The Grothendieck–Serre conjecture over valuation rings

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La conjecture de Grothendieck–Serre sur les anneaux de valuation

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à mes parents.

Et toute science, quand nous l'entendons non comme un instrument de pouvoir et de domination, mais comme aventure de connaissance de notre espèce à travers les âges, n'est autre chose que cette harmonie, plus ou moins vaste et plus ou plus riche d'une époque à l'autre, qui se déploie au cours des générations et des siècles, par le délicat contrepoint de tous les thèmes apparus tour à tour, comme appelés du néant, pour se joindre en elle et s'y entrelacer.

A. Grothendieck, *Recoltes et semailles : Réflexions et témoignage sur un passé de mathématicien*

一切科学，当我等不作为权力统治的工具，而是人类历代的知识冒险，则其和谐无他，随时代而增减，包罗万象。诸世代以精妙对应展开的主题依次出现，仿佛来自虚空，融会交并。

——格罗滕迪克，《播种与收获》

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Chapter 1

Introduction (en français)

1.1 La conjecture de Grothendieck–Serre

Schémas en groupes et leurs torseurs

Les schémas en groupes, objets Groupes dans la catégorie des schémas, sont une pratique des idées philosophiques de Grothendieck sur le point de vue relatif dans la théorie des groupes. En tant que grande famille dans les schémas en groupes, les schémas en groupes réductifs sont décrits par certaines propriétés géométriques : soit S un schéma, un S-groupe G est dit *réductif* s'il est affine et lisse sur S, à fibres géométriques connexes, et ne contient aucun sous-groupe distingué qui est une extension des groupes additifs. Cette description confère aux schémas en groupes réductifs des caracteristiques distinctifs. Par exemple, un groupe sur un corps de caractéristique zéro est réductif si et seulement si ses représentations sont toutes semi-simples. En ce sens, les schémas en groupes réductifs sont très proches de groupes finis. Donc, cette propriété souligne la position unique des schémas en groupes réductifs en mathématiques. Pour étudier l'action des schémas en groupes sur les schémas générales, nous considérons les torseurs. Un torseur X sous un schéma en groupes G sur un schéma S, est un S-schéma qui est localement isomorphe à G pour la topologie fpqc et est munie d'une action de G simplement transitive (c'est-à-dire, l'homomorphisme naturel $G \times_S X \to X \times_S X$ est un isomorphisme). Les torseurs ne sont en général pas isomorphes aux groupes, donc la trivialité des torseurs est un sujet intéressant en mathématiques. Cette thèse porte sur une étude des torseurs sous les schémas en groupes. Précisons nos notations cohomologiques : pour un schémas en groupes sur un schéma S, on notera $H^1_{\mathrm{fpqc}}(S, G)$ le premier ensemble de cohomologie de S à coefficients dans G, calculé pour la topologie fpqc ; c'est aussi l'ensemble des classes d'isomorphisme de torseurs sous G. On notera $H^1_{\text{ét}}(S, G)$ l'ensemble correspondant pour la topologie étale ; c'est donc la partie de $H^1_{\mathrm{fpqc}}(S, G)$ formée des classes de torseurs sous G qui sont localement triviaux pour la topologie étale. Si X est un torseur sous un schéma en groupes réductifs G, alors il y a toujours une section de X localement pour la topologie étale, donc on a $H^1_{\text{ét}}(S, G) = H^1_{\mathrm{fpqc}}(S, G)$.

Les concepts communs des schémas en groupes

Avant d'entrer formellement dans une discussion du théorème principal, nous rappelons une série de notions communes sur les schémas en groupes.

- Soient S un schéma. Pour M un groupe abélien de type fini sur $\mathbf{Z}$, on considère son schéma en groupes constant $\underline{M}$ sur S et le foncteur

$$\underline{\mathrm{Hom}}_{S\text{-gr.}}(\underline{M}, \mathbb{G}_{m,S})\colon \mathbf{Sch}_{/S} \to \mathbf{Ens}, \qquad S' \to \mathrm{Hom}_{S'\text{-gr.}}(\underline{M}_{S'}, \mathbb{G}_{m,S'}).$$

  Le foncteur $\underline{\mathrm{Hom}}_{S\text{-gr.}}(\underline{M}, \mathbb{G}_{m,S})$ est représentable par un schéma en groupes qui s'appelle un schéma en groupes *diagonalisable*. Soit G un S-schéma en groupes. On dit que G est un groupe *de type multiplicatif* si G est localement diagonalisable au sens de la topologie fpqc. Un tore sur S est un groupe S-lisse de type multiplicatif qui est localement isomorphe à un produit des $\mathbb{G}_m$ au sens de la topologie fpqc. Pour G un S-schéma en groupes réductifs, un sous-groupe T de G s'appelle un *tore maximal* de G, s'il n'est pas contenu dans un autre tore qui est un sous-groupe de G. Si G a un tore maximal diagonalisable (on dit aussi qu'il est déployé), on dit que G est *déployé*. Soient k un corps, H un k-schéma en groupes lisses connexes. Selon un théorème de Grothendieck, il existe un tore maximal de H. Notons que lorsque k n'est pas algébriquement clos, il peut y avoir plusieurs classes de conjugaison des tores maximaux. En outre, l'existence d'un tore déployé de G est également importante. Soit G un schéma en groupes réductifs sur S, on dit que G est *anisotrope* s'il ne contient aucun $\mathbb{G}_{m,S}$.

- Soient S un schéma, G un S-schéma en groupes lisse de présentation finie à fibres connexes. Un sous-groupe de Borel de G est un sous-groupe H lisse, de présentation finie sur S tel que pour chaque $s \in S$, la fibre géométrique $H_{\overline{s}}$ est maximale parmi tous les sous-groupes lisses, connexes, et solvables de $G_{\overline{s}}$.

- Soit G un schéma en groupes réductifs. Pour P un sous-S-schéma en groupes de G, on dit que P est un *sous-groupe parabolique* de G si
  - P est lisse sur S,
  - pour chaque $s \in S$, le $\overline{s}$-schéma quotient $G_{\overline{s}}/P_{\overline{s}}$ est propre (c'est-à-dire, $P_{\overline{s}}$ contient un sous-groupe de Borel de $G_{\overline{s}}$).

  Si un sous-groupe parabolique P ne contient aucun autre sous-groupe parabolique de G, on dit que P est *minimal*. Notons que, en général, les sous-groupes paraboliques minimaux ne sont pas les sous-groupes de Borel : il est possible que certaines fibres géométriques d'un sous-groupe parabolique minimal sont strictement plus grandes que les sous-groupes de Borel.

- Soit S un schéma; munissons $\mathbf{Sch}_{/S}$ de la topologie (fpqc) et considérons le S-faisceau en groupes $\underline{\mathrm{Aut}}_{S-\mathrm{gr.}}(G)$, où G est un S-schéma en groupes.

  Le *radical* $\mathrm{rad}(G)$ d'un groupe réductifs G est un sous-groupe caractéristique (c'est-à-dire, il est stable sous $\underline{\mathrm{Aut}}_{S-\mathrm{gr.}}(G)$).

- Le sous-groupe parabolique P de G possède un plus grand sous-schéma en groupes invariant, lisse et de présentation finie sur S, à fibres géométriques connexes et unipotentes. C'est un sous-groupe caractéristique de P, appelé le radical unipotent de P, noté $\mathrm{rad}^u(P)$.

- Un sous-groupe L de P tel que P est le produit semi-direct $L \cdot \mathrm{rad}^u(P)$ (c'est-à-dire, le morphisme canonique $L \to P/\mathrm{rad}^u(P)$ est un isomorphisme) est appelé un sous-groupe de Levi de P.

- Soit G un schéma en groupes réductifs sur un schéma S. On définit deux foncteurs

$$\underline{\mathrm{Tor}}(G)\colon \mathbf{Sch}_{/S} \to \mathbf{Ens}, \quad S' \mapsto \{\text{tores maximaux de } G_{S'}\}.$$

$$\underline{\mathrm{Par}}(G)\colon \mathbf{Sch}_{/S} \to \mathbf{Ens}, \quad S' \mapsto \{\text{sous-groupes paraboliques de } G_{S'}\}.$$

  Selon [SGA $3_{\mathrm{III\ new}}$, XXII, Cor. 5.8.3], ils sont représentables par des S-schémas lisses et de présentation finie, à fibres géométriques intègres, et respectivement affine, projectif sur S.

La conjecture de Grothendieck–Serre

En 1958, Serre et Grothendieck ont proposé une conjecture, qui prévoyait que pour un groupe algébrique G sur un corps algébriquement clos, un torseur sous G est Zariski localement trivial si et seulement s'il est génériquement trivial. Par la suite, Grothendieck a étendu la conjecture aux schémas en groupes semi-simples sur les schémas réguliers. Puisque GL_n est défini Zariski localement, le cas de $G = \mathrm{GL}_n$ est évident. En fait, dans le cadre susmentionné de Serre, il a prouvé que c'était le cas lorsque $G = \mathrm{PGL}_n$. A présent, la conjecture est reformulée sous la forme suivante.

Conjecture 1.1.1 (Grothendieck–Serre)**.** *Soient R un anneau local régulier et G un R-schéma en groupes réductifs. Alors le noyau de l'homomorphisme de restriction*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(\mathrm{Frac}(R), G)$$

est trivial.

Diverses variantes et cas de la Conjecture 1.1.1 ont été dérivés au cours des dernières décennies. Nous présentons ici un résumé historique.

(i) Le cas où G est un tore a été prouvée par Colliot-Thélène et Sansuc dans [CTS87, Thm. 4.1]. En fait, ils ont établi la conjecture de Grothendieck–Serre sur les anneaux semi-locaux réguliers. Le résultat est souvent utile pour diverses réductions de cas plus généraux. Ils ont défini la notion de la résolution flasque des tores sur un schéma général, et mis à profit cette technique pour obtenir plusieurs propriétés cohomologiques. D'autre part, ils ont également prouvé l'assertion sans utiliser la résolution flasque, voir [CTS78].

(ii) Le cas où G est semi-simple et R est un anneau de valuation discrète complet de corps résiduel parfait a éte abordée dans la thèse de doctorat de Nisnevich [Nis82, 2, Thm. 1] et son article de Comptes Rendus [Nis84, Thm. 4.2]. Là, il a réduit au cas complet qui avait été considérée par Tits dans un travail non publié. Cet travail était basé sur la théorie de Bruhat-Tits (voir [$\mathrm{BT}_{\mathrm{III}}$, Lem. 3.9], mais toujours avec des conditions auxiliaires). D'autres cas particuliers sont basés sur le cas de l'anneau de valuation discrète, par exemple, le cas où R est de dimension arbitraire et complet local régulier, voir [CTS79, 6.6.1]; en outre, le cas où R est henselian et G est déployée a été prouvé explicitement dans [BB70, Prop. 2].

(iii) Le cas où $\dim R = 2$ avec corps résiduel infini et où G est quasi-déployé a été considéré par Nisnevich, voir [Nis89, Thm. 6.3].

(iv) La variante où R est un anneau de Dedekind semi-local a été établie dans [Guo20]. Pour certains cas particuliers ou variantes, par exemple, lorsque G est une forme de PGL_n, PSp_n, ou PO_n (PO_n n'est pas connexe) a été prouvé par Beke et Van Geel dans [BVG14, Thm. 3.7]. D'ailleurs, le cas où G est une forme de GL_n, O_n et Sp_n a été récemment résolu par Bayer-Fluckiger et First dans [BFF17, Thm. 5.3], où ils ont également prouvé une variante pour les groupes non-réductifs avec des fibres génériques de la forme GL_n, O_n et Sp_n. Ils ont également étudié le problème correspondant pour PGL_n, PO_n et PSp_n dans [AFW19], où des contre-exemples pour G non-réductif apparaissent, même lorsque R est un anneau de valuation complet.

(v) Le cas où R contient un corps k est résumé comme suit. Quand k est algébriquement clos et G est défini sur k, la conjecture a été établie par Colliot-Thélène et Ojanguren dans [CTO92]. Pour R un anneau régulier semi-local contenant un corps k, si k est infini, la conjecture a été prouvée par Fedorov et Panin dans [FP15]; si k est fini, la conjecture a été prouvé à l'origine par Panin dans [Pan15], dont l'exposition a été organiséee dans son dernier œuvre [Pan17]. Pour une certaine simplificatioin d'*ibid.*, voir [Fed18].

(vi) Récemment, en utilisant la construction de bons voisinages d'Artin, les compactfications equivariantes des tores sur les bases de dimensions supérieures et la géometrie du Grassmannien affine en mauvaises caractéristiques, Česnavičius [Čes21] a prouvé le cas lorsque G est quasi-déployée et R est non-ramifiée (c'est-à-dire, quand R contient un corps ou $p := \mathrm{char}(R/\mathfrak{m}_R) \notin \mathfrak{m}_R^2$ ou les deux).

Une variante pour les anneaux de valuation et la résolution des singularités

Dans cet article, nous considérons une variante de la Conjecture 1.1.1 lorsque R est un anneau de valuation. Le théorèm ci-dessous est le résultat principal de cette thèse.

Theorem 1.1.2. *Soit V un anneau de valuation et soit G un V-schéma en groupes réductifs. Alors le noyau de l'homomorphisme de restriction*

$$H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(\operatorname{Frac} V, G)$$

est trivial.

Un anneau de valuation V est un anneau intègre qui est contenu dans un corps K tel que pour chaque élément $x \in K$, soit $x \in V$, soit $x^{-1} \in V$, soit les deux. Cette notion est une généralisation de la notion de degré ou d'ordre d'annulation d'un polynôme formel en algèbre, du degré de divisibilité par un nombre premier en théorie des nombres, de l'ordre d'un pôle en analyse complexe ou du nombre de points de contact entre deux variétés algébriques en géométrie algébrique. Bien que les anneaux de valuation sont typiquement non-noethériens, ils ont une signification géométrique particulière. Ils apparaissent dans la résolution de singularités et ils sont non-singuliers dans le sens suivant : les anneaux de valuation sont stables par éclatements. En fait, une de nos motivations pour considérer cette variante est la résolution des singularités. Ici, il s'agit de l'uniformisation locale, une forme faible de la résolution des singularités, indiquant en gros qu'une variété peut être désingularisée près de n'importe quelle valuation, ou en d'autres termes que l'espace Zariski–Riemann de la variété est en quelque sorte non singulier. L'uniformisation locale a été introduite par Zariski (1939, 1940), qui a réparti le problème de la résolution des singularités d'une variété en un problème d'uniformisation locale et un problème de combinaison des uniformisations locales en une désingularisation globale. Maintenant, nous mettons un cas particulier de l'uniformation locale ici.

Conjecture 1.1.3 (l'uniformization de Zariski)**.** *Chaque anneau de valuation est une limite directe filtrée des anneaux locaux réguliers.*

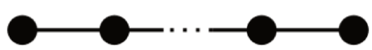
Par conséquent, en supposant cette Conjecture 1.1.3, et en utilisant un argument limite, nous écrivons $H^1_{\text{ét}}(V, G)$ comme une limite directe filtrée des $H^1_{\text{ét}}(R_i, G)$, où les R_i sont anneaux locaux réguliers. De la même manière, comme les limites directes sont interchangeables dans l'ordre, il y a aussi $H^1_{\text{ét}}(\operatorname{Frac} V, G) = \varinjlim_i H^1_{\text{ét}}(\operatorname{Frac}(R_i), G)$. Alors, notre variante est prédite par la combinaison de deux conjectures ouvertes : les Conjectures 1.1.1 et 1.1.3.

1.2 Les anneaux de valuation et leur géométrie

Comme le traitement de la Conjecture 1.1.1, pour notre variante Theorem 1.1.2, il est crucial de comprendre la géométrie des anneaux de valuation. Étant donné un anneau de valuation V, il existe une application $v\colon V \to \Gamma \cup \{\infty\}$, où Γ est un groupe abélien totalement ordonné, qui vérifie les propriétés suivantes :

- pour chaque $x \in V$, $v(x) = \infty$ si et seulement si $x = 0$;
- pour chaques $x, y \in V$, on a $v(xy) = v(x) + v(y)$; et
- pour chaques $x, y \in V$, on a $v(x + y) \geq \min(v(x), v(y))$, ce qui est relié à l'inégalité triangulaire dans les espaces métriques.

Le groupe Γ s'appelle le groupe des valuers de V. En fait, le groupe des valeurs reflète à son tour la structure (et la géométrie) de l'anneau de valuation. Pour être précis, nous considérons les sous-groupes propres non-vides Δ de Γ tel que pour chaque élément $\alpha \in \Delta$, toutes les élements dont les valeurs sont entre α et $-\alpha$ sont dans Δ. La cardinalité $\operatorname{rank}(\Gamma)$ de l'ensemble de ces Δ s'appelle le *rang* (ou la *hauteur*) de Γ. Les idéaux premiers non nuls de V et les Δ sont en correspondence biunivoque, donc le rang de Γ est égal à la dimension de Krull de V. En outre, les idéaux premiers de V sont totalement ordonnés, de sorte que le spectre de V est un ensemble totalement ordonné. Notez que $\operatorname{rang}(\Gamma)$ peut être infini. Cependant, chaque anneau de valuation est une union directe d'anneaux de valuation de rang fini. En fait, pour chaque sous-corps K_i de type fini de K, l'intersection $V_i := V \cap K_i$ est un sous-anneau de valuation de V et $V = \varinjlim_i V_i$. En particulier, pour prouver Theorem 1.1.2, il suffit de supposer que V est de rang fini. Dans ce cas, le spectre de V n'est qu'un ensemble fini totalement ordonné, comme le montre l'image suivante :



où chaque nœud de la ligne représente un idéal premier de V. En profitant de la structure de $\operatorname{Spec} V$, nous pouvons réduire de nombreuses propositions sur l'anneau de valuation au cas où le rang(Γ) est plus petit qu'auparavant : pour un idéal premier $\mathfrak{p}$ de V, le localisé de V en $\mathfrak{p}$ et le quotient $V/\mathfrak{p}$ sont des anneaux de valuation de rang rang$(\Gamma) - 1$. En outre, si V est hensélien, alors $V_{\mathfrak{p}}$ l'est aussi. Une autre façon de "décomposer" le spectre de V est de considérer le localisé $V[\frac{1}{a}]$ et la complété a-adique $\widehat{V}^a$ pour un élément a dans l'idéal maximal de V. Ces deux anneaux $\widehat{V}^a$ et $V[\frac{1}{a}]$ sont toujours des anneaux de valuation. En outre, ces localisés et quotients ont des propriétés de recollement :

$$V \xrightarrow{\sim} V_{\mathfrak{p}} \times_{V_{\mathfrak{p}}/\mathfrak{p}} V/\mathfrak{p} \quad \text{et} \quad V \xrightarrow{\sim} \widehat{V}^a \times_{\widehat{V}^a[\frac{1}{a}]} V[\tfrac{1}{a}].$$

Comme dans ce document, notre but initial est de réduire Theorem 1.1.2 au cas hensélien rang$(V) = 1$, nous prenons la deuxième option (la complétion a-adique de V).

1.3 Le cas toral

Le caractère non-noethérien des anneaux de valuation apporte des subtilités considérables, même dans le cas des tores. En particulier, dans ce cas, nous ne pouvons plus adopter la méthode de Colliot-Thélène–Sansuc et devons trouver des arguments alternatifs. Par exemple, un ingrédient crucial de *loc. cit.* est la suite exacte des faisceaux étale comme suit :

$$0 \to \mathbb{G}_{m,S} \to i_*(\mathbb{G}_{m,\xi}) \to \oplus_{x \in S^{(1)}} i_{x*}(\underline{\mathbf{Z}}_x) \to 0$$

où S est un schéma semi-local régulier qui est l'union des points génériques $i \colon \xi \to S$ et x parcourt tous les points de codimension 1. Cette suite ne s'applique pas aux anneaux de valuation, parce qu'on a une suite exacte suivante

$$0 \to V^\times \to K^\times \to \Gamma_V \to 0,$$

où V est un anneau de valuation de corps des fractions K et Γ_V est le groupe des valuers. Ce groupe est beaucoup plus compliqué et présente donc certaines difficultés techniques. Pour contourner cela, après avoir utilisé une résolution flasque du tore, nous profitons des techniques de cohomologie locale pour faire la récurrence sur le rang de l'anneau de valuation. Avant de commencer, nous rappelons quelques préliminaires sur la résolution flasque.

Flasque resolution

Les concepts des tores quasi-triviaux et des tores flasques sont ancrés dans deux modules Galois spéciaux qui servent ensuite de groupes des caractères : les modules de permutation et les modules flasques. Un module de permutation sur un groupe fini G est un $\mathbf{Z}$-module libre de type fini et possède une base sur $\mathbf{Z}$ stable sous l'action de G (c'est-à-dire, l'action de G sur cette base est une permutation) ; dans ce cas, M est de la forme $\oplus_i \mathbf{Z}[G/H_i]$ pour certaines sous-groupes $H_i \subset G$. Si, de plus, pour tout module de permutation Q on a $H^1(G, \operatorname{Hom}_{\mathbf{Z}}(M, Q)) = 0$, alors M s'appelle un module *flasque*. Par exemple, si Q est un réseau sur $\mathbf{Z}$ muni de l'action triviale de G, donc Q est de permutation et $H^1(G, \operatorname{Hom}(M, Q)) = 0$ pour tout module flasque M.

Définition 1.3.1. Soit T un tore sur un schéma S. T est dit **quasi-trivial** (resp., **flasque**) si pour chaque composante connexe Z de S et chaque revêtement galoisien connexe $Z' \to Z$ de groupe G, le G-module $\mathrm{Hom}(T_{Z'}, \mathbb{G}_{m,Z'})$ est de permutation (resp., flasque) sous G. Quand S est connexe, un tore de permutation T est de la forme $T = \prod_{1 \leq i \leq r} \mathrm{Res}_{S_i/S}(\mathbb{G}_m)$ où r est un entier, S_i/S est un revêtement galoisien fini et $\mathrm{Res}_{S_i/S}$ est la restriction de Weil.

Bien que la définition des tores flasques soit un peu compliquée, nous avons le fait suivant pour nous limiter à un cas plus simple.

Fait 1.3.2 ([SP, 0ASJ]). *Pour une extension Galoisienne d'anneaux de valuation V'/V, les deux groupes des valeurs sont identiques :* $\Gamma_{V'} = \Gamma_V$.

En utilisant ce Fait 1.3.2, nous pouvons vérifier que Γ_V est bien un module de permutation, car l'action de groupe Galoisien sur Γ_V est triviale. En d'autres termes, pour tout tore flasque T, la cohomologie Galoisienne suivante est triviale :

$$H^1(\mathrm{Gal}(V'/V), \mathrm{Hom}(X^*(T_{V'}), \Gamma_V)) = 0. \tag{1.1}$$

Cette équation est la clé qui nous permet de prouver qu'une cohomologie locale pour un tore flasque est triviale.

Les cohomologies locaux des tores flaques

Comme démontré dans [CTS87, 1.3.3], pour un tore T sur un schéma S qui n'a qu'un nombre fini de composantes connexes, il existe une suite exacte des S-tores, une résolution flasque de T :

$$1 \to F \to P \to T \to 1,$$

où F est flasque et P est quasi-trivial. On peut en déduire un diagramme commutatif suivant dont les lignes sont exactes.

$$\begin{tikzcd} H^1_{\text{ét}}(V,P) \arrow[r] \arrow[d] & H^1_{\text{ét}}(V,T) \arrow[r] \arrow[d] & H^2_{\text{ét}}(V,F) \arrow[d] \\ H^1_{\text{ét}}(K,P) \arrow[r] & H^1_{\text{ét}}(K,T) \arrow[r] & H^2_{\text{ét}}(K,F) \end{tikzcd} \tag{1.2}$$

Comme le tore quasi-trivial P est un produit des restrictions de Weil de $\mathbf{G}_m$, grâce au théorème 90 de Hilbert, l'injectivité de l'application $H^1_{\text{ét}}(V,T) \to H^1_{\text{ét}}(K,T)$ est réduite à l'injectivité de l'application $H^2_{\text{ét}}(V,F) \to H^2_{\text{ét}}(K,F)$. La dernière injectivité, par récurrence sur le rang de V, est en effet la disparition de la deuxième cohomologie locale. En fait, on a le lemme suivant.

Lemma 1.3.3. *Pour un tore flasque F sur un anneau de valuation V de rang fini, la cohomologie locale suiviante est triviale :*

$$H^2_{\mathfrak{m}_V}(V,F) = 0.$$

Après avoir utilisé la suite spectrale locale-globale, la partie essentielle de ce lemme est juste l'équation (1.1). Ainsi, nous obtenons le cas toral.

Proposition 1.3.4. *Pour un tore T sur un anneau de valuation V de corps des fractions K, l'application suivante*

$$H^1_{\text{ét}}(V, T) \to H^1_{\text{ét}}(K, T)$$

est injective.

En particulier, pour un tore flasque F sur un anneau de valuation V de corps des fractions K, l'application de restriction suivante

$$H^1_{\text{ét}}(V, F) \xrightarrow{\sim} H^1_{\text{ét}}(K, F)$$

est un isomorphisme.

Il convient de noter que la Proposition 1.3.4 ci-dessus implique la formule du produit toral : pour un anneau de valuation V de rang n, si $\mathfrak{p}$ l'idéal premier de la hauteur $n-1$ de V et a un élément dans l'idéal maximal de V et n'est pas contenu dans $\mathfrak{p}$, alors pour le complété a-adique $\widehat{V}^a$ et le localisé $V[\frac{1}{a}]$ de V et un tore T sur V on a

$$T(\mathrm{Frac}\,\widehat{V}^a) = \mathrm{Im}(T(V[\tfrac{1}{a}]) \to T(\mathrm{Frac}\,\widehat{V}^a)) \cdot T(\widehat{V}^a).$$

Cette formule est un élément crucial pour le recollement des torseurs.

1.4 Recollement des torseurs et l'approximation faible

Selon l'analyse de la structure du spectre d'un anneau de valuation de rang fini n, pour un élément $a \in \mathfrak{m}_V$ qui n'est pas contenu dans un idéal premier de rang $n-1$, le complété $\widehat{V}^a$ de V est hensélien de rang 1. Les anneaux de valuation henséliens présentent de nombreux avantages : les techniques les plus essentielles de la théorie de Bruhat-Tits ne sont disponibles que sur les anneaux de valuation henseliens, et les applications des points rationnels sur les anneaux henseliens sont généralement ouvertes, ce qui est pratique pour les approximations faibles. Par conséquent, nous utilisons le complété a-adique pour réduire le Theorem 1.1.2 au cas de Henselian de rang 1.

Pour un anneau de valuation de rang n, nous faisons deux hypothèses inductives :

1. le Theorem 1.1.2 vaut pour les anneaux de valuation de rang $\leq n$;

2. le Theorem 1.1.2 vaut pour les anneaux de valuation henséliens de rang 1.

Notons que lorsque V est de rang un, alors $\widehat{V}^a$ l'est aussi, et $V[\frac{1}{a}]$ est un corps sur lequel le Theorem 1.1.2 vaut trivialement. Pour un G-torseur X sur V tel que $X(\mathrm{Frac}\,V) \neq \emptyset$, nous devons prouver qu'il existe un V-point de X. Alors, le $\mathrm{Frac}\,V$-point de X donné par hypothèse donne une section $s_1\colon \mathrm{Spec}\,(V[\frac{1}{a}]) \to X$. Comme le complété a-adique de $V[\frac{1}{a}]$ est $\widehat{V}^a[\frac{1}{a}]$, la section s_1 nous donne une section $s \in X(\widehat{V}^a[\frac{1}{a}])$. Cette section s, par l'hypothèse de la récurrence, donne une section $s_2 \in X(\widehat{V}^a)$. Nous voulons coller ces deux sections s_1 et s_2 à une section globale de X sur $\mathrm{Spec}\,V$, mais le fait est que les images de s_1 et s_2 peuvent ne pas coïncider dans $X(\widehat{V}^a[\frac{1}{a}])$. Pour régler ce problème, nous considérons l'action des groupes : s'il existe deux éléments $g_1 \in G(V[\frac{1}{a}])$ et $g_2 \in G(\widehat{V}^a)$ tels que les images de $g_1 s_1$ et $g_2 s_2$ dans $X(\widehat{V}^a[\frac{1}{a}])$ coïncident, alors notre objectif est atteint. En d'autres termes, sous ces hypothèses, il suffit de prouver ce qui suit.

Proposition 1.4.1. *Pour un schéma en groupes réductifs G sur un anneau de valuation V de rang fini n, un élément a dans l'idéal maximal et n'est pas contenu dans l'idéal premier de hauteur $n-1$, et le complété a-adique $\widehat{V}^a$ de V, on a*

$$G(\mathrm{Frac}\,\widehat{V}^a) = \mathrm{Im}\left(G(V[\tfrac{1}{a}]) \to G(\mathrm{Frac}\,\widehat{V}^a)\right) \cdot G(\widehat{V}^a).$$

Notons que, en général, V n'est pas un corps mais juste un anneau de valuation. La question est donc la suivante :

Question. L'image de $G(V[\frac{1}{a}])$ dans $G(\mathrm{Frac}\,\widehat{V}^a)$ est-il assez grand ?

En fait, c'est ce à quoi nous répondons en utilisant l'approximation faible. Pour un corps K muni d'un nombre fini de valuations $v \in S$, les complétés K_v de K, et un schéma en groupes semi-simples G sur K, Harder a prouvé que l'adhérence de l'image de l'application diagonale

$$G(K) \hookrightarrow \prod_{v \in S} G(K_v)$$

contient un sous-groupe distingué ouvert. Dans notre cas, nous considérons l'application $G(V[\frac{1}{a}]) \to G(\mathrm{Frac}\,\widehat{V}^a)$, qui n'est même pas injective. Notre version de l'approximation faible est la suivante.

Proposition 1.4.2. *Soit V un anneau de valuation de rang fini n d'idéal premier $\mathfrak{p}$ de la hauteur $n-1$. Pour un élément $a \in \mathfrak{m}_V \backslash \mathfrak{p}$, le complété a-adique $\widehat{V}^a$ de V, un V-schéma en groupes réductifs G, on dote $G(\mathrm{Frac}\,\widehat{V}^a)$ la topologie a-adique. Alors, l'adhérence de l'image de l'application*

$$G(V[\tfrac{1}{a}]) \to G(\mathrm{Frac}\,\widehat{V}^a)$$

contient un sous-groupe distingué ouvert $N \subset G(\mathrm{Frac}\,\widehat{V}^a)$.

Comme pour la routine en cas d'approximation faible, il est essentiel de prouver l'approximation pour toutes les tores maximaux T de G. La stratégie consiste donc à tirer partie des techniques de l'algébrisation, développées dans [BČ20] par Bouthier et Česnavičius. Plus précisément, afin de voir la densité de $\mathrm{Im}\left(T(V[\frac{1}{a}])\right)$ dans $T(\mathrm{Frac}\,\widehat{V}^a)$, pour un nombre entier m, on considère les suites de Cauchy tronquées $(a)_{N \geq m}$ pour la topologie a-adique et l'anneau des suites de Cauchy $\mathbf{Cauchy}^{\geq m}\left(V[\frac{1}{a}]\right)$ formé par toutes les $(a_N)_{N\geq m}$. En utilisant un résultat dans l'algébrisation, nous pouvons prouver que l'application $T(\mathbf{Cauchy}^{\geq m}\left(V[\frac{1}{a}]\right))$ est surjective si nous pouvons le prouver par la propriété de relèvement qui suit.

Lemme 1.4.3. *Soit R un anneau local de corps résiduel κ. Soit G un R-schéma en groupes réductif. Si la cardinalité de κ n'est pas inférieure à la dimension du quotient adjoint G^{ad}, l'application suivante*

$$\underline{\mathrm{Tor}}(G)(R) \to \underline{\mathrm{Tor}}(G)(\kappa)$$

est surjective.

Ce résultat renforce un théorème d'existence de Grothendieck dans SGA3 qui ce suit.

Theorem 1.4.4 (Grothendieck)**.** *Pour un schéma en groupes réductifs G sur un schéma S, les tores maximaux de G existent localement pour la topologie de Zariski.*

La clé de notre résultat est de relever les sections régulières, qui correspondent aux tores maximaux. Pour y parvenir, nous avons besoin de l'existence de sous-algèbres de Cartan sur un corps. En effet, Barnes [Bar67] a prouvé que, la sous-algèbre de Cartan d'une algèbre de Lie $\mathfrak{g}$ sur un corps existe toujours si la cardinalité du corps est plus grande que $\dim \mathfrak{g} - 2$.

Le cas anisotrope de la formule du produit

Une étape très cruciale pour prouver la formule du produit est le cas anisotrope. Nous rappelons la notion de groupes anisotropes. Soient S un schéma semi-local connexe, G un S-groupe réductif. On dit que G est anisotrope si G ne contient aucun sous-tore déployé non réduit à la section neutral de G sur S.

Concrètement, nous pouvons démontrer ce qui suit. Dans la proposition suivante, notre objectif principal est le quatrième.

Proposition 1.4.5. *Soit G un schéma en groupes réductifs sur un anneau de valuation V de corps des fractions K.*

1. *Le V-rang déployé de G est égale au K-rang déployée de G_K.*
2. *Si V est hensélien, alors le V-rang déployé de G est égale à le $\widehat{V}$-rang déployé de $G_{\widehat{V}}$, où $\widehat{V}$ est le complété de V.*
3. *Si V est hensélien, alors pour chaque élément $a \in \mathfrak{m}_V$ et le complété a-adique $\widehat{V}^a$ de V, le V-rang déployé de G est égale au $\widehat{V}^a$-rang de $G_{\widehat{V}^a}$.*
4. *Si V est hensélien, alors G est anisotrope si et seulement si $G(V) = G(K)$.*

En fait, les nombreuses affirmations sur les rangs déployés peuvent être essentiellement réduites aux critères des anisotropies des schémas en groupes réductifs. Pour les critères des anisotropies, l'observation suivante est très utile.

Lemme 1.4.6 ([SGA $3_{\text{III new}}$, XXVI, 6.14])**.** *Soit S un schéma semi-local connexe. Pour que le S-groupe réductif G soit anisotrope, il faut et il suffit qu'il ne possède aucun sous-groupe parabolique $P \neq G$, et que son radical soit anisotrope.*

Donc, nous analysons le schéma $\underline{\mathrm{Par}}(G)$ paramétrant les sous-groupes paraboliques de G et le schéma $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))$ en utilisant des techniques d'algébrisation. Par exemple, pour un schéma en groupes réductifs sur un anneau de valuation hensélien V et le complété $\widehat{V}$, on prouve que si $\underline{\mathrm{Par}}(G)(\widehat{V})$ est non vide alors $\underline{\mathrm{Par}}(G)(V)$ est non vide. Avec les trois premières affirmations exposées, il est possible de prouver la formule du produit dans le cas anisotrope (en fait, il faut vérifier que G_K est anisotrope sur V). La base de la formule de produit est le résultat de Maculan dans le cas où V est complet ([Mac17, Thm. 1.1]), lorsque il a utilisé les outils analytiques de la géométrie non archimédienne pour pouver ce suit.

Lemma 1.4.7 (Maculan)**.** *Soit k un corps non archimédien d'anneau des entiers k°. Pour un schéma en groupes réductifs G sur k, le sous-groupe $G(k^\circ) \subset G(k)$ est un sous-groupe borné maximal.*

D'autre part, par la théorie de Bruhat–Tits, Rousseau prouve que pour un groupe réductif G sur un corps valué hensélian k le groupe $G(k)$ est borné. En combinant cette conclusion avec le théorème principal de Maculan, on obtient la formule de produit dans le cas anistrope. Avec un anneau de valuation hensélien V, il suffit de prendre le complété a-adique de V, ensuite on utilise le critère d'anisotropie Proposition 1.4.5 (3) et les produits fibrés

$$G(V) \xrightarrow{\sim} G(\mathrm{Frac}\, V) \times_{G(\mathrm{Frac}\, \widehat{V}^a)} G(\widehat{V}^a) \quad \text{et} \quad G(V) \xrightarrow{\sim} G(V_\mathfrak{p}) \times_{G(V_\mathfrak{p}/\mathfrak{p})} G(V/\mathfrak{p})$$

pour obtenir la formule du produit dans le cas hensélien.

1.5 Preuve du résultat principal

Cette section se compose de deux parties : la recurrence par les sous-groupes de Levi et la théorie de Bruhat–Tits. En ayant recours à la technique du recollement, nous avons prouvé que le Theorem 1.1.2 est réduit au cas hensélien de rang un. À ce stade, nous pouvons utiliser divers outils de la théorie de Bruhat–Tits.

Passage au cas anisotrope semi-simple

Dans cette section, rien n'est plus utile que la citation suivante de la SGA3.

Lemme 1.5.1 ([SGA $3_{\text{III new}}$, XXVI, Prop. 6.16])**.** *Soient S un schéma semi-local connexe, G un S-groupe réductif. Les sous-tores déployés maximaux de G sont les plus grands sous-tores centraux déployés des groupes de Levi des sous-groupes paraboliques minimaux de G. Deux tels tores sont conjugués par un élément de $G(S)$.*

Tout d'abord, nous considérons un sous-groupe parabolique minimal P de G. Par le critère valuatif de propreté, la preuve du Theorem 1.1.2 se réduit à prouver que l'application

$$H^1_{\text{ét}}(V, P) \to H^1_{\text{ét}}(\operatorname{Frac} V, P)$$

a un noyau trivial. Puis le critère d'affinité de Serre nous réduit à montrer le Theorem 1.1.2 pour un sous-groupe de Levi de P. Notons que L n'a pas de sous-groupe parabolique propre, il suffit de prendre le quotient $G/\operatorname{rad}(G)$ de G par le radical $\operatorname{rad}(G)$. D'où le cas des tores et le cas anisotrope de la formule du produit, la réduction est complète.

La théorie de Bruhat–Tits

Grâce aux réductions précédentes, il suffit de prouver le Theorem 1.1.2 dans le cas lorsque G est semi-simple anisotrope et V est hensélien de rang un. L'objectif de cette partie est de prouver ce qui suit.

Theorem 1.5.2. *Pour un schéma en groupes semi-simple anisotrope sur un anneau V de valuation hensélien de rang un de corps des fractions K, l'application suivante*

$$H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(K, G) \tag{1.3}$$

est injective.

Cette application est un composite des applications entre les ensembles de cohomologie galoisienne : on dénote $\Gamma := \operatorname{Gal}(V^{\text{sh}}/V)$, $\Gamma_{K^{\text{sh}}} := \operatorname{Gal}(K^{\text{sep}}/K^{\text{sh}})$ et $\Gamma_K := \operatorname{Gal}(K^{\text{sep}}/K)$, alors l'équation (1.3) est la suivante

$$H^1(\Gamma, G(V^{\text{sh}})) \xrightarrow{\alpha} H^1(\Gamma, G(K^{\text{sh}})) \xrightarrow{\beta} H^1(\Gamma_K, G(K^{\text{sep}})).$$

Selon la suite exacte d'inflation-restriction, β est injective.

Pour l'application α, on utilise les techniques de la théorie de Bruhat–Tit. Cette théorie concerne un schéma en groupes réductifs G sur un corps de valuation hensélien K. Grosso modo, la théorie de Bruhat–Tits attache le groupe $G(K)$ des points rationnels à un espace topologique appelé l'immeuble du $G(K)$. L'immeuble du $G(K)$ est constitué de complexes simpliciaux. Chaque facette d'immeuble correspond à un certain sous-groupe du $G(K)$, appelé sous-groupe parahorique. Le mot parahorique est la combinaison des mots parabolique et Iwahori, indiquant que ces sous-groupes sont des analogues des sous-groupes paraboliques qui généralisent les sous-groupes d'Iwahori. Cependant, il faut noter que les sous-groupes parahoriques ne sont pas leurs propres normalisateurs (telles propriétés sont vraies pour les sous-groupes hyperspéciaux). Nous revenons maintenant à la preuve de l'injectivité de α. Pour un cocyle $z \in H^1(\Gamma, G(V^{\mathrm{sh}}))$ qui est dans $\operatorname{Ker} \alpha$, il satifait que

$(*)$ il existe un $h \in G(K^{\mathrm{sh}})$ tel que pour tout $s \in \Gamma$, on a $z(s) = h^{-1}s(h) \in G(V^{\mathrm{sh}})$.

Le point essentiel est de considérer deux sous-groupes $G(V^{\mathrm{sh}})$ et $hG(V^{\mathrm{sh}})h^{-1}$ du $G(K^{\mathrm{sh}})$. En fait, les deux sous-groupes sont parahoriques et Γ-invariants (selon $(*)$), et $G(V^{\mathrm{sh}})$ est un sous-groupe parahorique hyperspecial. D'autre part, puisque G est anisotrope, la parite Γ-invariante de l'immeuble du $G(K^{\mathrm{sh}})$ est trivial, donc il n'y a qu'un seul sous-groupe parahorique Γ-invariant. Cela signifie que h est dans le normalisateur du $G(V^{\mathrm{sh}})$ dans $G(K^{\mathrm{sh}})$, on a

$$h \in G(V^{\mathrm{sh}}) = \operatorname{Norm}_{G(K^{\mathrm{sh}})}(G(V^{\mathrm{sh}})).$$

Donc z est trivial.

1.6 La conjecture de Nisnevich

Dans l'article [Nis89, 1.3], Nisnevich a adressé une preuve de la conjecture de Grothendieck–Serre dans le cas où R est de dimension 2 et G quasi-déployé. Pour un élément $u \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$, la stratégie de Nisnevich consiste à utiliser la technique du recollement en considérant $R[1/u]$ et le complété u-adique $\widehat{R}$ de R. Dans le processus, Nisnevitch a trouvé que la conjecture de pureté suivante était une partie importante de la preuve du théorème principal.

Conjecture 1.6.1. *Pour un schéma en groupes réductifs G sur un anneau local régulier R et un élément $u \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$, chaque torseur sous G sur $R[1/u]$ qui est localement trivial pour la topologie Zariski est trivial, c'est-à-dire, on a*

$$H^1_{\mathrm{Zar}}(R[1/u], G) = \{*\}.$$

Cette conjecture généralise la conjecture de Quillen [Qui76, Comments] lorsque $G = \mathrm{GL}_n$ et a été prouvée par Gabber [Gab81] pour $G = \mathrm{GL}_n$ ou PGL_n quand $\dim R \leq 3$. Dans cette article, nous acquérons une variante de la conjecture de pureté de Nisnevich : pour un anneau de valuation V et son anneau des séries formelles $V[\![t]\!]$, on considère R comme $V[\![t]\!]$ et $u = t$, donc $R[1/u] = V(\!(t)\!)$. L'énoncé est le suivant.

Proposition 1.6.2. *Pour un schéma en groupes réductifs G sur un anneau de valuation V, chaque torseur sous G sur $V(\!(t)\!)$ qui est localement trivial pour la topologie de Zariski est trivial, c'est-à-dire, on a*

$$H^1_{\mathrm{Zar}}(V(\!(t)\!), G) = \{*\}.$$

Cette Proposition 1.6.2 résulte immédiatement de la proposition suivante, lorsque on prouve que l'application $H^1_{\text{ét}}(V(\!(t)\!), G) \to H^1_{\text{ét}}(K(\!(t)\!), G)$ a un noyau trivial, où K est le corps des fractions de V.

Proposition 1.6.3. *Pour un schéma en groupes réductifs G sur un anneau de valuation V de corps des fractions K,*

1. *changement de base est une équivalence de catégories des torseurs $\mathcal{X}$ sous $G_{V[\![t]\!]}$ et des triplets*
$$(X, X', \iota\colon X_{K[\![t]\!]} \xrightarrow{\sim} X'_{V(\!(t)\!)}),$$
où X est un torseur sous $G_{V(\!(t)\!)}$, X' est un torseur sous $G_{K[\![t]\!]}$, et ι est un isomorphisme des torseurs indiqué ;
2. *l'application $H^1_{\text{ét}}(V(\!(t)\!), G) \to H^1_{\text{ét}}(K(\!(t)\!), G)$ a un noyau trivial ; et*
3. *l'application $H^1_{\text{ét}}(V[\![t]\!], G) \to H^1_{\text{ét}}(V(\!(t)\!), G)$ a un noyau trivial.*

La clé de cette proposition est le passage au cas $G = \mathrm{GL}_n$ en utilisant un monomorphisme $G \hookrightarrow \mathrm{GL}_n$ et la suite exacte de la cohomologie étale non abélienne
$$(\mathrm{GL}_n / G)(S) \to H^1_{\text{ét}}(S, G) \to H^1_{\text{ét}}(S, \mathrm{GL}_n),$$
où S est un schéma au-dessus de la base. En analysant les propriétés de faiseaux réflexifs sur $V[\![t]\!]$, on peut obtenir :
$$H^1_{\text{ét}}(V[\![t]\!], \mathrm{GL}_n) = \{*\}.$$

Chapter 2

Introduction

Originally conceived by A. Grothendieck [Gro58, pp. 26-27, Rem. 3] and J.-P. Serre [Ser58, p. 31, Rem.] in 1958, the prototype of the Grothendieck–Serre conjecture predicted that for an algebraic group G over an algebraically closed field k, a G-torsor over a nonsingular k-variety is Zariski-locally trivial if it is generically trivial. With its subsequent generalization to regular base schemes by A. Grothendieck [Gro68, Rem. 1.11.a] and the localization by spreading out, the conjecture became the following.

Conjecture 2.0.1 (Grothendieck–Serre)**.** *For a reductive group scheme G over a regular local ring R with fraction field K, the following map between nonabelian étale cohomology pointed sets has trivial kernel:*

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G);$$

in other words, a G-torsor over R is trivial if its restriction over K is trivial.

Diverse variants and cases of Conjecture 2.0.1 were derived in the last several decades. For the history of the topic, we refer to [Guo20, FP15]. The goal of this article is to settle the analogue of Conjecture 2.0.1 when R is instead assumed to be a valuation ring. This variant is expected because of the following consequence of the resolution of singularities conjecture, a weak form of Zariski's local uniformization.

Conjecture 2.0.2 (Zariski)**.** *Every valuation ring is a filtered direct limit of regular local rings.*

By assuming Conjecture 2.0.2, a limit argument [Gir71, VII, 2.1.6] reduces the Grothendieck–Serre conjecture over valuation rings to the original Conjecture 2.0.1. In particular, the combination of Conjecture 2.0.1 and 2.0.2 predicts our following main result.

Theorem 2.0.3. *For a reductive group scheme G over a valuation ring V with fraction field K, the map*

$$H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(K, G) \qquad \textit{is injective.} \qquad (\diamondsuit)$$

The special case of Theorem 2.0.3 when G is an orthogonal group for a nondegenerate quadratic form and V is a valuation ring in which 2 is invertible was proved in [CTS87, 6.4] and [CLRR80, Thm. 4.5].

The non-Noetherianness of general valuation rings introduces considerable subtleties, even when G is a torus. Namely, in this case we can no longer adopt the method of [CTS87, 4.1] and need to devise alternative arguments. For instance, a crucial ingredient of *loc. cit.* is the exact sequence of étale sheaves

$$0 \to \mathbb{G}_{m,S} \to i_*(\mathbb{G}_{m,\xi}) \to \oplus_{x\in S^{(1)}} i_{x*}(\underline{\mathbf{Z}}_x) \to 0, \tag{2.1}$$

where S is a semilocal regular scheme with the union of generic points $i : \xi \to S$ and x ranges over the points of codimension 1. Being used in the proof of [CTS87, 2.2], however, for valuation rings, we last nonzero term in the sequence above is the value group. It it difficult to utilise such short exact sequence in the case of valuation rings. To circumvent this, after using a flasque resolution of tori, we take advantage of local cohomology techniques to induct on the rank of the valuation ring. This reduces us to the following:

$$\text{for a flasque torus } F \text{ over a valuation ring } (V, \mathfrak{m}_V), \text{ we have} \qquad H^2_{\mathfrak{m}_V}(V, F) = 0. \tag{$*$}$$

For a flasque torus with character group Λ, by definition (§3.1), the Galois action on Λ has special properties, so certain Galois cohomology of Λ vanishes, which leads to the vanishing of local cohomology $(*)$ and therefore the case of tori:

Proposition 2.0.4 (Proposition 3.1.3)**.** *For a torus T over a valuation ring V with fraction field K,*

$$\textit{the map} \qquad H^1_{\text{ét}}(V, T) \hookrightarrow H^1_{\text{ét}}(K, T) \qquad \textit{is injective.}$$

This case of tori, in turn, yields the simplest case of the product formula stated in (2.2) below (or, see Lemma 5.1.5), which is essential for further reduction of Theorem 2.0.3.

A practical advantage of Henselian rank-one valuation rings is that several techniques of Bruhat–Tits theory, especially in [$\mathrm{BT_{II}}$, §4-5], become available. The goal of §4 and §5.1 is to reduce Theorem 2.0.3 to this case: after a limit argument that leads to the case of finite rank, we induct on the rank n of a valuation ring V by patching torsors. The induction hypothesis implies that our G-torsor over V is a gluing of trivial torsors. For this gluing, we choose an $a \in V$ such that the a-adic completion $\widehat{V}^a$ is a Henselian valuation ring of rank one with $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$; so that, $V[\frac{1}{a}]$ is a valuation ring of rank $n-1$. Similar to the Beauville–Laszlo's gluing of bundles, our patching is reformulated as the product formula

$$G(\widehat{K}^a) = \mathrm{Im}\big(G(V[\tfrac{1}{a}]) \to G(\widehat{K}^a)\big) \cdot G(\widehat{V}^a). \tag{2.2}$$

The strategy for proving this formula is a "dévissage" that establishes approximation properties of certain subgroups of $G_{\widehat{V}^a}$. In this procedure, techniques of algebraization [BČ20, §2] plays an important role, especially for a Harder-type approximation (see §4) and for the following integrality of rational points.

Proposition 2.0.5 (Proposition 5.1.3)**.** *For a reductive anisotropic group scheme G over a Henselian valuation ring V with fraction field K, we have $G(V) = G(K)$.*

Based on its special case when $K = \widehat{K}^a$ is complete due to Maculan [Mac17, Thm. 1.1], our approach to Proposition 2.0.5 is a reduction to completion that rests on techniques of algebraization to approximate schemes characterizing the anisotropicity of $G_{\widehat{V}^a}$. Indeed, Proposition 2.0.5 is an anisotropic version of the product formula (2.2). Proposition 2.0.5 is helpful, not only for the reduction to the Henselian rank-one case, but also for the induction on Levi subgroups when reducing to the semisimple anisotropic case in §6. After these reductions, we transfer Theorem 2.0.3 into the injectivity of a map of Galois cohomologies. We conclude by taking advantage of properties of parahoric subgroups in Bruhat–Tits theory, see Theorem 7.0.1.

In addition to techniques of algebraization, another crucial element of our reduction to the Henselian rank-one case is a lifting property of maximal tori of reductive group schemes.

Lemma 2.0.6 (Lemma 4.2.1). *Let G be a reductive group scheme over a local ring $(R, \mathfrak{m}_R)$ with a maximal $R/\mathfrak{m}_R$-torus T. If the cardinality of $R/\mathfrak{m}_R$ is at least $\dim(G)$, then G has a maximal R-torus $\mathscr{T}$ such that*

$$\mathscr{T}_{R/\mathfrak{m}_R} = T.$$

This strengthens a result of Grothendieck [SGA 3_{II}, XIV, 3.20] that a maximal torus of a reductive group scheme exists Zariski-locally on the base. By a correspondence of maximal tori and regular sections, the novelty is to lift regular sections instead of merely proving their existence Zariski-locally. Depending on inspection of the reasoning for *loc. cit.*, the key point is [Bar67], which guarantees that Lie algebras over fields with large cardinalities contain regular sections. For lifting regular sections, we need the functorial property of Killing polynomials. Indeed, Killing polynomials over rings were defined ambiguously in in the original literature, see [SGA 3_{II}, XIV, 2.2]. Therefore, to establish Lemma 2.0.6, we first add the supplementary details §4.1 for Killing polynomials over rings. Subsequently, for a Lie algebra with locally constant nilpotent rank, we use the functoriality of Killing polynomials to deduce the openness of the regular locus. This openness permits us to lift regular sections, which amounts to lifting maximal tori.

In §8, we acquire a variant of Nisnevich's purity conjecture [Nis89, 1.3], whose statement is the following.

Conjecture 2.0.7 (Nisnevich's purity). *For a reductive group scheme G over a regular local ring R with a regular parameter $f \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$, every Zariski-locally trivial G-torsor over $R[\frac{1}{f}]$ is trivial, that is, we have*

$$H^1_{\mathrm{Zar}}(R[\tfrac{1}{f}], G) = \{*\}.$$

This conjecture generalizes Quillen's conjecture [Qui76, Comments] when $G = \mathrm{GL}_n$ and was proved by Gabber [Gab81] for $G = \mathrm{GL}_n$ and PGL_n when $\dim R \leq 3$. In this article, we consider a variant: for a valuation ring V and its ring of formal power series $V[\![t]\!]$, we let $R = V[\![t]\!]$ and $f = t$, hence $R[\frac{1}{f}] = V(\!(t)\!)$.

Proposition 2.0.8 (Corollary 8.2.3). *For a reductive group scheme G over a valuation ring V, every Zariski-locally trivial G-torsor over $V(\!(t)\!)$ is trivial, that is, we have*

$$H^1_{\mathrm{Zar}}(V(\!(t)\!), G) = \{*\}.$$

This Proposition 2.0.8 follows from the injectivity of the map $H^1_{\text{ét}}(V(\!(t)\!), G) \to H^1_{\text{ét}}(K(\!(t)\!), G)$ proved in Proposition 8.2.2. In fact, by cohomological properties of reflexive sheaves (see 8.1), every étale GL_n-torsor over $V(\!(t)\!)$ is trivial. With an embedding $G \hookrightarrow \mathrm{GL}_n$, we obtain Proposition 2.0.8 by patching torsors.

2.1 Notation and conventions

We always assume that each fibre of a reductive group scheme is connected. For a valuation ring V, we denote by $\mathfrak{m}_V$ the maximal ideal of V. When V has finite rank n, for the prime $\mathfrak{p} \subset V$ of height $n-1$ and $a \in \mathfrak{m}_V \backslash \mathfrak{p}$, we denote by $\widehat{V}^a$ the a-adic completion of V. For a module M finitely generated over a topological ring A, we endow M with the *canonical topology* as the quotient of the product topology via $\pi \colon A^{\oplus n} \twoheadrightarrow M$. By [GR18, 8.3.34], this topology on M is independent of the choice of π. In particular, we endow each finitely generated $V[\frac{1}{a}]$-module with the "a-adic" topology. For a reductive group scheme G over a scheme S, by [SGA 3_{II}, XIV, 6.1], the functor

$$\underline{\mathrm{Tor}}(G) \colon \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}, \qquad S' \mapsto \{\text{maximal tori of } G_{S'}\}.$$

is representable by a smooth affine S-scheme.

Chapter 3

The case of tori

The goal of this section is to prove the Grothendieck–Serre conjecture over valuation rings for tori (Proposition 3.1.3), which is a non-Noetherian counterpart of Colliot-Thélène–Sansuc's result [CTS87, 4.1]. In [CTS87], the authors defined flasque resolutions of tori over arbitrary base schemes, which yielded several cohomological properties of tori over a regular scheme. In particular, they proved that for a torus T over a semilocal regular ring R with total ring of fractions K, the following map is injective:

$$H^1_{\text{ét}}(R, T) \hookrightarrow H^1_{\text{ét}}(K, T), \tag{3.1}$$

which is a stronger version of the Grothendieck–Serre conjecture for tori, see [CTS87, 4.1]. Nevertheless, if we substitute R in (3.1) with a valuation ring V, then the method in *loc. cit.* does not work any more because of the non-Noetherianness of V. Seeking an alternative argument in this case, we induct on the rank of V and use local cohomology. This case of tori obtained in Proposition 3.1.3 is crucial for subsequent steps of the proof of Theorem 2.0.3, such as for patching torsors (see Proposition 5.1.6 and 5.1.7).

3.1 Flasque resolution of tori

The concepts of quasitrivial and flasque tori are rooted in two special Galois modules that then serve as character groups: permutation and flasque modules. A *permutation* module M over a finite group G is a finite type $\mathbf{Z}$-free module with a $\mathbf{Z}$-basis on which G acts via permutations; in this case, M is of the form $\oplus_i \mathbf{Z}[G/H_i]$ for subgroups $H_i \subset G$. If, in addition, $H^1(G, \mathrm{Hom}_{\mathbf{Z}}(M, Q)) = 0$ for any permutation module Q, then M is *flasque*. For instance, if Q is a finite $\mathbf{Z}$-lattice with trivial G-action, then Q is permutation and

$$H^1(G, \mathrm{Hom}(M, Q)) = 0$$

for any flasque module M. Recall that the *Cartier dual* of a torus T over a scheme S is a sheaf $\mathscr{D}(T) := \mathcal{H}om_{S\text{-gr.}}(T, \mathbb{G}_{m,S})$. If for every connected component Z of S and every connected Galois cover $Z' \to Z$ with Galois group G splitting T, the G-module $(\mathscr{D}(T))(Z')$ is flasque (resp., permutation), then T is *flasque* (resp., *quasitrivial*). Indeed, when S is connected, every quasitrivial torus is a finite product of Weil restrictions $\mathrm{Res}_{S'_i/S}(\mathbb{G}_m)$ for finite étale connected covers $S'_i \to S$. As proved in [CTS87, 1.3.3], for a torus T over a scheme S with finitely many connected components, there is an exact sequence of S-tori, a *flasque resolution* of T:

$$1 \to F \to P \to T \to 1, \qquad \text{where } F \text{ is flasque and } P \text{ is quasitrivial.} \tag{3.2}$$

Lemma 3.1.1. *For a normal domain R, every multiplicative type R-group scheme M is isotrivial*[1].

Proof. Note that every normal domain is a direct limit of its normal subrings of finite type over $\mathbf{Z}$[2]. By a limit argument [SGA 3_{II}, XV, 3.6], there is a multiplicative type group M_0 defined over a Noetherian normal domain $R_0 \subset R$ that descends M over Spec R_0. By the isotriviality of tori over locally Noetherian geometrically unibranch bases [SGA 3_{II}, X, 5.16], after base change to R, we get a minimal Galois cover $R \to R'$ splitting M. □

Lemma 3.1.2. *For a flasque torus F over a valuation ring V of finite rank n, the following local cohomology vanishes:*

$$H^2_{\mathfrak{m}_V}(V, F) = 0.$$

Proof. We denote $X = \mathrm{Spec}(V)$ and $Z = \mathrm{Spec}(V/\mathfrak{m}_V)$. By excision [Mil80, III, 1.28], we may replace X by its Henselization X^{h}. For a variable X-étale scheme X' with preimage $Z' := X' \times_X Z$, let $\mathcal{H}^q_Z(-, F)$ be the étale sheafification of the presheaf $X' \mapsto H^q_{Z'}(X', F)$. By the local-to-global spectral sequence

$$H^p_{\text{ét}}(X, \mathcal{H}^q_Z(X, F)) \Rightarrow H^{p+q}_Z(X, F), \qquad \text{([SGA } 4_{\mathrm{II}}\text{, V, 6.4])}$$

the desired $H^2_Z(X, F) = 0$ follows from

$$H^0_{\text{ét}}(X, \mathcal{H}^2_Z(X, F)) = H^1_{\text{ét}}(X, \mathcal{H}^1_Z(X, F)) = H^2_{\text{ét}}(X, \mathcal{H}^0_Z(X, F)) = 0.$$

So first, we use [SGA 4_{II}, VII, 5.9] to calculate the $\mathcal{H}^i_Z(X, F)$ étale locally. In particular, it suffices to compute $H^q_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F)$ for $0 \le q \le 2$, where $\overline{\mathfrak{m}}_V$ is a geometric point above $\mathfrak{m}_V$ and V^{sh} is the strict Henselization of V at $\overline{\mathfrak{m}}_V$. The local ring map $V \to V^{\mathrm{sh}}$ is faithfully flat ([SP, 07QM]). Further, the strict Henselization of a valuation ring preserves the value group ([SP, 0ASK]). Therefore, for the prime $\mathfrak{p} \subset V$ of height $n-1$, there is a unique prime ideal $\mathfrak{P} \subset V^{\mathrm{sh}}$ lying over $\mathfrak{p}$ (that is, $\mathfrak{p}V^{\mathrm{sh}} = \mathfrak{P}$). By [SGA 4_{II}, V, 6.5], we have the following long exact sequence

$$\cdots \to H^i_{\text{ét}}(V^{\mathrm{sh}}, F) \to H^i_{\text{ét}}((V^{\mathrm{sh}})_{\mathfrak{P}}, F) \to H^{i+1}_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F) \to H^{i+1}_{\text{ét}}(V^{\mathrm{sh}}, F) \to \cdots. \tag{3.3}$$

First, we compute $H^q_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F)$ when $q = 0$ and 2. The injectivity of $H^0_{\text{ét}}(V^{\mathrm{sh}}, F) \to H^0_{\text{ét}}((V^{\mathrm{sh}})_{\mathfrak{P}}, F)$, combined with the vanishing of $H^1_{\text{ét}}((V^{\mathrm{sh}})_{\mathfrak{P}}, F)$ and $H^i_{\text{ét}}(V^{\mathrm{sh}}, F)$ for $i = 1, 2$ (see [SP, 03QO]) imply that

$$\mathcal{H}^0_Z(X, F) = \mathcal{H}^2_Z(X, F) = 0.$$

[1] Recall from [SGA 3_{II}, IX, 1.1] that a torus $\mathscr{T}$ over a scheme S is *isotrivial*, if there is a finite étale surjective morphism of schemes $S' \to S$ such that $\mathscr{T}_{S'}$ splits.

[2] By the limit argument in [EGA IV_3, §8], the normal domain is a direct union of its subdomains $R_i \subset R$ which are of finite type over $\mathbb{Z}$. Since R_i are Nagata rings ([SP, 035B]) and normalization maps of Nataga rings are finite maps ([SP, 035S]), the normalizations R_i^{ν} are Noetherian subdomains of R.

It remains to calculate $\mathcal{H}^1_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F)$. From (3.3), we obtain the following short exact sequence

$$0 \to H^0_{\text{ét}}(V^{\mathrm{sh}}, F) \to H^0_{\text{ét}}((V^{\mathrm{sh}})_{\mathfrak{P}}, F) \to H^1_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F) \to H^1_{\text{ét}}(V^{\mathrm{sh}}, F) = 0.$$

For the Cartier dual $\mathscr{D}(F)$ of F, we denote by $\Lambda := \mathscr{D}(F)(V^{\mathrm{sh}})$ the character group of F as a $\mathbf{Z}$-lattice and by $\Lambda^\vee := \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ its dual. The Cartier duality $F \cong \mathscr{D}(\mathscr{D}(F)) \cong \mathcal{H}om_{V\text{-gr.}}(\mathscr{D}(F), \mathbb{G}_m)$ provides

$$H^0_{\text{ét}}(V^{\mathrm{sh}}, F) \cong F\left(V^{\mathrm{sh}}\right) \cong \mathcal{H}om_{V\text{-gr.}}(\mathscr{D}(F), \mathbb{G}_m)(V^{\mathrm{sh}}) = \mathrm{Hom}_{\mathbf{Z}}(\Lambda, (V^{\mathrm{sh}})^\times) \cong \Lambda^\vee \otimes_{\mathbf{Z}} (V^{\mathrm{sh}})^\times,$$

$$\text{and similarly,} \qquad H^0_{\text{ét}}((V^{\mathrm{sh}})_{\mathfrak{P}}, F) \cong \Lambda^\vee \otimes_{\mathbf{Z}} (V^{\mathrm{sh}})^\times_{\mathfrak{P}}.$$

Subsequently, for the value group $\Gamma_{V^{\mathrm{sh}}/\mathfrak{P}}$ of $V^{\mathrm{sh}}/\mathfrak{P}$, we have the following isomorphism[3]

$$H^1_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F) = (\Lambda^\vee \otimes_{\mathbf{Z}} (V^{\mathrm{sh}})^\times_{\mathfrak{P}})/(\Lambda^\vee \otimes_{\mathbf{Z}} (V^{\mathrm{sh}})^\times) \cong \Lambda^\vee \otimes_{\mathbf{Z}} \Gamma_{V^{\mathrm{sh}}/\mathfrak{P}}.$$

The Henselianity of X permits us to view $\mathcal{H}^1_Z(X, F)$ as a sheaf over the site of profinite $\pi_1^{\text{ét}}(V) := \pi_1^{\text{ét}}(X, \overline{\mathfrak{m}}_V)$-sets. In particular, $\mathrm{Spec}(V/\mathfrak{m}_V)$ is the one-point set with trivial $\pi_1^{\text{ét}}(V)$-action. By [Sch13, 3.7 (iii)] derived from the Cartan–Leray spectral sequence, we obtain the first isomorphism below:

$$H^1_{\text{ét}}(X, \mathcal{H}^1_Z(X, F)) \cong H^1(\pi_1^{\text{ét}}(V), H^1_{\overline{\mathfrak{m}}_V}(V^{\mathrm{sh}}, F)) \cong H^1(\pi_1^{\text{ét}}(V), \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V^{\mathrm{sh}}/\mathfrak{P}})). \tag{3.4}$$

To see the action of $\pi_1^{\text{ét}}(V)$ on $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V^{\mathrm{sh}}/\mathfrak{P}})$, by Lemma 3.1.1, we first note that the $\pi_1^{\text{ét}}(V)$-action on Λ factors through its quotient $\mathrm{Gal}(Y/X)$, where Y is the minimal Galois cover of X splitting F. Besides, the following value groups are equal

$$\Gamma_{V^{\mathrm{sh}}/\mathfrak{P}} \overset{[\text{SP, 05WS}]}{=\!=\!=} \Gamma_{(V/\mathfrak{p})^{\mathrm{sh}}} \overset{[\text{SP, 0ASK}]}{=\!=\!=} \Gamma_{V/\mathfrak{p}},$$

so $\pi_1^{\text{ét}}(V)$ acts trivially on $\Gamma_{V^{\mathrm{sh}}/\mathfrak{P}} \cong \mathrm{Frac}(V/\mathfrak{p})^\times/(V/\mathfrak{p})^\times$. Thus, the action of $\pi_1^{\text{ét}}(V)$ on $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V/\mathfrak{p}})$ factors through $\mathrm{Gal}(Y/X)$. Since $\pi_1^{\text{ét}}(V)$ is a direct limit of the $\mathrm{Gal}(X_\alpha/X)$, where X_α are Galois covers of X, a limit argument [Ser02, §2.2, Cor. 1] reduces (3.4) to

$$H^1_{\text{ét}}(X, \mathcal{H}^1_Z(X, F)) \simeq \varinjlim_\alpha H^1\Big(\mathrm{Gal}(X_\alpha/X), \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V/\mathfrak{p}})^{\pi_1^{\text{ét}}(X_\alpha)}\Big). \tag{3.5}$$

We express $\Gamma_{V/\mathfrak{p}}$ as a direct limit of finitely generated $\mathbf{Z}$-submodules Γ_i. Since Λ is $\mathbf{Z}$-finitely presented, we have

$$\varinjlim_{i\in I} \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V/\mathfrak{p}}). \tag{3.6}$$

Combining the isomorphism (3.6) with a limit argument [Ser02, §2.2, Prop. 8], we reduce (3.5) to

$$\varinjlim_{\alpha} H^1\Big(\mathrm{Gal}(X_\alpha/X), \varinjlim_{i\in I} \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_i)^{\pi_1^{\text{ét}}(X_\alpha)}\Big) = \varinjlim_{\alpha; i\in I} H^1\Big(\mathrm{Gal}(X_\alpha/X), \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_i)^{\pi_1^{\text{ét}}(X_\alpha)}\Big).$$

[3]Here, we use the fact that for every valuation ring V with prime $\mathfrak{p}$, there is an isomorphism of nonunital rings $\mathfrak{p}V \xrightarrow{\sim} \mathfrak{p}V_{\mathfrak{p}}$. To see this, we write every element in $\mathfrak{p}V_{\mathfrak{p}}$ as the form a/b, where $a \in \mathfrak{p}V$ and $b \in V \backslash \mathfrak{p}$. Since V is the valuation ring of K, if $b/a \in V$ then $b \in \mathfrak{p}V$, which leads to a contradiction. Therefore, we have $a/b \in V$, so $a/b \in V \cap \mathfrak{p}V_{\mathfrak{p}} = \mathfrak{p}V$. In particular, we have $\mathfrak{P}V^{\mathrm{sh}} \xrightarrow{\sim} \mathfrak{P}(V^{\mathrm{sh}})_{\mathfrak{P}}$ so the fraction field of $V^{\mathrm{sh}}/\mathfrak{P}$ is $(V^{\mathrm{sh}})_{\mathfrak{P}}/\mathfrak{P} = V^{\mathrm{sh}}/\mathfrak{P}(V^{\mathrm{sh}})_{\mathfrak{P}}$.

It suffices to calculate for a large α_0 such that X_{α_0} splits F. In this situation, $\pi_1^{\text{ét}}(X_{\alpha_0})$ acts trivially on $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_i)$. Since F is a flasque torus, its character group Λ is a flasque $\mathrm{Gal}(X_{\alpha_0}/X)$-module. As aforementioned, $\mathrm{Gal}(X_{\alpha_0}/X)$ acts trivially on $\Gamma_{V/\mathfrak{p}}$, so the Γ_i are finite $\mathbf{Z}$-lattices with trivial $\mathrm{Gal}(X_{\alpha_0}/X)$-action. The example in §3.1 implies that

$$H^1(\mathrm{Gal}(X_{\alpha_0}/X), \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \Gamma_{V/\mathfrak{p}})) = 0,$$

which verifies that

$$H^1_{\text{ét}}(X, \mathcal{H}^1_Z(X, F)) = 0. \qquad \square$$

Now, we prove the case of tori, which, by Colliot-Thélène's remark, is generalized to multiplicative type group schemes under fpqc topology (for its smooth case, étale topology suffices).

Proposition 3.1.3. *For a multiplicative type group scheme M over a valuation ring V with fraction field K,*

(i) *the map*

$$H^2_{\text{fpqc}}(V, M) \hookrightarrow H^2_{\text{fpqc}}(K, M) \quad \textit{is injective; in particular,} \quad \mathrm{Br}(V) \hookrightarrow \mathrm{Br}(K) \quad \textit{is injective;}$$

(ii) *the map*

$$H^1_{\text{fpqc}}(V, M) \hookrightarrow H^1_{\text{fpqc}}(K, M) \quad \textit{is injective.}$$

Proof. As V is a filtered direct union of valuation subrings of finite rank ([BM20, 2.22]), a limit argument [SGA 4_{II}, VII, 5.7] reduces us to the case when V has finite rank n. Note that for a quasitrivial V-torus P, we have $P \simeq \prod_{S_i'} \mathrm{Res}_{S_i'/\operatorname{Spec} V}\mathbb{G}_m$ for finite étale connected V-schemes S_i', so [SGA $3_{\mathrm{III\ new}}$, XIX, 8.4] gives an isomorphism $H^1_{\text{ét}}(V, P) \cong \prod_{S_i'} H^1_{\text{ét}}(S_i', \mathbb{G}_m)$. The Grothendieck's version of Hilbert's 90 [SGA 3_{II}, VIII, Cor. 4.5] identifies $H^1_{\text{ét}}(S_i', \mathbb{G}_m) \simeq H^1_{\text{Zar}}(S_i', \mathbb{G}_m)$, which are trivial by [BouAC, II, §5, no. 3, Prop. 5]. Hence, we have

$$H^1_{\text{ét}}(V, P) \quad \text{is trivial for quasitrivial } V\text{-torus } P.$$

(i) First, we reduce to the case when M is a flasque V-torus. By the short exact sequence

$$1 \to M \to F \to P \to 1, \qquad \text{([CTS87, 1.3.2])}$$

where F is flasque and P is quasitrivial, we obtain the following commutative diagram

$$\begin{array}{ccccc} H^1_{\text{fpqc}}(V, P) & \longrightarrow & H^2_{\text{fpqc}}(V, M) & \longrightarrow & H^2_{\text{fpqc}}(V, F) \\ & & \downarrow & & \downarrow \\ & & H^1_{\text{fpqc}}(K, M) & \longrightarrow & H^2_{\text{fpqc}}(K, F). \end{array}$$

Now we are reduced to the case for flasque torus F and we induct on the rank n of V. The case of $V = K$ is trivial, so when $n \geq 1$, for the prime $\mathfrak{p}$ of V of height $n-1$, we assume that the assertion holds for $V_{\mathfrak{p}}$ (which has rank $n-1$). Denote $X = \mathrm{Spec}(V)$ and $Z = \mathrm{Spec}(V/\mathfrak{m}_V)$. By [SGA 4_{II}, V, 6.5], we have the following long exact sequence:

$$\cdots \to H^2_Z(X, F) \to H^2_{\text{ét}}(X, F) \to H^2_{\text{ét}}(X - Z, F) \to H^3_Z(X, F) \to \cdots. \qquad (3.7)$$

We conclude by the induction hypothesis and $H^2_Z(X, F) = 0$ proved in Lemma 3.1.2.

(ii) First, we reduce to the case when M is a torus. The isotriviality of M yields a short exact sequence
$$1 \to T \to M \to \mu \to 1,$$
where T is a V-torus and μ is a finite multiplicative type V-group scheme. This give the following commutative diagram

$$\begin{tikzcd}
\mu(V) \arrow[r] \arrow[d] & H^1_{\mathrm{fpqc}}(V,T) \arrow[r] \arrow[d] & H^1_{\mathrm{fpqc}}(V,M) \arrow[r] \arrow[d] & H^2_{\mathrm{fpqc}}(V,\mu) \arrow[d] \\
\mu(K) \arrow[r] & H^1_{\mathrm{fpqc}}(K,T) \arrow[r] & H^1_{\mathrm{fpqc}}(K,M) \arrow[r] & H^2_{\mathrm{fpqc}}(K,\mu).
\end{tikzcd}$$

The valuative criterion for properness of μ over V leads to $\mu(V) = \mu(K)$ and the injectivity of $H^1_{\mathrm{fpqc}}(V,\mu) \hookrightarrow H^1_{\mathrm{fpqc}}(K,\mu)$. Thus, by diagram chase, we are reduced to showing that

$$H^1_{\text{ét}}(V,T) \to H^1_{\text{ét}}(K,T) \quad \text{is injective.}$$

A flasque resolution $1 \to F \to P \to T \to 1$ of T as (3.2) leads to the following commutative diagram with exact rows

$$\begin{tikzcd}
H^1_{\text{ét}}(V,P) \arrow[r] & H^1_{\text{ét}}(V,T) \arrow[r] \arrow[d] & H^2_{\text{ét}}(V,F) \arrow[d] \\
 & H^1_{\text{ét}}(K,T) \arrow[r] & H^2_{\text{ét}}(K,F).
\end{tikzcd}$$

Since P is quasitrivial and $H^2_{\text{ét}}(V,F) \hookrightarrow H^2_{\text{ét}}(K,F)$ is injective by (i), the map

$$H^1_{\text{ét}}(V,T) \hookrightarrow H^1_{\text{ét}}(K,T) \qquad \text{is injective.} \qquad \square$$

With Proposition 3.1.3, we use Lemma 3.1.2 to the sequence (3.7) to obtain a corallary.

Corollary 3.1.4. *For a flasque torus F over a valuation ring V with fraction field K, the map*

$$H^1_{\text{ét}}(V,F) \xrightarrow{\sim} H^1_{\text{ét}}(K,F) \qquad \textit{is an isomorphism.}$$

Chapter 4

Algebraizations and a Harder-type approximation

In Proposition 4.2.9, we acquire a higher-height analog of Harder's weak approximation [Har68, Satz. 2.1] to reduce Theorem 2.0.3 to the case of Henselian rank one valuation rings. However, if we consider higher-height valuations, then Harder's argument no longer applies, see Remark 4.2.11. To nevertheless generalize Harder's result, we approximate by integral points and take advantage of techniques of algebraization from [BČ20, §2]. These procedures invoke a lifting Lemma 4.2.1 for maximal tori of reductive group schemes over local rings, which is essentially a lifting of Cartan subalgebras.

4.1 Regular sections, Cartan subalgebras and subgroups of type (C)

For a ring R, a Lie R-algebra $\mathfrak{h}$ that is locally free of rank n as a module and a variable R-algebra R' such that $\mathfrak{h}_{R'} := \mathfrak{h} \otimes_R R'$ has constant rank, every $a \in \mathfrak{h}_{R'}$ has an adjoint action on $x \in \mathfrak{h}_{R'}$, denoted by $\mathrm{ad}(a)(x) := [a, x]$ with characteristic polynomial

$$P_{\mathfrak{h}_{R'},a}(t) = t^n + c_1(a)t^{n-1} + \cdots + c_n(a) \in R'[t],$$

where n is the rank of $\mathfrak{h}_{R'}$. For the sheaf of symmetric algebras $\mathscr{A} := \underline{\mathrm{Sym}}_{\mathcal{O}_R}(\mathfrak{h}^\vee)$ of the dual R-module $\mathfrak{h}^\vee$, the associated vector bundle of $\mathfrak{h}$ is $\mathbf{W}(\mathfrak{h}) := \underline{\mathrm{Spec}}(\mathscr{A})$. By the universal property [EGA II, 1.7.4], we identify $\mathbf{W}(\mathfrak{h})(R')$ with $\mathfrak{h}_{R'}$. For each i, consider the assignment:

$$c_i : \quad \mathbf{W}(\mathfrak{h})(R') \to \mathbf{A}^1_R(R'), \qquad a \mapsto (R[\tau] \to R', \tau \mapsto c_i(a)).$$

Each c_i is a natural transformation and determines a morphism $R[\tau] \to \Gamma(\mathscr{A}) := \Gamma(\mathbf{W}(\mathfrak{h}), \mathscr{A})$ by sending τ to an element $c_i \in \Gamma(\mathscr{A})$. We define the *Killing polynomial* of $\mathfrak{h}$ as

$$P_{\mathfrak{h}}(t) := t^n + c_1 t^{n-1} + \cdots + c_n \in \Gamma(\mathscr{A})[t].$$

By the functorialities of the c_i, the formation of Killing polynomials commutes with base change. When R is a field k, the largest r such that $P_{\mathfrak{h}}(t)$ is divisible by t^r is the *nilpotent rank* of $\mathfrak{h}$. The nilpotent rank of the Lie algebra of a reductive group scheme is étale-locally constant (see

[SGA 3_{II}, XV, 7.3] and [SGA $3_{\mathrm{III\ new}}$, XXII, 5.1.2, 5.1.3]). Every $a \in \mathfrak{h}$ satisfying $c_{n-r}(a) \neq 0$ is called a *regular element*. Let G be a reductive group scheme over a scheme S. For the Lie algebra $\mathfrak{g}$ of G, if a subalgebra $\mathfrak{d} \subset \mathfrak{g}$ is Zariski-locally a direct summand such that its geometric fibre $\mathfrak{d}_{\bar{s}}$ at each $s \in S$ is nilpotent and equals to its own normalizer, then σ is a *Cartan subalgebra* of $\mathfrak{g}$ ([SGA 3_{II}, XIV, 2.4]). We say an S-subgroup $D \subset G$ is *of type (C)*, if D is S-smooth with connected fibres, and $\mathrm{Lie}(D) \subset \mathfrak{g}$ is a Cartan subalgebra. A section σ of $\mathfrak{g}$ is a *regular section*, if σ is in a Cartan subalgebra such that $\sigma(s) \in \mathfrak{g}_s$ is a regular element for all $s \in S$. A section of $\mathfrak{g}$ with regular fibres is *quasi-regular*, hence regular sections are quasi-regular.

4.2 Schemes of maximal tori

A lifting result

For a reductive group scheme G defined over a scheme S, the functor

$$\underline{\mathrm{Tor}}(G)\colon \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}, \qquad S' \mapsto \{\text{maximal tori of } G_{S'}\}.$$

is representable by a smooth affine S-scheme ([SGA 3_{II}, XIV, 6.1]). For an S-scheme S' and a maximal torus $T \in \underline{\mathrm{Tor}}(G)(S')$ of $G_{S'}$, by [SGA $3_{\mathrm{III\ new}}$, XXII, 5.8.3], the morphism

$$G_{S'} \to \underline{\mathrm{Tor}}(G_{S'}), \qquad g \mapsto gTg^{-1} \tag{4.1}$$

induces an isomorphism $G_{S'}/\underline{\mathrm{Norm}}_{G_{S'}}(T) \cong \underline{\mathrm{Tor}}(G_{S'})$. Here, $\underline{\mathrm{Norm}}_{G_{S'}}(T)$ is an S'-smooth scheme (see [SGA 3_{II}, XI, 2.4bis]). Now, we establish the lifting property of $\underline{\mathrm{Tor}}(G)$.

Lemma 4.2.1. *Let G be a reductive group scheme over a local ring R with residue field κ and let Z be the center of G. If the cardinality of κ is at least $\dim(G/Z)$, then the following map is surjective:*

$$\underline{\mathrm{Tor}}(G)(R) \twoheadrightarrow \underline{\mathrm{Tor}}(G)(\kappa).$$

Proof. By [SGA 3_{II}, XII, 4.7 c)], an isomorphism of schemes $\underline{\mathrm{Tor}}(G) \simeq \underline{\mathrm{Tor}}(G/Z)$ reduces us to the semisimple adjoint case, where the maximal tori of G are exactly the subgroups of type (C) ([SGA 3_{II}, XIV, 3.18]). These subgroups are bijectively assigned by $D \mapsto \mathrm{Lie}(D)$ to the Cartan subalgebras of $\mathfrak{g} := \mathrm{Lie}(G)$, see [SGA 3_{II}, XIV, 3.9]. It suffices to lift a Cartan subalgebra $\mathfrak{c}_0 \subset \mathfrak{g}_\kappa$ to that of $\mathfrak{g}$. Since the cardinality of κ is at least $\dim(G/Z) = \dim(G)$, by [Bar67, Thm. 1], $\mathfrak{c}_0$ is of the form

$$\mathrm{Nil}(a_s) := \bigcup_n \mathrm{Ker}(\mathrm{ad}(a_s)^n)$$

for some $a_s \in \mathfrak{c}_0$. Hence [SGA 3_{II}, XIII, 5.7] implies that $a_s \in \mathfrak{c}_0$ is a regular element of $\mathfrak{g}_s$. We take a section a of $\mathfrak{g}$ passing through a_0 and claim that

$$\mathcal{V} := \{s \in \mathrm{Spec}\, R \text{ such that } a_s \in \mathfrak{g}_s \text{ is regular}\}$$

is an open subset of $\mathrm{Spec}\, R$. We may assume that R is reduced. Since the nilpotent rank of $\mathfrak{g}$ is locally constant, the Killing polynomial of $\mathfrak{g}$ at every $s \in \mathrm{Spec}\, R$ is uniformly of the form

$$P_{\mathfrak{g}_s}(t) = t^r(t^{n-r} + (c_1)_s t^{n-r-1} + \cdots + (c_{n-r})_s)$$

such that $(c_{n-r})_s$ is nonzero. Thus, the regular locus in $\mathfrak{g}$ is the principle open subset

$$\{c_{n-r} \neq 0\} \subset \mathbf{W}(\mathfrak{g})$$

so $\mathcal{V}$ is nonempty and open, which implies that $\mathcal{V} = \operatorname{Spec} R$. In particular, the regular element $a_0 \in \mathfrak{c}_0$ is lifted to a quasi-regular section $a \in \mathfrak{g}$, which by [SGA $3_{\text{III new}}$, XIV, 3.7], is regular. By definition of regular sections, there is a Cartan subalgebra of $\mathfrak{g}$ containing a and is the desired lifting of $\mathfrak{c}_0$. □

Weak approximation of maximal tori

Next, we combine this lifting property with techniques of algebraization to deduce the density Lemma 4.2.5. Roughly speaking, this density permits us to "replace" maximal tori of $G_{\widehat{K}^a}$ by those of $G_{V[\frac{1}{a}]}$. Subsequently, we prove an approximation of $\widehat{K}^a$-points of a maximal torus of $G_{\widehat{K}^a}$ by $V[\frac{1}{a}]$-points (Lemma 4.2.8). First, we recall properties of Henselian pairs and the notion of Cauchy nets in [BČ20, §2].

Henselian pairs and Cauchy nets

Let (A, I) be a pair, where A is a ring and $I \subset A$ is an ideal. We say that (A, I) is *Zariski* if $1 + I \subset A^\times$; the pair is *Henselian* if for every affine étale A-scheme X, the map

$$X(A) \twoheadrightarrow X(A/I) \quad \text{is surjective,}$$

or equivalently (due to Gabber, see [SP, 09XI]), every polynomial

$$T^n(T-1) + a_n T^n + \cdots + a_1 T + a_0 \qquad \text{with } a_j \in I$$

has a (unique) root $\alpha \in 1 + I$.

Lemma 4.2.2. *For a Henselian valuation ring V and a prime ideal $\mathfrak{p} \subset V$, the localization $V_\mathfrak{p}$ and the quotient $V/\mathfrak{p}$ are Henselian valuation rings.*

Proof. By [SP, 05WQ], the valuation ring $V/\mathfrak{p}$ is Henselian. For $V_\mathfrak{p}$, by the footnote 2, we identify $\mathfrak{p}V$ with $\mathfrak{p}$. By [SP, 0DYD], the Henselianity of V implies that $(V, \mathfrak{p}V)$ is a Henselian pair, hence $(V_\mathfrak{p}, \mathfrak{p}V_\mathfrak{p})$ satisfies Gabber's criterion. □

Now we recall the notion of Cauchy nets for algebraizations. For a nonunital topological ring B, an open nonunital subring $B' \subset B$ whose induced topology has a neighbourhood base of 0 consisting of ideals of B', and a poset S given by some neighbourhood base of 0 in B with an order such that $U \leq U'$ if $U' \subset U$, we define

- The set of *null function* $\mathrm{Null}(S, B) := \{\text{functions } f\colon S \to B \text{ that converge to } 0\}$;
- The *Cauchy net* $\mathrm{Cauchy}(S, B) := \{\text{functions } f\colon S \to B \text{ that are Cauchy}\}$.

Note that $\mathrm{Null}(S, B) \subset \mathrm{Cauchy}(S, B)$ is an ideal. Further, we take direct limits

- $\mathrm{Null}_S(B) := \varinjlim_{s \in S} \mathrm{Null}(S_{\geq s}, B)$, which represents the "tail of null functions", and

- $\mathrm{Cauchy}_S(B) := \varinjlim_{s \in S} \mathrm{Cauchy}(S_{\geq s}, B)$ as the "tail of Cauchy nets".

Here $\mathrm{Null}_S(B) \subset \mathrm{Cauchy}_S(B)$ is a nonunital subring. Hence, the completion of B with respect to its topology is the inverse limit of quotients of B by open abelian subgroups, so by construction is

$$\widehat{B} \cong \mathrm{Cauchy}_S(B)/\mathrm{Null}_S(B). \tag{4.2}$$

In practice, it is convenient to compute $\widehat{B}$ by using B'. Since every null function eventually takes values in B', we have

$$\mathrm{Null}_S(B) \cong \mathrm{Null}_S(B').$$

In our setting, the topology of B and B' are "compatible" in the sense that the induced topology on B' is B'-*linear* (that is, B' has an open neighbourhood base of zero consisting of ideals of B'). Consequently, though B' does not appear explicitly in the definition of Cauchy nets, to compute $\widehat{B}$, it is convenient to specify B'.

Example 4.2.3.

1. For a pair (A, I), we let $B = A$ and $B' = I$ with the coarse topology (which means that $S = \{s_I, s_B\}$ where s_I represents the open neighbourhood I of zero and s_B is similar). Then $\mathrm{Null}(S, B)$ consists of functions f from $\{s_I, s_B\}$ to B that converge to zero, which means that $f(s_B) \in B$ and $f(s_I) \in I$. Similarly, $\mathrm{Cauchy}(S, B)$ consists of functions f such that $f(s_I) - f(s_B) \in I$. Therefore, the completion is $\widehat{B} = A/I$.

2. For a pair (A, I), we let $B = A$ and $B' = I$ with the I-adic topology (hence $S = \{s_{I^n}\}_{n \geq 0}$), then $\mathrm{Null}(S, B)$ consists of sequences of elements in B that converege to 0 in the I-adic topology and $\mathrm{Cauchy}(S, B)$ is just the set of classical Cauchy sequences in the I-adic topology. Therefore, the completion is $\widehat{B} = \widehat{A}^I$, the I-adic completion of A.

3. For a ring R, we denote the Henselization of $R[t]$ at $t = 0$ by $R\{t\}$. Let $B = R\{t\}[\frac{1}{t}]$ and $B' = tR\{t\}$. Then the completion of B is the Laurent formal power series $R(\!(t)\!)$.

4. For a valuation ring V, we let $B = \mathrm{Frac}\, V$ and B' be the maximal ideal $\mathfrak{m} \subset V$ equipped with the valuation topology (hence S consists of ideals determined by elements whose valuations are larger than γ for some γ in the value group Γ). Then the completion $\widehat{B}$ of B is $\mathrm{Frac}\, \widehat{V}$, which is the completion of the valued field with respect to the valuation topology.

Now, we consider a valuation ring V of finite rank n with prime $\mathfrak{p}$ of height $n-1$ and choose an $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ to form Cauchy net $\mathrm{Cauchy}_S(V[\frac{1}{a}])$, where S is the set of images of $a^n V$ in $V[\frac{1}{a}]$ via the canonical map $V \to V[\frac{1}{a}]$. The a-adic completion of $V[\frac{1}{a}]$ is $\widehat{K}^a$, by (4.2), there is an exact sequence of commutative rings

$$1 \to \mathrm{Null}_S(V[\tfrac{1}{a}]) \to \mathrm{Cauchy}_S(V[\tfrac{1}{a}]) \to \widehat{K}^a \to 1,$$

where $\widehat{K}^a$ is the fraction field of the a-adic completion $\widehat{V}^a$. Hence, $\mathrm{Null}_S(V[\frac{1}{a}]) \subset \mathrm{Cauchy}_S(V[\frac{1}{a}])$ is the maximal ideal. We obtain the following lemma.

Lemma 4.2.4. *For a valuation ring V of finite rank n with prime ideal of height $n-1$ and an element $a \in \mathfrak{m}_V \backslash \mathfrak{p}$, the nonunital ring $\mathrm{Cauchy}_S(V[\frac{1}{a}])$ (where S is defined by a^n as above) is a local ring with residue field $\widehat{K}^a$.*

Algebraization of the scheme of maximal tori

Lemma 4.2.5. *For a valuation ring V of finite rank n with prime $\mathfrak{p}$ of height $n-1$, we choose an $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ and form the a-adic completion $\widehat{V}^a$ with $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. Let G be a reductive group scheme over V and $\underline{\mathrm{Tor}}(G)$ the scheme of maximal tori of G. We endow $\underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ with the induced a-adic topology. Then,*

$$\text{the image of} \quad \underline{\mathrm{Tor}}(G)(V[\tfrac{1}{a}]) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a) \quad \text{is dense.}$$

Proof. To prove the density, we consider the direct limit of Cauchy nets $\mathrm{Cauchy}_S(V[\frac{1}{a}])$ as in Lemma 4.2.4, where we proved that it is a local ring with residue field $\widehat{K}^a$. Since $\underline{\mathrm{Tor}}(G)$ is finitely presented and affine over $V[\frac{1}{a}]$, by the lifting property Lemma 4.2.1, we obtain the following desired surjection

$$\varinjlim_{s\in S}\Big(\underline{\mathrm{Tor}}(G)\big(\mathrm{Cauchy}(S_{\geq s}, V[\tfrac{1}{a}])\big)\Big) \simeq \underline{\mathrm{Tor}}(G)\Big(\mathrm{Cauchy}_S(V[\tfrac{1}{a}])\Big) \twoheadrightarrow \underline{\mathrm{Tor}}(G)(\widehat{K}^a). \qquad \square$$

Corollary 4.2.6. *With the same assumption as in Lemma 4.2.5, for a maximal torus $T \subset G_{\widehat{K}^a}$ and for any open neighbourhood $W \subset G(\widehat{K}^a)$ of id, there is a maximal torus $T' \subset G_{V[\frac{1}{a}]}$ and a $g \in W$ such that*

$$T'_{\widehat{K}^a} = gTg^{-1}.$$

Proof. For the morphism (4.1), a criterion for openness [GGMB14, 3.2.8] applies and we have the open map:

$$\phi\colon G(\widehat{K}^a) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a), \qquad g \mapsto gTg^{-1}.$$

Consequently, ϕ sends every open neighborhood W of $\mathrm{id} \in G(\widehat{K}^a)$ to an open neighborhood of T. By Lemma 4.2.5, the density of $\underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])$ in $\underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ implies that

$$\phi(W) \cap \mathrm{Im}(\underline{\mathrm{Tor}}(G)(V[\tfrac{1}{a}]) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a)) \neq \emptyset.$$

Hence, there are $T' \in \underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])$ and $g \in W$ such that $gTg^{-1} = T'_{\widehat{K}^a} \in \phi(W)$. $\square$

Weak approximation of maximal tori

The weak approximation of maximal tori, as expected, is obtained from the image of a norm map. To be concrete, for a reductive group scheme G over a valuation ring of finite rank n, the prime ideal $\mathfrak{p}$ of height $n-1$, an element $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ inducing the a-adic completion $\widehat{V}^a$ with $\widehat{K}^a := \mathrm{Frac}\, \widehat{V}^a$, and a maximal torus $T \subset G_{\widehat{K}^a}$, we consider a minimal splitting field L_0 of T and the norm map

$$N_{L_0/\widehat{K}^a}\colon T(L_0) \to T(\widehat{K}^a).$$

Our goal is to calculate the norm map, via approximation, by using maximal tori of $G_{V[\frac{1}{a}]}$. Since the image of $V[\frac{1}{a}] \to \widehat{K}^a$ is dense, we wonder if there is a $V[\frac{1}{a}]$-algebra giving the similar density result with respect to L_0. We solve this by the following purely algebraic lemma.

Lemma 4.2.7. *With the notation above and a fixed maximal $V[\frac{1}{a}]$-torus $T' \subset G_{V[\frac{1}{a}]}$ found in Corollary 4.2.6, there is a minimal Galois cover $V[\frac{1}{a}] \to R$ splitting T', and the base change*

$$R \otimes_{V[\frac{1}{a}]} \widehat{K}^a \simeq \textstyle\prod_{i=1}^r L_i$$

is a product of a-adically complete fields L_i, which are minimal splitting fields of $T'_{\widehat{K}^a}$. Further, the image of the map

$$R^\times \to \prod_{i=1}^r L_i^\times$$

is dense with respect to the a-adic topology.

Proof. For the maximal torus $T' \in \underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])$ found in Corollary 4.2.6, by Lemma 3.1.1, there is a minimal Galois cover $V[\frac{1}{a}] \to R$ splitting T'. The base change $R \otimes_{V[\frac{1}{a}]} \widehat{K}^a \simeq \prod_{i=1}^r L_i$ is a product of fields L_i. Since R is a finite flat $V[\frac{1}{a}]$-module, it is free and we have $\widehat{R}^a = R \otimes_{V[\frac{1}{a}]} \widehat{K}^a$ so the L_i are a-adically complete. Let

$$\rho\colon \pi_1^{\text{ét}}(V[\tfrac{1}{a}]) \to \mathrm{GL}_n(\mathbf{Z})$$

be the $\pi_1^{\text{ét}}(V[\frac{1}{a}])$-action on the lattice $\mathbf{Z}^n \simeq \mathrm{Hom}_{R\text{-gr.}}(T'_R, \mathbb{G}_m)$ ([SGA 3_{II}, X, 1.2]) with image $Q := \rho(\pi_1^{\text{ét}}(V[\frac{1}{a}]))$. The minimality of R amounts to that $\mathrm{Ker}\,\rho = \pi_1^{\text{ét}}(R)$. The functoriality of étale fundamental groups [SGA 1_{new}, V, 6.1 *ff.*] yields $\tau\colon \pi_1^{\text{ét}}(\widehat{K}^a) \to \pi_1^{\text{ét}}(V[\frac{1}{a}])$. Therefore, the kernel of $\rho \circ \tau\colon \pi_1^{\text{ét}}(\widehat{K}^a) \to Q$ is the étale fundamental group of a connected component of $\mathrm{Spec}(R \otimes_{V[\frac{1}{a}]} \widehat{K}^a)$, that is, $\pi_1^{\text{ét}}(L_i)$ for all i. Consequently, L_i are minimal splitting fields of $T'_{\widehat{K}^a}$.

Now we consider a poset S defined the image of $a^n V$ via the map $V \to V[\frac{1}{a}] \to R$. The direct limit of Cauchy nets gives the following surjection

$$\mathrm{Cauchy}_S(R) \twoheadrightarrow \prod_{i=1}^r L_i.$$

Hence, the image of $R^\times \to \prod_{i=1}^r L_i^\times$ is dense. □

In the sequel, we may assume that $L_1 \simeq L_0$.

Lemma 4.2.8. *Let V be a valuation ring of finite rank n with prime $\mathfrak{p}$ of height $n-1$. For an $a \in \mathfrak{m}_V \backslash \mathfrak{p}$, let $\widehat{V}^a$ be the a-adic completion of V and $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. For a reductive group scheme G over V, we endow $G(\widehat{K}^a)$ with the a-adic topology. Let $\overline{G(V[\frac{1}{a}])}$ be the closure of the image of $G(V[\frac{1}{a}]) \to G(\widehat{K}^a)$. For a fixed maximal torus T of $G_{\widehat{K}^a}$ with minimal splitting field L_0, consider the norm map*

$$N_{L_0/\widehat{K}^a}\colon T(L_0) \to T(\widehat{K}^a).$$

Then, the image U of this norm map is an open subgroup of $T(\widehat{K}^a)$ and is contained in $\overline{G(V[\frac{1}{a}])}$.

Proof. Since $\underline{\mathrm{Tor}}(G)$ is of finite type over V, we endow $\underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ with the induced a-adic topology. First, we prove that U is open. Note that $\widehat{K}^a$ is a Henselian valued field, hence is étale-open in the sense of [Čes15, 2.8 (2)]. The kernel R^1T of the norm map $\mathrm{Res}_{L_0/\widehat{K}^a}(T_{L_0}) \to T$ is a torus[1] hence by [SGA 3_{II}, IX, 2.1 e)] is $\widehat{K}^a$-smooth. By the criterion [Čes15, 4.3], the norm map $N_{L_0/\widehat{K}^a}\colon T(L_0) \to T(\widehat{K}^a)$ is open so the image $U := N_{L_0/\widehat{K}^a}(T(L_0)) \subset T(\widehat{K}^a)$ is open. The proof for $U \subset \overline{G(V[\frac{1}{a}])}$ proceeds as follows.

[1]One can check that $R^1T'_{\widehat{K}^a}$ is a torus: when T' splits after a base change (of rank k), the associated $\mathbf{Z}$-module of the corresponding base change of $R^1T'_{\widehat{K}^a}$ has trivial Galois action and is the following

$$\mathrm{Coker}\left(\mathbf{Z}^k \to \mathbf{Z}[\mathrm{Gal}(L_0/\widehat{K}^a)]^k, (n_i) \mapsto (n_i \cdot \mathrm{id})\right) \simeq \mathbf{Z}[\mathrm{Gal}(L_0/\widehat{K}^a) - \{\mathrm{id}\}]^k, \qquad \text{which is torsion-free.}$$

(i) For the maximal torus $T' \in \underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])$ found in Corollary 4.2.6, by the argument for T in the very beginning, the image $U' := N_{L_0/\widehat{K}^a}(T'(L_0)) \subset T'(\widehat{K}^a)$ is open. By Lemma 4.2.7, the image of $R^\times \to \prod_{i=1}^r L_i^\times$ is dense. Note that T'_R is split, so the image of $T'(R) \to \prod_{i=1}^r T'(L_i) \overset{pr_1}{\to} T'(L_0)$ is dense. For a $u' \in U'$, there is $v' \in T'(L_0)$ such that $N_{L_0/\widehat{K}^a}(v') = u'$. Hence, there is a variable $r' \in T'(R)$ whose image approximates v'. Taking the norm image $N_{R/V[\frac{1}{a}]}(r') \in T'(V[\frac{1}{a}])$ of r', we see that $N_{R/V[\frac{1}{a}]}(r')$ approximates u'. Consequently, we have
$$U' \subset \overline{T'(V[\tfrac{1}{a}])}.$$

(ii) Let $(W_j)_{j\in J}$ be a filtered basis of open neighborhoods of $\mathrm{id} \in G(\widehat{K}^a)$. Just as the construction in (i), for each $j \in J$, there are $g_j \in W_j$ and $T'_j \in \underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])$ such that $T^{g_j} := g_j T g_j^{-1} = (T'_j)_{\widehat{K}^a} \in \phi(W_j)$. By transport of structure, the norm map $N_{L_0/\widehat{K}^a}$ sends $T^{g_j}(L_0)$ to $g_j(N_{L_0/\widehat{K}^a}(T(L_0)))g_j^{-1}$. Therefore, every $U_j := N_{L_0/\widehat{K}^a}(T^{g_j}(L_0))$ satisfies $U_j = g_j U g_j^{-1}$. For every open neighborhood $B_u \subset U$ of an element $u \in U$, the subsets $(W_j^{-1}uW_j)_{j\in J}$ form a basis of open neighborhood of u, hence $\{g_j^{-1}ug_j\}_{j\in J} \cap B_u \neq \emptyset$, which combined with $g_j B_u g_j^{-1} \subset U_j \subset \overline{G(V[\frac{1}{a}])}$ implies that $u \in \overline{G(V[\frac{1}{a}])}$. Thus, U is contained in $\overline{G(V[\frac{1}{a}])}$. □

Now, we establish the main result Proposition 4.2.9 of this section by constructing an open subgroup in the closure of $G(V[\frac{1}{a}])$. Moreover, by lumping together the approximations in toral cases (Lemma 4.2.8), we prove that the constructed open subgroup is normal. We will see, this normality is crucial for the dynamic argument for root groups when proving the product formula Proposition 5.1.6.

Proposition 4.2.9. *Let V be a valuation ring of finite rank n with prime $\mathfrak{p}$ of height $n-1$. For an $a \in \mathfrak{m}_V \backslash \mathfrak{p}$, let $\widehat{V}^a$ be the a-adic completion of V and $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. For a reductive group scheme G over V, we endow $G(\widehat{K}^a)$ with the a-adic topology. Let $\overline{G(V[\frac{1}{a}])}$ be the closure of the image of $G(V[\frac{1}{a}]) \to G(\widehat{K}^a)$. Then*
$$\overline{G(V[\tfrac{1}{a}])} \text{ contains an open normal subgroup } N \text{ of } G(\widehat{K}^a).$$

Proof.

(i) We fix a maximal torus $T \subset G_{\widehat{K}^a}$. We denote by $\mathfrak{g}$ the Lie algebra of $G_{\widehat{K}^a}$ and by $\mathfrak{h}$ the Lie algebra of T. For each $g \in G_{\widehat{K}^a}$, by [SGA 3_{II}, XIII, 2.6 b)], the subspace $\mathfrak{g}^{\mathrm{ad}(g)} \subset \mathfrak{g}$ fixed by $\mathrm{ad}(g)$ has dimension at least $\dim T$. We define the regular locus $G^{\mathrm{reg}} \subset G_{\widehat{K}^a}$ as the subscheme containing all $g \in G_{\widehat{K}^a}$ that satisfy $\dim(\mathfrak{g}^{\mathrm{ad}(g)}) = \dim T$. For a $t \in T$, by the following equation
$$\dim(\mathfrak{g}^{\mathrm{ad}(t)}) = \dim(\mathfrak{h}^{\mathrm{ad}(t)}) + \dim((\mathfrak{g}/\mathfrak{h})^{\mathrm{ad}(t)}),$$
t is regular in $G_{\widehat{K}^a}$ if and only if t is regular in T, in which case we have $(\mathfrak{g}/\mathfrak{h})^{\mathrm{ad}(t)} = 0$.

(ii) Recall the open subgroup $U \subset T(\widehat{K}^a)$ constructed in Lemma 4.2.8. We claim that $U \cap T^{\mathrm{reg}}(\widehat{K}^a) \neq \emptyset$. For the minimal splitting field L_0 of T, we consider the norm map $\mathrm{Nm}\colon \mathrm{Res}_{L_0/\widehat{K}^a}(T_{L_0}) \to T$. We note that $T_{L_0} \simeq \mathbb{G}_{m,L_0}^n$ is isomorphic to a dense open of

$\mathbb{A}^n_{L_0}$. Therefore, $\mathrm{Res}_{L_0/\widehat{K}^a}(T_{L_0})$ is also a dense open subset of $\mathbb{A}^{mn}_{\widehat{K}^a}$ for $m := [L_0 : \widehat{K}^a]$. The field $\widehat{K}^a$ is infinite, so $\left(\mathrm{Res}_{L_0/\widehat{K}^a}(T_{L_0})\right)(\widehat{K}^a)$ is Zariski dense and

$$\left(\mathrm{Res}_{L_0/\widehat{K}^a}(T_{L_0})\right)(\widehat{K}^a) \cap \mathrm{Nm}^{-1}(T^{\mathrm{reg}})(\widehat{K}^a) \neq \emptyset.$$

Applying Nm to this nonempty intersection proves the claim.

(iii) For a fixed $t_0 \in U \cap T^{\mathrm{reg}}(\widehat{K}^a)$, by (i), we have $(\mathfrak{g}/\mathfrak{h})^{\mathrm{ad}(t_0)} = 0$. So [SGA 3_{II}, XIII, 2.2] implies that

$$f\colon G_{\widehat{K}^a} \times T \to G_{\widehat{K}^a}, \qquad (g,t) \mapsto gtg^{-1}$$

is smooth at (id, t_0). Thus, there is an open neighborhood W of (id, t_0) such that $W(\widehat{K}^a) \subset G(\widehat{K}^a) \times U$ and $f|_W$ is smooth. Translate W by $G(\widehat{K}^a)$-action, there is an open neighborhood $U_0 \subset U$ of t_0 such that $U_0 \subset W(\widehat{K}^a)$ and $f|_{G(\widehat{K}^a)\times U_0}$ is smooth. The criterion [Čes15, 2.9 (a)] implies that $E := f(G(\widehat{K}^a) \times U_0)$ is an open subset of $G(\widehat{K}^a)$. Let N be the subgroup of $G(\widehat{K}^a)$ generated by E. Hence, N contains the open subset $f(G(\widehat{K}^a) \times U_0)$ so is an open subgroup.

(iv) Since E is stable under $G(\widehat{K}^a)$-conjugation, N is a normal subgroup of $G(\widehat{K}^a)$. For each $g \in G(\widehat{K}^a)$, we denote $T^g := gTg^{-1}$. The image $U^g := N_{L_0/\widehat{K}^a}(T^g(L_0))$ satisfies $U^g = gUg^{-1}$. Using Lemma 4.2.8 to T^g, we have $U^g \subset \overline{G(V[\frac{1}{a}])}$. Since E is the union of gUg^{-1} for all $g \in G(\widehat{K}^a)$, the closure of $G(V[\frac{1}{a}])$ contains E, hence contains N. □

Corollary 4.2.10. *With the notations in Proposition 4.2.9, then $\overline{G(V[\frac{1}{a}])}$ is an open subgroup of $G(\widehat{K}^a)$ and*

$$\overline{G(V[\tfrac{1}{a}])} \cdot G(\widehat{V}^a) = \mathrm{Im}\big(G(V[\tfrac{1}{a}]) \to G(\widehat{K}^a)\big) \cdot G(\widehat{V}^a).$$

Proof. The image of $G(V[\frac{1}{a}]) \to G(\widehat{K}^a)$ is a subgroup of $G(\widehat{K}^a)$, hence so is its closure $\overline{G(V[\frac{1}{a}])}$. Since $\overline{G(V[\frac{1}{a}])}$ contains an open subset N, it is an open (and therefore closed) subgroup of $G(\widehat{K}^a)$. Because G is affine, by [Con12, 2.2], the subgroup $G(\widehat{V}^a) \subset G(\widehat{K}^a)$ is open, hence the equality follows. □

Remark 4.2.11. Here, we compare Harder's original argument with the proof of Proposition 4.2.9. For a reductive group scheme H over a valued field F equipped with finitely many nontrivial nonequivalent valuations $v \in \mathcal{V}$ of rank one, we consider the completions F_v of F at $v \in \mathcal{V}$. The product $\prod_{v\in\mathcal{V}} H(F_v)$ is endowed with the product topology of v-adic topologies. For the diagonal map $i\colon H(F) \to \prod_{v\in\mathcal{V}} H(F_v)$, Harder proved that the closure of $i(H(F))$ contains an open normal subgroup of $\prod_{v\in\mathcal{V}} H(F_v)$ (see [Har68]). However, Harder's argument is only feasible for the case of rank one. Let v be a higher-rank valuation of F. The point is, F_v is not Henselian in general, see [EP05, 2.4.6]. Without the Henselianity, the criterion [Čes15, 4.3], which is important for Harder's rank-one case [Guo20, proof of 2.1], does not apply.

Chapter 5

Passage to the Henselian rank one case: patching by a product formula

5.1 Passage to the Henselian rank one case: patching by a product formula

The aim of this section is to reduce Theorem 2.0.3 to the case when V is a Henselian valuation ring of rank one. The key of our reduction is the following product formula for patching trivial torsors

$$G(\widehat{K^a}) = \mathrm{Im}\big(G(V[\tfrac{1}{a}]) \to G(\widehat{K^a})\big) \cdot G(\widehat{V^a}). \qquad \text{(Proposition 5.1.6)}$$

To show this, we establish several properties (Proposition 5.1.3) of an anisotropic group over a Henselian valuation ring. Based on Maculan's compactness [Mac17, Thm. 1.1] of anisotropic groups over complete nonarchimedean valued fields, we use the algebraization technique in [BČ20, §2] to prove the same for the Henselian case, generalizing its discrete valued case [Guo20, 3.6]. Subsequently, by the Harder-type approximation in Proposition 4.2.9, we prove the product formula, which gives the reduction Proposition 5.1.7.

First, we recall a criterion for anisotropicity [SGA $3_{\mathrm{III\ new}}$, XXVI, 6.14], which is practically useful.

Lemma 5.1.1. *A reductive group scheme G over a semilocal connected scheme S is anisotropic if and only if G has no proper parabolic subgroup and* $\mathrm{rad}(G)$ *contains no copy of* $\mathbb{G}_m$.

Precisely, to determine whether G is anisotropic, we consider the functor

$$\underline{\mathrm{Par}}(H)\colon \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}, \qquad S' \mapsto \{\text{parabolic subgroups of } H_{S'}\},$$

which is representable by a smooth projective S-scheme (see [SGA $3_{\mathrm{III\ new}}$, XXVI, 3.5]). To characterize the splitting behaviour of tori, recall the anti-equivalence of categories in [SGA 3_{II}, X, 1.2]

$$\big\{\text{isotrivial tori over } S\big\} \xrightarrow{\sim} \left\{\begin{matrix}\text{finite } \mathbb{Z}\text{-lattices with}\\ \text{continuous } \pi_1^{\text{ét}}(S)\text{-action}\end{matrix}\right\} \qquad T \mapsto M(T) := \mathrm{Hom}_{\overline{s}\text{-gr.}}(T_{\overline{s}}, \mathbb{G}_{m,\overline{s}}),$$

where $\overline{s}$ is a fixed geometric point of S that is used as the base point when forming the étale fundamental group $\pi_1^{\text{ét}}(S)$. Therefore, a split subtorus of T corresponds to a quotient lattice of $M(T)$ with trivial $\pi_1^{\text{ét}}(S)$-action. By Lemma 3.1.1, when S is a valuation ring, the left-hand side is the category of all S-tori.

Then, we use Lemma 5.1.1 to establish several equalities of split ranks of reductive group schemes over valuation rings and their completions, deducing useful properties of anisotropic groups.

Lemma 5.1.2. *For a reductive group scheme G over a Henselian valuation ring V with fraction field K and the completion $\widehat{K}$ of K with valuation ring $\widehat{V}$, we have*

(i) *the image of the map $\underline{\mathrm{Par}}(G)(K) \to \underline{\mathrm{Par}}(G)(\widehat{K})$ is dense.*

(ii) *if $\underline{\mathrm{Par}}(G)(\widehat{V}) \neq \{*\}$, then $\underline{\mathrm{Par}}(G)(V) \neq \{*\}$.*

Proof. Since $\underline{\mathrm{Par}}(G)$ is V-smooth, the assertion (i) follows from [BČ20, 2.2.9(iii)]. With (i), by the criterion of properness for $\underline{\mathrm{Par}}(G)$, we obtained (ii). □

Proposition 5.1.3. *Let G be a reductive group scheme over a valuation ring V with fraction field K.*

(a) *The V-split rank of G equals the K-split rank of G_K.*

(b) *For Henselian V and the completion of V, the V-split rank of G equals the $\widehat{V}$-split rank of $G_{\widehat{V}}$.*

(c) *For Henselian V, an arbitrary element $a \in \mathfrak{m}_V$, and the a-adic completion $\widehat{V}^a$ of V, the V-split rank of G equals the $\widehat{V}^a$-split rank of $G_{\widehat{V}^a}$.*

(d) *If V is Henselian, then G is anisotropic if and only if $G(V) = G(K)$.*

Proof.

(a) We consider a Levi subgroup L of a minimal parabolic $P \subset G$. The maximal V-split subtorus $\mathrm{rad}(L)_{split} \subset \mathrm{rad}(L)$ is central in L and $L/\mathrm{rad}(L)_{split}$ is anisotropic. By passing to the anisotropic quotient $L/\mathrm{rad}(L)_{split}$, we are reduced to show that if G is anisotropic if and only if so is G_K.

The "if" part follows, since the base change of $\mathbb{G}_{m,V} \subset G$ is $\mathbb{G}_{m,K} \subset G_K$. Given that G is anisotropic, by Lemma 5.1.1, G has no proper parabolic subgroup and $\mathrm{rad}(G)$ contains no copy of $\mathbb{G}_m$. The valuative criterion for the projectivity of $\underline{\mathrm{Par}}(G)$ implies that

$$\underline{\mathrm{Par}}(G)(K) = \underline{\mathrm{Par}}(G)(V) = \{*\},$$

where "$*$" denotes the trivial parabolic subgroup G or G_K. If $\mathrm{rad}(G_K) = \mathrm{rad}(G)_K$ contains a nontrivial split torus, then $M(\mathrm{rad}(G_K))$ has a quotient lattice permitting trivial $\pi_1^{\text{ét}}(K)$-action. The valuation ring V is normal, the surjectivity of $\pi_1^{\text{ét}}(K) \to \pi_1^{\text{ét}}(V)$ follows from [SP, 0BQM], giving $M(\mathrm{rad}(G)) = M(\mathrm{rad}(G_K))$ a nontrivial quotient lattice with trivial $\pi_1^{\text{ét}}(V)$-action. So $\mathrm{rad}(G)$ contains a copy of $\mathbb{G}_m$, which contradicts the assumption. Hence, G_K is also anisotropic.

(b) By a similar reduction in (a), we are reduced to show that G is anisotropic if and only if so is $G_{\widehat{V}}$. The deduction from the anisotropicity of $G_{\widehat{V}}$ to that of G is trivial. Assume that $G_{\widehat{V}}$ is not anisotropic, it remains to show that either $\underline{\mathrm{Par}}(G)(V) \neq \{*\}$, or $\mathrm{rad}(G)$ contains a copy of $\mathbb{G}_m$. The part for $\underline{\mathrm{Par}}(G)(V)$ has been processed in Lemma 5.1.2. Now, assume that $\mathrm{rad}(G_{\widehat{V}^a})$ contains a copy of $\mathbb{G}_m$. We consider the functor

$$\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))\colon \mathbf{Sch}^{\mathrm{op}}_{/V} \to \mathbf{Set}, \qquad R \mapsto \mathrm{Hom}_{R\text{-gr.}}(\mathbb{G}_{m,R}, \mathrm{rad}(G)_R),$$

which is representable by an étale locally constant group scheme (see [SGA 3_{II}, X, 5.6]). Since $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))(\widehat{V}^a) \neq \emptyset$, we have

$$\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))(V/\mathfrak{m}_V) = \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))(\widehat{V}^a/\mathfrak{m}_{\widehat{V}^a}) \neq \emptyset.$$

Since V is Henselian and $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))$ is V-smooth, the map

$$\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))(V) \to \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathrm{rad}(G))(V/\mathfrak{m}_V) \neq \emptyset$$

is surjective. Consequently, $\mathrm{rad}(G)$ contains a copy of $\mathbb{G}_m$, so by Lemma 5.1.1, G is not anisotropic.

(c) Among all prime ideals of V containing (a), by Zorn's lemma, there exists the minimum. This minimum becomes the height-one prime of the a-adic completion $\widehat{V}^a$ of V, making $\widehat{V}^a$ microbial (see [Hub96, I, Def. 1.1.4]) such that the valuation topology on $\widehat{V}^a$ coincides with the a-adic topology. Hence, $\widehat{V}^a$ is complete with respect to its valuation topology. So (c) follows from (b).

(d) If we have $G(K) = G(V)$, then $G(K)$ is bounded. Therefore, G does not contain any nontrivial split torus, so G is anisotropic. Now we assume that G is anisotropic. By [BM20, 2.22], V is a filtered direct union of valuation subrings V_i of finite rank, such that each $V_i \to V$ is a local ring map. Further, the Henselization V_i^{h} of each V_i is also a valuation subring of V and by [SP, 0ASK] is of finite rank. Since the filtered inductive limit of Henselian local rings along local ring maps is a Henselian local ring ([SP, 04GI]), V is a filtered direct union of Henselian valuation subrings V_i^{h} of finite rank. Similarly, K is a filtered direct union of $K_i^{\mathrm{h}} := \mathrm{Frac}(V_i^{\mathrm{h}})$. Since G is finitely presented over V, there is an index i_0 and an affine group scheme G_{i_0} smooth of finitely presented ([Nag66, Thm. 3']) over $V_{i_0}^{\mathrm{h}}$ such that the base change of G_{i_0} over V is isomorphic to G. Further, by [Con14, 3.1.11], G_{i_0} and hence $(G_i)_{i \geq i_0}$ are reductive group schemes. It is clear that all $(G_i)_{i \geq i_0}$ are anisotropic. By a limit argument [SP, 01ZC], we have $G(V) = \varinjlim_{i \geq i_0} G(V_i^{\mathrm{h}})$ and $G(K) = \varinjlim_{i \geq i_0} G(K_i^{\mathrm{h}})$. Subsequently, we reduce to the case when V is Henselian of finite rank n.

When $n = 0$, we have $V = K$ is a field and the assertion is trivial. Now, we prove the case when V is of rank one. For $a \in \mathfrak{m}_V \backslash \{0\}$, we form the a-adic completion $\widehat{V}^a$ of V with $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. By (c), $G_{\widehat{V}^a}$ is anisotropic. For the nonarchimedean complete valued field $\widehat{K}^a$, by [Mac17, Thm. 1.1], $G(\widehat{V}^a)$ is a maximal bounded subgroup of $G(\widehat{K}^a)$. On the other hand, a result of Bruhat–Tits–Rousseau [Rou77, Thm. 5.2.3] shows that $G(\widehat{K}^a)$ is bounded. Consequently, we have $G(\widehat{V}^a) = G(\widehat{K}^a)$. The rank-one assumption ensures that $V \hookrightarrow \widehat{V}^a$, so the map $G(V) \hookrightarrow G(\widehat{V}^a)$ is injective. The fibre product $V = K \times_{\widehat{K}^a} \widehat{V}^a$ (see [SP, 0BNR]) and the affineness of G yield

$$G(V) \xrightarrow{\sim} G(K) \times_{G(\widehat{K}^a)} G(\widehat{V}^a) \cong G(K).$$

When V is of rank $n > 1$, we assume the assertion holds for the case of rank $\le n-1$ and prove by induction. For the prime $\mathfrak{p} \subset V$ of height $n-1$, the localization $V_{\mathfrak{p}}$ and the quotient $V/\mathfrak{p}$ are Henselian valuation rings[1]. Since V is Henselian, a section of $\underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ over $V/\mathfrak{m}_V$ lifts to a section over V. Hence, $G_{V/\mathfrak{m}_V}$ is anisotropic and so is $G_{V/\mathfrak{p}}$. By (a), G is anisotropic implies that G_K and $G_{V_{\mathfrak{p}}}$ are anisotropic. By the settled rank-one case and the induction hypothesis, we have

$$G(V/\mathfrak{p}) = G(V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}}) \qquad \text{and} \qquad G(V_{\mathfrak{p}}) = G(K). \tag{5.1}$$

The affineness of G and the fibre product $V \xrightarrow{\sim} V_{\mathfrak{p}} \times_{V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}}} V/\mathfrak{p}$ imply the fibre product

$$G(V) \xrightarrow{\sim} G(V_{\mathfrak{p}}) \times_{G(V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}})} G(V/\mathfrak{p}V). \tag{5.2}$$

Therefore, the combination of (5.2) and (5.1) gives us the desired equation $G(V) = G(K)$. □

As an application of Proposition 5.1.3, we prove a higher-rank (non-Noetherian) counterpart of [Gil09, Lem. 4.5 (1)], which served as a useful ingredient when Fedorov and Panin proved the Grothendieck–Serre conjecture over regular local rings containing infinite fields (see [FP15, 5.1]).

Corollary 5.1.4. *For a semisimple simply-connected group scheme G over a Henselian valuation ring $(V, \mathfrak{m}_V)$ with fraction field K such that $G_{V/\mathfrak{m}_V}$ is isotropic, we consider all subgroups $U \subset G_K$ that are isomorphic to $\mathbb{G}_{a,K}$ and let $G(K)^+$ be the normal subgroup of $G(K)$ generated by all $U(K)$. We have*

$$G(K) = G(K)^+ \cdot G(V).$$

Proof. The group $G_{V/\mathfrak{m}_V}$ is isotropic and semisimple, hence by Lemma 5.1.1, there is a proper minimal parabolic subgroup $P \subset G_{V/\mathfrak{m}_V}$. Since $\underline{\mathrm{Par}}(G)$ is V-smooth ([SGA $3_{\mathrm{III\ new}}$, XXVI, 3.5]) and V is Henselian, P lifts to a proper minimal parabolic subgroup $\widetilde{P}$ of G. For the maximal split torus T of G_K contained in P, we let r be the rank of T and form the functor

$$\underline{\mathrm{Hom}}(\mathbb{G}_m^r, G) \colon \mathbf{Sch}_{/V}^{\mathrm{op}} \to \mathbf{Set}, \qquad R \mapsto \mathrm{Hom}_{R\text{-gr.}}(\mathbb{G}_{m,R}^r, G_R),$$

which by [SGA 3_{II}, XI, 2.1, 3.12] is representable by a smooth V-scheme. The Henselianity of V yields a surjection $\underline{\mathrm{Hom}}(\mathbb{G}_m^r, G)(V) \twoheadrightarrow \underline{\mathrm{Hom}}(\mathbb{G}_m^r, G)(V/\mathfrak{m}_V)$, which means that there exists a group homomorphism $\tau \colon \mathbb{G}_{m,V} \to G$ with special fibre $\tau_{V/\mathfrak{m}_V} \colon T \hookrightarrow G_{V/\mathfrak{m}_V}$. By [SGA 3_{II}, IX, 6.6.1], the homomorphism τ defines a subgroup $\widetilde{T} := \mathbb{G}_m^r \subset G$, which is a maximal split torus

[1] By [SP, 05WQ], the rank-one valuation ring $V/\mathfrak{p}$ is Henselian. For $V_{\mathfrak{p}}$, we use Gabber's criterion [SP, 09XI]: that is, we need to check that every monic polynomial of the form

$$f(T) = T^N(T-1) + a_N T^N + \cdots + a_1 T + a_0, \qquad \text{where } a_i \in \mathfrak{p}V_{\mathfrak{p}} \text{ for } i = 0, \cdots, N \text{ and } N \ge 1$$

has a root in $1 + \mathfrak{p}V_{\mathfrak{p}}$. Here, by the footnote 2, we identify $\mathfrak{p}V_{\mathfrak{p}}$ with $\mathfrak{p}$. By [SP, 0DYD], the Henselianity of V implies that $(V, \mathfrak{p}V)$ is a Henselian pair, hence $(V_{\mathfrak{p}}, \mathfrak{p}V_{\mathfrak{p}})$ satisfies Gabber's criterion.

of G lifting T. By [SGA $3_{\text{III new}}$, XXVI, 6.11], the centralizer $\widetilde{L} := \underline{\mathrm{Centr}}_G(\widetilde{T})$ of $\widetilde{T}$ is a Levi subgroup of $\widetilde{P}$. The quotient $\widetilde{L}/\widetilde{T}$ is anisotropic over V, hence Proposition 5.1.3 (d) implies that $\big(\widetilde{L}/\widetilde{T}\big)(V) = \big(\widetilde{L}/\widetilde{T}\big)(K)$, fitting into the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{T}(V) \arrow[r] \arrow[d] & \widetilde{L}(V) \arrow[r] \arrow[d] & (\widetilde{L}/\widetilde{T})(V) \arrow[r] \arrow[d, equal] & H^1(V, \widetilde{T}) \arrow[d] \\ 0 \arrow[r] & \widetilde{T}(K) \arrow[r] & \widetilde{L}(K) \arrow[r] & (\widetilde{L}/\widetilde{T})(K) \arrow[r] & H^1(K, \widetilde{T}), \end{tikzcd}$$

where two rightmost terms vanish by Hilbert's theorem 90. By diagram chase, we have a product

$$\widetilde{L}(K) = \widetilde{L}(V) \cdot \widetilde{T}(K). \tag{5.3}$$

On the other hand, by [BT73, Cor. 6.8], we have $\widetilde{T}(K) \subset G(K)^+ \cap \widetilde{T}(K)$, that is, $\widetilde{T}(K) \subset G(K)^+$. Combining this with (5.3), we obtain $\widetilde{L}(K) \subset G(V) \cdot G(K)^+$. By [Tit64, 3.1 (20)], the quotient $G(K)/G(K)^+$ is generated by $\widetilde{L}(K)$, hence $G(K) \subset G(V) \cdot G(K)^+$, as desired. $\square$

The following lemma provides the tori version of the product formula.

Lemma 5.1.5. *For a valuation ring V of rank $n > 0$ with prime $\mathfrak{p}$ of height $n-1$, we take an $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ and form the a-adic completion $\widehat{V}^a$ with $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. For a V-torus T, we have the product formula*

$$T(\widehat{K}^a) = \mathrm{Im}\big(T(V[\tfrac{1}{a}]) \to T(\widehat{K}^a)\big) \cdot T(\widehat{V}^a).$$

Proof. The left-hand side contains the right-hand side, so it remains to show that every element of $T(\widehat{K}^a)$ is a product of elements of $\mathrm{Im}\big(T(V[\frac{1}{a}]) \to T(\widehat{K}^a)\big)$ and $T(\widehat{V}^a)$. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T(V) \arrow[r] \arrow[d] & T(V[\frac{1}{a}]) \arrow[r] \arrow[d] & H^1_{V/(a)}(V,T) \arrow[r] \arrow[d] & H^1(V,T) \arrow[r] \arrow[d] & H^1(V[\frac{1}{a}],T) \arrow[d] \\ 0 \arrow[r] & T(V^{\mathrm{h}}) \arrow[r] \arrow[d] & T(V^{\mathrm{h}}[\frac{1}{a}]) \arrow[r] \arrow[d] & H^1_{V^{\mathrm{h}}/(a)}(V^{\mathrm{h}},T) \arrow[r] \arrow[d] & H^1(V^{\mathrm{h}},T) \arrow[r] \arrow[d] & H^1(V^{\mathrm{h}}[\frac{1}{a}],T) \arrow[d] \\ 0 \arrow[r] & T(\widehat{V}^a) \arrow[r] & T(\widehat{K}^a) \arrow[r] & H^1_{\widehat{V}^a/(a)}(\widehat{V}^a,T) \arrow[r] & H^1(\widehat{V}^a,T) \arrow[r] & H^1(\widehat{K}^a,T), \end{tikzcd}$$

where V^{h} is the Henselization of V and the rows are exact sequences of local cohomology [SGA 4_{II}, V, 6.5.3]. By [SP, 0F0L], V^{h} is also the a-Henselization of V, hence the a-adic completion of V^{h} is $\widehat{V}^a$ (see [FK18, 0, 7.3.5]). By the tori case Proposition 3.1.3, the horizontal morphisms in the two rightmost squares are injective. The excision [Mil80, III, 1.28] combined with a limit argument yield an isomorphism $H^1_{V/(a)}(V,G) \cong H^1_{V^{\mathrm{h}}/(a)}(V^{\mathrm{h}})$. Therefore, a diagram chase gives the following product

$$T(V^{\mathrm{h}}[\tfrac{1}{a}]) = \mathrm{Im}\,\big(T(V[\tfrac{1}{a}]) \to T(V^{\mathrm{h}}[\tfrac{1}{a}])\big) \cdot T(V^{\mathrm{h}}) \tag{5.4}$$

By [BČ20, 2.2.16], the image of $T(V^{\mathrm{h}}[\frac{1}{a}]) \to T(\widehat{K}^a)$ is dense. The openness of $T(\widehat{V}^a)$ in $T(\widehat{K}^a)$ implies

$$\mathrm{Im}\big(T(V^{\mathrm{h}}[\tfrac{1}{a}]) \to T(\widehat{K}^a)\big) \cdot T(\widehat{V}^a) = \overline{\mathrm{Im}\big(T(V^{\mathrm{h}}[\tfrac{1}{a}]) \to T(\widehat{K}^a)\big)} \cdot T(\widehat{V}^a) = T(\widehat{K}^a). \tag{5.5}$$

Combining (5.4) and (5.5), we obtain the product formula for the case of tori. $\square$

Proposition 5.1.6. *For a valuation ring V of rank $n > 0$ with prime $\mathfrak{p}$ of height $n-1$, we take $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ to form the a-adic completion $\widehat{V}^a$ of V with $\widehat{K}^a := \mathrm{Frac}(\widehat{V}^a)$. For a reductive V-group scheme G, we denote by $G(\widehat{V}^a)$ its image in $G(\widehat{K}^a)$ and by $\mathrm{Im}(G(V[\frac{1}{a}]))$ the image of $G(V[\frac{1}{a}])$ in $G(\widehat{K}^a)$. Then,*

$$G(\widehat{K}^a) = \mathrm{Im}\left(G(V[\tfrac{1}{a}])\right) \cdot G(\widehat{V}^a).$$

Proof. The right-hand side is contained in the left-hand side, so we only need to show that every element of $G(\widehat{K}^a)$ is a product of elements of $\mathrm{Im}\left(G(V[\frac{1}{a}])\right)$ and $G(\widehat{V}^a)$.

- The case when $G_{\widehat{V}^a}$ is anisotropic follows from Proposition 5.1.3. When $G_{\widehat{V}^a}$ contains no proper parabolic subgroup and $\mathrm{rad}(G_{\widehat{V}^a})$ contains a nontrivial split torus of $G_{\widehat{V}^a}$, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{rad}(G)(\widehat{V}^a) \arrow[r] \arrow[d] & G(\widehat{V}^a) \arrow[r] \arrow[d] & (G/\mathrm{rad}(G))(\widehat{V}^a) \arrow[r] \arrow[d, equal] & H^1(\widehat{V}^a, \mathrm{rad}(G)) \arrow[d] \\ 0 \arrow[r] & \mathrm{rad}(G)(\widehat{K}^a) \arrow[r] & G(\widehat{K}^a) \arrow[r] & (G/\mathrm{rad}(G))(\widehat{K}^a) \arrow[r] & H^1(\widehat{K}^a, \mathrm{rad}(G)), \end{tikzcd} \tag{5.6}$$

  where the rows are exact, and the equality follows from Lemma 5.1.1 and Proposition 5.1.3 (d). Since $\mathrm{rad}(G_{\widehat{V}^a})$ is a torus, by Proposition 3.1.3, the last vertical arrow is injective. Thus, a diagram chase gives $G(\widehat{K}^a) = \mathrm{rad}(G)(\widehat{K}^a) \cdot G(\widehat{V}^a)$ so the product formula for $\mathrm{rad}(G)$ (Lemma 5.1.5) leads to the assertion.

- By Lemma 5.1.1, the remaining case is when $G_{\widehat{V}^a}$ contains a proper parabolic subgroup. For a minimal parabolic subgroup P of $G_{\widehat{V}^a}$, we denote its unipotent radical by $U := \mathrm{rad}^u(P)$. As exhibited in [SGA $3_{\mathrm{III\ new}}$, XXVI, 6.11], the centralizer of a maximal split torus $T \subset G_{\widehat{V}^a}$ is a Levi subgroup L of P. By *op. cit.* 2.4 *ff.*, there is a maximal torus $\widetilde{T}$ of G containing T. We denote the base change by $\widehat{T} := \widetilde{T}_{\widehat{K}^a}$, which is a maximal torus of $G_{\widehat{K}^a}$. As mentioned in (i) of the proof of Lemma 4.2.8, the following map

$$\phi \colon G(\widehat{K}^a) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a), \qquad g \mapsto g\widehat{T}g^{-1}$$

  is open and sends an open neighborhood W of $\mathrm{id} \in G(\widehat{K}^a)$ to an open neighborhood $\phi(W)$ of $\widehat{T} \in \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$. By Proposition 4.2.9, there is an open normal subgroup $N \subset G(\widehat{K}^a)$ contained in $\mathrm{Im}\left(G(V[\frac{1}{a}])\right)$. Since G is affine, by [Con12, 2.2], $G(\widehat{V}^a) \subset G(\widehat{K}^a)$ is closed and open. So, $N \cap G(\widehat{V}^a)$ is an open neighborhood of identity. Similarly, the affineness of $\underline{\mathrm{Tor}}(G)$ implies that $\underline{\mathrm{Tor}}(G)(\widehat{V}^a) \subset \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ is closed and open. Now, we choose W such that $W \subset N \cap G(\widehat{V}^a)$. Therefore, every $g \in W$ is contained in $\mathrm{Im}\left(G(V[\frac{1}{a}])\right) \cap G(\widehat{V}^a)$ and $\phi(W) \cap \underline{\mathrm{Tor}}(G)(\widehat{V}^a)$ is an open subset of $\underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ containing $\widehat{T}$. The fibre product $V \xrightarrow{\sim} V[\frac{1}{a}] \times_{\widehat{K}^a} \widehat{V}^a$ ([SP, 0BNR]) and the affineness of $\underline{\mathrm{Tor}}(G)$ give the isomorphism

$$\underline{\mathrm{Tor}}(G)(V) \xrightarrow{\sim} \underline{\mathrm{Tor}}(G)(V[\tfrac{1}{a}]) \times_{\underline{\mathrm{Tor}}(G)(\widehat{K}^a)} \underline{\mathrm{Tor}}(G)(\widehat{V}^a). \tag{5.7}$$

  The density Lemma 4.2.5 of $\underline{\mathrm{Tor}}(G)(V[\frac{1}{a}]) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ implies that the intersection of $\phi(W) \cap \underline{\mathrm{Tor}}(G)(\widehat{V}^a)$ and $\mathrm{Im}\left(\underline{\mathrm{Tor}}(G)(V[\frac{1}{a}])\right)$ is nonempty, yielding a maximal torus

$T_0 \in \underline{\mathrm{Tor}}(G)(V)$ and a $g \in W$ such that $(T_0)_{\widehat{K}^a} = g\widehat{T}g^{-1} \in \phi(W)$. By the product formula for tori (Lemma 5.1.5), we have $T_0(\widehat{K}^a) = \mathrm{Im}\left(T_0(V[\frac{1}{a}])\right) \cdot T_0(\widehat{V}^a)$. Taking $\widehat{K}^a$-points of $\widehat{T} = g^{-1}(T_0)_{\widehat{K}^a}g$, we deduce that

$$\widehat{T}(\widehat{K}^a) = g^{-1}T_0(\widehat{K}^a)g = g^{-1}\mathrm{Im}\left(T_0(V[\tfrac{1}{a}])\right)T_0(\widehat{V}^a)g \subset g^{-1}\mathrm{Im}\left(G(V[\tfrac{1}{a}])\right)G(\widehat{V}^a)g. \quad (5.8)$$

Since $g \in W$ is contained in $\overline{\mathrm{Im}\left(G(V[\frac{1}{a}])\right)} \cap G(\widehat{V}^a)$, (5.8) implies that

$$\widehat{T}(\widehat{K}^a) \subset \overline{\mathrm{Im}\left(G(V[\tfrac{1}{a}])\right)}G(\widehat{V}^a).$$

Note that Corollary 4.2.10 gives us $\overline{\mathrm{Im}(G(V[\frac{1}{a}]))G(\widehat{V}^a)} = \mathrm{Im}(G(V[\frac{1}{a}]))G(\widehat{V}^a)$. Consequently, we have

$$T(\widehat{K}^a) \subset \widetilde{T}(\widehat{K}^a) = \widehat{T}(\widehat{K}^a) \subset \mathrm{Im}\left(G(V[\tfrac{1}{a}])\right)G(\widehat{V}^a). \quad (5.9)$$

- In this paragraph, we prove that $U(\widehat{K}^a) \subset \overline{\mathrm{Im}\left(G(V[\frac{1}{a}])\right)}$. The maximal split torus T acts on $G_{\widehat{V}^a}$

$$T \times G_{\widehat{V}^a} \to G_{\widehat{V}^a}, \qquad (t, g) \mapsto tgt^{-1},$$

inducing a weight decomposition $\mathrm{Lie}(G_{\widehat{V}^a}) = \bigoplus_{\alpha \in X^*(T)} \mathrm{Lie}(G_{\widehat{V}^a})^\alpha$, where $X^*(T)$ is the character lattice of T. The subset $\Phi \subset X^*(T) - \{0\}$ such that $\mathrm{Lie}(G_{\widehat{V}^a})^\alpha \neq 0$ is the relative root system of $(G_{\widehat{V}^a}, T)$. By [SGA $3_{\text{III new}}$, XXVI, 6.1; 7.4], $\mathrm{Lie}(L)$ is the zero-weight space of $\mathrm{Lie}(G_{\widehat{V}^a})$ and the set Φ_+ of positive roots of the relative root datum $((G_{\widehat{V}^a}, T), X^*(T), \Phi)$ fits into the decomposition

$$\mathrm{Lie}(P) = \mathrm{Lie}(L) \oplus \left(\bigoplus\nolimits_{\alpha \in \Phi_+} \mathrm{Lie}(G_{\widehat{V}^a})^\alpha\right) \qquad \text{with} \qquad \mathrm{Lie}(U) = \bigoplus\nolimits_{\alpha \in \Phi_+} \mathrm{Lie}(G_{\widehat{V}^a})^\alpha.$$

Let $\widetilde{K}/\widehat{K}^a$ be a Galois field extension splitting $G_{\widehat{V}^a}$. By *op. cit.* 2.4 *ff.*, there is a split maximal torus $T' \subset L_{\widetilde{K}} \subset P_{\widetilde{K}}$ of $G_{\widetilde{K}}$ containing $T_{\widetilde{K}}$. The centralizer of T' in $G_{\widetilde{K}}$ is itself, which is also a Levi subgroup of a Borel $\widetilde{K}$-subgroup $B \subset P_{\widetilde{K}}$. The adjoint action of T' on $G_{\widetilde{K}}$ induces a decomposition

$$\mathrm{Lie}(G_{\widetilde{K}}) = \bigoplus\nolimits_{\alpha \in X^*(T')} \mathrm{Lie}(G_{\widetilde{K}})^\alpha,$$

which is a coarsening of the base change of $\mathrm{Lie}(G_{\widehat{V}^a}) = \bigoplus_{\alpha \in X^*(T)} \mathrm{Lie}(G_{\widehat{V}^a})^\alpha$ over $\widetilde{K}$. For the root system Φ' with the positive set Φ'_+ for the Borel B, *op. cit.* 7.12 gives us a surjection $\eta\colon X^*(T') \twoheadrightarrow X^*(T)$ such that $\Phi_+ \subset u(\Phi'_+) \subset \Phi_+ \cup \{0\}$. By *op. cit.* 1.12, we have a decomposition

$$U_{\widetilde{K}} = \prod\nolimits_{\alpha \in \Phi''} U_{\widetilde{K},\alpha}, \qquad \mathrm{Lie}(U_{\widetilde{K}}) = \bigoplus\nolimits_{\alpha \in \Phi''} \mathrm{Lie}(G_{\widetilde{K}})^\alpha,$$

where $\Phi'' \subset \Phi'_+$ and we have isomorphisms $f_\alpha\colon U_{\widetilde{K},\alpha} \xrightarrow{\sim} \mathbb{G}_{a,\widetilde{K}}$. Since $\mathrm{Lie}(L) \subset \mathrm{Lie}(G_{\widehat{V}^a})$ is the zero-weight space for the T-action, the restriction to T of weights in $\mathrm{Lie}(U_{\widetilde{K}})$ must be nonzero, that is $\eta(\Phi'') \subset \Phi_+$. For a cocharacter $\xi\colon \mathbb{G}_m \to T$, the dual map $\eta^*\colon X_*(T) \hookrightarrow X_*(T')$ of u sends ξ to a cocharacter $\eta^*(\xi) \in X_*(T')$ of $T_{\widetilde{K}}$. The adjoint action of $\mathbb{G}_m$ on U induced by ξ is denoted by

$$\mathrm{ad}\colon \mathbb{G}_m(\widehat{K}^a) \times U(\widehat{K}^a) \to U(\widehat{K}^a), \qquad (t, u) \mapsto \xi(t)u\xi(t)^{-1}.$$

For the open normal subgroup $N \subset G(\widehat{K^a})$ constructed in Proposition 4.2.9, the intersection $N \cap U(\widehat{K^a})$ is open, nonempty and stable under $\mathbb{G}_m(\widehat{K^a})$-action. We consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m(\widehat{K^a}) \times (N \cap U(\widehat{K^a})) \arrow[r, "\xi \times \mathrm{id}"] \arrow[d] & T(\widehat{K^a}) \times (N \cap U(\widehat{K^a})) \arrow[r, "\mathrm{ad}"] \arrow[d] & N \cap U(\widehat{K^a}) \arrow[d] \\
\mathbb{G}_m(\widehat{K^a}) \times U(\widehat{K^a}) \arrow[r, "\xi \times \mathrm{id}"] \arrow[d] & T(\widehat{K^a}) \times U(\widehat{K^a}) \arrow[r, "\mathrm{ad}"] \arrow[d] & U(\widehat{K^a}) \arrow[d] \\
\mathbb{G}_m(\widetilde{K}) \times U(\widetilde{K}) \arrow[r, "\xi \times \mathrm{id}"] & T(\widetilde{K}) \times U(\widetilde{K}) \arrow[r, "\mathrm{ad}"] & U(\widetilde{K}).
\end{tikzcd}$$

Let ϖ be a pseudo-uniformizer[2] of $(\widehat{K^a})^\times = \mathbb{G}_m(\widehat{K^a})$. For an integer m, the action of ϖ^m on $u \in U(\widehat{K^a})$ is denoted by $(\varpi^m) \cdot u$. Let $\widetilde{u}$ be the image of u in $U(\widetilde{K})$. Since $\widetilde{u} = \prod_{\alpha \in \Phi''} f_\alpha(g_\alpha)$ with $g_\alpha \in \widetilde{K}$, the image of $(\varpi^m) \cdot u$ in $U(\widetilde{K})$ is $(\eta^*(\xi)(\varpi^m))\, \widetilde{u}\, (\eta^*(\xi)(\varpi^m))^{-1}$, expressed as the following

$$\prod_{\alpha \in \Phi''} (\eta^*(\xi)(\varpi^m))\, f_\alpha(g_\alpha)\, (\eta^*(\xi)(\varpi^m))^{-1} = \prod_{\alpha \in \Phi''} f_\alpha\big((\varpi^m)^{\langle \xi, \eta(\alpha)\rangle} g_\alpha\big).$$

Because $\eta(\Phi'') \subset \Phi_+$, we can choose a cocharacter ξ such that $\langle \xi, \eta(\alpha)\rangle$ are strictly positive. Then, when m grows, the element $(\varpi^m) \cdot y \in U(\widetilde{K})$ gets closer to identity, and so the same holds in $U(\widehat{K^a})$. Thus, since $N \cap U(\widehat{K^a})$ is an open neighbourhood of identity, every orbit of the T-action on $U(\widehat{K^a})$ intersects with $N \cap U(\widehat{K^a})$ nontrivially. So, we have $U(\widehat{K^a}) = \bigcup_{t \in T(\widehat{K^a})} t(N \cap U(\widehat{K^a}))t^{-1} = N \cap U(\widehat{K^a})$, which implies that $U(\widehat{K^a}) \subset N$. By combining with Proposition 4.2.9, we conclude that

$$U(\widehat{K^a}) \subset \overline{\mathrm{Im}\left(G(V[\tfrac{1}{a}])\right)}. \tag{5.10}$$

- Now we show that $P(\widehat{K^a}) \subset \mathrm{Im}\left(G(V[\frac{1}{a}])\right) G(\widehat{V^a})$. By Proposition 5.1.3, the quotient $H := L/T$ satisfies $H(\widehat{K^a}) = H(\widehat{V^a})$. Since T is split, Hilbert's theorem 90 gives the vanishing in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T(\widehat{V^a}) \arrow[r] \arrow[d] & L(\widehat{V^a}) \arrow[r] \arrow[d] & H(\widehat{V^a}) \arrow[r] \arrow[d, equal] & H^1(\widehat{V^a}, T) = 0 \arrow[d] \\
0 \arrow[r] & T(\widehat{K^a}) \arrow[r] & L(\widehat{K^a}) \arrow[r] & H(\widehat{K^a}) \arrow[r] & H^1(\widehat{K^a}, T) = 0
\end{tikzcd} \tag{5.11}$$

with exact rows. A diagram chase yields $L(\widehat{K^a}) = T(\widehat{K^a}) L(\widehat{V^a})$. Combining this with (5.9) and (5.10), we conclude that

$$P(\widehat{K^a}) \subset \mathrm{Im}\left(G(V[\tfrac{1}{a}])\right) G(\widehat{V^a}). \tag{5.12}$$

[2]We say that $\varpi \in \widehat{K^a}$ is a pseudo-uniformizer, if $\varpi \in \widehat{V^a}$ and $0 < |\varpi| < 1$, where $|\cdot|$ is the absolute value on $\widehat{K^a}$. Since $\widehat{K^a}$ is of rank one, the notion of pseudo-uniformizers coincides with that of topologically nilpotent units, which are elements of $(\widehat{K^a})^\times$ such that their powers converge to 0.

- By [SGA $3_{\text{III new}}$, XXVI, 4.3.2, 5.2], there is a parabolic Q such that $P \cap Q = L$ fitting into the surjection

$$\mathrm{rad}^u(P)(\widehat{K^a}) \cdot \mathrm{rad}^u(Q)(\widehat{K^a}) \twoheadrightarrow G(\widehat{K^a})/P(\widehat{K^a}). \tag{5.13}$$

Applying (5.10) to (5.13) gives $G(\widehat{K^a}) \subset \overline{\mathrm{Im}(G(V[\frac{1}{a}]))}P(\widehat{K^a})$, which combined with (5.12) yields $G(\widehat{K^a}) \subset \overline{\mathrm{Im}(G(V[\frac{1}{a}]))}G(\widehat{V^a})$. With the equality $\overline{\mathrm{Im}(G(V[\frac{1}{a}]))}G(\widehat{V^a}) = \mathrm{Im}(G(V[\frac{1}{a}]))G(\widehat{V^a})$ verified in Corollary 4.2.10, we obtain the desired product formula $G(\widehat{K^a}) = \mathrm{Im}(G(V[\frac{1}{a}]))G(\widehat{V^a})$. □

Proposition 5.1.7. *For Theorem 2.0.3, proving that $\diamondsuit$ has trivial kernel for rank one Henselian V suffices.*

Proof. A twisting technique [Gir71, III, 2.6.1 (1)] reduces us to showing that the map $\diamondsuit$ has trivial kernel. The valuation ring V is a filtered direct union of valuation subrings V_i of finite rank (see, for instance, [BM20, 2.22]). Since direct limits commute with localizations, the fraction field $K = \mathrm{Frac}(V)$ is also a filtered direct union of $K_i := \mathrm{Frac}(V_i)$. A limit argument [Gir71, VII, 2.1.6] gives compatible isomorphisms $H^1_{\text{ét}}(V, G) \cong \varinjlim_{i\in I} H^1_{\text{ét}}(V_i, G)$ and $H^1_{\text{ét}}(K, G) \cong \varinjlim_{i\in I} H^1_{\text{ét}}(K_i, G)$. Thus, it suffices to prove that $\diamondsuit$ has trivial kernel for V of finite rank, say $n \geq 0$. When $n = 0$, the valuation ring $V = K$ is a field, so this case is trivial. Our induction hypothesis is to assume that Theorem 2.0.3 holds for two kinds of valuation rings V': (1) for V' Henselian of rank one; (2) for V' of rank $n-1$. Indeed, (1) is only used for the case $n = 1$.

Let $\mathcal{X}$ be a G-torsor lying in the kernel of $H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(K, G)$. For the prime $\mathfrak{p} \subset V$ of height $n-1$, we choose an element $a \in \mathfrak{m}_V \backslash \mathfrak{p}$ and consider the a-adic completion $\widehat{V^a}$ of V with fraction field $\widehat{K^a}$. The induction hypothesis gives the triviality of $\mathcal{X}|_{V[\frac{1}{a}]}$ hence a section $s_1 \in \mathcal{X}(V[\frac{1}{a}])$. Consequently, $\mathcal{X}$ is trivial over $\widehat{K^a}$ and by induction hypothesis again, trivial over $\widehat{V^a}$ with $s_2 \in \mathcal{X}(\widehat{V^a})$. By the product formula $G(\widehat{K^a}) = G(V[\frac{1}{a}])G(\widehat{V^a})$ in Proposition 5.1.6, there are $g_1 \in G(V[\frac{1}{a}])$ and $g_2 \in G(\widehat{V^a})$ such that $g_1 s_1$ and $g_2 s_2$ have the same image in $\mathcal{X}(\widehat{K^a})$. Since $\mathcal{X}$ is affine over V, by [SP, 0BNR], we have the fibre product $\mathcal{X}(V) \xrightarrow{\sim} \mathcal{X}(V[\frac{1}{a}]) \times_{\mathcal{X}(\widehat{K^a})} \mathcal{X}(\widehat{V^a})$, which is nonempty, so the triviality of $\mathcal{X}$ follows. □

Chapter 6

Passage to the semisimple anisotropic case

After the passage to the Henselian rank one case (see Proposition 5.1.7), in this section, we further reduce Theorem 2.0.3 to the case when G is semisimple anisotropic, see Proposition 6.0.1. For this, by induction on Levi subgroups, we reduce to the case when G contains no proper parabolic subgroups. Subsequently, we consider the semisimple quotient of G, which is semisimple anisotropic. By using the integrality of rational points of anisotropic groups and a diagram chase, we obtain the desired reduction.

Proposition 6.0.1. *To prove Theorem 2.0.3, it suffices to show that $\diamondsuit$ has trivial kernel in the case when V is a Henselian valuation ring of rank one and G is semisimple anisotropic.*

Proof. First, we reduce to the case when G contains no proper parabolic subgroups. If G contains a proper minimal parabolic P with Levi L and unipotent radical $\mathrm{rad}^u(P)$, consider the commutative diagram

$$\begin{tikzcd} H^1_{\text{ét}}(V,L) \arrow[r] \arrow[d, "l_L"] & H^1_{\text{ét}}(V,P) \arrow[r] \arrow[d, "l_P"] & H^1_{\text{ét}}(V,G) \arrow[d, "l_G"] \\ H^1_{\text{ét}}(K,L) \arrow[r] & H^1_{\text{ét}}(K,P) \arrow[r] & H^1_{\text{ét}}(K,G). \end{tikzcd}$$

By [SGA $3_{\text{III new}}$, XXVI, 2.3], the left horizontal arrows are bijective. If a G-torsor $\mathcal{X}$ lies in $\mathrm{Ker}(l_G)$, then it satisfies $\mathcal{X}(K) \neq \emptyset$. By [SGA $3_{\text{III new}}$, XXVI, 3.3; 3.20], the fpqc quotient $\mathcal{X}/P$ is representable by a scheme which is projective over V. The valuative criterion of properness gives $(\mathcal{X}/P)(V) = (\mathcal{X}/P)(K) \neq \emptyset$, so we can form a fibre product $\mathcal{Y} := \mathcal{X} \times_{\mathcal{X}/P} \operatorname{Spec} V$ from a V-point of $\mathcal{X}/P$. Since $\mathcal{Y}(K) \neq \emptyset$, the class $[\mathcal{Y}] \in \mathrm{Ker}(l_P)$. On the other hand, the image of $[\mathcal{Y}]$ in $H^1_{\text{ét}}(V,G)$ coincides with $[\mathcal{X}]$. Consequently, the triviality of $\mathrm{Ker}(l_L)$ amounts to the triviality of $\mathrm{Ker}(l_G)$. By [SGA $3_{\text{III new}}$, XXVI, 1.20] and Proposition 5.1.7, we reduce to proving Theorem 2.0.3 for V Henselian of rank one and G without proper parabolic subgroup, more precisely, to showing that $\mathrm{Ker}(H^1(V,G) \to H^1(K,G)) = \{*\}$ for such V and G.

For the radical $\mathrm{rad}(G)$ of G, the quotient $G/\mathrm{rad}(G)$ is V-anisotropic and by Proposition 5.1.3 satisfies $(G/\mathrm{rad}(G))(V) = (G/\mathrm{rad}(G))(K)$, fitting into the following commutative diagram

$$\begin{tikzcd} (G/\mathrm{rad}(G))(V) \arrow[r] \arrow[d, equal] & H^1_{\text{ét}}(V,\mathrm{rad}(G)) \arrow[r] \arrow[d, "l(\mathrm{rad}(G))"] & H^1_{\text{ét}}(V,G) \arrow[r] \arrow[d, "l(G)"] & H^1_{\text{ét}}(V,G/\mathrm{rad}(G)) \arrow[d, "l(G/\mathrm{rad}(G))"] \\ (G/\mathrm{rad}(G))(K) \arrow[r] & H^1_{\text{ét}}(K,\mathrm{rad}(G)) \arrow[r] & H^1_{\text{ét}}(K,G) \arrow[r] & H^1_{\text{ét}}(K,G/\mathrm{rad}(G)) \end{tikzcd}$$

If $\mathrm{Ker}(l(G/\operatorname{rad}(G)))$ is trivial, then by the tori case Proposition 3.1.3 and Four Lemma, we conclude. $\square$

Chapter 7

Bruhat–Tits theory and the proof of the main theorem

In this section, we finish the proof of our main result Theorem 2.0.3. By the reduction of Proposition 6.0.1, it suffices to deal with semisimple anisotropic group schemes over Henselian valuation rings of rank one. In this situation, we argue by using techniques in Bruhat–Tits theory and Galois cohomology to conclude.

Theorem 7.0.1. *For a semisimple anisotropic group scheme G over a Henselian valuation ring V of rank one with fraction field K, the following map has trivial kernel*:

$$H^1_{\text{ét}}(V, G) \to H^1_{\text{ét}}(K, G).$$

Proof. Let $\widetilde{V}$ be a strict Henselization of V at $\mathfrak{m}_V$ with fraction field $\widetilde{K}$ as a subfield of a separable closure K^{sep}. We consider three Galois groups $\Gamma := \text{Gal}(\widetilde{V}/V)$, $\Gamma_{\widetilde{K}} := \text{Gal}(K^{\text{sep}}/\widetilde{K})$ and $\Gamma_K := \text{Gal}(K^{\text{sep}}/K)$. Since V is Henselian, an application [Sch13, 3.7 (iii)] of the Cartan–Leray spectral sequence yields an isomorphism $H^1_{\text{ét}}(V, G) \simeq H^1(\Gamma, G(\widetilde{V}))$. By [SGA 4_{II}, VIII, 2.1], we have $H^1_{\text{ét}}(K, G) \simeq H^1(\Gamma_K, G(K^{\text{sep}}))$. With these bijections, it suffices to show that each of the following maps has trivial kernel

$$H^1(\Gamma, G(\widetilde{V})) \xrightarrow{\alpha} H^1(\Gamma, G(\widetilde{K})) \xrightarrow{\beta} H^1(\Gamma_K, G(K^{\text{sep}})).$$

For $\beta\colon H^1(\Gamma, G(\widetilde{K})) \to H^1(\Gamma_K, G(K^{\text{sep}}))$, we invoke the inflation-restriction exact sequence [Ser02, 5.8 a)]:

$$0 \to H^1(G_1/G_2, A^{G_2}) \to H^1(G_1, A) \to H^1(G_2, A)^{G_1/G_2},$$

for which G_2 is a closed normal subgroup of a group G_1 and A is a G_1-group. It suffices to take

$$G_1 := \Gamma_K, \qquad G_2 := \Gamma_{\widetilde{K}}, \qquad \text{and } A := G(K^{\text{sep}}).$$

For $\alpha\colon H^1(\Gamma, G(\widetilde{V})) \to H^1(\Gamma, G(\widetilde{K}))$, let $z \in H^1(\Gamma, G(\widetilde{V}))$ be a cocycle in $\text{Ker}\,\alpha$, which signifies that

$$\text{there is an } h \in G(\widetilde{K}) \text{ such that for every } s \in \Gamma, \qquad z(s) = h^{-1}s(h) \in G(\widetilde{V}). \tag{7.1}$$

Now we come to Bruhat–Tits theory and consider $G(\widetilde{V})$ and $hG(\widetilde{V})h^{-1}$ as two subgroups of $G(\widetilde{K})$. Let $\mathscr{I}(G)$ be the building of $G_{\widetilde{K}}$. Since $G_{\widetilde{K}}$ is semisimple, the extended building

$\widetilde{\mathscr{I}}(G)^{ext} := \widetilde{\mathscr{I}}(G) \times (\mathrm{Hom}_{\widetilde{K}\text{-gr.}}(G, \mathbb{G}_{m,\widetilde{K}})^\vee \otimes_{\mathbf{Z}} \mathbf{R})$ has trivial vectorial part and equals to $\widetilde{\mathscr{I}}(G)$. The elements of $G(\widetilde{K})$ act on the building $\widetilde{\mathscr{I}}(G)$ and for each facet $F \subset \widetilde{\mathscr{I}}(G)$, we consider its stabilizer $P_F^\dagger$ and its connected pointwise stabilizer P_F^0. In fact, there are group schemes $\mathfrak{G}_F^\dagger$ and $\mathfrak{G}_F^0$ over $\widetilde{V}$ such that $\mathfrak{G}_F^\dagger(\widetilde{V}) = P_F^\dagger$ and $\mathfrak{G}_F^0(\widetilde{V}) = P_F^0$, see [BT$_{\mathrm{II}}$, 4.6.28]. Note that the residue field of $\widetilde{V}$ is separably closed and the closed fibre of $G_{\widetilde{V}}$ is reductive, so, by [BT$_{\mathrm{II}}$, 4.6.22, 4.6.31], there is a special point x in the building $\widetilde{\mathscr{I}}(G)$ such that the Chevalley group $G_{\widetilde{V}}$ is the stabilizer $\mathfrak{G}_x^\dagger = \mathfrak{G}_x^0$ of x with connected fibres. By definition [BT$_{\mathrm{II}}$, 5.2.6], $G(\widetilde{V})$ is a parahoric subgroup of $G(\widetilde{K})$. Therefore, its conjugate $hG(\widetilde{V})h^{-1}$ is also a parahoric subgroup $P^0_{h^{-1}\cdot x}$. Since $G(\widetilde{V})$ is Γ-invariant, every $s \in \Gamma$ acts on $hG(\widetilde{V})h^{-1}$ as follows

$$s(hG(\widetilde{V})h^{-1}) = s(h)G(\widetilde{V})s(h^{-1}) \overset{(7.1)}{=\!=} hG(\widetilde{V})h^{-1}.$$

The Γ-invariance of $G(\widetilde{V})$ and $hG(\widetilde{V})h^{-1}$ amounts to that x and $h \cdot x$ are two fixed points of Γ in $\widetilde{\mathscr{I}}(G)$. But by [BT$_{\mathrm{II}}$, 5.2.7], the anisotropicity of G_K gives the uniqueness of fixed points in $\widetilde{\mathscr{I}}(G)$. Thus, we have $G(\widetilde{V}) = hG(\widetilde{V})h^{-1}$, which means that for every $g \in G(\widetilde{V})$ its conjugate hgh^{-1} fixes x. This is equivalent to that g fixes $h^{-1} \cdot x$ and to the inclusion of stabilizers $P_x^\dagger \subset P^\dagger_{h^{-1}\cdot x}$. On the other hand, every $\tau \in P^\dagger_{h^{-1}\cdot x}$ satisfies $h\tau h^{-1} \cdot x = x$, so $h\tau h^{-1} \in P_x^\dagger = G(\widetilde{V})$. Since h normalizes $G(\widetilde{V})$, this inclusion implies that $\tau \in G(\widetilde{V})$ and $P^\dagger_{h^{-1}\cdot x} \subset G(\widetilde{V})$. Combined with $P_x^\dagger \subset P^\dagger_{h^{-1}\cdot x}$, this gives $P_x^\dagger = P^\dagger_{h^{-1}\cdot x} = G(\widetilde{V})$. Therefore, the stabilizer $P^\dagger_{h^{-1}\cdot x}$ is also a parahoric subgroup and equals to $P^0_{h^{-1}\cdot x}$. By [BT$_{\mathrm{II}}$, 4.6.29], the equality $P_x^0 = P^0_{h^{-1}\cdot x}$ implies that $h^{-1} \cdot x = x$, so $h \in P_x^0 = G(\widetilde{V})$, which gives the triviality of z. $\square$

Chapter 8

Torsors over $V((t))$ and Nisnevich's purity conjecture

In [Nis89, 1.3], Nisnevich proposed a purity conjecture that for a reductive group scheme G over a regular local ring R with a regular parameter $f \in \mathfrak{m}_R \backslash \mathfrak{m}_R^2$, every Zariski G-torsor over $R[\frac{1}{f}]$ is trivial, that is,

$$H^1_{\mathrm{Zar}}(R[\tfrac{1}{f}], G) = \{*\}.$$

In this section, we prove a variant of Nisnevich's purity conjecture when R is a formal power series over a valuation ring, see Corollary 8.2.3. For this, we devise a cohomological property Proposition 8.2.2 of $V((t))$ over a valuation ring V by taking advantage of techniques of reflexive sheaves.

8.1 Reflexive sheaves

For a locally coherent scheme X and an $\mathcal{O}_X$-module $\mathscr{F}$, we define the *dual* $\mathcal{O}_X$-module $\mathscr{F}^\vee := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$ and the *reflexive hull* of $\mathscr{F}$ as $\mathscr{F}^{\vee\vee}$. We say that $\mathscr{F}$ is *reflexive*, if it is coherent and the map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. Zariski-locally $\mathscr{F}$ has a presentation $\mathcal{O}_X^{\oplus m} \to \mathcal{O}_X^{\oplus n} \to \mathscr{F} \to 0$, whose dual is the exact sequence $0 \to \mathscr{F}^\vee \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m}$ exhibiting $\mathscr{F}^\vee$ as the kernel of maps between coherent sheaves, hence by [SP, 01BY] $\mathscr{F}^\vee$ is coherent, a priori finitely presented. If $\mathscr{F}$ is reflexive at a point $x \in X$, then the dual of a presentation $\mathcal{O}_{X,x}^{\oplus m'} \to \mathcal{O}_{X,x}^{\oplus n'} \to \mathscr{F}_x^\vee \to 0$ is a left exact sequence $0 \to \mathscr{F}_x \to \mathcal{O}_{X,x}^{\oplus n'} \to \mathcal{O}_{X,x}^{\oplus m'}$.

Lemma 8.1.1 ([GR18, 11.4.1])**.** *Let V be a valuation ring with spectrum $S := \operatorname{Spec} V$. For a flat finitely presented morphism $f\colon X \to S$ and a coherent $\mathcal{O}_X$-sheaf $\mathscr{F}$, we take a point $x \in X$ and let $y := f(x)$. Assume that $f^{-1}(y)$ is a regular scheme and $n := \dim \mathcal{O}_{f^{-1}y,x}$. Then*

(i) *If $\mathscr{F}$ is f-flat at x, then* $\mathrm{proj.dim}_{\mathcal{O}_{X,x}} \mathscr{F}_x \leq n$.

(ii) *We have* $\mathrm{proj.dim}_{\mathcal{O}_{X,x}} \mathscr{F}_x \leq n+1$.

(iii) *If $\mathscr{F}$ is reflexive at x, then* $\mathrm{proj.dim}_{\mathcal{O}_{X,x}} \mathscr{F}_x \leq \max(0, n-1)$.

Proof.

(i) Since $\mathcal{O}_X$ is coherent and $\mathscr{F}_x$ is finitely presented, there is free resolution of $\mathscr{F}_x$ by finite modules
$$\cdots \to P_2 \to P_1 \to P_0 \to \mathscr{F}_x \to 0.$$
It suffices to show that $L := \mathrm{Im}(P_n \to P_{n-1})$ is free. Now we have the following exact sequence
$$0 \to L \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathscr{F}_x \to 0.$$
Since $\mathscr{F}_x$ is flat and $\mathrm{Ker}(P_i \to P_{i-1})$ are flat for $1 \le i \le n-1$, the following sequence
$$0 \to L \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{f^{-1}(y),x} \to \cdots \to P_0 \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{f^{-1}(y),x} \to \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{f^{-1}(y),x} \to 0$$
is exact. For the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{f^{-1}(y),x}$ at x and the residue field $k(x)$ of x in $\mathcal{O}_{X,x}$, we note that $L \otimes_{\mathcal{O}_{X,x}} (\mathcal{O}_{f^{-1}(y),x}/\mathfrak{m}_x\mathcal{O}_{f^{-1}(y),x}) = L \otimes_{\mathcal{O}_{X,x}} k(x)$. For a free basis $(e_l)_{l\in I}$ generating $L \otimes_{\mathcal{O}_{X,x}} k(x)$, by Nakayama's lemma, there is a surjective map $u\colon \bigoplus_{l\in I} \mathcal{O}_{X,x}e_l \to L$. Since $f^{-1}(y)$ is regular (a priori locally Noetherian), the finite flat module $L \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{f^{-1}(y),x}$ is free. Therefore, the map $(u \otimes 1)_x : ((\bigoplus_{l\in I} \mathcal{O}_{X,x}e_l) \otimes_{\mathcal{O}_S} k(y))_x \to (L \otimes_{\mathcal{O}_S} k(y))_x$ is an isomorphism. By [EGA IV$_4$, 11.3.7], u is injective. Consequently, the $\mathcal{O}_{X,x}$-module L is free and $\mathrm{proj.dim}\mathscr{F}_x \le n$.

(ii) We prove the assertion Zariski-locally. There is a surjective map $\mathcal{O}_X^{\oplus m} \to \mathscr{F}$, whose kernel $\mathscr{G}$ is a torsion-free coherent $\mathcal{O}_X$-module. Since V is a valuation ring, $\mathscr{G}$ is f-flat, so by (i) we have $\mathrm{proj.dim}_{\mathcal{O}_X} \le n$. Therefore, [SP, 00O5] implies that $\mathrm{proj.dim}_{\mathcal{O}_X}(\mathscr{F}) = \mathrm{proj.dim}_{\mathcal{O}_X}\mathscr{G} + 1 \le n+1$.

(iii) By the analysis in §8.1, there is a left exact sequence $0 \to \mathscr{F}_x \to \mathcal{O}_{X,x}^{\oplus k} \xrightarrow{\phi} \mathcal{O}_{X,x}^{\oplus l}$. Hence we have
$$\mathrm{proj.dim}_{\mathcal{O}_{X,x}}(\mathscr{F}_x) \overset{\text{[SP, 00O5]}}{=\!=} \max(0, \mathrm{proj.dim}_{\mathcal{O}_{X,x}}(\mathrm{coker}\,\phi) - 2) \le \max(0, n-1),$$
where the inequality follows from (ii). □

8.2 Torsors over $V((t))$

Lemma 8.2.1. *For a valuation ring V, every vector bundle over $V((t))$ extends to a vector bundle over $V[[t]]$. In particular, we have the triviality of all GL_n-torsors over $V((t))$:*
$$H^1_{\text{ét}}(V((t)), \mathrm{GL}_n) = \{*\}.$$

Proof. The Henselization $V\{t\}$ of $V[t]$ along (t) is a filtered direct limit of étale ring extensions R_i over V with isomorphisms $V[t]/t \xrightarrow{\sim} R_i/t$. By [BČ20, 2.1.20], a vector bundle $\mathscr{V}$ over $V((t))$ descends to a vector bundle $\mathscr{V}'$ over $V\{t\}[\frac{1}{t}]$. By a limit argument [Gir71, VII, 2.1.6], we have $H^1_{\text{ét}}(V\{t\}[\frac{1}{t}], \mathrm{GL}_n) = \varinjlim_i H^1_{\text{ét}}(R_i[\frac{1}{t}], \mathrm{GL}_n)$ so $\mathscr{V}'$ descends to a vector bundle $\mathscr{V}_i$ over $R_{i_0}[\frac{1}{t}]$ for an i_0. By coherent extension [EGA I, 9.4.7], $\mathscr{V}_{i_0}$ extends to a coherent sheaf $\mathscr{W}_{i_0}$ on R_{i_0}, whose reflexive hull is $\mathscr{H}_{i_0} := \mathscr{W}_{i_0}^{\vee\vee}$. For the morphism $f\colon \mathrm{Spec}\, R_{i_0} \to \mathrm{Spec}\, V$, we exploit Lemma 8.1.1 (iii) to conclude that $\mathscr{H}_{i_0}$ is free. Consequently, the base change $(\mathscr{H}_{i_0})_{V[[t]]}$ is a trivial vector bundle on $V[[t]]$ such that $(\mathscr{H}_{i_0})_{V((t))} \cong \mathscr{V}$. □

In the following proposition, we analyse the triviality of torsors under restrictions. The anisotropic case of Proposition 8.2.2(c) is established in [FG21, Cor. 4.2], where the authors consider (Laurent) formal power series over general rings.

Proposition 8.2.2. *For a valuation ring V with fraction field K, a V-affine, smooth group scheme G permitting an embedding $G \hookrightarrow \mathrm{GL}(\mathcal{V})$ for a vector bundle $\mathcal{V}$ over V,*

(a) *base change is an equivalence from the category of $G_{V[[t]]}$-torsors $\mathscr{X}$ to that of triples*

$$(X, X', \iota\colon X_{K[[t]]} \xrightarrow{\sim} X'_{V((t))})$$

consisting of a $G_{V((t))}$-torsor X, a $G_{K[[t]]}$-torsor X', and an indicated torsor isomorphism ι;

(b) *the map $H^1_{\text{ét}}(V((t)), G) \to H^1_{\text{ét}}(K((t)), G)$ has trivial kernel; and*

(c) *If G is reductive, then the map $H^1_{\text{ét}}(V[[t]], G) \to H^1_{\text{ét}}(V((t)), G)$ has trivial kernel.*

Proof.

(a) First, we prove that the base change functor is fully faithful. For two $G_{V[[t]]}$-torsors $\mathscr{X}$ and $\mathscr{Y}$, the functor $\underline{\mathrm{Hom}}_{G_{V[[t]]}}(\mathscr{X}, \mathscr{Y})$ is representable by an affine scheme over $V[[t]]$, so we have

$$\underline{\mathrm{Hom}}_{G_{V[[t]]}}(\mathscr{X}, \mathscr{Y}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{G_{K[[t]]}}(X', Y') \times_{\underline{\mathrm{Hom}}_{G_{K((t))}}(\mathscr{X}_{K((t))}, \mathscr{Y}_{K((t))})} \underline{\mathrm{Hom}}_{G_{V((t))}}(X, Y).$$

Then, we prove that base change functor is essentially surjective. Since valuation rings are normal, [Tho87, Cor. 3.2 (3)] gives us a closed embedding $G \hookrightarrow \mathrm{GL}(\mathscr{V})$ for a vector bundle $\mathscr{V}$. By [Gir71, III, 3.2.2], we have the nonabelian cohomology exact sequence

$$(\mathrm{GL}(\mathscr{V})/G)\,(S) \to H^1_{\text{ét}}(S, G) \to H^1_{\text{ét}}(S, \mathrm{GL}(\mathscr{V})),$$

where S represents $V[[t]]$, $V((t))$, $K[[t]]$ or $K((t))$. The Zariski-triviality of $GL(\mathscr{V})_S$-torsors and Lemma 8.2.1 imply that each G_S-torsor class is in the image of $(GL(\mathscr{V})/G)\,(S)$. Thus, a triple (X, X', ι) is determined by

$$x \in (\mathrm{GL}(\mathscr{V})/G)\,(V((t))) \quad \text{and} \quad x' \in (\mathrm{GL}(\mathscr{V})/G)\,(K[[t]])$$

such that x, x' have the same image in $(\mathrm{GL}(\mathscr{V})/G)\,(K((t)))$. Because G is reductive, by [Alp14, 9.7.7], the quotient $\mathrm{GL}(\mathscr{V})/G$ is affine. Thus, the fibre product $V[[t]] \xrightarrow{\sim} V((t)) \times_{K((t))} K[[t]]$ induces

$$(\mathrm{GL}(\mathscr{V})/G)\,(V[[t]]) \xrightarrow{\sim} (\mathrm{GL}(\mathscr{V})/G)\,(K[[t]]) \times_{(\mathrm{GL}(\mathscr{V})/G)(K((t)))} (\mathrm{GL}(\mathscr{V})/G)\,(V((t))).$$

Consequently, there is an element in $(\mathrm{GL}(\mathscr{V})/G)\,(V[[t]])$ corresponding to a $G_{V[[t]]}$-torsor $\mathscr{G}$, whose base changes over $V((t))$, $K[[t]]$ and $K((t))$ give the triple (X, X', ι).

(b) For a $G_{V((t))}$-torsor X trivializes over $K((t))$, we take a trivial $G_{K[[t]]}$-torsor X' over $K[[t]]$ with an isomorphism $\iota\colon X|_{K((t))} \xrightarrow{\sim} X'|_{K((t))}$. By (a), there is a $G_{V[[t]]}$-torsor $\mathscr{X}$ restricts to X such that $\mathscr{X}_{K[[t]]}$ is trivial. By the main result Theorem 2.0.3 and [GR18, 5.8.14], the map $H^1_{\text{ét}}(V[[t]], G) \hookrightarrow H^1_{\text{ét}}(K[[t]], G)$ is injective. Hence, the torsor $\mathscr{X}$ that restricts to X is trivial.

(c) By the Grothendieck–Serre over valuation rings (Theorem 2.0.3) and [GR18, 5.8.14], the map
$$H^1_{\text{ét}}(V[\![t]\!], G) \to H^1_{\text{ét}}(K[\![t]\!], G)$$
is injective. Since $K[\![t]\!]$ is a discrete valuation ring, the map $H^1_{\text{ét}}(K[\![t]\!], G) \to H^1_{\text{ét}}(K(\!(t)\!), G)$ is injective. The injective map $H^1_{\text{ét}}(V[\![t]\!], G) \to H^1_{\text{ét}}(K(\!(t)\!), G)$ factors through
$$H^1_{\text{ét}}(V[\![t]\!], G) \to H^1_{\text{ét}}(V(\!(t)\!), G),$$
hence the later is injective. □

Now we prove a variant of the Nisnevich's purity conjecture for formal power series over valuation rings.

Corollary 8.2.3. *For a reductive group scheme G over a valuation ring V, every Zariski-locally trivial G-torsor over $V(\!(t)\!)$ is trivial, that is, we have*
$$H^1_{\text{Zar}}(V(\!(t)\!), G) = \{*\}.$$

Proof. A Zariski G-torsor over $V(\!(t)\!)$ is an étale G-torsor over $V(\!(t)\!)$ trivializing over $K(\!(t)\!)$. Hence the assertion follows from Proposition 8.2.2 (b). □

Remark 8.2.4. Indeed, in Proposition 8.2.2 and Corollary 8.2.3, the assertions remain true if one assume that G is defined over $V[\![t]\!]$. Since $(V[\![t]\!], t)$ is a Henselian pair, every pair of reductive group schemes (G, G') such that $G_{\{t=0\}} \simeq G'_{\{t=0\}}$ satisfies $G \simeq G'$. This follows from the corrected version of [Str77] proved in [BČ20, Thm. 2.1.5] (see also [Česk21, Cor. 3.1.3]).

Chapter 9

Appendix

9.1 Valuation rings and value groups

Ordered abelian groups

An *ordered abelian group* is an abelian group $(\Gamma, +, 0)$ with a binary relation $\leq$ on Γ, satisfying the following axioms: for all $\alpha, \beta, \gamma \in \Gamma$, we have

(i) $\alpha \leq \alpha$;

(ii) if $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$;

(iii) if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$;

(iv) $\alpha \leq \beta$ or $\beta \leq \alpha$ holds;

(v) if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

Convex subgroup

A subgroup Δ of an ordered abelian group Γ is called *convex* in Γ if for every $\delta \in \Delta$, elements $\gamma \in \Gamma$ satisfying $0 \leq \gamma \leq \delta$ are contained in Δ.

Rank of an ordered abelian group

By (iv) in the definition of ordered abelian groups, arbitrary two convex subgroups must have inclusion relation. Hence the colllection of all proper convex subgroups of Γ are ordered by inclusion relation. The order type of this collection is called the *rank* of Γ. In particular, if there are exactly n proper convex subgroups of Γ, then we say Γ is of rank n. For Γ, it is of rank 1 if and only if $\{0\}$ is the only proper convex subgroup.

Archimedean ordered abelian groups

An ordered abelian group Γ is *Archimedean* if for every $\alpha, \beta \in \Gamma$ where $\beta > 0$, there exists $n \in \mathbb{N}$ such that $\alpha \leq n\beta$. By definition, every element in Γ is contained in a nontrivial convex subgroup of Γ, hence every nontrivial convex subgroup of Γ is Γ itself. We conclude that Archimedean ordered abelian groups are of rank one. In fact, an ordered abelian group is of rank one if and only if it is an additive subgroup of $(\mathbb{R}, +)$, if and only if it is Archimedean.

Ordered fields

An ordering on a field K is a binary relation $\leq$ such that $(K, \leq)$ is an ordered abelian group and for all $0 \leq x, y \in K$ the product $xy \geq 0$. For a subring $R \subset K$, we define the $\leq$-convex hull as

$$\mathcal{O}_R(\leq) := \{x \in K | x, -x \leq a \quad \text{for some } a \in R\}.$$

Then for an ordered field K, the subring $\mathcal{O}(\leq) := \mathcal{O}_{\mathbb{Z}}(\leq)$ is a valuation ring: for an $x \in K$ which is not in $\mathcal{O}(\leq)$, without loss of generality, such $x \geq n$ for all $n \in \mathbb{Z}$. If $x^{-1} \notin \mathcal{O}(\leq)$, then $\epsilon x^{-1} \geq 1$ where ϵ is 1 or -1. Hence

$$x(\epsilon x^{-1} - 1) \geq 0, \text{ which implies that } \epsilon \geq x.$$

Similarly, for every subring $R \subset K$ containing 1, the $\leq$-convex hull $\mathcal{O}_R(\leq)$ is a valuation ring. Note that in many cases, the valuation ring is K itself. We have seen that all ordered

Absolute values on a field

For a field K, an *absolute value* on K is a map

$$|\cdot| : K \to \mathbb{R}$$

such that for every $x, y \in K$, we have

(i) $|x| = 0$ if and only if $x = 0$;

(ii) $|xy| = |x||y|$;

(iii) $|x + y| \leq |x| + |y|$ (triangle inequality).

If we strengthen (iii) to the ultra-metric inequality:

$$|x + y| \leq \max\{|x|, |y|\},$$

then $|\cdot|$ is called a *non-Archimedean* absolute value.

Lemma 9.1.1. *An absolute value $|\cdot|$ on a field K is non-Archimedean if and only if $|n| \leq 1$ for every $n := \sum_{i=1}^{n} 1 \in K$.*

Proof. For a non-Archimedean value $|\cdot|$, we have $|\cdot| = 1$. Then by induction, we conclude that $|n| \leq \max\{|n-1|, |1|\} = 1$. Conversely, if there are two elements $x, y \in K$ such that $|x + y| > \max\{|x|, |y|\}$ (we may assume that $|x| \geq |y|$), then for $\epsilon := y/x$, we have

$$|1 + \epsilon|^n = |(1 + \epsilon)^n| \leq 1 + |\epsilon| + \cdots + |\epsilon|^n \longrightarrow < +\infty,$$

But $|1 + \epsilon| > 1$, so $\lim |1 + \epsilon|^n = +\infty$, contradiction. Therefore, $|\cdot|$ satisfies the ultra-metric inequality. ∎

Corollary 9.1.2. *For a field K of positive characteristic, any absolute value on K is non-archimedean.*

Now we collect some properties of non-Archimedean valued fields.

Proposition 9.1.3. *For a field K equipped with a non-Archimedean absolute value $|\cdot|$,*

(i) *every triangle in K is isosceles.*

(ii) *two open balls have nonempty intersection if and only if they are nested.*

(iii) *every point in an open ball is a center of the ball.*

(iv) *all open balls are closed; in particular, every circle is clopen.*

(v) *the topology induced by $|\cdot|$ on K is totally disconnected.*

Proof. Since the longest side is not longer than the second longest side, hence every triangle is isosceles. Let $z \in U(x, r_1) \cap U(y, r_2)$, then $|x - y| < \max\{r_1, r_2\}$ and x, y are contained in an open ball with the larger radius. But the distance between any point in the smaller open ball and the other center is controled by $|x - y|$, hence the smaller ball is contained in the larger one. For a nonempty open ball $U(z, R)$ in K, the distance between two points $x, y \in U(z, R)$ is smaller than R, hence $U(x, R)$, $U(y, R)$, and $U(z, R)$ are pairwise nested. For an open ball $U(x, R)$ and a point y such that $|x - y| = R$, the open ball $U(y, r)$ for $r < R$ is disjoint with $U(x, R)$, so every open ball is closed. In particular, the non-Archimedean topology on K is totally disconnected. □

Valuations on a field

A *valuation* on a field K is a surjective map

$$v\colon K \to \Gamma \cup \{\infty\},$$

where Γ is an ordered abelian group, such that for all $x, y \in K$, we have

(i) if $v(x) = \infty$, then $x = 0$;

(ii) $v(xy) = v(x) + v(y)$;

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

Now, given a non-Archimedean valued field K, we define

$$v(x) := -\ln |x|,$$

then $v\colon K \to \mathbb{R} \cup \{\infty\}$ is a valuation of rank one. Conversely, given a valuation v of rank one, one can construct a non-Archimedean absolute valuation by taking a $q \in \mathbb{R}$ larger than 1 to define a map

$$|\cdot|_q\colon K \to \mathbb{R}, \qquad x \mapsto q^{-v(x)}$$

for nonzero x, and sends 0 to 0. The absolute value is independent of q, so we conclude as the following.

Theorem 9.1.4. *There is a one-to-one correspondence between non-Archimedean absolute values and valuations of rank one on a field K.*

"Non-Archimedean" in different senses

We clarify the relation among three notions: non-Archimedean ordered abelian groups, ordered fields, and valued fields. Non-Archimedean absolute values correspond to valuations of rank one, whose value groups are exactly additive subgroups of $(\mathbb{R}, +)$, and are exactly Archimedean ordered abelian groups. Therefore, the Archimedeanness of value groups is not compatible with that of absolute values.

As non-Archimedean absolute values are only specified by the ultra-metric inequality, the notion of valuaitons, who inherits the ultra-metric inequality, is merely a formal generalization of non-Archimedean absolute values to the higher rank case. Besides, valuations are akin to the "non-Archimedean" property in the following senses. First, non-Archimedean ordered fields are the fields with nontrivial valuations and orderable residue fields. Secondly, some authors (for example, Morel) use the generalized notion "multiplicative valuations": it suffices to replace $\mathbb{R}$ in the definition of non-Archimedean absolute values by an ordered abelian multiplicative group Γ. Such multiplicative valuations are also called "non-Archimedean absolute values"; in fact, additive and multiplicative valuations are interchangeable. Lastly, the following theorem shows that general valuations are "non-Archimedean", provided the existence of a height-one prime.

Theorem 9.1.5. *For a valued field (K, v) with the induced valuation topology and its valuation ring R, the following are equivalent:*

(i) *The topology on K is the same as the valuation topology defined by a rank one valuation on K;*

(ii) *there is a nonzero topological nilpotent element;*

(iii) *R has a prime of height 1.*

Valued fields or valuation rings satisfying the conditions in Theorem 9.1.5 are called *microbial*. Since valuations of rank one are non-Archimedean in the classical sense, we can also say a valuation of higher rank is *non-Archimedean* if its valuation topology comes from a rank-one valuation. Hence, a field K equipped with a valuation is microbial if and only if it is non-Archimedean. In Scholze's article on perfectoid spaces, he defined *a topological field is non-Archimedean if its topology is a valuation topology induced by a valuation of rank one.*

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ÉCOLE DOCTORALE
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Titre: La conjecture de Grothendieck–Serre sur les anneaux de valuation

Mots clés: schémas en groupes réductifs, espaces homogènes, torseurs, cohomologie galoisienne, approximation faible, anneaux de valuation.

Résumé: Dans cet article, nous établissons la conjecture de Grothendieck–Serre sur les anneaux de valuation : pour un schéma en groupes réductifs G sur un anneau de valuation V de corps des fractions K, un torseur sous G sur V est trivial s'il est trivial sur K. Ce résultat est prédit par la conjecture originale de Grothendieck-Serre et la résolution des singularités. La nouveauté de notre preuve réside dans le fait de surmonter les subtilités apportées par les anneaux de valuation généraux non discrets. En utilisant des résolutions flasques et en induisant avec la cohomologie locale, nous prouvons une contrepartie non-noethérienne du cas des tores de Colliot-Thélène–Sansuc. Ensuite, en profitant des techniques d'algébraization, nous obtenons le passage au cas hensélien de rang un. Enfin, nous induisons sur les sous-groupes de Levi et utilisons l'intégralité des points rationnels des groupes anisotropes pour réduire au cas anisotrope semi-simple, dans lequel nous faisons appel aux propriétés des sous-groupes parahoriques de la théorie de Bruhat–Tits pour conclure. Dans la dernière section, en utilisant des propriétés des faiseaux réflexifs, nous prouvons également une variante de la conjecture de pureté de Nisnevich.

Title: The Grothendieck–Serre conjecture over valuation rings

Keywords: reductive group schemes, torsors, weak approximation, valuation rings, homogeneous spaces, Galois cohomology.

Abstract: In this article, we establish the Grothendieck–Serre conjecture over valuation rings: for a reductive group scheme G over a valuation ring V with fraction field K, a G-torsor over V is trivial if it is trivial over K. This result is predicted by the original Grothendieck—Serre conjecture and the resolution of singularities. The novelty of our proof lies in overcoming subtleties brought by general nondiscrete valuation rings. By using flasque resolutions and inducting with local cohomology, we prove a non-Noetherian counterpart of Colliot-Thélène—Sansuc's case of tori. Then, taking advantage of techniques in algebraization, we obtain the passage to the Henselian rank one case. Finally, we induct on Levi subgroups and use the integrality of rational points of anisotropic groups to reduce to the semisimple anisotropic case, in which we appeal to properties of parahoric subgroups in Bruhat-–Tits theory to conclude. In the last section, by using properties of reflexive sheaves, we also prove a variant of Nisnevich's purity conjecture.

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