambda^s,\rho]\mathrm{div}\partial_j v-s\nabla\rho\cdot\Lambda^{s-2}(1-\Delta)\partial_j\mathcal{P}^\perp v\\&=[\Lambda^s,\rho]\mathrm{div}\partial_j v+s\nabla\rho\cdot\nabla\Lambda^{s-2}\mathrm{div}\partial_j v-s\nabla\rho\cdot\Lambda^{s-2}\partial_j\mathcal{P}^\perp v\\&=[\Lambda^s,\rho]\mathrm{div}\partial_j v-\frac{1}{\mathrm{i}}\{\langle\xi\rangle^s,\rho\}(D)(\mathrm{div}\partial_j v)-s\nabla\rho\cdot\Lambda^{s-2}\partial_j\mathcal{P}^\perp v\\&=l.o.t\end{aligned}$$

We thus have, by combining the fact $v = \mathcal{P}v + \mathcal{P}^\perp v$ and $\partial_j(\mathcal{P}^\perp v)_l = \partial_l(\mathcal{P}^\perp v)_j$, that

$$
\begin{aligned}
G_s^j &= \big((K\phi_s)'\rho - sK\phi_s - (s+2)\rho\phi_s K'\big)\nabla\rho\cdot\Lambda^s\partial_j(\mathcal{P}^\perp v) + \rho(K\phi_s)'\partial_j\rho\Lambda^s\mathrm{div}\mathcal{P}^\perp v \\
&\quad - (s+1)\rho\phi_s\nabla K\cdot\Lambda^s\nabla(\mathcal{P}v)_j + K\phi_s v\cdot\nabla\Lambda^s\partial_j n + \rho K\phi_s'\nabla\rho\cdot\partial_j\Lambda^s\mathcal{P}v + l.o.t \\
&= \rho(K\phi_s)'\big(\partial_j\rho\Lambda^s\mathrm{div}(\mathcal{P}^\perp v) - \nabla\rho\cdot\Lambda^s(\partial_j\mathcal{P}^\perp v)\big) \\
&\quad - (s+1)\rho\phi_s\nabla K\cdot\Lambda^s\nabla(\mathcal{P}v)_j + K\phi_s v\cdot\nabla\Lambda^s\partial_j n + \rho K\phi_s'\nabla\rho\cdot\partial_j\Lambda^s\mathcal{P}v + l.o.t
\end{aligned}
\tag{5.2.15}
$$

Note that in the second equality, we have used the definition of $\phi_s(\rho) = (\rho K(\rho))^{\frac{s}{2}}$ which satisfies:

$$
2(K\phi_s)'\rho = sK\phi_s + (s+2)\rho\phi_s K'.
$$

We first observe that the contribution of the term $K\phi_s v\cdot\nabla\Lambda^s\partial_j n$ of $G_s^j$ in the integral $I_5$ may be easily handled by integration by parts. Indeed,

$$
\begin{aligned}
\int \Lambda^s\nabla n\cdot(K\phi_s v\cdot\nabla\Lambda^s\nabla n)\,\mathrm{d}x &= -\frac{1}{2}\int \mathrm{div}(K\phi_s v)|\Lambda^s\nabla n|^2\,\mathrm{d}x \\
&\lesssim \|v\|_{W^{1,\infty}}(1+\|(\varrho,n)\|_{W^{1,\infty}})\|(\nabla n,v)\|_{H^s}^2.
\end{aligned}
\tag{5.2.16}
$$

Moreover, integrating by parts twice, using the fact that $\partial_j(\mathcal{P}^\perp v)_l = \partial_l(\mathcal{P}^\perp v)_j$, one gets

$$
\begin{aligned}
&\int \rho(K\phi_s)'\Lambda^s\partial_j n\big(\partial_j\rho\Lambda^s\mathrm{div}(\mathcal{P}^\perp v) - \nabla\rho\cdot\Lambda^s(\partial_j\mathcal{P}^\perp v)\big)\,\mathrm{d}x \\
&= \int \Lambda^s(\mathcal{P}^\perp v)_j\Big(\partial_l\big((K\phi_s)'\rho\partial_l\rho\big)\Lambda^s\partial_j n - \partial_j\big((K\phi_s)'\rho\partial_l\rho\big)\Lambda^s\partial_l n\Big)\,\mathrm{d}x \\
&\lesssim \|(n,\varrho)\|_{W^{2,\infty}}\|(\nabla n,v)\|_{H^s}^2.
\end{aligned}
\tag{5.2.17}
$$

Similarly, using that $\mathrm{div}(\mathcal{P}v) = 0$, we have by integrating by parts twice

$$
\begin{aligned}
&\int \rho\phi_s\nabla K\Lambda^s\partial_j n\cdot\Lambda^s\nabla(\mathcal{P}v)_j\,\mathrm{d}x \\
&= \int \Lambda^s(\mathcal{P}v)_j\big(\partial_j(\rho\phi_s\nabla K)\Lambda^s\nabla n - \mathrm{div}(\rho\phi_s\nabla K)\Lambda^s\partial_j n\big)\mathrm{d}x \\
&\lesssim \|(\varrho,n)\|_{W^{1,\infty}}\|(\nabla n,v)\|_{H^s}^2.
\end{aligned}
\tag{5.2.18}
$$

Gathering (5.2.14)-(5.2.17), we achieve that:

$$
I_5 = \sum_{j=1}^{3}\int \rho K\phi_s'\Lambda^s\partial_j n\nabla\rho\cdot\partial_j\Lambda^s\mathcal{P}v\mathrm{d}x + R
\tag{5.2.19}
$$

where

$$
R \lesssim (\|(n,\nabla n,v)\|_{H^s} + \|(\varrho,w)\|_{W^{s+3,\infty}})\|(n,\nabla n,v)\|_{H^s}^2.
$$

One see that $I_5$ is likely to lose one derivative if $\mathcal{P}v$ is not identical to zero. To overcome this difficulty, it is necessary to introduce another gauge function to find more cancellations. Performing $\Lambda^s$ on $(5.2.2)_2$, and multiplying it by a function $\varphi_s(\rho)$ (which will be determined later) that is positive on the interval $\bar\rho/2 \le \rho \le 3\bar\rho/2$, we get

$$
\begin{aligned}
&\partial_t(\varphi_s(\rho)\Lambda^s v) + g(\bar\rho)\nabla(\varphi_s\Lambda^s n) - \nabla[\varphi_s\Lambda^s\big(K(\rho)\Delta n + K'(\rho)\nabla n\cdot(\nabla n + 2\nabla\tilde\rho)\big)] \\
&= -\Lambda^s(K(\rho)\Delta n)\nabla\varphi_s - \varphi_s\Lambda^s(u\cdot\nabla v) + \partial_t\varphi_s\Lambda^s v + g(\bar\rho)\Lambda^s n\nabla\varphi_s - \varphi_s\Lambda^s(v\cdot\nabla w) \\
&\quad + \Lambda^s(K'(\rho)\nabla n\cdot(\nabla n + 2\nabla\tilde\rho))\nabla\varphi_s + \varphi_s\Lambda^s F \\
&= -\Lambda^s(K(\rho)\Delta n)\nabla\varphi_s - \varphi_s(u\cdot\nabla\Lambda^s v) + l.o.t,
\end{aligned}
\tag{5.2.20}
$$

where $l.o.t$ stands for the terms whose $L^2$ norm can be controlled by

$$
\|l.o.t\|_{L^2} \lesssim (\|(\varrho,w)\|_{W^{s+3,\infty}} + \|(n,\nabla n,v)\|_{H^s})\|(n,\nabla n,v)\|_{H^s}.
\tag{5.2.21}
$$

Multiplying (5.2.20) by $\mathcal{P}(\varphi_s(\rho)\Lambda^s v)$ and using integration by parts, one has

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathcal{P}(\varphi_s(\rho)\Lambda^s v)|^2\,\mathrm{d}x &= -\int \mathcal{P}(\varphi_s(\rho)\Lambda^s v)\Lambda^s(K(\rho)\Delta n)\nabla\varphi_s\,\mathrm{d}x\\ &\quad+\frac{1}{2}\int \mathrm{div}(\varphi_s^2 u)|\Lambda^s\mathcal{P}v|^2\,\mathrm{d}x+R\\ &=\sum_{j,l=1}^{3}\int \partial_l\varphi_s\Lambda^s(K(\rho)\partial_j n)\partial_j\left[\mathcal{P}\left(\varphi_s\Lambda^s v\right)\right]_l\mathrm{d}x+R\\ &=\sum_{j,l=1}^{3}\int \partial_l\varphi_s\varphi_s K(\rho)\Lambda^s\partial_j n\partial_j(\mathcal{P}\Lambda^s v)_l\,\mathrm{d}x+R\end{aligned}\tag{5.2.22}$$

where $R$ represents the terms that do not lose derivatives, that is

$$|R|\lesssim(\|(\varrho,w)\|_{W^{s+3,\infty}}+\|(n,\nabla n,v)\|_{H^s})\|(n,\nabla n,v)\|_{H^s}^2\tag{5.2.23}$$

We mention that we have used the estimate

$$\|[\partial_j(Id-(\Delta)^{-1}\partial_l\mathrm{div}),f]g\|_{L^2}\lesssim\|g\|_{L^2}\|f\|_{H^s},$$

due to (5.4.4).

We now choose $\varphi_s(\rho)$ satisfying the condition $(\varphi_s^2(\rho))'=-2\rho\phi_s'$ which cancels the the first terms of (5.2.19) and (5.2.22). More precisely, one can choose

$$\varphi_s(\rho)=\sqrt{A_s(\rho)}\tag{5.2.24}$$

where $A_s$ is one primitive of function: $\rho\to-2\rho\phi_s'(\rho)$ which has positive lower bound on the interval: $\bar{\rho}/2\le\rho\le3\bar{\rho}/2$. (For some special case, say $K(\rho)\equiv1$, one could write $\phi_s(\rho)$ explicitly by choosing $\varphi_s(\rho)=\sqrt{-\frac{2s}{s+2}\rho^{s+2}+M_s}$ with a constant $M_s=2\cdot(\frac{3}{2}\bar{\rho})^{\frac{s}{2}+1}+1$ which ensures that $\varphi_s(\rho)>1$ uniformly in $s$ on the interval $\bar{\rho}/2\le\rho(t,x)\le3\bar{\rho}/2$).

Gathering the estimates (5.2.19)-(5.2.23), we find that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathcal{P}(\varphi_s(\rho)\Lambda^s v)|^2\,\mathrm{d}x+I_5=R\tag{5.2.25}$$

where $R$ stands for those terms that do not lose derivatives, namely, we have that:

$$|R|\lesssim(\|(\varrho,w)\|_{W^{s+3,\infty}}+\|(n,\nabla n,v)\|_{H^s})\|(n,\nabla n,v)\|_{H^s}^2.$$

We define the modified energy by

$$E_s(t)=\frac{1}{2}\int\phi_s(\rho)\big(g(\bar{\rho})|\Lambda^s n|^2+K(\rho)|\Lambda^s\nabla n|^2+\rho|\Lambda^s v|^2\big)+|\mathcal{P}(\varphi_s\Lambda^s v)|^2\,\mathrm{d}x.$$

where $\phi_s(\rho)=(\rho K(\rho))^{\frac{s}{2}}$. We claim that $E_s(t)$ is equivalent to $\|(n,\nabla n,v)\|_{H^s}^2$ as long as $\|\varrho+n\|_{H^s}$ is sufficiently small. Since $K(\rho)\ge K_0>0$ for $\frac{1}{2}\bar{\rho}\le\rho\le\frac{3}{2}\bar{\rho}$, we see that the first three terms are equivalent to $\|(n,\nabla n,v)\|_{H^s}^2$, we thus only need to take care of the last term in $E_s(t)$. Indeed, by the identity

$$\mathcal{P}(\varphi_s\Lambda^s v)=\varphi_s(\mathcal{P}\Lambda^s v)+[\mathcal{P},\varphi_s]\Lambda^s v,$$

and the commutator estimate (5.4.1),

$$\begin{aligned}\|[\mathcal{P},\varphi_s]\Lambda^s v\|_{L^2}\lesssim\|\nabla(\varphi_s(\rho))\|_{H^{s-1}}\|v\|_{H^{s-1}}&=\|\nabla(\varphi_s(\rho)-\varphi_s(\bar{\rho}))\|_{H^{s-1}}\|v\|_{H^{s-1}}\\ &\le C(\|\rho-\bar{\rho}\|_{L^\infty})\|\rho-\bar{\rho}\|_{H^s}\|v\|_{H^{s-1}},\end{aligned}\tag{5.2.26}$$

for $s>\frac{5}{2}$. By recalling $\rho-\bar{\rho}=\varrho+n$, one easily see that, as long as $\|\varrho+n\|_{H^s}$ is sufficiently small, it holds that

$$\|\mathcal{P}(\varphi_s\Lambda^s v)\|_{L^2}^2\approx\|\varphi_s(\mathcal{P}\Lambda^s v)\|_{L^2}^2\approx\|\mathcal{P}\Lambda^s v\|_{L^2}^2.$$

Hence

$$\frac{1}{C_0}\|(n,\nabla n,v)\|_{H^s}^2 \le E_s(t) \le C_0\|(n,\nabla n,v)\|_{H^s}^2, \tag{5.2.27}$$

for some positive constant $C_0$.

We conclude from (5.2.7)-(5.2.10) and (5.2.25) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_s(t) &\lesssim \|(n,\nabla n,v)\|_{H^s}^{3/2} + \|(\varrho,w)\|_{W^{s+3,\infty}}\|(n,\nabla n,v)\|_{H^s}\\ &\le C_1\big(E_s^{3/2}(t) + \|(\varrho,w)\|_{W^{s+3,\infty}}E_s(t)\big),\end{aligned}$$

for some positive constant $C_1$. Moreover, by Theorem 5.2.1, one has that:

$$\|(\varrho,w)(t)\|_{W^{s+3,\infty}} \le C\delta_1(1+t)^{-\alpha}$$

which leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s(t) \le C_1\big(E_s^{3/2}(t) + C\delta_1(1+t)^{-\alpha}E_s(t)\big) \tag{5.2.28}$$

**Conclusion for Theorem 5.2.1.** Theorem 5.2.1 stems from a standard continuity argument. We define the maximal existence time $T_\star$ by

$$T_\star = \sup\{T\big|(n,\nabla n,u)\in C([0,T],H^s):\ E_s(T)\le 2MC_0\epsilon_1^2\}, \tag{5.2.29}$$

where $M = e^{\frac{CC_1}{\alpha-1}\delta_1}$, $C,\alpha$ are constants that appear in the statement of Theorem 5.2.1. In view of (5.2.27), as long as $\epsilon_1$ is sufficitenly small, we have: $\|n(t)\|_{L^\infty}\le\bar\rho/2$ for any $0\le t\le T_*$, which, combined with (5.2.4), yields $\bar\rho/2\le\rho\le 3\bar\rho/2$ for $0\le t\le T_*$.

Define further $T_0=\min\{T_\star,\kappa\epsilon_1^{-1}\}$ with $\kappa$ being small to be chosen later. By (5.2.28), one easily gets by Grönwall's inequality for $t\le T_0\le\kappa\epsilon_1^{-1}$ that

$$\begin{aligned}E_s(t) &\le \exp\Big(\int_0^t C_1C\delta_1(1+\tau)^{-\alpha}\mathrm{d}\tau\Big)\bigg(E_s(0)+C_1\int_0^t E_s^{3/2}(\tau)\,\mathrm{d}\tau\bigg)\\ &\le M\bigg(E_s(0)+C_1\int_0^t E_s^{3/2}(\tau)\,\mathrm{d}\tau\bigg)\\ &\le M\left(C_0\epsilon_1^2 + C_1T_0(2KC_0\epsilon_1^2)^{\frac{3}{2}}\right) \le \frac{3}{2}KC_0\epsilon_1^2,\end{aligned}$$

by choosing $\kappa = \big(2C_1C_0^{\frac12}(2K)^{\frac23}\big)^{-1}$, which leads to, by combining the local existence theory, $T_0=\kappa\epsilon_1^{-1} < T_\star$.

**Proof of the second part of Theorem 5.1.1** To finish the proof of Theorem 5.1.1, we suppose further that $\mathcal{P}u_0\in H^N$, we prove briefly that the solution belongs to (5.1.1) satisfies:

$$(\rho-\bar\rho,u)\in C([0,T_{\epsilon_1}],H^{N+1}(\mathbb{R}^3)\times H^N(\mathbb{R}^3)).$$

Let us define functional:

$$\mathscr{E}_N(t) = \frac12\int \phi_N(\rho)\big(g(\bar\rho)|\Lambda^N(\rho-\bar\rho)|^2 + K(\rho)|\Lambda^N\nabla\rho|^2 + \rho|\Lambda^N v|^2\big) + |\mathcal{P}(\varphi_N\Lambda^N v)|^2\,\mathrm{d}x.$$

where $\phi_N=(\rho K)^{\frac{N}{2}}$, $\varphi_N(\rho)$ is defined in (5.2.24). By calculations similar to that in the proof of Theorem 5.2.2, one could prove that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_N(t)\lesssim \|\rho-\bar\rho\|_{W^{2,\infty}\cap H^s}\mathscr{E}_N(t). \tag{5.2.30}$$

Note that we will always use tame estimate (5.4.6) and commutator estimate (5.4.8), (5.4.9) for the product terms and commutator terms in the expression of $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_N(t)$.

In light of the fact $\rho-\bar\rho\in C([0,T_{\epsilon_1}],H^{s+1}(\mathbb{R}^3))$ $(s>5/2)$, energy inequality (5.2.30) and Grönwall inequality, we have for any $0\le t\le T_{\epsilon_1}$

$$\mathscr{E}_N(t)\lesssim e^{ct}\mathscr{E}_N(0)$$

for some constant $c$.

## 5.3 Large time existence of two-fluid Euler-Maxwell equation

To show the versatility of the approach proposed in the Introduction, we will prove the similar results analogue to Theorem 5.1.1 for 'two-fluid' Euler-Maxwell system. As a physical model to describe the dynamics of plasma, namely electrons and ions, 'two-fluid' Euler-Maxwell system reads:

$$\begin{cases} \partial_t n + \mathrm{div}\big((1+n)v\big) = 0, \\ \iota(\partial_t v + v\cdot\nabla v) + d\nabla n + E + v\times B = 0, \\ \partial_t \varrho + \mathrm{div}\big((1+\varrho)u\big) = 0, \\ \partial_t u + u\cdot\nabla u + \nabla\varrho - E - u\times B = 0, \\ \partial_t B + \nabla\times E = 0, \\ \partial_t E - \frac{1}{\iota}\nabla\times B = [(n+1)v - (\varrho+1)u] \\ \mathrm{div}B = 0, \mathrm{div}E = \varrho - n \\ (n, v, \varrho, u, E, B)|_{t=0} = (n^0, v^0, \varrho^0, u^0, E^0, B^0). \end{cases} \tag{5.3.1}$$

where $1+n, v$(resp. $1+\varrho, u$) stand for the density and velocity of electrons (resp. ions), $E$, $B$ stand for the electric and magnetic field, $\iota, d$ are two constant parameters. Note that we have chosen the reference state being $(1,0,1,0,0,0)$. Before going further, we shall point out some observations. At first, from the equation for $B, E$, $\mathrm{div}B(t) = 0, \mathrm{div}E(t) = \varrho(t) - n(t)$ for any $t > 0$, as long as $\mathrm{div}B^0 = 0, \mathrm{div}E^0 = \varrho^0 - n^0$. Secondly, as indicated in [68] there are two important generalized vorticities $Y(t) =: B - \frac{1}{\iota}\nabla\times v$ and $W(t) =: B + \nabla\times u$ which satisfy the evolution equations:

$$\partial_t Y = \nabla\times(v\times Y) \qquad \partial_t W = \nabla\times(v\times W) \tag{5.3.2}$$

Note that by direct energy estimate, if $Y^0 = B^0 - \frac{1}{\iota}\nabla\times v^0 = 0$ and $W^0 = B^0 + \nabla\times u^0 = 0$, then this property will propagate, and we call this kind of flow as 'generalized irrotational flow'.

Unlike the compressible Euler equation for which singularity formation happens for finite time [75],[114] global small smooth solutions of (5.3.1) have been constructed in [57] with generalized irrotational initial datum by using 'space-time resonance' technique and delicate Fourier analysis.

**Theorem 5.3.1** (Theorem 1.1 of [57] ). *There exists $\delta_0 > 0$ small, $N_0$ large enough, $\beta > 1$, if the initial data satisfy the following:*

$$\|(n^0, v^0, \varrho^0, u^0, E^0, B^0)\|_{H^{N_0}\cap Z} \le \delta_0,$$

$$\mathrm{div}E^0 - \varrho^0 + n^0 = 0, \qquad B^0 - \frac{1}{\iota}\nabla\times v^0 = B^0 + \nabla\times u^0 = 0,$$

*then the system* (5.3.1) *has a unique global solution in $C([0,\infty), H^{N_0})$ and satisfies the following:*

$$\|(n, v, \varrho, u, E, B)\|_{H^{N_0}} + (1+t)^\beta \|(n, v, \varrho, u, E, B)\|_{W^{4,\infty}} \lesssim \delta_0. \tag{5.3.3}$$

**Remark 5.3.2.** *In the statement of last Theorem, $Z$ stands for the norm involving localization of both space and frequency which is compatible to the fractional weighted norm $\|x^{1^+} f\|_{L^2}$. For precise definition, one could refer to definition 4.1 of [57].*

**Remark 5.3.3.** *In contrast with the global existence for* ***small*** *irrotational solutions, the formation of singularity is likely to happen for large initial data. In [86], blow up result is obtained for spherically symmetric solutions to 'one-fluid' Euler-Poisson equations, which is a special case of two-fluid' Euler-Maxwell equations by neglecting the motion of ions and the effects of magnetic field. The method proposed by them seems to be extendable to the general case.*

In the following, we aim to study the long time existence of solution of (5.3.1) with general data, that is $B^0 - \frac{1}{\iota}\nabla\times v^0 \neq 0$ or $B^0 + \nabla\times u^0 \neq 0$. We have by (5.3.2) and identity: $\nabla\times(v\times Y) = Y\cdot\nabla v - v\cdot\nabla Y - Y\mathrm{div}v$ that

$$\partial_t Y = Y\cdot\nabla v - v\cdot\nabla Y - Y\mathrm{div}v, \quad \partial_t W = W\cdot\nabla u - u\cdot\nabla W - W\mathrm{div}u \tag{5.3.4}$$

for which there is no any dispersive or dissipation structure that can be used. It seems that one can only expect the lifespan of $Y, W$ in some Sobolev space (say $H^s, s > \frac{5}{2}$) is proportional to $1/\|Y^0\|_{H^s}$.

The following is the main result concerning to 'two-fluid' Euler-Maxwell equations:

**Theorem 5.3.4.** *There exists three constants $\delta, \epsilon > 0$ small, $\frac{5}{2} < s \leq 3$ and $N_0$ large. If the initial datum $(n^0, v^0, \varrho^0, u^0, E^0, B^0)$ satisfies the following:*

$$
\begin{aligned}
\mathrm{div} B^0 = 0, \qquad \mathrm{div} E^0 + n^0 - \varrho^0 = 0, \\
\|(n^0, \mathcal{P}^{\perp} v^0 + \frac{1}{\iota}(\Delta)^{-1}\nabla \times B^0, \varrho^0, \mathcal{P}^{\perp} u^0 - (\Delta)^{-1}\nabla \times B^0, E^0, B^0)\|_{H^{N_0} \cap Z} < \delta, \\
\|(0, \mathcal{P} v^0 - \frac{1}{\iota}(\Delta)^{-1}\nabla \times B^0, 0, \mathcal{P} u^0 + (\Delta)^{-1}\nabla \times B^0, 0)\|_{H^s} < \epsilon.
\end{aligned}
$$

*Then the Euler-Maxwell equation* (5.3.1) *admits a solution in $C([0, T_\epsilon], H^s)$ with $T_\epsilon \gtrsim \epsilon^{-1}$.*

*In addition if $\mathcal{P} v^0 - \frac{1}{\iota}(\Delta)^{-1}\nabla \times B^0, \mathcal{P} u^0 + (\Delta)^{-1}\nabla \times B^0$ belongs to $H^{N_0}$, then the solution lies in $C([0, T_\epsilon], H^{N_0})$ with an exponential growth:*

$$
\|(n, v, \varrho, u, E, B)(t)\|_{H^{N_0}} \lesssim e^{ct} \|(n, v, \varrho, u, E, B)(0)\|_{H^{N_0}}.
$$

*Proof of Theorem 5.3.4.* As explained in the introduction, we split the system into two systems. More precisely, we write

$$
(n, v, \varrho, u, E, B) = (n_1, v_1, \varrho_1, u_1, E_1, B_1) + (n_2, v_2, \varrho_2, u_2, E_2, B_2),
$$

where $(n_1, v_1, \varrho_1, u_1, E_1, B_1)$ is the global solution of system (5.3.1) provided by Theorem 5.3.1 with initial data $(n^0, \mathcal{P}^{\perp} v^0 + \frac{1}{\iota}(\Delta)^{-1}\nabla \times B^0, \varrho^0, \mathcal{P}^{\perp} u^0 - (\Delta)^{-1}\nabla \times B^0, E^0, B^0)$. Then $(n_2, v_2, \varrho_2, u_2, E_2, B_2)$ solves the equations which is a perturbation of the original system by that of $(n_1, v_1, \varrho_1, u_1, E_1, B_1)$. That is:

$$
\begin{cases}
\partial_t n_2 + \mathrm{div}((1+n)v_2 + n_2 v_1) = 0, \\
\iota(\partial_t v_2 + v \cdot \nabla v_2 + v_2 \cdot \nabla v_1) + d\nabla n_2 + E_2 + v \times B_2 + v_2 \times B_1 = 0, \\
\partial_t \varrho_2 + \mathrm{div}((1+\varrho)u_2 + \varrho_2 u_1) = 0, \\
\partial_t u_2 + u \cdot \nabla u_2 + u_2 \cdot \nabla u_1 + \nabla \varrho_2 - E_2 - u \times B_2 + u_2 \times B_1 = 0, \\
\partial_t B_2 + \nabla \times E_2 = 0, \\
\partial_t E_2 - \frac{1}{\iota}\nabla \times B_2 = [(n+1)v_2 + n_1 v_2 - (\varrho+1)u_2 - \varrho_1 u_2] \\
\mathrm{div} B_2 = 0, \mathrm{div} E_2 = \varrho_2 - n_2 \\
(n_2, v_2, \varrho_2, u_2, E_2, B_2)|_{t=0} = (0, \mathcal{P} v^0 - \dfrac{1}{\iota}(\Delta)^{-1}\nabla \times B^0, 0, \mathcal{P} u^0 + (\Delta)^{-1}\nabla \times B^0, 0, 0).
\end{cases}
\tag{5.3.5}
$$

We shall then prove Theorem 5.3.4 by direct energy estimate. Define the energy functional:

$$
\begin{aligned}
\mathcal{E}_s = \frac{1}{2}\Big( \int d|\Lambda^s n_2|^2 + \iota(1+n)|\Lambda^k v_2|^2 \mathrm{d}x + \int |\Lambda^s \varrho_2| + (1+\varrho)|\Lambda^s u|^2 \mathrm{d}x \\
+ \int |\Lambda^s E|^2 + \frac{1}{\iota}|\Lambda^s B|^2 \mathrm{d}x \Big)
\end{aligned}
$$

where $n = n_1 + n_2, \varrho = \varrho_2 + \varrho_2$.

Taking the time derivative of $\mathcal{E}_s$ and using the equations (5.3.5), we easily get:

$$
\begin{aligned}
\partial_t\mathcal{E}_s \quad = \quad & -\int d\Lambda^s n_2\Big(\Lambda^s\mathrm{div}\big((1+n)v_2\big)-\mathrm{div}\left((1+n)\Lambda^s v_2\right)\Big)\\
&\qquad\qquad +\Lambda^s\varrho_2\Big(\Lambda^s\mathrm{div}\left((1+\varrho)u_2\right)-\mathrm{div}\left((1+\varrho)\Lambda^s u_2\right)\Big)\mathrm{d}x\\
&-\int d\Lambda^s n_2\Lambda^s\mathrm{div}(n_2v_1)+\Lambda^s\varrho_2\Lambda^s\mathrm{div}(\varrho_2u_1)\mathrm{d}x\\
&-\Big(\int \iota(1+n)\Lambda^s v_2(v\cdot\nabla\Lambda^s v_2)+(1+\varrho)\Lambda^s u_2(u\cdot\nabla\Lambda^s u_2)\mathrm{d}x\\
&\qquad\qquad -\int \iota\partial_t n|\Lambda^s v_2|^2+\partial_t\varrho|\Lambda^s u_2|^2\mathrm{d}x\Big)\\
&-\int \iota(1+n)\Lambda^s v_2\Big([\Lambda^s,v_2]\nabla v_2+\Lambda^s(v_2\cdot\nabla v_1)\Big)\\
&\qquad\qquad +(1+\varrho)\Lambda^s u_2\Big([\Lambda^s,u_2]\nabla u_2+\Lambda^s(u_2\cdot\nabla u_1)\Big)\mathrm{d}x\\
&-\int (1+n)\Lambda^s v_2\Lambda^s(v\times B_2+v_2\times B_1)+(1+\varrho)\Lambda^s u_2\Lambda^s(u\times B_2+u_2\times B_1)\mathrm{d}x\\
&+\int \Lambda^s E_2\Big([\Lambda^s,n]v_2-[\Lambda^s,\varrho]u_2+n_2v_1-\varrho_2u_1\Big)\mathrm{d}x\\
=: \quad & J_1+J_2+\cdots J_6
\end{aligned}
$$

We now estimate $J_1$-$J_6$ rigorously. For $J_1$, we write:

$$
\Lambda^s\mathrm{div}\big((1+n)v_2\big)-\mathrm{div}\left((1+n)\Lambda^s v_2\right)=[\Lambda^s,n]\mathrm{div}v_2+v_2\cdot\Lambda^s\nabla n+[\Lambda^s,\nabla n,v_2],
$$

$$
\Lambda^s\mathrm{div}\big((1+\varrho)u_2\big)-\mathrm{div}\left((1+\varrho)\Lambda^s u_2\right)=[\Lambda^s,\varrho]\mathrm{div}u_2+u_2\cdot\Lambda^s\nabla\varrho+[\Lambda^s,\nabla\varrho,u_2],
$$

where we denote for $k\geq 1$, $[\Lambda^s,f,g]=\Lambda^s(fg)-g\Lambda^s f-f\Lambda^s g$.

For $J_1$, we use integration by parts and commutator estimates (5.4.7)-(5.4.8) to get:

$$
J_1\lesssim\big(\|\nabla(n_2,v_2,\varrho_2,u_2)\|_{L^\infty}+\|(n_1,v_1,\varrho_1,u_1)\|_{W^{s+1,\infty}}\big)\|(n,v,\varrho,u,E,B)\|_{H^s}^2. \tag{5.3.6}
$$

For example, by Lemma 5.4.3, we have:

$$
\|[\Lambda^s,n_1]\mathrm{div}v_2\|_{L^2}\lesssim\|n_1\|_{W^{s,\infty}}\|\nabla v_2\|_{H^{s-1}}.
$$

$J_2$, could be controlled in the same manner:

$$
\begin{aligned}
J_2 \quad = \quad & -\mathrm{Re}\int d\overline{\Lambda^s n_2}\big(v_1\cdot\nabla\Lambda^s n_2+[\Lambda^s,v_1]\nabla n_2+\Lambda^s(n_2\mathrm{div}v_1)\big)\\
&\qquad +\overline{\Lambda^s\varrho_2}\big(u_1\cdot\nabla\Lambda^s\varrho_2+[\Lambda^s,u_1]\nabla\varrho_2+\Lambda^s(\varrho_2\mathrm{div}u_1)\big)\mathrm{d}x\\
\lesssim \quad & \|n_2\|_{H^k}^2\|\nabla v_1\|_{W^{s,\infty}}+\|\varrho_2\|_{H^s}^2\|\nabla u_1\|_{W^{s,\infty}}.
\end{aligned} \tag{5.3.7}
$$

Next, from the equation satisfied by $n,\varrho$, $((5.3.1)_1,(5.3.1)_3)$: $\partial_t n+\mathrm{div}((1+n)v)=0,\partial_t\varrho+\mathrm{div}((1+\varrho)u)=0$ we have $J_3=0$.

For $J_4$, by commutator estimate (5.4.7)-(5.4.8) again, we have that:

$$
\begin{aligned}
J_4 \quad \lesssim \quad & (1+\|n\|_{L^\infty})\|\Lambda^s v_2\|_{L^2}(\|\nabla v_2\|_{L^\infty}\|v_2\|_{H^s}+\|\nabla v_1\|_{W^{s,\infty}}\|v_2\|_{H^s})\\
&+(1+\|\varrho\|_{L^\infty})\|\Lambda^s u_2\|_{L^2}(\|\nabla u_2\|_{L^\infty}\|u_2\|_{H^s}+\|\nabla u_1\|_{W^{s,\infty}}\|u_2\|_{H^s})\\
\lesssim \quad & \big(\|\nabla(v_2,u_2)\|_{L^\infty}+\|(v_1,u_1)\|_{W^{s+1,\infty}}\big)\|(v_2,u_2)\|_{H^s}^2
\end{aligned} \tag{5.3.8}
$$

Similarly:

$$
J_5\lesssim\big(\|(v_2,u_2,b_2)\|_{L^\infty}+\|(v_1,u_1,B_1)\|_{W^{s,\infty}}\big)\|(v_2,u_2,B_2)\|_{H^s}^2. \tag{5.3.9}
$$

$$J_6 \lesssim \quad \big(\|\nabla(n_2,v_2,\varrho_2,u_2)\|_{L^\infty} + \|(n_1,v_1,\varrho_1,u_1)\|_{W^{s,\infty}}\big) \\ \|(E_2,n_2,v_2,\varrho_2,u_2)\|_{H^s}^2. \tag{5.3.10}$$

We thus get by collecting the above estimates (5.3.6)-(5.3.10)

$$\begin{aligned}\mathcal{E}_s \quad \lesssim \quad & \|(n_2,v_2,\varrho_2,u_2,E_2,B_2)\|_{H^s}^3 \\ & + \|(n_1,v_1,\varrho_1,u_1,E_1,B_1)\|_{W^{s+1,\infty}} \|(n_2,v_2,\varrho_2,u_2,E_2,B_2)\|_{H^s}^2.\end{aligned}$$

Gronwall's inequality and continuation arguments then give the lower bound of lifespan $T_\epsilon \geq \epsilon^{-1}$. Since it is similar to the case of Euler-Korteweg, we omit the details.

□

## 5.4 Appendix

In this appendix, we recall and prove some basic commutator estimates frequently used in showing Theorem 5.1.1, based on a Littlewood-Paley decomposition. Let $\psi \in [0,1]$ be a cut-off function satisfying $\psi \equiv 1$ on $B(0,3/2)$ and $\psi \equiv 0$ on $B(0,2)^c$. Set $\phi_j(x) = \phi(2^{-j}x)$, with $\phi(x) = \psi(x) - \psi(2x)$ which is supported on the annulus $\{\frac{3}{4} \leq |x| \leq 2\}$. Then it holds that

$$1 = \psi(x) + \sum_{j\geq 1} \phi_j(x), \quad \text{for all } x \in \mathbb{R}^3.$$

We also recall the homogeneous dyadic block $\dot{\Delta}_k$ defined by

$$\dot{\Delta}_k f = \mathcal{F}^{-1}\big(\phi_k(\xi)\hat{f}(\xi)\big), \quad \text{for all } k \in \mathbb{Z},$$

and homogeneous low-frequency cut-off operator $\dot{S}_l$ given by

$$\dot{S}_l = \sum_{k\leq l-1} \dot{\Delta}_k, \quad \text{for all } l \in \mathbb{Z}.$$

Similarly, the nonhomogeneous dyadic block $\Delta_k$ and nonhomogeneous low-frequency cut-off operator $S_l$ are defined respectively by

$$\Delta_{-1} f = \mathcal{F}^{-1}\big(\psi(\xi)\hat{f}(\xi)\big),\ \Delta_k f = \mathcal{F}^{-1}\big(\phi_k(\xi)\hat{f}(\xi)\big), \quad \text{for all } k \in \mathbb{N}.$$

and

$$S_l = \sum_{-1\leq k\leq l-1} \Delta_k, \quad \text{for all } l \in \mathbb{Z}.$$

Let $\mathcal{R}$ be the Riesz potential, and $\Lambda^s$ $(s \geq 0)$ be the Fourier multiplier with symbol $(1+|\xi|)^{\frac{s}{2}}$. We first prove the following:

**Lemma 5.4.1.** *We have for $s > \frac{3}{2}$,*

$$\|[\mathcal{R}, f]\Lambda^s g\|_{L^2} \lesssim \|\nabla f\|_{H^s} \|g\|_{H^{s-1}}. \tag{5.4.1}$$

*Proof.* We use decomposition:

$$fg = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} f \dot{\Delta}_j g + \dot{\Delta}_j f \dot{S}_j g = \dot{T}_f g + \tilde{\dot{T}}_g f$$

to rewrite $[\mathcal{R}, f]\Lambda^s g$ as:

$$[\mathcal{R}, f]\Lambda^s g = [\mathcal{R}, \dot{T}_f]\Lambda^s g + \mathcal{R}(\tilde{\dot{T}}_{\Lambda^s g} f) - \tilde{\dot{T}}_{\mathcal{R}\Lambda^s g} f.$$

The last two terms can be easily treated by Bernstein's inequality:

$$
\begin{aligned}
\|\mathcal{R}(\tilde{\dot{T}}_{\Lambda^s g} f)\|_{L^2} + \|\tilde{\dot{T}}_{\mathcal{R}\Lambda^s g} f\|_{L^2} &\lesssim \sum_j \|\dot{S}_j \Lambda^{s-1} g\|_{L^2} \|\dot{\Delta}_j \nabla f\|_{L^\infty} \\
&\lesssim \|g\|_{H^{s-1}} \|\nabla f\|_{\dot{B}^0_{\infty,1}}.
\end{aligned}
\tag{5.4.2}
$$

We next handle the first term. Since the frequency of $\dot{T}_f \Lambda^s g$ localizes on the annulus, there exits a $C_c^\infty$ function $\tilde{\phi}$ supported away from origin such that

$$
[\mathcal{R}, \dot{T}_f]\Lambda^s g = \sum_j \tilde{\phi}(2^{-j}D)\big(\dot{S}_{j-1} f \dot{\Delta}_j \Lambda^s g\big) - \dot{S}_{j-1} f \big(\tilde{\phi}(2^{-j}D)\dot{\Delta}_j \Lambda^s g\big) =: \sum_j A_j.
$$

For $j \leq 0$, taking advantage of the Bernstein inequality, it is direct to see that:

$$
\|A_j\|_{L^2} \lesssim \|\dot{S}_{j-1} f\|_{L^\infty} \|\dot{\Delta}_j g\|_{L^2} \lesssim \|\nabla f\|_{H^{s-1}} \|\dot{\Delta}_j g\|_{L^2}
$$

Now for $j \geq 1$, denote $\tilde{\chi}_j = \mathcal{F}^{-1}(\tilde{\phi}(2^{-j}\xi)(1+|\xi|^2)^{\frac{s}{2}})$, one may write

$$
\begin{aligned}
A_j(x) &= \int \tilde{\chi}_j(y) \dot{\Delta}_j g(x-y)(\dot{S}_{j-1} f(x) - \dot{S}_{j-1}(x-y))\, \mathrm{d}y \\
&= \int \tilde{\chi}_j(y) \dot{\Delta}_j g(x-y) \int_0^1 y \cdot \nabla \dot{S}_{j-1} f(x-y+\tau y)\, \mathrm{d}\tau \mathrm{d}y.
\end{aligned}
$$

Hence

$$
\|A_j\|_{L^2} \lesssim \|\nabla \dot{S}_{j-1} f\|_{L^\infty} \|\dot{\Delta}_j g\|_{L^2} \| |\cdot| \tilde{\chi}_j\|_{L^1} \lesssim \|\nabla f\|_{H^s} \|\dot{\Delta}_j g\|_{L^2} 2^{j(s-1)}.
$$

Taking $l^2_j$ norm of sequence $(A_j)$, one has

$$
\|[\mathcal{R}, T_f]\Lambda^s g\|_{L^2} \lesssim \|\nabla f\|_{H^s} \|g\|_{H^{s-1}}. \tag{5.4.3}
$$

The desired result (5.4.1) follows from (5.4.2) and (5.4.3). □

**Corollary 5.4.2.** *For any $s > \frac{3}{2}$, one has also the commutator estimate:*

$$
\|[\nabla \mathcal{R}, f] g\|_{L^2} \lesssim \|\nabla f\|_{H^s} \|g\|_{L^2}. \tag{5.4.4}
$$

*Proof.* We have the following estimates similar to (5.4.1):

$$
\|[\mathcal{R}, f]\nabla g\|_{L^2} \lesssim \|\nabla f\|_{H^s} \|g\|_{L^2}
$$

Then (5.4.4) is the consequence of this estimate and the identity:

$$
[\nabla \mathcal{R}, f] g = \mathcal{R}(g \nabla f) + [\mathcal{R}, f]\nabla g.
$$

□

We will use also the following commutator and product estimates whose proof are standard and thus omitted. One could refer to [12] for example.

**Lemma 5.4.3.** *Let $s \geq 1$, we have*

$$
\|\Lambda^s(fg)\|_{L^2} \lesssim \|f\|_{W^{s,\infty}} \|g\|_{H^s}, \tag{5.4.5}
$$

$$
\|\Lambda^s(fg)\|_{L^2} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^s}, \tag{5.4.6}
$$

$$
\|[\Lambda^s, f] g\|_{L^2} \lesssim \|g\|_{H^{s-1}} \|f\|_{W^{s,\infty}}, \tag{5.4.7}
$$

$$
\|[\Lambda^s, f] g\|_{L^2} \leq \|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|g\|_{L^\infty} \|f\|_{H^s}. \tag{5.4.8}
$$

Given two functions $a(x,\xi), b(x,\xi)$, the Poisson brackets reads

$$\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_\xi b \cdot \partial_x a.$$

**Lemma 5.4.4.** *Let $2 \le s \le \frac{7}{2}$, we have*

$$\|[\Lambda^s, f]g - \mathrm{i}^{-1}\{\langle\xi\rangle^s, f\}(D)g\|_{L^2} \lesssim \|\nabla^2 f\|_{L^\infty \cap H^{\frac{3}{2}}} \|g\|_{H^{s-2}}, \tag{5.4.9}$$

*and*

$$\|[\Lambda^s, f]g - \mathrm{i}^{-1}\{\langle\xi\rangle^s, f\}(D)g\|_{L^2} \lesssim \|f\|_{W^{s+\epsilon,\infty}} \|g\|_{H^{s-2}}. \tag{5.4.10}$$

*Proof.* One can refer to [17, Lemma A.3] for the proof of (5.4.9). We only sketch the proof of (5.4.10).

We denote by $\partial_k, \partial^k$ the space derivative and frequency derivative respectively. We use decomposition:

$$\begin{aligned}
[\Lambda^s, f]g - \mathrm{i}^{-1}\{\langle\xi\rangle^s, f\}(D)g \qquad &= \underbrace{[\Lambda^s, T_f]g + \mathrm{i}T_{\partial_k f}(\partial^k \Lambda^s)(D)g}_{G_1} \\
&\quad + \underbrace{\Lambda^s(\tilde{T}_g f) - \tilde{T}_{\Lambda^s g}(f) + s\tilde{T}_{\Lambda^{s-2}\partial_k g}(\partial_k f)}_{G_2}
\end{aligned}$$

Taking $\tilde{\phi}$ (defined in the proof of Lemma 5.4.1), noticing that $\tilde{\phi}_j \equiv 1$ on the support of $\phi_j$, we may decompose $G_1$ as

$$\begin{aligned}
G_1 = \sum_{j\in\mathbb{Z}} \Big( &\Lambda^s \tilde{\phi}(2^{-j}D)\big(S_{j-1}f \Delta_j g\big) \\
&- S_{j-1}f\big(\tilde{\phi}(2^{-j}D)\Lambda^s \Delta_j g\big) + \mathrm{i}S_{j-1}\partial_k f\big(\partial^k(\tilde{\phi}(2^{-j}\cdot)\langle\cdot\rangle^s)(D)\Delta_j g\big)\Big) \\
=: &\sum_{j\in\mathbb{Z}} A^j.
\end{aligned}$$

where we denote $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$. For $j=0$, it is easy to see that:

$$\|A_0\|_{L^2} \lesssim \|f\|_{L^\infty} \|\Delta_0 g\|_{L^2}.$$

For $j \ge 1$, denote $\tilde{\chi}_j = \mathcal{F}^{-1}(\tilde{\phi}(2^{-j}\cdot)\langle\cdot\rangle^s)$. By Taylor expansion, one has

$$\begin{aligned}
A^j(x) &= \int \tilde{\chi}_j(y)\Delta_j g(x-y)\big(S_{j-1}f(x-y) - S_{j-1}f(x) + \partial_k S_{j-1}f(x)y_k\big)\,\mathrm{d}y \\
&= \int \tilde{\chi}_j(y)\Delta_j g(x-y)\int_0^1 y^T \cdot D^2 S_{j-1}f(x-ty)\cdot y(1-t)\,\mathrm{d}t\mathrm{d}y,
\end{aligned}$$

which yields, for $j \ge 1$

$$\begin{aligned}
\|A^j\|_{L^2} &\lesssim \||\cdot|^2\tilde{\chi}_j\|_{L^1}\|\Delta_j g\|_{L^2}\|D^2 S_{j-1}f\|_{L^\infty} \\
&\lesssim 2^{j(s-2)}\|\Delta_j g\|_{L^2}\|D^2 f\|_{L^\infty}.
\end{aligned}$$

Taking $l_j^2$ norm of of sequence $(A^j)$, one obtains

$$\|G_1\|_{L^2} \lesssim \|g\|_{H^{s-2}}\|f\|_{W^{s+\epsilon,\infty}}. \tag{5.4.11}$$

As for $G_2$, it can be estimated easily

$$\begin{aligned}
\|G_2\|_{L^2} &\lesssim \sum_{j\ge -1} 2^{js}\|S_j g\|_{L^2}\|\Delta_j f\|_{L^\infty} \\
&\lesssim \|g\|_{L^2}\|f\|_{B^s_{\infty,1}} \lesssim \|g\|_{L^2}\|f\|_{W^{s+\epsilon,\infty}}.
\end{aligned} \tag{5.4.12}$$

The desired result (5.4.10) follows from (5.4.11) and (5.4.12).

□

We recall the composition estimate whose proof could be found in [8] or [12].

**Lemma 5.4.5.** *Let $h : \mathbb{R} \to \mathbb{R}$ a smooth function with $h(0) = 0$. Suppose $u \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)(s > 0)$, then $h(u) \in H^s(\mathbb{R}^3)$, and the following holds:*

$$\|h(u)\|_{H^s(\mathbb{R}^3)} \leq C(s, |h|_{C^{[s]+1}}, \|u\|_{L^\infty})\|u\|_{H^s(\mathbb{R}^3)}. \tag{5.4.13}$$

*If in addition, $h'(0) = 0$, then*

$$\|h(u)\|_{H^s(\mathbb{R}^3)} \leq C(s, |h|_{C^{[s]+1}}, \|u\|_{L^\infty})\|u\|_{L^\infty(\mathbb{R}^3)}\|u\|_{H^s(\mathbb{R}^3)}. \tag{5.4.14}$$

*and for any $u, v \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)(s > 0)$, one has that:*

$$\|h(u) - h(v)\|_{H^s(\mathbb{R}^3)} \leq C(s, |h|_{C^{[s]+1}}, \|(u, v)\|_{L^\infty})\|u - v\|_{L^\infty \cap H^s}\|(u, v)\|_{L^\infty \cap H^s}. \tag{5.4.15}$$

**Corollary 5.4.6.** *Recall $F$ is defined in* (5.2.3)*, we have for $s > 5/2$,*

$$\|F\|_{H^s} \leq C(\|(\varrho, n)\|_{H^{s+1}})(\|n\|_{W^{1,\infty}} + \|\varrho\|_{W^{s+3,\infty}})(\|(n, \nabla n)\|_{H^s}. \tag{5.4.16}$$

*Proof.* We will control the term $F$ by product (5.4.5)-(5.4.6) and composition estimates (5.4.13)-(5.4.15). For instance, by product estimate (5.4.6) and composition estimates (5.4.14),

$$\begin{aligned}
\|(g(\rho) - g(\bar{\rho}))\nabla n\|_{H^s} &\leq g(\bar{\rho})\|(\varrho + n)\nabla n\|_{H^s} + \|h(\varrho + n)\nabla n\|_{H^s} \\
&\lesssim \|\nabla n\|_{H^s}\|n\|_{L^\infty} + \|\nabla n\|_{L^\infty}\|n\|_{H^s} + \|\varrho\|_{W^{s,\infty}}\|\nabla n\|_{H^s} \\
&\qquad + \|h(\varrho + n)\|_{L^\infty}\|\nabla n\|_{H^s} + \|h(\varrho + n)\|_{H^s}\|\nabla n\|_{L^\infty} \\
&\lesssim (\|(n, \nabla n)\|_{L^\infty} + \|\varrho\|_{W^{s,\infty}})\|(n, \nabla n)\|_{H^s}.
\end{aligned}$$

where $h(y) = g(\bar{\rho} + y) - g(\bar{\rho}) - g'(\bar{\rho})y$ satisfies $g(0) = g'(0) = 0$.

For the term

$$\nabla\big((K(\rho) - K(\tilde{\rho}))\Delta\varrho\big) = \big(K(\rho) - K(\tilde{\rho})\big)\nabla\Delta\varrho + \nabla\big(K(\rho) - K(\tilde{\rho})\big)\Delta\tilde{\varrho} =: (1) + (2),$$

We only estimate (1) as (2) is similar. Denote $h_1(x) = K(\bar{\rho} + x) - K(\bar{\rho}) - K'(\bar{\rho})x$, we have that:

$$(1) = \big(K'(\bar{\rho})n + h_1(\varrho + n) - h_1(\varrho)\big)\nabla\Delta\varrho \tag{5.4.17}$$

We thus have by (5.4.5) and (5.4.15):

$$\begin{aligned}
\|(1)\|_{H^s} &\lesssim (\|n\|_{H^s} + \|h_1(\varrho + n) - h_1(\varrho)\|_{H^s})\|\varrho\|_{W^{s+3,\infty}} \\
&\lesssim (1 + \|(\varrho, n)\|_{H^s \cap L^\infty})\|n\|_{H^s \cap L^\infty}\|\varrho\|_{W^{s+3,\infty}} \lesssim \|n\|_{H^s}\|\varrho\|_{W^{s+3,\infty}}.
\end{aligned} \tag{5.4.18}$$

The other terms in the expression of $F$ can be controlled in the same manner, we omit the proof. □

# Part II: Low Mach number limit for viscous fluids.

# Chapter 6

# Uniform regularity for the compressible Navier-Stokes system with low Mach number in bounded domains.

This chapter is a joint work with prof. Frédéric Rousset and Nader Masmoudi and is taken from [95] which is submitted and appears in arxiv abs/2106.06077.

**Abstract** We establish uniform with respect to the Mach number regularity estimates for the isentropic compressible Navier-Stokes system in smooth domains with Navier-slip condition on the boundary in the general case of ill-prepared initial data. To match the boundary layer effects due to the fast oscillations and the ill-prepared initial data assumption, we prove uniform estimates in an anisotropic functional framework with only one normal derivative close to the boundary. This allows to prove the local existence of a strong solution on a time interval independent of the Mach number and to justify the incompressible limit through a simple compactness argument.

## 6.1 Introduction

In this chapter, we consider the following scaled isentropic compressible Navier-Stokes system $(CNS)_\varepsilon$

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \operatorname{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \rho|_{t=0} = \rho_0^\varepsilon, \end{cases} \tag{6.1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\rho^\varepsilon(t,x)$ and $u^\varepsilon(t,x)$ are the density and the velocity of the fluid respectively, $P(\rho)$ is the pressure which is a given smooth function of the density that satisfies $\frac{\mathrm{d}P}{\mathrm{d}\rho} > 0$, for $\rho > 0$. The viscous stress tensor takes the form:

$$\mathcal{L}u^\varepsilon = 2\mu\mathbb{S}u^\varepsilon + \lambda\operatorname{div}u^\varepsilon\mathrm{Id}, \quad \mathbb{S}u^\varepsilon = \frac{1}{2}(\nabla u^\varepsilon + \nabla^t u^\varepsilon).$$

Here, $\mu, \lambda$ are viscosity parameters that are assumed to be constant and to satisfy the condition: $\mu > 0, 2\mu + 3\lambda > 0$. The parameter $\varepsilon$ is the scaled Mach number which is assumed small, that is $\varepsilon \in (0,1]$.

Since we are considering the system in a domain with boundaries, we shall supplement the system (6.1.1) with the Navier-slip boundary condition

$$u^\varepsilon \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^\varepsilon\mathbf{n}) + a\Pi u^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{6.1.2}$$

where $\mathbf{n}$ is the unit outward normal vector and $a$ is a constant related to a slip length (our analysis also holds if $a$ is a smooth function). We use the notation $\Pi f$ for the tangential part of a vector $f$, $\Pi f^\varepsilon = f^\varepsilon - (f^\varepsilon \cdot \mathbf{n}) \cdot \mathbf{n}$.

The aim of this paper is to study the uniform regularity (with respect to $\varepsilon$) and the low Mach number limit of system (6.1.1). Formally, due to the stiff term $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, the pressure (and hence the density $\rho^\varepsilon$) is expected to tend to a constant state. One thus expects to obtain in the limit a solution to the following incompressible Navier-Stokes system:

$$\begin{cases} \bar{\rho}(\partial_t u^0 + \mathrm{div}(u^0 \otimes u^0)) - \Delta u^0 + \nabla \pi = 0, \\ \mathrm{div} u^0 = 0, \qquad\qquad\qquad\qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^0|_{t=0} = u_0^0, \\ u^0 \cdot \mathbf{n} = 0, \quad \Pi(\mathbb{S}u^0\mathbf{n}) + a\Pi u^0 = 0 \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{6.1.3}$$

This limit process is therefore frequently referred to as the incompressible limit.

The rigorous justification of this limit process has been studied extensively in different contexts depending on the generality of the system (isentropic or non-isentropic), the type of the system (Navier-Stokes or Euler), the type of solutions (strong solutions or weak solutions), the properties of the domain (whole space, torus or bounded domain with various boundary conditions), as well as the type of the initial data considered (well-prepared or ill-prepared). Roughly speaking, in the case of the compressible Euler system, one proves first that the local strong solution exists on an interval of time independent of the Mach number, and then compactness arguments are developed to pass to the limit. In the case of the compressible Navier-Stokes system, one can either try to use the same approach as for the inviscid case (prove the existence of a strong solution on an interval of time independent of the Mach number and then try to pass to the limit) or try to pass to the limit directly from global weak solutions. Both approaches have been used in domains without boundaries (whole space or torus), nevertheless when a boundary is present the question of uniform regularity for general data is more subtle, as we shall see below, and has not been addressed.

More precisely, the mathematical justification of the low Mach number limit was initiated by Ebin [39], Klainerman-Majda [79, 80] for *local strong solutions* of compressible fluids (Navier-Stokes or Euler), in the whole space with well-prepared data ($\mathrm{div} u_0^\varepsilon = \mathcal{O}(\varepsilon), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon^2)$) and later, by Ukai [125] for ill-prepared data ($\mathrm{div} u_0^\varepsilon = \mathcal{O}(1), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon)$). In the latter case, there are acoustic waves of amplitude 1 and frequency $\varepsilon^{-1}$ in the system. These works were extended by several authors in different settings. For instance, one can refer to [2, 20, 100, 101] for the non-isentropic system and ill-prepared initial data whenever the domain is the whole space or the torus, and also [74, 109] for bounded domains with well-prepared initial data. Uniform (in Mach number) regularity estimates for the non-isentropic Euler equations in a bounded domain are established in [1]. The low Mach limit of *weak solutions* for the viscous fluid system (6.1.1) was studied by Lions and the first author [89], [90] where the convergence of the global weak solutions of the isentropic Navier-Stokes system towards a solution of the incompressible system is established. The result holds for ill-prepared initial data and several different domains (whole space, torus and bounded domain with suitable boundary conditions). In general, for ill-prepared data, one can only obtain weak convergence in time, nevertheless, by using the dispersion of acoustic waves in the whole space, Desjardin and Grenier [34] could get local strong convergence. There are also many other related works, one can see for example [14, 25, 30, 32, 41, 45, 54, 71, 91]. For more exhaustive information, one can refer for example to the well-written survey papers by Alazard [3], Danchin [31], Feireisl [43], Gallagher [46], Jiang-Masmoudi [73], Schochet [110].

Let us focus now more specifically on the study of the low Mach limit of the isentropic compressible Navier-Stokes $(CNS)_\varepsilon$ system in domains with boundaries with *ill-prepared* initial data, which is more related to the interest of the current paper. As mentioned above, Lions and Masmoudi [89] studied the convergence of *weak solutions* to $(CNS)_\varepsilon$ in bounded domains with Navier-slip boundary condition. Later on, for low Mach limit in bounded domains with Dirichlet boundary condition, the authors in [35, 72] noticed that, under some geometric assumption on the domain, the acoustic waves are damped in a boundary layer so that local in time strong convergence ($L^2_{t,x}$) holds. Recently, this result is extended by Feireisl et al [44] and Xiong [131] to the case of Navier-slip boundary conditions with $a$ of the order $\varepsilon^{-\frac{1}{2}}$. In this case, the boundary layer effect is comparable to the one in the Dirichlet case. One can also refer to [38, 41, 42] for the justification of convergence in unbounded domains by using the local energy decay for the acoustic system. Without one of the above properties of the domain, strong convergence does not hold for ill-prepared data.

In the current chapter, our aim is to obtain uniform (with respect to $\varepsilon$) high order regularity estimates for $(CNS)_\varepsilon$ in bounded domains with ill-prepared initial data, in order to get the existence of a local strong solution on a time interval independent of $\varepsilon$. There are only a few papers addressing this issue. In [105], the authors establish uniform global (for small data) $H^2$ estimates under a (very) *well-prepared* initial data assumption, namely the second time derivative of the velocity needs to be uniformly bounded initially. For ill-prepared initial data, the situation is more subtle and a uniform $H^2$ estimate, even locally in time, cannot be expected. Indeed, at leading order, after linearization and symmetrization, the system (6.1.1) becomes:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon} L U^\varepsilon - \begin{pmatrix} 0 \\ \mathrm{div}\mathcal{L}u^\varepsilon \end{pmatrix} = 0, \qquad L = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \quad U = (\sigma^\varepsilon, u^\varepsilon) \in \mathbb{R}\times\mathbb{R}^3_+. \tag{6.1.4}$$

Due to the presence of the diffusion term as well as the singular linear term, a boundary layer correction to the highly oscillating acoustic waves appear and create unbounded high order normal derivatives of the velocity. Note that here, we do not start from a small viscosity problem, nevertheless, at the scale $\tau = t/\varepsilon$ of the acoustic waves the system (6.1.4) behaves like a small viscosity perturbation of the acoustic system. For example, in the easiest case where the boundary is flat (for example $\Omega = \mathbb{R}^3_+$), we expect the following expansion of the solutions to (6.1.4) involving boundary layers

$$\begin{cases} \sigma^\varepsilon(t,x) = \sigma_0^I(\frac{t}{\varepsilon}, t, x) + \varepsilon^{\frac{3}{2}}\sigma^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots, \\ u^\varepsilon(t,x) = u_0^I(\frac{t}{\varepsilon}, t, x) + \sqrt{\varepsilon}\begin{pmatrix} u^B_{1,\tau}(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) \\ 0 \end{pmatrix} + \varepsilon u_2^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots \end{cases} \tag{6.1.5}$$

where $x = (y,z)$, $z > 0$, which suggests that $\|u_\tau\|_{L^2_t H^1}, \|u_3^\varepsilon\|_{L^2_t H^2}, \|\sigma^\varepsilon\|_{L^2_t H^3}$ can be uniformly bounded whereras $\|\partial_t(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_{t,x}}$ and $\|\partial_z^2 u_\tau^\varepsilon\|_{L^2_{t,x}}$ will blow up as $\varepsilon$ tends to 0.

In order to get uniform high order estimates, we shall thus need to use a functional framework based on conormal Sobolev spaces that minimize the use of normal derivatives close to the boundary in the spirit of [93], [94]. Nevertheless, note that here we have to handle simultaneously the fast oscillations in time and a boundary layer effect so that the difficulties and the analysis will be different from the ones in [106, 127] where compressible slightly viscous fluids are considered. Indeed, the energy estimates for conormal derivatives cannot be easily obtained since for example tangential vector fields do not commute with the singular part of the system, while in order to include ill-prepared data, it will be impossible to get uniform estimates for high order time derivatives as it is done in [106, 127] in the study of the inviscid limit. We shall explain more these two difficulties below after the introduction of the various norms used in this paper.

### 6.1.1 Conormal Sobolev spaces and notations.

To define the conormal Sobolev norms, we take a finite set of generators of vector fields that are tangent to the boundary of $\Omega$: $Z_j (1 \le j \le M)$. Due to the appearance in (6.1.5) of the 'fast scale' variable $\frac{t}{\varepsilon}$, it is also necessary to involve the scaled time derivative $Z_0 = \varepsilon\partial_t$. We set

$$Z^I = Z_0^{\alpha_0}\cdots Z_M^{\alpha_M}, \quad I = (\alpha_0, \alpha_1, \cdots \alpha_M) \in \mathbb{N}^{M+1}$$

Note that $Z^I$ contains not only spatial derivatives but also the scaled time derivative $\varepsilon\partial_t$. We introduce the following Sobolev conormal spaces: for $p = 2$ or $+\infty$,

$$L^p_t H^m_{co} = \{f \in L^p\big([0,t], L^2(\Omega)\big), Z^I f \in L^p\big([0,t], L^2(\Omega)\big), |I| \le m\},$$

equipped with the norm:

$$\|f\|_{L^p_t H^m_{co}} = \sum_{|I|\le m} \|Z^I f\|_{L^p([0,t], L^2(\Omega))}, \tag{6.1.6}$$

where $|I| = \alpha_0 + \cdots \alpha_M$. For the space modeled on $L^\infty$, we shall use the following notation for the norm:

$$|\!|\!| f |\!|\!|_{m,\infty,t} = \sum_{|I|\le m} \|Z^I f\|_{L^\infty([0,t]\times\Omega)}. \tag{6.1.7}$$

Since the number of time derivatives and spatial conormal derivatives need sometimes to be distinguished, we shall also use the notation:

$$\|f\|_{L^p_t\mathcal{H}^{j,l}} = \sum_{I=(k,\tilde{I}),k\leq j,|\tilde{I}|\leq l} \|Z^I f\|_{L^p([0,t],L^2(\Omega))} \tag{6.1.8}$$

and to simplify, we will use $\mathcal{H}^j = \mathcal{H}^{j,0}$. To measure pointwise regularity at a given time $t$ (in particular also with $t = 0$), we shall use the semi-norms

$$\|f(t)\|_{H^m_{co}} = \sum_{|I|\leq m} \|(Z^I f)(t)\|_{L^2(\Omega)}, \quad \|f(t)\|_{\mathcal{H}^{j,l}} = \sum_{I=(k,\tilde{I}),k\leq j,|\tilde{I}|\leq l} \|Z^I f(t)\|_{L^2(\Omega)}. \tag{6.1.9}$$

Finally, to measure regularity along the boundary, we use

$$|f|_{L^p_t\tilde{H}^s(\partial\Omega)} = \sum_{j=0}^{[s]} |(\varepsilon\partial_t)^j f|_{L^p([0,t],H^{s-j}(\partial\Omega))}. \tag{6.1.10}$$

Let us recall, how the vector fields $Z_j$, $1 \leq j \leq M$ can be defined. We consider $\Omega \in \mathbb{R}^3$ a smooth domain (the following construction and our results are actually valid as long as the boundary of $\Omega$ can be covered by a finite number of charts), therefore, there exists a covering such that :

$$\Omega \subset \Omega_0 \cup_{i=1}^N \Omega_i, \quad \Omega_0 \Subset \Omega, \quad \Omega_i \cap \partial\Omega \neq \varnothing, \tag{6.1.11}$$

and $\Omega_i \cap \Omega$ is the graph of a smooth function $z = \varphi_i(x_1, x_2)$.

In $\Omega_0$, we just take the vector fields $\partial_k$, $k = 1, 2, 3$. To define appropriate vector fields near the boundary, we use the local coordinates in each $\Omega_i$ :

$$\begin{aligned} \Phi_i : \quad (-\delta_i, \delta_i) \times (0, \epsilon_i) &\to \Omega_i \cap \Omega \\ (y, z)^t &\to \Phi_i(y, z) = (y, \varphi_i(y) + z)^t \end{aligned} \tag{6.1.12}$$

and we define the vector fields (up to some smooth cut-off functions compactly supported in $\Omega_i$) as :

$$Z^i_k = \partial_{y^k} = \partial_k + \partial_k\varphi_i\partial_3, \quad k = 1, 2 \qquad Z^i_3 = \phi(z)(\partial_1\varphi_1\partial_1 + \partial_2\varphi_1\partial_2 - \partial_3), \tag{6.1.13}$$

where $\phi(z) = \frac{z}{1+z}$, and $\partial_k, k = 1, 2, 3$ are the derivations with respect to the original coordinates of $\mathbb{R}^3$.

We shall denote by $\mathbf{n}$ the unit outward normal to the the boundary. In each $\Omega_i$, we can extend it to $\Omega_i$ by setting

$$\mathbf{n}(\Phi_i(y, z)) = \frac{1}{|\mathbf{N}|}\mathbf{N}, \quad \mathbf{N}(\Phi_i(y, z)) = (\partial_1\varphi_i(y), \partial_2\varphi_i(y), -1)^t.$$

In the same way, the projection on vector fields tangent to the boundary,

$$\Pi = \mathrm{Id} - \mathbf{n} \otimes \mathbf{n}$$

can be extended in $\Omega_i$ by using the extension of $\mathbf{n}$.

Let us observe that by identity

$$\Pi(\partial_{\mathbf{n}} u) = \Pi((\nabla u)\mathbf{n}) = 2\Pi(\mathbb{S}u) - \Pi((Du)\mathbf{n})$$

with $[(\nabla u)\mathbf{n}]_i = \sum_{j=1}^3 \mathbf{n}_j\partial_j u_i$, $[(Du)\mathbf{n}]_i = \sum_{j=1}^3 \partial_i u_j\mathbf{n}_j$, the boundary conditions (6.1.2) can be reformulated as:

$$u \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Pi(\partial_{\mathbf{n}} u) = \Pi[-2au + (D\mathbf{n})u] \tag{6.1.14}$$

where $[(D\mathbf{n})u]_i = \sum_{j=1}^3 \partial_i\mathbf{n}_j u_j$.

## 6.1.2 Main results and strategy of the proof

Let us introduce the new unknown

$$\sigma^\varepsilon = \frac{P(\rho^\varepsilon) - P(\bar\rho)}{\varepsilon},$$

where $\overline{\rho}$ is a positive constant state, we can rewrite the system (6.1.1) into the following form which is more convenient to perform energy estimates:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t\sigma^\varepsilon + u^\varepsilon\cdot\nabla\sigma^\varepsilon\big) + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon)(\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon\big) - \mathrm{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \sigma^\varepsilon|_{t=0} = \sigma_0^\varepsilon. \end{cases} \tag{6.1.15}$$

where the scalar functions $g_1$, $g_2$ are defined by

$$g_2(s) = \rho^\varepsilon = P^{-1}(\bar P + s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar P = -P(\bar\rho). \tag{6.1.16}$$

In order to establish uniform energy estimates, we shall use the following quantity

$$\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) = \mathcal{E}_{m,T}(\sigma^\varepsilon, u^\varepsilon) + \mathcal{A}_{m,T}(\sigma^\varepsilon, u^\varepsilon)$$

where $\mathcal{E}_{m,T}$ contains $L^2$ (in space) type quantities

$$\begin{aligned}\mathcal{E}_{m,T}(\sigma^\varepsilon, u^\varepsilon) &= \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty\mathcal{H}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^{m-2}\cap L_T^2 H_{co}^{m-1}} \\ &+ \varepsilon\big(\|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^{m-1}} + \|\nabla^2 u^\varepsilon\|_{L_T^\infty H_{co}^{m-2}}\big) + \varepsilon\|\nabla^2\sigma\|_{L_T^\infty L^2},\end{aligned} \tag{6.1.17}$$

and $\mathcal{A}_{m,T}$ involves $L^\infty$ (in space and time) type quantities

$$\begin{aligned}\mathcal{A}_{m,T}(\sigma^\varepsilon, u^\varepsilon) = \|\nabla u^\varepsilon\|_{0,\infty,T} &+ \|(\nabla\sigma^\varepsilon, \mathrm{div} u^\varepsilon, \varepsilon^{\frac12}\nabla u)\|_{[\frac{m-1}{2}],\infty,T} + \|(\sigma^\varepsilon, u^\varepsilon)\|_{[\frac{m+1}{2}],\infty,T} \\ &+ \varepsilon\|\nabla u^\varepsilon\|_{[\frac{m+1}{2}],\infty,T} + \varepsilon\|(\sigma^\varepsilon, u^\varepsilon)\|_{[\frac{m+3}{2}],\infty,T}.\end{aligned} \tag{6.1.18}$$

Note that the norms involved in the above definitions are defined in (6.1.6)-(6.1.8).

Before stating our main result, we introduce the following definition.

**Definition 3** (Compatibility conditions). *We say that $(\sigma_0^\varepsilon, u_0^\varepsilon)$ satisfy the compatibility conditions up to order $m$ if:*

$$(\varepsilon\partial_t)^j u^\varepsilon\big|_{t=0}\cdot\boldsymbol{n} = 0, \qquad \Pi\big[\mathbb{S}\big((\varepsilon\partial_t)^j u^\varepsilon|_{t=0}\big)\boldsymbol{n}\big] = -a\Pi\big[(\varepsilon\partial_t)^j u|_{t=0}\big] \quad on \quad \partial\Omega, j = 0, 1\cdots m-1.$$

Note that the restriction of the time derivatives of the solution at the initial time can be expressed inductively by using the equations. For example, we have

$$(\varepsilon\partial_t u^\varepsilon)(0) = \frac{1}{\rho_0^\varepsilon}(-\varepsilon u_0^\varepsilon\cdot\nabla u_0^\varepsilon + \varepsilon\mathrm{div}\mathcal{L}u_0^\varepsilon - \nabla\sigma_0^\varepsilon).$$

We thus define the admissible space for initial data as

$$Y_m = \Big\{(\sigma_0^\varepsilon, u_0^\varepsilon)\in H^2(\Omega)^4, \quad Y_m^\varepsilon(\sigma_0^\varepsilon, u_0^\varepsilon) < +\infty,$$
$$(\sigma_0^\varepsilon, u_0^\varepsilon) \text{ satisfy the compatibility conditions up to order } m\Big\}$$

where

$$\begin{aligned}Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) =: \varepsilon\|(\sigma_0^\varepsilon, u_0^\varepsilon)\|_{H^2(\Omega)} &+ \|(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^m} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^{m-1}} \\ &+ \sum_{|I|\le[\frac{m-1}{2}]}\|Z^I(\nabla\sigma^\varepsilon, \nabla u^\varepsilon)(0)\|_{L^\infty(\Omega)}\end{aligned} \tag{6.1.19}$$

by using our notation (6.1.9).

The following is our main uniform regularity result:

**Theorem 6.1.1** (Uniform estimates). *Given an integer $m \geq 6$ and a $C^{m+2}$ smooth bounded domain $\Omega$. Consider a family of initial data such that $(\sigma_0^\varepsilon, u_0^\varepsilon) \in Y_m$, and*

$$\sup_{\varepsilon\in(0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) < +\infty,$$

$$-\bar{c}\bar{P} \leq \varepsilon\sigma_0^\varepsilon(x) \leq \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0,1],$$

*where $0 < \bar{c} < 1/4$ is a fixed constant, $\bar{P} = P(\bar{\rho})$. There exist $\varepsilon_0 \in (0,1]$ and $T_0 > 0$, such that, for any $0 < \varepsilon \leq \varepsilon_0$, the system* (6.1.15),(6.1.2) *has a unique solution $(\sigma^\varepsilon, u^\varepsilon)$ which satisfies:*

$$-2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 2\bar{P}/\bar{c}, \quad \forall (t,x) \in [0,T_0] \times \Omega, \tag{6.1.20}$$

*and*

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \mathcal{N}_{m,T_0}(\sigma^\varepsilon, u^\varepsilon) < +\infty. \tag{6.1.21}$$

Let us begin with a few comments about the above assumptions and our result.

**Remark 6.1.2.** *In view of* (6.1.20)*, there exists $c_0 \in (0,1]$, such that:*

$$c_0 \leq \rho^\varepsilon(t,x) = g_2(\varepsilon\sigma) \leq 1/c_0 \quad \forall (t,x) \in [0,T_0] \times \Omega$$

*Moreover, as a consequence of* (6.1.21)*, the following uniform estimates hold:*

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \big( \|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_{T_0} H^{m-1}_{co} \cap L^2_{T_0} H^m_{co}} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_{T_0} H^{m-2}_{co} \cap L^2_{T_0} H^{m-1}_{co}} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{0,\infty,t} \big) < +\infty,$$

*in particular, we have a uniform estimate for $\|\nabla(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty([0,T_0]\times\Omega)}$.*

**Remark 6.1.3.** *Because of the compatibility conditions, the assumption $\sup_{\varepsilon\in(0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon) < +\infty$ imposes that the data are prepared (in the sense that it may depend on $\varepsilon$) on the boundary. Nevertheless, this is compatible with the fact that*

$$(\mathrm{div} u^\varepsilon, \nabla\sigma^\varepsilon) = \mathcal{O}(1)$$

*in the domain and thus ill-prepared data in the usual sense. Indeed, note that $Y_m$ clearly contains smooth functions which vanish identically near the boundary. This kind of compatiability conditions also appears in the study of the incompressible limit of the Euler system in bounded domains [1].*

**Remark 6.1.4.** *The control of the weighted time derivatives $(\varepsilon\partial_t)^k$ up to highest order $k = m$ : $\|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_T \mathcal{H}^m}$ is available since time derivation commute with the space derivation. Moreover,*

$$\|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_T H^{m-1}_{co} \cap L^2_T H^m_{co}} \lesssim \mathcal{E}_{m,T}(\sigma^\varepsilon, u^\varepsilon). \tag{6.1.22}$$

*In other words, we can control the highest number of derivatives in the $L^2_t L^2_x$ norm but lose the uniform control of the highest space conormal derivatives in $L^\infty_t L^2_x$. This is due to the bad commutation properties of the space conormal derivatives with the singular part of the system.*

**Remark 6.1.5.** *The solution constructed in Theorem 6.1.1 is a strong solution in the sense that for $\varepsilon > 0$ fixed $(\sigma^\varepsilon, u^\varepsilon) \in L^\infty([0,T_0], H^1 \times H^2)$, $u^\varepsilon \in L^2([0,T_0], H^3)$. Note that we further have a uniform control of the $L^\infty_t H^{m-1} \cap L^2_t H^m$ norms in every compact set in the interior of the domain. Nevertheless, due to boundary layer effects (see* (6.1.5)*), we cannot expect uniform estimates for higher order normal derivatives near the boundary.*

By combining the previous result with a compactness argument, we get the following convergence result:

**Theorem 6.1.6** (Convergence). *Under the assumptions of Theorem 6.1.1, let $(\sigma^\varepsilon, u^\varepsilon)$ the solution defined on $[0,T_0]$ given by Theorem 6.1.1 and assume that $u_0^\varepsilon$ converges strongly in $L^2(\Omega)$ to some $u_0^0$ when $\varepsilon$ tends to zero. Then, as $\varepsilon$ tends to zero, $\rho^\varepsilon$ (defined by* (6.1.16)*) converges to $\overline{\rho}$ in $L^\infty([0,T_0]\times\Omega)$ and $u^\varepsilon$ converges in $L^2_w([0,T_0], L^2(\Omega))$ (weak convergence in time) to $u^0$ such that*

$$u^0 \in L^\infty_{T_0} \mathcal{H}^{0,m-1} \cap L^2_{T_0} \mathcal{H}^{0,m}, \quad \nabla u^0 \in L^2_{T_0} \mathcal{H}^{0,m-1} \cap L^\infty([0,T_0]\times\Omega). \tag{6.1.23}$$

*Moreover, $u^0$ is the (unique in this class) weak solution to the incompressible Navier-Stokes system with Navier boundary condition* (6.1.3).

Note that $L^2_{T_0}\mathcal{H}^{0,m}$ is defined in (6.1.8) and involves only spatial conormal derivatives.

**Remark 6.1.7.** *Due to the absence of uniform estimate for the second order normal derivatives and thus also for the strong trace of the normal derivative, $u^0$ has to be interpreted as the weak solution to* (6.1.3) *in the following usual sense: for any $\psi \in C^\infty([0,T_0]\times \overline{\Omega})$ with $\mathrm{div}\psi = 0, \psi\cdot \boldsymbol{n}|_{\partial\Omega} = 0$, the following identity holds: for every $0 < t \leq T_0$,*

$$\begin{aligned}
&\bar{\rho}\int_\Omega (u^0\cdot\psi)(t,\cdot)\ \mathrm{d}x + \mu\iint_{Q_t} \nabla u^0\cdot\nabla\psi\ \mathrm{d}x\mathrm{d}s + \bar{\rho}\iint_{Q_t}(u^0\cdot\nabla u^0)\cdot\psi\ \mathrm{d}x\mathrm{d}s \\
&= \bar{\rho}\int_\Omega (u^0_0\cdot\psi)(0,\cdot)\ \mathrm{d}x + \bar{\rho}\iint_{Q_t} u^0\cdot\partial_t\psi\ \mathrm{d}x\mathrm{d}s + \mu\int_0^t\int_{\partial\Omega}\Pi(-2au^0 + (D\boldsymbol{n})u^0)\cdot\psi\ \mathrm{d}S_y\mathrm{d}s.
\end{aligned} \tag{6.1.24}$$

*where $Q_t = [0,t]\times\Omega$ and $\mathrm{d}S_y$ denotes the surface measure of $\partial\Omega$.*

**Remark 6.1.8.** *The convergence is weak in the time variable due to the lack of uniform estimate for $\partial_t(\sigma^\varepsilon, u^\varepsilon)$. This cannot be improved since in our bounded domain setting, there is no large time dispersion effect for the acoustic waves, and since because of our Navier boundary conditions with fixed slip length, there is no damping in the boundary layers of the acoustic waves.*

*Note that when $\varepsilon$ tends to zero, we have convergence of the whole family $u^\varepsilon$ and not only of subsequences due to the uniqueness for the limit system at this level of regularity.*

We shall now explain the main difficulties and the main strategies in order to prove Theorem 6.1.1. As already mentioned the main feature of our problem is the presence of both fast time oscillations and a boundary layer in space. These two aspects are well-understood when they occur separately, but in order to handle them simultaneously some new ideas will be needed. On the one hand, concerning the inviscid limit problem, one controls [93, 106, 127] the high order tangential derivatives by direct energy estimates, and then uses the vorticity to control the normal derivatives. Nevertheless, for the system with low Mach number, even the tangential derivative estimates are not easy to get, since the spatial tangential derivatives do not commute with $\nabla, \mathrm{div}$, defined with the standard derivations in $\mathbb{R}^3$, and thus create singular commutators. Without this a priori knowledge on the tangential derivatives, the estimate of the vorticity cannot be performed as in [93] [94] because of the consequent lack of information on its trace on the boundary. On the other hand, for the compressible Euler system with low Mach number, uniform high regularity estimates are established for example in [1]. One can get uniform $H^s (s > 5/2)$ estimates by using first $\varepsilon\partial_t$ derivatives and then recover space derivatives by using the equations to estimate the divergence of the velocity and the gradient of the pressure and a direct energy estimates for the vorticity which solves a transport equation with a characteristic vector field. Here, in the case of viscous fluids, we face again the fact that the estimates of the vorticity are challenging due to the lack of information on its trace on the boundary at this stage.

In order to get the missing information, we shall first use the Leray projection (the precise definition (6.3.2) is in Section 3) to split the velocity into a compressible part and an incompressible part: $u^\varepsilon = \nabla\Psi^\varepsilon + v^\varepsilon$. On the one hand, the compressible part $\nabla\Psi^\varepsilon$ of the velocity can be controlled by $\mathrm{div}u^\varepsilon$ thanks to standard elliptic theory and hence by using the mass conservation equation and the energy estimates for $\varepsilon\partial_t$ derivatives. On the other hand, the incompressible part $v^\varepsilon$ solves, up to the control of non-local commutators, a convection-diffusion equation without oscillations, and thus one can use direct energy estimates to get a control of $\|v^\varepsilon\|_{L^\infty_t H^{m-1}_{co}}$ and $\|\nabla v^\varepsilon\|_{L^2_t H^{m-1}_{co}}$. Note that we cannot estimate the maximal number of derivatives $m$ due to the lack of structure of the coupling terms involving the compressible part in the energy estimates. The key point here is that the diffusion (which on the other hand creates new difficulties in the control of the vorticity) allows to get the estimate of $\|\nabla v^\varepsilon\|_{L^2_t H^{m-1}_{co}}$. This is still not enough to close an estimate since, because of the time oscillations, we cannot use Sobolev embedding in time to control $\|\nabla v^\varepsilon\|_{L^\infty_t H^{m-2}_{co}}$ as it is done in small viscosity problems for compressible fluids (see for example [106], [127]). Here, we only have estimates for powers of $\varepsilon\partial_t$ instead of $\partial_t$. Nevertheless, with the additional information obtained from $v^\varepsilon$, we can then reduce the matter to the study of $\|\omega^\varepsilon\times\mathbf{n}\|_{L^\infty_t H^{m-2}_{co}}$ where $\omega^\varepsilon$ is the vorticity, which solves the heat equation with a non-homogeneous Dirichlet boundary condition which can be controlled from the previous estimates. We shall get the estimate by using the Green's function of the heat equation.

**Outline of the proof of Theorem 6.1.1.** The uniform energy estimates will be more precisely achieved in the following steps: (we shall skip the $\varepsilon$ dependence in the notations for the sake of simplicity).

**Step 1: Uniform high-order $\varepsilon\partial_t$ derivatives and $\varepsilon-$dependent high-order conormal derivatives.** In this step, we aim to prove two kinds of estimates. Namely, uniform estimates for high order $\varepsilon\partial_t$ derivatives, $\|(\sigma,u)\|_{L_t^\infty\mathcal{H}^m}$, and $\varepsilon-$dependent estimates: $\varepsilon\|(\sigma,u)\|_{L_t^\infty H_{co}^m}$, $\varepsilon\|(\nabla\sigma,\mathrm{div}u)\|_{L_t^\infty H_{co}^{m-1}}$. On the one hand, since the time derivative $\varepsilon\partial_t$ commutes with the spatial derivatives, we can get uniform estimates for high order time derivatives. Note that we use $\varepsilon\partial_t$ instead of $\partial_t$ since we are dealing with ill-prepared data. On the other hand, as the spatial conormal vector fields do not commute with $\nabla$, div, the singular part of the system, we need at this stage to add this additional $\varepsilon$ weight to control the commutator.

**Step 2: Uniform estimates for the incompressible part of the velocity.** Let us denote by $v=\mathbb{P}u$, and $\nabla\Psi=\mathbb{Q}u$ the incompressible and compressible part of the velocity respectively, where $\mathbb{P},\mathbb{Q}$ are defined in (6.3.2). By applying the projection $\mathbb{P}$ on the equation for the velocity and expanding the boundary conditions, we find that $v$ solves:

$$\begin{cases} \bar{\rho}\partial_t v-\mu\Delta v+\nabla q+\frac{g_2-\bar{\rho}}{\varepsilon}\varepsilon\partial_t u+g_2 u\cdot\nabla u=0 \quad \text{in} \quad \Omega \\ v\cdot\mathbf{n}=0, \quad \Pi(\partial_{\mathbf{n}}v)=\Pi(-2au+D\mathbf{n}\cdot\nabla\Psi+D\mathbf{n}\cdot u) \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.1.25}$$

where

$$\nabla q=-\mathbb{Q}(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t u+g_2 u\cdot\nabla u-\mu\Delta v).$$

Note that the first boundary condition $v\cdot\mathbf{n}=0$ is due to the definition of the projection $\mathbb{P}$ while the second boundary condition is deduced from (6.1.14). The incompressible part $v$ interacts with the compressible part $\nabla\Psi$ through the source term and the boundary condition. Due to the absence of singular terms, one can get the uniform estimates for $v$ (namely $\|v\|_{L_t^\infty H_{co}^{m-1}}$ and $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$) by direct energy estimates. Nevertheless, for latter use in the proof, we need to track in the energy estimates the counts of time and spatial conormal derivatives.

**Step 3: Uniform estimates for the compressible part of the system.** In this step, we aim to get the control of $\|(\nabla\sigma,\mathrm{div}u)\|_{L_t^\infty H_{co}^{m-2}\cap L_t^2 H_{co}^{m-1}}$. This can be done by using the equations and induction arguments. Indeed, by rewriting the system (6.1.15),

$$\begin{aligned} -\mathrm{div}u &= g_1\varepsilon\partial_t\sigma+\varepsilon g_1 u\cdot\nabla\sigma, \\ -\nabla\sigma &= g_2\varepsilon\partial_t u+\varepsilon(g_2 u\cdot\nabla u-\mathrm{div}\mathcal{L}u). \end{aligned}$$

In view of the above two equations, one can 'trade' one spatial derivative by one (small scale) time derivative $\varepsilon\partial_t$. We can thus recover the high order spatial (conormal) derivatives by using iteratively this observation.

**Step 4**: Control of $L_t^\infty H_{co}^{m-2}$ norm of $\nabla u$. In this step, we aim to get an uniform control of $\|\nabla u\|_{L_t^\infty H_{co}^{m-2}}$ which is quite useful to control $L_{t,x}^\infty$ type norms. The difficulty is the estimate close to the boundary. We can work in a local chart $\Omega_i$. In light of the identities

$$\partial_{\mathbf{n}}u\cdot\mathbf{n}=\mathrm{div}u-(\Pi\partial_{y_1}u)^1-(\Pi\partial_{y_2}u)^2, \quad \Pi(\partial_{\mathbf{n}}u)=\Pi(\omega\times\mathbf{n})-\Pi[(D\mathbf{n})u],$$

where $\mathbf{n}$ is an extension of the unit normal and $\Pi$ projects on $(\mathbf{n})^\perp$, it suffices to control $\|\omega\times\mathbf{n}\|_{L_t^\infty H_{co}^{m-2}}$. We remark that the advantage of working on $\omega\times\mathbf{n}$ rather than $\omega$ is that the boundary condition for $\omega\times\mathbf{n}$ (see (6.3.33)) only involves lower order terms on the boundary. To estimate $\omega\times\mathbf{n}$, a natural attempt, used in [93], is to perform energy estimates on the equation for the 'modified vorticity' $w=\omega\times\mathbf{n}+2\Pi(au-(D\mathbf{n})u)$ and to take advantage of the fact that $w$ vanishes on the boundary. However, the equations for $w$ still involve a stiff term $\frac{1}{\varepsilon}\nabla^\perp\sigma$, which is obviously an obstacle to obtain uniform energy estimates. We shall thus instead use a lifting of the boundary conditions by using Green's function for the solution of the heat equation with non-homogenous boundary conditions and estimate the remainder by energy estimates.

**Step 4: $L_{t,x}^\infty$ estimates.** The control of the $L_{t,x}^\infty$ norms contained in $\mathcal{A}_{m,T}$ mainly stems from the Sobolev embedding and the maximum principle for the system solved by the vorticity. Note that at this stage, it is crucial to use the direct $L_t^\infty H_{co}^{m-1}$ for $(\sigma,u)$ and $L_t^\infty H_{co}^{m-2}$ for $\nabla(\sigma,u)$ estimates obtained in the previous steps since because of the fast oscillations in time, uniform $L^\infty$ estimates in time cannot be deduced from a Sobolev embedding in time.

The case $\Omega = \mathbb{R}^3_+$ where the boundary is flat is easier to analyze. Indeed, the spatial tangential derivatives can be controlled directly through energy estimates without weight in $\varepsilon$, since in this case the derivatives $\partial_{y^i}$ commute with div or $\nabla$. The use of the step with the Helmholtz-Leray projection is thus not necessary, we shall give a sketch proof for this case in the Appendix of this Chapter.

**Organization of this chapter.** We will state the main uniform estimates in Section 2 which will be proven in Section 3 and Section 4. Section 5 is then devoted to the proof of Theorem 6.1.1. In Section 6, we will justify the incompressible limit. In the appendix I, we gather some useful product and commutator estimates as well as the proofs of some technical lemmas.

## 6.2 Uniform estimates.

In this section, we state the main uniform a priori estimate which is the heart of this paper and the crucial step towards the proof of Theorem 6.1.1:

**Proposition 6.2.1.** *Let $c_0 \in (0,1]$ be such that:*

$$\forall s \in \big[-3\bar{c}\bar{P}, 3\bar{P}/\bar{c}\big],\, c_0 \le g_i(s) \le 1/c_0,\, i=1,2, \quad |(g_1,g_2)|_{C^m\left(\left[-3\bar{c}\bar{P},3\bar{P}/\bar{c}\right]\right)} \le 1/c_0 \tag{6.2.1}$$

*where $\bar{c}$ is such that for some $T \in (0,1]$ the following assumption holds:*

$$-3\bar{c}\bar{P} \le \varepsilon\sigma^\varepsilon(t,x) \le 3\bar{P}/\bar{c} \qquad \forall (t,x) \in [0,T]\times\Omega, \forall \varepsilon \in [0,1]. \tag{6.2.2}$$

*Then, there exists $C(1/c_0) > 0$ and an increasing polynomial $\Lambda_0$ (whose coefficients are independent of $\varepsilon$), such that, for any $\varepsilon \in (0,1]$, we have for a smooth enough solution of* (6.1.15) *on* $[0,T]$ *the following estimate :*

$$\mathcal{N}^2_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le C\big(\frac{1}{c_0}\big)Y^2_m(\sigma^\varepsilon_0, u^\varepsilon_0) + (T+\varepsilon)^{\frac12}\Lambda_0\big(\frac{1}{c_0}, \mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon)\big), \tag{6.2.3}$$

*where $Y_m(\sigma^\varepsilon_0, u^\varepsilon_0)$ is defined in* (7.1.29).

*Proof.* This proposition is the consequence of Proposition 6.3.1 and 6.4.1, which will be established in Section 3 and Section 4 respectively. □

## 6.3 Uniform estimates-energy norm

In this section, we establish the a-priori estimates for the energy norm $\mathcal{E}_{m,T}$. Again, for notational convenience, we skip the $\varepsilon-$dependence of the solutions.

**Proposition 6.3.1.** *If the estimates* (6.2.2) (6.2.1) *are satisfied, then we can find a constant $C_1(1/c_0)$ that depends only on $1/c_0$ and a polynomial $\tilde{\Lambda}$ whose coefficients are independent of $\varepsilon$, such that for a smooth enough solution of* (6.1.15)*, the following estimate holds on* $[0,T]$ *for* $\varepsilon \in (0,1]$*:*

$$\mathcal{E}^2_{m,T} \le C_1\big(\frac{1}{c_0}\big)Y^2_m(\sigma_0,u_0) + (T+\varepsilon)^{\frac12}\tilde{\Lambda}\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{6.3.1}$$

As explained in the introduction, to overcome the difficulty due to the nontrivial commutators between the tangential spatial derivatives and the standard derivation $(\nabla, \mathrm{div})$, we need to split the velocity $u$ into $u = \nabla\Psi + v$, where $\nabla\Psi, v$ are the compressible part and the incompressible part respectively (see (6.3.2) the precisely definition). On the one hand, the compressible part $\nabla\Psi$ satisfies the elliptic equation $\Delta\Psi = \mathrm{div} u$ with Neumann boundary condition, from which one can deduce the estimate of $\nabla^2\Psi$ from that of $\mathrm{div} u$. On the other hand, since the incompressible part $v$ is governed by a convection diffusion equation without oscillations, we can control its conormal derivatives by direct energy estimates. The estimates for $\partial_{\mathbf{n}} v$ will then be deduced from the ones for $\omega \times \mathbf{n}$.

### 6.3.1 Preliminaries: Leray projection

To define the compressible or acoustic part and the incompressible part of the velocity field, we shall use the Leray projection. One has the decomposition,

$$L^2_x(\Omega)^3 = H \oplus G$$

where

$$H = \{v \in L^2_x(\Omega)^3, \mathrm{div} v = 0, v \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \quad G = \{\nabla\Psi, \nabla\Psi \in L^2(\Omega)^3\}.$$

We denote $\mathbb{P}, \mathbb{Q}$ the projectors that map $L^2_x(\Omega)^3$ to its subspaces $H$ and $G$ respectively, namely,

$$\begin{aligned} \mathbb{Q} : &L^2(\Omega)^3 \to G \qquad\qquad & \mathbb{P} : L^2(\Omega)^3 \to H \\ &f \mapsto \mathbb{Q}f = \nabla\Psi & f \mapsto f - \mathbb{Q}f \end{aligned} \tag{6.3.2}$$

where $\Psi$ is defined as the unique solution of

$$\begin{cases} \Delta\Psi = \mathrm{div} f \quad \text{in} \quad \Omega, \\ \partial_{\mathbf{n}}\Psi = f \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega, \\ \int_\Omega \Psi dx = 0. \end{cases} \tag{6.3.3}$$

Note that the solvability of the Neumann problem (6.3.3) in $H^1(\Omega)$ is well-known as an application of the Lax-Milgram theorem. Moreover, by Proposition (6.7.6), one has that for a $C^{k+1}$ bounded domain,

$$\|\nabla\Psi(t)\|_{H^k_{co}} \lesssim \|f(t)\|_{H^k_{co}}, \qquad \|\nabla^2\Psi(t)\|_{H^{k-1}_{co}} \lesssim \|\mathrm{div} f(t)\|_{H^{k-1}_{co}} + \|f(t)\|_{H^{k-1}_{co}}. \tag{6.3.4}$$

Note that in these estimates, the time variable is just an external parameter.

Since $[\mathbb{P}, \partial_t] = 0$, (6.1.15) is equivalent to the following system:

$$\begin{cases} g_1(\partial_t\sigma + u \cdot \nabla\sigma) + \frac{\Delta\Psi}{\varepsilon} = 0, \\ \bar\rho\partial_t\nabla\Psi + \mathbb{Q}\big(\frac{g_2 - \bar\rho}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u - \mu\Delta v - (2\mu + \lambda)\nabla\mathrm{div} u + \frac{\nabla\sigma}{\varepsilon}\big) = 0, \\ \bar\rho\partial_t v + \frac{g_2 - \bar\rho}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u - \mu\Delta v + \nabla q = 0, \end{cases} \tag{6.3.5}$$

where

$$v = \mathbb{P}u, \quad \nabla\Psi = \mathbb{Q}u, \quad \nabla q = -\mathbb{Q}\big(\frac{g_2 - \bar\rho}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u - \mu\Delta v\big), \quad \bar\rho = g_2(0).$$

By taking the divergence of the third equations of (6.3.5) and noting that $\mathrm{div} v = 0, \varepsilon\partial_t u \cdot \mathbf{n}|_{\partial\Omega} = 0$, we see that $\nabla q$ is governed by the following elliptic equation:

$$\begin{cases} \Delta q = -\mathrm{div}\big(\frac{g_2 - \bar\rho}{\varepsilon}\varepsilon\partial_t u + g_2 u \cdot \nabla u\big) \quad \text{in} \quad \Omega, \\ \partial_{\mathbf{n}} q = -(g_2 u \cdot \nabla u) \cdot \mathbf{n} + \mu\Delta v \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{6.3.6}$$

Proposition 6.3.1 can be shown by the first three steps outlined in the introduction, they will be handled in the following three subsections.

### 6.3.2 Step 1: highest conormal estimates.

For notational convenience, we denote $\Lambda$ for a polynomial which may differ from line to line, and use notation $\lesssim \cdot$ as $\le C\cdot$ for some generic constant $C = C(1/c_0)$ that depend on $1/c_0$ but not on $\varepsilon$.

Let us state the main result of this subsection.

**Lemma 6.3.2.** *Suppose that* (6.2.2) *is satisfied, then for any $m \ge 0$, any $0 < T \le 1$ and $\varepsilon \in (0, 1]$ we have:*

$$\begin{aligned} &\|(\sigma, u)\|^2_{L^\infty_T\mathcal{H}^m} + \varepsilon^2(\|(\sigma, u)\|^2_{L^\infty_T H^m_{co}} + \|(\nabla\sigma, \mathrm{div} u)\|^2_{L^\infty_T H^{m-1}_{co}}) \\ &\quad + \|\nabla u\|^2_{L^2_t\mathcal{H}^m} + \varepsilon^2(\|\nabla u\|^2_{L^2_T H^m_{co}} + \|\nabla\mathrm{div} u\|^2_{L^2_T H^{m-1}_{co}}) \\ &\lesssim Y^2_m(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,T}. \end{aligned} \tag{6.3.7}$$

*Proof.* The estimate (6.3.7) can be derived from the following two lemmas. □

Let us start with:

**Lemma 6.3.3.** *Under the same assumption as in Lemma 6.3.2, for any $0 < t \le T$, the following estimates hold:*

$$\|(\sigma,u)\|_{L_t^\infty\mathcal{H}^m}^2 + \|\nabla u\|_{L_t^2\mathcal{H}^m}^2 \lesssim \|(\sigma,u)(0)\|_{\mathcal{H}^m}^2 + \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)T^{\frac12}\mathcal{E}_{m,T}^2, \tag{6.3.8}$$

$$\varepsilon^2\big(\|(u,\sigma)(t)\|_{H_{co}^m}^2 + \|\nabla u\|_{L_t^2H_{co}^m}^2\big) \lesssim \varepsilon^2\|(\sigma,u)(0)\|_{H_{co}^m}^2 + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,T}^2 + \varepsilon^2\|\nabla\mathrm{div}u\|_{L_t^2H_{co}^{m-1}}^2. \tag{6.3.9}$$

We recall that in our notations the norms at $t=0$ involve the computation of powers of $\varepsilon\partial_t$ at $t=0$.

*Proof.* Define $\sigma^I = Z^I\sigma, u^I = Z^I u$. Then $(\sigma^I, u^I)$ satisfies:

$$\begin{cases} g_1(\partial_t\sigma^I + u\cdot\nabla\sigma^I) + \dfrac{\mathrm{div}u^I}{\varepsilon} = \mathcal{R}_\sigma^I,\\ g_2(\partial_t u^I + u\cdot\nabla u^I) - Z^I(\mathrm{div}\mathcal{L}u) + \dfrac{\nabla\sigma^I}{\varepsilon} = \mathcal{R}_u^I, \end{cases} \tag{6.3.10}$$

where

$$\mathcal{R}_\sigma^I = -[Z^I,\frac{g_1}{\varepsilon}]\varepsilon\partial_t\sigma - [Z^I, g_1u\cdot\nabla]\sigma - \frac{1}{\varepsilon}[Z^I,\mathrm{div}]u,$$
$$\mathcal{R}_u^I = -[Z^I,\frac{g_2}{\varepsilon}]\varepsilon\partial_t u - [Z^I, g_2u\cdot\nabla]u - \frac{1}{\varepsilon}[Z^I,\nabla]\sigma.$$

We first show (6.3.8) which is easier. Assuming that $I = (j,0,\cdots,0)$, $|j| \le m$ which means that $Z^I = (\varepsilon\partial_t)^j$ involves only time derivatives. The advantage of this case is that the commutators do not include singular terms, that is the third terms in $\mathcal{R}_\sigma^I$ and $\mathcal{R}_u^I$ vanish.

For the sake of notational simplicity, we denote $(\sigma^j, u^j) = (\varepsilon\partial_t)^j(\sigma,u)$. Taking the scalar product of (6.3.10) by $(\sigma^j,u^j)$ and taking benefits of the boundary conditions

$$u^j\cdot\mathbf{n} = 0,\quad \Pi(\partial_{\mathbf{n}}u^j) = \Pi(-2au^j + (D\mathbf{n})u^j)\quad\text{on}\quad\partial\Omega, \tag{6.3.11}$$

as well as the relation $\partial_t g_2 + \mathrm{div}(g_2u) = 0$, we get from standard integration by parts that:

$$\begin{aligned}&\frac12\int_\Omega(g_1|\sigma^j|^2 + g_2|u^j|^2)(t)\,\mathrm{d}x + \iint_{Q_t}\mu|\nabla u^j|^2 + (\mu+\lambda)|\mathrm{div}u^j|^2\,\mathrm{d}x\mathrm{d}s\\ &\le \frac12\int_\Omega\big(g_1|\sigma^j|^2 + g_2|u^j|^2\big)(0)\,\mathrm{d}x + \Big|\iint_{Q_t}\big(\partial_t g_1 + \mathrm{div}(g_1u)\big)|\sigma^j|^2\,\mathrm{d}x\mathrm{d}s\Big|\\ &\quad+\mu\Big|\int_0^t\int_{\partial\Omega}\Pi(\partial_{\mathbf{n}}u^j)\Pi u^j\mathrm{d}S_y\mathrm{d}s\Big| + \|\mathcal{R}_\sigma^I\|_{L^2(Q_t)}\|\sigma^j\|_{L^2(Q_t)} + \|\mathcal{R}_u^I\|_{L^2(Q_t)}\|u^j\|_{L^2(Q_t)},\end{aligned} \tag{6.3.12}$$

where we denote by $\mathrm{d}S_y$ the surface measure of $\partial\Omega$ and $Q_t = [0,t]\times\Omega$. The second term in the above right hand side can be controlled easily by $\Lambda_{1,\infty,t}\|\sigma^j\|_{L^2(Q_t)}^2$. Note that

$$|\!|\!|\partial_t g_1|\!|\!|_{0,\infty,t} \le \sup_{[-3\bar c\bar P, 3\bar P/\bar c]}(|g_1'(s)|)|\!|\!|\varepsilon\partial_t\sigma|\!|\!|_{0,\infty,t} \le \frac{1}{c_0}|\!|\!|\varepsilon\partial_t\sigma|\!|\!|_{0,\infty,t}.$$

The boundary term of the last line of (6.3.12) can be treated thanks to the boundary condition (6.3.11) and the trace inequality (6.7.11)

$$\mu\Big|\int_0^t\int_{\partial\Omega}\Pi(\partial_{\mathbf{n}}u^j)\cdot\Pi u^j\,\mathrm{d}S_y\mathrm{d}s\Big| \le \frac{\mu}{4}\|\nabla u^j\|_{L^2(Q_t)}^2 + C_\mu\|u^j\|_{L^2(Q_t)}^2. \tag{6.3.13}$$

We now detail the estimate of $(\mathcal{R}_\sigma^I, \mathcal{R}_u^I)$ which vanish unless $j \ne 0$. For $1 \le j \le m$, by the commutator estimate (6.7.4) and the estimate (6.7.5) for $g_1$,

$$\begin{aligned}\|\mathcal{R}_\sigma^I\|_{L^2(Q_t)} &\lesssim \|\partial_t g_1\|_{L_t^2\mathcal{H}^{m-1}}|\!|\!|(\varepsilon\partial_t)\sigma|\!|\!|_{[\frac m2]-1,\infty,t} + |\!|\!|\partial_t g_1|\!|\!|_{[\frac{m-1}{2}],\infty,t}\|(\varepsilon\partial_t)\sigma\|_{L_t^2\mathcal{H}^{m-1}}\\ &\quad+\|g_1u\|_{L_t^2\mathcal{H}^m}|\!|\!|\nabla\sigma|\!|\!|_{[\frac m2]-1,\infty,t} + |\!|\!|g_1u|\!|\!|_{[\frac{m+1}{2}],\infty,t}\|\nabla\sigma\|_{L_t^2\mathcal{H}^{m-1}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\big(\|\nabla\sigma\|_{L_t^2\mathcal{H}^{m-1}} + \|(\sigma,u)\|_{L_t^2\mathcal{H}^m}\big).\end{aligned} \tag{6.3.14}$$

In a similar way, we have:

$$\|\mathcal{R}_u^I\|_{L^2(Q_t)} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(\|\nabla(\sigma,u)\|_{L_t^2\mathcal{H}^{m-1}} + \|(\sigma,u)\|_{L_t^2\mathcal{H}^m}\big). \tag{6.3.15}$$

Therefore, (6.3.8) is the consequence of (6.3.12)-(6.3.15). Note that we have used the fact that

$$\|(\sigma,u)\|_{L_t^2\mathcal{H}^m} \lesssim T^{\frac{1}{2}}\|(\sigma,u)\|_{L_t^\infty\mathcal{H}^m} \lesssim T^{\frac{1}{2}}\mathcal{E}_{m,T}, \qquad \|\nabla(\sigma,u)\|_{L_t^2\mathcal{H}^{m-1}} \lesssim \mathcal{E}_{m,T}.$$

We are now ready to prove (6.3.9). Suppose now that $Z^I$ involves at least one spatial derivative and $1 \le |I| \le m$. In this case, it seems unlikely to get an uniform estimate with respect to $\varepsilon$ with this approach since $\mathcal{R}_\sigma^I, \mathcal{R}_u^I$ now contains singular terms. Taking the scalar product of system (6.3.10) by $\varepsilon^2(\sigma^I, u^I)$, and integrating by parts in space and time, we get in the same way as for (6.3.12) that:

$$\begin{aligned}\varepsilon^2 \int_\Omega & (g_1|\sigma^I|^2 + g_2|u^I|^2)(t)\ \mathrm{d}x \\ &\le \varepsilon^2 \int_\Omega (g_1|\sigma^I|^2 + g_2|u^I|^2)(0)\ \mathrm{d}x + \iint_{Q_t} (\partial_t g_1 + \mathrm{div}(g_1 u))|\sigma^I|^2\ \mathrm{d}x\mathrm{d}s \\ &+ 2\varepsilon^2 \iint_{Q_t} Z^I \mathrm{div}\mathcal{L}u \cdot u^I\ \mathrm{d}x\mathrm{d}s + \varepsilon^2\big(\|\mathcal{R}_\sigma^I\|_{L^2(Q_t)}\|\sigma^I\|_{L^2(Q_t)} + \|\mathcal{R}_u^I\|_{L^2(Q_t)}\|u^I\|_{L^2(Q_t)}\big).\end{aligned} \tag{6.3.16}$$

Before going further, it will be convenient to introduce the notation:

$$\|f\|_{E_t^m} = \|f\|_{L_t^2 H_{co}^m} + \|\nabla f\|_{L_t^2 H_{co}^{m-1}}. \tag{6.3.17}$$

Note that from the definition of $\mathcal{E}_{m,t}$ in (6.1.17), one has indeed that: $\|u\|_{E_t^m} \lesssim \mathcal{E}_{m,t}$.

Let us now estimate the terms in the last line of (6.3.16). It follows from the commutator estimate (6.7.3) that:

$$\varepsilon\|(\mathcal{R}_\sigma^I, \mathcal{R}_u^I)\|_{L^2(Q_t)} \lesssim \|\nabla(\sigma,u)\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{E_t^m}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big). \tag{6.3.18}$$

We remark that when controlling the extra term: $\frac{1}{\varepsilon}[Z^I, \nabla]\sigma$, we have used the following identity which can be shown by induction:

$$[Z^I, \partial_i] = \sum_{j=1}^3 \sum_{|J|\le|I|-1} c_{I,J} Z^J \partial_j = \sum_{j=1}^3 \sum_{|J|\le|I|-1} d_{I,J}\partial_j Z^J \tag{6.3.19}$$

where $J$ is an $(M+1)$ multi-index and $c_{I,J}, d_{I,J}$ are smooth functions that depend on $I$, $J$, $i$ and the derivatives (up to order $|I|$) of $\nabla\phi$, $\partial_i$ is the derivation in the standard Euclidean coordinates.

It remains to estimate the third term in the right hand side of (6.3.16). Since, we have

$$\mathrm{div}\mathcal{L}u = \mathrm{div}(2\mu\mathbb{S}u + \lambda\mathrm{div}u\mathrm{Id}) = \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u,$$

one has by integrating by parts that:

$$\begin{aligned}\iint_{Q_t} Z^I\mathcal{L}u\cdot u^I\ \mathrm{d}x\mathrm{d}s &= -\iint_{Q_t} \big(\mu[Z^I,\nabla]u\cdot\nabla u^I + (\mu+\lambda)[Z^I,\mathrm{div}]u\mathrm{div}u^I\big)\ \mathrm{d}x\mathrm{d}s \\ + \iint_{Q_t} &\big(\mu[Z^I,\mathrm{div}]\nabla u + (\mu+\lambda)[Z^I,\nabla]\mathrm{div}u\big)u^I\ \mathrm{d}x\mathrm{d}s - \iint_{Q_t}\mu|\nabla u^I|^2 + (\mu+\lambda)|\mathrm{div}u^I|^2\ \mathrm{d}x\mathrm{d}s \\ + \int_0^t &\int_{\partial\Omega} \mu u^I(Z^I\nabla u\cdot\mathbf{n}) + (\mu+\lambda)Z^I\mathrm{div}u(u^I\cdot\mathbf{n})\ \mathrm{d}S_y\mathrm{d}s =: \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4.\end{aligned} \tag{6.3.20}$$

Let us begin with the $\mathcal{K}_1$ term. By (6.3.19) and the Young inequality, we get

$$\mathcal{K}_1 \le \delta\mu\|\nabla u\|_{L_t^2H_{co}^m}^2 + C_{\delta,\mu,\lambda}\|\nabla u\|_{L_t^2H_{co}^{m-1}}^2 \tag{6.3.21}$$

for $\delta > 0$ to be chosen sufficiently small independent of $\varepsilon$. Next, by (6.3.19) and integration by parts, $\mathcal{K}_2$ can be written as a combination of the following two types of terms (up to some smooth coefficients that depending on $\phi, \mathbf{n}$ and their derivatives up to order $m+1$):

$$\mathcal{K}_2^1 = \iint_{Q_t} Z^{\tilde{I}}\partial_i u \cdot \partial_j u^I \,\mathrm{d}x\mathrm{d}s, \qquad \mathcal{K}_2^2 = \int_0^t \int_{\partial\Omega} Z^{\tilde{I}}\partial_i u \cdot u^I \mathbf{n}_j \,\mathrm{d}x\mathrm{d}s, \quad |\tilde{I}| \le |I| - 1.$$

The term $\mathcal{K}_2^1$ can be estimated in the same way as $\mathcal{K}_1$, we find again

$$\mathcal{K}_2^1 \le \delta\mu \|\nabla u^I\|_{L^2(Q_t)}^2 + C_{\delta,\mu,\lambda}\|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2.$$

For $\mathcal{K}_2^2$, we use the trace inequality (6.7.11) to get that:

$$\begin{aligned}\mathcal{K}_2^2 &\lesssim \int_0^t |Z^{\tilde{I}}\partial_i u|_{L^2(\partial\Omega)}|u^I \cdot \mathbf{n}_j|_{L^2(\partial\Omega)}\mathrm{d}s \lesssim \int_0^t (|u|_{\tilde{H}^m(\partial\Omega)} + |\mathrm{div} u|_{\tilde{H}^{m-1}(\partial\Omega)})|u^I \cdot \mathbf{n}_j|_{L^2(\partial\Omega)}\,\mathrm{d}s \\ &\le \delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu,\lambda}\big(\|u\|_{E_t^m} + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2\big).\end{aligned}$$

To get the second inequality, we have used that $\tilde{I}$ does not contain conormal derivatives of the type $Z_3^i$ since $Z_3^i$ vanishes on the boundary and the identity:

$$\partial_{\mathbf{n}} u \cdot \mathbf{n} = \mathrm{div} u - (\Pi\partial_{y^1} u)^1 - (\Pi\partial_{y^2} u)^2, \tag{6.3.22}$$

as well as the boundary condition (6.1.14).

To summarize, we have thus proven that there exists an absolute constant $C > 0$ (independent of $\delta$ and of course $\varepsilon$) such that

$$\mathcal{K}_2 \le C\delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu,\lambda}(\|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2 + \|u\|_{E_t^m}^2). \tag{6.3.23}$$

Finally, we handle the term $\mathcal{K}_4$ in the right hand side of (6.3.20) which is nontrivial only if $Z^I$ contains merely $\varepsilon\partial_t$ and tangential derivatives which read in local charts $\partial_{y^1}, \partial_{y2}$. For the second term of $\mathcal{K}_4$, since $Z^I$ is assumed to contain at least one spatial derivative, it can be written as $Z^I = \partial_y Z^{\tilde{I}}$ (we denote $\partial_y = \partial_{y^1}$ or $\partial_y = \partial_{y^2}$). Moreover, since $u \cdot \mathbf{n}|_{\partial\Omega} = 0$, $u^I \cdot \mathbf{n} = [Z^I, \mathbf{n}]u$. Integrating by parts along the boundary, and then use the trace inequality (6.7.12), we find that

$$\begin{aligned}\int_0^t \int_{\partial\Omega} Z^I \mathrm{div} u (u^I \cdot \mathbf{n})\,\mathrm{d}S_y\mathrm{d}s &\le \int_0^t |Z^{\tilde{I}}\mathrm{div} u|_{H^{\frac{1}{2}}(\partial\Omega)}|\partial_y[Z^I, \mathbf{n}]u|_{H^{-\frac{1}{2}}(\partial\Omega)}\,\mathrm{d}s \\ &\lesssim \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2 + \|u\|_{E_t^m}^2.\end{aligned} \tag{6.3.24}$$

For the first term of $\mathcal{K}_4$, we can split it into two terms:

$$\begin{aligned}&\mu \int_0^t \int_{\partial\Omega} -u^I([Z^I, \mathbf{n}]\nabla u) + [Z^I, \mathbf{n}]\partial_{\mathbf{n}} u (u^I \cdot \mathbf{n}) + [Z^I, \Pi]\partial_{\mathbf{n}} u \cdot \Pi u^I \,\mathrm{d}S_y\mathrm{d}s \\ &\quad - \mu \int_0^t \int_{\partial\Omega} Z^I(\partial_{\mathbf{n}} u \cdot \mathbf{n})(u^I \cdot \mathbf{n}) + Z^I(\Pi\partial_{\mathbf{n}} u) \cdot \Pi u^I)\,\mathrm{d}S_y\mathrm{d}s =: \mathcal{K}_{411} + \mathcal{K}_{412}.\end{aligned}$$

Thanks to the trace inequality and the Young's inequality, $\mathcal{K}_{411}$ can be bounded as:

$$\mathcal{K}_{411} \le \delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu}(\|u\|_{E_t^m}^2 + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2).$$

Next, for $\mathcal{K}_{412}$, we use again the identity (6.3.22), as well as the boundary conditions (6.1.14). Integrating by parts along the boundary for the first term of $\mathcal{K}_{412}$, we get that by writing $Z^I = \partial_y Z^{\tilde{I}}$

$$\begin{aligned}\mathcal{K}_{412} &= \mu \int_0^t |Z^{\tilde{I}}(\partial_{\mathbf{n}} u \cdot \mathbf{n})|_{H^{\frac{1}{2}}(\partial\Omega)}|\partial_y[Z^I, \mathbf{n}]u|_{H^{-\frac{1}{2}}(\partial\Omega)} + |Z^I \Pi\partial_{\mathbf{n}} u|_{L^2(\partial\Omega)}|u^I|_{L^2(\partial\Omega)}\,\mathrm{d}s \\ &\le \delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu}(\|u\|_{E_t^m}^2 + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2).\end{aligned}$$

To summarize, we get the following estimate for $\mathcal{K}_4$ :

$$\mathcal{K}_4 \leq 2\delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu}(\|u\|_{E_t^m}^2 + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \tag{6.3.25}$$

Inserting (6.3.21), (6.3.23), (6.3.25) into (6.3.20), we get that:

$$\begin{aligned} \int_{Q_t} Z^I \mathcal{L} u \cdot u^I \, \mathrm{d}x\mathrm{d}s \leq & -\iint_{Q_t} \mu|\nabla u^I|^2 + (\mu+\lambda)|\mathrm{div} u^I|^2 \, \mathrm{d}x\mathrm{d}s \\ & + (C+3)\delta\mu\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_{\delta,\mu}(\|u\|_{E_t^m}^2 + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2). \end{aligned} \tag{6.3.26}$$

Plugging (6.3.18) and (6.3.26) into (6.3.16) and summing up for $|I| \leq m$, we finally get (6.3.9) by choosing $\delta$ small enough (independent of $\varepsilon$). $\square$

**Lemma 6.3.4.** *Under the same assumption as in Lemma 6.3.2, for any $0 < t \leq T$, one has that:*

$$\begin{aligned} \varepsilon^2 \|(\nabla\sigma, \mathrm{div} u)(t)\|_{H_{co}^{m-1}(\Omega)}^2 + \varepsilon^2\|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}}^2 & \\ \lesssim \|(\nabla\sigma, \mathrm{div} u)(0)\|_{H_{co}^{m-1}}^2 + (T^{\frac{1}{2}} + \varepsilon^{\frac{2}{3}})\Lambda_{2,\infty,T}\mathcal{E}_{m,T}^2. & \end{aligned} \tag{6.3.27}$$

*Proof.* Applying the vector field $Z^I$ with $0 \leq |I| \leq m-1$, we then find that $((\nabla\sigma)^I, u^I) = (Z^I \nabla\sigma, Z^I u)$ solves the system:

$$\begin{cases} g_1(\partial_t + u\cdot\nabla)(\nabla\sigma)^I + \dfrac{\nabla \mathrm{div} u^I}{\varepsilon} =: \mathcal{C}_\sigma^I, \\ g_2\partial_t u^I - \mu\,\mathrm{curl}(Z^I\omega) - (2\mu+\lambda)\nabla\mathrm{div} u^I + \dfrac{(\nabla\sigma)^I}{\varepsilon} =: \mathcal{C}_u^I, \end{cases} \tag{6.3.28}$$

where $\omega = \mathrm{curl}\, u$ and

$$\begin{aligned} \mathcal{C}_\sigma^I &= -[Z^I\nabla, g_1/\varepsilon]\varepsilon\partial_t\sigma - [Z^I\nabla, g_1 u\cdot\nabla]\sigma - [Z^I, \nabla\mathrm{div}]u/\varepsilon, \\ \mathcal{C}_u^I &= -Z^I(g_2 u\cdot\nabla u) - [Z^I, g_2/\varepsilon]\varepsilon\partial_t u + \mu[Z^I, \mathrm{curl}]\omega + (2\mu+\lambda)[Z^I, \nabla\mathrm{div}]u. \end{aligned} \tag{6.3.29}$$

We take the scalar product of the equation $(6.3.28)_1$ by $(\nabla\sigma)^I$, and $(6.3.28)_2$ by $-\nabla\mathrm{div} u^I$, we then integrate in space and time and sum up the two equations to get that (note that the singular terms cancel):

$$\begin{aligned} &\frac{1}{2}\int_\Omega (g_1|(\nabla\sigma)^I|^2 + g_2|\mathrm{div} u^I|^2)(t)\,\mathrm{d}x + (2\mu+\lambda)\iint_{Q_t} |\nabla\mathrm{div} u^I|^2 \,\mathrm{d}x\mathrm{d}s \\ &\leq \frac{1}{2}\int_\Omega (g_1|\nabla\sigma^I|^2 + g_2|\mathrm{div} u^I|^2)(0)\,\mathrm{d}x + \frac{1}{2}\left|\iint_{Q_t} (\partial_t g_1 + \mathrm{div}(g_1 u))|\nabla\sigma^I|^2 \,\mathrm{d}x\mathrm{d}s\right| \\ &\quad + \left|\iint_{Q_t} (g_2'\varepsilon\partial_t u^I\cdot\nabla\sigma)\mathrm{div} u^I \,\mathrm{d}x\mathrm{d}s\right| + \left|\int_0^t\int_{\partial\Omega} g_2\partial_t u^I\cdot\mathbf{n}\,\mathrm{div} u^I \,\mathrm{d}S_y\mathrm{d}s\right| \\ &\quad + \mu\left|\iint_{Q_t} \mathrm{curl}\, Z^I\omega\nabla\mathrm{div} u^I \,\mathrm{d}x\mathrm{d}s\right| \\ &\quad + \|\mathcal{C}_\sigma^I\|_{L^2(Q_t)}\|\nabla\sigma^I\|_{L^2(Q_t)} + \frac{1}{(2\mu+\lambda)}\|\mathcal{C}_u^I\|_{L^2(Q_t)}^2 + \frac{2\mu+\lambda}{4}\|\nabla\mathrm{div} u^I\|_{L^2(Q_t)}^2. \end{aligned} \tag{6.3.30}$$

Among the terms in the right hand side, the second and the third terms can be bounded by:

$$\Lambda\big(\frac{1}{c_0}, |\!|\!|(\sigma,u)|\!|\!|_{1,\infty,t} + |\!|\!|(\nabla\sigma,\mathrm{div} u)|\!|\!|_{0,\infty,t}\big)\,\big\|\big((\nabla\sigma)^I, \mathrm{div} u^I, \varepsilon\partial_t u^I\big)\big\|_{L^2(Q_t)}^2. \tag{6.3.31}$$

Next, we note that the fourth term vanishes if $Z^I$ involves at least one conormal derivative $Z_3^i$ which vanishes on the boundary. We thus suppose that $I = (l, I'), |I'| \geq 1$ and $Z^I$ does not contain $Z_3^i$.

Consequently, the trace inequality (6.7.11) leads to

$$
\begin{aligned}
\Big|\int_0^t\int_{\partial\Omega} g_2\partial_t u^I\cdot\mathbf{n}\,\mathrm{div}u^I\,\mathrm{d}S_y\mathrm{d}s\Big| &\lesssim \frac{1}{\varepsilon}\int_0^t |[Z^I,\mathbf{n}]\varepsilon\partial_t u(s)|_{L^2(\partial\Omega)}|\mathrm{div}u^I(s)|_{L^2(\partial\Omega)}\,\mathrm{d}s\\
&\lesssim \frac{1}{\varepsilon}(\|\nabla u\|_{L_t^2H_{co}^{m-1}}+\|u\|_{L_t^2H_{co}^{m-1}})(\|\nabla\mathrm{div}u^I\|_{L^2(Q_t)}^{\frac12}\|\mathrm{div}u^I\|_{L^2(Q_t)}^{\frac12}+\|\mathrm{div}u^I\|_{L^2(Q_t)})\\
&\leq \frac{2\mu+\lambda}{4}\|\nabla\mathrm{div}u\|_{L^2(Q_t)}^2+C_{\mu,\lambda}(1+\varepsilon^{-\frac43})\|(u,\nabla u)\|_{L_t^2H_{co}^{m-1}}^2.
\end{aligned}
\tag{6.3.32}
$$

Note that since $\partial_t u\cdot\mathbf{n}|_{\partial\Omega}=0$, one has $(Z^I\partial_t u\cdot\mathbf{n})|_{\partial\Omega}=([Z^I,\mathbf{n}]\partial_t u)|_{\partial\Omega}$.

For the fifth term in the right hand side of (6.3.30) we first integrate by parts and then use the duality $\langle\cdot,\cdot\rangle_{H^{\frac12}(\partial\Omega)\times H^{-\frac12}(\partial\Omega)}$ to get that

$$
\begin{aligned}
\mu\Big|\iint_{Q_t}\mathrm{curl}\,Z^I\omega\cdot\nabla\mathrm{div}u^I\,\mathrm{d}x\mathrm{d}s\Big| &= -\mu\int_0^t\int_{\partial\Omega}(Z^I\omega\times\mathbf{n})\cdot\Pi\nabla\mathrm{div}u^I\,\mathrm{d}S_y\mathrm{d}s\\
&\leq \mu\int_0^t |Z^I\omega\times\mathbf{n}(s)|_{H^{\frac12}(\partial\Omega)}|\mathrm{div}u^I(s)|_{H^{\frac12}(\partial\Omega)}\,\mathrm{d}s
\end{aligned}
$$

We point out that for the derivation of the last line, the fact that $\Pi\nabla$ involves only tangential derivatives has been used. It remains to control $Z^I\omega\times\mathbf{n}$ on the boundary. One first deduces by (6.1.14) that on the boundary,

$$
\omega\times\mathbf{n}=\Pi(\omega\times\mathbf{n})=2\Pi(\mathbb{S}u)-2\Pi((\nabla u)^t\cdot\mathbf{n})=2\Pi(-au+D\mathbf{n}\cdot u)|_{\partial\Omega}. \tag{6.3.33}
$$

which leads to:

$$
\begin{aligned}
|Z^I\omega\times\mathbf{n}(s)|_{H^{\frac12}(\partial\Omega)} &\lesssim |Z^I(\omega(s)\times\mathbf{n})|_{H^{\frac12}(\partial\Omega)}+|[Z^I,\mathbf{n}]\times\omega|_{H^{\frac12}(\partial\Omega)}\\
&\lesssim |u(s)|_{\tilde{H}^{m-\frac12}}+|\omega(s)|_{\tilde{H}^{m-\frac32}}\lesssim |u(s)|_{\tilde{H}^{m-\frac12}}+|\mathrm{div}u(s)|_{\tilde{H}^{m-\frac32}}
\end{aligned}
$$

where we recall that we denote:

$$
|f(t)|_{\tilde{H}^r}:=\sum_{k\leq[r]}|(\varepsilon\partial_t)^k f(t)|_{H^{r-k}(\partial\Omega)}.
$$

Note that by using the boundary condition (6.1.14) and the identity (6.3.22), we have that:

$$
|\nabla u|_{\tilde{H}^s}\lesssim |u|_{\tilde{H}^{s+1}}+|\mathrm{div}u|_{\tilde{H}^s}.
$$

Finally, owing to the trace inequality (6.7.12) and Young's inequality, one obtains that:

$$
\begin{aligned}
&\mu\Big|\iint_{Q_t}\mathrm{curl}\,Z^I\omega\cdot\nabla\mathrm{div}u^I\,\mathrm{d}x\mathrm{d}s\Big|\\
&\leq C\mu(\|\nabla\mathrm{div}u\|_{L_t^2H_{co}^{m-2}}+\|\nabla u\|_{L_t^2H_{co}^{m-1}}+\|u\|_{L_t^2H_{co}^m})(\|\mathrm{div}u^I\|_{L^2(Q_t)}+\|\nabla\mathrm{div}u^I\|_{L^2(Q_t)})\\
&\leq \frac{2\mu+\lambda}{4}\|\nabla\mathrm{div}u^I\|_{L^2(Q_t)}^2+C_{\mu,\lambda}(\|\nabla\mathrm{div}u\|_{L_t^2H_{co}^{m-2}}^2+\|u\|_{E_t^m}^2)
\end{aligned}
\tag{6.3.34}
$$

where we use again the notation (6.3.17).

It remains to control the $L^2(Q_t)$ norm of $\mathcal{C}_\sigma^I,\mathcal{C}_u^I$ in (6.3.30). Let us begin with the estimate $\mathcal{C}_\sigma^I$. For the term:

$$
[Z^I\nabla,\frac{g_1}{\varepsilon}]\varepsilon\partial_t\sigma=Z^I((\nabla g_1/\varepsilon)\varepsilon\partial_t\sigma)+[Z^I,g_1/\varepsilon](\varepsilon\partial_t)\nabla\sigma,
$$

the product estimates (6.7.1) the commutator estimate (6.7.3) and the estimate (6.7.6) yield:

$$
\begin{aligned}
\|[Z^I\nabla,g_1/\varepsilon]\varepsilon\partial_t\sigma\|_{L^2(Q_t)} &\lesssim \|(\varepsilon\partial_t\sigma,\nabla\sigma)\|_{L_t^2H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\|\nabla\sigma\|_{[\frac{m}{2}]-1,\infty,t}+\|\sigma\|_{[\frac{m+1}{2}],\infty,t}\big)\\
&\lesssim \|\sigma\|_{E_t^m}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).
\end{aligned}
$$

For the term

$$[Z^I\nabla, g_1u\cdot\nabla]\sigma = Z^I\big(\nabla(g_1u)\nabla\sigma\big) + [Z^I, g_1u]\nabla\nabla\sigma,$$

since in the interior domain $\Omega_0$, the spatial conormal derivatives are equivalent to the derivations with respect to the standard coordinates in $\mathbb{R}^3$. The classical product and commutator estimates thus yield:

$$\|\chi_0[Z^I\nabla, g_1u\cdot\nabla]\sigma\|_{L^2(Q_t)} \lesssim (\|\tilde{\chi}_0(\sigma,u)\|_{L^2_tH^m} + \|\tilde{\chi}_0\nabla(\sigma,u)\|_{L^2_tH^{m-1}})\|\!|\tilde{\chi}_0(\nabla\sigma,u)\|\!|_{1,\infty,t}.$$

where $\mathrm{Supp}\,(\tilde{\chi}_0)\Subset\Omega$ and $\tilde{\chi}_0\chi_0=\chi_0$.

It suffices to focus on the case near the boundary. Direct computations show that, in the local coordinates (6.1.12),

$$u\cdot\nabla f = u_1\partial_{y^1}f + u_2\partial_{y^2}f + u\cdot\mathbf{N}\partial_z f, \tag{6.3.35}$$

which leads to:

$$\begin{aligned}
[Z^I\nabla, g_1u\cdot\nabla]\sigma &= Z^I\big(\nabla(g_1u)\nabla\sigma\big) + \sum_{j=1}^{2}[Z^I, g_1u_j]\partial_{y^j}\nabla\sigma \\
&\quad + [Z^I, (g_1u\cdot\mathbf{N})/\phi]\phi\partial_z\nabla\sigma + \big((g_1u\cdot\mathbf{N})/\phi\big)[Z^I,\phi]\partial_z\nabla\sigma + (g_1u\cdot\mathbf{N})[Z^I,\partial_z]\nabla\sigma.
\end{aligned}\tag{6.3.36}$$

With the help of the product and commutator estimates (6.7.1), (6.7.3) and the estimate (6.7.6) for $g_1$, the first two terms in the right hand side of (6.3.36) can be bounded as:

$$\begin{aligned}
&\varepsilon\|\chi_iZ^I\big(\nabla(g_1u)\nabla\sigma\big)\|_{L^2(Q_t)} + \sum_{j=1}^{2}\|\chi_i[Z^I, g_1u_j]\partial_{y^j}\nabla\sigma\|_{L^2(Q_t)} \\
&\lesssim \|(\sigma,u)\|_{E^m_t}\Lambda\big(\frac{1}{c_0}, \varepsilon\|\!|(\sigma,u)\|\!|_{[\frac{m}{2}],\infty,t} + \|\!|\nabla\sigma\|\!|_{[\frac{m-1}{2}],\infty,t} + \varepsilon\|\!|\nabla u\|\!|_{[\frac{m}{2}],\infty,t}\big) \\
&\lesssim \|(\sigma,u)\|_{E^m_t}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big).
\end{aligned}\tag{6.3.37}$$

To continue, we need to establish some estimates on $(g_1u\cdot\mathbf{N})/\phi$. At first, since $(u\cdot\mathbf{n})|_{\partial\Omega}=0$, one has by the fundamental theorem of calculus and the identity (6.3.22) that:

$$\begin{aligned}
\|\!|\chi_j(g_ju\cdot\mathbf{N})/\phi\|\!|_{k,\infty,t} &\lesssim (\|\!|\nabla(u\cdot\mathbf{N}))\|\!|_{k,\infty,t} + \|\!|u\|\!|_{k,\infty,t})\|\!|g\|\!|_{k,\infty,t} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \|\!|u\|\!|_{k+1,\infty,t} + \|\!|(\sigma,\mathrm{div}u)\|\!|_{k,\infty,t}\big), \quad j=1,2.
\end{aligned}\tag{6.3.38}$$

Next, thanks to Hardy inequality and product estimate (6.7.1), estimate (6.7.7) for $g_j$

$$\begin{aligned}
\|\chi_i(g_ju\cdot\mathbf{N})/\phi\|_{L^2_tH^{m-1}_{co}} &\lesssim \|\tilde{\chi}_i(u\cdot\mathbf{N})/\phi\|_{L^2_tH^{m-1}_{co}} + \|(g_j-g_j(0))(u\cdot\mathbf{N})/\phi\|_{L^2_tH^{m-1}_{co}}) \\
&\lesssim \big(\|\tilde{\chi}_i(u,\nabla u)\|_{L^2_tH^{m-1}_{co}} + \|g_j-g_j(0)\|_{L^2_tH^{m-1}_{co}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{E^m_t}, \qquad j=1,2,
\end{aligned}\tag{6.3.39}$$

where $\tilde{\chi}_i$ is a cut-off function supported on the vicinity of $\Omega_i$ and $\tilde{\chi}_i\chi_i=\chi_i$. Therefore, since $\phi\partial_z$ can be spanned by $Z_1^i, Z_2^i, Z_3^i$, it follows from (6.3.38), (6.3.39), (6.7.3), (6.7.6) that:

$$\begin{aligned}
&\varepsilon\|\chi_i[Z^I, (g_1u\cdot\mathbf{N})/\phi]\phi\partial_z\nabla\sigma\|_{L^2(Q_t)} \\
&\lesssim \|(\nabla\sigma, (g_1u\cdot\mathbf{N})/\phi)\|_{L^2_tH^{m-1}_{co}}\Lambda\big(\frac{1}{c_0}, \|\!|\nabla\sigma\|\!|_{[\frac{m-1}{2}],\infty,t} + \varepsilon\|\!|\tilde{\chi}_i(g_1u\cdot\mathbf{N})/\phi\|\!|_{[\frac{m}{2}],\infty,t}\big) \\
&\lesssim \|(\sigma,u)\|_{E^m_t}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).
\end{aligned}\tag{6.3.40}$$

Moreover, one gets by induction that (up to some coefficients that depend only on $\phi$ and its derivatives)

$$[Z^I,\phi](\partial_z f) = \sum_{|\tilde{I}|\le|I|-1} *_{\tilde{I}}Z^{\tilde{I}}(\phi\partial_z f), \qquad [Z^I,\partial_z] = \sum_{|\tilde{I}|\le|I|-1} *_{\tilde{I}}\partial_z Z^{\tilde{I}} \tag{6.3.41}$$

Hence, by (6.3.38), the last two terms in (6.3.36) can be controlled by $\|\nabla\sigma\|_{L_t^2H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)$, which, together with (6.3.37), (6.3.40) leads to:

$$\varepsilon\|\chi_i[Z^I\nabla, g_1 u\cdot\nabla]\sigma\|_{L^2(Q_t)} \lesssim \|(\sigma,u)\|_{E_t^m}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big). \tag{6.3.42}$$

We switch to the estimate of the third term of $\mathcal{C}_\sigma^I$ defined in (6.3.29), which is nontrivial only if $Z^I$ contains at least one spatial derivative, that is $|I'|\geq 1$. By induction, one has that (up to some coefficients which are regular enough)

$$[Z^I,\nabla\mathrm{div}] = \sum_{|\tilde{I}|\leq|I|-1,|\tilde{\tilde{I}}|\leq|I|-1}\sum_{j,k=1}^{3} *_{jk\tilde{I}}\partial_{jk}^2 Z^{\tilde{I}} + *_{j\tilde{I}}\partial_j Z^{\tilde{\tilde{I}}},$$

which yields that:

$$\frac{1}{\varepsilon}\|[Z^I,\nabla\mathrm{div}]u\|_{L^2(Q_t)} \lesssim \frac{1}{\varepsilon}(\|\nabla^2 u\|_{L_t^2H_{co}^{m-2}}+\|\nabla u\|_{L_t^2H_{co}^{m-2}}).$$

To summarize, we have thus obtained from the above estimates that:

$$\varepsilon\|\mathcal{C}_\sigma^I\|_{L^2(Q_t)} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{E_t^m} + \|(\nabla^2 u,\nabla u)\|_{L_t^2H_{co}^{m-2}}. \tag{6.3.43}$$

By using the same argument, $\mathcal{C}_u^I$ (defined in (6.3.29)) can be controlled as follows:

$$\varepsilon\|\mathcal{C}_u^I\|_{L^2(Q_t)} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{E_t^m} + \varepsilon\|\nabla^2 u\|_{L_t^2H_{co}^{m-2}}. \tag{6.3.44}$$

Plugging (6.3.31) (6.3.32) (6.3.34) (6.3.43) (6.3.44) in (6.3.30), we arrive at

$$\begin{aligned}
&\varepsilon^2\big(\|((\nabla\sigma)^I,\mathrm{div}u^I)(t)\|_{L^2(\Omega)}^2 + \|\nabla\mathrm{div}u^I\|_{L^2(Q_t)}^2\big)\\
&\lesssim \varepsilon^2\|((\nabla\sigma)^I,\mathrm{div}u^I)(0)\|_{L^2(\Omega)}^2 + \varepsilon^{\frac{2}{3}}\Lambda_{2,\infty,t}\|(\sigma,u)\|_{E_t^m}^2\\
&\quad + T^{\frac12}\|\varepsilon\nabla^2 u\|_{L_t^\infty H_{co}^{m-2}}(\|\varepsilon\nabla^2 u\|_{L_t^2H_{co}^{m-2}} + \|\nabla\sigma\|_{L_t^2H_{co}^{m-1}}).
\end{aligned} \tag{6.3.45}$$

We thus get (6.3.27) by summing up (6.3.45) for $0\leq|I|\leq m-1$. □

### 6.3.3 Step 2: Energy estimate for the incompressible part of velocity

In this subsection, we focus on the estimates of the incompressible part of the velocity $v=\mathbb{P}u$ which solves $(6.3.5)_3$.

In the following, we recall for convenience the definition of the $L_{t,x}^\infty$ norm:

$$\begin{aligned}
\mathcal{A}_{m,t} = \Lambda\big(\frac{1}{c_0}, &\|\nabla u\|_{0,\infty,t} + \|(u,\sigma)\|_{[\frac{m+1}{2}],\infty,t} + \|(\nabla\sigma,\mathrm{div}u,\varepsilon^{\frac12}\nabla u)\|_{[\frac{m-1}{2}],\infty,t}\\
&+ \|\varepsilon\nabla u\|_{[\frac{m+1}{2}],\infty,t} + \varepsilon\|(\sigma,u)\|_{[\frac{m+3}{2}],\infty,t}\big)
\end{aligned} \tag{6.3.46}$$

for some polynomial $\Lambda$ which may still differ from line to line.

We begin with some additional estimates on $\nabla\mathrm{div}u$ :

**Lemma 6.3.5.** *Suppose that* (6.2.2) *holds then for any* $0<t\leq T\leq 1$.

$$\|\nabla\mathrm{div}u\|_{L_t^2H_{co}^{m-2}} \lesssim \|\nabla\sigma\|_{L_t^2H_{co}^{m-1}} + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{E_t^m}, \tag{6.3.47}$$

$$\varepsilon\|\nabla\mathrm{div}u(t)\|_{H_{co}^{m-2}} \lesssim \varepsilon\|\nabla\sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}, \tag{6.3.48}$$

$$\|\nabla\mathrm{div}u(t)\|_{H_{co}^{m-3}} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{6.3.49}$$

*Proof.* By the equation for $\sigma$, we have that:

$$\nabla \text{div} u = g_1(0)\varepsilon\partial_t\nabla\sigma + \varepsilon\nabla\big(\frac{g_1(\varepsilon\sigma)-g_1(0)}{\varepsilon}\varepsilon\partial_t\sigma + g_1(\varepsilon\sigma)u\cdot\nabla\sigma\big). \tag{6.3.50}$$

we can control $\varepsilon\nabla\text{div}u$ as follows, for $p=2,+\infty$,

$$\|\nabla\text{div}u\|_{L_t^pH_{co}^{m-2}} \lesssim \|\nabla\sigma\|_{L_t^pH_{co}^{m-1}} + \varepsilon\|\nabla\big((g_1-g_1(0))\partial_t\sigma, g_1u\cdot\nabla\sigma\big)(t)\|_{L_t^pH_{co}^{m-2}} \tag{6.3.51}$$

Inequalities (6.3.47)-(6.3.48) can thus be derived from the following estimate:

$$\begin{aligned}&\varepsilon\|\nabla\big((g_1-g_1(0))\partial_t\sigma, g_1u\cdot\nabla\sigma\big)(t)\|_{L_t^pH_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\big(\|\varepsilon\nabla(\sigma,u)\|_{L_t^pH_{co}^{m-1}} + \varepsilon^{\frac12}\|(\sigma,u,\nabla\sigma,\nabla u)\|_{L_t^pH_{co}^{m-2}}\big).\end{aligned}$$

Let us show the estimate of the term $g_2u\cdot\nabla\nabla\sigma$, the other terms can be controlled in a similar way. Again, we focus only on the estimate near the boundary. Thanks to the identity (6.3.35), we have

$$\chi_i g_1 u\cdot\nabla\nabla\sigma = \chi_i g_1 u_y\cdot\partial_y\nabla\sigma + \chi_i g_1\frac{u\cdot\mathbf{N}}{\phi}\phi\partial_z\nabla\sigma.$$

Therefore, by applying the product estimate (6.7.2) and inequality (6.3.38), we find

$$\begin{aligned}\varepsilon\|\chi_i(u\cdot\nabla\nabla\sigma)\|_{L_t^pH_{co}^{m-2}} &\lesssim \varepsilon\|(u_y,\chi_i u\cdot\mathbf{N}/\phi)\|_{L_t^pH_{co}^{m-2}}\|g_1Z\nabla\sigma\|_{[\frac{m}{2}]-2,\infty,t}\\ &\quad+\varepsilon\|g_1Z\nabla\sigma(t)\|_{L_t^pH_{co}^{m-3}}\|(u_y,\chi_iu\cdot\mathbf{N}/\phi)\|_{[\frac{m-1}{2}],\infty,t}\\ &\quad+\|\varepsilon g_1Z\nabla\sigma(t)\|_{L_t^pH_{co}^{m-2}}\|(u_y,\chi_iu\cdot\mathbf{N}/\phi)\|_{\infty,t}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)(\|\varepsilon\nabla\sigma\|_{L_t^pH_{co}^{m-1}} + \varepsilon\|(u,\nabla\sigma,\nabla u)\|_{L_t^pH_{co}^{m-2}}).\end{aligned} \tag{6.3.52}$$

Finally, one gets (6.3.49) by using similar arguments as in the derivation of (6.3.48), we skip the details. □

**Remark 6.3.6.** *By* (6.3.7) *and* (6.3.48), *we have that:*

$$\varepsilon\|\nabla\text{div}u\|_{L_t^\infty H_{co}^{m-2}} \lesssim Y_m(\sigma_0,u_0) + (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{6.3.53}$$

**Lemma 6.3.7.** *Let*

$$f = -\frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t u - g_2u\cdot\nabla u \tag{6.3.54}$$

*and assume that* (6.2.2) *holds, then we have:*

$$\|f\|_{L_t^2H_{co}^{m-1}} + \|f\|_{L_t^\infty H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{6.3.55}$$

*Proof.* Since the the higher order $L_{t,x}^\infty$ norm of $\partial_{\mathbf{n}}u$ is not included in the definition of $\mathcal{A}_{m,t}$, we need to use again the fact that $u\cdot\mathbf{n}$ vanishes on the boundary. More precisely, by using the product estimate (6.7.1) and identity (6.3.35) and estimate (6.3.39), we get for $(p,k)=(2,1),(\infty,2)$,

$$\|g_2u\cdot\nabla u\|_{L_t^pH_{co}^{m-k}} \lesssim \|(\sigma,u,\nabla\sigma,\nabla u)\|_{L_t^pH_{co}^{m-k}}\Lambda\big(\frac{1}{c_0},\|(\nabla\sigma,\text{div}u)\|_{[\frac{m-1}{2}],\infty,t} + \|(\sigma,u)\|_{[\frac{m+1}{2}],\infty,t}\big).$$

The first term is a direct application of the product estimate (6.7.1), we omit the detail. □

We split the estimate for $v$ in the following three subsections.

#### 6.3.3.1 Estimate of $\nabla q$

We first give the estimate of $\nabla q$ that appears in $(6.3.5)_3$. Since $q$ is governed by the elliptic equation (6.3.6) without singular terms, it can be easily estimated by standard elliptic regularity theory.

**Lemma 6.3.8.** *Under the assumptions* (6.2.2), *we have the following estimates: for* $j+l\leq m-1, l\geq 1$,

$$\|\nabla q\|_{L_t^2\mathcal{H}^{j,l}}+\varepsilon^{\frac{1}{2}}\|\nabla q\|_{L_t^2\mathcal{H}^{m-1}}\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \tag{6.3.56}$$

*where* $E_t^m$ *is defined in* (6.3.17). *Moreover,*

$$\varepsilon\|\operatorname{curl}\omega(t)\|_{H_{co}^{m-2}}+\varepsilon\|\nabla q(t)\|_{H_{co}^{m-2}}\lesssim \|v(t)\|_{H_{co}^{m-1}}+Y_m(\sigma_0,u_0)+(T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{6.3.57}$$

*Proof.* Recall that $q$ is governed by (6.3.6), an elliptic equation with Neumann boundary conditions. We can apply (6.7.15) in the appendix by setting

$$f=-\frac{g_2-\bar{\rho}}{\varepsilon}\varepsilon\partial_t u-g_2u\cdot\nabla u,\quad g=\mu\Delta v\cdot\mathbf{n}$$

to get

$$\|\nabla q\|_{L_t^2\mathcal{H}^{j,l}}\lesssim \big\|f\big\|_{L_t^2H_{co}^{m-1}}+\sum_{|I|\leq m-1}\big|Z^I(\Delta v\cdot\mathbf{n})\big|_{L_t^2H^{-\frac{1}{2}}(\partial\Omega)} \tag{6.3.58}$$

The first term in the right hand side has been controlled in (6.3.55), it remains to estimate the boundary term. By using the identity

$$(\nabla\times a)\cdot b=\nabla\cdot(a\times b)+a\cdot(\nabla\times b), \tag{6.3.59}$$

we have that:

$$-\Delta v\cdot\mathbf{n}=(\nabla\times\omega)\cdot\mathbf{n}=\operatorname{div}(\omega\times\mathbf{n})+\omega\cdot\operatorname{curl}\mathbf{n}.$$

Near the boundary, it follows from (6.3.22) that:

$$\begin{aligned}\operatorname{div}(\omega\times\mathbf{n})&=\partial_{\mathbf{n}}(\omega\times\mathbf{n})\cdot\mathbf{n}+(\Pi\partial_{y^1}(\omega\times\mathbf{n}))^1+(\Pi\partial_{y^2}(\omega\times\mathbf{n}))^2\\&=-(\omega\times\mathbf{n})\cdot\partial_{\mathbf{n}}\mathbf{n}+(\Pi\partial_{y^1}(\omega\times\mathbf{n}))^1+(\Pi\partial_{y^2}(\omega\times\mathbf{n}))^2.\end{aligned} \tag{6.3.60}$$

Therefore, by using the boundary condition (6.3.33), one has that for $|I|\leq m-1$,

$$\big|Z^I(\operatorname{div}(\omega\times\mathbf{n}))\big|_{L_t^2H^{-\frac{1}{2}}(\partial\Omega)}\lesssim |u|_{L_t^2\tilde{H}^{m-\frac{1}{2}}(\partial\Omega)} \tag{6.3.61}$$

where $L_t^2\tilde{H}^s(\partial\Omega)$ is defined in (6.1.10). In view of the identity (6.3.22) and the boundary condition (6.1.14), we have for $l\geq 1$

$$\begin{aligned}\big|Z^I\omega\big|_{L_t^2H^{-\frac{1}{2}}(\partial\Omega)}\lesssim |u|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}+\big|Z^I(\partial_{\mathbf{n}}u)\big|_{L_t^2H^{-\frac{1}{2}}}&\lesssim |u|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}+\big|Z^I\operatorname{div}u\big|_{L_t^2H^{-\frac{1}{2}}}\\&\lesssim \|u\|_{E_t^m}+\|\nabla\operatorname{div}u\|_{L_t^2H_{co}^{m-2}}.\end{aligned} \tag{6.3.62}$$

Moreover, if $Z^I=(\varepsilon\partial_t)^{m-1}$, we have by $L^2(\partial\Omega)\hookrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ and the trace inequality (6.7.11)

$$\varepsilon^{\frac{1}{2}}\big|Z^I\operatorname{div}u\big|_{L_t^2H^{-\frac{1}{2}}}\lesssim \|(\operatorname{div}u,\varepsilon\nabla\operatorname{div}u)\|_{L_t^2\mathcal{H}^{m-1}} \tag{6.3.63}$$

Collecting (6.3.58)-(6.3.63), and using (6.3.47), (6.3.55), one obtains that:

$$\begin{aligned}&\|\nabla q\|_{L_t^2\mathcal{H}^{j,l}}+\varepsilon^{\frac{1}{2}}|\nabla q\|_{L_t^2\mathcal{H}^{m-1}}\\&\lesssim \|f\|_{L_t^2H_{co}^{m-1}}+\|u\|_{E_t^m}+\|\nabla\operatorname{div}u\|_{L_t^2H_{co}^{m-2}}+\varepsilon\|\nabla\operatorname{div}u\|_{L_t^2H_{co}^{m-1}}\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.\end{aligned}$$

We are now ready to prove (6.3.57). By using the equation $(6.3.5)_3$, the elliptic estimate (6.7.15) and the product estimate (6.7.1), one finds:

$$\begin{aligned}&\varepsilon\|\Delta v(t)\|_{H_{co}^{m-2}}+\varepsilon\|\nabla q(t)\|_{H_{co}^{m-2}}\\&\lesssim \|v(t)\|_{H_{co}^{m-1}}+\varepsilon\big\|f(t)\big\|_{H_{co}^{m-2}}+\varepsilon\sum_{|I|\leq m-2}\big|Z^I(\Delta v\cdot\mathbf{n})(t)\big|_{H^{-\frac{1}{2}}(\partial\Omega)}\end{aligned} \tag{6.3.64}$$

With the aid of the boundary condition (6.1.14), the identities (6.3.22), (6.3.60) and the estimates (6.3.7), (6.3.53), the boundary term can be treated as,

$$
\begin{aligned}
&\varepsilon \sum_{|I|\le m-2} |Z^I(\Delta v\cdot \mathbf{n})|_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\lesssim \varepsilon(\|\nabla u(t)\|_{H_{co}^{m-2}} + \|u(t)\|_{H_{co}^{m-1}}) + \varepsilon\|\nabla \mathrm{div} u(t)\|_{H_{co}^{m-2}} \\
&\lesssim Y_m(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.
\end{aligned}
\tag{6.3.65}
$$

Combined with (6.3.64) and the fact that $\Delta v = -\,\mathrm{curl}\,\omega$, this yields (6.3.57). $\square$

#### 6.3.3.2 High order regularity estimates for $v$

This subsection is devoted to the high order estimates for $v$ : $\|v\|_{L_t^\infty H_{co}^{m-1}}, \|\nabla v\|_{L_t^2 H_{co}^{m-1}}$.

**Lemma 6.3.9.** *Suppose that* (6.2.2) *is satisfied, then for any $j+l\le m-1, j,l\ge 0$ and for every $0<t\le T$, the following a-priori estimate holds:*

$$
\begin{aligned}
&\|v\|_{L_t^\infty\mathcal{H}^{j,l}}^2 + \varepsilon^2\|\nabla v\|_{L_t^\infty\mathcal{H}^{j,l}}^2 + \|\nabla v\|_{L_t^2\mathcal{H}^{j,l}}^2 + \varepsilon^2\|\,\mathrm{curl}\,\omega\|_{L_t^2\mathcal{H}^{j,l}}^2 \\
&\lesssim Y_m^2(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda_{2,\infty,T}\mathcal{E}_{m,T}^2 + \|\mathrm{div} u\|_{L_t^2\mathcal{H}^{j,l}\cap L_t^2\mathcal{H}^{j+1,l-1}}^2
\end{aligned}
\tag{6.3.66}
$$

*where we use the notation* (6.1.8).

**Remark 6.3.10.** *The estimate* (6.3.66) *will be used later (see Lemma 6.3.11) to get the high order spatial regularity for* $\mathrm{div} u$, *which in turn, together with* (6.3.66), *gives the control of* $v$.

*Proof.* In view of (6.1.22), (6.3.7), it suffices to show that the left hand side of (6.3.66) can be controlled by:

$$
C(1/c_0)\big(Y_m^2(\sigma_0,u_0) + \mathcal{W}_{m,T}^2 + \|\mathrm{div} u\|_{L_t^2\mathcal{H}^{j,l}\cap L_t^2\mathcal{H}^{j+1,l-1}}^2\big)
$$

where:

$$
\mathcal{W}_{m,T}^2 = \|u\|_{L_t^\infty\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{L_t^2\mathcal{H}^{m-1}}^2 + \varepsilon^2\|\nabla u\|_{L_t^2 H_{co}^m}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.
\tag{6.3.67}
$$

This estimate will be obtained as the direct consequence of the following three inequalities:

$$
\|v\|_{L_t^\infty\mathcal{H}^{m-1}}^2 + \|\nabla v\|_{L_t^2\mathcal{H}^{m-1}}^2 \lesssim \|u\|_{L_t^\infty\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{L_t^2\mathcal{H}^{m-1}}^2,
\tag{6.3.68}
$$

$$
\begin{aligned}
\|v\|_{L_t^\infty\mathcal{H}^{j,l}}^2 + \|\nabla v\|_{L_t^2\mathcal{H}^{j,l}}^2 &\lesssim \|v(0)\|_{H_{co}^{m-1}}^2 + \|\nabla u\|_{L_t^2\mathcal{H}^{m-1}}^2 \\
&\quad + \|\mathrm{div} u\|_{L_t^2\mathcal{H}^{j,l}}^2 + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2,\ l\ge 1,
\end{aligned}
\tag{6.3.69}
$$

$$
\begin{aligned}
\varepsilon^2\|\nabla v\|_{L_t^\infty\mathcal{H}^{j,l}}^2 + \varepsilon^2\|\Delta v\|_{L_t^2\mathcal{H}^{j,l}}^2 &\lesssim \varepsilon^2\|(\nabla v, v)(0)\|_{H_{co}^{m-1}} + \|\nabla v\|_{L_t^2\mathcal{H}^{j,l}\cap L_t^2\mathcal{H}^{j+1,l-1}}^2 \\
&\quad + \varepsilon^2\|\nabla u\|_{L_t^2 H_{co}^m}^2 + (T^{\frac{1}{2}}+\varepsilon)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.
\end{aligned}
\tag{6.3.70}
$$

Note that since the Leray projector $\mathbb{P}$ commutes with $\varepsilon\partial_t$, one has that: $\mathbb{P}((\varepsilon\partial_t)^j u) = (\varepsilon\partial_t)^j v$. Therefore, from the continuity of the projection, we have:

$$
\|v(0)\|_{H_{co}^{m-1}} \lesssim \|u(0)\|_{H_{co}^{m-1}}.
$$

The inequality (6.3.68) is a direct consequence of the definition of $v$ and the elliptic estimates in Proposition 6.7.6. We thus focus on the other two inequalities. Let us first prove (6.3.69) and then sketch the proof of (6.3.70). By (6.1.25), $v$ solves

$$
\bar\rho\partial_t v - \mu\Delta v + \nabla q = -\big(\frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t u + g_2 u\cdot\nabla u\big) =: f
\tag{6.3.71}
$$

supplemented with the boundary conditions:

$$v\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \Pi(\partial_{\mathbf{n}}v)=\Pi(-2av+D\mathbf{n}\cdot v)+2\Pi(-a\nabla\Psi+D\mathbf{n}\cdot\nabla\Psi). \tag{6.3.72}$$

We apply $Z^I$ to the equation (6.3.71) with $I=(j,I'), 0\le j+|I'|=j+l=k\le m-1, |I'|\ge 1$. Taking the scalar product by $Z^I v$, and then integrating in space and time, we get that:

$$\begin{aligned}\frac{1}{2}\bar\rho\int_\Omega |Z^I v(t)|^2\,\mathrm{d}x \le \frac{1}{2}\bar\rho\int_\Omega |(Z^I v)(0)|^2\,\mathrm{d}x+\mu\iint_{Q_t} Z^I(\Delta v)Z^I v\,\mathrm{d}x\mathrm{d}s\\ +\|Z^I v\|_{L^2(Q_t)}\big(\|\nabla q\|_{L_t^2\mathcal{H}^{j,l}}+\|f\|_{L_t^2H_{co}^{m-1}}\big).\end{aligned} \tag{6.3.73}$$

By (6.3.55) and (6.3.56), the second line in the above inequality can be bounded as:

$$\begin{aligned}\|Z^I v\|_{L^2(Q_t)}\big(\|\nabla q\|_{L_t^2\mathcal{H}^{j,l}}+\|f\|_{L_t^2H_{co}^{m-1}}\big)&\lesssim T^{\frac12}\|u\|_{L_t^\infty H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}\\ &\lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.\end{aligned} \tag{6.3.74}$$

It remains to control the second term in the right hand side of (6.3.73), which is the following task. We split it into three terms:

$$\begin{aligned}\mu\iint_{Q_t} Z^I(\Delta v)\cdot Z^I v\,\mathrm{d}x\mathrm{d}s=\mu\iint_{Q_t}[Z^I,\mathrm{div}]\nabla v\cdot Z^I v\,\mathrm{d}x\mathrm{d}s-\mu\iint_{Q_t} Z^I\nabla v\cdot\nabla Z^I v\,\mathrm{d}x\mathrm{d}s\\ +\mu\int_0^t\int_{\partial\Omega} Z^I\nabla v\cdot\mathbf{n}\,Z^I v\,\mathrm{d}S_y\mathrm{d}s=:\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3.\end{aligned} \tag{6.3.75}$$

The estimate of $\mathcal{T}_1-\mathcal{T}_3$ will be similar to that of $\mathcal{K}_1-\mathcal{K}_4$ in (6.3.20).

We first estimate $\mathcal{T}_2$. By integrating by parts, one has that:

$$\begin{aligned}\mathcal{T}_2&=-\mu\iint_{Q_t}|Z^I\nabla v|^2\,\mathrm{d}x\mathrm{d}s-\mu\iint_{Q_t} Z^I\nabla v[\nabla,Z^I]v\,\mathrm{d}x\mathrm{d}s\\ &\le-\frac{\mu}{2}\|Z^I\nabla v\|_{L^2(Q_t)}^2+\frac{\mu}{2}\|[\nabla,Z^I]v\|_{L^2(Q_t)}^2\le-\frac{\mu}{2}\|Z^I\nabla v\|_{L^2(Q_t)}^2+C\|\nabla v\|_{L_t^2\mathcal{H}^{j,l-1}}^2.\end{aligned} \tag{6.3.76}$$

Note that in the last estimate, by (6.3.19), we know that $[\nabla,Z^I]v$ involves only lower order ($\le k-1$) conormal derivatives of $\nabla v$.

We now switch to the estimate of the boundary term $\mathcal{T}_3$ in (6.3.75), which vanishes if $Z^I$ involves at least one weighted normal derivative $Z_3^i$. We thus assume that $Z^I$ contains only time derivatives and spatial tangential derivatives.

$$\begin{aligned}\mathcal{T}_3=-\mu\int_0^t\int_{\partial\Omega}\big(-[Z^I,\mathbf{n}]\nabla v\cdot Z^I v+[Z^I,\mathbf{n}]\cdot\partial_{\mathbf{n}}v(Z^I v\cdot\mathbf{n})+[Z^I,\Pi]\partial_{\mathbf{n}}v\cdot\Pi Z^I v\big)\,\mathrm{d}S_y\mathrm{d}s\\ +\mu\int_0^t\int_{\partial\Omega}\big(Z^I(\partial_{\mathbf{n}}v\cdot\mathbf{n})(Z^I v\cdot\mathbf{n})+Z^I(\Pi\partial_{\mathbf{n}}v)\cdot\Pi Z^I v\big)\,\mathrm{d}S_y\mathrm{d}s=:\mathcal{T}_{31}+\mathcal{T}_{32}.\end{aligned}$$

The first term $\mathcal{T}_{31}$ can be dealt with thanks to Hölder inequality and the trace inequality (6.7.11)

$$\begin{aligned}\mathcal{T}_{31}&\lesssim\int_0^t|(\varepsilon\partial_t)^j\nabla v(s)|_{H^{l-1}(\partial\Omega)}|Z^I v(s)|_{L^2(\partial\Omega)}\,\mathrm{d}s\\ &\lesssim\int_0^t(|(\varepsilon\partial_t)^j v|_{H^l(\partial\Omega)}+|(\varepsilon\partial_t)^j\nabla\Psi|_{H^l(\partial\Omega)})|Z^I v|_{L^2(\partial\Omega)}\,\mathrm{d}s\\ &\le\delta\mu\|\nabla v\|_{L_t^2\mathcal{H}^{j,l}}^2+C(\delta,\mu)\|(u,\mathrm{div}u)\|_{L_t^2H_{co}^k}^2.\end{aligned}$$

Note that in the second inequality, we have used the boundary condition (6.3.72) and the identity (since $\mathrm{div}v=0$):

$$\partial_{\mathbf{n}}v\cdot\mathbf{n}=-(\Pi\partial_{y^1}v)^1-(\Pi\partial_{y^2}v)^2, \tag{6.3.77}$$

to obtain that:

$$|(\varepsilon\partial_t)^j\nabla v(s)|_{H^{l-1}}\lesssim|(\varepsilon\partial_t)^j v(s)|_{H^l}+|(\varepsilon\partial_t)^j\nabla\Psi(s)|_{H^l}. \tag{6.3.78}$$

For the second term $\mathcal{T}_{32}$, since $l\geq 1$, we might as well assume that $Z^I=\partial_y Z^{\tilde{I}}$, where $\partial_y=\partial_{y^1}$ or $\partial_{y^2}$. In view of the boundary condition (6.3.72) and the identity (6.3.77), we have by integrating by parts along the boundary that:

$$\begin{aligned}\mathcal{T}_{32}&=\int_0^t\int_{\partial\Omega}Z^{\tilde{I}}(\partial_{\mathbf{n}}v\cdot\mathbf{n})\partial_y\cdot([Z^I,\mathbf{n}\cdot]v)+Z^I(\Pi\partial_{\mathbf{n}}v)\Pi Z^I v)\,\mathrm{d}S_y\mathrm{d}s\\&\lesssim\int_0^t|(\varepsilon\partial_t)^j v|^2_{H^l(\partial\Omega)}+|(\varepsilon\partial_t)^j(v,\nabla\Psi)|_{H^l(\partial\Omega)}|(\varepsilon\partial_t)^j v|_{H^l(\partial\Omega)}\,\mathrm{d}s\\&\lesssim\delta\mu\|\nabla v\|^2_{L_t^2\mathcal{H}^{j,l}}+C(\delta,\mu)\|(u,\mathrm{div}u)\|^2_{L_t^2\mathcal{H}^{j,l}}.\end{aligned}\tag{6.3.79}$$

It remains to control $\mathcal{T}_1$. Owing to (6.3.19) and (6.3.78), one obtains again by integrating by parts that:

$$\begin{aligned}\mathcal{T}_1&\lesssim\|\nabla v\|_{L_t^2\mathcal{H}^{j,l-1}}(\|v\|_{L_t^2\mathcal{H}^{j,l}}+\|\nabla v\|_{L_t^2\mathcal{H}^{j,l}})+|(\varepsilon\partial_t)^j\nabla v(s)|_{H^{l-1}(\partial\Omega)}|v|_{H^l(\partial\Omega)}\\&\lesssim\delta\mu\|\nabla v\|^2_{L_t^2\mathcal{H}^{j,l}}+C(\delta,\mu)(\|(u,\mathrm{div}u)\|^2_{L_t^2\mathcal{H}^{j,l}}+\|\nabla v\|^2_{L_t^2\mathcal{H}^{j,l-1}}).\end{aligned}\tag{6.3.80}$$

Plugging (6.3.75)-(6.3.80) into (6.3.73) and summing up for all $I=(j,I'),|I'|=l$, one has by choosing $\delta$ small enough that

$$\begin{aligned}\|v(t)\|^2_{\mathcal{H}^{j,l}}+\frac{\mu}{4}\|\nabla v\|^2_{L_t^2\mathcal{H}^{j,l}}\leq\|v(0)\|^2_{\mathcal{H}^{j,l}}+C(\delta,\mu)\|\nabla v\|^2_{L_t^2\mathcal{H}^{j,l-1}}+\|\mathrm{div}u\|^2_{L_t^2\mathcal{H}^{j,l}}\\+T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.\end{aligned}\tag{6.3.81}$$

In view of inequalities (6.3.68) and (6.3.81), we obtain (6.3.69) by induction on $l$.

We are now in position to prove (6.3.70). As before, we apply $Z^I$ to the equation (6.3.71) for $v$ and we take the scalar product by $-\varepsilon^2 Z^I\Delta v$. One gets by integration by parts and by using Young's inequality that:

$$\begin{aligned}&\frac{1}{2}\bar{\rho}\varepsilon^2\int_\Omega|\nabla Z^I v(t)|^2\,\mathrm{d}x+\frac{\mu}{2}\varepsilon^2\iint_{Q_t}|Z^I(\Delta v)|^2\,\mathrm{d}x\mathrm{d}s\\&\leq\frac{1}{2}\bar{\rho}\varepsilon^2\int_\Omega|\nabla Z^I v(0)|^2\,\mathrm{d}x+\varepsilon\iint_{Q_t}\varepsilon\partial_t Z^I v\cdot[Z^I,\Delta]v\,\mathrm{d}x\mathrm{d}s\\&+\varepsilon\int_0^t\int_{\partial\Omega}\varepsilon\partial_t Z^I v\cdot\partial_{\mathbf{n}}Z^I v\,\mathrm{d}S_y\mathrm{d}s+C_\mu\varepsilon^2\|(\nabla q,f)\|^2_{L_t^2H_{co}^{m-1}}.\end{aligned}\tag{6.3.82}$$

By induction, the following identity (up to some coefficients that depends on $\phi,\varphi$ and their derivatives up to order $m$) holds:

$$[Z^I,\Delta]=\sum_{\substack{|\tilde{I}|\leq|I|-1,|J|\leq|I|-1\\ \tilde{I}_0=j,J_0=j}}\sum_{i,k=1}^3(*Z^{\tilde{I}}\partial^2_{ik}+*Z^J\partial_k).$$

This identity, combined with elliptic regularity theory yields:

$$\begin{aligned}\|[Z^I,\Delta]v\|_{L^2(Q_t)}\lesssim\|\nabla^2 v\|_{L_t^2\mathcal{H}^{j,l-1}}+\|\nabla v\|_{L_t^2\mathcal{H}^{j,l-1}}&\lesssim\|\Delta v\|_{L_t^2\mathcal{H}^{j,l-1}}+|\partial_{\mathbf{n}}(\varepsilon\partial_t)^j v|_{H^{l-\frac{1}{2}}}\\&\lesssim\|\Delta v\|_{L_t^2\mathcal{H}^{j,l-1}}+\|(u,\nabla u)\|_{L_t^2\mathcal{H}^{j,l}}.\end{aligned}$$

Note that in the last inequality, we have used (6.3.78) and the trace inequality (6.7.11). We thus control the second term in (6.3.82) as follows:

$$\varepsilon\iint_{Q_t}\varepsilon\partial_t Z^I v\cdot[Z^I,\Delta]v\,\mathrm{d}x\mathrm{d}s\lesssim\varepsilon^2\|\Delta v\|^2_{L_t^2\mathcal{H}^{j,l-1}}+\|\varepsilon\partial_t v\|^2_{L_t^2\mathcal{H}^{j,l}}+\varepsilon\|u\|^2_{E_t^m}.\tag{6.3.83}$$

Moreover, the third term of (6.3.82) can be dealt with by arguments very similar to the ones for $\mathcal{T}_3$ :

$$\begin{aligned}
&\varepsilon \int_0^t \int_{\partial\Omega} \varepsilon\partial_t Z^I v \cdot \partial_{\mathbf{n}} Z^I v \,\mathrm{d}S_y \mathrm{d}s \\
&\lesssim \varepsilon \int_0^t |Z^I \varepsilon\partial_t v|_{L^2} \big( |(\varepsilon\partial_t)^j v|_{H^{l+1}} + |(\varepsilon\partial_t)^j \nabla\Psi|_{H^l} \big) \,\mathrm{d}s \\
&\lesssim \varepsilon (\|\nabla v\|^{\frac{1}{2}}_{L_t^2\mathcal{H}^{j+1,l}} \|v\|^{\frac{1}{2}}_{L_t^2\mathcal{H}^{j+1,l}} + \|v\|_{L_t^2\mathcal{H}^{j+1,l}}) \cdot \\
&\qquad (\|\nabla v\|^{\frac{1}{2}}_{L_t^2\mathcal{H}^{j,l+1}} \|v\|^{\frac{1}{2}}_{L_t^2\mathcal{H}^{j,l+1}} + \|v\|_{L_t^2\mathcal{H}^{j,l+1}} + \|u, \mathrm{div} u\|_{L_t^2\mathcal{H}^{j,l}}) \\
&\lesssim \varepsilon^2 \|\nabla v\|^2_{L_t^2 H_{co}^m} + \varepsilon \|(u, \mathrm{div} u)\|^2_{L_t^2 H_{co}^{m-1}} + \|\nabla v\|^2_{L_t^2\mathcal{H}^{j+1,l-1} \cap L_t^2\mathcal{H}^{j,l}} + \|v\|^2_{L_t^2\mathcal{H}^m}
\end{aligned} \tag{6.3.84}$$

Inserting (6.3.83) and (6.3.84) into (6.3.82), and use (6.3.55), (6.3.56) to find

$$\varepsilon^2 \|(\nabla q, f)\|^2_{L_t^2 H_{co}^{m-1}} \lesssim \varepsilon \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \mathcal{E}^2_{m,t},$$

we obtain (6.3.70) by induction. □

### 6.3.4 Step 3: Uniform estimates for $(\nabla\sigma, \mathrm{div} u)$

In this subsection, we aim to get uniform control of higher spatial conormal derivatives of $(\nabla\sigma, \mathrm{div} u)$. More precisely, we prove uniform boundedness of $\|(\nabla\sigma, \mathrm{div} u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}$. This will be achieved by using the equation iteratively.

**Lemma 6.3.11.** *Assume that* (6.2.2) *holds, we then have that for every* $0 < t \le T$,

$$\|(\nabla\sigma, \mathrm{div} u)\|^2_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}} \lesssim Y_m^2(\sigma_0, u_0) + (T + \varepsilon)^{\frac{1}{2}} \mathcal{E}^2_{m,T} \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big). \tag{6.3.85}$$

*Proof.* We will prove the following two inequalities:

- $L_t^2 H_{co}^{m-1}$ estimate: for any $j, k \ge 0, j + k \le m - 1$:

$$\begin{aligned}
\|(\nabla\sigma, \mathrm{div} u)\|_{L_t^2\mathcal{H}^{j,k}} \lesssim{}& Y_m(\sigma_0, u_0) + T^{\frac{1}{2}} \|(u, \sigma)\|_{L_t^\infty\mathcal{H}^m} \\
&+ \varepsilon \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}} + (T + \varepsilon)^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \mathcal{E}_{m,T}.
\end{aligned} \tag{6.3.86}$$

- $L_t^\infty H_{co}^{m-2}$ estimate: for any $j, l \ge 0$ and $j + l \le m - 2$ :

$$\begin{aligned}
\|(\nabla\sigma, \mathrm{div} u)\|_{L_t^\infty\mathcal{H}^{j,l}} \lesssim{}& Y_m(\sigma_0, u_0) + \varepsilon \|(\nabla \mathrm{div} u, \mathrm{curl}\, \omega)\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}} \\
&+ \|(\sigma, u)\|_{L_t^\infty\mathcal{H}^{m-1}} + \varepsilon \|\nabla\sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \mathcal{E}_{m,T}.
\end{aligned} \tag{6.3.87}$$

These two inequalities, combined with the estimates (6.3.7), (6.3.57), (6.3.66) and the definition (6.3.46), yield (6.3.85).

The inequality (6.3.86) can be obtained by induction on the number of space conormal derivatives. Let us first prove (6.3.86) for $k = 0, j \le m - 1$. By (6.3.50) and product estimate (6.7.1), we find that:

$$\|\mathrm{div} u\|_{L_t^2\mathcal{H}^{m-1}} \lesssim T^{\frac{1}{2}} \|\sigma\|_{L_t^\infty\mathcal{H}^m} + \varepsilon \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \mathcal{E}_{m,t}. \tag{6.3.88}$$

Moreover, by the equations $(6.1.15)_2$ for $u$,

$$\nabla\sigma = -\bar{\rho}\varepsilon\partial_t u + \varepsilon f - \varepsilon\mu\, \mathrm{curl}\, \omega + \varepsilon(2\mu + \lambda)\nabla \mathrm{div} u, \tag{6.3.89}$$

we thus have by (6.3.55), (6.3.66) that:

$$\begin{aligned}\|\nabla\sigma\|_{L_t^2\mathcal{H}^{m-1}} &\lesssim \|u\|_{L_t^2\mathcal{H}^m} + \varepsilon\|\operatorname{curl}\omega\|_{L_t^2\mathcal{H}^{m-1}} + \varepsilon\|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \\ &\lesssim T^{\frac{1}{2}}\|u\|_{L_t^\infty\mathcal{H}^m} + \|\mathrm{div} u\|_{L_t^2\mathcal{H}^{m-1}} + Y_m(\sigma_0, u_0) \\ &\quad + \varepsilon\|\nabla\mathrm{div} u\|_{L_t^2 H_{co}^{m-1}} + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t},\end{aligned} \tag{6.3.90}$$

which, together with (6.3.88), yields (6.3.86) for $k=0, j\le m-1$.

Now suppose that (6.3.86) holds for $k=k_0-1$ with $k_0\ge 1$, it suffices to prove that it is also true for $k=k_0$ and for every $j$ such that $j+k_0\le m-1$. We begin with the estimate of div$u$, which again follows from the equation (6.3.50) and product estimate (6.7.1):

$$\begin{aligned}\|\mathrm{div} u\|_{L_t^2\mathcal{H}^{j,k_0}} &\lesssim \|\varepsilon\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,k_0}} + \varepsilon\Lambda(\frac{1}{c_0}, \mathcal{A}_{m,t})\mathcal{E}_{m,t} \\ &\lesssim \|(\sigma,\nabla\sigma)\|_{L_t^2\mathcal{H}^{j+1,k_0-1}} + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \lesssim \text{R.H.S of (6.3.86)}.\end{aligned} \tag{6.3.91}$$

Next, one gets by equation (6.3.89), estimate (6.3.66) and the induction hypothesis that:

$$\begin{aligned}\|\nabla\sigma\|_{L_t^2\mathcal{H}^{j,k_0}} &\lesssim \|u\|_{L_t^2\mathcal{H}^{j+1,k_0}} + \varepsilon\|\operatorname{curl}\omega\|_{L_t^2\mathcal{H}^{j,k_0}} + \varepsilon\|\nabla\mathrm{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon\Lambda(\frac{1}{c_0}, \mathcal{A}_{m,T})\mathcal{E}_{m,t} \\ &\lesssim \|(\mathrm{div} u, \nabla v)\|_{L_t^2\mathcal{H}^{j+1,k_0-1}} + \varepsilon\|\operatorname{curl}\omega\|_{L_t^2\mathcal{H}^{j,k_0}} + \varepsilon\|\nabla\mathrm{div} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon\Lambda(\frac{1}{c_0}, \mathcal{A}_{m,T})\mathcal{E}_{m,t} \\ &\lesssim \text{R.H.S of (6.3.86)}.\end{aligned}$$

Let us switch to the proof of (6.3.87). By similar argument as in the derivation of (6.3.88), (6.3.90), one can find that:

$$\|(\nabla\sigma, \mathrm{div} u)\|_{L_t^\infty\mathcal{H}^{m-2}} \lesssim \|(\sigma,u)\|_{L_t^\infty\mathcal{H}^{m-1}} + \varepsilon\|(\nabla\mathrm{div} u, \operatorname{curl}\omega)\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}, \tag{6.3.92}$$

which proves (6.3.87) for $l=0$. Suppose that it is true for $l=l_0-1\le m-3$, we show that it also holds for $l=l_0$ and for any $j$, such that $j+l_0\le m-2$. Let us start with the estimate of div$u$. It follows from the equation (6.3.50), the product estimate (6.7.1) and the induction hypothesis that:

$$\begin{aligned}\|\mathrm{div} u\|_{L_t^\infty\mathcal{H}^{j,l_0}} &\lesssim \|\varepsilon\partial_t\sigma\|_{L_t^\infty\mathcal{H}^{j,l_0}} + \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \\ &\lesssim \|(\sigma,\nabla\sigma)\|_{L_t^\infty\mathcal{H}^{j+1,l_0-1}} + \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \\ &\lesssim \|\sigma\|_{L_t^\infty\mathcal{H}^{m-2}} + \|\nabla\sigma\|_{L_t^\infty\mathcal{H}^{j+1,l_0-1}} + \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t} \\ &\lesssim \text{R.H.S of (6.3.87)}.\end{aligned}$$

For the estimate of $\nabla\sigma$, we use the equation (6.3.89) and the product estimate (6.7.1) to obtain:

$$\begin{aligned}&\|\nabla\sigma\|_{L_t^\infty\mathcal{H}^{j,l_0}} \\ &\lesssim \|\varepsilon\partial_t u\|_{L_t^\infty\mathcal{H}^{j,l_0}} + \varepsilon\|(\nabla\mathrm{div} u, \operatorname{curl}\omega)\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.\end{aligned} \tag{6.3.93}$$

It remains to bound $\|\varepsilon\partial_t u\|_{L_t^\infty\mathcal{H}^{j,l_0}}$. We use that for $j+l_0\le m-2$,

$$\begin{aligned}\|\varepsilon\partial_t u\|_{L_t^\infty\mathcal{H}^{j,l_0}} &\lesssim \|v\|_{L_t^\infty H_{co}^{m-1}} + \|(\nabla\Psi, \nabla^2\Psi)\|_{L_t^\infty\mathcal{H}^{j+1,l_0-1}} \\ &\lesssim \|v\|_{L_t^\infty H_{co}^{m-1}} + \|(u, \mathrm{div} u)\|_{L_t^\infty\mathcal{H}^{j+1,l_0-1}} \\ &\lesssim \|u\|_{L_t^\infty\mathcal{H}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}} + \sum_{k=1}^{l_0}\|\mathrm{div} u\|_{L_t^\infty\mathcal{H}^{j+k,l_0-k}}.\end{aligned} \tag{6.3.94}$$

Plugging (6.3.48) and (6.3.94) into (6.3.93) and using the induction hypothesis, we get that:

$$\|\nabla\sigma\|_{L_t^\infty \mathcal{H}^{j,l_0}} \lesssim \text{R.H.S of (6.3.87)}.$$

We thus proved that (6.3.87) holds for $j+1, l_0$ which ends the proof. $\square$

**Remark 6.3.12.** *By Lemmas 6.3.9, 6.3.11, we get that:*

$$\|(\sigma,u)\|_{E_t^m}^2 \lesssim Y_m^2(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big). \tag{6.3.95}$$

### 6.3.5 Step 4: Uniform estimates for the gradient of the velocity.

In this section, we will bound $\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}$, which, combined with (6.3.3) (6.3.87), gives the control of $\|\nabla u\|_{L_t^\infty H_{co}^{m-2}}$.

**Lemma 6.3.13.** *Suppose that* (6.2.2) *holds, then for any* $0<t\le T$*, we have the following estimate,*

$$\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}^2 \lesssim Y_m^2(\sigma_0,u_0) + \|v\|_{L_t^\infty H_{co}^{m-1}}^2 + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big). \tag{6.3.96}$$

*Proof.* Since in the interior domain, the conormal spatial derivatives are equivalent to the standard spatial derivatives, we only have to estimate $\nabla v$ near the boundary, say $\|\chi_i\nabla v\|_{L_t^\infty H_{co}^{m-2}}$ where $\chi_i, (i=1\cdots N)$ are smooth functions associated to the covering (6.1.11) and are compactly supported in $\Omega_i$. Close to the boundary, it follows from the identity (6.3.77) and the following identity

$$\Pi(\partial_{\mathbf{n}}v) = \Pi((\nabla v - Dv)\mathbf{n}) + \Pi((Dv)\mathbf{n}) = \Pi(\omega\times\mathbf{n}) + \Pi(-(D\mathbf{n})v)$$

that:

$$\begin{aligned}\|\chi_i\nabla v\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \|\chi_i\Pi(\partial_{\mathbf{n}}v)\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}}\\ &\lesssim \|\chi_i(\omega\times\mathbf{n})\|_{L_t^\infty H_{co}^{m-2}} + \|v\|_{L_t^\infty H_{co}^{m-1}}.\end{aligned}$$

We thus reduce the problem to the estimate of $\chi_i(\omega\times\mathbf{n})$, which is the aim of the following lemma. $\square$

**Lemma 6.3.14.** *Under the assumption* (6.2.2)*, the following estimate holds: for every* $0<t\le T$,

$$\|\chi_i(\omega\times\boldsymbol{n})\|_{L_t^\infty H_{co}^{m-2}(\Omega)}^2 \lesssim \|\chi_i(\omega\times\boldsymbol{n})(0)\|_{H_{co}^{m-2}}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big). \tag{6.3.97}$$

*where* $\chi_i$ *is a smooth function compactly supported in* $\Omega_i$.

*Proof.* Note that the important feature of $\chi_i(\omega\times\mathbf{n})$ is that: it solves a transport-diffusion system **without** singular terms, with a non-homogeneous Dirichlet boundary condition. In order to perform the estimate, we split the system for $\chi_i(\omega\times\mathbf{n})$ into two parts, one which just solves the heat equation with the nontrivial Dirichlet boundary condition and a remainder which is amenable to energy estimates since it satisfies a convection-diffusion equation with homogeneous Dirichlet boundary condition. To deal with the first system, the explicit formula for heat equation will play an important role. It is thus helpful to transform the problem to the half-space.

Let us set $\eta_i = \chi_i\omega\times\mathbf{n}, i\ge 1$. Direct computations show that $\omega$ solves the following system:

$$g_2\partial_t\omega + g_2 u\cdot\nabla\omega - \mu\Delta\omega = g_2\omega\cdot\nabla u - g_2\omega\,\mathrm{div}u - \frac{\nabla g_2}{\varepsilon}\times(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u) =: G^\omega \tag{6.3.98}$$

from which we obtain the equations satisfied by $\eta_i$ (which is compactly supported in $\Omega_i$)

$$\begin{cases}\bar\rho\partial_t\eta_i - \mu\Delta\eta_i = F_i^\omega \quad \text{in} \quad \Omega_i\cap\Omega.\\ \eta_i = \chi_i\Pi(\omega\times\mathbf{n}) = 2\chi_i\Pi(-au + (D\mathbf{n})u) \quad \text{on} \quad \Omega_i\cap\partial\bar\Omega,\end{cases} \tag{6.3.99}$$

where

$$F_i^\omega =: \ -\Delta(\chi_i \mathbf{n}) \times \omega - 2\nabla\omega \times \nabla(\chi_i\mathbf{n}) - (g_2 u \cdot \nabla\omega) \times (\chi_i\mathbf{n}) + \frac{\bar{\rho}-g_2}{\varepsilon}\varepsilon\partial_t\omega \times (\chi_i\mathbf{n}) + G^\omega \times (\chi_i\mathbf{n}).$$

Since we will use the local coordinate (6.1.12), it is useful to know the expressions of Laplacian in this new coordinates. By direct computation, we find that:

$$(\nabla f)\circ\Phi_i = P\nabla(f\circ\Phi_i), \quad (\mathrm{div} F)\circ\Phi_i = \mathrm{div}\big(P^*(F\circ\Phi_i)\big) \quad (\Delta f)\circ\Phi_i = \mathrm{div}(E\nabla\big(f\circ\Phi_i)\big) \tag{6.3.100}$$

where $\nabla = (\partial_{y^1}, \partial_{y^2}, \partial_z)^t, \mathrm{div} = (\nabla)^*$ represent the gradient and the divergence in the new coordinates and

$$\begin{pmatrix} 1 & 0 & -\partial_{y^1}\varphi_i \\ 0 & 1 & -\partial_{y^2}\varphi_i \\ 0 & 0 & 1 \end{pmatrix}, \quad E = P^*P = \begin{pmatrix} 1 & 0 & -\partial_{y^1}\varphi_i \\ 0 & 1 & -\partial_{y^2}\varphi_i \\ -\partial_{y^1}\varphi_i & -\partial_{y^2}\varphi_i & |\mathbf{N}|^2 \end{pmatrix}. \tag{6.3.101}$$

Let us set $\tilde{\eta}_i(t,y,z) = \eta_i(t,\Phi_i(y,z)) := (\eta_i\circ\Phi_i)(y,z), (y,z)\in\Phi_i^{-1}(\Omega_i\cap\bar{\Omega})$. Denote also $\widetilde{F_i^\omega} = F_i^\omega\circ\Phi_i$. Since $\mathrm{Supp}\,\chi_i|_{\bar{\Omega}} \Subset \Omega_i\cap\bar{\Omega}$, We can extend the definition of $\tilde{\eta}_i$ and $\widetilde{F_i^\omega}$ from $\Phi_i^{-1}(\Omega_i\cap\bar{\Omega})$ to $\mathbb{R}_+^3$ by zero extension, which are still denoted by $\tilde{\eta}_i$, $\widetilde{F_i^\omega}$. Consequently, by (6.3.99) and (6.3.100), we find that $\tilde{\eta}_i$ satisfies:

$$\begin{cases} \bar{\rho}\partial_t\tilde{\eta}_i - \mu\mathrm{div}\big(E\nabla\tilde{\eta}_i\big) = F_{ni}^\omega \quad \text{in} \quad \mathbb{R}_+^3. \\ \tilde{\eta}_i|_{z=0} = 2[\chi_i\Pi(-au+(D\mathbf{n})u)]\circ\Phi_i\big|_{z=0}. \end{cases} \tag{6.3.102}$$

Let us set $\mathcal{Z}_0 = \varepsilon\partial_t$, $\mathcal{Z}_j = \partial_{y^j}, j=1,2$, $\mathcal{Z}_3 = \phi(z)\partial_z$ and define

$$\|\tilde{\eta}_i\|_{m,t} = \sum_{|\alpha|\le m}\|\mathcal{Z}^\alpha\tilde{\eta}_i\|_{L^2([0,t]\times\mathbb{R}_+^3)}, \quad \|\tilde{\eta}_i(t)\|_m = \sum_{|\alpha|\le m}\|(\mathcal{Z}^\alpha\tilde{\eta}_i)(t)\|_{L^2(\mathbb{R}_+^3)}. \tag{6.3.103}$$

where $\mathcal{Z}^\alpha = \mathcal{Z}_0^{\alpha_0}\mathcal{Z}_1^{\alpha_1}\mathcal{Z}_2^{\alpha_2}\mathcal{Z}_3^{\alpha_3}, \alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)$, by the definition of the conormal spaces (6.1.6) and the vector fields (6.1.13) we find that:

$$\|\tilde{\eta}_i\|_{m,t} \approx \|\eta_i\|_{L_t^2 H_{co}^m(\Omega)}, \quad \|\tilde{\eta}_i(t)\|_m \approx \|\eta_i(t)\|_{H_{co}^m(\Omega)}. \tag{6.3.104}$$

Therefore, our following task is to establish an estimate for $\sup_{0\le t\le T}\|\tilde{\eta}_i(t)\|_{m-2}$.

We shall write $\tilde{\eta}_i, \widetilde{F_i^\omega}$ by $\tilde{\eta}, \widetilde{F^\omega}$ for the sake of notational clarity. We write $\tilde{\eta} = \tilde{\eta}_h + \tilde{\eta}_{nh}$, where $\tilde{\eta}_h$ solves

$$\begin{cases} \bar{\rho}\partial_t\tilde{\eta}_h - \mu|\mathbf{N}|^2\partial_z^2\tilde{\eta}_h = 0 \quad \text{in} \quad \mathbb{R}_+^3, \\ \tilde{\eta}_h|_{t=0} = 0, \tilde{\eta}_h|_{z=0} = \tilde{\eta}|_{z=0} \end{cases} \tag{6.3.105}$$

while $\tilde{\eta}_{nh}$ satisfies

$$\begin{cases} \bar{\rho}\partial_t\tilde{\eta}_{nh} - \mu\mathrm{div}\big(E\nabla\tilde{\eta}_{nh}\big) = H(\tilde{\eta}_h) + F^\omega \quad \text{in} \quad \mathbb{R}_+^3, \\ \tilde{\eta}_{nh}|_{t=0} = \tilde{\eta}|_{t=0}, \tilde{\eta}_{nh}|_{z=0} = 0 \end{cases} \tag{6.3.106}$$

where

$$H(\tilde{\eta}_h) = \mu\sum_{i,j=1}^2\partial_{y^i}(E_{ij}\partial_{y^j}\tilde{\eta}_h) + \mu\sum_{i=1}^2\partial_{y^i}(E_{i3}\partial_z\tilde{\eta}_h) + \partial_z(E_{3i}\partial_{y^i}\tilde{\eta}_h).$$

Estimate (6.3.97) will be the consequence of the following two lemmas. □

**Lemma 6.3.15.** *Adopting the notation introduced in* (6.3.103)*, we have the following estimate: for any* $0 < t \le T$,

$$\sup_{0\le t\le T}\|\tilde{\eta}_h(t)\|_{m-2} + \|\tilde{\eta}_h\|_{m-1,T} \lesssim T^{\frac{1}{4}}\mathcal{E}_{m,T}. \tag{6.3.107}$$

*Proof.* Since $|\mathbf{N}|^2$ depends only on the tangential variable $y^1, y^2$, the equation (6.3.105) can be seen as a heat equation on the half line with Dirichlet boundary condition, which can be solved explicitly:

$$\tilde{\eta}_h(t,y,z) = -2\tilde{\mu}\int_0^t \frac{|\mathbf{N}|^2}{\big(4\pi\tilde{\mu}|\mathbf{N}|^2(t-s)\big)^{\frac{1}{2}}}\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2(t-s)}}\big)\tilde{\eta}|_{z=0}(s,y)\mathrm{d}s$$

where $\tilde{\mu} = \mu/\bar{\rho}$. Taking a multi-index $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, since time derivation commutes with $\partial_t, \partial_z^2$, we have that:

$$\big((\varepsilon\partial_t)^{\gamma_0}\tilde{\eta}_h\big)(t,y,z) = -2\tilde{\mu}\int_0^t \frac{|\mathbf{N}|^2}{\big(4\pi\tilde{\mu}|\mathbf{N}|^2(t-s)\big)^{\frac{1}{2}}}\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2(t-s)}}\big)\big((\varepsilon\partial_t)^{\gamma_0}\tilde{\eta}\big)|_{z=0}(s,y)\mathrm{d}s,$$

which, combined with (6.7.17) established in the appendix, yields that:

$$\|\mathcal{Z}^\gamma\tilde{\eta}_h(t)\|_{L^2_{y,z}(\mathbb{R}^3_+)} \lesssim \int_0^t (t-s)^{-\frac{3}{4}}\big|\tilde{\eta}|_{z=0}(s)\big|_{\tilde{H}^{|\gamma|}(\mathbb{R}^2_y)}\mathrm{d}s. \tag{6.3.108}$$

The above inequality, combined with the boundary condition $(6.3.102)_2$ and the trace inequality (6.7.10), yields that:

$$\|\tilde{\eta}_h(t)\|_{m-2} \lesssim T^{\frac{1}{4}}\sup_{0\le s\le t}|\tilde{\eta}(s)|_{\tilde{H}^{m-2}(\mathbb{R}^2_y)} \lesssim T^{\frac{1}{4}}\|(u,\nabla u)\|_{L^\infty_t H^{m-2}_{co}} \lesssim T^{\frac{1}{4}}\mathcal{E}_{m,T}.$$

Similarly, we apply a convolution inequality in the time variable (after extending $\tilde{\eta}(s)|_{z=0}$ to $s\in\mathbb{R}$ by zero extension) to (6.3.108), and use the boundary condition $(6.3.102)_2$ and the trace inequality (6.7.11) to obtain:

$$\|\tilde{\eta}_h(t)\|_{m-1,t} \lesssim T^{\frac{1}{4}}|\tilde{\eta}|_{L^2_t\tilde{H}^{m-1}(\mathbb{R}^2_y)} \lesssim T^{\frac{1}{4}}\|(u,\nabla u)\|_{L^2_t H^{m-1}_{co}} \lesssim T^{\frac{1}{4}}\mathcal{E}_{m,T}.$$

□

**Lemma 6.3.16.** *Using the notation* (6.3.103), *the following energy inequality holds: for any* $0 < t \le T$,

$$\|\tilde{\eta}_{nh}(t)\|^2_{m-2} + \|\nabla\tilde{\eta}_{nh}\|^2_{m-2,t} \lesssim \|\eta(0)\|^2_{H^{m-2}_{co}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big). \tag{6.3.109}$$

*Proof.* Suppose that $0 \le |\gamma| = k \le m-2$. Denote $\tilde{\eta}^\gamma_{nh} = \mathcal{Z}^\gamma\tilde{\eta}_{nh}$, then $\tilde{\eta}^\gamma_{nh}$ solves the system (note that $[\mathcal{Z}^\gamma, E] = 0$):

$$\begin{aligned}\bar{\rho}\partial_t\tilde{\eta}^\gamma_{nh} - \mu\mathrm{div}\big(E\nabla\tilde{\eta}^\gamma_{nh}\big) &= \mu[\mathcal{Z}^\gamma,\mathrm{div}](E\nabla\tilde{\eta}^\gamma_{nh}) + \mu\mathcal{Z}^\gamma H(\tilde{\eta}_h) + \mathcal{Z}^\gamma F^\omega\\ &=: \mathcal{R}^\gamma_1 + \mathcal{R}^\gamma_2 + \mathcal{Z}^\gamma F^\omega\end{aligned}$$

with the initial condition $\tilde{\eta}^\gamma_{nh}|_{t=0} = \mathcal{Z}^\gamma\tilde{\eta}|_{t=0}$ and the boundary condition $\tilde{\eta}^\gamma_{nh}|_{z=0} = 0$.

Standard energy estimates show that:

$$\begin{aligned}&\bar{\rho}\|\tilde{\eta}^\gamma_{nh}(t)\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^t\int_{\mathbb{R}^3_+} E\nabla\tilde{\eta}^\gamma_{nh}\cdot\nabla\tilde{\eta}^\gamma_{nh}\,\mathrm{d}x\mathrm{d}s\\ &\qquad = \bar{\rho}\|\tilde{\eta}^\gamma_{nh}(0)\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^t\int_{\mathbb{R}^3_+}(\mathcal{R}^\gamma_1 + \mathcal{R}^\gamma_2 + \mathcal{Z}^\gamma\widetilde{F^\omega})\tilde{\eta}^\gamma_{nh}\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{6.3.110}$$

At first, since we can find some $\kappa > 0$, such that $2|\mathbf{N}|^2 \le 1/\kappa$, one has that $EX\cdot X = |PX|^2 \ge \frac{1}{2|\mathbf{N}|^2}|X|^2 \ge \kappa|X|^2$ and hence, we deduce that:

$$\int_0^t\int_{\mathbb{R}^3_+} E\nabla\tilde{\eta}^\gamma_{nh}\cdot\nabla\tilde{\eta}^\gamma_{nh}\,\mathrm{d}x\mathrm{d}s \ge \kappa\|\nabla\tilde{\eta}^\gamma_{nh}\|^2_{0,t}. \tag{6.3.111}$$

For the second term of the right hand side of (6.3.110), one needs to integrate by parts to avoid involving additional normal derivatives. Let us first study $\mathcal{R}^\gamma_1$ which vanishes if $|\gamma| = 0$. By induction, one gets that for $k = |\gamma| \ge 1$.

$$[\mathcal{Z}^\gamma,\mathrm{div}] = [\mathcal{Z}^\gamma,\partial_z] = \sum_{\beta<\gamma} C_{\phi,\beta,\gamma}\partial_z\mathcal{Z}^\beta \tag{6.3.112}$$

Where $C_{\phi,\beta,\gamma}$ are smooth functions that depend on $\phi$ and its derivatives. Consequently, by integration by parts and Young's inequality, we obtain that:

$$\int_0^t\int_{\mathbb{R}^3_+}\mathcal{R}^\gamma_1\cdot\tilde{\eta}^\gamma_{nh}\,\mathrm{d}x\mathrm{d}s \le \delta\|\nabla\tilde{\eta}^\gamma_{nh}\|^2_{0,t} + C_\delta(\|\nabla\tilde{\eta}_{nh}\|^2_{k-1,t} + \|\tilde{\eta}_{nh}\|^2_{k,t}). \tag{6.3.113}$$

Similarly, by taking benefits of the zero boundary condition of $\tilde{\eta}_{nh}^{\gamma}$, one integrates by parts to get:

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{R}_2^{\gamma} \tilde{\eta}_{nh}^{\gamma} \,\mathrm{d}x\mathrm{d}s \leq \delta \|\nabla \tilde{\eta}_{nh}^{\gamma}\|_{0,t}^2 + C_{\delta}(\|\tilde{\eta}_h\|_{k+1,t}^2 + \|\tilde{\eta}_{nh}\|_{k,t}^2). \tag{6.3.114}$$

We are now left to deal with the term:

$$\int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^{\gamma} \widetilde{F^{\omega}} \tilde{\eta}_{nh}^{\gamma} \,\mathrm{d}x\mathrm{d}s = \sum_{j=1}^{5} \int_0^t \int_{\mathbb{R}_+^3} \mathcal{Z}^{\gamma} \widetilde{F_j^{\omega}} \tilde{\eta}_{nh}^{\gamma} \,\mathrm{d}x\mathrm{d}t =: \sum_{j=1}^{5} \mathcal{I}_j.$$

where we denote that:

$$\begin{aligned}\widetilde{F^{\omega}} &= -\widetilde{\Delta(\chi_i \mathbf{n})} \times \tilde{\omega} - 2\widetilde{\nabla\omega} \times \widetilde{\nabla(\chi_i \mathbf{n})} - (\widetilde{g_2 u \cdot \nabla\omega}) \times \widetilde{(\chi_i \mathbf{n})} + \frac{(\widetilde{\bar{\rho} - g_2})}{\varepsilon} \varepsilon \widetilde{\partial_t \omega} \times \widetilde{(\chi_i \mathbf{n})} + \widetilde{G^{\omega}} \times \widetilde{(\chi_i \mathbf{n})}.\\ &=: \widetilde{F_1^{\omega}} + \widetilde{F_2^{\omega}} + \widetilde{F_3^{\omega}} + \widetilde{F_4^{\omega}} + \widetilde{F_5^{\omega}}.\end{aligned}$$

Note that $G^{\omega}$ is defined in (6.3.98). Moreover, without much ambiguity, we denote $\tilde{f}$ as $(\tilde{\chi}_i f) \circ \Phi_i$ where $\tilde{\chi}_i$ is a smooth function such that $\tilde{\chi}_i \chi_i = \chi_i$.

By the Cauchy-Schwarz inequality and the fact (6.3.104), $\mathcal{I}_1$ can be controlled by:

$$\mathcal{I}_1 \lesssim \|\tilde{\omega}\|_{k,t} \|\tilde{\eta}_{nh}\|_{k,t} \lesssim T^{\frac{1}{2}} \|\nabla u\|_{L_t^{\infty} H_{co}^{m-2}} \|\tilde{\eta}_{nh}\|_{k,t}. \tag{6.3.115}$$

Nevertheless, for $\mathcal{I}_2$ and $\mathcal{I}_3$, as $\widetilde{F_2^{\omega}}$, $\widetilde{F_3^{\omega}}$ involve normal derivatives of $\omega$, it is necessary to use integration by parts. By doing so, we can bound the term $\mathcal{T}_2$ as follows:

$$\mathcal{I}_2 \leq \delta \|\nabla \tilde{\eta}_{nh}^{\gamma}\|_{0,t}^2 + C_{\delta}(\|\tilde{\eta}_{nh}\|_{k,t}^2 + \|\widetilde{\nabla u}\|_{k,t}^2). \tag{6.3.116}$$

Next, for $\mathcal{I}_3$, by noticing the expression

$$\begin{aligned}\widetilde{g_2 u \cdot \nabla\omega} &= \partial_{y^1}(\widetilde{g_2 u_1} \tilde{\omega}) + \partial_{y^2}(\widetilde{g_2 u_2} \tilde{\omega}) + \partial_z((\widetilde{g_2 u \cdot \mathbf{N}}) \tilde{\omega})\\ &\quad - \big(\partial_{y^1} \widetilde{g_2 u_1} + \partial_{y^2} \widetilde{g_2 u_2} + \partial_z (\widetilde{g_2 u \cdot \mathbf{N}})\big) \tilde{\omega},\end{aligned}$$

one performs an integration by parts again to get that:

$$\begin{aligned}\mathcal{I}_3 &\lesssim \|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} \|\nabla \tilde{\eta}_{nh}^{\gamma}\|_{0,t} + \|\tilde{\omega}(\partial_{y^j}(\widetilde{g_2 u_j}), \partial_z(\widetilde{g_2 u \cdot \mathbf{N}}))\|_{k,t} \|\tilde{\eta}_{nh}^{\gamma}\|_{0,t}\\ &\leq \delta \|\nabla \tilde{\eta}_{nh}^{\gamma}\|_{0,t}^2 + C_{\delta} \|\tilde{g}_2 \tilde{u} \tilde{\omega}\|_{k,t} + T^{\frac{1}{2}} (\sup_{s \in [0,t]} \|\tilde{\eta}_{nh}(s)\|_k) \|\tilde{\omega}(\partial_{y_j}(\widetilde{g_2 u_j}), \partial_z(\widetilde{g_2 u \cdot \mathbf{N}}))\|_{k,t}\end{aligned}$$

Here we used Einstein summation convention for $j = 1, 2$. By (6.3.104), (6.3.107) and the assumption $k \leq m - 2$, one can have that:

$$\sup_{s \in [0,t]} \|\tilde{\eta}_{nh}\|_k \lesssim \sup_{s \in [0,t]} \|(\tilde{\eta}, \tilde{\eta}_h)(s)\|_k \lesssim \|\nabla u\|_{L_t^{\infty} H_{co}^{m-2}} + T^{\frac{1}{4}} \mathcal{E}_{m,t} \lesssim \mathcal{E}_{m,t}. \tag{6.3.117}$$

Moreover, since $k \leq m - 2$, we have thanks to (6.3.104) that:

$$\begin{aligned}&\|\tilde{\omega}(\partial_{y^j}(\widetilde{g_2 u^j}), \partial_z(\widetilde{g_2 u \cdot \mathbf{N}}))\|_{m-2,t}\\ &\lesssim \|\!|\omega|\!\|_{0,\infty,t} \|Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})\|_{L_t^2 H_{co}^{m-2}}\\ &\qquad + \|\omega\|_{L_t^{\infty} H_{co}^{m-2}} \Big(\int_0^t \|\!|[Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})](s)|\!\|_{m-3,\infty}^2 \mathrm{d}s\Big)^{\frac{1}{2}}\end{aligned}$$

where $Z_i$ stands for the tangential vector fields in $\Omega_i$. By identity (6.3.22) and the Sobolev embedding (6.7.8) and estimate (6.3.47),

$$\begin{aligned}\Big(\int_0^t \|\!|Z_i(g_2 u_j), \nabla(g_2 u \cdot \mathbf{N})(s)|\!\|_{m-3,\infty}^2 \mathrm{d}s\Big)^{\frac{1}{2}} &\lesssim \|u\|_{E_t^m} + \|\nabla \mathrm{div} u\|_{L_t^2 H_{co}^{m-2}} + \varepsilon \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big)\\ &\lesssim \|(\sigma, u)\|_{E_t^m} + \varepsilon^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big),\end{aligned}$$

which together with the previous inequality, yields:

$$\|\tilde{\omega}(\partial_{y^j}(\widetilde{g_2 u_j}), \partial_z(\widetilde{g_2 u\cdot\mathbf{N}}))\|_{m-2,t} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big).$$

Similarly, we have that:

$$\|\tilde{g_2}\tilde{u}\tilde{\omega}\|_{k,t} \lesssim T^{\frac{1}{2}} |\!|\!|\omega|\!|\!|_{0,\infty,t}\|u\|_{L_t^\infty H_{co}^{m-2}} + \|u\|_{E_t^m}\|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Moreover, if $k \le [\frac{m}{2}]-2$,

$$\|\tilde{g_2}\tilde{u}\tilde{\omega}\|_{k,t} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\|\nabla u\|_{L_t^2 H_{co}^{[\frac{m}{2}]-2}} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

To summarize, we control $\mathcal{T}_3$ as follows:

$$\mathcal{T}_3 \le \delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big), \text{ if } k \le [\frac{m}{2}]-2, \tag{6.3.118}$$

and for $k \le m-2$,

$$\mathcal{T}_3 \le \delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big) + \|(\sigma,u)\|_{E_t^m}\|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}}. \tag{6.3.119}$$

For $\mathcal{I}_4$, the direct application of the Hölder inequality requires the control of the quantity $\|\frac{(\widetilde{\bar{\rho}-g_2})}{\varepsilon}\widetilde{\varepsilon\partial_t\omega}\|_{k,t}$, which further requires the estimate of $L_{t,x}^\infty$ type norm of $\partial_t\omega$ However, $|\!|\!|\varepsilon\partial_t\omega|\!|\!|_{\infty,t}$ (or $|\!|\!|\nabla u|\!|\!|_{1,\infty,t}$) seems out of control and does not appear in the $L_{t,x}^\infty$ type norms present in $\mathcal{A}_{m,T}$. To avoid this problem, since $\widetilde{\varepsilon\partial_t\omega} = (P\nabla)\times\widetilde{\varepsilon\partial_t u}$, we can integrate by parts in space before using product estimate. By doing so, we achieve that:

$$\begin{aligned}\mathcal{I}_4 &\le \delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta\|\tilde{\eta}_{nh}\|_{k,t}^2 + \|(\widetilde{\nabla\sigma}, \widetilde{\varepsilon\partial_t u})\|_{k,t}^2\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \\ &\lesssim \delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta\|\tilde{\eta}_{nh}\|_{k,t}^2 + T\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.\end{aligned} \tag{6.3.120}$$

Finally, regarding the term $\mathcal{T}_5$, we control it by Cauchy-Shwarz inequality as:

$$\mathcal{T}_5 \lesssim T^{\frac{1}{2}}\big(\sup_{s\in[0,t]}\|\tilde{\eta}_{nh}(s)\|_k\big)\|\widetilde{G^\omega}\|_{k,t}$$

By the estimate (6.3.117), the fact (6.3.104) and the Proposition 6.3.17, we get that:

$$\mathcal{T}_5 \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big). \tag{6.3.121}$$

To summarize, we have found by collecting (6.3.115)-(6.3.120) that for $0 \le k \le m-2$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma\widetilde{F^\omega}\tilde{\eta}_{nh}^\gamma \,\mathrm{d}x\mathrm{d}t \le 3\delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 \\ &+ C_\delta\big(\|(\tilde{\eta},\tilde{\eta}_h)\|_{k,t}^2 + +\|u\|_{E_t^m}\|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}}\big) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big) \\ &\le 3\delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + C_\delta\|u\|_{E_t^m}\|\omega\|_{L_t^\infty H_{co}^{[\frac{m}{2}]-2}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big),\end{aligned} \tag{6.3.122}$$

and also for $0 \le k \le [\frac{m}{2}]-2$,

$$\int_0^t\int_{\mathbb{R}_+^3} \mathcal{Z}^\gamma\widetilde{F^\omega}\tilde{\eta}_{nh}^\gamma \,\mathrm{d}x\mathrm{d}t \le 3\delta\|\nabla\tilde{\eta}_{nh}^\gamma\|_{0,t}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big). \tag{6.3.123}$$

Inserting (6.3.113)-(6.3.114) (6.3.122)-(6.3.123) in (6.3.110), we obtain by choosing $\delta$ small enough that for any $0 \le k \le m-2$,

$$\begin{aligned} &\|\tilde{\eta}_{nh}(t)\|_k^2 + \|\nabla\tilde{\eta}_{nh}\|_{k,t}^2 \lesssim \|\eta(0)\|_{H^k_{co}}^2 + \|\nabla\tilde{\eta}_{nh}\|_{k-1,t}^2 \\ &+ (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big) + \|(\sigma,u)\|_{E^m_t}\|\omega\|_{L^\infty_t H_{co}^{[\frac{m}{2}]-2}}\mathbb{I}_{\{k\ge[\frac{m}{2}]-1\}}. \end{aligned} \tag{6.3.124}$$

where the convention $\|\cdot\|_{l,t}=0$ if $l<0$ is used. We thus get by induction on $0\le k\le[\frac{m}{2}]-2$ that:

$$\|\tilde{\eta}_{nh}(t)\|_{[\frac{m}{2}]-2}^2 + \|\nabla\tilde{\eta}_{nh}\|_{[\frac{m}{2}]-2,t}^2 \lesssim \|\eta(0)\|_{H^{m-2}_{co}}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big), \tag{6.3.125}$$

which, together with (6.3.107) and (6.3.85) gives that:

$$\|\nabla u\|_{L^\infty_t H_{co}^{[\frac{m}{2}]-2}}^2 \lesssim Y_m^2(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big).$$

We then combine this estimate and (6.3.95) to obtain that:

$$\|u\|_{E^m_t}\|\omega\|_{L^\infty_t H_{co}^{[\frac{m}{2}]-2}} \lesssim Y_m^2(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big).$$

Therefore, we take benefits of the estimate (6.3.124) and the induction arguments to get (6.3.109). □

**Proposition 6.3.17.** *Assume that* (6.2.1) *holds and let*

$$G^\omega = g_2\omega\cdot\nabla u - g_2\omega\mathrm{div}u - \frac{\nabla g_2}{\varepsilon}\times(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u),$$

*then we have:*

$$\|\tilde{\chi_i}G^\omega\|_{L^2_t H^{m-2}_{co}} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big).$$

*Proof.* Let us show the estimate of $\tilde{\chi_i}\omega\cdot\nabla u$, which is not direct since the higher order $L^\infty_{t,x}$ norm (say $\|\nabla u\|_{[\frac{m}{2}]-1,\infty,t}$) is unlikely to be uniformly bounded. Nevertheless, thanks to identity (6.3.35), one can write this term as:

$$\tilde{\chi_i}\omega\cdot\nabla u = \tilde{\chi_i}\big(\omega_1\partial_{y^1}u + \omega_2\partial_{y^2}u + (\omega\cdot\mathbf{N})\partial_{\mathbf{n}}u\big).$$

Moreover, by identities (6.3.59) and (6.3.22),

$$\begin{aligned} \omega\cdot\mathbf{N} &= (\nabla\times u)\cdot\mathbf{N} \\ &= -(u\times\mathbf{N})\partial_{\mathbf{n}}\mathbf{n} + (\Pi\partial_{y^1}(u\times\mathbf{N}))^1 + (\Pi\partial_{y^2}(u\times\mathbf{N}))^2 + u\cdot\mathrm{curl}\,\mathbf{N} \end{aligned}$$

which gives that for any $t\in[0,T]$, any $k\ge0$,

$$\|(\omega\cdot\mathbf{N})(t)\|_{H^k_{co}} \lesssim \|u(t)\|_{H^{k+1}_{co}}. \qquad \|(\omega\cdot)\mathbf{N}(t)\|_{k,\infty} \lesssim \|u(t)\|_{k+1,\infty}$$

Therefore, by the Sobolev embedding (6.7.8), we have that:

$$\begin{aligned} &\|\tilde{\chi_i}\omega\cdot\nabla u\|_{L^2_t H^{m-2}_{co}} \\ &\lesssim \|\nabla u\|_{0,\infty,t}\|(\partial_{y^i}u,\omega\cdot\mathbf{N})\|_{L^2_t H^{m-2}_{co}} + \|\nabla u\|_{L^\infty_t H^{m-2}_{co}}\Big(\int_0^t \|(\partial_{y^i}u,\omega\cdot\mathbf{N})(s)\|_{m-3,\infty}^2\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\lesssim \|\nabla u\|_{0,\infty,t}\|u\|_{L^2_t H^{m-1}_{co}} + \|\nabla u\|_{L^\infty_t H^{m-2}_{co}}\|u\|_{E^m_t} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,t}\big). \end{aligned}$$

The other two terms in the definition of $G^\omega$ are similar or easier to treat, we omit the detail. □

**Remark 6.3.18.** *Collecting the results stated in Lemmas 6.3.4, 6.3.9, 6.3.13, 6.3.11, we find that:*

$$\begin{aligned} &\|\varepsilon\nabla(\sigma,u)\|_{L^\infty_t H^{m-1}_{co}}^2 + \|\nabla(\sigma,u)\|_{L^\infty_t H^{m-2}_{co}\cap L^2_t H^{m-1}_{co}}^2 + \|(\sigma,u)\|_{L^\infty_t H^{m-1}_{co}}^2 \\ &\lesssim Y_m^2(\sigma_0,u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \end{aligned} \tag{6.3.126}$$

### 6.3.6 $\varepsilon-$dependent estimate of $\nabla^2 u$

To finish the estimates for the energy norm, we are left to deal with $\|\varepsilon\nabla^2 u\|_{L_t^\infty H_{co}^{m-2}}, \varepsilon\|\nabla^2\sigma\|_{L_t^\infty L^2}$.

**Lemma 6.3.19.** *Under the assumption* (6.2.2)*, the following estimate holds:*

$$\|\varepsilon\nabla^2 u(t)\|^2_{H_{co}^{m-2}} \lesssim Y_m^2(\sigma_0, u_0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda(\frac{1}{c_0}, \mathcal{N}_{m,T}). \tag{6.3.127}$$

*Proof.* As $u$ satisfies the equation:

$$\varepsilon\mu\Delta u = -(\mu+\lambda)\varepsilon\nabla\mathrm{div}u + g_2(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u) + \nabla\sigma.$$

we have by elliptic regularity theory:

$$\begin{aligned}\|\varepsilon\nabla^2 u(t)\|_{H_{co}^{m-2}} \lesssim \varepsilon \sum_{|I|\le m-2} |Z^I\partial_{\mathbf{n}} u(t)|_{H^{\frac{1}{2}}} + \varepsilon\|\nabla\mathrm{div}u(t)\|_{H_{co}^{m-2}} \\ + \|u(t)\|_{H_{co}^{m-1}} + \|\nabla\sigma(t)\|_{H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\mathcal{E}_{m,T}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).\end{aligned} \tag{6.3.128}$$

It follows from the boundary condition (6.1.14), the identity (6.3.22) and the trace inequality (6.7.10) that:

$$\varepsilon \sum_{|I|\le m-2} |Z^I\partial_{\mathbf{n}} u(t)|_{H^{\frac{1}{2}}} \lesssim \varepsilon\|\nabla\mathrm{div}u(t)\|_{H_{co}^{m-2}} + \varepsilon\|(u,\nabla u)(t)\|_{H_{co}^{m-1}}. \tag{6.3.129}$$

Inserting (6.3.48) and (6.3.129) into (6.3.128), one arrives at:

$$\varepsilon\|\nabla^2 u(t)\|_{H_{co}^{m-2}} \lesssim \varepsilon\|\nabla(\sigma,u)(t)\|_{H_{co}^{m-1}} + \|\nabla\sigma(t)\|_{H_{co}^{m-2}} + \|u(t)\|_{H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\mathcal{E}_{m,T}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big),$$

which, combined with (6.3.126) leads to (6.3.127). □

**Lemma 6.3.20.** *Under the assumption* (6.2.2)*, we have the following estimate for* $\nabla^2\sigma$*:*

$$\|\varepsilon\nabla^2\sigma\|^2_{L_t^\infty L^2} + \|\nabla^2\sigma\|^2_{L^2(Q_t)} \lesssim Y_m^2(0) + (T+\varepsilon)\Lambda(\frac{1}{c_0}, \mathcal{N}_{m,T}). \tag{6.3.130}$$

*Proof.* By (6.3.50) and (6.3.89), one finds that $\nabla\sigma$ solves:

$$\varepsilon^2 g_1(\partial_t + u\cdot\nabla)\nabla\sigma + \frac{1}{(2\mu+\lambda)}\nabla\sigma = G \tag{6.3.131}$$

where

$$G = -\varepsilon^2(g_1' S\varepsilon\partial_t\sigma + \nabla(g_1 u_k)\cdot\partial_k\sigma) - \varepsilon\frac{\mu}{(2\mu+\lambda)}\,\mathrm{curl}\,\omega - \frac{1}{(2\mu+\lambda)}g_2(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u).$$

By taking the divergence of the equation (6.3.131), one arrives at:

$$\varepsilon^2 g_1(\partial_t + u\cdot\nabla)\Delta\sigma + \frac{1}{2\mu+\lambda}\Delta\sigma = \mathrm{div}G - \varepsilon^2\big[g_1'\nabla\sigma\cdot\varepsilon\partial_t\nabla\sigma + \sum_{i=1}^{3}\partial_i(g_1 u)\cdot\nabla\partial_i\sigma\big] =: \tilde{G}$$

From an energy estimate, we find

$$\varepsilon^2\|\Delta\sigma\|^2_{L_t^\infty L^2} + \|\Delta\sigma\|^2_{L^2(Q_t)} \lesssim T^{\frac{1}{2}}\|\Delta\sigma\|_{L^2(Q_t)}\|\tilde{G}\|_{L_t^\infty L^2} + T\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\|\varepsilon\Delta\sigma\|^2_{L_t^\infty L^2}. \tag{6.3.132}$$

We first observe that:

$$\|\tilde{G}\|_{L_t^\infty L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.$$

Moreover, since in the local coordinate, we can find some coefficients $a_{ij}$ that depends smoothly on $\mathbf{n}$, such that (we use the convention $\partial_{y^3} = \partial_{\mathbf{n}}$ ):

$$\Delta = \partial_{\mathbf{n}}^2 + \sum_{0\leq i,j\leq 3,(i,j)\neq(3,3).} \partial_{y^i}(a_{ij}\partial_{y^j}) \tag{6.3.133}$$

which yields:

$$\|\partial_{\mathbf{n}}\nabla\sigma\|_{L_t^\infty L^2} \lesssim \|\Delta\sigma\|_{L_t^\infty L^2} + \|\nabla\sigma\|_{L_t^\infty H_{co}^1}.$$

We thus obtain (6.3.130) from (6.3.132).

□

### 6.3.7 Proofs of Proposition 6.3.1

By collecting (6.3.7), (6.3.126) (6.3.127) and (6.3.130), we get (6.3.1).

**Remark 6.3.21.** *In view of the formal expansion* (6.1.5)*, one expects the first three normal derivatives of $\sigma$ to be bounded in $L^2(Q_t)$. This can be achieved in the following way. By imposing additional assumption on $\sigma_0$, namely $\varepsilon\nabla^2\sigma_0 \in H_{co}^1(\Omega), \nabla^3\sigma_0 \in L^2(\Omega)$, one can show by following similar computations as in the proof of Lemma 6.3.20 that: $\varepsilon\nabla^2\sigma \in L_t^\infty H_{co}^1, \nabla\sigma \in L_t^2 H_{co}^1$. These estimates at hand, one can carry out another energy estimate to control $\|\partial_{\mathbf{n}}\Delta\sigma\|_{L^2(Q_t)}$, which further leads to the boundedness of $\|\nabla^3\sigma\|_{L^2(Q_t)}$. We remark that in the latter energy estimate, the knowledge of $\|\varepsilon\nabla^3 u\|_{L^2(Q_t)}$ is needed. Nevertheless, this term can be bounded by all the controlled norms appearing in $\mathcal{N}_{m,T}$. More precisely, one has by equation for the velocity*

$$\varepsilon\mathrm{div}\mathcal{L}u = g_2(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u) + \nabla\sigma,$$

*and thus by* (6.3.133)*:*

$$\begin{aligned}\|\varepsilon\nabla^3 u\|_{L^2(Q_t)} &\lesssim \|\varepsilon\nabla\mathrm{div}\mathcal{L}u\|_{L^2(Q_t)} + \|\varepsilon\nabla^2 u\|_{L_t^2 H_{co}^1} \\ &\leq \Lambda(1/c_0, \mathcal{A}_{m,T})(\|(\sigma,u)\|_{L_t^2 H_{co}^2} + \|\nabla(\sigma,u)\|_{L_t^2 H_{co}^1} + \|\nabla^2\sigma\|_{L^2(Q_t)}).\end{aligned} \tag{6.3.134}$$

## 6.4 Uniform estimates- $L_{t,x}^\infty$ norms

In this section, we aim to control the $L_{t,x}^\infty$ norms appearing in $\mathcal{A}_{m,T}$. Part of them can be deduced directly from the Sobolev embedding in the conormal setting (see Proposition 6.7.4) and the norms controlled in the previous section. Moreover, we use the maximum principle for transport-diffusion equation (6.4.5) satisfied by $\omega$ and of the damped transport equation (6.3.131) for $\nabla\sigma$ to get the $L_{t,x}^\infty$ estimates of $\nabla u$ and $\nabla\sigma$ respectively.

We will prove the following proposition.

**Proposition 6.4.1.** *Assuming that* (6.2.2) (6.2.1) *hold, then there is a constant $C_2(1/c_0)$ depending only on $1/c_0$ and a polynomial $\bar{\Lambda}$ whose coefficients are independent of $\varepsilon$, such that:*

$$\mathcal{A}_{m,T} \leq C_2(1/c_0)\big(Y_m(\sigma_0,u_0) + \mathcal{E}_{m,T}\big) + (\varepsilon^{\frac{1}{2}} + T)\mathcal{A}_{m,T}\bar{\Lambda}(1/c_0, \mathcal{A}_{m,T}). \tag{6.4.1}$$

*Proof.* Let us recall that $\mathcal{A}_{m,T}$ is defined as:

$$\begin{aligned}\mathcal{A}_{m,T} = &\|\nabla u\|_{0,\infty,T} + \|(\nabla\sigma,\mathrm{div}u)\|_{[\frac{m-1}{2}],\infty,T} + \|(\sigma,u)\|_{[\frac{m+1}{2}],\infty,T} \\ &+ \|\varepsilon^{\frac{1}{2}}\nabla u\|_{[\frac{m-1}{2}],\infty,T} + \|\varepsilon\nabla u\|_{[\frac{m+1}{2}],\infty,T} + \varepsilon\|(\sigma,u)\|_{[\frac{m+3}{2}],\infty,T}.\end{aligned} \tag{6.4.2}$$

The last four terms of $\mathcal{A}_{m,T}$ can be controlled directly by the Sobolev embedding (6.7.8). For instance,

$$\|(\sigma,u)\|_{[\frac{m+1}{2}],\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|(\sigma,u)(s)\|_{H_{co}^{[\frac{m+5}{2}]}} + \|\nabla(\sigma,u)(s)\|_{H_{co}^{[\frac{m+3}{2}]}}\big) \lesssim \mathcal{E}_{m,T}, \tag{6.4.3}$$

$$\varepsilon^{\frac{1}{2}}\|\nabla u\|_{[\frac{m-1}{2}],\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|\nabla u(s)\|_{H_{co}^{[\frac{m+3}{2}]}} + \varepsilon\|\nabla^2 u(s)\|_{H_{co}^{[\frac{m+1}{2}]}}\big) \lesssim \mathcal{E}_{m,T},$$

$$\varepsilon\|\nabla u\|_{[\frac{m+1}{2}],\infty,T} \lesssim \varepsilon \sup_{0\leq s\leq T} \big(\|\nabla u(s)\|_{H_{co}^{[\frac{m+5}{2}]}} + \|\nabla^2 u(s)\|_{H_{co}^{[\frac{m+3}{2}]}}\big) \lesssim \mathcal{E}_{m,T}.$$

Note that we have $[\frac{m+3}{2}]+1 \leq m-2$, $[\frac{m+5}{2}] \leq m-1$ if $m \geq 6$.

We remark also that $\|\mathrm{div} u\|_{[\frac{m-1}{2}],\infty,T}$ can be estimated by the other quantities in the definition of $\mathcal{A}_{m,T}$. Indeed, by using the equation satisfied by $\sigma$, we have that:

$$\|\mathrm{div} u\|_{[\frac{m-1}{2}],\infty,T} \lesssim \|\sigma\|_{[\frac{m+1}{2}],\infty,T} + \varepsilon\mathcal{A}_{m,T}^2, \tag{6.4.4}$$

It thus remains to control $\|\nabla u\|_{0,\infty,T}, \|\nabla\sigma\|_{[\frac{m-1}{2}],\infty,T}$. We note that away from the boundaries where the conormal Sobolev norm is equivalent to the usual Sobolev norm, these two terms can be bounded by the standard Sobolev embedding. Therefore, it suffices to control $\|\chi_i\partial_{\mathbf{n}} u\|_{0,\infty,T}, \|\chi_i\partial_{\mathbf{n}}\sigma\|_{[\frac{m-1}{2}],\infty,T}$, where $\chi_i, (1\leq i\leq N)$ are smooth functions compactly supported in $\Omega_i$. Moreover, by identity (6.3.22) and

$$\Pi(\partial_{\mathbf{n}} u) = \omega\times\mathbf{n} + 2\Pi(-(D\mathbf{n})u),$$

we reduce our problem to the control of $\|\omega\|_{0,\infty,T}, \|\chi_i\partial_{\mathbf{n}}\sigma\|_{[\frac{m-1}{2}],\infty,T}$, which is the aim of the following two lemmas.

□

We begin with the estimate for $\|\omega\|_{0,\infty,T}$ which follows from the maximum principle of the transport-diffusion equation for the vorticity.

**Lemma 6.4.2.** *Under the assumption* (6.2.2)*, the following estimate holds:*

$$\|\omega\|_{0,\infty,T} \lesssim \|\omega(0)\|_{L^\infty(\Omega)} + \mathcal{E}_{m,T} + (T+\varepsilon)\mathcal{A}_{m,T}^2. \tag{6.4.5}$$

*Proof.* Recall that $\omega$ solves (6.3.98) which is rewritten below for convenience:

$$g_2(\partial_t + u\cdot\nabla)\omega - \mu\Delta\omega = g_2(\omega\cdot\nabla u - \omega\mathrm{div} u) + \nabla g_2 \times [(\partial_t + u\cdot\nabla)u] = G^\omega \quad x\in\Omega.$$

Since $g_2(\varepsilon\sigma)$ satisfies the transport equation: $\partial_t g_2 + \mathrm{div}(g_2 u) = 0$, by the maximum principle, (one can refer to Proposition 13 of [106])

$$\|\omega(t)\|_{L^\infty(\Omega)} \leq \|\omega(0)\|_{L^\infty(\Omega)} + |\omega(t)|_{L^\infty(\partial\Omega)} + \frac{1}{\inf g_2}\int_0^t \|G^\omega(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s. \tag{6.4.6}$$

For the second term in the right hand side of (7.8.9), we use the boundary condition (6.1.14), the identity (6.3.22) and (6.4.3), (6.4.4) to get that:

$$|\omega(t)|_{L^\infty(\partial\Omega)} \lesssim |(u, \partial_y u, \mathrm{div} u)(t)|_{L^\infty(\partial\Omega)} \lesssim \mathcal{E}_{m,T} + \varepsilon\mathcal{A}_{m,T}^2.$$

For the last term, we have by the assumption (6.2.2) and the property (6.2.1) that there is some $C(1/c_0)$, such that:

$$\frac{1}{\inf g_2}\int_0^t \|G^\omega(s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \leq C(1/c_0)T\mathcal{A}_{m,T}^2,$$

which ends the proof. □

In the following, we estimate $\|\chi_i\partial_{\mathbf{n}}\sigma\|_{[\frac{m-1}{2}],\infty,T}$ :

**Lemma 6.4.3.** *Under the assumption* (6.2.2)*, we have:*

$$\|\chi_i\partial_{\boldsymbol{n}}\sigma\|_{[\frac{m-1}{2}],\infty,T} \lesssim Y_m(\sigma_0, u_0) + \mathcal{E}_{m,T} + \varepsilon^{\frac{1}{2}}\mathcal{A}_{m,T}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \tag{6.4.7}$$

*where $\chi_i$ is a smooth function that is compactly supported in $\Omega_i$.*

*Proof.* We define $R = \chi_i \partial_{\mathbf{n}} \sigma = \chi_i \mathbf{n} \cdot \nabla \sigma$. By (6.3.131), $R$ solves the following equation:

$$\varepsilon^2 g_1(\partial_t R + u \cdot \nabla R) + \frac{1}{2\mu+\lambda} R = -\varepsilon^2 g_1 u \cdot \nabla(\chi_i \mathbf{n}_k)\partial_k \sigma + G \cdot \chi_i \mathbf{n} =: G_R \tag{6.4.8}$$

where

$$G = -\varepsilon^2 (g_1' R \varepsilon \partial_t \sigma + \nabla(g_1 u_k) \cdot \partial_k \sigma) - \varepsilon \frac{\mu}{(2\mu+\lambda)} \operatorname{curl} w - \frac{1}{(2\mu+\lambda)} g_2(\varepsilon \partial_t u + \varepsilon u \cdot \nabla u).$$

By applying $Z^I$ ($|I| \leq [\frac{m-1}{2}]$) to the equation (6.4.8), we get by setting $R^I = Z^I R$ that

$$\varepsilon^2 g_1(\partial_t R^I + u \cdot \nabla R^I) + \frac{1}{2\mu+\lambda} R^I = Z^I G_R + \mathcal{C}_{R,1}^I + \mathcal{C}_{R,2}^I =: \mathcal{H}^I$$

where $\mathcal{C}_{R,1}^I = -\varepsilon^2 [Z^I, g_1/\varepsilon]\varepsilon \partial_t R, \quad \mathcal{C}_{R,1}^I = -\varepsilon^2 [Z^I, g_1 u \cdot \nabla] R.$

It is convenient to use the Lagranian coordinates. Define the unique flow $X_t(x) = X(t,x)$ associated to $u$:

$$\begin{cases} \partial_t X(t,x) = u(t, X(t,x)) \\ X(0,x) = x \in \Omega. \end{cases} \tag{6.4.9}$$

Note that since $u \cdot \mathbf{n}|_{\partial\Omega} = 0$, and $u \in Lip([0,T] \times \Omega)$, we have for each $t \in [0,T]$, $X_t : \Omega \to \Omega$ is a diffeomorphism. By using the characteristics method, $R^I(t, X_t(x))$ can then be expressed in the following way:

$$R^I(t, X_t(x)) = e^{-\Gamma(t,x)} R^I(0) + \int_0^t e^{-\Gamma(t-s,x)} \big(\frac{1}{\varepsilon^2 g_1} \mathcal{H}^I\big)(s, X_s(x)) \, \mathrm{d}s \tag{6.4.10}$$

where $\Gamma(t,x) = \frac{1}{2\mu+\lambda} \int_0^t \frac{1}{\varepsilon^2 g_1(s, X_s(x))} \mathrm{d}s \geq \frac{c_0 t}{(2\mu+\lambda)\varepsilon^2}$. Note that we have used assumption (6.2.2) and property (6.2.1). Taking the supremum in $(t,x) \in [0,T] \times \Omega$ on both sides of (6.4.10), and using that $X(t,\cdot)(0 \leq t \leq T)$ is a diffeomorphism of $\Omega$, we arrive at:

$$\|R^I(t)\|_{L^\infty(\Omega)} \lesssim \|R^I(0)\|_{L^\infty(\Omega)} + \int_0^t e^{-\frac{t-s}{(2\mu+\lambda)c_1\varepsilon^2}} \frac{1}{c_0 \varepsilon^2} \mathrm{d}s \|\mathcal{H}^I\|_{\infty,T} \lesssim \|R^I(0)\|_{L^\infty(\Omega)} + \|\mathcal{H}^I\|_{\infty,T}. \tag{6.4.11}$$

We have thus reduced the problem to the estimate of $\|(\mathcal{C}_{R,1}^I, \mathcal{C}_{R,2}^I)\|_{\infty,T}$ and $\|G_R\|_{[\frac{m-1}{2}],\infty,T}$. By the identities (6.3.35) (6.3.41), and the definition of $\mathcal{A}_{m,T}$, we have:

$$\|(\mathcal{C}_{R,1}^I, \mathcal{C}_{R,2}^I)\|_{\infty,T} \leq \varepsilon \Lambda(\frac{1}{c_0}, \mathcal{A}_{m,T}) \mathcal{A}_{m,T}. \tag{6.4.12}$$

Moreover, $G_R$ (defined in (6.4.8)) can be controlled as:

$$\|G_R\|_{[\frac{m}{2}]-1,\infty,T} \lesssim \varepsilon^{\frac{1}{2}} \mathcal{A}_{m,T} \Lambda(1/c_0, \mathcal{A}_{m,T}) + \|(\sigma, u)\|_{[\frac{m+1}{2}],\infty,T} + \varepsilon \|\chi \operatorname{curl} \omega \cdot \mathbf{n}\|_{[\frac{m-1}{2}],\infty,T}.$$

Since $\operatorname{curl} \omega \cdot \mathbf{n} = \operatorname{div}(\omega \times \mathbf{n}) + \omega \cdot \operatorname{curl} \mathbf{n}$, the identity (6.3.60) yields

$$\varepsilon \|\chi \operatorname{curl} \omega \cdot \mathbf{n}\|_{[\frac{m-1}{2}],\infty,T} \lesssim \varepsilon \|\nabla u\|_{[\frac{m+1}{2}],\infty,T},$$

which further leads to:

$$\|G_R\|_{[\frac{m}{2}]-1,\infty,T} \lesssim \varepsilon^{\frac{1}{2}} \mathcal{A}_{m,T} \Lambda(\frac{1}{c_0}, \mathcal{A}_{m,T}) + \mathcal{E}_{m,T}. \tag{6.4.13}$$

Inserting (6.4.12)-(6.4.13) into (6.4.11), we get (6.4.7). □

## 6.5 Proof of Theorem 6.1.1

Based on the uniform estimates established in previous sections, Theorem 6.1.1 can be showed by combining a classical local existence results with a bootstrap argument.

By following similar arguments as in [27] [65], one can prove the following local existence result:

**Theorem 6.5.1.** *Assume that* $(\sigma_0^\varepsilon, u_0^\varepsilon) \in H^2(\Omega)$, *and*

$$-\bar{c}\bar{P} \le \varepsilon\sigma_0^\varepsilon(x) \le \bar{P}/\bar{c}, \quad \forall x \in \Omega, \varepsilon \in (0,1].$$

*there is some* $T_\varepsilon > 0$ *such that* (6.1.15) *has a unique strong solution which satisfies:* $(\sigma^\varepsilon, u^\varepsilon) \in C([0,T^\varepsilon], H^2), u^\varepsilon \in L^2([0,T^\varepsilon], H^3)$. *Moreover, the following property holds:*

$$-3\bar{c}\bar{P} \le \varepsilon\sigma^\varepsilon(t,x) \le 3\bar{P}/\bar{c} \quad \forall (t,x) \in [0,T^\varepsilon] \times \Omega. \tag{6.5.1}$$

By using this result, we can give the proof of Theorem 6.1.1.

## Proof of Theorem 6.1.1:

On the one hand, $(\sigma_0^\varepsilon, u_0^\varepsilon) \in H^2$, by Theorem 6.5.1, one can find some $T^\varepsilon > 0$ such that there is a unique solution of (6.1.15) satisfying: $(\sigma^\varepsilon, u^\varepsilon) \in C([0,T^\varepsilon], H^2), u^\varepsilon \in L^2([0,T^\varepsilon], H^3)$. Moreover, condition (6.5.1) holds. On the other hand, as $(\sigma_0^\varepsilon, u_0^\varepsilon) \in Y_m$, a higher regularity space, by standard propagation of regularity arguments (for example based on applying finite difference instead of derivatives) in the estimates of Section 3 and Section 4, we find that the estimates of Proposition 6.2.1 hold on $[0,T^\varepsilon]$. More specifically, we can find a constant $C(1/c_0)$ and an increasing polynomial $\Lambda_0$ that are independent of $\varepsilon$ and $T^\varepsilon$, such that for any $0 \le T \le \min\{1, T^\varepsilon\}, 0 < \varepsilon \le 1$,

$$\mathcal{N}_{m,T}^2(\sigma^\varepsilon, u^\varepsilon) \le C\big(\frac{1}{c_0}\big) Y_m^2(\sigma_0^\varepsilon, u_0^\varepsilon) + (T+\varepsilon)^{\frac{1}{2}} \Lambda_0\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{6.5.2}$$

Moreover, by using the characteristics method, we have that $\varepsilon\sigma$ can be expressed as,

$$\varepsilon\sigma^\varepsilon(t,x) = \varepsilon\sigma_0^\varepsilon(X^{-1}(t,x)) - \int_0^t (\mathrm{div} u^\varepsilon/g_1)(X(s, X^{-1}(t,x)))\,\mathrm{d}s \tag{6.5.3}$$

where $X(t,\cdot)$ is the flow associated to $u$.

Let us define

$$T_*^\varepsilon = \sup\{T \big| (\sigma^\varepsilon, u^\varepsilon) \in C([0,T], H^2), u^\varepsilon \in L^2([0,T], H^3)\},$$

$$\begin{aligned} T_0^\varepsilon = \sup\{T \le \min\{T_*^\varepsilon, 1\} \big| &\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le 2\sqrt{C(1/c_0)}M, \\ &-2\bar{c}\bar{P} \le \varepsilon\sigma^\varepsilon(t,x) \le 2\bar{P}/\bar{c} \quad \forall (t,x) \in [0,T] \times \Omega\} \end{aligned}$$

where $M > \sup_{\varepsilon \in (0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$.

We now choose successively two constants $0 < \varepsilon_0 \le 1$ and $0 < T_0 \le 1$ (uniform in $\varepsilon \in (0, \varepsilon_0]$) which are small enough, such that:

$$(T_0 + \varepsilon_0)^{\frac{1}{2}} \Lambda_0\big(1/c_0, 2\sqrt{C(1/c_0)}M\big) < 1/2, \quad 2\sqrt{C(1/c_0)}MT_0/c_0 \le \bar{c}\bar{P}.$$

In order to prove Theorem 6.1.1, it suffices to show that $T_0^\varepsilon \ge T_0$ for every $0 < \varepsilon \le \varepsilon_0$. Suppose otherwise $T_0^\varepsilon < T_0$ for some $0 < \varepsilon \le \varepsilon_0$, then in view of inequalities (6.5.2) and formula (6.5.3), we have by the definition of $\varepsilon_0$ and $T_0$ that:

$$\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le \sqrt{2C(1/c_0)}M, \qquad \forall T \le \tilde{T} = \min\{T_0, T_*^\varepsilon\}, \tag{6.5.4}$$

$$-2\bar{c}\bar{P} \le \varepsilon\sigma^\varepsilon(t,x) \le 2\bar{P}/\bar{c} \quad \forall (t,x) \in [0,\tilde{T}] \times \Omega. \tag{6.5.5}$$

We will prove that $\tilde{T} = T_0 \le T_*^\varepsilon$. This fact, combined with the definition of $T_0^\varepsilon$ and estimates (6.5.4), (6.5.5), yield $T_0^\varepsilon \ge T_0$, which is a contradiction with the assumption $T_0^\varepsilon < T_0$. To continue, we shall need the claim stated and proved below. Indeed, once the following claim holds, we have by (6.5.4) that $\|(\sigma^\varepsilon, u^\varepsilon)(T_0)\|_{H^2(\Omega)} < +\infty$. Combined with the local existence result stated in Theorem 6.5.1, this yields that $T_*^\varepsilon > T_0$.

**Claim.** For all $\varepsilon \in (0,1]$, if $\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) < +\infty$, then $(\sigma^\varepsilon, u^\varepsilon) \in C([0,T], H^2)$, $u^\varepsilon \in L^2([0,T], H^3)$.

*Proof of claim.* We see from the definition of $\mathcal{N}_{m,T}$ and the estimate (6.3.134) that:

$$\varepsilon u^\varepsilon \in L^2([0,T],H^3), \quad \varepsilon\partial_t u^\varepsilon \in L^2([0,T],H^1), \quad \varepsilon\sigma^\varepsilon \in L^\infty([0,T],H^2).$$

one deduces from interpolation that $\varepsilon u^\varepsilon \in C([0,T],H^2)$. Moreover, carrying out direct energy estimates for $\sigma^\varepsilon$ in $H^2(\Omega)$, one gets that:

$$|\partial_t R^\varepsilon(t)| \le K^\varepsilon\big(R^\varepsilon(t)+f^\varepsilon(t)\big) \tag{6.5.6}$$

where $K^\varepsilon = \Lambda(1/c_0, |\!|\!|(\nabla\sigma^\varepsilon, \nabla u^\varepsilon, \varepsilon\nabla^2 u^\varepsilon)|\!|\!|_{\infty,t})$ is uniformly bounded and

$$R^\varepsilon(t) = \|\varepsilon\sigma^\varepsilon(t)\|_{H^2}^2, \quad f^\varepsilon(t) = \|\varepsilon u^\varepsilon(t)\|_{H^3}\|\sigma^\varepsilon(t)\|_{H^2} \in L^1([0,T]).$$

Inequality (7.13.7) and the boundedness of $\|R^\varepsilon(\cdot)\|_{L^\infty([0,T])}$ leads to the fact that $R^\varepsilon(\cdot)\in C([0,T])$, which further yields that $\varepsilon\sigma^\varepsilon \in C([0,T],H^2)$. This ends the proof of the claim. Note that at this stage we do not require the norm $\|(\sigma^\varepsilon,u^\varepsilon)\|_{C([0,T],H^2)}$ to be bounded uniformly in $\varepsilon$. □

## 6.6 Proof of Theorem 6.1.6

The convergence result follows from compactness arguments. At first, since $\sigma^\varepsilon = \frac{P(\rho^\varepsilon)-P(\bar\rho)}{\varepsilon}$ is uniformly bounded in $L^\infty([0,T_0],W^{1,\infty}(\Omega))\cap L^2([0,T_0],H^1(\Omega))$, we have that: $P(\rho^\varepsilon)\to P(\bar\rho)$ in $L^\infty([0,T_0],W^{1,\infty}(\Omega))\cap L^2([0,T_0],H^1(\Omega))$, which yields that $\rho^\varepsilon\to\bar\rho$ in $L^2([0,T_0],H^1(\Omega))$.

For the convergence of $u^\varepsilon$, let us split the velocity into compressible part and incompressible part: $u^\varepsilon = \nabla\Psi^\varepsilon + v^\varepsilon$ by using the Leray decomposition (6.3.2). we shall prove that the compressible part $\nabla\Psi^\varepsilon$ tends to 0 in $L^2_{t,w}H^1(\Omega)$ whereras incompressible part of $u^\varepsilon$ tends to $u^0$ in $L^2(Q_{T_0})$. Since $\nabla\Psi^\varepsilon$ is uniformly bounded in $L^2_t H^2(\Omega)$, we have that, up to the extraction of a subsequence (that we do not mention explicitely) $\nabla\Psi^\varepsilon$ converges to $\mathbb{Q}u^0$ in $L^2_w([0,T_0],H^1(\Omega))$. Nevertheless, by the equation (6.3.50), $\mathrm{div}u^\varepsilon$ tends to 0 in the sense of distribution, which leads to $\mathbb{Q}u^0=0$. Because of this, one can indeed see that, without any extraction of the subsequences, $\nabla\Psi^\varepsilon\to 0$ in $L^2_w([0,T_0],H^1(\Omega))$.

We are now in position to prove the convergence of $v^\varepsilon$. By the equation of $v^\varepsilon$ : $(6.3.5)_3$, $\partial_t v^\varepsilon$ is uniformly bounded in $L^2([0,T_0],H^{-1}(\Omega))$ whereras $v^\varepsilon$ is uniformly bounded in $L^2([0,T_0],H^1(\Omega))$. Therefore, by Aubin-Lions lemma, $\{v^\varepsilon\}$ is compact in $L^2(Q_{T_0})$, which yields, up to extraction of subsequences, the convergence of $v^\varepsilon$ (say to $u^0$) in $L^2(Q_{T_0})$.

In the following, we aim to justify that $u^0$ is the unique weak solution of the incompressible Navier-Stokes equation (6.1.3) satisfying (6.1.23). Let us rewrite the equations of $v^\varepsilon$ as follows:

$$\bar\rho\partial_t v^\varepsilon - \mu\Delta v^\varepsilon + \nabla\pi^\varepsilon = F^\varepsilon = F_1^\varepsilon + F_2^\varepsilon. \tag{6.6.1}$$

where

$$F_1^\varepsilon = -(\rho^\varepsilon-\bar\rho)(\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon), \quad F_2^\varepsilon = -\bar\rho(v^\varepsilon\cdot\nabla u^\varepsilon + \nabla\Psi^\varepsilon\cdot\nabla v^\varepsilon),$$

Note that we put the gradient terms $\bar\rho\nabla(\partial_t\Psi^\varepsilon + \frac12|\nabla\Psi^\varepsilon|^2)$ into the pressure $\nabla\pi^\varepsilon$. Let us write down the weak formulation for (6.6.1). Multiplying equation (6.6.1) by a test function $\psi\in(C^\infty([0,T_0]\times\overline{\Omega}))^3$ which satisfies $\mathrm{div}\psi=0, \psi\cdot\mathbf{n}|_{\partial\Omega}=0$, we obtain that for each $0<t\le T_0$,

$$\begin{aligned}&\bar\rho\int_\Omega (v^\varepsilon\cdot\psi)(t,\cdot)\,\mathrm{d}x + \mu\iint_{Q_t}\nabla v^\varepsilon\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}s + \iint_{Q_t} F^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s\\ &= \bar\rho\int_\Omega (v^\varepsilon\cdot\psi)(0,\cdot)\,\mathrm{d}x + \bar\rho\iint_{Q_t} v^\varepsilon\cdot\partial_t\psi\,\mathrm{d}x\mathrm{d}s + \mu\int_0^t\int_{\partial\Omega}\Pi\partial_{\mathbf{n}}v^\varepsilon\cdot\psi\,\mathrm{d}S_y\mathrm{d}s.\end{aligned} \tag{6.6.2}$$

It remains to pass to the limit to show that $u^0$ satisfies (6.1.24). We shall only focus on the last terms in both sides of (6.6.2), as the other terms are direct. Since $\rho^\varepsilon = g_2(\varepsilon\sigma^\varepsilon)$, we have that $(\rho^\varepsilon-\bar\rho)/\varepsilon$ is uniformly bounded in $L^\infty(Q_{T_0})$, it then follows from the velocity equation in $(6.1.15)_2$ that

$$\iint_{Q_t} F_1^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s = \iint_{Q_t}\frac{\rho^\varepsilon-\bar\rho}{\rho^\varepsilon}(\mathrm{div}\mathcal{L}u^\varepsilon - \frac{\nabla\sigma^\varepsilon}{\varepsilon})\psi\,\mathrm{d}x\mathrm{d}s.$$

We then observe that
$$\frac{1}{\varepsilon}\iint_{Q_t}\frac{\rho^\varepsilon-\bar\rho}{\rho^\varepsilon}\nabla\sigma^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s=\frac{1}{\varepsilon}\iint_{Q_t}\frac{g_2(\varepsilon\sigma^\varepsilon)-g_2(0)}{g_2(\varepsilon\sigma^\varepsilon)}\nabla\sigma^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s=0$$
by integrating by parts since
$$\frac{g_2(\varepsilon\sigma^\varepsilon)-g_2(0)}{g_2(\varepsilon\sigma^\varepsilon)}\nabla\sigma^\varepsilon=\frac{1}{\varepsilon}\nabla\left(G(\varepsilon\sigma^\varepsilon)\right)$$
where $G(s)$ is such that
$$G'(s)=\frac{g_2(s)-g_2(0)}{g_2(s)}.$$
In a similar way, we have that
$$\iint_{Q_t}\frac{\rho^\varepsilon-\bar\rho}{\rho^\varepsilon}\nabla\mathrm{div}u^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s=-\varepsilon\iint_{Q_t}\mathrm{div}u^\varepsilon\,G''(\varepsilon\sigma^\varepsilon)\nabla\sigma^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s$$
$$\iint_{Q_t}\frac{\rho^\varepsilon-\bar\rho}{\rho^\varepsilon}\Delta u^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s=-\varepsilon\iint_{Q_t}G''(\varepsilon\sigma^\varepsilon)\left((\nabla\sigma^\varepsilon\cdot\nabla)u^\varepsilon\right)\cdot\psi\,\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\partial\Omega}\frac{\rho^\varepsilon-\bar\rho}{\rho^\varepsilon}\Pi\partial_n u^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s.$$
These three above terms tend to zero, for the last one, we use that $\|\rho^\varepsilon-\bar\rho\|_{L^\infty(Q_t)}=O(\varepsilon)$ while $\Pi\partial_n u^\varepsilon$ is uniformly bounded in $L^2(\partial\Omega)$ by using the Navier-boundary condition and the trace inequality. This yields
$$\iint_{Q_t}F_1^\varepsilon\cdot\psi\mathrm{d}x\mathrm{d}s\to 0.$$
Next, since $\nabla\Psi^\varepsilon\rightharpoonup 0, \nabla u^\varepsilon\rightharpoonup\nabla u^0, v^\varepsilon\to u^0$ in $L^2(Q_t)$ and $v^\varepsilon$ is uniformly bounded in $L^2([0,T_0],H^1(\Omega))$, we have that:
$$\iint_{Q_t}F_2^\varepsilon\cdot\psi\,\mathrm{d}x\mathrm{d}s\to\bar\rho\iint_{Q_t}(u^0\cdot\nabla u^0)\cdot\psi\,\mathrm{d}x\mathrm{d}s.$$
Finally, for the boundary term in (6.6.2), we use the boundary condition for $v^\varepsilon$ (see (6.3.72)):
$$\Pi(\partial_{\mathbf{n}}v^\varepsilon)=\Pi(-2av^\varepsilon+(D\mathbf{n})v^\varepsilon)+2\Pi(-a\nabla\Psi^\varepsilon+(D\mathbf{n})\nabla\Psi^\varepsilon).$$
As $v^\varepsilon\to u^0$ in $L^2(Q_t)$ and $v^\varepsilon$ is uniformly bounded in $L^2([0,t],H^1(\Omega))$, $\nabla\Psi^\varepsilon\to 0$ in $L^2_w([0,t],H^1(\Omega))$, it follows from the trace inequality and the Hölder inequality that: $v^\varepsilon|_{\partial\Omega}\to u^0|_{\partial\Omega}$ in $L^2([0,t],L^2(\partial\Omega))$, $\nabla\Psi^\varepsilon\to 0$ in $L^2_w([0,t],L^2(\partial\Omega))$. This yields:
$$\mu\int_0^t\int_{\partial\Omega}\Pi\partial_{\mathbf{n}}v^\varepsilon\cdot\psi\,\mathrm{d}S_y\mathrm{d}s\to\mu\int_0^t\int_{\partial\Omega}\Pi(-2au^0+(D\mathbf{n})u^0)\cdot\psi\,\mathrm{d}S_y\mathrm{d}s.$$
Therefore, $u^0$ satisfies the formulation (6.1.24) and hence is a weak solution to (6.1.3). Next, due to the uniform boundedness of $v^\varepsilon$ in $L^\infty_{T_0}H^{m-1}_{co}$ and $\nabla v^\varepsilon$ in $L^2_{T_0}H^{m-1}_{co}\cap L^\infty(Q_{T_0})$, we get that $u^0$ has the additional regularity property (6.1.23). The uniqueness result is easy owing to the boundedness of the Lipschitz norm. Since any subsequence of $u^\varepsilon$ will have an extracted subsequence that solves (6.1.24) and satisfies the additional regularity property (6.1.23), we finally get from the uniqueness that the whole family $u^\varepsilon$ converges to $u^0$. This ends the proof of Theorem 7.14.1.

## 6.7 Appendix

We state here the product and commutator estimates which are used throughout the paper:

**Lemma 6.7.1.** *For each $0\le t\le T$, and for any integer $k\ge 2$, one has the (rough) product estimates*
$$\|(fg)(t)\|_{H^k_{co}}\lesssim\|f(t)\|_{H^k_{co}}\|g\|_{[\frac{k-1}{2}],\infty,t}+\|g(t)\|_{H^k_{co}}\|f\|_{[\frac{k}{2}],\infty,t},\tag{6.7.1}$$
$$\|(fg)(t)\|_{H^k_{co}}\lesssim\|f(t)\|_{H^k_{co}}\|g\|_{[\frac{k}{2}]-1,\infty,t}+\|g(t)\|_{H^{k-1}_{co}}\|f\|_{[\frac{k+1}{2}],\infty,t}+\|g(t)\|_{H^k_{co}}\|f\|_{\infty,t}.\tag{6.7.2}$$
*and commutator estimates:*
$$\|[Z^I,f]g(t)\|_{L^2}\lesssim\|Zf(t)\|_{H^{k-1}_{co}}\|g\|_{[\frac{k}{2}]-1,\infty,t}+\|g(t)\|_{H^{k-1}_{co}}\|Zf\|_{[\frac{k-1}{2}],\infty,t},\quad |I|=k.\tag{6.7.3}$$
$$\|[(\varepsilon\partial_t)^k,f]g(t)\|_{L^2}\lesssim\|(\varepsilon\partial_t f)(t)\|_{\mathcal{H}^{k-1}}\|g\|_{[\frac{k}{2}]-1,\infty,t}+\|g(t)\|_{\mathcal{H}^{k-1}}\|\varepsilon\partial_t f\|_{[\frac{k-1}{2}],\infty,t}.\tag{6.7.4}$$

*Proof.* This lemma follows from simply counting the derivatives hitting on $f$ or $g$. For instance, to prove the product estimate (6.7.1) and (6.7.2), one can use the following expansion:

$$\begin{aligned} Z^I(fg) &= \Big(\sum_{|J|\leq[(k-1)/2]} + \sum_{|I-J|\leq[k/2]}\Big)(C_{I,J}Z^J g Z^{I-J} f)\\ &= \Big(\sum_{|J|\leq[k/2]-1} + \sum_{1\leq|I-J|\leq[(k+1)/2]}\Big)(C_{I,J}Z^J g Z^{I-J} f) + fZ^I g, \qquad |I| = k. \end{aligned}$$

□

As a corollary of Lemma 7.1 the following composition estimates hold:

**Corollary 6.7.2.** *Suppose that* $h \in C^0(Q_t) \cap L_t^2 H_{co}^m$ *with*

$$A_1 \leq h(t,x) \leq A_2, \quad \forall (t,x) \in Q_t.$$

*Let* $F(\cdot) : [A_1, A_2] \to \mathbb{R}$ *be a smooth function satisfying*

$$\sup_{s\in[A_1,A_2]} |F^{(m)}|(s) \leq B.$$

*Then we have the composition estimate, for* $p = 2, +\infty$

$$\|F(h(\cdot,\cdot)) - F(0)\|_{L_t^p H_{co}^m} \leq \Lambda(B, |\!|\!|h|\!|\!|_{[\frac{m}{2}],\infty,t})\|h\|_{L_t^p H_{co}^m},$$

*where* $\Lambda(B, |\!|\!|h|\!|\!|_{[\frac{m}{2}],\infty,t})$ *is a polynomial with respect to* $B$ *and* $|\!|\!|h|\!|\!|_{[\frac{m}{2}],\infty,t}$.

This Corollary, combined with Lemma 6.1 and Lemma 6.3, leads to the following estimates:

**Corollary 6.7.3.** *Let* $g_1(\varepsilon\sigma), g_2(\varepsilon\sigma)$ *defined in* (6.1.16) *and assume that* (6.2.2)*,* (6.2.1) *hold. Then one has the following estimates: for* $j = 1, 2$, $p = 2, +\infty$,

$$\|Zg_j\|_{L_t^p \mathcal{H}^{m-1}} \leq \varepsilon\Lambda\big(\frac{1}{c_0}, |\!|\!|\sigma|\!|\!|_{[\frac{m}{2}],\infty,t}\big)\|(\sigma, Z\sigma)\|_{L_t^p \mathcal{H}^{m-1}}, \tag{6.7.5}$$

$$\|Zg_j\|_{L_t^p H_{co}^{m-1}} \leq \varepsilon\Lambda\big(\frac{1}{c_0}, |\!|\!|\sigma|\!|\!|_{[\frac{m}{2}],\infty,t}\big)\|\sigma\|_{L_t^p H_{co}^m}, \tag{6.7.6}$$

$$\|g_j(\varepsilon\sigma) - g_j(0)\|_{L_t^p H_{co}^m} \lesssim \varepsilon\Lambda\big(\frac{1}{c_0}, |\!|\!|\sigma|\!|\!|_{[\frac{m}{2}],\infty,t}\big)\|\sigma\|_{L_t^p H_{co}^m}. \tag{6.7.7}$$

We will use often the following Sobolev embedding inequality whose proof is similar to that of Proposition 12 and Proposition 20 of [93].

**Proposition 6.7.4.** *Let* $\Omega = \mathbb{R}_+^3$ *or a smooth bounded domain, we have the following Sobolev embedding inequality*

$$\|f(t)\|_{L^\infty(\Omega)} \lesssim \|\nabla f(t)\|_{H_{co}^{k+1}}^{\frac{1}{2}}\|f(t)\|_{H_{co}^{k+2}}^{\frac{1}{2}} + \|f(t)\|_{H_{co}^{k+2}}. \tag{6.7.8}$$

*Proof.* For the case of the half-space, this is a consequence of the inequality: for a function $g$ defined on $\mathbb{R}_+^3$,

$$\|f(t)\|_{L^\infty(\mathbb{R}_+^3)} \lesssim \|\partial_z f(t)\|_{H_{co}^{s_1}(\mathbb{R}_+^3)}^{\frac{1}{2}}\|f(t)\|_{H_{co}^{s_2}(\mathbb{R}_+^3)}^{\frac{1}{2}} \tag{6.7.9}$$

where $s_1, s_2$ are positive and satisfy $s_1 + s_2 > 2$. One can refer to (Prop 2.2) of [94] for the proof. The case of general smooth bounded domains follows by working in local coordinates. □

The following trace inequalities are also used:

**Lemma 6.7.5.** *For multi-index $I = (I_0, \cdots, I_M)$ with $|I| = k$, we have the following trace inequalities:*

$$|Z^I f(t)|^2_{L^2(\partial\Omega)} \lesssim \|\nabla f(t)\|_{H^k_{co}} \|f(t)\|_{H^k_{co}} + \|f(t)\|^2_{H^k_{co}}. \tag{6.7.10}$$

$$\int_0^t |Z^I f(s)|^2_{L^2(\partial\Omega)} \,\mathrm{d}s \lesssim \|\nabla f\|_{L^2_t H^k_{co}} \|f\|_{L^2_t H^k_{co}} + \|f\|^2_{L^2_t H^k_{co}}. \tag{6.7.11}$$

$$\int_0^t |Z^I f(s)|^2_{H^{\frac{1}{2}}(\partial\Omega)} \mathrm{d}s \lesssim \|\nabla f\|^2_{L^2_t H^k_{co}} + \|f\|^2_{L^2_t H^k_{co}}. \tag{6.7.12}$$

In the next proposition, we state some elliptic estimates which are used frequently.

**Proposition 6.7.6.** *Given a bounded domain $\Omega$ with $C^{k+1}$ boundary. Consider the following elliptic equation with Neumann boundary condition:*

$$\begin{cases} \Delta q = \mathrm{div} f \quad in \quad \Omega \\ \partial_{\boldsymbol{n}} q = f \cdot \boldsymbol{n} + g \quad on \quad \partial\Omega \\ \int_\Omega q \mathrm{d}x = 0 \end{cases} \tag{6.7.13}$$

*The system (6.7.13) has a unique solution in $H^1(\Omega)$ which satisfies the following gradient estimate:*

$$\|\nabla q(t)\|_{L^2(\Omega)} \lesssim \|f(t)\|_{L^2(\Omega)} + |g(t)|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{6.7.14}$$

*Moreover, for $j + l = k$,*

$$\|\nabla q(t)\|_{\mathcal{H}^{j,l}(\Omega)} \lesssim \|f(t)\|_{\mathcal{H}^{j,l}(\Omega)} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}(\partial\Omega)}. \tag{6.7.15}$$

$$\|\nabla^2 q(t)\|_{\mathcal{H}^{j,l}(\Omega)} \lesssim \|(f(t), \mathrm{div} f(t))\|_{\mathcal{H}^{j,l}(\Omega)} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}(\partial\Omega)}. \tag{6.7.16}$$

*Proof.* The existence of the weak solution in $H =: \{q |\ q \in H^1(\Omega), \int_\Omega q \mathrm{d}x = 0\}$ as well as the gradient estimate (6.7.14) come from Lax-Milgram Lemma. The estimates (6.7.15)-(6.7.16) are then standard regularity estimates for elliptic equations, that take into account the number of time derivatives (the time variable being only a parameter in this Lemma). □

Finally, we state an elementary estimate of the heat kernel which is useful in the estimates of the vorticity.

**Lemma 6.7.7.** *Let*

$$K(s, y, z) = \tilde{\mu}|\boldsymbol{N}|^2 (4\pi\tilde{\mu}|\boldsymbol{N}|^2 s)^{-\frac{1}{2}} \partial_z \big(e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2 s}}\big), \quad \boldsymbol{N}(y) = (-\partial_1\varphi(y), -\partial_2\varphi(y), 1)^t$$

*where $(y, z) \in \mathbb{R}^3_+$ and set $\mathcal{Z}^\beta = \partial^{\beta_1}_{y^1} \partial^{\beta_2}_{y^2} \mathcal{Z}^{\beta_3}_3$, $\mathcal{Z}_3 = \frac{z}{1+z}\partial_z$. We have the following estimate:*

$$\|\mathcal{Z}^\beta K(s, y, \cdot)\|_{L^2_z(\mathbb{R}_+)} \le C(\beta, \tilde{\mu}, |\varphi|_{C^{|\beta|+1}}) s^{-\frac{3}{4}}. \tag{6.7.17}$$

*Proof.* It suffices to prove that, for any $l \in \mathbb{N}$, there is a polynomial $P_{2|\beta|+1}$ with $2|\beta| + 1$ degree, such that:

$$|\mathcal{Z}^\beta K(s, y, z)| \le C(\beta, \tilde{\mu}, |\varphi|_{C^{|\beta|+1}}) P_{2|\beta|+1}\big(\frac{z}{\sqrt{s}}\big) e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2 s}} s^{-1} \quad \forall s > 0, y \in \mathbb{R}^2. \tag{6.7.18}$$

By direct computation, one can see that, there exists a polynomial with degree $2(\beta_1 + \beta_2) + 1 : P_{2(\beta_1+\beta_2)+1}$, a smooth function depends on $\nabla_y\varphi$ and its derivatives up to order $\beta_1 + \beta_2 : F_{\beta_1+\beta_2}(\nabla_y\varphi)$ such that

$$\partial^{\beta_1}_{y_1} \partial^{\beta_2}_{y_2} K(s, y, z) = P_{2(\beta_1+\beta_2)+1}\big(\frac{z}{\sqrt{s}}\big) F_{\beta_1+\beta_2}(\nabla_y\varphi) e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2 s}} s^{-1}.$$

To prove (6.7.18), it suffices to show by induction arguments that, there exists a smooth function $F(|\mathbf{N}|^2)$, such that

$$\partial^{\beta_3}_z \left( P_{2(\beta_1+\beta_2)+1}\big(\frac{z}{\sqrt{s}}\big) e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^2 s}} \right) = F(|\mathbf{N}|^2) e^{-\frac{z^2}{4\tilde{\mu}|\mathbf{N}|^s}} P_{2|\beta|+1}\big(\frac{z}{\sqrt{s}}\big) z^{-\beta_3}.$$

□

## 6.8 Appendix II-Uniform regularity for the compressible Navier-Stokes system with low Mach number in half space.

The aim of this section is to get uniform energy estimate when $\Omega = \mathbb{R}^3_+$ on some time interval $[0,T]$ where $T<1$ is independent of Mach number $\varepsilon$.

The target system to be considered becomes:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t\sigma^\varepsilon + u^\varepsilon\cdot\nabla\sigma^\varepsilon\big) + \dfrac{\mathrm{div}u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon)(\partial_t u^\varepsilon + u^\varepsilon\otimes\nabla u^\varepsilon\big) - \mathrm{div}\mathcal{L}u^\varepsilon + \dfrac{\nabla\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+. \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \sigma|_{t=0} = \sigma_0^\varepsilon. \\ u_3^\varepsilon = 0, \mu\partial_3 u_\tau^\varepsilon = -a u_\tau^\varepsilon \qquad on \quad z=0. \end{cases} \tag{6.8.1}$$

where $x=(y,z)\in\mathbb{R}^2\times\mathbb{R}_+$, the scalar function $g_2(s) = \rho^\varepsilon = P^{-1}(P(\bar\rho)+s), g_1(s) = (\ln g_2)'(s); s\in\mathbb{R}$.

We first give the precise definition of vector fields and conormal norms. Due to the simple geometry, we could define (globally) the vector fields that is tangential to boundary: (suppose $x=(y,z)$)

$$Z_1 = \partial_{y_1} = \partial_1, \quad Z_2 = \partial_{y_2} = \partial_2, \quad Z_3 = \phi(z)\partial_z = \frac{z}{1+z}\partial_z.$$

Note that the weight in front of the normal derivative is tailored to the boundary layer effects. We denote the conormal Sobolev space

$$L^p_t H^m_{co}\big(\mathbb{R}^3_+\big) = \{Z^\alpha f\in L^p([0,t];L^2(\mathbb{R}^3_+)), |\alpha|\le m\},$$

$$L^p_t H^m_{tan}\big(\mathbb{R}^3_+\big) = \{Z^\alpha f\in L^p([0,t];L^2(\mathbb{R}^3_+)), |\alpha|\le m, \alpha_3 = 0\},$$

where $\alpha = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$, $Z^\alpha = (\varepsilon\partial_t)^{\alpha_0}Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}$. Throughout this chapter, we will only use $p=2,\infty$.

For the notational brevity, we shall sometimes use the following notation,

$$\|f\|_{m,t} = \|f\|_{L^2_t H^m_{co}}, \quad \|f\|_{E^m,t} = \|f\|_{m,t} + \|\nabla f\|_{m-1,t}, \quad \|f\|_{tan,m,t} = \|f\|_{L^2_t H^m_{tan}}.$$

We use also the notation:

$$\|\!|f|\!\|_{m,\infty,t} = \sum_{|\alpha|\le m}\|Z^\alpha f\|_{L^\infty([0,t]\times\mathbb{R}^3_+)}.$$

Further, for each $t>0$, we denote:

$$\|f(t)\|_{H^m_{co}} := \sum_{|\alpha|\le m}\|Z^\alpha f(t)\|_{L^2(\mathbb{R}^3_+)}, \qquad \|f(t)\|_{H^m_{tan}} := \sum_{|\alpha|\le m,\alpha_3=0}\|Z^\alpha f(t)\|_{L^2(\mathbb{R}^3_+)}.$$

We will use the following quantities

$$\mathcal{N}^\varepsilon_{m,T} = \mathcal{E}^\varepsilon_{m,T} + \mathcal{A}^\varepsilon_{m,T} \tag{6.8.2}$$

with the energy norm:

$$\begin{aligned} \mathcal{E}^\varepsilon_{m,T} &:= \varepsilon(\|(\sigma^\varepsilon,u^\varepsilon)\|_{L^\infty_T H^m_{co}} + \|\nabla(\sigma^\varepsilon,u^\varepsilon)\|_{L^\infty_T H^{m-1}_{co}}) \\ &\quad + \|(\sigma^\varepsilon,u^\varepsilon)\|_{L^\infty_T H^m_{tan}} + \|(\nabla\sigma^\varepsilon,\mathrm{div}u^\varepsilon)\|_{L^\infty_T H^{m-2}_{co}} + \|\omega^\varepsilon_\tau\|_{L^\infty_T H^{m-1}_{co}} + \|\nabla(\sigma^\varepsilon,u^\varepsilon)\|_{L^2_T H^{m-1}_{co}} \end{aligned}$$

and the $L^\infty_{t,x}$ norm defined in (6.1.18).

We elaborate now the main ideas to prove the uniform energy estimates. We are not able to control the weighted normal derivatives $Z_3$ by direct energy estimates since $Z_3$ does not commute with the spatial derivative $\partial_3$. Our strategy is illustrated in the following three steps:

- Step 1. Control all the tangential derivatives $(\varepsilon\partial_t,\partial_1,\partial_2)$ by energy estimates.

• Step 2. Recover the weighted normal derivatives by the equations and induction arguments.

• Step 3. Control of vorticity. Since the vorticity is governed by a transport-diffusion system without any oscillations, we expect to control the high-order conormal estimates by energy estimates. However, due to the absence of the boundary information of $\partial_z \omega_\tau$, we need split the system into two systems where one of them carry on all the nonlinear term and initial data whereas the other one is just heat equation with nontrivial Dirichlet boundary conditions which admits explicit formulae.

As for $L^\infty_{t,x}$ estimate, we can prove for $\|\partial_z \sigma\|_{[\frac{m}{2}]-1,\infty,t}$ by reformulating the equation of $\partial_z\sigma$ which can be considered as an ODE and has an explicit formulae in Lagranian coordinates. Finally, for $\|\omega_\tau\|_{\infty,t}$, we just use the maximal principle of transport-diffusion equations.

**Proposition 6.8.1** (Uniform estimates). *Define $c_0 \in (0,1]$ such that:*

$$\forall s \in \big[-3\bar{c}\bar{P}, 3\bar{P}/\bar{c}\big],\ c_0 \leq g_i(s) \leq 1/c_0,\ i=1,\,2, \quad |(g_1,g_2)|_{C^m\left(\left[-3\bar{c}\bar{P},3\bar{P}/\bar{c}\right]\right)} \leq 1/c_0 \tag{6.8.3}$$

*where $\bar{c}$ is such that for some $T\in(0,1]$ the following assumption holds:*

$$-3\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 3\bar{P}/\bar{c} \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^3_+, \forall \varepsilon\in[0,1]. \tag{6.8.4}$$

*Then, there exists $C(1/c_0)>0$ and an increasing polynomial $\Lambda_0$ (whose coefficients are independent of $\varepsilon$), such that, for any $\varepsilon\in(0,1]$, we have for a smooth enough solution of* (6.8.1) *on $[0,T]$ the following estimate:*

$$\mathcal{N}^2_{m,T}(\sigma^\varepsilon,u^\varepsilon) \leq C(1/c_0)Y^2_m(\sigma^\varepsilon_0,u^\varepsilon_0) + (T+\varepsilon)^{\frac12}\mathcal{N}^2_{m,T}(\sigma^\varepsilon,u^\varepsilon)\Lambda_0(1/c_0,\mathcal{A}_{m,T}), \tag{6.8.5}$$

*where $Y^\varepsilon_m(0)$ is defined by:*

$$Y^\varepsilon_m(0) = \|(\sigma^\varepsilon,u^\varepsilon)(0)\|_{H^m_{co}} + \|\nabla(\sigma^\varepsilon,u^\varepsilon)(0)\|_{H^{m-1}_{co}} + \sum_{|\alpha|\leq[\frac{m}{2}]-1}\|Z^\alpha\nabla(\sigma^\varepsilon,u^\varepsilon)(0)\|_{0,\infty,t}. \tag{6.8.6}$$

*In particular, we have the following uniform estimates:*

$$\|(\sigma^\varepsilon,u^\varepsilon)\|_{L^\infty_t H^{m-1}_{co}\cap L^2_t H^m_{co}} + \|\omega^\varepsilon_\tau\|_{L^\infty_t H^{m-1}_{co}} + \|(\nabla\sigma^\varepsilon,\mathrm{div}u^\varepsilon)\|_{L^\infty_t H^{m-2}_{co}\cap L^2_t H^{m-1}_{co}} + \|\nabla(\sigma,u)\|_{0,\infty,t} < +\infty.$$

### 6.8.1 Control of the energy norms

In this subsection, we establish the a-priori estimates for the energy norm $\mathcal{E}_{m,T}$. Again, for notational convenience, we skip the $\varepsilon-$dependence of the solutions.

**Proposition 6.8.2.** *Under the assumption* (6.8.4)*, the following energy inequality holds:*

$$\mathcal{E}_{m,T} \lesssim Y_m(0) + (T+\varepsilon)^{\frac14}\Lambda(\mathcal{A}_{m,T})\mathcal{E}_{m,T}. \tag{6.8.7}$$

*Proof.* This proposition shall be the consequence of the following Lemmas. □

• Step 1: Control of the tangential derivative. Analogous to (6.3.2), one can show

**Lemma 6.8.3.** *Suppose that* (6.8.4) *is satisfied, then for any $m\geq 0$, any $0<T\leq 1$ and $\varepsilon\in(0,1]$ we have:*

$$\begin{aligned}
&\|(\sigma,u)\|^2_{L^\infty_T H^m_{tan}} + \varepsilon^2(\|(\sigma,u)\|^2_{L^\infty_T H^m_{co}} + \|(\nabla\sigma,\mathrm{div}u)\|^2_{L^\infty_T H^{m-1}_{co}})\\
&\quad + \|\nabla u\|^2_{L^2_t H^m_{tan}} + \varepsilon^2(\|\nabla u\|^2_{L^2_T H^m_{co}} + \|\nabla\mathrm{div}u\|^2_{L^2_T H^{m-1}_{co}})\\
&\lesssim Y^2_m(\sigma_0,u_0) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,T}.
\end{aligned} \tag{6.8.8}$$

The only difference between this lemma an (6.3.2) is that here we can control the highest spatial tangential derivatives directly since they commute with the singular terms. The proof of these two lemmas are almost identical, we thus omit the proof.

In the next two steps, we aim to prove $L_t^\infty H_{co}^{m-2}$ and $L_t^2 H_{co}^{m-1}$ norms of $\nabla(\sigma,u)$. To estimate $\nabla u$, it suffices for us to control $\partial_z u$, or more precisely: $\mathrm{div}u$ and $\omega_\tau$. We thus shall establish the estimate of $(\nabla\sigma, \mathrm{div}u)$ and $\omega_\tau$ respectively in the following two steps.

• Step 2: Recovering the weighted normal derivatives of $(\nabla\sigma, \mathrm{div}u)$ by induction.

In this step, we shall use the equations and induction arguments to recover the weighted normal derivatives for $(\nabla\sigma, \mathrm{div}u)$ : $\|\nabla\sigma, \mathrm{div}u\|_{L_t^2 H_{co}^{m-1}\cap L_t^\infty H_{co}^{m-2}}$.

**Lemma 6.8.4.** *Suppose that* (6.8.4) *holds true, we have the estimates:*

$$\|(\nabla\sigma, \mathrm{div}u)\|^2_{L_t^2 H_{co}^{m-1}} \lesssim Y_m^2(0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,T}. \tag{6.8.9}$$

$$\|(\nabla\sigma, \mathrm{div}u)\|^2_{L_t^\infty H_{co}^{m-2}} \lesssim Y_m^2(0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,T} \tag{6.8.10}$$

*More precisely, we have the following $L_t^2 H_{co}^{m-1}$ control of* $(\nabla\sigma, \mathrm{div}u)$

$$\begin{aligned}\|(\nabla\sigma, \mathrm{div}u)\|_{L_t^2 H_{co}^{m-1}} &\lesssim T^{\frac{1}{2}}\|(\sigma,u)\|_{L_t^\infty H_{tan}^m} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\|(\sigma,u)\|_{E^m,t}\\ &\quad + \varepsilon(\|\nabla u\|_{L_t^2 H_{co}^m} + \|\nabla \mathrm{div}u\|_{L_t^2 H_{co}^{m-1}}).\end{aligned}$$

*Moreover, for any integer $j,l \geq 0$ and $j+l\leq m-2$, the following estimates hold:*

$$\begin{aligned}\|Z_3^l(\partial_z\sigma, \mathrm{div}u)(t)\|_{H^j_{tan}} &\lesssim \|(\sigma,u)\|_{L_t^\infty H_{tan}^{m-1}} + \varepsilon\|(\nabla\sigma, \mathrm{div}u)\|_{L_t^\infty H_{tan}^{m-1}}\\ &\quad + \varepsilon\|\omega_\tau\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.\end{aligned}$$

The proof of this Lemma is similar to (6.3.11), we thus skip the detail.

• Step 3: Conormal energy estimate for $\omega_\tau$.

The next lemma is devoted to the control of $L_t^\infty H_{co}^{m-1}$ norm of $\omega_\tau$.

**Lemma 6.8.5.** *Suppose that* (6.8.4) *holds, then for any $t\in[0,T]$,*

$$\|\omega_\tau\|^2_{L_t^\infty H_{co}^{m-1}} + \|Z_3\omega_\tau\|^2_{L_t^2 H_{co}^{m-1}} \lesssim \|\omega_\tau(0)\|^2_{H_{co}^{m-1}} + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}. \tag{6.8.11}$$

**Remark 6.8.6.** *This lemma indicates that one could expect the better estimate: $\|\omega_\tau\|_{L_t^\infty H_{co}^{m-1}}$. However, since* $\mathrm{div}u$ *only belongs to $L_t^\infty H_{co}^{m-2}$, we could only get the control of $\|\nabla u\|_{L_t^\infty H_{co}^{m-2}}$.*

*Proof.* Taking the curl of the velocity equation, and keeping the tangential part, we find that $\omega_\tau$ satisfies:

$$\bar\rho\partial_t\omega_\tau - \mu\Delta\omega_\tau = J = J_1 + J_2 + J_3 + J_4$$

where

$$\begin{aligned} J_1 &= -\frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t\omega_\tau, & J_2 &= -g_2 u\cdot\nabla\omega_\tau,\\ J_3 &= [(\nabla g_2/\varepsilon)\times(\varepsilon\partial_t u + \varepsilon u\cdot\nabla u)]_\tau, & J_4 &= g_2(\omega_\tau \mathrm{div}u - \omega\cdot\nabla u_\tau).\end{aligned}$$

We split $\omega_\tau$ into two pieces: $\omega_\tau = \omega_{\tau,nh} + \omega_{\tau,h}$, where $\omega_{\tau,h}$ satisfies the homogeneous heat equation:

$$\begin{cases} \bar\rho\partial_t\omega_{\tau,h} - \mu\Delta\omega_{\tau,h} = 0 \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\ \omega_{\tau,h}|_{t=0} = 0, \qquad \omega_{\tau,h}|_{z=0} = \omega_\tau|_{z=0},\end{cases}$$

while $\omega_{\tau,nh}$ solves the equations:

$$\begin{cases} \bar\rho\partial_t\omega_{\tau,nh} - \mu\Delta\omega_{\tau,nh} = J, \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+\\ \omega_{\tau,nh}|_{t=0} = \omega_\tau|_{t=0}, \quad \omega_{\tau,nh}|_{z=0} = 0.\end{cases}$$

Lemma 6.8.5 shall be the consequence of the following two lemmas. □

**Lemma 6.8.7.** *For any $0 \leq t \leq T$, the following estimates holds,*

$$\|\omega_{\tau,h}\|_{L_t^\infty H_{co}^{m-1}} \lesssim T^{\frac{1}{4}}\|(u_\tau, \nabla u_\tau)\|_{L_t^\infty H_{tan}^{m-1}}; \quad \|\omega_{\tau,h}\|_{L_t^2 H_{co}^m} \lesssim T^{\frac{1}{4}}\|(u_\tau, \nabla u_\tau)\|_{L_t^2 H_{tan}^m}. \tag{6.8.12}$$

*Proof.* For $\gamma = (\gamma', \gamma_3)$ and $|\gamma| \leq m-1$. Taking $Z_{tan}^{\gamma'}$ on the equation of $\omega_{\tau,h}$, we find that: $Z_{tan}^{\gamma'}\omega_{\tau,h}$ satisfies the equations:

$$\begin{cases} \bar{\rho}\partial_t(Z_{tan}^{\gamma'}\omega_{\tau,h}) - \mu\Delta(Z_{tan}^{\gamma'}\omega_{\tau,nh}) = 0 \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+^3 \\ \omega_{\tau,h}|_{t=0} = 0, \qquad \omega_{\tau,h}|_{z=0} = Z_{tan}^{\gamma'}\omega_\tau|_{z=0} = aZ_{tan}^{\gamma'}u_\tau^\perp|_{z=0}. \end{cases} \tag{6.8.13}$$

It is well known that equation (6.8.13) has the explicit formula:

$$Z_{tan}^{\gamma'}\omega_{\tau,h}(t,x) = 2a\tilde{\mu}\int_0^t\int_{\mathbb{R}^2} \partial_3 E(t-s, y-y', z)(Z_{tan}^{\gamma'}u_\tau^\perp|_{z=0})\mathrm{d}s\mathrm{d}y'$$

where $x = (y,z), \tilde{\mu} = \mu/\bar{\rho}$ and $E(t,x) = (4\pi\tilde{\mu}t)^{-\frac{3}{2}}e^{-\frac{|x|^2}{4\tilde{\mu}t}}$. We then take $Z_3^{\gamma_3}$ on this expression to get:

$$\begin{aligned} Z^\gamma\omega_{\tau,h}(t,x) &= 2a\tilde{\mu}\int_0^t\int_{\mathbb{R}^2} Z_3^{\gamma_3}\partial_3 E(t-s, y-y', z)(Z_{tan}^{\gamma'}u_\tau^\perp)\mathrm{d}s\mathrm{d}y' \\ &= 2a\tilde{\mu}\int_0^t K_{\gamma_3}(t-s, z)\int_{\mathbb{R}^2}(4\pi\tilde{\mu}(t-s))^{-1}e^{-\frac{|y-y'|^2}{4\tilde{\mu}(t-s)}}(Z_{tan}^{\gamma'}u_\tau^\perp|_{z=0})(s,y')\mathrm{d}y'\mathrm{d}s \end{aligned}$$

where $K_{\gamma_3}(s,z) = (4\pi\tilde{\mu}s)^{-\frac{1}{2}}Z_3^{\gamma_3}\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}s}}\big)$.

With the aid of the Young's inequality, trace inequality and (6.8.16) in the appendix, one has that:

$$\begin{aligned} \|\omega_{\tau,h}^\gamma(t)\|_{L^2(\mathbb{R}_+^3)} &\lesssim \int_0^t \|K_{\gamma_3}(t-s,\cdot)\|_{L_z^2(\mathbb{R}_+)}|(Z_{tan}^{\gamma'}u_\tau^\perp)(s,\cdot)|_{z=0}|_{L_y^2(\mathbb{R}^2)}\mathrm{d}s \\ &\lesssim \int_0^t (t-s)^{-\frac{3}{4}}\mathrm{d}s|Z_{tan}^{\gamma'}u_\tau^\perp|_{z=0}|_{L_t^\infty L_y^2} \\ &\lesssim T^{\frac{1}{4}}\|Z_{tan}^{\gamma'}u_\tau^\perp\|_{L_t^\infty L_x^2}^{\frac{1}{2}}\|\nabla Z_{tan}^{\gamma'}u_\tau^\perp\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \lesssim T^{\frac{1}{4}}\|(u_\tau, \nabla u_\tau)\|_{L_t^\infty H_{tan}^{m-1}}. \end{aligned}$$

which is the first inequality (6.8.12). The second one follows in the same fashion. □

**Lemma 6.8.8.** *We have the following energy estimates for $\omega_{\tau,nh}$:*

$$\|\omega_{\tau,nh}\|_{L_t^\infty H_{co}^{m-1}}^2 + \|\nabla\omega_{\tau,nh}\|_{L_t^2 H_{co}^{m-1}}^2 \lesssim \|\omega_\tau(0)\|_{H_{co}^{m-1}}^2 + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2. \tag{6.8.14}$$

The proof of this lemma bear some resemblance with Lemma 6.3.16, we thus skip the detail.

### 6.8.2 $L_{t,x}^\infty$ estimate.

We has the following estimate for the $L_{t,x}^\infty$ type norms whose proof is omitted.

**Proposition 6.8.9.** *Supposing that* (6.8.4) *holds, then one has the following control of the $L_{t,x}^\infty$ norm:*

$$\mathcal{A}_{m,T} \lesssim Y_m(0) + \mathcal{E}_{m,T} + (\varepsilon + T)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big). \tag{6.8.15}$$

### 6.8.3 An elementary lemma.

We state and prove an elementary but very useful lemma.

**Lemma 6.8.10.** *Let $K_l(s,y,z) = \tilde{\mu}(4\pi\tilde{\mu}s)^{-\frac{1}{2}}Z_3^l\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}s}}\big)$, where $l \in \mathbb{N}, (y,z) \in \mathbb{R}_+^3$, we have the following estimate:*

$$\|K_l(s,y,\cdot)\|_{L_z^2(\mathbb{R}_+)} \leq C(\beta,\tilde{\mu})s^{-\frac{3}{4}}. \tag{6.8.16}$$

*Proof.* At first, by induction, one has that: $Z_3^l \partial_z = \sum_{0\le j\le l} C_{l-j}(\phi)\phi^j(z)\partial_z^{j+1}$ where $C_{l-j}(\phi)$ are smooth functions depending on $\phi$ and its $l-j$ derivatives which are uniformly bounded. Thus, to prove (6.8.17), it suffices to show that, for any $j \ge 0$, there is a polynomial with degree $2j+1$, such that:

$$\partial_z^{j+1}(e^{-\frac{z^2}{4\tilde{\mu}s}}) = e^{-\frac{z^2}{4\tilde{\mu}s}} P_{2j+1}(\frac{z}{\sqrt{s}}) z^{-j} s^{-\frac{1}{2}}. \tag{6.8.17}$$

We will prove this claim by induction on $j$. At first, when $j=0$, we have that:

$$\partial_z(e^{-\frac{z^2}{4\tilde{\mu}s}}) = -\frac{z}{2\tilde{\mu}\sqrt{s}} e^{-\frac{z^2}{4\tilde{\mu}s}} s^{-\frac{1}{2}} := P_1(\frac{z}{\sqrt{s}}) e^{-\frac{z^2}{4\tilde{\mu}s}} s^{-\frac{1}{2}},$$

which yields (6.8.17) for $j=0$. Suppose now (6.8.17) holds for $j = j_0 - 1 \ge 0$, we show that it is true for $j = j_0$. By direct computations, one has that:

$$\begin{aligned}
\partial_z^{j_0+1}(e^{-\frac{z^2}{4\tilde{\mu}s}}) &= \partial_z(\partial_z^{j_0} e^{-\frac{z^2}{4\tilde{\mu}s}}) \\
&= \partial_z(e^{-\frac{z^2}{4\tilde{\mu}s}} P_{2j_0-1}(\frac{z}{\sqrt{s}}) z^{-(j_0-1)} s^{-\frac{1}{2}}) \\
&= \big(-\frac{z^2}{2\tilde{\mu}s} P_{2j_0-1}(\frac{z}{\sqrt{s}}) + P'_{2j_0-1}(\frac{z}{\sqrt{s}})\frac{z}{\sqrt{s}} - (j_0-1)P_{2j_0-1}(\frac{z}{\sqrt{s}})\big) e^{-\frac{z^2}{4\tilde{\mu}s}} z^{-j_0} s^{-\frac{1}{2}} \\
&:= P_{2j_0+1}(\frac{z}{\sqrt{s}}) e^{-\frac{z^2}{4\tilde{\mu}s}} z^{-j_0} s^{-\frac{1}{2}}
\end{aligned}$$

which proves (6.8.17) for $j = j_0$. We thus finish the proof of (6.8.17) and therefore (6.8.16). $\square$

# Chapter 7

# Incompressible limit for free surface Navier-Stokes equations

Taken from [96] which is to be submitted soon, this chapter is a joint work with Professors F. Rousset and N. Masmoudi.

**Abstract:** We establish uniform energy estimates with respect to the Mach number for the three dimensional free surface compressible Navier-Stokes system in the case of slightly well-prepared initial data and justify the low Mach number limit.

## 7.1 Introduction

We consider the motion of a slightly compressible viscous fluid with a free surface. It takes the following form:

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon w^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon w^\varepsilon) + \mathrm{div}(\rho^\varepsilon w^\varepsilon \otimes w^\varepsilon) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega_t^\varepsilon \\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \ w^\varepsilon|_{t=0} = w_0^\varepsilon \end{cases} \tag{7.1.1}$$

where $\rho^\varepsilon > 0, w^\varepsilon \in \mathbb{R}^3$ are the density and the velocity of the fluid, $P(\rho^\varepsilon)$, a smooth function of $\rho^\varepsilon$ with $\frac{\mathrm{d}P}{\mathrm{d}\rho} > 0$, stands for the pressure. The stress tensor $\mathcal{L}w^\varepsilon$ takes the form:

$$\mathcal{L}w^\varepsilon = 2\mu S w^\varepsilon + \lambda \mathrm{div} w^\varepsilon \mathrm{Id}, \quad S w^\varepsilon = \frac{1}{2}(\nabla w^\varepsilon + \nabla^t w^\varepsilon).$$

Here, $\mu, \lambda$ are viscosity parameters that are assumed to be constant and to satisfy the condition: $\mu > 0, 2\mu + 3\lambda > 0$. The parameter $\varepsilon$ is the scaled Mach number which is assumed small, that is $\varepsilon \in (0,1]$. We focus on the case where the fluid domain is given by:

$$\Omega_t^\varepsilon = \{x = (y,z) |\ y \in \mathbb{R}^2, -1 < z < h^\varepsilon(t,y)\},$$

where the upper surface is free and the bottom is fixed. Here $h^\varepsilon(t,y)$, the surface of the fluid domain, is unknown and needs to be solved together with $(\rho^\varepsilon, w^\varepsilon)$. Since the fluid particles do not cross the surface, $h^\varepsilon$ solves

$$\partial_t h^\varepsilon - w^\varepsilon(t,y,h^\varepsilon(t,y)) \cdot \mathbf{N}^\varepsilon = 0, \quad h^\varepsilon(0,y) = h_0^\varepsilon(y) \quad y \in \mathbb{R}^2 \tag{7.1.2}$$

where $\mathbf{N}^\varepsilon = (-\partial_1 h^\varepsilon, -\partial_2 h^\varepsilon, 1)^t$ denotes the outward normal vector on the surface $\Sigma_t^\varepsilon = \{x = (y,z), z = h^\varepsilon(t,y)\}$. We supplement the system (7.1.1) and (7.1.2) with the following physical condition: at the upper boundary, the continuity of the stress tensor reads:

$$\mathcal{L}u^\varepsilon \mathbf{N}^\varepsilon = \frac{1}{\varepsilon^2}\big(P(\rho^\varepsilon) - P(\bar\rho)\big)\mathbf{N}^\varepsilon \quad \text{on} \quad \Sigma_t^\varepsilon, \tag{7.1.3}$$

where $\bar{\rho} > 0$ is the reference constant density. At the bottom, we prescribe a slip boundary condition:

$$w_3^\varepsilon = 0, \quad \mu\partial_3 w_j^\varepsilon = a w_j^\varepsilon \quad (j = 1, 2), \quad \text{on} \quad \{z = -1\}, \tag{7.1.4}$$

where $a$ is a constant that quantifies the effects of the friction at the boundary. The case of Dirichlet boundary condition on the bottom is harder and would be left for the future's work. Note that we could also consider the case of an unbounded strip.

Involving one term that depends on the small parameter $\varepsilon$, the system (7.1.1) can be viewed as a result of a suitable scaling of the original physical variables. Indeed, we get (7.1.1), (7.1.2) by performing the following scaling:

$$\tilde{\rho}(t, x) = \rho^\varepsilon(\varepsilon t, x),\ \tilde{w}(t, x) = \varepsilon w^\varepsilon(\varepsilon t, x),\ \tilde{h} = h^\varepsilon(\varepsilon t, x),\ \tilde{\mu} = \varepsilon\mu,\ \tilde{\lambda} = \varepsilon\lambda,$$

where $\tilde{\rho}, \tilde{u}, \tilde{h}$ satisfies:

$$\begin{cases} \partial_t\tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{w}) = 0, \\ \partial_t(\tilde{\rho}\tilde{w}) + \mathrm{div}(\tilde{\rho}\tilde{w} \otimes \tilde{w}) - \mathrm{div}\tilde{\mathcal{L}}\tilde{w} + \nabla P(\tilde{\rho}) = 0, \\ \partial_t\tilde{h} + \tilde{w}(t, y, \tilde{h}(t, y)) \cdot \tilde{\mathbf{N}} = 0. \end{cases} \tag{7.1.5}$$

where $\tilde{\mathcal{L}}\tilde{w} = 2\tilde{\mu}S\tilde{w} + \tilde{\lambda}\mathrm{div}\tilde{w}$.

The aim of this chapter is to study the low Mach number limit problem, that is to study the behavior of (strong) solutions to (7.1.1) when $\varepsilon$ tends to 0. Formally, due to singular term $\frac{\nabla P(\rho^\varepsilon)}{\varepsilon^2}$, the pressure (and hence the density $\rho^\varepsilon$) is expected to tend to a constant state in some suitable space, one thus expect that the limit of the solutions to (7.1.1) will be the solution to the following incompressible free surface Navier-Stokes system:

$$\begin{cases} \bar{\rho}(\partial_t w^0 + w^0 \cdot \nabla w^0) - 2\mu \mathrm{div}\, S w^0 + \nabla\pi^0 = 0, \\ \mathrm{div}\, w^0 = 0, \qquad\qquad\qquad\qquad\qquad (t, x) \in \mathbb{R}_+ \times \Omega_t^0 \\ \partial_t h^0 - w^0(t, y, h^0(t, y)) \cdot \mathbf{N}^0 = 0, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ w^0|_{t=0} = w_0^0,\ h^0|_{t=0} = h_0^0. \end{cases} \tag{7.1.6}$$

supplemented with the boundary conditions:

$$S w^0 \mathbf{N}^0 = \pi^0 \mathbf{N}^0 \quad \text{on} \quad \{z = h^0(t, y)\},$$
$$w_3^0 = 0, \quad \partial_3 w_j^0 = a w_j^0\, (j = 1, 2) \quad \text{on} \quad \{z = -1\}.$$

The rigorous justification of the low Mach number limit has been drawn much research attention since the last four decades. It has been studied extensively in different contexts depending on the generality of the system (isentropic or non-isentropic), the type of the system (Navier-Stokes or Euler), the type of solutions (strong solutions or weak solutions), the properties of the domain (without boundaries, with fixed or free boundaries), as well as the type of the initial data considered (well-prepared or ill-prepared). There have been enormous works dealing with this issue when the fluid is occupied in the domain without boundary or with a fixed boundary. The mathematical justification of the low Mach number limit was initiated by Ebin [39], Klainerman-Majda [79, 80] for *local strong solutions* of compressible fluids (Euler or Navier-Stokes), in the whole space with well-prepared data ($\mathrm{div}\, u_0^\varepsilon = \mathcal{O}(\varepsilon), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon^2)$) and later, by Ukai [125] for ill-prepared data ($\mathrm{div}\, u_0^\varepsilon = \mathcal{O}(1), \nabla P_0^\varepsilon = \mathcal{O}(\varepsilon)$). These works are then extended by several authors in different settings. One can refer for instance to [2, 100, 101, 20] for non-isentropic (Euler or Navier-Stokes) equations under 'ill-prepared' initial data whenever the domain is the whole space or torus, and also [109, 74] for bounded domains with well-prepared initial data. There are also many other related works, one can see for example [1, 14, 30, 34, 32, 41, 45, 54, 71, 89, 90]. For more exhaustive information, one can refer for example to the well-written survey papers by Alazard [3], Danchin [31], Feireisl [43], Gallagher [46], Jiang-Masmoudi [73], Schochet [110].

Let us review several studies on the low Mach number limit problem of the isentropic compressible Navier-Stokes (CNS) system in domains with fixed boundaries, which is related to the interest of the current paper. Roughly speaking, for (CNS), one can either justify the limit process directly from global weak solutions, or prove the local strong solutions exists on a time interval independent of the Mach number and use compactness arguments to pass to the limit. For the first case, Lions and Masmoudi [89] investigated the convergence of *weak solutions* to (CNS) in bounded domains with Navier-slip boundary condition. Later on, for the same problem in bounded domains with Dirichlet boundary condition, the authors in [35, 72] noticed that under some geometric assumption on the domain, the acoustic waves are damped in a boundary layer so that local in time strong convergence ($L^2_{t,x}$) holds. One can also refer to [41, 42] for the justification of convergence in unbounded domains by using the local energy decay for the acoustic system. All these results hold true for the ill-prepared initial data. For the second case concerning the local strong solution, uniform high order energy estimates are established in [74] with Dirichlet boundary condition and in [105] with Navier-slip boundary conditions by assuming the initial data to be well-prepared. Recently, we establish in [95] the uniform high regularity estimates in bounded domains with Navier-slip boundary condition and ill-prepared initial data. To match the boundary layer effects due to the fast oscillations and the ill-prepared initial data assumption, we prove uniform estimates in an anisotropic functional framework with only one normal derivative close to the boundary.

Concerning the low Mach number limit problem for the systems in the presence of free boundaries, there are a few works on inviscid compressible systems. In [88], Lindblad-Luo prove uniform a-priori estimates for free boundary compressible Euler equations in the case of a bounded reference domain. More recently, this result is extended by Luo [92] for unbounded reference domains and by Disconzi-Luo [37] for a bounded reference domain but with surface tension. All of these results are based on the assumption that the initial datum is sufficiently well-prepared in the sense that the time derivatives up to at least order two are bounded initially. Regarding the viscous fluid, Ou considers in [104] the 1d compressible Navier-Stokes system with free boundaries and established the uniform estimates with the Mach number and the Froude number for both 'well-prepared' and 'ill-prepared' initial data. Nevertheless, within our knowledge, there is no related work for the multidimensional viscous system. Indeed, in the multi-dimension case, there will be some severe difficulties which do not appear in the 1d case. For instance, the uniform estimates of high order spatial derivatives cannot be obtained by those of (weighted) time derivatives and as will be explained later, a boundary layer will appear in the multi-dimension case due to the appearance of the boundaries. The objective of the current work is to investigate the low Mach number limit problem for the 3d viscous fluids. For the simplicity of presentation (compared to the case of general bounded domains ) and without losing generality (compared to the half-space case), we choose a channel with finite depth as the reference domain. Nevertheless, one can extend without much trouble to the cases where the reference domain is a half-space or a bounded domain, we shall explain more about this aspect in Section 7.15.

The main contribution of this chapter is to establish some uniform high regularity estimates, in order to get the existence of a local strong solution on a time interval independent of $\varepsilon$. Nevertheless, due to the presence of the diffusion term as well as the singular linear term, a boundary layer correction to the highly oscillating acoustic waves appear and create unbounded high order normal derivatives of the velocity. Therefore, we work in the functional framework based on conormal Sobolev space that minimizes the use of normal derivatives near the boundary in spirit of [94, 99]. Note that in the current situation, we have to handle simultaneously the fast oscillations in time and a boundary layer effect so that the difficulties and the analysis will be very different from the ones in [99], where the compressible slightly viscous fluids are considered. Indeed, the energy estimates for conormal derivatives cannot be easily obtained since tangential vector fields do not commute with the singular part of the system, while to include only slightly well-prepared data, it will be impossible for us to get uniform estimates for high order time derivatives. As a preparation, in the previous chapter, we get around these difficulties and establish uniform estimates for isentropic compressible Navier-Stokes equation with Navier boundary condition and ill-prepared initial data, in a fixed domain with smooth enough boundaries. Nevertheless, in a moving domain, there would be extra difficulties due to the appearance of the free surface. Indeed, because of the occurrence of the singular terms, the compressible part of the system behaves at time scale $\tau = t/\varepsilon$ like the small viscosity approximation of the Navier-Stokes equation, we thus cannot obtain extra regularity for the surface from the diffusion term. Therefore, we are forced to allow that the initial data to be slightly well-prepared in the sense that the first time derivative of the solution is still unbounded but has a size of order $\varepsilon^{-\frac{1}{2}}$ (slightly better than $\varepsilon^{-1}$ in the definition of ill-prepared data) see also Remark (7.1.2). We shall explain more below after the reformulation of the system and the statement of the main results.

### 7.1.1 Appropriate change of variable

Denote

$$\varrho^\varepsilon = \frac{P(\rho) - P(\bar\rho)}{\varepsilon},$$

the system (7.1.1) can be rewritten into the following symmetric form:

$$\begin{cases} g_1(\varepsilon\varrho^\varepsilon)\big(\partial_t\varrho^\varepsilon + w^\varepsilon\cdot\nabla\varrho^\varepsilon\big) + \dfrac{\mathrm{div} w^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\varrho^\varepsilon\big)(\partial_t w^\varepsilon + w^\varepsilon\cdot\nabla w^\varepsilon\big) - \mathrm{div}\mathcal{L}w^\varepsilon + \dfrac{\nabla\varrho^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega_t^\varepsilon \\ w^\varepsilon|_{t=0} = w_0^\varepsilon, \quad \varrho^\varepsilon|_{t=0} = \varrho_0^\varepsilon. \end{cases} \tag{7.1.7}$$

where the scalar functions $g_1, g_2$ are defined by:

$$g_2(s) = \rho^\varepsilon = P^{-1}(P(\bar\rho)+s), \quad g_1(s) = (\ln g_2)'(s); \quad s > -\bar P = -P(\bar\rho). \tag{7.1.8}$$

Moreover, the boundary condition (7.1.3) is transformed into

$$\mathcal{L}u^\varepsilon\mathbf{N}^\varepsilon = \frac{\varrho^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon \quad \text{on} \quad \Sigma_t^\varepsilon \tag{7.1.9}$$

Therefore, in the following, we shall work on the system (7.1.7), (7.1.2) with boundary conditions (7.1.4), (7.1.9).

We first choose some appropriate change of coordinates to reduce the free-surface domain to a fixed one. One natural possibility is to use Lagrangian coordinates, nevertheless, since we shall consider the problem conormal Sobolev setting, the Lagrangian transformation would be only bounded in the conormal setting. Therefore, instead of using Lagrangian coordinate, we shall use the following smoothing diffeomophism [83], where the map will be in the usual Sobolev spaces. Denote $\mathcal{S} = \mathbb{R}^2\times[-1,0]$, and consider the map

$$\begin{aligned} &\Phi_t^\varepsilon : \mathcal{S}\to\Omega_t^\varepsilon \\ &(y,z)\to\Phi^\varepsilon(t,y,z) = (y,\varphi^\varepsilon(t,y,z))^t, \end{aligned} \tag{7.1.10}$$

where

$$\varphi^\varepsilon(t,y,z) = z + \eta^\varepsilon(t,y,z)(1+z). \tag{7.1.11}$$

Here $\eta^\varepsilon$ is given by

$$(\mathcal{F}\eta^\varepsilon)(t,\xi,z) = e^{-\delta_0(1+|\xi|^2)z^2}(\mathcal{F}h^\varepsilon)(t,\xi) \tag{7.1.12}$$

where $\mathcal{F}$ stands for the Fourier transform with respect to the horizontal variable $y$. $\delta_0$ is a small parameter such that $\det(\mathrm{D}\Phi_0^\varepsilon) > 0$, which ensures that $\Phi_0^\varepsilon$ is a diffeomorphism. Note that

$$\det(\mathrm{D}\Phi_0^\varepsilon) = \partial_z\varphi^\varepsilon(0,x) = 1 + h^\varepsilon(0,x) + (\eta^\varepsilon - h^\varepsilon)(0,x) + \partial_z\eta^\varepsilon(0,x)(1+z) > 2c_0 > 0$$

as long as

$$1 + h^\varepsilon(0,x) \geq 3c_0 > 0,\ \forall x\in\mathcal{S}, \tag{7.1.13}$$

$$\|(\eta^\varepsilon - h^\varepsilon)(0)\|_{L^\infty(\mathcal{S})} + \|\partial_z\eta^\varepsilon(0)\|_{L^\infty(\mathcal{S})} < c_0, \tag{7.1.14}$$

where $c_0 > 0$ is a constant. Let us notice that (7.1.14) holds if $\|h^\varepsilon(0)\|_{H^{\frac{7}{2}}(\mathbb{R}^2)} < +\infty$ and $\delta_0$ is chosen sufficiently small. Moreover, we have that: $\|\nabla\varphi^\varepsilon(t)\|_{L^2(\mathcal{S})} \lesssim |h^\varepsilon(t)|_{H^{\frac{1}{2}}(\mathbb{R}^2)}$, which means that we gain one half derivative. Let us define

$$u^\varepsilon(t,y,z) = w^\varepsilon(t,y,\Phi^\varepsilon(t,y,z)), \quad \sigma^\varepsilon = \varrho^\varepsilon(t,y,\Phi^\varepsilon(t,y,z)).$$

Then we set, $\partial_j^{\varphi^\varepsilon}u^\varepsilon = (\partial_j w^\varepsilon)\circ\Phi^\varepsilon, \partial_j^{\varphi^\varepsilon}\sigma^\varepsilon = (\partial_j\varphi^\varepsilon)\circ\Phi^\varepsilon$, where $j = 0,1,2,3$ with $\partial_0 = \partial_t, \partial_3 = \partial_z$ and

$$\partial_i^{\varphi^\varepsilon} = \partial_i - \frac{\partial_i\varphi^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z, \quad i = 0,1,2, \quad \partial_z^{\varphi^\varepsilon} = \frac{1}{\partial_z\varphi^\varepsilon}\partial_z. \tag{7.1.15}$$

The equations (7.1.7), (7.1.2) and the boundary conditions (7.1.9), (7.1.4) are reformulated into the following systems:

$$\begin{cases} g_1(\varepsilon\sigma^\varepsilon)\big(\partial_t^{\varphi^\varepsilon}\sigma^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\sigma^\varepsilon\big) + \dfrac{\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon}{\varepsilon} = 0, \\ g_2\big(\varepsilon\sigma^\varepsilon\big)\big(\partial_t^{\varphi^\varepsilon} u^\varepsilon + u^\varepsilon\cdot\nabla^{\varphi^\varepsilon} u^\varepsilon\big) - \mathrm{div}^{\varphi^\varepsilon}\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon + \dfrac{\nabla^{\varphi^\varepsilon}\sigma^\varepsilon}{\varepsilon} = 0, \qquad (t,x)\in\mathbb{R}_+\times\Omega_t^\varepsilon \\ u^\varepsilon|_{t=0} = w_0^\varepsilon(\Phi_0^\varepsilon(x)) := u_0^\varepsilon, \quad \sigma^\varepsilon|_{t=0} = \varrho_0^\varepsilon(\Phi_0^\varepsilon(x)) := \sigma_0^\varepsilon. \end{cases} \tag{7.1.16}$$

$$\partial_t h^\varepsilon - u^\varepsilon(t,y,h^\varepsilon(t,y))\cdot\mathbf{N}^\varepsilon = 0. \tag{7.1.17}$$

$$\mathcal{L}^\varphi u^\varepsilon\mathbf{N}^\varepsilon = \frac{\sigma^\varepsilon}{\varepsilon}\mathbf{N}^\varepsilon \quad \text{on} \quad \{z=0\}. \tag{7.1.18}$$

$$u_3^\varepsilon = 0, \quad \mu\partial_z^\varphi u_j^\varepsilon = a u_j^\varepsilon \quad (j=1,2), \quad \text{on} \quad \{z=-1\} \tag{7.1.19}$$

### 7.1.2 Conormal spaces and notations.

Before stating our results, we need to introduce some notations. We define the conormal vector fields:

$$Z_0 = \varepsilon\partial_t,\ Z_1 = \partial_{y_1},\ Z_2 = \partial_{y_2},\ Z_3 = \phi(z)\partial_z$$

where the weight function $\phi(z) = z(1+z)/(2-z)^2$. We then introduce the space-time conormal space as follows, for $p = 2, +\infty$,

$$L_t^p H_{co}^m(\mathcal{S}) = \{f|\ Z^\alpha f \in L^p([0,t]; L^2(\mathcal{S})), |\alpha|\le m\},$$

with the corresponding norms:

$$\|f\|_{L_t^p H_{co}^m} = \sum_{|\alpha|\le m}\|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))}, \tag{7.1.20}$$

where $\alpha = (\alpha_0,\alpha') = (\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$. Moreover, we denote the $L_{t,x}^\infty$ type norm:

$$\|\!| f \|\!|_{k,\infty,t} = \sum_{|\alpha|\le k}\|Z^\alpha f\|_{L^\infty([0,t]\times\mathcal{S})}, \tag{7.1.21}$$

To distinguish the amounts of time and space derivatives, we introduce also the norm:

$$\|f\|_{L_t^p\mathcal{H}^{j,l}} = \sum_{\alpha_0\le j, |\alpha'|\le l}\|Z^\alpha f\|_{L^p([0,t],L^2(\mathcal{S}))}, \tag{7.1.22}$$

and to simplify, we use $\mathcal{H}^j = \mathcal{H}^{j,0}$. Moreover, to measure the regularity along the boundary, we use:

$$|f|_{L_t^p\tilde{H}^s} = \sum_{j=0}^{[s]}|(\varepsilon\partial_t)^j f|_{L_t^p H^{s-j}(\mathbb{R}^2)} \quad |f|_{k,\infty,t} = \sum_{|\alpha|\le k,\alpha_3=0}|Z^\alpha f|_{L^\infty([0,t]\times\mathbb{R}^2)}. \tag{7.1.23}$$

Finally, to measure pointwise regularity at a given time $t$ (in particular also with $t=0$), we shall use the semi-norms:

$$|f(t)|_{\tilde{H}^s} = \sum_{j=0}^{[s]}|(\varepsilon\partial_t)^j f(t)|_{H^{s-j}(\mathbb{R}^2)}, \tag{7.1.24}$$

$$\|f(t)\|_{H_{co}^m} := \sum_{|\alpha|\le m}\|Z^\alpha f(t)\|_{L^2(\mathcal{S})}, \quad \|f(t)\|_{\mathcal{H}^{j,l}} := \sum_{\alpha_0\le j,|\alpha'|\le l}\|Z^\alpha f(t)\|_{L^2(\mathcal{S})}, \tag{7.1.25}$$

$$\|f(t)\|_{k,\infty,\mathcal{S}} := \sum_{|\alpha|\le k}\|(Z^\alpha f)(t)\|_{L^\infty(\mathcal{S})}. \tag{7.1.26}$$

### 7.1.3 Main results

Before stating our main result, we introduce the following definition.

**Definition 7.1.1** (Compatibility condition)**.** *We say that* $(\sigma_0^\varepsilon, u_0^\varepsilon)$ *satisfy the compatibility condition up to order* $m$ *if for* $j = 0, 1 \cdots m-1$,

$$\begin{aligned}
(\varepsilon\partial_t)^j \big(\mathcal{L}^{\varphi^\varepsilon} u^\varepsilon \boldsymbol{n}^\varepsilon\big)|_{t=0} &= (\varepsilon\partial_t)^j (\sigma^\varepsilon/\varepsilon)\big|_{t=0}, \quad on \quad \{z=0\},\\
\varepsilon^j \partial_t^{j+1} h^\varepsilon|_{t=0} &= (\varepsilon\partial_t)^j (u^\varepsilon \cdot \boldsymbol{N}^\varepsilon)|_{t=0} \quad on \quad \{z=0\},\\
\big((\varepsilon\partial_t)^j u_3^\varepsilon\big)\big|_{t=0} = 0, \quad \big((\varepsilon\partial_t)^j \partial_z^{\varphi^\varepsilon} u_j^\varepsilon\big)|_{t=0} &= \frac{a}{\mu} (\varepsilon\partial_t)^j u_j^\varepsilon|_{t=0} \ (j=1,2) \quad on \quad \{z=-1\}.
\end{aligned} \tag{7.1.27}$$

Note that the restriction of time derivatives of the solution at the initial time is defined inductively by using the equations. For instance:

$$(\partial_t h^\varepsilon)(0) = u_0^\varepsilon|_{z=0} \cdot (-\nabla_y h_0^\varepsilon, 1)^t$$

$$(\varepsilon\partial_t u^\varepsilon)(0) = \frac{1}{g_2(\varepsilon\sigma_0^\varepsilon)} (-\varepsilon \underline{u}_0^\varepsilon \cdot \nabla u_0^\varepsilon + \varepsilon \mathrm{div}^{\varphi_0^\varepsilon} \mathcal{L}^{\varphi_0^\varepsilon} u_0^\varepsilon - \nabla^{\varphi_0^\varepsilon} \sigma_0^\varepsilon).$$

where $\underline{u}_0^\varepsilon = \big(u_{0,1}^\varepsilon, u_{0,2}^\varepsilon, (u_0^\varepsilon \cdot \mathbf{N}_0^\varepsilon - (\partial_t \varphi^\varepsilon)(0))/\partial_z \varphi_0^\varepsilon\big)^t, \varphi_0^\varepsilon(\cdot) = \varphi^\varepsilon(0, \cdot) = z + \eta^\varepsilon(0, \cdot)(1+z)$. We remark that $\partial_t \varphi^\varepsilon(0), \partial_z \varphi_0^\varepsilon$ is determined by $(\partial_t h)^\varepsilon(0)$ and $h_0^\varepsilon$ respectively through (7.1.11) and (7.1.12).

We now define the space for the initial data:

$$Y_m^\varepsilon = \left\{ (\sigma_0^\varepsilon, u_0^\varepsilon) \in H^3(\Omega_0)^4, h_0^\varepsilon \in H^{m-\frac{1}{2}}(\mathbb{R}^2) \middle| \begin{array}{l} Y_m^\varepsilon(0) < +\infty, (\sigma_0^\varepsilon, u_0^\varepsilon, h_0^\varepsilon) \text{ satisfy} \\ \text{compatibility condition up to order } m. \end{array} \right\} \tag{7.1.28}$$

where

$$\begin{aligned}
&Y_m^\varepsilon(0) =: \\
&\varepsilon^{\frac{1}{2}} \|(\sigma_0^\varepsilon, u_0^\varepsilon)\|_{H^3(\mathcal{S})} + \varepsilon^{\frac{1}{2}} \|\partial_t (\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^{m-1}(\mathcal{S})} + \|(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^m(\mathcal{S})} + \|\nabla(\sigma^\varepsilon, u^\varepsilon)(0)\|_{H_{co}^{m-1}} \\
&+ \varepsilon^{\frac{1}{2}} \|\partial_t \omega^\varepsilon(0)\|_{H_{co}^{m-4}(\mathcal{S})} + \varepsilon^{-\frac{1}{2}} \|\nabla\sigma^\varepsilon(0)\|_{m-5,\infty,\mathcal{S}} + \|\nabla u^\varepsilon(0)\|_{1,\infty,\mathcal{S}} + |h_0^\varepsilon|_{H^{m-\frac{1}{2}}} + \varepsilon |h_0^\varepsilon|_{H^{m+\frac{1}{2}}}.
\end{aligned} \tag{7.1.29}$$

where $\omega^\varepsilon = \mathrm{curl}^{\varphi^\varepsilon} u^\varepsilon$ denotes the vorticity. To prove Theorem (7.1.1), we introduce the following quantities:

$$\mathcal{N}_{m,T}^\varepsilon = \mathcal{E}_{m,T}^\varepsilon + \mathcal{A}_{m,T}^\varepsilon =: \mathcal{E}_{low,T}^\varepsilon + \tilde{\mathcal{E}}_{m,T}^\varepsilon + \mathcal{A}_{m,T}^\varepsilon. \tag{7.1.30}$$

Here, $\mathcal{E}_{m,T}^\varepsilon$ is composed of lower order energy norms $\mathcal{E}_{low,T}^\varepsilon$ and higher order energy norms $\tilde{\mathcal{E}}_{m,T}^\varepsilon$ :

$$\mathcal{E}_{low,T}^\varepsilon = \|\varepsilon^{\frac{1}{2}} \partial_t (\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty L^2} + \varepsilon^{\frac{1}{2}} \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_t^\infty H^3} + \varepsilon^{\frac{3}{2}} \|\nabla^4 u^\varepsilon\|_{L_t^2 L^2}.$$

$$\begin{aligned}
\tilde{\mathcal{E}}_{m,T}^\varepsilon = {}& \varepsilon^{-\frac{1}{2}} \|(\nabla^{\varphi^\varepsilon} \sigma^\varepsilon, \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon)\|_{L_T^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}} + \|\nabla u^\varepsilon\|_{L_T^\infty H_{co}^{m-4} \cap L_T^2 H_{co}^{m-1}} + |h^\varepsilon|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} \\
&+ \varepsilon^{\frac{1}{2}} \|\partial_t (\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty \mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}} \|\partial_t \nabla u^\varepsilon\|_{L_T^2 \mathcal{H}^{m-1} \cap L_T^2 H_{co}^{m-2} \cap L_T^\infty H_{co}^{m-4}} \\
&+ \varepsilon^{\frac{1}{2}} \|(\sigma^\varepsilon, u^\varepsilon)\|_{L_T^\infty H_{co}^m} + \varepsilon^{\frac{1}{2}} \|\nabla u^\varepsilon\|_{L_T^\infty H_{co}^{m-1} \cap L_T^2 H_{co}^m} + \varepsilon^{\frac{1}{2}} \|\nabla^2 u^\varepsilon\|_{L_T^\infty H_{co}^{m-2} \cap L_T^2 \mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}} |h^\varepsilon|_{L_t^\infty \tilde{H}^{m+\frac{1}{2}}},
\end{aligned}$$

whereas $\mathcal{A}_{m,T}^\varepsilon$ denotes the $L_{t,x}^\infty$ norms:

$$\begin{aligned}
\mathcal{A}_{m,T}^\varepsilon = {}& \|\nabla u^\varepsilon\|_{1,\infty,T} + \|(\varepsilon^{\frac{1}{2}} \partial_t (\sigma^\varepsilon, u^\varepsilon), \varepsilon^{-\frac{1}{2}} (\nabla^{\varphi^\varepsilon} \sigma, \mathrm{div}^{\varphi^\varepsilon} u)\|_{m-5,\infty,T} + \|(\mathrm{Id}, \varepsilon\partial_t)(\sigma^\varepsilon, u^\varepsilon)\|_{m-4,\infty,T} \\
&+ \varepsilon^{\frac{1}{2}} \|\nabla u^\varepsilon\|_{m-3,\infty,T} + \varepsilon^{\frac{1}{2}} \|(\sigma^\varepsilon, u^\varepsilon)\|_{m-2,\infty,T} + |h^\varepsilon|_{m-2,\infty,T},
\end{aligned} \tag{7.1.31}$$

where $[s]$ denotes the largest integer less or equal to $s$.

**Theorem 7.1.1** (Uniform estimates)**.** *Define* $0 < c_0 < \frac{1}{2}$ *such that*

$$\sup_{s \in [-3c_1 \bar{P}, 3\bar{P}/c_1]} |(g_1, g_2)(s)| \in [c_0, 1/c_0]$$

*where* $0 < c_1 < \frac{1}{4}$ *is a fixed constant. Given* $m \geq 7$ *an integer, suppose that the initial data belongs to* $Y_m^\varepsilon$ *and*

$$1 + h_0^\varepsilon(x) \geq 3c_0 > 0,$$

$$\sup_{\varepsilon \in (0,1]} Y_m^\varepsilon(0) < +\infty,$$

$$-c_1 \bar{P} \leq \varepsilon \sigma_0^\varepsilon(x) \leq \bar{P}/c_1, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1],$$

*and* $\delta_0$ *(the parameter appearing in* (7.1.12)*) is chosen such that* (7.1.14) *holds for* $t = 0$ *so that*

$$\partial_z \varphi_0^\varepsilon \geq 2c_0, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1].$$

*With this fixed parameter* $\delta$*, we also suppose for simplicity that (by assuming* $c_0$ *to be smaller if necessary)*

$$|(\nabla \varphi_0^\varepsilon, \nabla^2 \varphi_0^\varepsilon)(x)| \leq \frac{1}{2c_0}, \quad \forall x \in \mathcal{S}, \quad \forall \varepsilon \in (0,1].$$

*Then there exist* $0 < T_0 \leq 1, 0 < \varepsilon_0 \leq 1$*, such that for any* $0 < \varepsilon \leq \varepsilon_0$ *the system* (7.1.16)*-*(7.1.19) *has a unique solution which satisfies:* $\mathcal{N}_{m,T_0}^\varepsilon(\sigma^\varepsilon, u^\varepsilon) < +\infty$*. In particular, we have the uniform estimate*

$$\sup_{\varepsilon \leq \varepsilon_0} \big( \|(\sigma^\varepsilon, u^\varepsilon)\|_{L^2_{T_0} H^m_{co}(\mathcal{S})} + \|(\sigma^\varepsilon, u^\varepsilon)\|_{L^\infty_{T_0} H^{m-2}_{co}(\mathcal{S})} + \|(\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon, \nabla \sigma^\varepsilon)/\varepsilon^{\frac{1}{2}}\|_{L^\infty_{T_0} H^{m-2}_{co}(\mathcal{S}) \cap L^2_{T_0} H^{m-1}_{co}}$$
$$+ \|\nabla u^\varepsilon\|_{1,\infty,T_0} + \|(\nabla \sigma^\varepsilon, \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon)/\varepsilon^{\frac{1}{2}}\|_{[\frac{m}{2}]-1,\infty,T_0} \big) < +\infty.$$

*Moreover, the following properties hold: for any* $(t,x) \in [0,T_0] \times \mathcal{S}, \varepsilon \in (0, \varepsilon_0]$*,*

$$\partial_z \varphi^\varepsilon(t,x) \geq c_0, \quad |(\nabla \varphi^\varepsilon, \nabla^2 \varphi^\varepsilon)(t,x)| \leq 1/c_0, \quad -2c_1 \bar{P} \leq \varepsilon \sigma^\varepsilon(t,x) \leq 2\bar{P}/c_1.$$

**Remark 7.1.2.** *In view of the definition of* $Y_m^\varepsilon$*, we have assumed that the restriction of the first time derivative of the solution at* $t = 0$ *is of order* $\varepsilon^{-\frac{1}{2}}$*, which is slightly better than the well-prepared data case (where* $\partial_t(\sigma^\varepsilon, u^\varepsilon)|_{t=0}$ *is assumed to be order* 1*). This assumption has to be made due to some possible loss of regularity on the surface. We shall give more detail in subsection 7.1.5.*

**Remark 7.1.3.** *We can also prove the uniform estimates by imposing alternative assumption on the size of the acoustic waves, that is assuming them to be of size of order* $\varepsilon$ *in a rather low regularity* $H^1_{co}$ *but order* 1 *in the higher regularity* $H^m_{co}$*.*

**Theorem 7.1.4** (Convergence)**.** *Assuming that* $(u_0^\varepsilon, h_0^\varepsilon)$ *tends to* $(u_0^0, h_0^0)$ *in* $H^1(\mathcal{S}) \times L^2(\mathbb{R}^2)$ *and the assumptions made in Theorem 7.1.1 hold. Let* $(\sigma^\varepsilon, u^\varepsilon, h^\varepsilon)$ *the solution to* (7.1.16)*-*(7.1.19)*. Then* $(P(\bar{\rho}) + \varepsilon\sigma^\varepsilon, u^\varepsilon, h^\varepsilon)$ *converge in* $C^\gamma([0,T_0] \times \mathcal{S}) \times C([0,T_0], L^2_{loc}(\mathcal{S})) \times C([0,T_0], H^s_{loc}(\mathbb{R}^2))$ *to* $(P(\bar{\rho}), u^0, h^0)$ *where* $0 \leq \gamma < \frac{1}{2}$ *and* $0 < s < m - 1/2$*. Moreover,* $u^0$ *has the additional regularity:*

$$u^0 \in C\big([0,T_0], \mathcal{H}^{0,m-2}\big), \quad \nabla u^0 \in L^2([0,T_0], \mathcal{H}^{0,m-1}) \cap L^\infty([0,T_0] \times \mathcal{S}) \tag{7.1.32}$$

*and one can find* $\pi^0 \in L^2([0,T_0], \mathcal{H}^{0,m-1})$ *such that* $(u^0, \pi^0, h^0)$ *solves uniquely the following incompressible free-surface Navier-Stokes equations:*

$$\begin{cases} \bar{\rho}(\partial_t^{\varphi^0} u^0 + u^0 \cdot \nabla^{\varphi^0} u^0) - \mathrm{div}^{\varphi^0} S^{\varphi^0} u^0 + \nabla^{\varphi^0} \pi^0 = 0, \\ \mathrm{div}^{\varphi^0} u^0 = 0, \qquad\qquad (t,x) \in [0,T_0] \times \mathcal{S}, \\ \partial_t h^0 + u^0(t,y,0) \cdot \boldsymbol{N}^0 = 0, \\ u^0|_{t=0} = u_0^0, h^0|_{t=0} = h_0^0. \end{cases} \tag{7.1.33}$$

*with boundary conditions:*

$$S^{\varphi^0} u^0 \boldsymbol{N}^0 = \pi^0 \boldsymbol{N}^0 \quad \text{on } \{z=0\},$$
$$u_3^0 = 0, \quad \frac{\mu}{\partial_z \varphi_0} \partial_z u_j^0 = a u_j^0 \quad (j = 1,2) \quad \text{on } \{z=-1\}.$$

*Here* $\varphi^0$ *is defined in* (7.1.11) *(replacing* $h^\varepsilon$ *by* $h^0$*),* $\boldsymbol{N}^0 = (-\partial_1 h^0, -\partial_2 h^0, 1)^t$*.*

**Remark 7.1.5.** *Due to the absence of the estimate for the second order normal derivatives of the velocity $u^0$ (and thus for the strong trace of the normal derivative), the solution to* (7.1.33) *must be interpreted in the following sense:* $\mathrm{div}^{\varphi^0} u^0 = 0$ *holds in* $L^2([0, T_0] \times \mathcal{S})$ *and for any vector* $\psi = (\psi_1, \psi_2, \psi_3)^t \in \left[C_c^\infty(\overline{Q_{T_0}})\right]^3$ *with condition* $\psi_3|_{z=-1} = 0$*, the following identity holds: for any* $0 < t \le T_0$,

$$
\begin{aligned}
&\bar{\rho}\int_{\mathcal{S}} u^0 \cdot \psi(t,\cdot)\,\mathrm{d}\mathcal{V}_t^0 + 2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^0}u^0 \cdot \nabla^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s + \bar{\rho}\int_0^t\int_{\mathcal{S}}(u^0\cdot\nabla^{\varphi^0}u^0)\cdot\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s\\
&= \bar{\rho}\int_{\mathcal{S}} u^0\cdot\psi(0,\cdot)\,\mathrm{d}\mathcal{V}_0^0 + \bar{\rho}\int_0^t\int_{\mathcal{S}} u^0\cdot\partial_t^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s + \int_0^t\int_{\mathcal{S}}\pi^0\mathrm{div}^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}_s^0\mathrm{d}s. \qquad (7.1.34)\\
&\quad + \int_0^t\int_{z=0}(u^0\cdot\boldsymbol{N}^0)u^0\cdot\psi\,\mathrm{d}y\mathrm{d}s + a\int_0^t\int_{z=-1}(u_1^0\cdot\psi_1 + u_2^0\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

*where* $\mathrm{d}\mathcal{V}_t^0 = \frac{1}{\partial_z\varphi^0}(t,\cdot)\,\mathrm{d}y\mathrm{d}z$.

### 7.1.4 Main difficulties, general strategies.

Due to the simultaneous presence of boundaries and the singular term and the viscous term in the equation, we are confronted with both difficulties resulting from boundary layer effects and fast time oscillations. These two phenomena are well understood when they occur separately, but to deal with them simultaneously, some new ideas need to be introduced. On the one hand, regarding the vanishing viscosity limit problem, one controls [99, 94] the high order tangential derivatives by direct energy estimates, and then uses the vorticity to control the normal derivatives. Nevertheless, for the system with a low Mach number, even the tangential derivative estimates are not easy to get, since the spatial tangential derivatives $\partial_y$ do not commute with $\nabla^{\varphi^\varepsilon}, \mathrm{div}^{\varphi^\varepsilon}$ and thus create singular commutators. Without a priori knowledge on the tangential derivatives, the estimate of the vorticity cannot be conducted. On the other hand, for the compressible free boundary Euler system with a low Mach number, uniform estimates are established for example in [37, 88, 92]. If we understand correctly, besides the difficulties arising from the Taylor sign condition and regularity of the surface, the idea behind getting a uniform estimate is to control first weighted time derivatives $(\varepsilon\partial_t)^k$ and then to recover space derivatives by using the equations to estimate the divergence of the velocity and by direct energy estimates for the vorticity which solves a transport equation. Here, in the case of viscous fluids, we face again the fact that the estimates of the vorticity are challenging due to the lack of information on the trace of $\omega^\varepsilon$ on the boundary at this stage. We shall explain more precisely in the following. For the sake of notational convenience, we will drop the $\varepsilon-$ dependence of the solution.

By computations, $\omega$ solves a transport-diffusion equation, with the Dirichlet boundary condition (see (7.4.5), (7.4.8))

$$\omega|_{\partial\mathcal{S}} \approx \partial_y u + \mathrm{div}^{\varphi} u|_{\partial\mathcal{S}}. \qquad (7.1.35)$$

Let us consider the simplest case, the heat equation with homogeneous force and initial data but with nonhomogeneous Dirichlet condition in half space (note that we can always reduce the problem to the half space by multiplying some suitable cut-off functions since there is no singular term in the equation of $\omega$):

$$\bar{\rho}\partial_t f - \mu\Delta f = 0, \quad f|_{t=0} = 0, \quad f|_{z=0} = f^{b,1}, \quad (t,x)\in[0,T]\times\mathbb{R}^3_-, \qquad (7.1.36)$$

which can be solved explicitly:

$$f(t,x) = 2\tilde{\mu}\int_0^t\int_{\mathbb{R}^2}\partial_z E_3(t-s, y-y', z) f^{b,1}(s,y')\mathrm{d}s\mathrm{d}y \qquad (7.1.37)$$

where $E_3(t,y,z) = \frac{1}{4\pi\tilde{\mu}(t-s)^{3/2}} e^{-\frac{|y|^2+|z|^2}{4\tilde{\mu}t}}$ $(\tilde{\mu} = \mu/\bar{\rho})$ is the fundamental solution of the three dimensional heat equation. By Young's inequality, one has that

$$\|f\|_{L_t^2 H_{co}^{m-1}} \lesssim T^{\frac14}|f^{b,1}|_{L_t^2\tilde{H}^{m-1}}.$$

By applying this estimate to $\omega$, we see that the boundary contribution when estimating $\|\omega\|_{L_t^2 H_{co}^{m-\frac12}}$ is more or less $|(\partial_y u, \mathrm{div} u)|_{L_t^2\tilde{H}^{m-1}}$, which requires the foreknowledge of the tangential derivatives and which indicates the loss of half derivative. One may also expect to use the (tangential) smoothing effects of the heat equation to overcome this loss of derivative. Nevertheless, in this way, it seems impossible to extract the extra $\varepsilon$ or $T$ which are essential to close the estimate. More precisely, by using Laplace transform, one gets that

$$\begin{aligned}\|\omega\|_{L_t^2 H_{co}^{m-1}} &\leq C|(\partial_y u, \mathrm{div}^{\varphi} u)|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}} + \text{other terms} \\ &\leq C(\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + \|\nabla \mathrm{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-2}}) + \text{other terms}\end{aligned}$$

which leads to the circular argument. Note that the constant $C$ is independent of $T$ and $\varepsilon$.

To overcome these problems, we split the velocity $u$ into a compressible part $\nabla^{\varphi}\Psi$ and a incompressible part $v$ (see definition (7.5.2), (7.5.3)). On the one hand, the compressible part is governed by the elliptic equation $\Delta^{\varphi}\Psi = \mathrm{div}^{\varphi} u$ with mixed boundary conditions (with homogeneous Dirichlet boundary condition on the upper boundary and homogeneous Neumann boundary condition on the bottom). Hence the estimate for its gradient $\nabla^2\Psi$ can be deduced from the estimate of $\mathrm{div}^{\varphi} u$. We then use induction arguments and the equations to establish the high-order estimates of $\mathrm{div}^{\varphi} u$. On the other hand, the incompressible part $v$, solves, up to the control of non-local commutators, a transport-diffusion equation and hence one can use direct energy estimates to get a control of the the tangential derivatives: $\|\partial_y^{m-1} v\|_{L_t^\infty L^2}$ and $\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$, which, together with the estimates on $\mathrm{div}^{\varphi} u$, lead to the uniform control of $\|\partial_y^{m-1} u\|_{L_t^\infty L^2(\mathcal{S})}$ and $\|\nabla u\|_{L_t^2 H_{co}^{m-1}}$. The final task is to estimate $\|\nabla v\|_{L_t^\infty H_{co}^{m-4}}$ which stems from a careful study on $\omega \times \mathbf{n}$. We remark that this strategy has been employed in the previous chapter where the uniform in low Mach number estimates are established in the case of fixed bounded domain with Navier boundary condition and ill-prepared initial data.

### 7.1.5 Remarks on the slightly well-prepared data assumption.

As mentioned before, compared to the fixed domain case, there would be some extra difficulties for the free boundary problem, arising from the regularity of the surface. This is the reason that we have to allow the initial data to be slightly well-prepared. Indeed, since the incompressible part $v^{\varepsilon}$ satisfies the boundary condition (see (7.5.6))

$$(2\mu S^{\varphi} v - \pi \mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^{\varphi} u \mathrm{Id} - \nabla^{\varphi}\nabla^{\varphi}\Psi)\mathbf{N}|_{z=0},$$

in order to perform energy estimates for $v$ at order $m-1$, it requires the information of $\|\nabla^3\Psi\|_{L_t^2 H_{co}^{m-3}}$, which, by elliptic estimates, can be controlled by $\|\nabla \mathrm{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-2}}$ and $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$. Nevertheless, due to the fast oscillation, we can not expect $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$ to be uniformly bounded. The similar problem occurs also when one recovers the $L_t^2 H_{co}^{m-1}$ norm of $\nabla^2\Psi$ from the one of $\mathrm{div}^{\varphi} u$ by elliptic estimate. To overcome this problem, we assume the data to be slightly well-prepared so that $\|\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^1}$ can be proved to be of order $\varepsilon^{\vartheta}$, $(0 < \vartheta < 1$ to be chosen). This can make an extra $\varepsilon^{\vartheta}$ appear in front of $|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$ in the process of the elliptic estimates (one can refer to Step 3 of the following subsection for more details). In turn, to control uniformly the term $\varepsilon^{\vartheta}|h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}$, which reduces to the estimate of $\varepsilon^{\vartheta}\|\nabla u\|_{L_t^2 H_{co}^m}$, we must assume that the compressible part is $(\mathrm{div}^{\varphi} u, \nabla\sigma) = \mathcal{O}(\varepsilon^{1-\vartheta})$ in $L_t^2 H_{co}^{m-1}$. Indeed, when performing the highest-order energy estimates, we need to be careful of the singular term:

$$\varepsilon^{2\vartheta-1}\int_0^t \int_{\mathcal{S}} Z^{\alpha}\sigma \underbrace{[Z^{\alpha}, \mathrm{div}^{\varphi}]u}_{[Z^{\alpha}, \frac{\mathbf{N}}{\partial_z\varphi}\cdot\partial_z]u} + Z^{\alpha}u \cdot \underbrace{[Z^{\alpha}, \nabla^{\varphi}]\sigma}_{[Z^{\alpha}, \frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\sigma} \,\mathrm{d}\mathcal{V}_s \mathrm{d}s, \quad |\alpha| = m \tag{7.1.38}$$

By direct computations, these term can be bounded by (up to other good terms)

$$\varepsilon^{\vartheta-1}|\varepsilon^{\vartheta} h|_{L_t^2 \tilde{H}^{m+\frac{1}{2}}}\big(\|Z\sigma\|_{L_t^2 H_{co}^{m-1}} \|\!|\nabla u\|\!|_{0,\infty,t} + \|u\|_{L_t^2 H_{co}^m} \|\!|\nabla\sigma\|\!|_{0,\infty,t}\big)\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big),$$

which can be uniformly bounded if

$$\|Z\sigma\|_{L_t^2 H_{co}^{m-1}} = \mathcal{O}(\varepsilon^{1-\vartheta}), \quad \|\!|\nabla\sigma\|\!|_{0,\infty,t} = \mathcal{O}(\varepsilon^{1-\vartheta}).$$

For the clarity of the presentation, we present our result and prove the uniform estimates by assuming that $(\nabla\sigma, \mathrm{div}^{\varphi} u)|_{t=0} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$ (1/2 is obtained by letting $\vartheta = 1-\vartheta$). Nevertheless, it is not hard to establish the uniform estimates by assuming generally the compressible part to be of size at $\mathcal{O}(\vartheta)$ ($\frac{1}{2} < \vartheta \le 1$) in a low regularity space (say $H^1_{co}$) and at $\mathcal{O}(1-\vartheta)$ in a higher regularity space (say $H^{m-1}_{co}$).

One may be curious that if the introduction of the good unknown which is used frequently in the free boundary problems can help us to avoid losing derivative on the surface and to get uniform estimates without any size assumption on the data. It turns out not the case. Indeed, the use of the Alinahc good unknown requires the validity of the Taylor sign condition ($\partial^{\varphi}_{\mathbf{n}}\sigma|_{z=0} > 0$), which seems out of reach for ill-prepared data since $\sigma$ solves a transport equation with a source terms at size of $\mathcal{O}(\varepsilon^{-1})$.

### 7.1.6 Sketch of the proof

Let us give more details concerning the proof in this subsection. The uniform energy estimates shall be more precisely established in the following steps:

**Step 1: $\varepsilon-$dependent high-order energy estimates and $\varepsilon-$independent high-order temporary estimates.**

In this step, we aim to obtain two kinds of energy estimates. The first one is the estimate of $\varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{L^\infty_t H^m_{co}}$ and $\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{L^\infty_t \mathcal{H}^{m-1}}$. Since the spatial tangential vector fields $Z_1, Z_2, Z_3$ do not commute with $\nabla^{\varphi}$ and $\mathrm{div}^{\varphi}$, it seems hard to get the uniform estimate of $\|(\sigma,u)\|_{L^\infty_t H^m_{co}}$ by direct energy estimates. Nevertheless, it is easy to get an $\varepsilon-$dependent estimate involving the control of $\|\nabla(\sigma,u)\|_{L^2_t H^{m-1}_{co}}$. This can be done by taking $Z^{\alpha}(|\alpha|\le m)$ derivative on the equations (7.1.16) and then multiply them by $\varepsilon Z^{\alpha}(\sigma,u)$. We remark that in this stage we do not lose regularity on the surface. Indeed, besides the term list in (7.1.38) (setting $\vartheta = \frac{1}{2}$), the possible problematic commutator term is

$$\varepsilon\int_0^t\int_{\mathcal{S}} Z^{\alpha}\mathbf{N}\cdot\partial_z\mathcal{L}^{\varphi}u Z^{\alpha}u \,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\ \mathrm{d}\mathcal{V}_s = \frac{1}{\partial_z\varphi}\,\mathrm{d}y\mathrm{d}z$$

which can be bounded by: $\varepsilon^{\frac{1}{2}}|h|_{L^2_t\tilde{H}^{m+\frac{1}{2}}}\|u\|_{L^2_t H^m_{co}}\|\varepsilon^{\frac{1}{2}}\partial_z\mathcal{L}^{\varphi}u\|_{\infty,t}$. Note that the estimate of $\varepsilon^{\frac{1}{2}}|h|_{L^2_t\tilde{H}^{m+\frac{1}{2}}}$ is available owing to the control of $\varepsilon^{\frac{1}{2}}\|\nabla u\|_{L^2_t H^m_{co}}$ and $\|\varepsilon^{\frac{1}{2}}\partial_z\mathcal{L}^{\varphi}u\|_{\infty,t}$ can be bounded by the terms appearing in $\mathcal{A}_{m,t}$ by using the equation of the velocity.

The norm $\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{L^\infty_t\mathcal{H}^{m-1}}$ can be derived also by straightforward energy estimates. The main observation is that: although the weighted time derivative $(\varepsilon\partial_t)^k$ do not commute with $\nabla^{\varphi}$, their commutator can be uniformly controlled. Indeed, direct computation shows that for $k\le m-1$,

$$\varepsilon^{-\frac{1}{2}}[(\varepsilon\partial_t)^k\partial_t, \mathrm{div}^{\varphi}]u = \varepsilon^{k-\frac{1}{2}}\big[\partial_t^{k+1}, \frac{\mathbf{N}}{\partial_z\varphi}\big]\cdot\partial_z u$$

whose $L^2_t L^2(\mathcal{S})$ norm is uniformly controlled as long as $k\ge 1$ thanks to the boundedness of $|\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L^2_t\tilde{H}^{m-\frac{3}{2}}}$ (see (7.6.2)). We remark that in view of definition (7.1.12), the boundedness of $\mathbf{N}$ can be derived from that of $h$. The case $k=0$ needs to be treated differently and is explained in the next step.

The second kind of estimate is $\varepsilon^{\frac{1}{2}}\|(\nabla^{\varphi}\sigma, \mathrm{div}^{\varphi}u)\|_{L^\infty_t H^{m-1}_{co}}$, $\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L^2_t H^{m-1}_{co}}$ These estimates follow again from the direct energy estimates, we thus do not detail here.

**Step 2. Uniform lower order energy estimates.** In this step, we aim to show the boundedness of $\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{L^\infty_t L^2}$. We remark that a naive energy estimate fails due to bad commutators. Actually, the $L^2_t L^2(\mathcal{S})$ norm of the term $\varepsilon^{-\frac{1}{2}}[\partial_t, \mathrm{div}^{\varphi}]u = \varepsilon^{-\frac{1}{2}}\partial_t(\mathbf{N}/\partial_z\varphi)\cdot\partial_z u$ is out of control. The small trick to avoid this problem is to test $\partial_t^{\varphi}(7.1.16)_1$ by $\varepsilon\partial_t\sigma$ and test $\partial_t(7.1.16)_2$ by $\varepsilon\partial_t^{\varphi}u$. In this way, the singular term can be dealt with as:

$$\begin{aligned}&\int_0^t\int_{\mathcal{S}}\partial_t\nabla^{\varphi}\sigma\partial_t^{\varphi}u + \partial_t^{\varphi}\mathrm{div}^{\varphi}u\partial_t\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ &= \int_0^t\int_{z=0}\partial_t^{\varphi}u\cdot\mathbf{N}\partial_t\sigma\,\mathrm{d}y\mathrm{d}s + \int_0^t\int_{\mathcal{S}}\partial_t^{\varphi}u[\partial_t,\nabla^{\varphi}]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\end{aligned}\tag{7.1.39}$$

where $\mathrm{d}\mathcal{V}_s = \partial_z\varphi\ \mathrm{d}y\mathrm{d}z$. The first boundary term combined with another boundary term which comes from the integration by parts of the viscous term, result in good term that can be controlled. Namely

$$\varepsilon\int_0^t\int_{z=0}\partial_t\big[-\mathcal{L}^{\varphi}u+\frac{\sigma}{\varepsilon}\mathrm{Id}\big]\mathbf{N}\cdot\partial_t^{\varphi}u\ \mathrm{d}y\mathrm{d}s=-\varepsilon\int_0^t\int_{z=0}(-\mathcal{L}^{\varphi}u+\frac{\sigma}{\varepsilon}\mathrm{Id})\partial_t\mathbf{N}\cdot\partial_t^{\varphi}u\ \mathrm{d}y\mathrm{d}s$$

Note that the trace of $\frac{\sigma}{\varepsilon}$ on the upper boundary can be expressed as the spatial tangential derivatives of the velocity (see (7.4.1)) which can be easily treated by the trace inequality. The second term in (7.1.39) is also manageable since $\varepsilon^{-\frac{1}{2}}\|[\partial_t,\nabla^{\varphi}]\sigma\|_{L_t^2L^2(\mathcal{S})}$ can be roughly bounded by $\|\varepsilon^{-\frac{1}{2}}\nabla^{\varphi}\sigma\|_{L_t^2L^2(\mathcal{S})}$.

It should be mentioned that the above strategy does not apply for the control of $\varepsilon^{\frac{1}{2}}\|(\partial_y,Z_3)\partial_t(\sigma,u)\|_{L_t^{\infty}L^2}$ due to the bad commutator terms. We thus use the strategy of the splitting mentioned before to deal with them. See Step 3-Step 5.

**Step 3. Recovering high order spatial derivatives of $(\nabla\sigma,\nabla\nabla^{\varphi}\Psi)$ by induction.** Denote $\nabla^{\varphi}\Psi$ compressible part of the velocity which is defined by the unique solution to the elliptic equation with mixed boundary conditions:

$$\begin{cases}-\mathrm{div}^{\varphi}\nabla^{\varphi}\Psi=-\mathrm{div}^{\varphi}u,\\ \Psi|_{z=0}=0,\\ \partial_{\mathbf{n}}\Psi|_{z=-1}=0.\end{cases}\tag{7.1.40}$$

In this step, we aim to control $L_t^2H_{co}^{m-1}$ norm of $\nabla^{\varphi}(\sigma,\nabla^{\varphi}\Psi)$, which can be reduced to the control of $\varepsilon^{-\frac{1}{2}}\|(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)\|_{L_t^2H_{co}^{m-1}}$. We will use the equation and induction arguments to recover the high order spatial derivatives of $(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)$. Let us rewrite the system (7.1.16) as follows:

$$\begin{cases}-\mathrm{div}^{\varphi}u=g_1\varepsilon\partial_t\sigma+\varepsilon g_1\underline{u}\cdot\nabla\sigma,\\ -\mu\varepsilon\,\mathrm{curl}^{\varphi}\,\omega-\nabla^{\varphi}\big(\sigma-(2\mu+\lambda)\varepsilon\mathrm{div}^{\varphi}u\big)=g_2\varepsilon\partial_t u+\varepsilon g_2\underline{u}\cdot\nabla u.\end{cases}\tag{7.1.41}$$

where $\underline{u}=(u_1,u_2,u_z)=:(u_1,u_2,\frac{u\cdot\mathbf{N}-\partial_t\varphi}{\partial_z\varphi})$. It view of (7.1.41), one could expect that for $j+l\leq m-1$,

$$\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{j,l}}\lesssim\|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,l}}+\mathcal{O}(\varepsilon^{\frac{1}{2}})\lesssim\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j+1,l-1}}+\mathcal{O}(\varepsilon^{\frac{1}{2}}),\tag{7.1.42}$$

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j,l}}&\lesssim\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\Psi\|_{L_t^2\mathcal{H}^{j+1,l}}+\mathcal{X}_{m,t}+\mathcal{O}(\varepsilon^{\frac{1}{2}})\\&\lesssim\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{j+1,l-1}}+\mathcal{X}_{m,t}+\mathcal{O}\big((T+\varepsilon)^{\frac{1}{2}}\big)\end{aligned}\tag{7.1.43}$$

where

$$\mathcal{X}_{m,t}=(\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}u\|_{L_t^2H_{co}^{m}})\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)$$

which has been controlled in the first step. These two inequalities in hand, we can conclude by induction arguments. Note that the inequality (7.1.42) results from the equality $(7.1.41)_1$ and the product estimates. To obtain (7.1.43), we take $\mathrm{div}^{\varphi}$ of the $(7.1.41)_2$ and use the boundary condition (7.1.18) to get the following elliptic equation:

$$\begin{cases}\Delta^{\varphi}(\varepsilon\theta)=\mathrm{div}^{\varphi}\big[\bar{\rho}\varepsilon\partial_t\nabla^{\varphi}\Psi+\varepsilon(\frac{g_2-\bar{\rho}}{\varepsilon}\varepsilon\partial_t+g_2\underline{u}\cdot\nabla)u\big]=:\mathrm{div}^{\varphi}\tilde{G}\\ \varepsilon\theta|_{z=0}=-2\varepsilon\mu(\partial_1u_1+\partial_2u_2)+\varepsilon(\omega\times\mathbf{N})_3\\ \partial_{\mathbf{n}}\theta|_{z=-1}=\tilde{G}\cdot\mathbf{n}+\mu\varepsilon\,\mathrm{curl}^{\varphi}\,\omega\times\mathbf{n}|_{z=-1}.\end{cases}\tag{7.1.44}$$

where $\theta=\sigma/\varepsilon-(2\mu+\lambda)\mathrm{div}^{\varphi}u$. Inequality (7.1.43) is thus the consequence of the elliptic estimates in the conormal setting (see Section 5). We remark that the trace of $\omega\times\mathbf{N}$ involves only the tangential derivatives of the velocity (see (7.4.2)).

Now that $\mathrm{div}^{\varphi}u$ has been bounded, we can control the compressible part of the velocity $\nabla^2\Psi$ by again elliptic estimates. Nevertheless, there will be a loss of one derivative on the surface if no smallness condition made on the compressible part. Indeed, as $\nabla^{\varphi}\Psi$ is solves equation (7.1.40), we have by the elliptic estimates that

$$\|\nabla^2\Psi\|_{L_t^2\mathcal{H}^{0,m-1}}\lesssim(|h|_{L_t^2H^{m+\frac{1}{2}}}+\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{0,m-1}})\Lambda(\|\mathrm{div}^{\varphi}u\|_{L_t^{\infty}H_{tan}^1}+|h|_{2,\infty,t}),\tag{7.1.45}$$

where $\Lambda$ denotes a polynomial with its arguments that can change from line to line. It thus requires one more regularity of the surface than that we can expect (note that we has only the control of $|h|_{L_t^2 H^{m-\frac{1}{2}}}$ ). Nevertheless, we are aware of that when performing the variational arguments, the main problematic term is indeed $\nabla\Psi Z^\alpha \nabla \mathbf{N}$ ($|\alpha| = m-1, \alpha_0 = 0$), whose $L_t^2 L^2(\mathcal{S})$ norm can be bounded by

$$\|\nabla\Psi\|_{L^\infty_{t,x}} |h|_{L_t^2 H^{m+\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^\infty H^1_{tan}} |h|_{L_t^2 H^{m+\frac{1}{2}}}.$$

The right hand side can be controlled only if $\|\mathrm{div}^\varphi u\|_{L_t^2 H^1_{tan}} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$. Hopefully, once we assume $\varepsilon^{\frac{1}{2}}(\partial_t\sigma, \partial_t u)(0)$ to be bounded uniformly in $H^1_{co}(\mathcal{S})$, it can be shown that $\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H^1_{co}} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$. This is one reason that we need the initial data to be slightly well-prepared.

**Step 4. Uniform energy estimate of the incompressible part of the velocity.** Denote $v = u - \nabla^\varphi\Psi$ the incompressible part of the velocity. By the computations in Section 5, we find that $v$ solves the following system:

$$\begin{cases} \bar\rho\partial_t^\varphi v - \mu\Delta^\varphi v + \nabla^\varphi\pi = -(f + \nabla^\varphi q + \bar\rho[\mathbb{P}_t, \partial_t^\varphi]u), \\ \mathrm{div}^\varphi v = 0, \\ (2\mu S^\varphi v - \pi\mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^\varphi u\mathrm{Id} - (\nabla^\varphi)^2\Psi)\mathbf{N}|_{z=0}, \\ v_3|_{z=-1} = 0,\ \mu\partial_z^\varphi v_j = a u_j|_{z=-1},\ j = 1, 2. \end{cases} \tag{7.1.46}$$

where $\mathbb{Q}_t, \mathbb{P}_t$, are time-dependent projectors defined in (7.5.2) (7.5.3) and

$$f = (g_2 u\cdot\nabla^\varphi u + \frac{g_2 - \bar\rho}{\varepsilon}\partial_t^\varphi u), \nabla^\varphi q = -\mathbb{Q}_t(f - \mu\Delta^\varphi v), \nabla^\varphi\pi = \mathbb{P}_t[\nabla^\varphi(\frac{\sigma}{\varepsilon} - (2\mu+\lambda)\mathrm{div}^\varphi u)].$$

Note that $\nabla^\varphi\pi$ does not vanish identically since $\mathbb{Q}\nabla^\varphi \neq \nabla^\varphi$. Nevertheless, although including singular term $\sigma/\varepsilon$, it can indeed be estimated uniformly since it solves a Laplace equation with amenable boundary conditions. In view of (7.1.46), we expect to perform the energy estimates to get the a priori control of $\|v\|_{L_t^\infty H^{m-1}_{co}}, \|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^\infty H^{m-2}_{co}}$ and $\|\nabla^\varphi v\|_{L_t^2 H^{m-1}_{co}}, \|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|_{L_t^\infty H^{m-2}_{co}}$. Of course, due to the interaction with the compressible part through the boundary, its control relies also on the information of compressible part $\nabla^\varphi\Psi$ and we cannot get higher order estimates.

**Step 5. Control of the normal derivative of the velocity.** We have obtained the estimates of $\|\nabla^\varphi u\|_{L_t^2 H^{m-1}_{co}}$ in Step 3 and Step 4. It remains to control $\varepsilon^{-\frac{1}{2}}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H^{m-2}_{co}}$ and $\|(\nabla v, \varepsilon^{\frac{1}{2}}\partial_t\nabla v)\|_{L_t^\infty H^{m-4}_{co}}$, which is useful to control the $L^\infty_{t,x}$ norm of the solution. The former quantity can be obtained by again induction arguments while the latter quantity can be deduced from that of $\omega\times\mathbf{n}$. Indeed, we have roughly the estimate:

$$\|(\nabla v, \varepsilon^{\frac{1}{2}}\partial_t\nabla v)\|_{L_t^\infty H^{m-4}_{co}} \lesssim \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\omega\times\mathbf{n}\|_{L_t^\infty H^{m-4}_{co}} + \|(v, \varepsilon^{\frac{1}{2}}\partial_t v)\|_{L_t^\infty H^{m-3}_{co}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde H^{m-2}}.$$

Let us explain the estimate of $\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)(\omega\times\mathbf{n})\|_{L_t^\infty H^{m-4}_{co}}$. Direct computations show that:

$$\omega\times\mathbf{n}|_{\partial\mathcal{S}} = -2\Pi(\partial_1 u\cdot\mathbf{n}, \partial_2 u\cdot\mathbf{n}, 0)^t. \tag{7.1.47}$$

where $\Pi = \mathrm{Id}_{3\times3} - \mathbf{n}\otimes\mathbf{n}$. We define the modified vorticity $\omega_{\mathbf{n}} = \omega\times\mathbf{n} + 2\Pi(\partial_1 v\cdot\mathbf{n}, \partial_2 v\cdot\mathbf{n}, 0)$, so that:

$$\omega_{\mathbf{n}}|_{\partial\mathcal{S}} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n}, \partial_2\nabla^\varphi\Psi\cdot\mathbf{n}, 0)^t.$$

The advantage of working on $\omega_{\mathbf{n}}$ rather than $\omega\times\mathbf{n}$ is that the former one only involves the compressible part of velocity on the boundary, whose estimates has been established in Step 3. To estimate $\omega_{\mathbf{n}}$, we shall thus instead use a lifting of the boundary conditions by using Green's function for the solution of the heat equation with non-homogenous boundary conditions and estimate the remainder by energy estimates. More precisely, let $\omega^h$ solves the heat equation (7.1.36) with boundary condition $\omega^{b,1}|_{z=0} = \omega_{\mathbf{n}}|_{z=0}$, we use formulae (7.1.37) to get roughly that:

$$\begin{aligned}\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\omega^h_{\mathbf{n}}\|_{L_t^\infty H^{m-4}_{co}} &\lesssim T^{\frac{1}{4}}\big(|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\nabla\Psi|_{L_t^\infty \tilde H^{m-3}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)h|_{L_t^\infty \tilde H^{m-3}}\big) \\ &\lesssim T^{\frac{1}{4}}(\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)\mathrm{div}^\varphi u\|_{L_t^\infty H^{m-3}_{co}} + |(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t)h|_{L_t^\infty \tilde H^{m-3}}).\end{aligned}$$

The remainder $\omega_{\mathbf{n}} - \omega_{\mathbf{n}}^h$ can then be controlled by direct energy estimates.

**Step 6. $L^\infty_{t,x}$ estimates.** This final step is dedicated to the estimates of $L^\infty_{t,x}$ type norms defined in $\mathcal{A}_{m,T}$. Most of them can be controlled thanks to the Sobolev embedding and the energy norms established in $\mathcal{E}_{m,T}$. The remaining terms $\varepsilon^{-\frac{1}{2}}\|\nabla\sigma\|_{m-5,\infty,T}$ and $\|\nabla u\|_{1,\infty,t}$ will rely on the maximum principle of damped transport equation satisfied by $\nabla\sigma$ and the careful analysis of explicit formula for the heat equation satisfied by $\omega$.

**Structure of the current chapter:** We state the uniform a-priori estimates in Section 2, which are shown in the following sections. We make some preliminaries (useful lemmas, identities Projection, and elliptic estimates) in Section 3- Section 5. The control of the energy norms is achieved in Section 6-Section 11. The $L^\infty_{t,x}$ type estimates are established in Section 12. Theorem 7.1.1 and Theorem 7.1.4 are then proved in Section 13 and Section 14 respectively. In Section 15, we explain how our results can be extended to the case when the reference domain is changed to a channel with infinite depth. Finally, We provide the proof of one product estimate in the appendix.

**Further notations**

- We denote $\Lambda(\cdot,\cdot)$ a polynomial that may differ from line to line.
- The traces on the upper boundary {z=0} and lower boundary {z=-1} for a function $f \in H^1(\mathcal{S})$ are denoted by $f^{b,1}$ and $f^{b,2}$ respectively.
- We use notation $\lesssim$ as $\leq C(1/c_0)$ for some number $C(1/c_0)$ that depends only on $1/c_0$.
- We use the notation $L^2_t L^2 = L^2([0,t]\times\mathcal{S})$.
- We denote $\|f\|_{E^k,t} = \|f\|_{L^2_t H^k_{co}} + \|\nabla f\|_{L^2_t H^{k-1}_{co}}$.

## 7.2 Uniform a-priori estimates

**Theorem 7.2.1.** *Let $c_0 \in [0, \frac{1}{2}]$ such that:*

$$\sup_{s\in[-3c_1\bar{P}, 3\bar{P}/c_1]} |(g_1, g_2)(s)| \in [c_0, 1/c_0]. \tag{7.2.1}$$

*where $0 < c_1 < \frac{1}{4}$ is a fixed constant. Suppose that for some $0 < T \leq 1$, for all $(t,x) \in [0,T]\times\mathcal{S}, \varepsilon\in[0,1]$, it holds that:*

$$\partial_z\varphi^\varepsilon(t,x) \geq c_0, \quad |(\nabla\varphi^\varepsilon, \nabla^2\varphi^\varepsilon)(t,x)| \leq 1/c_0, \quad -3c_1\bar{P} \leq \varepsilon\sigma^\varepsilon(t,x) \leq 3\bar{P}/c_1, \tag{7.2.2}$$

*Then there exist two continuous functions $P_1, P_2 : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$ which are independent of $\varepsilon$ and a constant $\vartheta > 0$, such that the following estimate holds:*

$$\mathcal{N}^\varepsilon_{m,T} \leq P_1\big(\frac{1}{c_0}, Y^\varepsilon_m(0)\big) + (T+\varepsilon)^\vartheta P_2\big(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}\big). \tag{7.2.3}$$

*where $\mathcal{N}^\varepsilon_{m,T}$ is defined in* (7.1.30).

This theorem is a direct consequence of the following two propositions.

**Proposition 7.2.2.** *Under the assumption of Theorem 7.2.1, there exist two $\varepsilon-$independent continuous functions $P_3, P_4 : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$, such that:*

$$\mathcal{E}^\varepsilon_{m,T} \leq P_3\big(\frac{1}{c_0}, Y^\varepsilon_m(0)\big) + (T+\varepsilon)^{\vartheta_1} P_4\big(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}\big). \tag{7.2.4}$$

*Proof.* This proposition is obtained by energy estimates, we split it into several sections (Section 6-11). By Lemmas 7.6.1, 7.7.1, 7.7.4, 7.9.1, 7.9.5, 7.10.1, 7.11.1, 7.11.3, 7.11.10, we can find two polynomials $\Lambda_5, \Lambda_6$ whose coefficients are independent of $\varepsilon$, such that:

$$(\tilde{\mathcal{E}}^\varepsilon_{m,T})^2 \leq \Lambda_5\big(\frac{1}{c_0}, |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}} + Y^\varepsilon_m(0)^2\big) Y^\varepsilon_m(0)^2 + (T+\varepsilon)^{\frac{1}{4}}\Lambda_6\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{7.2.5}$$

By Lemma 7.8.1, there exist polynomials $\Lambda_7, \Lambda_8$ whose coefficients are independent of $\varepsilon$, such that:

$$\tilde{\mathcal{E}}^2_{low,T} \lesssim \Lambda_7\big(\frac{1}{c_0}, |h^\varepsilon|^2_{3,\infty,T}\big)(Y^\varepsilon_m(0)^2 + (\tilde{\mathcal{E}}^\varepsilon_{m,T})^2) + (T+\varepsilon)^{\frac{1}{2}}\Lambda_8\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big).$$

By (7.12.4) and Sobolev embedding,

$$|h^\varepsilon|^2_{3,\infty,T} \lesssim |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}},$$

we thus find two polynomials $\Lambda_9$ and $\Lambda_{10}$ such that:

$$(\mathcal{E}^\varepsilon_{m,T})^2 \leq \Lambda_9\big(\frac{1}{c_0}, |h^\varepsilon|^2_{L^\infty_T \tilde{H}^{m-\frac{1}{2}}} + Y^\varepsilon_m(0)^2\big)Y^\varepsilon_m(0)^2 + (T+\varepsilon)^{\frac{1}{4}}\Lambda_{10}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{7.2.6}$$

By (7.6.3), there exists a polynomial $\Lambda_{11}$, such that:

$$|h^\varepsilon|^2_{L^\infty_t \tilde{H}^{m-\frac{1}{2}}} \leq Y^\varepsilon_m(0)^2 + T^{\frac{1}{2}}\Lambda_{11}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big)$$

Plugging this inequality into (7.2.5), one finds another two polynomials $\Lambda_{12}, \Lambda_{13}$, and a constant $\vartheta_2 > 0$, such that:

$$(\tilde{\mathcal{E}}^\varepsilon_{m,T})^2 \lesssim \Lambda_{12}(\frac{1}{c_0}, Y^\varepsilon_m(0)^2) + (T+\varepsilon)^{\vartheta_2}\Lambda_{13}(\frac{1}{c_0}, Y^\varepsilon_m(0) + \mathcal{N}^\varepsilon_{m,T}).$$

We thus finish the proof by inserting the above inequality into (7.2.6). □

**Proposition 7.2.3.** *Assume that* (7.2.2) *holds, we have the a-priori estimate for* $L^\infty_t L^\infty(\mathcal{S})$ *norms,*

$$\mathcal{A}^\varepsilon_{m,T} \leq \Lambda\big(\frac{1}{c_0}, Y_m(0)\big) + \Lambda\big(\frac{1}{c_0}, |h^\varepsilon|_{3,\infty,t}\big)\tilde{\mathcal{E}}^\varepsilon_{m,T} + (\tilde{\mathcal{E}}^\varepsilon_{m,T})^4 + (T+\varepsilon^{\frac{1}{4}})\Lambda_{14}\big(\frac{1}{c_0}, \mathcal{N}^\varepsilon_{m,T}\big). \tag{7.2.7}$$

*where* $\Lambda_{14}$ *is a polynomial with* $\varepsilon-$*independent coefficients.*

*Proof.* Its proof is presented in Section 12. □

## 7.3 Preliminaries I: Useful lemmas.

In this section, we list some elementary lemmas which shall be frequently used throughout this paper. For notational convenience, we will skip the $\varepsilon$-dependence of the solution.

### 7.3.1 Product and commutator estimates.

We begin with the following product and commutator estimates in $\mathbb{R}^2$.

**Lemma 7.3.1.** *Let* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *belong to the spaces appearing in below. For any* $s \geq 1$,

$$|\Lambda^s(fg)| \lesssim |f|_{H^s}|g|_{L^\infty} + |g|_{H^s}|f|_{L^\infty} \tag{7.3.1}$$

$$|[\Lambda^s, f]g|_{L^2(\mathbb{R}^2)} \lesssim |f|_{H^{s-1}}|g|_{L^\infty} + |f|_{W^{1,\infty}}|g|_{H^{s-1}} \tag{7.3.2}$$

*For any* $-1 < s \leq 1$,

$$|[\Lambda^s, g]f|_{L^2(\mathbb{R}^2)} \lesssim |f|_{H^{s-1}}|g|_{H^{2+}}, \tag{7.3.3}$$

$$|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s}\min\{|g|_{H^{1+}}, |g|_{W^{1,\infty}}\}. \tag{7.3.4}$$

*where* $\Lambda^s = \mathcal{F}^{-1}_{\xi\to y}\big((1+|\xi|^2)^{\frac{s}{2}}\big)$, $a^+$ *denotes a real number that is larger but arbitrary close to* $a$.

Product (7.3.1) and commutator estimate (7.3.2) can be found in [23], (7.3.3) is indeed a restatement of (A.6) in [17]. The proof of (7.3.4) is presented in the appendix.

**Corollary 7.3.2.** *Let $k \geq 2$ be an integer, one has the following estimates:*

$$|(fg)(t)|_{\tilde{H}^{k+\frac{1}{2}}} \lesssim |f(t)|_{\tilde{H}^{[\frac{k}{2}]+}} |g(t)|_{\tilde{H}^{k+\frac{1}{2}}} + |g(t)|_{\tilde{H}^{[\frac{k+1}{2}]+1+}} |f(t)|_{\tilde{H}^{k+\frac{1}{2}}}, \tag{7.3.5}$$

$$|[Z^\alpha, f]g(t)|_{H^{\frac{1}{2}}} \lesssim |f(t)|_{\tilde{H}^{[\frac{k}{2}]+}} |g(t)|_{\tilde{H}^{k-\frac{1}{2}}} + |g(t)|_{\tilde{H}^{[\frac{k+1}{2}]+1+}} |f(t)|_{\tilde{H}^{k+\frac{1}{2}}}. \tag{7.3.6}$$

*where $\tilde{H}^s$ is defined in* (7.1.24), *and commutator $[Z^\alpha, f]g = Z^\alpha(fg) - fZ^\alpha g$.*

*Proof.* For any $|\alpha| \leq k$, we write

$$Z^\alpha(fg)(t) = \Big(\sum_{|\beta| \leq [\frac{|\alpha|}{2}]-1} + \sum_{|\alpha-\beta| \leq [\frac{|\alpha|+1}{2}]}\Big) Z^\beta f(t) Z^{\alpha-\beta} g(t) \tag{7.3.7}$$

Inequality (7.3.5) can then be derived from product estimate (7.3.4). The proof of (7.3.6) follows in the same fashion. □

The following (crude) product estimates in $L_t^\infty \mathcal{H}^{j,l}$ will be useful for instance in the elliptic estimates.

**Lemma 7.3.3.** *Let $Z^\alpha = (\varepsilon\partial_t)^j \mathcal{Z}^{\alpha'}$ with $\mathcal{Z} = (Z_1, Z_2, Z_3), |\alpha'| \leq l = k - j, k \geq 2$. One has the crude estimates: for any integer $n \in [0, k-1]$*

$$\|(fg)(t)\|_{\mathcal{H}^{j,l}} \leq \|f(t)\|_{\mathcal{H}^{j,l}} \|\!|g|\!\|_{n,\infty,t} + \|g(t)\|_{\mathcal{H}^{j,l}} \|\!|f|\!\|_{k-n-1,\infty,t}, \tag{7.3.8}$$

$$\|[Z^\alpha, f]g(t)\|_{L^2(\mathcal{S})} \lesssim \big(\sum_{j' \leq j, l' \leq l, j'+l' \leq k-n} \|f(t)\|_{\mathcal{H}^{j',l'}}\big) \|\!|g|\!\|_{n,\infty,t} + (\|g(t)\|_{\mathcal{H}^{j-1,l}} + \|g(t)\|_{\mathcal{H}^{j,l-1}}) \|\!|f|\!\|_{k-n-1,\infty,t}. \tag{7.3.9}$$

This lemma in hand, we have the following composition estimates:

**Corollary 7.3.4.** *Suppose that $\psi \in C^0(Q_t) \cap L_t^2 H_{co}^m$ with*

$$A_1 \leq \psi(t,x) \leq A_2, \quad \forall (t,x) \in Q_t.$$

*Let $F(\cdot) : [A_1, A_2] \to \mathbb{R}$ be a smooth function satisfying*

$$\sup_{s \in [A_1, A_2], j \leq m} |F^{(j)}|(s) \leq B.$$

*Then we have the composition estimate:*

$$\|F(\psi(\cdot,\cdot)) - F(0)\|_{L_t^p H_{co}^m} \leq \Lambda(B, \|\!|\psi|\!\|_{[\frac{m}{2}],\infty,t}) \|\psi\|_{L_t^p H_{co}^m}, \tag{7.3.10}$$

**Corollary 7.3.5.** *Let $g_1(\varepsilon\sigma), g_2(\varepsilon\sigma)$ defined in* (7.1.8) *and assume property* (7.2.1) *and assumption* (7.2.2) *hold. Then one has the following estimates: for $j = 1, 2$*

$$\|g_j(\varepsilon\sigma) - g_j(0)\|_{L_t^p H_{co}^m} \lesssim \varepsilon\Lambda\big(\frac{1}{c_0}, \|\!|\sigma|\!\|_{[\frac{m}{2}],\infty,t}\big) \|\sigma\|_{L_t^p H_{co}^m}. \tag{7.3.11}$$

$$\|Zg_j\|_{L_t^p \mathcal{H}^{m-1}} \leq \varepsilon\Lambda\big(\frac{1}{c_0}, \|\!|\sigma|\!\|_{[\frac{m}{2}],\infty,t}\big) \|(\sigma, Z\sigma)\|_{L_t^p \mathcal{H}^{m-1}}, \tag{7.3.12}$$

$$\|Zg_j\|_{L_t^p H_{co}^{m-1}} \leq \varepsilon\Lambda\big(\frac{1}{c_0}, \|\!|\sigma|\!\|_{[\frac{m}{2}],\infty,t}\big) \|\sigma\|_{L_t^p H_{co}^m}, \tag{7.3.13}$$

*Proof.* Inequality (7.3.11) is a direct consequence of the composition estimate (7.3.10). To get (7.3.12), (7.3.13), one can apply (7.3.8) for $n = [\frac{m-1}{2}] - 1$ and use again (7.3.10). □

The next lemma states the generalized product estimate and commutator estimate [52].

**Lemma 7.3.6.** *For $|\alpha| \leq m, \alpha_0 = 0$ we have the product estimate and commutator estimates:*

$$\|Z^\alpha(fg)\|_{L_t^2 L^2} \lesssim \|f\|_{L_t^2\mathcal{H}^{0,m}} \|g\|_{0,\infty,t} + \|g\|_{L_t^2\mathcal{H}^{0,m}} \|f\|_{0,\infty,t}, \tag{7.3.14}$$

$$\|[Z^\alpha, f]g\|_{L_t^2 L^2} \lesssim \|f\|_{L_t^2\mathcal{H}^{0,m}} \|g\|_{0,\infty,t} + \|g\|_{L_t^2\mathcal{H}^{0,m-1}} \|f\|_{1,\infty,t}. \tag{7.3.15}$$

We state the following Sobolev embedding and trace inequality whose proofs can be found in Proposition 2.2 of [94].

**Lemma 7.3.7.** *For each $t \in [0,T]$, we have:*

$$\|f(t)\|_{L^\infty(\mathcal{S})} \lesssim \|(f,\nabla f)(t)\|_{H^{s_1}_{tan}(\mathcal{S})}^{\frac{1}{2}} \|f(t)\|_{H^{s_2}_{tan}(\mathcal{S})}^{\frac{1}{2}}, \quad s_1+s_2>2, s_1,s_2\geq 0, \tag{7.3.16}$$

$$\begin{aligned} &|f(t,\cdot,0)|_{H^s(\mathbb{R}^2)} + |f(t,\cdot,-1)|_{H^s(\mathbb{R}^2)} \\ &\lesssim \|\partial_z f(t)\|_{H^{s-1/2}_{tan}(\mathcal{S})}^{\frac{1}{2}} \|f(t)\|_{H^{s+1/2}_{tan}(\mathcal{S})}^{1/2} + \|f(t)\|_{H^{s+1/2}_{tan}(\mathcal{S})}, \quad s \geq \frac{1}{2}. \end{aligned} \tag{7.3.17}$$

*where we have used the notation $\|f(t)\|_{H^s_{tan}(\mathcal{S})} = \|\Lambda^s f(t)\|_{L^2(\mathcal{S})}$ where $\Lambda^s$ is the Fourier multiplier with symbol $(1+|\xi|^2)^{\frac{s}{2}}$.*

### 7.3.2 Regularity of the extension and some further commutator estimates.

We first aim to show that the diffeomorphism $\Phi$ has the same regularity as $u$ in $\mathcal{S}$, which stems from the fact that the extension function $\varphi$ gains half-space derivative from that of $h$.

**Lemma 7.3.8.** *Recall that $\varphi$ and $\eta$ are defined in* (7.1.11), (7.1.12)*. For any integers $j,k\geq 0$, we have the following estimates:*

$$\|[(\varepsilon\partial_t)^j \nabla\varphi](t)\|_{H^k(\mathcal{S})} \lesssim |[(\varepsilon\partial_t)^j h](t,\cdot)|_{H^{k+\frac{1}{2}}(\mathbb{R}^2)} \tag{7.3.18}$$

$$\|\nabla\varphi\|_{L_t^2\mathcal{H}^{j,k}(\mathcal{S})} \lesssim |h|_{L_t^2\tilde{H}^{k+j+\frac{1}{2}}(\mathbb{R}^2)}. \tag{7.3.19}$$

*Moreover, we have the $L^\infty_{t,x}$ estimates for $\eta$ which is defined in* (7.1.12)

$$\|[(\varepsilon\partial_t)^j\eta](t)\|_{W^{k,\infty}(\mathcal{S})} \lesssim |[(\varepsilon\partial_t)^j h](t)|_{W^{k,\infty}(\mathbb{R}^2)} \lesssim |h|_{k+j,\infty,t}. \tag{7.3.20}$$

*Proof.* These estimates can be deduced from the convolution inequality and the following inequalities:

$$\int_{-1}^0 e^{-2z^2\langle\xi\rangle^2} \mathrm{d}z \lesssim \langle\xi\rangle; \quad \|\mathcal{F}^{-1}(e^{-z^2\langle\xi\rangle^2})\|_{L_z^\infty L_y^1} \lesssim 1.$$

One can refer to Proposition 3.1 of [94] for the detail of the case $j=0$. The case for $j>0$ follows from the observation that time derivatives commute with the actions $\varphi(h)$ and $\eta(h)$. □

**Lemma 7.3.9.** *Suppose that: $\partial_z\varphi(t,x)\geq c_0$ for $(t,x)\in[0,T]\times\mathcal{S}$. Then for any $k\in\boldsymbol{N}$,*

$$\Big\|\frac{f}{\partial_z\varphi}\Big\|_{L_t^p H^k_{co}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t} + \|f\|_{[\frac{k}{2}],\infty,t}\Big)\big(\|f\|_{L_t^p H^k_{co}} + |h|_{L_t^p\tilde{H}^{k+\frac{1}{2}}}\big), \quad p=2,+\infty. \tag{7.3.21}$$

*Proof.* Let us write:

$$\frac{f}{\partial_z\varphi} = \frac{f}{1+\eta+\partial_z\eta(1+z)} = f - f\frac{\eta+\partial_z\eta(1+z)}{1+\eta+\partial_z\eta(1+z)}$$

Therefore, one obtains (7.3.21) by applying product estimate (7.3.8) for $n=[\frac{k}{2}]$ and composition estimate (7.3.10) for $F(x)=\frac{x}{1+x}$ $(0<x<1)$. □

**Remark 7.3.10.** *Similar to* (7.3.21)*, the following estimate also holds true:*

$$\Big\|\frac{f}{\partial_z\varphi}\Big\|_{L_t^p\mathcal{H}^{0,k}} \lesssim \Lambda\Big(\frac{1}{c_0}, |h|_{1,\infty,t} + \|f\|_{0,\infty,t}\Big)(\|f\|_{L_t^p\mathcal{H}^{0,k}} + |h|_{L_t^p\tilde{H}^{k+\frac{1}{2}}}),\ p=2,+\infty. \tag{7.3.22}$$

The next lemma contains some commutator estimates which are used very often.

**Lemma 7.3.11.** *Under the assumption (7.2.2), the following commutator estimates hold, for $j = 1,2,3, |\alpha| \le k$*

$$\begin{aligned} \|[Z^\alpha, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h|_{L_t^2\tilde{H}^{k-n+\frac12}}\|\nabla f\|_{n,\infty,t} \\ + \Lambda\big(\frac{1}{c_0}, |h|_{k-n,\infty,t}\big)\|\nabla f\|_{L_t^2H_{co}^{k-1}} \quad (0 \le n \le k-1). \end{aligned} \tag{7.3.23}$$

*If $\alpha_0 = 0$, we have that:*

$$\|[Z^\alpha, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{1,\infty,t}\big)\|\nabla f\|_{L_t^2H_{co}^{k-1}} + \Lambda\big(\frac{1}{c_0}, \|\nabla f\|_{0,\infty,t}\big)|h|_{L_t^2\tilde{H}^{k+\frac12}}. \tag{7.3.24}$$

*Moreover, for $k \ge 3$,*

$$\begin{aligned} \|[Z_0^k\partial_t, \partial_j^\varphi]f\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \|(\partial_z f, \varepsilon\partial_t\partial_z f)\|_{0,\infty,t} + |(h,\partial_t h)|_{k-2,\infty,t} + (\int_0^t |\varepsilon\partial_t^2 h(s)|_{k-2,\infty}^2 \mathrm{d}s)^{\frac12}\big) \\ \cdot\big(\varepsilon\sum_{l\le k-1}|Z_0^l\partial_t^2 h|_{L_t^2H^{\frac12}} + \|Z_0\partial_z f\|_{L_t^2\mathcal{H}^{k-1}} + \|Z_0\partial_z f\|_{L_t^\infty\mathcal{H}^1}\big). \end{aligned} \tag{7.3.25}$$

*Proof.* By the definition of $\nabla^\varphi$:

$$[Z^\alpha, \partial_j^\varphi]f = [Z^\alpha, \mathbf{N}_j/\partial_z\varphi]\partial_z f + (\mathbf{N}_j/\partial_z\varphi)[Z^\alpha, \partial_z]f. \tag{7.3.26}$$

Moreover, there exist smooth functions $C_{\phi,\beta,\alpha}, C_{\phi,\gamma,\alpha}$ which depend on $\phi$ and its derivatives, such that:

$$[Z^\alpha, \partial_z] = \sum_{|\beta|\le|\alpha|-1} C_{\phi,\beta,\alpha}Z^\beta\partial_z = \sum_{|\gamma|\le|\beta|-1} C_{\phi,\gamma,\alpha}\partial_z Z^\gamma. \tag{7.3.27}$$

Therefore, we get (7.3.23) by (7.3.9), (7.3.21). and get (7.3.24) by (7.3.15), (7.3.22).

Next, for (7.3.25), we use the following direct expansion

$$[Z_0^k\partial_t, g]w = \Big(\sum_{0\le l\le 1} + \sum_{0\le k-l\le k-3}\Big)\big(C_{k,l}Z_0^{k-l}\partial_t g\, Z_0^l w\big) + C_{k,2}Z_0^{k-2}\partial_t g Z_0^2 w. \tag{7.3.28}$$

to obtain:

$$\begin{aligned} \|[Z_0^k\partial_t, g]w\|_{L^2L^2} \lesssim \|Z_0\partial_t g\|_{L_t^2\mathcal{H}^{k-1}}\|w\|_{1,\infty,t} + \|Z_0 w\|_{L_t^2\mathcal{H}^{k-1}}\|\partial_t g\|_{k-3,\infty,t} \\ + \|Z_0 w\|_{L_t^\infty\mathcal{H}^1}\|Z_0^{k-2}\partial_t g\|_{L_t^2L^\infty} \end{aligned} \tag{7.3.29}$$

Applying (7.3.29) with $g = \frac{\mathbf{N}_j}{\partial_z\varphi}, w = \partial_z f$, and using (7.3.18), we get (7.3.25). $\square$

### 7.3.3 Energy identities and Korn inequality

We present some identities which are often used in the energy estimates:

**Lemma 7.3.12.** *It holds that:*

$$\int_{\mathcal{S}} g_1(\partial_t^\varphi + u\cdot\nabla^\varphi)\sigma(t)\cdot\sigma(t)\,\mathrm{d}\mathcal{V}_t = \frac12\partial_t\int_{\mathcal{S}} g_1|\sigma|^2(t)\mathrm{d}\mathcal{V}_t - \frac12\int_{\mathcal{S}}(\partial_t^\varphi g_1 + \mathrm{div}^\varphi(g_1 u))|\sigma|^2(t)\ \mathrm{d}\mathcal{V}_t, \tag{7.3.30}$$

$$\int_{\mathcal{S}} g_2(\partial_t^\varphi + u\cdot\nabla^\varphi)u(t)\cdot u(t)\ \mathrm{d}\mathcal{V}_t = \frac12\partial_t\int_{\mathcal{S}} g_2|u|^2(t)\ \mathrm{d}\mathcal{V}_t, \tag{7.3.31}$$

$$\begin{aligned} &\int_{\mathcal{S}}(-\mathrm{div}^\varphi\mathcal{L}^\varphi u + \nabla^\varphi\sigma/\varepsilon)\cdot u(t)\ \mathrm{d}\mathcal{V}_t \\ &= \int_{\mathcal{S}} 2\mu|S^\varphi u(t)|^2 + \lambda|\mathrm{div}^\varphi u(t)|^2\ \mathrm{d}\mathcal{V}_t - \int_{\mathcal{S}}\sigma\mathrm{div}^\varphi u(t)\ \mathrm{d}\mathcal{V}_t + a\int_{z=-1}|u_\tau|^2\ \mathrm{d}y. \end{aligned} \tag{7.3.32}$$

*where $u_\tau = (u_1, u_2, 0)^t$ denotes the tangential components of $u$, $\mathrm{d}\mathcal{V}_t = \partial_z\varphi\ \mathrm{d}y\mathrm{d}z$ represents the volume measure.*

*Proof.* By direct computations, one can obtain the following identities:

$$\int_{\mathcal{S}} \partial_j^\varphi f(t) g(t)\,\mathrm{d}\mathcal{V}_t = -\int_{\mathcal{S}} f(t)\partial_j^\varphi g(t)\,\mathrm{d}\mathcal{V}_t + \int_{\partial\mathcal{S}} f(t)g(t)\mathbf{N}_j\,\mathrm{d}y, \quad j=1,2,3$$

$$\int_{\mathcal{S}} \partial_t^\varphi f(t)g(t)\,\mathrm{d}\mathcal{V}_t = \partial_t \int_{\mathcal{S}} fg(t)\,\mathrm{d}\mathcal{V}_t - \int_{\mathcal{S}} f(t)\partial_t^\varphi g(t)\,\mathrm{d}\mathcal{V}_t + \int_{z=0} f(t)g(t)\partial_t h\,\mathrm{d}y,$$

which, along with the equation (7.1.17)-(7.1.19) lead to (7.3.30)-(7.3.32). Note that in the derivation of (7.3.31), we have used the fact that $\partial_t^\varphi g_2 + \mathrm{div}^\varphi(g_2 u) = 0$ in $[0,t]\times\mathcal{S}$. □

The next lemma shows that one can control the gradient of the velocity by $S^\varphi u$.

**Lemma 7.3.13** (Korn's inequality)**.** *Suppose that* (7.2.2) *is true, then there exists* $\Lambda_0(\frac{1}{c_0}), \Lambda_1(\frac{1}{c_0})$ *such that:*

$$\int_{\mathcal{S}} |\nabla u|^2(t)\,\mathrm{d}\mathcal{V}_t \le \Lambda_0\big(\frac{1}{c_0}\big)\int_{\mathcal{S}} |\nabla^\varphi u|^2(t)\,\mathrm{d}\mathcal{V}_t \le \Lambda_1\big(\frac{1}{c_0}\big)\int_{\mathcal{S}} (|S^\varphi u|^2 + |u|^2)\,\mathrm{d}\mathcal{V}_t. \tag{7.3.33}$$

*As a consequence,*

$$\int_0^t\int_{\mathcal{S}} |\nabla u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s \le \Lambda_1\big(\frac{1}{c_0}\big)\int_0^t\int_{\mathcal{S}} (|S^\varphi u|^2 + |u|^2)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s. \tag{7.3.34}$$

These two inequalities can be proved in the similar manner that of Proposition 2.9 in [94].

## 7.4 Preliminaries II: Reformulations of the boundary conditions

**Proposition 7.4.1.** *The following boundary condition on* $\{z=0\}$ *hold:*

$$\frac{\sigma}{\varepsilon} = (2\mu+\lambda)\mathrm{div}^\varphi u - 2\mu(\partial_1 u_1 + \partial_2 u_2) + \mu(\omega\times\boldsymbol{N})_3, \tag{7.4.1}$$

$$\omega\times\boldsymbol{n} = -2\Pi(\partial_1 u\cdot\boldsymbol{n}, \partial_2 u\cdot\boldsymbol{n}, 0)^t, \tag{7.4.2}$$

$$\Pi(\partial_{\boldsymbol{n}}^\varphi u) = -\Pi(\partial_1 u\cdot\boldsymbol{n}, \partial_2 u\cdot\boldsymbol{n}, 0)^t, \tag{7.4.3}$$

$$\begin{aligned}\partial_{\boldsymbol{n}}^\varphi u\cdot\boldsymbol{n} &= |\boldsymbol{N}|^2\partial_z^\varphi u\cdot\boldsymbol{n} - (\boldsymbol{n}_1\partial_1 u\cdot\boldsymbol{n} + \boldsymbol{n}_2\partial_2 u\cdot\boldsymbol{n})\\ &= |\boldsymbol{N}|(\mathrm{div}^\varphi u - \partial_1 u_1 - \partial_2 u_2) - (\boldsymbol{n}_1\partial_1 u\cdot\boldsymbol{n} + \boldsymbol{n}_2\partial_2 u\cdot\boldsymbol{n})\end{aligned} \tag{7.4.4}$$

*where* $\omega = \mathrm{curl}^\varphi u, \Pi = Id_{3\times3} - \boldsymbol{n}\otimes\boldsymbol{n}$.

*Proof.* The first identity can be deduced from condition (7.1.18). Indeed, by taking the third component of (7.1.18), one gets that on the upper boundary $\{z=0\}$,

$$\begin{aligned}\frac{\sigma}{\varepsilon} &= \lambda\mathrm{div}^\varphi u + 2\mu\partial_z^\varphi u\cdot\mathbf{N} + \mu\big[(\nabla^\varphi u - (\nabla^\varphi u)^t)\cdot\mathbf{N}\big]_3\\ &= (2\mu+\lambda)\mathrm{div}^\varphi u - 2\mu(\partial_1 u_1 + \partial_2 u_2) + \mu(\omega\times\mathbf{N})_3.\end{aligned}$$

Note that we have used the identity

$$\partial_z^\varphi u\cdot\mathbf{N} = \mathrm{div}^\varphi u - \partial_1 u_1 - \partial_2 u_2. \tag{7.4.5}$$

which holds indeed in the whole domain $\mathcal{S}$. For the second identity, we have that on the upper boundary:

$$\begin{aligned}\mu\omega\times\mathbf{N} = \mu\Pi(\omega\times\mathbf{N}) &= 2\mu\Pi\big(-(\nabla^\varphi u)^t\mathbf{N} + S^\varphi u\mathbf{N}\big)\\ &= \Pi\big(-2\mu(\nabla^\varphi u)^t\mathbf{N} + (\sigma/\varepsilon - \lambda\mathrm{div}^\varphi u)\mathbf{N}\big)\\ &= -2\mu\Pi(\partial_1 u\cdot\mathbf{N}, \partial_2 u\cdot\mathbf{N}, 0)^t.\end{aligned} \tag{7.4.6}$$

Note that $(\nabla^\varphi u)^t\cdot\mathbf{N} = (\partial_1 u\cdot\mathbf{N}, \partial_2 u\cdot\mathbf{N}, 0)^t + (\partial_z^\varphi u\cdot\mathbf{N})\mathbf{N}$. The third inequality can be derived in the similar way:

$$\mu\Pi(\partial_{\mathbf{n}}^\varphi u) = \mu\Pi(2S^\varphi u\mathbf{n} - (\nabla^\varphi u)^t\cdot\mathbf{n}) = -\mu\Pi\big((\nabla^\varphi u)^t\cdot\mathbf{n}\big). \tag{7.4.7}$$

The inequality (7.4.4) follows from direct computations and identity (7.4.5).

**Remark 7.4.2.** *By the identity:* $|\boldsymbol{N}|\partial_z^{\varphi} u = \partial_{\boldsymbol{n}}^{\varphi} u - \boldsymbol{n}_1 \partial_1 u - \boldsymbol{n}_2 \partial_2 u$*, we have also:*

$$|\boldsymbol{N}|\Pi\partial_z^{\varphi} u = \Pi(\partial_1 u \cdot \boldsymbol{n}, \partial_2 u \cdot \boldsymbol{n}, 0)^t - \Pi(\boldsymbol{n}_1\partial_1 u + \boldsymbol{n}_2 \partial_2 u). \tag{7.4.8}$$

**Remark 7.4.3.** *In view of* (7.4.5), (7.4.8)*, we have that* $\partial_z^{\varphi} u \approx \mathrm{div}^{\varphi} u + \partial_y u$ *on* $\{z=0\}$*, so that:*

$$\begin{aligned} |(\nabla^{\varphi} u)^{b,1}|_{L_t^2 \tilde{H}^k} \lesssim \Lambda\big(\frac{1}{c_0}, \|\mathrm{div}^{\varphi} u\|_{0,\infty,t} + \|u\|_{1,\infty,t} + |h|_{1,\infty,t}\big) \\ \big(|(\mathrm{div}^{\varphi} u)^{b,1}|_{L_t^2 \tilde{H}^k} + |u^{b,1}|_{L_t^2 \tilde{H}^{k+1}} + |h|_{L_t^2 \tilde{H}^{k+1}}\big), \end{aligned} \tag{7.4.9}$$

*where we denote* $f^{b,1} = f|_{z=0}$.

□

# 7.5 Preliminaries III: Projection operators.

## 7.5.1 Definition of the projection and the reformulation of equations.

We define the projection operator $\mathbb{Q}_t$:

$$\begin{aligned} \mathbb{Q}_t : \quad L^2(\mathcal{S}\, \mathrm{d}\mathcal{V}_t)^3 &\to L^2(\mathcal{S}\, \mathrm{d}\mathcal{V}_t)^3 \\ f &\to \mathbb{Q}_t f = \nabla^{\varphi} \varrho \end{aligned} \tag{7.5.1}$$

where $\varrho$ satisfies the elliptic equation with mixed boundary condition:

$$\begin{cases} -\Delta^{\varphi} \varrho = -\mathrm{div}^{\varphi} f \quad \text{in } \mathcal{S} \\ \varrho|_{z=0} = 0 \\ \partial_z^{\varphi} \varrho|_{z=-1} = f \cdot e_3 \end{cases} \tag{7.5.2}$$

Denote also the projection

$$\mathbb{P}_t = \mathrm{Id} - \mathbb{Q}_t. \tag{7.5.3}$$

**Remark 7.5.1.** *Let us notice that the definition of the projection* $\mathbb{Q}_t$ *is not the same as the standard Leary projection where only the Neumann boundary condition is involved. Nevertheless, the definition* (7.5.2) *is classical in free boundary problems, one can refer for example to [13].*

**Remark 7.5.2.** *We remark that these two projectors are time-dependent since* $\varphi$ *depends on* $t$*. One thus has in general,* $\mathbb{P}_t \nabla^{\varphi} \neq 0, \mathbb{Q}_t \nabla^{\varphi} \neq \nabla^{\varphi}$*. These facts would lead to some extra commutators when we act the projection on the equations* $(7.1.16)_2$.

Denote $v = \mathbb{P}_t u, \nabla^{\varphi}\Psi = \mathbb{Q}_t u$. Applying $\mathbb{P}_t$ projection on the velocity equation $(7.1.16)_2$, one gets:

$$\bar{\rho}\partial_t^{\varphi} v + \mathbb{P}_t \nabla^{\varphi}(\sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^{\varphi} u) = -\mathbb{P}_t(f - \mu\Delta^{\varphi} v) - \bar{\rho}[\mathbb{P}_t, \partial_t^{\varphi}]u$$

where $f = \frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t^{\varphi} u + g_2 u \cdot \nabla^{\varphi} u$. By definition $\mathbb{P}_t \nabla^{\varphi}$ can be expressed as a gradient, we thus denote $\nabla^{\varphi}\pi = \mathbb{P}_t \nabla^{\varphi}(\sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^{\varphi} u)$. To shorten the notation, we denote further $\nabla^{\varphi} q = -\mathbb{Q}_t(f - \mu\Delta^{\varphi} v)$. Therefore, the above equations read:

$$\bar{\rho}\partial_t^{\varphi} v - \mu\Delta^{\varphi} v + \nabla^{\varphi}\pi = -(f + \nabla^{\varphi} q + \bar{\rho}[\mathbb{P}_t, \partial_t^{\varphi}]u). \tag{7.5.4}$$

We are now in position to compute the boundary condition of $v$. On the bottom, in light of (7.1.19) and the fact $\partial_z^{\varphi}\Psi = u_3$, we get that

$$v_3|_{z=-1} = 0, \quad \partial_z^{\varphi} v_{\tau}|_{z=-1} = \partial_z^{\varphi} u_{\tau}|_{z=-1} - \nabla_{\tau}^{\varphi}\partial_z^{\varphi}\Psi|_{z=-1} = \frac{a}{\mu} u_{\tau}|_{z=-1}. \tag{7.5.5}$$

where $\nabla_{\tau}^{\varphi} = (\partial_1^{\varphi}, \partial_2^{\varphi}, 0)^t, f_{\tau} = (f_1, f_2, 0)^t$. Note that $\nabla_{\tau}^{\varphi} = (\partial_1, \partial_2, 0)^t$ on the boundary $\{z=-1\}$ since $\partial_{\tau}\varphi|_{z=-1} = 0$.

On the upper boundary, one first notices that by definition, $\pi|_{z=0} = \sigma/\varepsilon - 2(\mu+\lambda)\mathrm{div}^{\varphi} u$. Therefore, with the aid of the condition (7.1.18), we find that:

$$(2\mu S^{\varphi} v - \pi \mathrm{Id})\mathbf{N}|_{z=0} = 2\mu(\mathrm{div}^{\varphi} u \mathrm{Id} - (\nabla^{\varphi})^2\Psi)\mathbf{N}|_{z=0}. \tag{7.5.6}$$

### 7.5.2 Elliptic estimates

In this section, we establish some useful elliptic estimates in the conormal setting. We first consider the problem:

$$\begin{cases} -\Delta^{\varphi}\varrho = -\mathrm{div}^{\varphi}\tilde{F} \\ \varrho|_{z=0} = 0 \\ \partial_z^{\varphi}\varrho|_{z=-1} = \tilde{F}\cdot e_3 + g \end{cases} \tag{7.5.7}$$

where $e_3 = (0,0,1)^t$. To perform elliptic estimates, it would be convenient to write it in a more explicit way. By a straightforward calculation, one finds that:

$$\mathrm{div}^{\varphi}(\cdot) = \frac{1}{\partial_z\varphi}\mathrm{div}(P\cdot), \quad \nabla^{\varphi} = \frac{1}{\partial_z\varphi}P^*\nabla^{\varphi}, \quad \Delta^{\varphi} = \frac{1}{\partial_z\varphi}\mathrm{div}(E\nabla)$$

where

$$P = \begin{pmatrix} \partial_z\varphi & 0 & 0 \\ 0 & \partial_z\varphi & 0 \\ -\partial_1\varphi & -\partial_2\varphi & 1 \end{pmatrix}, \quad E = \frac{1}{\partial_z\varphi}PP^{\star} \tag{7.5.8}$$

Note that $E$ is uniformly elliptic. Indeed, if $\|\nabla\varphi\|_{\infty,t} \le 1/c_0, \partial_z\varphi \ge c_0$, then there exists $\delta(1/c_0)$ such that for any vectors $X \in \mathbb{R}^3$, $EX\cdot X \ge \delta|X|^2$.

Denote $F = P\tilde{F}$, the equation (7.5.7) is then equivalent to the following elliptic problem:

$$\begin{cases} -\mathrm{div}(E\nabla\varrho) = -\mathrm{div}F \\ \varrho|_{z=0} = 0 \\ (E\nabla\varrho\cdot e_3)|_{z=-1} = F_3^{b,2} + g \end{cases} \tag{7.5.9}$$

where $F_3^{b,2} = F^{b,2}\cdot e_3$. One should keep in mind that the above elliptic equation is satisfied for each time $t$. The following lemma are dedicated to various elliptic estimates.

**Lemma 7.5.3** (Elliptic estimates). *Suppose that $j+l=k, l\ge 1, j\ge 0$, we have the following $L_t^{\infty}L^2(\mathcal{S})$ estimates:*

$$\|\nabla\varrho(t)\|_{H_{co}^{k+1}} + \|\nabla^2\varrho(t)\|_{H_{co}^k} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+2,\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{H_{co}^k} + |(F_3^{b,2}+g)(t)|_{\tilde{H}^{k+\frac{1}{2}}}\big), \tag{7.5.10}$$

$$\begin{aligned}\|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\ &+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big),\end{aligned} \tag{7.5.11}$$

$$\begin{aligned}\|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\ &+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{\mathcal{H}^{j,l-1}} + |(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big),\end{aligned} \tag{7.5.12}$$

$$\begin{aligned}\|\nabla^2\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+5}{2}],\infty,t}\big)\big(\|\mathrm{div}F(t)\|_{\mathcal{H}^{j,l}} + |(F_3^{b,2}, g)(t)|_{\tilde{H}^{k+\frac{1}{2}}}\big) \\ &+ \Lambda\big(\frac{1}{c_0}\|\nabla\varrho\|_{[\frac{k-1}{2}],\infty,t} + |h|_{[\frac{k+5}{2}],\infty,t}\big)\big(\|\nabla\varrho\|_{0,\infty,t}|h(t)|_{\tilde{H}^{k+\frac{3}{2}}} + |h(t)|_{\tilde{H}^{k+\frac{1}{2}}}\big),\end{aligned} \tag{7.5.13}$$

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, |h|_{k+1,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t F(t)\|_{\mathcal{H}^{j,l}} + \varepsilon^{\frac{1}{2}}|\partial_t g(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big) \\ &+ \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{1,\infty,t} + |\partial_t h|_{k-1,\infty,t} + |h|_{k,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^k}),\end{aligned} \tag{7.5.14}$$

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}F(t)\|_{\mathcal{H}^{j,l-1}} + |\varepsilon^{\frac{1}{2}}\partial_t(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big) \\ &+ \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac{1}{2}}\nabla\varrho\|_{[\frac{k}{2}],\infty,t} + |(h, \partial_t h,)|_{[\frac{k+3}{2}],\infty,t}\big)(|(\varepsilon\partial_t h, h)(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|\nabla\varrho(t)\|_{H_{co}^k}).\end{aligned} \tag{7.5.15}$$

**Remark 7.5.4.** *We shall use* (7.5.14) *when* $k \le m-3$ *since as will be seen later,* $|h|_{m-2,\infty,t}$ *can be uniformly controlled. The inequality* (7.5.15) *will be used when* $m-3 \le k \le m-1$.

*Proof.* The first inequality can be proved easily by the variational arguments and the use of Poincaré inequality:

$$\|\varrho(t)\|_{L^2(\mathcal{S})} \le C\|\nabla\varrho(t)\|_{L^2(\mathcal{S})}.$$

Note that the generic constant $C$ is independent of $t$ and $\varepsilon$. More precisely, by testing (7.5.9) by $\varrho(t)$, we easily get that:

$$\begin{aligned}\delta\|\nabla\varrho(t)\|_{L^2(\mathcal{S})} \le \int_{\mathcal{S}} E\nabla\varrho(t)\cdot\nabla\varrho(t)\,\mathrm{d}x &= -\int_{\mathcal{S}}\varrho(t)\mathrm{div}F(t)\,\mathrm{d}x + \int_{z=-1}(F_3^{b,2}+g)(t)\varrho(t)\,\mathrm{d}y\\ &\le \frac{\delta}{2}\|\nabla\varrho(t)\|_{L^2(\mathcal{S})} + C_\delta(\|\mathrm{div}F(t)\|_{L^2(\mathcal{S})} + |(F_3^{b,2},g)(t)|_{H^{-\frac12}}).\end{aligned}$$

The higher-order norm in $\|\nabla\varrho(t)\|_{H^{k+1}}$ follows again from the variational arguments and the commutator estimates. We skip them since they are essentially included in the proof of other inequalities (for instance (7.5.11) and (7.5.13)).

We now begin to prove (7.5.11). Let $\alpha = (j,\alpha'), Z^\alpha = (\varepsilon\partial_t)^j Z_1^{\alpha_1}Z_2^{\alpha_2}Z_3^{\alpha_3}$. If $\alpha_3 \ne 0$, taking $Z^\alpha$ derivatives on the equation shall destroy the divergence form. The trick to avoid this problem is to use another vector field $\tilde{Z}_3 = Z_3 + \partial_z\phi\mathrm{Id}$, such that: $\tilde{Z}_3\partial_z = \partial_z Z_3$. By induction, we have for any $\alpha_3 \ge 1$, $\tilde{Z}_3^{\alpha_3}\partial_z = \partial_z Z_3^{\alpha_3}$, which yields

$$\tilde{Z}^\alpha\partial_z =: (\varepsilon\partial_t)^j Z_1^{\alpha_1}Z_2^{\alpha_2}\tilde{Z}_3^{\alpha_3}\partial_z = \partial_z Z^\alpha.$$

It is useful to notice that

$$\|(\tilde{Z}^\alpha - Z^\alpha)f(t)\|_{L^2(\mathcal{S})} \lesssim \|f(t)\|_{\mathcal{H}^{j,l-1}}. \tag{7.5.16}$$

Taking $\tilde{Z}^\alpha$ derivative on the equation (7.5.9), we find that:

$$\begin{cases} -\mathrm{div}\big(E(Z^\alpha\nabla\varrho)\big) = \mathrm{div}([Z^\alpha,E]\nabla\varrho - Z^\alpha F) + \mathrm{div}\big(\tilde{Z}^\alpha - Z^\alpha)[(E\nabla\varrho)_\tau - F_\tau]\big),\\ Z^\alpha\varrho|_{z=0} = 0,\\ Z^\alpha(E\nabla\varrho)\cdot e_3|_{z=-1} = \mathbb{I}_{\{\alpha_3=0\}}Z^\alpha(F_3^{b,2}+g).\end{cases} \tag{7.5.17}$$

Note that we denote $X_\tau = (X_1,X_2,0)^t$ the horizontal components of a three dimensional vector $X$. Testing equation (7.5.17) by $Z^\alpha\varrho$ in $L^2(\mathcal{S})$, we obtain (we drop the $t$ dependence temporarily):

$$\begin{aligned}&\delta\|Z^\alpha\nabla\varrho\|_{L^2}^2 \le \int_{\mathcal{S}} EZ^\alpha\nabla\varrho Z^\alpha\nabla\varrho\,\mathrm{d}x\\ &= \int_{\mathcal{S}} EZ^\alpha\nabla\varrho\cdot[Z^\alpha,\nabla]\varrho\,\mathrm{d}x - \int_{\mathcal{S}}[Z^\alpha,E]\nabla\varrho\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x\\ &-\int_{\mathcal{S}}(\tilde{Z}^\alpha - Z^\alpha)\big((E\nabla\varrho)_\tau - F_\tau\big)\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x + \int_{\mathcal{S}} Z^\alpha F\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x - \int_{z=-1}\mathbb{I}_{\{\alpha_3=0\}}Z^\alpha g Z^\alpha\varrho\,\mathrm{d}y.\end{aligned} \tag{7.5.18}$$

Combined with Young's inequality, property (7.5.16) and the trace inequality (7.3.17), this yields

$$\|Z^\alpha\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \lesssim \|F(t)\|_{\mathcal{H}^{j,l}}^2 + |g(t)|_{\tilde{H}^{k-\frac12}}^2 + \|(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[Z^\alpha,E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2. \tag{7.5.19}$$

It follows from the product and commutator estimates (7.3.8) (7.3.9) that:

$$\begin{aligned}\|\nabla\varrho(t)\|_{\tilde{H}^{j,l}} &\le \Lambda(1/c_0)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac12}} + \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}}\\ &\quad + \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\|E\|_{[\frac{k+1}{2}],\infty,t} + \|E(t)\|_{\tilde{H}^{k+\frac12}}\|\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t}\big).\end{aligned} \tag{7.5.20}$$

By Lemma 7.3.8 and the expression of $E$ (7.5.8), we get

$$\|E\|_{k,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+1,\infty,t}\big), \quad \|E(t)\|_{\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac12}}. \tag{7.5.21}$$

Inserting (7.5.21) into (7.5.20), we arrive at:

$$
\begin{aligned}
\|\nabla\varrho(t)\|_{\tilde{H}^{j,l}} &\leq \Lambda\big(\frac{1}{c_0}, |\!|\!|\nabla\varrho|\!|\!|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\
&+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}} + \|\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\big).
\end{aligned} \tag{7.5.22}
$$

The inequality (7.5.11) then follows by induction. To get (7.5.12), it suffices to observe that the last three terms in (7.5.18) can indeed be replaced by:

$$
\begin{aligned}
&\int_{\mathcal{S}} Z^{\tilde{\alpha}}\mathrm{div}F\partial_y Z^\alpha\varrho\,\mathrm{d}x - \int_{z=-1} Z^\alpha(F_3+g)Z^\alpha\varrho\,\mathrm{d}y, \quad \text{if} \quad \alpha_3 = 0, Z^\alpha = \partial_y Z^{\tilde{\alpha}}. \\
&-\int_{\mathcal{S}}(\tilde{Z}^\alpha - Z^\alpha)(E\nabla\varrho)_h\cdot\nabla Z^\alpha\varrho\,\mathrm{d}x\int_{\mathcal{S}} Z^{\tilde{\alpha}}\mathrm{div}F(Z_3+\partial_z\phi)(Z^\alpha\varrho)\,\mathrm{d}x, \quad \text{if} \quad \alpha_3\neq 0, Z^\alpha = Z_3 Z^{\tilde{\alpha}}
\end{aligned}
$$

To prove (7.5.13), we first estimate $\|\partial_y\nabla\varrho(t)\|_{\mathcal{H}^{j,l}}$ and then use the equation itself to recover $\|\partial_z^2\varrho(t)\|_{\mathcal{H}^{j,l}}$. The estimate of $\|\partial_y\nabla\varrho(t)\|_{\mathcal{H}^{j,l}}$ is almost identical but slightly different to that of (7.5.12) since for the latter we need to distinguish the highest derivatives hitting on $E$ (or finally on $h$). Hence, when estimating the term $[Z^\alpha\partial_y, E]\nabla\varrho$, we write

$$
[Z^\alpha\partial_y, E]\nabla\varrho = (Z^\alpha\partial_y E)\nabla\varrho + \text{ other terms}
$$

and control the first term as

$$
\|(Z^\alpha\partial_y E)\nabla\varrho(t)\|_{L^2(\mathcal{S})} \lesssim |\!|\!|\nabla\varrho|\!|\!|_{0,\infty,t}\Lambda(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+2,\infty,t})|h(t)|_{\tilde{H}^{k+\frac{3}{2}}}.
$$

We now sketch the proof of (7.5.14) and (7.5.15). For (7.5.14), we first have the following inequality analogues to (7.5.19).

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}Z^\alpha\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t F(t)\|_{\mathcal{H}^{j,l}}^2 + |\varepsilon^{\frac{1}{2}}\partial_t g(t)|_{\tilde{H}^{k-\frac{1}{2}}}^2 \\
&+ \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2,
\end{aligned}
$$

where the last two terms can be bounded in a rather rough way:

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\partial_t(E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}}\Lambda(\frac{1}{c_0}, |h|_{k,\infty,t}) \\
&+ \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, |\!|\!|\nabla\varrho|\!|\!|_{0,\infty,t} + |\partial_t h|_{k-1,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k-\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^{k-1}}), \\
\varepsilon^{\frac{1}{2}}\|[\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})} &\leq \varepsilon^{\frac{1}{2}}\|Z^\alpha(\partial_t E\nabla\varrho)(t)\|_{L^2(\mathcal{S})} + \varepsilon^{\frac{1}{2}}\|[Z^\alpha, E]\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})} \\
&\lesssim |\!|\!|\varepsilon^{\frac{1}{2}}\partial_t\nabla\varrho|\!|\!|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\Lambda\big(\frac{1}{c_0}, |h|_{[k,\infty,t}\big) \\
&+ \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, |\!|\!|\nabla\varrho|\!|\!|_{1,\infty,t} + |\partial_t h|_{k-1,\infty,t}\big)(|\partial_t h(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^k}),
\end{aligned}
$$

The inequality (7.5.14) then follows from the induction. For (7.5.15), analogues to (7.5.12), we have:

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}Z^\alpha\partial_t\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}F(t)\|_{\mathcal{H}^{j,l}}^2 + |\varepsilon^{\frac{1}{2}}(F_3^{b,2}, \partial_t g)(t)|_{\tilde{H}^{k-\frac{1}{2}}}^2 \\
&+ \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2.
\end{aligned}
$$

The last two terms are bounded as

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t(\nabla\varrho, E\nabla\varrho)(t)\|_{\mathcal{H}^{j,l-1}}^2 + \|[\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha, E]\nabla\varrho(t)\|_{L^2(\mathcal{S})}^2 \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l-1}\cap\mathcal{H}^{j-1,l}}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big) \\
&+ \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac{1}{2}}\nabla\varrho\|_{[\frac{k}{2}],\infty,t} + |(\partial_t h, h)|_{[\frac{k+3}{2}],\infty,t}\big)\big(|(\varepsilon\partial_t h, h)(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|\nabla\varrho(t)\|_{H_{co}^k}\big).
\end{aligned}
$$

Equipped with the above two inequalities, we obtain (7.5.15) again by induction.

□

**Remark 7.5.5.** *Analogue to* (7.5.11), (7.5.15) *the following estimate also hold, for* $j+l=k\geq 3$,

$$\begin{aligned}\|\nabla\varrho(t)\|_{H^k_{co}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k-n,\infty,t}\big)\big(\|F(t)\|_{H^k_{co}} + |g(t)|_{\tilde{H}^{k-\frac12}}\big)\\ &\quad + \|\!|\nabla\varrho|\!\|_{n,\infty,t}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac12}}\ (n=0,1),\end{aligned}\tag{7.5.23}$$

$$\begin{aligned}&\varepsilon^{\frac12}\|\partial_t\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} + \varepsilon^{\frac12}\|\partial_t\nabla^2\varrho(t)\|_{\mathcal{H}^{j,l-1}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k,\infty,t}\big)\big(\|\varepsilon^{\frac12}\partial_t F(t)\|_{\mathcal{H}^j} + \|\varepsilon^{\frac12}\partial_t \mathrm{div} F(t)\|_{\mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\geq 1\}} + |\varepsilon^{\frac12}\partial_t(F_3^{b,2}, g)(t)|_{\tilde{H}^{k-\frac12}}\big)\\ &+ \Lambda\big(\frac{1}{c_0}, |h|_{k,\infty,t}\big)\|\!|\varepsilon^{\frac12}\partial_t\nabla\varrho|\!\|_{0,\infty,t}|h(t)|_{\tilde{H}^{k+\frac12}}\\ &+ \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \|\!|\varepsilon^{-\frac12}\nabla\varrho|\!\|_{[\frac{k+1}{2}],\infty,t} + |(h,\partial_t h)|_{[\frac{k+3}{2}],\infty,t}\big)\big(|\partial_t h(t)|_{\tilde{H}^{k+\frac12}} + \|\nabla\varrho(t)\|_{H^k_{co}}\big).\end{aligned}\tag{7.5.24}$$

**Corollary 7.5.6.** *Let* $\nabla^{\varphi}\Psi = \mathbb{Q}_t u$ *be the compressible part of the velocity, we have the following two estimates:*

$$\|\nabla\nabla^{\varphi}\Psi\|_{L^2_t H^{m-1}_{co}} + \|\nabla^{\varphi}\Psi\|_{L^2_t H^m_{co}} \lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),\tag{7.5.25}$$

$$\|\varepsilon^{\frac12}\partial_t\nabla^{\varphi}\Psi\|_{L^2_t H^{m-1}_{co}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L^\infty_t\tilde{H}^{m-\frac12}}\big)\|\varepsilon^{\frac12}\partial_t\mathrm{div}^{\varphi}u\|_{L^2_t H^{m-2}_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),\tag{7.5.26}$$

$$\begin{aligned}&\varepsilon^{\frac12}\|\partial_t\nabla^{\varphi}\Psi\|_{L^\infty_t H^{m-2}_{co}} + \varepsilon^{\frac12}\|\partial_t\nabla\nabla^{\varphi}\Psi\|_{L^\infty_t H^{m-3}_{co}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L^\infty_t\tilde{H}^{m-\frac12}}\big)\big(\|\varepsilon^{\frac12}\partial_t\mathrm{div}^{\varphi}u\|_{L^\infty_t H^{m-3}_{co}} + \|\varepsilon^{\frac12}\partial_t u\|_{L^\infty_t\mathcal{H}^{m-2}}\big) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}\tag{7.5.27}$$

*Proof.* We begin with the proof of (7.5.25). Let us detail the estimate of $\|\nabla\nabla^{\varphi}\Psi\|_{L^2_t H^{m-1}_{co}}$, the other term is similar. It suffices to show that:

$$\begin{aligned}\|\nabla\nabla^{\varphi}\Psi\|_{L^2_t H^{m-1}_{co}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\mathrm{div}^{\varphi}u\|_{L^2_t H^{m-1}_{co}}\\ &\quad + \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)(|h|_{L^2_t\tilde{H}^{m-\frac12}} + |\varepsilon^{\frac12}h|_{L^2_t\tilde{H}^{m+\frac12}}).\end{aligned}\tag{7.5.28}$$

which leads to (7.5.25). By definition, $\Psi$ solves the elliptic equation:

$$\begin{cases}\mathrm{div}(E\nabla\Psi) = \mathrm{div}(Pu),\\ \Psi|_{z=0} = 0,\\ \partial_{\mathbf{n}}\Psi|_{z=-1} = 0.\end{cases}\tag{7.5.29}$$

We apply (7.5.13) for $F = Pu$, $\mathrm{div}F = \partial_z\varphi\,\mathrm{div}^{\varphi}u$, $F_3^{b,2} = g = 0$ to get:

$$\begin{aligned}\|\nabla^2\Psi\|_{L^2_t H^{m-1}_{co}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\partial_z\varphi\,\mathrm{div}^{\varphi}u\|_{L^2_t H^{m-1}_{co}}\\ &\quad + \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t} + \|\!|\varepsilon^{-\frac12}\nabla\Psi|\!\|_{[\frac{m}{2}]-1,\infty,t}\big)(|h|_{L^2_t\tilde{H}^{m-\frac12}} + |\varepsilon^{\frac12}h|_{L^2_t\tilde{H}^{m+\frac12}}).\end{aligned}$$

By product estimate (7.3.8), one finds

$$\begin{aligned}&\|\nabla\nabla^{\varphi}\Psi\|_{L^2_t H^{m-1}_{co}} \lesssim \|\nabla^2\Psi\|_{L^2_t H^{m-1}_{co}} + \|\nabla\big(\frac{\mathbf{N}}{\partial_z\varphi}\partial_z\Psi\big)\|_{L^2_t H^{m-1}_{co}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\mathrm{div}^{\varphi}u\|_{L^2_t H^{m-1}_{co}} + \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t} + \|\!|\varepsilon^{-\frac12}(\nabla\Psi, \mathrm{div}^{\varphi}u)|\!\|_{[\frac{m}{2}]-1,\infty,t}\big)\cdot\\ &\qquad (|h|_{L^2_t\tilde{H}^{m-\frac12}} + \varepsilon^{\frac12}|h|_{L^2_t\tilde{H}^{m+\frac12}}).\end{aligned}\tag{7.5.30}$$

Moreover, Sobolev embedding (7.3.16) combined with inequality (7.5.10) gives, for $k\geq 0$,

$$\begin{aligned}\varepsilon^{-\frac12}\|\!|\nabla\Psi|\!\|_{[\frac{m}{2}]-1,\infty,t} &\lesssim \varepsilon^{-\frac12}(\|\nabla^2\Psi\|_{L^\infty_t H^{[\frac{m}{2}]}_{co}} + \|\nabla\Psi\|_{L^\infty_t H^{[\frac{m}{2}]+1}_{co}})\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+2,\infty,t}\big)\|\varepsilon^{-\frac12}\mathrm{div}^{\varphi}u\|_{L^\infty_t H^{[\frac{m}{2}]}_{co}}.\end{aligned}\tag{7.5.31}$$

Plugging the above two inequalities into (7.5.30), we arrive at (7.5.28). Moreover, by applying (7.5.15), (7.5.24), (7.5.31), we get (7.5.26) and (7.5.27). ☐

**Corollary 7.5.7.** *Consider elliptic system with nontrivial Dirichlet upper boundary condition:*

$$\begin{cases} -\mathrm{div}(E\nabla\varrho) = -\mathrm{div}F, \\ \varrho|_{z=0} = b, \\ (E\nabla\varrho)\cdot e_3|_{z=-1} = F_3^{b,2} + g. \end{cases} \tag{7.5.32}$$

*The following estimates hold:*

$$\|\nabla\varrho\|_{\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\big(\|\mathrm{div}F\|_{L_t^\infty H_{co}^1} + |b|_{L_t^\infty H^{\frac{5}{2}}} + |g|_{L_t^\infty H^{\frac{3}{2}}}\big), \tag{7.5.33}$$

$$\begin{aligned} \varepsilon^{-\frac{1}{2}}\|\nabla\varrho(t)\|_{\mathcal{H}^{j,l}} &\lesssim \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{-\frac{1}{2}}\nabla\varrho\|_{[\frac{k}{2}]-1,\infty,t} + |h|_{[\frac{k+3}{2}],\infty,t} + \varepsilon^{-\frac{1}{2}}|b|_{L_t^\infty \tilde{H}^{[\frac{k}{2}]+1+}}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}} \\ &\quad + \varepsilon^{-\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, |h|_{[\frac{k+3}{2}],\infty,t}\big)\big(\|F(t)\|_{\mathcal{H}^{j,l}} + |b(t)|_{\tilde{H}^{k+\frac{1}{2}}} + |g(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big), \end{aligned} \tag{7.5.34}$$

$$\begin{aligned} \|\nabla\varrho(t)\|_{H_{co}^k} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k-j,\infty,t}\big)\big(\|F(t)\|_{H_{co}^k} + |b(t)|_{H^{k+\frac{1}{2}}} + |g(t)|_{L_t^\infty H^{k-\frac{1}{2}}}\big) \\ &\quad + \Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{j,\infty,t} + |h|_{[\frac{k}{2}]+1,\infty,t}\big)|h(t)|_{\tilde{H}^{k+\frac{1}{2}}},\ k\geq 2, j = 0 \text{ or } 1. \end{aligned} \tag{7.5.35}$$

$$\begin{aligned} \varepsilon^{\frac{1}{2}}\|\partial_t\nabla\varrho(t)\|_{H_{co}^k} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{k+1,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t F(t)\|_{H_{co}^k} + |\varepsilon^{\frac{1}{2}}\partial_t b(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_t g(t)|_{\tilde{H}^{k-\frac{1}{2}}}\big) \\ &\quad + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \|\nabla\varrho\|_{1,\infty,t} + |\partial_t h|_{k-1,\infty,t} + |h|_{k,\infty,t}\big)\big(|\partial_t h(t)|_{\tilde{H}^{k+\frac{1}{2}}} + \|\nabla\varrho(t)\|_{H_{co}^k}\big), \end{aligned} \tag{7.5.36}$$

*Proof.* We introduce the lifting:

$$\varrho^H(t,y,z) = \mathcal{F}^{-1}_{\xi\to y}\big(e^{-z^2\langle\xi\rangle^2}\hat{b}(t,\xi)\big)(1+z),$$

and reformulate the problem as:

$$\begin{cases} -\mathrm{div}(E\nabla\varrho^L) = -\mathrm{div}(F - E\nabla\varrho^H) \\ \varrho^L|_{z=0} = 0 \\ \partial_z\varrho^L|_{z=-1} = (F - E\nabla\varrho^H)\cdot e_3 + g. \end{cases}$$

We apply Lemma 7.5.3 with $F - E\nabla\varrho^H$. Note that we use again the product estimate (7.3.8) to bound $E\nabla\varrho^H$. Moreover, Young's inequality and the definition of $\varrho^H$ give:

$$\|\nabla\varrho^H(t)\|_{\mathcal{H}^{j,l}} \lesssim |b(t)|_{\tilde{H}^{j+l+\frac{1}{2}}}, \quad \|\nabla\varrho^H\|_{[\frac{k}{2}]-1,\infty,t} \lesssim |b|_{[\frac{k}{2}],\infty,t} \lesssim |b|_{L_t^\infty\tilde{H}^{[\frac{k}{2}]+1+}}.$$

☐

## 7.6 Regularity of the surface

In this section, we prove some regularity properties for the surface $h$. Here and in the sequel, we will denote $m\geq 7$ an integer and denote $\Lambda$ a polynomial with its arguments and may differ from line to line.

**Lemma 7.6.1.** *Recall that* $\mathcal{N}_{m,T}, \mathcal{E}_{m,T}, \mathcal{A}_{m,T}$ *are defined in* (7.1.30). *The following regularity estimates hold: for any* $0 < t \leq T \leq 1$,

$$|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{3}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} \lesssim \mathcal{E}_{m,T} + \mathcal{E}_{m,T}^2, \tag{7.6.1}$$

$$\varepsilon^{\frac{1}{2}}|\partial_t^2 h|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} + |\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty\tilde{H}^{m-\frac{5}{2}}} + \sum_{k\leq m-1}|\varepsilon^{\frac{1}{2}}(\varepsilon\partial_t)^k\partial_t^2 h|_{L_t^2 H^{-\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.6.2}$$

$$|h|^2_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}} + \varepsilon|h|^2_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}} \lesssim Y_m^2(0) + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.6.3}$$

*where* $\Lambda$ *denotes a polynomial that may change according to the contexts.*

*Proof.* Proof of (7.6.1): We have by using the equation (7.1.17), the product estimate (7.3.5) the trace inequality (7.3.17) and the definition of $\mathcal{E}_{m,T}$ that:

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} &= \varepsilon|(u\cdot\mathbf{N})|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\\
&\lesssim \big(1+|u|_{L_t^\infty \tilde{H}^{[\frac{m-1}{2}]+\frac{1}{2}}}+|h|_{L_t^\infty \tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}}\big)|\varepsilon^{\frac{1}{2}}(u,\nabla_y h)|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\\
&\lesssim (1+\mathcal{E}_{m,T})(\|\varepsilon^{\frac{1}{2}}(u,\nabla u)\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}|h|_{L_t^\infty \tilde{H}^{m+\frac{1}{2}}})\lesssim \mathcal{E}_{m,T}+\mathcal{E}_{m,T}^2.
\end{aligned}$$

Note that we have $[\frac{m-1}{2}]+1\leq m-2, [\frac{m}{2}]+\frac{3}{2}\leq m-\frac{1}{2}$ for $m\geq 5$. The quantity $|\partial_t h|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}}$ can be dealt with in the same manner, we thus omit the proof.

Proof of (7.6.2): Let us detail the estimates of the first two terms, the last one can be controlled by similar calculations. Again, we use the equation (7.1.17) for $h$, the product estimate (7.3.5), the trace inequality (7.3.17) to obtain that

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}|\partial_t h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}} &\lesssim |(\varepsilon^{\frac{1}{2}}\partial_t u\cdot\mathbf{N}, u\cdot\varepsilon^{\frac{1}{2}}\partial_t\mathbf{N})|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}}\\
&\lesssim |\varepsilon^{\frac{1}{2}}\partial_t u|_{L_t^2 \tilde{H}^{[\frac{m-1}{2}]+\frac{1}{2}}}|h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}+(1+|h|_{L_t^\infty \tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}})|\varepsilon^{\frac{1}{2}}\partial_t u|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}}\\
&\quad+|\varepsilon^{\frac{1}{2}}\partial_t h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}|u|_{L_t^2 \tilde{H}^{[\frac{m}{2}]+\frac{3}{2}}}+|\varepsilon^{\frac{1}{2}}\partial_t h|_{L_t^\infty \tilde{H}^{[\frac{m}{2}]+\frac{1}{2}}}|u|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}}\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

For the second term, we use equation (7.1.17) and the trace inequality to get:

$$|\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty \tilde{H}^{m-\frac{5}{2}}}\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\partial_z(u\cdot\mathbf{N})\|_{L_t^\infty H_{co}^{m-3}}+\|\varepsilon^{\frac{1}{2}}\partial_t(u\cdot\mathbf{N})\|_{L_t^\infty H_{co}^{m-2}}.$$

With the aid of identity (7.4.4) and the product estimate (7.3.8), we then find that:

$$\begin{aligned}
&|\varepsilon^{\frac{1}{2}}\partial_t^2 h|_{L_t^\infty \tilde{H}^{m-\frac{5}{2}}}\\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t \mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^{m-3}}+\|(u,\varepsilon^{\frac{1}{2}}\partial_t u)\|_{L_t^\infty H_{co}^{m-2}}+|(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}}\big)\\
&\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

Proof of (7.6.3). We would explain the estimate of $|h|_{L_t^\infty H^{m-\frac{1}{2}}}$, the control of $\varepsilon^{\frac{1}{2}}|h|_{L_t^\infty H^{m+\frac{1}{2}}}$, the second one being similar. Acting $Z^\alpha\Lambda_y^{\frac{1}{2}}(|\alpha|\leq m-1,\alpha_3=0)$ on (7.1.17), one obtains:

$$(\partial_t+u_y\partial_y)(Z^\alpha\Lambda_y^{\frac{1}{2}}h)-Z^\alpha\Lambda_y^{\frac{1}{2}}u_3=f=:[\Lambda_y^{\frac{1}{2}},u_y]Z^\alpha\partial_y h-\Lambda_y^{\frac{1}{2}}\big([Z^\alpha,u_y]\partial_y h\big).$$

Multiplying this equation by $Z^\alpha\Lambda_y^{\frac{1}{2}}h$ and integrating in space and time, we get that:

$$\begin{aligned}
&|Z^\alpha\Lambda_y^{\frac{1}{2}}h(t)|_{L_y^2}^2\lesssim |Z^\alpha h(0)|_{H^{\frac{1}{2}}}^2\\
&\quad+T^{\frac{1}{2}}\Lambda(\|u\|_{1,\infty,t})\big(|u_3|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}^2+|f|_{L_t^2 L_y^2}^2+|h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}^2\big)
\end{aligned}\tag{7.6.4}$$

By trace inequality,

$$|u_3|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}^2\lesssim \|(u,\nabla u)\|_{L_t^2 H_{co}^{m-1}}^2.\tag{7.6.5}$$

To estimate the first term in $f$, we apply commutator estimate (7.3.3) to get that:

$$\begin{aligned}
|[\Lambda_y^{\frac{1}{2}},u_y]Z^\alpha\partial_y h|_{L_t^2 L_y^2}&\lesssim |Z^\alpha\partial_y h|_{L_t^2 H^{-\frac{1}{2}}}|u_y|_{L_t^\infty H^{2.5}}\\
&\lesssim |h|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}\|(u,\nabla u)\|_{L_t^\infty H^2(\mathcal{S})}\lesssim T^{\frac{1}{2}}\mathcal{E}_{m,T}^2.
\end{aligned}\tag{7.6.6}$$

For the second term in $f$, we have by the commutator estimate (7.3.6) and the trace inequality (7.3.17) that:

$$|[Z^\alpha,u_y]\partial_y h|_{L_t^2 H^{\frac{1}{2}}}\lesssim |u|_{L_t^2 \tilde{H}^{[\frac{m}{2}]+\frac{1}{2}}}|h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}+|h|_{L_t^\infty \tilde{H}^{[\frac{m}{2}]+\frac{5}{2}}}|u|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}\lesssim \mathcal{E}_{m,T}^2.\tag{7.6.7}$$

Inserting (7.6.5)-(7.6.7) into (7.6.4), we achieve (7.6.3). $\square$

## 7.7 High order energy estimates

In this section, we prove two kinds of energy estimates, namely the $\varepsilon-$dependent high order conormal energy estimates involving at least one spatial derivative, and the higher order estimates when only the time derivatives are involved.

### 7.7.1 Energy estimate I: Highest order energy estimates.

**Lemma 7.7.1.** *Suppose that (7.2.2) holds for some $T > 0$ then for any $0 < t \leq T$, then we have the following energy estimates:*

$$\varepsilon\|(\sigma, u)\|^2_{L^\infty_t H^m_{co}} + \varepsilon\|\nabla u\|^2_{L^2_t H^m_{co}} \lesssim \varepsilon\|(\sigma, u)(0)\|^2_{H^m_{co}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.7.1}$$

*Proof.* Let us start with (7.7.1) for $m = 0$ which is standard. Performing direct energy estimates for (7.1.16) we get by identities (7.3.30)-(7.3.32) that:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{S}} (g_1|\sigma|^2 + g_2|u|^2)(t)\,\mathrm{d}\mathcal{V}_t + \int_0^t\int_{\mathcal{S}} 2\mu|S^\varphi u|^2 + \lambda|\mathrm{div}^\varphi u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s \\
&= \frac{1}{2}\int_{\mathcal{S}} (g_1|\sigma|^2 + g_2|u|^2)(0)\,\mathrm{d}\mathcal{V}_0 + \frac{1}{2}\int_{\mathcal{S}} (\partial_t^\varphi g_1 + \mathrm{div}^\varphi(g_1 u))|\sigma|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s - a\int_0^t\int_{z=-1} |u_\tau|^2\,\mathrm{d}y\mathrm{d}s
\end{aligned} \tag{7.7.2}$$

where $u_\tau = (u_1, u_2, 0)^t$. Thanks to (7.2.1) and assumption (7.2.2), we have:

$$\begin{aligned}
\|\partial_t^\varphi g_1 + \mathrm{div}^\varphi(g_1 u)\|_{0,\infty,t} &\leq \Lambda\big(\frac{1}{c_0}, \|(\sigma, u)\|_{1,\infty,t} + \|\nabla(\sigma, u)\|_{0,\infty,t} + |h|_{1,\infty,t}\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
\end{aligned}$$

In view of the Korn inequality (7.3.34), the trace inequality (7.3.17), one gets by using Young's inequality that:

$$\begin{aligned}
\|(\sigma, u)\|^2_{L^\infty_t L^2} + \|\nabla u\|^2_{L^2_t L^2} &\lesssim \|(\sigma_0, u_0)\|^2_{L^2(\mathcal{S})} + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\|(\sigma, u)\|^2_{L^2_t L^2} \\
&\lesssim \|(\sigma_0, u_0)\|^2_{L^2(\mathcal{S})} + T\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\|(\sigma, u)\|^2_{L^\infty_t L^2}.
\end{aligned} \tag{7.7.3}$$

We now detail the high order estimates in (7.7.1) (7.7.19). Let $\alpha$ be a multi-index with $1 \leq |\alpha| \leq m$, applying $Z^\alpha$ on the equation (7.1.16), and denoting $(\sigma^\alpha, u^\alpha) = Z^\alpha(\sigma, u)$, one obtains the system:

$$\begin{cases}
g_1(\partial_t^\varphi + u\cdot\nabla^\varphi)\sigma^\alpha + \frac{\mathrm{div}^\varphi u^\alpha}{\varepsilon} = \mathcal{C}^\alpha_\sigma - \frac{1}{\varepsilon}[Z^\alpha, \mathrm{div}^\varphi]u, \\
g_2(\partial_t^\varphi + u\cdot\nabla^\varphi)u^\alpha - \mathrm{div}^\varphi Z^\alpha\mathcal{L}^\varphi u + \frac{\nabla^\varphi\sigma}{\varepsilon} = \mathcal{C}^\alpha_u - \frac{1}{\varepsilon}[Z^\alpha, \nabla^\varphi]\sigma + [Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u.
\end{cases} \tag{7.7.4}$$

where the commutators are given by:

$$\begin{aligned}
\mathcal{C}^\alpha_\sigma &= \big[Z^\alpha, \frac{g_1}{\varepsilon}\big]\varepsilon\partial_t\sigma + [Z^\alpha, g_1 u_y]\nabla_y\sigma + [Z^\alpha, g_1 U_z\partial_z]\sigma, \\
\mathcal{C}^\alpha_u &= \big[Z^\alpha, \frac{g_2}{\varepsilon}\big]\varepsilon\partial_t u + [Z^\alpha, g_2 u_y]\nabla_y u + [Z^\alpha, g_2 U_z\partial_z]u.
\end{aligned} \tag{7.7.5}$$

with

$$U_z = \frac{u\cdot\mathbf{N} - \partial_t\varphi}{\partial_z\varphi}. \tag{7.7.6}$$

Note that we have from (7.1.15) that

$$\partial_t^\varphi + u\cdot\nabla^\varphi = \partial_t + u_y\nabla_y + U_z\partial_z. \tag{7.7.7}$$

The energy equality then reads:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{S}} (g_1|\sigma^\alpha|^2 + g_2|u^\alpha|^2)(t)\,\mathrm{d}\mathcal{V}_t + \int_0^t\int_{\mathcal{S}} 2\mu|Z^\alpha S^\varphi u|^2 + \lambda|Z^\alpha\mathrm{div}^\varphi u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s \\
&= F_0^\alpha + F_1^\alpha + \cdots + F_7^\alpha.
\end{aligned} \tag{7.7.8}$$

where

$$F_0^\alpha = \frac{1}{2}\int_{\mathcal{S}} \big(g_1|\sigma^\alpha|^2 + g_2|u^\alpha|^2\big)\,\mathrm{d}\mathcal{V}_0, \quad F_1^\alpha = \frac{1}{2}\int_0^t\int_{\mathcal{S}} \big(\partial_t^\varphi g_1 + \mathrm{div}^\varphi(g_1 u)\big)|\sigma^\alpha|^2 \,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

$$F_2^\alpha = -\int_0^t\int_{z=0} [Z^\alpha, \mathbf{N}](\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})\cdot u^\alpha \,\mathrm{d}y\mathrm{d}s\ \mathbb{I}_{\{\alpha_3=0\}},$$

$$F_3^\alpha = \int_0^t\int_{\mathcal{S}} Z^\alpha\mathcal{L}^\varphi u\cdot[Z^\alpha,\nabla^\varphi]u \,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \quad F_4^\alpha = -\int_0^t\int_{\mathcal{S}}[Z^\alpha,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\cdot u^\alpha \,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

$$F_5^\alpha = \int_0^t\int_{\mathcal{S}} \mathcal{C}_\sigma^\alpha\sigma^\alpha + \mathcal{C}_u^\alpha\cdot u^\alpha \,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \quad F_6^\alpha = -\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}}\sigma^\alpha[Z^\alpha,\mathrm{div}^\varphi]u + u^\alpha\cdot[Z^\alpha,\nabla^\varphi]\sigma \,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$

$$F_7^\alpha = -a\int_0^t\int_{z=-1}|Z^\alpha u_\tau|^2 \,\mathrm{d}y\mathrm{d}s.$$

The first two terms can be controlled directly by:

$$\varepsilon(|F_0^\alpha| + |F_1^\alpha|) \lesssim \varepsilon\|Z^\alpha(\sigma,u)(0)\|_{L^2(\mathcal{S})}^2 + T\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\varepsilon\|Z^\alpha\sigma\|_{L_t^\infty L^2(\mathcal{S})}^2. \tag{7.7.9}$$

For the boundary term $F_2^\alpha$, which vanishes identically if $\alpha_3 = 0$, we split it as:

$$F_2^\alpha = -\int_0^t\int_{z=0}(\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})Z^\alpha\mathbf{N}\cdot u^\alpha + [Z^\alpha, (\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id}), \mathbf{N}]u^\alpha \,\mathrm{d}y\mathrm{d}s =: F_{21}^\alpha + F_{22}^\alpha$$

By duality and (7.3.4), $F_{21}^\alpha$ can be bounded as:

$$|F_{21}^\alpha| \lesssim |(\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})^{b,1}|_{L_t^\infty W_y^{1,\infty}}|(u^\alpha)^{b,1}|_{L_t^2 H_y^{\frac{1}{2}}}|Z^\alpha\mathbf{N}|_{L_t^2 H_y^{-\frac{1}{2}}}$$

By the identities (7.4.1), (7.4.3), (7.4.4) and the definition (7.1.31), we have:

$$\begin{aligned}|(\mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id})^{b,1}|_{[\frac{m}{2}]-1,\infty,t} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}],\infty,t} + \|\mathrm{div}^\varphi u\|_{[\frac{m}{2}]-1,\infty,t} + \|u\|_{[\frac{m}{2}],\infty,t}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).\end{aligned} \tag{7.7.10}$$

Hence, by trace inequality and Young's inequality, we get that:

$$\varepsilon|F_{21}^\alpha| \le \delta\varepsilon\|\nabla u\|_{L_t^2 H_{co}^m}^2 + \varepsilon\big(|Z^\alpha h|_{L_t^2 H^{\frac{1}{2}}}^2 + \|u^\alpha\|_{L_t^2 L^2}^2\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big).$$

For $F_{22}^\alpha$, we use successively the Cauchy-Schwarz inequality, the estimate (7.7.10) and the trace inequality (7.3.17) to get:

$$\begin{aligned}|F_{22}^\alpha| &\lesssim |(u^\alpha)^{b,1}|_{L_t^2 L_y^2}\big|[Z^\alpha, \mathcal{L}^\varphi u - (\sigma/\varepsilon)\mathrm{Id}, \mathbf{N}]\big|_{L_t^2 L_y^2}\\ &\lesssim |(u^\alpha)^{b,1}|_{L_t^2 L_y^2}\big(|(\mathcal{L}^\varphi u, \sigma/\varepsilon)|_{[\frac{m}{2}]-1,\infty,t}|h|_{L_t^2\tilde{H}^m} + |(\mathcal{L}^\varphi u,\sigma/\varepsilon)|_{L_t^2\tilde{H}^{m-1}}|\mathbf{N}|_{[\frac{m+1}{2}]+1,\infty,t}\big)\\ &\le \delta\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_\delta\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\|u\|_{E^m,t}^2 + \|\nabla\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}\|\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2\tilde{H}^m}^2\big).\end{aligned}$$

To summarize, we can control $\varepsilon F_2^\alpha$ as:

$$\varepsilon|F_2^\alpha| \le 2\delta\varepsilon\|\nabla u\|_{L_t^2 H_{co}^m}^2 + C_\delta\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(T\varepsilon|h|_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}}^2 + \varepsilon^{\frac{1}{2}}(\|u\|_{E^m,t}^2 + \varepsilon\|\nabla\mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-1}}^2)\big). \tag{7.7.11}$$

Let us detail the estimate of $F_3^\alpha$. We use estimate (7.3.23) for $n = 2$ and Young's inequality to get that:

$$\begin{aligned}|\varepsilon F_3^\alpha| &\le \varepsilon\|Z^\alpha\mathcal{L}^\varphi u\|_{L_t^2 L^2}\big(\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + |h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}\big)\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t} + \varepsilon^{\frac{1}{2}}\|\nabla u\|_{2,\infty,t}\big)\\ &\le \delta\varepsilon\|\nabla u\|_{L_t^2 H_{co}^m}^2 + \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\varepsilon\|\nabla u\|_{L_t^2 H_{co}^{m-1}}^2 + T\varepsilon|h|_{L_t^\infty\tilde{H}^{m+\frac{1}{2}}}^2\big).\end{aligned} \tag{7.7.12}$$

Similarly, for $F_4$, by Hölder's inequality, the commutator estimate (7.3.23) and the definition (7.1.31), we find

$$\begin{aligned}|\varepsilon F_4^\alpha| &\le \varepsilon\|u^\alpha\|_{L_t^2L^2}\|[Z^\alpha,\mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2L^2}\\ &\lesssim \varepsilon^{\frac12}\|u^\alpha\|_{L_t^2L^2}\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\varepsilon^{\frac12}\|\nabla\mathcal{L}^\varphi u\|_{2,\infty,t}\big)\big(|h|_{L_t^2\tilde{H}^{m+\frac12}}+\|\varepsilon^{\frac12}\nabla\mathcal{L}^\varphi u\|_{L_t^2H_{co}^{m-1}}\big)\\ &\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\end{aligned}\tag{7.7.13}$$

Next, we control $F_5$ as:

$$\varepsilon|F_5^\alpha|\le T^{\frac12}\|\varepsilon^{\frac12}(\sigma^\alpha,u^\alpha)\|_{L_t^\infty L^2}\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^\alpha,\mathcal{C}_u^\alpha)\|_{L_t^2L^2}.$$

It thus remains to estimate $(\mathcal{C}_\sigma^\alpha,\mathcal{C}_u^\alpha)$ defined in (7.7.5). Taking benefits of the commutator estimate (7.3.9) and the estimate (7.3.13) for $g_1,g_2$, we obtain:

$$\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^\alpha,\mathcal{C}_u^\alpha)\|_{L_t^2L^2}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)(\|(\sigma,u)\|_{E^m,t}+\varepsilon^{\frac12}|h|_{L_t^2H^{m+\frac12}}).$$

Therefore:

$$|\varepsilon F_5^\alpha|\lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\tag{7.7.14}$$

Let us split $F_6^\alpha$ as: $F_6^\alpha=F_{6,1}^\alpha+F_{6,2}^\alpha$ with

$$F_{6,1}^\alpha=-\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}}\sigma^\alpha[Z^\alpha,\mathrm{div}^\varphi]u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\qquad F_{6,2}^\alpha=-\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}}u^\alpha\cdot[Z^\alpha,\nabla^\varphi]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

For $F_{6,1}^\alpha$, thanks to the commutator estimate (7.3.23),

$$\begin{aligned}|\varepsilon F_{6,1}^\alpha| &\lesssim \|\varepsilon^{-\frac12}\sigma^\alpha\|_{L_t^2L^2}\varepsilon^{\frac12}\|[Z^\alpha,\mathrm{div}^\varphi]u\|_{L_t^2L^2}\\ &\lesssim (\|\varepsilon^{\frac12}\partial_t\sigma\|_{L_t^2\mathcal{H}^{m-1}}+\|\varepsilon^{-\frac12}\nabla\sigma\|_{L_t^2H_{co}^{m-1}})(\varepsilon^{\frac12}|h|_{L_t^2\tilde{H}^{m+\frac12}}+\varepsilon^{\frac12}\|\nabla u\|_{L_t^2H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\\ &\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\end{aligned}\tag{7.7.15}$$

Similarly, by using the fact that (recall $m\ge 7$,)

$$\varepsilon^{-\frac12}\|\nabla\sigma\|_{2,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big),$$

we find:

$$\begin{aligned}|\varepsilon F_{6,2}^\alpha| &\lesssim \|u\|_{L_t^2H_{co}^m}\big(\|\nabla\sigma\|_{L_t^2H_{co}^{m-1}}+|h|_{L_t^2\tilde{H}^{m+\frac12}}\|\nabla\sigma\|_{2,\infty,t}\big)\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\\ &\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\end{aligned}\tag{7.7.16}$$

Gathering (7.7.15) and (7.7.16), we find that:

$$|\varepsilon F_6^\alpha|\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\tag{7.7.17}$$

Finally, for the boundary term $F_7^\alpha$, we apply the trace inequality (7.3.17) and Young's inequality to get that:

$$\varepsilon|F_7^\alpha|\lesssim\delta\varepsilon\|\nabla Z^\alpha u_\tau\|_{L_t^2L^2}^2+C_\delta T\varepsilon\|Z^\alpha u_\tau\|_{L_t^\infty L^2(\mathcal{S})}^2.\tag{7.7.18}$$

Collecting (7.7.9), (7.7.11), (7.7.12), (7.7.13), (7.7.14), (7.7.17), (7.7.18) and summing up for $|\alpha|\le m$, we find by Korn inequality (7.3.34) and by choosing $\delta$ small enough,

$$\varepsilon\|(\sigma,u)\|_{L_t^\infty H_{co}^m}^2+\varepsilon\|\nabla u\|_{L_t^2H_{co}^m}^2\lesssim\varepsilon\|(\sigma,u)(0)\|_{H_{co}^m}^2+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.$$

$\square$

**Theorem 7.7.2** (Estimates for High-order temporal derivatives). *Under the same assumption of Lemma 7.7.1, we have the following estimates: for any* $0 < t \leq T$,

$$\varepsilon\|\partial_t(\sigma,u)\|^2_{L^\infty_t\mathcal{H}^{m-1}} + \varepsilon\|\partial_t\nabla u\|^2_{L^2_t\mathcal{H}^{m-1}} \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|^2_{\mathcal{H}^{m-1}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.7.19}$$

*Proof.* Due to the oscillation terms in the equations, we need to deal with the zero order and the higher order estimates for $\varepsilon^{\frac12}\partial_t(\sigma,u)$ differently. We will prove in (7.8.2) the zero order estimate:

$$\varepsilon\|\partial_t(\sigma,u)\|^2_{L^\infty_t L^2} + \varepsilon\|\partial_t\nabla u\|^2_{L^2_t L^2} \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|^2_{L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

We thus focus on the higher order estimates. Substituting $Z^\alpha$ by $\varepsilon^{\frac12}Z_0^k\partial_t$ $(1\leq k\leq m-1)$ in (7.7.8), we find that:

$$\begin{aligned}&\frac{\varepsilon}{2}\int_{\mathcal{S}}(g_1|Z_0^k\partial_t\sigma|^2 + g_2|Z_0^k\partial_t u|^2)(t)\,\mathrm{d}\mathcal{V}_t + \varepsilon\int_0^t\int_{\mathcal{S}} 2\mu|Z_0^k\partial_t S^\varphi u|^2 + \lambda|Z_0^k\partial_t\mathrm{div}^\varphi u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ &= F_0^k + F_1^k + \cdots + F_7^k.\end{aligned} \tag{7.7.20}$$

where $F_0^k - F_7^k$ are defined in the same way as $F_0^\alpha - F_7^\alpha$ (defined in (7.7.8)) by changing $Z^\alpha$ by $\varepsilon^{\frac12}Z_0^k\partial_t$. Our following task is to control $F_0^k - F_7^k$ one by one. The first two terms can be controlled by:

$$|F_0^k + F_1^k| \lesssim \varepsilon\|\partial_t(\sigma,u)\|^2_{L^2_t\mathcal{H}^k} + T\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\varepsilon\|\partial_t\sigma\|^2_{L^\infty_t\mathcal{H}^k}. \tag{7.7.21}$$

Now, for the term $F_2^k = -\varepsilon\int_0^t\int_{z=0}[Z_0^k\partial_t,\mathbf{N}](\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id})Z_0^k\partial_t u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s$, we first use the duality, Cauchy-Schwarz inequality and the estimate (7.7.10) to control it as:

$$|F_2^k| \lesssim |\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t H^{\frac12}}|\varepsilon^{\frac12}Z_0^k\partial_t h|_{L^2_t H^{\frac12}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big) + |\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t L^2_y}\big|\varepsilon^{\frac12}[Z_0^k\partial_t,\mathbf{N},(\mathcal{L}^\varphi u - \frac{\sigma}{\varepsilon}\mathrm{Id})]\big|_{L^2_t L^2_y}$$

By (7.6.1), the trace inequality (7.3.17) and the Young's inequality, the first term in the right hand side of the above inequality is bounded by:

$$\delta\|\varepsilon^{\frac12}Z_0^k\nabla^\varphi\partial_t u\|^2_{L^2_t L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,t}.$$

Moreover, we use the expansion (7.3.28), the estimates (7.4.9), (7.6.1), the trace inequality (7.3.17) successively to control the second one as:

$$\begin{aligned}&C|\varepsilon^{\frac12}Z_0^k\partial_t u|_{L^2_t L^2_y}\big(\varepsilon^{\frac12}|\partial_t h|_{L^2_t\tilde{H}^k}|(\mathcal{L}^\varphi u,\sigma/\varepsilon)^{b,1}|_{[\frac{k-1}{2}],\infty,t} + \varepsilon^{\frac12}|(\mathcal{L}^\varphi u,\sigma/\varepsilon)^{b,1}|_{L^2_t\tilde{H}^k}|\partial_t h|_{[\frac{k}{2}],\infty,t}\big)\\ &\leq \delta\|\varepsilon^{\frac12}Z_0^k\nabla^\varphi\partial_t u\|^2_{L^2_t L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}.\end{aligned}$$

Note that by (7.4.1), (7.4.3), (7.4.4), one has that:

$$\begin{aligned}&|\varepsilon^{\frac12}(\mathcal{L}^\varphi u,\sigma)^{b,1}|_{L^2_t\tilde{H}^{m-1}} \lesssim \big(|\varepsilon^{\frac12}(\partial_y u,\mathrm{div}^\varphi u)^{b,1}|_{L^2_t\tilde{H}^{m-1}} + \varepsilon^{\frac12}|h|_{L^2_t\tilde{H}^m}\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\\ &\lesssim \varepsilon^{\frac14}\big(\|\nabla u\|_{L^2_t H^{m-1}_{co}} + \varepsilon^{\frac12}\|\nabla u\|_{L^2_t H^m_{co}} + \varepsilon^{\frac12}\|\nabla\mathrm{div}u\|_{L^2_t H^{m-1}_{co}}\big)\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big) + T^{\frac12}|\varepsilon^{\frac12}h|_{L^\infty_t\tilde{H}^m}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\\ &\lesssim (\varepsilon^{\frac14} + T^{\frac12})\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}.\end{aligned}$$

We thus find that:

$$|F_2^k| \lesssim 2\delta\|\varepsilon^{\frac12}Z_0^k\nabla^\varphi\partial_t u\|^2_{L^2_t L^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t}. \tag{7.7.22}$$

Next, with the aid of commutator estimate (7.3.25) and estimate (7.6.2), we can control the commutator $[Z_0^k\partial_t,\nabla^\varphi]u$ as:

$$\begin{aligned}\|[Z_0^k\partial_t,\nabla^\varphi]u\|_{L^2_t L^2} \lesssim&\big(|\varepsilon\partial_t^2 h|_{L^2_t\tilde{H}^{m-\frac32}} + \|\varepsilon\partial_t\partial_z u\|_{L^2_t\mathcal{H}^{m-2}\cap L^\infty_t\mathcal{H}^1}\big)\\ &\Lambda\big(\frac{1}{c_0},\|\partial_z u\|_{1,\infty,t} + |(h,\varepsilon^{\frac12}\partial_t h)|_{m-2,\infty,t} + (\int_0^t|\varepsilon^{\frac12}\partial_t^2 h(s)|_{m-2,\infty}\mathrm{d}s)^{\frac12}\big)\\ \lesssim&\,\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, we bound the term

$$F_3^k = \varepsilon \int_0^t \int_{\mathcal{S}} Z_0^k \partial_t \mathcal{L}^\varphi u \cdot [Z_0^k \partial_t, \nabla^\varphi] u \ \mathrm{d}\mathcal{V}_s \mathrm{d}s$$

by using Young's inequality and the assumption $k \leq m-1$,

$$|F_3^k| \leq \delta\varepsilon \|Z_0^k \partial_t \nabla^\varphi u\|_{L_t^2 L^2}^2 + \varepsilon \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.7.23}$$

We proceed to estimate $F_4^k = -\varepsilon \int_0^t \int_{\mathcal{S}} [Z_0^k \partial_t, \mathrm{div}^\varphi] \mathcal{L}^\varphi u \cdot Z_0^k \partial_t u \, \mathrm{d}\mathcal{V}_s \mathrm{d}s$. By expansion (7.7.29), estimate (7.6.2), and assumption $k \leq m-1$,

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}} [Z_0^k \partial_t, \mathrm{div}^\varphi] \mathcal{L}^\varphi u\|_{L_t^2 L^2} &\lesssim (|\varepsilon \partial_t^2 h|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}} + \varepsilon^{\frac{1}{2}} \|\partial_z \mathcal{L}^\varphi u\|_{L_t^2 \mathcal{H}^{m-1}}) \cdot \\
&\quad \Lambda\big(\frac{1}{c_0}, \varepsilon^{\frac{1}{2}} \|\!|\partial_z \mathcal{L}^\varphi u\|\!|_{[\frac{m}{2}]-1, \infty, t} + |\partial_t h|_{[\frac{m-1}{2}], \infty, t} + |h|_{[\frac{m+1}{2}], \infty, t}\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

We thus control $F_4^k$ by Cauchy-Schwarz inequality:

$$\begin{aligned}
|F_4^k| &\leq T^{\frac{1}{2}} \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{L_t^\infty \mathcal{H}^k} \|\varepsilon^{\frac{1}{2}} [Z_0^k \partial_t, \mathrm{div}^\varphi] \mathcal{L}^\varphi u\|_{L_t^2 L^2} \\
&\lesssim T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.7.24}$$

The next term $F_5^k$ is defined by

$$F_5^k = \varepsilon \int_0^t \int_{\mathcal{S}} C_\sigma^k Z_0^k \partial_t \sigma + C_u^k \cdot Z_0^k \partial_t u \, \mathrm{d}\mathcal{V}_s \mathrm{d}s.$$

We will control the commutators $\varepsilon^{\frac{1}{2}} (\mathcal{C}_\sigma^k, \mathcal{C}_u^k)$ (see the definition (7.7.28) ) in Proposition 7.7.3 that:

$$\|\varepsilon^{\frac{1}{2}} (\mathcal{C}_\sigma^k, \mathcal{C}_u^k)\|_{L_t^2 L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Therefore, $F_5^k$ can be estimated as:

$$|F_5^k| \lesssim T^{\frac{1}{2}} \|\varepsilon^{\frac{1}{2}} \partial_t (\sigma, u)\|_{L_t^\infty \mathcal{H}^{m-1}} \|\varepsilon^{\frac{1}{2}} (\mathcal{C}_\sigma^k, \mathcal{C}_u^k)\|_{L_t^2 L^2} \lesssim T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.7.25}$$

For the term $F_6^k = -\int_0^t \int_{\mathcal{S}} Z_0^k \partial_t \sigma \cdot [Z_0^k \partial_t, \mathrm{div}^\varphi] u + Z_0^k \partial_t u \cdot [Z_0^k \partial_t, \nabla^\varphi] \sigma \, \mathrm{d}\mathcal{V}_s \mathrm{d}s$, we can apply commutator estimate (7.3.25) to obtain:

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}} (\|[Z_0^k \partial_t, \mathrm{div}^\varphi] u\|_{L_t^2 L^2} + \|[Z_0^k \partial_t, \nabla^\varphi] \sigma\|_{L_t^2 L^2}) \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big) \big(|\varepsilon^{\frac{1}{2}} \partial_t^2 h|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}} + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla (\sigma, u)\|_{L_t^2 \mathcal{H}^{m-2} \cap L_t^\infty \mathcal{H}^1}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

This estimate, together with Cauchy-Schwarz inequality, yields:

$$|F_6^k| \lesssim T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.7.26}$$

Finally, we control the last term $F_7^k = -a\varepsilon \int_0^t \int_{z=-1} |Z_0^k \partial_t u_\tau|^2 \, \mathrm{d}y \mathrm{d}s$ by the trace inequality (7.3.17) and Young's inequality:

$$F_7^k \leq \delta\varepsilon \int_0^t \int_{\mathcal{S}} |Z_0^k \partial_t \nabla^\varphi u|^2 \mathrm{d}\mathcal{V}_s \mathrm{d}s + (T + \varepsilon) \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) \mathcal{E}_{m,t}^2. \tag{7.7.27}$$

Collecting (7.7.21)-(7.7.27), summing up for $k \leq m-1$ and choosing $\delta$ small enough, we find (7.7.19). $\square$

**Proposition 7.7.3.** *For commutators*

$$\begin{aligned}\mathcal{C}_\sigma^k &= \big[Z_0^k\partial_t, \frac{g_1}{\varepsilon}\big]\varepsilon\partial_t\sigma + [Z_0^k\partial_t, g_1u_y]\nabla_y\sigma + [Z_0^k\partial_t, g_1U_z\partial_z]\sigma,\\ \mathcal{C}_u^k &= \big[Z_0^k\partial_t, \frac{g_1}{\varepsilon}\big]\varepsilon\partial_t u + [Z_0^k\partial_t, g_1u_y]\nabla_y u + [Z_0^k\partial_t, g_1U_z\partial_z]u.\end{aligned} \tag{7.7.28}$$

*we have the estimate: for $k \le m-1$*

$$\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^k, \mathcal{C}_u^k)\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

*Proof.* We use the following two expansions

$$\varepsilon^{\frac12}[Z_0^{m-1}\partial_t, f]g = \sum_{0\le l\le[\frac m2]-1}(C_m^l Z_0^l g Z_0^{m-1-l}\varepsilon^{\frac12}\partial_t f) + \sum_{[\frac m2]\le l\le m-1}(C_m^l Z_0^{l-1}\varepsilon^{\frac12}\partial_t g Z_0^{m-l} f) \tag{7.7.29}$$

$$\begin{aligned}\varepsilon^{\frac12}[Z_0^{m-1}\partial_t, f]g = &\sum_{0\le l\le 1}(C_m^l Z_0^l g Z_0^{m-1-l}\varepsilon^{\frac12}\partial_t f) + C_m^2 Z_0^1\varepsilon^{\frac12}\partial_t g Z_0^{m-2} f\\ &+ \sum_{3\le l\le m-1}(C_m^l Z_0^{l-1}\varepsilon^{\frac12}\partial_t g Z_0^{m-l} f)\end{aligned}$$

In light of the second expansion, we control the third term of $\mathcal{C}_u^{m-1}$ as follows:

$$\begin{aligned}&\varepsilon^{\frac12}\|[Z_0^{m-1}\partial_t, g_1U_z]\partial_z u\|_{L_t^2L^2} \lesssim \|\!|\partial_z u|\!\|_{1,\infty,t}\|\varepsilon^{\frac12}\partial_t(g_1U_z)\|_{L_t^2\mathcal{H}^{m-1}}\\ &\quad + \|\varepsilon^{\frac12}\partial_t\partial_z u\|_{L_t^\infty\mathcal{H}^1}\|Z_0^{m-2}(g_1U_z)\|_{L_t^\infty L^2} + \|\varepsilon^{\frac12}\partial_t\partial_z u\|_{L_t^2\mathcal{H}^{m-1}}\|\!|g_1U_z|\!\|_{m-3,\infty,t}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

The other terms appearing in $\mathcal{C}_\sigma^{m-1}, \mathcal{C}_u^{m-1}$ can be estimated by using the first expansion:

$$\begin{aligned}&\|\varepsilon^{\frac12}(\mathcal{C}_\sigma^{m-1}, \mathcal{C}_u^{m-1} - [Z_0^{m-1}\partial_t, g_1U_z]\partial_z u)\|_{L_t^2L^2}\\ &\lesssim \sum_{j=1}^2\big[\|\varepsilon^{\frac12}\partial_t(g_j/\varepsilon, g_ju_y, g_jU_z)\|_{L_t^2\mathcal{H}^{m-1}}(\|\!|(\sigma,u)|\!\|_{[\frac m2],\infty,t} + \|\!|\nabla\sigma|\!\|_{[\frac m2]-1,\infty,t})\\ &\qquad + \|\!|\varepsilon\partial_t(g_j/\varepsilon, g_ju_y, g_jU_z)|\!\|_{[\frac{m-1}{2}],\infty,t}\|\varepsilon^{\frac12}\partial_t(Z_0,\nabla)(\sigma,u)\|_{L_t^2\mathcal{H}^{m-2}}\big]\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

□

### 7.7.2 Energy estimates II: High-order energy estimate for the compressible part of the system.

In this step, we estimate the compressible part $(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)$ :

**Lemma 7.7.4.** *Under the same assumption of Lemma 7.7.1, the following estimates hold:*

$$\begin{aligned}&\varepsilon(\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|^2_{L_t^\infty H_{co}^{m-1}} + \|\nabla^\varphi\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}})\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)Y_m^2(0) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.7.30}$$

*Proof.* Let $\beta$ be multi-index satisfying $|\beta| \le m-1$. Applying $Z^\beta\nabla^\varphi$ (resp. $Z^\beta$) to the equation for $\sigma$ (resp. $u$), we find that:

$$\begin{cases} g_1(\partial_t^\varphi + u\cdot\nabla^\varphi)Z^\beta\nabla^\varphi\sigma + \frac{1}{\varepsilon}Z^\beta\nabla^\varphi\mathrm{div}^\varphi u = \mathcal{R}_\sigma^\beta\\ g_2(\partial_t^\varphi + u\cdot\nabla^\varphi)Z^\beta u + \mu\,\mathrm{curl}^\varphi Z^\beta\omega - (2\mu+\lambda)\nabla^\varphi Z^\beta\mathrm{div}^\varphi u + \frac{1}{\varepsilon}Z^\beta\nabla^\varphi\sigma = \mathcal{R}_u^\beta.\end{cases} \tag{7.7.31}$$

where

$$\mathcal{R}_{\sigma}^{\beta} = \mathcal{R}_{\sigma,1}^{\beta} + \mathcal{R}_{\sigma,2}^{\beta} + \mathcal{R}_{\sigma,3}^{\beta}, \quad \mathcal{R}_{u}^{\beta} = \mathcal{R}_{u,1}^{\beta} + \cdots \mathcal{R}_{u,3}^{\beta} \tag{7.7.32}$$

with

$$\begin{aligned}
&\mathcal{R}_{\sigma,1}^{\beta} = Z^{\beta}(\nabla^{\varphi} g_1 \partial_t^{\varphi}\sigma + \nabla^{\varphi}(g_1 u)\cdot\nabla^{\varphi}\sigma), && \mathcal{R}_{u,1}^{\beta} = [Z^{\beta}, g_2/\varepsilon]\varepsilon\partial_t u + [Z^{\beta}, g_1 u_y]\nabla_y u,\\
&\mathcal{R}_{\sigma,2}^{\beta} = [Z^{\beta}, g_1/\varepsilon]\varepsilon\partial_t\nabla^{\varphi}\sigma + [Z^{\beta}, g_1 u_y]\nabla_y\nabla^{\varphi}\sigma, && \mathcal{R}_{u,2}^{\beta} = [Z^{\beta}, g_1 U_z\partial_z]u,\\
&\mathcal{R}_{\sigma,3}^{\beta} = [Z^{\beta}, g_1 U_z\partial_z]\nabla^{\varphi}\sigma, && \mathcal{R}_{u,3}^{\beta} = -\mu[Z^{\beta}, \mathrm{curl}^{\varphi}]\omega + (2\mu+\lambda)[Z^{\beta}, \nabla^{\varphi}]\mathrm{div}^{\varphi} u.
\end{aligned}$$

with $U_z$ defined in (7.7.6). Multiplying (7.7.31) by $(Z^{\beta}\nabla^{\varphi}\sigma, -\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u)^t$ and integrating in space and time, one gets the following energy identity:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{S}}\big(g_1|Z^{\beta}\nabla^{\varphi}\sigma|^2 + g_2|Z^{\beta}\mathrm{div}^{\varphi}u|^2\big)(t)\,\mathrm{d}\mathcal{V}_t + (2\mu+\lambda)\|\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u\|_{L_t^2L^2}^2\\
&= J_0^{\beta} + J_1^{\beta} + \cdots J_7^{\beta}.
\end{aligned} \tag{7.7.33}$$

with:

$$\begin{aligned}
J_0^{\beta} &= \frac{1}{2}\int_{\mathcal{S}}\big(g_1|Z^{\beta}\nabla^{\varphi}\sigma|^2 + g_2|Z^{\beta}\mathrm{div}^{\varphi}u|^2\big)(0)\,\mathrm{d}\mathcal{V}_0,\\
J_1^{\beta} &= \frac{1}{2}\int_0^t\int_{\mathcal{S}}(\partial_t^{\varphi}g_1 + \mathrm{div}^{\varphi}(g_1u))|Z^{\beta}\nabla^{\varphi}\sigma|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
J_2^{\beta} &= \int_0^t\int_{\mathcal{S}}\big(\nabla^{\varphi}g_2\cdot\partial_t^{\varphi}Z^{\beta}u + \nabla^{\varphi}(g_2u)\otimes\nabla^{\varphi}Z^{\beta}u\big)Z^{\beta}\mathrm{div}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
J_3^{\beta} &= \int_0^t\int_{\mathcal{S}}g_2(\partial_t^{\varphi} + u\cdot\nabla^{\varphi})([Z^{\beta},\mathrm{div}^{\varphi}]u)Z^{\beta}\mathrm{div}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
J_4^{\beta} &= \int_0^t\int_{z=0}g_2(\partial_t + u_y\partial_y)Z^{\beta}u\cdot\mathbf{N}Z^{\beta}\mathrm{div}^{\varphi}u\,\mathrm{d}y\mathrm{d}s\mathbb{I}_{\{\beta_3=0\}},\\
J_5^{\beta} &= -\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{S}}Z^{\beta}\nabla^{\varphi}\sigma[Z^{\beta},\nabla^{\varphi}]\mathrm{div}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
J_6^{\beta} &= \mu\int_0^t\int_{\mathcal{S}}\mathrm{curl}^{\varphi}\,Z^{\beta}\omega\cdot\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\
J_7^{\beta} &= \int_0^t\int_{\mathcal{S}}\mathcal{R}_{\sigma}^{\beta}\cdot Z^{\beta}\nabla^{\varphi}\sigma + \mathcal{R}_u^{\beta}\cdot\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.
\end{aligned}$$

The first three terms can be controlled directly:

$$\varepsilon J_0^{\beta} \le \varepsilon\|(\nabla^{\varphi}\sigma, \mathrm{div}^{\varphi}u)(0)\|_{H_{co}^{m-1}}^2, \tag{7.7.34}$$

$$\varepsilon(J_1^{\beta} + J_2^{\beta}) \lesssim \varepsilon(\|(\sigma,u)\|_{E^m,t}^2 + |h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}^2). \tag{7.7.35}$$

In order to bound $J_3^{\beta}$, we need to control $(\partial_t^{\varphi} + u\cdot\nabla^{\varphi})[Z^{\beta},\mathrm{div}^{\varphi}]u$. By the identity (7.7.7), we can write

$$\partial_t^{\varphi} + u\cdot\nabla^{\varphi} = \partial_t + u_1\partial_1 + u_2\partial_2 + \frac{U_z}{\phi}Z_3.$$

Since $U_z|_{\partial\mathcal{S}} = \frac{u\cdot\mathbf{N}-\partial_t\varphi}{\partial_z\varphi}\big|_{\partial S} = 0$, we have by the fundamental theorem of calculus and (7.3.20) that:

$$\|U_z/\phi\|_{0,\infty,t} \lesssim \|(U_z,\partial_z U_z)\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \|(u,\nabla u)\|_{0,\infty,t} + |h|_{2,\infty,t}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big). \tag{7.7.36}$$

Therefore, we see that:

$$\|(\partial_t^{\varphi} + \underline{u}\cdot\nabla^{\varphi})[Z^{\beta},\mathrm{div}^{\varphi}]u\|_{L_t^2L^2} \lesssim \frac{1}{\varepsilon}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\|(\varepsilon\partial_t,\varepsilon Z)[Z^{\beta},\mathrm{div}^{\varphi}]u\|_{L_t^2L^2}. \tag{7.7.37}$$

Let us first consider:

$$\varepsilon\partial_t[Z^\beta, \mathrm{div}^\varphi]u = \varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}[Z^\beta, \partial_z]u\big) + \big[Z^\beta, \varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\big]\partial_z u + \big[Z^\beta, \frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon\partial_t\partial_z u$$

In view of Lemma 7.3.9, identity (7.3.27), commutator estimate (7.3.9), the first two terms in the right hand side of the above identity can be bounded as:

$$\big(\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + |(h, \varepsilon\partial_t h)|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big)\Lambda\big(\frac{1}{c_0}, \|\nabla u\|_{1,\infty,t} + |h|_{[\frac{m}{2}]+2,\infty,t}\big).$$

For the third one, we control it as:

$$\begin{aligned}\|\big[Z^\beta, \frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon\partial_t\partial_z u\|_{L_t^2L^2} &\lesssim \|\frac{\mathbf{N}}{\partial_z\varphi}\|_{L_t^2H_{co}^{m-1}}\|\nabla u\|_{1,\infty,t} + \|\varepsilon^{\frac{1}{2}}\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\|_{m-2,\infty,t}\|\varepsilon^{\frac{1}{2}}\partial_t\partial_z u\|_{L_t^2H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}.\end{aligned}$$

Gathering the previous two estimates, we find that:

$$\|\varepsilon\partial_t[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}.$$

Similarly, we have:

$$\|\varepsilon Z[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}.$$

Plugging the above two estimates into (7.7.37), we can then control $J_3^\beta$ as:

$$\begin{aligned}\varepsilon J_3^\beta &\lesssim \varepsilon^{\frac{1}{2}}\|\mathrm{div}^\varphi u/\varepsilon^{\frac{1}{2}}\|_{L_t^2H_{co}^{m-1}}\|\varepsilon(\partial_t^\varphi + \underline{u}\cdot\nabla^\varphi)[Z^\beta, \mathrm{div}^\varphi]u\|_{L_t^2L^2}\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2.\end{aligned} \tag{7.7.38}$$

We now switch to estimate $J_4^\beta$. On one hand, if $Z^\beta = Z_0^k, k \leq m-1$, we have by the trace inequality (7.3.17) that:

$$\begin{aligned}&\varepsilon^{\frac{1}{2}}|(\partial_t + u_y\partial_y)Z_0^k u|_{L_t^2L_y^2}\\ &\lesssim (\|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u)\|_{L_t^2\mathcal{H}^{m-1}} + \|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^2H_{co}^m})\Lambda(\|u\|_{0,\infty,t}) \lesssim \Lambda(\|u\|_{0,\infty,t})\mathcal{E}_{m,t}\end{aligned}$$

Therefore, by the trace inequality (7.3.17), we get that in this case:

$$\begin{aligned}\varepsilon J_4^\beta &\lesssim \varepsilon^{\frac{1}{2}}|Z_0^k\mathrm{div}^\varphi u|_{L_t^2L_y^2}|\varepsilon^{\frac{1}{2}}(\partial_t + u_y\partial_y)Z_0^k u|_{L_t^2L_y^2}|\mathbf{N}|_{0,\infty,t}\\ &\lesssim \varepsilon^{\frac{1}{2}}(\|\mathrm{div}^\varphi u/\varepsilon^{\frac{1}{2}}\|_{L_t^2\mathcal{H}^{m-1}}^2 + \|\varepsilon^{\frac{1}{2}}\nabla\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{m-1}}^2 + \mathcal{E}_{m,t}^2)\Lambda(\|u\|_{0,\infty,t} + |h|_{1,\infty,t})\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda(\|u\|_{0,\infty,t} + |h|_{1,\infty,t})\mathcal{E}_{m,t}^2.\end{aligned} \tag{7.7.39}$$

On the other hand, if $Z^\beta$ contains at least one spatial tangential derivatives $\partial_{y_1}, \partial_{y_2}$, we control $\varepsilon J_3^\beta$ as follows. By equation $(7.1.16)_2$ and identity (7.4.1), we can express $(\mathrm{div}^\varphi u)$ on the boundary $\{z=0\}$ by:

$$\mathrm{div}^\varphi u = \varepsilon g_1(\partial_t + u_y\partial_y)\big(\varepsilon\mathrm{div}^\varphi u + 2\mu\varepsilon(\partial_1 u_1 + \partial_2 u_2) - \mu\varepsilon(\omega\times\mathbf{N})_3\big) \text{ on } \{z=0\}$$

This, together with the product estimate (7.3.14), identity (7.4.2) and the trace inequality (7.3.17) yields that:

$$\begin{aligned}&|Z^\beta\mathrm{div}^\varphi u|_{L_t^2H^{-\frac{1}{2}}}\\ &\lesssim |(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2\tilde{H}^{m-\frac{3}{2}}} \lesssim \varepsilon\big|\big((\mathrm{div}^\varphi u)^{b,1}, \partial_y u^{b,1}, (\omega\times\mathbf{N})_3\big)\big|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\\ &\lesssim \varepsilon^{\frac{1}{2}}\big(\varepsilon^{\frac{1}{2}}(|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}} + \|\nabla u\|_{L_t^2H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) + \varepsilon^{\frac{1}{2}}\|\nabla\mathrm{div}u\|_{L_t^2H_{co}^{m-1}}\\ &\qquad + \varepsilon^{\frac{1}{2}}\|\nabla u\|_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)\big),\end{aligned}$$

which, combined with the Young's inequality, allows us to control $\varepsilon J_4^\beta$ as:

$$
\begin{aligned}
|\varepsilon J_4^\beta| &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big)|\varepsilon^{\frac12}(\varepsilon\partial_t, \varepsilon Z)Z^\beta u|_{L_t^2H^{\frac12}}|\varepsilon^{-\frac12}Z^\beta \mathrm{div}^\varphi u|_{L_t^2H^{-\frac12}} \\
&\leq \delta\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}} + C_\delta\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big) + T^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,t}.
\end{aligned}
\tag{7.7.40}
$$

In view of (7.7.39) and (7.7.40), we find that:

$$
|\varepsilon J_4^\beta| \leq \delta\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-1}} + C_\delta\|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2H_{co}^m}\Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,t}. \tag{7.7.41}
$$

Next, thanks to (7.3.23), (7.9.4), $J_5^\beta$ can be bounded by:

$$
\begin{aligned}
\varepsilon J_5^\beta &\lesssim \varepsilon^{\frac12}\|\nabla\sigma/\varepsilon^{\frac12}\|_{L_t^2H_{co}^{m-1}}\big(\|\nabla\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-2}} + |h|_{L_t^2\tilde{H}^{m-\frac12}}\big)\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t} + \|\partial_z \mathrm{div}^\varphi u\|_{1,\infty,t}\big) \\
&\lesssim \varepsilon^{\frac12}\big(\|\nabla\sigma/\varepsilon^{\frac12}\|^2_{L_t^2H_{co}^{m-1}} + \|\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-2}} + |h|^2_{L_t^2\tilde{H}^{m-\frac12}}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \lesssim \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,t}.
\end{aligned}
\tag{7.7.42}
$$

Note that by the equation $(7.1.16)_1$ $\partial_z\mathrm{div}^\varphi u = \partial_z(g_1\varepsilon\partial_t + \varepsilon u_y\partial_y + \varepsilon U_z\partial_z)\sigma$, we get

$$
\|\partial_z\mathrm{div}^\varphi u\|_{1,\infty,t} \lesssim \Lambda\big(1/c_0, \|(\sigma,\nabla\sigma)\|_{2,\infty,t} + \|(u,\nabla u)\|_{1,\infty,t} + |h|_{3,\infty,t}\big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
$$

For the next term $J_6^\beta$, we assume $\beta_3 = 0$, since otherwise it vanishes identically. It follows from integration by parts that:

$$
\begin{aligned}
J_6^\beta &= \mu\int_0^t\int_{z=0}(Z^\beta\omega\times\mathbf{n})\Pi\nabla^\varphi Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}y\mathrm{d}s + \mu\int_0^t\int_{z=-1}Z^\beta(\omega_2,-\omega_1,0)^t\cdot(\partial_y,0)Z^\beta\mathrm{div}^\varphi u\,\mathrm{d}y\mathrm{d}s \\
&= J_{6,1}^\beta + J_{6,2}^\beta.
\end{aligned}
$$

where $\omega = \mathrm{curl}^\varphi\,\omega = (\omega_1,\omega_2,\omega_3)^t$. In light of the boundary condition (7.1.19), we have by integration by parts along the boundary and the trace inequality (7.3.17) that:

$$
\begin{aligned}
&\varepsilon J_{6,2}^\beta \lesssim \varepsilon|u^{b,2}|_{L_t^2\tilde{H}^m}|Z^\beta(\mathrm{div}^\varphi u)^{b,2}|_{L_t^2L^2} \\
&\lesssim \varepsilon^{\frac12}(\|u\|^{\frac12}_{L_t^2H_{co}^m}\|\varepsilon^{\frac12}\nabla u\|^{\frac12}_{L_t^2H_{co}^m} + \|u\|_{L_t^2H_{co}^m})(\|\mathrm{div}^\varphi u\|^{\frac12}_{L_t^2H_{co}^m}\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^{\frac12}_{L_t^2H_{co}^{m-1}} + \|\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-1}}) \\
&\lesssim \varepsilon^{\frac12}\mathcal{E}^2_{m,t}.
\end{aligned}
\tag{7.7.43}
$$

For $J_{6,1}^\beta$, since $\Pi\nabla^\varphi = \Pi(\partial_1,\partial_2,0)^t$, we also integrate by parts along the boundary to get:

$$
\begin{aligned}
\varepsilon J_{6,1}^\beta \lesssim \varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\Big(&\big|\big(Z^\beta(\omega^{b,1}\times\mathbf{n}), Z^\beta\mathbf{n}\big)\big|_{L_t^2H^{\frac12}}|Z^\beta(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2H^{\frac12}} \\
&+ \big|\big(Z^\beta\omega^{b,1}, [\partial_y Z^\beta, \mathbf{n}, \omega^{b,1}]\big)\big|_{L_t^2L_y^2}|Z^\beta(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2L_y^2}\Big).
\end{aligned}
$$

Thanks to the boundary condition (7.4.2), we have that

$$
|Z^\beta(\omega^{b,1}\times\mathbf{n})|_{L_t^2H^{\frac12}} \lesssim |u^{b,1}|_{L_t^2\tilde{H}^{m+\frac12}}\Lambda\big(\frac{1}{c_0}, |h|_{2,\infty,t}\big) + (|u^{b,1}|_{L_t^2\tilde{H}^{m-\frac12}} + |h|_{L_t^2\tilde{H}^{m+\frac12}})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
$$

Moreover, by (7.4.5) (7.4.8), we have:

$$
\begin{aligned}
|Z^\beta\omega^{b,1}|_{L_t^2L_y^2} &\lesssim \big(|(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2\tilde{H}^{m-1}} + |(u^{b,1},h)|_{L_t^2\tilde{H}^m}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \\
\big|[\partial_y Z^\beta, \mathbf{n}, \omega^{b,1}]\big|_{L_t^2L_y^2} &\lesssim (|\omega^{b,1}|_{L_t^2\tilde{H}^{m-1}} + |h|_{L_t^2\tilde{H}^m})\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \\
&\lesssim \big(|(\mathrm{div}^\varphi u)^{b,1}|_{L_t^2\tilde{H}^{m-1}} + |(u^{b,1},h)|_{L_t^2\tilde{H}^m}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
\end{aligned}
$$

Hence, by the trace inequality and Young's inequality, we end up with that:

$$\begin{aligned}\varepsilon J_{6,1}^{\beta} &\leq \delta\varepsilon\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}^2+\varepsilon\|\nabla u\|_{L_t^2H_{co}^m}^2\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)\\ &\quad+\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)(\varepsilon\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-2}}^2+\varepsilon|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}^2+\varepsilon\|u\|_{E^m,t}).\end{aligned} \tag{7.7.44}$$

Summing up (7.7.43) and (7.7.44), and using (7.9.4), we obtain:

$$\varepsilon J_6^{\beta}\leq 2\delta\varepsilon^2\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}^2+C_{\delta}\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^2H_{co}^m}^2)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2. \tag{7.7.45}$$

Finally, for $J_7^{\beta}$, by Young's inequality,

$$\varepsilon J_7^{\beta}\leq\delta\varepsilon\|\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u\|_{L_t^2L^2}^2+C_{\delta}\varepsilon\|\mathcal{R}_u^{\beta}\|_{L_t^2L^2}^2+\varepsilon^{\frac{1}{2}}(\|\nabla^{\varphi}\sigma/\varepsilon^{\frac{1}{2}}\|_{L_t^2H_{co}^{m-1}}^2+\varepsilon\|\mathcal{R}_{\sigma}^{\beta}\|_{L_t^2L^2}^2). \tag{7.7.46}$$

The remaining task is thus to control $\varepsilon^{\frac{1}{2}}\|(\mathcal{R}_{\sigma}^{\beta},\mathcal{R}_u^{\beta})\|_{L_t^2L^2}$. Let us first see the estimate of $\varepsilon\mathcal{R}_{\sigma}^{\beta}$. In view of the definition (7.7.32), we have by product estimate (7.3.8) and Corollary 7.3.5 that

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,1}^{\beta}\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\|u\|_{E^m,t}+\|(\varepsilon^{\frac{1}{2}}\partial_t\sigma,\varepsilon^{-\frac{1}{2}}\nabla\sigma)\|_{L_t^2H_{co}^{m-1}}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big). \tag{7.7.47}$$

Similarly, by the commutator estimate (7.3.9) and Corollary 7.3.5,

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,2}^{\beta}\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\|u\|_{E^m,t}+\|\varepsilon^{-\frac{1}{2}}\sigma\|_{E^m,t}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big). \tag{7.7.48}$$

For $\mathcal{R}_{\sigma,3}^{\beta}$, we split it as:

$$\mathcal{R}_{\sigma,3}^{\beta}=[Z^{\beta},g_1U_z/\phi]Z_3\nabla^{\varphi}\sigma+(g_1U_z/\phi)[Z^{\beta},\phi]\partial_z\nabla^{\varphi}\sigma+g_1U_z[Z^{\beta},\partial_z]\nabla^{\varphi}\sigma=:(1)+(2)+(3).$$

Thanks to the commutator estimate (7.3.9), we have:

$$\varepsilon^{\frac{1}{2}}\|(1)\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2H_{co}^{m-1}}\|g_1U_z/\phi\|_{[\frac{m+1}{2}],\infty,t}+\varepsilon^{\frac{1}{2}}\|g_1U_z/\phi\|_{L_t^2H_{co}^{m-1}}\|\nabla^{\varphi}\sigma\|_{[\frac{m}{2}]-1,\infty,t}.$$

Note that as $U_z$ vanishes on the boundary, we have by Hardy's inequality,

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|g_1U_z/\phi\|_{L_t^2H_{co}^{m-1}}&\lesssim\varepsilon^{\frac{1}{2}}\|\partial_z(g_1U_z)\|_{L_t^2H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\|g_1U_z\|_{L_t^2H_{co}^{m-1}}\\ &\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)(\|(\sigma,u,\nabla\sigma,\mathrm{div}u)\|_{L_t^2H_{co}^{m-1}}+|\varepsilon^{\frac{1}{2}}h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}+|\varepsilon^{\frac{1}{2}}\partial_th|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}).\end{aligned}$$

Moreover, as for (7.7.36), the fundamental theorem of calculus leads to:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|g_1U_z/\phi\|_{[\frac{m+1}{2}],\infty,t}&\lesssim\varepsilon^{\frac{1}{2}}\|(U_z,\partial_zU_z)\|_{[\frac{m+1}{2}],\infty,t}(1+\|Zg_1\|_{[\frac{m-1}{2}],\infty,t})\\ &\lesssim\Lambda\big(\frac{1}{c_0},\varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{[\frac{m+3}{2}],\infty,t}+\varepsilon^{\frac{1}{2}}\|\mathrm{div}^{\varphi}u)\|_{[\frac{m+1}{2}],\infty,t}\\ &\quad+|\varepsilon^{\frac{1}{2}}h|_{[\frac{m+5}{2}],\infty,t}+|\varepsilon^{\frac{1}{2}}\partial_th|_{[\frac{m+3}{2}],\infty,t}+\|(\sigma,u)\|_{[\frac{m}{2}],\infty,t}+|(h,\partial_th)|_{[\frac{m}{2}]+1}\big).\end{aligned}$$

In view of Equation (7.1.17) and the definition (7.1.31), we conclude:

$$\varepsilon^{\frac{1}{2}}\|g_1U_z/\phi\|_{[\frac{m+1}{2}],\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.7.49}$$

We thus obtain that:

$$\varepsilon^{\frac{1}{2}}\|(1)\|_{L_t^2L^2}\lesssim\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.7.50}$$

It remains to estimate (2), (3). By induction, one has up to some smooth function which depends only on $\phi$ and its derivatives,

$$[Z^{\beta},\phi]=\sum_{\gamma<\beta}*_{\beta,\gamma}Z^{\gamma}\phi,$$

The above identity, combined with (7.3.27), (7.7.49) yields:

$$\varepsilon^{\frac{1}{2}}\|(2)+(3)\|_{L_t^2L^2} \lesssim \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma/\varepsilon^{\frac{1}{2}}\|_{L_t^2H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).$$

To summarize, we have obtained:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma,3}^{\beta}\|_{L_t^2L^2} \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\varepsilon^{\frac{1}{2}}\mathcal{E}_{m,t}. \tag{7.7.51}$$

Collecting (7.7.47)-(7.7.51), we thus arrive at:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{\sigma}^{\beta}\|_{L_t^2L^2} \lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.7.52}$$

To finish the estimates of right hand side of (7.7.46), it remains to control $\mathcal{R}_u^{\beta}$ which is defined in (7.7.32). We first find, in a similar way as the control of $\mathcal{R}_{\sigma}^{\beta}$, that:

$$\varepsilon^{\frac{1}{2}}\|(\mathcal{R}_{u,1}^{\beta}+\mathcal{R}_{u,2}^{\beta})\|_{L_t^2L^2} \lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}. \tag{7.7.53}$$

From the identity:

$$[Z^{\beta},\mathrm{curl}^{\varphi}]\omega=[Z^{\beta},\frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\times\omega, \quad [Z^{\beta},\nabla^{\varphi}]\mathrm{div}^{\varphi}u=[Z^{\beta},\frac{\mathbf{N}}{\partial_z\varphi}\partial_z]\mathrm{div}^{\varphi}u,$$

$\mathcal{R}_{u,3}^{\beta}$ can be treated thanks to (7.3.23) as:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\mathcal{R}_{u,3}^{\beta}\|_{L_t^2L^2} &\lesssim \Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\varepsilon^{\frac{1}{2}}\|\partial_z(\omega,\mathrm{div}^{\varphi}u)\|_{1,\infty,t}\big)\big(\varepsilon^{\frac{1}{2}}\|\partial_z(\omega,\mathrm{div}^{\varphi}u)\|_{L_t^2H_{co}^{m-2}}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\big(\varepsilon^{\frac{1}{2}}\|\nabla^2u\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac{1}{2}}\|u\|_{E^m,t}+|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big).\end{aligned} \tag{7.7.54}$$

Combining (7.7.53) and (7.7.54), one finds that:

$$\varepsilon^{\frac{1}{2}}\|\mathcal{R}_u^{\beta}\|_{L_t^2L^2} \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}. \tag{7.7.55}$$

Plugging (7.7.52) and (7.7.55) into (7.7.46), we finally get that:

$$\varepsilon|J_7^{\beta}| \le \delta\varepsilon^2\|\nabla^{\varphi}Z^{\beta}\mathrm{div}^{\varphi}u\|_{L_t^2L^2}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}^2. \tag{7.7.56}$$

Collecting (7.7.34)-(7.7.42), (7.7.45), (7.7.56), and summing up for $k\le m-1$, we find that by choosing $\delta$ small enough,

$$\begin{aligned}&\varepsilon\|(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)\|_{L_t^{\infty}H_{co}^{m-1}}^2+\varepsilon\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}^2\\ &\lesssim \varepsilon\|(\nabla^{\varphi}\sigma,\mathrm{div}^{\varphi}u)(0)\|_{H_{co}^{m-1}}^2+\Lambda\big(\frac{1}{c_0},|h|_{2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla^{\varphi}u\|_{L_t^2H_{co}^m}^2+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

This inequality, combined with (7.7.1) leads to (7.7.30).

□

## 7.8 Control of lower-order energy norms

This section is devoted to the control of the lower order term $\mathcal{E}_{low,T}$. and

$$\mathcal{E}_{low,T}=\varepsilon^{\frac{1}{2}}\|\partial_t(\sigma,u)\|_{L_t^{\infty}L^2}+\varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{L_t^{\infty}H^3}+\varepsilon^{\frac{3}{2}}\|\nabla^4u\|_{L_t^2L^2}. \tag{7.8.1}$$

Except the first norm, the other norms appearing in $\mathcal{E}_{low,T}$ are indeed not crucial to get an estimate uniformly in $\varepsilon$. Nevertheless, their presence allows us to take benefit of the known local existence results in Sobolev-Slobodeskii spaces [111, 121, 132] (see Theorem (7.13.1) in Section 13).

**Lemma 7.8.1.** *Under the assumption* (7.2.2), *the following estimate holds:*

$$\mathcal{E}_{low,T}^2 \leq \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,T}^2\big)(Y_m^2(0) + \tilde{\mathcal{E}}_{m,T}^2) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.8.2}$$

*Proof.* This lemma is the consequence of the following three lemmas. □

Let us first control the first term in $\mathcal{E}_{low,T}$. Before stating the lemma, we introduce the notation:

$$\Lambda_{2,\infty,t} = \Lambda\big(\frac{1}{c_0}, \|(\sigma,u)\|_{2,\infty,t} + \varepsilon^{-\frac{1}{2}}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{1,\infty,t} + \varepsilon^{\frac{1}{2}}\|\nabla^2 u\|_{0,\infty,t} + |h|_{3,\infty,t}\big), \tag{7.8.3}$$

where $\Lambda$ denotes a polynomial that may differ from line to line. Note that by the equation for $h$ (7.1.17), we have:

$$|\partial_t h|_{2,\infty,t} \lesssim \Lambda_{2,\infty,t}. \tag{7.8.4}$$

**Lemma 7.8.2.** *Supposing that* (7.2.2) *holds true, then for every* $0 < t \leq T$, *for* $r = 0, 1$, *we have the following estimate,*

$$\varepsilon\|\partial_t(\sigma,u)\|_{L_t^\infty L^2}^2 + \varepsilon\|\nabla\partial_t u\|_{L_t^2 L^2}^2 \lesssim \varepsilon\|\partial_t(\sigma,u)(0)\|_{L^2(\mathcal{S})}^2 + (T+\varepsilon)^{\frac{1}{2}}\Lambda_{2,\infty,T}\mathcal{E}_{m,T}^2. \tag{7.8.5}$$

*Proof.* Denote $Z_0 = \varepsilon\partial_t$. Applying $\partial_t^\varphi$ (resp. $\partial_t$) on $(7.1.16)_1$ (resp.$(7.1.16)_2$), one gets that:

$$\begin{cases} g_1(\partial_t^\varphi + \underline{u}\cdot\nabla)(\partial_t\sigma) + \dfrac{1}{\varepsilon}\partial_t^\varphi \mathrm{div}^\varphi u = \mathcal{T}_\sigma \\ g_2(\partial_t + \underline{u}\cdot\nabla)(\partial_t^\varphi u) + \dfrac{1}{\varepsilon}\partial_t\nabla^\varphi\sigma - \mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u) = \mathcal{T}_u \end{cases} \tag{7.8.6}$$

where

$$\mathcal{T}_\sigma = \mathcal{T}_\sigma^1 + \mathcal{T}_\sigma^2 + \mathcal{T}_\sigma^3,\ \mathcal{T}_u = \mathcal{T}_u^1 + \mathcal{T}_u^2 + \mathcal{T}_u^3 + \mathcal{T}_u^4 \tag{7.8.7}$$

with the following definitions:

$$\mathcal{T}_\sigma^1 = \big(\frac{\partial_t^\varphi g_1}{\varepsilon}\big)(\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)\sigma, \quad \mathcal{T}_\sigma^2 = g_1[\partial_t, \underline{u}\cdot\nabla]\sigma, \quad \mathcal{T}_\sigma^3 = -\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z(\underline{u}\cdot\nabla\sigma),$$

$$\begin{aligned} \mathcal{T}_u^1 &= \big(\frac{\partial_t g_2}{\varepsilon}\big)(\partial_t + \underline{u}\cdot\nabla)u, \quad \mathcal{T}_u^2 = g_2\partial_t\underline{u}\cdot\nabla u, \\ \mathcal{T}_u^3 &= [\partial_t, \mathrm{div}^\varphi]\mathcal{L}^\varphi u, \quad \mathcal{T}_u^4 = -g_2(\partial_t + \underline{u}\cdot\nabla)\big(\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z u\big). \end{aligned}$$

where $\underline{u} = (u_1, u_2, U_z)$ and $U_z$ is defined in (7.7.6). Taking the scalar product of (7.8.6) and $\varepsilon\big(\partial_t\sigma, \partial_t^\varphi u\big)^t$, integrating in space and time, we get by using Lemma 7.3.12 that

$$\begin{aligned} &\frac{\varepsilon}{2}\int_{\mathcal{S}} g_1|\partial_t\sigma|^2(t) + g_2|\partial_t^\varphi u|^2(t)\,\mathrm{d}\mathcal{V}_t - \varepsilon\int_0^t\int_{\mathcal{S}} \mathrm{div}^\varphi(\partial_t\mathcal{L}^\varphi u)\partial_t^\varphi u(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s \\ &= I_0 + I_1 + \cdots I_4 \end{aligned} \tag{7.8.8}$$

where

$$\begin{aligned} I_0 &= \frac{\varepsilon}{2}\int_{\mathcal{S}} g_1|\partial_t\sigma|^2(0) + g_2|\partial_t^\varphi u|^2(0)\,\mathrm{d}\mathcal{V}_0, \quad I_1 = \frac{\varepsilon}{2}\int_0^t\int_{z=0} g_1\partial_t h|\partial_t\sigma|^2\,\mathrm{d}y\mathrm{d}s, \\ I_2 &= \frac{\varepsilon}{2}\int_0^t\int_{\mathcal{S}} \big(\partial_t^\varphi g_1 + \frac{1}{\partial_z\varphi}\mathrm{div}(g_1\underline{u}\partial_z\varphi)\big)|\partial_t\sigma|^2(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \\ I_3 &= \int_0^t\int_{\mathcal{S}} \partial_t\sigma\partial_t^\varphi\mathrm{div}^\varphi u + \partial_t\nabla^\varphi\sigma\cdot\partial_t^\varphi u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \\ I_4 &= \varepsilon\int_0^t\int_{\mathcal{S}} \partial_t\sigma\mathcal{T}_\sigma + \partial_t^\varphi u\cdot\mathcal{T}_u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s. \end{aligned}$$

We focus on the control of $I_1 - I_4$ in the following. The estimates of $I_1, I_2$ are direct:

$$|I_1| \lesssim \varepsilon|\partial_t h|_{\infty,t}|\partial_t\sigma|_{z=0}|_{L_t^2L_y^2}^2.$$

$$|I_2| \lesssim \Lambda(\|\nabla(\sigma,u)\|_{\infty,t} + \|(\sigma,u)\|_{1,\infty,t} + |h|_{2,\infty,t})\|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2L^2}^2. \tag{7.8.9}$$

We remark that in view of the boundary condition (7.4.1),

$$\partial_t\sigma|_{z=0} = \partial_t(\sigma|_{z=0}) = \varepsilon\partial_t\big((2\mu+\lambda)\mathrm{div}^{\varphi}u - 2\mu(\partial_1u_1+\partial_2u_2) + \mu(\omega\times\mathbf{N})_3|_{z=0}\big).$$

Therefore, by the trace inequality (7.3.17), we have:

$$\begin{aligned}|I_1| &\lesssim \varepsilon|\partial_t h|_{\infty,t}|\partial_t\sigma|_{L_t^2L_y^2}^2 \lesssim \varepsilon\big(\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^1} + \|(u,\nabla u)\|_{L_t^2H_{co}^2} + |h|_{L_t^2\tilde{H}^2}\big)\Lambda_{2,\infty,t}\\ &\lesssim \varepsilon\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\end{aligned} \tag{7.8.10}$$

Note that $\Lambda_{2,\infty,t}$ is defined in (7.8.3).

To deal with $I_3$, we integrate by parts in space to get:

$$I_3 = \int_0^t\int_{\mathcal{S}}\partial_t^{\varphi}u\cdot[\partial_t,\nabla^{\varphi}]\sigma\,\mathrm{d}\mathcal{V}_s\mathrm{d}s + \int_0^t\int_{z=0}\partial_t\sigma\partial_t^{\varphi}u\cdot\mathbf{N}\,\mathrm{d}y\mathrm{d}s =: I_{31}+B_1.$$

Since $[\partial_t,\nabla^{\varphi}]\sigma = [\partial_t,\frac{\mathbf{N}}{\partial_z\varphi}]\partial_z\sigma$, it follows from the Cauchy-Schwarz inequality that:

$$\begin{aligned}|I_{31}| &\lesssim \|\partial_t^{\varphi}u\|_{L_t^2L^2}\big\|\partial_z\sigma\big\|_{L_t^2L^2}\|\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\|_{0,\infty,t}\\ &\lesssim T^{\frac{1}{2}}\Lambda_{2,\infty,t}\varepsilon^{\frac{1}{2}}\|(\partial_t u,\nabla u)\|_{L_t^\infty L^2}\big\|\varepsilon^{-\frac{1}{2}}\nabla\sigma\big\|_{L_t^2L^2}.\end{aligned} \tag{7.8.11}$$

The boundary term $B_1$ combined with the boundary term arising from the integration by parts of the viscous term, lead to some cancellations, we thus first rewrite the viscous term:

$$\begin{aligned}-\varepsilon\int_0^t\int_{\mathcal{S}}\mathrm{div}^{\varphi}(\partial_t\mathcal{L}^{\varphi}u)\cdot\partial_t^{\varphi}u(s)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s &= \varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^{\varphi}u\cdot\nabla^{\varphi}\partial_t^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ +\varepsilon a\int_0^t\int_{z=-1}|\partial_t u_\tau|^2\,\mathrm{d}y\mathrm{d}s &- \varepsilon\underbrace{\int_0^t\int_{z=0}\partial_t\mathcal{L}^{\varphi}u\mathbf{N}\cdot\partial_t^{\varphi}u\,\mathrm{d}y\mathrm{d}s}_{=:\,B_2}.\end{aligned} \tag{7.8.12}$$

In view of the boundary condition (7.1.18), identities (7.4.9), (7.4.1) as well as the trace inequality (7.3.17), we have:

$$\begin{aligned}B_1+B_2 &= -\varepsilon\int_0^t\int_{z=0}\partial_t^{\varphi}u\cdot\partial_t\mathbf{N}\big(\mathcal{L}^{\varphi}u - \frac{\sigma}{\varepsilon}\mathrm{Id}\big)\,\mathrm{d}y\mathrm{d}s\\ &\lesssim \varepsilon\big|\partial_t\mathbf{N}\big(\mathcal{L}^{\varphi}u-\frac{\sigma}{\varepsilon}\mathrm{Id}\big)\big|_{L_t^2L_y^2}|\partial_t^{\varphi}u|_{L_t^2L_y^2}\\ &\lesssim \varepsilon^{\frac{1}{2}}\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^2H^1}^2 + \|\varepsilon^{\frac{1}{2}}u\|_{L_t^2H^2}^2 + \|\nabla u\|_{L_t^2H_{co}^1}^2 + \|\nabla\mathrm{div}u\|_{L_t^2L^2}^2\big)\Lambda\big(\frac{1}{c_0},|\partial_t h|_{0,\infty,t}+|h|_{1,\infty,t}\big)\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}^2.\end{aligned}$$

Before proceeding to estimate $I_4$, we would estimate further the first two term in the right hand side of (7.8.12). By Young's inequality:

$$\begin{aligned}&\varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^{\varphi}u\cdot\nabla^{\varphi}\partial_t^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ =&\varepsilon\int_0^t\int_{\mathcal{S}}\partial_t\mathcal{L}^{\varphi}u\cdot\big(\partial_t\nabla^{\varphi}u - \frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\nabla^{\varphi}u)\mathrm{d}\mathcal{V}_s\mathrm{d}s\\ \geq&\varepsilon\int_0^t\int_{\mathcal{S}}2\mu|\partial_t S^{\varphi}u|^2 + \lambda|\partial_t\mathrm{div}^{\varphi}u|^2\,\mathrm{d}\mathcal{V}_s\mathrm{d}s - \Lambda_{2,\infty,t}\|\varepsilon^{\frac{1}{2}}\partial_t\mathcal{L}^{\varphi}u\|_{L_t^2L^2}\|\varepsilon^{\frac{1}{2}}\partial_z\nabla^{\varphi}u\|_{L_t^2L^2}\\ \geq&\varepsilon\int_0^t\int_{\mathcal{S}}\mu|\partial_t S^{\varphi}u|^2 + \frac{\lambda}{2}|\partial_t\mathrm{div}^{\varphi}u|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s - C_{\mu,\lambda}T\Lambda_{2,\infty,t}\|\varepsilon^{\frac{1}{2}}\nabla^{\varphi}u\|_{L_t^\infty H^1}^2\end{aligned} \tag{7.8.13}$$

Moreover, by trace inequality,

$$\varepsilon \int_0^t \int_{z=-1} |\partial_t u_\tau|^2 \,\mathrm{d}y\mathrm{d}s \le \delta\varepsilon \|\partial_t \nabla^\varphi u\|_{L_t^2 L^2}^2 + TC_\delta(\varepsilon \|(\partial_t u_\tau, \nabla u_\tau)\|_{L_t^\infty L^2}^2)\Lambda_{2,\infty,t}. \tag{7.8.14}$$

Therefore, we get by collecting (7.8.11)-(7.8.14) that:

$$\begin{aligned} &I_3 + \varepsilon \int_0^t \int_{\mathcal{S}} \mathrm{div}^\varphi(\partial_t \mathcal{L}^\varphi u) \cdot \partial_t^\varphi u(s) \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \\ &\le -\varepsilon \int_0^t \int_{\mathcal{S}} \mu |\partial_t S^\varphi u|^2 + \frac{\lambda}{2} |\partial_t \mathrm{div}^\varphi u|^2 \mathrm{d}\mathcal{V}_s \mathrm{d}s + \delta\varepsilon \|\partial_t \nabla^\varphi u\|_{L_t^2 L^2}^2 + (T+\varepsilon)^{\frac{1}{2}} \Lambda_{2,\infty,t} \mathcal{E}_{m,t}^2. \end{aligned} \tag{7.8.15}$$

We now go back to the estimate of $I_4$ which can be bounded directly by

$$|I_4| \lesssim T^{\frac{1}{2}} \big( \|\varepsilon^{\frac{1}{2}} \mathcal{T}_\sigma\|_{L_t^2 L^2} \|\varepsilon^{\frac{1}{2}} \partial_t \sigma\|_{L_t^\infty L^2} + \|\varepsilon^{\frac{1}{2}} \mathcal{T}_u\|_{L_t^2 L^2} \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{L_t^\infty L^2} \big). \tag{7.8.16}$$

It thus remains to control the commutators $\mathcal{T}_\sigma, \mathcal{T}_u$ defined in (7.8.7). By the explicit expression of $\mathcal{T}_\sigma, \mathcal{T}_u$, we can obtain that:

$$\varepsilon^{\frac{1}{2}} \|(\mathcal{T}_\sigma, \mathcal{T}_u)\|_{L_t^2 L^2} \lesssim \Lambda_{2,\infty,t} (\|\varepsilon^{\frac{1}{2}} \partial_t (\sigma, u)\|_{L_t^2 L^2} + \|\nabla(\sigma, u)\|_{L_t^2 H_{co}^1}) \lesssim \Lambda_{2,\infty,t} \mathcal{E}_{m,t} \tag{7.8.17}$$

For instance, we control $\varepsilon^{\frac{1}{2}} \mathcal{T}_\sigma^1 = \varepsilon^{\frac{1}{2}} \partial_t^\varphi (g_1/\varepsilon)(\varepsilon\partial_t + \varepsilon \underline{u} \cdot \nabla)\sigma$, $\varepsilon^{\frac{1}{2}} \mathcal{T}_u^1 = \varepsilon^{\frac{1}{2}} \partial_t (g_2/\varepsilon)(\varepsilon\partial_t + \varepsilon \underline{u} \cdot \nabla) u$ by:

$$\|\varepsilon^{\frac{1}{2}} (\mathcal{T}_\sigma^1 + \mathcal{T}_u^1)\| \lesssim \Lambda_{1,\infty,t} (\|\varepsilon^{\frac{1}{2}} \partial_t (\sigma, u)\|_{L_t^2 L^2} + \|\nabla(\sigma, u)\|_{L_t^2 L^2}) \lesssim \Lambda_{2,\infty,t} \mathcal{E}_{m,t}.$$

Collecting (7.8.16)-(7.8.17), we find that

$$|I_4| \le T^{\frac{1}{2}} \Lambda_{2,\infty,t} \mathcal{E}_{m,t}^2. \tag{7.8.18}$$

Now, in view of estimates: (7.8.9)-(7.8.10), (7.8.15) (7.8.18), by choosing $\delta$ small enough,

$$\begin{aligned} &\frac{1}{2} \varepsilon \int_{\mathcal{S}} g_1 |\partial_t \sigma|^2(t) + g_2 |\partial_t^\varphi u|^2(t) \,\mathrm{d}\mathcal{V}_t + \varepsilon \int_0^t \int_{\mathcal{S}} \mu |\partial_t S^\varphi u|^2 + \frac{\lambda}{2} |\partial_t \mathrm{div}^\varphi u|^2 \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \\ &\le \frac{1}{2} \varepsilon \int g_1 |\partial_t \sigma|^2(0) + g_2 |\partial_t^\varphi u|^2(0) \,\mathrm{d}\mathcal{V}_0 + \delta\varepsilon \|\nabla^\varphi \partial_t u\|_{L_t^2 L^2}^2 + (T+\varepsilon)^{\frac{1}{2}} \Lambda_{2,\infty,t} \mathcal{E}_{m,t}^2. \end{aligned} \tag{7.8.19}$$

We can write by the direct computation that:

$$\begin{aligned} &\int_0^t \int_{\mathcal{S}} \mu |\partial_t S^\varphi u|^2 + \frac{\lambda}{2} |\partial_t \mathrm{div}^\varphi u|^2 \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \\ &\ge \int_0^t \int_{\mathcal{S}} \mu |S^\varphi \partial_t u|^2 + \frac{\lambda}{2} |\mathrm{div}^\varphi \partial_t u|^2 \,\mathrm{d}\mathcal{V}_s \mathrm{d}s - \Lambda_{2,\infty,t} T^{\frac{1}{2}} \|\nabla u\|_{L_t^\infty L^2}^2. \end{aligned}$$

Hence, by using Korn's inequality (7.3.34) and by choosing $\delta$ small enough, we finally obtain (7.8.5). $\square$

The following two lemmas are devoted to the estimates of the other norms appearing in $\mathcal{E}_{low,T}$.

**Lemma 7.8.3.** *Suppose that* (7.2.2) *are holds, then we have for any* $0 < t \le T$,

$$\varepsilon \|\nabla^3 \sigma\|_{L_t^\infty L^2}^2 + \varepsilon^{-1} \|\nabla^3 \sigma\|_{L_t^2 L^2}^2 + \varepsilon \|\nabla^2 \sigma\|_{L_t^\infty H_{co}^1}^2 + \varepsilon^{-1} \|\nabla^2 \sigma\|_{L_t^2 H_{co}^1}^2 \lesssim Y_m^2(0) + (T+\varepsilon) \Lambda_{2,\infty,t} \mathcal{E}_{m,t}^2. \tag{7.8.20}$$

*Proof.* Taking $\varepsilon^2 \nabla^\varphi$ on the equation $(7.1.16)_1$ and expressing the term $\varepsilon \nabla^\varphi \mathrm{div}^\varphi u$ by the velocity equations $(7.1.16)_2$, we find that $\nabla^\varphi \sigma$ solves

$$\varepsilon^2 g_1 (\partial_t + \underline{u} \cdot \nabla) \nabla^\varphi \sigma + \frac{1}{2\mu + \lambda} \nabla^\varphi \sigma = \mathcal{Q}_1 \tag{7.8.21}$$

where

$$\mathcal{Q}_1 = -\varepsilon^2 g_1' \nabla^\varphi \sigma (\varepsilon \partial_t + \varepsilon \underline{u} \cdot \nabla) \sigma - \varepsilon^2 g_1 \nabla^\varphi u \cdot \nabla^\varphi \sigma - \frac{\mu\varepsilon}{2\mu + \lambda} \mathrm{curl}^\varphi \omega - \frac{1}{2\mu + \lambda} g_2 (\varepsilon \partial_t + \varepsilon \underline{u} \cdot \nabla) u.$$

By taking $\mathrm{div}^\varphi$ of the equation (7.8.21), we find that $\Delta^\varphi\sigma$ solves:

$$\begin{aligned}\varepsilon^2 g_1(\partial_t+\underline{u}\cdot\nabla)\Delta^\varphi\sigma+\frac{1}{2\mu+\lambda}\Delta^\varphi\sigma&=\mathrm{div}^\varphi\mathcal{Q}_1-\varepsilon^2 g_1'\nabla^\varphi\sigma\cdot\varepsilon\partial_t\nabla^\varphi\sigma-\varepsilon^2\nabla^\varphi(g_1\underline{u})\cdot\nabla\nabla^\varphi\sigma\\&=:\mathcal{H}\end{aligned}\tag{7.8.22}$$

Performing direct energy estimates for (7.8.22), we get:

$$\begin{aligned}&\varepsilon\|\Delta^\varphi\sigma\|^2_{L_t^\infty H_{co}^1}+\|\varepsilon^{-\frac12}\Delta^\varphi\sigma\|^2_{L_t^2H_{co}^1}\\&\lesssim\varepsilon\|\Delta^\varphi\sigma(0)\|^2_{H_{co}^1}+T\Lambda_{1,\infty,t}\varepsilon\|\Delta^\varphi\sigma\|^2_{L_t^\infty H_{co}^1}+T^{\frac12}\|\varepsilon^{-\frac12}\Delta^\varphi\sigma\|_{L_t^2H_{co}^1}(\|\varepsilon^{-\frac12}\mathcal{H}\|_{L_t^\infty H_{co}^1}+\varepsilon^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t})\\&\lesssim T\Lambda_{1,\infty,t}\mathcal{E}^2_{low,t}+T^{\frac12}(\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}+\varepsilon^{\frac12}\Lambda_{2,\infty,t}\mathcal{E}_{m,t})\|\varepsilon^{-\frac12}\Delta^\varphi\sigma\|_{L_t^2H_{co}^1}\end{aligned}$$

It thus follows from Young's inequality that

$$\|\varepsilon\Delta^\varphi\sigma\|^2_{L_t^\infty H_{co}^1}+\|\Delta^\varphi\sigma\|^2_{L_t^2H_{co}^1}\lesssim Y_m^2(0)+T\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.$$

Moreover, we can get also that:

$$\begin{aligned}&\varepsilon\|\partial_z\Delta^\varphi\sigma\|^2_{L_t^\infty L^2}+\varepsilon^{-1}\|\partial_z\Delta^\varphi\sigma\|^2_{L_t^2L^2}\\&\lesssim\varepsilon\|\partial_z\Delta^\varphi\sigma(0)\|^2_{L^2}+T\Lambda_{2,\infty,t}\big(\|\varepsilon\nabla^3\sigma\|^2_{L_t^\infty L^2}+\|\varepsilon^{\frac12}\partial_t\nabla\mathrm{div}u\|^2_{L_t^\infty L^2}+\varepsilon\mathcal{E}^2_{m,t}\big)\\&\lesssim Y_m^2(0)+T\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.\end{aligned}$$

By the expressions of $\Delta^\varphi\sigma$,

$$\Delta^\varphi\sigma=\frac{|\mathbf{N}|^2}{\partial_z\varphi}\partial_z^2\sigma+\Delta_y\sigma+\partial_1(\mathbf{N}_1\partial_z^\varphi\sigma)+\partial_2(\mathbf{N}_2\partial_z^\varphi\sigma)+\mathbf{N}_1\partial_z^\varphi\partial_1\sigma+\mathbf{N}_2\partial_z^\varphi\partial_2\sigma+\frac12\partial_z\sigma\partial_z\Big|\frac{\mathbf{N}}{\partial_z\varphi}\Big|^2,\tag{7.8.23}$$

we see that:

$$\begin{aligned}\varepsilon\|\nabla^2\sigma\|^2_{L_t^\infty H_{co}^1}&\lesssim\varepsilon\|\nabla\sigma\|^2_{L_t^\infty H_{co}^2}+\varepsilon\|\partial_z^2\sigma\|^2_{L_t^\infty L^2}\\&\lesssim\varepsilon\Lambda(1/c_0,|h|_{3,\infty,t})\|\nabla\sigma\|^2_{L_t^\infty H_{co}^2}+\varepsilon\|\Delta^\varphi\sigma\|^2_{L_t^\infty H_{co}^1}\\&\lesssim Y_m^2(0)+(T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.\end{aligned}$$

Note that $|h|_{3,\infty,t}$ is included in the definition of $\Lambda_{2,\infty,t}$ (7.8.3). We have further that:

$$\varepsilon\|\nabla^3\sigma\|^2_{L_t^\infty L^2}\lesssim\varepsilon\|\partial_z\Delta^\varphi\sigma\|^2_{L_t^\infty L^2}+\Lambda_{2,\infty,t}\varepsilon\|\nabla^2\sigma\|^2_{L_t^\infty H_{co}^1}\lesssim Y_m^2(0)+(T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.$$

In a similar way, the following estimate holds also:

$$\varepsilon^{-1}\|\nabla^3\sigma\|^2_{L_t^2L^2}+\varepsilon^{-1}\|\nabla^2\sigma\|^2_{L_t^2H_{co}^1}\lesssim Y_m^2(0)+T\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.$$

The proof of (7.8.20) is now finished. □

**Lemma 7.8.4.** *Assume that* (7.2.2) *holds, then we have for any* $0<t\le T$ :

$$\begin{aligned}&\varepsilon^{-1}\|\nabla^2\sigma\|^2_{L_t^\infty L^2}+\varepsilon\|\nabla^3u\|^2_{L_t^\infty L^2}+\varepsilon^3\|\nabla^4u\|^2_{L_t^2L^2}\\&\lesssim\Lambda\big(\frac{1}{c_0},|h|^2_{3,\infty,t}\big)(\varepsilon\|\nabla^2\sigma\|^2_{L_t^\infty H_{co}^1}+\tilde{\mathcal{E}}_{m,t})+(T+\varepsilon)\Lambda_{2,\infty,t}\mathcal{E}^2_{m,t}.\end{aligned}\tag{7.8.24}$$

*Proof.* By taking $\mathrm{div}^\varphi$ on $(7.1.16)_2$, we see that $\sigma$ solves the following elliptic problem:

$$\begin{cases}-\Delta^\varphi(\sigma/\varepsilon)=\mathrm{div}^\varphi G,\\ \sigma/\varepsilon=(2\mu+\lambda)\mathrm{div}^\varphi u-2\mu(\partial_1u_1+\partial_2u_2)-\mu(\omega\times\mathbf{N})_3\quad\text{on}\quad\{z=0\},\\ \partial_z^\varphi\sigma/\varepsilon=-G\cdot e_3+\mu\,\mathrm{curl}^\varphi\,\omega\cdot e_3\quad\text{on}\quad\{z=-1\},\end{cases}\tag{7.8.25}$$

where

$$G=\bar\rho\partial_t^\varphi u+g_2u\cdot\nabla^\varphi u+\frac{g_2-\bar\rho}{\varepsilon}\varepsilon\partial_t^\varphi u-(2\mu+\lambda)\nabla^\varphi\mathrm{div}^\varphi u.\tag{7.8.26}$$

Note that on the upper boundary we have boundary identity (7.4.2) for $\omega \times \mathbf{N}$ and on the bottom, we have

$$\mu \operatorname{curl}^{\varphi} \omega \times e_3 = \mu(\partial_1^{\varphi}\omega_2 - \partial_2^{\varphi}\omega_1) = a(\partial_1 u_1 + \partial_2 u_2). \tag{7.8.27}$$

Applying the elliptic estimate (7.5.10), we find that:

$$\begin{aligned}
\varepsilon^{-\frac{1}{2}}\|\nabla^2\sigma\|_{L_t^\infty L^2} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\big(\varepsilon^{\frac{1}{2}}\|(\operatorname{div}^{\varphi} G, G)\|_{L^\infty L^2} + |\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L_t^\infty H^{\frac{3}{2}}} + |\varepsilon^{-\frac{1}{2}}(\partial_{\mathbf{n}}\sigma)^{b,2}|_{L_t^\infty H^{\frac{1}{2}}}\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)(\varepsilon^{\frac{1}{2}}\|\operatorname{div}^{\varphi} u\|_{L_t^\infty H^2} + \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^1} + \|\varepsilon^{\frac{1}{2}}\partial_t \operatorname{div}^{\varphi} u\|_{L_t^\infty L^2}) + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)(\varepsilon^{\frac{1}{2}}\|\nabla^2\sigma\|_{L_t^\infty H_{co}^1} + \tilde{\mathcal{E}}_{m,t}) + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t},
\end{aligned}$$

where $G$ is defined in (7.8.26). Note that by $(7.1.16)_1$ and the definition of $\mathcal{E}_{m,t}$,

$$\varepsilon^{\frac{1}{2}}\|\operatorname{div}^{\varphi} u\|_{L_t^\infty H^2} \lesssim \varepsilon^{\frac{1}{2}}\|\nabla^2\sigma\|_{L_t^\infty H_{co}^1} + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.$$

Next, we get by the equation of velocity:

$$\varepsilon \operatorname{div}^{\varphi}\mathcal{L}^{\varphi} u = g_2(\varepsilon\partial_t + \underline{u}\cdot\nabla)u + \nabla\sigma$$

that:

$$\begin{aligned}
\varepsilon^{\frac{1}{2}}\|\nabla^3 u\|_{L_t^\infty L^2} &\lesssim \varepsilon^{\frac{1}{2}}\|\partial_z \operatorname{div}^{\varphi}\mathcal{L}^{\varphi} u\|_{L_t^\infty L^2} + \varepsilon^{\frac{1}{2}}\|\nabla^2 u\|_{L_t^\infty H_{co}^1}\Lambda_{2,\infty,t} \\
&\lesssim \varepsilon^{-\frac{1}{2}}\|\nabla\sigma\|_{L_t^\infty H^1} + |h|_{L_t^\infty \tilde{H}^{\frac{3}{2}}} + \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H^1} + \varepsilon^{\frac{1}{2}}\Lambda_{2,\infty,t}\mathcal{E}_{m,t}.
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{\frac{3}{2}}\|\nabla^4 u\|_{L_t^2 L^2} &\lesssim \varepsilon^{\frac{3}{2}}\|\nabla^2 \operatorname{div}^{\varphi}\mathcal{L}^{\varphi} u\|_{L_t^2 L^2} + \varepsilon^{\frac{3}{2}}\big(\|\nabla^3 u\|_{L_t^2 H_{co}^1} + |h|_{L_t^2 \tilde{H}^{\frac{7}{2}}}\big)\Lambda_{2,\infty,t} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\nabla^3\sigma\|_{L_t^2 L^2} + \varepsilon^{\frac{1}{2}}\|\nabla^2 u\|_{L_t^2\mathcal{H}^1} + \varepsilon^{\frac{3}{2}}\|\nabla^3 u\|_{L_t^2 H_{co}^1}\Lambda_{2,\infty,t} + \varepsilon\Lambda_{2,\infty,t}\mathcal{E}_{m,t} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\partial_t u\|_{L_t^2\mathcal{H}^1} + (T^{\frac{1}{2}} + \varepsilon)\Lambda_{2,\infty,t}\mathcal{E}_{m,t} \lesssim (T^{\frac{1}{2}} + \varepsilon)\Lambda_{2,\infty,t}\mathcal{E}_{m,t}
\end{aligned}$$

□

Gathering (7.8.5), (7.8.20) and (7.8.24) we finally obtain (7.8.2).

## 7.9 Uniform control of high order energy norms-I

In this section, we focus on the uniform $L_t^2 H_{co}^{m-1}$ estimates for $\nabla^{\varphi}(\sigma, u)$. We first bound the higher order norms for $(\nabla^{\varphi}\sigma, \operatorname{div}^{\varphi} u)$ by using the elliptic estimates for $\sigma$ and the equations to recover spatial derivatives from time derivatives iteratively. Then, we perform direct energy estimates for the incompressible part $v$ ($v = \mathbb{P}_t u$ solves (7.5.4)) to get the uniform control for $\|\nabla^{\varphi} v\|_{L_t^2 H_{co}^{m-1}}$ (and also $\|v\|_{L_t^\infty H_{co}^{m-1}}$ as a by-product).

### 7.9.1 Uniform estimates for the compressible part

In this subsection, we focus on the uniform estimates of the compressible part of the solution. More precisely, we shall establish the estimate of $\|(\nabla^{\varphi}\sigma, \operatorname{div}^{\varphi} u)\|_{L_t^2 H_{co}^{m-1}}$.

**Lemma 7.9.1.** *Suppose that* (7.2.2) *is true, we can find some polynomial* $\Lambda$*, such that, for any* $0 < t \le T$*,*

$$\begin{aligned}
&\varepsilon^{-1}\|(\nabla^{\varphi}\sigma, \operatorname{div}^{\varphi} u)\|_{L_t^2 H_{co}^{m-1}}^2 + \varepsilon^{-1}\|\nabla \operatorname{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-2}}^2 \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L_T^\infty \tilde{H}^{m-\frac{1}{2}}}^2\big) Y_m^2(0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.9.1}$$

*More precisely, for any* $j, l$ *with* $j + l \le m - 1, m \ge 8$,

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}}\|(\nabla^{\varphi}\sigma, \operatorname{div}^{\varphi} u)\|_{L_t^2\mathcal{H}^{j,l}} \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\
&\quad + \big(\varepsilon^{\frac{1}{2}}\|\nabla \operatorname{div}^{\varphi} u\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi} u\|_{L_t^2 H_{co}^m} + \varepsilon^{\frac{1}{2}}\|\partial_t(\sigma, u)\|_{L_t^2\mathcal{H}^{m-1}}\big)\Lambda\big(\frac{1}{c_0}, |h|_{L_T^\infty \tilde{H}^{m-\frac{1}{2}}}\big).
\end{aligned} \tag{7.9.2}$$

**Remark 7.9.2.** *By using the equation* $(7.1.16)_1$ *for* $\sigma$*, we have:*

$$\nabla \mathrm{div}^{\varphi} u = g_1(0)\varepsilon\partial_t\nabla\sigma + \varepsilon\nabla\big((\frac{g_1 - g_1(0)}{\varepsilon}\varepsilon\partial_t\sigma) + g_1\underline{u}\cdot\nabla\sigma\big), \tag{7.9.3}$$

*which, combined with product estimate* (7.3.8)*, leads to:*

$$\varepsilon^{-\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2 H_{co}^{m-2}} \lesssim \varepsilon^{-\frac{1}{2}}\|\nabla\sigma\|_{L_t^2 H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}. \tag{7.9.4}$$

**Remark 7.9.3.** *By* (7.7.1), (7.7.19), (7.7.30), (7.9.4) *we can derive* (7.9.1) *from* (7.9.2).

*Proof.* We shall establish (7.9.2) by induction on the amount of conormal spatial derivatives. Firstly, let us rewrite the equation $(7.1.16)_1$ as:

$$\mathrm{div}^{\varphi}u = g_1(0)\varepsilon\partial_t\sigma + \varepsilon\big(\frac{g_1 - g_1(0)}{\varepsilon}\varepsilon\partial_t\sigma + g_1 u\cdot\nabla\sigma\big), \tag{7.9.5}$$

By the product estimate (7.3.8) for

$$\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{m-1}} \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Moreover, as $\sigma$ solves by the elliptic problem (7.8.25), we can apply the elliptic estimate (7.5.34) with

$$b = \sigma^{b,1},\ g = (\varepsilon\mu\,\mathrm{curl}^{\varphi}\,\omega\cdot e_3)^{b,2}\ F = \varepsilon PG\,(\text{vector } G \text{ defined in } (7.8.26),\ \text{matrix } P \text{ defined in } (7.5.3))$$

and the identity (7.8.27) to get:

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{m-1}} \lesssim\ & \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+1,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}G\|_{L_t^2\mathcal{H}^{m-1}} + |\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_y u^{b,2}|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}\big)\\ &+ \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+1,\infty,t} + \|\!|(\varepsilon^{-\frac{1}{2}}\nabla\sigma, \varepsilon^{\frac{1}{2}}G)\|\!|_{[\frac{m}{2}]-1,\infty,t}\big)|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}.\end{aligned}$$

By the definition (7.8.26) for $G$ and the product estimate:

$$\|\!|\varepsilon^{\frac{1}{2}}G\|\!|_{[\frac{m}{2}]-1,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big),$$

$$\|\varepsilon^{\frac{1}{2}}G\|_{L_t^2\mathcal{H}^{m-1}} \lesssim \varepsilon^{\frac{1}{2}}\big(\|\partial_t u\|_{L_t^2\mathcal{H}^{m-1}} + \|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2 H_{co}^{m-1}}\big) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}.$$

Moreover, thanks to the identity (7.4.1), the trace inequality (7.3.17), we have that:

$$\begin{aligned}&|\varepsilon^{-\frac{1}{2}}\sigma^{b,1}|_{L_t^2\tilde{H}^{m-\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}|\partial_y u^{b,2}|_{L_t^2\tilde{H}^{m-\frac{3}{2}}}\\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{L_t^{\infty}\tilde{H}^{m-\frac{1}{2}}}\big)\varepsilon^{\frac{1}{2}}\big(\|\nabla u\|_{L_t^2 H_{co}^m} + \|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2 H_{co}^{m-1}}\big) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Gathering the previous four inequalities, we get (7.9.2) for $j \le m-1, l = 0$. Assuming now that (7.9.2) holds for $j+1, l-1$ with $j+l \le m-1, l \ge 1$, we then prove that it is also true for $j, l$. By equation (7.9.5) and the product estimate (7.3.8), we get:

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^2\mathcal{H}^{j,l}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)\\ &\lesssim \|\varepsilon^{-\frac{1}{2}}\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j+1,l-1}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \lesssim \text{R.H.S of (7.9.2)}.\end{aligned}$$

As for the estimate of $\nabla^{\varphi}\sigma$, we first remark that in the elliptic equation (7.8.25), $G$ (defined in (7.8.26)) can indeed be changed slightly through replacing $\partial_t^{\varphi}u$ by $\partial_t^{\varphi}\nabla\Psi$, since $\mathrm{div}^{\varphi}v = 0, \partial_t^{\varphi}v_3|_{z=-1} = 0$. Denote thus

$$\tilde{G} = \bar{\rho}\partial_t^{\varphi}\nabla^{\varphi}\Psi + g_2 u\cdot\nabla^{\varphi}u + \frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t^{\varphi}u - (2\mu+\lambda)\nabla^{\varphi}\mathrm{div}^{\varphi}u.$$

Now, we use again the elliptic estimate (7.5.34) to get that:

$$
\begin{aligned}
\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j,l}} &\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big) \\
&\quad + \Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}\big)\big(\|\varepsilon^{\frac{1}{2}}\tilde{G}\|_{L_t^2\mathcal{H}^{j,l}}+\varepsilon^{\frac{1}{2}}(\|\nabla u\|_{L_t^2H_{co}^m}+\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}})\big) \\
&\lesssim \Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}\big)\varepsilon^{\frac{1}{2}}\big(\|\partial_t\nabla^{\varphi}\Psi\|_{L_t^2\mathcal{H}^{j,l}}+\|\nabla^{\varphi}\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}}+\|\nabla^{\varphi}u\|_{L_t^2H_{co}^m}\big)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Since $\Psi$ is governed by the elliptic problem (7.5.29), we can apply the elliptic estimate (7.5.15) and the estimate (7.5.31) to get that:

$$
\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\Psi\|_{L_t^2\mathcal{H}^{j,l}} \lesssim \Lambda\big(\frac{1}{c_0},|h|_{[\frac{m}{2}]+1,\infty,t}\big)\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi}u\|_{L_t^2\mathcal{H}^{j+1,l-1}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

Combining the two previous inequalities and induction arguments, one finds:

$$
\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^2\mathcal{H}^{j,l}} \lesssim \text{R.H.S of (7.9.2)}.
$$

$\square$

### 7.9.2 Energy estimates: Incompressible part

In this subsection, we focus on the analysis of the incompressible part of the velocity $v=\mathbb{P}_t u$ whose estimates can be obtained from direct energy estimates. By (7.5.4)-(7.5.6), it satisfies the following system:

$$
\begin{cases}
\bar{\rho}\partial_t^{\varphi}v-\mu\Delta^{\varphi}v+\nabla^{\varphi}\pi=-(f+\nabla^{\varphi}q+\bar{\rho}[\mathbb{P}_t,\partial_t^{\varphi}]u),\\
(2\mu S^{\varphi}v-\pi Id)\mathbf{N}|_{z=0}=2\mu(\mathrm{div}^{\varphi}u\mathrm{Id}-(\nabla^{\varphi})^2\Psi)\mathbf{N}|_{z=0},\\
v_3|_{z=-1}=0,\quad \mu\partial_z^{\varphi}v_j|_{z=-1}=au_j|_{z=-1},\quad j=1,2.
\end{cases}
\tag{7.9.6}
$$

where

$$
\nabla^{\varphi}\pi=\mathbb{P}_t\nabla^{\varphi}(\sigma/\varepsilon-2(\mu+\lambda)\mathrm{div}^{\varphi}u)=:\mathbb{P}_t\nabla^{\varphi}\theta,
\tag{7.9.7}
$$

$$
f=\frac{g_2-\bar{\rho}}{\varepsilon}(\varepsilon\partial_t^{\varphi}u+\varepsilon u\cdot\nabla^{\varphi}u)+\bar{\rho}u\cdot\nabla^{\varphi}u,\qquad \nabla^{\varphi}q=-\mathbb{Q}_t(f-\mu\Delta^{\varphi}v).
$$

Before stating the main result for $v$, it is useful to formulate some auxiliary estimates for $\nabla^{\varphi}\pi, f, \nabla^{\varphi}q$.

**Proposition 7.9.4.** *Under the assumption* (7.2.2), *the following* $L_t^2L^2(\mathcal{S})$ *type estimates hold: for any* $m\geq 7$,

$$
\|f\|_{L_t^2H_{co}^{m-1}}+\|\mathrm{div}^{\varphi}f\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac{1}{2}}\|\partial_t f\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac{1}{2}}\|f\|_{L_t^\infty H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),
\tag{7.9.8}
$$

$$
\|\nabla q\|_{L_t^2H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}q\|_{L_t^\infty H_{co}^{m-2}}+\varepsilon^{\frac{1}{2}}\|\partial_t\nabla^{\varphi}q\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),
\tag{7.9.9}
$$

$$
\|\!|\nabla\pi|\!\|_{1,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{5,\infty,t}\big)\mathcal{E}_{m,T},
\tag{7.9.10}
$$

$$
\|\nabla\pi\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla u\|_{L_t^2H_{co}^{m-1}}+T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),
\tag{7.9.11}
$$

$$
\varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\pi\|_{L_t^\infty H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-1}}+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),
\tag{7.9.12}
$$

$$
\varepsilon^{\frac{1}{2}}\|\partial_t\nabla\pi\|_{L_t^2H_{co}^{m-3}}\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t(u,\nabla u)\|_{L_t^2H_{co}^{m-2}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),
\tag{7.9.13}
$$

$$
\|[\mathbb{P}_t,\partial_t^{\varphi}]u\|_{L_t^2H_{co}^{m-1}}+\|[\mathbb{P}_t,\partial_t^{\varphi}]u\|_{L_t^\infty H_{co}^{m-2}}+\|\varepsilon^{\frac{1}{2}}\partial_t[\mathbb{P}_t,\partial_t^{\varphi}]u\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\tag{7.9.14}
$$

*Proof.* Proof of (7.9.8). We detail the estimate of $u\cdot\nabla^\varphi u$ and $\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)$, the other terms can be controlled in a similar way. First, for the $L_t^\infty H_{co}^{m-2}$ norm, we have thanks to the product estimate (7.3.8) that:

$$\begin{aligned}\varepsilon^{\frac12}(\|u\cdot\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \Lambda\big(\frac{1}{c_0}, \|u\|_{[\frac m2],\infty,t}+\varepsilon^{\frac12}\|\nabla u\|_{[\frac m2]-1,\infty,t}\big)\|(u,\varepsilon^{\frac12}\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,T}.\end{aligned}$$

For the first three norms, we first have by the product estimate (7.3.14),

$$\begin{aligned}&\|u\cdot\nabla^\varphi u\|_{L_t^2\mathcal{H}^{0,m-1}}+\|\mathrm{div}^\varphi(u\cdot\nabla u)\|_{L_t^2\mathcal{H}^{0,m-2}}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\|(u,\nabla^\varphi u)\|_{0,\infty,t}+\|\nabla\mathrm{div}^\varphi u\|_{1,\infty,t}\big)\big(\|(u,\nabla^\varphi u)\|_{L_t^2\mathcal{H}^{0,m-1}}+\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2\mathcal{H}^{0,m-2}}\big).\end{aligned}$$

It remains to control $\|\varepsilon\partial_t\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)\|_{L_t^2H_{co}^{m-3}}$ and $\varepsilon^{\frac12}\|\partial_t(u\cdot\nabla^\varphi u)\|_{L_t^2H_{co}^{m-2}}$ We can estimate them in a rather rough way:

$$\begin{aligned}&\|\varepsilon\partial_t\mathrm{div}^\varphi(u\cdot\nabla^\varphi u)\|_{L_t^2H_{co}^{m-3}}\lesssim\|(\varepsilon\partial_t\nabla^\varphi u\cdot\nabla^\varphi u,\varepsilon\partial_t(u\cdot\nabla^\varphi\mathrm{div}^\varphi u))\|_{L_t^2H_{co}^{m-3}}\\ &\lesssim\|\nabla^\varphi u\|_{0,\infty,t}\|\nabla^\varphi u\|_{L_t^2H_{co}^{m-2}}+\|\varepsilon\partial_t\nabla^\varphi u\|_{0,\infty,t}\|\nabla^\varphi u\|_{L_t^2H_{co}^{m-3}}\\ &\quad+\varepsilon^{\frac12}\|\partial_t\nabla^\varphi u\|_{L_t^2H_{co}^{m-4}}\|\varepsilon^{\frac12}\nabla^\varphi u\|_{m-4,\infty,t}\\ &\quad+\|\nabla^\varphi\mathrm{div}^\varphi u\|_{[\frac m2]-2,\infty,t}\|u\|_{L_t^2H_{co}^{m-2}}+\|u\|_{[\frac{m-1}{2}],\infty,t}\|\nabla^\varphi\mathrm{div}^\varphi u\|_{L_t^2H_{co}^{m-2}}\\ &\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\mathcal{E}_{m,T},\end{aligned}$$

$$\begin{aligned}&\varepsilon^{\frac12}\|\partial_t(u\cdot\nabla^\varphi u)\|_{L_t^2H_{co}^{m-2}}\lesssim\|(u\cdot\varepsilon^{\frac12}\partial_t\nabla^\varphi u,\varepsilon^{\frac12}\partial_t u\cdot\nabla^\varphi u)\|_{L_t^2H_{co}^{m-2}}\\ &\lesssim\|u\|_{1,\infty,t}\|\varepsilon^{\frac12}\partial_t\nabla^\varphi u\|_{L_t^2H_{co}^{m-2}}+\|\varepsilon^{\frac12}\partial_t\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}}\big(\int_0^t\|u(s)\|_{m-2,\infty}^2\mathrm{d}s\big)^{\frac12}\\ &\quad+\|\varepsilon^{\frac12}\partial_t u\|_{L_t^2H_{co}^{m-2}}\|\nabla^\varphi u\|_{0,\infty,t}+\|\nabla^\varphi u\|_{L_t^2H_{co}^{m-2}}\big(\int_0^t\|\varepsilon^{\frac12}\partial_t u(s)\|_{m-3,\infty}\mathrm{d}s\big)^{\frac12}\\ &\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Proof of (7.9.9) Let us now show the estimate (7.9.9) for $q$. By the definition of $\mathbb{Q}_t$ in (7.5.2), the fact that $\mathrm{div}^\varphi\Delta^\varphi v=0$, $q$ solves the elliptic problem:

$$\begin{cases}\mathrm{div}(E\nabla q)=-\mathrm{div}(Pf),\\ q|_{z=0}=0,\\ \partial_z^\varphi q|_{z=-1}=(-f\cdot e_3+\mu\Delta^\varphi v_3)|_{z=-1}\end{cases}$$

where $P$ and $E$ are defined in (7.5.8). To shorten the notation, we set $(\Delta^\varphi v_3)^{b,2}=\Delta^\varphi v_3|_{z=-1}$.

Applying elliptic estimate (7.5.35), (7.5.10) for $F=f, g=(\Delta^\varphi v_3)^{b,2}$, we find:

$$\begin{aligned}\|\nabla q\|_{L_t^2H_{co}^{m-1}}&\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\|\mathrm{div}^\varphi f\|_{L_t^\infty H_{tan}^2}+\big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac52}}\big)\\ &\quad(\|f\|_{L_t^2H_{co}^{m-1}}+|h|_{L_t^2\tilde H^{m-\frac12}}+\big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^2\tilde H^{m-\frac32}}),\end{aligned}\tag{7.9.15}$$

$$\begin{aligned}\varepsilon^{\frac12}\|\nabla q\|_{L_t^\infty H_{co}^{m-2}}&\lesssim\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\|\mathrm{div}^\varphi f\|_{L_t^\infty H_{tan}^1}+\big|(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac32}}\big)\\ &\quad(\varepsilon^{\frac12}\|f\|_{L_t^\infty H_{co}^{m-2}}+\varepsilon^{\frac12}|(\Delta^\varphi v)^{b,2}|_{L_t^\infty\tilde H^{m-\frac52}}+\varepsilon^{\frac12}|h|_{L_t^\infty\tilde H^{m-\frac32}}),\end{aligned}\tag{7.9.16}$$

$$\begin{aligned}&\varepsilon^{\frac12}\|\partial_t\nabla q\|_{L_t^2H_{co}^{m-2}}\\ &\lesssim\Lambda\big(\frac{1}{c_0},|(h,\varepsilon^{\frac12}\partial_t h)|_{m-2,\infty,t}+\|\varepsilon^{-\frac12}\mathrm{div}^\varphi f\|_{L_t^\infty H_{co}^2}+\big|(\mathrm{Id},\varepsilon^{\frac12}\partial_t)(\Delta^\varphi v_3)^{b,2}\big|_{L_t^\infty H_{tan}^{\frac52}}\big)\cdot\\ &\quad(\varepsilon^{\frac12}\|\partial_t f\|_{L_t^2H_{co}^{m-2}}+\varepsilon^{\frac12}|\partial_t(\Delta^\varphi v)^{b,2}|_{L_t^\infty\tilde H^{m-\frac32}}+|(h,\varepsilon^{\frac12}\partial_t h)|_{L_t^2\tilde H^{m-\frac32}}+\|\nabla q\|_{L_t^2H_{co}^{m-2}}).\end{aligned}\tag{7.9.17}$$

It follows from direct computations that:

$$\Delta^\varphi v_3 = \Delta^\varphi u_3 - \partial_z^\varphi \mathrm{div}^\varphi u = (\partial_1^\varphi)^2 u_3 + (\partial_2^\varphi)^2 u_3 - (\partial_1^\varphi \partial_z^\varphi u_1 + \partial_2^\varphi \partial_z^\varphi u_2).$$

This, combined with the identities

$$\partial_1^\varphi|_{z=-1} = \partial_1, \quad \partial_2^\varphi|_{z=-1} = \partial_2$$

as well as the boundary condition (7.1.19), yields:

$$(\Delta^\varphi v_3)^{b,2} = -\frac{a}{\mu}(\partial_1 u_1 + \partial_2 u_2)^{b,2}. \tag{7.9.18}$$

In light of (7.9.8), (7.9.15)-(7.9.17), (7.9.18), we find (7.9.9) by the trace inequality (7.3.17).

Proof of (7.9.11)-(7.9.13). Let us switch to the estimate of $\pi$. By definition, $\pi$ satisfies the following elliptic problem:

$$\begin{cases} \mathrm{div}(E\nabla\pi) = 0, \\ \pi|_{z=0} = \theta^{b,1}, \\ \partial_z^\varphi \pi|_{z=-1} = 0. \end{cases}$$

where $\theta^{b,1} = \theta|_{z=0}$. Therefore, to prove (7.9.10), we apply (7.5.33) to get that:

$$\begin{aligned} \|\nabla\pi\|_{1,\infty,t} &\lesssim \|\nabla^2\pi\|_{L_t^\infty H_{tan}^2} + \|\nabla\pi\|_{L_t^\infty H_{tan}^3} \\ &\lesssim \Lambda(\frac{1}{c_0}, |h|_{4,\infty,t})|\theta^{b,1}|_{L_t^\infty H^{\frac{7}{2}}}. \end{aligned}$$

By using boundary conditions (7.4.1) (7.4.2), we have that on the upper boundary,

$$\theta = -2\mu(\partial_1 u_1 + \partial_2 u_2) - 2\mu(\Pi(\partial_1 u\cdot\mathbf{N}, \partial_2 u\cdot\mathbf{N}, 0)^t)_3, \tag{7.9.19}$$

hence, by the product estimate (7.3.4) and trace inequality (7.3.17), we get:

$$|\theta^{b,1}|_{L_t^\infty H^{\frac{7}{2}}} \lesssim (\|\nabla u\|_{L_t^\infty H_{co}^4} + \|u\|_{L_t^\infty H_{co}^5})\Lambda(\frac{1}{c_0}, |h|_{5,\infty,t}).$$

We thus finish the proof of (7.9.10). Now, we can apply (7.5.35) and (7.9.10) to get that for $p = 2, +\infty$,

$$\|\nabla^\varphi\pi\|_{L_t^p H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)|\theta^{b,1}|_{L_t^p \tilde{H}^{m-\frac{3}{2}}} + |h|_{L_t^p \tilde{H}^{m-\frac{3}{2}}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{7.9.20}$$

In view of (7.9.19), one has by the product estimate (7.3.4) and the trace inequality (7.3.17) that

$$|\theta^{b,1}|_{L_t^p \tilde{H}^{m-k+\frac{1}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{[\frac{m}{2}]+1,\infty,t}\big)\|\nabla u\|_{L_t^p H_{co}^{m-1}} + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)|h|_{L_t^p \tilde{H}^{m-\frac{1}{2}}}, \tag{7.9.21}$$

which, combined with (7.9.20), yields (7.9.11)-(7.9.12). Finally, for the estimate of (7.9.13), we use elliptic estimate (7.5.36) to obtain that:

$$\varepsilon^{\frac{1}{2}}\|\partial_t\nabla^\varphi\pi\|_{L_t^2 H_{co}^{m-3}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)(|\varepsilon^{\frac{1}{2}}\partial_t\theta|_{L_t^2\tilde{H}^{m-\frac{5}{2}}} + |\varepsilon^{\frac{1}{2}}(\Delta^\varphi v)^{b,2}|_{L_t^2\tilde{H}^{m-\frac{7}{2}}}) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),$$

we thus obtain (7.9.13) by noting that:

$$|\varepsilon^{\frac{1}{2}}\partial_t\theta|_{L_t^2\tilde{H}^{m-\frac{7}{2}}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\varepsilon^{\frac{1}{2}}\|\partial_t(u, \nabla^\varphi u)\|_{L_t^2 H_{co}^{m-2}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Proof of (7.9.14). Finally, we estimate the commutator between the projection and the time derivative. Denote $\nabla^\varphi\Psi_1 = \mathbb{Q}_t\partial_t^\varphi u$, then

$$[\mathbb{P}_t, \partial_t^\varphi] = [\mathbb{Q}_t, \partial_t^\varphi] = \nabla^\varphi(\Psi_1 - \Psi).$$

By definition, $\Psi_1 - \Psi$ solves the elliptic problem:

$$\Delta^\varphi(\Psi_1 - \partial_t^\varphi\Psi) = 0, \quad (\Psi_1 - \partial_t^\varphi\Psi)|_{z=0} = \frac{\partial_t h}{\partial_z\varphi}\partial_z\Psi, \quad \partial_z^\varphi(\Psi_1 - \partial_t^\varphi\Psi)|_{z=-1} = 0.$$

It follows from (7.5.23) and the product estimate (7.3.14) that:

$$
\begin{aligned}
&\|\nabla^{\varphi}(\Psi_1-\partial_t^{\varphi}\Psi)\|_{L_t^2H_{co}^{m-1}}\\
&\lesssim \Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+\big|\frac{\partial_t h}{\partial_z\varphi}\partial_z\pi_1\big|_{L_t^\infty\tilde{H}^{\frac{5}{2}}}\big)\big|(h,\frac{\partial_t h}{\partial_z\varphi}\partial_z\Psi)\big|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\\
&\lesssim \Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}+|\partial_t h|_{3,\infty,t}+\|\mathrm{div}^{\varphi}u\|_{L_t^\infty H_{co}^2}\big)\\
&\qquad\big(|\partial_t h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\|\nabla\Psi\|_{3,\infty,t}+\|(\nabla\Psi,\nabla^2\Psi)\|_{L_t^2H_{co}^{m-1}}|\partial_t h|_{m-3,\infty,t}\big).
\end{aligned}
\tag{7.9.22}
$$

which, combined with (7.5.25), (7.5.27), (7.5.31), gives the control of the first quantity in (7.9.14). The second quantity can be controlled in a similar way, we omit the proof. □

**Lemma 7.9.5.** *Suppose that $m\geq 7$ and (7.2.2) holds. We can find some polynomials $\Lambda$ such that the following high order energy estimate for $v$ holds: for every $0<t\leq T$,*

$$
\|v\|_{L_t^\infty H_{co}^{m-1}}^2+\|\nabla^{\varphi}v\|_{L_t^2H_{co}^{m-1}}^2\leq \Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}^2+Y_m^2(0)\big)Y_m^2(0)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.23}
$$

*where $\mathcal{Z}=Z_1,Z_2,Z_3$ denote the spatial conormal derivatives.*

**Remark 7.9.6.** *By elliptic estimate (7.5.11) and (7.5.31), we have:*

$$
\|\nabla^{\varphi}\Psi(0)\|_{H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},\tilde{Y}_{[\frac{m}{2}]}(0)\big)(\|u(0)\|_{H_{co}^{m-1}}+|h(0)|_{\tilde{H}^{m-\frac{1}{2}}})
$$

*where $\tilde{Y}_{[\frac{m}{2}]}(0)=\|(\mathrm{div}^{\varphi}u)(0)\|_{H_{co}^{[\frac{m}{2}]}(\mathcal{S})}+\sum_{|\alpha|\leq[\frac{m}{2}]+1}|(Z^{\alpha}h)(0)|_{L^\infty(\mathbb{R}^2)}\lesssim Y_m(0)$.*

*Since $v=u-\nabla^{\varphi}\Psi$, we thus get:*

$$
\|(v,\nabla^{\varphi}\Psi)(0)\|_{H_{co}^{m-1}}\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big)Y_m(0).
$$

*that is the reason why $\|v(0)\|_{H_{co}^{m-1}}$ does not appear in the right hand side of (7.9.23).*

**Remark 7.9.7.** *By the control of normal derivative of the compressible part (7.5.25), (7.9.1) and of the incompressible part (7.9.23), one has that:*

$$
\|\nabla^{\varphi}u\|_{L_t^2H_{co}^{m-1}}^2\lesssim\Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac{1}{2}}}^2\big)Y_m^2(0)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.24}
$$

*Proof.* Let $\alpha=(\alpha_0,\alpha'),|\alpha|=k\leq m-1$. We might as well assume that $Z^{\alpha}$ contains at least one spatial vector field (ie. $|\alpha'|\neq 0$), since $\|v\|_{L_t^\infty\mathcal{H}^{m-1}}$ and $\|\nabla^{\varphi}v\|_{L_t^2\mathcal{H}^{m-1}}$ can be derived directly from the norms that have been bounded. Indeed, one has by elliptic estimates (7.5.23) and (7.5.13) that

$$
\|v\|_{L_t^\infty\mathcal{H}^{m-1}}\lesssim\|(u,\nabla^{\varphi}\Psi)\|_{L_t^\infty\mathcal{H}^{m-1}}\lesssim\|u\|_{L_t^\infty\mathcal{H}^{m-1}}\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

$$
\|\nabla v\|_{L_t^2\mathcal{H}^{m-1}}\lesssim\|(u,\nabla^{\varphi}\Psi)\|_{L_t^2\mathcal{H}^{m-1}}\lesssim\Lambda(\frac{1}{c_0},|h|_{m-2,\infty,t})\|\nabla u\|_{L_t^2\mathcal{H}^{m-1}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

Applying $Z^{\alpha}$ to $(7.9.6)_1$, we obtain:

$$
\begin{aligned}
&\bar{\rho}\partial_t^{\varphi}Z^{\alpha}v-2\mu\mathrm{div}^{\varphi}Z^{\alpha}S^{\varphi}v+\nabla^{\varphi}Z^{\alpha}\pi\\
&=-Z^{\alpha}(f+\nabla^{\varphi}q+\bar{\rho}[\mathbb{P}_t,\partial_t^{\varphi}]u)-[Z^{\alpha},\nabla^{\varphi}]\pi+2\mu[Z^{\alpha},\mathrm{div}^{\varphi}]S^{\varphi}u-\bar{\rho}[Z^{\alpha},\partial_t^{\varphi}]v.
\end{aligned}
$$

Performing standard energy estimates, we obtain the energy identity:

$$
\begin{aligned}
&\frac{1}{2}\bar{\rho}\int_{\mathcal{S}}|Z^{\alpha}v|^2(t)\mathrm{d}\mathcal{V}_t+2\mu\int_0^t\int_{\mathcal{S}}|Z^{\alpha}S^{\varphi}v|^2\mathrm{d}\mathcal{V}_s\mathrm{d}s+a\int_0^t\int_{z=-1}|Z^{\alpha}v_\tau|^2\mathrm{d}y\mathrm{d}s\\
&=:\mathcal{K}_0+\mathcal{K}_1+\cdots\mathcal{K}_8,
\end{aligned}
\tag{7.9.25}
$$

where

$$\mathcal{K}_0 = \frac{1}{2}\bar\rho\int_{\mathcal{S}} |Z^\alpha v|^2(0)\,\mathrm{d}\mathcal{V}_0, \qquad \mathcal{K}_1 = \frac{1}{2}\bar\rho\int_0^t\int_{z=0} \partial_t h|Z^\alpha v|^2\,\mathrm{d}y\mathrm{d}s,$$
$$\mathcal{K}_2 = 2\mu\int_0^t\int_{\mathcal{S}} Z^\alpha S^\varphi v\cdot[Z^\alpha,\nabla^\varphi]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \quad \mathcal{K}_3 = \int_0^t\int_{z=0} Z^\alpha(2\mu S^\varphi v-\pi\mathrm{Id})\mathbf{N}\cdot Z^\alpha v\,\mathrm{d}y\mathrm{d}s,$$
$$\mathcal{K}_4 = \int_0^t\int_{\mathcal{S}} Z^\alpha \pi[\mathrm{div}^\varphi, Z^\alpha]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \qquad \mathcal{K}_5 = -\int_0^t\int_{\mathcal{S}} Z^\alpha v\cdot[Z^\alpha,\nabla^\varphi]\pi\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$\mathcal{K}_6 = -\bar\rho\int_0^t\int_{\mathcal{S}} Z^\alpha v\cdot[Z^\alpha,\partial_t^\varphi]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \qquad \mathcal{K}_7 = 2\mu\int_0^t\int_{\mathcal{S}}[Z^\alpha,\mathrm{div}^\varphi]S^\varphi v\cdot Z^\alpha v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,$$
$$\mathcal{K}_8 = -\int_0^t\int_{\mathcal{S}} Z^\alpha v\cdot\big(Z^\alpha(f+\nabla^\varphi q+\bar\rho[\mathbb{P}_t,\partial_t^\varphi]u)\big)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s.$$

By the trace inequality,

$$a\int_0^t\int_{z=-1}|Z^\alpha v_\tau|^2\mathrm{d}y\mathrm{d}s \geq -\delta\|\nabla v\|^2_{L_t^2H_{co}^k} - C_\delta(\|\nabla v\|^2_{L_t^2H_{co}^{k-1}}+\|v\|^2_{L_t^2H_{co}^k}). \tag{7.9.26}$$

Our following task is to estimate $\mathcal{K}_0-\mathcal{K}_8$ one by one. By Remark 7.9.6, we get that:

$$\mathcal{K}_0 \lesssim \Lambda\big(\frac{1}{c_0}, Y_m^2(0)\big)Y_m^2(0). \tag{7.9.27}$$

Thanks to the trace inequality and Young's inequality, $\mathcal{K}_1$ can be treated as:

$$\begin{aligned}\mathcal{K}_1 &\lesssim |\partial_t h|_{0,\infty,t}(\|\nabla Z^\alpha v\|_{L_t^2L^2}\|Z^\alpha v\|_{L_t^2L^2}+\|Z^\alpha v\|^2_{L_t^2L^2})\\ &\leq \delta\|\nabla v\|^2_{L_t^2H_{co}^k}+C_\delta\|\nabla v\|^2_{L_t^2H_{co}^{k-1}}+\Lambda\big(\frac{1}{c_0},|\partial_t h|_{0,\infty,t}\big)\|v\|^2_{L_t^2H_{co}^k}.\end{aligned} \tag{7.9.28}$$

For the term $\mathcal{K}_2$, to deal with the commutator term $[Z^\alpha,\nabla^\varphi]v$, we apply (7.3.24) if $\alpha_0=0$ and (7.3.23) if $\alpha_0\geq 1$ and find that:

$$\begin{aligned}\|[Z^\alpha,\nabla^\varphi]v\|_{L^2} &\lesssim \Lambda\big(\frac{1}{c_0},\|\!|\nabla v\|\!|_{1,\infty,t}+|(h,\varepsilon\partial_t h)|_{m-2,\infty,t}\big)(|h|_{L_t^2\tilde H^{m-\frac12}}+|\varepsilon h|_{L_t^2\tilde H^{m-\frac32}})\\ &\quad+\Lambda\big(\frac{1}{c_0},|(h,\varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2H_{co}^{m-2}}\\ &\lesssim T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)+\Lambda\big(\frac{1}{c_0},|(h,\varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2H_{co}^{m-2}}.\end{aligned} \tag{7.9.29}$$

Note that by the estimate (7.5.31), we have:

$$\begin{aligned}\|\!|\nabla v\|\!|_{1,\infty,t} &\lesssim \|\!|\nabla(u,\nabla^\varphi\Psi)\|\!|_{1\infty,t}\\ &\lesssim \Lambda\big(\frac{1}{c_0},\|\!|\nabla u\|\!|_{1\infty,t}+|h|_{4,\infty,t}+\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^2}\big)\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by Young's inequality, one can control $\mathcal{K}_2$ by:

$$\mathcal{K}_2 \leq \delta\|\nabla v\|^2_{L_t^2H_{co}^k}+\Lambda\big(\frac{1}{c_0},|(h,\varepsilon\partial_t h)|_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2H_{co}^{k-1}}+T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.30}$$

For the boundary term $\mathcal{K}_3$, we use the boundary condition $(7.9.6)_2$ to split it into two terms:

$$\begin{aligned}\mathcal{K}_3 &= \int_0^t\int_{z=0} Z^\alpha\big(2\mu(\mathrm{div}^\varphi u\mathrm{Id}-\nabla^\varphi\nabla^\varphi\Psi)\mathbf{N}\big)\cdot Z^\alpha v-[Z^\alpha,\mathbf{N}](2\mu S^\varphi v-\pi\mathrm{Id})\cdot Z^\alpha v\,\mathrm{d}y\mathrm{d}s\\ &=: \mathcal{K}_{31}+\mathcal{K}_{32}\end{aligned}$$

Since this term vanishes if $\alpha_3\neq 0$, we may assume that $Z^\alpha=\partial_y Z^{\tilde\alpha}$. It then follows by duality that:

$$\mathcal{K}_{31}\lesssim |Z^\alpha v|_{L_t^2H^{\frac12}}|Z^{\tilde\alpha}\big(2\mu(\mathrm{div}^\varphi u\mathrm{Id}-\nabla^\varphi\nabla^\varphi\Psi)\mathbf{N}\big)|_{L_t^2H^{\frac12}}$$

Thanks to product estimate (7.3.5), we obtain for $k \le m-1$,

$$
\begin{aligned}
|Z^{\tilde{\alpha}}\big(2\mu(\mathrm{div}^{\varphi}u\mathrm{Id}-(\nabla^{\varphi})^2)\mathbf{N}\big)|_{L_t^2H^{\frac12}} &\lesssim |(\mathrm{div}^{\varphi}u,(\nabla^{\varphi})^2\Psi)|_{L_t^2\tilde{H}^{k-\frac12}}|h|_{L_t^\infty\tilde{H}^{[\frac{k-1}{2}]+2+}}\\
&\quad+|h|_{L_t^2\tilde{H}^{k+\frac12}}|(\mathrm{div}^{\varphi}u,(\nabla^{\varphi})^2\Psi)|_{L_t^\infty\tilde{H}^{[\frac{k-1}{2}]+1+}}\\
&\lesssim (\|\nabla\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-2}}+\|\mathrm{div}^{\varphi}u\|_{L_t^2H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0},|h|_{L_t^\infty\tilde{H}^{m-\frac12}}+\|\nabla\Psi\|_{2,\infty,t}\big)\\
&\quad+T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\big(|h|_{L_t^\infty\tilde{H}^{m-\frac12}}+\varepsilon^{\frac12}|h|_{L_t^\infty\tilde{H}^{m+\frac12}}\big)\\
&\lesssim (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

We remark that by the estimate (7.5.31), one has that for $l \le [\frac{k-1}{2}]+1^+ \le [\frac{m}{2}]^+ \le m-3$ (since $k \le m-1, m \ge 7$),

$$
\begin{aligned}
|(\nabla^{\varphi})^2\Psi|_{L_t^\infty\tilde{H}^l} &\lesssim \|\nabla(\nabla^{\varphi})^2\Psi\|_{L_t^\infty\tilde{H}^l}+\|(\nabla^{\varphi})^2\Psi\|_{L_t^\infty\tilde{H}^l}\\
&\lesssim \big(\|(\nabla\mathrm{div}^{\varphi}u,\mathrm{div}^{\varphi}u)\|_{L_t^\infty\tilde{H}^l}+|h|_{L_t^\infty\tilde{H}^{l+5/2}}\big)\Lambda\big(\frac{1}{c_0},\|\nabla^{\varphi}\Psi\|_{[\frac{m}{2}]-1,\infty,t}+|h|_{[\frac{m}{2}]+2,\infty,t}\big)\\
&\lesssim \Lambda(\frac{1}{c_0},\mathcal{N}_{m,T}).
\end{aligned}
$$

Therefore, by the trace inequality and Young's inequality, we get:

$$
\mathcal{K}_{31} \le \delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2+(T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.31}
$$

For $\mathcal{K}_{32}$, in order not to lose derivative on the surface, we write it further as:

$$
\begin{aligned}
\mathcal{K}_{32} &= -\int_0^t\int_{z=0}(2\mu S^{\varphi}v-\pi\mathrm{Id})Z^{\alpha}\mathbf{N}\cdot Z^{\alpha}v+[Z^{\alpha},(S^{\varphi}v-\pi\mathrm{Id}),\mathbf{N}]Z^{\alpha}v\,\mathrm{d}y\mathrm{d}s\\
&=: \mathcal{K}_{321}+\mathcal{K}_{322}.
\end{aligned}
$$

By the definition (7.9.7) for $\pi$ we have that on the upper boundary,

$$
\pi=\theta=-2\mu(\partial_1u_1+\partial_2u_2)-2\mu(\Pi(\partial_1u\cdot\mathbf{N},\partial_2u\cdot\mathbf{N},0)^t)_3. \tag{7.9.32}
$$

Moreover, thanks to the boundary condition (7.4.8), we can indeed express $\partial_z^{\varphi}v$ on the upper boundary. On the one hand, we have the identity:

$$
\partial_z^{\varphi}v\cdot\mathbf{N}=\mathrm{div}^{\varphi}v-\partial_1v_1-\partial_2v_2=-(\partial_1v_1+\partial_2v_2). \tag{7.9.33}
$$

On the other hand, by the identity (7.4.8), one deduces:

$$
\begin{aligned}
|\mathbf{N}|\Pi\partial_z^{\varphi}v &= |\mathbf{N}|\Pi\partial_z^{\varphi}u-|\mathbf{N}|\Pi\nabla^{\varphi}\partial_z^{\varphi}\Psi\\
&= \Pi(\partial_1u\cdot\mathbf{n},\partial_2u\cdot\mathbf{n},0)^t-\Pi(\mathbf{n}_1\partial_1u+\mathbf{n}_2\partial_2u)-|\mathbf{N}|\Pi(\partial_1,\partial_2,0)^t\partial_z^{\varphi}\Psi,
\end{aligned} \tag{7.9.34}
$$

One thus has that:

$$
|(S^{\varphi}v,\pi)|_{1,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0},\|(v,\nabla^{\varphi}\Psi)\|_{2,\infty,t}+|h|_{2,\infty,t}\big) \lesssim \Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
$$

Therefore, by the duality and the trace inequality (7.3.17)

$$
\begin{aligned}
\mathcal{K}_{321} &\le |2\mu S^{\varphi}v-\pi\mathrm{Id}|_{\infty,t}|Z^{\alpha}\mathbf{N}|_{L_t^2H^{-\frac12}}|Z^{\alpha}v|_{L_t^2H^{\frac12}}\\
&\le \delta\|\nabla v\|_{L_t^2H_{co}^k}^2+C_\delta\|\nabla v\|_{L_t^2H_{co}^{k-1}}^2+(\|v\|_{L_t^2H_{co}^{m-1}}^2+T|h|_{L_t^\infty\tilde{H}^{m-\frac12}}^2)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.9.35}
$$

Next, we can control $\mathcal{K}_{322}$, in the following way:

$$
\mathcal{K}_{322} \lesssim |Z^{\alpha}v|_{L_t^2L_y^2}\big(|h|_{L_t^2\tilde{H}^{m-1}}|(S^{\varphi}v,\pi)|_{1,\infty,t}+|(S^{\varphi}v,\pi)|_{L_t^2\tilde{H}^{k-1}}|h|_{m-2,\infty,t}\big).
$$

By virtue of the boundary conditions (7.9.32)-(7.9.34), we obtain that:

$$|(S^{\varphi}v,\pi)|_{L_t^2\tilde{H}^{k-1}} \lesssim \Lambda\big(|h|_{m-2,\infty,t} + \|\!|(v,\nabla^{\varphi}\Psi)\|\!|_{2,\infty,t}\big)\big(|(v,\nabla^{\varphi}\Psi)|_{L_t^2\tilde{H}^k} + |h|_{L_t^2\tilde{H}^k}\big).$$

Combined with the trace inequality (7.3.17), Young's inequality and the elliptic estimate (7.5.25), we find:

$$\mathcal{K}_{322} \le \delta\|\nabla v\|^2_{L_t^2H^k_{co}} + C_\delta\|\nabla v\|^2_{L_t^2H^{k-1}_{co}} + \big(\|v\|^2_{L_t^2H^{m-1}_{co}} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

This estimate, together with (7.9.35), (7.9.31), gives (with possibly another $C_\delta$)

$$\mathcal{K}_3 \le 3\delta\|\nabla v\|^2_{L_t^2H^k_{co}} + C_\delta\|\nabla v\|^2_{L_t^2H^{k-1}_{co}} + \big(\|v\|^2_{L_t^2H^k_{co}} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.36}$$

For the term $\mathcal{K}_4$, since $Z^\alpha$ contains at least one spatial derivative, we can estimate it as:

$$\mathcal{K}_4 \lesssim \|\nabla\pi\|_{L_t^2H^{k-1}_{co}}\big(\|\nabla v\|_{L_t^2H^{k-1}_{co}}\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big) + |h|_{L_t^2\tilde{H}^{k+\frac12}}big(\frac{1}{c_0},\|\!|\nabla v\|\!|_{1,\infty,t} + |h|_{m-2,\infty,t}\big)\big).$$

We apply (7.9.11) and the elliptic estimate (7.5.25) to estimate $\nabla^{\varphi}\pi$ as:

$$\begin{aligned}\|\nabla^{\varphi}\pi\|_{L_t^2H^{k-1}_{co}} &\lesssim \Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla u\|_{L_t^2H^k_{co}} + T^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &\lesssim \Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2H^k_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by Young's inequality, we get:

$$\mathcal{K}_4 \le \delta\|\nabla v\|^2_{L_t^2H^k_{co}} + \Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2H^{k-1}_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Similarly, for $\mathcal{K}_5$, by applying (7.3.21), (7.9.11), (7.9.10), we obtain:

$$\begin{aligned}\mathcal{K}_5 &\lesssim \|v\|_{L_t^2H^k_{co}}\bigg(\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla\pi\|_{L_t^2H^k_{co}} + \Lambda\big(\frac{1}{c_0},\|\!|\nabla\pi\|\!|_{1,\infty,t} + |h|_{m-2,\infty,t}\big)|h|_{L_t^2\tilde{H}^{m-\frac12}}\bigg)\\ &\lesssim \|v\|_{L_t^2H^k_{co}}\bigg(\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\nabla v\|_{L_t^2H^k_{co}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\bigg).\end{aligned}$$

Combined with the Young's inequality, this yields:

$$\mathcal{K}_5 \le \delta\|\nabla v\|^2_{L_t^2H^k_{co}} + C_\delta\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\|v\|^2_{L_t^2H^k_{co}} + (T+\varepsilon)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.37}$$

For the term $\mathcal{K}_6$, we follow the similar arguments as in (7.9.29) to deal with the commutator term:

$$\big\|[Z^\alpha,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z]v\big\|_{L_t^2L^2} \lesssim \big(\|\nabla v\|_{L_t^2H^{k-1}_{co}} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Therefore, we control $\mathcal{K}_6$ by the Cauchy-Schwarz inequality that:

$$\begin{aligned}\mathcal{K}_6 &\le \|Z^\alpha v\|_{L_t^2L^2}\big\|[Z^\alpha,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z]v\big\|_{L_t^2L^2}\\ &\lesssim \|\nabla v\|^2_{L_t^2H^{k-1}_{co}} + \big(\|v\|^2_{L_t^2H^{m-1}_{co}} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned} \tag{7.9.38}$$

We are now ready to estimate $\mathcal{K}_7$. In order not to lose normal derivative, we split it into three terms:

$$\mathcal{K}_7 = \mathcal{K}_{71} + \mathcal{K}_{72} + \mathcal{K}_{73}.$$

with

$$
\begin{aligned}
\mathcal{K}_{71} &= 2\mu \int_0^t \int_{\mathcal{S}} [Z^\alpha, \partial_z]\big(\frac{1}{\partial_z \varphi} S^\varphi v \mathbf{N}\big) \cdot Z^\alpha v \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \\
\mathcal{K}_{72} &= 2\mu \int_0^t \int_{\mathcal{S}} \bigg(\partial_z Z^\alpha\big(\frac{1}{\partial_z \varphi} S^\varphi v \mathbf{N}\big) - \big(\partial_z Z^\alpha S^\varphi v\big)\frac{\mathbf{N}}{\partial_z \varphi}\bigg) \cdot Z^\alpha v \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \\
\mathcal{K}_{73} &= -2\mu \int_0^t \int_{\mathcal{S}} Z^\alpha \bigg(S^\varphi v \partial_z\big(\frac{\mathbf{N}}{\partial_z \varphi}\big)\bigg) \cdot Z^\alpha v \,\mathrm{d}\mathcal{V}_s \mathrm{d}s
\end{aligned}
$$

To deal with $\mathcal{K}_{71}$, we can use the identity (7.3.27) to integrate by parts in space. By doing so, we are led to control the following type of terms (up to some smooth functions that depends only on $\phi$ and its derivatives)

$$
\int_0^t \int_{\mathcal{S}} Z^\gamma\big(\frac{1}{\partial_z \varphi} S^\varphi v \mathbf{N}\big)\partial_z(Z^\alpha v \partial_z \varphi)\,\mathrm{d}x\mathrm{d}s, \quad \int_0^t \int_{\partial\mathcal{S}} Z^\gamma\big(\frac{1}{\partial_z \varphi} S^\varphi v \mathbf{N}\big) Z^\alpha v \partial_z \varphi \,\mathrm{d}y\mathrm{d}s, \quad |\gamma| \le k-1.
$$

The first type of term can be controlled easily by:

$$
\delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} + T\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),
$$

while the second type of terms can be bounded by:

$$
\begin{aligned}
& |v|_{L_t^2 \tilde{H}^k}\big(|S^\varphi v|_{L_t^2 \tilde{H}^{k-1}} + T^{\frac12}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\
& \lesssim |v|_{L_t^2 \tilde{H}^k}\big(|(v, \nabla^\varphi \Psi)|_{L_t^2 \tilde{H}^k} + T^{\frac12}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\
& \le \delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + \big(\|v\|^2_{L_t^2 H_{co}^k} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Hence, we get that:

$$
\begin{aligned}
\mathcal{K}_{71} \le 2\delta \|\nabla v\|^2_{L_t^2 H_{co}^k} &+ C_\delta \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} \\
&+ \big(\|v\|^2_{L_t^2 H_{co}^k} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.9.39}
$$

For $\mathcal{K}_{72}$, we use again integration by parts to split it into three terms: $\mathcal{K}_{72} = \mathcal{K}_{721} + \mathcal{K}_{722} + \mathcal{K}_{723}$, with

$$
\begin{aligned}
\mathcal{K}_{721} &= -2\mu \int_0^t \int_{\mathcal{S}} \big[Z^\alpha, \frac{\mathbf{N}}{\partial_z \varphi}\big] S^\varphi v \cdot \partial_z (Z^\alpha v \partial_z \varphi)\,\mathrm{d}x\mathrm{d}s, \\
\mathcal{K}_{722} &= 2\mu \int_0^t \int_{\mathcal{S}} Z^\alpha S^\varphi v \cdot \partial_z\big(\frac{\mathbf{N}}{\partial_z \varphi}\big) Z^\alpha v \,\mathrm{d}x\mathrm{d}s, \\
\mathcal{K}_{723} &= 2\mu \int_0^t \int_{\partial\mathcal{S}} \big[Z^\alpha, \frac{\mathbf{N}}{\partial_z \varphi}\big] S^\varphi v \cdot Z^\alpha v \partial_z \varphi \,\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

In view of the expressions of these three terms, one can show by the commutator estimate (7.3.9) that

$$
\begin{aligned}
\mathcal{K}_{72} \le \delta \|\nabla v\|^2_{L_t^2 H_{co}^k} &+ C_\delta \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} \\
&+ \big(\|v\|^2_{L_t^2 H_{co}^k} + (T+\varepsilon)^{\frac12}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.9.40}
$$

Note that the boundary term $\mathcal{K}_{723}$ can be controlled in a similar way as in the proof of $\mathcal{K}_{32}$. We thus skip the details.

For $\mathcal{K}_{73}$, to avoid losing regularity on the surface, we use the assumption that $|\alpha'| \ge 1$ to integrate by parts in space. By doing so, we find that it can be bounded as:

$$
\mathcal{K}_{73} \le \delta \|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.9.41}
$$

We remark that there is no boundary contribution in the process of integration by parts since the spatial vector fields are tangent to the boundary. Collecting (7.9.39)-(7.9.41), we finally find that:

$$\begin{aligned}\mathcal{K}_7 &\leq 4\delta\|\nabla v\|^2_{L_t^2 H_{co}^k} + C_\delta \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla v\|^2_{L_t^2 H_{co}^{k-1}} \\ &\quad + \big(\|v\|^2_{L_t^2 H_{co}^{m-1}} + (T+\varepsilon)^{\frac{1}{2}}\big)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.9.42}$$

It remains to treat the last term $\mathcal{K}_8$. By (7.9.8) (7.9.9), (7.9.14), we have:

$$\begin{aligned}\mathcal{K}_8 &\lesssim \|v\|_{L_t^2 H_{co}^k}(\|(f, \nabla^\varphi q)\|_{L_t^2 H_{co}^k} + \|[\mathbb{P}_t, \partial_t^\varphi]u\|_{L_t^2 H_{co}^k}) \\ &\lesssim \|v\|_{L_t^2 H_{co}^{m-1}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.9.43}$$

Gathering (7.9.26)-(7.9.30), (7.9.36)-(7.9.38), (7.9.42), (7.9.43), we find by using Korn's inequality (7.3.34) and by choosing $\delta$ small enough that for any $0 \leq k \leq m-1$,

$$\begin{aligned}\|v\|^2_{L_t^\infty H_{co}^k} + \|\nabla^\varphi v\|^2_{L_t^2 H_{co}^k} &\lesssim Y_m^2(0) + \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\nabla^\varphi v\|^2_{L_t^2 H_{co}^{k-1}} \\ &\quad + (\|v\|_{L_t^2 H_{co}^{m-1}} + (T+\varepsilon)^{\frac{1}{2}})\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by induction (on the number of the derivatives), we get (up to changing possibly the polynomial)

$$\begin{aligned}\|v\|^2_{L_t^\infty H_{co}^{m-1}} + \|\nabla^\varphi v\|^2_{L_t^2 H_{co}^{m-1}} &\lesssim (Y_m^2(0) + \|\nabla v\|^2_{L_t^2 L^2})\Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\big) \\ &\quad + (\|v\|_{L_t^2 H_{co}^{m-1}} + (T+\varepsilon)^{\frac{1}{2}})\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.9.44}$$

By (7.5.27), we can extract extra $T^{\frac{1}{2}}$ from $\|v\|_{L_t^2 H_{co}^{m-1}}$ :

$$\|v\|_{L_t^2 H_{co}^{m-1}} \lesssim \|(u, \nabla^\varphi \Psi)\|_{L_t^2 H_{co}^{m-1}} \lesssim T^{\frac{1}{2}}\|(u, \nabla^\varphi \Psi)\|_{L_t^\infty H_{co}^{m-1}} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Moreover, thanks to the elliptic estimate (7.5.10) and the definition $v = \mathbb{P}_t u = u - \nabla^\varphi \Psi$,

$$\|\nabla v\|_{L_t^2 L^2} \leq \|\nabla u\|_{L_t^2 L^2}\Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big).$$

Inserting the above two estimates and (7.7.19) into (7.9.44), we arrive at (7.9.23). □

We need also to show the following lemma.

**Lemma 7.9.8.** *Under the assumption* (7.2.2)*, the following estimate for $v$ holds:*

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t v\|^2_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla v\|^2_{L_t^2 H_{co}^{m-2}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.9.45}$$

*Proof.* The proof of this Lemma is very similar to the previous one, we thus only sketch its proof. We have by the elliptic estimate (7.5.15) that:

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla v\|_{L_t^2 \mathcal{H}^{m-2}} \\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla^\varphi \Psi)\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla(u, \nabla^\varphi \Psi)\|_{L_t^2 \mathcal{H}^{m-2}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla u\|_{L_t^2 \mathcal{H}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \mathrm{div}^\varphi u\|_{L_t^2 \mathcal{H}^{m-2}}\big) \\ &\quad + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\big(|\partial_t h|_{L_t^\infty \tilde{H}^{m-\frac{3}{2}}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\big).\end{aligned}$$

For any multi-index $\beta$ with $|\beta| \le m-2$, direct energy estimates yield:

$$
\begin{aligned}
&\frac{1}{2}\bar{\rho}\varepsilon \int_{\mathcal{S}} |Z^\beta \partial_t v|^2(t)\, \mathrm{d}\mathcal{V}_t + 2\mu\varepsilon \int_0^t \int_{\mathcal{S}} |Z^\beta \partial_t S^\varphi v|^2 \,\mathrm{d}\mathcal{V}_s \mathrm{d}s + a\varepsilon \int_0^t \int_{z=-1} |Z^\beta \partial_t v_\tau|^2 \,\mathrm{d}y\mathrm{d}s \\
&=: \tilde{\mathcal{K}}_0 + \tilde{\mathcal{K}}_1 + \cdots \tilde{\mathcal{K}}_8,
\end{aligned}
\tag{7.9.46}
$$

where $\tilde{\mathcal{K}}_0 - \tilde{\mathcal{K}}_8$ are terms analogues to $\mathcal{K}_0 - \mathcal{K}_8$ defined in (7.9.25) in which $Z^\alpha$ is replaced by $\varepsilon^{\frac{1}{2}} Z^\beta \partial_t$.

At first, thanks to the trace inequality (7.3.17), Korn's inequality (7.3.34) and Young's inequality

$$
a\varepsilon \int_0^t \int_{z=-1} |Z^\beta \partial_t v_\tau|^2 \,\mathrm{d}y\mathrm{d}s \ge -\delta\varepsilon \|Z^\beta \partial_t S^\varphi v\|_{L_t^2 L^2}^2 - C_\delta(\varepsilon \|\partial_t \nabla v\|_{L_t^2 H_{co}^{m-3}}^2 + \varepsilon \|\partial_t v\|_{L_t^2 H_{co}^{m-2}}^2). \tag{7.9.47}
$$

The remaining task is thus to estimate $\tilde{\mathcal{K}}_1 - \tilde{\mathcal{K}}_8$. We assume that $Z^\beta$ contains at least one spatial tangential derivatives $Z_1, Z_2, Z_3$.

$\underline{\tilde{\mathcal{K}}_1}$ : Similar to (7.9.47), by the trace inequality (7.3.17), Young's inequality and Korn's inequality (7.3.34), we have that:

$$
\begin{aligned}
\tilde{\mathcal{K}}_1 &= \frac{1}{2}\varepsilon \int_0^t \int_{z=0} \partial_t h |Z^\beta \partial_t v|^2 \,\mathrm{d}y\mathrm{d}s \\
&\le \delta\varepsilon \|Z^\beta \partial_t S^\varphi v\|_{L_t^2 L^2}^2 + C_\delta \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)(T\varepsilon \|Z^\beta \partial_t v\|_{L_t^\infty L^2}^2 + \varepsilon \|\partial_t \nabla v\|_{L_t^2 H_{co}^{m-3}}^2).
\end{aligned}
\tag{7.9.48}
$$

$\underline{\tilde{\mathcal{K}}_2}$ : By Young's inequality, $\mathcal{K}_2$ can be controlled similarly:

$$
\tilde{\mathcal{K}}_2 = 2\mu\varepsilon \int_0^t \int_{\mathcal{S}} Z^\beta \partial_t S^\varphi v \cdot [Z^\beta \partial_t, \nabla^\varphi] v \,\mathrm{d}\mathcal{V}_s \mathrm{d}s \le \delta\varepsilon \|Z^\beta \partial_t S^\varphi v\|_{L_t^2 L^2}^2 + C_\delta \varepsilon \|[Z^\beta \partial_t, \nabla^\varphi] v\|_{L_t^2 L^2}^2
$$

Since

$$
[Z^\beta \partial_t, \partial_j^\varphi] f = Z^\beta \big(\partial_t \big(\frac{\mathbf{N}_j}{\partial_z \varphi}\big) \cdot \partial_z f\big) + \big[Z^\beta, \frac{\mathbf{N}_j}{\partial_z \varphi}\big] \partial_t \partial_z f + \frac{\mathbf{N}_j}{\partial_z \varphi} [Z^\beta, \partial_z] \partial_t \partial_z f, \; j = 1, 2, 3,
$$

we can use the fact that $|\beta| \le m-2$ to get that:

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}} \|[Z^\beta \partial_t, \partial_j^\varphi] f\|_{L_t^2 L^2} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \varepsilon^{\frac{1}{2}} \|\partial_t \nabla f\|_{L_t^2 H_{co}^{m-3}} \\
&+ \Lambda\big(\frac{1}{c_0}, \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t) \partial_z f\|_{0,\infty,t} + |(h, \partial_t h)|_{m-3,\infty,t}\big)(\varepsilon^{\frac{1}{2}} \|\partial_z f\|_{L_t^2 H_{co}^{m-2}} + |(h, \varepsilon^{\frac{1}{2}} \partial_t h)|_{L_t^2 \tilde{H}^{m-\frac{3}{2}}}).
\end{aligned}
\tag{7.9.49}
$$

We thus obtain that:

$$
\tilde{\mathcal{K}}_2 \le \delta\varepsilon \|Z^\beta \partial_t S^\varphi v\|_{L_t^2 L^2}^2 + C_\delta \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \|\varepsilon^{\frac{1}{2}} \partial_t \nabla v\|_{L_t^2 H_{co}^{m-3}} + (T + \varepsilon)\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
$$

$\underline{\tilde{\mathcal{K}}_3}$ : Regarding to the estimate of

$$
\tilde{\mathcal{K}}_3 = \varepsilon \int_0^t \int_{z=0} Z^\beta \partial_t (2\mu S^\varphi v - \pi \mathrm{Id}) \mathbf{N} \cdot Z^\beta \partial_t v \,\mathrm{d}y\mathrm{d}s,
$$

similar to that of $\mathcal{K}_3$, we write:

$$
\begin{aligned}
Z^\beta \partial_t (2\mu S^\varphi v - \pi \mathrm{Id}) \mathbf{N} &= 2\mu Z^\beta \partial_t \big((\mathrm{div}^\varphi u \mathrm{Id} - (\nabla^\varphi)^2 \Psi) \mathbf{N}\big) + [Z^\beta \partial_t, \mathbf{N}](2\mu S^\varphi v - \pi \mathrm{Id}) \\
&= 2\mu Z^\beta \partial_t (\mathrm{div}^\varphi u \mathrm{Id} - (\nabla^\varphi)^2 \Psi) \mathbf{N} + \varepsilon^{\frac{1}{2}} [Z^\beta \partial_t, \mathbf{N}](2\mu(\mathrm{div}^\varphi u \mathrm{Id} - (\nabla^\varphi)^2 \Psi) + 2\mu S^\varphi v - \pi \mathrm{Id})
\end{aligned}
$$

By using the trace inequality (7.3.17) and Lemma 7.5.3, we get in a similar way to (7.9.49) that:

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}} \big|Z^\beta \partial_t (\mathrm{div}^\varphi u \mathrm{Id} - (\nabla^\varphi)^2 \Psi) \mathbf{N}\big|_{L_t^2 H_y^{-\frac{1}{2}}} \\
&\lesssim |h|_{2,\infty,t} \|\varepsilon^{\frac{1}{2}} \partial_t (\mathrm{div}^\varphi u, (\nabla^\varphi)^2 \Psi)\|_{L_t^2 H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla (\mathrm{div}^\varphi u, (\nabla^\varphi)^2 \Psi)\|_{L_t^2 H_{co}^{m-3}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big) \big(\|\varepsilon^{\frac{1}{2}} \partial_t \nabla \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-3}} + \|\varepsilon^{\frac{1}{2}} \partial_t \mathrm{div}^\varphi u\|_{L_t^2 H_{co}^{m-2}}\big) + \varepsilon^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Moreover, by the boundary conditions (7.9.32)-(7.9.34), we have

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}}\big|[Z^\beta\partial_t,\mathbf{N}](2\mu(\mathrm{div}^\varphi u\mathrm{Id}-(\nabla^\varphi)^2\Psi)+2\mu S^\varphi v-\pi\mathrm{Id})\big|_{L_t^2L_y^2}\\
&\lesssim|\varepsilon^{\frac{1}{2}}\partial_t(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla^\varphi)^2\Psi)^{b,1}|_{L_t^2\tilde{H}^{m-3}}\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\\
&+\Lambda\big(\frac{1}{c_0},|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,Z)(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla)^2\Psi)^{b,1}|_{0,\infty,t}+|\partial_t h|_{m-3,\infty,t}\big)\cdot\\
&\qquad\big(\varepsilon^{\frac{1}{2}}|(S^\varphi v,\pi,\mathrm{div}^\varphi u,(\nabla)^2\Psi)|_{L_t^2\tilde{H}^{m-2}}+|(h,\varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^2\tilde{H}^{m-1}}\big)\\
&\lesssim\big(|\varepsilon^{\frac{1}{2}}\partial_t v|_{L_t^2\tilde{H}^{m-2}}+\|\varepsilon^{\frac{1}{2}}\partial_t(\mathrm{div}^\varphi u,\nabla\mathrm{div}^\varphi u)\|_{L_t^2H_{co}^{m-3}}\big)\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Therefore, by duality, Cauchy-Schwarz inequality and Young's inequality (7.3.34), we obtain that:

$$
\begin{aligned}
\tilde{\mathcal{K}}_3&\leq\delta\|\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\nabla v\|^2_{L_t^2H_{co}^{m-2}}+(T+\varepsilon)\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\\
&+C_\delta\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t(v,\mathrm{div}^\varphi u)\|^2_{L_t^2H_{co}^{m-2}}+\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-3}}\big).
\end{aligned}
\tag{7.9.50}
$$

<u>$\tilde{\mathcal{K}}_4$</u> : Recall that:

$$\mathcal{K}_4=\varepsilon\int_0^t\int_{\mathcal{S}}Z^\beta\partial_t\pi[\mathrm{div}^\varphi,Z^\beta\partial_t]v\,\mathrm{d}\mathcal{V}_s\mathrm{d}s$$

By Hölder inequality, the estimate (7.9.13) for $\varepsilon^{\frac{1}{2}}\nabla\partial_t\pi$, the Korn's inequality (7.3.34), the commutator estimate (7.9.49) we get:

$$
\begin{aligned}
\tilde{\mathcal{K}}_4&\leq\|\varepsilon^{\frac{1}{2}}\nabla\partial_t\pi\|_{L_t^2H_{co}^{m-3}}\|\varepsilon^{\frac{1}{2}}[\mathrm{div}^\varphi,Z^\beta\partial_t]v\|_{L_t^2L^2}\leq\delta\|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|^2_{L_t^2H_{co}^{m-3}}\\
&+C_\delta\Lambda\big(\frac{1}{c_0},|h|^2_{m-2,\infty,t}\big)(\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi v\|^2_{L_t^2H_{co}^{m-3}}+\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^\varphi u\|^2_{L_t^2H_{co}^{m-2}})+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

<u>$\tilde{\mathcal{K}}_5$.</u> By Cauchy-Schwarz inequality and estimates (7.9.11), (7.9.13),

$$
\begin{aligned}
\tilde{\mathcal{K}}_5&\lesssim\|\varepsilon^{\frac{1}{2}}Z^\beta\partial_t v\|_{L_t^2L^2}\|\varepsilon^{\frac{1}{2}}[Z^\beta\partial_t,\nabla^\varphi]\pi\|_{L_t^2L^2}\\
&\lesssim\|\varepsilon^{\frac{1}{2}}\partial_t v\|_{L_t^2H_{co}^{m-2}}\big(\Lambda\big(\frac{1}{c_0},|h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\nabla\pi\|_{L_t^2H_{co}^{m-3}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\big)\\
&\leq\delta\|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|^2_{L_t^2H_{co}^{m-2}}+(T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
\tag{7.9.51}
$$

<u>$\tilde{\mathcal{K}}_6,\tilde{\mathcal{K}}_8$</u> : By (7.9.8), (7.9.9), (7.9.14), we have:

$$\varepsilon^{\frac{1}{2}}\|\partial_t(f+\nabla^\varphi q+\partial_t[\mathbb{P}_t,\partial_t^\varphi]u)\|_{L_t^2H_{co}^{m-2}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

In addition, the following estimate holds:

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\|\big[Z^\beta\partial_t,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\|_{L_t^2L^2}&\lesssim\Lambda\big(\frac{1}{c_0},|(h,\partial_t h)|_{m-3,\infty,t}+\|\nabla v,\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|_{0,\infty,t}\big)\\
&\quad\big(\|(\varepsilon^{\frac{1}{2}}\partial_t\nabla v,\nabla v)\|_{L_t^2H_{co}^{m-3}}\|(\varepsilon^{\frac{1}{2}}\partial_t\nabla v,\nabla v)\|_{L_t^\infty H_{co}^{m-4}}|(\partial_t h,\varepsilon^{\frac{1}{2}}\partial_t^2 h)|_{L_t^2\tilde{H}^{m-3}}\big)\\
&\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
$$

Therefore, similar to $\mathcal{K}_5$, we control $\tilde{\mathcal{K}}_6+\tilde{\mathcal{K}}_8$ as:

$$
\begin{aligned}
\tilde{\mathcal{K}}_6+\tilde{\mathcal{K}}_8&\lesssim\|\varepsilon^{\frac{1}{2}}Z^\beta\partial_t v\|_{L_t^2L^2}\bigg(\varepsilon^{\frac{1}{2}}\big\|\big[Z^\beta\partial_t,\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\big]v\big\|_{L_t^2L^2}+\varepsilon^{\frac{1}{2}}\big\|Z^\beta\partial_t(f+\nabla^\varphi q+[\mathbb{P}_t,\partial_t^\varphi]u)\big\|_{L_t^2L^2}\bigg)\\
&\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}
\tag{7.9.52}
$$

$\underline{\tilde{\mathcal{K}}_7}$. For this term, one needs to integrate by parts to avoid losing normal derivatives. By following the same lines as the control of $\mathcal{K}_7$ in the last lemma, we find that:

$$\tilde{\mathcal{K}}_7 \leq \delta\|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|^2_{L_t^2H_{co}^{m-3}} + \Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}v\|^2_{L_t^2H_{co}^{m-3}} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.9.53}$$

Plugging (7.9.48)-(7.9.53) into (7.9.46), we get by choosing $\delta$ small enough and by using Korn's inequality (7.3.34) that:

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t v\|^2_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla v\|^2_{L_t^2H_{co}^{m-2}} \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t v(0)\|^2_{H_{co}^{m-2}(\mathcal{S})} + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\\ &+\Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}v\|^2_{L_t^2H_{co}^{m-3}}\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\mathrm{div}^{\varphi}u\|^2_{L_t^2H_{co}^{m-3}} + \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi}u\|^2_{L_t^2H_{co}^{m-2}}\big).\end{aligned}$$

This, combined with (7.6.3), (7.8.5), (7.9.1) and the induction arguments yields (7.9.45). □

## 7.10 $\varepsilon-$dependent high order energy estimate-II

In this subsection, we aim to control $\varepsilon^{\frac{1}{2}}\|\nabla u\|_{L_t^\infty H_{co}^{m-1}}$, which is useful for the control of $L^\infty$ type norms.

**Lemma 7.10.1.** *Under the assumption* (7.2.2)*, we have for any* $0 < t \leq T$,

$$\varepsilon\|\nabla u\|^2_{L_t^\infty H_{co}^{m-1}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.10.1}$$

*Proof.* We will prove the following estimates:

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\nabla u\|^2_{L_t^\infty H_{co}^{m-1}} \lesssim Y_m^2(0) + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0}, |h|^2_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\nabla u\|^2_{L_t^2H_{co}^{m}} + \|\varepsilon^{\frac{1}{2}}\nabla\mathrm{div}^{\varphi}u\|^2_{L_t^2H_{co}^{m-1}} + \|\varepsilon^{\frac{1}{2}}\partial_t(u,\nabla u)\|^2_{L_t^2\mathcal{H}^{m-1}\cap L_t^2H_{co}^{m-2}}\big).\end{aligned} \tag{7.10.2}$$

By (7.7.1), (7.7.19), (7.7.30), (7.9.45), we can then find a polynomial $\Lambda$, such that (7.10.1) holds.

The inequality (7.10.2) can be obtained by the direct energy estimates. Applying $Z^\alpha$ ($|\alpha| \leq m-1$) on $(7.1.16)_2$, taking the scalar product with $-\varepsilon^2 Z^\alpha(\mathrm{div}^{\varphi}\mathcal{L}^{\varphi}u)$ and integrating in space and time, we get by integration by parts that:

$$\begin{aligned}&\varepsilon\mu\int_{\mathcal{S}}|Z^\alpha S^{\varphi}u|^2(t)\,\mathrm{d}\mathcal{V}_t + \frac{1}{2}\varepsilon\lambda\int_{\mathcal{S}}|Z^\alpha\mathrm{div}^{\varphi}u|^2(t)\,\mathrm{d}\mathcal{V}_t + \varepsilon\|Z^\alpha\mathrm{div}^{\varphi}\mathcal{L}^{\varphi}u\|^2_{L_t^2L^2}\\ &+\frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(t)\,\mathrm{d}y = K_0 + K_1 + \cdots + K_5.\end{aligned} \tag{7.10.3}$$

where

$$\begin{aligned}&K_0 = \varepsilon\mu\int_{\mathcal{S}}|Z^\alpha S^{\varphi}u|^2(0)\,\mathrm{d}\mathcal{V}_0 + \frac{1}{2}\varepsilon\lambda\int_{\mathcal{S}}|Z^\alpha\mathrm{div}^{\varphi}u|^2(0)\,\mathrm{d}\mathcal{V}_0 + \frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(0)\,\mathrm{d}y,\\ &K_1 = -\varepsilon\int_0^t\int_{\mathcal{S}}\big(\partial_t[\nabla^{\varphi}, Z^\alpha]u + [\nabla^{\varphi},\partial_t]Z^\alpha u\big)\cdot Z^\alpha\mathcal{L}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ &K_2 = \varepsilon\int_0^t\int_{\mathcal{S}}\partial_t Z^\alpha u\cdot[Z^\alpha,\mathrm{div}^{\varphi}]\mathcal{L}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \quad K_3 = \varepsilon\int_0^t\int_{\mathcal{S}}Z^\alpha\big(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t u\big)\cdot Z^\alpha\mathrm{div}^{\varphi}\mathcal{L}^{\varphi}u\,\mathrm{d}\mathcal{V}_s\mathrm{d}s,\\ &K_4 = \varepsilon\int_0^t\int_{\mathcal{S}}Z^\alpha(\nabla^{\varphi}\sigma)Z^\alpha(\mathrm{div}^{\varphi}\mathcal{L}^{\varphi}u)\,\mathrm{d}\mathcal{V}_s\mathrm{d}s, \quad K_5 = -\varepsilon\int_0^t\int_{\partial\mathcal{S}}Z^\alpha\mathcal{L}^{\varphi}u\mathbf{N}\cdot\partial_t Z^\alpha u\,\mathrm{d}y\mathrm{d}s.\end{aligned}$$

First, by the trace inequality (7.3.17):

$$\frac{a}{2}\varepsilon\int_{z=1}|Z^\alpha u_\tau|^2(t)\,\mathrm{d}y \geq -\delta\|\varepsilon^{\frac{1}{2}}\nabla u(t)\|^2_{H_{co}^{m-1}} - C_\delta\varepsilon\|u\|^2_{L_t^\infty H_{co}^{m-1}}. \tag{7.10.4}$$

Next, for the term $K_1$, we use (7.3.23) to find that:

$$K_1 \lesssim \|\nabla^\varphi u\|_{L_t^2 H_{co}^{m-1}}\big(\varepsilon\Lambda\big(\frac{1}{c_0}, |\partial_t h|_{0,\infty,t}\big)\|\nabla u\|_{L_t^2 H_{co}^{m-1}} + \|\varepsilon\partial_t[\nabla^\varphi, Z^\alpha]u\|_{L_t^2 L^2}\big).$$

By using the identity (7.3.26), we find that:

$$\varepsilon\partial_t[Z^\alpha, \nabla^\varphi]u = \varepsilon^{\frac{1}{2}}\big[Z^\alpha, \varepsilon^{\frac{1}{2}}\partial_t(\frac{\mathbf{N}}{\partial_z\varphi})\big]\partial_z u + \varepsilon\partial_t\big(\frac{\mathbf{N}}{\partial_z\varphi}[Z^\alpha, \partial_z]u\big) + \varepsilon^{\frac{1}{2}}\big[Z^\alpha, \frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon^{\frac{1}{2}}\partial_t\partial_z u.$$

The $L_t^2 L^2$ norm of the first two terms in the right hand side can be controlled by:

$$\begin{aligned}&\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + \|\!|\nabla u|\!\|_{1,\infty,t}\big)\big(\|\nabla u\|_{L_t^2 H_{co}^{m-2}} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{L_t^2 \tilde{H}^{m-\frac{1}{2}}}\big)\\ &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Moreover, the third term can be bounded as:

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\big[Z^\alpha, \frac{\mathbf{N}}{\partial_z\varphi}\big]\varepsilon^{\frac{1}{2}}\partial_t\partial_z u\|_{L_t^2 L^2} &\lesssim T^{\frac{1}{2}}\|\!|\varepsilon^{\frac{1}{2}}\partial_t\nabla u|\!\|_{L_t^2 H_{co}^1}\big|\varepsilon^{\frac{1}{2}}\big(\frac{\mathbf{N}}{\partial_z\varphi}\big)\big|_{m-2,\infty,t}\\ &\quad + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}\|\!|\varepsilon^{\frac{1}{2}}\partial_t\nabla u|\!\|_{0,\infty,t} + |h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{L_t^2 H_{co}^{m-2}} + |h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\big)\\ &\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

The previous two estimates then lead to that:

$$\|\varepsilon\partial_t[Z^\alpha, \nabla^\varphi]u\|_{L_t^2 L^2} \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),$$

from which we find that:

$$K_1 \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.10.5}$$

Thanks to the commutator estimate (7.3.23), we control the term $\varepsilon^{\frac{1}{2}}[Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u$ in the term $K_2$ as follows:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|[Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2 L^2} &\lesssim \Lambda\big(\frac{1}{c_0}, \|\!|\varepsilon^{\frac{1}{2}}\partial_z\mathcal{L}^\varphi u|\!\|_{1,\infty,t} + |h|_{m-2,\infty,t}\big)|h|_{L_t^2\tilde{H}^{m-\frac{1}{2}}}\\ &\quad + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla\mathcal{L}^\varphi u\|_{L_t^2 H_{co}^{m-2}} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality, $K_2$ can be bounded by:

$$K_2 \lesssim \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^2 H_{co}^{m-1}}\|\varepsilon^{\frac{1}{2}}[Z^\alpha, \mathrm{div}^\varphi]\mathcal{L}^\varphi u\|_{L_t^2 L^2} \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.10.6}$$

Moreover, by the product estimate (7.3.8), we obtain:

$$K_3 + K_4 \le \delta\|\varepsilon^{\frac{1}{2}}Z^\alpha \mathrm{div}^\varphi\mathcal{L}^\varphi u\|_{L_t^2 L^2}^2 + C_\delta\varepsilon\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\|(\sigma, u)\|_{E^m,t}. \tag{7.10.7}$$

As for the term $K_5$, we use the boundary condition (7.1.18) to split it as :

$$K_5 = -\varepsilon\int_0^t\int_{z=0} Z^\alpha(\sigma/\varepsilon)\partial_t Z^\alpha u\cdot\mathbf{N} + [Z^\alpha, \mathbf{N}]\mathcal{L}^\varphi u\cdot\partial_t Z^\alpha u\,\mathrm{d}y\mathrm{d}s =: K_{51} + K_{52}.$$

Thanks to the trace inequality (7.3.17) and the boundary conditions (7.4.5), (7.4.8), $K_{52}$ can be bounded as:

$$\begin{aligned}K_{52} &\lesssim |\varepsilon^{\frac{1}{2}}\partial_t Z^\alpha u|_{L_t^2 H^{-\frac{1}{2}}}|\varepsilon^{\frac{1}{2}}[Z^\alpha, \mathbf{N}]\mathcal{L}^\varphi u|_{L_t^2 H^{\frac{1}{2}}}\\ &\lesssim \big(\|\varepsilon^{\frac{1}{2}}\partial_t(u, \nabla u)\|_{L_t^2\mathcal{H}^{m-1}} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla u\|_{L_t^2 H_{co}^{m-2}}\big)\big(\varepsilon^{\frac{1}{2}}|h|_{L_t^2\tilde{H}^{m+\frac{1}{2}}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big) + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,t}\big)\big)\\ &\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}_{m,t}^2.\end{aligned}$$

For $K_{51}$, we take benefits of the boundary condition (7.4.1) and the trace inequality (7.3.17) to find that, if $Z^\alpha = (\varepsilon\partial_t)^j, j \le m-1$,

$$K_{51} \lesssim \varepsilon^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\big(\|(\varepsilon^{\frac12}\partial_t(u,\nabla u), \mathrm{div}^\varphi u, \varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u)\|^2_{L_t^2\mathcal{H}^{m-1}} + \|\nabla u\|^2_{L_t^2 H_{co}^{m-1}} + |h|^2_{L_t^2\tilde{H}^{m-\frac12}}\big)$$
$$\lesssim \varepsilon^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}^2_{m,t},$$

and if $Z^\alpha = \partial_y Z^{\tilde\alpha}$,

$$K_{51} \lesssim |\varepsilon^{\frac12} Z^\alpha(\sigma/\varepsilon)|_{L_t^2 H_y^{\frac12}} |\varepsilon^{\frac12}\partial_t Z^{\tilde\alpha} u|_{L_t^2 H_y^{\frac12}} |h|_{2,\infty,t}$$
$$\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\big(\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2 H_{co}^{m-1}} + \|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2 H_{co}^m} + \|\varepsilon^{\frac12}\partial_t(u,\nabla u)\|^2_{L_t^2 H_{co}^{m-2}}\big)$$
$$+ T^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)|h|_{L_t^2\tilde{H}^{m-\frac12}}.$$

The previous three inequalities yield:

$$K_5 \lesssim (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,t}\big)\mathcal{E}^2_{m,t} + \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\cdot$$
$$\big(\|\varepsilon^{\frac12}\nabla\mathrm{div}^\varphi u\|^2_{L_t^2 H_{co}^{m-1}} + \|\varepsilon^{\frac12}\nabla u\|^2_{L_t^2 H_{co}^m} + \|\varepsilon^{\frac12}\partial_t(u,\nabla u)\|^2_{L_t^2 H_{co}^{m-2}\cap L_t^2\mathcal{H}^{m-1}}\big).$$

Inserting this inequality and (7.10.4)-(7.10.7) into (7.10.3), using Korn's inequality (7.3.33) and choosing $\delta$ small enough, we obtain (7.10.2). $\square$

## 7.11 Uniform control of high order energy norms-II

### 7.11.1 $L_t^\infty L^2$ type norm for compressible part

In this section, we aim to get the a-priori estimates for $\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-2}}$. This would be done by the induction arguments.

**Lemma 7.11.1.** *Suppose that* (7.2.2) *is true, we can find some polynomials such that, for any* $0 < t \le T, m \ge 7$,

$$\varepsilon^{-1}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|^2_{L_t^\infty H_{co}^{m-2}}$$
$$\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty\tilde{H}^{m-\frac12}} + Y_m^2(0)\big)Y_m^2(0) + (\varepsilon+T)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.1}$$

*Proof.* We shall prove for for $j+l \le m-2$ that:

$$\varepsilon^{-\frac12}\|(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{L_t^\infty\mathcal{H}^{j,l}}$$
$$\lesssim (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac12}\partial_t\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac32}}$$
$$+ \Lambda\big(\frac{1}{c_0}, |h|_{L_t^\infty\tilde{H}^{m-\frac12}}\big)(\|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty\mathcal{H}^{m-2,0}} + \|\varepsilon^{\frac12}\nabla(\sigma,u)\|_{L_t^\infty H_{co}^{m-1}}). \tag{7.11.2}$$

and also for the lower order term:

$$\|\varepsilon^{\frac12}\partial_t(\mathrm{div}^\varphi u, \nabla^\varphi\sigma)\|_{L_t^\infty H_{co}^1} \lesssim \|\varepsilon^{\frac12}\partial_t(\sigma,u)\|_{L_t^\infty\mathcal{H}^2} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.3}$$

These two inequalities, together with (7.7.19), (7.7.30) and (7.10.1) would lead to (7.9.1). Indeed, thanks to the estimate (7.7.19), we derive that:

$$\|\varepsilon^{\frac12}\partial_t(\mathrm{div}^\varphi u, \nabla^\varphi\sigma)\|_{L_t^\infty H_{co}^1} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)Y_m(0) + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Inserting this inequality into (7.11.2), and use the estimate (7.7.19), (7.7.30), (7.10.1), we find (7.11.1).

First of all, for any non-negative integers $j, l$ such that $j + l \le m - 2$, it follows from the equation (7.9.5) that:

$$
\begin{aligned}
\varepsilon^{-\frac{1}{2}}\|\mathrm{div}^{\varphi} u\|_{L_t^\infty \mathcal{H}^{j,l}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} + \varepsilon^{\frac{1}{2}}\|\big(\frac{g_1 - g_1(0)}{\varepsilon}\varepsilon\partial_t + g_1\underline{u}\cdot\nabla\big)\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} \\
&\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{-\frac{1}{2}}\nabla^{\varphi}\sigma\|_{L_t^\infty \mathcal{H}^{j+1,l-1}}\mathbb{I}_{\{l\ge 1\}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)\mathcal{E}_{m,t}.
\end{aligned}
\tag{7.11.4}
$$

Let us switch to the control of $\|\nabla^{\varphi}\sigma\|_{L_t^\infty \mathcal{H}^{j,l}}$. As before, we denote

$$\theta = \nabla^{\varphi}\sigma/\varepsilon - 2(\mu+\lambda)\nabla^{\varphi}\mathrm{div}^{\varphi} u.$$

By the equation of velocity,

$$\nabla^{\varphi}\theta = -\partial_t^{\varphi} u - f + \mu\Delta^{\varphi} v,$$

where

$$f = \frac{g_2 - \bar{\rho}}{\varepsilon}\varepsilon\partial_t^{\varphi} u + u\cdot\nabla^{\varphi} u, v = \mathbb{P}_t u.$$

We thus get that:

$$
\begin{aligned}
\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^\infty \mathcal{H}^{j,l}} &\lesssim \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\|(\mathbb{P}_t, \mathbb{Q}_t)\nabla^{\varphi}\theta\|_{L_t^\infty \mathcal{H}^{j,l}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\|\partial_t^{\varphi}\nabla^{\varphi}\Psi\|_{L_t^\infty \mathcal{H}^{j,l}} \\
&\quad + \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\pi\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\|[\mathbb{Q}_t, \partial_t^{\varphi}]u\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi} q\|_{L_t^\infty H_{co}^{m-2}}.
\end{aligned}
\tag{7.11.5}
$$

Recall that $\mathbb{P}_t\nabla^{\varphi}\theta = \nabla^{\varphi}\pi, \nabla^{\varphi} q = -\mathbb{Q}_t(f - \mu\Delta^{\varphi} v)$. By the elliptic estimate (7.5.24),

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t\nabla^{\varphi}\Psi\|_{L_t^\infty \mathcal{H}^{j,l}} \\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}} \\
&+ \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi} u\|_{L_t^\infty \mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\ge 1\}}\big).
\end{aligned}
\tag{7.11.6}
$$

Next, by the elliptic estimate (7.5.13),

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}}\|\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\nabla^{\varphi}\Psi\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)\big(\|\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-2}} + |h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + |\partial_t h|_{L_t^\infty \tilde{H}^{m-2}}\big) \\
&\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

This, together with (7.11.6), yields:

$$
\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}\partial_t^{\varphi}\nabla^{\varphi}\Psi\|_{L_t^\infty \mathcal{H}^{j,l}} \\
&\lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}} \\
&+ \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{\frac{1}{2}}\partial_t\mathrm{div}^{\varphi} u\|_{L_t^\infty \mathcal{H}^{j,l-1}}\mathbb{I}_{\{l\ge 1\}}\big).
\end{aligned}
\tag{7.11.7}
$$

Let us control the other four terms appearing in (7.11.5):

- $\varepsilon^{\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-2}}$. Thanks to the equation (7.9.3), we have:

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}}\|\nabla\mathrm{div}^{\varphi} u\|_{L_t^\infty H_{co}^{m-2}} \\
&\le \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon^{\frac{3}{2}}\big\|\big(\frac{\partial_t\varphi}{\partial_z\varphi}\partial_z\sigma, \nabla^{\varphi}(\frac{g_2-1}{\varepsilon}\varepsilon\partial_t\sigma), \nabla^{\varphi}(g_2\underline{u}\cdot\nabla\sigma)\big)\big\|_{L_t^\infty H_{co}^{m-2}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
\tag{7.11.8}
$$

- $\varepsilon^{\frac{1}{2}}\|(\nabla^\varphi q, \nabla^\varphi \pi, [\mathbb{Q}_t, \partial_t^\varphi]u)\|_{L_t^\infty H_{co}^{m-3}}$. By (7.9.9), (7.9.12), (7.9.14), we have that:

$$\varepsilon^{\frac{1}{2}}\|(\nabla^\varphi q, \nabla^\varphi \pi, [\mathbb{Q}_t, \partial_t^\varphi]u)\|_{L_t^\infty H_{co}^{m-2}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-1}} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \quad (7.11.9)$$

Inserting (7.11.7)-(7.11.9) into (7.11.5), we achieve that (by changing $\Lambda$ accordingly)

$$\begin{aligned}\|\nabla^\varphi \sigma\|_{L_t^\infty \mathcal{H}^{j,l}} &\lesssim \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t \mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}|h|_{L_t^\infty H_{co}^{m-\frac{3}{2}}} \\ &+ (\|\mathrm{div}^\varphi u\|_{L_t^\infty \mathcal{H}^{j+1,l-1}} + \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty \mathcal{H}^{m-2,0}} + \|\varepsilon^{\frac{1}{2}}\nabla(\sigma, u)\|_{L_t^\infty H_{co}^{m-1}})\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big).\end{aligned} \quad (7.11.10)$$

This, together with (7.11.4) and induction arguments, yields (7.11.2). $\square$

**Remark 7.11.2.** *By the estimates* (7.5.27) (7.5.26) *and* (7.9.1), (7.11.1), *we find*

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t \nabla^\varphi \Psi\|_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t (\nabla^\varphi)^2 \Psi\|_{L_t^\infty H_{co}^{m-3} \cap L_t^2 H_{co}^{m-2}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (\varepsilon + T)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

*which further, together with* (7.9.45), *yields that:*

$$\begin{aligned}&\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla u\|_{L_t^2 H_{co}^{m-2}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (\varepsilon + T)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \quad (7.11.11)$$

### 7.11.2 Uniform control of the gradient of the velocity-II

In this subsection, we aim to control the $L_t^\infty H_{co}^{m-4}$ norm of $(\nabla u, \varepsilon^{\frac{1}{2}}\partial_t \nabla u)$ More precisely, the following lemma will be proved.

**Lemma 7.11.3.** *Under the assumption* (7.2.2), *for any* $0 < t \le T$, *we have the following estimate:*

$$\begin{aligned}&\|\nabla u\|^2_{L_t^\infty H_{co}^{m-4}} + \|\varepsilon^{\frac{1}{2}}\partial_t \nabla u\|^2_{L_t^\infty H_{co}^{m-4}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (T + \varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \quad (7.11.12)$$

*Proof.* By identities (7.4.8) and

$$\begin{aligned}|\mathbf{N}|\Pi(\partial_z^\varphi u) &= \Pi(\partial_{\mathbf{n}}^\varphi u - \mathbf{n}_1\partial_1 u - \mathbf{n}_2\partial_2 u) \\ &= \omega \times \mathbf{n} + \Pi\big((\nabla^\varphi u)^t \cdot \mathbf{n} - \mathbf{n}_1\partial_1 u - \mathbf{n}_2\partial_2 u\big) \\ &= \omega \times \mathbf{n} + \Pi(\partial_1 u \cdot \mathbf{n}, \partial_2 u \cdot \mathbf{n}, 0)^t - \Pi(\mathbf{n}_1\partial_1 u + \mathbf{n}_2\partial_2 u),\end{aligned}$$

we have that:

$$\|\nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|u\|_{L_t^\infty H_{co}^{m-3}} + \|(\omega \times \mathbf{n}, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-4}},$$

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_t \nabla^\varphi u\|_{L_t^\infty H_{co}^{m-4}} &\lesssim \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-3}} + \|\varepsilon^{\frac{1}{2}}\partial_t(\omega \times \mathbf{n}, \mathrm{div}^\varphi u)\|_{L_t^\infty H_{co}^{m-4}} \\ &+ (T + \varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

Therefore, (7.11.12) can be derived from the estimate (7.11.11), Lemma 7.7.4 for $\mathrm{div}^\varphi u$, Lemma 7.9.5 for $v$, Lemma 7.6.1 for $h$ as well as the next lemma for $\omega \times \mathbf{n}$. $\square$

**Lemma 7.11.4.** *Suppose that assumption* (7.2.2) *is true, then the following estimate holds:*

$$\begin{aligned}\|\omega\times \boldsymbol{n}\|^2_{L^\infty_t H^{m-4}_{co}} + \|\varepsilon^{\frac12}\partial_t(\omega\times \boldsymbol{n})\|^2_{L^\infty_t H^{m-4}_{co}} &\lesssim Y_m^2(0) + (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|(v, \varepsilon^{\frac12}\partial_t v, \varepsilon^{\frac12}\nabla u)\|^2_{L^\infty_t H^{m-2}_{co}}.\end{aligned} \tag{7.11.13}$$

*Proof.* As explained in the introduction, although $\omega\times\mathbf{n}$ satisfies a transport-diffusion equation without singular terms, one can not control it by direct energy estimates due to the lack of the information of the trace of $\omega\times\mathbf{n}$ on the boundary. Since

$$(\omega\times\mathbf{n})|_{z=0} = 2\Pi(\partial_1 u\cdot\mathbf{n}, \partial_2 u\cdot\mathbf{n}, 0)^t|_{z=0}.$$

A natural attempt in order to do energy estimates is to introduce the modified vorticity: $\tilde\omega = \omega\times\mathbf{n} - \Pi(\partial_1 u\cdot\mathbf{n}, \partial_2 u\cdot\mathbf{n}, 0)^t$. Nevertheless, if taking this way, we are confronted with the original difficulty due to the presence of the singular term in the equation of $\omega\times\mathbf{n}$. However, since the singular term appears only in the equation of the compressible part of velocity, it is still useful to introduce the following quantity:

$$\omega_{\mathbf{n}} = \omega\times\mathbf{n} - 2\Pi(\partial_1 v\cdot\mathbf{n}, \partial_2 v\cdot\mathbf{n}, 0)^t. \tag{7.11.14}$$

where $v$ is the incompressible part of the velocity. As will be seen later, the main advantage to work on $\omega_{\mathbf{n}}$ rather than $\omega\times\mathbf{n}$ is that up to remainders, one can reduce the estimate of $\omega_{\mathbf{n}}$ to that of the compressible part of the velocity and one can extract some extra power of $T$ in the estimates, which is essential to establish the local existence in an uniform time interval.

Since away from the boundary, the conormal space is equivalent to the standard Sobolev space, it suffices to get the estimate of $\omega_{\mathbf{n}}$ near the boundary. In the following, we shall focus on its control near the surface, the case near the bottom is similar (and is even simpler, one can refer to [95] for details). To overcome the difficulty resulting from the nontrivial boundary condition, the general strategy to get a uniform estimate for $\omega_{\mathbf{n}}$ is to split its system into two systems, one carries on the nonlinear terms and the initial data but with trivial Dirichlet boundary condition, while the other one is just a free heat equation with zero initial data and nontrivial Dirichlet boundary condition. The first system can be treated by direct energy estimates because of the homogeneous Dirichlet boundary condition. The analysis of the second system relies on the explicit formulae for the heat equation in the half-space.

To use the explicit formulae of the heat equation in the half-space, it would be convenient to use a coordinate in which the Laplacian has a good form. We thus use the following normal geodesic coordinates:

$$\begin{aligned}\tilde\Phi_t:\quad \mathcal{S}_\kappa =&\mathbb{R}^2\times[-\kappa,0]\longrightarrow\Omega_t\\ &(y,z)\to\begin{pmatrix} y\\ h(t,y)\end{pmatrix} + z\mathbf{n}^{b,1}(y)\end{aligned} \tag{7.11.15}$$

where $\mathbf{n}^{b,1} = \frac{\mathbf{N}^{b,1}}{|\mathbf{N}^{b,1}|} = (-\partial_1 h, -\partial_2 h, 1)/\sqrt{1+|\nabla h|^2}$ denotes the outward normal vector. Straightforward computations show that:

$$\mathrm{D}\tilde\Phi_t = \begin{pmatrix} 1 & 0 & \mathbf{n}_1^{b,1}\\ 0 & 1 & \mathbf{n}_2^{b,1}\\ \partial_1 h & \partial_2 h & \mathbf{n}_3^{b,1}\end{pmatrix} + z\begin{pmatrix} \partial_1\mathbf{n}_1^b & \partial_2\mathbf{n}_1^b & 0\\ \partial_1\mathbf{n}_2^b & \partial_2\mathbf{n}_2^b & 0\\ \partial_1\mathbf{n}_3^b & \partial_2\mathbf{n}_3^b & 0\end{pmatrix}$$

Therefore, as long as $|h|_{2,\infty,T} < +\infty$, and $\kappa$ small enough, we have that: $\det(D\tilde\Phi_t) > 0$ for any $[0,T]\times\mathcal{S}_\kappa$, hence $\tilde\Phi_t$ is a diffeomorphism between $\mathcal{S}_\kappa$ and $\tilde\Phi_t(\mathcal{S}_\kappa)$. The Riemann metric induced by the pullback of the Euclidean metric in $\Omega_t$ through $\tilde\Phi_t^{-1}$ has the block structure:

$$g(y,z) = \begin{pmatrix} \tilde g(y,z) & 0\\ 0 & 1\end{pmatrix}$$

where $\tilde g$ is a function that depends on the gradient of $\tilde\Phi_t$. Therefore, the Laplacian in this metric has the form:

$$\Delta_g f = \partial_z^2 f + \frac12\partial_z(\ln|g|)\partial_z f + \Delta_{\tilde g} f, \tag{7.11.16}$$

where

$$\Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1\leq i,j\leq 2} \partial_{y^i}(\tilde{g}^{ij}|\tilde{g}|^{\frac{1}{2}}\partial_{y^j} f) \quad |g| = \det g.$$

Taking a cut off function $\chi = \chi_0(\frac{z}{C(\kappa)})$, where $\chi_0(s) : \mathbb{R}_- \to \mathbb{R}$ is a smooth function supported on $[-\frac{3}{4}, 0]$ and equal to 1 on the interval $[-\frac{1}{2}, 0]$. Moreover, $C(\kappa)$ is chosen such that $\Phi_t(\mathbb{R}^2 \times [-C_\kappa, 0]) \subset \tilde{\Phi}_t(\mathcal{S}_\kappa)$. The following task is thus to establish the estimate of $\chi\omega_{\mathbf{n}}$. Let us begin with the derivation of the $\chi\omega_{\mathbf{n}}$. First of all, by taking the curl of $(7.1.16)_2$, we find that $\omega = \mathrm{curl}^\varphi u$ solves:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)\omega = G^\omega \tag{7.11.17}$$

with

$$G^\omega = -u\cdot\nabla^\varphi\omega + \omega\cdot\nabla^\varphi u - \omega \mathrm{div}^\varphi u - \frac{\nabla g_2}{\varepsilon} \times ((\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)u) + \frac{\bar{\rho} - g_2}{\varepsilon}((\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)\omega).$$

Hence $\chi\omega \times \mathbf{n}$ is governed by:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)(\chi\omega\times\mathbf{n}) = G_\chi^\omega$$

with

$$G_\chi^\omega = \chi G^\omega \times \mathbf{n} - \mu\Delta^\varphi(\chi\mathbf{n})\omega - 2\mu\nabla^\varphi\omega\times\nabla^\varphi(\chi\mathbf{n}) + \bar{\rho}\omega\times\partial_t^\varphi(\chi\mathbf{n}). \tag{7.11.18}$$

Recall that $v$ satisfies the equation:

$$\bar{\rho}\partial_t^\varphi v - \mu\Delta^\varphi v = -(f + \nabla^\varphi q + \bar{\rho}[\mathbb{P}_t, \partial_t^\varphi]u) - \nabla^\varphi\pi =: H.$$

which gives:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)(\partial_j v\cdot\mathbf{N}) = L_j$$

with

$$L_j = [\partial_j H + \partial_j(\frac{\partial_t\varphi}{\partial_z\varphi})\partial_z v - \mu[\partial_j, \Delta^\varphi]v]\cdot\mathbf{N} + \bar{\rho}\partial_j v\cdot\partial_t^\varphi\mathbf{N} - 2\mu\nabla^\varphi\partial_j v\cdot\nabla^\varphi\mathbf{N} - \mu\Delta^\varphi\mathbf{N}\cdot\partial_j v.$$

Denote $\varsigma = 2(\partial_1 v\cdot\mathbf{N}, \partial_2 v\cdot\mathbf{N}, 0)^t, L = (L_1, L_2, 0)^t$. Therefore, it holds that:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)(\chi\Pi\varsigma) = G_\chi^\varsigma$$

where

$$G_\chi^\varsigma = 2\chi\Pi L + \bar{\rho}\chi[\partial_t, \Pi]\varsigma - \bar{\rho}\chi\frac{\partial_t\varphi}{\partial_z\varphi}[\partial_z, \Pi]\varsigma + \mu\chi[\Pi, \Delta^\varphi]\varsigma + \bar{\rho}[\partial_t^\varphi, \chi]\Pi\varsigma + \mu[\chi, \Delta^\varphi]\Pi\varsigma. \tag{7.11.19}$$

We thus finally find that:

$$(\bar{\rho}\partial_t^\varphi - \mu\Delta^\varphi)(\chi\omega_n) = G_\chi^\varsigma + G_\chi^\omega. \tag{7.11.20}$$

For the sake of notational simplicity, we would denote $\zeta = \chi\omega_{\mathbf{n}}, G_\chi^\zeta = G_\chi^\varsigma + G_\chi^\omega$. Consider

$$\tilde{\zeta}(t,x) = \zeta(t, \Phi_t^{-1}\circ\tilde{\Phi}_t(x)),$$

then $\tilde{\zeta} : [0,T]\times\mathcal{S}_\kappa \to \mathbb{R}$ solves the system:

$$\begin{cases} (\bar{\rho}\partial_t - \mu\Delta_g)\tilde{\zeta} = \widetilde{G_\chi^\zeta} + \bar{\rho}(\mathrm{D}\tilde{\Phi}_t)^{-1}\partial_t\tilde{\Phi}_t\cdot\nabla\tilde{\zeta}, \\ \tilde{\zeta}|_{t=0} = \zeta(\Phi_0^{-1}\circ\tilde{\Phi}_0), \\ \tilde{\zeta}|_{z=0} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n}, \partial_2\nabla^\varphi\Psi\cdot\mathbf{n}, 0)^t|_{z=0}. \end{cases}$$

where $\Delta_g$ is defined in (7.11.16). Since $\tilde{\zeta}$ vanishes on the vicinity of the $\{z = -\kappa\}$, we thus can extend it by zero to the whole lower half space $\mathbb{R}_-^3$. Denote

$$\|f\|_{L_t^p H_{co}^k(\mathbb{R}_-^3)} = \sum_{|\alpha|\leq k} \|Z^\alpha f\|_{L_t^p L^2(\mathbb{R}_-^3)}.$$

By Proposition 7.11.5,

$$\|\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim \|\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} \lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\|\tilde{\zeta}\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)},$$

$$\begin{aligned}
&\|\varepsilon^{\frac12}\partial_t\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim \|\varepsilon^{\frac12}\partial_t\zeta\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)}\\
&\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})(\|\varepsilon^{\frac12}\partial_t\tilde{\zeta}\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} + \varepsilon^{\frac12}\|\tilde{\zeta}\|_{L_t^\infty H_{co}^{m-3}(\mathbb{R}^3_-)}) + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\frac12}\|\tilde{\zeta}\|_{L_t^\infty H_{co}^{m-3}(\mathbb{R}^3_-)}) &\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\|\varepsilon^{\frac12}\zeta\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})}\\
&\lesssim \Lambda(\frac{1}{c_0}, |h|_{m-2,\infty,t})\|\varepsilon^{\frac12}\nabla u\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})} + (T+\varepsilon)^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

Therefore, (7.11.13) follows from the estimate:

$$\|(\tilde{\zeta}, \varepsilon^{\frac12}\partial_t\tilde{\zeta})\|_{L_t^\infty H_{co}^{m-4}(\mathcal{S})} \lesssim Y_m^2(0) + (T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{7.11.21}$$

which are the consequences of lemma 7.11.7 and Lemma 7.11.8. □

**Proposition 7.11.5.** *Suppose that $\mathcal{T}_t : \mathbb{R}^3_- \to \mathbb{R}^3_-$ a $C^{m-3}$ diffeomorphism with condition $\mathcal{T}_t(y,0) = y$. For any function $f(t,\cdot)$ which supported on $\mathcal{S}_\kappa$, and for $p = 2, +\infty$, it holds that*

$$\|f(s, \mathcal{T}_s\cdot)\|_{k,\infty,t} \lesssim \Lambda(\|(\mathcal{T}, \partial_z\mathcal{T})\|_{k,\infty,t})\|f\|_{k,\infty,t}, \tag{7.11.22}$$

$$\|f(s, \mathcal{T}_s\cdot)\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)} \lesssim \Lambda(\|(\mathcal{T}, \partial_z\mathcal{T})\|_{k,\infty,t})\|f\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)}, \tag{7.11.23}$$

$$\begin{aligned}
&\|\varepsilon^{\frac12}\partial_s[f(s, \mathcal{T}_s\cdot)]\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)} \lesssim \Lambda(\|(\mathcal{T}, \partial_z\mathcal{T})\|_{k,\infty,t})\|\varepsilon^{\frac12}(\partial_t, \mathcal{Z})f\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)}\\
&+ \varepsilon^{\frac12}\Lambda(\|\partial_t(\mathcal{T}, \partial_z\mathcal{T})\|_{k-1,\infty,t})\|f\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)} + \|\mathcal{Z}\tilde{\zeta}\|_{0,\infty,t}\Lambda(\|\partial_t\partial_z\mathcal{T}\|_{L_t^\infty H_{co}^k})
\end{aligned} \tag{7.11.24}$$

*where we denote $\mathcal{Z} = (\partial_{y_1}, \partial_{y_2}, Z_3)$ the spatial tangential derivatives.*

**Remark 7.11.6.** *Since $\Phi_t^{-1}\circ\tilde{\Phi}_t = \Phi_t^{-1}(t, y_1 + z\boldsymbol{n}_1^{b,1}, y_2 + z\boldsymbol{n}_2^{b,1}, h + z\boldsymbol{n}_3^{b,1})$, and $|D\Phi_t^{-1}| \lesssim |h|_{1,\infty,t}$, we have that:*

$$\|(\Phi_t^{-1}\circ\tilde{\Phi}_t, \partial_z(\Phi_t^{-1}\circ\tilde{\Phi}_t))\|_{k,\infty,t} \lesssim \Lambda(\frac{1}{c_0}, |h|_{k+1,\infty,t}).$$

*Proof.* The proof of this Lemma follows from the Leibniz rule, we thus omit the proof. □

As explained before, to show (7.11.21), we write $\tilde{\zeta} = \tilde{\zeta}_1 + \tilde{\zeta}_2$, where $\tilde{\zeta}_1, \tilde{\zeta}_2$ satisfy the following two systems:

$$\begin{cases}
(\bar{\rho}\partial_t - \mu\partial_z^2)\tilde{\zeta}_1 = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^3_-,\\
\tilde{\zeta}_1|_{t=0} = 0, \quad \tilde{\zeta}_1|_{z=0} = \tilde{\zeta}|_{z=0} = -2\Pi(\partial_1\nabla^\varphi\Psi\cdot\mathbf{n}, \partial_2\nabla^\varphi\Psi\cdot\mathbf{n}, 0)^t|_{z=0}.
\end{cases} \tag{7.11.25}$$

$$\begin{cases}
(\bar{\rho}\partial_t - \mu\Delta_g)\tilde{\zeta}_2 = \widetilde{G_\chi^\zeta} + \bar{\rho}\partial_t\tilde{\Phi}_t(\mathrm{D}\tilde{\Phi}_t)^{-1}\nabla\tilde{\zeta} + \frac12\mu\partial_z(\ln|g|)\partial_z\tilde{\zeta}_1 - \mu\Delta_{\tilde{g}}\tilde{\zeta}_1,\\
\tilde{\zeta}_2|_{t=0} = \tilde{\zeta}|_{t=0}, \quad \tilde{\zeta}_2|_{z=0} = 0.
\end{cases} \tag{7.11.26}$$

For the first equation, we can use the explicit formulae of the heat equation in the half-space.

**Lemma 7.11.7.** *Under the assumption (7.2.2), it holds that, for any $0 < t \le T$,*

$$\|(\tilde{\zeta}_1, \varepsilon^{\frac12}\partial_t\tilde{\zeta}_1)\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} + \|(\tilde{\zeta}_1, \varepsilon^{\frac12}\partial_t\tilde{\zeta}_1)\|_{L_t^2 H_{co}^{m-3}(\mathbb{R}^3_-)} \lesssim T^{\frac14}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \tag{7.11.27}$$

$$\|(Id, \varepsilon^{\frac12}\partial_t, \partial_y, Z_3)\tilde{\zeta}_1\|_{L^\infty([0,T]\times\mathbb{R}^3_-)} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.28}$$

*Proof.* We present the estimates for $\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1$ appearing in the inequality (7.11.27), the estimates for $\tilde{\zeta}_1$ is similar and easier. Let $\gamma = (\gamma', \gamma_3)$ a multi-index such that $|\gamma| \le m-4$, $Z^\gamma = Z_{tan}^{\gamma'} Z_3^{\gamma_3}$ where $Z_{tan}^{\gamma'} = Z_0^{\gamma_0} Z_1^{\gamma_1} Z_2^{\gamma_2}$. Taking $Z_{tan}^{\gamma'}$ on the equation of (7.11.25), we get:

$$\begin{cases} (\bar{\rho}\partial_t - \mu\partial_z^2)(Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}_-^3, \\ Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1|_{t=0} = 0, \quad Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}_1|_{z=0} = Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}|_{z=0}. \end{cases}$$

By the explicit formulae of the heat equation in the half-line, we have that:

$$\varepsilon^{\frac{1}{2}} Z^\gamma \partial_t\tilde{\zeta}_1(t,y,z) = 2\tilde{\mu}\varepsilon^{\frac{1}{2}} \int_0^t \frac{1}{(4\pi\tilde{\mu}(t-s))^{\frac{1}{2}}} Z_3^{\gamma_3}\partial_z\big(e^{-\frac{z^2}{4\tilde{\mu}(t-s)}}\big) Z_{tan}^{\gamma'}\partial_t\tilde{\zeta}|_{z=0}(s,y)\,\mathrm{d}s \tag{7.11.29}$$

where $\tilde{\mu} = \mu/\bar{\rho}$. To continue, we need the following claim which is essentially proved in Lemma 7.7 of [95].

**Claim** For any $l \ge 0$

$$\|Z_3^l\partial_z(e^{-\frac{z^2}{4\tilde{\mu}(t-s)}})\|_{L_z^2(0,\infty)} \lesssim (t-s)^{-\frac{1}{4}}. \tag{7.11.30}$$

Now, taking the $L_z^2 L_y^2$ norm of (7.11.29) and applying (7.11.30), we find that for any $0 < t \le T$,

$$\varepsilon^{\frac{1}{2}}\|Z^\gamma\partial_t\tilde{\zeta}_1\|_{L_t^\infty L^2(\mathbb{R}_+^3)} \lesssim T^{\frac{1}{4}}|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}|_{z=0}|_{L_t^\infty\tilde{H}^{m-4}}.$$

By the trace inequality (7.3.17) and the estimate (7.5.27), we get that:

$$\begin{aligned} |\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1|_{z=0}|_{L_t^\infty\tilde{H}^{m-4}} &\lesssim |(\nabla^\varphi\Psi, \varepsilon^{\frac{1}{2}}\partial_t\nabla^\varphi\Psi)|_{L_t^\infty\tilde{H}^{m-3}}\Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-3,\infty,t}\big) \\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla^\varphi\Psi, \nabla\nabla^\varphi\Psi), (\nabla^\varphi\Psi, \nabla\nabla^\varphi\Psi)\|_{L_t^\infty H_{co}^{m-3}(\mathcal{S})}\Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-3,\infty,t}\big) \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big), \end{aligned}$$

which, combined with the last inequality, yields:

$$\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1\|_{L_t^\infty H_{co}^{m-4}(\mathbb{R}_-^3)} \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Similarly, by employing Young's inequality (after extending $\tilde{\zeta}|_{z=0}$ by zero to $t > T$ and $t < 0$) and the estimate (7.5.25), we obtain that:

$$\begin{aligned} &\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1\|_{L_t^2 H_{co}^{m-3}(\mathbb{R}_-^3)} \lesssim T^{\frac{1}{4}}|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}|_{z=0}|_{L_t^2\tilde{H}^{m-3}} \\ &\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, |(h, \varepsilon\partial_t h|_{m-2,\infty,t}\big) \cdot \|\varepsilon^{\frac{1}{2}}\partial_t(\nabla^\varphi\Psi, \nabla\nabla^\varphi\Psi), \varepsilon^{-\frac{1}{2}}(\nabla^\varphi\Psi, \nabla\nabla^\varphi\Psi)\|_{L_t^2 H_{co}^{m-2}(\mathcal{S})} \\ &\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned}$$

The above inequality then leads to (7.11.27). We now show the $L_{t,x}^\infty$ estimate. It results from the explicit formulae that: for any $t > 0, z > 0, j = 1, 2, Z^0 = Id, Z^1 = (\varepsilon^{\frac{1}{2}}\partial_t, \partial_y)$,

$$\begin{aligned} \|Z_{tan}^j\tilde{\zeta}_1(t,\cdot,z)\|_{L_y^\infty} &\le \big|Z_{tan}^j\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty}\int_0^t \sqrt{2\pi^{-1}\tilde{\mu}^2}z^{-2}\big(\frac{z^2}{2\tilde{\mu}(t-s)}\big)^{\frac{3}{2}}e^{-\frac{z^2}{4\tilde{\mu}(t-s)}}\,\mathrm{d}s \\ &\le C(\tilde{\mu})\big|Z_{tan}^j\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty} \lesssim \Lambda(\varepsilon^{-\frac{1}{2}}\|\nabla^\varphi\Psi\|_{2,\infty,t} + |h|_{2,\infty,t} + |\varepsilon^{\frac{1}{2}}\partial_t h|_{1,\infty,t}). \end{aligned} \tag{7.11.31}$$

where $C(\tilde{\mu})$ is a constant that depends only on $\tilde{\mu}$. In the same fashion, we have

$$\begin{aligned} \|Z_3\tilde{\zeta}_1(t,\cdot,z)\|_{L_y^\infty} &\le \big(\sqrt{2\pi^{-1}\tilde{\mu}^2}\phi(z)z^{-1}\int_0^t z^{-2}P\big(\frac{z}{\sqrt{2\tilde{\mu}s}}\big)\mathrm{d}s\big)\big|\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty} \\ &\le C(\tilde{\mu})\big|\tilde{\zeta}_1|_{z=0}\big|_{L_t^\infty L_y^\infty} \lesssim \Lambda(\|\nabla^\varphi\Psi\|_{1,\infty,t} + |h|_{1,\infty,t}), \end{aligned} \tag{7.11.32}$$

where $P(z) = |(1-z^2)|z^3 e^{-z^2}$. Note that $\phi(z)z^{-1} = (1+z)/(2-z)^2$ is uniformly bounded. The proof of (7.11.28) is now finished. $\square$

**Lemma 7.11.8.** *Suppose that (7.2.2) holds, for any $0 < t \le T$, we have the following estimate:*

$$\|\tilde{\zeta}_2\|^2_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} + \|(\nabla\tilde{\zeta}_2, \varepsilon^{\frac{1}{2}}\partial_t\nabla\tilde{\zeta}_2)\|^2_{L_t^2 H_{co}^{m-4}(\mathbb{R}^3_-)} \lesssim Y_m^2(0) + T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.33}$$

$$\|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|^2_{L_t^\infty H_{co}^{m-4}(\mathbb{R}^3_-)} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla\tilde{\zeta}_2\|^2_{L_t^2 H_{co}^{m-4}(\mathbb{R}^3_-)} \lesssim \Lambda\big(Y_m^2(0) + \widetilde{\mathcal{E}}^2_{m,t}\big)Y_m^2(0) + T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.34}$$

*Proof.* Again, we only give the detail for the estimate of $\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2$, since $\tilde{\zeta}_2$ is similar to deal with. Let $\beta$ be a multi-index such that $|\beta| = k \le m-4$. Since $\Delta_{\tilde{g}} = \partial_i(\tilde{g}^{ij}\partial_j\cdot) - \partial_i(|\tilde{g}|^{-\frac{1}{2}})\tilde{g}^{ij}|\tilde{g}|^{\frac{1}{2}}\partial_i f$, to avoid losing derivatives on the surface, it is convenient to rewrite the system (7.11.26) as:

$$\begin{cases} \big(\bar{\rho}\partial_t - \mu\partial_z^2 - \mu\partial_i(\tilde{g}^{ij}\partial_j\cdot)\big)\tilde{\zeta}_2 = F_\chi^{\tilde{\zeta}}, \\ \tilde{\zeta}_2|_{t=0} = \tilde{\zeta}|_{t=0}, \quad \tilde{\zeta}_2|_{z=0} = 0. \end{cases} \tag{7.11.35}$$

where

$$F_\chi^{\tilde{\zeta}} = \widetilde{G_\chi^\zeta} - \bar{\rho}\partial_t\tilde{\Phi}_t(\mathrm{D}\tilde{\Phi}_t)^{-1}\nabla\tilde{\zeta} + \frac{1}{2}\mu\partial_z(\ln|g|)\partial_z\tilde{\zeta} + \mu\partial_i(\ln|g|)g^{\tilde{i}j}\partial_j\tilde{\zeta} + \mu\partial_i(\tilde{g}^{ij}\partial_j\tilde{\zeta}_1).$$

Note that we have used the summation convention for $i, j = 1, 2$. Applying $Z^\beta$ on the equation (7.11.35), we get that:

$$\varepsilon^{\frac{1}{2}}\big(\bar{\rho}\partial_t - \mu\partial_z^2 - \mu\partial_i(\tilde{g}^{ij}\partial_j)\big)(Z^\beta\partial_t\tilde{\zeta}_2) = Z^\beta\varepsilon^{\frac{1}{2}}\partial_t F_\chi^{\tilde{\zeta}} + \mu[Z^\beta\varepsilon^{\frac{1}{2}}\partial_t, \partial_z^2]\tilde{\zeta} + \mu\partial_i[Z^\beta\varepsilon^{\frac{1}{2}}\partial_t, \tilde{g}^{ij}]\tilde{\zeta},$$

from which we get energy inequality:

$$\begin{aligned} &\bar{\rho}\varepsilon\|Z^\beta\partial_t\tilde{\zeta}_2(t)\|^2_{L^2(\mathbb{R}^3_-)} + \mu\varepsilon\|\partial_z Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + \mu\int_0^t\int_{\mathbb{R}^3_-}\tilde{g}_{ij}\partial_i Z^\beta\partial_t\tilde{\zeta}_2\cdot\partial_j Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s \\ &\le \bar{\rho}\varepsilon\|Z^\beta\partial_t\tilde{\zeta}(0)\|^2_{L^2(\mathbb{R}^3_-)} + \mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta\partial_t, \partial_z^2]\tilde{\zeta}_2\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| \\ &+ \mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta, \tilde{g}^{ij}]\partial_j\tilde{\zeta}_2\partial_i Z^\beta\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| + \varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}Z^\beta F_\chi^{\tilde{\zeta}}\cdot Z^\beta\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big|. \end{aligned} \tag{7.11.36}$$

As long as $\kappa$ is chosen small enough, the matrix $\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix}$ is positive definite, so that the last two terms in the first line of (7.11.36) control $C_\kappa\|\varepsilon^{\frac{1}{2}}\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)}$. In the sequel, to lighten the notation load and without much ambiguity, we shall denote

$$\|\tilde{f}\|_{L_t^p H_{co}^k} = \|\tilde{f}\|_{L_t^p H_{co}^k(\mathbb{R}^3_-)}, \; \|f\|_{L_t^p H_{co}^k} = \|f\|_{L_t^p H_{co}^k(\mathcal{S})}, \qquad p = 2, +\infty.$$

We begin now to estimate the last three terms of the right hand side of (7.11.36). At first, we have up to some smooth functions depending on $\phi$,

$$[Z^\beta, \partial_z^2] = \sum_{|\tilde{\beta}| \le |\beta|-1} *_{\beta,\tilde{\beta}}\partial_z^2 Z^{\tilde{\beta}} + \sum_{|\gamma| \le |\beta|-1} *_{\beta,\gamma}\partial_z Z^\gamma,$$

Therefore, thanks to the integration by parts and Young's inequality:

$$\mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta, \partial_z^2]\partial_t\tilde{\zeta}\cdot Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| \le \delta\varepsilon\|\partial_z Z^\beta\partial_t\tilde{\zeta}\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + C_\delta\varepsilon(\|\partial_z\partial_t\tilde{\zeta}_2\|^2_{L_t^2 H_{co}^{k-1}} + \|\partial_t\tilde{\zeta}_2\|^2_{L_t^2 H_{co}^{m-5}}). \tag{7.11.37}$$

Similarly, by Young's inequality:

$$\begin{aligned} &\mu\varepsilon\Big|\int_0^t\int_{\mathbb{R}^3_-}[Z^\beta\partial_t, \tilde{g}^{ij}]\partial_j\tilde{\zeta}_2\cdot\partial_i Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\Big| \le \delta\varepsilon\|\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} \\ &+ C_\delta\Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + |\partial_t h|_{2,\infty,t}\big)(\|(\tilde{\zeta}_2, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2)\|^2_{L_t^2 H_{co}^{m-4}} + \varepsilon\|\tilde{\zeta}_2\|^2_{L_t^2 H_{co}^{m-3}}). \end{aligned} \tag{7.11.38}$$

We are now in position to control the last term in (7.11.36). We split it into several terms:

$$\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t F_\chi^{\tilde{\zeta}} \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4.$$

with

$$\mathcal{J}_1 = \varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \widetilde{G_\chi^\zeta} \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s, \quad \mathcal{J}_2 = \bar{\rho}\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \big( (\mathrm{D}\tilde{\Phi}_s)^{-1} \partial_s \tilde{\Phi}_s \cdot \nabla \tilde{\zeta} \big) \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s,$$

$$\mathcal{J}_3 = \mu\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \partial_i (\tilde{g}^{ij} \partial_j \tilde{\zeta}_1) \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s, \quad \mathcal{J}_4 = \frac{1}{2}\mu\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \big( \partial_z (\ln |g|) \partial_z \tilde{\zeta} \big) \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s,$$

$$\mathcal{J}_5 = \frac{1}{2}\mu\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \big( \partial_i (\ln |g|) \tilde{g}^{ij} \partial_j \tilde{\zeta} \big) \cdot Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s.$$

To estimate $\mathcal{J}_2$, let us split it into two terms $\mathcal{J}_2 = \mathcal{J}_{21} + \mathcal{J}_{22}$ :

$$\mathcal{J}_{21} = \bar{\rho}\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \Big( \mathrm{div}\big( (\mathrm{D}\tilde{\Phi}_s)^{-1} \partial_s \tilde{\Phi}_s \big) \tilde{\zeta} \Big) Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s,$$

$$\mathcal{J}_{22} = \bar{\rho}\varepsilon \int_0^t \int_{\mathbb{R}^3_-} Z^\beta \partial_t \partial_l \Big( \big( (\mathrm{D}\tilde{\Phi}_s)^{-1} \partial_s \tilde{\Phi}_s \big)_l \tilde{\zeta} \Big) Z^\beta \partial_t \tilde{\zeta}_2 \, \mathrm{d}x\mathrm{d}s.$$

We emphasize that since there is no gain of the regularity of $\tilde{\Phi}$ from that of $h$, the careful attention needs to be paid to the regularity of the surface, which makes the following computations not that direct. To estimate $\mathcal{J}_{21}$, in order not to lose regularity on the surface, we consider two cases. If $Z^\beta$ contains at least one spatial conormal derivative, we integrate by parts in space, and then use the Young's inequality to get:

$$\begin{aligned}\mathcal{J}_{21} \leq \delta\varepsilon \|\nabla Z^\beta \partial_t \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + \big( \|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta})\|^2_{L_t^2 H_{co}^{m-4}} + |\varepsilon^{\frac{1}{2}} \partial_t^2 h|^2_{L_t^2 \tilde{H}^{m-3}} \big) \cdot \\ \Lambda\big( \|\tilde{\zeta}\|_{1,\infty,t} + |(h, \varepsilon^{\frac{1}{2}} \partial_t h)|_{m-2,\infty,t} + |\partial_t h|_{m-3,\infty,t} + |\varepsilon^{\frac{1}{2}} \partial_t^2 h|_{m-5,\infty,t} \big).\end{aligned}$$

Moreover, we have by Proposition 7.11.5 and estiamte (7.11.27) that for $l = 3, 4$

$$\begin{aligned}&\|(\tilde{\zeta}_2, \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2)\|_{L_t^2 H_{co}^{m-l}} \leq \|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}), (\tilde{\zeta}_1, \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_1)\|_{L_t^2 H_{co}^{m-l}} \\ &\lesssim \|(\nabla u, \varepsilon^{\frac{1}{2}} \partial_t \nabla u)\|_{L_t^2 H_{co}^{m-l}} \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}} \partial_t h)|_{m-l+1,\infty,t}\big) + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\ &(\lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \text{ for } l = 4, \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \text{ for } l = 3)\end{aligned} \tag{7.11.39}$$

and by (7.11.28) that:

$$\begin{aligned}&\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t, \partial_y, Z_3)\tilde{\zeta}\|_{L^\infty([0,T]\times\mathbb{R}^3_-)} \\ &\lesssim \|(\mathrm{Id}, \varepsilon^{\frac{1}{2}} \partial_t, \partial_y, Z_3)\zeta\|_{0,\infty,t} \Lambda\big( |\varepsilon^{\frac{1}{2}} \partial_t h|_{2,\infty,t} + |h|_{3,\infty,t} \big) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).\end{aligned} \tag{7.11.40}$$

Therefore, by noting (7.6.2), we obtain that in this case,

$$\mathcal{J}_{21} \leq \delta\varepsilon \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{2}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.41}$$

If $Z^\beta = (\varepsilon\partial_t)^k$, $(k \leq m-4)$, thanks to (7.6.2), (7.11.27), (7.11.39), (7.11.40), we can control it as:

$$\begin{aligned}\mathcal{J}_{21} &\lesssim \|\varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-4}} \Lambda\big(\frac{1}{c_0}, \|(\tilde{\zeta}, \varepsilon\partial_t \tilde{\zeta})\|_{0,\infty,t} + \mathcal{G}_{\infty,t}(h)\big) \cdot \\ &\quad (|(\varepsilon^{\frac{1}{2}} \partial_t^2 h, \varepsilon^{\frac{3}{2}} \partial_t^3 h)|_{L_t^2 \tilde{H}^{m-3}} + \|(\tilde{\zeta}_2, \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2)\|_{L_t^2 H_{co}^{m-4}}) \\ &\lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned} \tag{7.11.42}$$

where

$$\mathcal{G}_{\infty,t}(h) := |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t} + |\partial_t h|_{m-3,\infty,t} + |(\varepsilon^{\frac{1}{2}}\partial_t^2 h, \varepsilon^{\frac{3}{2}}\partial_t^3 h)|_{m-5,\infty,t}.$$

Note that by (7.6.1)-(7.6.2), and the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$,

$$\mathcal{G}_{\infty,t}(h) \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

Collecting (7.11.41) and (7.11.42), we finally get that

$$\mathcal{J}_{21} \leq \delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.43}$$

For $\mathcal{J}_{22}$, write $Z^\beta \partial_l = [Z^\beta, \partial_l] + \partial_l Z^\beta$, we integrate by parts and follow the similar arguments as in the estimate of $\mathcal{J}_{21}$ that:

$$\mathcal{J}_{22} \leq \delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

This, combined with (7.11.43), yields:

$$\mathcal{J}_{2} \leq 2\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.44}$$

For $\mathcal{J}_3$, we integrate by parts again and use Cauchy-Schwarz inequality to get:

$$\begin{aligned}
\mathcal{J}_3 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\tilde{g}^{ij}\partial_j\tilde{\zeta}_1)\|_{L_t^2 H_{co}^{m-4}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-3}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big) \|(\tilde{\zeta}_1, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_1)\|_{L_t^2 H_{co}^{m-3}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-3}}
\end{aligned}$$

By using and estimates (7.11.27), (7.11.39), we thus find that:

$$\mathcal{J}_3 \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.45}$$

We begin now to estimate $\mathcal{J}_4$. By writing

$$\partial_z(\ln|g|)\partial_z\tilde{\zeta} = -\partial_z^2(\ln|g|)\tilde{\zeta} + \partial_z\big(\partial_z(\ln|g|)\tilde{\zeta}\big),$$

we can follow the similar computations as in the estimates of $\mathcal{J}_2$ to obtain (it is indeed easier in the sense that $\partial_z^2(\ln|g|), \partial_z(\ln|g|)$ involve only two derivatives of $h$)

$$\begin{aligned}
\mathcal{J}_4 &\leq \delta \|\varepsilon^{\frac{1}{2}}\partial_t\nabla Z^\beta\tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big) \|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|^2_{L_t^2 H_{co}^{m-4}} \\
&\leq \delta \|\varepsilon^{\frac{1}{2}}\partial_t\nabla Z^\beta\tilde{\zeta}_2\|^2_{L_t^2 L^2(\mathbb{R}^3_-)} + T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.11.46}$$

We proceed to estimate $\mathcal{J}_5$. If $Z^\beta = (\varepsilon\partial_t)^k$, we control it by inequalities (7.6.2), (7.11.27), (7.11.39):

$$\begin{aligned}
\mathcal{J}_5 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-4}} \Lambda\big(\frac{1}{c_0}, \|\varepsilon^{\frac{1}{2}}\tilde{\zeta}\|_{1,\infty,t} + |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big)\big(\|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|_{L_t^2 H_{co}^{m-3}} + |\varepsilon\partial_t^2 h|_{L_t^2 \tilde{H}^{m-2}}\big) \\
&\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

If $Z^\beta$ contains at least one spatial conormal derivative, we integrate by parts in space and control it in a similar way with $\mathcal{J}_3$ :

$$\begin{aligned}
\mathcal{J}_5 &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t(\partial_i(\ln|g|)\tilde{g}^{ij}\partial_j\tilde{\zeta}))\|_{L_t^2 H_{co}^{m-5}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-3}} \\
&\lesssim \Lambda\big(\frac{1}{c_0}, |(h, \varepsilon^{\frac{1}{2}}\partial_t h)|_{m-2,\infty,t}\big) \|(\tilde{\zeta}, \varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta})\|_{L_t^2 H_{co}^{m-4}} \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L_t^2 H_{co}^{m-3}} \\
&\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}$$

To summarize, we get that:

$$\mathcal{J}_5 \lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.47}$$

We are now left to control the term $\mathcal{J}_1$. After checking every term of $G^\omega_\chi$ and $G^\varsigma_\chi$ defined in (7.11.18) and (7.11.19), we find that the problematic terms that may lead to the loss of derivatives are the following:

$$G^\omega_{\chi,1} = (u\cdot\nabla^\varphi\omega)\times\chi\mathbf{N}, \quad G^\omega_{\chi,2} = \nabla^\varphi\omega\times\nabla^\varphi(\chi\mathbf{n}), \quad G^\varsigma_{\chi,1} = \chi\Pi([\partial_1,\Delta^\varphi]v\cdot\mathbf{N},[\partial_2,\Delta^\varphi]v\cdot\mathbf{N},0)^t.$$

All the other terms can be controlled directly Cauchy-Schwarz inequality, estimate (7.11.39) and proposition (7.11.9):

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3_-}\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\big(\widetilde{G^\zeta_\chi}-\widetilde{G^\omega_{\chi,1}}-\widetilde{G^\omega_{\chi,2}}-\widetilde{G^\varsigma_{\chi,1}}\big)\cdot\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\\
&\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L^2_tH^{m-4}_{co}}\|\varepsilon^{\frac{1}{2}}\partial_t\big(\widetilde{G^\zeta_\chi}-\widetilde{G^\omega_{\chi,1}}-\widetilde{G^\omega_{\chi,2}}-\widetilde{G^\varsigma_{\chi,1}}\big)\|_{L^2_tH^{m-4}_{co}}\\
&\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\tilde{\zeta}_2\|_{L^2_tH^{m-4}_{co}}\|\varepsilon^{\frac{1}{2}}\partial_t(G^\zeta_\chi-G^\omega_{\chi,1}-G^\omega_{\chi,2}-G^\varsigma_{\chi,1})\|_{L^2_tH^{m-4}_{co}(\mathcal{S})}\\
&\lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).
\end{aligned}$$

Note that by Proposition 7.11.9

$$\|G^\zeta_\chi-G^\omega_{\chi,1}-G^\omega_{\chi,2}-G^\varsigma_{\chi,1}\|_{L^2_tH^{m-3}_{co}(\mathcal{S})}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

It remains to control the remaining three terms. We shall explain the estimates of the term involving $G^\omega_{\chi,1}$. Let us first rewrite:

$$u\cdot\nabla^\varphi\omega = u_1\partial_{y_1}\omega+u_2\partial_{y_2}\omega+(u\cdot\mathbf{N})\cdot\partial_z\omega = R_1-R_2.$$

where

$$R_1=\partial_{y_1}(u_1\omega)+\partial_{y_2}(u_2\omega)+\partial_z\big((\frac{u\cdot\mathbf{N}}{\partial_z\varphi})\omega\big),\quad R_2=\partial_{y_1}u_1\cdot\omega+\partial_{y_2}u_2\cdot\omega+\partial_z\big(\frac{u\cdot\mathbf{N}}{\partial_z\varphi}\big)\cdot\omega$$

Since

$$\partial_z\big(\frac{u\cdot\mathbf{N}}{\partial_z\varphi}\big)=\partial_z^\varphi u\cdot\mathbf{N}+u\cdot\partial_z(\frac{\mathbf{N}}{\partial_z\varphi})=\mathrm{div}^\varphi u-\partial_{y_1}u_1-\partial_{y_2}u_2+u\cdot\partial_z(\frac{\mathbf{N}}{\partial_z\varphi}),$$

there is no term like $\partial_z u\cdot\partial_z u$ appearing in $R_2$, we thus can show by using the similar arguments as in Proposition 7.11.9 that:

$$\|\varepsilon^{\frac{1}{2}}\partial_t R_2\|_{L^2_tH^{m-4}_{co}}\lesssim\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),$$

which further yields:

$$\int_0^t\int_{\mathbb{R}^3_-}\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\widetilde{R_2}\cdot\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\lesssim T^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.11.48}$$

Next, by the change of variable, we have:

$$\widetilde{R_1}=(\mathrm{D}(\Phi_t\circ\widetilde{\Phi_t}^{-1})^{-1})_{jl}\partial_l\big[\widetilde{I_j(u)\omega}\big],\qquad \text{where } I(u)=(u_1,u_2,\frac{u\cdot\mathbf{N}}{\partial_z\varphi}).$$

Therefore, using the similar strategy as employed in the estimate of $\mathcal{J}_2$, we find that:

$$\int_0^t\int_{\mathbb{R}^3_-}\varepsilon^{\frac{1}{2}}\partial_t Z^\beta\widetilde{R_1}\cdot\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\le\delta\|\varepsilon^{\frac{1}{2}}\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)}+T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big),$$

which, together with (7.11.48), leads to that:

$$\int_0^t\int_{\mathbb{R}^3_-}\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\widetilde{G^\omega_{\chi,1}}\varepsilon^{\frac{1}{2}}Z^\beta\partial_t\tilde{\zeta}_2\,\mathrm{d}x\mathrm{d}s\le\delta\|\varepsilon^{\frac{1}{2}}\nabla Z^\beta\partial_t\tilde{\zeta}_2\|^2_{L^2_tL^2(\mathbb{R}^3_-)}+T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).$$

Following the similar arguments, one can also show that:

$$\int_0^t \int_{\mathbb{R}^3_-} \varepsilon^{\frac{1}{2}} \partial_t Z^\beta (\widetilde{G^\omega_{\chi,2}} + \widetilde{G^\varsigma_{\chi,1}}) \cdot \varepsilon^{\frac{1}{2}} Z^\beta \partial_t \tilde{\zeta}_2 \,\mathrm{d}x\mathrm{d}s \leq \delta \|\nabla Z^\beta \varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

To summarize, we have obtained that:

$$\mathcal{J}_1 \leq 2\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.49}$$

Gathering (7.11.44)-(7.11.47),(7.11.49) and using (7.11.39), we obtain:

$$\Big| \int_0^t \int_{\mathbb{R}^3_-} Z^\beta F^{\tilde{\zeta}}_\chi \cdot Z^\beta \tilde{\zeta}_2 \,\mathrm{d}x\mathrm{d}s \Big| \leq 10\delta \|\nabla Z^\beta \tilde{\zeta}_2\|^2_{L^2_t L^2(\mathbb{R}^3_-)} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.50}$$

Inserting (7.11.37), (7.11.38) and (7.11.50) into (7.11.36), we get that by choosing $\delta$ small enough that for any $0 \leq k \leq m-4$,

$$\|\varepsilon^{\frac{1}{2}} \partial_t \tilde{\zeta}_2\|^2_{L^\infty_t H^k_{co}} + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla \tilde{\zeta}_2\|^2_{L^2_t H^k_{co}} \lesssim Y^2_m(0) + \|\varepsilon^{\frac{1}{2}} \partial_t \nabla \tilde{\zeta}_2\|^2_{L^2_t H^{k-1}_{co}} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.51}$$

Note that in the above, we use the convention that $\|\cdot\|_{L^2_t H^l_{co}} = 0$ if $l < 0$. Moreover, we can show by repeating the procedure to prove (7.11.51) that:

$$\|\tilde{\zeta}_2\|^2_{L^\infty_t H^k_{co}} + \|\nabla \tilde{\zeta}_2\|^2_{L^2_t H^k_{co}} \lesssim Y^2_m(0) + \|\nabla \tilde{\zeta}_2\|^2_{L^2_t H^{k-1}_{co}} + T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.52}$$

The estimate (7.11.33) then stems from (7.11.52) and induction on $k \in [0, m-4]$, the estimate of (7.11.34) can be derived from (7.11.33) and the induction arguments. □

**Proposition 7.11.9.** *Assume that* (7.2.2) *holds, then for any* $0 < t \leq T$,

$$\|(\mathrm{Id}\,, \varepsilon^{\frac{1}{2}} \partial_t)(G^\zeta_\chi - G^\omega_{\chi,1} - G^\omega_{\chi,2} - G^\varsigma_{\chi,1})\|_{L^2_t H^{m-4}_{co}(\mathcal{S})} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).$$

*Proof.* One can show this estimate by bounding each term appearing in $G^\zeta_\chi - G^\omega_{\chi,1} - G^\omega_{\chi,2} - G^\varsigma_{\chi,1}$. We will give the detail for one term, namely $\omega \cdot \nabla^\varphi u$, which is most indirect, the other terms can be controlled without much trouble. Let us write

$$\omega \cdot \nabla^\varphi u = \omega_1 \partial_{y_1} u + \omega_2 \partial_{y_2} u + (\omega \cdot \mathbf{N}) \partial^\varphi_z u.$$

Further,

$$\begin{aligned} \omega \cdot \mathbf{N} &= \mathrm{div}^\varphi (u \times \mathbf{N}) + u \cdot (\nabla^\varphi \times \mathbf{N}) \\ &= -(u \times \mathbf{N}) \cdot \partial^\varphi_z \mathbf{N} - \partial_{y_1} (u \times \mathbf{N})_1 - \partial_{y_2} (u \times \mathbf{N})_2 + u \cdot (\nabla^\varphi \times \mathbf{N}). \end{aligned}$$

We thus see that $\omega \cdot \nabla^\varphi u = \partial_z u \cdot F_1(\partial_y u, \nabla^\varphi \mathbf{N}, u, \mathbf{N}, \frac{1}{\partial_z \varphi}) + F_2(\partial_y u, \partial_y u)$ where $F_1, F_2$ are some polynomials with degree 4. Let us control $\varepsilon^{\frac{1}{2}} \partial_t (\partial_z u \partial_y u)$ for example, the other ones can be bounded in a similar way. By counting the derivatives hitting on each term, one finds that:

$$\begin{aligned} &\|(\varepsilon^{\frac{1}{2}} \partial_t \partial_z u \cdot \partial_y u, \partial_z u \cdot \varepsilon^{\frac{1}{2}} \partial_t \partial_y u)\|_{L^2_t H^{m-4}_{co}} \\ &\lesssim \|\varepsilon^{\frac{1}{2}} \partial_t \partial_z u\|_{0,\infty,t} \|u\|_{L^2_t H^{m-3}_{co}} + \|\varepsilon^{\frac{1}{2}} \partial_t \partial_z u\|_{L^2_t H^{m-4}_{co}} \|u\|_{m-4,\infty,t} \\ &\quad + \|\nabla u\|_{1,\infty,t} \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{L^2_t H^{m-3}_{co}} + \|\varepsilon^{\frac{1}{2}} \partial_t u\|_{m-5,\infty,t} \|\nabla u\|_{L^\infty_t H^{m-4}_{co}} \\ &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \end{aligned}$$

□

### 7.11.3 Estimate of second normal derivative of the velocity

To finish the a-priori estimates for the energy norms, we are left to the (non-uniform) high order estimates for $\nabla^2 u$ which is the object of the following lemma.

**Lemma 7.11.10.** *Assume that* (7.2.2) *holds for some* $T > 0$ *then for any* $0 < t \le T$*, the following estimate hold,*

$$\|\varepsilon^{\frac{1}{2}}\nabla^2 u\|^2_{L_t^\infty H_{co}^{m-2}\cap L_t^2\mathcal{H}_{co}^{m-1}} \lesssim \Lambda\big(\frac{1}{c_0}, |h|^2_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}} + Y_m^2(0)\big)Y_m^2(0) + (T+\varepsilon)^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.11.53}$$

*Proof.* We will prove the following two inequalities:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\nabla^2 u\|_{L_t^\infty H_{co}^{m-2}} \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-1}} \\ + \varepsilon^{\frac{1}{2}}\|\partial_t u\|_{L_t^\infty H_{co}^{m-2}} + \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi \sigma\|_{L_t^\infty H_{co}^{m-2}},\end{aligned} \tag{7.11.54}$$

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\nabla^2 u\|_{L_t^2\mathcal{H}^{m-1}} \lesssim (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \\ + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^2 H_{co}^m} + \varepsilon^{-\frac{1}{2}}\|\nabla^\varphi \sigma\|_{L_t^2\mathcal{H}^{m-1}}\end{aligned} \tag{7.11.55}$$

where $\mathcal{N}_{m,T}$ is defined in (7.1.30). These two estimates, together with (7.7.1), (7.7.19), (7.9.1), (7.10.1), (7.11.1), would yield (7.11.53). To prove (7.11.54) and (7.11.55), it suffices to control $\varepsilon^{\frac{1}{2}}\partial_z^2 u$. Let us rewrite the equations $(7.1.16)_2$ as

$$\varepsilon^{\frac{1}{2}}\Delta^\varphi u = \varepsilon^{\frac{1}{2}} g_2(\partial_t + \underline{u}\cdot\nabla)u + \varepsilon^{-\frac{1}{2}}\nabla^\varphi\sigma - \varepsilon^{\frac{1}{2}}\nabla^\varphi \mathrm{div}^\varphi u. \tag{7.11.56}$$

Moreover, a direct computation shows that:

$$\Delta^\varphi u = \frac{|\mathbf{N}|^2}{\partial_z\varphi}\partial_z^2 u + \Delta_y u + \partial_1(\mathbf{N}_1\partial_z^\varphi u) + \partial_2(\mathbf{N}_2\partial_z^\varphi u) + \mathbf{N}_1\partial_z^\varphi\partial_1 u + \mathbf{N}_2\partial_z^\varphi\partial_2 u + \frac{1}{2}\partial_z u\partial_z\big|\frac{\mathbf{N}}{\partial_z\varphi}\big|^2. \tag{7.11.57}$$

In view of (7.11.56), (7.11.57), we have by the product estimate (7.3.8) and the definition of $\mathcal{E}_{m,t}$ that:

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\partial_z^2 u\|_{L_t^\infty H_{co}^{m-2}} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{-\frac{1}{2}}\nabla\sigma\|_{L_t^\infty H_{co}^{m-2}} \\ &\quad + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\big(\varepsilon^{\frac{1}{2}}\|(\sigma, u, \nabla\sigma, \nabla u)\|_{L_t^\infty H_{co}^{m-1}} + \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)|\varepsilon^{\frac{1}{2}} h|_{L_t^\infty \tilde{H}^{m-\frac{1}{2}}}\big) \\ &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t u\|_{L_t^\infty H_{co}^{m-2}} + \|\varepsilon^{-\frac{1}{2}}\nabla\sigma\|_{L_t^\infty H_{co}^{m-2}} + \Lambda\big(\frac{1}{c_0}, |h|_{m-2,\infty,t}\big)\|\varepsilon^{\frac{1}{2}}\nabla u\|_{L_t^\infty H_{co}^{m-1}} \\ &\quad + (T+\varepsilon)^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).\end{aligned}$$

We thus finish the proof of (7.11.54). The inequality (7.11.55) can be shown in the similar fashion, we thus omit the proof. □

## 7.12 Control of $L^\infty_{t,x}$ norm

In this section, we prove Proposition 7.2.3, the a-priori estimate for $\mathcal{A}_{m,T}$ :

$$\begin{aligned}\mathcal{A}_{m,T}(\sigma, u) = |h|_{m-2,\infty,t} + \|\nabla u\|_{1,\infty,T} + \|\varepsilon^{-\frac{1}{2}}(\nabla^\varphi\sigma, \mathrm{div}^\varphi u)\|_{m-5,\infty,T} + \|\varepsilon^{\frac{1}{2}}\partial_t(\sigma, u)\|_{m-5,\infty,T} \\ + \|(\mathrm{Id}, \varepsilon\partial_t)(\sigma, u)\|_{m-4,\infty,T} + \|\varepsilon^{\frac{1}{2}}\nabla u\|_{m-3,\infty,T} + \|\varepsilon^{\frac{1}{2}}(\sigma, u)\|_{m-2,\infty,T}.\end{aligned}$$

**Remark 7.12.1.** *By identity* (7.12) *and the equation* $(7.1.16)_2$ *for* $u$*, we have that:*

$$\varepsilon^{\frac{1}{2}}\|\partial_z^2 u\|_{m-5,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big). \tag{7.12.1}$$

**Remark 7.12.2.** *As $[\frac{m}{2}] \leq m-4$ if $m \geq 7$, we thus have:*

$$\|\varepsilon^{-\frac{1}{2}}(\nabla^{\varphi}\sigma, \mathrm{div}^{\varphi} u)\|_{[\frac{m}{2}]-1,\infty,T} + \|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)\|_{[\frac{m}{2}]-1,\infty,T} + \varepsilon^{\frac{1}{2}}\|\partial_z^2 u\|_{[\frac{m}{2}]-1,\infty,t}$$
$$+ \|(Id, \varepsilon\partial_t)(\sigma,u)\|_{[\frac{m}{2}],\infty,T} + \|\varepsilon^{\frac{1}{2}}\nabla u\|_{[\frac{m}{2}]+1,\infty,T} + \|\varepsilon^{\frac{1}{2}} u\|_{[\frac{m}{2}]+2,\infty,T} \lesssim \mathcal{A}_{m,T}.$$

The $L^{\infty}_{t,x}$ estimate of $\nabla\sigma$ results from the maximum principle of the damped transport equation (7.8.21) satisfied by $\nabla\sigma$. To control $L^{\infty}_{t,x}$ norm of $\nabla u$, we reduce the matter to the estimate of vorticity $\omega$ which is done by using the Green function. The other terms appearing in $\mathcal{A}_{m,T}$ can be obtained by the Sobolev embedding (7.3.16).

*Proof of Proposition 7.2.3.* We control the first term directly by the Sobolev embedding $H^{\frac{3}{2}}(\mathbb{R}^2) \hookrightarrow L^{\infty}(\mathbb{R}^2)$

$$|h|_{m-2,\infty,T} \lesssim |h|_{L^{\infty}_T \tilde{H}^{m-\frac{1}{2}}} \lesssim \tilde{\mathcal{E}}_{m,T}. \tag{7.12.2}$$

Further, thanks to Sobolev embedding (7.3.16), the last four terms can be controlled by the ones appearing in $\mathcal{E}_{m,T}$. Indeed,

$$\varepsilon^{\frac{1}{2}}\|\partial_t(\sigma,u)\|_{m-5,\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac{1}{2}}\partial_t u(s)\|_{H^{m-3}_{co}} + \|\varepsilon^{\frac{1}{2}}\partial_t\nabla u(s)\|_{H^{m-4}_{co}}\big) \lesssim \tilde{\mathcal{E}}_{m,T}, \tag{7.12.3}$$

$$\varepsilon^{\frac{1}{2}}\|\nabla u(s)\|_{m-3,\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac{1}{2}}\nabla u(s)\|_{H^{m-1}_{co}} + \|\varepsilon^{\frac{1}{2}}\nabla^2 u(s)\|_{H^{m-2}_{co}}\big) \lesssim \tilde{\mathcal{E}}_{m,T},$$

$$\|(\sigma,u)\|_{m-4,\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|(\sigma,u)(s)\|_{H^{m-1}_{co}} + \|\nabla(\sigma,u)(s)\|_{H^{m-4}_{co}}\big) \lesssim \tilde{\mathcal{E}}_{m,T}, \tag{7.12.4}$$

$$\|\varepsilon\partial_t(\sigma,u)\|_{m-4,\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma,u)(s)\|_{H^{m-2}_{co}} + \varepsilon^{\frac{1}{2}}\|\varepsilon\partial_t\nabla(\sigma,u)(s)\|_{H^{m-3}_{co}}\big) \lesssim \tilde{\mathcal{E}}_{m,T},$$

$$\varepsilon^{\frac{1}{2}}\|(\sigma,u)\|_{m-2,\infty,T} \lesssim \sup_{0\leq s\leq T}\big(\|(\sigma,u)(s)\|_{H^{m}_{co}} + \|\varepsilon^{\frac{1}{2}}\nabla(\sigma,u)(s)\|_{H^{m-1}_{co}}\big) \lesssim \tilde{\mathcal{E}}_{m,T}. \tag{7.12.5}$$

For the first term, we can use the equation for $\sigma$ to get that:

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|\mathrm{div}^{\varphi} u\|_{m-5,\infty,T} &\lesssim \|\varepsilon^{\frac{1}{2}}\partial_t\sigma\|_{m-5,\infty,T} + \varepsilon^{\frac{1}{2}}\big(\|(u,\varepsilon\partial_t\sigma,\nabla\sigma\|_{m-5,\infty,T} + |h|_{m-4,\infty,T}\big)^2 \\ &\lesssim \tilde{\mathcal{E}}_{m,T} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).\end{aligned} \tag{7.12.6}$$

Moreover, in view of (7.12.2) (7.12.4) and identity $\Pi\nabla^{\varphi} = \Pi(\partial_1,\partial_2,0)^t$,

$$\begin{aligned}\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{m-5,\infty,T} &\lesssim \varepsilon^{-\frac{1}{2}}\|\partial_y\sigma\|_{m-5,\infty,T}(1+|h|_{m-4,\infty,T})^2 + \varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\cdot\mathbf{n}\|_{m-5,\infty,T}|h|_{m-4,\infty,T} \\ &\lesssim \tilde{\mathcal{E}}_{m,T} + \tilde{\mathcal{E}}^3_{m,T} + \|\nabla^{\varphi}\sigma\cdot\mathbf{n}\|^2_{[\frac{m}{2}]-1,\infty,T},\end{aligned}$$

By the Sobolev embedding (7.3.16),

$$\varepsilon^{-\frac{1}{2}}\|\partial_y\sigma\|_{m-5,\infty,T} \lesssim \varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\|_{L^{\infty}_t H^{m-3}_{co}} \lesssim \tilde{\mathcal{E}}_{m,T}.$$

Therefore, it remains to control $\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\cdot\mathbf{n}\|_{[\frac{m}{2}]-1,\infty,T}$, which is the aim of the following lemma. □

**Lemma 7.12.3.** *Suppose that* (7.2.2) *holds, then:*

$$\varepsilon^{-\frac{1}{2}}\|\nabla^{\varphi}\sigma\cdot\boldsymbol{n}\|_{[\frac{m}{2}]-1,\infty,T} \lesssim Y^2_m(0) + \tilde{\mathcal{E}}^2_{m,T} + \varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big). \tag{7.12.7}$$

*Proof.* By (7.8.21), we find that $\nabla^{\varphi}\sigma$ is governed by

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot\nabla)\nabla^{\varphi}\sigma + \frac{1}{2\mu+\lambda}\nabla^{\varphi}\sigma = \mathcal{Q}_1 \tag{7.12.8}$$

where $\mathcal{Q}_1 = \mathcal{Q}_{11} + \mathcal{Q}_{12} + \mathcal{Q}_{13}$, with

$$\mathcal{Q}_{11} = -\varepsilon^2 g_1' \nabla^\varphi \sigma (\varepsilon\partial_t + \varepsilon \underline{u}\cdot\nabla)\sigma - \varepsilon^2 g_1 \nabla^\varphi u \cdot \nabla^\varphi \sigma,$$

$$\mathcal{Q}_{12} = -\frac{\mu\varepsilon}{2\mu+\lambda}\operatorname{curl}^\varphi \omega, \quad \mathcal{Q}_{13} = -\frac{1}{2\mu+\lambda} g_2(\varepsilon\partial_t + \varepsilon\underline{u}\cdot\nabla)u.$$

Denote $R = \nabla^\varphi \sigma \cdot \mathbf{n}$, then by (7.8.21), $R$ solves:

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot)R + \frac{1}{2\mu+\lambda}R = \varepsilon^2 g_1 \nabla^\varphi\sigma(\partial_t + \underline{u}\cdot)\mathbf{n} + \mathcal{Q}_1\cdot\mathbf{n} =: \mathcal{Q}_2 + \mathcal{Q}_1\cdot\mathbf{n}$$

For any multi-index with $|\beta| \le m-5$, denote $R^\beta = Z^\beta R$, then $R^\beta$ satisfies:

$$\varepsilon^2 g_1(\partial_t + \underline{u}\cdot)R^\beta + \frac{1}{2\mu+\lambda}R^\beta = Z^\beta(\mathcal{Q}_2 + \mathcal{Q}_1\cdot\mathbf{n}) + \mathcal{C}^\beta_{R,1} + \mathcal{C}^\beta_{R,2} =: \mathcal{Q}^\beta,$$

where

$$\mathcal{C}^\beta_{R,1} = -\varepsilon^2[Z^\beta, g_1/\varepsilon]\varepsilon\partial_t R, \quad \mathcal{C}^\beta_{R,2} = -\varepsilon^2[Z^\beta, g_1\underline{u}\cdot\nabla]R.$$

Define $X_t(x) = X(t,x)$ the unique flow associated to $\underline{u}$ :

$$\partial_t X(t,x) = \underline{u}(t, X(t,x)), \quad X(0,x) = x.$$

Note that since $\underline{u}\cdot\mathbf{n}|_{z=0} = 0$, and $u \in \mathrm{Lip}\,([0,T]\times\Omega)$, we have for each $t\in[0,T]$, $X_t : \mathcal{S}\to\mathcal{S}$ is a diffeomorphism. Denote $f^X = f(t, X(t,x))$, then $R^{\beta,X}$ is governed by:

$$\varepsilon^2(g_1\partial_t R^\beta)(t, X_t(x)) + \frac{1}{2\mu+\lambda}R^\beta(t, X_t(x)) = Q^\beta(t, X_t(x))$$

from which, we deduce that:

$$R^\beta(t, X_t(x)) = e^{-\int_0^t \frac{1}{\varepsilon^2 g_1(s, X_s(x))}\mathrm{d}s} R^\beta(0) + \int_0^t e^{-\int_\tau^t \frac{1}{\varepsilon^2 g_1(x, X_s(x))}\mathrm{d}s}\frac{1}{\varepsilon^2}Q^\beta(\tau, X_\tau(x))\mathrm{d}\tau.$$

By assumption (7.2.2), $c_0 \le g_1(t, X_t(x)) \le \frac{1}{c_0}$ for any $(t,x)\in[0,T]\times\mathcal{S}$. Therefore,

$$\begin{aligned}
\varepsilon^{-\frac12}\|R^\beta\|_{0,\infty,T} &\lesssim \varepsilon^{-\frac12}\sup_{(t,x)\in[0,T]\times\mathcal{S}}|R^\beta(t, X_t(x))| \\
&\lesssim \varepsilon^{-\frac12}\|R^\beta(0)\|_{L^\infty(\mathcal{S})} + \varepsilon^{-\frac12}\int_0^T e^{-c_0(t-s)/\varepsilon^2}\frac{1}{\varepsilon^2}\mathrm{d}s\|Q^\beta\|_{0,\infty,T} \\
&\lesssim Y_m(0) + \varepsilon^{-\frac12}\|Q^\beta\|_{0,\infty,T}.
\end{aligned} \tag{7.12.9}$$

It thus suffices to control the term $\varepsilon^{-\frac12}\|Q^\beta\|_{0,\infty,T}$. First of all, by property (7.2.1), we get that:

$$\varepsilon^{-\frac12}\|\mathcal{C}^\beta_{R,1}\|_{0,\infty,T} \lesssim \varepsilon^{\frac32}\big(\|(\sigma,\nabla\sigma)\|_{m-5,\infty,T} + |h|_{m-4,\infty,T}\big)^2 \lesssim \varepsilon^{\frac32}\mathcal{A}^2_{m,T}. \tag{7.12.10}$$

Next, by using $\underline{u}\cdot\nabla = u_y\partial_y + \frac{U_z}{\phi}Z_3 R$, we can control the second commutator term as:

$$\varepsilon^{\frac32}\|\mathcal{C}^\beta_{R,2}\|_{0,\infty,T} \lesssim \varepsilon\big(\|(\sigma, u, \nabla\sigma, \varepsilon^{\frac12}\nabla u)\|_{m-5,\infty,T} + |h|_{m+3,\infty,T}\big)^2 \lesssim \varepsilon\mathcal{A}^2_{m,T}. \tag{7.12.11}$$

Similarly, we can find some polynomial $\Lambda$, such that

$$\begin{aligned}
\varepsilon^{-\frac12}\|Z^\beta(\mathcal{Q}_2 + \mathcal{Q}_{11}\cdot\mathbf{n})\|_{0,\infty,T} &\lesssim \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \|(\sigma, u, \nabla\sigma, \varepsilon^{\frac12}\nabla u)\|_{[\frac{m}{2}]-1,\infty,T} + |h|_{[\frac{m}{2}]+1,\infty,T}\big) \\
&\lesssim \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
\end{aligned} \tag{7.12.12}$$

Moreover, in light of (7.12.2) and (7.12.4), we have by a possible change of $\Lambda$,

$$\begin{aligned}
\varepsilon^{-\frac12}\|Z^\beta(\mathcal{Q}_{13}\cdot\mathbf{n})\|_{0,\infty,T} &\lesssim \|(\varepsilon^{\frac12}\partial_t u\cdot\mathbf{n}, \varepsilon^{\frac12}\underline{u}\cdot\nabla u)\|_{m-5,\infty,T} + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big) \\
&\lesssim \tilde{\mathcal{E}}^2_{m,T} + \varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big).
\end{aligned} \tag{7.12.13}$$

Finally, since

$$\begin{aligned}\operatorname{curl}^\varphi \omega\cdot\mathbf{n} &= \operatorname{div}^\varphi(\omega\times\mathbf{n})+\omega\cdot\operatorname{curl}^\varphi\mathbf{n}\\ &= -(\omega\times\mathbf{n})\cdot\partial_z^\varphi\mathbf{N}+\partial_1(\omega\times\mathbf{n})_1+\partial_2(\omega\times\mathbf{n})_2+\omega\cdot\operatorname{curl}^\varphi\mathbf{n}\end{aligned}$$

involves only tangential derivatives of $\nabla^\varphi u$, one has again by (7.12.2) and (7.12.4) that:

$$\varepsilon^{-\frac12}\|Z^\beta(\mathcal{Q}_{12}\cdot\mathbf{n})\|_{0,\infty,T}\lesssim\big(\|\varepsilon^{\frac12}\nabla u\|_{m-4,\infty,T}+|h|_{m-3,\infty,T}\big)^2\lesssim\tilde{\mathcal{E}}^2_{m,T}. \tag{7.12.14}$$

Collecting (7.12.10)-(7.12.14), we find that:

$$\|\mathcal{Q}\|_{[\frac{m}{2}]-1,\infty,T}\lesssim\tilde{\mathcal{E}}^2_{m,T}+\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big).$$

Inserting this inequality into (7.12.9), we eventually get (7.12.7). □

The following Lemma consists in the $L^\infty_{t,x}$ estimates of $\nabla u$, namely $\|\varepsilon^{\frac12}\partial_t\nabla u\|_{0,\infty,t}$, $\|\nabla u\|_{1,\infty,t}$.

**Lemma 7.12.4.** *Assume that* (7.2.2) *holds, then we have that for any* $0<t\leq T$,

$$\|\varepsilon^{\frac12}\partial_t\nabla u\|_{0,\infty,t}+\|\nabla u\|_{1,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big)+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,T}+(T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.12.15}$$

*Proof.* In view of identities (7.4.5) and

$$\begin{aligned}\Pi(\partial_z^\varphi u)&=\frac{1}{|\mathbf{N}|}\Pi\big(\omega\times\mathbf{N}+(\nabla^\varphi u)^t\cdot\mathbf{n}-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u\big)\\ &=\frac{1}{|\mathbf{N}|}(\omega\times\mathbf{N})+\Pi\nabla^\varphi(u\cdot\mathbf{n})-\Pi\big((\nabla^\varphi\mathbf{n})^t u-\mathbf{n}_1\partial_1 u-\mathbf{n}_2\partial_2 u\big)\end{aligned}$$

one gets that:

$$\begin{aligned}&\|\nabla u\|_{1,\infty,t}+\varepsilon^{\frac12}\|\partial_t\nabla u\|_{0,\infty,t}\lesssim\varepsilon^{\frac12}\Lambda\big(\frac{1}{c_0},\mathcal{A}_{m,T}\big)\\ &+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\big(\|u\|_{2,\infty,t}+\|\varepsilon^{\frac12}\partial_t u\|_{1,\infty,t}+\|\varepsilon^{-\frac12}\operatorname{div}^\varphi u\|_{1,\infty,t}+\|\omega\|_{1,\infty,t}+\|\varepsilon^{\frac12}\partial_t\omega\|_{0,\infty,t}\big).\end{aligned}$$

The inequality (7.12.15) then follows from (7.12.3), (7.12.4), (7.12.6) and the next lemma for the estimates of $\omega$. □

**Lemma 7.12.5.** *Under the same assumption as in Lemma* (7.12.4),

$$\|\omega\|_{1,\infty,t}+\|\varepsilon^{\frac12}\partial_t\omega\|_{0,\infty,t}\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big)+\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,T}+(T+\varepsilon)^{\frac14}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big). \tag{7.12.16}$$

*Proof.* Away from the boundary, the conormal spaces are equivalent to the usual Sobolev space, the $L^\infty_{t,x}$ norm of $\omega$ can be obtained directly from the usual Sobolev embedding. It thus suffices establish corresponding estimates near the boundaries. In what follows, we will detail their estimates near the upper boundary (or surface), the case near the bottom being easier. As in the proof of Lemma 7.11.4, we will employ the normal geodesic coordinates (7.11.15) to take the benefit of the explicit formulae of the heat equation in the half line. Taking the same cut off function $\chi=\chi_0(\frac{z}{C(\kappa)})$ introduced there (which satisfies $\Phi_t(\operatorname{Supp}\chi)\Subset\tilde{\Phi}_t(\mathcal{S}_k)$), we use the equation (7.11.17) to obtain that:

$$(\bar\rho\partial_t-\mu\Delta^\varphi)(\chi\omega)=\chi G^\omega-\mu\Delta^\varphi\chi\omega-\mu\partial_z\chi(\mathbf{N}\cdot\nabla^\varphi)\omega=:G^{\chi,\omega}$$

where

$$G^\omega=-u\cdot\nabla^\varphi\omega+\omega\cdot\nabla^\varphi u-\omega\operatorname{div}^\varphi u-\frac{\nabla g_2}{\varepsilon}\times((\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)u)+\frac{\bar\rho-g_2}{\varepsilon}((\varepsilon\partial_t+\varepsilon\underline{u}\cdot\nabla)\omega).$$

For a function $f(t,\cdot)$ supported on $\mathbb{R}^2\times[-C(\kappa),0]$, we use the notation

$$\tilde{f}(t,x)=f(t,\Phi_t^{-1}\circ\tilde{\Phi}_t(x)).$$

By the change of the variable, we find that $\widetilde{\chi\omega}$ satisfies the system:

$$\begin{aligned}(\bar{\rho}\partial_t-\mu\partial_z^2)\widetilde{\chi\omega}=\widetilde{F^{\chi,\omega}}=:\widetilde{G^{\chi,\omega}}+\bar{\rho}(D\tilde{\Phi}_t)^{-1}\partial_t\tilde{\Phi}_t\cdot\nabla\widetilde{\chi\omega}\\ +\mu\big[\frac{1}{2}\partial_z(\ln|g|)\partial_z+\partial_i(\ln|g|)\tilde{g^{ij}}\partial_j+\partial_i(\tilde{g}^{ij}\partial_j\cdot)\big](\widetilde{\chi\omega})\end{aligned}\tag{7.12.17}$$

supplemented with the initial and the boundary conditions:

$$\widetilde{\chi\omega}|_{t=0}=\chi\omega|_{t=0}(\Phi_0^{-1}\circ\tilde{\Phi}_0),\qquad \widetilde{\chi\omega}|_{z=0}=\omega|_{z=0}=:\omega^{b,1}.\tag{7.12.18}$$

Let

$$E(t,z,z')=\tilde{\mu}\frac{1}{(4\pi\tilde{\mu}t)^{\frac{1}{2}}}\big(e^{-\frac{|z-z'|^2}{4\tilde{\mu}t}}-e^{-\frac{|z+z'|^2}{4\tilde{\mu}t}}\big),\qquad \tilde{\mu}=\bar{\rho}/\mu.$$

the solution to the system (7.12.17)-(7.12.18) can be expressed explicitly:

$$\begin{aligned}\widetilde{\chi\omega}(t,y,z)=-\int_0^t(\partial_{z'}E)(t-s,z,0)\omega^{b,1}(s,y)\,\mathrm{d}s+\int_{-\infty}^0E(t,z,z')\widetilde{\chi\omega}|_{t=0}(y,z')\,\mathrm{d}z'\\ +\int_0^t\int_{-\infty}^0E(t-s,z,z')\widetilde{F^{\chi,\omega}}(s,y,z')\,\mathrm{d}z'\mathrm{d}s=(1)+(2)+(3).\end{aligned}\tag{7.12.19}$$

Control of the boundary term (1). Analogues to (7.11.31) and (7.11.32), we can bound the boundary term as:

$$\|\!|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y,Z_3)(1)\|\!|_{0,\infty,t}\leq C(\mu)\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)\omega^{b,1}\|_{L^\infty_{t,y}}$$

By (7.4.4), (7.4.3) on the boundary,

$$\omega^{b,1}\approx F(u^{b,1},\partial_yu^{b,1},(\mathrm{div}^\varphi u)^{b,1},\mathbf{n}^{b,1},\nabla\mathbf{n}^{b,1}),$$

which, together with the previous inequality, yields that:

$$\begin{aligned}&\|\!|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y,Z_3)(1)\|\!|_{0,\infty,t}\\ &\lesssim\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\big(\|\!|\varepsilon^{\frac{1}{2}}\partial_tu\|\!|_{1,\infty,t}+\|\!|\varepsilon^{-\frac{1}{2}}\mathrm{div}^\varphi u\|\!|_{1,\infty,t}+\|\!|u\|\!|_{2,\infty,t}\big)+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\\ &\lesssim\Lambda\big(\frac{1}{c_0},|h|_{3,\infty,t}\big)\widetilde{\mathcal{E}}_{m,t}+\varepsilon^{\frac{1}{2}}\Lambda\big(\frac{1}{c_0},\mathcal{N}_{m,T}\big).\end{aligned}\tag{7.12.20}$$

Control of the initial evolution (2). Since $\partial_t,\partial_y$ commute with the operator $\bar{\rho}\partial_t-\mu\partial_z^2$, the following identity holds:

$$(\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)(2)=\int_{-\infty}^0E(t,z,z')(\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)(\widetilde{\chi\omega})|_{t=0}(y,z')\,\mathrm{d}z',$$

from which we derive that:

$$\begin{aligned}&\|\!|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)(2)\|\!|_{0,\infty,t}\lesssim\big\|\int_{-\infty}^0|E(t,z,z')|\mathrm{d}z'\big\|_{L^\infty_tL^\infty_z}\|(\mathrm{Id},\varepsilon^{\frac{1}{2}}\partial_t,\partial_y)\widetilde{\chi\omega})|_{t=0}\|_{L^\infty(\mathcal{S}_\kappa)}\\ &\lesssim\Lambda\big(\frac{1}{c_0},|h_0|_{2,\infty}+|\varepsilon^{\frac{1}{2}}\partial_th|_{t=0}|_{1,\infty}\big)\big(\|(\varepsilon^{\frac{1}{2}}\partial_t\omega)|_{t=0}\|_{L^\infty(\mathcal{S})}+\|(\mathrm{Id},\partial_y,Z_3)\omega_0\|_{L^\infty(\mathcal{S})}\big)\\ &\lesssim\Lambda\big(\frac{1}{c_0},Y_m(0)\big).\end{aligned}\tag{7.12.21}$$

To control $Z_3(2)$, we denote $E_\pm(t,z,z')=\tilde{\mu}\frac{1}{(4\pi\tilde{\mu}t)^{\frac{1}{2}}}e^{-\frac{|z\pm z'|^2}{4\tilde{\mu}t}}$. By writing $z=z-z'+z'$ or $z=z+z'-z'$, one can split $Z_3(2)$ into two terms:

$$Z_3(2)=\int_{-\infty}^0\phi(z)\partial_z(E_--E_+)(t,z,z')(\widetilde{\chi\omega})|_{t=0}\,\mathrm{d}z'=(Z_3(2))_1+(Z_3(2))_2$$

with

$$
\begin{aligned}
(Z_3(2))_1 &= \phi_1(z)\int_{-\infty}^0 \big((z-z')\partial_z E_- - (z+z')\partial_z E_+\big)(t,z,z')(\widetilde{\chi\omega})|_{t=0}\,\mathrm{d}z',\\
(Z_3(2))_2 &= \phi_1(z)\int_{-\infty}^0 E(t,z,z')\partial_{z'}(z'(\widetilde{\chi\omega})|_{t=0})\,\mathrm{d}z',
\end{aligned}
$$

where we use the notation $\phi(z) = \frac{z(1-z)}{(2-z)^2} = z\phi_1(z)$. By the straightforward calculation,

$$
\big|\phi_1(z)\int_{-\infty}^0 \big((z-z')\partial_z E_- - (z+z')\partial_z E_+\big)(t,z,z')\mathrm{d}z'\big| \le C(\tilde{\mu})
$$

where $C(\tilde{\mu})$ is a constant depending only on $\tilde{\mu}$ ( in particular, independent of $z$ and $t$). The first term can thus be bounded as:

$$
\|\!|(Z_3(2))_1\|\!|_{0,\infty,t} \lesssim \|(\widetilde{\chi\omega})|_{t=0}\|_{L^\infty(\mathcal{S}_\kappa)} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big).
$$

Next, by writing

$$
\partial_{z'}(z'(\widetilde{\chi\omega})|_{t=0}) = (\widetilde{\chi\omega})|_{t=0} + \frac{1}{\phi_1(z)} Z_3(\widetilde{\chi\omega})|_{t=0},
$$

and noting that $\phi_1(z)$ has uniform lower bound in $[-\kappa, 0]$, we control the second term as:

$$
\|\!|(Z_3(2))_2\|\!|_{0,\infty,t} \lesssim \|(\mathrm{Id}, Z_3)(\widetilde{\chi\omega})|_{t=0}\|_{L^\infty(\mathcal{S}_\kappa)} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big).
$$

To summarize, we obtain that

$$
\|\!|Z_3(2)\|\!|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big),
$$

which, together with (7.12.21), yields that:

$$
\|\!|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3)(3)\|\!|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big). \tag{7.12.22}
$$

Control of the nonlinear term (3). We need to distinguish the terms appearing in $\widetilde{F^{\chi,\omega}}$ that involves one normal derivative of the vorticity and the others. Therefore, let us denote

$$
\widetilde{F^{\chi,\omega}} = \bar{\rho}\chi\widetilde{u\cdot\nabla^\varphi}\omega + \bar{\rho}\partial_t\tilde{\Phi}_t\cdot\nabla(\widetilde{\chi\omega}) - \mu\partial_z\chi\widetilde{\mathbf{N}\cdot\nabla^\varphi}\omega + \frac{1}{2}\mu\partial_z(\ln|g|)\partial_z\widetilde{\chi\omega} + R. \tag{7.12.23}
$$

where the remainder term $R$ satisfy the estimate

$$
\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\partial_t R\|_{L_t^2 H_{co}^2} + \|R\|_{L_t^2 H_{co}^3} &\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{A}_{m,T}\big)(\|\varepsilon^{\frac{1}{2}}\partial_t(\sigma, u, \nabla u)\|_{L_t^2 H_{co}^4} + \|(\sigma, u, \nabla u)\|_{L_t^2 H_{co}^5})\\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

By using the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L_y^\infty(\mathbb{R}^2)$ we can deal with the term $\int_0^t\int_{-\infty}^0 E(t-s,z,z')R(s,y,z')\,\mathrm{d}z'\mathrm{d}s$ as follows:

$$
\begin{aligned}
&\int_0^t\int_{-\infty}^0 E(t-s,z,z')(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y)R(s,y,z')\,\mathrm{d}z'\mathrm{d}s\\
&\lesssim \big(\int_0^t\int_{-\infty}^0 |E(t-s,z,z')|^2\mathrm{d}z'\mathrm{d}s\big)^{\frac{1}{2}}\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y)R\|_{L_t^2 L_{z'}^2, L_y^\infty}\\
&\lesssim \big(\int_0^t (t-s)^{-\frac{1}{2}}\mathrm{d}s\big)^{\frac{1}{2}}\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y)R\|_{L_t^2 H_{co}^2} \lesssim T^{\frac{1}{4}}\Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Moreover, similar to the control of $Z_3(2)$, we have that:

$$
\begin{aligned}
&Z_3 \int_0^t \int_{-\infty}^0 E(t-s,z,z')R(s,y,z')\,\mathrm{d}z'\mathrm{d}s \lesssim \|(\mathrm{Id}, Z_3)R\|_{L_t^2 H_{co}^2} \cdot \\
&\Big( \big( \int_0^t \int_{-\infty}^0 |E|^2 \mathrm{d}z'\mathrm{d}s\big)^{\frac{1}{2}} + \big( \int_0^t \int_{-\infty}^0 (|(z-z')\partial_z E_-|^2 + |(z+z')\partial_z E_+|^2)\mathrm{d}z'\mathrm{d}s\big)^{\frac{1}{2}} \Big) \\
&\lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.12.24}
$$

We are left to treat the first four terms appearing in (7.12.23), for which we need to integrate by parts in order not to lose normal derivative. Let us explain the estimate for term:

$$
\bar{\rho} \int_0^t \int_{-\infty}^0 E(t-s,z,z')\widetilde{\chi u \cdot \nabla^{\varphi}\omega}\,\mathrm{d}z'\mathrm{d}s
$$

By the change of variable, we find that

$$
\begin{aligned}
\widetilde{\chi u \cdot \nabla^{\varphi}\omega} &= \widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{jk} \partial_j (\widetilde{\chi_1 \omega}) + \widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k} \partial_z (\widetilde{\chi_1 \omega}) \\
&= \widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{jk} \partial_j (\widetilde{\chi_1 \omega}) - \partial_z (\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k}) \widetilde{\chi_1 \omega} + \partial_z (\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k} \widetilde{\chi_1 \omega})
\end{aligned}
$$

where $\chi_1$ is a cut-off function supported on $[-C(\kappa), 0]$ that satisfies $\chi_1 \chi = \chi$. The Einstein summation convention is used for $j = 1,2, k = 1,2,3$. As the first two terms in right hand side of the above identity does not involve normal derivative of $(\widetilde{\chi_1 \omega})$, we have by following the same procedure as in the estimate of $R$ that:

$$
\bar{\rho} \int_0^t \int_{-\infty}^0 E(t-s,z,z') \big( \widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{jk} \partial_j (\widetilde{\chi_1 \omega}) - \partial_z (\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k}) \widetilde{\chi_1 \omega} \big)\,\mathrm{d}z'\mathrm{d}s \lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
$$

For the last one, we integrate by parts in $z'$ to get that:

$$
\begin{aligned}
&\bar{\rho} \int_0^t \int_{-\infty}^0 E(t-s,z,z') \partial_{z'} (\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k} \widetilde{\chi_1 \omega}) \mathrm{d}z'\mathrm{d}s \\
&\lesssim \int_0^t \|\partial_{z'} E(t-s,z,\cdot)\|_{L_{z'}^2} \mathrm{d}s \, \|\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k} \widetilde{\chi_1 \omega}\|_{L_t^\infty L_{z'}^2 L_y^\infty} \\
&\lesssim T^{\frac{1}{4}} \|\widetilde{\chi u}_k (\mathrm{D}\tilde{\Phi})_{3k} \widetilde{\chi_1 \omega}\|_{L_t^\infty H_{co}^2} \lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned}
$$

Apart from the above two inequalities, we have also analogues to (7.12.24) that:

$$
\begin{aligned}
&\bar{\rho}(\varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3) \int_0^t \int_{-\infty}^0 E(t-s,z,z') \widetilde{\chi u \cdot \nabla^{\varphi}\omega}\,\mathrm{d}z'\mathrm{d}s \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \int_0^t \|\partial_{z'} (E(t-s,z\cdot), \, (z-z')\partial_z E_-, \, (z+z')\partial_z E_+)\|_{L_{z'}^2} \mathrm{d}s \\
&\lesssim \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big) \int_0^t (t-s)^{-\frac{3}{4}} \mathrm{d}s \lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big).
\end{aligned} \tag{7.12.25}
$$

We thus finish the estimate of term $\int_0^t \int_{-\infty}^0 E(t-s,z,z')\widetilde{\chi u \cdot \nabla^{\varphi}\omega}\,\mathrm{d}z'\mathrm{d}s$. The other three in (7.12.23) can be dealt with in the same way. Consequently, we find that for any $t \in (0,T], z < 0$,

$$
\int_0^t \int_{-\infty}^0 E(t-s,z,z') \widetilde{F^{\chi,\omega}} \mathrm{d}z'\mathrm{d}s \lesssim T^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big). \tag{7.12.26}
$$

Collecting (7.12.20), (7.12.22) and (7.12.26), we find that:

$$
\|(\mathrm{Id}, \varepsilon^{\frac{1}{2}}\partial_t, \partial_y, Z_3)(\widetilde{\chi\omega})\|_{0,\infty,t} \lesssim \Lambda\big(\frac{1}{c_0}, Y_m(0)\big) + \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\tilde{\mathcal{E}}_{m,t} + (T+\varepsilon)^{\frac{1}{4}} \Lambda\big(\frac{1}{c_0}, \mathcal{N}_{m,T}\big),
$$

which, by the property (7.11.22), leads to (7.12.16). $\square$

## 7.13 Proof of Theorem 7.1.1.

This section is devoted to the proof of Theorem 7.1.1 which is based on the known local existence results (non-uniform with respect to $\varepsilon$) and the uniform estimates established in the previous sections. The local existence in the Sobolev-Slobodeskii spaces is established $H^{4,2}$ (defined later in the statement of the following theorem) in [111] [132] (see also [122] the local existence in Hölder spaces). By following the similar arguments as in these papers, we have the following theorem which is analogs to Theorem B of [111] or Theorem 6.2 in [132].

**Theorem 7.13.1.** *Assume that the first order compatibility condition* ((7.1.27) *with* $j = 1$) *holds and*

$$(\sigma_0^\varepsilon, u_0^\varepsilon) \in (H^3(\mathcal{S}))^4, \quad h_0^\varepsilon \in H^{\frac{7}{2}}(\mathbb{R}^2), \quad 1 + h_0^\varepsilon \geq 3c_0 > 0,$$

$\delta$ *is chosen sufficiently small such that*

$$\partial_z \varphi_0^\varepsilon(x) = 1 + \partial_z \eta_0^\varepsilon(1+z) + \eta_0^\varepsilon \geq 2c_0 > 0, \forall x \in \mathcal{S},$$

*where* $\eta_0^\varepsilon$ *is the extension of* $h_0^\varepsilon$ *defined in* (7.1.12). *Then for any* $\varepsilon \leq 1$, *we can find* $T^\varepsilon > 0$ *(which is generally very small) such that:*

$$(\sigma^\varepsilon, u^\varepsilon) \in C([0, T^\varepsilon], H^3(\mathcal{S})), \quad h^\varepsilon \in C([0, T^\varepsilon], H^{\frac{7}{2}}(\mathbb{R}^2)).$$

*Moreover,* $u^\varepsilon \in H^{4,2}([0, T^\varepsilon] \times \mathcal{S}) = \{u \big| \partial_t^j u \in L^2\big([0, T^\varepsilon], H^{4-2j}(\mathcal{S})\big), j = 0, 1, 2\}$ *and* (7.2.2) *holds.*

We shall combine this theorem with the uniform regularity estimates established in the previous sections. Set

$$T_*^\varepsilon = \sup\big\{T | (\sigma^\varepsilon, u^\varepsilon) \in C([0, T], H^3(S)), u^\varepsilon \in W^{4,2}([0, T^\varepsilon] \times \mathcal{S}) \text{ and (7.2.2) holds}\big\}.$$

Since the initial datum is assumed to belong to $Y_m^\varepsilon$, the space with higher regularity, by standard propagation of regularity arguments (for example based on applying finite difference instead of derivatives) and the computations presented in Section 6-Section 12, we get that for $T < \min\{T_*^\varepsilon, 1\}$, the following uniform estimate analogues to Theorem 7.2.1 holds:

$$\mathcal{N}_{m,T}^\varepsilon \leq P_1\big(\frac{1}{c_0}, Y_m^\varepsilon(0)\big) + (T + \varepsilon)^\vartheta P_2\big(\frac{1}{c_0}, Y_m^\varepsilon(0) + \mathcal{N}_{m,T}^\varepsilon\big). \tag{7.13.1}$$

where $0 < \vartheta < 1$ and $P_1, P_2$ are two continuous functions that are independent of $\varepsilon$. By the fundamental theorem of calculus and Lemma 7.3.8, one finds for $0 \leq t \leq T$

$$\begin{aligned} \partial_z \varphi(t, x) &= \partial_z \varphi(0, x) + \int_0^t (\partial_t \eta + (1+z)\partial_t \partial_z \eta)(s, x)\, \mathrm{d}s \\ &\geq \partial_z \varphi(0, x) - C_1 T |\partial_t h(t)|_{L^\infty(\mathbb{R}^2)}, \end{aligned} \tag{7.13.2}$$

$$\|(\nabla\varphi, \nabla^2\varphi)(t)\|_{L^\infty(\mathcal{S})} \leq \|(\nabla\varphi, \nabla^2\varphi)(0)\|_{L^\infty(\mathcal{S})} + C_2 T |h(t)|_{W^{2,\infty}(\mathbb{R}^2)}. \tag{7.13.3}$$

where $C_1, C_2$ are two constants independent of $\varepsilon$. Moreover, $\varepsilon\sigma^\varepsilon$ can be expanded by using the characteristic method:

$$\varepsilon\sigma^\varepsilon(t, x) = \varepsilon\sigma_0^\varepsilon(X^{-1}(t, x)) - \int_0^t (\mathrm{div} u^\varepsilon / g_1)(X(s, X^{-1}(t, x)))\mathrm{d}s \tag{7.13.4}$$

where $X(t, x)$ is the unique flow associated to $\underline{u}$. Let us define

$$T_*^\varepsilon = \sup\{T \big| (\sigma^\varepsilon, u^\varepsilon) \in C([0, T], H^3), u^\varepsilon \in W^{4,2}([0, T] \times \mathcal{S})\},$$

$$\begin{aligned} T_0^\varepsilon = \sup\{T \leq \min\{T_*^\varepsilon, 1\} \big| &\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \leq 2P_1\big(1/c_0, M\big) \\ &- 2\bar{c}\bar{P} \leq \varepsilon\sigma^\varepsilon(t, x) \leq 2\bar{P}/\bar{c} \quad \forall (t, x) \in [0, T] \times \mathcal{S}\}. \end{aligned}$$

where $M \geq \sup_{\varepsilon \in (0,1]} Y_m(\sigma_0^\varepsilon, u_0^\varepsilon)$.

We now choose successively two constants $0 < \varepsilon_0 \le 1$ and $0 < T_0 \le 1$ (uniform in $\varepsilon \in (0, \varepsilon_0]$) which are small enough, such that:

$$(T_0 + \varepsilon_0)^{\vartheta} P_2\big(1/c_0, M + 2P_1(1/c_0, M)\big) < \frac{1}{2} P_1(1/c_0, M) \quad 2P_1(1/c_0, M)T_0/c_0\big) \le \bar{c}\bar{P}.$$

In order to prove Theorem 7.1.1, it suffices to show that $T_0^\varepsilon \ge T_0$ for every $0 < \varepsilon \le \varepsilon_0$. Suppose otherwise $T_0^\varepsilon < T_0$ for some $0 < \varepsilon \le \varepsilon_0$, then in view of inequalities (7.13.1) and formula (7.13.4), we have by the definition of $\varepsilon_0$ and $T_0$ that:

$$\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) \le \frac{3}{2} P_1(1/c_0, M) \qquad \forall T \le \tilde{T} = \min\{T_0, T_*^\varepsilon\}, \tag{7.13.5}$$

$$-2\bar{c}\bar{P} \le \varepsilon \sigma^\varepsilon(t, x) \le 2\bar{P}/\bar{c} \quad \forall (t, x) \in [0, \tilde{T}] \times \Omega. \tag{7.13.6}$$

We will prove that $\tilde{T} = T_0 \le T_*^\varepsilon$. This fact, combined with the definition of $T_0^\varepsilon$ and estimates (7.13.5), (7.13.6), yields $T_0^\varepsilon \ge T_0$, which is a contradiction with the assumption $T_0^\varepsilon < T_0$. To continue, we shall need the claim stated and proved below. Indeed, once the following claim holds, we have by (7.13.5) that $\|(\sigma^\varepsilon, u^\varepsilon)(T_0)\|_{H^3(\Omega)} < +\infty$, which, combined with the local existence result stated in Theorem 7.13.1, yields that $T_*^\varepsilon > T_0$.

**Claim.** For all $\varepsilon \in (0, 1]$, if $\mathcal{N}_{m,T}(\sigma^\varepsilon, u^\varepsilon) < +\infty$, then $(\sigma^\varepsilon, u^\varepsilon) \in C([0, T], H^3)$, $u^\varepsilon \in H^{4,2}([0, T] \times \mathcal{S})$.

*Proof of claim.* By the definition of $\mathcal{N}_{m,T}$, we derive that:

$$\varepsilon^{\frac{3}{2}} u^\varepsilon \in L^2([0, T], H^4), \quad \varepsilon^{\frac{3}{2}} \partial_t u^\varepsilon \in L^2([0, T], H^2), \quad \varepsilon^{\frac{3}{2}} \partial_t^2 u \in L^2([0, T], L^2) \quad \varepsilon^{\frac{1}{2}} \sigma^\varepsilon \in L^\infty([0, T], H^3),$$

which yields by interpolation that $\varepsilon^{\frac{3}{2}} u^\varepsilon \in C([0, T], H^3) \cap H^{4,2}([0, T] \times \mathcal{S})$. Moreover, carrying out direct energy estimates for $\sigma^\varepsilon$ in $H^3(\Omega)$, one gets that:

$$|\partial_t R^\varepsilon(t)| \le K^\varepsilon f^\varepsilon(t) \tag{7.13.7}$$

where $K^\varepsilon = \Lambda(1/c_0, \|(\sigma^\varepsilon, \nabla\sigma^\varepsilon, \nabla u^\varepsilon, \varepsilon^{\frac{1}{2}} \nabla^2 u^\varepsilon)\|_{\infty,t})$ is uniformly bounded and

$$R^\varepsilon(t) = \|\varepsilon^{\frac{1}{2}} \sigma^\varepsilon(t)\|_{H^3}^2, \quad f^\varepsilon(t) = \|\varepsilon^{\frac{3}{2}} u^\varepsilon(t)\|_{H^4}^2 + \|\varepsilon^{\frac{1}{2}} u^\varepsilon(t)\|_{H^3}^2 + \|(\sigma^\varepsilon, \varepsilon^{-\frac{1}{2}} \nabla\sigma^\varepsilon)(t)\|_{H^2}^2 \in L^1([0, T]).$$

Inequality (7.13.7) and the boundedness of $\|R^\varepsilon(\cdot)\|_{L^\infty([0,T])}$ leads to the fact that $R^\varepsilon(\cdot) \in C([0, T])$, which further yields that $\varepsilon^{\frac{1}{2}} \sigma^\varepsilon \in C([0, T], H^3)$. This ends the proof of the claim. Note that at this stage we do not require the norm $\|(\sigma^\varepsilon, u^\varepsilon)\|_{C([0,T],H^3)}$ to be bounded uniformly in $\varepsilon$. □

## 7.14 Convergence

This section aims to show Theorem 7.1.4, which is essentially the consequence of uniform estimates established in Theorem 7.1.1 and compactness arguments. In the following, we denote $Q_{T_0} = [0, T_0] \times \mathcal{S}$, $\Gamma_{T_0} = [0, T_0] \times \mathbb{R}^2$.

First of all, for the surface, since $\partial_t h^\varepsilon$ is uniformly bounded in $L^\infty([0, T_0], H^{m-3/2}(\mathbb{R}^2))$, $h^\varepsilon$ is uniformly bounded in $L^\infty([0, T_0], H^{m-1/2}(\mathbb{R}^2))$, one has that $h^\varepsilon$ converges (say to $h^0$) in $C([0, T_0], H^s_{loc}(\mathbb{R}^2))$ for any $0 \le s < m - 1/2$. Further, from the definition of $\varphi^\varepsilon$ (7.1.11) and Lemma (7.3.8), we conclude also that $\varphi^\varepsilon \to \varphi_0$ in $C([0, T_0], H^s_{loc}(\mathcal{S})), 0 \le s < m$ where $\varphi^0$ is defined in a similar way as (7.1.11) by replacing $h^\varepsilon$ with $h^0$.

Next, since $(\varepsilon^{\frac{1}{2}} \partial_t \sigma^\varepsilon, \varepsilon^{\frac{1}{2}} \sigma^\varepsilon)$ are uniformly bounded in $L^\infty([0, T_0], H^1(\mathcal{S})) \times L^\infty([0, T_0], H^3(\mathcal{S}))$, we have that $\varepsilon^{\frac{1}{2}} \sigma^\varepsilon$ is uniformly bounded in $C^\gamma(Q_{T_0}), 0 < \gamma < \frac{1}{2}$. In view of the definition of $\sigma^\varepsilon : \sigma^\varepsilon = (P(\rho) - P(\bar{\rho}))/\varepsilon$, we have that $P(\rho^\varepsilon) \to P(\bar{\rho})$ in $C^\gamma(Q_{T_0})$, which, combined with the uniform boundedness of $\|\nabla P(\rho^\varepsilon)\|_{\infty,t}$, yields the convergence of $\rho^\varepsilon$ to $\bar{\rho}$ in $C^\gamma(Q_{T_0})$.

Let us see the convergence of the velocity. We write $u^\varepsilon = \nabla^{\varphi^\varepsilon} \Psi^\varepsilon + v^\varepsilon$, where $\nabla^{\varphi^\varepsilon} \Psi^\varepsilon$ and $v^\varepsilon$ denote the compressible and incomprssible part of the velocity (see definitions (7.5.2), (7.5.3)). On the one hand, since $\varepsilon^{-\frac{1}{2}} \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon$, $\varepsilon^{\frac{1}{2}} \partial_t \mathrm{div}^{\varphi^\varepsilon} u^\varepsilon$ are both uniformly bounded in $L^\infty([0, T_0], H^1(\mathcal{S}))$, we get that $\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon \to 0$ in $C^\gamma([0,T_0],H^1(\mathcal{S})), 0<\gamma<\frac{1}{2}$. By elliptic estimates (7.5.10), $\nabla^{\varphi^\varepsilon}\Psi^\varepsilon \to 0$ in $C^\gamma([0,T_0],H^2(\mathcal{S}))$. On the other hand, due to the uniform boundedness of $\partial_t v^\varepsilon$ in $L^2([0,T_0],H^{-1}(\mathcal{S}))$, and of $v^\varepsilon$ in $L^\infty([0,T_0],H^1(\mathcal{S}))$, we obtain by Aubin-Lions lemma that up to extraction of subsequences, $v^\varepsilon$ converges (say to $u^0$) in $C([0,T_0],L^2_{loc}(\mathcal{S}))$. Since we will prove that $u^0$ is the *unique* solution (in conormal spaces with additional regularity property), to the incompressible free-surface Navier-Stokes equations this convergence holds indeed for the whole sequences. We thus proved that $u^\varepsilon$ converges to $u^0$ in $C^\gamma([0,T_0],H^1(\mathcal{S}))+C([0,T_0],L^2_{loc}(\mathcal{S}))$.

To conclude, we have achieved that

$$\sigma^\varepsilon \to 0 \quad \rho^\varepsilon \to \bar{\rho} \quad \nabla^{\varphi^\varepsilon}\Psi^\varepsilon \to 0 \text{ in } \quad C^\gamma(Q_{T_0}) \quad v^\varepsilon \to u^0 \quad \text{in} \quad C\big([0,T_0],L^2_{loc}\big), \tag{7.14.1}$$

$$\varphi^\varepsilon \to \varphi^0 \text{ in } C\big([0,T_0],H^s_{loc}(\mathcal{S})\big) \qquad h^\varepsilon \to h^0 \quad \text{in} \quad C\big([0,T_0],H^s_{loc}(\mathbb{R}^2)\big), \quad 0\le s\le m-\frac{1}{2}. \tag{7.14.2}$$

We now show that there exists $\pi_0 \in L^2([0,T_0],\mathcal{H}^{0,m-1})$ such that $(u^0,\pi^0,h^0)$ is the (unique) solution to the incompressible free surface system (7.1.33). Let us rewrite the equations for the incompressible part of the velocity (see (7.9.6)) as follows:

$$\bar{\rho}(\partial_t^{\varphi^\varepsilon} v^\varepsilon + v^\varepsilon\cdot\nabla^{\varphi^\varepsilon} v^\varepsilon) - \mu\Delta^{\varphi^\varepsilon} v^\varepsilon + \nabla^{\varphi^\varepsilon}\tilde{\pi}^\varepsilon = F^\varepsilon. \tag{7.14.3}$$

where

$$\begin{aligned}
&\nabla^{\varphi^\varepsilon}\tilde{\pi}^\varepsilon = \nabla^{\varphi^\varepsilon}(\pi^\varepsilon - q^\varepsilon) - [\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon,\\
&F^\varepsilon = \varepsilon\frac{g_2-1}{\varepsilon}(\partial_t + \underline{u^\varepsilon}\cdot\nabla)u^\varepsilon - \bar{\rho}(v^\varepsilon\cdot(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon + \nabla^{\varphi^\varepsilon}\Psi^\varepsilon\cdot\nabla^{\varphi^\varepsilon}u^\varepsilon).
\end{aligned}$$

with $\nabla^{\varphi^\varepsilon}\pi^\varepsilon, \nabla^{\varphi^\varepsilon}q^\varepsilon$ defined in (7.9.7). Note that by the definition (7.5.2), (7.5.3) for $\mathbb{Q}_t,\mathbb{P}_t$, the commutator $-[\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon$ can be expressed as a gradient:

$$-[\partial_t^{\varphi^\varepsilon},\mathbb{P}_t]u^\varepsilon = [\partial_t^{\varphi^\varepsilon},\mathbb{Q}_t]u^\varepsilon = \nabla^{\varphi^\varepsilon}(\partial_t^{\varphi^\varepsilon}\Psi^\varepsilon - \tilde{\Psi}^\varepsilon) \tag{7.14.4}$$

where we denote $\nabla^{\varphi^\varepsilon}\tilde{\Psi}^\varepsilon = \mathbb{Q}_t(\partial_t^{\varphi^\varepsilon}u^\varepsilon)$. By estimates established in (7.9.9), (7.9.13) and (7.9.14), we readily see that $\nabla\tilde{\pi}^\varepsilon$ is uniformly bounded in $L^2([0,T_0],\mathcal{H}^{0,m-2})$. Therefore, there exists $\pi^0\in L^2([0,T_0],\mathcal{H}^{0,m-1})$ such that $\nabla\tilde{\pi}^\varepsilon$ tends (up to subsequences) to $\nabla\pi^0$ in $L^2_w(Q_{T_0})$ and $\tilde{\pi}^\varepsilon$ converges to $\pi_0$ in $L^2_w([0,T_0],L^2_{loc}(\mathcal{S}))$. Next, by boundary conditions $(7.9.6)_2-(7.9.6)_3$ as well as the fact (7.14.4) that:

$$(2\mu S^{\varphi^\varepsilon}u^\varepsilon - \tilde{\pi}^\varepsilon\mathrm{Id})\mathbf{N}^\varepsilon = 2\mu(\mathrm{div}^{\varphi^\varepsilon}u\mathrm{Id} - (\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon)\mathbf{N}^\varepsilon + (\frac{\partial_t h^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z\Psi^\varepsilon)\mathbf{N}^\varepsilon \quad \text{ on } z=0, \tag{7.14.5}$$

$$v_3^\varepsilon = 0,\ \mu\partial_z^{\varphi^\varepsilon}v_j^\varepsilon = au_j^\varepsilon \quad (j=1,2) \qquad \text{on } z=-1. \tag{7.14.6}$$

Let us now choose a smooth vector $\psi=(\psi_1,\psi_2,\psi_3)^t\in[C^\infty_{loc}(Q_{T_0})]^3$ with condition $\psi_3|_{z=-1}=0$. Multiplying the equations (7.14.3) by $\psi$ and integrating by parts in space and time, we find by using the boundary conditions (7.14.5), (7.14.6) that:

$$\begin{aligned}
&\bar{\rho}\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^\varepsilon + 2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^\varepsilon}v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s + \bar{\rho}\int_0^t\int_{\mathcal{S}}(v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}v^\varepsilon)\cdot\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s\\
&= \bar{\rho}\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^\varepsilon + \int_0^t\int_{\mathcal{S}}F^\varepsilon\cdot\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s + \bar{\rho}\int_0^t\int_{\mathcal{S}}v^\varepsilon\cdot\partial_t^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s + \int_0^t\int_{\mathcal{S}}\tilde{\pi}^\varepsilon\mathrm{div}^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}_s^\varepsilon\mathrm{d}s\\
&\quad + a\int_0^t\int_{z=-1}(u_1^\varepsilon\cdot\psi_1 + u_2^\varepsilon\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s + \int_0^t\int_{z=0}(v^\varepsilon\cdot\mathbf{N}^\varepsilon)(v^\varepsilon\cdot\psi)\,\mathrm{d}y\mathrm{d}s\\
&\quad + \int_0^t\int_{z=0}(2\mu\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon + \frac{\partial_t h^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z\Psi^\varepsilon)(\psi\cdot\mathbf{N}^\varepsilon) - (\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon\mathbf{N}^\varepsilon\cdot\psi\,\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{7.14.7}$$

where $\mathrm{d}\mathcal{V}_t^\varepsilon = \frac{1}{\partial_z\varphi^\varepsilon}(t,\cdot)\,\mathrm{d}y\mathrm{d}z$. Since $v^\varepsilon\to v^0$ in $C([0,T_0],L^2(\mathcal{S}))$, $\partial_z\varphi^\varepsilon$ converges to $\partial_z\varphi^0$ in $C([0,T_0],C_{loc}(\mathcal{S}))$, we see that:

$$\bar{\rho}\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^\varepsilon \to \bar{\rho}\int_{\mathcal{S}}(u^0\cdot\psi)(t,\cdot)\,\mathrm{d}\mathcal{V}_t^0, \quad \bar{\rho}\int_{\mathcal{S}}(v^\varepsilon\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^\varepsilon \to \bar{\rho}\int_{\mathcal{S}}(u^0\cdot\psi)(0,\cdot)\,\mathrm{d}\mathcal{V}_0^0 \tag{7.14.8}$$

Let us now show the convergence of the last two terms in the left hand side of the above identity. Since

$$v^\varepsilon \to u^0 \text{ in } L^2([0,T_0],L^2_{loc}(\mathcal{S})),\ \nabla v^\varepsilon \rightharpoonup \nabla u^0 \text{ in } L^2(Q_{T_0}),\ v^\varepsilon \text{ uniformly bounded in } L^2([0,T_0],H^1(\mathcal{S})) \tag{7.14.9}$$

$$\varphi^\varepsilon \to \varphi^0 \text{ in } C([0,T_0],C^1_{loc}(\mathcal{S})), \qquad (\partial_z\varphi^\varepsilon,\partial_z\varphi_0)(t,x)\geq c_0>0,\forall (t,x)\in Q_{T_0} \tag{7.14.10}$$

one gets that: $\mathcal{S}^{\varphi^\varepsilon}v^\varepsilon \rightharpoonup \mathcal{S}^{\varphi_0}v^0$, $\nabla^{\varphi^\varepsilon}\psi \to \nabla^{\varphi^0}\psi$ in $L^2(Q_{T_0})$, which leads to the fact:

$$\begin{aligned}&2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^\varepsilon}v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}^\varepsilon_s\mathrm{d}s+\bar\rho\int_0^t\int_{\mathcal{S}}(v^\varepsilon\cdot\nabla^{\varphi^\varepsilon}v^\varepsilon)\cdot\psi\,\mathrm{d}\mathcal{V}^\varepsilon_s\mathrm{d}s\\ &\to 2\mu\int_0^t\int_{\mathcal{S}} S^{\varphi^0}u^0\cdot\nabla^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s+\bar\rho\int_0^t\int_{\mathcal{S}}(u^0\cdot\nabla^{\varphi^0}u^0)\cdot\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s\end{aligned} \tag{7.14.11}$$

It suffices to deal with convergence of the the last four terms in the right hand side (7.14.7). As $\nabla^{\varphi^\varepsilon}\psi^\varepsilon=\mathcal{O}(\varepsilon^{\frac12})$ in $L^2_tH^1$ and $(u^\varepsilon,\varepsilon^{\frac12}\partial_t u^\varepsilon)$ uniformly bounded in $L^2([0,T_0],H^1(\mathcal{S}))$, one readily see that $F^\varepsilon\to 0$ in $L^2(Q_{T_0})$, which gives that:

$$\int_0^t\int_{\mathcal{S}} F^\varepsilon\cdot\psi\,\mathrm{d}\mathcal{V}^\varepsilon_s\mathrm{d}s\to 0. \tag{7.14.12}$$

Next, since $\partial_t\varphi^\varepsilon\to\partial_t\varphi^0$ in $L^2_w([0,T_0],L^2(\mathcal{S}))$, we have by combining (7.14.10) that $\partial_t^{\varphi^\varepsilon}\psi\rightharpoonup\partial_t^{\varphi^0}\psi$ in $L^2(Q_{T_0})$ This, together with (7.14.9) gives that:

$$\bar\rho\int_0^t\int_{\mathcal{S}} v^\varepsilon\cdot\partial_t^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}^\varepsilon_s\mathrm{d}s\to\bar\rho\int_0^t\int_{\mathcal{S}} u^0\cdot\partial_t^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s. \tag{7.14.13}$$

Analogues to (7.14.11), we have also that:

$$\int_0^t\int_{\mathcal{S}}\tilde\pi^\varepsilon\mathrm{div}^{\varphi^\varepsilon}\psi\,\mathrm{d}\mathcal{V}^\varepsilon_s\mathrm{d}s\to\int_0^t\int_{\mathcal{S}}\pi^0\mathrm{div}^{\varphi^0}\psi\,\mathrm{d}\mathcal{V}^0_s\mathrm{d}s. \tag{7.14.14}$$

To proceed, we prove that $(u^\varepsilon)^{b,j},(v^\varepsilon)^{b,j}$ both convergent to $(u^0)^{b,j}$ in $L^2_{loc}([0,T_0]\times\mathbb{R}^2)$ where $j=1,2$. Indeed, by the trace inequality and the fact (7.14.9), one has for any $K\subset\mathbb{R}^2$ compact,

$$|(v^\varepsilon)^{b,j}-(u^0)^{b,j}|_{L^2([0,T_0]\times K)}\lesssim\|v^\varepsilon-u^0|^{\frac12}_{L^2([0,T_0],L^2(\tilde K\times[-1,0]))}\|v^\varepsilon-u^0|^{\frac12}_{L^2([0,T_0],H^1(\mathcal{S})}\to 0.$$

where $\tilde K\subset\mathbb{R}^2$ a compact set such that $K\Subset\tilde K$. The same argument applies also for $u^\varepsilon$. Therefore, one deduces that:

$$\begin{aligned}&a\int_0^t\int_{z=-1}(u_1^\varepsilon\cdot\psi_1+u_2^\varepsilon\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s+\int_0^t\int_{z=0}(v^\varepsilon\cdot\mathbf{N}^\varepsilon)(v^\varepsilon\cdot\psi)\,\mathrm{d}y\mathrm{d}s\\ &\to a\int_0^t\int_{z=-1}(u_1^0\cdot\psi_1+u_2^0\cdot\psi_2)\,\mathrm{d}y\mathrm{d}s+\int_0^t\int_{z=0}(u^0\cdot\mathbf{N}^0)(u^0\cdot\psi)\,\mathrm{d}y\mathrm{d}s\end{aligned} \tag{7.14.15}$$

Finally, by the trace inequality $\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon,\nabla^{\varphi^\varepsilon}\Psi^\varepsilon(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon=\mathcal{O}(\varepsilon^{\frac12})$ in $L^2([0,T_0],L^2(\mathbb{R}^2))$, which yields that:

$$\int_0^t\int_{z=0}(2\mu\mathrm{div}^{\varphi^\varepsilon}u^\varepsilon+\frac{\partial_t h^\varepsilon}{\partial_z\varphi^\varepsilon}\partial_z\Psi^\varepsilon)(\psi\cdot\mathbf{N}^\varepsilon)-(\nabla^{\varphi^\varepsilon})^2\Psi^\varepsilon\mathbf{N}^\varepsilon\cdot\psi\,\mathrm{d}y\mathrm{d}s\to 0. \tag{7.14.16}$$

Plugging (7.14.8) and (7.14.11)-(7.14.16) into (7.14.7), we find that $(u^0,\pi^0,h^0)$ satisfies (7.1.34). Finally, it is direct to see that $u^0$ admits additional regularity (7.1.32). In particular, $u^0$ is Lipschitz continuous, which is sufficient to verify the uniqueness.

## 7.15 Remarks for other reference domain.

In this section, we shall explain how to extend the uniform estimates results established in section 5-12 to the case when the reference domain is a channel with infinite depth or the bounded domain. We will only explain the former case since the latter one can be dealt with by working on the local coordinates which is based on the former case.

Assume now that $\Omega_t^\varepsilon$ is given by:

$$\Omega_t^\varepsilon = \{x = (y, z) |\, y \in \mathbb{R}^2, z < h^\varepsilon(t, y)\}.$$

The first step is still to use the so-called harmonic extension transformation to reduce the problem to a fixed domain. Consider the map

$$\begin{aligned} \Phi_t^\varepsilon : \mathbb{R}^3_- &\to \Omega_t^\varepsilon \\ (y, z) &\to \Phi^\varepsilon(t, y, z) = (y, \varphi^\varepsilon(t, y, z))^t \end{aligned} \tag{7.15.1}$$

where

$$\varphi^\varepsilon(t, y, z) = Az + \eta^\varepsilon(t, x) \tag{7.15.2}$$

Here $\eta$ is given by (7.1.12) and $A$ is a constant which is chosen sufficiently large such that $\partial_z\varphi^\varepsilon > 0$. We introduce the conormal vector fields

$$Z_0 = \varepsilon\partial_t, \quad Z_1 = \partial_{y_1}, \quad Z_2 = \partial_{y_2}, \quad Z_3 = \phi(z)\partial_z.$$

where the weight function $\phi(z) = z/(1-z)$. Define the conormal spaces analogue to those in Section 1.2. Furthermore, we would use the quantity $\mathcal{N}_{m,T}^\varepsilon$ defined in (7.1.30) (with the conormal norms being changed accordingly in the current definition). The projections $\mathbb{Q}_t, \mathbb{P}_t$ that send a vector field in $(L^2(\mathbb{R}^3_- \,\mathrm{d}\mathcal{V}_t))^3, (\mathrm{d}\mathcal{V}_t = \partial_z\varphi \,\mathrm{d}y\mathrm{d}z)$ to its compressible part and incompressible part are defined as: $\mathbb{P}_t = \mathrm{Id} - \mathbb{Q}_t$ and

$$\begin{aligned} \mathbb{Q}_t : \quad L^2(\mathbb{R}^3_- \,\mathrm{d}\mathcal{V}_t)^3 &\to L^2(\mathbb{R}^3_- \,\mathrm{d}\mathcal{V}_t)^3 \\ f &\to \mathbb{Q}_t f = \nabla^{\varphi^\varepsilon}\varrho \end{aligned} \tag{7.15.3}$$

where $\varrho$ satisfies the elliptic equation with trivial Dirichlet boundary condition:

$$\begin{cases} -\Delta^{\varphi^\varepsilon}\varrho = -\mathrm{div}^{\varphi^\varepsilon} f \quad \text{in } \mathbb{R}^3_- \\ \varrho|_{z=0} = 0 \end{cases} \tag{7.15.4}$$

Denote further $v^\varepsilon = \mathbb{P}_t u^\varepsilon, \nabla^{\varphi^\varepsilon}\Psi^\varepsilon = \mathbb{Q}_t u^\varepsilon$.

Following the similar (and even easier since there is no lower boundary) computations done in Section 5-12, we can prove the uniform estimates analogue to Theorem 7.2.1. The statements are the same with Theorem 7.2.1 (with probably other continuous functions appearing in (7.2.3)), we thus do not detail them. We comment that one crucial point that we have used in the computations is that $\|\nabla\Psi^\varepsilon\|_{0,\infty,t}$ can be controlled by the $L_t^\infty H_{co}^1$ norm of $\mathrm{div}^{\varphi^\varepsilon} u^\varepsilon$ (rather than $u^\varepsilon$) which has a size of $\varepsilon$. This is achieved by Sobolev embedding and elliptic estimate similar to (7.5.10). In the current situation, due to the lack of suitable Poincaré inequality, only $\|\nabla^2\Psi\|_{L_t^\infty H_{co}^1}$ (but not $\|\nabla\Psi^\varepsilon\|_{L_t^\infty H_{co}^2}$ ) can be controlled by $\|\mathrm{div}^\varphi u^\varepsilon\|_{L_t^\infty H_{co}^1}$. Nevertheless, in the current situation, one has the following Sobolev embedding:

$$\|f\|_{L^\infty(\mathbb{R}^3_-)} \lesssim \|\nabla f\|_{H^1_{tan}(\mathbb{R}^3_-)}$$

which leads to that:

$$\|\nabla\Psi^\varepsilon\|_{0,\infty,t} \lesssim \|\nabla^2\Psi^\varepsilon\|_{L_t^\infty H_{co}^1} \lesssim \Lambda\big(\frac{1}{c_0}, |h|_{3,\infty,t}\big)\|\mathrm{div}^\varphi u\|_{L_t^\infty H_{co}^1}.$$

## 7.16 Appendix

We give a short proof of (7.3.4). The proof of $|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s}|g|_{W^{1,\infty}}, (0 \le s \le 1)$ can be found in Theorem 15.2 of [94]. The case for $-1 < s < 0$ is derived by duality. We thus focus on the proof of inequality: $|fg|_{H^s(\mathbb{R}^2)} \lesssim |f|_{H^s}|g|_{H^{1+}}, (-1 < s \le 1)$. We shall use Bony's decomposition: $fg = T_g f + \tilde{T}_f g = \sum_{j\ge 0} S_{j-1}g\Delta_j f + \sum_{k\ge -1} S_{k+2}f\Delta_k g$. One can refer to [P. 61, [6]] for the definition of nonhomogeneous dyadic block $\Delta_k$ and nonhomogeneous low-frequency cut-off operator $S_k$. It is not hard to show that for any $s \in \mathbb{R}$,

$$|T_g f|_{H^s(\mathbb{R}^2)} \lesssim |g|_{L^\infty} |f|_{H^s} \lesssim |g|_{H^{1+}} |f|_{H^s}.$$

As for $\tilde{T}_f g$, if $s < 0$, we control it with the aid of Bernstein inequality:

$$\begin{aligned}
\big(2^{js} |\Delta_j \tilde{T}_f g|_{L^2}\big)_{l^2} &\lesssim \left(2^{j(s+1)} |\Delta_j \big(\sum_k S_{k+2} f \Delta_k g\big)|_{L^1}\right)_{l^2_j} \\
&\lesssim \big(2^{js} \sum_{k \leq j+5} |\Delta_k g|_{L^2}\big)_{l^2_j} \sup_k (2^{ks} |S_{k+2} f|_{L^2}) \lesssim |g|_{H^1} |f|_{H^s},
\end{aligned}$$

and if $s > 0$,

$$|\tilde{T}_f g| \lesssim \sup_k \big(2^{k(s-1-\kappa)} |S_{k+2} f|_{L^\infty}\big) |g|_{H^{1+\kappa}} \lesssim |f|_{H^s} |g|_{H^{1+\kappa}},$$

where $\kappa > 0$ is a number that can be arbitrarily closed to 0. The proof is now complete.

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ÉCOLE DOCTORALE
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**Titre:** Stabilité uniforme pour certains systèmes issus de la mécanique des fluides et de la physique des plasmas.

**Mots clés:** stabilité uniforme, limite incompressible, résonance espace-temps, division.

**Résumé:** Cette thèse est consacrée aux problèmes de stabilité uniforme et de limites singulières des systèmes fluides issus de la physique des plasmas et de la mécanique des fluides. Dans la première partie, nous étudions le problème de stabilité uniforme (par rapport au nombre de Reynolds) pour le système de Navier-Stokes-Poisson (NSP), qui est un modèle décrivant la dynamique des plasmas. Plus précisément, pour 3D NSP, nous construisons une solution globale unique autour de l'équilibre constant avec une hypothèse de petitesse sur la perturbation qui est indépendante du nombre de Reynolds sauf pour la partie rotationnelle de la vitesse. Sous hypothèse similaire, nous obtenons également l'estimation de la durée de vie pour 2d NSP qui est plus délicate en raison de la dispersion plus faible. La méthode de la "résonance espace-temps" et les estimations paraboliques classiques sont les principaux ingrédients de la preuve. De plus, l'idée de "splitting" proposée dans ces travaux peut également être utilisée pour obtenir la borne inférieure de la durée de vie des solutions de certains systèmes fluides. Dans la seconde partie de la thèse, nous étudions le problème de la limite du bas nombre de Mach pour les équations de Navier-Stokes compressibles isentropiques (CNS) dans des domaine à frontières fixes et libres. Nous établissons des estimations uniformes (par rapport au nombre de Mach) à haute régularité et justifions que la solution forte de (CNS) converge (dans un sens approprié) vers celle des équations incompressibles de Navier-Stokes. Dans le domaine aux frontières fixes, on obtient des estimations uniformes dans un cadre général de données initiales mal préparées où l'apparition simultanée d'effets de couche limite et l'oscillation rapide sont le principal obstacle de la preuve. Pour le (CNS) avec une surface libre, en raison de la difficulté supplémentaire résultant de la régularité uniforme de la surface, nous prouvons des estimations uniformes en permettant aux données initiales d'être légèrement bien préparées.

**Title:** Uniform stability for some systems arising from fluid mechanics and plasma physics.

**Keywords:** uniform stability, incompressible limit, space-time resonance, splitting.

**Abstract:** This thesis is devoted to the uniform stability and singular limit problems for some fluid systems arising from plasma physics and fluid mechanics. In the first part, we investigate the uniform (with respect to the Reynolds number) stability problem for the Navier-Stokes-Poisson (NSP) system, which is a model describing the dynamics of plasmas. More precisely, for 3d NSP, we construct a unique global solution around the constant equilibrium with a smallness assumption on the perturbation which is independent of the Reynolds number except for the curl part of the velocity. Under a similar assumption, we also obtain the lifespan estimate for 2d NSP which is more delicate due to the weaker dispersion. The 'space-time resonance' method and the classical parabolic estimates are the main ingredients of the proof. Moreover, the 'splitting' idea proposed in these works can also be used to get the lower bound for the lifespan of solutions to some fluid systems with nontrivial vorticity. In the second part of the thesis, we study the low Mach number limit problem for the isentropic compressible Navier-Stokes equations (CNS) in a domain with fixed and free boundaries. We establish the uniform (with respect to Mach number) high regularity estimates and justify that as the Mach number vanishes, the strong solution of (CNS) converges (in some suitable sense) to that of the incompressible Navier-Stokes equations. In the domain with fixed boundaries, we obtain uniform estimates in a general setting of ill-prepared initial data where the simultaneous appearance of boundary layer effects and the fast oscillation are the main obstacle of the proof. For the (CNS) with a free surface, due to the extra difficulty arising from the uniform regularity of the surface, we prove uniform estimates by allowing the initial data to be slightly well-prepared.