About the type of modal logics for the unification problem

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Abstract

In this thesis, we shall investigate on the unification problem in ordinary modal logics, fusions of two modal logics and multi-modal epistemic logics. With respect to a propositional logic L, given a formula A, we have to find substitutions s such that s(A) is in L. When they exist, these substitutions are called unifiers of A in L. We study different methods for the construction of minimal complete sets of unifiers of a given formula A and according to the cardinality of these minimal complete sets, we shall discuss on the unification type of A. Then, we determine the unification types of several propositional logics.

Résumé

Dans cette thèse, nous étudierons le problème de l'unification dans les logiques modales ordinaires, les fusions de deux logiques modales et les logiques épistémiques multi-modales. Relativement à une logique propositionnelle L, étant donnée une formule A, nous devons trouver des substitutions s telles que s(A) est dans L. Lorsqu'elles existent, ces substitutions sont appelées unifieurs de A dans L. Nous étudions différentes méthodes pour construire des ensembles minimaux complets d'unifieurs d'une formule donnée A et, en fonction de la cardinalité des ces ensembles minimaux complets, nous discutons du type de l'unification de A. Enfin, nous déterminons les types de l'unification de plusieurs logiques propositionnelles.

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Contents

1 Introduction

In many research area of computer science and artificial intelligence, non-classical logics are considered: temporal logics, epistemic logics, etc. The main task to be solved for the applicability of these logics is their mechanization. In Propositional logic, there exists an important problem which is called admissibility of rules. A rule of inference is admissible in a given logic L if the set of theorems in L does not change when that rule is added to the existing rules in L. In other words, every formula that can be derived using that rule is already derivable without that rule. Decision problem in admissibility of rules is the most problem. In Classical Propositional Logic, each admissible rule is derivable but in general, the opposite of this phrase is not true. For example, In Intuitionistic logic there are some rules which are admissible but are not derivable. Admissible rules were studied by Lorenzen [42], Harrop [32] and Mints [50] who has found interesting examples of admissible rules that are not derivable in Intuitionistic logic, in $S4$, etc. The question whether algorithms exist for recognising whether rules in Intuitionistic Propositional Logic IPC are admissible was asked by Friedman [28]. This problem was solved by V. Rybakov [45] and [46] for IPC and for modal logic $S4$. He also proved the same approach can be used for a broad range of propositional modal logics, for example $K4$, $S4$, GL [44]. Unification theory provides a systematic approach to some important logical problems, in particular, to the admissibility problem of inference rules. Whenever the unification type of a logic is unitary or finitary there exists an algorithm to recognize if a given inference rule is admissible in that logic. Two relation between admissibility of rules and unification problem is defined as follows:

- Let L is a consistent logic. The following are equivalent:
  1. The formula A is unifiable,
  2. The rule $r = \dfrac{A}{\perp}$ is non-admissible.
- If L is finitary then the following are equivalent:

1. The rule $r = \dfrac{A_1, ..., A_n}{B}$ is admissible
2. The formula $\sigma(B) \in L$ for every maximal unifier σ for formulas $A_1, ..., A_n$.

As you see, To reduce admissibility to unification problem we need to know about unification type of logic L.
Unification which is the problem of making terms syntactically equal by replacing their variables by some new terms was introduced in automated deduction by Robinson [51]. He showed that unifiable terms have a most one general unifier. A unification problem is usually solved by substitution, which is the mapping of a symbolic value to every variable involved in the problem. In other words, the unification problem essentially focus to look for a substitution in order to unify two given terms. At the next step it is expected to provide a minimal and complete set of substitutions for a given problem. The unification in logic also is related to find a substitution that makes a formula into theorem or tautology. In general, the unification problem in a normal modal logic is to determine, given a formula φ whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. We shall say that a set of unifiers of a unifiable formula φ is complete if for all unifiers σ of φ, there exists a unifier τ of φ in that set such that τ is more general than σ. Now, an important question is to determine whether a given unifiable formula has minimal complete sets of unifiers [5], [22]. When such sets exist, they all have the same cardinality. In that case, a unifiable formula is either infinitary, or finitary, or unitary, depending whether its complete sets of unifiers are either infinite, or finite, or with cardinality 1. Otherwise, the formula is nullary. F. Baader, W. Snyder studied E-unification theory [6] where the terms are no longer required to become syntactically equal, but only equivalent modulo the equational theory. For example, if we consider the theory $C = \{f(x, y) = f(y, x)\}$, which says that the binary function symbol f is commutative, then the unification problem $f(x, y) =^? f(a, b)$ (for constants a, b) has the syntactic unifier $\sigma = \{x \mapsto a, y \mapsto b\}$, which is also a C-unifier, but the substitution $\sigma = \{x \mapsto b, y \mapsto a\}$ is another C-unifier, which is not a syntactic one.
F. Wolter and M. Zakharyaschev in [52] proved that unification problem is undecidable for modal logics K^u and $K4^u$ which are modal logic K and $K4$ extended with the universal modality. They also proved that the admissibility problem for inference rules is undecidable for these logics. In fact, these logics were the first simple examples showing that the decidability of modal logics does not guarantee decidability of unification and admissibility problems. V. Rybakov in

[47] answered to the question whether admissibility in the logic $S4^u$ is decidable. Also, admissibility rules in $S4$ have been studied in [4] by S. Babenyshev et al. They made a sound, complete and terminating tableau calculus deciding both admissibility and derivability of a given rule in modal logic $S4$. Ç. Gencer proved that a modal logic λ such that $\lambda \supseteq K4$ and λ possesses finite model property inherits all admissible rules in $K4$ iff λ satisfies the so-called co-cover property which is a semantic property about $K4$-models [29].
For first time, S. Ghilardi introduced the notion of projectivity in [31] to determine that the unification type is finitary in $S4$ and $K4$ (Also see [36]). Jěrábek in [34] showed that the unification type is nullary in basic modal logic K. P. Balbiani et al. proved that unification type of modal logic $K + \Box\Box\bot$ is finitary, or unitary [12].
S. Babenyshev, V. Rybakov proved that unification type of a propositional Linear Temporal Logic is unitary. Moreover, they presented an algorithm for constructing a most general unifier for unifiable formulas in Linear Temporal Logic (see [3]).
W. Dzik in [24] proved that if a logic has projective unifiers then it is almost structurally complete. W. Dzik in [25] proved that every unifiable formula has a projective unifier in L iff L contains $S4.3$ where L is a normal modal logic containing $S4$. S. Kost [37] showed that a transitive normal modal logic L have projective unification iff L contains $K4D1$.
P. Balbiani and Ç. Gencer in [7] proved that unification type of modal logics KD is nullary. They used the similar arguments of Jěrábek in [34]. In addition, P. Balbiani and Ç. Gencer in [10] proved that unification type of Modal Logics Between KB and KTB are nullary. And they also proved that unification type of several non-symmetric non-transitive modal logics are nullary [9].

The thesis presents results on unification and unification types in modal logics $KD5$, $K5$ and $Alt_1 + \Box\Box\bot$, in fusions of modal logics and in Dynamic Epistemic logic.

Chapter 2 contains necessary basic notions of modal logic.
In Chapter 3, the admissibility of rules in modal logic $S4$ is investigated. In this chapter we define a general reduced normal form. Then we transform an inference rule to a general reduced normal form. we present an algorithm inspired by [44] for recognizing non-admissibility rules in logic $S4$. In this chapter, we also consider sets of admissible rules and investigate about some properties of them.
Chapter 4 contains the basic notion of unification. In this chapter, we review

some of previous works and show that the unification type of modal logics $KD5$ and $K5$ is unitary or nullary.
In Chapter 5, we prove that unification type of the logic $Alt_1 + \Box\Box\bot$ is unitary. The proof follows from two statements. On the one hand, we prove that the logic $Alt_1 + \Box\Box\bot$ is filtering hence it is nullary or unitary. On the other hand, we prove that the logic $Alt_1 + \Box\Box\bot$ is reasonable then it is finitary or unitary. Therefore, the logic $Alt_1 + \Box\Box\bot$ is unitary. In general, P. Balbiani et al. proved that unification types of the modal logics determined by classes of deterministic frames is unitary (see [11]). These results partly answer to an open problem of S. Ghilardi (private communication, 2018). they will be presented during the workshop UNIF [12].
Chapter 6 contains unification problem in fusion of two modal logics. Fusion of modal logics are everywhere in computer science and artificial intelligence. K. Fine and G. Schurz proved that some properties such as completeness and decidability of modal logics L_1 and L_2 are inherited to the fusion $L_1 \otimes L_2$ [26]. See further about combining modal logics in [40]. In this chapter, we consider fusion $L_1 \otimes L_2$ and prove that if L_1 is nullary and L_2 is a consistent modal logic then the unification type of fusion $L_1 \otimes L_2$ is neither unitary nor finitary. For instance, we prove that the unification type of fusion $K_1 \otimes K_2$ is nullary. As well, in this chapter we prove that the unification type of multi-epistemic logic (fusion of $S5_1 \otimes S5_2$) is nullary. (see [13]). This last result about $S5_1 \otimes S5_2$ is an answer to an open problem of W. Dzik [22].
Chapter 7 contains unification in simple epistemic planning problem. In this chapter, we solve the simple epistemic planning problem with unification technique. In this respect, we consider the associated formula $A \to \langle x \rangle B$ where A and B are epistemic formula and x is a variable, we find a public announcement ψ by unification technique such that $A \to \langle \psi \rangle B$ is valid in public announcement logic. Then, we have to find a most general unifier for the associated formula $A \to \langle x \rangle B$.

2 Basic Modal Logic

Contents

Modal logic is a type of formal logic primarily developed in the beginning of the 20th century [41] and in the 1960 by [33]. It extended Classical Propositional Logic by operators expressing modalities. The most well-known modal propositions are propositions about what is a necessary case and what is a possible case. For example, the following sentences are modal propositions:

- It is possible that it will rain tomorrow.
- A proposition p is not possible if and only if the negation of p is necessary.

The operators "it is possible that" and "it is necessary that" are called "modal" operators.

2.1 Syntax

The language of Basic Modal Logic is an extension of the classical propositional syntax. The two unary connectives $\Box$ and $\Diamond$ are added to the language of classical propositional logic. Let P is a countable set of atoms and we use the notation $p, q, r, ...$ for elements of P. The elements of P are also called atomic formulas or propositional letters.

Definition 1 *Formulas of basic modal logic are given by the following rule*

$$\varphi ::= \; p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$$

where p is any atomic formula. We will also write formulas with lower case Greek letter α, β, etc or with upper case Latin letter A, B, etc. We will write $\varphi(p_1, ..., p_n)$ (or $\alpha(p_1, ..., p_n), A(p_1, ..., p_n)$) to insist on the fact that a formula only contains the atomic formulas $p_1, ..., p_n$. We will also write $\varphi(\overline{p})$ (or $\alpha\overline{p}, A(\overline{p})$) where $\overline{p}$ denotes a tuple of atomic formulas. For all tuples $\overline{x}$ of atomic formulas, let $F(\overline{x})$ be the set of all formulas of the form $\varphi(\overline{x})$.

The Boolean connective $\top, \vee, \rightarrow$ and $\leftrightarrow$ are defined as usual. In this case, the diamond ("possible") connective is $\Diamond\varphi ::= \neg\Box\neg\varphi$. The new connectives $\Box$ and $\Diamond$ are read "box" and "diamond" respectively and are dual of each other.
Substitution: Throughout this thesis we will use the notion of substitution. A substitution is a function σ from P to the set of all formulas. By induction on the formula φ, we can define the formula $\sigma(\varphi)$ as follows:

- $\sigma(p) = p$,
- $\sigma(\bot) = \bot$,
- $\sigma(\neg\varphi) = \neg\sigma(\varphi)$,
- $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$,
- $\sigma(\Box\varphi) = \Box\sigma(\varphi)$.

Definition 2 ***(Degree)*** *We define the degree of modal formulas as follows.*

- $deg(p) = 0$,
- $deg(\bot) = 0$,
- $deg(\neg\varphi) = deg(\varphi)$,
- $deg(\varphi \wedge \psi) = max\{deg(\varphi), deg(\psi)\}$,
- $deg(\Box(\varphi)) = 1 + deg(\varphi)$.

An axiomatic system for a modal logic L consists of axioms and inference rules. Axioms contain at least the Boolean tautologies and the axiom K:

- $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

The rules contain at least modus ponens and necessitation:

- $\dfrac{\varphi, \varphi \to \psi}{\psi}$
- $\dfrac{\varphi}{\Box\varphi}$.

The theorems of a logic are all the formulas which can be derived from the axioms by the inference rules. To make a new axiomatic system we need to add axioms and inference rules to the above minimal axiomatic system. Let us define inference rules and we will investigate about admissible rules in chapter 3.

Definition 3 *An inference rule is usually given as a finite set of premise and a conclusion. The rule is denoted as follows:*

$$r = \frac{\alpha_1(x_1, ..., x_n), ..., \alpha_k(x_1, ..., x_n)}{\beta(x_1, ..., x_n)}$$

Where $\alpha_1, ..., \alpha_k, \beta$ are formulas. We often use $r = \dfrac{\alpha}{\beta}$ briefly.

Definition 4 *For a formula φ, we denote by $sub(\varphi)$ the set of all sub-formulas of φ. For a rule $r = \dfrac{\alpha}{\beta}$, we denote by $sub(\alpha, \beta)$ the set of all sub-formulas of α and β.*

Definition 5 *A rule $r = \dfrac{\alpha}{\beta}$ is admissible for the modal logic L, if for every substitution from $\sigma(\alpha) \in L$ it follows $\sigma(\beta) \in L$.*

2.2 Semantics

In this section, we introduce frames and models and we explain how to determine whether a given formula is true or false in a given model.

Definition 6 *A frame $\mathscr{F}$ in basic modal logic is a pair $\langle W, R\rangle$ such that*

1. *W is a non-empty set.*
2. *R is a binary relation on W.*

That is, a frame for the basic modal language is simply a relational structure bearing a single binary relation. The elements of W are called "possible worlds" or "states". The binary relation R is called "accessibility relation".

Definition 7 *A model for the basic modal language is a pair $\mathcal{M} = (\mathcal{F}, v)$, where $\mathcal{F} = (W, R)$ is a frame for the basic modal language and v is a function assigning to each proposition letter p in P a subset $v(p)$ of W. Formally $v : P \to \mathcal{P}(W)$, where $\mathcal{P}(W)$ denotes the power set of W.*

Definition 8 *Suppose w is a state in a model $\mathcal{M} = \langle W, R, v\rangle$. Then we inductively define the notion of a formula φ being satisfied (or true) in $\mathcal{M}$ at state w as follows:*

- *$\mathcal{M}, w \vDash p$ iff $w \in v(p)$, where $p \in P$,*
- *$\mathcal{M}, w \nvDash \bot$,*
- *$\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$,*
- *$\mathcal{M}, w \vDash \varphi \wedge \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$,*
- *$\mathcal{M}, w \vDash \Box\varphi$ if and only if for all $v \in W$, wRv and $\mathcal{M}, v \vDash \varphi$.*

It follows from this definition that $\mathcal{M}, w \vDash \Diamond\varphi$ iff for some $v \in W$ we have wRv and $\mathcal{M}, v \vDash \varphi$.

If $\mathcal{M}$ does not satisfy φ at w we often write $\mathcal{M}, w \nvDash \varphi$, and say that φ is false or refuted in w. For all formulas φ, let $v(\varphi) = \{w \in W : \mathcal{M}, w \vDash \varphi\}$.

Definition 9 *A formula φ is valid at a state w in a frame $\mathcal{F}$ (notation: $\mathcal{F}, w \vDash \varphi$) if φ is true at w in every model $(\mathcal{F}, v)$ based on $\mathcal{F}$; φ is valid in a frame $\mathcal{F}$ (notation $\mathcal{F} \vDash \varphi$) if it is valid at every state in $\mathcal{F}$. A formula φ is valid in a class of frames F (notation: $F \vDash \varphi$) if it is valid in every frame $\mathcal{F}$ in F; and it is valid (notation: $\vDash \varphi$) if it is valid in the class of all frame. The set of all formulas that are valid in a class of frames F is called the logic of F (notation: Λ_F).*

Definition 10 *The inference rule $r = \dfrac{\alpha}{\beta}$ is valid in model $\mathcal{M}$ iff $\mathcal{M} \vDash \alpha$ implies $\mathcal{M} \vDash \beta$.*

In this thesis we will consider the modal logics $K, KD, S4, S5$, etc. For example, accessibility relation in logic $S5$ is transitive, reflexive and Euclidean.

Proposition 1 *Let $\mathcal{F} = (W, R)$ be a frame, then*

1. *R is reflexive if and only if $\mathcal{F} \vDash \Box p \to p$,*

2. *R is transitive if and only if $\mathscr{F} \models \Box p \to \Box\Box p$,*

3. *R is Euclidean if and only if $\mathscr{F} \models \Diamond p \to \Box\Diamond p$*

Proof 1 *refer to [18], Example* 3.6.

Definition 11 *We first define "disjoint unions" for the basic modal language. We say that two models are disjoint if their domains contain no common elements. For disjoint models $\mathscr{M}_i = (W_i, R_i, v_i)(i \in I)$, their disjoint union is the structure $\biguplus_i \mathscr{M}_i = (W, R, v)$, where W is the union of the sets W_i, R_is the union of the relations R_i, and for each proposition letter p, $v(p) = \biguplus_{i \in I} v_i(p)$.*

Proposition 2 *For each modal formula φ, for each $i \in I$, and each element w of $\mathscr{M}_i$, we have $\mathscr{M}_i, w \models \varphi$ iff $\biguplus_{i \in I} \mathscr{M}_i, w \models \varphi$.*

Proof 2 *Refer to [18], proposition* 2.3.

Definition 12 ***(Generated Submodels)****We define generated submodels for the basic modal language. Let $\mathscr{M} = (W, R, v)$ and $\mathscr{M}' = (W', R', v')$ be two models; we say that $\mathscr{M}'$ is a sub-model of $\mathscr{M}$ if $W' \subseteq W$, R' is the restriction of R to W' (that is: $R' = R \cap (W' \times W')$), and v' is the restriction of v to $W\prime$ (that is: for each p, $v'(p) = v(p) \cap W'$). We say that $\mathscr{M}'$ is a generated submodel of $\mathscr{M}$ (notation:$\mathscr{M}' \rightarrowtail \mathscr{M}$) if $\mathscr{M}'$ is a submodel of $\mathscr{M}$ and for all points w the following condition holds:*

if w is in $\mathscr{M}'$ and wRv, then v is in $\mathscr{M}'$.

Proposition 3 *Let $\mathscr{M}$ and $\mathscr{M}'$ be models such that $\mathscr{M}'$ is a generated submodel of $\mathscr{M}$. Then, for each modal formula φ and each element w of $\mathscr{M}'$ we have that $\mathscr{M}, w \models \varphi$ iff $\mathscr{M}', w \models \varphi$.*

Proof 3 *Refer to [18], proposition* 2.6.

Definition 13 *(Bounded Morphisms) Let $\mathscr{M} = (W, R, v)$ and $\mathscr{M}' = (W', R', v')$ be models. for the basic modal language. A mapping $f : \mathscr{M} \to \mathscr{M}'$ is a bounded morphism if it satisfies the following conditions:*

1. *w and $f(w)$ satisfy the same proposition letters.*

2. *(**The forth condition**) f is a homomorphism with respect to the relation R (that is, if wRv then $f(w)R'f(v)$.*

3. *(**The back condition**) if $f(w)R'v'$ then there exists v such that wRv and $f(v) = v'$.*

If there is a surjective bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$, then we say that $\mathcal{M}'$ is a bounded morphic image of $\mathcal{M}$, and write $\mathcal{M} \twoheadrightarrow \mathcal{M}'$.

Proposition 4 *Let $\mathcal{M}$ and $\mathcal{M}'$ be models such that $f : \mathcal{M} \to \mathcal{M}'$ is a bounded morphism. Then, for each modal formula φ, and each element $w \in \mathcal{M}$, we have $\mathcal{M}, w \vDash \varphi$ iff $\mathcal{M}', f(w) \vDash \varphi$.*

Proof 4 *Refer to [18], proposition* 2.14.

Definition 14 *Let $\mathcal{M} = (W, R, V)$ be a model. A subset X of W is called definable (or expressible) iff there exists a formula α such that*

$$X = V(\alpha).$$

An element $x \in W$ is definable (or expressible) if the set $\{x\}$ is definable. Let S be a new valuation of certain propositional variables on the frame (W, R). The valuation S is called definable (or expressible) if and only if for any letter p_i from the domain of S, there exists a formula α_i such that $S(p_i) = V(\alpha_i)$.

2.3 Normal Modal Logic

A normal modal logic is simply a set of formulas satisfying certain syntactic closure conditions. Which conditions? We will define a Hilbert-style axiom system called K. K is the "minimal" (or weakest) system for reasoning about frames; stronger systems are obtained by adding extra axioms. We discuss K in some detail, and then, at the end of the section, define normal modal logics.

A formula φ is K-provable if it occurs as the last item of some K-proof, and if this is the case we write $\vdash_K \varphi$. K is the minimal modal Hilbert system in the following sense. As we have seen, its axioms are all valid, and all three rules of inference preserve validity, hence all K-provable formulas are valid. (K is sound with respect to the class of all frames.) Moreover, the converse is also true: if a basic modal formula is valid, then it is K-provable. (That is, K is complete with respect to the class of all frames.) In short, K generates precisely the valid formulas.

Definition 15 ***(Normal Modal Logics)*** *A normal modal logic Λ is a set of formulas that contains all Boolean tautologies, all formula of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, modus ponens $\dfrac{\varphi, \varphi \to \psi}{\psi}$ and necessitation $\dfrac{\varphi}{\Box\varphi}$. We call the smallest normal modal logic K.*

Definition 16 *A proof is a finite sequence of formulas, each of which is an axiom, or follows from one or more earlier items in the sequence by applying a rule of proof . The axioms of K are all instances of propositional tautologies plus: $(K)\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. Its rules of proof are modus ponens and necessitation.*

Example 1 *1. $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$ is K-provable.*

2. $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$ is K-provable.

In this thesis, we will consider the following modal logics:

$K4$	$K \oplus \Box p \to \Box\Box p$
$S4$	$K4 \oplus \Box p \to p$
$S5$	$S4 \oplus \Diamond p \to \Box\Diamond p$
KD	$K \oplus \Box p \to \Diamond p$
$KD5$	$KD \oplus \Diamond p \to \Box\Diamond p$
$K45$	$K4 \oplus \Diamond p \to \Box\Diamond p$

Definition 17 *Let $n \geqslant 0$. A Kripke model $K_n = (W, R, V)$ is called n-characterizing for a modal logic L (any normal modal logic)if the domain of the valuation V from K_n is the set P which consists of n different propositional variables, and if the following holds: for any formula α which is build up of variables from P*

$$\alpha \in L \Leftrightarrow K_n \models \alpha$$

Let L be a logic. Let Γ be a set of formulas and A be a formula. A derivation of A from Γ in L is a finite sequence $A_1, ..., A_n$ of formulas such that $A_n = A$ and every formula in the sequence either is in L, or is in Γ, or is obtained by means of modus ponens rule from previous formulas in the sequence, or is obtained by means of necessitation rule from a previous formula in the sequence. We will write $\Gamma \vdash_L A$ if there exists a derivation of A from Γ in L.
If $\Gamma = \{B_1, ..., B_m\}$ is finite that we will write $B_1, ..., B_m \vdash_L A$.

Definition 18 *An inference rule* $r = \dfrac{\alpha}{\beta}$ *is called derivable in logic* L *if* $\alpha \vdash_L \beta$.

We say that a frame F is a frame for modal logic L (or is an L-frame) if $F \vDash L$.

Definition 19 *A rule* r_1 *is semantically equivalent to a rule* r_2 *in modal logic* L *iff* $F \vDash r_1$ *iff* $F \vDash r_2$ *for any* L*-frame* F.

3 Admissibility in the logic $S4$

Contents

The concept of an admissible rule was introduced by Paul Lorenzen (1955). The admissible rules of a logic are the rules that can be added to the logic as inference rules without producing any new theorems. Equivalently, they are rules such that if the premises are made into theorems by any substitution then this substitution also makes the conclusion into a theorem. Admissible rules have been studied by many authors in particular, V. Rybakov. One important question about admissible rules of a given logic is whether the set of all admissible rules is decidable. Note that the problem is non-trivial even if the logic itself is decidable. For instance, the basic modal logic K is decidable and the decidability of the problem of admissibility in K is a major open problem. Modal Logic $S4$ is decidable and the problem of admissibility in $S4$ is decidable as proved by V. Rybakov (1985). In fact, admissibility of rules is known to be decidable in many modal and superintuitionistic logics. The first algorithm or decision procedures to recognize admissibility of rules was introduced by V. Rybakov (1984, 1985).

Example 2 *The rule $\frac{\Box x}{x}$ is admissible in logic $S4$ since if $\vdash_{S4} \Box\sigma(x)$ then $\vdash \sigma(x)$ for arbitrary substitution σ.*

There is a strong relation between admissibility and unification. Suppose L is a modal logic ($K4, S4, etc$). Let $\dfrac{A}{B}$ be an inference rule. So, $\dfrac{A}{B}$ is non-admissible iff there exists a substitution σ such that $\sigma(A) \in L$ and $\sigma(B) \notin L$. When L is unitary or finitary, unifiable formulas possess finite minimal complete sets of unifiers. As a result, when L is decidable and when minimal complete sets of unifiers can be computed for any arbitrary given unifiable formula, then the non-admissibility problem in L can be decided as follows:

- given a rule $\frac{A}{B}$,
- check whether A in L-unifiable,
- if A is no L-unifiable then answer "rule $\frac{A}{B}$ is L-admissible"
- otherwise, compute a minimal complete set $\Sigma = \{\sigma_1, ..., \sigma_n\}$ of L-unifiers of A and for all $i = 1, ..., n$ check whether $\sigma_i(B) \in L$,
- if for all $i = 1, ..., n$, $\sigma_i(B) \in L$ then answer "rule $\frac{A}{B}$ is L-admissible",
- otherwise answer "rule $\frac{A}{B}$ is non-L-admissible".

As can be seen from this algorithm, when L is decidable, when L-unification is decidable, when L is either unitary or finitary and when one can compute the minimal complete sets of L-unifiers for any given unifiable formula than the above algorithm decides L-admissibility.
Conversely, L-unification can be reduced to non-L-admissibility seeing that for all consistent modal logic L (it does not matter what is the unification type of L), a given formula A is unifiable in L iff the inference rule $\frac{A}{\perp}$ is non-admissible in L.

3.1 Syntactic criteria for admissibility in $S4$

In this section first we introduce the notion of admissibility for inference rules and also some properties in the logic $S4$. Then we provide some theorems and an algorithm introduced by V. Rybakov [44] which are used for recognizing the admissibility of inference rules in modal logic $S4$. Historically this algorithm is the first algorithm for recognising admissible rules of modal logic $S4$. In this respect, we need to define the notion of reduced normal form.

Definition 20 *A rule r is said to be in reduced normal form if it has the form*

$$r = \frac{\bigvee_{1\leqslant j\leqslant s} \phi_j}{x_0}$$

where each disjunct ϕ_j has the form

$$\phi_j = \bigwedge_{0\leqslant i\leqslant n} x_i^{t(i,j,0)} \wedge \bigwedge_{0\leqslant i\leqslant n} (\Diamond x_i)^{t(i,j,1)}$$

and

1. *s and t are integers,*
2. *All ϕ_j are different,*
3. *$x_0,\ldots,x_n$ are propositional variables,*
4. *t is a Boolean function $t : \{0,\ldots,n\} \times \{1,\ldots,s\} \times \{0,1\} \to \{0,1\}$,*
5. *$\alpha^0 = \neg\alpha$ and $\alpha^1 = \alpha$ for any formula α.*

Example 3 *The rule $r = \dfrac{(x_2 \wedge \Diamond x_2) \vee (\neg x_3 \wedge \Diamond x_3)}{x_1}$ is a rule in reduced normal form.*

We usually use the notation rf(r) when a rule in reduced normal form obtained from r. Let us see, how to convert an inference rule to its reduced normal form. Also this method has been introduced by V. Rybakov.

Proposition 5 *If an inference rule is derivable in L then the rule is admissible for L.*

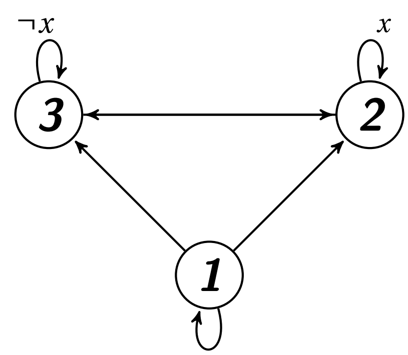
Proof 5 *Suppose that $\alpha_1(x_1,\ldots,x_n),\ldots,\alpha_m(x_1,\ldots,x_n) \vdash_L \beta(x_1,\ldots,x_n)$. Consider a substitution ν, $\nu(x_i) = \gamma_i$ (γ_i is a L-formula) such that for every j, the inclusion $\alpha_j(x_1,\ldots,x_n) \in L$ holds. We take an arbitrary derivation $\mathscr{S}$ of β from $\alpha_1,\ldots,\alpha_m$ in L. Furthermore, we choose the substitution ω which coincides with ν on the domain $Dom(\nu)$ of ν and maps any letter lying not in $Dom(\nu)$ onto β, say. The sequence $\mathscr{S}^\omega$, obtained from $\mathscr{S}$ by applying ω to each their members, will be a derivation in L from the empty set of hypothesis. Indeed, under substitution ω all hypothesis will turn into theorems of L, the set of theorems of is closed with respect to substitutions, and all inference rules are structural (consistent with substitutions). Thus $\vdash_L \beta^\nu$, that is $\beta(\gamma_1,\ldots,\gamma_n) \in L$.*

Lemma 1 *Any valid inference rule* $r = \dfrac{\alpha}{\beta}$ *in modal logic* $S4$ *is admissible for* $S4$.

Proof 6 *Refer to* [44], *Lemma* 3.1.5.

Example 4 *Let x be a propositional variable*

- *The rule* $r = \dfrac{\Box x}{x}$ *is derivable and admissible in* $S4$. *Be more concise: since* $\vdash_{S4} \Box x \to x$ *then the rule r is derivable. Thus the rule r is admissible by Lemma 1.*
- *The rule* $r = \dfrac{\Box\Diamond x}{\Diamond\Box x}$ *is not admissible and not derivable in* $S4$. *A substitution which can be used to show that the rule non-admissible is* $\sigma(x) = x \to \Box x$.
  Hence, $\vdash_{S4} \Box\Diamond\sigma(x)$ *and* $\nvdash_{S4} \Diamond\Box\sigma(x)$.
  Let us see why $\vdash_{S4} \Box\Diamond(x \to \Box x)$ *and* $\nvdash_{S4} \Diamond\Box(x \to \Box x)$. *Since,* $\Diamond(x \to \Box x)$ *is equivalent to* $\Box x \to \Diamond\Box x$ *and*
  - $\vdash_{S4} \Box x \to \Diamond\Box x$. *Then,*
  - $\vdash_{S4} \Diamond(x \to \Box x)$. *By necessitation,*
  - $\vdash_{S4} \Box\Diamond(x \to \Box x)$. *But*
  - $\nvdash_{S4} \Diamond\Box(x \to \Box x)$. *Let us see why* $\nvdash_{S4} \Diamond\Box(x \to \Box x)$. *Consider the following model M where the accessibility relation is the reflexive and transitive closure of the one shown by the arrows. Hence,* $M, 1 \nvDash \Diamond\Box(x \to \Box x)$.



Therefore, the rule $r = \dfrac{\Box\Diamond x}{\Diamond\Box x}$ *is not admissible and by Lemma 1 is not derivable.*

– *The rule* $r = \dfrac{\Diamond x \wedge \Diamond \neg x}{\bot}$ *is admissible in* S4 *but it is not derivable in* S4 *[27].*

Lemma 2 *There exists an algorithm which for any given inference rule r in the language of modal propositional logic, constructs a suitable reduced normal form rf(r).*

Proof 7 *Let* $r = \frac{\alpha}{\beta}$ *be a rule. We need a set of new variables* $\{z_\gamma \mid \gamma \in sub(\alpha,\beta)\}$. *Let us consider the following steps:*

- *Step* 1*: replace* $r = \frac{\alpha}{\beta}$ *with* $r_1 = \frac{\alpha \wedge (z_\beta \leftrightarrow \beta)}{z_\beta}$.

- *Inductive step: suppose the rule* $r_i = \dfrac{\gamma_i}{z_\beta}$ *was obtained in the i-th step. Find* $\delta \in sub(\gamma_i) \cap sub(\alpha,\beta)$ *when* δ *is not a variable and not a proper subformula of any other formula in* $sub(\gamma_i) \cap sub(\alpha,\beta)$. δ *is called final. At the end, replace the rule* r_i *with a new one* $r_{i+1} = \frac{\gamma_{i+1}}{z_\beta}$*, namely*

$$\gamma_{i+1} = z_\alpha \wedge \bigwedge_{\gamma \in Sub(\alpha,\beta) \setminus Var(r)} (z_\gamma \leftrightarrow \gamma^{\circledast})$$

where

$$\gamma^{\circledast} = \begin{cases} z_\delta \vee z_\epsilon, & when\ \gamma = \delta \vee \epsilon \\ *\delta, & when\ \gamma = *\delta for * \in \{\neg, \Diamond\} \end{cases}$$

Therefore after a finite number of steps we get a premise γ_k*, which is a Boolean combination of literals of the form* x *or* $\Diamond x$*, where* x *is propositional variable.*

- *Final step: we transform the premise of the obtained rule* $r_N = \dfrac{\gamma_k}{z_\beta}$ *into an equivalent disjunctive normal form over literals*

It is easy to show that the reduced normal form of inference rule is equivalent to the original rule.

Definition 21 *A rule r_1 is equivalent by admissibility to a rule r_2 in a logic $S4$ if r_1 is admissible in $S4$ iff r_2 is admissible in $S4$.*

Lemma 3 *For any inference rule r, r is semantically equivalent to rf(r) in $S4$.*

Proof 8 *Refer to* [44], *Lemma* 3.1.8.

Corollary 1 *A rule r is valid in modal logic $S4$ iff the rule rf(r) is valid in $S4$.*

Proof 9 *Refer to* [44], *corollary* 3.1.9.

Suppose $r = \dfrac{\bigvee\limits_{1\leqslant j\leqslant s} \phi_j}{x_0}$ is a rule as defined in 20. Let $\Theta(r) = \{\phi_1, ..., \phi_s\}$ be the set of all disjuncts of premise of r. Notice that if ϕ_i and ϕ_j are distinct elements of $\Theta(r)$ then $\phi_i \wedge \phi_j$ is logically equivalent to $\perp$. For every $\phi_j \in \Theta(r)$, let

$$\theta(\phi_j) = \{x_i \mid t(i, j, 0) = 1\} \text{ and } \theta_{\Diamond}(\phi_j) = \{x_i \mid t(i, j, 1) = 1\}$$

In fact, $\theta(\phi_j)$ is the set of variables of r with positive occurrence in ϕ_j, and $\theta_{\Diamond}(\phi_j)$ is the set of variables x_i of r with the positive occurrence of $\Diamond x_i$ in ϕ_j.

To express the main theorem, we need to define a new model that is associated to r and to an arbitrary non-empty subset W of $\Theta(r)$. We construct $\mathscr{M}(\Theta(r)) = (W, R, v)$ for every non-empty subset W of $\Theta(r)$ as follows:

- $W \subseteq \Theta(r)$,
- $\phi_i R \phi_j \Leftrightarrow \theta_{\Diamond}(\phi_j) \subseteq \theta_{\Diamond}(\phi_i)$ for any $\phi_i, \phi_j \in W$,
- $p_i \in v(\phi_j) \Leftrightarrow x_i \in \theta(\phi_j)$ for any $\phi_i \in W$.

Now, we have all required tools to express main theorem which says a rule is admissible or non-admissible in logic $S4$.

Theorem 1 *A rule $r = \dfrac{\bigvee\limits_{1\leqslant j\leqslant s} \phi_j}{x_0}$ in reduced normal form is admissible for modal logic $S4$ iff for any non-empty set $W \subseteq \Theta(r)$, the model $\mathscr{M}(\Theta(r))$ fails to have at least one of the following properties.*

1. *There is $\phi_j \in W$ such that $\mathscr{M}(\Theta(r)), \phi_j \nvDash x_0$.*
2. *$\mathscr{M}(\Theta(r)), \phi_j \vDash \phi_j$ for all $\phi_j \in W$.*

3. *For any subset D of W there exists $\phi_j \in W$ such that*

$$\theta_{\Diamond}(\phi_j) = \theta(\phi_j) \cup \bigcup_{\phi \in D} \theta_{\Diamond}(\phi).$$

Proof 10 *refer to [44], Theorem* 3.9.6.

Thanks to Theorem 1 it is possible to construct an algorithm for deciding admissibility in $S4$. Let us use an example to illustrate Theorem 1:

Example 5 *Consider the rule $r = \dfrac{\Diamond x \wedge \Diamond \neg x}{\perp}$. We show that this rule is admissible. In order to use 1, we should transform the rule r to a rule in reduced normal form. Hence, we use Lemma 2 to find rf(r). Then, we have $rf(r) = \dfrac{\neg x \wedge \neg y_0 \wedge y_1 \wedge y_2 \wedge \Diamond x \wedge \Diamond y_2}{y_0}$. Let $\phi_1 = \neg x \wedge \neg y_0 \wedge y_1 \wedge y_2 \wedge \Diamond x \wedge \Diamond y_2$ and $W = \{\neg x \wedge \neg y_0 \wedge y_1 \wedge y_2 \wedge \Diamond x \wedge \Diamond y_2\}$. Let us check the conditions of Theorem* 1.

1. *As you see, y_0 is as conclusion of the rule rf(r) and $\mathcal{M}(\Theta(r)), \phi_1 \nvDash y_0$.*

2. *let us prove that $\mathcal{M}(\Theta(r)), \phi_1 \nvDash \phi_1$. Suppose $\mathcal{M}(\Theta(r)), \phi_1 \vDash \Diamond x$ then, we must have $\mathcal{M}(\Theta(r)), \phi_1 \vDash x$. This is in contradiction to $\mathcal{M}(\Theta(r)), \phi_1 \vDash \neg x$.*

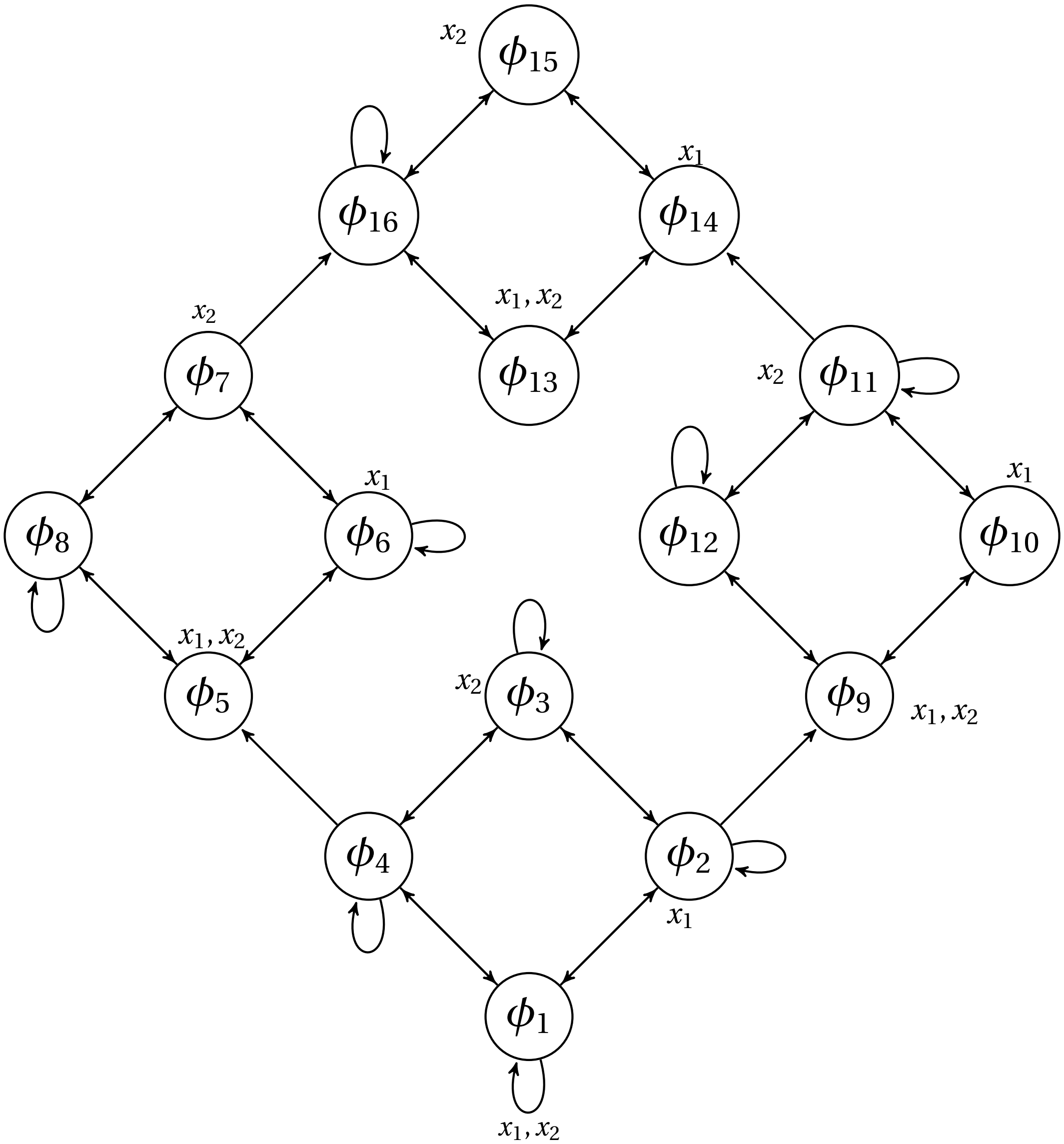
3. *Let $D = \emptyset$. We have $\theta_{\Diamond}(\phi_1) = \{x, y_2\}$ and $\theta(\phi_1) = \{y_1, y_2\}$ then, $\theta_{\Diamond}(\phi_1) \neq \theta(\phi_1)$. Thus, $\theta_{\Diamond}(\phi_1) \neq \theta(\phi_1)$ i.e the third condition of Theorem 1 failed.*

First condition of Theorem 1 *holds but second and third conditions of Theorem* 1 *do not hold. Therefore, the rule r is admissible.*

Let us consider a general form when we transform a given rule r to reduced normal form rf(r) and rf(r) has only two variables. In this case, the premise of rule rf(r) will be subset of the following 16 formulas. Let the rule rf(r) be as $rf(r) = \dfrac{\bigvee_{i \in I} \phi_i}{x_1}$ where $I \subseteq \{1, ..., 16\}$ and ϕ_1 to ϕ_{16} are as follows:

$\phi_1 = x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$
$\phi_2 = x_1 \wedge \neg x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$
$\phi_3 = \neg x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$
$\phi_4 = \neg x_1 \wedge \neg x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$
$\phi_5 = x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \neg \Diamond x_2$
$\phi_6 = x_1 \wedge \neg x_2 \wedge \Diamond x_1 \wedge \neg \Diamond x_2$
$\phi_7 = \neg x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \neg \Diamond x_2$

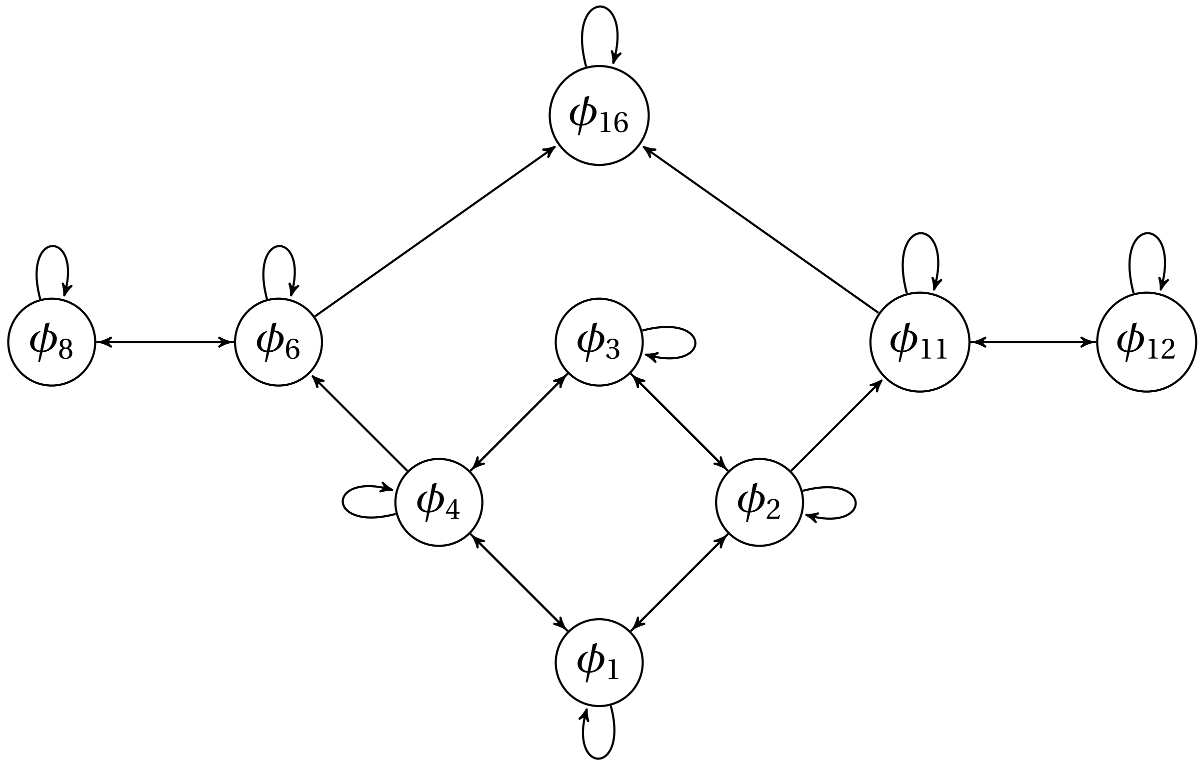
$\phi_8 = \neg x_1 \land \neg x_2 \land \Diamond x_1 \land \neg \Diamond x_2$
$\phi_9 = x_1 \land x_2 \land \neg \Diamond x_1 \land \Diamond x_2$
$\phi_{10} = x_1 \land \neg x_2 \land \neg \Diamond x_1 \land \Diamond x_2$
$\phi_{11} = \neg x_1 \land x_2 \land \neg \Diamond x_1 \land \Diamond x_2$
$\phi_{12} = \neg x_1 \land \neg x_2 \land \neg \Diamond x_1 \land \Diamond x_2$
$\phi_{13} = x_1 \land x_2 \land \neg \Diamond x_1 \land \neg \Diamond x_2$
$\phi_{14} = x_1 \land \neg x_2 \land \neg \Diamond x_1 \land \neg \Diamond x_2$
$\phi_{15} = \neg x_1 \land x_2 \land \neg \Diamond x_1 \land \neg \Diamond x_2$
$\phi_{16} = \neg x_1 \land \neg x_2 \land \neg \Diamond x_1 \land \neg \Diamond x_2$
According to the definition of model $\mathscr{M}(\Theta(r))$, we have



This model is the model $\mathcal{M}(\Theta(r))$ associated to the rule $r = \dfrac{\bigvee_{i \in I} \phi_i}{x_1}$ when $I = \{1, \ldots, 16\}$ and $W = \{\phi_1, \ldots, \phi_{16}\}$.

Remark 1 *In the model of the rule* $r = \dfrac{\bigvee_{i \in I} \phi_i}{x_1}$ *when* $I = \{1, \ldots, 16\}$*, notice that when* $\mathcal{M}(\Theta(r)), \phi_i \vDash \phi_i$*, we show* ϕ_i *like a reflexive point, otherwise it is irreflexive point.*

Remark 2 *The rule* $r = \dfrac{\bigvee_{i\in I}\phi_i}{x_1}$ *when* $I = \{1,\ldots,16\}$ *is non-admissible.*

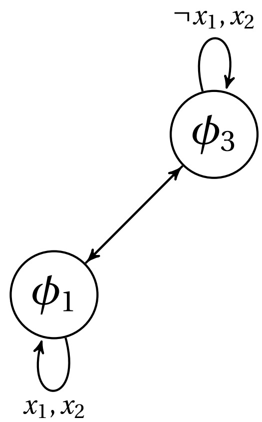


This model is the model $\mathscr{M}(\Theta(r))$ associated to the rule $r = \dfrac{\bigvee_{i\in I}\phi_i}{x_1}$ when $I = \{1,2,3,4,6,8,11,12,16\}$ and $W = \{\phi_1,\phi_2,\phi_3,\phi_4,\phi_6,\phi_8,\phi_{11},\phi_{12},\phi_{16}\}$.

Remark 3 *Notice that any rule in reduced normal form as* $rf(r) = \dfrac{\bigvee_{i\in I}\phi_i}{x_1}$ *such that* $\{\phi_i : i \in I\}$ *contains at least one of sets* $\{\phi_1,\phi_3\}$, $\{\phi_1,\phi_4\}$, $\{\phi_3,\phi_6\}$, $\{\phi_6,\phi_8\}$, $\{\phi_{11}\}$ *and* $\{\phi_{16}\}$ *is non-admissible. Since these rules satisfy all conditions of Theorem* 1. *Let us see an example.*

Example 6 *Consider a rule in reduced normal form as* $rf(r) = \dfrac{\bigvee\phi_i}{x_1}$ *which contains one of the above sets. Let us find which substitutions are appropriate to make this rule non-admissible.*

1. *Let* $rf(r) = \dfrac{\bigvee_{i\in I}\phi_i}{x_1}$ *be a rule in reduced normal form such that its premise contains* $\{\phi_1,\phi_3\}$. *Hence the rule* $rf(r) = \dfrac{\bigvee\phi_j \vee \phi_1 \vee \phi_3}{x_1}$ *where* $\phi_1 = x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$ *and* $\phi_3 = \neg x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$. *Let us check all conditions of Theorem* 1. *Let* $W = \{\phi_1,\phi_3\}$.
   Since $\theta_\Diamond(\phi_1) = \theta_\Diamond(\phi_3)$ *then,* ϕ_1 *and* ϕ_3 *can see each other. Also,* $x_1, x_2 \in \nu(\phi_1)$ *and* $\neg x_1, x_2 \in \nu(\phi_3)$. *Hence, we have the model* $\mathcal{M}(\Theta(r))$ *associated to the rule and* W *ia as follows:*



   - *First condition holds. Since,* $p_1 \notin \nu(\phi_3)$ *then by definition of model* $\mathcal{M}(\Theta(r))$ *we have,* $\mathcal{M}(\Theta(r)), \phi_3 \nvDash x_1$.
   - *Second condition holds. Since,* $x_1, x_2 \in \nu(\phi_1)$ *then,* $\mathcal{M}(\Theta(r)), \phi_1 \vDash x_1 \wedge x_2$ *and* $\mathcal{M}(\Theta(r)), \phi_1 \vDash \Diamond x_1 \wedge \Diamond x_2$. *Then* $\mathcal{M}(\Theta(r)), \phi_1 \vDash x_1 \wedge x_2 \wedge \Diamond x_1 \wedge \Diamond x_2$. *Thus,*$\mathcal{M}(\Theta(r)), \phi_1 \vDash \phi_1$. *Also since* $\neg x_1, x_2 \in \nu(\phi_3)$ *then,* $\mathcal{M}(\Theta(r)), \phi_3 \vDash \neg x_1 \wedge x_2$. *Since,* $\phi_3 R \phi_1$ *and* $\mathcal{M}(\Theta(r)), \phi_1 \vDash x_1$, $\mathcal{M}(\Theta(r)), \phi_1 \vDash x_2$ *thus,*$\mathcal{M}(\Theta(r)), \phi_3 \vDash \phi_3$.
   - *Third condition holds. We only check the case* $D = \emptyset$. *Since,* $\theta_\Diamond(\phi_1) = \{x_1, x_2\}$, $\theta(\phi_1) = \{x_1, x_2\}$ *then* $\theta_\Diamond(\phi_1) = \theta(\phi_1)$. *As the reader can see, for all other* $D \subseteq W$, *the third condition holds.*

   *This means that the rule is not admissible in S*4. *So, now, it is time to find an appropriate substitution showing that the rule is not S*4*-admissible. Obviously,* $\phi_1 \vee \phi_3$ *is S*4*-equivalent to* $x_2 \wedge \Diamond x_1$. *Hence, we need a substitution* σ *such that* $\nvdash \sigma(x_1)$ *and* $\vdash \sigma(x_2) \wedge \Diamond\sigma(x_1)$. *It is possible to consider*

$\sigma(x_2) = \top$ *and also the following tablet lists some substitutions* σ *which satisfy the conditions* $\nvdash \sigma(x_1), \vdash \Diamond\sigma(x_1)$.
For example, if σ *is a substitution such that* $\sigma(x_2) = \top$ *and* $\sigma(x_1) = x \to \Box x$, *then,* $\vdash \sigma(x_2), \vdash \Diamond\sigma(x_1)$ *and* $\nvdash \sigma(x_1)$ *in S4. So,* σ *is a substitution showing the rule* $\dfrac{\bigvee \phi_j \vee \phi_1 \vee \phi_3}{x_1}$ *is not S4-admissible.*

$x \to \Box x$	$\Diamond x \to \Diamond\Box\Diamond x$	$\Box(\Diamond x \to \Box\Diamond x)$
$x \to \Box\Diamond x$	$\Diamond\Box x \to \Box\Diamond\Box x$	$\Box(\Diamond x \to \Diamond\Box\Diamond x)$
$x \to \Diamond\Box x$	$\Diamond\Box\Diamond x \to \Box\Diamond x$	$\Box(\Diamond\Box x \to \Box\Diamond\Box x)$
$x \to \Box\Diamond\Box x$	$\Box(x \to \Box\Diamond x)$	$\Box(\Diamond\Box\Diamond x \to \Box\Diamond x)$
$x \to \Diamond\Box\Diamond x$	$\Box(x \to \Diamond\Box\Diamond x)$	$\Box\Diamond(x \to \Box\Diamond\Box x)$
$\Diamond x \to \Box\Diamond x$	$\Box(\Box x \to \Box\Diamond\Box x)$	$\Box\Diamond(\Diamond x \to \Box\Diamond x)$

2. *Let* $rf(r) = \dfrac{\bigvee \phi_i}{x_1}$ *such that* $\bigvee \phi_i$ *contains* $\phi_1 \vee \phi_4$. *The rule rf(r) is non-admissible and satisfies the condition* 1 *to* 3 *of Theorem 1. But which substitution is appropriate for this rule?*

   Obviously, $\phi_1 \vee \phi_4$ *is S4-equivalent to* $(\Diamond x_1 \wedge \Diamond x_2) \wedge (x_1 \leftrightarrow x_2)$. *We need a substitution* σ *such that* $\nvdash \sigma(x_1), \vdash \Diamond\sigma(x_1)$ *and* $\vdash \Diamond\sigma(x_2)$. *For this case, we can use the above table as well.*

3. *Let* $rf(r) = \dfrac{\bigvee \phi_i}{x_1}$ *such that* $\bigvee \phi_i$ *contains* $\phi_3 \vee \phi_6$. *The rule rf(r) is non-admissible and satisfies the condition* 1 *to* 3 *of Theorem 1. But which substitution is appropriate for this rule?*

   Obviously, $\phi_3 \vee \phi_6$ *is S4-equivalent to* $\Diamond x_1 \wedge (\neg x_1 \leftrightarrow x_2) \wedge (\neg x_1 \leftrightarrow \Diamond x_2) \wedge x_1 \leftrightarrow \Box x_1)$. *Hence we need a substitution* σ *which has the properties* $\nvdash \sigma(x_1), \vdash \Diamond\sigma(x_1), \vdash \neg\sigma(x_1) \leftrightarrow \sigma(x_2)$ *and* $\vdash \sigma(x_1) \leftrightarrow \Box\sigma(x_1)$ *and* $\vdash \neg\sigma(x_1) \leftrightarrow \Diamond\sigma(x_2)$.

4. *Let* $rf(r) = \dfrac{\bigvee \phi_i}{x_1}$ *such that* $\bigvee \phi_i$ *contains* $\phi_6 \vee \phi_8$. *The rule rf(r) is non-admissible and satisfies the condition* 1 *to* 3 *of Theorem 1. But which substitution is appropriate for this rule?*

   Obviously, $\phi_6 \vee \phi_8$ *is S4-equivalent to* $\Diamond x_1 \wedge \neg\Diamond x_2$. *Hence we need a substitution* σ *which has the properties* $\nvdash \sigma(x_1), \vdash \Diamond\sigma(x_1)$ *and* $\vdash \neg\Diamond\sigma(x_2)$. $\sigma(x_1)$ *can be any member of the above table and* $\sigma(x_2) = \bot$.

5. *Let* $rf(r) = \dfrac{\bigvee \phi_i}{x_1}$ *such that* $\bigvee \phi_i$ *contains* ϕ_{11}. *The rule rf(r) is non-admissible and satisfies the condition* 1 *to* 3 *of Theorem 1. But which substitution is appropriate for this rule?*

   Obviously, ϕ_{11} *is* $S4$*-equivalent to* $\neg\Diamond x_1 \wedge x_2$. *We need a substitution* σ *such that* $\nvDash \sigma(x_1)$, $\vdash \neg\Diamond\sigma(x_1)$ *and* $\vdash \sigma(x_2)$. *For this case, we can consider* $\sigma(x_1) = \bot$ *and* $\sigma(x_2) = \top$.

6. *Let* $rf(r) = \dfrac{\bigvee \phi_i}{x_1}$ *such that* $\bigvee \phi_i$ *contains* ϕ_{16}. *The rule rf(r) is non-admissible and satisfies the condition* 1 *to* 3 *of Theorem 1. But which substitution is appropriate for this rule?*

   Obviously, ϕ_{16} *is* $S4$*-equivalent to* $\neg\Diamond x_1 \wedge \neg\Diamond x_2$. *We need a substitution* σ *such that* $\nvDash \sigma(x_1)$, $\vdash \neg\Diamond\sigma(x_1)$ *and* $\vdash \neg\Diamond\sigma(x_2)$. *For this case, we can consider* $\sigma(x_1) = \sigma(x_2) = \bot$.

3.2 Generalized reduced normal form

At the previous section, we discussed Rybakov's results on the admissibility condition of any given rule in reduced normal form as $rf(r) = \dfrac{\bigvee_{i\in I} \phi_i}{x_0}$. In this section, we generalize the definition of reduced normal form. Also, in this section, we express some criteria that a set of rules in this general reduced normal form may have and see how these criteria can help to decide $S4$-admissibility. In this respect, we define general reduced normal form as follows:

Definition 22 *A rule* r *is in general reduced normal form if it has the form*

$$r = \frac{\bigvee_{j\in I} \phi_j}{\bigvee_{j\in J} \phi_j}$$

where each disjunct ϕ_j *has the form*

$$\phi_j = \bigwedge_{0\leqslant i\leqslant n} x_i^{t(i,j,0)} \wedge \bigwedge_{0\leqslant i\leqslant n} (\Diamond x_i)^{t(i,j,1)}$$

and

1. $J \subset I$
2. *All* ϕ_j *are different,*
3. $x_0, ..., x_n$ *are propositional variable,*
4. t *is a Boolean function* $t : \{0, ..., n\} \times \{1, ..., s\} \times \{0, 1\} \to \{0, 1\}$,
5. $\alpha^0 = \neg\alpha$ *and* $\alpha^1 = \alpha$ *for any formula* α.
6. $\Theta(r) = \{\phi_j : j \in I\}$.

Lemma 4 *There exists an algorithm which for any given inference rule* r *in the language of modal propositional logic, constructs a suitable general reduced normal form rf(r).*

Proof 11 *Let* $r = \frac{\alpha}{\beta}$ *be a rule. We need a set of new variables* $\{z_\gamma \mid \gamma \in Sub(\alpha)\}$ *and* $\{z'_\gamma \mid \gamma \in Sub(\beta)\}$. *Let us consider the following steps:*

- *Step* 1*: replace* $r = \frac{\alpha}{\beta}$ *with* $r_1 = \frac{\alpha\wedge(z_\alpha\leftrightarrow\alpha)}{\beta\wedge(z'_\beta\leftrightarrow\beta)}$.
- *Suppose the rule* $r_i = \frac{\gamma_i}{\chi_i}$ *was obtained in the* i*th step. Find* $\delta \in sub(\gamma_i) \cap sub(\alpha)$ *and* $\delta' \in sub(\chi_i) \cap sub(\beta)$ *when* δ *and* δ' *are not a variable and not a proper sub-formula of any other formula in* $Sub(\gamma_i) \cap sub(\alpha)$ *and* $Sub(\chi_i) \cap sub(\beta)$. δ *and* δ' *are called final. At the end, replace the rule* r_i *with the new one* $r_{i+1} = \frac{\gamma_{i+1}}{\chi_{i+1}}$*, namely*

$$\gamma_{i+1} = z_\alpha \wedge \bigwedge_{\gamma\in Sub(\alpha)\setminus Var(r)} (z_\gamma \leftrightarrow \gamma^{\circledast}) \text{ and}$$
$$\chi_{i+1} = z'_\beta \wedge \bigwedge_{\gamma\in Sub(\beta)\setminus Var(r)} (z_\gamma \leftrightarrow \gamma^{\oplus})$$

where

$$\gamma^{\circledast} = \begin{cases} z_\delta \vee z_\epsilon, & \text{when } \gamma = \delta \vee \epsilon \\ *\delta, & \text{when } \gamma = *\delta \text{ for } * \in \{\neg, \Diamond\} \end{cases}$$

and

$$\gamma^{\oplus} = \begin{cases} z'_{\delta} \vee z'_{\epsilon}, & when\, \gamma = \delta' \vee \epsilon' \\ *\delta', & when\, \gamma = *\delta' for * \in \{\neg, \Diamond\} \end{cases}$$

Therefore after a finite number of steps we get a premise γ_k and a conclusion χ_k, which is a Boolean combination of literals of the form x or $\Diamond x$, where x is propositional variable.

- *Final step: we transform the premise and conclusion of the obtained rule $r_N = \dfrac{\gamma_k}{q_\beta}$ into an equivalent disjunctive normal form over literals.*

We have seen in Theorem 1 that V. Rybakov gave a simple criterion for the admissibility of inference rules. We want to extend Theorem 1 to inference rules in general reduced normal form.

Definition 23 *Let $M = (W, R, V)$ be an $S4$-model. Let S be a valuation on W. We say that S is a definable valuation if there exists a substitution σ such that for all propositional variable x, for all $w \in W$, $w \in S(x)$ iff $M, w \vDash \sigma(x)$.*

Lemma 5 *Let $\mathscr{M} = (M, R, V)$ be a $S4$-model. Let S be a valuation on M.*

1. *If σ is a substitution and $S(x_i) = V(\sigma(x_i))$, then S is a definable valuation such that $S(\alpha) = V(\sigma(\alpha))$ for each formula α, that is $(M, R, S), w \vDash \alpha$ iff $(M, R, V), w \vDash \sigma(\alpha)$ for each $w \in M$.*

2. *If for each variable x_i there is a formula ϕ_i such that for all $w \in M$, $(M, R, S), w \vDash x_i$ iff $(M, R, V), w \vDash \phi_i$, and if σ is the substitution such that for each x_i, $\sigma(x_i) = \phi_i$ then for each formula α, $S(\alpha) = V(\sigma(\alpha))$, that is to say $(M, R, S), w \vDash \alpha$ iff $(M, R, V), w \vDash \sigma(\alpha)$ for each $w \in M$.*

3. *If σ is a substitution, S is a definable valuation for which $S(\alpha) = V(\sigma(\alpha))$ for each formula α, $r := \frac{\alpha_1, \ldots, \alpha_m}{\beta}$ is a rule and $\sigma(r) := \frac{\sigma(\alpha_1), \ldots, \sigma(\alpha_m)}{\sigma(\beta)}$ then r is valid in (M, R, S) iff $\sigma(r)$ is valid in (M, R, V).*

Proof 12 1. *By induction on α:*

- $(\Rightarrow)$ *Let* $\alpha = x_i$. *Let* $(M,R,S), w \vDash x_i$. *Since,* $S(x_i) = V(\sigma(x_i))$ *then,* $(M,R,V), w \vDash \sigma(x_i)$.

  $(\Leftarrow)$ *Let* $(M,R,V), w \vDash \sigma(x_i)$.*Since,* $S(x_i) = V(\sigma(x_i))$ *then,* $(M,R,S), w \vDash x_i$.
- $(\Rightarrow)$ *Let* $\alpha = (\varphi \wedge \psi)$. *Let* $(M,R,S), w \vDash (\varphi \wedge \psi)$. *Hence,* $(M,R,S), w \vDash \varphi$ *and* $(M,R,S), w \vDash \psi$. *By induction hypothesis,* $(M,R,V), w \vDash \sigma(\varphi)$ *and* $(M,R,V), w \vDash \sigma(\psi)$. *Then,* $(M,R,V), w \vDash (\varphi \wedge \psi)$.

  $(\Leftarrow)$ *Let* $(M,R,V), w \vDash \sigma(\varphi\wedge\psi)$. *Then,* $(M,R,V), w \vDash \sigma(\varphi)$ *and* $(M,R,V), w \vDash \sigma(\psi)$. *By induction hypothesis,* $(M,R,S), w \vDash \varphi$ *and* $(M,R,S), w \vDash \psi$. *Hence,* $(M,R,S), w \vDash (\varphi \wedge \psi)$.
- $(\Rightarrow)$ *Let* $\alpha = \Diamond\varphi$. *Let* $(M,R,S), w \vDash \Diamond\varphi$. *Let* $w' \in M$ *such that* wRw' *and,* $(M,R,S), w' \vDash \varphi$. *By induction hypothesis,* $(M,R,V), w' \vDash \sigma(\varphi)$. *Then,* $(M,R,V), w \vDash \Diamond\sigma(\varphi)$.

  $(\Leftarrow)$ *Let* $(M,R,V), w \vDash \Diamond\sigma(\varphi)$. *Let* $w' \in M$ *such that* wRw' *and,* $(M,R,V), w' \vDash \sigma(\varphi)$. *By induction hypothesis,* $(M,R,S), w' \vDash \varphi$. *Then,* $(M,R,S), w \vDash \Diamond\varphi$.

The proof of (2) *can be done by induction on* α *and the proof of* (3) *by using* (1) *and* (2).

Theorem 2 *Let* $(K_n, R_n, V_n)_{n\in\mathbb{N}}$ *be a sequence of* n*-characterizing models for* S4 *(see Definition* 17*). Inference rules* $r_1 := \frac{\alpha_{11},\ldots,\alpha_{1m_1}}{\beta_1}, \ldots, r_k := \frac{\alpha_{k1},\ldots,\alpha_{km_k}}{\beta_k}$ *are inadmissible in* S4 *with the same substitution* σ *iff* $r_1,\ldots,r_k$ *are invalid in* (K_n, R_n, S) *for some* $n \in N$ *and some definable valuation* S *of variables from* $r_1,\ldots,r_k$ *in* K_n *(that is, If* $S(\alpha_{ij}) = K_n$ *and* $S(\beta_i) \neq K_n$ *for* $i = 1,\ldots,k$ *and* $j = 1,\ldots,m_i$*).*

Proof 13 $(\Rightarrow)$ *Suppose* $r_1,\ldots,r_k$ *are not admissible in* S4 *with the same substitution* σ. *Let* σ *be a substitution such that* $\vdash_{S4} \sigma(\alpha_{ij})$ *and* $\nvdash_{S4} \sigma(\beta_i)$ *for* $i = 1,\ldots,k$ *and* $j = 1,\ldots,m_i$. *Let the number of propositional variable occurring in* $\sigma(\alpha_{ij})$ *and* $\sigma(\beta_i)$ *be* n. *Hence we have for* $i = 1,\ldots,k$ *and* $j = 1,\ldots,m_i$, $(K_n,R_n,V_n) \vDash_{S4} \sigma(\alpha_{ij})$ *and* $(K_n,R_n,V_n) \nvDash_{S4} \sigma(\beta_i)$ *by definition of* n*-characterizing models. Let* S *be a valuation on* K_n *such that* $S(x_i) = V(\sigma(x_i))$ *for* $i = 1,\ldots,n$. *Hence, since* S *is definable and* $(K_n,R_n,V_n) \vDash_{S4} \sigma(\alpha_{ij})$ *and* $(K_n,R_n,V_n) \nvDash_{S4} \sigma(\beta_i)$, *then,* S *invalidate* r_i. *Therefore,* $r_1,\ldots,r_k$ *are invalid in* (K_n,R_n,S).
$(\Leftarrow)$ *Suppose* $r_1,\ldots,r_k$ *are invalid in* (K_n,R_n,S) *with the definable valuation* S *for some* $n \in \mathbb{N}$. *Then by the part* 2 *of Lemma* 5*, there is a substitution* σ *for*

each variable x_i in $r_1,...,r_k$ such that $S(x_i) = V_n(\sigma(x_i))$. Hence by definition of the truth of formulas in Kripke models we obtain, $S(\alpha_{ij}) = V_n(\sigma(\alpha_{ij}))$ and $S(\beta_i) = V_n(\sigma(\beta_i))$ for $i = 1,...,k$ and $j = 1,...,m_i$. Thus, $(K_n, R_n, V_n) \vDash_{S4} \sigma(\alpha_{ij})$ and $(K_n, R_n, V_n) \nvDash_{S4} \sigma(\beta_i)$. Since, K_n is n-characterizing model then, $\vdash_{S4} \sigma(\alpha_{ij})$ and $\nvdash_{S4} \sigma(\beta_i)$. Therefore $r_1,...,r_k$ are not admissible in $S4$ with the same substitution σ.

Before presenting our main theorems that gives a characterisation of admissible rules in reduced normal form, we need the following technical important lemma.

In Lemma 6, we consider the rule $\dfrac{\bigvee\limits_{i\in I}\phi_i}{\bigvee\limits_{j\in J}\phi_j}$ in general reduced normal form.

Lemma 6 *Let $\mathcal{N} = (N, R, V)$ be a $S4$-model. Let $I \subseteq \{1,...,s\}$. Assume $\mathcal{N} \vDash \bigvee\limits_{i\in I}\phi_i$ and let $W = \{\phi_i \in \Theta(r) \mid i \in I,\ \ \exists a \in N\ \ s.t\ \ \mathcal{N}, a \models \phi_i\}$. Notice that $W \neq \emptyset$. Let $\mathcal{M}(\Theta(r))$ be the $S4$-model associated to r and W. Then*

1. *If $\mathcal{N}, a \models \phi_i$ then $\mathcal{N}, a \models \phi$ iff $\mathcal{M}(\Theta(r)), \phi_i \models \phi$ for each formula $\phi \in \Theta(r)$.*
2. *$W \subseteq \{\phi_i \in \Theta(r) \mid i \in I, \mathcal{M}(\Theta(r)), \phi_i \models \phi_i\}$.*
3. *Let $I' \subseteq I$. Then, $\mathcal{M}(\Theta(r)) \models \bigvee\limits_{i\in I'}\phi_i$ iff $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I'\}$.*
4. *Let $I' \subseteq I$. Then, $\mathcal{N} \vDash \bigvee\limits_{i\in I'}\phi_i$ iff $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I'\}$.*
5. *$\mathcal{N} \nvDash x_k$ iff $\mathcal{M}(\Theta(r)) \nvDash x_k$ for $k = 1,...,n$.*
6. *If for each subset D of N there exists $a \in N$ such that*

$$\theta_\diamond(a) = \theta(a) \cup \bigcup_{d\in D}\theta_\diamond(d)$$

then for each subset D of W there exists $\phi_j \in W$ such that

$$\theta_\diamond(\phi_j) = \theta(\phi_j) \cup \bigcup_{\phi\in D}\theta_\diamond(\phi)$$

where for all $a \in N$, $\Theta(a) = \{x_i \mid \mathcal{N}, a \vDash x_i\}$ and $\Theta_\diamond(a) = \{x_i \mid \mathcal{N}, a \vDash \diamond x_i\}$.

Proof 14 *Remind that if ϕ_i and ϕ_j are distinct elements in $\Theta(r)$ then $\phi_i \wedge \phi_j$ is logically equivalent to $\perp$. Since $\mathcal{N} \vDash \bigvee_{i \in I} \phi_i$ then for all $a \in N$, there exists exactly one $i \in I$ such that $\mathcal{N}, a \vDash \phi_i$. We define a surjective function $f : N \longrightarrow W$ such that for all $a \in N$, $f(a) = \phi_i$ where $i \in I$ and $\mathcal{N}, a \vDash \phi_i$. We claim that f is a homomorphism. Let $b, a \in N$ such that bRa. Let $i, j \in N$ such that $f(b) = \phi_i$ and $f(a) = \phi_j$. Let $x_k \in \theta_\diamond(\phi_j)$. Then $\mathcal{N}, a \vDash \diamond x_k$ and $\mathcal{N}, b \vDash \diamond x_k$. Therefore, $x_k \in \theta_\diamond(\phi_i)$. As a result, $f(b)$ can see $f(a)$ in $\mathcal{M}(\Theta(r))$.*

1. *Let $\mathcal{N}, b \models \phi_i$. We prove by induction on ϕ.*

   - *($\Rightarrow$) Let $\phi = x_k$. Let $\mathcal{N}, b \vDash x_k$. Since $\mathcal{N}, b \vDash \phi_i$ then, $x_k \in \theta(\phi_i)$. Then, $x_k \in V_n(\phi_i)$. Therefore, $\mathcal{M}(\Theta(r)), \phi_i \vDash x_k$.*

     ($\Leftarrow$) Let $\mathcal{M}(\Theta(r)), \phi_i \vDash x_k$. Then, $x_k \in V_n(\phi_i)$. Thus, $x_k \in \theta(\phi_i)$. Since $\mathcal{N}, b \vDash \phi_i$ then, $\mathcal{N}, b \vDash x_k$.

   - *($\Rightarrow$) Let $\phi = \neg x_k$. Let $\mathcal{N}, b \vDash \neg x_k$. Since $\mathcal{N}, b \vDash \phi_i$ then, $\neg x_k \in \theta(\phi_i)$. Then, $\neg x_k \in V_n(\phi_i)$. Therefore, $\mathcal{M}(\Theta(r)), \phi_i \vDash \neg x_k$.*

     ($\Leftarrow$) Let $\mathcal{M}(\Theta(r)), \phi_i \vDash \neg x_k$. Then, $\neg x_k \in V_n(\phi_i)$. Thus, $\neg x_k \in \theta(\phi_i)$. Since $\mathcal{N}, b \vDash \phi_i$ then, $\mathcal{N}, b \vDash \neg x_k$.

   - *Let $\phi = \varphi \wedge \psi$. $\mathcal{N}, b \vDash \varphi \wedge \psi$ iff $\mathcal{N}, b \vDash \varphi$ and $\mathcal{N}, b \vDash \psi$ iff $\mathcal{M}(\Theta(r)), \phi_i \vDash \varphi$ and $\mathcal{M}(\Theta(r)), \phi_i \vDash \psi$ (by induction hypothesis) iff $\mathcal{M}(\Theta(r)), \phi_i \vDash \varphi \wedge \psi$.*

   - *($\Rightarrow$) Let $\phi = \Diamond p_k$. Let $\mathcal{N}, b \vDash \Diamond x_k$. Let $a \in N$ and bRa such that $\mathcal{N}, a \vDash x_k$. Let $f(b) = \phi_i$ and $f(a) = \phi_j$. Since, $\mathcal{N}, a \vDash \phi_j$ then, $x_k \in \theta(\phi_j)$. Then, $\mathcal{M}(\Theta(r)), \phi_j \vDash x_k$. Since, bRa and the function f is a homomorphism then, $\phi_i R \phi_j$. Therefore, $\mathcal{M}(\Theta(r)), \phi_i \vDash \Diamond x_k$.*

     ($\Leftarrow$) Let $\mathcal{M}(\Theta(r)), \phi_i \vDash \Diamond x_k$. Then, $\Diamond x_k \in V_n(\phi_i)$. By our assumption, $\mathcal{N}, b \vDash \phi_i$ then, $\mathcal{N}, b \vDash \Diamond x_k$.

2. *By item* 1.

3. $(\Rightarrow)$ *Suppose* $\mathscr{M}(\Theta(r)) \vDash \bigvee_{i \in I'} \phi_i$. *Let* $\phi_j \in W$. *Let* $b \in N$ *such that* $\mathscr{N}, b \vDash \phi_j$. *We have* $\mathscr{M}(\Theta(r)), \phi_j \vDash \phi_j$ *by* 1. *By our assumption* $\mathscr{M}(\Theta(r)) \vDash \bigvee_{i \in I'} \phi_i$ *then there exists* $i \in I'$ *such that* $\mathscr{M}(\Theta(r)), \phi_j \vDash \phi_i$. *Then,* $\phi_j = \phi_i$. *Then,* $j \in I'$. *Therefore,* $\phi_j \in \{\phi_i \in \Theta_n \mid i \in I'\}$.
   $(\Leftarrow)$ *Suppose* $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I'\}$. *For all* $\phi_j \in W$ *we have to show that* $\mathscr{M}(\Theta(r)), \phi_i \vDash \bigvee_{i \in I'} \phi_i$. *Let* $b \in N$ *such that* $\mathscr{N}, b \vDash \phi_j$. *Then,* $\mathscr{M}(\Theta(r)), \phi_j \vDash \phi_j$ *by part* 1. *Since,* $\phi_j \in W \subseteq \{\phi_i \in \Theta_n \mid i \in I'\}$ *and* $\mathscr{M}(\Theta(r)), \phi_j \vDash \phi_j$ *then,* $\mathscr{M}(\Theta(r)), \phi_j \vDash \bigvee_{i \in I'} \phi_i$. *Therefore,* $\mathscr{M}(\Theta(r)) \vDash \bigvee_{i \in I'} \phi_i$.

4. *Suppose* $(\Rightarrow)$ $\mathscr{N} \vDash \bigvee_{i \in I'} \phi_i$. *Let* $\phi_j \in W$. *Let* $b \in N$ *be such that* $\mathscr{N}, b \vDash \phi_j$. *Since* $\mathscr{N} \vDash \bigvee_{i \in I'} \phi_i$ *then* $\mathscr{N}, b \vDash \bigvee_{i \in I'} \phi_i$. *Thus there is* $i \in I'$ *such that* $\mathscr{N}, b \vDash \phi_i$. *Let* $i \in I'$ *and* $\mathscr{N}, b \vDash \phi_i$. *Since* $\mathscr{N}, b \vDash \phi_j$ *then* $\phi_i = \phi_j$ *and* $j \in I'$. *So we have proved that* $W \subseteq \{\phi_i : i \in I'\}$.
   $(\Leftarrow)$ *Suppose* $W \subseteq \{\phi_i : i \in I'\}$. *We have to prove that* $\mathscr{N} \vDash \bigvee_{i \in I'} \phi_i$. *Let* $b \in \mathscr{N}$ *and let us prove* $\mathscr{N}, b \vDash \bigvee_{i \in I'} \phi_i$. *Since* $\mathscr{N} \vDash \bigvee_{i \in I} \phi_i$ *then let* $i \in I$ *be such that* $\mathscr{N}, b \vDash \phi_i$. *Then* $\phi_i \in W$. *Since* $W \subseteq \{\phi_i : i \in I'\}$ *then* $i \in I'$. *Hence,* $\mathscr{N}, b \vDash \bigvee_{i \in I'} \phi_i$.

5. $(\Rightarrow)$ *Suppose* $\mathscr{N} \nvDash x_k$. *Let* $b \in N$ *such that* $\mathscr{N}, b \vDash \neg x_k$. *Since* $\mathscr{N} \vDash \bigvee_{i \in I} \phi_i$ *then let* $\phi_i \in \Theta(r)$ *such that* $\mathscr{N}, b \vDash \phi_i$. *Consequently,* $x_k \notin \theta(\phi_i)$. *Moreover,* $\mathscr{M}(\Theta(r)), \phi_i \vDash \phi_i$ *by part* 1. *Since,* $\mathscr{M}(\Theta(r)), \phi_i \vDash \phi_i$ *and* $x_k \notin \theta(\phi_i)$ *then,* $\mathscr{M}(\Theta(r)), \phi_i \nvDash x_k$. *Therefore,* $\mathscr{M}(\Theta(r)) \nvDash x_k$.

   $(\Leftarrow)$ *Suppose* $\mathscr{M}(\Theta(r)) \nvDash x_k$. *Let* $\phi_i \in W$ *such that* $\mathscr{M}(\Theta(r)), \phi_i \nvDash x_k$. *Hence,* $x_k \notin \theta(\phi_i)$. *Since* $\phi_i \in W$ *then there is* $b \in N$ *such that* $\mathscr{N}, b \vDash \phi_i$. *Since,* $\mathscr{N}, b \vDash \phi_i$ *and* $x_k \notin \theta(\phi_i)$ *then,* $\mathscr{N}, b \nvDash x_k$. *Therefore,* $\mathscr{N} \nvDash x_k$.

6. *Let* $D' = \{\phi_1, ..., \phi_k\} \subseteq W$. *Since* $f : N \to W$ *is surjective then there are* $b_1, ..., b_k \in N$ *such that,* $f(b_i) = \phi_i$ *for* $1 \leqslant i \leqslant k$. *Then we have* $\mathscr{N}, b_i \vDash \phi_i$ *for* $1 \leqslant i \leqslant k$ *by definition of* f. *Let* $D = \{b_1, ..., b_k\}$. *Let* $a \in N$ *be such that* $\theta_\diamond(a) = \theta(a) \cup \bigcup_{a_k \in D} \theta_\diamond(a_k)$ *by our assumption.*

   Claim 1 *Let* $b_i \in N$. *We have* *(a)* $\theta(b_i) = \theta(\phi_i)$

(b) $\theta_\diamond(b_i) = \theta_\diamond(\phi_i)$

Proof 15 *(a) Suppose $x_k \in \theta(b_i)$. Then, $\mathcal{N}, b_i \models x_k$. Since $f(b_i) = \phi_i$ then $\mathcal{N}, b_i \models \phi_i$. Since $\mathcal{N}, b_i \models x_k$ such that $x_k \in \phi_i$.*

Reciprocally, suppose $p_k \in \phi_i$. Since $\mathcal{N}, b_i \models \phi_i$ then $\mathcal{N}, b_i \models x_k$. Hence, $x_k \in \theta(b_i)$.

(b) Suppose $x_k \in \theta_\Diamond(x_i)$. Then, $\mathcal{N}, b_i \models \Diamond x_k$. Let $a \in N$ be such that $b_i R a$ and $\mathcal{N}, a \models x_k$. Then $f(b_i) R f(a_i)$. Then $x_k \in \theta(a)$ and $x_k \in \theta(f(a))$. Since$f(b_i) R f(a_i)$ then $\Diamond x_k \in \theta(f(b_i))$. Since $f(b_i) = \phi_i$ then $x_k \in \theta_\Diamond(\phi_i)$.

Reciprocally, suppose $x_k \in \theta_\Diamond(\phi_i)$. Then $\mathcal{M}(\Theta(r)), \phi_i \models \Diamond x_k$. Let $\phi_j \in W$ be such that $\phi_i R \phi_j$ and $\mathcal{M}(\Theta(r)), \phi_i \models p_k$. Since f is surjective, let $a \in N$ be such that $f(a) = \phi_i$. We have $\mathcal{M}(\Theta(r)), f(a) \models x_k$, therefore $x_k \in \phi_j$ and $x_k \in \theta(a)$.

Since,$\theta(b) = \theta(\phi_i)$ and $\theta_\diamond(b) = \theta_\diamond(\phi_i)$ then, $\theta_\diamond(\phi_j) = \theta(\phi_j) \cup \bigcup_{\phi \in D} \theta_\diamond(\phi)$.

This ends the proof of Lemma 6.

Now, we are prepared to express our results as follows. We firstly determine under which conditions a rule in general reduced normal form is invalid and then, we discuss about admissibility of such rules.

Theorem 3 *A rule $r = \dfrac{\bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j}{\bigvee_{j \in J} \phi_j}$ is invalid for S4-models iff there is a non-empty set $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I \cup J\}$ such that the model $\mathcal{M}(\Theta(r))$ associated to r and W satisfies the following conditions:*

1. *$\mathcal{M}(\Theta(r)), \phi_j \models \phi_j$ for all $\phi_j \in W$.*
2. *$\mathcal{M}(\Theta(r)) \models \bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j$.*
3. *There exists $i \in I$ such that $\phi_i \in W$ and $\mathcal{M}(\Theta(r)), \phi_i \not\models \bigvee_{j \in J} \phi_j$.*

Proof 16 $(\Rightarrow)$ *Suppose r is invalid in $\mathcal{N} = (N, R, V)$. Then $\mathcal{N} \models \bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j$ and $\mathcal{N} \not\models \bigvee_{j \in J} \phi_j$. Let $W = \{\phi_i \in \Theta(r) \mid i \in I, \exists w \in N \text{ s.t } \mathcal{N}, w \models \phi_i\}$. Obviously, W is non-empty. Let $\mathcal{M}(\Theta(r))$ be the model associated to r and W.*

1. *Let $\phi_j \in W$. Hence there is $w \in N$ such that we have $\mathcal{N}, w \models \phi_j$ by definition of W. Then $\mathcal{M}(\Theta(r)), \phi_j \models \phi_j$ by Lemma 6.*

2. *Let $\phi_k \in W$. We have to show that $\mathcal{M}(\Theta(r)), \phi_k \models \bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j$. Since $\phi_k \in W$ then let $w \in N$ be such that $\mathcal{N}, w \models \phi_k$. Then, by part 2 of Lemma 6, $\mathcal{M}(\Theta(r)), \phi_k \models \phi_k$. Moreover, since $\mathcal{N}, w \models \bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j$ then $\mathcal{M}(\Theta(r)), \phi_k \models \bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j$.*

3. *By our assumption, $\mathcal{N} \not\models \bigvee_{j \in J} \phi_j$. Then $W \not\subseteq \{\phi_i \in \Theta(r) \mid i \in J\}$ by part (4) of Lemma 6. Since $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I \cup J\}$ then there exists an $i \in I - J$ such that $\phi_i \in W$. By $\models \phi_i \to \neg \bigvee_{j \in J} \phi_j$ and $\mathcal{M}(\Theta(r)), \phi_i \models \phi_i$ then we obtain $\mathcal{M}(\Theta(r)), \phi_i \models \neg \bigvee_{j \in J} \phi_j$. Therefore, $\mathcal{M}(\Theta(r)), \phi_i \not\models \bigvee_{j \in J} \phi_j$.*
   Therefore, r is invalid in $\mathcal{M}(\Theta(r))$.
   $(\Leftarrow)$ *The model $\mathcal{M}(\Theta(r))$ has all properties of Lemma 3.4.9 of [44] then by Lemma 3.4.10 of [44] there exists a definable valuation S of the rule r such that r is not provable in $Ch_{S4}(n)$. Therefore the rule r is invalid in $Ch_{S4}(n)$.*

Theorem 4 *A rule $r = \dfrac{\bigvee_{i \in I} \phi_i \vee \bigvee_{j \in J} \phi_j}{\bigvee_{j \in J} \phi_j}$ is inadmissible for S4 iff there is a set $W \subseteq \{\phi_i \in \Theta(r) \mid i \in I \cup J\}$ such that*

1. *$\phi_i \in W$ for some $i \in I$.*

2. *$\mathcal{M}(\Theta(r)), \phi_j \models \phi_j$ for all $\phi_j \in W$.*

3. *For each subset (D) of $\mathcal{M}$ there exists $\phi_j \in W$ such that*

$$\theta_\diamond(\phi_j) = \theta(\phi_j) \cup \bigcup_{\phi \in D} \theta_\diamond(\phi).$$

Proof 17 $(\Rightarrow)$ *Proof of this direction is similar to Theorem 1.*
$(\Leftarrow)$ *The rule r is invalid in $\mathscr{M}(\Theta(r))$ by Theorem* 3. *By Lemma* 3.4.10 *of [44] there exists a definable valuation S of the rule r such that invalidate r in $Ch_{S4}(n)$. Therefore the rule r is inadmissible in $S4$ by Lemma* 2.

4 Unification and Unification Types in modal Logic

Contents

In logic and computer science, unification means solving logical equations. Unification in logic is the problem of finding a substitution that transform a given formula into a theorem (or a tautology). For instance consider, $(\varphi_1 \leftrightarrow \psi_1) \wedge ... \wedge (\varphi_n \leftrightarrow \psi_n)$. If we can find a substitution σ such that $\vdash_L (\sigma(\varphi_1) \leftrightarrow \sigma(\psi_1)) \wedge ... \wedge (\sigma(\varphi_n) \leftrightarrow \sigma(\psi_n))$ then we can say that this formula is unifiable in the considered logic L.
Chapter 4 presents already existing results on unification in propositional logic and modal logic.

- Classical Propositional Logic has projective unification (Proposition 9, p. 45) and thus is unitary (see Proposition 8).
- Jerábek proved that modal logic K is nullary.
- P. Balbiani and Ç. Gencer adapted Jeřábek's argument to KD and proved that KD is nullary too.

- Ghilardi proved that $K4$ and its extensions are finitary and given a formula, its finite complete sets of unifiers can be computed.
- W. Dzik showed that $S5$ is unitary.
- P. Balbiani and T. Tinchev showed that $KD45$ is unitary.
- $KD5$ and $K5$ are both filtering hence either unitary or nullary. The exact type is open.

4.1 Fundamental Notions of Unification

To discuss about unification type of modal logics, first we give some basic definitions and then we consider unification type of modal logics.

Definition 24 *We define some features of substitution:*

- *A substitution σ is a mapping from variables to a formulas. It is denoted by $\sigma : \overline{x} \to F(\overline{y})$. Substitutions will generally be represented by σ, θ, λ, τ and so on.*

- *Composition of two substitutions $\sigma : \overline{x} \to F(\overline{y})$ and $\tau : \overline{y} \to F(\overline{z})$ is the substitution $\tau \circ \sigma : \overline{x} \to F(\overline{z})$ defined by*

  $$\tau \circ \sigma(x) = \tau(\sigma(x)).$$

  for each $x \in \overline{x}$.

- *A substitution $\sigma : \overline{x} \to F(\overline{y})$ is equivalent in a logic L to a substitution $\tau : \overline{x} \to F(\overline{z})$ if*

  $$\sigma(\overline{x}) \leftrightarrow \tau(\overline{x}) \in L$$

  for each $x \in \overline{x}$. We will denote it by $\sigma \simeq_L \tau$

Definition 25 *Let $A(x_1, ..., x_n)$ is a formula built up from variables $x_1, ..., x_n$ and denoted by $A(\overline{x})$. Let L be a logic.*

- *a substitution σ is an L-unifier of A if $A(\sigma(x_1), ..., \sigma(x_n)) \in L$.*

- *A is unifiable in L if there exists a substitution σ such that $A(\sigma(x_1),...,\sigma(x_n)) \in L$. In this case, A is called unifiable in L.*

- *A substitution $\sigma : \overline{x} \to F(\overline{y})$ is more general (or less specific) than a substitution $\tau : \overline{x} \to F(\overline{z})$ in L iff there exist a substitution $\lambda : \overline{y} \to F(\overline{z})$ such that $\lambda(\sigma(x)) \leftrightarrow \tau(x) \in L$ for all variable $x \in \overline{x}$. We will denote it by $\sigma \preccurlyeq_L \tau$.*

- *A substitution σ of the form $\sigma : \overline{x} \to F(\emptyset)$ is called ground unifier or closed unifier.*

- *Let $U_L(A)$ be the set of all unifiers for the formula A in a logic L. A set $U \subseteq U_L(A)$ is said to be complete set of unifier for A, if for every unifier τ for A there is a unifier from the set U which is more general than τ.*

- *A complete set of unifiers for A in L is a minimal complete set if its members are pairwise incomparable with respect to $\leqslant_L$.*

- *A unifier σ for A in logic L is called a most general unifier (mgu) in L for a formula A, if $\{\sigma\}$ is a complete set of unifiers for A.*

Example 7 *The formula $\Box x \vee \Box\neg x$ is unifiable in K. The ground substitutions $\sigma_\top$ and $\sigma_\bot$ defined by $\sigma_\top = \top$ and $\sigma_\bot = \bot$ are K-unifiers of $\Box x \vee \Box\neg x$.*

Note that a most general unifier is not unique. There are always more than 1 most general unifiers. Nevertheless, of course, if τ_1 and τ_2 are two most general unifiers, then there exists substitutions λ_1 and λ_2 such that $\lambda_1 \circ \tau_1 \simeq_L \tau_2$ and $\lambda_2 \circ \tau_2 \simeq_L \tau_1$ hence, $\tau_1 \preccurlyeq_L \tau_2$ and $\tau_2 \preccurlyeq_L \tau_1$. This means they are equivalent instances of each other.

Lemma 7 *If a given formula L has two minimal complete sets Σ and Σ' in logic L then $card(\Sigma) = card(\Sigma')$.*

Proof 18 *Let $f : \Sigma \to \Sigma'$ and $g : \Sigma' \to \Sigma$ such that*

- *For all $\sigma \in \Sigma$, $f(\sigma) \preccurlyeq_L \sigma$,*

- *For all $\sigma' \in \Sigma'$, $g(\sigma') \preccurlyeq_L \sigma'$.*

The functions f and g exist because Σ and Σ' are complete. We show that f is injective. Let $\sigma, \tau \in \Sigma$ such that $f(\sigma) = f(\tau)$. Notice that $g(f(\sigma)) \preccurlyeq_L f(\sigma) \preccurlyeq_L \sigma$ and $g(f(\tau)) \preccurlyeq_L f(\tau) \preccurlyeq_L \tau$. Since $f(\sigma) = f(\tau)$ then $g(f(\sigma)) = g(f(\tau))$. But

$g(f(\sigma)) \in \Sigma$, $\sigma, \tau \in \Sigma$. Since $g(f(\sigma)) \preccurlyeq_L \sigma$ then $g(f(\sigma)) = \sigma$ by minimality of Σ. Since $g(f(\tau)) \preccurlyeq_L \tau$ then similarly $g(f(\tau)) = \tau$. Since $g(f(\sigma)) = g(f(\tau))$ then $\sigma = \tau$.
By the same way we can prove g is injective. Consequently, $card(\Sigma) = card(\Sigma')$.

Definition 26 *Let $A(\overline{x})$ be a unifiable formula in logic L.*

- *The formula A has unification type unitary if it has a minimal complete set of unifiers of cardinality 1.*
- *The formula A has unification type finitary if it has a finite minimal complete set of unifiers for formula A and cardinality fo this minimal complete set is strictly greater than 1.*
- *The formula A has unification type infinitary if it has a infinite minimal complete set of unifiers.*
- *The formula A has unification type nullary if it does not have any minimal complete set of unifiers.*

Definition 27 *Let L be a logic*

- *L is unitary if every L-unifiable formula is of type unitary.*
- *L is finitary if there exists a L-unifiable formula of type finitary and every L-unifiable formula is either of type unitary, or of type finitary.*
- *L is infinitary if there exists a L-unifiable formula of type infinitary and every L-unifiable formula is either of type unitary, or of type finitary, or of type infinitary,*
- *L is nullary if there exists a L-unifiable formula of type nullary.*

Let us see these definitions at the following example:

Example 8 *Let $A = x \vee y$. Consider the substitutions σ_1 defined by $\sigma_1(x) = x$ and $\sigma_1(y) = \neg x$. After applying σ_1, we have: $\sigma_1(A) = \sigma_1(x) \vee \sigma_1(y) = x \vee \neg x$. Therefore, $\sigma_1(A)$ is a tautology. In this case, A is unifiable and σ_1 is unifier of A in Classical Propositional Logic.*
Consider the substitution σ_2 defined by $\sigma_2(x) = \top$ and $\sigma_2(y) = \top$. Hence, $\sigma_2(A)$ is a tautology and σ_2 is also a unifier of A.

Lemma 8 *Let A be a unifiable formula in logic L. Then the formula A possesses a ground unifier in logic L.*

Proof 19 *Let A be a unifiable formula in logic L. Let σ be a unifier of A such that $\sigma(A) \in L$. Let τ be a ground substitution. Since $\sigma(A) \in L$ then $\tau(\sigma(A)) \in L$. Since τ is ground substitution thus, $\tau \circ \sigma$ is a ground unifiers of A.*

Definition 28 *A unifier σ for a formula A is said to be projective in logic L if for each $x \in \overline{x}$*

$$A \vdash_L \sigma(x) \leftrightarrow x$$

A formula is projective in logic L iff there exists a projective unifier for the formula. If each unifiable formula is projective in logic L, then we say that L has projective unification.

Lemma 9 *Each projective unifier for A is a most general unifier for A.*

Proof 20 *Let σ is a projective unifier of unifiable formula A. Then $A \vdash_L \sigma(x) \leftrightarrow x$. Let τ be a unifier of A then, $\vdash_L \tau(A)$. By applying τ on $A \vdash_L \sigma(x) \leftrightarrow x$, we obtain $\tau(A) \vdash_L \tau(\sigma(x)) \leftrightarrow \tau(x)$. Since $\vdash_L \tau(A)$ then, $\vdash_L (\tau(\sigma(x)) \leftrightarrow \tau(x))$. Therefore, $\sigma \preccurlyeq_L \tau$ (since τ was arbitrary) and σ is a most general unifier of A.*

Proposition 6 *If L has projective unification then, unification type L is unitary.*

4.2 Unification in Classical Propositional Logic

In this section, we are going to review unification problem and unification type of Classical Propositional Logic [49]. It has been proved that all unifiable formulas in Classical Propositional Logic have a most general unifier. To prove that Classical Propositional Logic has unitary unification, we will use Löwenheim formula.
Let us prove that unification type of Classical Propositional Logic is unitary.

Syntax and semantic of Classical Propositional Logic are as usual.
Consider a formula A and a substitution γ. Let λ be the substitution defined by $\lambda(x) = (A \wedge x) \vee (\neg A \wedge \gamma(x))$. The substitution λ is the so-called Löwenheim substitution associated to A and γ. Variant of it in modal logics have been used by [22] and [31].

Lemma 10 *Le v be a valuation. For any formula B*

1. *If $v(A) = \top$ then, $v(B) = v(\lambda(B))$.*
2. *If $v(A) = \bot$ then, $v(\gamma(B)) = v(\lambda(B))$.*

Proof 21

1. *Suppose $v(A) = \top$. We prove by induction on B:*
   - *Let $B = x$. We have to prove $v(x) = v(\lambda(x))$. Since $v(A) = \top$ and $v(\lambda(x)) = (v(A) \wedge v(x)) \vee (\neg v(A) \wedge v(\gamma(x)))$ then, $v(x) = v(\lambda(x))$.*
   - *Let $B = \bot$. We have to prove $v(\bot) = v(\lambda(\bot))$. Since $\bot$ is Boolean constant and λ is substitution hence, $\lambda(\bot) = \bot$. Then $v(\lambda(\bot)) = \bot$ hence, $v(\bot) = v(\lambda(\bot))$.*
   - *Let $B = \neg B'$. We have to prove $v(\neg B') = v(\lambda(\neg B'))$. By induction hypothesis, $v(B') = v(\lambda(B'))$. Therefore, $v(\neg B') = v(\lambda(\neg B'))$.*
   - *Let $B = B' \wedge B''$. We have to prove $v(B' \wedge B'') = v(\lambda(B' \wedge B'))$. By induction hypothesis, $v(B') = v(\lambda(B'))$ and $v(B'') = v(\lambda(B''))$. Therefore, $v(B' \wedge B'') = v(\lambda(B' \wedge B''))$.*
2. *Let $v(A) = \bot$. We prove by induction on B:*
   - *Let $A = x$. We have to prove $v(\gamma(x)) = v(\lambda(x))$. Since $v(A) = \bot$ and $v(\lambda(x)) = (v(A) \wedge v(x)) \vee (\neg v(A) \wedge v(\gamma(x)))$ then, $v(\gamma(x)) = v(\lambda(x))$.*
   - *Let $B = \bot$. We have to prove $v(\gamma(\bot)) = \bot = v(\lambda(\bot))$. Since $\lambda(\bot) = \bot$ and $\gamma(\bot) = \bot$ hence $v(\lambda(\bot)) = v(\gamma(\bot))$.*
   - *Let $B = \neg B'$. We have to prove $v(\gamma(\neg B')) = v(\lambda(\neg B'))$. By induction hypothesis, $v(\gamma(B')) = v(\lambda(B'))$. Hence, $\neg v(\gamma(B')) = \neg v(\lambda(B'))$. Therefore, $v(\gamma(\neg B')) = v(\lambda(B'))$.*
   - *Let $B = B' \wedge B''$. We have to prove $v(\gamma(B' \wedge B'')) = v(\lambda(B' \wedge B'))$. By induction hypothesis, $v(\gamma(B')) = v(\lambda(B'))$ and $v(\gamma(B'')) = v(\lambda(B''))$. Therefore, $v(\gamma(B' \wedge B'')) = v(\lambda(B' \wedge B''))$.*

Theorem 5 *Let A be a unifiable formula and γ unifier of A. The substitution λ defined above is a most general unifier of A.*

Proof 22 *First, we prove that λ is a unifier for A. Suppose λ is not a unifier of A. Then $\nvdash \lambda(A)$. Hence, there exists a v such that $v(\lambda(A)) = \bot$. Hence we have two cases:*

1. *If* $v(A) = \top$*: then by Lemma* 10*,* $v(\lambda(A)) = v(A)$*. Then* $\bot = \top$*. This is a contradiction.*

2. *If* $v(A) = \bot$*: then by Lemma* 10*,* $v(\lambda(A)) = v(\gamma(A))$*. Then* $\bot = v(\gamma(A))$*. Then* $\nvdash \gamma(A)$*: this is a contradiction with the fact that* γ *is a unifier of* A*. Therefore* λ *is a unifier of* A*.*

Second, we prove that λ *is most general:*
Let τ *be a unifier of* A*. Then,* $\vdash \tau(A)$*. Let* x *be an arbitrary variable. By part* (1) *of Lemma* 10 *we have* $\vdash A \to (\lambda(x) \leftrightarrow x)$*. Then by applying* τ *on* $\vdash A \to (\lambda(x) \leftrightarrow x)$ *we get,* $\vdash \tau(A) \to (\tau(\lambda(x)) \leftrightarrow \tau(x))$*. Since* $\vdash \tau(A)$ *and* $\vdash \tau(A) \to (\tau(\lambda(x)) \leftrightarrow \tau(x))$ *then,* $\vdash \tau(\lambda(x)) \leftrightarrow \tau(x)$*. Therefore* λ *is most general than* τ $(\lambda \preccurlyeq \tau)$ *in Classical Propositional Logic.*

Lemma 11 *The substitution* λ *defined above is a projective unifier for* A*.*

Proof 23 *Firstly,* $\vdash \lambda(A)$ *by Theorem* 5*.*
Secondly, $\vdash A \to (\lambda(x) \leftrightarrow x)$ *by part* 1 *of Lemma* 10*.*
Therefore, λ *is projective unifier.*

From the above results, it follows:

Proposition 7 *Classical Propositional Logic has projective unification.*

Proposition 8 *Every unifiable formula in Classical Propositional Logic has a most general unifier.*

Example 9 *Consider the formula* $A = x \vee y$*. The substitution* σ *such that* $\sigma(x) = x$ *and* $\sigma(y) = \neg x$ *is one of the unifiers of* A*. Let* λ *be the substitution defined by* $\lambda(x) = (A \wedge x) \vee (\neg A \wedge \sigma(x))$ *and* $\lambda(y) = (A \wedge y) \vee (\neg A \wedge \sigma(y))$*. Hence we have* $\lambda(x) = (A \wedge x) \vee (\neg A \wedge x) = x$ *and* $\lambda(y) = (A \wedge y) \vee (\neg A \wedge \neg x) = y \vee \neg x$*. By Lemma* 5*, we know that* λ *is a most general unifier of* A*.*

4.3 Unification in Modal Logic

In this section, we consider some modal logic such as K, $S4$, $S5$ and so on and we explain their unification type; Unification type of these modal logics respectively, are nullary, finitary and unitary.

4.3.1 Unification in Modal Logic K

Emil Jeřábek in [34] has proved that unification type in normal modal logic K is nullary. In this respect he considered a formula and introduced some substitutions. First, he proved that these substitutions are K-unifiers of that formula. Then, he proved that these unifiers are not more general than each other. Let us see which formula and substitutions he considered and how he proved that modal logic K is nullary. In Chapter 6, we will adapt the argument of Jeřábek show that the fusion $S5 \otimes S5$ is nullary. Jeřábek considered the formula $\varphi = x \to \Box x$. He introduced the substitutions $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ (for each $n \geqslant 0$) and $\sigma_\top(x) = \top$ and then proved that

Lemma 12 *For each $n \in \mathbb{N}$*

1. *The substitution σ_n is a K-unifier of the formula φ.*
2. *The substitution $\sigma_\top$ is a K-unifier of the formula φ.*

Proof 24 1. *By the inference rule $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$ which is derivable in K, and by the distributivity of $\Box$ over $\wedge$, we have*

$$\Box^{<n} x \wedge \Box^n \bot \to \Box\Box^{<n} x \in K \text{ and}$$
$$\Box^n \bot \to \Box^{n+1} \bot \in K \text{ thus,}$$
$$\Box^{<n} x \wedge \Box^n \bot \to \Box(\Box^{<n} x \wedge \Box^n \bot) \in K$$

2. *Clearly, $\top \to \Box\top \in K$.*

The Lemmas 13 and 14 show that $\sigma_n \preccurlyeq_K \sigma_{n-1} \preccurlyeq_K \ldots \preccurlyeq_K \sigma_0$ and $\sigma_0 \not\preccurlyeq_K \sigma_1 \not\preccurlyeq_K \ldots \not\preccurlyeq_K \sigma_n$.

Lemma 13 *Let $k, l \in \mathbb{N}$. If $k \leqslant l$ then $\sigma_l \preccurlyeq_K \sigma_k$.*

Proof 25 *Suppose $k \leqslant l$. Let ν be the substitution defined by $\nu(x) = x \wedge \Box^k \bot$. It is easy to check that $\nu \circ \sigma_l \simeq_K \sigma_k$. Hence, $\sigma_l \preccurlyeq_K \sigma_k$.*

Lemma 14 *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preccurlyeq_K \sigma_l$.*

Proof 26 *Suppose $k < l$ and $\sigma_k \preccurlyeq_K \sigma_l$. Let ν be a substitution such that $\nu \circ \sigma_k \simeq_K \sigma_l$. Hence, $\vdash_K \nu(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. Thus, $\vdash_K \Box^{<l} x \wedge \Box^l \bot \to \Box^{<k} \nu(x) \wedge \Box^k \bot$. Consequently, after replacing x by $\top$, $\vdash_K \Box^l \bot \to \Box^k \bot$: a contradiction.*

Then Jeřábek [34] showed that some σ_n or $\sigma_\top$ are more general in some cases than a given unifier σ of φ as follows:

Lemma 15 *If σ is a unifier of $\varphi = x \to \Box x$ and $n \in \mathbb{N}$, the following are equivalent:*

1. $\sigma \circ \sigma_n \simeq_K \sigma$,
2. $\sigma_n \preccurlyeq_K \sigma$,
3. $\vdash_K \sigma(x) \to \Box^n \bot$.

Proof 27

1. $(1 \Rightarrow 2)$ *By definition of* $\preccurlyeq_K$.
2. $2 \Rightarrow 3$ *Suppose* $\sigma_n \preccurlyeq_K \sigma$. *Let* ν *be a substitution such that* $\nu \circ \sigma_n \simeq_K \sigma$. *Hence,* $\vdash_K \nu(\sigma_n(x)) \leftrightarrow \sigma(x)$. *Then,* $\vdash_K \sigma(x) \to \Box^n \bot$.
3. $(3 \Rightarrow 1)$ *Suppose* $\vdash_K \sigma(x) \to \Box^n \bot$. *Since* σ *is a unifier of* φ *then,* $\vdash_K \sigma(x) \to \Box\sigma(x)$. *Hence,* $\vdash_K \sigma(x) \to \Box^{<n}\sigma(x)$. *Since* $\vdash_K \sigma(x) \to \Box^n \bot$ *and* $\vdash_K \sigma(x) \to \Box^{<n}\sigma(x)$ *then* $\vdash_K \sigma(x) \to \Box^{<n}\sigma(x) \wedge \Box^n \bot$. *Thus,* $\vdash_K \sigma(x) \to \sigma(\sigma_n(x))$. *Now, we consider two following cases:*

   - *If* $n = 0$ *then* $\vdash_K \Box^n \bot \to \sigma(x)$ *and*
   - *If* $n \geqslant 1$ *then* $\vdash_K \Box^{<n}\sigma(x) \to \sigma(x)$.

   Therefore, $\vdash_K \Box^{<n}\sigma(x) \wedge \Box^n \bot \to \sigma(x)$. *Hence,* $\vdash_K \sigma(\sigma_n(x)) \to \sigma(x)$. *Since,* $\vdash_K \sigma(x) \to \sigma(\sigma_n(x))$ *and* $\vdash_K \sigma(\sigma_n(x)) \to \sigma(x)$ *therefore,* $\sigma \circ \sigma_n \simeq_K \sigma$.

Lemma 16 *If σ is a substitution, the following are equivalent:*

1. $\sigma_\top \preccurlyeq_K \sigma$,
2. $\sigma \circ \sigma_\top \simeq_K \sigma$,
3. $\vdash_K \sigma(x)$.

Proof 28 *The proof is similar to the proof of Lemma* 15.

At the next step, Jeřábek stated that the unifiers σ_n or $\sigma_\top$ are more general than any unifier σ of φ.

Theorem 6 *Let σ be a K-unifier of $\varphi = x \to \Box x$ then one of the following conditions holds:*

1. $\sigma_\top \preccurlyeq_K \sigma$.

2. There exist $n \in \mathbb{N}$ such that $\sigma_n \preccurlyeq_K \sigma$.

Proof 29 *Let $n \geqslant deg(\sigma(x))$. Suppose none of the above conditions holds. Hence, $\sigma_n \not\preccurlyeq_K \sigma$ and $\sigma_\top \not\preccurlyeq_K \sigma$. By Lemmas 15 and 16 we have that $\nvdash_K \sigma(x)$ and $\nvdash_K \sigma(x) \to \Box^n \bot$. Then, there are models $M_1 = (W_1, R', v_1)$ and $M_2 = (W_2, R'', v_2)$ and there are $s_1 \in W_1, s_2 \in W_2$ such that, $M_1, s_1 \nvDash \sigma(x)$, $M_2, s_2 \vDash \sigma(x)$ and $M_2, s_2 \nvDash \Box^n \bot$. Let $s''_2, ..., s''_{n+1} \in W_2$ be such that $s_2 R'' s''_2 ... R'' s''_{n+1}$. By the tree-model property of K, we can assume without loss of generality that $s_2, s''_2, ..., s''_{n+1}$ are pairwise different. Let us construct the model $M = (W, R, v)$ which is an extension of the disjoint union of models M_1 and M_2 and we define the model M as follows:*

- $W = W_1 \cup W_2$,
- $R = R' \cup R'' \cup \{(s''_{n+1}, s_1)\}$,
- $v = v_1 \cup v_2$ *that is to say for all proposition letters y, $v(y) = v_1(y) \cup v_2(y)$.*

Since $M_1, s_1 \nvDash \sigma(x)$, then $M, s_1 \nvDash \sigma(x)$. Since $M_2, s_2 \vDash \sigma(x)$ and $n \geqslant deg(\sigma(x))$, then $M, s_2 \vDash \sigma(x)$. Since $\vdash_K \sigma(x) \to \Box\sigma(x)$ then $M, s''_2 \vDash \sigma(x), ..., M, s''_{n+1} \vDash \sigma(x)$. Then $M, s_1 \vDash \sigma(x)$: a contradiction.

The main result about unification type of modal logic K has been proved as follows.

Lemma 17 *The set of substitutions $\Sigma = \{\sigma_n : n \in \omega\} \cup \{\sigma_\top\}$ is a complete set of K-unifiers of $\varphi = x \to \Box x$.*

Proof 30 *By Theorem 6, Σ constitutes a complete set of unifiers of the formula $\varphi = x \to \Box x$.*

Lemma 18 *The formula $\varphi = x \to \Box x$ does not possess a minimal complete set of K-unifiers.*

Proof 31 *Suppose φ possesses a minimal complete set Σ' of K-unifiers. Since Σ' is complete, let $\sigma' \in \Sigma'$ be such that $\sigma' \preccurlyeq_K \sigma_0$. Since σ' is a K-unifier of φ, then $\sigma'(x) \to \Box\sigma'(x) \in K$. Hence, by the rule of margin [34] either $\sigma'(x) \in K$ or $\sigma'(x) \to \Box^{deg(\sigma'(x))} \bot \in K$.*
In the former case, by Lemma 16, $\sigma_\top \preccurlyeq_K \sigma'$. Since $\sigma' \preccurlyeq_K \sigma_0$, then $\sigma_\top \preccurlyeq_K \sigma_0$.

Thus, $\top \leftrightarrow \bot \in K$*: a contradiction.*
In the latter case, by Lemma 15 *since* Σ' *is a set of unifiers of* φ*, then* $\sigma_{deg(\sigma'(x))} \preccurlyeq_K \sigma'$*. Since* Σ' *is complete, let* $\sigma'' \in \Sigma'$ *be such that* $\sigma'' \preccurlyeq_K \sigma_{deg(\sigma'(x))+1}$*. Since* $\sigma_{deg(\sigma'(x))+1} \preccurlyeq_K \sigma_{deg(\sigma'(x))} \preccurlyeq_K \sigma'$*, then* $\sigma'' \preccurlyeq_K \sigma'$*. Since* Σ' *is minimal, then* $\sigma'' = \sigma'$*. Since* $\sigma_{deg(\sigma'(x))} \preccurlyeq_K \sigma'$ *and* $\sigma'' \preccurlyeq_K \sigma_{deg(\sigma'(x))+1}$*, then* $\sigma_{deg(\sigma'(x))} \preccurlyeq_K \sigma_{deg(\sigma'(x))+1}$*: a contradiction with Lemma* 14.

Proposition 9 *Unification type is nullary in modal logic* K.

Proof 32 *By Lemma* 18.

We shall adapt this method in Chapter 5 to investigate on unification type of modal logic $K_1 \otimes K_2$ and $S5_1 \otimes S5_2$ for instance

4.3.2 Unification in Modal Logic KD

P. Balbiani and Ç. Gencer have adapted Jeřábek argument to KD. They proved that unification type of modal logic KD is nullary [7] too. In this respect, they used a special kind of atomic formulas called parameters and they considered the formula $\varphi = (x \to p) \wedge (x \to [p]x)$. Parameters are atomic formulas that are not replaced by formulas when a substitution is applied. For all parameters p, the modal connective $[p]$ is defined as follows:

- $[p]x ::= \Box(p \to x)$.

For all parameters p, the modal connective $[p]^k$ is inductively defined as follows for each $n \in \mathbb{N}$:

- $[p]^0\varphi ::= \varphi$,
- $[p]^{k+1} ::= [p][p]^k\varphi$.

A parameter is a propositional letter that is not moved by substitutions. Parameters will be denoted by p, q, etc. A parameter is like a constant proposition letter. For instance, if σ is the substitution defined by $\sigma(x) = \Box p \vee \Box y$ and $\sigma(y) = \Box y$, then $\sigma(\Box(x \to \Box p \vee y)) = \Box(\Box p \vee \Box y \to \Box p \vee \Box y)$.Parametrized unification (as well as parametrized admissibility) have been considered by several authors, but mainly considered by V. Rybakov [44].
For all parameters p, the modal connective $[p]^{<k}$ is inductively defined as follows for each $n \in \mathbb{N}$:

- $[p]^{<0}\varphi ::= \top$.
- $[p]^{<k+1}\varphi ::= [p]^{<k}\varphi \wedge [p]^{k}\varphi$.

Consider the formula $\varphi = (x \to p) \wedge (x \to [p]x)$ and substitutions $\sigma_p(x) = p$ and $\sigma_n(x) = p \wedge [p]^{<n}x \wedge [p]^{n}\bot$ where $n \in \mathbb{N}$. P. Balbiani and Ç. Gencer proved that

Lemma 19 *For all $n \in \mathbb{N}$*

1. $\sigma_n(x) = p \wedge [p]^{<n}x \wedge [p]^{n}\bot$ *is a KD-unifier of φ.*
2. $\sigma_p(x) = p$ *is a KD-unifier of φ.*

Proof 33 1. *It is clear that $\vdash p \wedge [p]^{<n}x \wedge [p]^{n}\bot \to p$. Since*
$\vdash_{KD} p \to [p]p$. *Hence,*
$\vdash_{KD} p \wedge [p]^{<n}x \wedge [p]^{n}\bot \to [p]p$. *Since*
$\vdash_{KD} [p]^{<n}x \wedge [p]^{n}\bot \to [p][p]^{<n}x$ *and*
$\vdash_{KD} [p]^{n}\bot \to [p]^{n+1}\bot$ *then,*
$\vdash_{KD} p \wedge [p]^{<n}x \wedge [p]^{n}\bot \to [p]p \wedge [p][p]^{<n}x \wedge [p]^{n+1}\bot$.
Therefore, σ_n is a KD-unifier of φ.

2. *Since, $\vdash (p \to p)$ and $\vdash p \to [p]p$ then, $\vdash (p \to p) \wedge (p \to [p]p)$. Therefore, σ_p is a KD-unifier of φ.*

In order to adapt Jeřábek argument, it is needed to prove that the sequence of substitutions σ_n for $n \in \mathbb{N}$ satisfies the property $\sigma_n \preccurlyeq_{KD} \dots \preccurlyeq_{KD} \sigma_0$ and $\sigma_0 \not\preccurlyeq_{KD} \dots \not\preccurlyeq_{KD} \sigma_n$

Lemma 20 *Let $k, l \in \mathbb{N}$. If $k \leqslant l$ then $\sigma_l \preccurlyeq_{KD} \sigma_k$.*

Proof 34 *Suppose $k \leqslant l$. Let $\nu(x) = x \wedge [p]^{k}\bot$. Since,*
$\vdash_{KD} [p]^{<l}x \to [p]^{<k}x$ *and*
$\vdash_{KD} [p]^{<l}[p]^{k}\bot \to [p]^{k}\bot$. *Therefore,*
$\vdash_{KD} p \wedge [p]^{<l}(x \wedge [p]^{k}\bot) \wedge [p]^{l}\bot \to p \wedge [p]^{<k}x \wedge [p]^{k}\bot$. *Since,*
$\vdash_{KD} [p]^{k}\bot \to [p]^{l}\bot$ *and*
$\vdash_{KD} [p]^{<k}x \wedge [p]^{k}\bot \to [p]^{<l}x$ *and,*
$\vdash_{KD} [p]^{k}\bot \to [p]^{<l}[p]^{k}\bot$ *therefore,*
$\vdash_{KD} (p \wedge [p]^{<k}x \wedge [p]^{k}\bot) \to p \wedge [p]^{<l}(x \wedge [p]^{k}\bot) \wedge [p]^{l}\bot$. *Hence,*
$\vdash_{KD} p \wedge [p]^{<l}(x \wedge [p]^{k}) \wedge [p]^{l}\bot \leftrightarrow p \wedge [p]^{<k}x \wedge [p]^{k}\bot$.
Consequently, $\sigma_l \preccurlyeq_{KD} \sigma_k$.

Lemma 21 *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preceq_{KD} \sigma_l$.*

Proof 35 *Suppose $k < l$ and $\sigma_k \preceq_{KD} \sigma_l$. Let ν be a substitution such that $\vdash_{KD} \nu(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. Then, $\vdash_{KD} p \wedge [p]^{<k}\nu(x) \wedge [p]^k\bot \leftrightarrow p \wedge [p]^{<l}x \wedge [p]^l\bot$. Then, $\vdash_{KD} [p]^l\bot \to [p]^k\bot$. This is contradiction.*

P. Balbiani and Ç. Gencer proved that if there exists a unifier σ of the formula $\varphi = (x \to P) \wedge (x \to [p]x)$ then either $\sigma_p \preccurlyeq_{KD} \sigma$ or there exists $n \in \mathbb{N}$ such that $\sigma_n \preccurlyeq_{KD} \sigma$. In this respect, they proved that

Lemma 22 *Let σ be a KD-unifier of φ. The following conditions are equivalent:*

1. $\sigma_p \circ \sigma \simeq_{KD} \sigma$.
2. $\sigma_p \preccurlyeq_{KD} \sigma$.
3. $\vdash_{KD} \sigma(x) \leftrightarrow p$.

Proof 36 $(1 \Rightarrow 2)$*: By definition of $\preccurlyeq_{KD}$.*
$(2 \Rightarrow 3)$*: Suppose $\sigma_p \preccurlyeq_{KD} \sigma$. Let ν be a substitution such that $\vdash_{KD} \nu(\sigma_p(x)) \leftrightarrow \sigma(x)$. Then, $\vdash_{KD} p \leftrightarrow \sigma(x)$.*
$(3 \Rightarrow 1)$*: Suppose $\vdash_{KD} \sigma(x) \leftrightarrow p$. Then, $\vdash_{KD} \sigma(x) \leftrightarrow \sigma(\sigma_p(x))$. Hence, $\sigma_p \circ \sigma \simeq_{KD} \sigma$.*

Lemma 23 *Let σ be a KD-unifier of $\varphi = (x \to p) \wedge (x \to [p]x)$. Let $k \in \mathbb{N}$. The following conditions are equivalent:*

1. $\sigma_k \circ \sigma \simeq_{KD} \sigma$
2. $\sigma_k \preccurlyeq_{KD} \sigma$
3. $\vdash_{KD} \sigma(x) \to [p]^k\bot$.

Proof 37 $(1 \Rightarrow 2)$*: By definition of $\preccurlyeq_{KD}$.*
$(2 \Rightarrow 3)$*: Suppose $\sigma_n \preccurlyeq_{KD} \sigma$. Let ν be a substitution such that $\vdash_{KD} \nu(\sigma_n(x)) \leftrightarrow \sigma(x)$. Then $\vdash_{KD} \sigma(x) \to [p]^k\bot$.*
$(3 \Rightarrow 1)$*: Suppose $\vdash_{KD} \sigma(x) \to [p]^k\bot$. Since σ is a unifier of φ then, $\vdash_{KD} \sigma(x) \to p \wedge [p]^{<k}\sigma(x)$. Hence, $\vdash_{KD} \sigma(x) \to p \wedge [p]^{<k}\sigma(x) \wedge [p]^k\bot$. Consider two following case:*
If $n = 0$ then $\vdash_{KD} [p]^k\bot \to \sigma(x)$ and
If $n \geqslant 1$ then, $\vdash_{KD} p \wedge [p]^{<k}\sigma(x) \wedge [p]^k\bot \to \sigma(x)$.
Therefore, $\vdash_{KD} p \wedge [p]^{<k}\sigma(x) \wedge [p]^k\bot \leftrightarrow \sigma(x)$. Thus, $\sigma_n \circ \sigma \simeq_{KD} \sigma$.

Lemma 24 *Let σ is a unifier of φ. Let $n \geqslant deg(\sigma(x))$. Then one of the following conditions holds:*

- $\sigma_p \preccurlyeq_{KD} \sigma$ *or*
- $\sigma_n \preccurlyeq_{KD} \sigma$.

Proof 38 *Suppose none of the above conditions hold. Hence by Lemma* 22 *and* 23, $\nvdash_{KD} \sigma(x) \leftrightarrow p$ *and* $\nvdash_{KD} \sigma(x) \to [p]^n\bot$. *Since,* σ *is a unifier of* φ *then,* $\vdash_{KD} \sigma(x) \to p$. *Hence,* $\nvdash_{KD} p \to \sigma(x)$. *Consider two models* $M_1 = (W_1, R', v_1)$ *and* $M_2 = (W_2, R'', v_2)$ *and* $s_1 \in W_1$, $s_2 \in W_2$ *such that* $M_1, s_1 \nvDash p \to \sigma(x)$ *and* $M_2, s_2 \nvDash \sigma(x) \to [p]^n\bot$. *Thus* $M_1, s_1 \vDash p$, $M_1, s_1 \nvDash \sigma(x)$, $M_2, s_2 \vDash \sigma(x)$ *and* $M_2, s_2 \nvDash [p]^n\bot$. *Hence, there exists* $t_3, ..., t_{n+2} \in W_2$ *such that* $s_2R''t_3R''t_4...R''t_{n+2}$. *By the tree-model property of KD, we can assume that* $s_2, t_3, ..., t_{n+2}$ *are pairwise distinct. Notice that* $t_3, ..., t_{n+2} \in v(p)$. *Let model* $M = \langle W, R, v\rangle$ *be an extension of the disjoint union of* M_1 *and* M_2 *and defined as follows:*

- $W = W_1 \cup W_2$,
- $R = R' \cup R'' \cup \{(t_{n+2}, s_1)\}$,
- $v = v_1 \cup v_2$.

By our assumption σ *is a unifier of* φ *and* $\vdash_{KD} \sigma(x) \to [p]\sigma(x)$. *By proposition* 2, $M, s_1 \vDash p$ *and* $M, s_1 \nvDash \sigma(x)$. *Moreover, since* $n \geqslant deg(\sigma(x))$, *then* $M, s_2 \vDash \sigma(x)$. *Since* σ *is a unifier of* $(x \to p) \wedge (x \to [p]x)$, *then* $M, t_2 \vDash \sigma(x)$, ..., $M, t_{n+2} \vDash \sigma(x)$. *Then,* $M, t_{n+2} \vDash [p]\sigma(x)$. *Since* $M, s_1 \vDash p$ *and* $t_{n+2}Rs_1$, *then* $M, s_1 \vDash \sigma(x)$*: a contradiction.*

At the end step, P. Balbiani and Ç. Gencer showed that

Lemma 25 *The set of substitutions* $\Sigma = \{\sigma_p\} \cup \{\sigma_n \mid n \in \mathbb{N}\}$ *is a complete set of* KD*-unifiers of* φ.

Proof 39 *By Lemmas* 19 *to* 24.

Lemma 26 *The formula* φ *does not possess a minimal complete set of* KD*-unifiers.*

Proof 40 *Refer to [7], Lemma* 7. *The proof of this Lemma is similar to the proof of lemma* 18.

Proposition 10 *Unification type is nullary in modal logic KD.*

Proof 41 *By Lemma* 26.

We shall use the same method in chapter 6 to discuss on unification type of the fusion $KD_1 \otimes KD_2$.

4.3.3 Unification in Modal Logics Extending $K4$

S. Ghilardi showed that some modal logics extending $K4$ (like $K4$ itself, $S4$, GL, Grz, etc.) are finitary and that finite complete sets of unifiers can be effectively computed [31].
As we already said the most important role of most general unifiers in unification theory is generating all unifiers of a formula. In classical propositional logic every unifiable formula has a most general unifier. S. Ghilardi investigated whether every unifiable formula in modal logic L has a most general unifier. The answer was negative for many modal logics L enjoying disjunction property. For example, consider the formula $\Box x \vee \Box\neg x$. This formula has unifiers in $K4$, $S4$ and GL:

- $\sigma_1(x) = \top$,
- $\sigma_2(x) = \bot$.

and there is no unifier more general than both of them because if
$\vdash_L \Box\sigma(x) \vee \Box\neg\sigma(x)$
then by the modal disjunction property, we have: either $\vdash_L \sigma(x)$ (so that σ is equivalent to σ_1) or $\vdash_L \neg\sigma(x)$ (so that σ is equivalent to σ_2). Thus this formula has no most general unifier. Moreover, $\Sigma = \{\sigma_1, \sigma_2\}$ is a minimal complete set of unifiers for $\Box x \vee \Box\neg x$ in $K4$, $S4$ and GL. Hence, Ghilardi in [31] proved many transitive modal logics have finitary unification type and that finite complete set of unifiers can be effectively computed.
S. Ghilardi investigated which modal logics are unitary in [30]. Hence he introduced a significant characterisation of modal logic that called filtering unification. See [35] for further discussion about filtering unification.

Definition 29 *A given logic L is filtering iff for all L-unifiable formulas ϕ and for all L-unifiers σ, τ of ϕ there exists a L-unifier μ of ϕ such that $\mu \preceq_L \sigma$ and $\mu \preceq_L \tau$.*

S.Ghilardi proved that filtering unification in modal logic is characterized by the fact that finitely presented projective algebras are closed under binary products. Then he used this characterization to the case of normal extensions L of the modal system $K4$ and showed that a normal modal logic $K4 \subseteq L$ has filtering unification iff L extends $K4.2^+$. The logic $K4.2^+$ is the logic obtained from $K4$ by adding the axiom $\Diamond^+\Box^+ A \rightarrow \Box^+\Diamond^+ A$ where $\Box^+$ and $\Diamond^+$ are defined by $\Box^+ = \Box B \wedge B$ and $\Diamond^+ = \Diamond B \vee B$. At the next step, he proved that unification is unitary in $K4.2^+$. Hence, S. Ghilardi proved that

Lemma 27 *If L is filtering then either L is unitary, or L is nullary.*

Proof 42 *Suppose L is filtering and neither L is unitary, nor L is nullary. Hence, either L is finitary, or L is infinitary. Let ϕ be a L-unifiable formula either of type finitary, or of type infinitary. Since unification type of ϕ is either finitary or infinitary then let Σ be a minimal complete set of L-unifiers of ϕ such that $Card(\Sigma) \geq 2$. Since $Card(\Sigma) \geq 2$, we can suppose that there exist $\sigma, \tau \in \Sigma$ such that $\sigma \neq \tau$. Let μ be a L-unifier of ϕ such that $\mu \preccurlyeq \sigma$ and $\mu \preccurlyeq \tau$. Such L-unifier of ϕ exists because L is filtering. Since Σ is a complete set of L-unifier of ϕ then there exists a $\nu \in \Sigma$ such that $\nu \preccurlyeq \mu$. Since $\mu \preccurlyeq \sigma$ and $\mu \preccurlyeq \tau$ therefore, $\nu \preccurlyeq \sigma$ and $\nu \preccurlyeq \tau$. Since Σ is a minimal set therefore, $\nu = \sigma$ and $\nu = \tau$ then $\sigma = \tau$ and this is a contradiction.*

4.3.4 Unification In the modal logic $S5$

W. Dzik in [23] showed that the modal logic $S5$ and all extensions of the modal logic $S5$ have unitary unification type. Dzik discussed on unification and unification types in four areas of logic: non-Fregean logics, intermediate logics (extensions of intuitionistic logic), modal and multimodal logics, including Tense Logics and Epistemic Logics (Logics of Knowledge) in [22]. Let us see how Dzik proved that the unification type of Epistemic logic $S5$ is unitary.
Consider an $S5$-unifiable formula A and a substitution σ. Suppose σ is an $S5$-unifier of A. Let λ be the substitution defined as follows for all variables x occurring in A:

$$\lambda(x) = (\Box A \wedge x) \vee (\neg\Box A \wedge \sigma(x))$$

Notice how λ is similar to the Löwenheim substitution used in Section 4.2.

Lemma 28 *For any formula B*

1. $\vdash_{S5} \Box A \to (\lambda(B) \leftrightarrow B)$.

2. $\vdash_{S5} \neg\Box A \to (\lambda(B) \leftrightarrow \sigma(B))$.

Proof 43 1. *By induction on* B

- *Let* $B = x$. *We have to prove* $\vdash \Box A \to (\lambda(x) \leftrightarrow x)$. *Since*
  - $\Box A \to ((\Box A \wedge x) \vee (\neg\Box A \wedge \sigma(x)) \to x)$ *is equivalent to* $(\Box A \wedge x \to x)$. *Thus,*
  - $\vdash_{S5} \Box A \to (\lambda(x) \to x)$. *Since,*
  - $\vdash_{S5} (\Box A \wedge x) \to \lambda(x)$ *hence,*
  - $\vdash_{S5} \Box A \to (x \to \lambda(x))$. *Since,*
  - $\vdash_{S5} \Box A \to (\lambda(x) \to x)$ *and*
  - $\vdash_{S5} \Box A \to (x \to \lambda(x))$ *therefore,*
  - $\vdash_{S5} \Box A \to (\lambda(x) \leftrightarrow x)$.
- *Let* $B = \bot$. *We have to prove* $\vdash_{S5} \Box A \to (\lambda(\bot) \leftrightarrow \bot)$. *Since,* $\lambda(\bot) = \bot$ *hence,* $\vdash_{S5} \Box A \to (\lambda(\bot) \leftrightarrow \bot)$.
- *Let* $B = \neg B'$. *We have to prove* $\vdash_{S5} \Box A \to (\lambda(\neg B') \leftrightarrow \neg B')$. *By induction hypothesis,*
  - $\vdash_{S5} \Box A \to (\lambda(B') \leftrightarrow B')$. *Since,*
  - $\vdash_{S5} (\lambda(B') \leftrightarrow B') \to (\neg\lambda(B') \leftrightarrow \neg B')$. *Then,*
  - $\vdash_{S5} \Box A \to (\lambda(\neg B') \leftrightarrow \neg B')$.
- *Let* $B = B' \wedge B''$. *We have to prove* $\vdash_{S5} \Box A \to (\lambda(B' \wedge B'') \leftrightarrow ((B' \wedge B'')))$. *By induction hypothesises,*
  - $\vdash_{S5} \Box A \to (\lambda(B') \leftrightarrow B')$ *and*
  - $\vdash_{S5} \Box A \to (\lambda(B'') \leftrightarrow B'')$. *Therefore,*
  - $\vdash_{S5} \Box A \to (\lambda(B' \wedge B'') \leftrightarrow ((B' \wedge B'')))$.
- *Let* $B = \Box B'$. *We have to prove* $\vdash_{S5} \Box A \to (\lambda(\Box B') \leftrightarrow \Box B')$. *By induction hypothesis,*
  - $\vdash_{S5} \Box A \to (\lambda(B') \leftrightarrow B')$. *By necessitation and axiom* K,
  - $\vdash_{S5} \Box\Box A \to (\lambda(\Box B') \leftrightarrow \Box B')$. *Since,*
  - $\vdash_{S5} \Box A \to \Box\Box A$ *then,*
  - $\vdash_{S5} \Box A \to (\lambda(\Box B') \leftrightarrow \Box B')$.

2. *By induction on B*

   - *Let* $B = x$. *We have to prove,* $\vdash \neg\Box A \to (\lambda(x) \leftrightarrow \sigma(x))$. *Since,*
     - $\neg\Box A \to ((\Box A \wedge x) \vee (\neg\Box A \wedge \sigma(x)) \to \sigma(x))$ *is equivalent to* $\neg\Box A \wedge \sigma(x) \to \sigma(x)$ *then,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(x) \to \sigma(x))$. *Since,*
     - $\vdash_{S5} (\neg\Box A \wedge \sigma(x)) \to \lambda(x)$ *hence,*
     - $\vdash_{S5} \neg\Box A \to (\sigma(x) \to \lambda(x))$. *Since,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(x) \to \sigma(x))$ *and*
     - $\vdash_{S5} \neg\Box A \to (\sigma(x) \to \lambda(x))$ *therefore,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(x) \leftrightarrow \sigma(x))$.
   - *Let* $B = \bot$. *We have to prove,* $\vdash_{S5} \neg\Box A \to (\lambda(\bot) \leftrightarrow \sigma(\bot))$. *Since,* $\lambda(\bot) = \sigma(\bot) = \bot$. *Therefore,* $\vdash_{S5} \neg\Box A \to (\lambda(\bot) \leftrightarrow \sigma(\bot))$.
   - *Let* $B = \neg B'$. *We have to prove,* $\vdash_{S5} \neg\Box A \to (\lambda(\neg B') \leftrightarrow \sigma(\neg B'))$. *By induction hypothesis,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(B') \leftrightarrow \sigma(B'))$. *Since,*
     - $(\lambda(B') \leftrightarrow \sigma(B')) \to (\neg\lambda(B') \leftrightarrow \neg\sigma(B'))$ *then,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(\neg B') \leftrightarrow \sigma(\neg B'))$.
   - *Let* $B = B' \wedge B''$. *We have to prove* $\vdash_{S5} \Box A \to (\lambda(B' \wedge B'') \leftrightarrow (\sigma(B') \wedge \sigma(B''))$. *By induction hypothesises,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(B') \leftrightarrow \sigma(B'))$ *and*
     - $\vdash_{S5} \neg\Box A \to (\lambda(B'') \leftrightarrow \sigma(B''))$. *Therefore,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(B' \wedge B'') \leftrightarrow \neg(\sigma(B') \wedge \sigma(B'')))$.
   - *Let* $B = \Box B'$. *We have to prove* $\vdash_{S5} \neg\Box A \to (\lambda(\Box B') \leftrightarrow \Box B')$. *By induction hypothesis,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(B') \leftrightarrow \sigma(B'))$. *By necessitation and axiom* K,
     - $\vdash_{S5} \neg\Diamond\Box A \to (\lambda(\Box B') \leftrightarrow \sigma(\Box B'))$. *By axiom* 5,
     - $\vdash_{S5} \neg\Box A \to \neg\Diamond\Box A$ *then,*
     - $\vdash_{S5} \neg\Box A \to (\lambda(\Box B') \leftrightarrow \sigma(\Box B'))$.

Lemma 29 *λ is a most general unifier.*

Proof 44 *First, we prove that λ is a unifier. By part* (1) *of Lemma* 28,

1. $\vdash_{S5} \Box A \to (\lambda(A) \leftrightarrow A)$. *Since,*

2. $\vdash_{S5} \Box A \to A$ *then,*

3. $\vdash_{S5} \Box A \to \lambda(A)$. *By part* (2) *of Lemma* 28*,*

4. $\vdash_{S5} \neg\Box A \to (\lambda(A) \leftrightarrow \sigma(A))$. *Since, σ is a unifier of A then,*

5. $\vdash_{S5} \neg\Box A \to \lambda(A)$. *By lines* 3 *and* 5*,*

6. $\vdash_{S5} \lambda(A)$.

Therefore, λ is a unifier of A. Second, we have to prove that λ is a most general unifier.
Let τ be a unifier of A. Then, $\vdash_{S5} \tau(\Box A)$. Let x is an arbitrary variable. We have $\vdash_{S5} \Box A \to (\lambda(x) \leftrightarrow x)$ by part (1) *of Lemma* 28*. Then, $\vdash_{S5} \tau(\Box A) \to (\tau(\lambda(x)) \leftrightarrow \tau(x))$. Since $\vdash_{S5} \tau(\Box A)$ and $\vdash_{S5} \tau(\Box A) \to (\tau(\lambda(x)) \leftrightarrow \tau(x))$ then, $\vdash_{S5} \tau(\lambda(x)) \leftrightarrow \tau(x)$. Therefore λ is a most general unifier of A ($\lambda \preccurlyeq_{S5} \tau$).*

Example 10 *Consider the formula $A = \Box\neg x \vee \Box y$. The substitution σ such that $\sigma(x) = x$ and $\sigma(y) = \Diamond x$ is one of the $S5$-unifiers of A. Let λ be the substitution defined by $\lambda(x) = (\Box A \wedge x) \vee (\neg\Box A \wedge \sigma(x))$ and $\lambda(y) = (\Box A \wedge y) \vee (\neg\Box A \wedge \sigma(y))$. Hence we have $\lambda(x) = (\Box A \wedge x) \vee (\neg\Box A \wedge x) = x$ and $\lambda(y) = (\Box A \wedge y) \vee (\neg\Box A \wedge \Diamond x) = y \vee \Diamond x$. By Lemmas* 29*, this means that λ is a most general unifier of A.*

4.3.5 Unification in Modal Logics $KD5$ and $K5$

In this section, we interest in the logics $KD5$ and $K5$. Notice that $K4 \not\subseteq KD5$ and $K4 \not\subseteq K5$. P. Balbiani and T. Tinchev showed that unification type of modal logic $KD45$ is unitary [15]. Hence we need to express some Lemmas as follows:

Lemma 30 *Every variable-free formula in $KD5$, is $KD5$-equivalent to $\bot$ or $KD5$-equivalent to $\top$.*

Proof 45 *Let φ be a variable-free formula. We have to prove $\vdash \varphi \leftrightarrow \bot$ or $\vdash \varphi \leftrightarrow \top$. We prove by induction on φ. We only consider the case $\varphi := \Box\varphi'$. We remind that $\vdash_{KD5} \Box\bot \leftrightarrow \bot$ and $\vdash_{KD5} \Box\top \leftrightarrow \top$.*
By induction hypothesis we have that $\vdash_{KD5} \varphi' \leftrightarrow \bot$ or $\vdash_{KD5} \varphi' \leftrightarrow \top$. Then $\vdash_{KD5} \Box\varphi' \leftrightarrow \bot$ or $\vdash_{KD5} \Box\varphi' \leftrightarrow \top$. Therefore, $\vdash_{KD5} \varphi \leftrightarrow \bot$ or $\vdash_{KD5} \varphi \leftrightarrow \top$.

Lemma 31 *Every closed substitution in $KD5$ is $KD5$-equivalent to a substitution σ such that for every variable x, either $\sigma(x) = \top$ or $\sigma(x) = \bot$.*

Proof 46 *Let σ be a closed substitution. Then for all variables x, $\sigma(x)$ is variable-free formula hence, by Lemma 30, $\sigma(x)$ is $KD5$-equivalent to $\top$ or $KD5$-equivalent to $\bot$.*

Lemma 32 *Let ϕ be a formula, then the following conditions are equivalent in $KD5$:*

1. *ϕ is $KD5$-unifiable,*
2. *There exists a $KD5$-unifier σ of ϕ such that for all variable x, either $\sigma(x) = \top$ or $\sigma(x) = \bot$.*

Proof 47
1. $(1 \Rightarrow 2)$ *Suppose ϕ is $KD5$-unifiable. Let the substitution σ such that $\vdash_{KD5} \sigma(\phi)$, by Lemma 8, ϕ possesses a closed $KD5$-unifier σ' and for all variables x, $\sigma'(x)$ is variable-free formula hence, by Lemma 31, $\sigma'(x)$ is $KD5$-equivalent to $\top$ or $\bot$.*
2. $(2 \Leftarrow 1)$ *It is easy.*

Lemma 33 *Every variable-free formula in $K5$ is either $K5$-equivalent to $\top$ or $K5$-equivalent to $\bot$ or $K5$-equivalent to $\Box\bot$ or $K5$-equivalent to $\Diamond\top$.*

Proof 48 *Let φ be a variable-free formula. We have to prove $\vdash_{K5} \varphi \leftrightarrow \bot$ or $\vdash_{K5} \varphi \leftrightarrow \top$ or $\vdash_{K5} \varphi \leftrightarrow \Box\bot$ or $\vdash_{K5} \varphi \leftrightarrow \Diamond\top$. We prove by induction on φ. We only consider the case $\varphi := \Box\varphi'$.*
By induction hypothesis we have that $\vdash_{K5} \varphi' \leftrightarrow \bot$ or $\vdash_{K5} \varphi' \leftrightarrow \top$ or $\vdash_{K5} \varphi' \leftrightarrow \Box\bot$ or $\vdash_{K5} \varphi' \leftrightarrow \Diamond\top$. Hence,
If $\vdash_{K5} \varphi' \leftrightarrow \bot$ then $\vdash_{K5} \Box\varphi' \leftrightarrow \Box\bot$. If $\vdash_{K5} \varphi' \leftrightarrow \top$ then $\vdash_{K5} \Box\varphi' \leftrightarrow \top$. If $\vdash_{K5} \varphi' \leftrightarrow \Box\bot$ then $\vdash_{K5} \Box\varphi' \leftrightarrow \Box\Box\bot$. Thus, $\vdash_{K5} \Box\varphi' \leftrightarrow \Box\bot$. If $\vdash_{K5} \varphi' \leftrightarrow \Diamond\top$ then $\vdash_{K5} \Box\varphi' \leftrightarrow \Box\Diamond\top$. Hence, $\vdash_{K5} \Box\varphi' \leftrightarrow \top$.

Lemma 34 *Every closed substitution in $K5$ is $K5$-equivalent to a substitution σ such that for every variable x, either $\sigma(x) = \top$ or $\sigma(x) = \bot$ or $\sigma(x) = \Box\bot$ or $\sigma(x) = \Diamond\top$.*

Proof 49 *Let σ be a closed substitution. Then for all variables x, $\sigma(x)$ is variable-free formula hence, by Lemma 33, $\sigma(x)$ is either $K5$-equivalent to $\top$ or $K5$-equivalent to $\bot$ or $K5$-equivalent to $\Diamond\top$ or $K5$-equivalent to $\Box\bot$.*

Lemma 35 *Let ϕ be a formula, then the following conditions are equivalent in $K5$:*

1. *ϕ is $K5$-unifiable,*
2. *There exists a $K5$-unifier σ of ϕ such that for all variable x, either $\sigma(x) = \top$ or $\sigma(x) = \bot$ or $\sigma(x) = \Box\bot$ or $\sigma(x) = \Diamond\top$.*

Proof 50 1. $(1 \Rightarrow 2)$ *Suppose ϕ is $K5$-unifiable. Let the substitution σ be such that $\vdash_{K5} \sigma(\phi)$. By Lemma 8, ϕ possesses a closed $K5$-unifier σ'. Then for all variables x, $\sigma'(x)$ is variable-free formula hence, by Lemma 31, $\sigma'(x)$ is $K5$-equivalent to either $\top$ or $\bot$ or $\Box\bot$ or $\Diamond\top$.*

2. $(2 \Rightarrow 1)$ *It is easy.*

With Lemma 32 and 35, we can only conclude that given a modal formula ϕ, it is relatively simple to determine whether ϕ is $KD5$-unifiable or $K5$-unifiable. For instance, given ϕ, to determine if ϕ is $KD5$-unifiable it suffices to non-deterministically replace in ϕ each variable either by $\bot$, or by $\top$ and then to see if the resulting closed formula is in $KD5$. In $K5$, it suffices to non-deterministically replace variables in ϕ either by $\bot$, or by $\bot$, or by $\Box\bot$, or by $\Diamond\top$. Now, let us try to determine the unification type of $KD5$ and $K5$ which is still unknown. Notice that the result of Ghilardi mentioned after Definition 29 cannot be used for $KD5$ and $K5$ because $K4 \nsubseteq KD5$ and $K4 \nsubseteq KD5$.
Let ϕ be a modal formula and σ, τ be substitutions. Let y be a new variable. This means that y does not occur in ϕ. Moreover, for all variables x occurring in ϕ, y does not occur in $\sigma(x)$ for variables x in ϕ. Let $\alpha^{\sigma,\tau}_{KD5}$ be substitution defined by

$$\alpha^{\sigma,\tau}_{KD5}(x) = (\Box\Box y \wedge \sigma(x)) \vee (\diamond\diamond\neg y \wedge \tau(x)).$$

Lemma 36 *Let σ be a substitution of a given formula ϕ.*

1. *$\alpha^{\sigma,\tau}_{KD5} \preceq_{KD5} \sigma$.*
2. *$\alpha^{\sigma,\tau}_{KD5} \preceq_{KD5} \tau$.*

Proof 51 1. *Let ν be the substitution defined by $\nu(y) = \top$ and for all other variable x, $\nu(x) = x$. Since the variable y is new hence, $\nu \circ \alpha^{\sigma,\tau}_{KD5}(x) \simeq_{KD5} \sigma(x)$. Then, $\alpha^{\sigma,\tau}_{KD5} \preceq_{KD5} \sigma$.*

2. *Let ν be the substitution defined by $\nu(y) = \bot$ and for all other variable x, $\nu(x) = x$. Since the variable y is new hence, $\nu \circ \alpha^{\sigma,\tau}_{KD5}(x) \simeq_{KD5} \tau(x)$. Then, $\alpha^{\sigma,\tau}_{KD5} \preceq_{KD5} \tau$.*

Lemma 37 *Let ψ be a formula not containing y.*

1. $\vdash_{KD5} \Box\Box y \to (\alpha^{\sigma,\tau}_{KD5}(\psi) \leftrightarrow \sigma(\psi))$.

2. $\vdash_{KD5} \Diamond\Diamond\neg y \to (\alpha^{\sigma,\tau}_{KD5}(\psi) \leftrightarrow \tau(\psi))$.

Proof 52 *We will do the proof by using the semantics of $KD5$. Remind that $KD5$-models are of the form (W, R, V) where R is serial and Euclidean.Notice that if $w \in W$ is such that $M, w \models \Box\Box y$ then for all v in the sub-model of M generated from w, we have $M, v \models \Box\Box y$. Similarly, if $w \in W$ is such that $M, w \models \Diamond\Diamond\neg y$ then for all v in the sub-model of M generated from w, we have $M, v \models \Diamond\Diamond\neg y$. Suppose $M = (W, R, V)$ is $KD5$-model. Then for all formulas ψ we prove by induction on ψ that:*

1. *If $M, w \models \Box\Box y$ then $M, w \models \alpha^{\sigma,\tau}_{KD5}(\psi)$ iff $M, w \models \sigma(\psi)$.*

2. *If $M, w \models \Diamond\Diamond\neg y$ then $M, w \models \alpha^{\sigma,\tau}_{KD5}(\psi)$ iff $M, w \models \tau(\psi)$.*

1. *Suppose $M, w \models \Box\Box y$ we want to show that $M, w \models \alpha^{\sigma,\tau}_{KD5}(\psi)$ iff $M, w \models \sigma(\psi)$. The proof is done by induction on ψ.*

   - *$\psi = x$. Hence, $M, w \models \alpha^{\sigma,\tau}_{KD5}(x)$ iff $M, w \models (\Box\Box y \wedge \sigma(x)) \vee (\diamond\diamond\neg y \wedge \tau(x))$ iff $M, w \models \sigma(x)$ since $M, w \models \Box\Box y$.*

   - *$\psi = \neg\psi'$. By our assumption,*
     (1) $M, w \models \Box\Box y$. *By induction hypothesis*
     (2) $M, w \models (\alpha^{\sigma,\tau}_{KD5}(\psi') \leftrightarrow \sigma(\psi'))$. *By* 2
     (3) $M, w \models \neg\alpha^{\sigma,\tau}_{KD5}(\psi') \leftrightarrow \neg\sigma(\psi')$. *Then*
     (4) $M, w \models \alpha^{\sigma,\tau}_{KD5}(\neg\psi') \leftrightarrow \sigma(\neg\psi')$.

   - *The case when $\psi = \psi' \wedge \psi''$. By our assumption,*
     (1) $M, w \models \Box\Box y$. *By induction hypothesis,*
     (2) $M, w \models (\alpha^{\sigma,\tau}_{KD5}(\psi') \leftrightarrow \sigma(\psi'))$ *and*
     (3) $M, w \models (\alpha^{\sigma,\tau}_{KD5}(\psi'') \leftrightarrow \sigma(\psi''))$. *Then,*
     (4) $M, w \models \alpha^{\sigma,\tau}_{KD5}(\psi' \wedge \psi'') \to \sigma(\psi' \wedge \psi'')$.

- $\psi = \Box\psi'$. *By induction hypothesis, we know that for all* v *in the submodel of* M *generated from* w,
  (1) $M, v \models (\alpha^{\sigma,\tau}_{KD5}(\psi') \leftrightarrow \sigma(\psi'))$. *Then*
  (2) $M, w \models \alpha^{\sigma,\tau}_{KD5}(\Box\psi') \leftrightarrow \sigma(\Box\psi')$.

2. *if we suppose that* $M, w \models \Diamond\Diamond\neg y$ *then the argument is similar.*

Lemma 38 *Let* ϕ *be a formula. If* σ *and* τ *are* $KD5$*-unifiers of* ϕ *then* $\alpha^{\sigma,\tau}_{KD5}$ *is a* $KD5$*-unifier of* ϕ.

Proof 53 *Suppose* σ *and* τ *are* $KD5$*-unifiers of* ϕ. *Then,*
(1) $\models_{KD5} \sigma(\phi)$ *and*
(2) $\models_{KD5} \tau(\phi)$. *By Lemma* 37
(3) $\models_{KD5} \Box\Box y \rightarrow (\alpha^{\sigma,\tau}_{KD5}(\phi) \leftrightarrow \sigma(\phi))$ *and*
(4) $\models_{KD5} \Diamond\Diamond\neg\phi \rightarrow (\alpha^{\sigma,\tau}_{KD5}(\phi) \leftrightarrow \tau(\phi))$. *By* 1 *and* 3,
(5) $\models_{KD5} \Box\Box y \rightarrow \alpha^{\sigma,\tau}_{KD5}(\phi)$. *By* 2 *and* 4
(6) $\models_{KD5} \Diamond\Diamond\neg\phi \rightarrow \alpha^{\sigma,\tau}_{KD5}(\phi)$. *By* 5 *and* 6,
(7) $\models_{KD5} \alpha^{\sigma,\tau}_{KD5}(\phi)$.

Proposition 11 *Unification in* $KD5$ *is filtering.*

Proof 54 *Let* ϕ *be a* $KD5$*-unifiable formula. Let* σ, τ *be* $KD5$*-unifiers of* ϕ. *By Lemmas 36 and 38,* $\alpha^{\sigma,\tau}_{KD5}$ *is a* $KD5$*-unifier of* ϕ *such that* $\alpha^{\sigma,\tau}_{KD5}(\phi) \preceq_{KD5} \sigma$ *and* $\alpha^{\sigma,\tau}_{KD5}(\phi) \preceq_{KD5} \tau$. *As* ϕ *is an arbitrary* $KD5$*-unifiable formula,* $KD5$ *is filtering.*

As a consequence, $KD5$ is either of type unitary or of type nullary (see Lemma 27). We conjecture that, like $K5$, $KD5$ is unitary. Now, let us adapt our line or reasoning to the case of modal logic $K5$.
Consider a modal formula ϕ and substitutions σ, τ. Let y be a new variable. Let $\alpha^{\sigma,\tau}_{K5}$ be the substitution defined for all variable x occurring in ϕ,

$$\alpha^{\sigma,\tau}_{K5}(x) = ((\Box\Box y \wedge (y \vee \Diamond\top)) \wedge \sigma(x)) \vee ((\Diamond\Diamond\neg y \vee (\neg y \wedge \Box\bot)) \wedge \tau(x))$$

Lemma 39 *Let* σ *be a substitution of a given formula* ϕ.

- $\alpha^{\sigma,\tau}_{K5} \preceq_{K5} \sigma$.
- $\alpha^{\sigma,\tau}_{K5} \preceq_{K5} \tau$.

Proof 55 *The proof is similar to the proof of Lemma 36*

Lemma 40 *Let ψ be a formula*
(1) $\vdash_{K5} (\Box\Box y \wedge (y \vee \Diamond\top)) \rightarrow (\alpha^{\sigma,\tau}_{K5}(\psi) \leftrightarrow \sigma(\psi))$.
(2) $\vdash_{K5} (\Diamond\Diamond\neg y \vee (\neg y \wedge \Box\bot)) \rightarrow (\alpha^{\sigma,\tau}_{K5}(\psi) \leftrightarrow \tau(\psi))$.

Proof 56 *The proof can be done by using the semantics of $K5$. We remind that models of $K5$ are of the form (W,R,V) where R is Euclidean. Notice that if $w \in W$ is such that $M, w \models \Box\Box y \wedge (y \vee \Diamond\top)$ then for all v in the sub-model of M generated from w, we have $M, v \models \Box\Box y \wedge (y \vee \Diamond\top)$. Similarly, if $w \in W$ is such that $M, w \models \Diamond\Diamond\neg y \vee (\neg y \wedge \Box\bot)$ then for all v in the sub-model of M generated from w, we have $M, v \models \Diamond\Diamond\neg y \vee (\neg y \wedge \Box\bot)$.*

Lemma 41 *If σ and τ are $K5$-unifiers of ϕ then $\alpha^{\sigma,\tau}_{K5}$ is a $K5$-unifier of ϕ.*

Proof 57 *The proof is similar to the proof of Lemma 38*

Proposition 12 *Unification in $K5$ is filtering.*

Proof 58 *The proof is similar to the proof of Proposition 11*

Proposition 13 (1) *Either $K5$ is unitary, or $K5$ is nullary.*
(2) *Either $KD5$ is unitary, or $KD5$ is nullary.*

Proof 59 *By Lemma 27 and Lemmas 11 and 12.*

The exact unification type of $KD5$ and $K5$ is still unknown. The main difficulty in determining this type is that neither $KD5$ nor $K5$ contain $K4$. Hence, the techniques developed by Ghilardi [31] for showing that $K4$ and some of its extension like $S4$ and GL are finitary cannot be used. In other respect, Ghilardi's results [30] saying that an extension L of $K4$ has filtering unification iff L contains $K4.2^+$ cannot be applied in the case of $KD5$ and $K5$ for the same reason (neither $KD5$ nor $K5$ contains $K4$).

5 Unification in the Logic $Alt_1 + \Box\Box\bot$

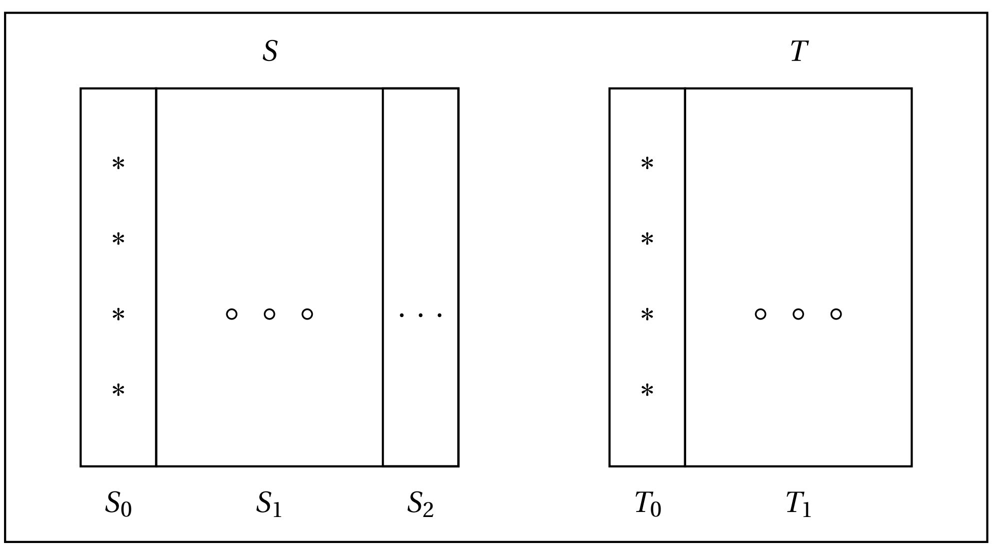
Contents

In chapter 4, we have seen that the unification in modal logics $KD5$ and $K5$ are either unitary or nullary, the exact unification type of these logics are still unknown. This is quite surprising, considering the fact that $KD5$ and $K5$ are relatively simple logics. Another simple modal logic is $Alt_1 = K \oplus \Diamond A \to \Box A$. Balbiani and Tinchev [14] have proved that Alt_1 is nullary. Models of Alt_1 are of the form (W, R, v) where R is deterministic relation (every possible world has at most one successor). In this section, after a suggestion of Silvio Ghilardi, we investigate the unification type of logics $Alt_1 + \Box\Box\bot$. Models of $Alt_1 + \Box\Box\bot$ are very simple. They are structures of the form (W, R, v) where W contains exactly 1 world and $R = \emptyset$, or W contains exactly two worlds w and v and $R = \{(w, v)\}$ (that is to say w can see v and w, v are irreflexive). In this Chapter, we show that unification type of $Alt_1 + \Box\Box\bot$ is unitary. It is obvious that some results have to be proven. Now, we introduce a result which will prove to be very useful in Section 5.6. For all set S, notation $\parallel S \parallel$ will be used as the cardinality of the set S. For all non-empty sets S, for all equivalence relations $\sim$ on S and for all $\alpha \in S$, notation $[\alpha]$ will denote the equivalence class modulo $\sim$ with α as its representative. For all non-empty sets S, for all equivalence relations $\sim$ on S and for all $T \subseteq S$, $T/\sim$ will denote the quotient set of T modulo $\sim$. Notice that for all non-empty sets S, for all equivalence relations $\sim$ on S and for all $\alpha, \beta \in S$, $\alpha \sim \beta$ iff $\alpha \in [\beta]$ iff $[\alpha] \cap [\beta] \neq \emptyset$. Now, we introduce a result which will be very useful in Section 5.6.

Proposition 14 *Let S, T be finite non-empty sets. Let $\sim$ be an equivalence relation on S. The following conditions are equivalent:*

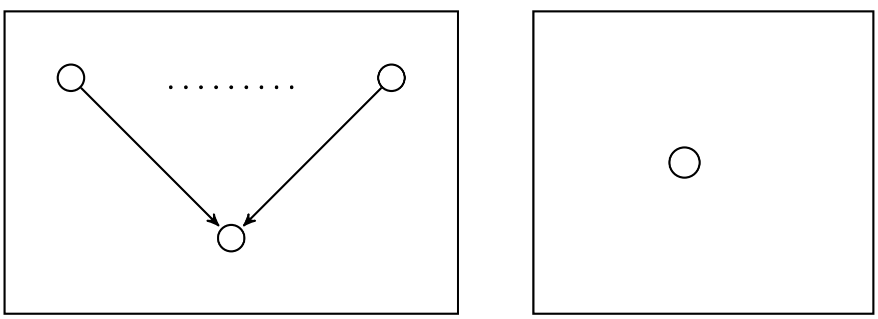
1. $\parallel S/\sim \parallel \leqslant \parallel T \parallel \leqslant \parallel S \parallel$,
2. *there exists a surjective function f from S to T such that for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$.*

Proof 60 $(1 \Rightarrow 2)$ *Suppose $\parallel S/\sim \parallel \leqslant \parallel T \parallel \leqslant \parallel S \parallel$. Let h be a function from $S/\sim$ to S such that for all $\alpha \in S$, $h[\alpha] \in [\alpha]$. h is injective. Let $S_0 = \{h[\alpha] : \alpha \in S\}$. Since h is injective, therefore $\parallel S/\sim \parallel = \parallel S_0 \parallel$. Since $\parallel S/\sim \parallel \leqslant \parallel T \parallel$, therefore $\parallel S_0 \parallel \leqslant \parallel T \parallel$. Let T_0 be a subset of T such that $\parallel T_0 \parallel = \parallel S_0 \parallel$. Let f_0 be a one-to-one correspondence between S_0 and T_0. Let $T_1 = T \setminus T_0$. Notice that T_0 and T_1 make a partition of T. Since $\parallel T \parallel \leqslant \parallel S \parallel$ and $\parallel T_0 \parallel = \parallel S_0 \parallel$, therefore $\parallel T_1 \parallel \leqslant \parallel S \setminus S_0 \parallel$. Let S_1 be a subset of $S \setminus S_0$ such that $\parallel T_1 \parallel = \parallel S_1 \parallel$. Let f_1 be a one-to-one correspondence between S_1 and T_1. Let $S_2 = (S \setminus S_0) \setminus S_1$. Let f_2 be the function from S_2 to T such that for all $\alpha \in S_2$, $f_2(\alpha) = f_0(h([\alpha]))$. Let f be the function from S to T defined by $f \mid S_0 = f_0$, $f \mid S_1 = f_1$ and $f \mid S_2 = f_2$. By construction of f, it is easy to show that f is surjective and for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$.*



(2 ⇒ 1) *Suppose f is a surjective function from S to T such that for all $\alpha, \beta \in S$, if $f(\alpha) = f(\beta)$ then $\alpha \sim \beta$. For the sake of the contradiction, suppose either $\| S/\sim \| > \| T \|$, or $\| T \| > \| S \|$. Since f is surjective, therefore $\| T \| \leqslant \| S \|$. Since either $\| S/\sim \| > \| T \|$, or $\| T \| > \| S \|$, therefore $\| S/\sim \| > \| T \|$. Let $p \in \mathbb{N}$ and $\beta^1, \ldots, \beta^p \in S$ be such that $p > \| T \|$ and for all $q, r \in \mathbb{N}$, if $1 \leqslant q$, $r \leqslant p$ and $q \neq r$ then $\beta^q \nsim \beta^r$. Hence, for all $q, r \in \mathbb{N}$, if if $1 \leqslant q$, $r \leqslant p$ and $q \neq r$ then $f(\beta^q) \neq f(\beta^r)$. Thus, $p \leqslant \| T \|$: a contradiction.*

We remind that P is a countably infinite set of propositional variables (with typical members denoted x, y, etc). Let $(x_1, x_2, \ldots)$ be an enumeration of P without repetitions. For all $n \in \mathbb{N}$, let $\mathbf{FOR}_n$ be the set of all formulas based on the variables $x_1, \ldots, x_n$.
We shall say that a frame (W, R) is deterministic if for all $s, t, u \in W$, if sRt and sRu then $t = u$.
We shall say that a frame (W, R) is bounded if for all $s_0, s_1, s_2 \in W$ either $s_0 \not R s_1$ or $s_1 \not R s_2$.
Let $\mathscr{C}^b_{det}$ be the class of all deterministic bounded frame. Let L be the logic characterized by $\mathscr{C}^b_{det}$. As is well-known, $L = Alt_1 + \Box\Box\bot$. For all $n \geqslant 1$, an n-tuple of bits (denoted α, β, etc) is a function from $\{1, \ldots, n\}$ to $\{0, 1\}$. For all $n \geqslant 1$, let $\mathbf{BIT}_n$ be the set of all n-tuples of bits. For all $\alpha \in \mathbf{BIT}_n$, we will write $\alpha = (\alpha_1, \ldots, \alpha_n)$.
Obviously, frames in $\mathscr{C}^b_{det}$ are disjoint unions of the following structures where circles represent irreflexive possible worlds.



In this chapter, when we write "frame $F = (W, R)$" we mean "frame $F = (W, R)$ in $\mathscr{C}^b_{det}$".

5.1 Semantics

Instead of considering models giving truth values to any kind of formulas, we will use models giving values to formulas based on a restricted (finite) set of variables. Let $n \geqslant 1$. An n-model based on a frame (W, R) is a triple (W, R, v) where v is a function assigning for all $i \in 1, ..., n$, a subset $v(x_i)$ of W to the variable x_i. Given an n-model (W, R, V), the n-satisfiability of $\varphi \in \mathbf{FOR}_n$ at $s \in W$ (in symbols $s \vDash_n \varphi$) is inductively defined as follows:

- $s \vDash_n x_i$ iff $s \in V(x_i)$,
- $s \nvDash_n \bot$,
- $s \vDash_n \neg\varphi$ iff $s \nvDash_n \varphi$,
- $s \vDash \varphi \wedge \psi$ iff $s \vDash_n \varphi$ and $s \vDash_n \psi$,
- $s \vDash_n \Box\varphi$ iff for all $t \in W$ if sRt then $t \vDash_n \varphi$.

We shall say that $\varphi \in \mathbf{FOR}_n$ is n-true in a n-model (W, R, v) if φ is n-satisfied at all $s \in W$.
We shall say that $\varphi \in \mathbf{FOR}_n$ is n-valid in a frame (W, R) if φ is n-true in all n-models based on (W, R).
We shall say that $\varphi \in \mathbf{FOR}_n$ is n-valid (in symbol $\vDash \varphi$) if φ is n-valid in all frames.
Remind that, in this Chapter, all frames are bounded deterministic.
Let $\equiv^n$ be the equivalence relation on $\mathbf{FOR}_n$ defined by

- $\varphi \equiv^n \psi$ iff $\vDash \varphi \leftrightarrow \psi$,

where φ and ψ range over $\mathbf{FOR}_n$. The next result follows from the fact that in the logic $Alt_1 + \Box\Box\bot$ for all $\varphi \in \mathbf{FOR}_n$, there exists $\psi \in \mathbf{FOR}_n$ such that $deg(\psi) \leqslant 1$ and $\vDash \varphi \leftrightarrow \psi$.

Proposition 15 *$\mathscr{C}^b_{det}$ is locally tabular. That is to say for all $n \geqslant 1$, the equivalence relation $\equiv^n$ possesses finitely many equivalence classes.*

Proof 61 *Refer to [18], Proposition* 2.29.

Let $\mathscr{G}\mathscr{C}^b_{det}$ be the class consisting of all frames of the form (W, R) where $W = \{s_0, s_1\}$ and $s_0 \neq s_1$ and $R = \{(s_0, s_1)\}$. Notice that $\mathscr{G}\mathscr{C}^b_{det} \subseteq \mathscr{C}^b_{det}$. The next result shows that $\mathscr{G}\mathscr{C}^b_{det}$ and $\mathscr{C}^b_{det}$ determine the same modal logic: $Alt1 + \Box\Box\bot$. Its proof is standard.

Proposition 16 *For all $\varphi \in \boldsymbol{FOR}_n$, $\varphi \in Alt_1 + \Box\Box\bot$ iff φ is n-valid in all frames of $\mathscr{C}^b_{det}$ iff ϕ is valid in all frames of $\mathscr{G}\mathscr{C}^b_{det}$.*

5.2 Unification

In order to show that $Alt_1 + \Box\Box\bot$ is unitary, we will use a special notation for substitution. We remind from Chapter 4 that a substitution is a mapping $\sigma : \overline{x} \to F(\overline{y})$ where $\overline{x}, \overline{y}$ are finite tuples of variables and $F(\overline{y})$ is the set of formulas based on the variables in $\overline{y}$. In this chapter, we will use a different notation for substitutions. Let $n \geqslant 1$. An n-substitution is a pair (k, σ) where $k \geqslant 1$ and σ is a homomorphism from $\mathbf{FOR}_n$ to $\mathbf{FOR}_k$, i.e. $\sigma : \mathbf{FOR}_n \to \mathbf{FOR}_k$ is such that

- $\sigma(\bot) = \bot$,
- $\sigma(\neg\varphi) = \neg\sigma(\varphi)$,
- $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$,
- $\sigma(\Box\varphi) = \Box\sigma(\varphi)$.

Let $\mathbf{SUB}_n$ be the set of all n-substitutions. The equivalence relation $\simeq^n$ on $\mathbf{SUB}_n$ is defined by

- $(k, \sigma) \simeq^n (l, \tau)$ iff for all $i \in \{1, \dots, n\}, \models \sigma(x_i) \leftrightarrow \tau(x_i)$,

where $(k, \sigma), (l, \tau)$ range over $\mathbf{SUB}_n$. The pre-order $\preccurlyeq^n$ on $\mathbf{SUB}_n$ is defined by

- $(k, \sigma) \preccurlyeq^n (l, \tau)$ iff there exists a k-substitution (m, ν) such that for all $i \in \{1, \dots, n\}, \models \nu(\sigma(x_i)) \leftrightarrow \tau(x_i)$,

where $(k, \sigma), (l, \tau)$ range over $\mathbf{SUB}_n$. Obviously, $\simeq^n$ is contained in $\preccurlyeq^n$. An n-unifier of $\varphi \in \mathbf{FOR}_n$ is an n-substitution (k, σ) such that $\models \sigma(\varphi)$. We say that $\varphi \in \mathbf{FOR}_n$ is n-unifiable if there exists a n-unifier of φ. We say that a set Σ of n-unifiers of a n-unifiable $\varphi \in \mathbf{FOR}_n$ is n-complete if for all n-unifiers (k, σ) of φ, there exists $(l, \tau) \in \Sigma$ such that $(l, \tau) \preccurlyeq^n (k, \sigma)$.

Definition 30 *For all n-unifiable $\varphi \in \mathbf{FOR}_n$, we shall say that*

- *φ is n-filtering if for all n-unifiers $(k, \sigma), (l, \tau)$ of φ, there exists a n-unifier (m, ν) such that $(m, \nu) \preccurlyeq^n (k, \sigma)$ and $(m, \nu) \preccurlyeq^n (l, \tau)$.*

The next result is standard and Lemma 27 rephrases itself as the following Lemma.

Lemma 42 *Let $\varphi \in \mathbf{FOR}_n$ be n-unifiable. If φ is n-filtering then either φ is n-nullary, or φ is n-unitary.*

Proof 62 *The proof is similar to the proof of Lemma 27.*

Definition 31 *For all n-unifiable $\varphi \in \boldsymbol{FOR}_n$ and for all $\pi \geqslant 1$, we shall say that*

- *φ is n-π-reasonable if for all n-unifiers (k,σ) of φ, if $k > \pi$ then there exists a n-unifier (l,τ) of φ such that $(l,\tau) \preccurlyeq^n (k,\sigma)$ and $l \leqslant \pi$.*

In other words, an n-unifiable $\varphi \in \mathbf{FOR}_n$ will be n-π-reasonable where every n-unifier of φ is an instance of an n-unifier (l,τ) such that $l \leqslant \pi$, that is to say the variables occurring in $\tau(x_1),\dots,\tau(x_n)$ belong to the set $\{x_1,\dots,x_l\} \subseteq \{x_1,\dots,x_\pi\}$. The next result is new. Combined with Proposition 42, it will be very useful in Section 5.4 for showing that unifiable n-formulas are unitary in $Alt_1 + \Box\Box\bot$,

Proposition 17 *Let $\varphi \in \boldsymbol{FOR}_n$ be n-unifiable and $\pi \geqslant 1$. If φ is n-π-reasonable then either φ is n-finitary, or φ is n-unitary.*

Proof 63 *Suppose φ is n-π-reasonable. Let Σ be the set of all n-unifiers of φ. Notice that Σ is n-complete. Let Σ' be the set of n-substitutions obtained from Σ by keeping only the n-substitutions (k,σ) such that $k \leqslant \pi$. Since Σ is n-complete and φ is n-π-reasonable, therefore Σ' is n-complete. Let Σ'' be the set of n-substitutions obtained from Σ' by keeping only one representative of each equivalence class modulo $\simeq^n$. Since Σ' is n-complete, therefore Σ'' is n-complete. Moreover, since $\mathscr{C}^b_{det}$ is locally π-tabular, therefore Σ'' is finite. Hence, either φ is n-finitary, or φ is n-unitary.*

5.3 About bounded deterministic frames

Let $n \geqslant 1$. The next result implies that in $Alt_1 + \Box\Box\bot$, unifiable n-formulas are either nullary, or unitary.

Proposition 18 *For all $\varphi \in \boldsymbol{FOR}_n$, if φ is n-unifiable then φ is n-filtering.*

Proof 64 *Let $\varphi \in \boldsymbol{FOR}_n$. Suppose φ is n-unifiable. Let (k,σ), (l,τ) be n-unifiers of φ. Let $m = max\{k,l\} + 1$. Notice that x_m does not occur in φ,σ and τ. Let (m,μ) be the n-substitution defined by*

- $\mu(x_i) = \Big((\Diamond x_m \vee (x_m \wedge \Box\bot)) \wedge \sigma(x_i)\Big) \vee \Big((\Box\neg x_m \wedge (\neg x_m \vee \Diamond\top)) \wedge \tau(x_i)\Big),$

where i ranges over $\{1,\dots,n\}$. Let $(m,\lambda_\top)$ and $(m,\lambda_\bot)$ be the m-substitutions defined by

- *if $i < m$ then $\lambda_\top(x_i) = x_i$ else $\lambda_\top(x_i) = \top$,*

- *if $i < m$ then $\lambda_\bot(x_i) = x_i$ else $\lambda_\bot(x_i) = \bot$,*

*where i ranges over $\{1,...,m\}$. Notice that for all $i \in \{1,...,n\}$, $\vDash \lambda_\top(\mu(x_i)) \leftrightarrow \sigma(x_i)$ and $\vDash \lambda_\bot(\mu(x_i)) \leftrightarrow \tau(x_i)$. Hence, $(m,\mu) \preccurlyeq^n (k,\sigma)$ and $(m,\mu) \preccurlyeq^n (l,\tau)$. Moreover, by induction on $\psi \in$ **FOR**$_n$ the reader may show that $\vDash \Diamond x_m \vee (x_m \wedge \Box\bot) \rightarrow (\mu(\psi) \leftrightarrow \sigma(\psi))$ and $\vDash \Box\neg x_m \wedge (\neg x_m \vee \Diamond\top) \rightarrow (\mu(\psi) \leftrightarrow \tau(\psi))$. Thus $\vDash \Diamond x_m \vee (x_m \wedge \Box\bot) \rightarrow \mu(\varphi)$ and $\vDash \Box\neg x_m \wedge (\neg x_m \vee \Diamond\top) \rightarrow \mu(\varphi)$. Consequently, $\vDash \mu(\varphi)$ and (m,μ) is a n-unifier of φ. Since $(m,\mu) \preccurlyeq^n (k,\sigma)$ and $(m,\mu) \preccurlyeq^n (l,\tau)$, therefore φ is n-filtering.*

In order to show that in $Alt_1 + \Box\Box\bot$, unifiable n-formulas are reasonable (Proposition 25), we introduce an alternative semantics as follows.
A n-model is a structure of the form (α, A) where α is an n-tuple of bits and A is a set of n-tuples of bits of cardinality 0 or 1.
Let $\mathbf{MOD}_n^{=\emptyset}$ be the set of all n-models (α, A) such that $A = \emptyset$ hence $\mathbf{MOD}_n^{=\emptyset} = \{(\alpha, A) : \alpha \in \mathbf{BIT}_n, A = \emptyset\}$ and $\mathbf{MOD}_n^{\neq\emptyset}$ be the set of all n-models (α, A) such that $A \neq \emptyset$ hence, $\mathbf{MOD}_n^{\neq\emptyset} = \{(\alpha, A) : \alpha \in \mathbf{BIT}_n, \parallel A \parallel = 1\}$. Let $\mathbf{MOD}_n$ be the set of all n-models hence, $\mathbf{MOD}_n = \mathbf{MOD}_n^{=\emptyset} \cup \mathbf{MOD}_n^{\neq\emptyset}$. The binary relation $\vDash_n$ of satisfiability between $\mathbf{MOD}_n$ and $\mathbf{FOR}_n$ is defined in two following cases.
If $(\alpha, A) \in \mathbf{MOD}_n^{\neq\emptyset}$ then,

- $(\alpha, A) \vDash_n x_i$ iff $\alpha_i = 1$,

- $(\alpha, A) \nvDash_n \bot$,

- $(\alpha, A) \vDash_n \neg\varphi$ iff $(\alpha, A) \nvDash_n \varphi$,

- $(\alpha, A) \vDash \varphi \wedge \psi$ iff $(\alpha, A) \vDash_n \varphi$ and $(\alpha, A) \vDash_n \psi$,

- $(\alpha, A) \vDash_n \Box\varphi$ iff for the unique $\alpha' \in A$, $(\alpha', \emptyset) \vDash_n \varphi$.

If $(\alpha, A) \in \mathbf{MOD}_n^{=\emptyset}$ then,

- $(\alpha, A) \vDash_n x_i$ iff $\alpha_i = 1$,

- $(\alpha, A) \nvDash_n \bot$,

- $(\alpha, A) \vDash_n \neg\varphi$ iff $(\alpha, A) \nvDash_n \varphi$,

- $(\alpha, A) \models \varphi \wedge \psi$ iff $(\alpha, A) \models_n \varphi$ and $(\alpha, A) \models_n \psi$,
- $(\alpha, A) \models_n \Box\varphi$.

The next result shows that $\mathcal{GC}^b_{det}$ and n-models determine the same modal logic. Its proof is standard.

Proposition 19 *For all $\varphi \in \boldsymbol{FOR}_n$, $\mathcal{GC}^b_{det} \models_n \varphi$ iff for all $(\alpha, A) \in \boldsymbol{MOD}_n$, $(\alpha, A) \models_n \varphi$.*

We remind that for all formulas ψ, ψ^0 denotes $\neg\psi$ and ψ^1 denotes ψ.

Definition 32 *The function $\boldsymbol{for}_n$ from $\boldsymbol{MOD}_n$ to $\boldsymbol{FOR}_n$ is inductively defined as follows:*

- *if $A \neq \emptyset$ then $\boldsymbol{for}_n((\alpha, A)) = x_1^{\alpha_1} \wedge \ldots \wedge x_n^{\alpha_n} \wedge \Diamond\boldsymbol{for}_n((\alpha', \emptyset))$, where $A = \{\alpha'\}$,*
- *if $A = \emptyset$ then $\boldsymbol{for}_n((\alpha, A)) = x_1^{\alpha_1} \wedge \ldots \wedge x_n^{\alpha_n} \wedge \Box\bot$*

where (α, A) ranges over $\boldsymbol{MOD}_n$.

Proposition 20 *Let $(k, \sigma) \in \boldsymbol{SUB}_n$. Let $(\alpha, A) \in \boldsymbol{MOD}_k$ and $(\beta, B) \in \boldsymbol{MOD}_n$. If $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$ then $\parallel A \parallel = \parallel B \parallel$.*

Proof 65 *Let $\parallel A \parallel \neq \parallel B \parallel$. Hence either $\parallel A \parallel < \parallel B \parallel$ or $\parallel B \parallel < \parallel A \parallel$. Assume $\parallel A \parallel = \emptyset$ and $\parallel B \parallel \neq \emptyset$. By our assumption we have $(\alpha, \emptyset) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$. Hence, $(\alpha, \emptyset) \models_k \Diamond\top$ because $B \notin \emptyset$ and this is contradiction.*
Let $\parallel B \parallel < \parallel A \parallel$. In this case, we can do similar to the case $\parallel A \parallel < \parallel B \parallel$.
Therefore $\parallel A \parallel = \parallel B \parallel$.

Proposition 21 *Let $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_n$. The following conditions are equivalent:*

1. *$(\alpha, A) = (\beta, B)$,*
2. *$(\alpha, A) \models_n \boldsymbol{for}_n((\beta, B))$.*

Proof 66 $(1 \Rightarrow 2)$ *Let $(\alpha, A) = (\beta, B)$. Hence, $\alpha = \beta$ and $A = B$. We consider two following cases:*

- *Let $A = \emptyset$ and $B = \emptyset$. By definition $(\alpha, A) \models_n x_1^{\alpha_1} \wedge \ldots \wedge x_n^{\alpha_n}$. Since $\alpha = \beta$ then, $\alpha_i = \beta_i$ for each $i = 1, \ldots, n$, hence, $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n}$. Since, $A = \emptyset$ hence, $(\alpha, A) \models_n \Box\bot$. Hence, $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n} \wedge \Box\bot$. Since $A = B = \emptyset$ then, $(\alpha, A) \models_n \boldsymbol{for}_n((\beta, B))$.*

- *Let $A \neq \emptyset \neq B$. By definition $(\alpha, A) \models_n x_1^{\alpha_1} \wedge \ldots \wedge x_n^{\alpha_n}$. Since $\alpha = \beta$ then, $\alpha_i = \beta_i$ for each $i = 1, \ldots, n$, hence, $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n}$. Since, $A \neq \emptyset$ let $\alpha' \in A$ and $(\alpha', \emptyset) \models_n x_1^{\alpha'_1} \wedge \ldots \wedge x_n^{\alpha'_n}$. By our assumption, $A = B$ and $B \neq \emptyset$. Let $\beta' \in \boldsymbol{BIT}_n$ such that $B = \{\beta'\}$. Hence $\alpha' = \beta'$ and $(\alpha', \emptyset) \models_n x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n}$. Then, $(\alpha, A) \models_n \Diamond(x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n})$. Since $(\alpha, A) \models_n \Diamond(x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n})$ and $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n}$ therefore, $(\alpha, A) \models_n \boldsymbol{for}_n((\beta, B))$.*

$(2 \Rightarrow 1)$ *Let $(\alpha, A) \models_n \boldsymbol{for}_n((\beta, B))$. Hence, $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n} \wedge \Box\bot$ where $B = \emptyset$ and $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n} \wedge \Diamond(x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n})$ where $B = \{\beta'\}$. It follows that $A = \emptyset$ where $B = \emptyset$ and $A \neq \emptyset$ where $B \neq \emptyset$. We consider two following cases:*

- *Let $A = \emptyset$ and $B = \emptyset$. Hence $A = B$. It is enough to show $\alpha = \beta$. Since $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n}$ then, $(\alpha, A) \models_n x_i^{\beta_i}$ for each $i = 1, \ldots, n$. Then $\alpha_i = \beta_i$ for each $i = 1, \ldots, n$. Thus, $\alpha = \beta$.*

- *Let $A \neq \emptyset$ and $B \neq \emptyset$. Let $\beta' \in \boldsymbol{BIT}_n$ such that $B = \{\beta'\}$. Let $\alpha' \in \boldsymbol{BIT}_n$ such that $A = \{\alpha'\}$. Since $(\alpha, A) \models_n x_1^{\beta_1} \wedge \ldots \wedge x_n^{\beta_n}$ then $(\alpha, A) \models_n x_i^{\beta_i}$ for $i \in \{1, \ldots, n\}$ hence, $\alpha = \beta$. Since $(\alpha, A) \models_n \Diamond(x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n})$ then $(\alpha', \emptyset) \models_n x_1^{\beta'_1} \wedge \ldots \wedge x_n^{\beta'_n}$ and $\alpha' = \beta'$. Since $\alpha = \beta$ and $\alpha' = \beta'$ therefore, $(\alpha, A) = (\beta, B)$.*

Proposition 22 *Let $(k, \sigma) \in \boldsymbol{SUB}_n$. Let $(\alpha, A) \in \boldsymbol{MOD}_k$. There exists $(\beta, B) \in \boldsymbol{MOD}_n$ such that $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$.*

Proof 67 *We consider two following cases:*

- *Case $A = \emptyset$. Let $\beta \in \boldsymbol{BIT}_n$ be such that for all $i \in \{1, \ldots, n\}$, if $(\alpha, A) \models_k \sigma(x_i)$ then $\beta_i = 1$ else $\beta_i = 0$. Consequently, $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n}$. Since $A = \emptyset$ then, $(\alpha, A) \models_k \Box\bot$. Thus $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Box\bot$. Therefore, $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, \emptyset)))$.*

- *Case $A \neq \emptyset$. Let $\alpha' \in \boldsymbol{BIT}_n$ be such that $A = \{\alpha'\}$. Let $\beta \in \boldsymbol{BIT}_n$ be such that for all $i \in \{1, \ldots, n\}$, if $(\alpha, A) \models_k \sigma(x_i)$ then $\beta_i = 1$ else $\beta_i = 0$. Consequently, $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n}$. Moreover, since $A \neq \emptyset$ let $\beta' \in \boldsymbol{BIT}_n$ be such that $(\alpha', \emptyset) \models_k \sigma(x_1)^{\beta'_1} \wedge \ldots \wedge \sigma(x_n)^{\beta'_n}$. Since $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n}$ then $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, \{\beta'\})))$.*

Proposition 23 *Let $(k, \sigma) \in \boldsymbol{SUB}_n$. Let $(\alpha, A) \in \boldsymbol{MOD}_k$. For all $(\beta, B), (\gamma, C) \in \boldsymbol{MOD}_n$, if $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$ and $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\gamma, C)))$ then $(\beta, B) = (\gamma, C)$.*

Proof 68 *We consider two following cases.*

- *Case $A = \emptyset$. Let $(\beta, B), (\gamma, C) \in \boldsymbol{MOD}_n$ be such that $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$ and $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\gamma, C)))$. Hence, if $B \neq \emptyset$ then $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Diamond(\sigma(x_1)^{\beta'_1} \wedge \ldots \wedge \sigma(x_n)^{\beta'_n})$ where $B = \{\beta'\}$ else $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Box\bot$. If $C \neq \emptyset$ then $(\alpha, A) \models_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n} \wedge \Diamond(\sigma(x_1)^{\gamma'_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma'_n})$ where $C = \{\gamma'\}$ else $(\alpha, A) \models_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n} \wedge \Box\bot$. Since, $A = \emptyset$, therefore $B = \emptyset$, $C = \emptyset$ and for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \models_k \sigma(x_i)^{\beta_i}$ and $(\alpha, A) \models_k \sigma(x_i)^{\gamma_i}$. Thus $\beta = \gamma$. Since $B = C = \emptyset$ and $\beta = \gamma$ therefore $(\beta, B) = (\gamma, C)$.*

- *Case $A \neq \emptyset$. Let $(\beta, B), (\gamma, C) \in \boldsymbol{MOD}_n$ be such that $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\beta, B)))$ and $(\alpha, A) \models_k \sigma(\boldsymbol{for}_n((\gamma, C)))$. Hence, if $B \neq \emptyset$ then $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta}_n \wedge \Diamond(\sigma(x_1)^{\beta'_1} \wedge \ldots \wedge \sigma(x_n)^{\beta'_n})$ where $B = \{\beta'\}$ else $(\alpha, A) \models_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Box\bot$ and if $C \neq \emptyset$ then $(\alpha, A) \models_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n} \wedge \Diamond(\sigma(x_1)^{\gamma'_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma'_n})$ where $C = \{\gamma'\}$ else $(\alpha, A) \models_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n} \wedge \Box\bot$. Since $A \neq \emptyset$ therefore $B \neq \emptyset$ and $C \neq \emptyset$ and for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \models_k \sigma(x_i)^{\beta_i}$ and $(\alpha, A) \models_k \sigma(x_i)^{\gamma_i}$. Hence, $\beta = \gamma$. Moreover $(\alpha', \emptyset) \models_k \sigma(x_1)^{\beta'_1} \wedge \ldots \wedge \sigma(x_n)^{\beta'_n}$ and $(\alpha', \emptyset) \models_k \sigma(x_1)^{\gamma'_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma'_n}$. Hence, $\beta' = \gamma'$. Since $\beta = \gamma$ and $\beta' = \gamma'$ consequently, $(\beta, B) = (\gamma, C)$.*

From proposition 22 and 23, we conclude that for all $(k, \sigma) \in \mathbf{SUB}_n$ and for all $(\alpha, A) \in \mathbf{MOD}_k$, there exists exactly one $(\beta, B) \in \mathbf{MOD}_n$ such that $(\alpha, A) \models_k \sigma(\mathbf{for}_n((\beta, B)))$.
For all $k \geqslant 1$, a (k, n)-morphism is a function f from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that for all $(\alpha, A) \in \mathbf{MOD}_k$ and for all $(\beta, B) \in \mathbf{MOD}_n$, if $f((\alpha, A)) = (\beta, B)$ then

- **Forward condition:** if $A \neq \emptyset$ then $B \neq \emptyset$ and there exists $\alpha' \in \mathbf{BIT}_k$, $\beta' \in \mathbf{BIT}_n$ such that $A = \{\alpha'\}$, $B = \{\beta'\}$ and $f((\alpha', \emptyset)) = (\beta', \emptyset)$.

- **backward condition:** if $B \neq \emptyset$ then $A \neq \emptyset$ and there exists $\alpha' \in \mathbf{BIT}_k$, $\beta' \in \mathbf{BIT}_n$ such that $A = \{\alpha'\}$, $B = \{\beta'\}$ and $f((\alpha', \emptyset)) = (\beta', \emptyset)$.

This kind of morphisms is different from the bounded morphisms usually considered in modal logic (see [18], definition 2.10). In particular, in the above definition, there is no condition related to the valuation of variables. The next result is a good example of what the properties of morphism are like.

Proposition 24 *Let $k \geqslant 1$. Let f be a (k, n)-morphism. Let $(\beta, B) \in \boldsymbol{MOD}_k$ and $(\gamma, C) \in \boldsymbol{MOD}_n$. If the following conditions hold then $f((\beta, B)) = (\gamma, C)$:*

- *for all* $i \in \{1,...,n\}$, $f((\beta,B)) \models_n x_i^{\gamma_i}$,
- *if* $B \neq \emptyset$ *then* $C \neq \emptyset$ *and there exists* $\beta' \in \textbf{\textit{BIT}}_k$ *and* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$,
- *if* $C \neq \emptyset$ *then* $B \neq \emptyset$ *and there exists* $\beta' \in \textbf{\textit{BIT}}_k$ *and* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$.

Proof 69 *Suppose for all* $i \in \{1,...,n\}$, $f((\beta,B)) \models_n x_i^{\gamma_i}$. *Moreover, suppose if* $B \neq \emptyset$ *then* $C \neq \emptyset$ *and there exists* $\beta' \in \textbf{\textit{BIT}}_k$ *and* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$ *and if* $C \neq \emptyset$ *then* $B \neq \emptyset$ *and there exists* $\beta' \in \textbf{\textit{BIT}}_k$ *and* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$. *For the sake of the contradiction, suppose* $f((\beta,B)) \neq (\gamma,C)$. *Let* $(\delta,D) \in \textbf{\textit{MOD}}_n$ *be such that* $f((\beta,B)) = (\delta,D)$. *Consequently,* $(\gamma,C) \neq (\delta,D)$. *Since for all* $i \in \{1,...,n\}$, $f((\beta,B)) \models_n x_i^{\gamma_i}$, *therefore for all* $i \in \{1,...,n\}$, $(\delta,D) \models_n x_i^{\gamma_i}$. *Since for all* $i \in \{1,...,n\}$, $(\delta,D) \models_n x_i^{\delta_i}$ *therefore* $\gamma = \delta$. *Since* $f((\beta,B)) \neq (\gamma,C)$ *and* $f((\beta,B)) = (\delta,D)$, *therefore* $(\gamma,C) \neq (\delta,D)$. *Since* $\gamma = \delta$, *therefore* $C \neq D$. *It follows that either* $C \neq \emptyset$ *or* $D \neq \emptyset$. *We consider the following two cases:*

- $C \neq \emptyset$. *Hence* $B \neq \emptyset$ *and there exists* $\beta' \in \textbf{\textit{BIT}}_k$ *and* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$. *Since* f *is a* $(k,n)-$*morphism and* $f((\beta,B)) = (\delta,D)$, *therefore* $D \neq \emptyset$ *and* $f((\beta',\emptyset)) = (\delta',\emptyset)$ *for some* $\delta' \in \textbf{\textit{BIT}}_n$ *such that* $D = \{\delta'\}$. *Since* $\gamma = \delta$ *and* $f((\beta',\emptyset)) = (\delta',\emptyset)$, *therefore* $(\gamma,C) = (\delta,D)$*: a contradiction.*
- $D \neq \emptyset$. *Since* f *is a* $(k,n)-$*morphism and* $f((\beta,B)) = (\delta,D)$, *therefore* $B \neq \emptyset$ *and* $f((\beta',\emptyset)) = (\delta',\emptyset)$ *for some* $\beta' \in \textbf{\textit{BIT}}_k$ *such that* $B = \{\beta'\}$ *and some* $\delta' \in \textbf{\textit{BIT}}_n$ *such that* $D = \{\delta'\}$. *Thus,* $C \neq \emptyset$ *and* $f((\beta',\emptyset)) = (\gamma',\emptyset)$ *for some* $\gamma' \in \textbf{\textit{BIT}}_n$ *such that* $C = \{\gamma'\}$. *Since* $\gamma = \delta$ *and* $f((\beta',\emptyset)) = (\delta',\emptyset)$, *therefore* $(\gamma,C) = (\delta,D)$*: a contradiction.*

5.4 Main Results

Let $\pi = n$. The next result implies that in $Alt_1 + \Box\Box\bot$, unifiable n-formulas are either finitary or unitary.

Proposition 25 *For all* $\varphi \in \textbf{\textit{FOR}}_n$, *if* φ *is* n*-unifiable then* φ *is* n*-*π*-reasonable.*

Proof 70 *Let $\varphi \in \boldsymbol{FOR}_n$. Suppose φ is n-unifiable. Let (k,σ) be a n-unifier of φ such that $k > \pi$. Hence, $\vDash \sigma(\varphi)$. Moreover, since $n \leqslant \pi$, therefore $k \geqslant n$. Let g be a (k,n)-morphism such that for all $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_k$, if $g((\alpha, A)) = g((\beta, B))$ then for all $i \in \{1,\dots,n\}$, $(\alpha, A) \vDash_k \sigma(x_i)$ iff $(\beta, B) \vDash_k \sigma(x_i)$. The proof of the existence of g is presented in Section* 5.5.
Let f be a surjective (k,n)-morphism such that for all $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_k$, if $f((\alpha, A)) = f((\beta, B))$ then $g((\alpha, A)) = g((\beta, B))$. The proof of the existence of f is presented in Section 5.6.
Let $(n,\tau), (k,\nu)$ be the n-substitution defined by

- $\tau(x_i) = \bigvee\{\boldsymbol{for}_n(f((\alpha, A))) : (\alpha, A) \in \boldsymbol{MOD}_k$ *is such that* $(\alpha, A) \vDash_k \sigma(x_i)\}$,
- $\nu(x_i) = \bigvee\{\boldsymbol{for}_k((\alpha, A)) : (\alpha, A) \in \boldsymbol{MOD}_k$ *is such that* $f((\alpha, A)) \vDash_n x_i\}$,

where i ranges over $\{1,\dots,n\}$. Now, we show that φ is $n-\pi$-reasonable. In this respect, we have to prove Lemmas 43, 44, 45 *and* 46.

In actual fact, the purpose of Lemmas 43, 44, 45 and 46 is to show that (n,τ) is an n-unifier of φ such that $(n,\tau) \preccurlyeq_n (k,\sigma)$.

Lemma 43 *Let $\psi \in \boldsymbol{FOR}_n$. For all $(\beta, B) \in \boldsymbol{MOD}_n$, the following conditions are equivalent:*

1. *there exists $(\alpha, A) \in \boldsymbol{MOD}_k$ such that $f((\alpha, A)) = (\beta, B)$ and $(\alpha, A) \vDash_k \sigma(\psi)$,*
2. *for all $(\alpha, A) \in \boldsymbol{MOD}_k$ if $f((\alpha, A)) = (\beta, B)$ then $(\alpha, A) \vDash_k \sigma(\psi)$,*
3. *$(\beta, B) \vDash_n \tau(\psi)$.*

Proof 71 *By induction on $\psi \in \boldsymbol{FOR}_n$. We consider the following cases $\psi = x_i$ and $\psi = \Box x$.*

- *Let $\psi = x_i$. Let $(\beta, B) \in \boldsymbol{MOD}_n$.*
  $(1 \Rightarrow 2)$ *Suppose $(\alpha, A) \in \boldsymbol{MOD}_k$ is such that $f((\alpha, A)) = (\beta, B) \in \boldsymbol{MOD}_n$ and $(\alpha, A) \vDash_k \sigma(x_i)$. Let $(\gamma, C) \in \boldsymbol{MOD}_k$ be such that $f((\gamma, C)) = (\beta, B)$. Since $f((\alpha, A)) = (\beta, B)$, therefore $f((\alpha, A)) = f((\gamma, C))$. Hence, $g((\alpha, A)) = g((\gamma, C))$. Thus, $(\alpha, A) \vDash_k \sigma(x_i)$ iff $(\gamma, C) \vDash_k \sigma(x_i)$. Since $(\alpha, A) \vDash_k \sigma(x_i)$ therefore $(\gamma, C) \vDash_k \sigma(x_i)$.*
  $(2 \Rightarrow 3)$ *Suppose for all $(\alpha, A) \in \boldsymbol{MOD}_k$ if $f((\alpha, A)) = (\beta, B)$ then $(\alpha, A) \vDash_k \sigma(x_i)$. Since f is surjective therefore let $(\gamma, C) \in \boldsymbol{MOD}_k$ be such that $f((\gamma, C)) = (\beta, B)$. Since for all $(\alpha, A) \in \boldsymbol{MOD}_k$, if $f((\alpha, A)) = (\beta, B)$ then $(\alpha, A) \vDash_k \sigma(x_i)$,*

therefore $(\gamma, C) \vDash_k \sigma(x_i)$. *Consequently* $(\beta, B) \vDash_n \boldsymbol{for}_n(f((\gamma, C))) \rightarrow \tau(x_i)$. *Since* $f((\gamma, C)) = (\beta, B)$, *therefore* $(\beta, B) \vDash_n \boldsymbol{for}_n((\beta, B)) \rightarrow \tau(x_i)$. *Since by Proposition* 22, $(\beta, B) \vDash_n \boldsymbol{for}_n((\beta, B))$, *therefore* $(\beta, B) \vDash_n \tau(x_i)$.
$(3 \Rightarrow 1)$ *Suppose* $(\beta, B) \vDash_n \tau(x_i)$. *Let* $(\alpha, A) \in \boldsymbol{MOD}_k$ *be such that* $(\alpha, A) \vDash_k \sigma(x_i)$ *and* $(\beta, B) \vDash_n \boldsymbol{for}_n(f((\alpha, A)))$. *Such* (α, A) *exists by the definition of* τ. *Hence, by Proposition* 21, $f((\alpha, A)) = (\beta, B)$.

- $\psi = \Box\chi$. *Let* $(\beta, B) \in \boldsymbol{MOD}_n$.
  $(1 \Rightarrow 2)$ *Suppose* $(\alpha, A) \in \boldsymbol{MOD}_k$ *is such that* $f((\alpha, A)) = (\beta, B) \in \boldsymbol{MOD}_n$ *and* $(\alpha, A) \vDash_k \sigma(\Box\chi)$. *Let* $(\gamma, C) \in \boldsymbol{MOD}_k$ *be such that* $f((\gamma, C)) = (\beta, B)$. *Suppose* $(\gamma, C) \nvDash_k \sigma(\Box\chi)$. *Thus,* $C \neq \emptyset$ *and* $(\gamma', \emptyset) \nvDash_k \sigma(\chi)$ *for* $\gamma' \in \boldsymbol{BIT}_k$ *such that* $C = \{\gamma'\}$. *Since* f *is a* (k, n)*-morphism and* $f((\gamma, C)) = (\beta, B)$, *therefore* $B \neq \emptyset$ *and* $f((\gamma', \emptyset)) = (\beta', \emptyset)$ *for some* $\beta' \in \boldsymbol{BIT}_n$ *such that* $B = \{\beta'\}$. *Since* f *is a* (k, n)*-morphism and* $f((\alpha, A)) = (\beta, B)$, *therefore* $A \neq \emptyset$ *and* $f((\alpha', \emptyset)) = (\beta', \emptyset)$ *for some* $\alpha' \in \boldsymbol{BIT}_k$ *such that* $A = \{\alpha'\}$. *Since* $(\gamma', \emptyset) \nvDash_k \sigma(\chi)$ *and* $f((\gamma', \emptyset)) = (\beta', \emptyset)$, *therefore,* $(\alpha', \emptyset) \nvDash_k \sigma(\chi)$. *Hence,* $(\alpha, A) \nvDash_k \sigma(\Box\chi)$*: a contradiction.*
  $(2 \Rightarrow 3)$ *Suppose for all* $(\alpha, A) \in \boldsymbol{MOD}_k$, *if* $f((\alpha, A)) = (\beta, B)$ *then* $(\alpha, A) \vDash_k \sigma(\Box\chi)$. *Suppose* $(\beta, B) \nvDash_k \tau(\Box\chi)$. *Consequently,* $B \neq \emptyset$ *and* $(\beta', \emptyset) \nvDash_n \tau(\chi)$. *Since* f *is surjective, therefore let* $(\gamma, C) \in \boldsymbol{MOD}_k$ *be such that* $f((\gamma, C)) = (\beta, B)$. *Since for all* $(\alpha, A) \in \boldsymbol{MOD}_k$, *if* $f((\alpha, A)) = (\beta, B)$ *then* $(\beta, B) \vDash_k \sigma(\Box\chi)$, *therefore* $(\gamma, C) \vDash_{k\sigma(\Box\chi)}$. *Since* f *is a* (k, n)*-morphism,* $B \neq \emptyset$ *and* $f((\gamma, C)) = (\beta, B)$, *therefore* $C \neq \emptyset$ *and* $f((\gamma', \emptyset)) = (\beta', \emptyset)$ *for* $\gamma' \in \boldsymbol{BIT}_k$ *such that* $C = \{\gamma'\}$. *Since* $(\beta', \emptyset) \nvDash_n \tau(\chi)$, *therefore by induction hypothesis,* $(\gamma', \emptyset) \nvDash_k \sigma(\chi)$. *Thus,* $(\gamma, C) \nvDash_k \sigma(\Box\chi)$*: a contradiction.*
  $(3 \Rightarrow 1)$ *Suppose* $(\beta, B) \vDash_n \tau(\Box\chi)$. *Since* f *is surjective, therefore let* $(\alpha, A) \boldsymbol{MOD}_k$ *be such that* $f((\alpha, A)) = (\beta, B)$. *Suppose* $(\alpha, A) \nvDash_k \sigma(\Box\chi)$. *Consequently,* $A \neq \emptyset$ *and* $(\alpha', \emptyset) \nvDash_k \sigma(\chi)$ *for some* $\alpha' \in \boldsymbol{BIT}_k$ *such that* $A = \{\alpha'\}$. *Since* f *is a* (k, n)*-morphism and* $f((\alpha, A)) = (\beta, B)$, *therefore* $B \neq \emptyset$ *and* $f((\alpha', \emptyset)) = (\beta', \emptyset)$ *for some* $\beta' \in \boldsymbol{BIT}_n$ *such that* $B = \{\beta'\}$. *Since* $(\alpha', \emptyset) \nvDash_k \sigma(\chi)$, *therefore by induction hypothesis,* $(\beta', \emptyset) \nvDash_n \tau(\chi)$. *Hence,* $(\beta, B) \nvDash_n \tau(\Box\chi)$*: a contradiction.*

Lemma 44 *For all* $(\beta, B) \in \boldsymbol{MOD}_k$ *and for all* $i \in \{1, \ldots, n\}$, *the following conditions are equivalent:*

1. $(\beta, B) \vDash_k \nu(x_i)$,
2. $f((\beta, B)) \vDash_n x_i$.

Proof 72 *Let $(\beta, B) \in \boldsymbol{MOD}_k$ and $i \in \{1, ..., n\}$.*
$(1 \Rightarrow 2)$ *Suppose $(\beta, B) \vDash_k \nu(x_i)$. Let $(\alpha, A) \in \boldsymbol{MOD}_k$ be such that $f((\alpha, A)) \vDash_n x_i$ and $(\beta, B) \vDash_k \boldsymbol{for}_k((\alpha, A))$. Such (α, A) exists by the definition of ν. Thus, by proposition 21, $(\beta, B) = (\alpha, A)$. Since $f((\alpha, A)) \vDash_n x_i$, therefore, $f((\beta, B)) \vDash_n x_i$.*
$(2 \Rightarrow 1)$ *Suppose $f((\beta, B)) \vDash_n x_i$. Consequently, by the definition of ν, $(\beta, B) \vDash_k \boldsymbol{for}_k((\beta, B)) \to \nu(x_i)$. Since by Proposition 21 , $(\beta, B) \vDash_k \boldsymbol{for}_k((\beta, B))$, therefore $(\beta, B) \vDash_k \nu(x_i)$.*

Lemma 45 *Let $(\beta, B) \in \boldsymbol{MOD}_k$ and $(\gamma, C) \in \boldsymbol{MOD}_n$. The following conditions are equivalent:*

1. *$f((\beta, B)) = (\gamma, C)$,*
2. *$(\beta, B) \vDash_k \nu(\boldsymbol{for}_n((\gamma, C)))$.*

Proof 73 *Obviously, if $f((\beta, B)) = (\gamma, C)$ then $B = \emptyset$ iff $C = \emptyset$. Similarly, if $(\beta, B) \vDash_k \nu(\boldsymbol{for}_n((\gamma, C)))$ then $B = \emptyset$ iff $C = \emptyset$. For this reason we consider two following cases.*

- *Case $B = \emptyset$ and $C = \emptyset$.*
  $(1 \Rightarrow 2)$ *Suppose $f((\beta, \emptyset)) = (\gamma, \emptyset)$. Since for all $i \in \{1, ..., n\}$, $(\gamma, \emptyset) \vDash_n x_i^{\gamma_i}$, therefore for all $i \in \{1, ..., n\}$, $f((\beta, \emptyset)) \vDash_n x_i^{\gamma_i}$. Thus, for all $i \in \{1, ..., n\}$, by Lemma 44, $(\beta, \emptyset) \vDash_k \nu(x_i)^{\gamma_i}$. Hence, $(\beta, \emptyset) \vDash_k \nu(x_1)^{\gamma_1} \wedge ... \wedge \nu(x_n)^{\gamma_n}$. Since $B = \emptyset$ and $C = \emptyset$, therefore $(\beta, B) \vDash_k \nu(\boldsymbol{for}_n((\gamma, C)))$.*
  $(2 \Rightarrow 1)$ *Suppose $(\beta, \emptyset) \vDash_k \nu(\boldsymbol{for}_n((\gamma, \emptyset)))$. Consequently, $(\beta, \emptyset) \vDash_k \nu(x_1)^{\gamma_1} \wedge ... \wedge \nu(x_n)^{\gamma_n}$. Hence for all $i \in \{1, ..., n\}$, $(\beta, \emptyset) \vDash_k \nu(x_i)^{\gamma_i}$. Thus for all $i \in \{1, ..., n\}$, by Lemma 44, $f((\beta, \emptyset)) \vDash_n x_i^{\gamma_i}$. Since $B = \emptyset$ and $C = \emptyset$, therefore by Proposition 24, $f((\beta, \emptyset)) = (\gamma, \emptyset)$.*

- *Case $B \neq \emptyset$ and $C \neq \emptyset$.*
  $(1 \Rightarrow 2)$ *Suppose $f((\beta, B)) = (\gamma, C)$. Since for all $i \in \{1, ..., n\}$, $(\gamma, C) \vDash_n x_i^{\gamma_i}$, therefore for all $i \in \{1, ..., n\}$, $f((\beta, B)) \vDash_n x_i^{\gamma_i}$. Moreover, since f is a (k, n)-morphism $B \neq \emptyset$ and $C \neq \emptyset$, therefore $f((\beta', \emptyset)) = (\gamma', \emptyset)$ for $\beta' \in \boldsymbol{BIT}_k$, $\gamma' \in \boldsymbol{BIT}_n$ such that $B = \{\beta'\}$, $C = \{\gamma'\}$. Hence for all $i \in \{1, ..., n\}$, by Lemma 44, $(\beta, B) \vDash_k \nu(x_i)^{\gamma_i}$. Moreover, by the first case above, since $f((\beta', \emptyset)) = (\gamma', \emptyset)$, $(\beta', \emptyset) \vDash_k \nu(\boldsymbol{for}_n((\gamma', \emptyset)))$. Consequently, $(\beta, B) \vDash_k \nu(x_1)^{\gamma_1} \wedge ... \wedge \nu(x_n)^{\gamma_n}$. Moreover, $(\beta, B) \vDash_k \Diamond\nu(\boldsymbol{for}_n((\gamma', \emptyset)))$. Thus, $(\beta, B) \vDash_k \nu(\boldsymbol{for}_n((\gamma, C)))$.*
  $(2 \Rightarrow 1)$ *Suppose $(\beta, B) \vDash_k \Diamond\nu(\boldsymbol{for}_n((\gamma, C)))$. since $C \neq \emptyset$ then $(\beta, B) \vDash_k \nu(x_1)^{\gamma_1} \wedge ... \wedge \nu(x_n)^{\gamma_n} \wedge \Diamond\nu(\boldsymbol{for}_n((\gamma', \emptyset)))$ thus, for all $i \in \{1, ..., n\}$, $(\beta, B) \vDash_k$*

$v(x_i)^{\gamma_i}$. *Moreover,* $(\beta',\emptyset) \models_k v(\boldsymbol{for}_n((\gamma',\emptyset)))$ *for* $\beta' \in \boldsymbol{BIT}_k$, $\gamma' \in \boldsymbol{BIT}_n$ *such that* $B = \{\beta'\}$, $C = \{\gamma'\}$. *Thus for all* $i \in \{1,...,n\}$, *by Lemma* 44, $f((\beta,B)) \models_n x_i^{\gamma_i}$. *Moreover, by the first case above since* $(\beta',\emptyset) \models_k v(\boldsymbol{for}_n((\gamma',\emptyset)))$ *then,* $f((\beta',\emptyset)) = (\gamma',\emptyset)$. *Consequently, by Proposition* 24, $f((\beta,B)) = (\gamma,C)$.

Lemma 46 *For all* $(\beta,B) \in \boldsymbol{MOD}_k$ *and for all* $i \in \{1,...,n\}$, *The following conditions are equivalent:*

1. $(\beta,B) \models_k v(\tau(x_i))$,
2. $(\beta,B) \models_k \sigma(x_i)$.

Proof 74 *Let* $(\beta,B) \in \boldsymbol{MOD}_k$ *and* $i \in \{1,...,n\}$.
$(1 \Rightarrow 2)$ *Suppose* $(\beta,B) \models_k v(\tau(x_i))$. *Let* $(\alpha,A) \in \boldsymbol{MOD}_k$ *be such that* $(\alpha,A) \models_k \sigma(x_i)$ *and* $(\beta,B) \models_k v(\boldsymbol{for}_n(f((\alpha,A))))$. *Such* (α,A) *exists by the definition of* τ. *Hence, by Lemma* 45, $f((\beta,B)) = f((\alpha,A))$. *Thus,* $g((\beta,B)) = g((\alpha,A))$. *Since* $(\alpha,A) \models_k \sigma(x_i)$, *therefore* $(\beta,B) \models_k \sigma(x_i)$.
$(2 \Rightarrow 1)$ *Suppose* $(\beta,B) \models_k \sigma(x_i)$. *Consequently, by the definition of* τ, $(\beta,B) \models_k v(\boldsymbol{for}_n(f((\beta,B)))) \to v(\tau(x_i))$. *Since by Lemma* 45, $(\beta,B) \models_k v(\boldsymbol{for}_n(f((\beta,B))))$, *therefore,* $(\beta,B) \models_k v(\tau(x_i))$.

Since $\models \sigma(\varphi)$, therefore by Proposition 19, for all $(\alpha,A) \in \mathbf{MOD}_k$, $(\alpha,A) \models_k \sigma(\varphi)$. Thus by Lemma 43, for all $(\beta,B) \in \mathbf{MOD}_n$, $(\beta,B) \models_n \tau(\varphi)$. Consequently, by Proposition 19, $\models \tau(\varphi)$. Hence, (n,τ) is a n-unifier of φ. Moreover, by Lemma 46, $(n,\tau) \preccurlyeq_n (k,\sigma)$. Since $n \leqslant \pi$, therefore φ is $n-\pi$-reasonable. This is the end of the proof of Proposition 25.
The next result follows from Propositions 15, 17, 19 and 25.

Proposition 26 *For all* $\varphi \in \boldsymbol{FOR}_n$, *if* n*-unifiable then* φ *is* n*-unitary.*

Now, our main result can be state as follows.

Proposition 27 *Unification in* $Alt_1 + \Box\Box\bot$ *is unitary.*

Proof 75 *By Proposition* 26 *and* 18.

5.5 Definition of the function g used in section 5.4

Let $n \geqslant 1$. Let $(k, \sigma) \in \mathbf{SUB}_n$. Now, we define the function g used in Section 5.4 Let g be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that

- $g((\alpha, A))$ is the unique $(\beta, B) \in \mathbf{MOD}_n$ such that $(\alpha, A) \vDash_k \sigma(\mathbf{for}_n((\beta, B)))$,

where (α, A) ranges over $\mathbf{MOD}_k$. Notice that by Propositions 22 and 23, g is well-defined. Propositions 28 and 29 show that g possesses the properties required in Section 5.4.

Proposition 28 *g is a $(k, n)-$morphism.*

Proof 76 *Let $(\alpha, A) \in \boldsymbol{MOD}_k$ and $(\beta, B) \in \boldsymbol{MOD}_n$ be such that $g((\alpha, A)) = (\beta, B)$. Hence, $(\alpha, A) \vDash_k \sigma(\boldsymbol{for}_n((\beta, B)))$. Thus, if $B \neq \emptyset$ then $(\alpha, A) \vDash_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Diamond\sigma(\boldsymbol{for}_n((\beta', \emptyset)))$ where $\beta' \in \boldsymbol{BIT}_n$ is such that $B = \{\beta'\}$ else $(\alpha, A) \vDash_k \sigma(x_1)^{\beta_1} \wedge \ldots \wedge \sigma(x_n)^{\beta_n} \wedge \Box\bot$. Consequently, if $A \neq \emptyset$ then $B \neq \emptyset$ and $(\alpha', \emptyset) \vDash_k \sigma(\boldsymbol{for}_n((\beta', \emptyset)))$, (where $\alpha' \in \boldsymbol{BIT}_k$ is such that $A = \{\alpha'\}$) and $g((\alpha', \emptyset)) = (\beta', \emptyset)$. Moreover, if $B \neq \emptyset$ then $A \neq \emptyset$ and $(\alpha', \emptyset) \vDash_k \sigma(\boldsymbol{for}_n((\beta', \emptyset)))$, i.e. $g((\alpha', \emptyset)) = (\beta', \emptyset)$.*

Proposition 29 *For all $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_k$, if $g((\alpha, A)) = g((\beta, B))$ then for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \vDash_k \sigma(x_i)$ iff $(\beta, B) \vDash_k \sigma(x_i)$.*

Proof 77 *Let $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_k$. Suppose $g((\alpha, A)) = g((\beta, B))$. Hence, let $(\gamma, C) \in \boldsymbol{MOD}_n$ be such that $g((\alpha, A)) = (\gamma, C)$ and $g((\beta, B)) = (\gamma, C)$. Thus, $(\alpha, A) \vDash_k \sigma(\boldsymbol{for}_n((\gamma, C)))$ and $(\beta, B) \vDash_k \sigma(\boldsymbol{for}_n((\gamma' C)))$. Consequently, $(\alpha, A) \vDash_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n}$ and $(\beta, B) \vDash_k \sigma(x_1)^{\gamma_1} \wedge \ldots \wedge \sigma(x_n)^{\gamma_n}$. Hence, for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \vDash_k \sigma(x_i)^{\gamma_i}$ and $(\beta, B) \vDash_k \sigma(x_i)^{\gamma_i}$. Thus, for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \vDash_k \sigma(x_i)$ iff $(\beta, B) \vDash_k \sigma(x_i)$.*

5.6 Definition of the function f used in section 5.4

Let $n \geqslant 1$. Let $(k, \sigma) \in \mathbf{SUB}_n$ be such that $k \geqslant n$. Let g be a (k, n)-morphism such that for all $(\alpha, A), (\beta, B) \in \mathbf{MOD}_k$, if $g((\alpha, A)) = g((\beta, B))$ then for all $i \in \{1, \ldots, n\}$, $(\alpha, A) \vDash_k \sigma(x_i)$ iff $(\beta, B) \vDash_k \sigma(x_i)$. The proof of existence of g has been presented in Section 5.4. In order to define the function f used in Section 5.4, we need define the function f_0 and f_1 such that f_0 is a function from $\mathbf{MOD}_k^{=\emptyset}$

to $\mathbf{MOD}_n^{=\emptyset}$ and f_1 is a function from $\mathbf{MOD}_k^{\neq\emptyset}$ to $\mathbf{MOD}_n^{\neq\emptyset}$. Firstly, we define the function f_0 and then we define the function f_1 based on f_0. Let $U = \{g((\alpha,\emptyset)) : (\alpha,\emptyset) \in \mathbf{MOD}_k^{=\emptyset}\}$. By Proposition 20, $U \subseteq \mathbf{MOD}_n^{=\emptyset}$. Let h be a function from U to $\mathbf{MOD}_k^{=\emptyset}$ such that $g(h(g((\alpha,\emptyset)))) = g((\alpha,\emptyset))$. Obviously, h is injective. Hence we have, $\parallel U \parallel = \parallel \{h(g((\alpha,\emptyset))) : \alpha \in \mathbf{BIT}_k\} \parallel$. Since $k \geqslant n$, therefore, $\parallel \mathbf{MOD}_n^{=\emptyset} \backslash U \parallel \leqslant \parallel \mathbf{MOD}_k^{=\emptyset} \backslash \{h(g((\alpha,\emptyset))) : \alpha, \in \mathbf{BIT}_k\} \parallel$. Let S be a subset of $\mathbf{MOD}_k^{=\emptyset} \backslash \{h(g((\alpha,\emptyset))) : \alpha \in \mathbf{BIT}_k\}$ such that $\parallel S \parallel = \parallel \mathbf{MOD}_n^{=\emptyset} \backslash U \parallel$. Let f_0^* be a one-to-one correspondence between S and $\mathbf{MOD}_n^{=\emptyset} \backslash U$.
Now, we define the function f_0. Let f_0 be the function from $\mathbf{MOD}_k^{=\emptyset}$ to $\mathbf{MOD}_n^{=\emptyset}$ such that

- if $(\alpha,\emptyset) \in S$ then $f_0((\alpha,\emptyset)) = f_0^*((\alpha,\emptyset))$ else $f_0((\alpha,\emptyset)) = g((\alpha,\emptyset))$,

where $(\alpha,\emptyset)$ ranges over $\mathbf{MOD}_k^{=\emptyset}$. Lemma 47 and 48 show that f_0 possesses interesting properties as follows.

Lemma 47 *f_0 is surjective*

Proof 78 *Let $(\beta,\emptyset) \in \boldsymbol{MOD}_n^{=\emptyset}$. We consider the following two cases:*

- *$(\beta,\emptyset) \in \boldsymbol{MOD}_n^{=\emptyset} \backslash U$. Since f_0^* is one-to-one, therefore let $(\alpha,\emptyset) \in \boldsymbol{MOD}_k^{=\emptyset}$ be such that $(\alpha,\emptyset) \in S$ and $f_0^*((\alpha,\emptyset)) = (\beta,\emptyset)$. Consequently, $f_0((\alpha,\emptyset)) = f_0^*((\alpha,\emptyset))$. Since $f_0^*((\alpha,\emptyset)) = (\beta,\emptyset)$, therefore $f_0((\alpha,\emptyset)) = (\beta,\emptyset)$.*

- *$(\beta,\emptyset) \notin \boldsymbol{MOD}_n^{=\emptyset} \backslash U$. Thus, $(\beta,\emptyset) \in U$. Consequently, let $(\alpha,\emptyset) \in \boldsymbol{MOD}_k^{=\emptyset}$ be such that $g((\alpha,\emptyset)) = (\beta,\emptyset)$ and $(\alpha,\emptyset) = h((\beta,\emptyset))$. Hence, $f_0((\alpha,\emptyset)) = g((\alpha,\emptyset))$. Since $g((\alpha,\emptyset)) = (\beta,\emptyset)$, therefore $f_0((\alpha,\emptyset)) = (\beta,\emptyset)$.*

Lemma 48 *For all $(\alpha,\emptyset),(\beta,\emptyset) \in \boldsymbol{MOD}_k^{=\emptyset}$, if $f_0((\alpha,\emptyset)) = f_0((\beta,\emptyset))$ then $g((\alpha,\emptyset)) = g((\beta,\emptyset))$.*

Proof 79 *Let $(\alpha,\emptyset),(\beta,\emptyset) \in \boldsymbol{MOD}_k^{=\emptyset}$. Suppose $f_0((\alpha,\emptyset)) = f_0((\beta,\emptyset))$. We consider the following three cases.*

- *$(\alpha,\emptyset) \in S$ and $(\beta,\emptyset) \in S$. Hence, $f_0((\alpha,\emptyset)) = f_0^*((\alpha,\emptyset))$ and $f_0((\beta,\emptyset)) = f_0^*((\beta,\emptyset))$. Since $f_0((\alpha,\emptyset)) = f_0((\beta,\emptyset))$, therefore $f_0^*((\alpha,\emptyset)) = f_0^*((\beta,\emptyset))$. Since f_0^* is one-to-one, therefore $(\alpha,\emptyset) = (\beta,\emptyset)$. Consequently, $g((\alpha,\emptyset)) = g((\beta,\emptyset))$.*

- *$(\alpha,\emptyset) \in S$ and $(\beta,\emptyset) \notin S$. Hence, $f_0((\alpha,\emptyset)) = f_0^*((\alpha,\emptyset))$ and $f_0((\beta,\emptyset)) = g((\beta,\emptyset))$. Since $f_0((\alpha,\emptyset)) = f_0((\beta,\emptyset))$, therefore $f_0^*((\alpha,\emptyset)) = g((\beta,\emptyset))$. Since $f_0^*((\alpha,\emptyset)) \in \boldsymbol{MOD}_n^{=\emptyset}\setminus U$ and $g((\beta,\emptyset)) \in U$, therefore $\boldsymbol{MOD}_n^{=\emptyset}\setminus U$ and U do not make a partition of $\boldsymbol{MOD}_n^{=\emptyset}$: a contradiction.*

- *$(\alpha,\emptyset) \notin S$ and $(\beta,\emptyset) \notin S$. Hence, $f_0((\alpha,\emptyset)) = g((\alpha,\emptyset))$ and $f_0((\beta,\emptyset)) = g((\beta,\emptyset))$. Since $f_0((\alpha,\emptyset)) = f_0((\beta,\emptyset))$, therefore $g((\alpha,\emptyset)) = g((\beta,\emptyset))$.*

The surjective function f_0 from $\mathbf{MOD}_k^{=\emptyset}$ to $\mathbf{MOD}_n^{=\emptyset}$ has been defined such that for all $(\alpha',\emptyset),(\beta',\emptyset) \in \mathbf{MOD}_n^{=\emptyset}$, if $f_0((\alpha',\emptyset)) = f_0((\beta',\emptyset))$ then $g((\alpha',\emptyset)) = g((\beta',\emptyset))$. For $\delta' \in \mathbf{BIT}_n$, let $S((\delta',\emptyset)) = \{(\beta,\{\beta'\}) : \beta,\beta' \in \mathbf{BIT}_k$, and $f_0((\beta',\emptyset)) = (\delta',\emptyset)\}$ and $T((\delta',\emptyset)) = \{(\epsilon,\{\epsilon'\}) : \epsilon,\epsilon' \in \mathbf{BIT}_n, \epsilon' = \delta'\}$. Notice that For $\delta' \in \mathbf{BIT}_n$, $S((\delta',\emptyset)) \subseteq \mathbf{MOD}_k^{\neq\emptyset}$ and $T((\delta',\emptyset)) \subseteq \mathbf{MOD}_n^{\neq\emptyset}$. Also notice that $\parallel T((\delta',\emptyset)) \parallel = 2^n$. For $\delta' \in \mathbf{BIT}_n$ let $\sim_{(\delta',\emptyset)}$ be the equivalence relation on $S((\delta',\emptyset))$ such that

- $(\beta,B) \sim_{(\delta',\emptyset)} (\gamma,C)$ iff $g((\beta,B)) = g((\gamma,C))$,

where $(\beta,B),(\gamma,C)$ range over $S((\delta',\emptyset))$. The next result will allow us to use Proposition 14.

Proposition 30 *For all $\delta' \in \boldsymbol{BIT}_n$,*

1. *$\parallel S((\delta',\emptyset))/\sim_{(\delta',\emptyset)} \parallel \leqslant \parallel T((\delta',\emptyset)) \parallel$,*
2. *$\parallel T((\delta',\emptyset)) \parallel \leqslant \parallel S((\delta',\emptyset)) \parallel$.*

Proof 80 *Let $\delta' \in \boldsymbol{BIT}_n$. Obviously, $\parallel T((\delta',\emptyset)) \parallel = 2^n$.*

1. *For the sake of contradiction, suppose $\parallel S((\delta',\emptyset))/\sim_{(\delta',\emptyset)} \parallel > \parallel T((\delta',\emptyset)) \parallel$. Let $p \in \mathbb{N}$ and $(\beta^1,\beta'^1),\ldots,(\beta^p,\beta'^p) \in S((\delta',\emptyset))$ be such that $p > \parallel T((\delta',\emptyset)) \parallel$ and for all $q,r \in \mathbb{N}$, if $1 \leqslant q$, $r \leqslant p$ and $q \neq r$ then $(\beta^q,\beta'^q) \nsim_{(\delta',\emptyset)} (\beta^r,\beta'^r)$. Thus, $f_0((\beta'^1,\emptyset)) = (\delta',\emptyset),\ldots,f_0((\beta'^p,\emptyset)) = (\delta',\emptyset)$. Consequently, let $\epsilon' \in \boldsymbol{BIT}^n$ be such that $g((\beta'^1,\emptyset)) = (\epsilon',\emptyset),\ldots,g((\beta'^p,\emptyset)) = (\epsilon',\emptyset)$. Since g is a (k,n)-morphism, therefore let $\epsilon^1,\ldots,\epsilon^p \in \boldsymbol{BIT}_n$ be such that $g((\beta^1,\beta'^1)) = (\epsilon^1,\epsilon'),\ldots,g((\beta^p,\beta'^p)) = (\epsilon^p,\epsilon')$. Since for all $q,r \in \mathbb{N}$, if $1 \leqslant q$, $r \leqslant p$ and $p \neq r$ then $(\beta^q,\beta'^p) \nsim_{(\delta',\emptyset)} (\beta^r,\beta'^r)$, thus for all $q,r\mathbb{N}$, if $1 \leqslant q$, $r \leqslant p$ and $q \neq r$ then $g((\beta^q,\beta'^q)) \neq g((\beta^r,\beta'^r))$. Since $g((\beta^1,\beta'^1)) = (\epsilon^1,\epsilon'),\ldots,g((\beta^p,\beta'^p)) = (\epsilon^p,\epsilon')$, thus for all $q,r \in \mathbb{N}$, if $1 \leqslant q$, $r \leqslant p$ and $q \neq r$ then $\epsilon^p \neq \epsilon^r$. Hence $p \leqslant 2^n$. Since $\parallel T((\delta',\emptyset)) \parallel = 2^n$, therefore $p \leqslant \parallel T((\delta',\emptyset)) \parallel$: a contradiction.*

2. *Since f_0 is surjective, therefore obviously, $\| S((\delta',\emptyset)) \| \geqslant 2^k$. Since $k \geqslant n$ and $\| T((\delta',\emptyset)) \| = 2^n$, therefore $\| T((\delta',\emptyset)) \| \leqslant \| S((\delta',\emptyset)) \|$.*

Hence, for all $\delta' \in \mathbf{BIT}_n$, by Proposition 14 and 30, let $f_1^{(\delta',\emptyset)}$ be a surjective function from $S((\delta',\emptyset))$ to $T((\delta',\emptyset))$ such that for all $(\beta,B),(\gamma,C) \in S((\delta',\emptyset))$, if $f_1^{(\delta',\emptyset)}((\beta,B)) = f_1^{(\delta',\emptyset)}((\gamma,C))$ then $(\beta,B) \sim_{(\delta',\emptyset)} (\gamma,C)$.

Now, we define the function f_1. Let f_1 be the function from $\mathbf{MOD}_k^{\neq\emptyset}$ to $\mathbf{MOD}_n^{\neq\emptyset}$ such that

- $f_1((\beta,B)) = f_1^{f_0(\beta',\emptyset)}((\beta,B))$,

where (β,B) ranges over$\mathbf{MOD}_k^{\neq\emptyset}$ and $\beta' \in \mathbf{BIT}_k$ is such that $B = \{\beta'\}$. Lemma 49 and 50 show that f_1 possesses interesting properties.

Lemma 49 *f_1 is surjective.*

Proof 81 *Let $(\delta,D) \in \boldsymbol{MOD}_n^{\neq\emptyset}$. Let $\delta' \in \boldsymbol{BIT}_n$ is such that $D = \{\delta'\}$. Hence, $(\delta,D) \in T((\delta',\emptyset))$. Since $f_1^{(\delta',\emptyset)}$ is surjective, therefore let $(\beta,B) \in S((\delta',\emptyset))$ be such that $f_1^{(\delta',\emptyset)}((\beta,B)) = (\delta,D)$. Let $\beta' \in \boldsymbol{BIT}_k$ is such that $B = \{\beta'\}$. Thus, $f_0((\beta',\emptyset)) = (\delta',\emptyset)$. Moreover, $f_1((\beta,B)) = f_1^{f_0((\beta',\emptyset))}((\beta,B))$. Consequently, $f_1((\beta,B)) = f_1^{(\delta',\emptyset)}((\beta,B))$. Since $f_1^{(\delta',\emptyset)}((\beta,B)) = (\delta,D)$, Therefore $f_1((\beta,B)) = (\delta,D)$.*

Lemma 50 *For all $(\alpha,A),(\beta,B) \in \boldsymbol{MOD}_k^{\neq\emptyset}$, if $f_1((\alpha,A)) = f_1((\beta,B))$ then $g((\alpha,A)) = g((\beta,B))$.*

Proof 82 *Let $(\alpha,A),(\beta,B) \in \boldsymbol{MOD}_k^{\neq\emptyset}$. Suppose $f_1((\alpha,A)) = f_1((\beta,B))$. Let $\alpha' \in \boldsymbol{BIT}_k$ be such that $A = \{\alpha'\}$. Let $\beta' \in \boldsymbol{BIT}_k$ be such that $B = \{\beta'\}$. Hence $f_1((\alpha,A)) = f_1^{f_0((\alpha',\emptyset))}((\alpha,A))$ and $f_1((\beta,B)) = f_1^{f_0((\beta',\emptyset))}((\beta,B))$. Since $f_1((\alpha,A)) = f_1((\beta,B))$ therefore $f_1^{f_0((\alpha',\emptyset))}((\alpha,A)) = f_1^{f_0((\beta',\emptyset))}((\beta,B))$. Let $(\gamma',\emptyset),(\delta',\emptyset) \in \boldsymbol{MOD}_n^{=\emptyset}$ be such that $f_0((\alpha',\emptyset)) = (\gamma',\emptyset)$ and $f_0((\beta',\emptyset)) = (\delta',\emptyset)$. Since $f_1^{f_0((\alpha',\emptyset))}((\alpha,A)) = f_1^{f_0((\beta',\emptyset)}((\beta,B))$, therefore $f_1^{(\gamma',\emptyset)}((\alpha,A)) = f_1^{(\delta',\emptyset)}((\beta,B))$. Since $f_1^{(\gamma',\emptyset)}((\alpha,A)) \in T((\gamma',\emptyset))$ and $f_1^{(\delta',\emptyset)}((\beta,B)) \in T((\delta',\emptyset))$, therefore $(\gamma',\emptyset) = (\delta',\emptyset)$. Since $f_1^{(\gamma',\emptyset)}((\alpha,A)) = f_1^{(\delta',\emptyset)}((\beta,B))$, therefore $(\alpha,A) \sim_{(\gamma',\emptyset)} (\beta,B)$ and $(\alpha,A) \sim_{(\delta',\emptyset)} (\beta,B)$. Consequently, $g((\alpha,A)) = g((\beta,B))$.*

Now, we define the function f used in Section 5.4 Let f be the function from $\mathbf{MOD}_k$ to $\mathbf{MOD}_n$ such that

- $f((\beta,B)) = f_0((\beta,\emptyset))$ when $B = \emptyset$,

- $f((\beta, B)) = f_1((\beta, B))$ when $B \neq \emptyset$.

where (β, B) ranges over $\mathbf{MOD}_k$. Propositions 31-33 show that f possesses the properties required in Section ...

Proposition 31 *f is a $(k, n)-$morphism.*

Proof 83 *Suppose f is not a $(k, n)-$morphism. let $(\alpha, A) \in \boldsymbol{MOD}_k$ and $(\beta, B) \in \boldsymbol{MOD}_n$ be such that $f((\alpha, A)) = (\beta, B)$ and either forward condition does not hold, or backward condition does not hold. In the former case, $A \neq \emptyset$ and $B = \emptyset$, or there exists $\alpha' \in \boldsymbol{BIT}_k$, $\beta' \in \boldsymbol{BIT}_n$ such that $A = \{\alpha'\}$, $B = \{\beta'\}$ and $f((\alpha', \emptyset)) \neq (\beta', \emptyset)$. Since $A \neq \emptyset$ and $f((\alpha, A)) = (\beta, B)$, then $B \neq \emptyset$. Thus let $\alpha' \in \boldsymbol{BIT}_k$, $\beta' \in \boldsymbol{BIT}_n$ such that $A = \{\alpha'\}$, $B = \{\beta'\}$ and $f_0((\alpha', \emptyset)) \neq (\beta', \emptyset)$. Since $f((\alpha', A)) = (\beta', B)$ then $f_1^{f_0((\alpha', \emptyset))}((\alpha, \{\alpha'\})) = (\beta, \{\beta'\})$. Then $(\beta, \{\beta'\}) \in T(f_0((\alpha', \emptyset)))$, then $f_0((\alpha', \emptyset)) = (\beta', \emptyset)$: a contradiction.*
In the latter case, $B \neq \emptyset$ and $A = \emptyset$ or there exists $\alpha' \in \boldsymbol{BIT}_k$, $\beta' \in \boldsymbol{BIT}_n$ such that $A = \{\alpha'\}$, $B = \{\beta'\}$ and $f((\alpha', \emptyset)) \neq (\beta', \emptyset)$. Since $B \neq \emptyset$ and $f((\alpha, A)) \neq (\beta, B)$ then $A \neq \emptyset$. And the rest of the argument is similar to the one used in the former case.

Proposition 32 *For all $(\alpha, A), (\beta, B) \in \boldsymbol{MOD}_k$, if $f((\alpha, A)) = f((\beta, B))$ then $g((\alpha, A)) = g((\beta, B))$.*

Proof 84 *By Lemmas* 48 *and* 50.

Proposition 33 *f is surjective.*

Proof 85 *By Lemmas* 47 *and* 49.

In this Chapter, we have shown that $Alt_1 + \Box\Box\bot$ is unitary (Proposition 27). The adaptation of this proof to $K + \Box\Box\bot$ (showing $K + \Box\Box\bot$) will be presented during the workshop UNIF 2020.

6 Unification in Fusion of Two Modal Logics

Contents

The fusion $L_1 \otimes L_2$ of two normal modal logics L_1 and L_2 formulated in languages $\mathscr{L}_1$ and $\mathscr{L}_2$ with disjoint sets of modal operators is the smallest normal modal logic containing $L_1 \cup L_2$. It is easy to see that if each L_i is axiomatized by a set $\sum_i$ of axioms (written in the respective language) then $L_1 \otimes L_2$ is axiomatized by the union $\sum_1 \cup \sum_2$. Fusion of modal logics are everywhere in computer science and artificial intelligence. For instance Public Announcement Logic is like a fusion of finitary many $S5$ logics. In this chapter we consider some fusions of two modal logics and discuss about their unification type.

6.1 Syntax

Let VAR be a countable set of atomic formulas called variables (denoted x, y, ...). Formula of modal languages $\mathscr{L}_1$ and $\mathscr{L}_2$ are respectively defined as follows

$$\varphi ::= \; x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box_1\varphi,$$
$$\varphi ::= \; x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box_2\varphi.$$

Definition 33 *Formulas of the fusion $\mathscr{L}$ of $\mathscr{L}_1$ and $\mathscr{L}_2$ are given by the following rule*

$$\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box_1\varphi \mid \Box_2\varphi.$$

As usual, the rest of the connectives are defined from the ones given. In this case, we have $\Diamond_1\varphi ::= \neg\Box_1\neg\varphi$ and $\Diamond_2\varphi ::= \neg\Box_2\neg\varphi$.

Definition 34 *Let L_1 be a normal modal logic in $\mathscr{L}_1$ and L_2 be a normal modal logic in $\mathscr{L}_2$. The fusion of L_1 and L_2 (denoted $L_1 \otimes L_2$) is the least normal modal logic in $\mathscr{L}$ containing L_1 and L_2.*

A number of transfer results have been obtained. For instance, if L_1 is decidable and L_2 is decidable then $L_1 \otimes L_2$ is decidable [38] and [53]. For us, in this chapter, it will be important to remember that when L_1 is consistent and L_2 is consistent, the fusion $L_1 \otimes L_2$ is a conservative extension of L_1 and L_2 respectively, that is to say: for all $i \in \{1,2\}$ and for all formulas φ in $\mathscr{L}_i$, $\varphi \in L_1 \otimes L_2$ iff $\varphi \in L_i$.

6.2 Semantic

In this Section, we will see Semantics of fusion of two modal logic $L_1 \otimes L_2$.

Definition 35 *A Frame $\mathscr{F}$ for $\mathscr{L}$ is a triple $\langle W, R_1, R_2\rangle$ where W is a non-empty set of possible worlds and R_1 and R_2 are binary relations on W.*

Definition 36 *A model $\mathscr{M}$ is a structure (W, R_1, R_2, v), where*
• W is a set of possible worlds,
• R_1 and R_2 are binary relations on W to evaluate $\Box_1$ and $\Box_2$ respectively and
• v is a function $v : W \to \mathscr{P}(Var)$.

We define the notion of a formula φ being true in model $\mathscr{M} = (W, R_1, R_2, v)$ at a world $w \in W$ (in symbols $\mathscr{M}, w \models \varphi$) as follows:

- $\mathscr{M}, w \models x$ iff $w \in V(x)$,
- $\mathscr{M}, w \not\models \bot$,
- $\mathscr{M}, w \models \neg\varphi$ iff $\mathscr{M}, w \not\models \varphi$,

- $\mathcal{M}, w \models \varphi \vee \psi$ iff either $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models \Box_1\varphi$ iff for all $w' \in W$, if wR_1w' then, $\mathcal{M}, w' \models \varphi$,
- $\mathcal{M}, w \models \Box_2\varphi$ iff for all $w' \in W$, if wR_2w' then, $\mathcal{M}, w' \models \varphi$.

As a result,

- $\mathcal{M}, w \models \Diamond_1\varphi$ iff there exists $w' \in W$ such that wR_1w' and $\mathcal{M}, w' \models \varphi$,
- $\mathcal{M}, w \models \Diamond_2\varphi$ iff there exists $w' \in W$ such that wR_2w' and $\mathcal{M}, w' \models \varphi$.

Example 11 *Consider the formula $\varphi = \Diamond_1(x \wedge \Box_2 y)$. Let $\mathcal{M} = \langle W, R_1, R_2, v\rangle$ be a model of $K \otimes K$. $\mathcal{M}$ satisfies $\Diamond_1(x \wedge \Box_2 y)$ at a world $w_0 \in W$ iff there exists $w_1 \in W$ such that $w_0 R_1 w_1$ and $\mathcal{M}$ satisfies $x \wedge \Box_2 y$ at w_1. But this means $w_1 \in v(x)$ and $w_1 \in v(\Box_2 y)$. $\mathcal{M}$ satisfies $\Box_2 v(y)$ at w_1 iff for every $w_2 \in W$ such that $w_1 R_2 w_2$ we have $\mathcal{M}$ satisfies $v(y)$ at w_2.*

6.3 Unification Type in fusion $K_1 \otimes K_2$

Dzik proved that the fusion $K_1 \otimes K_2$ of K with itself provides the rule of disjunction [22]. In this section, we mention shortly about the rule of disjunction in the fusion $K_1 \otimes K_2$.

Definition 37 *Let L_1, L_2 be normal modal logics in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. The fusion $L = L_1 \otimes L_2$ provides the rule of disjunction if the following condition hold for any $A_1, A_2 \in \mathcal{L}$:*

- $\vdash \Box_1 A_1 \vee \Box_2 A_2$ *then,* $\vdash A_i$ *for some* $i \in \{1, 2\}$.

At the below Lemma, we claim that fusions $K_1 \otimes K_2$ satisfies the rule of disjunction.

Theorem 7 *The fusion $K_1 \otimes K_2$ provides the rule of disjunction.*

- $\vdash_{K_1 \otimes K_2} \Box_1 A \vee \Box_2 B \Rightarrow \vdash_{K_1 \otimes K_2} A$ *or* $\vdash_{K_1 \otimes K_2} B$

Proof 86 *Suppose $\nvdash_{K_1 \otimes K_2} A$ and $\nvdash_{K_1 \otimes K_2} B$. We have to show that $\nvdash_{K_1 \otimes K_2} \Box_1 A_1 \vee \Box_2 B$. Let $\mathcal{M}_1 = \langle W_1, R_1', R_2', v_1\rangle$ and $\mathcal{M}_2 = \langle W_2, R_1'', R_2'', v_2\rangle$ be $K_1 \otimes K_2$-models. Let $t_1 \in W_1$ and $s_1 \in W_2$ such that $\mathcal{M}_1, t_1 \not\models A$ and $\mathcal{M}_2, s_1 \not\models B$. Let us construct the model $\mathcal{M} = \langle W, R_1, R_2, v\rangle$ which is the disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$ together with a new state w_0. We define the model as follows:*

- $W = W_1 \cup W_2 \cup \{w_0\}$,
- $R_1 = R_1' \cup R_1'' \cup \{(w_0, t_1)\}$,
- $R_2 = R_2' \cup R_2'' \cup \{(w_0, s_1)\}$ *and*
- $v = v_1 \cup v_2$.

Obviously, the sub-model of $\mathcal{M}$ generated from t_1 is equal to the sub-model of $\mathcal{M}_1$ generated from t_1 and the sub-model of $\mathcal{M}$ generated from s_1 is equal to the sub-model of $\mathcal{M}_2$ generated from s_1. Since, $\mathcal{M}_1, t_1 \nvDash A$ then, $\mathcal{M}, t_1 \nvDash A$. Since, $\mathcal{M}_2, s_1 \nvDash B$ then, $\mathcal{M}, s_1 \nvDash B$. Since $w_0 R_1 t_1$ and $\mathcal{M}, t_1 \nvDash A$ then, $\mathcal{M}, t_1 \nvDash \Box_1 A$. Since $w_0 R_2 s_1$ and $\mathcal{M}, s_1 \nvDash B$ then, $\mathcal{M}, s_1 \nvDash \Box_2 B$. Then, $\mathcal{M}, w_0 \nvDash \Box_1 A \vee \Box_2 B$. Therefore, $\nvdash_{K_1 \otimes K_2} \Box_1 A \vee \Box_2 B$.

Since we know that some logics providing the rule of disjunction (for example K and $K4$) does not possess a unitary unification type. For example consider the formula $\Box_1 x \vee \Box_2 \neg x$. This formula has unifiers

- $\sigma_1(x) = \top$,
- $\sigma_2(x) = \bot$.

and there is no unifier more general than both of them because if
$\vdash_{K_1 \otimes K_2} \Box_1 \sigma(x) \vee \Box_2 \neg \sigma(x)$
then either $\vdash_{K_1 \otimes K_2} \sigma(x)$ (so that σ is equivalent to σ_1) or $\vdash_{K_1 \otimes K_2} \neg \sigma(x)$ (so that σ is equivalent to σ_2). Thus this formula has no most general unifier.

Theorem 8 *Unification type of the fusion $K_1 \otimes K_2$ is not unitary.*

Before discussing on unification type of fusion $K_1 \otimes K_2$ we consider a general form of logic $L_1 \otimes L_2$ when L_1 has nullary unification type and L_2 is a consistent modal logic. Then we show unification type of fusion $L_1 \otimes L_2$ is not finitary and not unitary.
Consider two unimodal logics **Triv** $= K + \{\Box p \leftrightarrow p\}$ and **Ver** $= K + \{\Box p\}$. D. Makinson proved a property of consistent unimodal logics[48]. This well-known property is as follows:

- If unimodal logic L is consistent then $L \subseteq$ **Triv** or $L \subseteq$ **Ver**.

For instance, $S5 \subseteq$ **Triv**, $S5 \subsetneq$ **Ver**, $K4 \subseteq$ **Triv** and $K4 \subseteq$ **Ver**. We define a translation t from the language $\mathcal{L}$ to the language $\mathcal{L}_1$ as:

Definition 38 *Since L_2 is consistent, we have $L_2 \subseteq$ **Triv**$_2$ or $L_2 \subseteq$ **Ver**$_2$. Let $t : \mathscr{L} \to \mathscr{L}_1$ be a function defined as follows:*

- $t(x) = x$,
- $t(p) = p$,
- $t(\bot) = \bot$,
- $t(\neg\varphi) = \neg\, t(\varphi)$,
- $t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$,
- $t(\Box_1\varphi) = \Box_1 t(\varphi)$,
- $t(\Box_2\varphi) = t(\varphi)$, *when* $L_2 \subseteq \mathrm{Triv}_2$.
- $t(\Box_2\varphi) = \top$ *when* $L_2 \subseteq \mathrm{Ver}_2$.

The below lemmas show that if L_1 is nullary and L_2 is consistent then $L_1 \otimes L_2$ is not unitary and not finitary.

Lemma 51 *Let $L_1 \subseteq \mathscr{L}_1$ and $L_2 \subseteq \mathscr{L}_2$ be normal modal logics. if L_1 is nullary and L_2 is consistent then $L_1 \otimes L_2$ is not unitary.*

Proof 87 *Suppose L_1 is nullary and L_2 is consistent. Suppose $L_1 \otimes L_2$ is unitary. Since, L_1 is nullary, therefore let $\varphi \in \mathscr{L}_1$ be such that φ is L_1-unifiable and φ has no minimal complete set of L_1-unifiers. Let σ be an $\mathscr{L}_1$-substitution such that $\sigma(\varphi) \in L_1$. Since the fusion $L_1 \otimes L_2$ contains both L_1 and L_2 hence, $\sigma(\varphi) \in L_1 \otimes L_2$. Thus, φ is $(L_1 \otimes L_2)$-unifiable. Since $L_1 \otimes L_2$ is unitary, Let τ be an $\mathscr{L}$-substitution such that τ is an $(L_1 \otimes L_2)$-unifier of φ and for all $\mathscr{L}$-substitution σ', if σ' is an $(L_1 \otimes L_2)$-unifier of φ then $\tau \preccurlyeq_{L_1 \otimes L_2} \sigma'$. Since L_2 is consistent therefore either* $L_2 \subseteq \mathrm{Triv}_2$ *or* $L_2 \subseteq \mathrm{Ver}_2$.
Let $t : \mathscr{L} \to \mathscr{L}_1$ be the function defined in Definition 38. *For all $\mathscr{L}$-substitutions θ, let θ^t be the $\mathscr{L}_1$-substitution such that for all variable x, $\theta^t(x) = t(\theta(x))$.*

Claim 2 *For all $\psi \in \mathscr{L}_1$, $\theta^t(\psi) = t(\theta(\psi))$ for all $\mathscr{L}$-substitutions θ.*

Proof 88 *By induction on $\psi \in \mathscr{L}_1$:*

- *Let* $\psi = x$. *We have,* $\theta^t(x) = t(\theta(x))$.
- *Let* $\psi = \bot$. *We have,* $\theta^t(\bot) = \bot$.
- *Let* $\psi = \varphi \vee \varphi'$. *By induction hypothesis* $\theta^t(\varphi) = t(\theta(\varphi))$ *and* $\theta^t(\varphi') = t(\theta(\varphi'))$. *Hence,* $\theta^t(\varphi \vee \varphi') = t(\theta(\varphi \vee \varphi'))$.
- *Let* $\psi = \Box_1\varphi'$. *By induction hypothesis* $\theta^t(\varphi') = t(\theta(\varphi'))$. *Then,* $\Box_1\theta^t(\varphi') = \Box_1 t(\theta(\varphi')) = t(\Box_1\theta(\varphi')) = t(\theta(\Box_1\varphi'))$.

Claim 3 *For all* $\psi \in \mathscr{L}$

1. *If* $L_2 \subseteq \boldsymbol{Triv}_2$ *then* $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Triv}_2$.
2. *If* $L_2 \subseteq \boldsymbol{Ver}_2$ *then* $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Ver}_2$.

Proof 89 *by induction on* ψ.

1. *Suppose* $L_2 \subseteq \boldsymbol{Triv}_2$. *We only explain the cases* $\psi = \Box_1\psi'$ *and* $\psi = \Box_2\psi'$.
   - *Let* $\psi = \Box_1\psi'$. *By induction hypothesis* $\psi' \leftrightarrow t(\psi') \in L_1 \otimes \boldsymbol{Triv}_2$. *By necessitation* $\Box_1\psi' \leftrightarrow \Box_1 t(\psi') \in L_1 \otimes \boldsymbol{Triv}_2$. *Thus,* $\Box_1\psi' \leftrightarrow t(\Box_1\psi') \in L_1 \otimes \boldsymbol{Triv}_2$.
   - *Let* $\psi = \Box_2\psi'$. *By induction hypothesis* $\psi' \leftrightarrow t(\psi') \in L_1 \otimes \boldsymbol{Triv}_2$. *By necessitation* $\Box_2\psi' \leftrightarrow \Box_2 t(\psi') \in L_1 \otimes \boldsymbol{Triv}_2$. *Consequently,* $\Box_2\psi' \leftrightarrow t(\psi') \in L_1 \otimes \boldsymbol{Triv}_2$. *Since* $t(\Box_2\psi') = t(\psi')$ *thus,* $\Box_2\psi' \leftrightarrow t(\Box_2\psi') \in L_1 \otimes \boldsymbol{Triv}_2$.
2. *The proof of this item is similar to the proof of item* 1.

Claim 4 *For all* $\psi \in \mathscr{L}$, *if* $\psi \in L_1 \otimes L_2$ *then* $t(\psi) \in L_1$.

Proof 90 *We consider the two following cases:*

1. *Suppose* $L_2 \subseteq \boldsymbol{Triv}_2$. *Let* $\psi \in L_1 \otimes L_2$. *Then by Claim* 3, $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Triv}_2$. *Since,* $\psi \in L_1 \otimes L_2$ *and* $L_1 \otimes L_2 \subseteq L_1 \otimes \boldsymbol{Triv}_2$ *then,* $\psi \in L_1 \otimes \boldsymbol{Triv}_2$. *Since,* $\psi \in L_1 \otimes \boldsymbol{Triv}_2$ *and* $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Triv}_2$ *then,* $t(\psi) \in L_1 \otimes \boldsymbol{Triv}_2$. *Therefore, knowing that* $L_1 \otimes \boldsymbol{Triv}_2$ *is a conservative extension of* L_1, $t(\psi) \in L_1$.

2. *Suppose* $L_2 \subseteq \boldsymbol{Ver}_2$. *Let* $\psi \in L_1 \otimes L_2$. *Then by Claim* 3, $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Ver}_2$. *Since,* $\psi \in L_1 \otimes L_2$ *and* $L_1 \otimes L_2 \subseteq L_1 \otimes \boldsymbol{Ver}_2$ *then,* $\psi \in L_1 \otimes \boldsymbol{Ver}_2$. *Since,* $\psi \in L_1 \otimes \boldsymbol{Ver}_2$ *and* $(\psi \leftrightarrow t(\psi)) \in L_1 \otimes \boldsymbol{Ver}_2$ *then,* $t(\psi) \in L_1 \otimes \boldsymbol{Ver}_2$. *Therefore, knowing that* $L_1 \otimes \boldsymbol{Ver}_2$ *is a conservative extension of* L_1, $t(\psi) \in L_1$.

Since, τ is an $(L_1 \otimes L_2)$-unifier of φ, therefore $\tau(\varphi) \in L_1 \otimes L_2$. Hence, by Claim 4, $t(\tau(\varphi)) \in L_1$. Thus by Claim 2, $\tau^t(\varphi) \in L_1$. Consequently, τ^t is an L_1-unifier of φ.

Claim 5 *For all $\psi \in \mathscr{L}$ and for all $\mathscr{L}$-substitution λ, $t(\lambda(\psi)) = \lambda^t(t(\psi))$.*

Proof 91 *By induction on ψ.*

Let θ be an $\mathscr{L}_1$-substitution such that θ is an L_1-unifier of φ. Hence, $\theta(\varphi) \in L_1$. Hence, $\theta(\varphi) \in L_1 \otimes L_2$. Thus, knowing that the τ defined before Claim 2 is a most general unifier of φ in $L_1 \otimes L_2$, $\tau \preccurlyeq_{L_1 \otimes L_2} \theta$. Let λ be an $\mathscr{L}$-substitution such that for all variables x, $\lambda(\tau(x)) \leftrightarrow \theta(x) \in L_1 \otimes L_2$. Hence, by Claim 4, for all variable x, $t(\lambda(\tau(x))) \leftrightarrow t(\theta(x)) \in L_1$. Thus by Claim 5, for all variables x, $\lambda^t(\tau^t(x)) \leftrightarrow \theta(x) \in L_1$. Consequently, $\tau^t \preccurlyeq_{L_1} \theta$.
As a result, τ^t is an L_1-unifier of φ (by the remark preceding Claim 5) and for all $\mathscr{L}_1$-substitutions θ, if θ is an L_1-unifier of φ then $\tau^t \preccurlyeq_{L_1} \theta$. Thus, $\{\tau^t\}$ is a minimal complete set of L_1-unifiers of φ and this is contradiction with assumption that φ has no minimal complete set of L_1-unifiers. This ends the proof of Lemma 51.

Lemma 52 *Let $L_1 \subseteq \mathscr{L}_1$ and $L_2 \subseteq \mathscr{L}_2$ be normal modal logics. if L_1 is nullary and L_2 is consistent then $L_1 \otimes L_2$ is not finitary.*

Proof 92 *Suppose L_1 is nullary and L_2 is consistent. Suppose $L_1 \otimes L_2$ is finitary. Since, L_1 is nullary, therefore let $\varphi \in \mathscr{L}_1$ be such that φ is L_1-unifiable and φ has no minimal complete set of L_1-unifiers. Let σ be an $\mathscr{L}_1$-substitusion such that $\sigma(\varphi) \in L_1$. Hence, $\sigma(\varphi) \in L_1 \otimes L_2$. Thus, φ is $(L_1 \otimes L_2)$-unifiable. Since, $L_1 \otimes L_2$ is finitary, let $\tau_1, ..., \tau_n$ be $\mathscr{L}$-substitutions such that $\tau_1, ..., \tau_n$ are $(L_1 \otimes L_2)$-unifiers of φ and for all $\mathscr{L}$-substitutions σ', if σ' is an $(L_1 \otimes L_2)$-unifier of φ then $\tau_i \preccurlyeq_{L_1 \otimes L_2} \sigma'$ for some $i \in \{1, ..., n\}$. Since L_2 is consistent therefore either $L_2 \subseteq \mathrm{Triv}_2$ or $L_2 \subseteq \mathrm{Ver}_2$.*
Let $t : \mathscr{L} \to \mathscr{L}_1$ be a function defined as in Definition 38.
Since, τ_i for $i \in \{1, ..., n\}$ is an $(L_1 \otimes L_2)$-unifier of φ, therefore $\tau_i(\varphi) \in L_1 \otimes L_2$ for $i \in \{1, ..., n\}$. Hence, $t(\tau_i(\varphi)) \in L_1$. Thus $\tau_i^t(\varphi) \in L_1$ for $i \in \{1, ..., n\}$. Consequently, τ_i^t is an L_1-unifier of φ.
Let θ be an $\mathscr{L}_1$-substitution such that θ is an L_1-unifier of φ. Hence, $\theta(\varphi) \in L_1$. Hence, $\theta(\varphi) \in L_1 \otimes L_2$. Thus, remembering that $\{\tau_1, ..., \tau_n\}$ is a complete set of unifiers of φ in $L_1 \otimes L_2$, $\tau_i \preccurlyeq_{L_1 \otimes L_2} \theta$ for some $i \in \{1, ..., n\}$. Let λ be an $\mathscr{L}$-substitution such that for all variables x, $\lambda(\tau_i(x)) \leftrightarrow \theta(x) \in L_1 \otimes L_2$. Hence, by Claim 4 for all variable x, $t(\lambda(\tau_i(x))) \leftrightarrow t(\theta(x)) \in L_1$. Thus, by Claim 5, for all variables x,

$\lambda^t(\tau_i^t(x)) \leftrightarrow \theta(x) \in L_1$. Consequently, $\tau_i^t \preccurlyeq_{L_1} \theta$ for some $i \in \{1,...,n\}$.
As a result, $\{\tau_1^t,...,\tau_n^t\}$ in a complete set of L_1-unifiers of φ and this is contradiction with our assumption.

Since K_1 is nullary and K_2 is consistent hence fusion $K_1 \otimes K_2$ is not unitary by Theorem 51 and not finitary by Theorem 52. At the following, we shall show that there exists a $(K_1 \otimes K_2)$-unifiable formula which has no minimal complete set. Hence $K_1 \otimes K_2$ is nullary. In this respect, we shall use Jeřábek's method in [34] in order to show that the unification type of the fusion $(K_1 \otimes K_2)$ is nullary. We need to define

- $(\Box_1\Box_2)^0\varphi ::= \varphi$
- $(\Box_1\Box_2)^{n+1}\varphi ::= (\Box_1\Box_2)(\Box_1\Box_2)^n\varphi$
- $(\Box_1\Box_2)^{<0}\varphi ::= \top$
- $(\Box_1\Box_2)^{<n+1}\varphi ::= (\Box_1\Box_2)^{<n}\varphi \wedge (\Box_1\Box_2)^n\varphi$

where n is a non-negative integer.
The next Lemma expresses some required facts that we will use to prove $K_1 \otimes K_2$ is nullary.

Lemma 53 *Let $k, l \in \mathbb{N}$ and φ, ψ be $\mathscr{L}$-formula.*

1. *If $\vdash \varphi \to \psi$ then $\models \Box_1\Box_2\varphi \to \Box_1\Box_2\psi$.*
2. *If $k \leqslant l$ then $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^l\bot$.*
3. *If $k < l$ then $\vdash (\Box_1\Box_2)^{<l}\varphi \to (\Box_1\Box_2)^k\varphi$.*
4. *If $k \leqslant l$ then $\vdash (\Box_1\Box_2)^{<k}\varphi \wedge (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}\varphi$.*
5. *If $k < l$ then, $\nvdash (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^k\bot$.*
6. *If $k \leqslant l$ then $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot$.*

Proof 93 *Let $k, l \in \mathbb{N}$.*

1. *Suppose $\vdash \varphi \to \psi$. Then, $\vdash \Box_2(\varphi \to \psi)$ by necessitation. Hence we obtain $\vdash \Box_2\varphi \to \Box_2\psi$ by axiom K_2. Since $\vdash \Box_2\varphi \to \Box_2\psi$ hence, $\vdash \Box_1(\Box_2\varphi \to \Box_2\psi)$ by necessitation. Thus, $\vdash \Box_1\Box_2\varphi \to \Box_1\Box_2\psi$ by axiom K_1.*

2. *Suppose $k \leqslant l$. Since, $\vdash \bot \to (\Box_1\Box_2)^{l-k}\bot$ then $\vdash \Box_2\bot \to \Box_2(\Box_1\Box_2)^{l-k}\bot$ by necessitation and axiom K_2. Then we obtain, $\vdash \Box_1\Box_2\bot \to \Box_1\Box_2(\Box_1\Box_2)^{l-k}\bot$ by necessitation and axiom K_1. We can use k-times axiom K_1 and K_2. Thus, $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^k(\Box_1\Box_2)^{l-k}\bot$. Therefore, $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^l\bot$.*

3. *Suppose, $k < l$. By definition we have, $(\Box_1\Box_2)^{<l}\varphi = \varphi \wedge (\Box_1\Box_2)\varphi \wedge \ldots \wedge (\Box_1\Box_2)^k\varphi \wedge \ldots \wedge (\Box_1\Box_2)^{l-1}\varphi$ Then, $\vdash (\Box_1\Box_2)^{<l}\varphi \to (\Box_1\Box_2)^k\varphi$.*

4. *We have,*
   - $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^k\varphi \wedge \ldots \wedge (\Box_1\Box_2)^{l-1}\varphi$ *(since $k \leqslant l$) and*
   - $\vdash (\Box_1\Box_2)^{<k}\varphi \to \varphi \wedge \ldots \wedge (\Box_1\Box_2)^{k-1}\varphi$. *Hence,*
   - $\vdash (\Box_1\Box_2)^{<k}\varphi \wedge (\Box_1\Box_2)^k\bot \to \Big((\Box_1\Box_2)^k\varphi \wedge \ldots \wedge (\Box_1\Box_2)^{l-1}\varphi\Big) \wedge \Big(\varphi \wedge \ldots \wedge (\Box_1\Box_2)^{k-1}\varphi\Big)$. *Therefore,*
   - $\vdash (\Box_1\Box_2)^{<k}\varphi \wedge (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}\varphi$.

5. *Consider a $K_1 \otimes K_2$-model $M = (W, R_1, R_2, v)$ such that $W = \{w_1, \ldots., w_{2k+1}\}$ and $w_1 R_1 w_2 R_2 \ldots R_2 w_{2k+1}$. Hence, $M, w_1 \models (\Box_1\Box_2)^l\bot$ (since $k < l$) and $M, w_1 \not\models (\Box_1\Box_2)^k\bot$. Thus, $M, w_1 \not\models (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^k\bot$.*

6. *Suppose $k \leqslant l$. Since, $(\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot = (\Box_1\Box_2)^k\bot \wedge \ldots \wedge (\Box_1\Box_2)^{l+k-1}\bot$ hence by part (2), $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^k\bot \wedge \ldots \wedge (\Box_1\Box_2)^{l+k-1}\bot$. Therefore, $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot$.*

Consider the formula $\varphi = x \to \Box_1\Box_2 x$ and the substitutions $\sigma_\top(x) = \top$ and $\sigma_n(x) = (\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot$. We will show that φ is unifiable in $K_1 \otimes K_2$ and nullary.

Lemma 54 *For all $n \in \mathbb{N}$,*

1. *$\sigma_n(x) = (\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot$ is a $K_1 \otimes K_2$-unifier of φ.*

2. *$\sigma_\top(x) = \top$ is a $K_1 \otimes K_2$-unifier of φ.*

Proof 94 *Let $n \in \mathbb{N}$.*

1. *We have to prove* σ_n *is a unifier of* φ. *By part* (4) *of Lemma* 53 *we have* $\vdash (\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot \to (\Box_1\Box_2)(\Box_1\Box_2)^{<n}x$. *By part* (2) *of Lemma* 53 $\vdash (\Box_1\Box_2)^n\bot \to (\Box_1\Box_2)^{n+1}\bot$ *then,* $\vdash (\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot \to (\Box_1\Box_2)(\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)(\Box_1\Box_2)^n\bot$. *Thus,* $\vdash (\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot \to (\Box_1\Box_2)((\Box_1\Box_2)^{<n}x \wedge (\Box_1\Box_2)^n\bot)$. *Therefore,* $\vdash \sigma_n(x) \to \Box_1\Box_2\sigma_n(x)$. *Consequently,* σ_n *is a unifier of* φ.

2. *Since,* $\vdash \top \to \Box_1\Box_2\top$ *it is clear that* $\sigma_\top$ *is a unifier of* φ.

Lemma 55 *Let* $k, l \in \mathbb{N}$. *If* $k \leqslant l$ *then* $\sigma_l \preccurlyeq \sigma_k$.

Proof 95 *Suppose* $k \leqslant l$. *We have to prove* $\sigma_l \preccurlyeq \sigma_k$. *Let* $\nu(x) = x \wedge (\Box_1\Box_2)^k\bot$. *Since,* $\vdash (\Box_1\Box_2)^{<l}x \to (\Box_1\Box_2)^{<k}x$ *(since* $k \leqslant l$*) and* $\vdash (\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^k\bot$ *then,* $\vdash (\Box_1\Box_2)^{<l}x \wedge (\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot \wedge (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot$. *Hence,* $\vdash (\Box_1\Box_2)^{<l}(x \wedge (\Box_1\Box_2)^k\bot) \wedge (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^k x \wedge (\Box_1\Box_2)^k\bot$. *For the other direction we shall prove as follows. By part* 6 *of Lemma* 53 *we have,* $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}(\Box_1\Box_2)^k\bot$ *and by part* (5) *of Lemma* 53 $\vdash (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}x$ *and by part* (2) *of Lemma* 53, $\vdash (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^l\bot$. *Thus,* $\vdash (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}(x \wedge (\Box_1\Box_2)^k\bot) \wedge (\Box_1\Box_2)^{<l}\bot$. *Since,* $\vdash (\Box_1\Box_2)^{<l}(x \wedge (\Box_1\Box_2)^k\bot) \wedge (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot$ *and* $\vdash (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot \to (\Box_1\Box_2)^{<l}(x \wedge (\Box_1\Box_2)^k\bot) \wedge (\Box_1\Box_2)^l\bot$ *therefore,* $\vdash (\Box_1\Box_2)^{<l}(x \wedge (\Box_1\Box_2)^k\bot) \wedge (\Box_1\Box_2)^l\bot \leftrightarrow (\Box_1\Box_2)^{<k}x \wedge (\Box_1\Box_2)^k\bot$.

Thus, $\nu \circ \sigma_l \simeq \sigma_k$. *Consequently,* $\sigma_l \preccurlyeq \sigma_k$.

Lemma 56 *Let* $k, l \in \mathbb{N}$. *If* $k < l$ *then* $\sigma_k \not\preccurlyeq \sigma_l$

Proof 96 *Suppose* $k < l$. *Suppose* $\sigma_k \preceq \sigma_l$. *Let* ν *be a substitution such that* $\vdash \nu(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. *Then,* $\vdash (\Box_1\Box_2)^{\leqslant k}\nu(x) \wedge (\Box_1\Box_2)^k\bot \leftrightarrow (\Box_1\Box_2)^{\leqslant l}x \wedge (\Box_1\Box_2)^l\bot$. *Hence,* $\vdash (\Box_1\Box_2)^{<l}x \wedge (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^{<k}\nu(x) \wedge (\Box_1\Box_2)^k\bot$. *Then by replacing* x *by* $\top$, $\vdash (\Box_1\Box_2)^l\bot \to (\Box_1\Box_2)^k\bot$. *This is a contradiction with part* (5) *of Lemma* 53.

Lemma 57 *Let* σ *be a substitution. The following conditions are equivalent:*

1. $\sigma_\top \circ \sigma \simeq \sigma$.

2. $\sigma_\top \preccurlyeq \sigma$.

3. $\vdash \sigma(x)$.

Proof 97 $(1 \Rightarrow 2)$*: By definition of* $\preccurlyeq$.
$(2 \Rightarrow 3)$*: Suppose* $\sigma_\top \preccurlyeq \sigma$. *Let* ν *be a substitution such that* $\vdash \nu(\sigma_\top(x)) \leftrightarrow \sigma(x)$. *Then,* $\vdash \top \leftrightarrow \sigma(x)$. *Therefore,* $\vdash \sigma(x)$.
$(3 \Rightarrow 1)$*: Suppose* $\vdash \sigma(x)$. *Then* $\vdash \top \leftrightarrow \sigma(x)$. *Hence,* $\vdash \sigma(x) \leftrightarrow \sigma(\sigma_\top(x))$. *Therefore,* $\sigma_\top \circ \sigma \simeq \sigma$.

Lemma 58 *Let* $k \in \mathbb{N}$. *Let* σ *be a unifier of* φ. *The following conditions are equivalent:*

1. $\sigma_n \circ \sigma \simeq \sigma$.

2. $\sigma_n \preccurlyeq \sigma$.

3. $\vdash \sigma(x) \to (\Box_1\Box_2)^n \bot$.

Proof 98 $(1 \Rightarrow 2)$ *By definition of* $\preccurlyeq$.
$(2 \Rightarrow 3)$ *Suppose* $\sigma_n \preccurlyeq \sigma$. *Let* ν *be a substitution such that* $\vdash \nu(\sigma_n(x)) \leftrightarrow \sigma(x)$. *Then,* $\vdash (\Box_1\Box_2)^{<n}\nu(x) \wedge (\Box_1\Box_2)^n\bot \leftrightarrow \sigma(x)$. *Hence,* $\vdash \sigma(x) \to (\Box_1\Box_2)^{<n}\nu(x) \wedge (\Box_1\Box_2)^n\bot$. *Therefore,* $\vdash \sigma(x) \to (\Box_1\Box_2)^n\bot$.
$(3 \Rightarrow 1)$ *Suppose* $\vdash \sigma(x) \to (\Box_1\Box_2)^n\bot$. *Since* σ *is a unifier of* φ *then,*
$\vdash \sigma(x) \to (\Box_1\Box_2)\sigma(x)$. *Hence by necessitation and axiom* K_1 *and* K_2,
$\vdash (\Box_1\Box_2)\sigma(x) \to (\Box_1\Box_2)(\Box_1\Box_2)\sigma(x)$. *Hence,*
$\vdash \sigma(x) \to (\Box_1\Box_2)(\Box_1\Box_2)\sigma(x)$. *By necessitation, axiom* K_1 *and* K_2 *(n−times),*
$\vdash \sigma(x) \to (\Box_1\Box_2)^{<n}\sigma(x)$. *By our assumption,*
$\vdash \sigma(x) \to (\Box_1\Box_2)^n\bot$ *and*
$\vdash \sigma(x) \to (\Box_1\Box_2)^{<n}\sigma(x)$ *then,*
$\vdash \sigma(x) \to (\Box_1\Box_2)^{<n}\sigma(x) \wedge (\Box_1\Box_2)^n\bot$.
We consider two cases:

- *If* $n = 0$ *then* $\vDash (\Box_1\Box_2)^n\bot \to \sigma(x)$.

- *If* $n \leqslant 1$ *then* $\vdash (\Box_1\Box_2)^{<n}\sigma(x) \to \sigma(x)$.

Therefore in both cases, $\vdash (\Box_1\Box_2)^{<n}\sigma(x) \wedge (\Box_1\Box_2)^n\bot \to \sigma(x)$. *Since,*
$\vdash \sigma(x) \to (\Box_1\Box_2)^{<n}\sigma(x) \wedge (\Box_1\Box_2)^n\bot$ *and*
$\vdash (\Box_1\Box_2)^{<n}\sigma(x) \wedge (\Box_1\Box_2)^n\bot \to \sigma(x)$ *thus,*
$\vdash (\Box_1\Box_2)^{<n}\sigma(x) \wedge (\Box_1\Box_2)^n\bot \leftrightarrow \sigma(x)$. *Thus,*
$\vdash \sigma(\sigma_n(x)) \leftrightarrow \sigma(x)$. *Therefore,*
$\sigma_n \circ \sigma \simeq \sigma$.

Theorem 9 *Let σ be a unifier of $\varphi = x \to \Box_1\Box_2 x$ then either $\vdash \sigma(x)$ or $\vdash \sigma(x) \to (\Box_1\Box_2)^n\bot$ where $n \geqslant deg(\sigma(x))$.*

Proof 99 *Suppose neither $\nvdash \sigma(x)$ nor $\nvdash \sigma(x) \to (\Box_1\Box_2)^n\bot$. Let $M_1 = \langle W_1, R_1', R_2', v_1\rangle$ be a model and $t_1 \in W_1$ such that $M_1, t_1 \nvDash \sigma(x)$. Let $M_2 = \langle W_2, R_1'', R_2'', v_2\rangle$ be a model and $w_1 \in W_2$ such that $M_2, w_1 \vDash \sigma(x) \wedge (\Diamond_1\Diamond_2)^n\top$. Since, $M_2, w_1 \vDash (\Diamond_1\Diamond_2)^n\top$ then there exists a sequence $w_1, ..., w_{2n+1} \in W_2$ such that $w_1 R_1'' w_2 R_2'' w_3 R_1'' ... R_2'' w_{2n+1}$. By the tree-model property of $K_1 \otimes K_2$, we can assume that $w_1, w_2, w_3, ..., w_{2n+1}$ are pairwise distinct and that the path $w_1 R_1'' w_2 R_2'' w_3 R_1'' ... R_2'' w_{2n+1}$ is the shortest path in M_2 between w_1 and w_{2n+1}. Let $M = (W, R_1, R_2, v)$ where:*

- *$W = W_1 \cup W_2 \cup \{t_0\}$ where t_0 is a new possible worlds,*
- *$R_1 = R_1' \cup R_1'' \cup \{(w_{2n+1}, t_0)\}$,*
- *$R_2 = R_2' \cup R_2'' \cup \{(t_0, t_1)\}$,*
- *$v = v_1 \cup v_2$.*

Since M is a disjoint union of M_1, M_2 and the state t_0 and $M_1, t_1 \nvDash \sigma(x)$ then, $M, t_1 \nvDash \sigma(x)$. Since $n \geqslant deg(\sigma(x))$, M is a disjoint union of M_1, M_2 and the state t_0 and $M_2, w_1 \vDash \sigma(x)$ then $M, w_1 \vDash \sigma(x)$. By our assumption σ is a unifier of φ then $\vdash \sigma(x) \to \Box_1\Box_2\sigma(x)$. Since $M, w_1 \vDash \sigma(x)$ therefore $M, w_{2i+1} \vDash \sigma(x)$ for all $i = 1, ..., n$. Thus, $M, w_{2n+1} \vDash \Box_1\Box_2\sigma(x)$. Since, $w_{2n+1} R_1 t_0 R_2 t_1$ therefore $M, t_1 \vDash \sigma(x)$. This is contradiction .

Lemma 59 *The set of substitutions $\Sigma = \{\sigma_\top\} \cup \{\sigma_n \mid n \in \mathbb{N}\}$ forms a complete set of $K_1 \otimes K_2$-unifiers of $\varphi = x \to \Box_1\Box_2 x$.*

Proof 100 *By Lemmas* 54, 55, 56, 57 *and* 58 *and Theorem* 9.

Lemma 60 *The formula $\varphi = x \to \Box_1\Box_2 x$ does not possess a minimal complete set of $K_1 \otimes K_2$-unifiers.*

Proof 101 *Let Γ be a minimal complete set of unifiers of φ and $\sigma \in \Gamma$. Since Γ is complete, then let $\sigma \in \Gamma$ be such that $\sigma \preccurlyeq \sigma_0$. Since σ is a unifier of φ hence $\sigma_\top \preccurlyeq_{K_1\otimes K_2} \sigma$ or for some $n \in \mathbb{N}$, $\sigma_n \preccurlyeq_{K_1\otimes K_2} \sigma$ by Theorem* 9.
Suppose $\sigma_n \preccurlyeq_{k_1\otimes k_2} \sigma$. By definition of Σ, $\sigma_{n+1} \in \Sigma$. Let $\sigma' \in \Gamma$ such that $\sigma' \preccurlyeq \sigma_{n+1}$. Since $\sigma' \preccurlyeq \sigma_{n+1} \preccurlyeq \sigma_n \preccurlyeq \sigma$ then $\sigma' \preccurlyeq \sigma$. Since Γ is minimal complete set and

its members are pairwise incomparable then $\sigma' = \sigma$. Since $\sigma_n \preccurlyeq \sigma$ and $\sigma' \preccurlyeq \sigma_{n+1}$ then, $\sigma_n \preccurlyeq \sigma_{n+1}$. Since $n < n+1$ by lemma 56, $\sigma_n \not\preccurlyeq \sigma_{n+1}$ and this is a contradiction.
Suppose $\sigma_\top \preccurlyeq_{K_1 \otimes K_2} \sigma$. Since $\sigma \preccurlyeq \sigma_0$ then, $\sigma_\top \preccurlyeq \sigma_0$. Therefore $\vdash \top \leftrightarrow \bot$. This is contradiction.

Lemma 61 *Unification type is nullary in fusion $K \otimes K$*

Proof 102 *By Lemma* 60.

6.4 Unification in Fusion $KD_1 \otimes KD_2$

In this section we will discuss on unification type of the fusion $KD_1 \otimes KD_2$. In order to show the unification type of the fusion $KD_1 \otimes KD_2$ is nullary we use the method mentioned in [7]. In this respect, we need to define the following abbreviation where p is a parameter:

- $[p]x ::= \Box_1\Box_2(p \to x)$.

For all parameters p, the modal connective $[p]^k$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $[p]^0\varphi ::= \varphi$,
- $[p]^{k+1}\varphi ::= [p][p]^k\varphi$.

For all parameters p, the modal connective $[p]^{<k}$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $[p]^{<0}\varphi ::= \top$.
- $[p]^{<k+1}\varphi ::= [p]^{<k}\varphi \wedge [p]^k\varphi$.

Consider the formula $\varphi = (x \to p) \wedge (x \to [p]x)$ and substitutions $\sigma_p(x) = p$ and $\sigma_n(x) = p \wedge [p]^{<n}x \wedge [p]^n\bot$ where $n \in \mathbb{N}$.

Lemma 62

1. $\vdash p \to [p]p$.
2. $\vdash [p](\varphi \wedge \psi) \leftrightarrow [p]\varphi \wedge [p]\psi$.

3. *If* $k \leqslant l$ *then,* $\vdash [p]^{<l}[p]^k\bot \to [p]^k\bot$.

4. *If* $k \leqslant l$ *then,* $\vdash [p]^{<k}x \wedge [p]^k\bot \to [p]^{<l}x$.

5. *If* $k \leqslant l$ *then,* $\vdash [p]^k\bot \to [p]^{<l}[p]^k\bot$.

6. *If* $k \leqslant l$ *then,* $\vdash [p]^k\bot \to [p]^l\bot$.

7. *If* $k < l$ *then,* $\nvdash p \wedge [p]^l\bot \to [p]^k\bot$

Proof 103 *The proof of this Lemma is similar to the proof of Lemma* 53.

Lemma 63 *For all* $n \in \mathbb{N}$

1. $\sigma_n(x) = p \wedge [p]^{<n}x \wedge [p]^n\bot$ *is a* $KD_1 \otimes KD_2$*-unifier of* φ.

2. $\sigma_p(x) = p$ *is a* $KD_1 \otimes KD_2$*-unifier of* φ.

Proof 104

1. *We have to prove* $\vdash (\sigma_n(x) \to p) \wedge (\sigma_n(x) \to [p]\sigma_n(x))$. *Hence we have to prove* $\vdash \sigma_n(x) \to p$ *and* $\vdash \sigma_n(x) \to [p]\sigma_n(x)$. *Since,*
$\vdash p \wedge [p]^{<n}x \wedge [p]^n\bot \to p$ *thus,*
$\vdash \sigma_n(x) \to p$.
Let us prove $\vdash (p \wedge [p]^{<n}x \wedge [p]^n\bot) \to [p](p \wedge [p]^{<n}x \wedge [p]^n\bot)$. *Hence, By part* (2) *of Lemma* 62 *we have to prove*
$\vdash p \wedge [p]^{<n}x \wedge [p]^n\bot \to [p]p \wedge [p][p]^{<n}x \wedge [p][p]^n\bot$. *By part* (1) *of Lemma* 62 *we have*
$\vdash p \to [p]p$. *Hence,*
$\vdash p \wedge [p]^{<n}x \wedge [p]^n\bot \to [p]p$.*By part* (5) *of Lemma* 62 *we have*
$\vdash [p]^{<n}x \wedge [p]^n\bot \to [p][p]^{<n}x$. *By part* (7) *of Lemma* 62 *we have*
$\vdash [p]^n\bot \to [p]^{n+1}\bot$. *Then,*
$\vdash p \wedge [p]^{<n}x \wedge [p]^n\bot \to [p]p \wedge [p][p]^{<n}x \wedge [p]^{n+1}\bot$. *Thus,*
$\vdash p \wedge [p]^{<n}x \wedge [p]^n\bot \to [p](p \wedge [p]^{<n}x \wedge [p]^n\bot)$.
Therefore, σ_n *is an* $KD_1 \otimes KD_2$*-unifier of* φ.

2. *Since,* $\vdash (p \to p)$ *and* $\vdash p \to [p]p$ *then,* $\vdash (p \to p) \wedge (p \to [p]p)$. *Therefore,* σ_p *is a* $KD_1 \otimes KD_2$*-unifier of* φ.

Lemma 64 *Let* $k, l \in \mathbb{N}$. *If* $k \leqslant l$ *then* $\sigma_l \preccurlyeq_{KD_1 \otimes KD_2} \sigma_k$.

Proof 105 *Suppose $k \leqslant l$. We have to prove $\sigma_l \preccurlyeq_{KD_1 \otimes KD_2} \sigma_k$. Let $\nu(x) = x \wedge [p]^k\bot$. Hence we have to show that $\vdash p \wedge [p]^{<l}(x \wedge [p]^k\bot) \wedge [p]^l\bot \leftrightarrow p \wedge [p]^{<k}x \wedge [p]^k\bot$. To prove $\vdash p \wedge [p]^{<l}x \wedge [p]^{<l}[p]^k\bot \wedge [p]^l\bot \to p \wedge [p]^{<k}x \wedge [p]^k\bot$ we shall do as follows. By part* (3) *of Lemma* 62 *we have*
$\vdash [p]^{<l}x \to [p]^{<k}x$. *By part* (4) *of Lemma* 62
$\vdash [p]^{<l}[p]^k\bot \to [p]^k\bot$. *Thus,*
$\vdash p \wedge [p]^{<l}x \wedge [p]^{<l}[p]^k\bot \wedge [p]^l\bot \to p \wedge [p]^{<k}x \wedge [p]^k\bot$. *Therefore,*
$\vdash p \wedge [p]^{<l}(x \wedge [p]^k\bot) \wedge [p]^l\bot \to p \wedge [p]^{<k}x \wedge [p]^k\bot$. *For the other direction, we shall do as follows:By part* (7) *of Lemma* 62*,*
$\vdash [p]^k\bot \to [p]^l\bot$. *By part* (5) *of Lemma* 62*,*
$\vdash [p]^{<k}x \wedge [p]^k\bot \to [p]^{<l}x$. *By part* (6) *of Lemma* 62*,*
$\vdash [p]^k\bot \to [p]^{<l}[p]^k\bot$. *Then,*
$\vdash (p \wedge [p]^{<k}x \wedge [p]^k\bot) \to (p \wedge [p]^{<l}x \wedge [p]^{<l}[p]^k\bot \wedge [p]^l\bot)$. *Therefore,*
$\vdash (p \wedge [p]^{<k}x \wedge [p]^k\bot) \to p \wedge [p]^{<l}(x \wedge [p]^k\bot) \wedge [p]^l\bot$. *Since,*
$\vdash p \wedge [p]^{<l}(x \wedge [p]^k\bot) \wedge [p]^l\bot \to p \wedge [p]^{<k}x \wedge [p]^k\bot$ *and*
$\vdash p \wedge [p]^{<k}x \wedge [p]^{<k}\bot \to p \wedge [p]^{<l}(x \wedge [p]^k\bot) \wedge [p]^{<l}\bot$ *therefore,*
$\vdash p \wedge [p]^{<l}(x \wedge [p]^k) \wedge [p]^l\bot \leftrightarrow p \wedge [p]^{<k}x \wedge [p]^k\bot$.
Thus, $\vdash \nu(\sigma_l(x)) \leftrightarrow \sigma_k(x)$. *Consequently,* $\sigma_l \preccurlyeq_{KD_1 \otimes KD_2} \sigma_k$.

Lemma 65 *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preccurlyeq_{KD_1 \otimes KD_2} \sigma_l$.*

Proof 106 *Suppose $k < l$ and $\sigma_k \preceq \sigma_l$. Let ν be a substitution such that $\vdash \nu(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. Then, $\vdash p \wedge [p]^{<k}\nu(x) \wedge [p]^k\bot \leftrightarrow p \wedge [p]^{<l}x \wedge [p]^l\bot$. Hence, $\vdash p \wedge [p]^{<l}x \wedge [p]^l\bot \to p \wedge [p]^{<k}\nu(x) \wedge [p]^k\bot$. Then by replacing x by $\top$, $\vdash p \wedge [p]^l\bot \to [p]^k\bot$. This is contradiction with part* (7) *of Lemma* 62.

Lemma 66 *Let σ be a $KD_1 \otimes KD_2$-unifier of φ. The following conditions are equivalent:*

1. $\sigma_p \circ \sigma \simeq \sigma$.
2. $\sigma_p \preccurlyeq \sigma$.
3. $\vdash \sigma(x) \leftrightarrow p$.

Proof 107 $(1 \Rightarrow 2)$*: By definition of* $\preccurlyeq$.
$(2 \Rightarrow 3)$*: Suppose $\sigma_p \preccurlyeq \sigma$. Let ν be a substitution such that $\vdash \nu(\sigma_p(x)) \leftrightarrow \sigma(x)$. Then, $\vdash p \leftrightarrow \sigma(x)$.*
$(3 \Rightarrow 1)$*: Suppose $\vdash \sigma(x) \leftrightarrow p$. Then, $\vdash \sigma(x) \leftrightarrow \sigma(\sigma_p(x))$. Hence, $\sigma_p \circ \sigma \simeq \sigma$.*

Lemma 67 *Let σ be a $(KD_1 \otimes KD_2)$-unifier of φ. Let $n \geqslant 0$. The following conditions are equivalent:*

1. $\sigma_n \circ \sigma \simeq \sigma$
2. $\sigma_n \preccurlyeq \sigma$
3. $\vdash \sigma(x) \to [p]^n \bot$.

Proof 108 $(1 \Rightarrow 2)$*: By definition of* $\preccurlyeq$.
$(2 \Rightarrow 3)$*: Suppose* $\sigma_n \preccurlyeq \sigma$. *Let* ν *be a substitution such that* $\vdash \nu(\sigma_n(x)) \leftrightarrow \sigma(x)$. *Then,* $\vdash p \wedge [p]^{<n}\nu(x) \wedge [p]^n\bot \leftrightarrow \sigma(x)$. *Hence,* $\vdash \sigma(x) \to p \wedge [p]^{<n}\nu(x) \wedge [p]^n\bot$. *Therefore we have,* $\vdash \sigma(x) \to [p]^n\bot$.
$(3 \Rightarrow 1)$*: Suppose* $\vdash \sigma(x) \to [p]^n\bot$. *Since* σ *is a unifier of* φ *then,* $\vdash \sigma(x) \to p$ *and* $\vdash \sigma(x) \to [p]\sigma(x)$. *Since,* $\vdash \sigma(x) \to [p]\sigma(x)$ *hence by necessitation and axiom K we have* $\vdash \sigma(x) \to [p]^{<n}\sigma(x)$. *Since,* $\vdash \sigma(x) \to [p]^n\bot$, $\vdash \sigma(x) \to p$ *and* $\vdash \sigma(x) \to [p]^{<n}\sigma(x)$ *then,* $\vdash \sigma(x) \to p \wedge [p]^{<n}\sigma(x) \wedge [p]^n\bot$.
For the converse implication, we consider two cases:

- *If* $n = 0$ *then* $\vdash [p]^n\bot \to \sigma(x)$ *and*
- *If* $n \geqslant 1$ *then,* $\vdash [p]^{<n}\sigma(x) \to \sigma(x)$.

Hence in both cases, $\vdash p \wedge [p]^{<n}\sigma(x) \wedge [p]^n\bot \to \sigma(x)$. *Therefore,* $\vdash p \wedge [p]^{<n}\sigma(x) \wedge [p]^n\bot \leftrightarrow \sigma(x)$. *Thus,* $\sigma_n \circ \sigma \simeq \sigma$.

Lemma 68 *Let* σ *is a unifier of* φ. *Let* $n \geqslant deg(\sigma(x))$. *Then one of the following conditions holds*

- $\vdash \sigma_p \preccurlyeq \sigma$ *or*
- $\vdash \sigma_n \preccurlyeq \sigma$.

Proof 109 *Suppose none of the above conditions holds. Then, neither* $\vdash \sigma_p \preccurlyeq \sigma$ *nor* $\vdash \sigma_n \preccurlyeq \sigma$. *Hence by Lemma* 66 *and* 67, $\nvdash \sigma(x) \leftrightarrow p$ *and* $\nvdash \sigma(x) \to [p]^n\bot$. *Since,* σ *is a unifier of* φ *then,* $\vdash \sigma(x) \to p$. *Hence,* $\nvdash p \to \sigma(x)$. *Let* $M_1 = \langle W_1, R'_1, R'_2, \nu_1\rangle$ *and* $M_2 = \langle W_2, R''_1, R''_2, \nu_2\rangle$ *be* $KD_1 \otimes KD_2$*-models and* $s \in W_1$ *and* $t_1 \in W_2$ *such that* $M_1, s \nvDash p \to \sigma(x)$ *and* $M_2, t_1 \nvDash \sigma(x) \to [p]^n\bot$. *Since,* $M_2, t_1 \nvDash [p]^n\bot$ *then there exists a sequence of* $t_1, ..., t_{2n+1} \in W_2$ *such that* $t_1 R''_1 t_2 R''_2 t_3 R''_1 ... R''_2 t_{2n+1}$ *and* $M_2, t_{2i+1} \vDash p$ *for* $0 \leqslant i \leqslant n$. *Again, as in the proof of Theorem* 11, *by the tree-model property of* $KD_1 \otimes KD_2$, *we can assume that* $t_1, ..., t_{2n+1}$ *are pairwise distinct and that the path* $t_1 R''_1 t_2 R''_2 t_3 R''_1 ... R''_2 t_{2n+1}$ *is the shortest path in* M_2 *between* t_1 *and* t_{2n+1}. *Let* $M = \langle W, R_1, R_2, \nu\rangle$ *be the model defined as follows:*

- $W = W_1 \cup W_2 \cup s_0$ *where* s_0 *is a new possible world,*
- $R_1 = R_1' \cup R_1'' \cup \{(t_{2n+1}, s_0)\}$,
- $R_2 = R_2' \cup R_2'' \cup \{(s_0, s)\}$,
- $v = v_1 \cup v_2$.

Since M *is a disjoint union of* M_1, M_2 *and* s_0 *and* $M_1, s \nvDash \sigma(x)$ *and* $M_1, s \vDash p$ *then,* $M, s \nvDash \sigma(x)$ *and* $M, s \vDash p$. *Since* $n \geqslant deg(\sigma(x))$, M *is a disjoint union of* M_1, M_2 *and* s_0 *and* $M_2, t_1 \vDash \sigma(x)$ *and* $M_2, t_{2i+1} \vDash p$ *for* $0 \leqslant i \leqslant n$ *then* $M, t_1 \vDash \sigma(x)$ *and* $M, t_{2i+1} \vDash p$ *for* $0 \leqslant i \leqslant n$. *By our assumption* σ *is a unifier of* φ *then* $\vdash \sigma(x) \rightarrow [p]\sigma(x)$. *Since* $M, t_1 \vDash \sigma(x)$ *therefore* $M, t_{2i+1} \vDash \sigma(x)$ *for* $0 \leqslant i \leqslant n$. *Then,* $M, t_{2n+1} \vDash [p]\sigma(x)$. *Since,* $t_{2n+1} R_1 s_0 R_2 s$ *therefore* $M, s \vDash (p \rightarrow \sigma(x))$. *Since,* $M, s \vDash p$ *thus* $M, s \vDash \sigma(x)$. *This is contradiction .*

Lemma 69 *The set of substitutions* $\Sigma = \{\sigma_p\} \cup \{\sigma_n \mid n \in \mathbb{N}\}$ *is a complete set of* $KD_1 \otimes KD_2$*-unifiers of* φ.

Proof 110 *By Lemmas* 63, 64, 65, 66 *and* 67 *and Theorem* 68.

Lemma 70 *The formula* φ *does not possess a minimal complete set of* $KD_1 \otimes KD_2$*-unifiers.*

Proof 111 ***Proof 112*** *Let* Γ *be a minimal complete set of unifiers of* φ *and* $\sigma \in \Gamma$. *Since* Γ *is complete then let* $\sigma \in \Gamma$ *be such that* $\sigma \preccurlyeq \sigma_0$. *Let* $n = deg(\sigma(x))$ *Since* σ *is a unifier of* φ *hence* $\sigma_\top \preccurlyeq_{KD_1 \otimes KD_2} \sigma$ *or* $\sigma_n \preccurlyeq_{KD_1 \otimes KD_2} \sigma$ *by Theorem* 68.
Suppose $\sigma_n \preccurlyeq_{KD_1 \otimes KD_2} \sigma$. *By definition of* Σ, $\sigma_{n+1} \in \Sigma$. *Let* $\sigma' \in \Gamma$ *such that* $\sigma' \preccurlyeq \sigma_{n+1}$. *Since* $\sigma' \preccurlyeq \sigma_{n+1} \preccurlyeq \sigma_n \preccurlyeq \sigma$ *then* $\sigma' \preccurlyeq \sigma$. *Since* Γ *is minimal complete set and its members are pairwise incomparable then* $\sigma' = \sigma$. *Since* $\sigma_n \preccurlyeq \sigma$ *and* $\sigma' \preccurlyeq \sigma_{n+1}$ *then,* $\sigma_n \preccurlyeq \sigma_{n+1}$. *Since* $n < n+1$ *by lemma 65,* $\sigma_n \not\preccurlyeq \sigma_{n+1}$ *and this is a contradiction.*
Suppose $\sigma_p \preccurlyeq_{KD_1 \otimes KD_2} \sigma$. *Since* $\sigma \preccurlyeq \sigma_0$ *then,* $\sigma_p \preccurlyeq \sigma_0$. *Therefore* $\vdash \neg p$. *This is contradiction.*

Lemma 71 *Unification type is nullary in fusion* $KD_1 \otimes KD_2$

Proof 113 *By Lemma* 70.

6.5 Unification in Fusion $S5 \otimes S5$

In this section we will discuss on unification type of the fusion $S5_1 \otimes S5_2$ and we will show that unification type of fusion $S5_1 \otimes S5_2$ is nullary. By doing so, we are answering an open question of Dzik [22] (2007) who conjectures that $S5_1 \otimes S5_2$ is nullary or infinitary. In this respect, we consider the formula $\varphi_0 = (x \to \boxslash x) \wedge (\neg x \to \boxbslash \neg x)$ where,

- $\boxslash \psi = \Box_1(\overline{p}\,\overline{q}\,\overline{r} \to \Box_2(\overline{p}\,\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\,\overline{r} \to \Box_2(p\overline{q}r \to \psi))))))$ and
- $\boxbslash \psi = (p\overline{q}r \to \Box_2(p\overline{q}\,\overline{r} \to \Box_1(\overline{p}qr \to \Box_2(\overline{p}q\overline{r} \to \Box_1(\overline{p}\,\overline{q}r \to \Box_2(\overline{p}\,\overline{q}\,\overline{r} \to \Box_1\psi))))))$.

We will show that φ_0 is nullary for $S5_1 \otimes S5_2$. In order to prove the unification type of the fusion $S5_1 \otimes S5_2$ is nullary we need to define the modal connective $\boxslash^k$ and $\boxbslash^k$ inductively as follows for each $k \in \mathbb{N}$:

- $\boxslash^0 \varphi ::= \varphi$,
- $\boxslash^{k+1} \varphi ::= \boxslash \boxslash^k \varphi$.
- $\boxbslash^0 \varphi ::= \varphi$,
- $\boxbslash^{k+1} \varphi ::= \boxbslash \boxbslash^k \varphi$.

As a result, we define also:

- $\Diamond^0 \varphi ::= \varphi$
- $\Diamond^{k+1} \varphi ::= \Diamond \Diamond^k \varphi$.

The modal connective $\boxslash^{<k}$ and $\boxbslash^{<k}$ are inductively defined as follows for each $k \in \mathbb{N}$:

- $\boxslash^{<0} \varphi ::= \top$.
- $\boxslash^{<k+1} \varphi ::= \boxslash^{<k} \varphi \wedge \boxslash^k \varphi$.
- $\boxbslash^{<0} \varphi ::= \top$.
- $\boxbslash^{<k+1} \varphi ::= \boxbslash^{<k} \varphi \wedge \boxbslash^k \varphi$.

As a result, we define also:

- $\diamondslash^{0}\varphi ::= \bot$.
- $\diamondslash^{<k+1}\varphi ::= \diamondslash^{<k}\varphi \vee \diamondslash^{k}\varphi$.

Initially, we need to prove the following Lemma:

Lemma 72 *For all formulas φ,*

1. *If* $\vdash \varphi$ *then,* $\vdash \boxslash\varphi$.
2. $\vdash \boxslash(\varphi \to \psi) \to (\boxslash\varphi \to \boxslash\psi)$.
3. $\vdash \boxslash^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxslash\boxslash^{<k}\varphi$.
4. $\vdash \boxslash^{<k}\varphi \to \varphi$ *where* $k \geqslant 1$.
5. *If* $k \leqslant l$ *then,* $\vdash \boxslash^{<k}\bot \to \boxslash^{<l}\bot$.

Proof 114 1. *Suppose,*

1$-\vdash \psi$ *hence,*

2$-\vdash (p\overline{q}r \to \psi)$ *by* 1 *and CP*

3$-\vdash \Box_2(p\overline{q}r \to \psi)$ *by* 2 *and necessitation*

4$-\vdash p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)$ *by* 3 *and CP*

5$-\vdash \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))$ *by* 4 *and necessitation*

6$-\vdash \overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))$ *by* 5 *and Cp*

7$-\vdash \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))$ *by* 6 *and necessitation*

8$-\vdash \overline{p}q\overline{r} \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))))$ *by* 7 *and CP*

9$-\vdash \Box_1(\overline{p}q\overline{r} \to\to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))))$ *by* 8 *and necessitation*

10$-\vdash \overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))$ *by* 9 *and CP*

11$-\vdash \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))))$ *by* 10 *and necessitation*

12$-\vdash \overline{p}\overline{q}\overline{r} \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))))$ *by* 11 *and CP*

13$-\vdash \Box_1(\overline{p}\overline{q}\overline{r} \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))))$ *by* 12 *and necessitation.*

Therefore, $\vdash \boxslash\psi$.

2. *We have by tautology*
   $(1) \vdash (\varphi \to \psi) \to (\varphi \to \psi)$ *hence,*
   $(2) \vdash (p\overline{q}r \to (\varphi \to \psi)) \to (p\overline{q}r \to (\varphi \to \psi))$. *By* 2 *and CP*
   $(3) \vdash (p\overline{q}r \to (\varphi \to \psi)) \to ((p\overline{q}r \to \varphi) \to (p\overline{q}r \to \psi))$. *By* 3*, necessitation and axiom k*
   $(4) \vdash \Box_2(p\overline{q}r \to (\varphi \to \psi)) \to (\Box_2(p\overline{q}r \to \varphi) \to \Box_2(p\overline{q}r \to \psi))$. *By* 4 *and CP*
   $(5) \vdash \big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big) \to \big(\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big) \to \big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)$. *By* 5*, necessitation and axiom k.*
   $(6) \vdash \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big) \to \big(\Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big) \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)$. *By* 6*, CP, necessitation and axiom k*
   $(7) \vdash \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big)\big) \to$
   $\Big(\Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big)\big) \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)\Big)$. *By* 7*, CP, necessitation and axiom k*
   $(8) \vdash \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big)\big)\Big) \to$
   $\Big(\Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big)\big)\Big) \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)\Big)\Big)$.
   By 8*, CP, necessitation and axiom k*
   $(9) \vdash \Box_2\Big(\overline{p}\overline{q}r \to \Box_1\big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big)\big)\big)\Big) \to$
   $\Big(\Box_2\Big(\overline{p}\overline{q}r \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big)\big)\Big)\Big) \to$
   $\Box_2\Big(\overline{p}\overline{q}r \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)\Big)\Big)\Big)$. *By* 9*, CP, necessitation and axiom k*
   $(10) \vdash \Box_1\Big(\overline{p}\overline{q}\overline{r} \to \Box_2\Big(\overline{p}\overline{q}r \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to (\varphi \to \psi))\big)\big)\Big)\Big)\Big) \to$
   $\Big(\Box_1\Big(\overline{p}\overline{q}\overline{r} \to \Box_2\Big(\overline{p}\overline{q}r \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \varphi)\big)\big)\Big)\Big)\Big) \to$
   $\Box_1\Big(\overline{p}\overline{q}\overline{r} \to \Box_2\Big(\overline{p}\overline{q}r \to \Box_1\Big(\overline{p}q\overline{r} \to \Box_2\big(\overline{p}qr \to \Box_1\big(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)\big)\big)\Big)\Big)\Big)\Big)$.
   Therefore, $\vdash \boxslash(\varphi \to \psi) \to (\boxslash\varphi \to \boxslash\psi)$.

3. *Since,* $\vdash \boxslash^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxslash\varphi \wedge \ldots \wedge \boxslash^{k}\varphi$ *and* $\vdash \varphi \wedge \boxslash\varphi \wedge \ldots \wedge \boxslash^{k}\varphi \leftrightarrow \varphi \wedge \boxslash(\varphi \wedge \ldots \wedge \boxslash^{k-1}\varphi)$ *then,* $\vdash \boxslash^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxslash\,\boxslash^{<k}\varphi$.

4. *Soppose* $k \geqslant 1$. *Since* $\vdash \boxslash^{<k}\varphi \leftrightarrow \varphi \wedge \boxslash\varphi \wedge \ldots \wedge \boxslash^{k-1}\varphi$ *and* $\vdash \varphi \wedge \boxslash\varphi \wedge \ldots \wedge \boxslash^{k-1}\varphi \to \varphi$ *thus,* $\vdash \boxslash^{<k}\varphi \to \varphi$.

5. *Let* $k \leqslant l$. *Since,* $\vdash \bot \to \boxslash^{l-k}\bot$ *then we have* $\vdash \boxslash^{k}(\bot \to \boxslash^{l-k}\bot)$ *by part* (1). *Since,* $\vdash \boxslash^{k}(\bot \to \boxslash^{l-k}\bot)$ *thus we have* $\vdash \boxslash^{k}\bot \to \boxslash^{l}\bot$ *by part* (3).

Lemma 73 *Let $k, l \in \mathbb{N}$. For all formulas φ, ψ,*

1. *If* $\vdash \varphi$ *then,* $\vdash \boxbslash\varphi$.
2. $\vdash \boxbslash(\varphi \to \psi) \to (\boxbslash\varphi \to \boxbslash\psi)$.
3. $\vdash \boxbslash^{<k+1}\varphi \leftrightarrow \varphi \wedge \boxbslash \boxbslash^{<k} \varphi$.
4. $\vdash \boxbslash^{<k}\varphi \to \varphi$ *where* $k \geqslant 1$.
5. *If* $k \leqslant l$ *then,* $\vdash \boxbslash^{<k}\bot \to \boxslash^{<l}\bot$.
6. *If* $k \leqslant l$ *then,* $\vdash \Diamond^{l}\top \to \Diamond^{k}\top$.
7. *If* $k < l$ *then* $\nvdash \boxslash^{l}\bot \to \boxslash^{k}\bot$.
8. *If* $k < l$ *then* $\nvdash \Diamond^{k}\top \to \Diamond^{l}\top$.
9. $\nvdash \Diamond^{l}\top \to \boxslash^{k}\bot$.
10. $\nvdash \Diamond^{k}\top \to \boxslash^{l}\bot$.

Proof 115 *We prove items* 7 *to* 10.
7. *Let $M = (W, R_1, R_2, v)$ be the modal defined as follows:*
$W = \{s_0, s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}, s_{1,6} \ldots s_{k,1}, s_{k,2}, s_{k,3}, s_{k,4}, s_{k,5}, s_{k,6}\}$,
R_1 *is the least equivalence relation on W such that* $s_0 R_1 s_{1,1}$, $s_{1,2} R_1 s_{1,3}$, $s_{1,4} R_1 s_{1,5}$, $s_{1,6} R_1 s_{2,1}$, ..., $s_{k-1,6} R_1 s_{k,1}$, $s_{k,2} R_1 s_{k,3}$, $s_{k,4} R_1 s_{k,5}$,
R_2 *is the least equivalence relation on W such that* $s_{1,1} R_2 s_{1,2}$, $s_{1,3} R_2 s_{1,4}$, $s_{1,5} R_2 s_{1,6}$, ..., $s_{k,1} R_2 s_{k,2}$, $s_{k,3} R_2 s_{k,4}$, $s_{k,5} R_2 s_{k,6}$,
$v(p) = \{s_{1,5}, s_{1,6}, \ldots, s_{k,5}, s_{k,6}\}$,
$v(q) = \{s_{1,3}, s_{1,6}, \ldots, s_{k,5}, s_{k,6}\}$,
$v(r) = \{s_{1,2}, s_{1,4}, s_{1,6}, \ldots, s_{k,2}, s_{k,4}, s_{k,6}\}$.
Obviously, $M, s_0 \models \boxslash^{l}\bot$ *but* $M, s_0 \not\models \boxslash^{k}\bot$. *Thus,* $\nvdash \boxslash^{l}\bot \to \boxslash^{k}\bot$.
8. *Let $M = (W, R_1, R_2, v)$ be the modal defined as follows:*
$W = \{s_0, s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}, s_{1,6}, \ldots s_{k,1}, s_{k,2}, s_{k,3}, s_{k,4}, s_{k,5}, s_{k,6}\}$,
R_1 *is the least equivalence relation on W such that* $s_{1,1} R_1 s_{1,2}$, $s_{1,3} R_1 s_{1,4}$, $s_{1,5} R_1 s_{1,6}$, ..., $s_{k,1} R_1 s_{k,2}$, $s_{k,3} R_1 s_{k,4}$, $s_{k,5} R_1 s_{k,6}$,
R_2 *is the least equivalence relation on W such that* $s_0 R_2 s_{1,1}$, $s_{1,2} R_2 s_{1,3}$, $s_{1,4} R_2 s_{1,5}$, $s_{1,6} R_2 s_{2,1}$, ..., $s_{k-1,6} R_2 s_{k,1}$, $s_{k,2} R_2 s_{k,3}$, $s_{k,4} R_2 s_{k,5}$,
$v(p) = \{s_0, s_{1,1}, \ldots, s_{2,1}, s_{2,6}\}$,
$v(q) = \{s_{1,2}, s_{1,3}, \ldots, s_{k,2}, s_{k,3}\}$,

$v(r) = \{s_0, s_{1,2}, s_{1,4}, ..., s_{k,2}, s_{k,4}, s_{k,6}\}$.
Obviously, $M, s_0 \vDash \diamondslash^k \top$ *but* $M, s_0 \nvDash \diamondslash^l \top$. *Thus,* $\nvdash \diamondslash^k \top \to \diamondslash^l \top$.
9. *Suppose* $\vdash \diamondslash^l \top \to \boxslash^k \bot$. *Hence, by Lemma* 72, $\vdash \diamondslash^{l-i} \top \to \boxslash^{k+1} \bot$. *Thus, by using Lemma* $extralemmas$, $l-1$ *times, we obtain* $\vdash \top \to \boxslash^{k+1} \bot$. *Hence,* $\vdash \boxslash^{k+1} \bot$. *Thus,* $\vdash \boxslash^{k+l+1} \bot \to \boxslash^{k+l} \bot$*: a contradiction with item* (7).
10. *Suppose* $\vdash \diamondslash^k \top \to \boxslash^l \bot$. *Hence, by using Lemma* 72, $\diamondslash^{k-1} \top \to \boxslash^{l+1} \bot$. *Thus by Lemma* 72, $k-1$ *times, we obtain* $\vdash \top \to \boxslash^{l+1} \bot$. *Here,* $\vdash \boxslash^{l+1} \bot$. *Thus,* $\vdash \boxslash^{k+l+1} \bot \to \boxslash^{k+l} \bot$*: a contradiction with item* (7).

Consider substitutions $\sigma_k(x) = \boxslash^{<k} x \wedge \boxslash^k \bot$ and $\tau_k(x) = \neg(\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot)$. We will show that $\sigma_k(x)$ and $\tau_k(x)$ are $(S5_1 \otimes S5_2)$-unifiers of φ_0. Notice that $\sigma_k(x)$ and $\tau_k(x)$ can be written as follows:
$\sigma_0(x) = \bot$ and,
$\sigma_{k+1}(x) = \boxslash^{<k+1} x \wedge \boxslash^{k+1} \bot = x \wedge \boxslash \boxslash^{<k} x \wedge \boxslash^{k+1} \bot = x \wedge \boxslash(\boxslash^{<k} x \wedge \boxslash^k \bot) = x \wedge \boxslash \sigma_k(x)$.
$\tau_0(x) = \top$ and,
$\tau_{k+1}(x) = \neg(\boxbslash^{<k+1} \neg x \wedge \boxbslash^{k+1} \bot) = \neg(\neg x \wedge \boxbslash \boxbslash^{<k} \neg x \wedge \boxbslash^{k+1} \bot) = \neg(\neg x \wedge \boxbslash(\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot)) = x \vee \diamondslash \tau_k(x)$.
It is well-known that in $S5_1 \otimes S5_2$, we have for all formula φ, ψ

- $\vdash \varphi \to \Box_1 \psi$ iff $\vdash \Diamond_1 \varphi \to \psi$ and
- $\vdash \varphi \to \Box_2 \psi$ iff $\vdash \Diamond_2 \varphi \to \psi$. Moreover,

Lemma 74 *For all formulas* ψ*, the following conditions are equivalent:*

1. $\vdash \varphi \to \boxslash \psi$.
2. $\vdash \neg \psi \to \boxbslash \neg \varphi$.

Proof 116 *Suppose* $\vdash_L \varphi \to \boxslash \psi$. *Then,*
$\vdash_L \varphi \to \Box_1(\overline{p}\overline{q}\overline{r} \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))))$
$\vdash_L \Diamond_1 \varphi \to (\overline{p}\overline{q}\overline{r} \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))))$
$\vdash_L (\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi) \to \Box_2(\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))))$
$\vdash_L \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi) \to (\overline{p}\overline{q}r \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))))$
$\vdash_L \overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi) \to \Box_1(\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))$
$\vdash_L \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi)) \to (\overline{p}q\overline{r} \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))))$
$\vdash_L \overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi)) \to \Box_2(\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))$
$\vdash_L \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1 \varphi))) \to (\overline{p}qr \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi)))$

$\vdash_L \overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi))) \to \Box_1(p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))$
$\vdash_L \Diamond_1(\overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi)))) \to (p\overline{q}\overline{r} \to \Box_2(p\overline{q}r \to \psi))$
$\vdash_L p\overline{q}\overline{r} \wedge \Diamond_1(\overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi)))) \to \Box_2(p\overline{q}r \to \psi)$
$\vdash_L \Diamond_2(p\overline{q}\overline{r} \wedge \Diamond_1(\overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi))))) \to (p\overline{q}r \to \psi)$
$\vdash_L (p\overline{q}r \wedge \Diamond_2(p\overline{q}\overline{r} \wedge \Diamond_1(\overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi)))))) \to \psi$
$\vdash_L \neg\psi \to \neg(p\overline{q}r \wedge \Diamond_2(p\overline{q}\overline{r} \wedge \Diamond_1(\overline{p}qr \wedge \Diamond_2(\overline{p}q\overline{r} \wedge \Diamond_1(\overline{p}\overline{q}r \wedge \Diamond_2(\overline{p}\overline{q}\overline{r} \wedge \Diamond_1\varphi))))))$
$\vdash_L \neg\psi \to (p\overline{q}r \to \Box_2(p\overline{q}\overline{r} \to \Box_1(\overline{p}qr \to \Box_2(\overline{p}q\overline{r} \to \Box_1(\overline{p}\overline{q}r \to \Box_2(\overline{p}\overline{q}\overline{r} \to \Box_1\psi))))))$
$\vdash_L \neg\psi \to \boxbslash\neg\varphi$. *The proof of the converse direction is similar.*

Lemma 75 *For all $k \in \mathbb{N}$,*

1. *σ_k is an $(S5_1 \otimes S5_2)$-unifier of φ_0.*

2. *τ_k is an $(S5_1 \otimes S5_2)$-unifier of φ_0.*

Proof 117 *Let $k \in \mathbb{N}$.*

1. *By Lemma 74, it suffices to prove $\vdash \sigma_k(x) \to \boxslash\sigma_k(x)$. In fact, we have to prove*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to \boxslash(\boxslash^{<k}x \wedge \boxslash^k\bot)$ *or equivalently*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to \boxslash\boxslash^{<k}x \wedge \boxslash\boxslash^k\bot$. *We know that*
   $\boxslash\boxslash^{<k}x = \boxslash(x \wedge \boxslash x \wedge \ldots \wedge \boxslash^{(k-1)}x) = \boxslash x \wedge \boxslash\boxslash x \wedge \ldots \wedge \boxslash^k x$ *and*
   $\boxslash^{<k}x = (x \wedge \boxslash x \wedge \boxslash\boxslash x \wedge \ldots \wedge \boxslash^{(k-1)}x)$. *Since,*
   $\vdash (x \wedge \boxslash x \wedge \boxslash\boxslash x \wedge \ldots \wedge \boxslash^{(k-1)}x) \to (\boxslash x \wedge \boxslash\boxslash x \wedge \ldots \wedge \boxslash^{(k-1)}x)$ *and*
   $\vdash \boxslash^k\bot \to \boxslash^k x$ *then we have,*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to (\boxslash x \wedge \boxslash\boxslash x \wedge \ldots \wedge \boxslash^{(k-1)}x \wedge \boxslash^k x)$. *Thus,*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to \boxslash\boxslash^{<k}x$. *By part* (5) *of Lemma* 72*,*
   $\vdash \boxslash^k\bot \to \boxslash^{k+1}\bot$. *Since,*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to \boxslash\boxslash^{<k}x$ *therefore,*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to (\boxslash\boxslash^{<k}x \wedge \boxslash^{k+1}\bot)$ *or equivalently*
   $\vdash (\boxslash^{<k}x \wedge \boxslash^k\bot) \to \boxslash(\boxslash^{<k}x \wedge \boxslash^k\bot)$. *Therefore,*
   $\vdash \sigma_k(x) \to \boxslash\sigma_k(x)$.

2. *By Lemma 74, it suffices to prove $\vdash \neg\tau_k(x) \to \boxbslash\neg\tau_k(x)$. In fact, we have to prove $\vdash (\boxbslash^{<k}\neg x \wedge \boxbslash^k\bot) \to \boxbslash(\boxbslash^{<k}\neg x \wedge \boxbslash^k\bot)$. We know that*
   $\boxbslash\boxbslash^{<k}\neg x = \boxbslash(\neg x \wedge \boxbslash\neg x \wedge \ldots \wedge \boxbslash^{(k-1)}\neg x) = \boxbslash\neg x \wedge \boxbslash\boxbslash\neg x \wedge \ldots \wedge \boxbslash^k\neg x$ *and*
   $\boxbslash^{<k}\neg x = (\neg x \wedge \boxbslash\neg x \wedge \boxbslash\boxbslash\neg x \wedge \ldots \wedge \boxbslash^{(k-1)}\neg x)$. *Since,*
   $\vdash (\neg x \wedge \boxbslash\neg x \wedge \boxbslash\boxbslash\neg x \wedge \ldots \wedge \boxbslash^{(k-1)}\neg x) \to (\boxbslash\neg x \wedge \boxbslash\boxbslash\neg x \wedge \ldots \wedge \boxbslash^{(k-1)}\neg x)$
   and

$\vdash \boxbslash^k \bot \to \boxbslash^k \neg x$ *then we have,*
$\vdash (\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot) \to (\boxbslash \neg x \wedge \boxbslash \boxbslash \neg x \wedge \ldots \wedge \boxbslash^{(k-1)} \neg x \wedge \boxbslash^k \neg x)$. *Thus,*
$\vdash (\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot) \to \boxbslash \boxbslash^{<k} \neg x$. *By part* (5) *of Lemma* 73*,*
$\vdash \boxbslash^k \bot \to \boxbslash^{k+1} \bot$. *Since*
$\vdash (\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot) \to \boxbslash \boxbslash^{<k} \neg x$ *therefore,*
$\vdash (\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot) \to (\boxbslash \boxbslash^{<k} \neg x \wedge \boxbslash^{k+1} \bot)$ *or equivalently*
$\vdash (\boxbslash^{<k} \neg x \wedge \boxbslash^k \bot) \to \boxbslash(\boxbslash^{<k} \neg x \wedge \boxslash^k \bot)$. *Then,*
$\vdash \neg \tau_k(x) \to \boxbslash \neg \tau_k(x)$.

Lemma 76 *For all $k, l \in \mathbb{N}$, if $k \leqslant l$ then $\sigma_l \preccurlyeq_{S5_1 \otimes S5_2} \sigma_k$ and $\tau_l \preccurlyeq_{S5_1 \otimes S5_2} \tau_k$.*

Proof 118 *Let $k, l \in \mathbb{N}$. Suppose $k \leqslant l$. We have to prove $\sigma_l \preccurlyeq_{S5_1 \otimes S5_2} \sigma_k$. Let the substitution υ be defined by $\upsilon(x) = x \wedge \boxslash^k \bot$. We want to show that $\sigma_l \circ \nu \simeq_{S5_1 \otimes S5_2} \sigma_k$. Hence we have to prove*
$\vdash (\boxslash^{<l}(x \wedge \boxslash^k \bot) \wedge \boxslash^l \bot) \leftrightarrow (\boxslash^{<k} x \wedge \boxslash^k \bot)$. *Let us prove*
$\vdash (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot) \wedge \boxslash^l \bot \to (\boxslash^{<k} x \wedge \boxslash^k \bot)$. *Since*
$\boxslash^{<l} x = x \wedge \boxslash x \wedge \ldots \wedge \boxslash^{l-1} x$ *and* $\boxslash^{<k} x = x \wedge \boxslash x \wedge \ldots \wedge \boxslash^{k-1} x$ *hence,*
$\vdash \boxslash^{<l} x \to \boxslash^{<k} x$. *Thus*
$\vdash (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot) \to \boxslash^{<k} x$. *Since*
$\vdash \boxslash^{<l} \boxslash^k \bot = \boxslash^k \bot \wedge \boxslash \boxslash^k \bot \ldots \wedge \boxslash^{l-1} \boxslash^k \bot$ *then*
$\vdash \boxslash^{<l} \boxslash^k \bot \to \boxslash^k \bot$. *Hence,*
$\vdash (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot) \to \boxslash^k \bot$. *Since,*
$\vdash (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot) \to \boxslash^{<k} x$ *therefore*
$\vdash (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot) \wedge \boxslash^l \bot \to (\boxslash^{<k} x \wedge \boxslash^k \bot)$.

For the other direction we have to prove
$\vdash (\boxslash^{<k} x \wedge \boxslash^k \bot) \to (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot \wedge \boxslash^l \bot)$. *Since*
$\boxslash^{<l} x = x \wedge \boxslash x \wedge \ldots \wedge \boxslash^{l-1} x$ *and* $\boxslash^{<k} x = x \wedge \boxslash x \wedge \ldots \wedge \boxslash^{k-1} x$ *hence,*
$\vdash (\boxslash^{<k} x \wedge \boxslash^k \bot) \to \boxslash^{<l} x$. *By part* (5) *of Lemma* 72*,*
$\vdash \boxslash^k \bot \to \boxslash^l \bot$. *Since,* $\boxslash^{<l} \boxslash^k \bot = \boxslash^k \bot \wedge \ldots \wedge \boxslash^{l-1} \boxslash^k \bot$ *hence, by part* (5) *of Lemma* 72*,*
$\vdash \boxslash^k \bot \to \boxslash^{<l} \boxslash^k \bot$. *Since*
$\vdash (\boxslash^{<k} x \wedge \boxslash^k \bot) \to \boxslash^{<l} x$ *and*
$\vdash \boxslash^k \bot \to \boxslash^l \bot$ *and*
$\vdash \boxslash^k \bot \to \boxslash^{<l} \boxslash^k \bot$ *therefore,*
$\vdash (\boxslash^{<k} x \wedge \boxslash^k \bot) \to (\boxslash^{<l} x \wedge \boxslash^{<l} \boxslash^k \bot \wedge \boxslash^l \bot)$.
Consequently, $\vdash (\boxslash^{<l}(x \wedge \boxslash^k \bot) \wedge \boxslash^l \bot) \leftrightarrow (\boxslash^{<k} x \wedge \boxslash^k \bot)$. *Thus,* $\sigma_l \circ \nu \simeq_{S5_1 \otimes S5_2} \sigma_k$. *Therefore,* $\sigma_l \preccurlyeq_{S5_1 \otimes S5_2} \sigma_k$.

We have to prove $\tau_l \preccurlyeq_{S5_1 \otimes S5_2} \tau_k$. Let the substitution υ be defined by $\upsilon(x) = x \vee \Diamond^k \top$. Let us show $\tau_l \circ \nu \simeq_{S5_1 \otimes S5_2} \tau_k$. Hence we have to show $\vdash (\Diamond^{<l}(x \vee$

$\Diamond^k\top) \vee \Diamond^l\top) \leftrightarrow (\Diamond^{<k}x \vee \Diamond^k\top)$. *Let us prove*
$\vdash (\Diamond^{<l}(x \vee \Diamond^k\top) \vee \Diamond^l\top) \to (\Diamond^{<k}x \vee \Diamond^k\top)$ *or equivalently*
$\vdash (\Diamond^{<l}x \vee \Diamond^{<l}\Diamond^k\top \vee \Diamond^l\top) \to (\Diamond^{<k}x \vee \Diamond^k\top)$. *By part* (6) *of Lemma* 73,
$\vdash \Diamond^l\top \to \Diamond^k\top$.*Since,* $\Diamond^{<l}\Diamond^k\top = \Diamond^k\top \vee \ldots \vee \Diamond^{l-1}\Diamond^k\top$ *hence by part* (6) *of Lemma* 73
$\vdash \Diamond^{<l}\Diamond^k\top \to \Diamond^k\top$. *Since,* $\Diamond^{<l}x = x \vee \ldots \vee \Diamond^{l-1}x$ *and* $\Diamond^{<k}x = x \vee \ldots \vee \Diamond^{k-1}x$ *hence*
$\vdash x \vee \ldots \vee \Diamond^{k-1}x \vee \Diamond^k x \vee \ldots \vee \Diamond^{l-1}x \to x \vee \ldots \vee \Diamond^{k-1}x \vee \Diamond^k\top$ *then,*
$\vdash \Diamond^{<l}x \to \Diamond^{<k}x \vee \Diamond^k\top$. *Since,*
$\vdash \Diamond^l\top \to \Diamond^k\top$ *and*
$\vdash \Diamond^{<l}\Diamond^k\top \to \Diamond^k\top$ *and*
$\vdash \Diamond^{<l}x \to \Diamond^{<k}x \vee \Diamond^k\top$ *thus,*
$\vdash (\Diamond^{<l}x \vee \Diamond^{<l}\Diamond^k\top \vee \Diamond^l\top) \to (\Diamond^{<k}x \vee \Diamond^k\top)$. *Therefore,*
$\vdash (\Diamond^{<l}(x \vee \Diamond^k\top) \vee \Diamond^l\top) \to (\Diamond^{<k}x \vee \Diamond^k\top)$.

For the other direction we have to prove
$\vdash \Diamond^{<k}x \vee \Diamond^k\top \to \Diamond^{<l}x \vee \Diamond^{<l}\Diamond^k\top \vee \Diamond^l\top$. *Since*
$\vdash \Diamond^{<k}x \to \Diamond^{<l}x$ *and* $\Diamond^{<l}\Diamond^k\top = \Diamond^k\top \vee \ldots \vee \Diamond^{l-1}\Diamond^k\top$ *then,*
$\vdash \Diamond^k\top \to \Diamond^{<l}\Diamond^k\top$. *Therefore,*
$\vdash \Diamond^{<k}x \vee \Diamond^k\top \to \Diamond^{<l}x \vee \Diamond^{<l}\Diamond^k\top \vee \Diamond^l\top$. *Since,*
$\vdash (\Diamond^{<l}(x \vee \Diamond^k\top) \vee \Diamond^l\top) \to (\Diamond^{<k}x \vee \Diamond^k\top)$ *and*
$\vdash (\Diamond^{<k}x \vee \Diamond^k\top) \to \Diamond^{<l}(x \vee \Diamond^k\top) \vee \Diamond^l\top$ *then,*
$\vdash (\Diamond^{<l}(x \vee \Diamond^k\top) \vee \Diamond^l\top) \leftrightarrow (\Diamond^{<k}x \vee \Diamond^k\top)$. *Hence,* $\tau_l \circ \nu \simeq_{S5_1 \otimes S5_2} \tau_k$. *Therefore,* $\tau_l \preccurlyeq_{S5_1 \otimes S5_2} \tau_k$.

Lemma 77 *For all* $k, l \in \mathbb{N}$, *if* $k < l$ *then* $\sigma_k \not\preccurlyeq_{S5_1 \otimes S5_2} \sigma_l$ *and* $\tau_k \not\preccurlyeq_{S5_1 \otimes S5_2} \tau_l$.

Proof 119 *Suppose* $k < l$ *and* $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \sigma_l$. *Let* υ *be a substitution such that* $\sigma_k \circ \upsilon \simeq_{S5_1 \otimes S5_2} \sigma_l$. *Hence,* $\vdash \upsilon(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. *Thus,* $\vdash (\boxslash^{<k}\upsilon(x) \wedge \boxslash^k\bot) \leftrightarrow (\boxslash^{<l}x \wedge \boxslash^l\bot)$. *Hence,* $\vdash (\boxslash^{<l}x \wedge \boxslash^l\bot) \to \boxslash^k\bot$. *Thus by replacing* x *by* $\top$, $\vdash \boxslash^l\bot \to \boxslash^k\bot$. *This is a contradiction with the part* (7) *of Lemma* 73.
Suppose $k < l$ *and* $\tau_k \preccurlyeq_{S5_2 \otimes S5_2} \tau_l$. *Let* υ *be a substitution such that* $\tau_k \circ \upsilon \simeq_{S5_1 \otimes S5_2} \tau_l$. *Hence,* $\vdash \upsilon(\tau_k(x)) \leftrightarrow \tau_l(x)$. *Thus,* $\vdash (\Diamond^{<k}\upsilon(x) \vee \Diamond^k\top) \leftrightarrow (\Diamond^{<l}x \vee \Diamond^l\top)$. *Hence,* $\vdash \Diamond^{<k}\upsilon(x) \vee \Diamond^k\top \to \Diamond^{<l}x \vee \Diamond^l\top$. *Then,* $\vdash \Diamond^k\top \to \Diamond^{<l}x \vee \Diamond^l\top$. *Hence, by replacing* x *by* $\bot$, $\vdash \Diamond^k\top \to \Diamond^l\top$. *This is a contradiction with the part* (8) *of Lemma* 73.

Lemma 78 *For all* $k, l \in \mathbb{N}$, *if* $k < l$ *then* $\sigma_k \not\preccurlyeq_{S5_1 \otimes S5_2} \tau_l$ *and* $\tau_k \not\preccurlyeq_{S5_1 \otimes S5_2} \sigma_l$.

Proof 120 *Suppose* $k < l$ *and* $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \tau_l$. *Let* υ *be a substitution such that* $\sigma_k \circ \upsilon \simeq_{S5_1 \otimes S5_2} \tau_l$. *Hence,* $\vdash \upsilon(\sigma_k(x)) \leftrightarrow \tau_l(x)$. *Then,* $\vdash (\boxslash^{<k}\upsilon(x) \wedge \boxslash^k \bot) \leftrightarrow (\neg \boxbslash^{<l} \neg x \vee \neg \boxbslash^l \bot)$. *Hence,* $\vdash (\Diamond^{<l} x \vee \Diamond^l \top) \rightarrow \boxslash^{<k}\upsilon(x) \wedge \boxslash^k \bot$. *Thus,* $\vdash \Diamond^l \top \rightarrow \boxslash^k \bot$. *This is a contradiction with the part* (9) *of Lemma* 73.
Suppose $k < l$ *and* $\tau_k \preccurlyeq_L \sigma_l$. *Let* υ *be a substitution such that* $\tau_k \circ \upsilon \simeq_L \sigma_l$. *Hence,* $\vdash \upsilon(\tau_k(x)) \leftrightarrow \sigma_l(x)$. *Then,* $\vdash (\Diamond^{<k}\upsilon(x) \vee \Diamond^k \top) \leftrightarrow (\boxslash^{<l} x \wedge \boxslash^l \bot)$. *Hence,* $\vdash (\Diamond^{<k}\upsilon(x) \vee \Diamond^k \top) \rightarrow (\boxslash^{<l} x \wedge \boxslash^l \bot)$. *Thus,* $\vDash \Diamond^k \top \rightarrow \boxslash^l \bot$. *This is a contradiction with the part* (10) *of Lemma* 73.

Lemma 79 *Let* μ *be an* $(S5_1 \otimes S5_2)$*-unifier of* φ_0. *For all* $k \in \mathbb{N}$, *the following conditions are equivalent:*

1. $\sigma_k \circ \mu \simeq_{S5_1 \otimes S5_2} \mu$,
2. $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$,
3. $\vdash_{S5_1 \otimes S5_2} \mu(x) \rightarrow \boxslash^k \bot$.

Proof 121 $(1 \Rightarrow 2)$ *By definition of* $\preccurlyeq_{S5_1 \otimes S5_2}$.
$(2 \Rightarrow 3)$ *Suppose* $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$. *Let* ν *be a substitution such that* $\sigma_k \circ \upsilon \simeq_{S5_1 \otimes S5_2} \mu$. *Hence,* $\vdash \nu(\sigma_k(x)) \leftrightarrow \mu(x)$. *By definition of* σ_k *we have,* $\vdash (\boxslash^{<k}\upsilon(x) \wedge \boxslash^k \bot) \leftrightarrow \mu(x)$. *therefore,* $\vdash \mu(x) \rightarrow \boxslash^k \bot$.
$(3 \Rightarrow 1)$ *Suppose* $\vdash \mu(x) \rightarrow \boxslash^k \bot$. *Since* μ *is a unifier of* φ_0 *then* $\vdash \mu(x) \rightarrow \boxslash \mu(x)$. *Since,* $\vdash \mu(x) \rightarrow \boxslash \mu(x)$ *then we have,* $\vdash \mu(x) \rightarrow \mu(x) \wedge \boxslash \mu(x) \wedge \ldots \wedge \boxslash^{<k-1}\mu(x)$ *by part* (2) *and* (3) *of Lemma* 72. *Thus,* $\vdash \mu(x) \rightarrow \boxslash^{<k}\mu(x)$. *Since* $\vdash \mu(x) \rightarrow \boxslash^k \bot$ *then* $\vdash \mu(x) \rightarrow \boxslash^k \bot \wedge \boxslash^{<k}\mu(x)$. *Therefore* $\vdash \mu(x) \rightarrow \mu(\sigma_k(x))$. *Now, it is enough to prove* $\vdash \mu(\sigma_k(x)) \rightarrow \mu(x)$. *In this respect, we consider two cases,*
Case $k = 0$: *Since* $\sigma_0(x) = \bot$ *then,* $\mu(\sigma_0(x)) = \bot$.
Case $k \geqslant 1$: *Since,* $\vdash \boxslash \mu^{<k}(x) \rightarrow \mu(x)$ *then,* $\vdash \boxslash \mu^{<k}(x) \wedge \boxslash^k \bot \rightarrow \mu(x)$. *Thus in both cases we have,*
$\vdash \mu(\sigma_k(x)) \rightarrow \mu(x)$. *Since* $\vdash \mu(x) \rightarrow \mu(\sigma_k(x))$ *and* $\vdash \mu(\sigma_k(x)) \rightarrow \mu(x)$ *therefore,* $\vdash \mu(\sigma_k(x)) \leftrightarrow \mu(x)$. *Consequently,* $\mu \circ \sigma_k \simeq_{S5_1 \otimes S5_2} \mu$.

Lemma 80 *Let* μ *be an* $S5_1 \otimes S5_2$*-unifier of* φ_0. *For all* $k \in \mathbb{N}$, *the following conditions are equivalent:*

1. $\tau_k \circ \mu \simeq_{S5_1 \otimes S5_2} \mu$,

2. $\tau_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$,

3. $\vdash_{S5_1 \otimes S5_2} \neg\mu(x) \to \boxbslash^k \bot$.

Proof 122 $(1 \Rightarrow 2)$ *By definition of* $\preccurlyeq_{S5_1 \otimes S5_2}$.
$(2 \Rightarrow 3)$ *Suppose* $\tau_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$. *Let* v *be a substitution such that* $\tau_k \circ v \simeq_{S5_1 \otimes S5_2} \mu$. *Hence,* $\vdash v(\tau_k(x)) \leftrightarrow \mu(x)$. *By definition of* τ_k *we have,* $\vdash \neg(\boxbslash^{<k}\neg v(x) \wedge \boxbslash^k\bot) \leftrightarrow \mu(x)$. *Then,* $\vdash \neg\boxbslash^{<k}\neg v(x) \vee \neg\boxbslash^k\bot \to \mu(x)$. *Hence,* $\vdash \neg\boxbslash^k\bot \to \mu(x)$. *Therefore,* $\vdash \neg\mu(x) \to \boxbslash^k\bot$.
$(3 \Rightarrow 1)$ *Suppose* $\vdash \neg\mu(x) \to \boxbslash^k\bot$. *Then,* $\vdash \neg\boxbslash^k\bot \to \mu(x)$. *Since* μ *is a unifier of* φ_0 *then* $\vdash \neg\mu(x) \to \boxbslash\neg\mu(x)$. *Hence,* $\vdash \neg\mu(x) \to \boxbslash^{<k}\neg\mu(x)$ *by part* (2) *and (3)of Lemma* 73. *Thus,* $\vdash \neg\boxbslash^{<k}\neg\mu(x) \to \mu(x)$. *Since,* $\vdash \neg\boxbslash^k\bot \to \mu(x)$ *hence,* $\vdash \neg(\boxbslash^k\bot \wedge \boxbslash^{<k}\neg\mu(x)) \to \mu(x)$. *Therefore,* $\vdash \mu(\tau_k(x)) \to \mu(x)$. *Now, it is enough to prove* $\vdash \mu(x) \to \mu(\tau_k(x))$, *in this respect, we consider two cases,*
Case $k = 0$: *Since,* $\tau_0(x) = \top$ *thus,* $\vdash \mu(x) \to \mu(\tau_0(x))$.
Case $k \geqslant 1$: *Since,* $\vdash \mu(x) \to \neg\boxbslash^{<k}\neg\mu(x)$ *then,* $\vdash \mu(x) \to \neg\boxbslash\neg\mu(x) \vee \neg\boxbslash^k\bot$. *Thus,* $\vdash \mu(x) \to \mu(\tau_k(x))$.
Since, $\vdash \mu(\tau(x)) \to \mu(x)$ *and* $\vdash \mu(x) \to \mu(\tau_k(x))$ *therefore,* $\vdash \mu(\tau(x)) \leftrightarrow \mu(x)$. *Consequently* $\mu \circ \tau_k \simeq_{S5_1 \otimes S5_2} \mu$.

Theorem 10 *Let* μ *be an* $(S5_1 \otimes S5_2)$*-unifier of* φ. *Then there exists* $k \in \mathbb{N}$ *such that either* $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$ *or* $\tau_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$.

Proof 123 *Let* $k \geqslant deg(\mu(x))$. *Suppose neither* $\sigma_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$ *nor* $\tau_k \preccurlyeq_{S5_1 \otimes S5_2} \mu$ *for all* $k \in \mathbb{N}$. *Let* $k \geqslant deg(\mu(x))$. *Then by Lemmas* 79 *and* 80, $\nvdash_{S5_1 \otimes S5_2} \mu(x) \to \boxslash^k\bot$ *and* $\nvdash_{S5_1 \otimes S5_2} \neg\mu(x) \to \boxbslash^k\bot$. *Let* $\mathcal{M}_1 = \langle W_1, R'_1, R'_2, v_1\rangle$ *and* $\mathcal{M}_2 = \langle W_2, R''_1, R''_2, v_2\rangle$ *be* $(S5_1 \otimes S5_2)$*-models such that* $t_1 \in W_1$, $s_2 \in W_2$ *and* $\mathcal{M}_1, t_1 \nvDash \mu(x) \to \boxslash^k\bot$ *and* $\mathcal{M}_2, s_1 \nvDash \neg\mu(x) \to \boxbslash^k\bot$. *We will define now the unravelling* $\mathcal{M}'_1$ *of* $\mathcal{M}_1$ *around* t_1 *and the unravelling* $\mathcal{M}'_2$ *of* $\mathcal{M}_2$ *around* s_2 *as follows:*
The unravelling of $\mathcal{M}_1$ *around* t_1 *is the model* $\mathcal{M}'_1 = (X_1, S'_1, S'_2, v_1)$ *where*

- X_1 *is the set of all finite sequences of the form* $(m_0, a_1, m_1, ..., a_k, m_k)$ *where* $k \in \mathbb{N}$, $m_0, m_1, ..., m_k \in W_1$, $a_1, ..., a_k \in \{1, 2\}$, $m_0 = t_1$ *and for all* $i \in \mathbb{N}$, *if* $i < k$ *then* $m_i R'_{a_{i+1}} m_{i+1}$.

- S'_1 *is the binary relation on* X_1 *such that* $(u_0, a_1, u_1, ..., a_k, u_k) S'_1 (v_0, b_1, v_1, ..., b_l, v_l)$ *iff there is* $m \in \mathbb{N}$ *such that* $m \leqslant k$, $m \leqslant l$, $(u_0, a_1, u_1, ..., a_m, u_m) = (v_0, b_1, v_1, ..., b_m, v_m)$ *and for all* $i \geqslant m$, *if* $i < k$ *then* $a_{i+1} = 1$ *and if* $i < l$ *then* $b_{i+1} = 1$.

- *S'_2 is the binary relation on X_1 such that $(u_0, a_1, u_1, ..., a_k, u_k) S'_2 (v_0, b_1, v_1, ..., b_l, v_l)$ iff there is $m \in \mathbb{N}$ such that $m \leqslant k$, $m \leqslant l$, $(u_0, a_1, u_1, ..., a_m, u_m) = (v_0, b_1, v_1, ..., b_m, v_m)$ and for all $i \geqslant m$, if $i < k$ then $a_{i+1} = 2$ and if $i < l$ then $b_{i+1} = 2$.*

- *v'_1 is the valuation on X_1 such that for all propositional variables or parameters α, $v'_1(\alpha) = \{(u_0, a_1, u_1, ..., a_k, u_k) \in X_1 : u_k \in v_1(\alpha)\}$.*

The unravelling $\mathcal{M}'_2 = (X_2, S''_1, S''_2, v_2)$ of $\mathcal{M}_2$ around s_2 can be defined in a similar way. Notice that $(t_1) \in X_1$ and $(s_2) \in X_2$. Moreover, notice that S'_1 and S'_2 are equivalence relations on X_1 whereas S''_1 and S''_2 are equivalence relations on X_2. Let $f_1 : X_1 \to W_1$ and $f_2 : X_2 \to W_2$ be defined as follows:

- *For all $(u_0, a_1, u_1, ..., a_k, u_k) \in X_1$, let $f_1(u_0, a_1, u_1, ..., a_k, u_k) = u_k$,*

- *For all $(u_0, a_1, u_1, ..., a_k, u_k) \in X_2$, let $f_2(u_0, a_1, u_1, ..., a_k, u_k) = u_k$.*

Obviously, f_1 is a bounded morphism from $\mathcal{M}'_1$ to $\mathcal{M}_1$ and f_2 is a bounded morphism from $\mathcal{M}'_2$ to $\mathcal{M}_2$. By the bounded morphism Lemma ([18], Theorem 3.14), since $\mathcal{M}_1, t_1 \not\models \mu(x) \to \boxslash^k \bot$ and $\mathcal{M}_2, s_2 \not\models \neg\mu(x) \to \boxbslash^k \bot$, we have $\mathcal{M}'_1, (t_1) \models \mu(x)$, $\mathcal{M}'_1, (t_1) \not\models \boxslash^k \bot$, $\mathcal{M}'_2, (s_2) \not\models \mu(x)$ and $\mathcal{M}'_2, (s_2) \not\models \boxbslash^k \bot$.
Consequently, there exists $t_{2,1}, t_{2,2}, t_{2,3}, t_{2,4}, t_{2,5}, t_{2,6}, ..., t_{k,1}, t_{k,2}, t_{k,3}, t_{k,4}, t_{k,5}, t_{k,6} \in W_1$ such that

- $t_1 R'_1 t_{2,1} R'_2 t_{2,2} R'_1 t_{2,3} R'_2 t_{2,4} R'_1 t_{2,5} R'_2 t_{2,6} ... R'_1 t_{k,1} R'_2 t_{k,2} R'_1 t_{k,3} R'_2 t_{k,4} R'_1 t_{k,5} R'_2 t_{k,6}$,

- $\mathcal{M}_1, t_{2,1} \models \overline{p}\overline{q}\overline{r}$, $\mathcal{M}_1, t_{2,2} \models \overline{p}\overline{q}r$, $\mathcal{M}_1, t_{2,3} \models \overline{p}q\overline{r}$, $\mathcal{M}_1, t_{2,4} \models \overline{p}qr$, $\mathcal{M}_1, t_{2,5} \models p\overline{q}\overline{r}$, $\mathcal{M}_1, t_{2,6} \models p\overline{q}r$, ..., $\mathcal{M}_1, t_{k,1} \models \overline{p}\overline{q}\overline{r}$, $\mathcal{M}_1, t_{k,2} \models \overline{p}\overline{q}r$, $\mathcal{M}_1, t_{k,3} \models \overline{p}q\overline{r}$, $\mathcal{M}_1, t_{k,4} \models \overline{p}qr$, $\mathcal{M}_1, t_{k,5} \models p\overline{q}\overline{r}$, $\mathcal{M}_1, t_{k,6} \models p\overline{q}r$.

Similarly, there exists $s_{2,1}, s_{2,2}, s_{2,3}, s_{2,4}, s_{2,5}, s_{2,6}, ..., s_{k,1}, s_{k,2}, s_{k,3}, s_{k,4}, s_{k,5}, s_{k,6} \in W_2$ such that

- $\mathcal{M}_2, s_2 \models p\overline{q}r$, $\mathcal{M}_2, s_{2,1} \models p\overline{q}\overline{r}$, $\mathcal{M}_2, s_{2,2} \models \overline{p}qr$, $\mathcal{M}_2, s_{2,3} \models \overline{p}q\overline{r}$, $\mathcal{M}_2, s_{2,4} \models \overline{p}\overline{q}r$, $\mathcal{M}_2, s_{2,5} \models \overline{p}\overline{q}\overline{r}$, $\mathcal{M}_2, s_{2,6} \models p\overline{q}r$, ..., $\mathcal{M}_2, s_{k,1} \models p\overline{q}\overline{r}$, $\mathcal{M}_2, s_{k,2} \models \overline{p}qr$, $\mathcal{M}_2, s_{k,3} \models \overline{p}q\overline{r}$, $\mathcal{M}_2, s_{k,4} \models \overline{p}\overline{q}r$, $\mathcal{M}_2, s_{k,5} \models \overline{p}\overline{q}\overline{r}$,

- $s_2 R''_2 s_{2,1} R''_1 s_{2,2} R''_2 s_{2,3} R''_1 s_{2,4} R''_2 s_{2,5} R''_1 s_{2,6} ... R''_2 s_{k,1} R''_1 s_{k,2} R''_2 s_{k,3} R''_1 s_{k,4} R''_2 s_{k,5} R''_1 s_{k,6}$.

Let $\mathcal{M}^0 = (W^0, R^0_1, R^0_2, v^0)$ be the disjoint union of $\mathcal{M}'_1$ and $\mathcal{M}'_2$. By Theorem 3.14 in [18], we have:

$\mathcal{M}^0,(t_1)\vDash\mu(x)$, $\mathcal{M}^0,(s_2)\nvDash\mu(x)$. Let $\mathcal{M}^\oplus=(W^\oplus,R_1^\oplus,R_2^\oplus,v^\oplus)$ be the model obtained from $\mathcal{M}^0$ by adding new possible worlds w_1,w_2,w_3,w_4,w_5 and such that

$$(t_1,1,t_{2,1},2,t_{2,2},1,t_{2,3},2,t_{2,4},1,t_{2,5},2,t_{2,6},...,1,t_{k,1},2,t_{k,2},1,t_{k,3},2,t_{k,4},1,t_{k,5},2,t_{k,6})$$
$$R_1^\oplus w_1 R_2^\oplus w_2 R_1^\oplus w_3 R_2^\oplus w_4 R_1^\oplus w_5 R_2^\oplus$$
$$(s_2,2,s_{2,1},1,s_{2,2},2,s_{2,3},1,s_{2,4},2,s_{2,5},1,s_{2,6},...,2,s_{k,1},1,s_{k,2},2,s_{k,3},1,s_{k,4},2,s_{k,5},1,s_{k,6})$$

and $\mathcal{M}^\oplus,w_1\vDash\overline{pqr}$, $\mathcal{M}^\oplus,w_2\vDash\overline{pq}r$, $\mathcal{M}^\oplus,w_3\vDash\overline{p}q\overline{r}$, $\mathcal{M}^\oplus,w_4\vDash\overline{p}qr$ and $\mathcal{M}^\oplus,w_5\vDash p\overline{qr}$.
Since the shortest path from (t_1) and $(t_1,1,t_{2,1},2,t_{2,2},1,t_{2,3},2,t_{2,4},1,t_{2,5},2,t_{2,6},...,1,t_{k,1},2,t_{k,2},1,t_{k,3},2,t_{k,4},1,t_{k,5},2,t_{k,6})$ is of length $6.k$ and the shortest path from (s_2) and $(s_2,2,s_{2,1},1,s_{2,2},2,s_{2,3},1,s_{2,4},2,s_{2,5},1,s_{2,6},...,2,s_{k,1},1,s_{k,2},2,s_{k,3},1,s_{k,4},2,s_{k,5},1,s_{k,6})$ is of length $6.K$ that (knowing that $k\geqslant deg(\mu(x))$), we have $\mathcal{M}^\oplus,(t_1)\vDash\mu(x)$ and $\mathcal{M}^\oplus,(s_2)\nvDash\mu(x)$. Now, $\vdash\mu(x)\to\boxslash\mu(x)$, because μ is a unifier of φ_0. It follows that $\vdash\mu(x)\to\boxslash^{k+1}\mu(x)$. Since $\mathcal{M}^\oplus,(t_1)\vDash\mu(x)$ then, $\mathcal{M}^\oplus,(t_1)\vDash\boxslash^{k+1}\mu(x)$. Consequently, $\mathcal{M}^\oplus,(s_2)\vDash\mu(x)$. This is contradiction.

Lemma 81 *The set of substitutions $\Sigma=\{\sigma_n\mid n\in\mathbb{N}\}\cup\{\tau_n\mid n\in\mathbb{N}\}$ form a complete set of $S5_1\otimes S5_2$-unifiers of φ_0.*

Proof 124 *By Lemmas 75 to 78 and Theorem 10.*

Lemma 82 *φ_0 does not possess a minimal complete set of $(S5_1\otimes S5_2)$- unifiers.*

Proof 125 *Let Γ be a minimal complete set of unifiers of φ_0. Let $\mu\in\Gamma$ be such that $\mu\preccurlyeq\sigma_0$. Then by Theorem 10, there exists $k\in\mathbb{N}$ such that $\sigma_k\preccurlyeq_{S5_1\otimes S5_2}\mu$ or $\tau_k\preccurlyeq_{S5_1\otimes S5_2}\mu$. Consider firstly the case $\sigma_k\preccurlyeq_{S5_1\otimes S5_2}\mu$. By definition of Σ, $\sigma_{k+1}\in\Sigma$. Let $\mu'\in\Gamma$ such that $\mu'\preccurlyeq\sigma_{k+1}$. Since $\mu'\preccurlyeq\sigma_{k+1}\preccurlyeq\sigma_k\preccurlyeq\mu$ then $\mu'\preccurlyeq\mu$. Since Γ is minimal complete set and its members are pairwise incomparable then $\mu'=\mu$. Since $\sigma_k\preccurlyeq\mu$ and $\mu'\preccurlyeq\sigma_{k+1}$ then, $\sigma_k\preccurlyeq\sigma_{k+1}$. Since $k<k+1$ by lemma 77, $\sigma_k\not\preccurlyeq\sigma_{k+1}$ and this is a contradiction with lemma 77.*
Consider the second case $\tau_k\preccurlyeq_{(S5_1\otimes S5_2)}\mu$. Since $\mu\preccurlyeq\sigma_0$ then $\tau_k\preccurlyeq\sigma_0$, a contradiction with Lemma 78.

Lemma 83 *Unification type is nullary in fusion $S5\otimes S5$*

Proof 126 *By Lemma 82.*

6.6 Unification in fusion $K4_1 \otimes K4_2$

Dzik in Chapter 6 of [22] proved that the fusion $K4 \otimes K4$ has the rule of disjunction. Hence, unification type of the fusion $K4 \otimes K4$ is not unitary. In this section, we shall prove that unification type of the fusion $K4 \otimes K4$ nullary. Consider the formula $\varphi = x \to \Box_1\Box_2 x$ and substitutions $\sigma_\top(x) = \top$ and $\sigma_n(x) = (\Box_1\Box_2)^{<n} x \wedge (\Box_1\Box_2)^n \bot$ where $n \in \mathbb{N}$.

Lemma 84 *For all $n \in \mathbb{N}$,*

1. $\sigma_n(x) = (\Box_1\Box_2)^{<n} x \wedge (\Box_1\Box_2)^n \bot$ *is a $K4_1 \otimes K4_2$-unifier of φ.*

2. $\sigma_\top(x) = \top$ *is a $K4_1 \otimes K4_2$-unifier of φ.*

Proof 127 *The proof is similar to the proof of Lemma* 54.

Lemma 85 *Let $k, l \in \mathbb{N}$. If $k \leqslant l$ then $\sigma_l \preccurlyeq \sigma_k$.*

Proof 128 *The proof is similar to the proof of Lemma* 55.

Lemma 86 *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preccurlyeq \sigma_l$*

Proof 129 *The proof is similar to the proof of Lemma* 56. *Since we consider $K4 \otimes K4$, we will use the fact that if $k < l$ then $\nvdash_{K4 \otimes K4} (\Box_1\Box_2)^l \bot \to (\Box_1\Box_2)^k \bot$. The proof of this fact is similar to the proof of item* 5 *in Lemma* 53.

Lemma 87 *Let σ be a substitution. The following conditions are equivalent:*

1. $\sigma_\top \circ \sigma \simeq \sigma$.
2. $\sigma_\top \preccurlyeq \sigma$.
3. $\vdash \sigma(x)$.

Proof 130 *The proof is similar to the proof of Lemma* 57.

Lemma 88 *Let $n \in \mathbb{N}$. Let σ be a unifier of φ. The following conditions are equivalent:*

1. $\sigma_n \circ \sigma \simeq \sigma$.

2. $\sigma_n \preccurlyeq \sigma$.

3. $\vdash \sigma(x) \to (\Box_1\Box_2)^n\bot$.

Proof 131 *The proof is similar to the proof of Lemma* 58.

Theorem 11 *Let σ be a unifier of $\varphi = x \to \Box_1\Box_2 x$ then either $\vdash_{K4\otimes K4} \sigma(x)$ or $\vdash_{K4_1\otimes K4_2} \sigma(x) \to (\Box_1\Box_2)^n\bot$ where $n \geqslant deg(\sigma(x))$.*

Proof 132 *Suppose neither $\nvdash \sigma(x)$ nor $\nvdash \sigma(x) \to (\Box_1\Box_2)^n\bot$. Let $M_1 = \langle W_1, R'_1, R'_2, v_1\rangle$ be a model and $t_1 \in W_1$ such that $M_1, t_1 \nvDash \sigma(x)$. Let $M_2 = (W_2, R''_1, R''_2, v_2)$ be a model and $t_2 \in W_2$ be such that $M_2, t_2 \nvDash \sigma(x) \to (\Box_1\Box_2)^n\bot$. We will define the unravelling M'_1 of M_1 around t_1 and the unravelling M'_2 of M_2 around t_2. Let $M'_1 = (X_1, S'_1, S'_2, v'_1)$ where*

- *X_1 is the set of all finite sequences of the form $(u_0, a_1, u_1, ..., a_k, u_k)$ where $k \in \mathbb{N}, u_0, u_1, ..., u_k \in W_1, a_1, ..., a_k \in \{1,2\}, u_0 = t_1$ and for all $i \in \mathbb{N}$, if $i < k$ then $u_i R'_{a_{i+1}} u_{i+1}$,*

- *S'_1 is the binary relation on X_1 such that $(u_0, a_1, u_1, ..., a_k, u_k) S'_1 (v_0, b_1, v_1, ..., b_l, v_l)$ iff $k < l$, $(u_0, a_1, u_1, ..., a_k, u_k) = (v_0, b_1, v_1, ..., b_k, v_k)$ and for all $i \geqslant k$, if $i < k$ then $b_{i+1} = 1$,*

- *S'_2 is the binary relation on X_1 such that $(u_0, a_1, u_1, ..., a_k, u_k) S'_2 (v_0, b_1, v_1, ..., b_l, v_l)$ iff $k < l$, $(u_0, a_1, u_1, ..., a_k, u_k) = (v_0, b_1, v_1, ..., b_k, v_k)$ and for all $i \geqslant k$, if $i < k$ then $b_{i+1} = 2$,*

- *v'_1 is the valuation on X_1 such that for all propositional variable or parameters α, $v'_1(\alpha) = \{(u_0, a_1, u_1, ..., a_k, u_k) \in X_1 : u_k \in v_1(\alpha)\}$.*

The unravelling $M'_2 = (X_2, S''_1, S''_2, v'_2)$ of M_2 around t_2 is described in a similar way. Notice that $(t_1) \in X_1$ and $(t_2) \in X_2$. Notice also that S'_1 and S'_2 are transitive relations on X_1 and S''_1 and S''_2 are transitive relations on X_2. In other respect, let $f_1 : X_1 \to W_1$ and $f_2 : X_2 \to W_2$ be defined as in the proof of Theorem 10. *The functions f_1 and f_2 being bounded morphism, it follows from [18] (Theorem* 3.14*) that $M'_1, (t_1) \vDash \sigma(x)$ and $M'_2, (t_2) \vDash \sigma(x)$ and $M'_2, (t_2) \nvDash (\Box_1\Box_2)^n\bot$. Consequently, there exists $u_1, v_1, ..., u_n, v_n \in W_2$ such that $t_2 R_1 u_1 R_2 v_1 ... R_1 u_n R_2 v_n$. Notice that therefore $(t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n) \in X_2$. Let $M^0 = (W^0, R^0_1, R^0_2, v^0)$ be the disjoint union of M'_1 and M'_2. By theorem* 3.14 *in [18], we have $M^0, (t_1) \nvDash \sigma(x)$ and $M^0, (t_2) \vDash \sigma(x)$. Moreover, $(t_2)(R_1 \circ R_2)^n (t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n)$.*

Notice that the shortest path between (t_2) *and* $(t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n)$ *is of length* $2n$. *Let* $M^\oplus = (W^\oplus, R_1^\oplus, R_2^\oplus, v^\oplus)$ *be obtained from* M^0 *by adding a new possible world* w *such that* $(t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n) R_1^\oplus w R_2^\oplus (t_1)$. *Since* $n \geqslant deg(\sigma(x))$, $M^0, (t_1) \not\models \sigma(x)$ *and* $M^0, (t_2) \models \sigma(x)$, *then* $M^\oplus, (t_1) \not\models \sigma(x)$ *and* $M^\oplus, (t_2) \models \sigma(x)$. *Since* σ *is a unifier of* $x \to \Box_1 \Box_2 x$, *then* $\vdash \sigma(x) \to \Box_1 \Box_2 \sigma(x)$. *Hence,* $\vdash \sigma(x) \to (\Box_1 \Box_2)^{n+1} \sigma(x)$. *It follows from* $M^\oplus, (t_2) \models \sigma(x)$ *that* $M^\oplus, (t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n) \models \Box_1 \Box_2 \sigma(x)$. *Since* $(t_2, 1, u_1, 2, v_1, ..., 1, u_n, 2, v_n) R_1^\oplus w R_2^\oplus (t_1)$, *it follows that* $M^\oplus, (t_1) \models \sigma(x)$. *This is a contradiction.*

Lemma 89 *The set of substitutions* $\Sigma = \{\sigma_\top\} \cup \{\sigma_n \mid n \in \mathbb{N}\}$ *forms a complete set of* $K_1 \otimes K_2$*-unifiers of* $\varphi = x \to \Box_1 \Box_2 x$.

Proof 133 *By Lemmas* 84, 85, 86, 87 *and* 88 *and Theorem* 11.

Lemma 90 *The formula* $\varphi = x \to \Box_1 \Box_2 x$ *does not possess a minimal complete set of* $K4_1 \otimes K4_2$*-unifiers.*

Proof 134 *The proof is similar to the proof of Lemma* 60.

Lemma 91 *Unification type is nullary in fusion* $K4_1 \otimes K4_2$.

Proof 135 *By Lemma* 90.

7 Unification in Simple Epistemic Planning problem

Contents

In general, epistemic planning extends automated planning with epistemic notions such as knowledge and belief. When the number of agents is one, it is called epistemic planning for single agent and this kind of epistemic planning consider the following problem:

An agent's current state of knowledge,

a desirable state of knowledge,

how does it get from one to the other by executing a finite sequence of action?

But in the case of epistemic planning for multi-agents, the current and desirable states of knowledge might also refer to the states of knowledge of other agents.

In this chapter, we define a kind of simple epistemic planning problem where atomic actions are public announcements and then, we struggle to find some appropriate announcements by unification technique.

7.1 Simple epistemic planning problem with public announcement logic

We are going to solve some simple epistemic planning problems with unification technique. In this section, actions are public announcements. In this respect we need to know syntax and semantic of public announcement logic. Dynamic Epistemic Logic (DEL) considers information change and the information change is modeled by transforming Kripke models. In fact, in Dynamic Epistemic Logic an agent's information change during communication. In terms of Kripke models, that means that the accessibility relations of the agents have to change (and consequently the set of states of the model might change as well). Language of Dynamic Epistemic Logic is an extension of the language of Epistemic Logic by announcements. The first extension of the language of Epistemic Logic was called public announcement logic and was introduced by Plaza [43](1989).

At the following, we consider syntax and semantic of public announcement logic based on [20].

7.1.1 Syntax of the public announcement logic

Let A be a finite set of agents and P be a countable set of atoms.

Definition 39 *The language $\mathscr{L}_{k[]}$ is inductively defined by*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box_a \varphi \mid [\psi]\varphi$$

Besides the usual propositional language, $\Box_a\varphi$ is read as agent a knows that φ, and $[\psi]\varphi$ is read as after announcement of ψ, it holds that φ. We will use the following abbreviation:

- $\Diamond_a\varphi = \neg\Box_a\neg\varphi$,
- $\langle\psi\rangle\varphi = \neg[\psi]\neg\varphi$.

We will also write A, B, etc for formula.

7.1.2 Semantics of the logic of announcements

The public announcement of ψ restricts the epistemic state to all (factual) states where φ holds, including access between states.

Definition 40 *An epistemic model is a triple $M = (W, \sim, V)$ where $W \neq \emptyset$, for each $a \in A$, $\sim_a$ is an equivalence relation on W and for each $p \in P$, $V(p) \subseteq W$.*

Definition 41 *Let an epistemic model $\mathcal{M} = \langle W, \sim, v \rangle$ for set of agents A and set of atoms P be given, the truth conditions for the formulas in $\mathcal{L}_{k[]}$ are defined as follows:*

- $\mathcal{M}, w \models p$ *iff* $w \in v(p)$
- $\mathcal{M}, w \models \neg\varphi$ *iff* $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ *iff* $\mathcal{M}, w \models \varphi$ *and* $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box_a \varphi$ *iff* *for all v such that $w \sim_a v$, $\mathcal{M}, v \models \varphi$*
- $\mathcal{M}, w \models [\psi]\varphi$ *iff* $\mathcal{M}, w \models \psi$ *implies* $\mathcal{M}|_\psi, w \models \varphi$

where $\mathcal{M}|_\psi = \langle W', \sim', v' \rangle$ is defined as follows (where $[\psi]_\mathcal{M}$ is the set of all states $v \in W$ such that $\mathcal{M}, v \models \psi$

$$
\begin{aligned}
&W' = [\psi]_\mathcal{M} \\
&\sim'_a = \sim_a \cap ([\psi]_\mathcal{M} \times [\psi]_\mathcal{M}) \\
&v'_p = v_p \cap [\psi]_\mathcal{M}
\end{aligned}
$$

As a result:

$$\mathcal{M}, w \models \Diamond_a \varphi \text{ if there exists } v \text{ such that } w \sim_a v \text{ and } \mathcal{M}, w \models \varphi.$$
$$\mathcal{M}, w \models \langle\psi\rangle\varphi \text{ iff } \mathcal{M}, w \models \psi \text{ and } \mathcal{M}|_\psi, w \models \varphi.$$

Since $\Box_a$ and $\Diamond_a$ are interpreted by equivalence relation, the formulas like

- $\Box_a(A \vee \Diamond_a B) \leftrightarrow \Box_a A \vee \Diamond_a B$ and
- $\Box_a(A \wedge \Diamond_a B) \leftrightarrow \Box_a A \wedge \Diamond_a B$

are valid.

7.1.3 Axiomatisation of Public Announcement Logic

The axiomatisation PAL of Public Announcement Logic has been introduced in [20] and it consists of the following axioms and rules:

- all instantiations of propositional tautologies
- $\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$ (distribution of $\Box_a$ over $\to$)
- $\Box_a\varphi \to \varphi$ (truth)
- $\Box_a\varphi \to \Box_a\Box_a\varphi$ (positive introspection)
- $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$ (negative introspection)
- $[\varphi]p \leftrightarrow (\varphi \to p)$ (atomic permanence)
- $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$ (announcement and negation)
- $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$ (announcement and conjunction)
- $[\varphi]\Box_a\psi \leftrightarrow (\varphi \to \Box_a[\varphi]\psi)$ (announcement and knowledge)
- $[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$ (announcement composition)
- From φ and $\varphi \to \psi$, infer ψ (modus ponens)
- From φ, infer $\Box_a\varphi$ (necessitation of $\Box_a$)

Now, we present a simple epistemic planning problem then we will solve it by unification technique. Bolander and Anderson have introduced different epistemic planning problem [19].
Let us define our main problem. Our problem is a special kind of epistemic planning problem. Let us define our problem and see how it will be solved by unification technique as follows:

Definition 42 *A simple epistemic planning problem is a pair* (A, B) *where*

- ***Input****: A and B are formulas in* $\mathcal{L}_{k[]}$*.*
- ***Question****: is there a public announcement* ψ *such that each time A holds,* ψ *can be announced and, after announcing* ψ*, B becomes true.*

In this chapter, we will also consider the other following problems:

- **Input**: formulas A, B in $\mathscr{L}_{k[]}$, an agent i,
- **Question**: is there a public announcement ψ such that $A \to \langle\psi\rangle\Box_i B$ is valid?
- **Input**: formulas A, C in $\mathscr{L}_{k[]}$, an agent j,
- **Question**: is there a public announcement ψ such that $A \to \langle\psi\rangle\Diamond_j C$ is valid?
- **Input**: formulas $A, B_1, ..., B_m$ in $\mathscr{L}_{k[]}$, agents $i_1, ..., i_m$,
- **Question**: is there a public announcement ψ such that $A \to \langle\psi\rangle(\Box_{i_1} B_1 \wedge ... \wedge \Box_{i_m} B_m)$ is valid?
- **Input**: formulas $A, C_1, ..., C_n$ in $\mathscr{L}_{k[]}$, agents $j_1, ..., j_n$,
- **Question**: is there a public announcement ψ such that $A \to \langle\psi\rangle(\Diamond_{j_1} C_1 \wedge ... \wedge \Diamond_{j_n} C_n)$ is valid?
- **Input**: formulas $A, B_1, ..., B_m, C_1, ..., C_n$ in $\mathscr{L}_{k[]}$, agents $i_1, ..., i_m, j_1, ..., j_n$,
- **Question**: is there a public announcement ψ such that $A \to \langle\psi\rangle(\Box_{i_1} B_1 \wedge ... \wedge \Box_{i_m} B_m \wedge \Diamond_{j_1} C_1 \wedge ... \wedge \Diamond_{j_n} C_n)$ is valid?

We propose to use unification tools for solving such problems. How?
For instance, for a given input (A, B, i) of the first problem, we will consider the PAL-formula $P = A \to \langle x\rangle\Box_i B$. Here we assume A, B do not contain the variable x. In fact, we suppose A, B only contain parameters. Then, we will use the reduction axiom of PAL to obtain a PAL- formula P_1 which has the same unifiers as P and for which it seems easier to compute a most general unifier. In this respect, we will also use the fact that the modalities $\Box_1, \Box_2, ...$ are interpreted in models by equivalence relations and, consequently, the following inference rules are admissible:

$$\frac{\varphi \to \Box_i \psi}{\Diamond_i \varphi \to \psi} \qquad\qquad \frac{\Diamond_i \varphi \to \psi}{\varphi \to \Box_i \psi}$$

Then, considering P_1, we will find a necessary and sufficient condition for the unifiability of P_1 and then of P.
Finally, assuming this necessary and sufficient condition holds, we will construct a most general unifier of P_1 and then of P.

Example 12 *Consider the case $A = \Box_1 p$ and $B = \Box_2 p$ the planning problem is a unification problem. It is the problem of unifying the formula $\Box_1 p \to \langle x \rangle \Box_2 p$. Let the agent 1 knows that p is true. Is there any announcement ψ such that after announcing ψ, the agent 2 knows p? Our answer to this question is **positive**. Since we can announce agent 1 knows p or $\Box_1 p$. In this case, after announcing $\Box_1 p$ then the agent 2 knows that p is true that is to say $\Box_2 p$ becomes true.*

To solve such problems by unification technique, we consider the associated formula $A \to \langle x \rangle B$. Hence, we apply the following steps to solve the associated formula $A \to \langle x \rangle B$.

1. Use axiomatisation of public announcement logics in order to simplify the formula $A \to \langle x \rangle B$.

2. Determine a necessary and sufficient condition in order to be able to unify to the formula $A \to \langle x \rangle B$.

3. When condition of item 2 holds, compute or find one unifier or solution of the formula $A \to \langle x \rangle B$.

4. If there exists a unifier, can we find a most general unifier?

7.2 Simple epistemic planning problem $A \to \langle x \rangle B$

In this part, we consider all possible cases as $A \to \langle x \rangle B$ and we have to find an appropriate public announcement ψ such that the formula $A \to \langle \psi \rangle B$ is valid.

Lemma 92 *Let $P = A \to \langle x \rangle B$ where B is Boolean formula. Then, $\models A \to B$ iff P possesses a unifier.*

Proof 136 *We have to do the steps 1 to 4.*

1. *Use axiomatisation of public announcement to simplify P .*

    - $A \to \langle x \rangle B$
    - $A \to x \wedge [x]B$
    - $(A \to x) \wedge (A \to [x]B)$
    - $(A \to x) \wedge (A \to (x \to B))$
    - $(A \to x) \wedge (A \to B)$

Hence, let $P_1 = (A \to x) \wedge (A \to B)$. *Notice that* P *and* P_1 *are equivalent in PAL. Hence, they have the same unifiers.*

2. *Now, let us show that* P_1 *is unifiable iff* $\models A \to B$. *If* $\models A \to B$ *then* $\sigma(x) = \top$ *is a unifier of* P_1. *Reciprocally, if* τ *is a unifier of* P_1 *then* $\models A \to B$. *We remind that* A, B *contain only parameters.*

3. *Now, assuming that* $\models A \to B$, *let us find a unifier of* P_1. *Since* $\models A \to B$, *it is clear that* $\sigma(x) = B$ *is a unifier of* P_1.

4. *Now, assuming that* $\models A \to B$, *let us find a most general unifier of* P_1 *if it exists.*

Notice that in P_1, *all occurrences of* x *are at the level* 0. *Consider Löwenheim substitution* ε *associated to* P_1 *and* σ *as follows:* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$. *Since,* $\sigma(x) = B$ *hence we have* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge B)$ *or equivalently* $\varepsilon(x) = (P_1 \vee B) \wedge (x \vee \neg P_1) \wedge (x \vee B)$. *In order to check,* $\varepsilon(x)$ *is a most general unifier of* P_1, *we have to prove first,* ε *is a unifier of* P_1 *and second* ε *is a most general of* P_1. *First, let us prove* ε *is a unifier of* P_1. *Hence, we have to prove* $\models A \to \varepsilon(x)$. *Since,*

1. $\models \neg(A \to x) \to \neg P_1$ *then,*

2. $\models (A \wedge \neg x) \vee x \to (\neg P_1 \vee x)$. *Hence,*

3. $\models (A \vee x) \to (\neg P_1 \vee x)$ *thus,*

4. $\models A \to (\neg P_1 \vee x)$. *Since by our assumption,* $\models A \to B$ *then,*

5. $\models A \to (P_1 \vee B) \wedge (x \vee B)$. *By steps* (4) *and* (5) *we have,*

6. $\models A \to (\neg P_1 \vee x) \wedge (P_1 \vee B) \wedge (x \vee B)$. *Therefore*

7. $\models A \to \varepsilon(x)$.

Therefore, ε *is a unifier of* P_1.
Second, let σ' *be a unifier of* P_1. *We have to prove* $\varepsilon \preccurlyeq \sigma'$. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg\sigma'(P_1) \wedge B)$ *is logically equivalent to* $\sigma'(x)$. *Thus,* $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *are equivalent and* P *and* P_1 *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 93 *Let* $P = A \to \langle x \rangle \Box B$ *where* B *is Boolean formula. Then,* P *possesses a unifier iff* $\models A \to B$.

Proof 137 *We have to do the steps* $1-4$ *described in the proof of Lemma* 92.

1. *Use axiomatisation of public announcement to simplify* P*. by the reduction axioms of PAL,* P *is logically equivalent to* $P' = (A \to x) \wedge (A \to \Box(x \to B))$*.Since* $\Box$ *is interpreted by an equivalence relation, then* P' *has the same unifiers as* $P'' = (A \to x) \wedge (\Diamond A \to (x \to B))$ *which has itself the same unifiers as* $P_1 = (A \to x) \wedge (x \to (\Diamond A \to B))$*. Hence, let* $P_1 = (A \to x) \wedge (x \to (\Diamond A \to B))$.

2. *Now, let us prove that* P_1 *is unifiable iff* $\models A \to B$*. Suppose* $\models A \to B$*. Hence, obviously* $\sigma(x) = B$ *is a unifier of* P_1*. Now, suppose* P_1 *has a unifier* τ*. Thus,* $\models A \to \tau(x)$ *and* $\models \tau(x) \to (\Diamond A \to B)$*. Hence,* $\models A \to (\Diamond A \to B)$ *and* $\models A \to B$*. We remind that* $\models A \to \Diamond A$.

3. *Suppose* $\models A \to B$*. Since* $\models A \to B$*, it is clear that* $\sigma(x) = B$ *is a unifier of* P_1.

4. *Let us find a most general unifier of* P_1.

Notice that in P_1*, all occurrences of* x *are at the level* 0*. Consider Löwenheim's formula* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$*. Since,* $\sigma(x) = B$ *hence we have* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge B)$ *or equivalently* $\varepsilon(x) = (P_1 \vee B) \wedge (x \vee \neg P_1) \wedge (x \vee B)$*. In order to check,* $\varepsilon(x)$ *is a most general unifier of* P_1*, we have to prove first,* ε *is a unifier of* P_1 *and second* ε *is a most general unifier of* P_1*. First, let us prove* ε *is a unifier of* P_1*. Hence, we have to prove*

1. $\models A \to \varepsilon(x)$ *and*

2. $\models \varepsilon(x) \to (\Diamond A \to B)$.

1. *We have to prove* $\models A \to \varepsilon(x)$*. Since,*

    (a) $\models \neg(A \to x) \to \neg P_1$ *then,*
    (b) $\models (A \wedge \neg x) \vee x \to (\neg P_1 \vee x)$*. Hence,*
    (c) $\models (A \vee x) \to (\neg P_1 \vee x)$ *thus,*
    (d) $\models A \to (\neg P_1 \vee x)$*. Since by our assumption,* $\models A \to B$ *then,*
    (e) $\models A \to (P_1 \vee B) \wedge (x \vee B)$*. By steps* (c) *and* (e) *we have,*
    (f) $\models A \to (\neg P_1 \vee x) \wedge (P_1 \vee B) \wedge (x \vee B)$*. Therefore*
    (g) $\models A \to \varepsilon(x)$.

2. *To prove* $\models \varepsilon(x) \to (\Diamond A \to B)$ *we have to prove* $\models (\neg P_1 \wedge B) \to (\Diamond A \to B)$ *and* $\models (P_1 \wedge x) \to (\Diamond A \to B)$.

   It is clear that $\models (\neg P_1 \wedge B) \to (\Diamond A \to B)$. *Since* $\models P_1 \to (x \to (\Diamond A \to B))$ *then,* $\models (P_1 \wedge x) \to (\Diamond A \to B)$.

Therefore, ε *is a unifier of* P_1.
Second, let σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg\sigma'(P_1) \wedge B)$ *is logically equivalent to* $\sigma'(x)$. *We remind that* B *contains only parameters;* B *contains no occurrence of* x. *Thus,* $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *are equivalent and* P *and* P_1 *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 94 *Let* $P = A \to \langle x \rangle(\Box_1 B_1 \wedge ... \wedge \Box_n B_n)$ *where* $B_1, ..., B_n$ *are Boolean formulas. Then,* $\models A \to (B_1 \wedge ... \wedge B_n)$ *iff* P *possesses a unifier.*

Proof 138 *We have to do the steps* $1-4$ *as before.*

1. *Simplification of* P *in this Lemma is similar to simplification of* P *in* 93. *Hence, let* $P_1 = (A \to x) \wedge (x \to (\Diamond_1 A \to B_1)) \wedge ... \wedge (x \to (\Diamond_n A \to B_n))$. *By an argument similar to the argument used in the proof of Lemma* 93, *we know that* P *and* P_1 *have the same unifiers.*

2. *As well, one can show that* P_1 *is unifiable iff* $\models A \to (B_1 \wedge ... \wedge B_n)$.

3. *Suppose* $\models A \to (B_1 \wedge ... \wedge B_n)$. *Since* $\models A \to (B_1 \wedge ... \wedge B_n)$, *it is clear that* $\sigma(x) = A$ *is a unifier of* P_1.

4. *Now, let us find a most general unifier of* P_1.

Notice that all occurrences of x *in* P_1 *are at the level* 0. *Consider Löwenheim's formula* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$. *Since,* $\sigma(x) = A$ *hence we have* $\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge A)$. *In order to check,* $\varepsilon(x)$ *is a most general unifier of* P_1, *we have to prove first,* ε *is a unifier of* P_1 *and second* ε *is a most general unifier of* P_1. *First, let us prove* ε *is a unifier of* P_1. *Hence, we have to prove*

1. $\models A \to \varepsilon(x)$ *and*

2. $\models \varepsilon(x) \to ((\Diamond_1 A \to B_1) \wedge ... \wedge (\Diamond_n A \to B_n))$.

1. *to prove* $\models A \to \varepsilon(x)$, *we use similar method as in the proof of Lemma* 93.

2. *To prove* $\models \varepsilon(x) \to (\Diamond_i A \to B_i)$ *for* $1 \leqslant i \leqslant n$, *we have to prove* $\models (\neg P_1 \land A) \to (\Diamond_i A \to B_i)$ *and* $\models (P_1 \land x) \to (\Diamond_i A \to B_i)$ *for* $1 \leqslant i \leqslant n$.
   Since, $\vdash A \to (B_1 \land \ldots \land B_n)$ *then,* $\models (\neg P_1 \land A) \to (\Diamond_i A \to B_i)$ *for* $1 \leqslant i \leqslant n$.
   Since,
   $\models P_1 \to (x \to (\Diamond_i A \to B_i))$ *then,*
   $\models (P_1 \land x) \to (\Diamond_i A \to B_i)$ *for all* $1 \leqslant i \leqslant n$.
   Thus, $\models \varepsilon(x) \to (\Diamond_1 A \to B_1) \land \ldots \land (\Diamond_n A \to B_n)$.

Therefore, ε *is a unifier of* P_1.
Second, let σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\sigma'(P_1) \land \sigma'(x)) \lor (\neg\sigma'(P_1) \land A)$ *is logically equivalent to* $\sigma'(x)$. *We remind that* A *contains only parameters. Thus,* $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *are equivalent and* P *and* P_1 *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 95 *Let* $P = A \to \langle x \rangle \Diamond C$ *where* C *is Boolean formula. Then,* $\models A \to \Diamond C$ *iff* P *possesses a unifier.*

Proof 139 *We have to do the steps* $1-4$ *as before.*

1. *Use axiomatisation of public announcement to simplify* P.
   - $(A \to \langle x \rangle \Diamond C)$
   - $(A \to x \land [x]\neg\Box\neg C)$
   - $\Big((A \to x) \land (A \to [x]\neg\Box\neg C)\Big)$
   - $\Big((A \to x) \land (A \to (x \to \neg[x]\Box\neg C))\Big)$
   - $\Big((A \to x) \land (A \to \neg[x]\Box\neg C)\Big)$
   - $\Big((A \to x) \land (A \to \neg(x \to \Box[x]\neg C))\Big)$
   - $\Big((A \to x) \land (A \to \neg\Box[x]\neg C)\Big)$
   - $\Big((A \to x) \land (A \to \neg\Box(x \to \neg C))\Big)$
   - $\Big((A \to x) \land (A \to \Diamond(x \land C))\Big)$.

   Hence, let $P_1 = (A \to x) \land (A \to \Diamond(x \land C))$. P *and* P_1 *are logically equivalent. More importantly, they have the same unifiers.*

2. *Now, let us show that P_1 is unifiable iff $\models A \to \Diamond C$. Suppose $\models A \to \Diamond C$. then, obviously $\sigma(x) = \top$ is a unifier of P_1. Reciprocally if some substitution τ are unifiers of P_1 then, $\models A \to \Diamond(\tau(x) \wedge C)$. Thus, $\models A \to \Diamond C$.*

3. *Suppose $\models A \to \Diamond C$. Since, $\models A \to \Diamond C$ it is clear that $\sigma(x) = \top$ is a unifier of P_1.*

4. *Now, let us find a most general unifier of P_1.*

Notice that, contrary to the cases of Lemmas 92, 93 and 94, in P_1 there is one occurrence of x at the level 0 and one occurrence of x in the scope of $\Diamond$. So, in Löwenheim's formula we will replace P_1 by $\Box P_1$ as in modal logic S5 in Chapter 4. Consider Löwenheim's formula $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$. Since, $\sigma(x) = \top$ hence we have $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \top)$ or equivalently $\varepsilon(x) = \neg \Box P_1 \vee x$ which is equivalent to $\varepsilon(x) = (\Box P_1 \to x)$. In order to check, $\varepsilon(x)$ is a most general unifier of P_1, we have to prove first, ε is a unifier of P_1 and second ε is a most general unifier of P_1. First, let us prove ε is a unifier of P_1. Hence, we have to prove

1. $\models A \to \varepsilon(x)$ *and*

2. $\models A \to \Diamond(\varepsilon(x) \wedge C)$.

1. *We have to prove $\models A \to \varepsilon(x)$. Since,*
   $\models \Box P_1 \to (A \to x)$ *then,*
   $\models (\Box P_1 \wedge A) \to x$ *hence,*
   $\models A \to (\Box P_1 \to x)$.

2. *We have to prove that $\models A \to \Diamond(\varepsilon(x) \wedge C)$. Since $\varepsilon(x) = \Box P_1 \to x$, it is equivalent to prove that $\models A \wedge \Box(C \to \Box P_1) \to \Diamond(x \wedge C)$. we remind that $\Box$ and $\Diamond$ are interpreted in models by equivalence relations. Since,*

   - $\models A \wedge \Box(C \to \Box P_1) \to A \wedge (\Diamond C \to \Box P_1)$ *and by our assumption,*
   - $\models A \to \Diamond C$ *then,*
   - $\models A \wedge \Box(C \to \Box P_1) \to \Box P_1 \wedge A$. *Since,*
   - $\models \Box P_1 \to (A \to \Diamond(x \wedge C))$ *hence,*
   - $\models \Box P_1 \wedge A \to \Diamond(x \wedge C))$. *Since,*
   - $\models A \wedge \Box(C \to \Box P_1) \to \Box P \wedge A$, *and*

- $\models \Box P_1 \wedge A \rightarrow \Diamond(x \wedge C))$ *thus,*
- $\models A \wedge \Box(C \rightarrow \Box P_1) \rightarrow \Diamond(x \wedge C)$.

Therefore, ε is a unifier of P_1.

Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box\sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = \sigma'(x) \vee \neg\Box\sigma'(P_1)$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$. Consequently, ε is a most general unifier of P_1. Since P_1 and P are equivalent and P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 96 *Let $P = A \rightarrow \langle x\rangle(\Diamond_1 C_1 \wedge ... \wedge \Diamond_n C_n)$ where C_i is a Boolean formula for all $i = 1, ..., n$. Then $\models A \rightarrow (\Diamond_1 C_1 \wedge ... \wedge \Diamond_n C_n)$ iff P possesses a unifier.*

Proof 140 *We have to do the steps $1-4$ as for the proof of the previous Lemma.*

1. *Simplification of P in this lemma is similar to simplification of P In Lemma 95. Hence, let $P_1 = (A \rightarrow x) \wedge (A \rightarrow \Diamond_1(x \wedge C_1)) \wedge ... \wedge (A \rightarrow \Diamond_n(x \wedge C_n))$. By the reduction axioms of PAL, P and P_1 are logically equivalent.*
2. *Obviously, as well, P_1 is unifiable iff $\models A \rightarrow (\Diamond_1 C_1 \wedge ... \wedge \Diamond_n C_n)$.*
3. *Moreover, if we assume $\models A \rightarrow \Diamond_1 C_1 \wedge ... \wedge \Diamond_n C_n$ it is clear that $\sigma(x) = \top$ is a unifier of P_1.*
4. *Now, let us find a most general unifier of P_1.*

Notice that in P_1, there is one occurrence of x at the level 0 and for all $i = 1, ..., n$, there is one occurrence of x in the scope of $\Diamond_i$. For this reason, we adapt Löwenheim's formula to the context of P_1. Consider Löwenheim's formula $\varepsilon(x) = (\Box_1 P_1 \wedge ... \wedge \Box_n P_1 \wedge x) \vee (\neg(\Box_1 P_1 \wedge ... \wedge \Box_n P_1) \wedge \sigma(x))$. Since, $\sigma(x) = \top$ hence $\varepsilon(x)$ is logically equivalent to $\Box_1 P_1 \wedge ... \wedge \Box_n P_1 \rightarrow x$. In order to check, $\varepsilon(x)$ is a most general unifier of P_1, we have to prove first, ε is a unifier of P_1 and second ε is a most general unifier of P_1. First, let us prove ε is a unifier of P_1. Hence, we have to prove

1. $\models A \rightarrow \varepsilon(x)$ *and*
2. $\models A \rightarrow \Diamond_i(\varepsilon(x) \wedge C_i)$ *for* $1 \leqslant i \leqslant n$.

1. *The proof of this part is similar to the proof of Lemma 95.*

2. *We have to prove that* $\vDash A \to \Diamond_i(\varepsilon(x) \wedge C_i)$ *for* $1 \leqslant i \leqslant n$ *or equivalently* $\vDash A \wedge \Box_i(C_i \to \Box_1 P_1 \wedge \ldots \wedge \Box_n P_1) \to \Diamond_i(x \wedge C_i)$. *Since,*

   - $\vDash A \wedge \Box_i(C_i \to \Box_1 P_1 \wedge \ldots \wedge \Box_n P_1) \to A \wedge (\Diamond_i C_i \to \Diamond_i \Box_i P_1)$ *and by assumption*
   - $\vDash A \to \Diamond_i C_i$ *then,*
   - $\vDash A \wedge \Box_i(C_i \to \Box_1 P_1 \wedge \ldots \wedge \Box_n P_1) \to A \wedge \Box_i P_1$. *Since,*
   - $\vDash \Box_i P_1 \to ((A \to \Diamond_1(x \wedge C_1)) \wedge \ldots \wedge (A \to \Diamond_n(x \wedge C_n)))$ *then for* $1 \leqslant i \leqslant n$,
   - $\vDash \Box_i P_1 \wedge A \to \Diamond_i(x \wedge C_i)$. *Since,*
   - $\vDash A \wedge \Box_i(C_i \to \Box_1 P_1 \wedge \ldots \wedge \Box_n P_1) \to A \wedge \Box_i P_1$ *and*
   - $\vDash \Box_i P_1 \wedge A \to \Diamond_i(x \wedge C_i)$ *thus,*
   - $\vDash A \wedge \Box_i(C_i \to \Box_1 P_1 \wedge \ldots \wedge \Box_n P_1) \to \Diamond_i(x \wedge C_i)$.

   Therefore, ε *is a unifier of* P_1.

Second, let σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \Box_1 \sigma'(P_1), \ldots, \vdash \Box_n \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = \Box_1 \sigma'(P_1) \wedge \ldots \wedge \Box_1 \sigma'(P_1) \to \sigma'(x)$ *is logically equivalent to* $\sigma'(x)$. *Thus,* $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *are equivalent and* P *and* P_1 *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 97 *Let* $P = A \to \langle x \rangle \Big((\Box_{k_1} B_1 \wedge \ldots \wedge \Box_{k_m} B_m) \wedge (\Diamond_{l_1} C_1 \wedge \ldots \wedge \Diamond_{l_n} C_n) \Big)$ *where* B_i *and* C_j *are Boolean formula for* $1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$. *Let* $B' = (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Then,* $\vDash A \to B' \wedge \Diamond_{l_1}(C_1 \wedge B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge B')$ *iff* P *possesses a unifier.*

Proof 141 *We have to do the steps* $1-4$ *as for the previous Lemmas.*

1. *Simplification of* P *in this Lemma is similar to simplification of Lemmas* 94 *and* 96. *Hence, let Let* $P_1 = (A \to x) \wedge (x \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)) \wedge (A \to \Diamond_{l_1}(x \wedge C_1) \wedge \ldots \wedge (A \to \Diamond_{l_n}(x \wedge C_n))$. P_1 *and* P *are not logically equivalent. Nevertheless, they have exactly the same unifiers.*

2. *As before, it happens that* P_1 *is unifiable iff* $\vDash A \to B' \wedge \Diamond_{l_1}(C_1 \wedge B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge B')$ *where* $B' = (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$.

3. *Assuming* $\vDash A \to B' \wedge \Diamond_{l_1}(C_1 \wedge B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge B')$ *where* $B' = (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$ *it is clear that* $\sigma(x) = B'$ *is a unifier of* P_1.

4. *Now, let us find a most general unifier of* P_1.

Notice that there are two occurrences of x *in* P_1 *at level* 0 *and* n *occurrences in the scopes of* $\Diamond_{l_1},...,\Diamond_{l_n}$. *Consider Löwenheim's formula* $\varepsilon(x) = (\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x) \vee (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge \sigma(x))$. *Since,* $\sigma(x) = B'$ *hence we have* $\varepsilon(x) = (\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x) \vee (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge B')$. *Notice that* $\varepsilon(x)$ *is equivalent with* $\Big(((\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee B') \wedge (x \vee B') \wedge (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee x)\Big)$. *In order to check,* $\varepsilon(x)$ *is a most general unifier of* P_1, *we have to prove first,* ε *is a unifier of* P_1 *and second* ε *is a most general unifiers of* P_1. *First, let us prove* ε *is a unifier of* P_1. *Hence, we have to prove*

1. $\models A \rightarrow \varepsilon(x)$ *and*

2. $\models (\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x) \vee (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge B') \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$.

3. $\models (A \rightarrow \Diamond_{l_1}(\varepsilon(x) \wedge C_1) \wedge ... \wedge (A \rightarrow \Diamond_{l_n}(\varepsilon(x) \wedge C_n))$.

1. *To prove* $\models A \rightarrow \varepsilon(x)$ *we consider the following steps: Since,*
   - $\models \neg(A \rightarrow x) \rightarrow \neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1)$ *then,*
   - $\models A \rightarrow \neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee x$. *Since,*
   - $\models A \rightarrow B'$ *then,*
   - $\models A \rightarrow ((\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee B') \wedge (x \vee B')$. *Since,*
   - $\models A \rightarrow \neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee x$ *then,*
   - $\models A \rightarrow ((\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee B') \wedge (x \vee B') \wedge (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \vee x)$. *Thus,*
   - $\models A \rightarrow \varepsilon(x)$.

2. *Obviously,* $\models (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge B') \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$. *Since,*
   $\models \Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \rightarrow (x \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m))$ *then,*
   $\models \Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$. *Since,*
   $\models \Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$ *and*
   $\models (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge B') \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$ *then,*
   $\models (\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1 \wedge x) \vee (\neg(\Box_{l_1}P_1 \wedge ... \wedge \Box_{l_n}P_1) \wedge B') \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$. *Therefore,*
   $\models \varepsilon(x) \rightarrow (\Diamond_{k_1}A \rightarrow B_1) \wedge ... \wedge (\Diamond_{k_m}A \rightarrow B_m)$.

3. *We have to prove that* $\models (A \to \Diamond_{l_1}(\varepsilon(x) \wedge C_1) \wedge \ldots \wedge (A \to \Diamond_{l_n}(\varepsilon(x) \wedge C_n))$. *In this respect, we will only prove,* $\models A \to \Diamond_{l_1}(\varepsilon(x) \wedge C_1)$. *Let us prove* $\models A \to \Diamond_{l_1}(\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1 \wedge x \wedge C_1) \vee \Diamond_{l_1}(\neg(\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1) \wedge B' \wedge C_1)$ *or equivalently* $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1 \wedge x \wedge C_1)$. *Since,* $\models A \to \Diamond_{l_1}(B' \wedge C_1)$, *then*

   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Box_{l_1} P_1$ *and*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(x \wedge C_1)$. *Then,*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(x \wedge C_1) \wedge \Box_{l_1} P_1$. *Hence,*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(x \wedge C_1 \wedge P_1)$. *Since,*
   - $\models P_1 \wedge x \to B'$ *hence,*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(x \wedge B' \wedge C_1 \wedge P_1)$. *Since,*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1))$ *and*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(x \wedge B' \wedge C_1 \wedge P_1)$ *then,*
   - $\models A \wedge \Box_{l_1}(B' \wedge C_1 \to (\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1)) \to \Diamond_{l_1}(\Box_{l_1} P_1 \wedge \ldots \wedge \Box_{l_n} P_1 \wedge x \wedge C_1)$.

   Therefore, ε *is a unifier of* P_1.

Second, let σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \Box_{l_1}\sigma'(P_1), \ldots, \vdash \Box_{l_n}\sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\Box_{l_1}\sigma'(P_1) \wedge \ldots \wedge \Box_{l_n}\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg(\Box_{l_1}\sigma'(P_1) \wedge \ldots \wedge \Box_{l_n}\sigma'(P_1)) \wedge B')$ *is logically equivalent to* $\sigma'(x)$. *Thus,* $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P *too.*

The last Lemma contains simple epistemic planning problem of the form $A \to \langle x \rangle \Big((\Box_{k_1} B_1 \wedge \ldots \wedge \Box_{k_m} B_m) \wedge (\Diamond_{l_1} C_1 \wedge \ldots \wedge \Diamond_{l_n} C_n) \Big)$ where $B_1, \ldots, B_m$ and $C_j 1, \ldots, C_n$ are Boolean formulas. The solutions of these problem are formulas ψ such that if A holds then ψ can be announced and after ψ is announced, agent k_i knows B_i hold ($1 \leqslant i \leqslant m$) and agent l_j considers it is possible that C_j holds ($1 \leqslant j \leqslant n$).

$A \to \langle \psi \rangle B$	Necessary condition	$\sigma(x)$	mgu
$A \to \langle x \rangle B$	$\models A \to B$	$\top$	$\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$
$A \to \langle x \rangle \Box B$	$\models A \to B$	$\top$	$\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$
$A \to \langle x \rangle (\Box_1 B_1 \wedge \ldots \wedge \Box_n B_n)$	$\models A \to (B_1 \wedge \ldots \wedge B_n)$	$B_1 \wedge \ldots \wedge B_n$	$\varepsilon(x) = (P_1 \wedge x) \vee (\neg P_1 \wedge \sigma(x))$
$A \to \langle x \rangle \Diamond C$	$\models A \to \Diamond C$	$\top$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$
$A \to \langle x \rangle (\Diamond_1 C_1 \wedge \ldots \wedge \Diamond_n C_n)$	$\models A \to (\Diamond_1 C_1 \wedge \ldots \wedge \Diamond_n C_n)$	$\top$	$\varepsilon(x) = (\bigwedge_{i=1}^{n} \Box_i P_1 \wedge x) \vee (\neg \bigwedge_{i=1}^{n} \Box_i P_1 \wedge \sigma(x))$
$A \to \langle x \rangle (\bigwedge_{i=1}^{m} \Box_{k_i} B_i \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} C_j)$	$\models A \to B' \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} (C_j \wedge B')$	B'	$\varepsilon(x) = (\bigwedge_{j=1}^{n} \Box_{l_j} P_1 \wedge x) \vee (\neg (\bigwedge_{j=1}^{n} \Box_{l_j} P_1) \wedge \sigma(x))$

Table 7.1: Simple epistemic planning problem $A \to \langle x \rangle B$

7.3 Simple epistemic planning problem $A \to \langle \Box x \rangle B$

In this section, the solution of the simple epistemic planning problems that we will consider should be of the form $\Box\psi$.

Lemma 98 *Let $P = A \to \langle \Box x \rangle B$ Where B is Boolean formula. Then, $\vDash A \to B$ iff P possesses a unifier.*

Proof 142 *We have to do the steps* 1 *to* 4.

1. *We use axiomatisation of public announcement to simplify P. Hence, let $P_1 = (\Diamond A \to \Box x) \wedge (A \to B)$. By the reduction oxioms pf PAL and by the fact that $\Box$ and $\Diamond$ are interpreted in models by equivalence relations, we obtain that P and P_1 have the same unifiers.*

2. *If P_1 is unifiable then $\vDash A \to B$. we remind that A, B contain only parameters. Reciprocally, suppose $\vDash A \to B$. Then $\sigma(x) = \top$ is a unifier of P_1.*

3. *Now, let us find a unifier of P_1. Since $\vDash A \to B$, it is clear that $\sigma(x) = \top$ is a unifier of P_1.*

4. *Now, let us find a most general unifier of P_1.*

Consider Löwenheim's formula $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg\Box P_1 \wedge \sigma(x))$. Since, $\sigma(x) = \top$ hence we have $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg\Box P_1 \wedge \top)$ or equivalently $\varepsilon(x) = (\Box P_1 \to x)$. In order to check, $\varepsilon(x)$ is a most general unifier of P_1, we have to prove first, ε is a unifier of P_1 and second ε is a most general general unifier of P_1. First, let us prove ε is a unifier of P_1. Hence, we have to prove $\vDash \Diamond A \to \Box\varepsilon(x)$. Since,

1. $\vDash \Box P_1 \to (\Diamond A \to \Box x)$ *then,*

2. $\vDash \Box P_1 \wedge \Diamond A \to \Box x$. *Since,*

3. $\vDash \Box x \to x$ *hence,*

4. $\vDash \Box P_1 \wedge \Diamond A \to x$. *Then,*

5. $\vDash \Diamond A \to (\Box P_1 \to x)$.

Therefore, ε is a unifier of P_1.
Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box\sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = (\Box\sigma'(P_1) \to \sigma'(x))$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε is a most general unifier of P_1. Since P_1 and P are equivalent and P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 99 *Let $P = A \to \langle \Box x \rangle \Box B$ where B is a Boolean formula. then, $\models A \to \Box B$ iff P has a unifier.*

Proof 143 *We have to do the steps* $1-4$.

1. *Use axiomatisation of public announcement to simplify P. By the reduction axioms of PAL, P is logically equivalent to $P' = \Big((A \to \Box x) \wedge (A \to \Box[\Box x]B)\Big)$. Since $\Box$ is interpreted by an equivalence relation, then P' has the same unifiers as $P'' = \Big((\Diamond A \to \Box x) \wedge (\Diamond A \to (\Box x \to B))\Big)$ which has itself the same unifiers as $P_1 = (\Diamond A \to \Box x) \wedge (\Diamond A \to B)$.*

2. *Obviously, P_1 is unifiable iff $\models \Diamond A \to B$. We remind that A, B contain only parameters.*

3. *Suppose $\models \Diamond A \to B$. It is clear that $\sigma(x) = \top$ is a unifier of P_1.*

4. *Let us find a most general unifier of P_1.*

Consider Löwenheim's formula $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$. Since, $\sigma(x) = \top$ hence we have $\varepsilon(x) = (\Box P_1 \to x)$. In order to check, $\varepsilon(x)$ is a most general unifier of P_1, we have to prove first, ε is a unifier of P_1 and second ε is a most general unifier of P_1. First, let us prove ε is a unifier of P_1. Hence, we have to prove $\models \Diamond A \to \Box \varepsilon(x)$. Since,

1. $\models \Box P_1 \to (\Diamond A \to \Box x)$ *then,*

2. $\models \Box P_1 \wedge \Diamond A \to \Box x$. *Since,*

3. $\models \Box x \to x$ *hence,*

4. $\models \Box P_1 \wedge \Diamond A \to x$. *Then,*

5. $\models \Diamond A \to (\Box P_1 \to x)$. *Then,*

6. $\models \Diamond A \to \Box(\Box P_1 \to x)$

Since $\models \Diamond A \to B$, we obtain that ε is a unifier of P_1.
Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box \sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = (\Box \sigma'(P_1) \to \sigma'(x))$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$.
Consequently, ε is a most general unifier of P_1. Since P_1 and P are equivalent and P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 100 *Let $P = A \to \langle \Box_1 x \rangle \Box_2 B$ where B is Boolean formula. Then, $\models \Diamond_1 A \to (\Diamond_2 A \to B)$ iff P possesses a unifier.*

Proof 144 *We have to do at the following steps:*

1. *Simplify P by axiomatisation of public announcement logic. By the reduction axioms of PAL, P is logically equivalent to $P' = \Big((A \to \Box_1 x) \wedge (A \to \Box_2 [\Box_1 x] B)\Big)$. Since $\Box_1$ and $\Box_2$ are interpreted by equivalence relations, then P' has the same unifiers as $P'' = \Big((A \to \Box_1 x) \wedge (\Diamond_2 A \to [\Box_1 x] B)\Big)$ which has itself the same unifiers as $P_1 = (\Diamond_1 A \to \Box_1 x) \wedge (\Box_1 x \to (\Diamond_2 A \to B))$.*

2. *Assume, $\models \Diamond_1 A \to (\Diamond_2 A \to B)$. Hence, $\sigma(x) = \Diamond_1 A$ is a unifier of P_1. Reciprocally, it is obvious that if τ is a unifier of P_1 then $\models \Diamond_1 A \to (\Diamond_2 A \to B)$.*

3. *Let us find a most general unifier of P_1. We claim Löwenheim's formula $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$ is a most genearl unifier of P_1.*

   (a) *Let us prove ε is a unifier of P_1. We need to make sure that ε is a unifier of P_1 hence, we have to prove $\models \Diamond_1 A \to \Box_1 \varepsilon(x)$ and $\models \Box_1 \varepsilon(x) \to (\Diamond_2 A \to B)$.*
   To prove first part: Since

      i. $\models \Box_1 P_1 \to (\Diamond_1 A \to \Box_1 x)$
      ii. $\models \Box_1 P_1 \wedge \Diamond_1 A \to \Box_1 x$
      iii. $\models \Box_1 x \to x$
      iv. $\models \Box_1 P_1 \wedge \Diamond_1 A \to x$
      v. $\models \Diamond_1 A \to (\Box_1 P_1 \to x)$
      vi. $\models \Diamond_1 A \to \Box_1 (\Box_1 P_1 \to x)$
      vii. $\models \Diamond_1 A \to \Box_1 \Big((\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)\Big)$.

   To prove second part: Let us prove $\models \Big((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)\Big) \to (\Diamond_2 A \to B)$. Since,

      i. $\models \Box_1 P_1 \to (\Box_1 x \to (\Diamond_2 A \to B))$ *then*
      ii. $\models (\Box_1 P_1 \wedge \Box_1 x_1) \to (\Diamond_2 A \to B)$. *Since*
      iii. $\models \Diamond_1 A \to (\Diamond_2 A \to B)$ *Then,*
      iv. $\models (\neg \Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 A \to B)$. *Thus,*

v. $\models \Big((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A)\Big) \to (\Diamond_2 A \to B)$.
Therefore, ε is a unifier of P_1.

(b) *Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box_1\sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = (\Box_1\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg\Box_1\sigma'(P_1) \wedge \Diamond_1 A)$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$.*
Consequently, ε is a most general unifier of P_1. Since P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 101 *Let $P = A \to \langle\Box_1 x\rangle(\Box_2 B_2 \wedge \ldots \wedge \Box_n B_n)$ where B_i are Boolean formulas for $2 \leqslant i \leqslant n$. Then, $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$ iff P possesses a unifier.*

Proof 145 *Simplify P by axiomatisation of public announcement logic. We proceed as in Lemma 100. Hence, let $P_1 = (\Diamond_1 A \to \Box_1 x) \wedge (\Box_1 x \to (\Diamond_2 A \to B_2)) \wedge \ldots \wedge (\Box_1 x \to (\Diamond_n A \to B_n))$. Suppose $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$. Since $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$, it is clear that $\sigma(x) = \Diamond_1 A$ is a unifier of P_1. Reciprocally, when P_1 has a unifier, then $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$. Let us find a most general unifier of P_1. We claim that Löwenheim's formula $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$ is a most general unifier. Since, $\sigma(x) = \Diamond_1 A$ hence, $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A)$.*

- *Let us prove ε is a unifier of P_1. We need to make sure that ε is a unifier of P_1 hence, we have to prove $\models \Diamond_1 A \to \Box_1\varepsilon(x)$ and $\models \Box_1\varepsilon(x) \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$.*
  Notice that $\Box_1\varepsilon(x)$ is logically equivalent to $(\Box_1 P_1 \wedge \Box_1 x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A)$.
  To prove first part: Since

  1. $\models \Box_1 P_1 \to (\Diamond_1 A \to \Box_1 x)$ *then,*
  2. $\models \Diamond_1 A \to (\Box_1 P_1 \to \Box_1 x)$. *Therefore,*
  3. $\models \Diamond_1 A \to \Big((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A)\Big)$

  To prove second part: Let us prove $\models \Big((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A)\Big) \to \Big((\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)\Big)$. *Since,*

  1. $\models \Box_1 P_1 \to \Big(\Box_1 x \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)\Big)$ *then*
  2. $\models (\Box_1 P_1 \wedge \Box_1 x) \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$. *Since*

3. $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$ *Then,*
4. $\models (\neg \Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$. *Thus,*
5. $\models \Big((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A) \Big) \to \Big((\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n) \Big)$.

Therefore, ε is a unifier of P_1.

- *Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box_1 \sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = (\Box_1 \sigma'(P_1) \wedge \sigma'(x)) \vee (\neg \Box_1 \sigma'(P_1) \wedge \Diamond_1 A)$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$.*
  Consequently, ε is a most general unifier of P_1. Since P_1 and P are equivalent and P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 102 *Let $P = A \to \langle \Box x \rangle \Diamond C$ where C is a Boolean formula. Then, $\models A \to \Diamond C$ iff P possesses a unifier.*

Proof 146 *Simplify P by axiomatisation of public announcement logic. Hence*

1. $(A \to \langle \Box x \rangle \Diamond C)$
2. $\Big((A \to \Box x) \wedge (A \to [\Box x] \neg \Box \neg C) \Big)$
3. $(A \to \Box x) \wedge (A \to \Diamond(\Box x \wedge C))$.

Let $P_1 = (A \to \Box x) \wedge (A \to \Diamond(\Box x \wedge C))$. Then P and P_1 have exactly the same unifiers. If, $\models A \to \Diamond C$ hence, $\sigma(x) = \top$ is a unifier of P_1. Reciprocally, obviously, if P_1 is unifiable then $\models A \to \Diamond C$. Let us find a most general unifier of P_1. We claim that $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$ is a most general unifier of P_1. Since, $\sigma(x) = \top$ hence, $\varepsilon(x) = \neg \Box P_1 \vee x$ is equivalent to $(\Box P_1 \to x)$. In this respect,:

1. *We prove ε is a unifier of P_1. In this respect, we need to prove $\models A \to \Box \varepsilon(x)$ and $A \to \Diamond(\Box \varepsilon(x) \wedge C)$. Let us prove $\models A \to \Box(\Box P_1 \to x)$: Since,*
   $\models \Box P_1 \to (A \to \Box x)$ *then,*
   $\models (\Box P_1 \wedge A) \to \Box x$. *Hence,*
   $\models A \to (\Box P_1 \to x)$. *Therefore,*
   $\models A \to \Box \varepsilon(x)$.
   Let us prove prove that $\models A \to \Diamond(\Box(\Box P_1 \to x) \wedge C)$ or equivalently $\models A \to \Diamond(\neg \Box P_1 \wedge C) \vee \Diamond(\Box x \wedge C)$. In this respect, it is enough to show that $\models A \wedge \Box(C \to \Box P_1) \to \Diamond(\Box x \wedge C)$. Since,

$\models A \wedge \Box(C \to \Box P_1) \to A \wedge (\Diamond C \to \Box P_1)$ *and by our assumption,*
$\models A \to \Diamond C$ *Hence,*
$\models A \wedge \Box(C \to \Box P_1) \to \Box P_1 \wedge A$.*We know that*
$\models \Box P_1 \to (A \to \Diamond(\Box x \wedge C))$ *then,*
$\models \Box P_1 \wedge A \to \Diamond(\Box x \wedge C)$. *Since,*
$\models A \wedge \Box(C \to \Box P_1) \to \Box P_1 \wedge A$ *and*
$\models \Box P_1 \wedge A \to \Diamond(\Box x \wedge C)$ *then,*
$\models A \wedge \Box(C \to \Box P_1) \to \Diamond(\Box x \wedge C)$. *Thus,*
$\models A \to \Diamond(\Box \varepsilon(x) \wedge C)$.

Therefore, ε is a unifier of P_1.

2. *Second, let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box_1 \sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = \neg\Box\sigma'(P_1) \vee \sigma'(x)$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'$.*
   Consequently, ε is a most general unifier of P_1. Since P and P_1 have the same unifiers then, ε is a most general unifier of P.

Lemma 103 *Let $P = A \to \langle \Box_1 x \rangle \Diamond_2 C$ where C is a Boolean formula. Then, $\models A \to \Diamond_2 C$ iff P possesses a unifier.*

Proof 147 *Use axiomatisation of public announcement logic. Hence, let $P_1 = (A \to \Box_1 x) \wedge (A \to \Diamond_2(\Box_1 x \wedge C))$. P and P_1 have exactly the same unifiers. Suppose $\models A \to \Diamond_2 C$. Since, $\models A \to \Diamond_2 C$ hence, $\sigma(x) = \top$ is a unifier of P_1. Reciprocally, if P_1 has a unifier then $\models A \to \Diamond_2 C$. Let us find a most general unifier of P_1. We claim that $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge x) \vee (\neg\Box_1\Box_2 P_1 \wedge \sigma(x))$ is a most general unifier of P_1. Since, $\sigma(x) = \top$ hence, $\varepsilon(x) = \neg\Box_1\Box_2 P_1 \vee x$. In this respect, we consider the following steps:*

1. *We prove ε is a unifier of P_1. In this respect, we need to prove $\models A \to \Box_1 \varepsilon(x)$ and $\models A \to \Diamond_2(\Box_1 \varepsilon(x) \wedge C)$. Let us prove first one: Since,*
   $\models \Box_1\Box_2 P_1 \to (A \to \Box_1 x)$ *then,*
   $\models (\Box_1\Box_2 P_1 \wedge A) \to \Box_1 x$ *hence,*
   $\models A \to (\Box_1\Box_2 P_1 \to \Box_1 x)$. *Therefore*
   $\models A \to \Box_1 \varepsilon(x)$.
   Second, we have to prove that $\models A \to \Diamond_2(\Box_1 \varepsilon(x) \wedge C)$ or equivalently $\models A \wedge \Box_2(C \to \Box_1\Box_2 P_1) \to \Diamond_2(\Box_1 x \wedge C)$. Since,
   $\models A \wedge \Box_2(C \to \Box_1\Box_2 P_1) \to A \wedge (\Diamond_2 C \to \Diamond_2\Box_1\Box_2 P_1)$ *and by assumption*
   $\models A \to \Diamond_2 C$ *then,*

$\models A \wedge \Box_2(C \to \Box_1\Box_2 P_1) \to \Diamond_2\Box_1\Box_2 P_1 \wedge A$. *We know that,*
$\models \Diamond_2\Box_1\Box_2 P_1 \wedge A \to \Box_2 P_1 \wedge A$. *Since,*
$\models \Box_2 P_1 \to (A \to \Diamond_2(\Box_1 x \wedge C))$ *hence,*
$\models \Box_2 P_1 \wedge A \to \Diamond_2(\Box_1 x \wedge C)$. *Since,*
$\models A \wedge \Box_2(C \to \Box_1\Box_2 P_1) \to \Diamond_2\Box_1\Box_2 P_1 \wedge A$ *and*
$\models \Diamond_2\Box_1\Box_2 P_1 \wedge A \to \Box_2 P_1 \wedge A$ *and*
$\models \Box_2 P_1 \wedge A \to \Diamond_2(\Box_1 x \wedge C)$ *thus,*
$\models A \wedge \Box_2(C \to \Box_1\Box_2 P_1)) \to \Diamond_2(\Box_1 x \wedge C)$.

Therefore, ε is a unifier of P_1.

2. *Let us prove that ε is more general than any unifier of P_1. Let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box_1\Box_2\sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = \neg\Box_1\Box_2\sigma'(P_1) \vee \sigma'(x)$ is logically equivalent to $\sigma'(x)$. Thus, $\varepsilon \preccurlyeq \sigma'(x)$. Consequently, ε is a most general unifier of P_1. Since validity of P_1 and P are equivalent then, ε is a most general unifier of P.*

Lemma 104 *Let $P = A \to \langle\Box_1 x\rangle(\Diamond_2 C_2 \wedge ... \wedge \Diamond_n C_n)$ where C_i are Boolean formulas for $2 \leqslant i \leqslant n$. Then, $\models A \to \Diamond_2 C_2 \wedge ... \wedge \Diamond_n C_n$ iff P possesses a unifier.*

Proof 148 ($\Rightarrow$) *Simplify P by axiomatisation of public announcement logic. Hence, let $P_1 = (A \to \Box_1 x) \wedge (A \to \Diamond_2(\Box_1 x \wedge C_2)) \wedge ... \wedge (A \to \Diamond_n(\Box_1 x \wedge C_n))$. P and P_1 have the same unifiers. Suppose $\models A \to \Diamond_2 C_2 \wedge ... \wedge \Diamond_n C_n$. Since, $\models A \to \Diamond_2 C_2 \wedge ... \wedge \Diamond_n C_n$ hence, $\sigma(x) = \top$ is a unifier of P_1. Reciprocally, if P_1 has a unifier then it is clear that $\models A \to \Diamond_2 C_2 \wedge ... \wedge \Diamond_n C_n$. Let us find a most general unifier of P_1. We claim that $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1 \wedge x) \vee (\neg(\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1) \wedge \sigma(x))$ is a most general unifier of P_1. Notice that $\varepsilon(x) = \neg(\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1) \vee x$. In this respect, we will do the following steps:*

1. *We prove that ε is a unifier. In this respect, we have to prove $\models A \to \Box_1\varepsilon(x)$ and $\models A \to \Diamond_i(\Box_1\varepsilon(x) \wedge C_i)$ for all $2 \leqslant i \leqslant n$ or equivalently $\models A \wedge \Box_i(C_i \to \Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1) \to \Diamond_i(\Box_1 x \wedge C_i)$.Since,*
$\models \Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1 \to (A \to \Box_1 x)$ *then,*
$\models (\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1 \wedge A) \to \Box_1 x$ *hence,*
$\models A \to (\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1 \to \Box_1 x)$. *Therefore,*
$\models A \to \Box_1\varepsilon(x)$.

Let us prove the second one: Since,
$\models A \wedge \Box_i(C_i \to \Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1) \to A \wedge (\Diamond_i C_i \to \Diamond_i(\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1))$ *and by assumption*

$\models A \to \Diamond_i C_i$ *then,*
$\models A \wedge \Box_i (C_i \to \Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \to A \wedge \Diamond_i (\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1)$.
We know that for all $2 \leqslant i \leqslant n$,
$\models \Diamond_i (\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \wedge A \to \Box_i P_1 \wedge A$. *Since for all* $2 \leqslant i \leqslant n$,
$\models \Box_i P_1 \to ((A \to \Diamond_2(\Box_1 x \wedge C_1)) \wedge \ldots \wedge (A \to \Diamond_n(\Box_1 x \wedge C_n)))$ *then,*
$\models \Box_i P_1 \wedge A \to \Diamond_i (\Box_1 x \wedge C_i)$. *Since,*
$\models A \wedge \Box_i (C_i \to \Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \to A \wedge \Diamond_i (\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1)$,
$\models \Diamond_i (\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \wedge A \to \Box_i P_1 \wedge A$ *and*
$\models \Box_i P_1 \wedge A \to \Diamond_i (\Box_1 x \wedge C_i)$ *thus,*
$\models A \wedge \Box_i (C_i \to \Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \to \Diamond_i (\Box_1 x \wedge C_i)$.

2. *Let us prove that* ε *is more general than any unifier of* P_1. *Let* σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$ *and* $\models \Box_1\Box_i \sigma'(P_1)$ *for each* $i = 2, \ldots, n$. *Thus,* $\sigma'(\varepsilon(x)) = \neg(\Box_1\Box_2\sigma'(P_1) \wedge \ldots \wedge \Box_1\Box_n\sigma'(P_1)) \vee \sigma'(x)$ *is logically equivalent to* $\sigma'(x)$. *Therefore,* $\varepsilon \preccurlyeq \sigma'(x)$.

   Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 105 *Let* $P = A \to \langle \Box_1 x \rangle \Big((\Box_{k_1} B_1 \wedge \ldots \wedge \Box_{k_m} B_m) \wedge (\Diamond_{l_1} C_1 \wedge \ldots \wedge \Diamond_{l_n} C_n) \Big)$ *where* B_i *and* C_j *are Boolean formulas for* $1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$. *Let* $B' = (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Then,* $\models A \to \Box_1 B' \wedge \Diamond_{l_1}(C_1 \wedge \Box_1 B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge \Box_1 B')$ *iff* P *possesses a unifier.*

Proof 149 *Simplify* P *by axiomatisation of public announcement logic. Hence, let* $P_1 = (\Diamond_1 A \to \Box_1 x) \wedge (\Box_1 x \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)) \wedge (A \to \Diamond_{l_1}(\Box_1 x \wedge C_1)) \wedge \ldots \wedge (A \to \Diamond_{l_n}(\Box_1 x \wedge C_n))$. *Clearly,* P *and* P_1 *have exactly the same set of unifiers. Suppose,* $\models A \to \Box_1 B' \wedge \Diamond_{l_1}(C_1 \wedge \Box_1 B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge \Box_1 B')$. *Hence,* $\sigma(x) = \Box_1 B'$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *We claim that Löwenheim's formula* $\varepsilon(x) = (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \sigma(x))$ *is a most general unifier. Since,* $\sigma(x) = \Box_1 B'$ *then,* $\varepsilon(x) = (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B')$. *In this respect, we have to do the following steps:*

1. *We should prove that* ε *is a unifier of* P_1. *Then we should prove* $\models \Diamond_1 A \to \Box_1 \varepsilon(x)$ *and* $\models \Box_1 \varepsilon(x) \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$ *and* $\models (A \to \Diamond_{l_1}(\Box_1 \varepsilon(x) \wedge C_1) \wedge \ldots \wedge (A \to \Diamond_{l_n}(\Box_1 \varepsilon(x) \wedge C_n))$. *Let us prove*
   first one: *Since,*

$\vDash \neg(\Diamond_1 A \to \Box_1 x) \to \neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1)$ *then,*
$\vDash \Diamond_1 A \to \neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \vee \Box_1 x$. *Since,*
$\vDash A \to \Box_1 B'$ *then,*
$\vDash \Diamond_1 A \to \Box_1 B'$ *hence,*
$\vDash \Diamond_1 A \to ((\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \vee \Box_1 B') \wedge (\Box_1 x \vee \Box_1 B')$. *Since,*
$\vDash \Diamond_1 A \to \neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \vee \Box_1 x$ *then,*
$\vDash \Diamond_1 A \to ((\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \vee \Box_1 B') \wedge (\Box_1 x \vee \Box_1 B') \wedge (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \vee \Box_1 x)$. *Therefore,*
$\vDash \Diamond_1 A \to \Box_1 \varepsilon(x)$.

Second one: *we have to prove that* $\vDash \Box_1\varepsilon(x) \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$. *In this respect, we need to prove*
$\vDash (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge \Box_1 x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1\Box_1 B') \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$. *It is obvious,*
$\vDash (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B') \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Since,*
$\vDash \Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \to (\Box_1 x \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$ *then,*
$\vDash \Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge \Box_1 x \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Since,*
$\vDash \Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge \Box_1 x \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$ *and*
$\vDash (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B') \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$ *then,*
$\vDash (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B') \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Since,*
$\vDash (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B') \to (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$ *hence,*
$\vDash \Box_1((\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B')) \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$ *then,*
$\vDash (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B') \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$. *Therefore,*
$\vDash \Box_1\varepsilon(x) \to \Box_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m))$.

Third one: *we have to prove* $\vDash (A \to \Diamond_{l_1}(\Box_1\varepsilon(x) \wedge C_1) \wedge \ldots \wedge (A \to \Diamond_{l_n}(\Box_1\varepsilon(x) \wedge C_n))$. *In this respect, we will prove,* $\vDash A \to \Diamond_{l_1}(\Box_1\varepsilon(x) \wedge C_1)$ *and the proof of rest of parenthesis are similar. Let us prove* $\vDash A \to \Diamond_{l_1}(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge \Box_1 x \wedge C_1) \vee \Diamond_{l_1}(\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Box_1 B' \wedge C_1)$ *or equivalently* $\vDash A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1)) \to \Diamond_{l_1}(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge \Box_1 x \wedge C_1)$. *Since,*
$\vDash A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1)) \to \Box_{l_1} P_1$ *and* $\vDash A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1)) \to \Diamond_{l_1}(\Box_1 x \wedge C_1)$ *then,*
$\vDash A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1)) \to \Diamond_{l_1}(\Box_1 x \wedge C_1) \wedge \Box_{l_1} P_1$

hence,
$\models A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1)) \to \Diamond_{l_1}(\Box_1 x \wedge C_1 \wedge P_1)$.
Since,
$\models P_1 \wedge \Box_1 x \to \Box_1 B'$ *hence,*
$\models A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1)) \to \Diamond_{l_1}(\Box_1 x \wedge \Box_1 B' \wedge C_1 \wedge P_1)$. *Since,*
$\models A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1)) \to \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1))$ *and*
$\models A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1)) \to \Diamond_{l_1}(\Box_1 x \wedge \Box_1 B' \wedge C_1 \wedge P_1)$ *then,*
$\models A \wedge \Box_{l_1}(\Box_1 B' \wedge C_1 \to (\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1)) \to \Diamond_{l_1}(\Box_1\Box_{l_1}P_1 \wedge \ldots \wedge \Box_1\Box_{l_n}P_1 \wedge \Box_1 x \wedge C_1)$. *Therefore,*
ε *is a unifier of* P_1.

2. *Let us prove that* ε *is more general than any unifier of* P_1. *Let* σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$ *and* $\vdash \Box_1\Box_{l_1}\sigma'(P_1) \wedge \ldots \wedge \Box_1\Box_{l_n}\sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\Box_1\Box_{l_1}\sigma'(P_1) \wedge \ldots \wedge \Box_1\Box_{l_n}\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg(\Box_1\Box_{l_1}\sigma'(P_1) \wedge \ldots \wedge \Box_1\Box_{l_n}\sigma'(P_1)) \wedge \Box_1 B')$ *is logically equivalent to* $\sigma'(x)$. *Therefore,* $\varepsilon \preccurlyeq \sigma'$.

   Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 106 *Let* $P = A \to \langle\Box_2 x\rangle\Box_1\Box_2 B$ *where* B *is a Boolean formula. Then,* $\models A \to \Box_2 B$ *iff* P *possesses a unifier.*

Proof 150 *We simplify* P *by axiomatisation of public announcement logic. Hence, Let* $P_1 = (A \to \Box_2 x) \wedge (\Box_2 x \to (\Diamond_1 A \to \Box_2 B))$. *Suppose* $\models A \to \Box_2 B$. *Since,* $\models A \to \Box_2 B$ *then,* $\sigma(x) = B$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *We claim that Löwenheim's formula* $\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg\Box_2 P_1 \wedge \sigma(x))$ *is a most general unifier of* P_1. *Since,* $\sigma(x) = B$ *hence,* $\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg\Box_2 P_1 \wedge B)$. *In this respect, we will do the following steps:*

1. *We have to prove* ε *is a unifier of* P_1. *Let us prove* $\models A \to \Box_2\varepsilon(x)$ *and* $\models \Box_2\varepsilon(x) \to (\Diamond_1 A \to \Box_2 B)$.

   The proof of first one: it is equivalent to prove $\models A \to (\Box_2 P_1 \vee \Box_2 B) \wedge (\Box_2 x \vee \neg\Box_2 P_1) \wedge (\Box_2 x \vee \Box_2 B)$. *By our assumption,*
   $\models A \to \Box_2 B$. *Then,*
   $\models A \to (\Box_2 P_1 \vee \Box_2 B) \wedge (\Box_2 x \vee \Box_2 B)$. *Since,*

$\models \neg(A \to \Box_2 x) \to \neg\Box_2 P_1$ *then,*
$\models A \to \neg\Box_2 P_1 \vee \Box_2 x$. *Since,*
$\models A \to (\Box_2 P_1 \vee \Box_2 B) \wedge (\Box_2 x \vee \Box_2 B)$ *and*
$\models A \to \neg\Box_2 P_1 \vee \Box_2 x$ *thus,*
$\vdash A \to (\Box_2 P_1 \vee \Box_2 B) \wedge (\Box_2 x \vee \neg\Box_2 P_1) \wedge (\Box_2 x \vee \Box_2 B)$. *Therefore,*
$\models A \to \Box\varepsilon(x)$.
Let us prove $\models \Box_2\varepsilon(x) \to (\Diamond_1 A \to \Box_2 B)$. *It is enough to show that* $\models (\Box_2 P_1 \wedge \Box_2 x) \to (\Diamond_1 A \to \Box_2 B)$. *Since,*
$\models \Box_2 P_1 \to (\Box_2 x \to (\Diamond_1 A \to \Box_2 B))$ *then,*
$\models (\Box_2 P_1 \wedge \Box_2 x) \to (\Diamond_1 A \to \Box_2 B)$. *Thus,*
$\models \Box_2\varepsilon(x) \to (\Diamond_1 A \to \Box_2 B)$.
Therefore, $\varepsilon(x)$ *is a unifier of* P_1.

2. *Let us prove that* ε *is more general than any unifier of* P_1. *Let* σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\Box_2\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg\Box_2\sigma'(P_1) \wedge B)$ *is logically equivalent to* $\sigma'(x)$. *Therefore,* $\varepsilon \preccurlyeq \sigma$.

   Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P.

$(\Leftarrow)$ *Let* σ *be a unifier of* P_1. *Then,*

1. $\models A \to \Box_2\sigma(x)$ *and*

2. $\models \Box_2\sigma(x) \to (\Diamond_1 A \to \Box_2 B)$. *Hence,*

3. $\models A \to (\Diamond_1 A \to \Box_2 B)$. *Therefore,*

4. $\models A \to \Box_2 B$.

Lemma 107 *Let* $P = A \to \langle \Box_2 x \rangle (\Box_1\Box_2 B_1 \wedge \ldots \wedge \Box_n\Box_2 B_n)$ *where* B_i *are Boolean formula for* $1 \leqslant i \leqslant n$. *Then,* $\models A \to \Box_2 B_1 \wedge \ldots \wedge \Box_2 B_n$ *iff* P *possesses a unifier.*

Proof 151 *Let use axiomatisation of public announcement logic in order to simplify* P. *Hence, let* $P_1 = (A \to \Box_2 x) \wedge (\Box_2 x \to (\Diamond_1 A \to \Box_2 B_1)) \wedge \ldots \wedge (\Box_2 x \to (\Diamond_n A \to \Box_2 B_n))$. *Clearly,* P *and* P_1 *have exactly the same set of unifiers. Suppose* $\models A \to \Box_2 B_1 \wedge \ldots \wedge \Box_2 B_n$. *Since,* $\models A \to \Box_2 B_1 \wedge \ldots \wedge \Box_2 B_n$ *then,* $\sigma(x) = B_1 \wedge \ldots \wedge B_n$ *is a unifier of* P_1. *And of course, reciprocally, when* P_1 *has a unifier then* $\models A \to \Box_2 B_1 \wedge \ldots \wedge \Box_2 B_n$. *We claim that* $\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg\Box_2 P_1 \wedge \sigma(x))$ *is a most general unifier of* P_1. *Since,* $\sigma(x) = B_1 \wedge \ldots \wedge B_n$ *then,* $\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg\Box_2 P_1 \wedge (B_1 \wedge \ldots \wedge B_n))$. *We can use the method of 106 to prove* ε *is a most general unifier.*

Lemma 108 *Let* $P = A \to \langle \Box_1 x \rangle \Box_1 \Box_2 B$ *where* B *is a Boolean formula. Then,* $\models \Diamond_1 A \to B$ *iff* P *possesses a unifier.*

Proof 152 *We have to do the following steps:*

1. *We can simplify* P *by axiomatisation of public announcement logic as before. Let* $P_1 = (\Diamond_1 A \to \Box_1 x) \wedge (\Box_1 x \to (\Diamond_2 \Diamond_1 A \to B))$. *Clearly,* P *and* P_1 *have exactly the same set of unifiers. Assume,* $\models \Diamond_1 A \to B$. *Hence,* $\sigma(x) = \Diamond_1 A$ *is a unifier of* P_1. *Reciprocally, if* P_1 *is unifiable then* $\models \Diamond_1 A \to B$.

2. *Let us find a most general unifier of* P_1. *We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)$ *is a most general unifier of* P_1.

   (a) *Let us prove* ε *is a unifier of* P_1. *We need to make sure that* $\varepsilon(x)$ *is a unifier of* P_1 *hence, we have to prove* $\models \Diamond_1 A \to \Box_1 \varepsilon(x)$ *and* $\models \Box_1 \varepsilon(x) \to (\Diamond_2 \Diamond_1 A \to B)$.
   To prove first part: Since

      i. $\models \Box_1 P_1 \to (\Diamond_1 A \to \Box_1 x)$
      ii. $\models \Box_1 P_1 \wedge \Diamond_1 A \to \Box_1 x$
      iii. $\models \Diamond_1 A \to (\Box_1 P_1 \to \Box_1 x)$. *Therefore,*
      iv. $\models \Diamond_1 A \to (\Box_1 P_1 \wedge \Box_1 x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)$

   To prove second part: Let us prove $\models ((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)) \to (\Diamond_2 \Diamond_1 A \to B)$. *Since,*

      i. $\models \Box_1 P_1 \to (\Box_1 x_1 \to (\Diamond_2 \Diamond_1 A \to B))$ *then*
      ii. $\models (\Box_1 P_1 \wedge \Box_1 x_1) \to (\Diamond_2 \Diamond_1 A \to B)$. *Since*
      iii. $\models \Diamond_1 A \to \to B$ *then,*
      iv. $\models (\neg \Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 \Diamond_1 A \to B)$. *Thus,*
      v. $\models ((\Box_1 P_1 \wedge \Box_1 x) \vee (\neg \Box_1 P_1 \wedge \Diamond_1 A)) \to (\Diamond_2 \Diamond_1 A \to B)$.
      Therefore, ε *is a unifier of* P_1.

   (b) *Let us prove that* ε *is more general than any unifier of* P_1. *Let* σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \Box_1 \sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\Box_1 \sigma'(P_1) \wedge \sigma'(x)) \vee (\neg \Box_1 \sigma'(P_1) \wedge \Diamond_1 A)$ *is logically equivalent to* $\sigma'(x)$. *Therefore,* $\varepsilon \preccurlyeq \sigma'$.
   Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 109 *Let* $P = A \to \langle \Box_1 x\rangle(\Box_1\Box_2 B_2 \wedge ... \wedge \Box_1\Box_n B_n)$ *where* B_i *are Boolean formulas for* $2 \leqslant i \leqslant n$*. Then,* $\models \Diamond_1 A \to (B_2 \wedge ... \wedge B_n)$ *iff* P *possesses a unifier.*

Proof 153 *We simplify* P *by axiomatisation of public announcement logic as before. Hence, let* $P_1 = (\Diamond_1 A \to \Box_1 x) \wedge (\Box_1 x \to (\Diamond_2\Diamond_1 A \to B_2)) \wedge ... \wedge (\Box_1 x \to (\Diamond_n\Diamond_1 A \to B_n))$*. Clearly* P *and* P_1 *have exactly the same unifiers. Now, suppose* $\models \Diamond_1 A \to (B_2 \wedge ... \wedge B_n)$ *hence,* $\sigma(x) = \Diamond_1 A$ *is a unifier of* P_1*. Reciprocally, if* P_1 *has a unifier then obviously* $\models \Diamond_1 A \to (B_2 \wedge ... \wedge B_n)$*. We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$ *is a most general unifier of* P_1*. The method of proving* ε *is a most general unifier is similar to* 108.

Lemma 110 *Let* $P = A \to \langle \Box_1 x\rangle\Box_1\Diamond_2 B$ *where* B *is a Boolean formula. Then,* $\models A \to \Box_1\Diamond_2 B$ *iff* P *possesses a unifier.*

Proof 154 *We use axiomatisation of public announcement logic. Hence let* $P_1 = (A \to \Box_1 x) \wedge (\Diamond_1 A \to \Diamond_2(\Box_1 x \wedge B))$*. Suppose* $\models A \to \Box_1\Diamond_2 B$*. Since* $\models A \to \Box_1\Diamond_2 B$ *then,* $\sigma(x) = \top$ *is a unifier of* P_1*. Reciprocally if* P_1 *has a unifier then it is clear that* $\models A \to \Box_1\Diamond_2 B$*. Let us find a most general unifier of* P_1*. We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge x) \vee (\neg\Box_1\Box_2 P_1 \wedge \sigma(x))$ *is a most general unifier of* P_1*. We can use the similar method of Lemma* 103 *to check* ε *is a most general unifier.*

Lemma 111 *Let* $P = A \to \langle \Box_1 x\rangle(\Box_1\Diamond_2 B_2 \wedge ... \wedge \Box_1\Diamond_n B_n)$ *where* B_i *are Boolean formulas for* $2 \leqslant i \leqslant n$*. Then,* $\models A \to \Box_1\Diamond_2 B_2 \wedge ... \wedge \Box_1\Diamond_n B_n$ *iff* P *possesses a unifier.*

Proof 155 *We use axiomatisation of public announcement logic as before. Hence we assume* $P_1 = (A \to \Box_1 x) \wedge (\Diamond_1 A \to \Diamond_2(\Box_1 x \wedge B_2)) \wedge ... \wedge (\Diamond_1 A \to \Diamond_n(\Box_1 x \wedge B_n))$*. Suppose* $\models A \to \Box_1\Diamond_2 B_2 \wedge ... \wedge \Box_1\Diamond_n B_n$*. Since* $\models A \to \Box_1\Diamond_2 B_2 \wedge ... \wedge \Box_1\Diamond_n B_n$ *then,* $\sigma(x) = \top$ *is a unifier of* P_1*. Let us find a most general unifier of* P_1*. We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1 \wedge x) \vee (\neg(\Box_1\Box_2 P_1 \wedge ... \wedge \Box_1\Box_n P_1) \wedge \sigma(x))$ *is a most general unifier. To check* ε *is a most general unifier, we use the similar method of Lemma* 104.

The last Lemma contains simple epistemic planning problem of the form $A \to \langle \Box_1 x\rangle(\Box_1\Diamond_2 B_2 \wedge ... \wedge \Box_1\Diamond_n B_n)$. The solutions of these problems are formula ψ such that if A holds then agent 1 can announce ψ and after this announcement, agent k_i knows B_i $(1 \leqslant i \leqslant m)$ and agent l_j consider it is possible that C_j holds $(1 \leqslant j \leqslant n)$.

$A \to \langle \psi \rangle B$	Necessary condition	$\sigma(x)$	mgu
$A \to \langle \Box x \rangle B$	$\models A \to B$	$\top$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$
$A \to \langle \Box x \rangle \Box B$	$\models A \to \Box B$	$\top$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle \Box_2 B$	$\models \Diamond_1 A \to (\Diamond_2 A \to \Box B)$	$\Diamond_1 A$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle (\Box_2 B_2 \wedge \ldots \wedge \Box_n B_n)$	$\models \Diamond_1 A \to \bigwedge_{i=2}^{n} (\Diamond_i A \to B_i)$	$\Diamond_1 A$	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$
$A \to \langle \Box x \rangle \Diamond C$	$\models A \to \Diamond C$	$\top$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle \Diamond_2 C$	$\models A \to \Diamond C$	$\top$	$\varepsilon(x) = (\Box_1 \Box_2 P_1 \wedge x) \vee (\neg \Box_1 \Box_2 P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$	$\models A \to (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$	$\top$	$\varepsilon(x) = (\bigwedge_{i=2}^{n} \Box_1 \Box_i P_1 \wedge x) \vee (\neg (\bigwedge_{i=2}^{n} \Box_1 \Box_i P_1) \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle (\bigwedge_{i=1}^{m} \Box_{k_i} B_i \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} C_j)$	$\models A \to \Box_1 B' \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} (C_j \wedge \Box_1 B')$	$\Box_1 B'$	$\varepsilon(x) = (\bigwedge_{j=1}^{n} \Box_1 \Box_{l_j} P_1 \wedge x) \vee (\neg (\bigwedge_{j=1}^{n} \Box_1 \Box_{l_j} P_1) \wedge \sigma(x))$
$A \to \langle \Box_2 x \rangle \Box_1 \Box_2 B$	$\models A \to \Box_2 B$	B	$\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg \Box_2 P_1 \wedge \sigma(x))$
$A \to \langle \Box_2 x \rangle (\Box_1 \Box_2 B_1 \wedge \ldots \wedge \Box_n \Box_2 B_n)$	$\models A \to \Box_2 B_1 \wedge \ldots \wedge \Box_2 B_n$	$B_1 \wedge \ldots \wedge B_n$	$\varepsilon(x) = (\Box_2 P_1 \wedge x) \vee (\neg \Box_2 P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle \Box_1 \Box_2 B$	$\models \Diamond_1 A \to \to B$	$\Diamond_1 A$	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle (\Box_1 \Box_2 B_2 \wedge \ldots \wedge \Box_1 \Box_n B_n)$	$\models \Diamond_1 A \to B_2 \wedge \ldots \wedge B_n$	$\Diamond_1 A$	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle \Box_1 \Diamond_2 B$	$\models A \to \Box_1 \Diamond_2 B$	$\top$	$\varepsilon(x) = (\Box_1 \Box_2 P_1 \wedge x) \vee (\neg \Box_1 \Box_2 P_1 \wedge \sigma(x))$
$A \to \langle \Box_1 x \rangle (\Box_1 \Diamond_2 B_2 \wedge \ldots \wedge \Box_1 \Diamond_n B_n)$	$\models A \to (\Box_1 \Diamond_2 B_2 \wedge \ldots \wedge \Box_1 \Diamond_n B_n)$	$\top$	$\varepsilon(x) = (\bigwedge_{i=2}^{n} \Box_1 \Box_i P_1 \wedge x) \vee (\neg (\bigwedge_{i=2}^{n} \Box_1 \Box_i P_1) \wedge \sigma(x))$

Table 7.2: Simple epistemic planning problem $A \to \langle \Box x \rangle B$

7.4 Simple epistemic planning problem $A \to \langle \Diamond x \rangle B$

In this section, the solution of the simple epistemic planning problem that we will consider should be of the form $A \to \langle \Diamond x \rangle B$.

Lemma 112 *Let $P = A \to \langle \Diamond x \rangle B$ where B is a Boolean formula. Then, $\models A \to B$ iff P possesses a unifier.*

Proof 156 *We use axiomatisation of public announcement logic to simplify P. Let $P_1 = (A \to \Diamond x) \wedge (A \to B)$. By the reduction axiom of PAL, P nd P_1 have the same unifiers. Suppose $\models A \to B$. Since $\models A \to B$ then $\sigma(x) = \top$ is a unifier of P_1. Reciprocally, we have $\models A \to B$ if P_1 has a unifier. Let us find a most general unifier of P_1. We claim that Löwenheim's formula $\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg \Box P_1 \wedge \sigma(x))$ is a most general unifier. Notice that since $\sigma(x) = \top$ then $\varepsilon(x)$ is equivalent to $\Box P_1 \to x$. In order to prove ε is a most general unifier, we proceed the following steps:*

1. *We prove ε is a unifier of P_1. In this respect, we have to prove $\models A \to \Diamond \varepsilon(x)$. Since,*
   $\models \Box P_1 \to (A \to \Diamond x)$ *then,*
   $\models \Box P_1 \wedge A \to \Diamond x$ *hence,*
   $\models A \to (\Box P_1 \to \Diamond x)$. *Therefore,*
   $\models A \to \Diamond(\Box P_1 \to x)$.

2. *Let us prove that ε is more general than any unifier of P_1. Let σ' be a unifier of P_1. Since σ' is a unifier of P_1 then $\vdash \Box \sigma'(P_1)$. Hence, $\sigma'(\varepsilon(x)) = \sigma'(x) \vee \neg \Box \sigma'(P_1)$ is logically equivalent to $\sigma'(x)$. Therefore, $\varepsilon \preccurlyeq \sigma'$.*

   Consequently, ε is a most general unifier of P_1. Since P_1 and P has the same unifier then, ε is a most general unifier of P.

Lemma 113 *Let $P = A \to \langle \Diamond_1 x \rangle \Box_2 B$ where B is a Boolean formula. Then, $\models \Diamond_1 A \to (\Diamond_2 A \to B)$ iff P possesses a unifier.*

Proof 157 *We use axiomatisation of public announcement logic. Let $P_1 = (\Diamond_1 A \to \Diamond_1 x) \wedge (\Diamond_1 x \to (\Diamond_2 A \to B)$. Suppose $\models \Diamond_1 A \to (\Diamond_2 A \to B)$. Since $\models \Diamond_1 A \to (\Diamond_2 A \to B)$ then, $\sigma(x) = A$ is a unifier of P_1. Moreover, if P_1 has unifiers then obviously,*

$\models \Diamond_1 A \to (\Diamond_2 A \to B)$. *Let us find a most general unifier of* P_1. *We use Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$. *Since,* $\sigma(x) = A$ *is a unifier of* P_1 *then* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge A)$. *To check* ε *is a most general unifier, we will do the following steps:*

1. *We prove that* ε *is a unifier of* P_1. *In this respect, we have to prove* $\models \Diamond_1 A \to \Diamond_1\varepsilon(x)$ *and* $\models \Diamond_1\varepsilon(x) \to (\Diamond_2 A \to B)$. *Let us prove first one. Notice that* $\Diamond_1\varepsilon(x)$ *is logically equivalent to* $(\Box_1 P_1 \vee \Diamond_1 A) \wedge (\neg\Box_1 P_1 \vee \Diamond_1 x) \wedge (\Diamond_1 x \vee \Diamond_1 A)$. *Since* $\models \Diamond_1 A \to \Diamond_1 A$ *then* $\models \Diamond_1 A \to (\Box_1 P_1 \vee \Diamond_1 A) \wedge (\Diamond_1 x \vee \Diamond_1 A)$. *Since* $\models \Box_1 P_1 \to (\Diamond_1 A \to \Diamond_1 x)$ *then,*
   $\models \Diamond_1 A \to (\Box_1 P_1 \to \Diamond_1 x)$. *Therefore,*
   $\models \Diamond_1 A \to \Diamond_1\Big((\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge A)\Big)$.

   Let us prove $\models \Diamond_1\varepsilon(x) \to (\Diamond_2 A \to B)$. *Since,* $\models \Diamond_1 A \to (\Diamond_2 A \to B)$ *then,*
   $\models (\neg\Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 A \to B)$. *Since,*
   $\models \Box_1 P_1 \to (\Diamond_1 x \to (\Diamond_2 A \to B))$ *then,*
   $\models \Box_1 P_1 \wedge \Diamond_1 x \to (\Diamond_2 A \to B)$. *Since,*
   $\models (\neg\Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 A \to B)$ *then,*
   $\models (\Box_1 P_1 \wedge \Diamond_1 x) \vee (\neg\Box_1 P_1 \wedge \Diamond_1 A) \to (\Diamond_2 A \to B)$.
   Therefore, ε *is a unifier of* P_1.

2. *Let us prove that* ε *is more general than any unifier of* P_1. *Let* σ' *be a unifier of* P_1. *Since* σ' *is a unifier of* P_1 *then* $\vdash \Box_1\sigma'(P_1)$. *Hence,* $\sigma'(\varepsilon(x)) = (\Box_1\sigma'(P_1) \wedge \sigma'(x)) \vee (\neg\Box_1\sigma'(P_1) \wedge A)$ *is logically equivalent to* $\sigma'(x)$. *Therefore,* $\varepsilon \preccurlyeq \sigma'$.

   Consequently, ε *is a most general unifier of* P_1. *Since* P_1 *and* P *have the same unifiers then,* ε *is a most general unifier of* P.

Lemma 114 *Let* $P = A \to \langle\Diamond_1 x\rangle(\Box_2 B_2 \wedge \ldots \wedge \Box_n B_n)$ *where* B_i *are Boolean formulas for* $2 \leqslant i \leqslant n$. *Then,* $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$ *iff* P *possesses a unifier.*

Proof 158 *We use axiomatisation of public announcement logic to simplify* P *as before. Hence, let* $P_1 = (\Diamond_1 A \to \Diamond_1 x) \wedge (\Diamond x_1 \to (\Diamond_2 A \to B_2)) \wedge \ldots \wedge (\Diamond x_1 \to (\Diamond_n A \to B_n))$. *Suppose* $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$. *Since* $\models \Diamond_1 A \to (\Diamond_2 A \to B_2) \wedge \ldots \wedge (\Diamond_n A \to B_n)$ *then,* $\sigma(x) = A$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *We claim that Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$ *is a most general unifier. Since,* $\sigma(x) = A$ *hence,* $\varepsilon(x) =$

$(\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge A)$. *To check* ε *is a most general unifier, we use the similar method of the proof of Lemma* 113.

Lemma 115 *Let* $P = A \to \langle\Diamond_1 x\rangle\Diamond_2 C$. *Then,* $\vDash A \to \Diamond_2 C$ *iff* P *possesses a most general unifier.*

Proof 159 *We use axiomatisation of public announcement logic to simplify* P. *Let* $P_1 = (A \to \Diamond_1 x) \wedge (A \to \Diamond_2(\Diamond_1 x \wedge C))$. *Suppose* $\vDash A \to \Diamond_2 C$. *Since,* $\vDash A \to \Diamond_2 C$ *hence,* $\sigma(x) = \top$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge x) \vee (\neg\Box_1\Box_2 P_1 \wedge \sigma(x))$ *is a most general unifier. To check* ε *is a most general unifier, we use the similar method of the proof of Lemma* 103.

Lemma 116 *Let* $P = A \to \langle\Diamond_1 x\rangle(\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$ *where* C_i *are Boolean formulas. Then,* $\vDash A \to (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$ *iff* P *possesses a most general unifier.*

Proof 160 *Use axiomatisation of public announcement logic. Let* $P_1 = (A \to \Diamond_1 x) \wedge (A \to \Diamond_2(\Diamond_1 x \wedge C_2)) \wedge \ldots \wedge (A \to \Diamond_n(\Diamond_1 x \wedge C_n))$. *Suppose* $\vDash A \to (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$. *Since,* $\vDash A \to (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$ *hence,* $\sigma(x) = \top$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *we claim Löwenheim's* $\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1 \wedge x) \vee (\neg(\Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1) \wedge \sigma(x))$ *is a most general unifier. Since,* $\sigma(x) = \top$ *hence,* $\varepsilon(x) = \Box_1\Box_2 P_1 \wedge \ldots \wedge \Box_1\Box_n P_1 \to x$ *is a most general unifier of* P_1. *To check* ε *is a most general unifier, we use the similar method of the proof of Lemma* 104.

Lemma 117 *Let* $P = A \to \langle\Diamond_1 x\rangle\Big((\Box_{k_1} B_1 \wedge \ldots \wedge \Box_{k_m} B_m) \wedge (\Diamond_{l_1} C_1 \wedge \ldots \wedge \Diamond_{l_n} C_n)\Big)$ *where* B_i *and* C_j *are Boolean formulas for* $1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$. *Let* $B' = (\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)$. *Then,* $\vDash A \to \Diamond_1 B' \wedge \Diamond_{l_1}(C_1 \wedge \Diamond_1 B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge \Diamond_1 B')$ *iff* P *possesses a unifier.*

Proof 161 *We use axiomatisation of public announcement logic in order to simplify* P. *Let* $P_1 = (\Diamond_1 A \to \Diamond_1 x) \wedge (\Diamond_1 x \to \Diamond_1((\Diamond_{k_1} A \to B_1) \wedge \ldots \wedge (\Diamond_{k_m} A \to B_m)) \wedge (A \to \Diamond_{l_1}(\Diamond_1 x \wedge C_1)) \wedge \ldots \wedge (A \to \Diamond_{l_n}(\Diamond_1 x \wedge C_n))$. *Suppose* $\vDash A \to \Diamond_1 B' \wedge \Diamond_{l_1}(C_1 \wedge \Diamond_1 B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge \Diamond_1 B')$. *Since,* $\vDash A \to \Diamond_1 B' \wedge \Diamond_{l_1}(C_1 \wedge \Diamond_1 B') \wedge \ldots \wedge \Diamond_{l_n}(C_n \wedge \Diamond_1 B')$ *hence,* $\sigma(x) = \Diamond_1 B'$ *is a unifier of* P_1. *Let us find a most general unifier of* P_1. *In this respect, we claim Löwenheim's formula* $\varepsilon(x) = (\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1 \wedge x) \vee (\neg(\Box_1\Box_{l_1} P_1 \wedge \ldots \wedge \Box_1\Box_{l_n} P_1) \wedge \Diamond_1 B')$ *is a most general unifier of* P_1. *To check* ε *is a most general unifier, we use the similar method of Lemmas* 97 *and* 104.

Lemma 118 *Let* $P = A \to \langle \Diamond_1 x \rangle \Box_2 \Box_1 B$ *where* B *is a Boolean formula. Then,* $\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B)$ *iff* P *possesses a unifier.*

Proof 162 *Let us use axiomatisation of public announcement logic in order to simplify* P*. Hence let* $P_1 = (A \to \Diamond_1 x) \wedge (\Diamond_1 x \to (\Diamond_2 A \to \Box_1 B))$*. Suppose* $\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B)$*. Since* $\Diamond_1 A \to (\Diamond_2 A \to \Box_1 B)$ *hence,* $\sigma(x) = A$ *is a unifier of* P_1*. Let us find a most general unifier of* P_1*. We claim Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$ *is a most general unifier. To check* ε *is a most general unifier, we use the similar method of the proof of Lemma* 106.

Lemma 119 *Let* $P = A \to \langle \Diamond_1 x \rangle (\Box_2 \Box_1 B_2 \wedge \ldots \wedge \Box_n \Box_1 B_n)$ *where* B_i *are Boolean formulas for* $2 \leqslant i \leqslant n$*. Then,* $\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B_1) \wedge \ldots \wedge (\Diamond_n A \to \Box_1 B_n)$ *iff* P *possesses a unifier.*

Proof 163 *Let us simplify* P *by using axiomatisation of public announcement logic. Let,* $P_1 = (A \to \Diamond_1 x) \wedge (\Diamond x_1 \to (\Diamond_2 A \to \Box_1 B_1)) \wedge \ldots \wedge (\Diamond x_1 \to (\Diamond_n A \to \Box_1 B_n))$*. Suppose* $\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B_1) \wedge \ldots \wedge (\Diamond_n A \to \Box_1 B_n)$*. Since* $\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B_1) \wedge \ldots \wedge (\Diamond_n A \to \Box_1 B_n)$*, it is clear that* $\sigma(x) = A$ *is a unifier of* P_1*. Let us find a most general unifier of* P_1*. We use Löwenheim's formula* $\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg \Box_1 P_1 \wedge \sigma(x))$ *as a most general unifier. To check* ε *is a most general unifier, we use the similar method of Lemma* 107.

$A \to \langle\psi\rangle B$	Necessary condition	$\sigma(x)$	mgu
$A \to \langle\Diamond x\rangle B$	$\vDash A \to B$	$\top$	$\varepsilon(x) = (\Box P_1 \wedge x) \vee (\neg\Box P_1 \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle \Box_2 B$	$\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box B)$	A	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle (\Box_2 B_2 \wedge \ldots \wedge \Box_n B_n)$	$\vDash \Diamond_1 A \to \bigwedge_{i=2}^{n} (\Diamond_i A \to B_i)$	A	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle \Diamond_2 C$	$\vDash A \to \Diamond_2 C$	$\top$	$\varepsilon(x) = (\Box_1\Box_2 P_1 \wedge x) \vee (\neg\Box_1\Box_2 P_1 \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$	$\vDash A \to (\Diamond_2 C_2 \wedge \ldots \wedge \Diamond_n C_n)$	$\top$	$\varepsilon(x) = (\bigwedge_{i=2}^{n} \Box_1\Box_i P_1 \wedge x) \vee (\neg(\bigwedge_{i=2}^{n} \Box_1\Box_i P_1) \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle \bigwedge_{i=1}^{m} \Box_{k_i} B_i \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} C_j$	$\vDash A \to \Diamond_1 B' \wedge \bigwedge_{j=1}^{n} \Diamond_{l_j} (C_j \wedge \Box_1 B')$	$\Box_1 B'$	$\varepsilon(x) = (\bigwedge_{j=1}^{n} \Box_1\Box_{l_j} P_1 \wedge x) \vee (\neg(\bigwedge_{j=1}^{n} \Box_1\Box_{l_j} P_1) \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle \Box_2\Box_1 B$	$\vDash \Diamond_1 A \to (\Diamond_2 A \to \Box_1 B)$	A	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$
$A \to \langle\Diamond_1 x\rangle (\Box_2\Box_1 B_2 \wedge \ldots \wedge \Box_n\Box_1 B_n)$	$\vDash \Diamond_1 A \to \bigwedge_{i=2}^{n} (\Diamond_i A \to \Box_1 B_i)$	A	$\varepsilon(x) = (\Box_1 P_1 \wedge x) \vee (\neg\Box_1 P_1 \wedge \sigma(x))$

Table 7.3: Simple epistemic planning problem $A \to \langle\Diamond x\rangle B$

8 Conclusion

The unification problem and the admissibility problems are strongly related, as explained at the beginning of the thesis. As seen in Chapters 4 to 7, there are many different ways to study the unification types of modal logics. There are still many open problem about unification types of modal logics. In Chapter 4 of this Thesis, we have proved that unification type of modal logics $KD5$ and $K5$ are unitary or nullary. Here, there are some open question as follows:

- What is exact unification type of logics $KD5$ and $K5$?
- What is unification type of logics $KD5$ and $K5$ with constant?
- What is unification type of every logic extending $K5$?

In Chapter 5, we have proved that unification type of $Alt_1 + \Box\Box\bot$ is unitary. We have in [12] that $K + \Box\Box\bot$ is finitary. Here also there are open questions as follows:

- What is unification type of $Alt_1 + \Box^d\bot$ and $K + \Box^d\bot$ when $d \geqslant 3$?
- What is unification type of $Alt_1 + \Diamond\top$?
- What is unification type of $Alt_1 + \Box^d\bot$ and $K + \Box^d\bot$ when $d \geqslant 2$ for unification with constant?

In chapter 6, we have proved that if L_1 is nullary and L_2 is consistent modal logic then unification type of the fusion $L_1 \otimes L_2$ is not unitary and not finitary. Also we have proved that unification type of fusion $S5_1 \otimes S5_2$ with constants is nullary. Now, there are open questions as follows:

- is unification type of the fusion of two consistent modal logics always nullary when these logics are different from **Triv** and **Ver**?
- What is the unification type of the fusion $S5_1 \otimes S5_2$ without constants?

- What is the unification type of the fusion $KD_1 \otimes KD_2$ without constants?

In chapter 7, we have considered simple epistemic planning problem with associated formula $A \to \langle x \rangle B$, $A \to \langle \Box x \rangle B$, $A \to \langle \Diamond x \rangle B$ and found necessary and sufficient condition for existence of unifier when announcements are public announcements. Here also there are some open problems concerning what is necessary and sufficient condition for existence of unifier when announcement ψ is a group announcements [1], semi-private announcement [21], complete private announcement [16][17] etc. For example, one may ask, given epistemic variable-free formulas A, B and C whether there exists a semi-private announcement ψ to agent 1 such that the following formula is valid in the logic of semi-private announcement [16]: $A \to \langle \frac{1}{2}(1, \psi) \rangle (\Box_j B \wedge \Diamond_k C)$.
In natural language, such planning problem consists in computing a formula ψ in the language of semi-privately announced to agent 1 and, after announcement, agent j knows B holds and agent k considers that C is possible.
Can we adapt the approach developed in Chapter 7 when announcements are lies [2]?

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