Analyse mathématique et asymptotique de modèles couplés fluide-cinétique issus de la mécanique des fluides et des sciences du vivant

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ANALYSE MATHÉMATIQUE ET ASYMPTOTIQUE DE MODÈLES COUPLÉS FLUIDE-CINÉTIQUE ISSUS DE LA MÉCANIQUE DES FLUIDES ET DES SCIENCES DU VIVANT

Résumé

Dans ce manuscrit, nous nous intéressons à des modèles, dits fluide-cinétique, décrivant l'évolution de particules en suspension dans un fluide porteur. Un tel système physique est représenté mathématiquement par le couplage d'équations aux dérivées partielles multi-échelles issues de la mécanique des fluides. Plus précisément, nous faisons l'hypothèse que le fluide porteur peut être décrit par des quantités macroscopiques, sa vitesse et sa pression, grâce aux équations de Navier-Stokes incompressible. Le spray de particules est quant à lui décrit à l'échelle mésoscopique, comme classiquement en théorie cinétique des gaz, par sa fonction de densité dans l'espace des phases, régie par une équation de type Vlasov. La prise en compte de l'accélération de traînée fournie par le fluide aux particules et, en retour, la force de rétroaction subie par le fluide entraîne un couplage fort du système d'équations étudié.
Dans un premier temps, nous portons notre attention sur un modèle fluide-cinétique récemment proposé pour décrire le mouvement d'un aérosol thérapeutique au sein des voies respiratoires supérieures. En plus des interactions présentées ci-dessus, les effets de l'humidité de l'air ambiant sur la taille et la température des particules sont pris en compte par l'introduction d'équations de convection-diffusion décrivant la fraction massique de vapeur d'eau dans l'air et sa température, ainsi que l'intégration de la variation du rayon et de la température des particules dans l'équation de transport régissant l'aérosol. Nous démontrons l'existence de solutions faibles globales dans un domaine borné dépendant du temps pour ce système d'équations puis nous présentons quelques résultats d'expérimentations numériques.
Enfin, nous étudions plusieurs régimes de haute friction pour le système de Vlasov-Navier-Stokes présenté précédemment. Nous définissons un cadre permettant de traiter rigoureusement ces limites hydrodynamiques lorsque les particules sont légères ou petites par rapport au fluide. Nous obtenons à la limite les systèmes Transport-Navier-Stokes ou Navier-Stokes inhomogène, respectivement.

Mots clés : fluide-cinétique, Vlasov-Navier-Stokes, solutions faibles, simulation numérique, limite hydrodynamique

Abstract

In this manuscript, we consider so-called fluid-kinetic models that describe the evolution of particles flowing through a fluid. Such a physical system is represented mathematically by the coupling of multiscale partial differential equations stemming from fluid mechanics. More precisely, we assume that the sustaining fluid can be described by macroscopic quantities, its velocity and pressure, thanks to the incompressible Navier-Stokes equations. As for the particle spray, it is described at the mesoscopic scale, as usual in the kinetic theory of gases, by its density function in the phase space, which obeys a Vlasov-type equation. Taking into account the drag acceleration exerted by the fluid on the particles and, conversely, the drag force applied on the fluid by the spray leads to a strong coupling of the system of equations under study.
First, we focus on a fluid-kinetic model recently proposed to describe the motion of a therapeutic aerosol in the superior regions of the airways. In addition to the interactions presented above, the effects of the airway humidity on the particle size and temperature are taken into account by introducing convection-diffusion equations describing the water vapor mass fraction and the temperature of the air, as well as integrating the size and temperature variations into the transport equation satisfied by the aerosol density function. We prove the existence of global weak solutions in a time-dependent domain for this system of equations and present some results of numerical experiments.
Finally, we study several high-friction regimes for the Vlasov-Navier-Stokes system presented above. We define a framework allowing to properly justify these hydrodynamic limits in the case where the particles are light (resp. small) with respect to the fluid. At the limit, we derive the Transport-Navier-Stokes (resp. Inhomogeneous Navier-Stokes) systems.

Keywords: fluid-kinetic, Vlasov-Navier-Stokes, weak solutions, numerical simulation, hydrodynamic limit

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Introduction

Sommaire du présent chapitre

1.1 Systèmes couplés fluide-cinétique

Ce manuscrit présente des études mathématiques et numériques de modèles décrivant l'évolution d'un ensemble de particules liquides ou gazeuses, appelé aérosol ou spray, au sein d'un fluide porteur. Dans des travaux fondateurs [ORo81 ; Wil85], O'Rourke puis Williams s'intéressent à la combustion de particules liquides en suspension dans un gaz, par exemple au sein d'un moteur Diesel idéalisé. Afin de pouvoir étudier un grand nombre de telles particules (d'un point de vue numérique pour le premier, plus théorique pour le second), ces deux auteurs adoptent une approche statistique et les décrivent par une équation cinétique. Le fluide environnant est quant à lui modélisé selon les principes de la mécanique des milieux continus. L'ensemble constitue un système *fluide-cinétique* d'équations aux dérivées partielles.

Comme le mentionne [Des10], ce n'est pas la seule approche possible. En effet, selon les propriétés physiques que l'on considère, on peut distinguer trois échelles spatiales : les échelles microscopique, mésoscopique et macroscopique.

À l'échelle microscopique, il est par exemple possible de modéliser avec précision des interfaces liquide-gaz. Cette question est cruciale pour comprendre, notamment, certains phénomènes influant sur la dynamique de l'atmosphère et des océans ou encore le comportement des plasmas. On décrit alors l'ensemble par deux systèmes équations de Navier-Stokes, l'un pour le liquide à l'intérieur des particules et l'autre pour le milieu gazeux, avec une condition de bord libre à l'interface. Bien sûr, ce type de modélisation s'avère très vite trop coûteux à mesure que le nombre de particules formant l'aérosol augmente.

Lorsque le spray n'est pas très dilué, on peut travailler à l'échelle macroscopique. On considère dans ce cas le système fluide-particules comme un mélange de deux fluides miscibles. L'interface microscopique considérée précédemment disparaît et laisse place à la fraction volumique occupée par l'ensemble des particules. Un exemple de tel système d'équations est présenté dans [Des10]. Il est par exemple employé pour développer des outils industriels pour extraire du gaz naturel de puits océaniques [Suh+18]. Les auteurs de [DV05] utilisent également cette approche pour modéliser l'évaporation des gouttelettes évoluant dans un gaz.

Un bon compromis entre les deux échelles consiste à conserver une description macroscopique de l'évolution du milieu ambiant (le fluide) et à travailler à l'échelle mésoscopique pour le spray.

L'échelle macroscopique permet en effet de décrire le comportement du fluide dans une large variété de situations physiques, selon qu'il est compressible/incompressible, visqueux ou non, homogène ou non, parfait, newtonien, *etc.* On rencontre alors les équations de Stokes, Euler, Navier-Stokes...

L'échelle mésoscopique s'appuie sur les principes de la théorie cinétique des gaz. On abandonne l'idée de décrire de nombreuses particules individuellement et l'on considère plutôt une fonction de distribution de probabilité $f(t, x, v)$ dépendant du temps $t \in \mathbb{R}_+$, de la position $x \in \mathbb{R}^3$ et la vitesse $v \in \mathbb{R}^3$ des particules. D'autres paramètres peuvent être ajoutés comme le rayon, la température ou l'énergie interne. Ceci permet la modélisation de phénomènes complexes comme les échanges thermiques, l'abrasion, la fusion, les interactions entre particules (collisions), au prix d'une complexité accrue tant sur le plan théorique que numérique. Comme nous le verrons dans ce mémoire, c'est l'une des approches mises en œuvre pour modéliser l'évolution d'un aérosol thérapeutique au sein des voies respiratoires, *cf.* notamment [GCC02 ; LH11 ; Bou+15].

Une fois adoptée l'échelle mésoscopique, il reste donc encore à préciser les phénomènes physiques à prendre en compte, et notamment l'interaction entre le fluide et le spray. L'un des processus les plus étudiés, depuis l'article fondateur de Brinkman [Bri49], est la friction entre ces deux phases. De manière générale, cette *force* dite *de Brinkman* tend à aligner les vitesses du fluide et des gouttelettes, ce qui se traduit mathématiquement par la présence dans les équations du fluide d'un facteur de type $(v - u)$, où v et u désignent respectivement la vitesse des particules et du fluide, *cf.* [Duf05] par exemple. La justification mathématique de cette force à partir d'équations « fondamentales » fait l'objet de nombreux travaux comme [DGR08 ; Hil18 ; HMS19 ; CH20] pour une approche de champ moyen ou [Ber+17 ; Ber+18] *via* l'équation de Boltzmann, et est encore un sujet de recherche ouvert. En vertu de la troisième loi de Newton, si le fluide subit la force de Brinkman, le spray subit quant à lui une force opposée. Cela se traduit dans l'équation cinétique par la présence d'un terme, appelé *accélération de traînée*, de la forme $\mathrm{div}_v\,(f(u - v))$.

Il est également possible de tenir compte des fluctuations de la vitesse thermique des particules à l'échelle microscopique, le mouvement brownien [Ein05], ce qui se traduit à l'échelle mésoscopique par l'apparition du terme $-\Delta_v f$ dans l'équation cinétique, alors appelée équation de Vlasov-Fokker-Planck. Nous renvoyons les lecteurs à [CG06] pour une présentation détaillée de ce type de modèles.

Lorsque le spray est *épais* [Des10], au sens où le volume occupé par les particules est du même ordre de grandeur que celui occupé par le fluide, il convient de prendre également en compte les phénomènes de collision entre les particules qui le composent, le plus souvent *via* un opérateur de collision $Q(f, f)$ dans l'équation cinétique. Ces collisions ont un rôle non négligeable dans un certain nombre d'applications, au sein de la chambre de combustion d'un moteur Diesel comme dans un réacteur nucléaire. Nous ferons cependant ici toujours l'hypothèse d'un spray *fin*, sans interaction entre les gouttelettes, et renvoyons les lecteurs à [Mat06] par exemple, pour une discussion sur les modèles fluide-cinétique avec collision.

Dans les mêmes contextes, mais encore, comme on l'a découvert plus récemment, dans le cadre de l'étude de la séquestration du dioxyde de carbone par les océans [BDC20], les particules sont soumises à un processus de fragmentation, qui fait varier leur dimension. Ce phénomène a fait l'objet d'une étude poussée du point de vue numérique, au cours du développement de logiciels industriels modélisant la chambre de combustion d'un moteur [AOB89 ; Ams92], mais a également été considéré de façon plus théorique dans [BDM14].

Les lecteurs trouveront dans [Duf05] un panorama plus large de la grande variété des phénomènes physiques que le cadre fluide-cinétique permet de modéliser.

1.2 Questions et outils liés aux systèmes fluide-cinétique

Après cette présentation générale des modèles fluide-cinétique, nous nous attachons dans cette section à mettre en évidence les principales questions et difficultés que leur étude soulève et comment celles-ci sont traitées habituellement, tant sur le plan de l'analyse mathématique que de leur approximation et simulation numérique.

Débutons par une remarque sur la proximité des systèmes fluide-cinétique avec les équations décrivant l'évolution de plasmas, comme les systèmes de Vlasov-Poisson [Dol02] ou Vlasov-Maxwell [GS86]. Dans ces modèles, l'équation cinétique est couplée avec les équations de l'électromagnétisme. Leur étude nécessite de se confronter à certaines des difficultés inhérentes aux systèmes couplés que nous décrivons ci-dessous. Nous nous concentrons cependant exclusivement dans la suite sur les modèles fluide-cinétique.

1.2.1 Point de vue théorique

La première difficulté que l'on peut remarquer vient des échelles différentes auxquelles on considère le fluide et le spray. En effet, à l'échelle macroscopique, le fluide est décrit par des paramètres variant, en général, en fonction du temps $t \in \mathbb{R}_+$ et de la position $x \in \Omega$, où Ω est le domaine spatial choisi. Dans le cas d'un fluide newtonien homogène incompressible, on utilise par exemple la vitesse $u(t, x)$ et la pression $p(t, x)$. En revanche, à l'échelle mésoscopique, on considère d'une part le temps et la position, mais également la vitesse $v \in \mathbb{R}^3$ des particules, voire sa taille (rayon, surface, volume), sa température, son énergie interne, *etc.* La fonction de distribution décrivant le spray dépend donc de variables supplémentaires, et l'équation régissant son évolution doit donc être posée sur un domaine différent, appelé espace des phases. Les argumentations mises en œuvre pour étudier les équations du fluide et du spray devront nécessairement en tenir compte et présenteront souvent des traitements spécifiques pour chacune d'elles.

Choix du domaine spatial. Après avoir décidé des phénomènes physiques à modéliser, le dernier choix à faire est celui du domaine spatial d'étude. On peut considérer une région de l'espace physique, en imposant alors des conditions au bord du domaine. Mais il est bien connu que la prise en compte de ces contraintes complexifie en général l'analyse mathématique. C'est la raison pour laquelle il est commun de débuter l'étude dans un domaine spatial sans bord. Le tore plat est un candidat idéal puisqu'il a en plus l'avantage d'être borné. C'est le choix de nombreux auteurs abordant pour la première fois un modèle fluide-cinétique. Citons par exemple [Bou+09] (en dimensions 2 ou 3) pour la démonstration de l'existence de solutions au problème de Cauchy du système de Vlasov-Navier-Stokes, [CK15] et [HMM20] pour l'étude du comportement en temps long des solutions, ou encore [Han+20] pour étudier la question de l'unicité en dimension 2. Nous adopterons également cette stratégie au Chapitre 3 pour l'étude de certaines limites hydrodynamiques). On peut également considérer le cas de l'espace tout entier, comme [BD06] pour démontrer l'existence de solutions régulières du système de Vlasov-Euler, ou [Han20] pour décrire le comportement en temps long des solutions du système de Vlasov-Navier-Stokes.

Il est cependant possible d'étudier mathématiquement des conditions au bord pertinentes d'un point de vue physique. On impose en général au fluide une condition de Dirichlet au bord. Pour le spray, on peut considérer la réflexion spéculaire comme, par exemple, dans [AB97 ; Ham98 ; Yu13] pour un domaine fixe, ou l'absorption comme dans [EHM21] pour un domaine fixe ou [BGM17] pour un domaine dépendant du temps. Cela pose notamment la question des propriétés de la trace des solutions faibles de l'équation cinétique sur la frontière, à commencer par leur existence. Le choix de la condition de bord est également déterminant pour les éventuelles propriétés de conservation/dissipation satisfaites par les solutions et qui, comme nous le verrons, sont d'une importance capitale pour l'analyse mathématique.

Pour conclure sur les difficultés liées au domaine spatial considéré, décrivons les conséquences de la prise en compte de sa variation au cours du temps. La résolution des équations pour le fluide est rendue plus complexe, indépendamment de tout couplage, mais ce cas est connu depuis bien longtemps. Les auteurs de [FS70] ont en effet présenté une méthode de pénalisation permettant de se ramener à un domaine spatio-temporel cylindrique. Par ailleurs, des résultats classiques de compacité comme le lemme d'Aubin-Lions (voir par exemple [BF06]) ne peuvent être appliqués directement. Une solution peut être de s'appuyer sur des théorèmes de compacité spécifiquement pensés et démontrés dans [Mou16], comme nous le verrons au Chapitre 2.

Structure des équations considérées individuellement. La question de l'existence de solutions faibles ou fortes des équations décrivant le fluide est bien connue sous réserve d'un contrôle sur le terme source. Dans le cas des équations de Navier-Stokes incompressible homogène, nous renvoyons les lecteurs à [Che+06 ; RRS16] pour une présentation des principaux résultats, et à [AKM90 ; Lio96] dans le cas d'un fluide inhomogène incompressible. L'étude des équations de Navier-Stokes compressible est exposée dans [Lio98 ; Nov02]. Concernant les équations d'Euler, nous renvoyons à [Wol33 ; EM70 ; DiP83 ; LW07].

Le problème de l'unicité des solutions fortes pour le système de Navier-Stokes incompressible ou compressible est également bien compris [RRS16 ; Dan05]. Des résultats de non-unicité sont également connus pour les équations d'Euler, *cf.* par exemple [DS12]. L'unicité des solutions faibles pour les équations de Navier-Stokes reste à ce jour un problème ouvert en dimension 3, alors qu'elle est démontrée en dimension 2 [Che+06]).

À l'inverse, le problème de l'existence et l'unicité de solution d'équations de transport (dont l'équation de Vlasov est un cas particulier) pour un champ de vecteurs peu régulier est bien connu depuis les résultats de [DL89] sur les solutions renormalisées d'une équation de transport dans tout l'espace. Le cas d'un domaine borné est présenté notamment dans [Mis00 ; Boy05 ; BF13] (et ses références) et est étendu dans un domaine variable par les auteurs de [BGM17].

Couplage des équations. Il s'agit d'une des principales caractéristiques des modèles fluide-cinétique. Notons tout d'abord que le couplage par la force de Brinkman et l'accélération de traînée est non linéaire. De plus, il est local pour l'équation cinétique mais non local pour l'équation du fluide. Le système est fortement couplé, au sens où l'équation pour l'une des phases fait intervenir la solution de l'équation de l'autre, et réciproquement. Il n'est donc pas possible de commencer par résoudre l'une d'elles puis de se servir de ce résultat pour résoudre la seconde. Une stratégie générale pour surmonter cette difficulté consiste à utiliser une méthode de point fixe. Schématiquement, on peut considérer des candidats $\widetilde{u}$ et $\widetilde{f}$ pour remplacer u et f dans les équations cinétique et fluide, respectivement. On démontre alors que l'application qui à de tels $(\widetilde{u}, \widetilde{f})$, pris dans de bons espaces fonctionnels, associe l'unique solution du problème découplé (en remplaçant u par $\widetilde{u}$ dans l'équation cinétique et f par $\widetilde{f}$ dans celle du fluide) admet un point fixe.

Encore faut-il que le système découplé admette bien une unique solution... ce qui est compliqué par la non-linéarité du couplage. La force de Brinkman (et sa contrepartie dans l'équation de Vlasov) est en effet non linéaire puisqu'elle fait intervenir le produit $(u-v)f$. Cela engendre une difficulté liée au contrôle nécessaire du terme source des équations de Navier-Stokes mentionné au paragraphe précédent. Avant de découpler le problème, il est donc nécessaire de le régulariser. Toutes les contributions à l'étude de l'existence de solutions pour des systèmes fluide-cinétique [AB97 ; Ham98 ; MV07 ; Bou+09 ; Gou+10 ; CKL11 ; CKL13 ; Yu13 ; WY15 ; LMW15 ; BGM17] adaptent cette stratégie aux équations étudiées. Pour le système de Vlasov-Navier-Stokes, les autrices de [AB97] et, par améliorations successives, [Bou+09 ; Yu13 ; BGM17] ont défini une stratégie qui consiste à introduire une troncature de la vitesse relative $v - u(t, x)$ du fluide et des particules dans l'expression de la force de Brinkman. Afin de préserver la structure du bilan d'énergie du système, cette même troncature est introduite dans l'expression de l'accélération de traînée pour l'équation cinétique. Quitte à simplifier, on ne s'embarrasse également plus du terme convectif des équations de Navier-Stokes qui peut être régularisé par différentes méthodes.

Estimations *a priori*. Comme indiqué au paragraphe précédent, la démonstration de l'existence de solutions pour les modèles fluide-cinétique repose sur une méthode d'approximation. Celle-ci est toujours conclue par un argument de compacité, permettant d'obtenir finalement l'existence de solutions du problème initial. La compacité découle quant à elle de contrôles uniformes en les paramètres du problème. La clé pour obtenir de telles inégalités est l'utilisation d'estimations *a priori*. Dans notre cadre, les auteurs utilisent abondamment l'énergie cinétique et sa dissipation pour obtenir des inégalités que toute solution raisonnable devrait satisfaire. Les schémas d'approximation évoqués ci-dessus sont construits pour que leurs solutions vérifient toujours de telles estimations, qui permettent d'obtenir des bornes uniformes et donc de la compacité.

Unicité des solutions. Après avoir obtenu des résultats d'existence, il est naturel d'étudier la question de l'unicité des solutions obtenues. Dans le cas du système de Vlasov-Euler compressible isentropique, les auteurs de [BD06] utilisent les mêmes estimations *a priori* de haute régularité ayant servi à construire une solution forte dans $\mathbb{R}^N$ (pour tout $N \in \mathbb{N}^*$) pour en démontrer l'unicité. De même, considérer l'équation de Vlasov-Fokker-Planck permet de préserver la régularité lors de la construction itérative d'une solution. Les estimations *a priori* permettent également de démontrer l'unicité des solutions fortes du système Vde lasov-Fokker-Planck-Navier-Stokes dans $\mathbb{R}^3$ ou $\mathbb{T}^3$ autour de l'équilibre maxwellien [CDM11]. Le cas des équations de Navier-Stokes est significativement plus compliqué. À notre connaissance, des résultats d'unicité de solutions faibles pour les systèmes Vlasov-Fokker-Planck-Navier-Stokes et Vlasov-Navier-Stokes ne sont obtenus en toute généralité qu'en dimension 2, par [CKL11] et [Han+20] respectivement. Les auteurs de [LMW15] démontrent également l'existence et l'unicité de solutions pour le système de Vlasov-Fokker-Planck-Navier-Stokes dans $\mathbb{R}^3$ et $\mathbb{T}^3$ pour des perturbations régulières de l'équilibre maxwellien.

Cas de particules de rayons distincts. Dans la première étude mathématique, à notre connaissance, du système de Vlasov-Navier-Stokes, les autrices de [AB97] considèrent un aérosol dispersé en rayon (mais le rayon d'une particule ne varie pas avec le temps contrairement au modèle que nous étudions dans ce mémoire). Elles ne normalisent donc plus le rayon apparaissant dans l'expression de la force de Brinkman. Cela se traduit par l'apparition d'un facteur r dans l'expression de la force de Brinkman et d'un facteur $1/r^2$ dans l'équation de Vlasov. Cette singularité les contraint à restreindre l'ensemble des rayons et d'introduire un rayon minimal admissible. Nous justifions physiquement et mathématiquement cette restriction dans le Chapitre 2.

Comportement en temps long. Après avoir obtenu des résultats d'existence, il est naturel de chercher à décrire le comportement asymptotique des solutions obtenues. Lorsque l'on considère l'équation de Vlasov-Fokker-Planck, la distribution maxwellienne permet de construire un état d'équilibre de systèmes fluide-cinétique comme ceux de Vlasov-Fokker-Planck-Navier-Stokes ou Vlasov-Fokker-Planck-Euler. Il convient alors de s'intéresser à sa stabilité, comme dans [Gou+10] ou [CDM11]. Un tel équilibre régulier n'est pas connu pour le système de Vlasov-Navier-Stokes mais l'on peut néanmoins discuter du comportement des solutions en temps long. L'analyse est initiée sur le tore dans [CK15] par l'obtention d'un résultat conditionnel. Il repose essentiellement sur la décroissance exponentielle d'une fonctionnelle mesurant les fluctuations des solutions et appelée *énergie modulée*, que les auteurs obtiennent sous la condition d'une borne uniforme *a priori* sur la densité de l'aérosol au cours du temps. La démonstration de cette condition fait l'objet de [HMM20] et repose sur la formule de représentation des solutions de l'équation de Vlasov et un changement de variables suivant la caractéristique en vitesse. Cependant, sa mise en œuvre est nettement compliquée par la présence de l'accélération de traînée due au fluide dans l'équation de Vlasov. La justification du changement de variable est ardue et requiert des arguments sophistiqués sur la régularité des solutions des équations de Navier-Stokes ainsi que l'obtention d'estimations fines pour la force de Brinkman. Les auteurs démontrent ainsi la convergence vers un état dans lequel les particules ont un comportement monocinétique. Ce résultat est étendu dans l'espace tout entier par [Han20], en considérant une nouvelle famille de fonctionnelles qu'il nomme *dissipations d'ordre supérieur* et permettant l'étude de la force de Brinkman, puis enfin dans un domaine borné avec une condition de Dirichlet/absorption dans [EHM21].

Ordres de grandeur des coefficients des équations. Notons enfin que les travaux mathématiques cités précédemment considèrent des équations « épurées », où les coefficients physiques ont été considéré comme égaux à 1. On peut cependant se placer dans des *régimes* physiques pour lesquels ces coefficients ne sont plus tous du même ordre de grandeur. La question est alors de décrire le comportement asymptotique des solutions lorsque certains paramètres tendent vers 0 (comprendre : deviennent négligeables devant les autres). On peut prendre la limite formelle des équations du modèle et obtenir un système d'équations aux dérivées partielles devant être satisfait à la limite. Ce type de raisonnement n'est bien sûr pas exclusif des modèles fluide-cinétique, ayant été introduit par Hilbert comme une manière de résoudre le problème de l'axiomatisation de la physique. Nous renvoyons les lecteurs à la présentation générale dans le cadre de l'étude de fluides qu'en donne [Gol05]. Nous considérons dans ce mémoire des régimes dans lesquels la friction devient prédominante. Dans ce contexte, la question de la justification rigoureuse du comportement asymptotique a été étudiée pour la première fois pour l'équation de Vlasov par [Jab00b] puis [GP04]. Des modèles complets fluide-cinétique ont également été considérés sous cette angle, notamment par [Gou01] en dimension 1 pour un modèle *jouet* du système de Vlasov-Navier-Stokes, [GJV04a ; GJV04b] pour le système de Vlasov-Fokker-Planck-Navier-Stokes ou [Höf18b] pour le système de Vlasov-Stokes. Si les méthodes développées par ces auteurs sont assez spécifiques aux problèmes qu'ils étudient, relevons néanmoins l'introduction par [GJV04b] d'une fonctionnelle appelée *entropie relative* mesurant la distance entre les solutions dans le régime étudié et les solutions du système limite obtenu formellement.

Au Chapitre 3, nous démontrons un comportement monocinétique pour certains types de sprays. On le retrouve également dans d'autres contextes comme par exemple, dans le cadre de l'électromagnétisme, dans la limite quasi-neutre (faible longueur de Debye) pour le système de Vlasov-Poisson [Bre00] ou, plus récemment, pour l'équation cinétique de Cucker-Smale [KV15 ; FK19].

1.2.2 Point de vue de l'expérimentation numérique

Du fait de l'origine même des systèmes fluide-cinétique, à savoir la modélisation d'un phénomène physique, il est naturel de s'intéresser à la résolution numérique des équations mises en jeu et de décrire certaines propriétés des solutions. L'objet des travaux numériques de ce mémoire étant de présenter des expériences numériques en trois dimension pour un modèle fluide-cinétique déjà étudié en dimension deux, nous nous contentons ici de décrire les principales difficultés pratiques identifiées dans ce type d'étude, sans rentrer dans les détails des schémas numériques implémentés, pour lesquels nous renvoyons les lecteurs à, par exemple, [GCC02 ; Mat06 ; Mou09 ; Bou+15 ; Fan18 ; Bou+20].

Grand nombre de particules. La difficulté la plus évidente consiste à trouver des schémas numériques adaptés permettant de traiter, à moindre coût, des aérosols contenant de très nombreuses particules (comme l'ont calculé les auteurs de [Bou+15], un nébuliseur grand public peut injecter jusqu'à 10^{10} particules par minute dans les voies respiratoires). Dès les premières contributions à l'étude numérique des modèles fluide-cinétique (voir par exemple [ORo81 ; Ams92 ; AOB89 ; GCC02] pour le code KIVA), on utilise une méthode dite *particulaire* : on choisit de suivre la trajectoire de *macro-particules*, représentant chacune un grand nombre de gouttelettes réelles censées être suffisamment proches pour que leur comportement soit similaire.

Calcul des interactions entre le fluide et les particules. Dans un schéma numérique naïf, on peut donc imaginer discrétiser le temps et, à chaque itération, résoudre un problème pour le fluide (par des méthodes d'éléments finis par exemple) et suivre la trajectoire des macro-particules en résolvant les équations différentielles déduites de l'équation de Vlasov par la méthode des caractéristiques. Nous retrouvons le pendant numérique du constat initial de la section précédente sur le fait que les équations sont de natures différentes. Alors que l'on suit la trajectoire des macro-particules dans un espace des phases continu, on ne connaît les quantités macroscopiques décrivant le fluide qu'aux points du maillage. Se pose alors la question du calcul des termes d'interaction. On utilise une méthode dite PIC (*particle-in-cell*, *cf.* [CR86 ; DM89a ; DM89b ; HE81 ; Wes94]) dans laquelle on interpole les quantités macroscopiques calculées aux points du maillage, tandis que l'on calcule les moments de la fonction de distribution (qui interviennent dans les termes d'interaction) uniquement sur ces points.

Équations du fluide. La résolution en trois dimensions des équations décrivant le fluide par des méthodes d'éléments finis est très coûteuse en temps de calcul. Par exemple, dans une implémentation naïve dans un maillage cylindrique plutôt grossier en trois dimensions, la résolution de chaque itération temporelle pour les équations de Navier-Stokes incompressible homogène sur un ordinateur grand public récent nous a demandé plus d'une heure de calcul (et il est nécessaire d'en traiter plusieurs centaines). Une idée qui, à notre connaissance, n'a pas encore été mise en œuvre dans le cadre de modèles fluide-cinétique consiste à utiliser des méthodes telles que la décomposition de domaines pour paralléliser la résolution de ces équations. Mais, à l'heure actuelle, on se contente, comme l'a justifié [Bou+15] dans le cas d'un aérosol constitué de particules très petites, de négliger la force de Brinkman. Il n'y a alors plus de couplage et les équations de Navier-Stokes peuvent être résolues indépendamment, le résultat stocké et réutilisé sans refaire systématiquement les calculs.

Problème raide pour la température. Dans l'étude numérique en dimension 2 d'un modèle d'aérosol thérapeutique dans les voies respiratoires prenant en compte les effets hygroscopiques, les auteurs de [Bou+20] constatent que le problème numérique d'évolution de la température d'une macro-particule est *raide* au cours des premières itérations temporelles (globales). Il est donc nécessaire d'en tenir compte dans le choix de la méthode de résolution des équations différentielles régissant les macro-particules.

1.3 Prise en compte des effets hygroscopiques

Nous nous intéressons dans la première partie du présent mémoire à l'étude mathématique du modèle introduit par [Bou+20], que nous rappelons dans la section suivante. Ce modèle fait l'objet de deux articles que nous présentons dans ce mémoire au Chapitre 2.

1.3.1 Présentation du modèle

Une partie des travaux que nous présentons a trait à la description de l'évolution d'un aérosol dans les voies respiratoires. L'ambition d'un tel modèle est d'améliorer la compréhension des thérapies par aérosol, utilisées dans les traitements de bronchopneumopathies chroniques obstructives. La problématique principale dans ce type d'approche est d'assurer l'administration d'un médicament dans la région obstruée des voies respiratoires le plus efficacement possible, au plus près de la zone obstruée.

Pour cela, le principe actif est mélangé à un excipient et de l'eau *via* un nébuliseur, afin de créer un grand nombre de gouttelettes, formant l'aérosol. Les voies respiratoires étant un milieu humide, il est naturel de considérer les propriétés hygroscopiques (c'est-à-dire liée à l'humidité du milieu) de l'ensemble des particules ainsi constitué [LK05 ; LH10 ; LH11]. En particulier, les gouttelettes peuvent échanger de la matière aqueuse avec l'air saturé en vapeur d'eau présent dans les poumons, ce qui pourrait avoir un effet sur leur taille.

Une première idée pour décrire l'évolution de l'aérosol dans l'air consiste à considérer les particules individuellement. C'est l'approche adoptée par [ZKK08 ; Oak+14]. Elle constitue un véritable défi algorithmique du fait du très grand nombre de particules à considérer comme nous l'avons indiqué précédemment. Une deuxième idée serait de considérer l'aérosol comme un fluide, à l'instar de [AHB01 ; CBW10]. Cependant, ce point de vue semble compliquer l'obtention de propriétés quantitatives telles que les zones de dépôt des particules.

Les auteurs de [Bou+20] ont opté pour un modèle fluide-cinétique, tel que défini au début de cette introduction. Celui-ci étant l'objet de notre étude, nous le présentons ici en simplifiant les expressions et renvoyons les lecteurs à [Bou+20] pour le modèle physique précis.

Nous nous plaçons dans une région du haut de l'arbre bronchique, dans laquelle l'air s'écoule encore à une vitesse telle que les phénomènes de convection dominent sur la diffusion et nous considérons une phase d'inhalation. Pour simplifier, nous nous plaçons dans un domaine fixe Ω dont la frontière $\Gamma = \partial\Omega$ peut être divisée en trois sous-ensembles : la paroi Γ^{wall}, l'entrée Γ^{in} et la sortie Γ^{out}.

L'aérosol est décrit par sa fonction de distribution, dépendant au moins des variables de temps, position et vitesse. L'objet de ce nouveau modèle étant d'améliorer la prise en compte

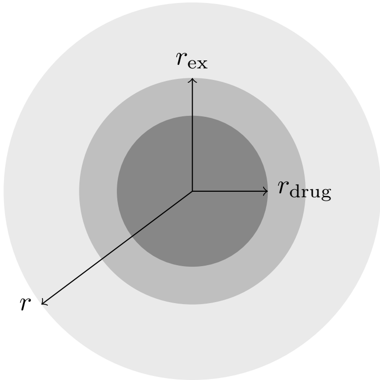


FIGURE 1.1 – Modélisation d'une particule de l'aérosol.

des effets hygroscopiques au sein des voies respiratoires, on considère également les variables de *taille* et de température. Les auteurs de [LH10 ; LH11] réalisent en effet la première analyse des conséquences de l'humidité de l'air et expliquent que la vapeur d'eau présente dans l'air est susceptible de se condenser à la surface des particules de l'aérosol et, réciproquement, l'eau à la surface des particules peut s'évaporer, selon un mécanisme régi par la comparaison des fractions massiques d'eau dans l'air et à la surface des gouttelettes. Ces changements de phase s'accompagnent naturellement d'échanges thermiques.

Pour simplifier l'écriture du modèle, on suppose que les particules formant l'aérosol sont sphériques. De plus, on divise formellement chaque gouttelette en trois couches, *cf.* Figure 1.1 : le principe actif, l'excipient et l'eau. La boule de rayon $r_{\text{drug}} > 0$, au centre de la particule, contient le médicament. La couronne suivante, de rayon extérieur $r_{\text{ex}} \geq r_{\text{drug}}$, contient l'excipient (en l'absence d'excipient, $r_{\text{ex}} = r_{\text{drug}}$). Enfin, le reste de la particule est composé d'eau et a un rayon $r \geq r_{\text{ex}}$. On suppose que le phénomène d'évaporation ne concerne pas le principe actif et l'excipient, ce qui se traduit par le fait que r_{drug} et r_{ex} sont constants. Si l'on note ρ_{w}, ρ_{drug} et ρ_{ex} les masses volumiques de l'eau, du principe actif et de l'excipient, alors, en supposant qu'elles sont constantes, la masse et la masse volumique d'une gouttelette ne varient qu'avec r et sont données par

$$m(r) = \frac{4}{3}\pi \left[r_{\text{drug}}^3 \rho_{\text{drug}} + (r_{\text{ex}}^3 - r_{\text{drug}}^3)\rho_{\text{ex}} + (r^3 - r_{\text{ex}}^3)\rho_{\text{w}} \right],$$

$$\rho_{\text{d}}(r) = \frac{1}{r^3} \left[r_{\text{drug}}^3 \rho_{\text{drug}} + (r_{\text{ex}}^3 - r_{\text{drug}}^3)\rho_{\text{ex}} + (r^3 - r_{\text{ex}}^3)\rho_{\text{w}} \right].$$

Écrivons à présent les équations régissant l'évolution du système composé de l'air et de l'aérosol. Il est habituel de considérer le gaz comme un fluide newtonien incompressible, décrit par sa vitesse u et sa pression p, dépendant du temps $t \geq 0$ et de la position $x \in \Omega$. Comme nous l'avons indiqué ci-dessus, la fraction massique $Y_{\text{v,air}}(t, x)$ de vapeur d'eau dans l'air, ainsi que la température $T_{\text{air}}(t, x)$ interviendront dans notre description des interactions entre le fluide et l'aérosol.

La fonction de distribution f représentant les particules satisfait l'équation de Vlasov

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v[\alpha(u - v) + g)f] + \partial_r(af) + \partial_T(bf) = 0, \tag{1.3.1}$$

où g est le champ gravitationnel, $\alpha(u-v)$ est l'accélération de traînée due au mouvement de l'air, et a et b décrivent respectivement les variations de rayon et de température des gouttelettes. On a

$$\alpha(r) = \frac{c_1 r}{m(r)},$$

où $c_i > 0$, $1 \leq i \leq 7$, désigne ci-dessus et dans tout ce qui suit une constante physique dont on trouvera l'expression dans [Bou+20],

$$\begin{aligned} a(r,T,Y_{\mathrm{v,air}}(t,x)) &= -c_2 N_{\mathrm{d}}(r,T,Y_{\mathrm{v,air}}(t,x)), \\ b(r,T,Y_{\mathrm{v,air}}(t,x),T_{\mathrm{air}}(t,x)) &= \frac{c_3}{\rho_{\mathrm{d}}(r)r}\left[-Q_{\mathrm{d}}(r,T,T_{\mathrm{air}}(t,x)) - L_{\mathrm{v}} N_{\mathrm{d}}(r,T,Y_{\mathrm{v,air}}(t,x))\right], \end{aligned}$$

où N_{d} est le flux massique à la surface d'une particule, Q_{d} est le flux de chaleur convectif entre l'air et les particules et L_{v} la chaleur latente de vaporisation de l'eau. Le flux Q_{d} est donné par

$$Q_{\mathrm{d}}(r,T,T_{\mathrm{air}}(t,x)) = \frac{c_4}{r}(T - T_{\mathrm{air}}(t,x)).$$

L'expression de N_{d} est plus complexe. On définit d'abord la pression de vapeur $P_{\mathrm{v,sat}}(T)$, une fonction $K(r,T)$ décrivant l'effet Kelvin sur la concentration de la surface des particules en vapeur d'eau induit par la courbure des particules, ainsi que l'activité de l'eau $S(r)$ qui vérifie la propriété fondamentale $S(r_{\mathrm{ex}}) = 0$. La fraction massique de vapeur d'eau à la surface d'une particule est alors donnée par

$$Y_{\mathrm{v,surf}}(r,T) = \frac{S(r)K(r,T)P_{\mathrm{v,sat}}(T)}{\rho_{\mathrm{d}}(r)R_{\mathrm{v}}T},$$

où R_{v} est un paramètre physique. Notons que $0 \leq Y_{\mathrm{v,surf}} \leq 1$ et que $Y_{\mathrm{v,surf}}(r_{\mathrm{ex}},\cdot) = 0$. On peut enfin donner l'expression du flux de matière :

$$N_{\mathrm{d}}(r,T,Y_{\mathrm{v,air}}(t,x)) = c_5 \frac{D_{\mathrm{v}}(T_{\mathrm{air}})}{r} \frac{Y_{\mathrm{v,surf}}(r,T) - Y_{\mathrm{v,air}}(t,x)}{1 - Y_{\mathrm{v,surf}}(r,T)},$$

où le coefficient de diffusion de la vapeur d'eau dans l'air D_{v} est donné par

$$D_{\mathrm{v}}(T_{\mathrm{air}}(t,x)) = c_6 T_{\mathrm{air}}(t,x)^{c_7}.$$

Comme nous l'avons indiqué précédemment, le fluide est décrit par sa vitesse et sa pression, solutions des équations de Navier-Stokes incompressible

$$\begin{cases} \rho_{\mathrm{air}}(\partial_t u + (u\cdot\nabla_x)u) - \eta\Delta_x u + \nabla_x p = F, \\ \operatorname{div}_x u = 0, \end{cases} \tag{1.3.2}$$

où ρ_{air} désigne la masse volumique de l'air et η sa viscosité dynamique. La force de Brinkman F décrivant la rétroaction des particules sur le fluide a pour expression

$$F(t,x) = c_1 \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(v - u(t,x)) f(t,x,v,r,T)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}T.$$

Le modèle (1.3.1)–(1.3.2) est présenté par [LH11] en supposant que $Y_{\mathrm{v,air}}$ et T_{air} sont des constantes uniformes dans tout le domaine considéré. Les auteurs de [Bou+20] relaxent cette hypothèse et décrivent la dynamique de ces quantités par les équations suivantes. La fraction massique de vapeur d'eau dans l'air satisfait l'équation de convection-diffusion

$$\rho_{\mathrm{air}}(\partial_t Y_{\mathrm{v,air}} + u\cdot\nabla_x Y_{\mathrm{v,air}}) - \mathrm{div}_x(D_{\mathrm{v}}(T_{\mathrm{air}})\nabla_x Y_{\mathrm{v,air}}) = S_Y, \tag{1.3.3}$$

où le terme source S_Y représente les transferts de masse entre l'air et l'aérosol et est donné par

$$S_Y(t,x) = \rho_{\mathrm{w}} \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} 4\pi r^2 N_{\mathrm{d}}(r,T,Y_{\mathrm{v,air}}(t,x)) f(t,x,v,r,T)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}T.$$

La température de l'air satisfait quant à elle

$$\rho_{\mathrm{air}}(\partial_t T_{\mathrm{air}} + u\cdot\nabla_x T_{\mathrm{air}}) - \kappa_{\mathrm{air}}\Delta_x T_{\mathrm{air}} = S_T, \tag{1.3.4}$$

où κ_{air} est la conductivité thermique de l'air et le terme source S_T représente les transferts thermiques entre l'air et l'aérosol et est donné par

$$S_T(t,x) = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} 4\pi r^2 Q_{\mathrm{d}}(r,T,T_{\mathrm{air}}(t,x)) f(t,x,v,r,T)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}T.$$

Les équations (1.3.1)–(1.3.4) sont complétées par des conditions au bord du domaine. Pour l'équation (1.3.1) on considère une condition d'absorption sur la paroi, qui s'écrit, pour tous $(t,x,v,r,T)\in\mathbb{R}_+\times\Gamma^{\mathrm{wall}}\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*$,

$$f(t,x,v,r,T) = 0 \qquad \text{si } v\cdot n \le 0, \tag{1.3.5}$$

où n est le vecteur unitaire normal à Γ^{wall}. On suppose de plus que l'air satisfait une condition de Dirichlet sur la paroi et que la dynamique est prescrite à l'entrée du domaine :

$$\begin{cases} u = u^{\mathrm{in}} & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{in}},\\ u = 0 & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{wall}},\\ \sigma(u,p)\cdot n = 0 & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{out}}, \end{cases} \tag{1.3.6}$$

où $\sigma(u,p) = \nabla_x u + (\nabla_x u)^{\intercal} - p\,\mathrm{Id}$ est le tenseur des contraintes et $u^{\mathrm{in}}:\mathbb{R}_+\times\Gamma^{\mathrm{in}}\to\mathbb{R}^3$ est donnée. De même, on fixe la fraction massique de vapeur d'eau ainsi que la température à l'entrée du domaine et sur la paroi, et on impose une condition de Neumann homogène à la sortie :

$$\begin{cases} Y_{\mathrm{v,air}} = Y_{\mathrm{v,air}}^{\mathrm{in}} & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{in}},\\ Y_{\mathrm{v,air}} = Y_{\mathrm{wall}} & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{wall}},\\ \nabla_x Y_{\mathrm{v,air}}\cdot n = 0 & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{out}}, \end{cases} \tag{1.3.7}$$

où $Y_{\mathrm{v,air}}^{\mathrm{in}}, Y_{\mathrm{wall}}\in(0,1)$ et

$$\begin{cases} T_{\mathrm{air}} = T_{\mathrm{air}}^{\mathrm{in}} & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{in}},\\ T_{\mathrm{air}} = T_{\mathrm{air}}^{\mathrm{wall}} & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{wall}},\\ \nabla_x T_{\mathrm{air}}\cdot n = 0 & \text{sur } \mathbb{R}_+\times\Gamma^{\mathrm{out}}, \end{cases} \tag{1.3.8}$$

où $T_{\mathrm{air}}^{\mathrm{in}}, T_{\mathrm{wall}}\in\mathbb{R}_+^*$ sont donnés.

1.3.2 Étude numérique en dimension 3

La présentation du modèle décrit à la section précédente dans [Bou+20] s'accompagne d'une étude numérique en dimension 2. En collaboration avec Laurent Boudin, nous avons étendu cette étude en dimension 3, dans deux domaines distincts. Le premier est un cylindre et le second une bifurcation modélisant, de façon idéalisée, la trachée et la première génération des bronches, comme à la Figure 1.2. Afin de simuler l'utilisation d'un aérosol thérapeutique, nous considérons que celui-ci est injecté dans les voies respiratoires par vagues contenant un nombre équivalent de gouttelettes, à des temps prédéfinis. Notre objectif est de décrire l'évolution de cet ensemble, et notamment du rayon des particules, au cours du passage dans les voies respiratoires. L'enjeu des aérosols thérapeutiques étant le dépôt du principe actif dans la zone obstruée, nous nous attachons à déterminer, dans le cas de la bifurcation, le nombre de gouttelettes qui se déposent sur la paroi.

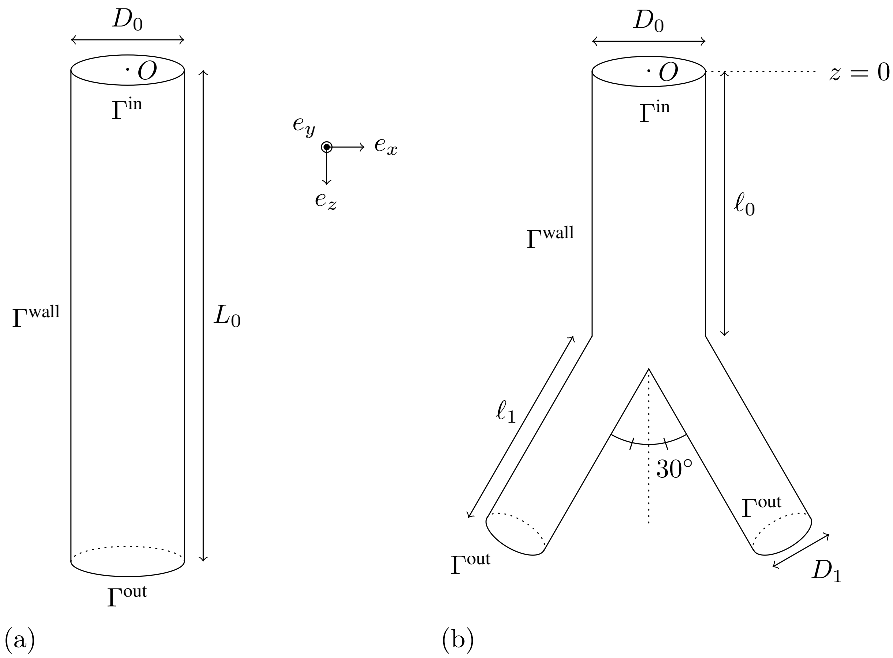


FIGURE 1.2 – Schéma (a) du domaine cylindrique, (b) de la bifurcation.

Le domaine cylindrique sert essentiellement d'outil de validation de notre code tridimensionnel. En effet, si l'on néglige la force de Brinkman dans les équations de Navier-Stokes (1.3.2), nous devons retrouver la stationnarité du profil de Poiseuille. Nous vérifions également que si l'on injecte des particules en respectant la symétrie cylindrique, alors cette dernière est préservée dans le temps. Enfin, nous utilisons également le cylindre pour tester d'éventuelles simplifications du modèle dans un cadre géométrique et numérique plus favorable que la bifurcation.

Ces travaux ont donné lieu à [BM20], accepté pour publication dans *Journal of Computational and Theoretical Transport*.

Comme nous l'avons indiqué à la Section 1.2, la résolution des équations de Navier-Stokes en trois dimensions est très coûteuse et nous faisons donc l'hypothèse d'un spray très fin (au sens de [Bou+15]) et négligeons la force de Brinkman [1]. Les équations de Navier-Stokes sont alors entièrement découplées des autres équations et peuvent être résolues indépendamment. Ce long calcul est réalisé en amont et nous constatons qu'un état d'équilibre est atteint très rapidement dans les deux domaines de calcul étudiés. C'est cette solution qui est utilisée dans toutes les autres équations. Notons qu'à ce stade les deux équations de convection-diffusion et l'équation de Vlasov restent couplées.

La principale difficulté que nous rencontrons en dimension 3 est la durée des calculs. Notre objectif étant d'obtenir des résultats statistiques sur le rayon et le dépôt des particules injectées, nous sommes prêts à discuter d'hypothèses raisonnables si elles conduisent à un gain de temps facilitant la répétition des expériences numériques.

Pour cela, nous commençons par comparer les ordres de grandeur des coefficients de l'équation (1.3.3) sur la fraction massique de vapeur d'eau dans l'air $Y_{\mathrm{v,air}}$. Dans le cadre de la bifurcation et dans le régime de température/vitesse choisi, on obtient :

$$\frac{\rho_{\mathrm{air}}}{\Delta t} \sim 7 \times 10^{-1}\,\mathrm{g.cm^{-3}.s^{-1}}, \qquad \frac{\rho_{\mathrm{air}}|u|}{\Delta x} \sim 5 \times 10^{-1}\,\mathrm{g.cm^{-3}.s^{-1}},$$

$$\frac{D_{\mathrm{v}}(T_{\mathrm{air}}^{\mathrm{in}})}{(\Delta x)^2} \sim 11\,\mathrm{g.cm^{-3}.s^{-1}}, \qquad |S_Y| \sim 5 \times 10^{-5}\,\mathrm{g.cm^{-3}.s^{-1}}.$$

Nous nous attendons donc à ce que le terme prépondérant soit le terme de diffusion, et donc à ce que $Y_{\mathrm{v,air}}$ atteigne rapidement un équilibre. Nous avons pu le vérifier numériquement. De plus, nous avons constaté que la variation du coefficient de diffusion $D_{\mathrm{v}}(T_{\mathrm{air}})$ avec la température T_{air} n'a pas d'influence sur l'équilibre de $Y_{\mathrm{v,air}}$ (dans des intervalles de températures physiquement pertinent). Nous adoptons donc pour cette quantité la même stratégie que pour la vitesse du fluide : nous calculons l'équilibre en amont (sans particules), et nous l'utilisons par la suite dans les autres équations. Cela induit encore un gain de temps notable.

Une autre façon de réduire le temps de calcul est de traiter certaines tâches en parallèle. Nous avons pu mettre en pratique cette stratégie pour le suivi des particules, sur 100 cœurs. Lors de l'injection des gouttelettes, chacune d'elle est attribuée à l'un des processeurs, qui sera chargé de tous les calculs liés à son évolution. Cela est rendu possible par le fait que les équations décrivant le comportement des particules sont indépendantes les unes des autres. En effet, d'après la méthode des caractéristiques, chaque gouttelette p est régie par le système

$$\begin{cases} \dot{x}_p(t) = v_p(t), \\ \dot{v}_p(t) = \alpha(r_p(t))(u(t, x_p(t)) - v_p(t)) + g, \\ \dot{r}_p(t) = a(r_p(t), T_p(t), Y_{\mathrm{v,air}}(t, x_p(t))), \\ \dot{T}_p(t) = b(r_p(t), T_p(t), Y_{\mathrm{v,air}}(t, x_p(t)), T_{\mathrm{air}}(t, x_p(t))). \end{cases}$$

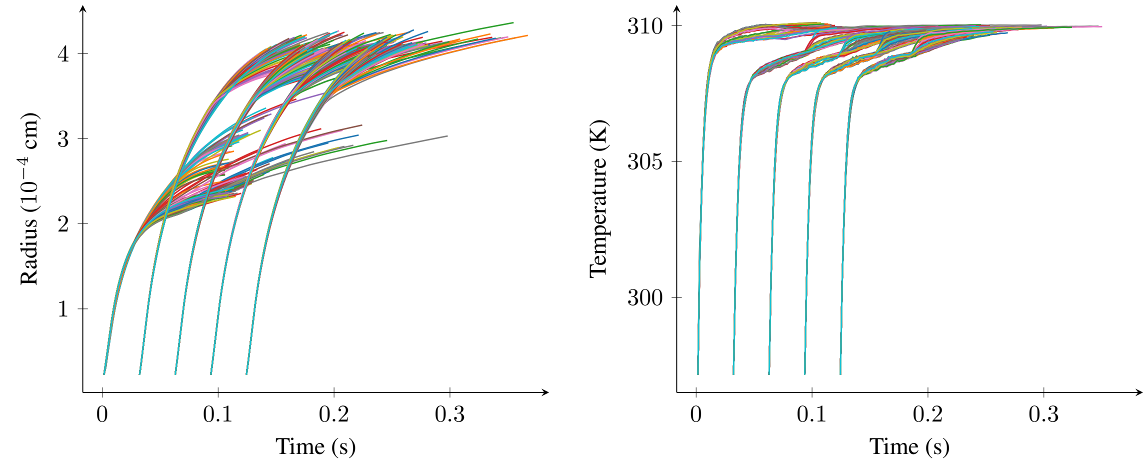
Cette approche nous permet de considérer un nombre de macro-particules comparable au cas de [Bou+20] en dimension 2.

Les expériences numériques que nous avons menées consistent à injecter cinq vagues de 100 particules numériques, représentant chacune 10^4 gouttelettes. Elles sont initialement constituées uniquement de principe actif (on néglige l'excipient) et leur position initiale est tirée aléatoirement dans un disque restreint autour du centre de la trachée.

La Figure 1.3 représente l'évolution du rayon et de la température des particules entre leur injection et leur sortie/dépôt. Comme dans [Bou+20], nous constatons que la première injection

1. Notons que les travaux présentés au Chapitre 3 pourraient apportent une justification mathématique à ce choix physiquement pertinent.

a un comportement différent des suivantes, qui s'explique par la présence initiale d'air chaud à l'intérieur des bronches, les particules de la première vague étant alors sur le front froid entrant dans la trachée, tandis que celles des vagues suivantes baignent dans un air froid. Nous observons également la dispersion des températures et des rayons lorsque les particules quittent la trachée et entrent dans les bronches.



(a) (b)

FIGURE 1.3 – Évolution du (a) rayon et de la (b) température des particules.

Nous obtenons aussi des statistiques pour le taux de dépôt des particules, le rayon de dépôt/sortie ou encore la température de dépôt/sortie. Ces résultats sont cohérents avec ceux obtenus en dimension 2 par [Bou+20]. Nous avons également étudié une simplification du modèle, consistant à considérer que $Y_{\mathrm{v,air}}$ est constant dans le domaine (en plus de l'hypothèse de stationnarité), et obtenu des résultats identiques sur la statistique de dépôt et un écart de l'ordre de 3% pour le rayon des particules, ce qui semble acceptable, compte tenu de la simplification apportée.

1.3.3 Étude de l'existence de solutions faibles globales

En collaboration avec Laurent Boudin et Ayman Moussa, nous avons étudié le problème de Cauchy pour le modèle prenant en compte les effets hygroscopiques, présenté à la Section 1.3.1 et résolu numériquement à la Section 1.3.2. Ces travaux font l'objet de l'article [BMM20] publié dans *Mathematical Models and Methods in Applied Sciences*.

1.3.3.1 Le système d'équations considéré

Nous considérons en réalité un cadre un peu plus complexe, en autorisant le domaine spatial à varier au cours du temps comme dans [BGM17]. Soit un domaine de référence $\Omega_0 \subset \mathbb{R}^3$ ouvert borné ayant une frontière lipschitzienne. Ses variations temporelles sont représentées par une application $\mathcal{A} \in \mathscr{C}^2(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3)$, $(t,x) \mapsto \mathcal{A}(t,x) = \mathcal{A}_t(x)$ telle que, pour tout $t \geq 0$, $\mathcal{A}_t$ est un $\mathscr{C}^1$-difféomorphisme et $\mathcal{A}_0 = \mathrm{Id}_{\mathbb{R}^3}$. Pour tout $t \geq 0$, notons $\Omega_t = \mathcal{A}_t(\Omega_0)$ le domaine borné au temps t et

$$\widehat{\Omega}_t = \bigcup_{0<s<t} \{s\} \times \Omega_s.$$

Notre étude porte sur un temps fini arbitrairement grand, noté $\tau > 0$. Pour $t = \tau$, notons simplement $\widehat{\Omega} = \widehat{\Omega}_\tau$. Posons également

$$\widehat{\Gamma} = \bigcup_{0<t<\tau} \{t\} \times \partial\Omega_t$$

et, pour tout $t \in [0,\tau]$, appelons n_t le vecteur unitaire sortant normal à $\partial\Omega_t$. Nous utiliserons abondamment la vitesse eulérienne w associée au flot $t \mapsto \mathcal{A}_t$, définie par

$$\forall (t,x) \in [0,\tau] \times \mathbb{R}^3, \qquad w(t, \mathcal{A}_t(x)) = \partial_t \mathcal{A}(t,x).$$

Nous supposons que $\operatorname{div}_x w = 0$, ce qui implique que le jacobien de la transformation $\mathcal{A}_t$ ne dépend pas du temps, et est donc constant égal à 1. Définissons enfin l'espace des phases et ses frontières. Pour tout $t \in [0,\tau]$, posons

$$\Pi_t = \Omega_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Pi}_t = \widehat{\Omega}_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

$$\Sigma_t = \partial\Omega_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Sigma}_t = \bigcup_{0<s<t} \{s\} \times \partial\Omega_s \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

$$\widehat{\Pi} = \widehat{\Omega} \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Sigma} = \widehat{\Gamma} \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*.$$

et

$$\widehat{\Sigma}^\pm = \{(t,x,v,r,T) \in \widehat{\Sigma},\ \pm(v - w(t,x)) \cdot n_t(x) > 0\},$$
$$\widehat{\Sigma}^0 = \{(t,x,v,r,T) \in \widehat{\Sigma},\ (v - w(t,x)) \cdot n_t(x) = 0\}.$$

Sans que cela n'ait d'incidence sur les résultats obtenus, nous simplifions les constantes physiques apparaissant dans les équations. En particulier, négligeons la présence d'excipient ($r_{\mathrm{ex}} = r_{\mathrm{drug}}$) et considérons que la masse volumique de l'eau ρ_{w} est égale à 1 et celle du principe actif ρ_{drug} est égale à 2. La masse d'une gouttelette s'exprime alors comme

$$m(r) = r^3 + r_{\mathrm{drug}}^3. \tag{1.3.9}$$

Nous écrivons donc l'accélération de traînée des particules

$$A(t,x,v,r) = \frac{r}{r^3 + r_{\mathrm{drug}}^3}(u(t,x) - v).$$

Sa contrepartie, la force de Brinkman, est donnée par

$$F(t,x) = -\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r(u(t,x) - v) f(t,x,v,r,T)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}T.$$

Grâce à l'ordre de grandeur $Y_{\mathrm{v,surf}} \sim 3\%$ (*cf.* [Bou+20]), nous simplifions également l'expression de la fonction décrivant les flux de matières entre les particules et l'air et considérons que

$$a(t,x,r,T) = \frac{Y(t,x) - Y_{\mathrm{v,surf}}(r,T)}{r},$$

où nous allégeons le nom de la variable représentant la fraction massique de vapeur d'eau dans l'air $Y := Y_{\mathrm{v,air}}$ et où nous ne nous préoccupons pas de l'expression de la fonction $Y_{\mathrm{v,surf}}$, si ce n'est pour garantir sa régularité ainsi que les bornes $0 \le Y_{\mathrm{v,surf}} \le 1$ et l'égalité $Y_{\mathrm{v,surf}}(r_{\mathrm{drug}}, \cdot) = 0$.

La fonction décrivant les flux thermiques est également simplifiée de la façon suivante :

$$b(t,x,r,T) = \frac{Y(t,x) - Y_{\mathrm{v,surf}}(r,T)}{r^2} + \frac{\Theta(t,x) - T}{r^2},$$

où nous avons noté $\Theta := T_{\mathrm{air}}$. Dans le régime de température considéré, le coefficient de diffusion de la vapeur d'eau dans l'air $D_v(T_{\mathrm{air}})$ est majoré et minoré par une constante strictement positive. L'analyse que nous présentons dans ce manuscrit peut être facilement adaptée pour n'importe quelle fonction D_v vérifiant ces conditions. Sans perte de généralité, nous considérons donc $D_v(T_{\mathrm{air}}) \equiv 1$. Cette hypothèse est à mettre en parallèle avec la remarque faite à la Section 1.3.2 selon laquelle la dépendance du coefficient de diffusion en la température de l'air a une influence négligeable sur l'évolution de $Y_{\mathrm{v,air}}$.

Nous considérons également des conditions au bord différentes. La frontière physique du domaine n'est plus divisée en entrée/paroi/sortie(s) et nous imposons une condition de Dirichlet sur l'intégralité du bord pour la vitesse du fluide. Pour la température de l'air et la fraction massique de vapeur d'eau, nous imposons la nullité du flux au bord. La fonction de distribution de l'aérosol vérifie quant à elle une condition d'absorption sur toute la frontière.

Le système d'équations aux dérivées partielles que nous étudions est donc le suivant :

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left(\frac{r}{r^3 + r_{\mathrm{drug}}^3}(u - v) f \right) + \partial_r \left(\frac{Y - Y_{v,\mathrm{surf}}}{r} f \right) + \partial_T \left(\left(\frac{Y - Y_{v,\mathrm{surf}}}{r^2} + \frac{\Theta - T}{r^2} \right) f \right) = 0 \quad \text{sur } \widehat{\Pi}, \tag{1.3.10}$$

$$f = 0 \quad \text{sur } \widehat{\Sigma}^-, \tag{1.3.11}$$

$$\partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \Delta_x u = -\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r(u - v) f \quad \text{sur } \widehat{\Omega}, \tag{1.3.12}$$

$$\mathrm{div}_x\, u = 0 \quad \text{sur } \widehat{\Omega}, \tag{1.3.13}$$

$$u = w \quad \text{sur } \widehat{\Gamma}, \tag{1.3.14}$$

$$\partial_t Y + u \cdot \nabla_x Y - \Delta_x Y + \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r f \right) Y = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r Y_{v,\mathrm{surf}} f \quad \text{sur } \widehat{\Omega}, \tag{1.3.15}$$

$$\nabla_x Y \cdot n_t = 0 \quad \text{sur } \widehat{\Gamma}. \tag{1.3.16}$$

$$\partial_t \Theta + u \cdot \nabla_x \Theta - \Delta_x \Theta + \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r f \right) \Theta = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r T f \quad \text{sur } \widehat{\Omega}, \tag{1.3.17}$$

$$\nabla_x \Theta \cdot n_t = 0 \quad \text{sur } \widehat{\Gamma}, \tag{1.3.18}$$

auquel nous ajoutons les conditions initiales

$$f(0,\cdot,\cdot,\cdot) = f^{\mathrm{in}} \quad \text{sur } \Pi_0, \tag{1.3.19}$$

$$u(0,\cdot) = u^{\mathrm{in}} \quad \text{sur } \Omega_0, \tag{1.3.20}$$

$$Y(0,\cdot) = Y^{\mathrm{in}} \quad \text{sur } \Omega_0, \tag{1.3.21}$$

$$\Theta(0,\cdot) = \Theta^{\mathrm{in}} \quad \text{sur } \Omega_0. \tag{1.3.22}$$

Sous cette forme ainsi épurée, nous remarquons que si f est positive, alors les solutions Y et Θ vérifient, au moins formellement, le principe du maximum faible. Comme $0 \leq Y_{\mathrm{v,surf}} \leq 1$, on obtient $0 \leq Y \leq 1$ et $\Theta \geq 0$ en tout temps $t \in [0, \tau]$ sous réserve d'avoir $0 \leq Y^{\mathrm{in}} \leq 1$ et $\Theta^{\mathrm{in}} \geq 0$.

Notre étude du système fluide-cinétique (1.3.10)–(1.3.22) débute par l'écriture formelle du bilan d'énergie. Rappelons au préalable la formule de Reynolds : pour toute application régulière $k : \widehat{\Omega} \to \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} k = \int_{\Omega_t} \partial_t k + \int_{\partial\Omega_t} k\, w \cdot n_t.$$

Selon la méthode classique, adaptée au cas d'un domaine variable par [BGM17] pour le système de Vlasov-Navier-Stokes, nous multiplions scalairement (1.3.12) par $u - w$ et intégrons par parties sur $\widehat{\Omega}_t$, pour $t \in [0, \tau]$ fixé. De même, nous multiplions (1.3.10) par $m(r)|v|^2/2$ et intégrons par parties sur $\widehat{\Pi}_t$. Grâce à (1.3.9), (1.3.13)–(1.3.14) et (1.3.19)–(1.3.20), nous obtenons [2]

$$\begin{aligned}
\frac{1}{2} \int_{\Omega_t} |u(t)|^2 &+ \frac{1}{2} \iint_{\Pi_t} (r^3 + r_{\mathrm{drug}}^3)|v|^2 f(t) + \iint_{\widehat{\Omega}_t} |\nabla_x u|^2 + \iiint_{\widehat{\Pi}_t} r|u - v|^2 f \\
&= \frac{1}{2} \int_{\Omega_0} |u^{\mathrm{in}}|^2 + \frac{1}{2} \iint_{\Pi_0} (r^3 + r_{\mathrm{drug}}^3)|v|^2 f^{\mathrm{in}} + \int_{\Omega_t} u(t) \cdot w(t) - \int_{\Omega_0} u^{\mathrm{in}} \cdot w(0) \\
&\quad - \iint_{\widehat{\Omega}_t} u \cdot \partial_s w - \iint_{\widehat{\Omega}_t} ((u \cdot \nabla_x)\, w) \cdot u + \iint_{\widehat{\Omega}_t} \nabla_x u : \nabla_x w \\
&\quad + \iiint_{\widehat{\Pi}_t} r\, w \cdot (u - v) f + \frac{3}{2} \iiint_{\widehat{\Pi}_t} r(Y - Y_{v,\mathrm{surf}}) f \\
&\quad + \frac{1}{2} \iiint_{\widehat{\Sigma}_t} (r^3 + r_{\mathrm{drug}}^3)|v|^2 f (w - v) \cdot n_t.
\end{aligned}$$

Supposons que l'eau entourant le principe actif (voir Figure 1.1 page 9) ne s'évapore jamais complètement. Le rayon des gouttelettes est alors minoré par r_{drug}. Mathématiquement, cela se traduit par l'hypothèse, pour presque tout $r < r_{\mathrm{drug}}$,

$$f(\cdot, \cdot, \cdot, r, \cdot) = 0. \tag{1.3.23}$$

Si $f \geq 0$, on en déduit, comme $0 \leq Y_{v,\mathrm{surf}}, Y \leq 1$ (par définition d'une fraction massique ou, comme nous l'avons indiqué ci-dessus, par application formelle du principe du maximum), que

$$\iiint_{\widehat{\Pi}_t} r\, w \cdot (u - v) f + \iiint_{\widehat{\Pi}_t} r(Y - Y_{v,\mathrm{surf}}) f \leq \frac{1}{2} \iiint_{\widehat{\Pi}_t} r|u - v|^2 f + \frac{\|w\|_{\mathrm{L}^\infty(\widehat{\Omega})} + 2}{r_{\mathrm{drug}}^2} \iiint_{\widehat{\Pi}_t} r^3 f.$$

Mais en multipliant (1.3.10) par r^3 et en intégrant sur $\widehat{\Pi}_t$, nous obtenons, grâce à la condition d'absorption (1.3.11) et le lemme de Grönwall,

$$\iint_{\Pi_t} r^3 f(t) \lesssim 1, \tag{1.3.24}$$

où la notation $\lesssim$ indique que l'inégalité est valable à une constante multiplicative près qui peut dépendre de τ, w, r_{drug} et la donnée initiale. En utilisant à nouveau la condition d'absorption (1.3.11), nous déduisons de cette inégalité et du lemme de Grönwall l'estimation d'énergie

2. Les variables (v, r, T) d'une part, x d'autre part mais également t ayant un statut distinct, nous écrivons le signe intégral une fois pour la variable temporelle, une fois pour la variable spatiale, et une fois pour les variables (v, r, T). Il n'y alors plus de confusion possible et nous omettons d'écrire les variables dans l'intégrande.

suivante :

$$\frac{1}{2}\int_{\Omega_t}|u(t)|^2+\iint_{\Pi_t}(r^3+r_{\mathrm{drug}}^3)|v|^2f(t)+\iint_{\widehat{\Omega}_t}|\nabla_x u|^2+\iiint_{\widehat{\Pi}_t} r|u-v|^2f\lesssim 1. \tag{1.3.25}$$

Intéressons-nous à présent au lien entre les températures de l'air et de l'aérosol. Pour cela, nous multiplions (1.3.17) par Θ et intégrons par parties sur $\widehat{\Omega}_t$, à $t\in(0,\tau)$ fixé. De même, nous multiplions (1.3.10) par $r^3T^2/2$ et intégrons par parties sur $\widehat{\Pi}_t$. Grâce à (1.3.13)–(1.3.14), (1.3.18)–(1.3.19), et (1.3.22), nous obtenons

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t}|\Theta(t)|^2+\frac{1}{2}\iint_{\Pi_t} r^3T^2f(t)+\iint_{\widehat{\Omega}_t}|\nabla_x\Theta|^2+\iiint_{\widehat{\Pi}_t} r|\Theta-T|^2f&\\ \leq 3\iiint_{\widehat{\Pi}_t} rT^2f+\iiint_{\widehat{\Pi}_t} rTf+\int_{\Omega_0}|\Theta^{\mathrm{in}}|^2+\iint_{\Pi_0} r^3T^2f^{\mathrm{in}}.&\end{aligned}$$

En utilisant l'hypothèse (1.3.23), l'inégalité de Cauchy-Schwarz, et l'estimation (1.3.24), nous déduisons du lemme de Grönwall que

$$\frac{1}{2}\int_{\Omega_t}|\Theta(t)|^2+\frac{1}{2}\iint_{\Pi_t} r^3T^2f(t)+\iint_{\widehat{\Omega}_t}|\nabla_x\Theta|^2+\iiint_{\widehat{\Pi}_t} r|\Theta-T|^2f\lesssim 1. \tag{1.3.26}$$

1.3.3.2 Énoncé du résultat principal

Les estimations (1.3.25)–(1.3.26) ainsi que les conditions de bord motivent la définition des espaces fonctionnels suivants. Pour $p,q\in[1,+\infty]$ et $m\in\mathbb{N}$, on note

$$\mathrm{L}^p(0,\tau;\mathrm{W}^{m,q}(\Omega_t))=\left\{\zeta \text{ mesurable}, \ \|\zeta\|_{\mathrm{L}^p(0,\tau;\mathrm{W}^{m,q}(\Omega_t))}<\infty\right\},$$

où

$$\|\zeta\|_{\mathrm{L}^p(0,\tau;\mathrm{W}^{m,q}(\Omega_t))}=\left\|t\mapsto\|\zeta(t)\|_{\mathrm{W}^{m,q}(\Omega_t)}\right\|_{\mathrm{L}^p(0,\tau)}.$$

On définit également

$$\begin{aligned}\mathcal{V}_0&=\left\{\varphi\in\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t)),\ \mathrm{div}_x\,\varphi=0,\ \varphi=0 \text{ sur } \widehat{\Gamma}\right\},\\ \mathscr{V}&=\left\{\varphi\in\mathscr{C}^1(\overline{\widehat{\Omega}}),\ \mathrm{div}_x\,\varphi=0 \text{ sur } \widehat{\Omega},\ \varphi=0 \text{ sur } \widehat{\Gamma},\ \varphi(\tau)=0\right\},\\ \mathscr{X}&=\left\{\zeta\in\mathscr{C}^1(\overline{\widehat{\Omega}}),\ \zeta=0 \text{ sur } \widehat{\Gamma},\ \zeta(\tau)=0\right\},\end{aligned}$$

puis, en notant $\overline{g}$ l'extension par zéro à $[0,\tau]\times\mathbb{R}^3$ d'une fonction g définie sur $\widehat{\Omega}$,

$$\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\Pi_t))=\left\{f \text{ mesurable}, \ \overline{f}\in\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*))\right\},$$

et enfin

$$\mathscr{W}=\left\{\psi\in\mathscr{C}_c^1\left(\overline{\widehat{\Omega}}\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*\right),\ \psi=0 \text{ sur } \widehat{\Sigma}^+\cup\widehat{\Sigma}^0,\ \psi(\tau)=0\right\}.$$

Nous pouvons à présent énoncer notre définition de solution faible.

Définition 1. *On dit qu'un quadruplet* (u,Y,Θ,f) *est une solution faible du système* (1.3.10)–(1.3.17) *muni des conditions initiales* (1.3.20)–(1.3.22) *si les conditions suivantes sont satisfaites.*

La fonction de distribution doit vérifier :

- $\overline{f} \in \mathrm{L}^\infty((0,\tau)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*)\cap\mathscr{C}^0([0,\tau];\mathrm{L}^p(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*))$ *pour tout* $p \in [1,\infty[$,
- $(1+r^3+r^3|v|^2)f \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^1(\Pi_t))$,
- $(Y - Y_{v,\mathrm{surf}})\left(\frac{1}{r}+\frac{1}{r^2}\right) f \in \mathrm{L}^1_{\mathrm{loc}}(\widehat{\Pi})$,
- $\frac{\Theta - T}{r^2} f \in \mathrm{L}^1_{\mathrm{loc}}(\widehat{\Pi})$.

Les quantités relatives au fluide doivent vérifier :

- $u \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$,
- $u - w \in \mathcal{V}_0$,
- $Y \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$,
- $\Theta \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$.

Enfin, les formulations faibles suivantes doivent être satisfaites pour tout $\psi \in \mathscr{W}$, $\varphi \in \mathscr{V}$, *et* $\zeta \in \mathscr{X}$ *:*

$$\iiint_{\widehat{\Pi}} f \left(\partial_t \psi + v \cdot \nabla_x \psi + \frac{u - v}{r^2 + \frac{r_{\mathrm{drug}}^3}{r}} \cdot \nabla_v \psi \right) + \iiint_{\widehat{\Pi}} f \left(\frac{Y - Y_{v,\mathrm{surf}}}{r} \left(\partial_r \psi + \frac{1}{r} \partial_T \psi \right) + \frac{\Theta - T}{r^2} \partial_T \psi \right) = - \iint_{\Pi_0} f^{\mathrm{in}} \psi(0,\cdot), \quad (1.3.27)$$

$$\iint_{\widehat{\Omega}} (u \cdot \partial_t \varphi + (u \otimes u) : \nabla_x \varphi - \nabla_x u : \nabla_x \varphi) = - \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(v-u) f \right) \cdot \varphi - \int_{\Omega_0} u^{\mathrm{in}} \cdot \varphi(0,\cdot), \quad (1.3.28)$$

$$\iint_{\widehat{\Omega}} (-Y \partial_t \zeta + \zeta u \cdot \nabla_x Y + \nabla_x Y \cdot \nabla_x \zeta) = \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(Y_{v,\mathrm{surf}} - Y) f \right) \zeta + \int_{\Omega_0} Y^{\mathrm{in}} \zeta(0,\cdot), \quad (1.3.29)$$

$$\iint_{\widehat{\Omega}} (-\Theta \partial_t \zeta + \zeta u \cdot \nabla_x \Theta + \nabla_x \Theta \cdot \nabla_x \zeta) = \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(T - \Theta) f \right) \zeta + \int_{\Omega_0} \Theta^{\mathrm{in}} \zeta(0,\cdot). \quad (1.3.30)$$

Nous faisons de plus les hypothèses suivantes sur les données initiales.

Hypothèse 1. $Y^{\mathrm{in}} \in \mathrm{L}^2(\Omega_0)$ *et* $0 \leq Y^{\mathrm{in}} \leq 1$ *sur* Ω_0.

Hypothèse 2. $u^{\mathrm{in}} \in \mathrm{L}^2(\Omega_0)$ *et* $\mathrm{div}_x\, u^{\mathrm{in}} = 0$.

Hypothèse 3. $\Theta^{\mathrm{in}} \in \mathrm{L}^2(\Omega_0)$ *et* $\Theta^{\mathrm{in}} > 0$ *sur* Ω_0.

Hypothèse 4. $f^{\mathrm{in}} \in \mathrm{L}^\infty(\Pi_0)$ *est positive, pour presque tout* $r \leq r_{\mathrm{drug}}$, $f^{\mathrm{in}}(\cdot,\cdot,r,\cdot) = 0$, *et*

$$\left(1 + r^3 + r^3|v|^2 + \left|Y^{\mathrm{in}} - Y_{v,\mathrm{surf}}\right| \left(\frac{1}{r} + \frac{1}{r^2} \right) + \frac{|\Theta^{\mathrm{in}} - T|}{r^2} \right) f^{\mathrm{in}} \in \mathrm{L}^1(\Pi_0).$$

Le résultat principal peut désormais être énoncé très simplement.

Théorème 1. *Sous les Hypothèses 1–4, le système* (1.3.12)–(1.3.22) *admet une solution faible au sens de la Définition 1.*

1.3.3.3 Stratégie de démonstration

Nous suivons la stratégie présentée dans la Section 1.2 et considérons donc un problème approché (dans lequel, notamment, la force de Brinkman et l'accélération de traînée sont tronquées) que nous résolvons par une méthode de point fixe, puis nous en déduisons l'existence d'une solution au système initial par des propriétés de compacité découlant des estimations *a priori* exposées ci-dessus. Décrivons les spécificités liées à notre modèle.

Afin de démontrer qu'une estimation *a priori* similaire à (1.3.25) est valable, les auteurs de [BGM17] s'appuient sur des inégalités d'interpolation sur les moments de f en la vitesse v. Dans notre cas, nous aurions besoin de telles inégalités sur des moments croisés de f faisant intervenir à la fois la vitesse v et le rayon r, mais nous ne parvenons pas à en démontrer. Nous devons plutôt nous reposer sur l'hypothèse, justifiée par le modèle, que f est initialement nulle pour des rayons inférieurs à r_{drug}. Nous considérons donc $\eta(r)f$ plutôt que f, où η est une application nulle au voisinage de 0. Afin d'adapter le résultat d'existence et d'unicité pour l'équation de Vlasov posée dans un domaine variable obtenu par [BGM17] ainsi que démontrer un tel résultat pour les équations de convection-diffusion, nous nous restreignons par ailleurs, dans la phase de régularisation, à une fonction de distribution initiale à support compact.

Bien qu'il aurait sans doute été possible d'adopter la même stratégie de régularisation des équations de Navier-Stokes que dans [BGM17] en modifiant le terme convectif, nous choisissons d'utiliser l'approximation de Friedrichs, en combinant projecteur de Leray et troncature des hautes fréquences dans la transformée de Fourier de u.

Notons que la procédure de pénalisation introduite pour se ramener à l'espace tout entier dans les équations de Navier-Stokes ne permet pas de préserver la relation de bord $u = w$, nécessaire à l'étude des équations pour Y et Θ (qui ne font pas l'objet de cette pénalisation). Pour y remédier, nous définissons, pour $N \in \mathbb{N}^*$, les ensembles

$$K_N = \{(t,x) \in \widehat{\Omega},\, d((t,x),\widehat{\Gamma}) < 1/N\},$$

où $d(\cdot,\widehat{\Gamma})$ désigne la distance euclidienne à la frontière $\widehat{\Gamma}$, définie sur $\mathbb{R}_+ \times \mathbb{R}^3$. Afin de préserver la condition d'incompressibilité, nous considérons également le projecteur π_{Ω_t} défini, pour tout $t \in [0,\tau]$, par

$$\pi_{\Omega_t} : \mathrm{L}^2(\Omega_t) \to \{u \in \mathrm{H}^1(\Omega_t),\, \mathrm{div}_x\, u = 0,\, u_{|\partial\Omega_t} = 0\}.$$

Le champ de vitesse u est alors remplacé dans (1.3.15)–(1.3.17) par

$$z_N = w + \pi_{\Omega_t}((u-w)\mathbf{1}_{K_N}).$$

Contrairement à [BGM17], nous n'utilisons pas le théorème de point fixe de Schaefer mais le théorème de Schauder dans un espace vectoriel normé (voir [Bon62] pour un énoncé ne requérant pas la complétude de l'espace). Pour ce faire, nous appliquons des résultats d'existence obtenus par [BGM17] pour l'équation de Vlasov et le théorème de Cauchy-Lipschitz pour notre approximation des équations de Navier-Stokes. À l'inverse, à notre connaissance, la résolution des équations de convection-diffusion dans un domaine variable n'est pas connue. Nous proposons une stratégie reposant sur un changement de variable eulérien, permettant de se ramener à un domaine fixe, dans lequel nous pouvons utiliser la procédure d'approximation de Galerkin.

Une fois résolu le système approché, nous relaxons les diverses régularisations et considérons une suite d'approximations, dont notre objectif est de démontrer qu'elle converge vers une solution faible du problème initial. Nous gardons cependant, à ce stade, la troncature en rayon $\eta(r)$. En effet, comme nous l'avons indiqué ci-dessus, nous ne pouvons pas procéder comme [BGM17] pour démontrer des estimations d'énergie et obtenir la compacité nécessaire pour passer à la limite. Nous les obtenons cependant grâce à des bornes comme

$$\iint_{\Pi_t} r^3 \eta(r) f_n(t) \lesssim 1, \qquad \iint_{\Pi_t} r^3 T^2 \eta(r) f_n(t) \lesssim 1,$$

démontrées à l'aide de la troncature. Afin de passer à la limite dans les termes non linéaires apparaissant dans la Définition 1, nous obtenons de la compacité forte grâce à la variante du lemme d'Aubin-Lions démontrée dans un domaine variable par [Mou16]. Nous obtenons ainsi une solution faible du système initial, à ceci près que f est toujours remplacée par $\eta(r)f$. Mais, à présent, en appliquant la méthode des caractéristiques (après régularisation grâce à la théorie de DiPerna-Lions [DL89]), nous garantissons que si la fonction de distribution initiale est nulle pour des rayons inférieurs à r_{drug}, c'est aussi le cas de la solution obtenue, de sorte que $\eta f = f$. Nous avons donc bien résolu le système (1.3.12)–(1.3.22).

1.4 Limites hydrodynamiques pour le système de Vlasov-Navier-Stokes incompressible

Nous abordons dans [HM21], en collaboration avec Daniel Han-Kwan (CMLS, CNRS, École Polytechnique), une autre question usuelle dans l'étude de problèmes fluide-cinétique, celle de limites hydrodynamiques. Comme nous l'indiquions à la Section 1.2, cette problématique date de la volonté de Hilbert de présenter une théorie axiomatique de la physique. Deux choix de principes fondamentaux, les *axiomes*, ont été retenus : ceux de la dynamique moléculaire et du problème à N corps d'une part, ceux de la théorie cinétique des gaz de Boltzmann d'autre part. Dans les deux cas, l'objectif est de retrouver, dans le cadre de certaines limites, des modèles macroscopiques bien connus par ailleurs. Notons par exemple que la justification rigoureuse de l'équation de Boltzmann à partir des principes de la dynamique moléculaire est exposée dans [Lan75]. On se sert ainsi des limites hydrodynamiques pour établir des liens entre les différentes approches (microscopique, macroscopique, mésoscopique) décrites à la Section 1.1.

Dans le cadre du système de Vlasov-Navier-Stokes, les deux approches sont en cours d'étude.

- Il est possible de chercher à obtenir le système de Vlasov-Navier-Stokes comme une limite dite *de champ moyen* pour des systèmes de N particules en interaction avec un fluide. Le lecteur pourra trouver des résultats rigoureux partiels dans [All90 ; DGR08 ; Hil18 ; Höf18a ; Mec19 ; HMS19 ; GH19 ; CH20], mais la question d'une justification complète et rigoureuse est encore ouverte ;
- La justification du système de Vlasov-Navier-Stokes à partir d'équations de Boltzmann pour un mélange gazeux est abordée dans [Ber+17].

Nous considérons un autre problème : celui d'identifier des situations physiques pour lesquels le comportement des solutions du système de Vlasov-Navier-Stokes se rapproche de celui de solutions d'autres systèmes d'équations aux dérivées partielles et d'en faire la démonstration. Cette problématique a été abordée dans [Ben12 ; BDM14] dans le cas d'un spray bidispersé en rayon. Les auteurs considèrent en effet la limite lorsque le plus petit rayon devient négligeable devant le plus grand. À la limite, il convient de remplacer la description cinétique du spray de petit rayon en considérant désormais cette phase comme étant mélangée au fluide. Les auteurs justifient formellement le système d'équations fluide-cinétique résultant et se consacrent à son étude. Les méthodes développées au Chapitre 3 permettent de traiter rigoureusement cette limite, ce que nous faisons dans [HM21].

1.4.1 Régimes de haute friction

Dans les sections précédentes, nous avons adopté le point de vue qui consiste à écrire les équations en prenant les paramètres physiques comme étant égaux à 1. Comme dans [GJV04a], une étape intermédiaire consiste à écrire des équations sous forme adimensionnée. Le système de Vlasov-Navier-Stokes devient alors

$$\begin{cases} \partial_t u + (u\cdot\nabla_x)u - K\Delta_x u + \nabla_x p = C(j_f - \rho_f u), \\ \operatorname{div}_x u = 0, \\ \partial_t f + Av\cdot\nabla_x f + B\operatorname{div}_v\left[f\left(\dfrac{1}{A}u - v\right)\right] = 0, \end{cases} \tag{1.4.1}$$

avec

$$\rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\mathrm{d}v, \qquad j_f(t,x) = A\int_{\mathbb{R}^3} vf(t,x,v)\mathrm{d}v,$$

et

- $A = \frac{\sqrt{\theta}}{U}$, où θ est la vitesse quadratique moyenne des particules (agitation thermique) et la vitesse typique du fluide U est égale à $\frac{L}{T}$, avec L la distance caractéristique d'observation et T la durée d'observation ;
- $B = \frac{T}{\tau}$, où τ est le temps de relaxation de Stokes, donné par

$$\tau = \frac{2a^2\rho_{\text{part}}}{9\mu},$$

  avec a (resp. ρ_{part}) le rayon renormalisé [3] (resp. la masse volumique) d'une particule et μ la viscosité dynamique du fluide ;
- $C = \frac{T}{\tau}\frac{\rho_{\text{part}}}{\rho_{\text{fluide}}} = B\frac{\rho_{\text{part}}}{\rho_{\text{fluide}}}$, où ρ_{fluide} est la masse volumique du fluide ;
- $K = \frac{2}{9}\left(\frac{a}{L}\right)^2\frac{T}{\tau}\frac{\rho_{\text{part}}}{\rho_{\text{fluide}}} = \frac{2}{9}\left(\frac{a}{L}\right)^2 C$.

Comme dans [GJV04a ; GJV04b] pour le système de Vlasov-Fokker-Planck-Navier-Stokes, nous avons identifié deux types de régimes d'intérêt, que nous présentons ci-dessous.

1.4.1.1 Régimes des particules *légères* et *légères et rapides*

Dans un premier temps, nous considérons des particules légères et distinguons selon que leur vitesse est de même ordre de grandeur ou plus grande que celle du fluide. L'étude du premier régime pour le système de Vlasov-Stokes fait l'objet de [Höf18b]. Le second cas est étudié dans [GJV04a] pour le système de Vlasov-Fokker-Planck-Navier-Stokes.

Régime des particules légères. On considère les ordres de grandeur

$$A = 1, \qquad B = \frac{1}{\varepsilon}, \qquad C = 1, \qquad K = 1,$$

3. Formellement, le système de Vlasov-Navier-Stokes peut être écrit comme la limite $N \to +\infty$ d'un système de N particules dont le rayon r tend également vers 0, avec l'ordre de grandeur $r \sim \frac{1}{N}a$.

avec $\varepsilon > 0$ ayant vocation à être petit. Cela correspond à une situation physique dans laquelle :

- $U \sim \sqrt{\theta}$, c'est-à-dire que les particules et le fluide ont des vitesses caractéristiques comparables,
- $\tau \ll T$, c'est-à-dire que le temps de relaxation de Stokes est très petit par rapport à la durée d'observation,
- $\rho_{\text{part}} \ll \rho_{\text{fluide}}$, ce qui implique que les particules sont légères en comparaison avec le fluide. Asymptotiquement, elles n'ont plus d'inertie propre.

Le système (1.4.1) devient alors

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f(u - v) \right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \quad j_f(t,x) = \int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v. \end{cases} \tag{1.4.2}$$

Régime des particules légères et rapides. Une variante du régime précédent consiste à considérer des particules légères et rapides, pour lesquelles

$$A = \frac{1}{\varepsilon^\alpha}, \qquad B = \frac{1}{\varepsilon}, \qquad C = 1, \qquad K = 1,$$

où $\alpha > 0$. La seule différence avec la situation précédente est que $U \ll \sqrt{\theta}$, c'est-à-dire que la vitesse du fluide est négligeable devant celle des particules. Notons qu'il n'est pas nécessaire de fixer $\alpha = 1/2$ comme [GJV04a], bien que ce cas particulier est un intérêt que nous discuterons ci-dessous.

Le système (1.4.1) devient alors

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0, \\ \partial_t f + \dfrac{1}{\varepsilon^\alpha} v \cdot \nabla_x f + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f\left(\varepsilon^\alpha u - v\right) \right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \quad j_f(t,x) = \frac{1}{\varepsilon^\alpha} \int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v. \end{cases} \tag{1.4.3}$$

1.4.1.2 Régime des particules fines

Le régime des particules fines correspond à

$$A = 1, \qquad B = \frac{1}{\varepsilon}, \qquad C = \frac{1}{\varepsilon}, \qquad K = 1,$$

c'est-à-dire une situation dans laquelle

- $U \sim \sqrt{\theta}$, c'est-à-dire que le fluide et les particules ont des vitesses comparables,
- $\tau \ll T$, c'est-à-dire que le temps de relaxation de Stokes est négligeable devant la durée d'observation,
- $\rho_{\text{part}} \sim \rho_{\text{fluide}}$, c'est-à-dire que les particules et le fluide ont des masses volumiques comparables,

- $a \ll L$, c'est-à-dire que la taille des particules est négligeable devant la longueur d'observation.

Le système (1.4.1) devient alors

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = \dfrac{1}{\varepsilon}(j_f - \rho_f u), \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \dfrac{1}{\varepsilon} \operatorname{div}_v [f(u-v)] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \quad j_f(t,x) = \int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v. \end{cases} \tag{1.4.4}$$

Dans le cadre des travaux [Ben12 ; BDM14], on retrouve également ce système pour décrire l'interaction du fluide et de la portion de spray constituée des particules de rayon négligeable. L'étude de ce régime pour le système de Vlasov-Fokker-Planck-Navier-Stokes fait l'objet de [GJV04b].

1.4.1.3 Écriture unifiée des équations

Une partie de l'analyse étant commune aux trois systèmes décrits ci-dessus, nous introduisons des notations permettant de donner des démonstrations, valables dans chacun des cas, de façon concise. Nous considérons donc le système

$$\begin{cases} \partial_t u_{\varepsilon,\gamma,\sigma} + (u_{\varepsilon,\gamma,\sigma} \cdot \nabla_x) u_{\varepsilon,\gamma,\sigma} - \Delta_x u_{\varepsilon,\gamma,\sigma} + \nabla_x p = \dfrac{1}{\gamma}(j_{f_{\varepsilon,\gamma,\sigma}} - \rho_{f_{\varepsilon,\gamma,\sigma}} u_{\varepsilon,\gamma,\sigma}), \\ \operatorname{div}_x u_{\varepsilon,\gamma,\sigma} = 0, \\ \partial_t f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\sigma} v \cdot \nabla_x f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\varepsilon} \operatorname{div}_v [f_{\varepsilon,\gamma,\sigma} (\sigma u_{\varepsilon,\gamma,\sigma} - v)] = 0, \\ \rho_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \displaystyle\int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma}(t,x,v) \mathrm{d}v, \quad j_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \frac{1}{\sigma}\int_{\mathbb{R}^3} v f_{\varepsilon,\gamma,\sigma}(t,x,v) \mathrm{d}v. \end{cases} \tag{1.4.5}$$

où les paramètres (γ, σ) peuvent valoir :

- $\sigma = 1$, $\gamma = 1$, dans le régime des particules légères,
- $\sigma = \varepsilon^\alpha$, $\gamma = 1$, avec $\alpha > 0$, dans le régime des particules légères et rapides,
- $\sigma = 1$, $\gamma = \varepsilon$, dans le régime des particules fines.

Nous étudions le système (1.4.5) pour $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$, où $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3$ est le tore de dimension 3, muni de la mesure de Lebesgue normalisée.

Comme nous l'avons vu dans les sections précédentes, le système de Vlasov-Navier-Stokes possède une structure remarquable qui permet d'écrire, du moins formellement, la dissipation d'énergie. Dans les régimes étudiés, nous définissons donc

$$\mathrm{E}_{\varepsilon,\gamma,\sigma}(t) = \frac{\varepsilon}{\sigma^2\gamma}\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t,x)|^2 \mathrm{d}x,$$
$$\mathrm{D}_{\varepsilon,\gamma,\sigma}(t) = \int_{\mathbb{T}^3} |\nabla_x u_{\varepsilon,\gamma,\sigma}(t,x)|^2 \mathrm{d}x + \frac{1}{\gamma}\int_{\mathbb{T}^3\times\mathbb{R}^3} \left|\frac{v}{\sigma} - u_{\varepsilon,\gamma,\sigma}(t,x)\right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Ces fonctionnelles vérifient la relation (formelle) fondamentale

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{\varepsilon,\gamma,\sigma} + \mathrm{D}_{\varepsilon,\gamma,\sigma} = 0. \tag{1.4.6}$$

Afin d'alléger les notations, nous définissons également

$$\rho_{\varepsilon,\gamma,\sigma} := \rho_{f_{\varepsilon,\gamma,\sigma}}, \qquad j_{\varepsilon,\gamma,\sigma} := j_{f_{\varepsilon,\gamma,\sigma}},$$

la force de Brinkman étant alors donnée par

$$F_{\varepsilon,\gamma,\sigma} := \frac{1}{\gamma}\left(j_{\varepsilon,\gamma,\sigma} - \rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma}\right).$$

Dans le régime des particules légères et rapides, nous nous restreindrons au cas où le paramètre α appartient à $]0,\frac{1}{2}]$. Il serait possible de traiter les cas $\alpha > \frac{1}{2}$, mais l'expression de l'énergie $\mathrm{E}_{\varepsilon,\gamma,\sigma}$ nous indique *a priori* qu'une hypothèse de données bien préparées sera systématiquement nécessaire pour assurer que l'énergie cinétique est bornée. Notons qu'il semble naturel de considérer le cas $\alpha = \frac{1}{2}$, puisqu'il correspond au régime où les énergies cinétiques du fluide et de l'aérosol sont du même ordre de grandeur dans $\mathrm{E}_{\varepsilon,\gamma,\sigma}$.

1.4.2 Justification formelle des limites

Cette section est consacrée au passage à la limite $\varepsilon \to 0$ dans les régimes présentés ci-dessus. Notre objectif est de déterminer les équations décrivant le système fluide-cinétique limite.

1.4.2.1 Limites dans les régimes de particules *légères* et *légères et rapides*

Régime des particules légères. Nous considérons $(\gamma,\sigma) = (1,1)$ et le système (1.4.2). En intégrant l'équation de Vlasov (resp. v fois cette équation), nous obtenons les lois de conservation de la masse et de la quantité de mouvement :

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \mathrm{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v\, f_\varepsilon \mathrm{d}v \right) = \dfrac{1}{\varepsilon}(\rho_\varepsilon u_\varepsilon - j_\varepsilon). \end{cases}$$

L'équation de dissipation d'énergie (1.4.6) suggère que le terme

$$\int_{\mathbb{R}^3} v \otimes v\, f_\varepsilon \mathrm{d}v$$

reste borné. On en déduit que

$$\rho_\varepsilon u_\varepsilon - j_\varepsilon \xrightarrow[\varepsilon \to 0]{} 0.$$

Si l'on admet de plus que si le triplet $(\rho_\varepsilon, j_\varepsilon, u_\varepsilon)$ converge, on en déduit formellement, puisqu'il s'agit de passer à la limite (faible) dans un produit, que sa limite (ρ, u, j) vérifie

$$j = \rho u,$$

et que le couple (ρ, u) est solution du système d'équations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t u + u \cdot \nabla_x u - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0, \end{cases} \tag{1.4.7}$$

qui décrit le transport de matière par un champ de vitesse satisfaisant aux équations de Navier-Stokes incompressible et que nous appellerons Transport-Navier-Stokes. Ainsi, dans ce régime, l'influence des particules sur le fluide devient négligeable et celui-ci est uniquement régi par les équations de Navier-Stokes. Les vitesses des particules s'alignent sur celle du fluide et l'aérosol est transporté en suivant le courant du fluide.

Régime des particules légères et rapides. Dans ce cas, on considère $(\gamma, \sigma) = (1, \varepsilon^\alpha)$ et le système (1.4.3). L'équation de dissipation d'énergie (1.4.6) assure que

$$\int_0^{+\infty} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t,x,v) |v - \varepsilon^\alpha u_\varepsilon(t,x)|^2 \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} 0.$$

Si l'on suppose de plus que $(\rho_\varepsilon, u_\varepsilon)$ converge vers (ρ, u), on en déduit formellement que

$$f_\varepsilon(t,x,v) \xrightarrow[\varepsilon \to 0]{} \rho(t,x) \otimes \delta_{v=0}.$$

Ceci nous suggère que

$$\int_{\mathbb{R}^3} v \otimes v f_\varepsilon(t,x,v)\, \mathrm{d}v \xrightarrow[\varepsilon \to 0]{} 0.$$

Ainsi, les lois de conservation de la masse et de la quantité de mouvement

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \dfrac{1}{\varepsilon^\alpha} \mathrm{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v\, f_\varepsilon \mathrm{d}v \right) = \dfrac{1}{\varepsilon}(\rho_\varepsilon u_\varepsilon - j_\varepsilon), \end{cases}$$

donnent

$$\rho_\varepsilon u_\varepsilon - j_\varepsilon \xrightarrow[\varepsilon \to 0]{} 0.$$

On conclut comme précédemment que la limite (ρ, u) est solution du système Transport-Navier-Stokes (1.4.7).

1.4.2.2 Limite dans le régime des particules fines

On considère enfin $(\gamma, \sigma) = (\varepsilon, 1)$ et le système (1.4.4). Comme nous l'avons indiqué précédemment, la limite formelle pour ces équations a été présentée dans [Ben12]. Par souci de complétude, nous reprenons tout de même l'analyse. De même que dans le régime précédent, l'équation de dissipation d'énergie (1.4.6) assure que

$$\int_0^{+\infty} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t,x,v) |v - u_\varepsilon(t,x)|^2 \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} 0$$

et donc, si $(\rho_\varepsilon, u_\varepsilon)$ converge vers (ρ, u),

$$f_\varepsilon(t,x,v) \xrightarrow[\varepsilon\to 0]{} \rho(t,x)\otimes\delta_{v=u(t,x)}. \tag{1.4.8}$$

Par ailleurs, les équations de conservation de la masse et de la quantité de mouvement s'écrivent

$$\begin{cases} \partial_t\rho_\varepsilon + \mathrm{div}_x\, j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \mathrm{div}_x\left(\displaystyle\int_{\mathbb{R}^3} v\otimes v\, f_\varepsilon \mathrm{d}v\right) = -F_\varepsilon. \end{cases}$$

Grâce à (1.4.8), on en déduit que

$$\partial_t\rho + \mathrm{div}_x(\rho u) = 0.$$

De plus, (1.4.8) suggère également que

$$\int_{\mathbb{R}^3} v\otimes v\, f_\varepsilon \mathrm{d}v \xrightarrow[\varepsilon\to 0]{} \rho\, u\otimes u,$$

donc, si l'on suppose que F_ε converge vers F (et donc, nécessairement j_ε converge vers ρu), on obtient

$$\partial_t(\rho u) + \mathrm{div}_x(\rho\, u\otimes u) = -F.$$

En passant à la limite dans les équations de Navier-Stokes, on a

$$\partial_t u + \mathrm{div}_x(u\otimes u) + \nabla_x p = F, \qquad \mathrm{div}_x\, u = 0.$$

Ainsi, (ρ, u) vérifie

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \nabla_x((1+\rho)u\otimes u) - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0, \end{cases} \tag{1.4.9}$$

qui correspond au système de Navier-Stokes incompressible inhomogène. Ainsi, contrairement aux régimes précédents, les particules ont toujours une influence sur le fluide à la limite et se présentent sous la forme d'inhomogénéités transportées par le courant.

Ce type de système limite présente un intérêt majeur pour l'approximation numérique des solutions de modèles fluide-cinétique. En effet, comme nous l'avons vu en 1.2.2 et 1.3.2, la dualité entre le problème fluide (Navier-Stokes) et le problème particulaire (Vlasov) complique considérablement la résolution numérique des équations. La considération d'un système d'équations exclusivement macroscopiques, résolu sur un maillage, permet d'alléger les schémas mis en œuvre.

Notons que le problème de Cauchy pour le système (1.4.9), pour des solutions faibles globales à la Leray ou des solutions fortes à la Fujita-Kato a été largement étudié. Nous renvoyons les lecteurs à [AKM90] et [Lio96] ou [Sim90 ; Dan04 ; AGZ11 ; AGZ12 ; PZZ13 ; Pou15 ; DM19].

1.4.3 Stratégie de démonstration de la convergence

Comme nous l'avons vu à la Section 1.4.1, nous nous attendons à ce que la fonction de distribution adopte un comportement monocinétique. Les fluctuations en vitesse apparaissant dans l'*énergie modulée* introduite par [CK15], il paraît naturel de s'intéresser à cette fonctionnelle. Les auteurs ont en particulier démontré que sous l'hypothèse d'un contrôle uniforme de ρ_ε dans $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))$, on obtient une décroissance en temps qui est exponentielle et d'autant plus rapide que le paramètre $\varepsilon > 0$ est petit. Si un tel contrôle ne garantit malheureusement pas la convergence, il s'avère crucial comme outil d'interpolation et permet d'obtenir de nombreux résultats d'intégrabilité et de petitesse, qui permettront à leur tour d'obtenir les bornes uniformes dont on pourra déduire la convergence. Suivant l'analyse de [HMM20], on sait que le contrôle de ρ_ε découle, grâce à un changement de variables en vitesse, d'une estimation de la forme

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \ll 1. \tag{1.4.10}$$

La démonstration de (1.4.10) constitue le point essentiel permettant le passage à la limite sous les conditions les moins restrictives (et, en contrepartie, avec des résultats moins précis, par exemple en temps court et sans taux de convergence explicite). Elle repose sur un argument de *bootstrap* [4] pour lequel nous aurons besoin d'estimations régulières pour u_ε. Elles découlent des inégalités de régularité maximale pour l'équation de Stokes. La difficulté principale consiste à obtenir des bornes uniformes sur la force de Brinkman dans un espace $\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{T}^3))$, pour une valeur de p assez grande. Pour ce faire, nous décrivons deux stratégies de *désingularisation* (par rapport à ε). Nous mettons en œuvre la première (changement de variables et intégration par parties dans la méthode des caractéristiques) pour traiter les régimes de particules *légères* et *légères et rapides* puis nous changeons d'approche pour le régime, plus singulier, des particules fines, bien que l'on puisse, à ce stade, se contenter de la première méthode. Cette seconde stratégie se fonde sur une famille de fonctionnelles que [Han20] appelle *dissipations d'ordre supérieur* et dont l'étude permet de comprendre finement la structure de la force de Brinkman et ainsi obtenir de meilleures estimations. C'est de plus cette approche qui permet d'obtenir les résultats de convergence.

L'obtention de taux de convergence explicites repose, pour les régimes de particules *légères* et *légères et rapides*, sur la structure des équations de Navier-Stokes et de l'équation de conservation de la masse pour l'aérosol. Ceci n'est plus efficace dans le régime des particules fines. Nous utilisons alors, pour la première fois à notre connaissance dans l'étude du système de Vlasov-Navier-Stokes, une méthode d'entropie relative qui permet d'obtenir de tels taux explicites (mais qui, contrairement au cas du système de Vlasov-Fokker-Planck-Navier-Stokes [GJV04a ; GJV04b], ne suffit pas à elle seule pour démontrer la convergence).

1.4.4 Description des résultats principaux

Concluons cette introduction par la présentation des résultats principaux obtenus, en commençant par les hypothèses de travail.

Hypothèse 5. *Soient des données initiales* $(u^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0} \subset \mathrm{H}^1(\mathbb{T}^3)$ *et* $(f^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0} \subset \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *vérifiant les conditions suivantes :*

- *il existe* $q > 4$ *tel que, pour tout* $\varepsilon > 0$*,*

$$(1 + |v|^q) f^0_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{T}^3)) \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3);$$

4. Nous emploierons le terme anglais, bien que d'aucuns suggèrent de le traduire par « à la Cyrano » en référence au passage suivant de l'œuvre d'Edmond Rostand [Ros97, acte III, scène 13] dans lequel le protagoniste expose ses stratégies pour atteindre la Lune :

> Enfin, me plaçant sur un plateau de fer,
> Prendre un morceau d'aimant et le lancer en l'air !
> Ça, c'est un bon moyen : le fer se précipite,
> Aussitôt que l'aimant s'envole, à sa poursuite ;
> On relance l'aimant bien vite, et cadédis !
> On peut monter ainsi indéfiniment.

- *il existe* $r \in (2,3)$ *et* $p \in \left(3, \frac{3(2+r)}{4}\right]$ *tels que* $u^0_{\varepsilon,\gamma,\sigma} \in \mathrm{B}^{1,3}_2(\mathbb{T}^3) \cap \mathrm{B}^{1-2/r,3}_r(\mathbb{T}^3) \cap \mathrm{B}^{s,p}_p(\mathbb{T}^3)$, *où* $s = 2 - 2/p$,
- *il existe* $M > 1$ *tel que, pour tout* $\varepsilon > 0$,

$$\left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{H}^1\cap \mathrm{B}^{s,p}_p(\mathbb{T}^3)} + \left\|f^0_{\varepsilon,\gamma,\sigma}|v|^q\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} + \sup_{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3} (1+|v|^q) f^0_{\varepsilon,\gamma,\sigma}(x,v) \leq M.$$

On pourra se placer dans le cadre défini par Fujita et Kato [FK64] avec l'hypothèse suivante de petitesse de la donnée initiale pour le fluide.

Hypothèse 6. *La donnée initiale* $(u^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0}$ *satisfait*

$$\forall \varepsilon > 0, \qquad \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)} \leq \frac{C^*}{2},$$

où $C^* > 0$ *est une constante universelle, donnée dans le Corollaire C.2 page 155.*

Pour simplifier la présentation dans cette introduction, nous faisons l'hypothèse qu'il existe $(u^0, \rho^0) \in \mathrm{L}^2(\mathbb{T}^3) \times \mathrm{L}^\infty(\mathbb{T}^3)$ tels que

$$\left\|u^0_{\varepsilon,\gamma,\sigma} - u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \qquad W_1(\rho^0_{\varepsilon,\gamma,\sigma}, \rho^0) \xrightarrow[\varepsilon\to 0]{} 0,$$

où $\rho^0_{\varepsilon,\gamma,\sigma} = \rho_{f^0_{\varepsilon,\gamma,\sigma}}$ et W_1 désigne la distance de Wasserstein. Nous verrons au Chapitre 3 que l'on peut relaxer ces hypothèses.

Pour une fonction $g \in \mathrm{L}^1(\mathbb{T}^3)$, nous notons $\langle g \rangle$ sa valeur moyenne sur $\mathbb{T}^3$.

1.4.4.1 Régime des particules légères

Dans le régime des particules légères, nous considérons $(\gamma, \sigma) = (1,1)$ dans (1.4.5) et obtenons le système limite Transport-Navier-Stokes (1.4.7) sous trois jeux d'hypothèses.

- Si l'on ne fait que l'Hypothèse 5 (cas *général* ci-après), on démontre l'existence d'un temps $T > 0$ avant lequel $u_{\varepsilon,1,1}(t)$ converge vers $u(t)$ dans $\mathrm{L}^2(\mathbb{T}^3)$ et $f_{\varepsilon,1,1}$ tend faiblement vers $\rho \otimes \delta_{v=u}$ sous réserve d'intégrer en temps, où (u,ρ) est solution de (1.4.7) sous la condition initiale (u^0, ρ^0).
- Sous des hypothèses supplémentaires (cas *moyennement bien préparé*), nous démontrons que les convergences ci-dessus ont lieu pour tout temps $T > 0$.
- Enfin, en considérant encore une autre hypothèse (cas *bien préparé*), nous obtenons la convergence ponctuelle de $f_{\varepsilon,1,1}$.

L'énoncé suivant affine la description ci-dessus mais, afin de préserver la lisibilité de cette introduction, nous renvoyons les lecteurs cherchant des énoncés précis au Chapitre 3.

Théorème 2 (Régime des particules légères)**.** *Soit* $(u_{\varepsilon,1,1}, f_{\varepsilon,1,1})$ *une solution faible globale de* (1.4.5) *associée à la donnée initiale* $(u^0_{\varepsilon,1,1}, f^0_{\varepsilon,1,1})$.

1. Cas général. *Sous l'Hypothèse 5, il existe* $T > 0$ *tel que*

$$\int_0^T W_1(f_{\varepsilon,1,1}(t), \rho(t) \otimes \delta_{v=u(t)})\,\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0 \tag{1.4.11}$$

et, pour tout $t \in [0,T]$,

$$\|u_{\varepsilon,1,1}(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \tag{1.4.12}$$

où (ρ, u) *est solution du système Transport-Navier-Stokes* (1.4.7)*.*
2. Cas moyennement bien préparé. *Sous l'Hypothèse supplémentaire 6, il existe* $\eta > 0$ *suffisamment petit tel que si*

$$\left\| u^0_{\varepsilon,1,1} - \langle u^0_{\varepsilon,1,1} \rangle \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

alors les convergences (1.4.11) *et* (1.4.12) *sont valables pour tout temps* $T > 0$*.*
3. Cas bien préparé. *Enfin, si l'on suppose de plus que*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v - u^0_{\varepsilon,1,1}(x)| f^0_{\varepsilon,1,1}(x,v) \, \mathrm{d}x \, \mathrm{d}v \xrightarrow[\varepsilon \to 0]{} 0,$$

alors, pour tout $t > 0$*,*

$$W_1(f_{\varepsilon,1,1}(t), \rho(t) \otimes \delta_{v=u(t)}) \xrightarrow[\varepsilon \to 0]{} 0. \tag{1.4.13}$$

Tous les résultats de convergence sont quantitatifs en T *et* ε*.*

1.4.4.2 Régime des particules légères et rapides

Dans le régime des particules légères, nous considérons $(\gamma, \sigma) = (1, \varepsilon^\alpha)$ dans (1.4.5) et obtenons à nouveau le système limite Transport-Navier-Stokes (1.4.7).

La différence principale avec le régime précédent vient du fait que la force de Brinkman n'est pas nécessairement bornée uniformément en ε au temps initial (du fait du facteur $1/\sigma$ dans la définition de $j_{\varepsilon,\gamma,\sigma}$). Il est cependant possible de démontrer qu'à cause de la friction, la vitesse des particules devient instantanément d'ordre 1 (si $\alpha < 1/2$). Dans le cas général, il y a donc une couche initiale dont la contribution, une fois intégrée en temps, devient négligeable lorsque $\varepsilon \to 0$.

Nous ne présentons toutefois pas ce résultat dans ce mémoire. En effet, sous une hypothèse de donnée bien préparée supplémentaire, il est possible de traiter de façon unifiée les régimes des particules *légères* et *légères et rapides*. Nous faisons ce choix afin d'alléger le Chapitre 3. Comme nous l'avons indiqué à la Section 1.4.3, le raisonnement repose sur l'estimation de la force de Brinkman dans $\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{T}^3))$. Dans le cas général du régime des particules légères et rapides, il convient de s'intéresser également aux espaces anisotropes $\mathrm{L}^{p'}(0,T;\mathrm{L}^p(\mathbb{T}^3))$, où $1/p + 1/p' = 1$. Les arguments étant très similaires, nous ne les reproduisons pas dans ce mémoire et renvoyons les lecteurs intéressés à la publication correspondante [HM21]. Nous démontrons ici les résultats suivants et renvoyons à nouveau les lecteurs au Chapitre 3 pour des énoncés précis.

Théorème 3 (Régime des particules légères et rapides)**.** *Soit* $(u_{\varepsilon,1,\varepsilon^\alpha}, f_{\varepsilon,1,\varepsilon^\alpha})$ *une solution faible globale de* (1.4.5) *associée à la donnée initiale* $(u^0_{\varepsilon,1,\varepsilon^\alpha}, f^0_{\varepsilon,1,\varepsilon^\alpha})$*. On suppose qu'il existe* $\kappa \in (0,1)$ *tel que, pour tout* $\varepsilon > 0$*,*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^p f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v) \, \mathrm{d}x \, \mathrm{d}v \lesssim \varepsilon^{\alpha p + \kappa - 1}. \tag{1.4.14}$$

A. *Cas* $\boldsymbol{\alpha < 1/2}$*.*

1. Cas général. *Sous l'Hypothèse 5, il existe* $T > 0$ *tel que*

$$\int_0^T W_1(f_{\varepsilon,1,\varepsilon^\alpha}(t), \rho(t) \otimes \delta_{v=0}) \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} 0 \tag{1.4.15}$$

et, pour tout $t \in [0,T]$*,*

$$\| u_{\varepsilon,1,\varepsilon^\alpha}(t) - u(t) \|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon \to 0]{} 0, \tag{1.4.16}$$

où (ρ, u) *est solution du système Transport-Navier-Stokes* (1.4.7)*.*
2. Cas moyennement bien préparé. *Sous l'Hypothèse supplémentaire 6, il existe* $\eta > 0$ *suffisamment petit tel que si*

$$\left\| u^0_{\varepsilon,1,\varepsilon^\alpha} - \langle u^0_{\varepsilon,1,\varepsilon^\alpha} \rangle \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

alors les convergences (1.4.15) *et* (1.4.16) *sont valables pour tout* $T > 0$*.*
3. Cas bien préparé. *Enfin, si l'on suppose de plus que*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v)\, \mathrm{d}x\, \mathrm{d}v \xrightarrow[\varepsilon \to 0]{} 0,$$

alors, pour tout $t > 0$*,*

$$W_1(f_{\varepsilon,1,\varepsilon^\alpha}(t), \rho(t) \otimes \delta_{v=0}) \xrightarrow[\varepsilon \to 0]{} 0. \tag{1.4.17}$$

Tous les résultats de convergence sont quantitatifs en T *et* ε*.*

B. ***Cas*** $\boldsymbol{\alpha = 1/2}$*.*

Les résultats précédents sont encore valables sous l'hypothèse supplémentaire, dans les cas général et moyennement bien préparé,

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f^0_{\varepsilon,1,\varepsilon^{1/2}}(x,v) \mathrm{d}x\, \mathrm{d}v \leq \eta,$$

pour $\eta > 0$ *suffisamment petit. De plus, les résultats sont quantitatifs sous l'hypothèse*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| f^0_{\varepsilon,1,\varepsilon^{1/2}} \mathrm{d}x\, \mathrm{d}v \xrightarrow[\varepsilon \to 0]{} 0.$$

Remarque 1. *Nous maintenons la distinction entre les cas général, moyennement bien préparé et bien préparé, pour suggérer au lecteur les résultats qu'il est possible d'obtenir sans l'Hypothèse* (1.4.14) *(*cf. *[HM21]), bien que le cas bien préparé soit, ici, identique à celui moyennement bien préparé.*

Notons que, sous l'Hypothèse 5, la condition (1.4.14) est automatiquement satisfaite pour $\alpha < 1/3$.

1.4.4.3 Le régime des particules fines

Dans le régime des particules fines, nous considérons $(\gamma, \sigma) = (1/\varepsilon, 1)$ dans (1.4.5) et obtenons comme limite le système de Navier-Stokes incompressible inhomogène (1.4.9). Ce régime se révèle être beaucoup plus singulier que les deux précédents. En conséquence, nous avons besoin d'hypothèses plus fortes sur la donnée initiale pour justifier les convergences. En particulier, nous n'obtenons plus le résultat sous la seule Hypothèse 5 et devons toujours considérer des données bien préparées. De plus, une condition de petitesse sur la fonction de distribution initiale s'avère nécessaire.

Théorème 4 (Régime des particules fines)**.** *Soit* $(u_{\varepsilon,\varepsilon,1}, f_{\varepsilon,\varepsilon,1})$ *une solution faible globale de* (1.4.5) *associée à la donnée initiale* $(u^0_{\varepsilon,\varepsilon,1}, f^0_{\varepsilon,\varepsilon,1})$*.*
1. Cas moyennement bien préparé. *Sous l'Hypothèse 5, il existe* ε_0*,* $\eta > 0$*,* $M' > 0$ *tels que si pour tout* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\left\| f^0_{\varepsilon,\varepsilon,1} \right\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{T}^3))} \leq \eta$$

et

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v-u^0_{\varepsilon,\varepsilon,1}(x)|^p}{\varepsilon^{p-1}} f^0_{\varepsilon,\varepsilon,1}(x,v)\mathrm{d}x\,\mathrm{d}v \leq M',$$

alors il existe $T>0$ tel que, pour tout $t\in[0,T]$,

$$W_1(f_{\varepsilon,\varepsilon,1}(t), \rho(t)\otimes\delta_{v=u(t)}) \xrightarrow[\varepsilon\to 0]{} 0, \tag{1.4.18}$$

et, pour tout $t\in[0,T]$,

$$\|u_{\varepsilon,1,1}(t)-u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \tag{1.4.19}$$

où (ρ,u) est solution du système de Navier-Stokes incompressible inhomogène (1.4.9)*.*
2. Cas bien préparé. *Sous l'Hypothèse supplémentaire 6, il existe $\eta'>0$ suffisamment petit tel que si*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \left| v - \frac{\langle j^0_{\varepsilon,\varepsilon,1}\rangle}{\langle \rho^0_{\varepsilon,\varepsilon,1}\rangle}\right|^2 f^0_{\varepsilon,\varepsilon,1}\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{T}^3} |u^0_{\varepsilon,\varepsilon,1} - \langle u^0_{\varepsilon,\varepsilon,1}\rangle|^2\mathrm{d}x + \left|\frac{\langle j^0_{\varepsilon,\varepsilon,1}\rangle}{\langle \rho^0_{\varepsilon,\varepsilon,1}\rangle} - \langle u^0_{\varepsilon,\varepsilon,1}\rangle\right|^2 \leq \eta,$$

alors les convergences (1.4.18) *et* (1.4.19) *sont valables pour tout $T>0$.*
Si l'on suppose que la solution (ρ,u) de (1.4.9) *est suffisamment régulière, alors les convergences* (1.4.18) *et* (1.4.19) *sont quantitatives en T et ε.*

Remarquons que les hypothèses des Théorèmes 2–4 sont comparables à celles de [HMM20] pour le comportement en temps long du système de Vlasov-Navier-Stokes sauf que nous imposons en plus dans l'Hypothèse 5 une borne dans l'espace de Besov $\mathrm{B}^{s,p}_p(\mathbb{T}^3)$. Nous ne pouvons en effet plus compter sur des estimations grossières sur les moments au voisinage de $t=0$ car de telles estimations divergent pour $\varepsilon\to 0$.

Chapitre 2

Système de Vlasov-Navier-Stokes avec effets hygroscopiques

Ce chapitre est dédié à l'étude théorique et numérique d'un modèle fluide-cinétique décrivant les effets hygroscopiques au sein des voies respiratoires. Nous y regroupons les travaux suivants :

[BMM20] publié dans *Mathematical Models and Methods in Applied Sciences*, rédigé en collaboration avec Laurent Boudin et Ayman Moussa,

[BM20] accepté pour publication dans *Journal of Computational and Theoretical Transport*, rédigé en collaboration avec Laurent Boudin.

Sommaire du présent chapitre

2.1 Introduction

The motion of a dispersed phase made of small particles inside a fluid can be described through a fluid-kinetic model (see [ORo81; Wil85]). The unknowns corresponding to the fluid are the usual macroscopic quantities such as the velocity and pressure while the dispersed phase is described by a distribution function. Under the so-called *thin spray* hypothesis, [Des10] proposes a model in which the fluid and kinetic equations are coupled through a drag term. It depends on the fluid unknowns as well as the distribution function and describes the energy and momentum exchanges between the two phases.

Here, we focus on the variation of the size of the particles due to the evaporation of the water they contain or, on the contrary, to the condensation of the water vapor in the air around the particle. As in [LH11], we consider a fluid-kinetic model that takes into account the variation of temperature of the fluid due to the transport of the particles, whose own temperature can also vary. In the kinetic part of the model, the aerosol distribution function has then not only the usual time, space and velocity variable set, but also the particle size and temperature, inducing new difficulties to be dealt with.

The model studied in this chapter describes the transport and deposition of a therapeutic aerosol evolving in a Newtonian viscous incompressible fluid, the air, in the first generations of the human bronchial tree. The mathematical study of this phenomenon related to respiration began with [GCC02] and [Bar+05] and the system we analyze in this paper was presented in [Bou+20] in a fixed domain. Numerous works have contributed to this study for several decades while taking into account various parameters and physical processes such as the compressibility or viscosity of the fluid, the transport of the particles, the interaction between the particles and the fluid or the walls, *etc.* But, up to our knowledge, the existence of a solution taking into account the variation of the size and temperature of the particles has not been studied yet.

Indeed, the following contributions all assume that the effects of the variation of the size of the particles are negligible. In a first contribution to the case of a compressible fluid, the local-in-time existence of classical solutions is proven in [BD06]. The same kind of result is obtained in [Mat10] for the Euler-Vlasov-Boltzmann system. In [MV07], the authors prove the global existence of weak solutions to the compressible Navier-Stokes-Vlasov-Fokker-Planck system. The global existence of solutions around the equilibrium is studied in [CKL13] and [LMW15] and an asymptotic analysis is conducted in [MV08]. More recently, under the assumption of existence of local weak solutions, the study of a hydrodynamic limit for a system of the type Navier-Stokes-Vlasov-Fokker-Planck in the compressible case is conducted in [CJ19].

The study of the Vlasov-Navier-Stokes system in the case of an incompressible fluid began with [AB97] and [Ham98]. Existence of weak solutions is proven in [Bou+09] in a periodic framework. This result is extended to a bounded domain by [Yu13] and then to a time-dependent bounded domain by [BGM17]. In dimension two, [GHM18] provides the existence and stability of regular equilibria for the Vlasov-Navier-Stokes system, and the uniqueness of weak solutions in the whole space or a periodic domain is obtained in [Han+20]. More recently, the asymptotic behavior of the weak solutions of the Vlasov-Navier-Stokes system in a periodic framework in two or three dimensions is studied in [HMM20].

In [DV05], the authors propose to simulate the variation of the particle size due to the evaporation or condensation of the water thanks to a multi-fluid model which relies on equations for the first and second moments of the distribution function. Another static point of view is assumed in [LH11] to tackle these hygroscopic effects : they are considered globally in space and are described by a differential equation on the mass fraction of the water vapor in the air. This model is made more precise in [Bou+20] by taking into account the fact that these effects are local and time-dependent. The aim of this chapter is twofold : first, we prove the existence of weak solutions for this system of partial differential equations ; we then present an extension to the three-dimensional case of the numerical results obtained in two dimensions by [Bou+20].

The work presented in this chapter is therefore a natural continuation of the contributions above, and especially of [BGM17 ; Bou+20]. In the theoretical part of this chapter, we follow the same penalization strategy (though another classical approximation strategy) as [BGM17] to account for the time-dependence of the domain when dealing with the Navier-Stokes equations. For the Vlasov equation, we apply existence and stability results of DiPerna-Lions type (see [DL89]) which the authors of [BGM17] obtain. The approximation procedure requires to prove existence and uniqueness for convection-diffusion equations in a time-dependent domain using strong compactness results from [Mou16]. As opposed to [BGM17], we were not able to prove interpolation estimates on the moments of the distribution function and therefore we have to assume that there exists a minimal radius for the particles. This hypothesis consists in considering that all the spherical particles are composed of a dry core with common radius (containing the drug) that is surrounded by water. We prove that if all the particles at initial time have this property, then the evaporation of water is not complete and the radius of any given particle remains greater than that of the dry core. It allows to derive fruitful energy estimates linking all the unknowns and to use the same arguments as in [BGM17].

This chapter is organized as follows. We begin by presenting the simplified model and the main result, and describing the approximation strategy we implement. In Section 2.3, we recover existence for the approximated system by a fixed-point method, using the Schauder theorem. In order to invoke the latter, we recall results on the Vlasov and Navier-Stokes equations and prove the existence and uniqueness of the global solution to a convection-diffusion equation with Neumann boundary condition in a time-dependent domain. We conclude the proof of the main result in Section 2.4, which consists in passing to the limit in the approximated problem in order to obtain a solution to the initial one. In Section 2.5, we discuss how to implement in three dimensions the numerical scheme introduced in [Bou+20] in two dimensions. Section 2.6 is dedicated to the presentation of numerical experiments for the full model and the investigation of potential simplifications.

2.2 Presentation of the model

We begin by presenting the model we investigate in this chapter. It extends and simplifies the one proposed in [Bou+20] to describe the behavior of an aerosol in the respiratory system to a time-dependent domain.

Let $\tau > 0$ and $\Omega_0 \subset \mathbb{R}^3$ be an open bounded domain with Lipschitz boundary. The variation of the spatial domain is taken into account by means of a mapping $\mathcal{A} \in \mathscr{C}^2(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3)$, $(t, x) \mapsto \mathcal{A}(t, x) = \mathcal{A}_t(x)$ such that, for all $t \geq 0$, $\mathcal{A}_t$ is a $\mathscr{C}^1$-diffeomorphism and $\mathcal{A}_0 = \mathrm{Id}_{\mathbb{R}^3}$. For every $0 \leq t \leq \tau$, we set $\Omega_t = \mathcal{A}_t(\Omega_0)$ the bounded domain at time t and

$$\widehat{\Omega}_t = \bigcup_{0<s<t} \{s\} \times \Omega_s.$$

For $t = \tau$, we simply write $\widehat{\Omega} = \widehat{\Omega}_\tau$. Furthermore, let

$$\widehat{\Gamma} = \bigcup_{0<t<\tau} \{t\} \times \partial\Omega_t$$

and, for all $t \in [0, \tau]$, n_t be the outgoing unit normal vector field of $\partial\Omega_t$. We also use the Eulerian velocity w associated to the flow $t \mapsto \mathcal{A}_t$, defined by

$$\forall (t, x) \in [0, \tau] \times \mathbb{R}^3, \qquad w(t, \mathcal{A}_t(x)) = \partial_t \mathcal{A}(t, x).$$

We shall assume that $\mathrm{div}_x\, w = 0$, so that the Jacobian of the transformation $\mathcal{A}_t$ does not depend on t and is therefore constant equal to 1. Finally, we need to consider the phase space and its boundaries. For any $t \in [0, \tau]$, let us set

$$\Pi_t = \Omega_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Pi}_t = \widehat{\Omega}_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

$$\Sigma_t = \partial\Omega_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Sigma}_t = \bigcup_{0<s<t} \{s\} \times \partial\Omega_s \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

$$\widehat{\Pi} = \widehat{\Omega} \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \qquad \widehat{\Sigma} = \widehat{\Gamma} \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*.$$

and

$$\widehat{\Sigma}^{\pm} = \{(t, x, v, r, T) \in \widehat{\Sigma},\ \pm(v - w(t, x)) \cdot n_t(x) > 0\},$$
$$\widehat{\Sigma}^{0} = \{(t, x, v, r, T) \in \widehat{\Sigma},\ (v - w(t, x)) \cdot n_t = 0\},$$

The particles are composed of an active substance and water. and we neglect the excipient introduced in Chapter 1 ($r_{\text{ex}} = 0$). More precisely, we suppose that the active substance lies in a ball the radius of which, denoted by $r_{\text{drug}} > 0$, does not change. We assume that this ball is surrounded by water and the whole creates a ball of radius r. In particular, all the particles have a radius greater than r_{drug}. If we denote by, respectively, ρ, ρ_{w} and ρ_{drug} the mass densities of a particle, water and drug, the mass conservation at the microscopic level writes

$$\rho r^3 = \rho_{\text{w}}(r^3 - r_{\text{drug}}^3) + \rho_{\text{drug}} r_{\text{drug}}^3.$$

In the rest of this work, we choose $\rho_w = 1$ and $\rho_{\text{drug}} = 2$. Consequently, the radius dependent mass of a particle becomes, up to a multiplication by a positive constant,

$$m(r) = r^3 + r_{\text{drug}}^3. \tag{2.2.1}$$

In addition to the variation of radius, we consider that the temperature of the particles may vary. Therefore, a particle is characterized at time $t \geq 0$ by its position $x \in \Omega_t$, its velocity $v \in \mathbb{R}^3$, its radius $r > r_{\text{drug}}$ and its temperature $T > 0$. Consequently, we describe the aerosol by a density function $f(t, x, v, r, T)$ which solves the Vlasov-like equation

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v (Af) + \partial_r(af) + \partial_T(bf) = 0 \quad \text{in } \widehat{\Pi},$$

where A is the drag acceleration exerted by the air on the particles, a describes the variation of the radii of the particles and b describes the variation of their temperatures. We prescribe the following absorption condition :

$$f = 0 \quad \text{on } \widehat{\Sigma}^-,$$

which means that all the particles reaching the physical boundary are deposited.

We classically assume the air to be a viscous Newtonian incompressible fluid [Bou+15]. It can therefore be described by its pressure $p(t, x)$ and velocity $u(t, x)$, which satisfy the incompressible Navier-Stokes equations, with constant mass density and viscosity both taken equal to 1,

$$\partial_t u + (u \cdot \nabla_x)\, u + \nabla_x p - \Delta_x u = F \quad \text{in } \widehat{\Omega},$$

$$\operatorname{div}_x u = 0 \quad \text{in } \widehat{\Omega},$$

where F is the Brinkman force applied by the aerosol on the fluid. We prescribe the Dirichlet boundary condition :

$$u = w \quad \text{on } \widehat{\Gamma}.$$

The coupling terms A and F are given by the Stokes law [Bou+15] as

$$A(t,x,v,r) = \frac{r}{r^3 + r_{\text{drug}}^3}(u(t,x) - v)$$

and

$$F(t,x) = -\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(u(t,x)-v)f(t,x,v,r,T)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}T.$$

The variation of the radius of a particle stems from the evaporation of the water it contains or the condensation of the water vapor from the surrounding air. If the water vapor mass fraction at the surface of the particle $Y_{v,\text{surf}}(r,T)$ is smaller than the water vapor mass fraction in the air $Y(t,x)$, then a part of the surrounding vapor condensates and the radius of the particle increases. In the opposite case, a part of the water contained in the particle evaporates and the radius decreases. More precisely, [Bou+20, formulas (6)–(13)] provide the following expression for the function a, when all physical parameters are normalized :

$$a(t,x,r,T) = \frac{Y(t,x) - Y_{v,\text{surf}}(r,T)}{r}.$$

Note that, given the expression for $Y_{v,\text{surf}}$ in [Bou+20], we have $Y_{v,\text{surf}}(r_{\text{drug}},\cdot) = 0$.

The evaporation and condensation of water also give rise to heat fluxes between the air and the particles, which result in temperature variation. There is also a convective heat flux. Again, [Bou+20] gives the following expression for b :

$$b(t,x,r,T) = \frac{Y(t,x) - Y_{v,\text{surf}}(r,T)}{r^2} + \frac{\Theta(t,x) - T}{r^2},$$

where Θ is the air temperature.

The water vapor in the air is subject to transport and diffusion phenomena and also interacts with the particles. Therefore, we assume that Y solves the following convection-diffusion equation in $\widehat{\Omega}$

$$\partial_t Y + u\cdot\nabla_x Y - \operatorname{div}_x\left(D_v(\Theta)\nabla_x Y\right) = -\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r^2 a f,$$

where $D_v(\Theta)$ is the diffusion coefficient of water vapor in the air at temperature Θ. In fact, we can assume $D_v \equiv 1$. Indeed, the following analysis remains true as long as $D_v \in \mathrm{L}^\infty(\mathbb{R}_+)$ and D_v is bounded from below by a positive constant, and those assumptions are physically relevant in the temperature range for which the model is expected to hold. Note that in [Bou+20], the numerical simulations lead to a variation of D_v of only 2% and that in Section 2.5, we shall obtain that this variation has almost no influence on Y. We prescribe the Neumann boundary condition.

$$\nabla_x Y\cdot n_t = 0 \quad \text{on } \widehat{\Gamma}.$$

The variation of the air temperature is also described by a convection-diffusion equation in $\widehat{\Omega}$:

$$\partial_t\Theta + u\cdot\nabla_x\Theta - \Delta_x\Theta = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(T-\Theta)f,$$

with the Neumann boundary condition :

$$\nabla_x\Theta\cdot n_t = 0 \quad \text{on } \widehat{\Gamma}.$$

To summarize, the system under study in the first part of this chapter is the following one :

$$\partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\left(\frac{r}{r^3+r_{\mathrm{drug}}^3}(u-v)f\right) + \partial_r\left(\frac{Y-Y_{v,\mathrm{surf}}}{r}f\right) + \partial_T\left(\left(\frac{Y-Y_{v,\mathrm{surf}}}{r^2}+\frac{\Theta-T}{r^2}\right)f\right) = 0 \quad \text{in } \widehat{\Pi}, \tag{2.2.2}$$

$$f = 0 \quad \text{on } \widehat{\Sigma}^-, \tag{2.2.3}$$

$$\partial_t u + (u\cdot\nabla_x)\,u + \nabla_x p - \Delta_x u = -\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r(u-v)f \quad \text{in } \widehat{\Omega}, \tag{2.2.4}$$

$$\mathrm{div}_x\, u = 0 \quad \text{in } \widehat{\Omega}, \tag{2.2.5}$$

$$u = w \quad \text{on } \widehat{\Gamma}, \tag{2.2.6}$$

$$\partial_t Y + u\cdot\nabla_x Y - \Delta_x Y + \left(\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} rf\right) Y = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} rY_{v,\mathrm{surf}} f, \tag{2.2.7}$$

$$\nabla_x Y\cdot n_t = 0 \quad \text{on } \widehat{\Gamma}. \tag{2.2.8}$$

$$\partial_t\Theta + u\cdot\nabla_x\Theta - \Delta_x\Theta + \left(\int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} rf\right)\Theta = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} rTf, \tag{2.2.9}$$

$$\nabla_x\Theta\cdot n_t = 0 \quad \text{on } \widehat{\Gamma}. \tag{2.2.10}$$

It is supplemented with initial conditions for f, u, Y, and Θ :

$$f(0,\cdot,\cdot,\cdot) = f^0 \quad \text{in } \Pi_0, \tag{2.2.11}$$

$$u(0,\cdot) = u^0 \quad \text{in } \Omega_0, \tag{2.2.12}$$

$$Y(0,\cdot) = Y^0 \quad \text{in } \Omega_0, \tag{2.2.13}$$

$$\Theta(0,\cdot) = \Theta^0 \quad \text{in } \Omega_0. \tag{2.2.14}$$

Remark 2.2.1. *Note that, since f is nonnegative, solutions Y and Θ to the previous convection-diffusion equations satisfy, at least formally, the weak maximum principle. Since $0 \le Y_{v,\mathrm{surf}} \le 1$, we obtain $0 \le Y \le 1$ and $\Theta \ge 0$ for all time $t \in [0,\tau]$ if we assume $0 \le Y^0 \le 1$ and $\Theta^0 \ge 0$.*

Let us now formally compute an energy equality relating the energy dissipation and the exchanges between the air and the particles. Recall the Reynolds formula, for any real-valued function $k : \widehat{\Omega} \to \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_t} k = \int_{\Omega_t}\partial_t k + \int_{\partial\Omega_t} k\, w\cdot n_t.$$

Following the argument in [BGM17], we multiply (2.2.4) by $u-w$ and integrate by parts over $\widehat{\Omega}_t$ for a fixed value of $t \in [0,\tau]$. Similarly, we multiply (2.2.2) by $m(r)|v|^2/2$ and integrate by parts over $\widehat{\Pi}_t$. Using (2.2.1), (2.2.5)–(2.2.6) and (2.2.11)–(2.2.12), we obtain

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_t}|u|^2 &+ \frac{1}{2}\iint_{\Pi_t}(r^3+r_{\mathrm{drug}}^3)|v|^2 f + \iint_{\widehat{\Omega}_t}|\nabla_x u|^2 + \iiint_{\widehat{\Pi}_t} r|u-v|^2 f \\
&= \frac{1}{2}\int_{\Omega_0}|u^0|^2 + \frac{1}{2}\iint_{\Pi_0}(r^3+r_{\mathrm{drug}}^3)|v|^2 f^0 + \int_{\Omega_t} u(t)\cdot w(t) - \int_{\Omega_0} u^0\cdot w(0) \\
&\quad - \iint_{\widehat{\Omega}_t} u\cdot\partial_s w - \iint_{\widehat{\Omega}_t}\left((u\cdot\nabla_x)\,w\right)\cdot u + \iint_{\widehat{\Omega}_t}\nabla_x u : \nabla_x w \\
&\quad + \iiint_{\widehat{\Pi}_t} r\, w\cdot(u-v) f + \frac{3}{2}\iiint_{\widehat{\Pi}_t} r(Y - Y_{v,\mathrm{surf}}) f \\
&\quad + \frac{1}{2}\iiint_{\widehat{\Sigma}_t}(r^3+r_{\mathrm{drug}}^3)|v|^2 f(w-v)\cdot n_t,
\end{aligned}$$

We assume that the water surrounding the active substance does not entirely evaporate. Therefore, the radii of the particles are bounded from below by r_{drug}. This translates into the following assumption :

$$f(\cdot,\cdot,\cdot,r,\cdot) = 0, \qquad \text{a.e } r < r_{\mathrm{drug}}. \tag{2.2.15}$$

Therefore, if f is nonnegative, since $0 \le Y_{v,\mathrm{surf}}, Y \le 1$, we have

$$\iiint_{\widehat{\Pi}_t} r\, w\cdot(u-v) f + \iiint_{\widehat{\Pi}_t} r(Y - Y_{v,\mathrm{surf}}) f \le \frac{1}{2}\iiint_{\widehat{\Pi}_t} r|u-v|^2 f + \frac{\|w\|_{\mathrm{L}^\infty(\widehat{\Omega})}+2}{r_{\mathrm{drug}}^2}\iiint_{\widehat{\Pi}_t} r^3 f.$$

But, if we multiply (2.2.2) by r^3 and integrate over $\widehat{\Pi}_t$, thanks to the absorption condition (2.2.3) and the Grönwall lemma, we obtain

$$\iint_{\Pi_t} r^3 f \lesssim 1, \tag{2.2.16}$$

where the notation $\lesssim$ indicates that the inequality holds up to a multiplicative constant which can depend on τ, w, r_{drug} and the initial data. Using again the absorption condition (2.2.3), we deduce from the previous estimations and the Grönwall lemma again that

$$\frac{1}{2}\int_{\Omega_t}|u|^2 + \iint_{\Pi_t}(r^3+r_{\mathrm{drug}}^3)|v|^2 f + \iint_{\widehat{\Omega}_t}|\nabla_x u|^2 + \iiint_{\widehat{\Pi}_t} r|u-v|^2 f \lesssim 1. \tag{2.2.17}$$

Let us now derive an estimate relating the air temperature and the particle distribution. We multiply (2.2.9) by Θ and integrate by parts over $\widehat{\Omega}_t$, for fixed $t\in(0,\tau)$. Similarly, we multiply (2.2.2) by $r^3T^2/2$ and integrate by parts over $\widehat{\Pi}_t$. Using (2.2.5)–(2.2.6) and (2.2.10))–(2.2.11) as well as (2.2.14), we get

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_t}|\Theta|^2 &+ \frac{1}{2}\iint_{\Pi_t} r^3T^2 f + \iint_{\widehat{\Omega}_t}|\nabla_x\Theta|^2 + \iiint_{\widehat{\Pi}_t} r|\Theta - T|^2 f \\
&\le 3\iiint_{\widehat{\Pi}_t} rT^2 f + \iiint_{\widehat{\Pi}_t} rTf + \int_{\Omega_0}|\Theta^0|^2 + \iint_{\Pi_0} r^3T^2 f^0.
\end{aligned}$$

Thanks to (2.2.15), the Cauchy-Schwarz inequality, and (2.2.16), the Grönwall lemma yields

$$\frac{1}{2}\int_{\Omega_t}|\Theta|^2 + \frac{1}{2}\iint_{\Pi_t} r^3T^2 f + \iint_{\widehat{\Omega}_t}|\nabla_x\Theta|^2 + \iiint_{\widehat{\Pi}_t} r|\Theta - T|^2 f \lesssim 1.$$

The previous estimates and the boundary conditions motivate the introduction of the following functional spaces. For $p, q \in [1, +\infty]$ and $m \in \mathbb{N}$, denote

$$\mathrm{L}^p(0,\tau;W^{m,q}(\Omega_t)) = \Big\{\zeta \text{ measurable, } \|\zeta\|_{\mathrm{L}^p(0,\tau;W^{m,q}(\Omega_t))} < \infty\Big\}$$

where we set

$$\|\zeta\|_{\mathrm{L}^p(0,\tau;W^{m,q}(\Omega_t))} = \Big\| t \mapsto \|\zeta(t)\|_{W^{m,q}(\Omega_t)} \Big\|_{\mathrm{L}^p(0,\tau)},$$

and

$$\begin{aligned}
\mathcal{V}_0 &= \Big\{\varphi \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t)),\ \mathrm{div}_x\, \varphi = 0,\ \varphi = 0 \text{ on } \widehat{\Gamma}\Big\},\\
\mathscr{V} &= \Big\{\varphi \in \mathscr{C}^1(\overline{\widehat{\Omega}}),\ \mathrm{div}_x\, \varphi = 0 \text{ in } \widehat{\Omega},\ \varphi = 0 \text{ on } \widehat{\Gamma},\ \varphi(\tau) = 0\Big\},\\
\mathscr{X} &= \Big\{\zeta \in \mathscr{C}^1(\overline{\widehat{\Omega}}),\ \zeta = 0 \text{ on } \widehat{\Gamma},\ \zeta(\tau) = 0\Big\},
\end{aligned}$$

and, denoting by $\overline{g}$ the extension on $[0,\tau] \times \mathbb{R}^3$ by zero of a function g defined on $\widehat{\Omega}$, let

$$\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\Pi_t)) = \{f \text{ measurable, } \overline{f} \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*))\},$$

$$\mathscr{W} = \Big\{\psi \in \mathscr{C}_c^1(\overline{\widehat{\Omega}} \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*),\ \psi = 0 \text{ on } \widehat{\Sigma}^+ \cup \widehat{\Sigma}^0,\ \psi(\tau) = 0\Big\}.$$

Let us now state the assumptions on the initial data.

Assumption 2.1. *$f^0 \in \mathrm{L}^\infty(\Pi_0)$ is nonnegative, and for almost every $r \le r_{\mathrm{drug}}$, $f^0(\cdot,\cdot,r,\cdot) = 0$, and*

$$\left(1 + r^3 + r^3|v|^2 + \left|Y^0 - Y_{v,\mathrm{surf}}\right|\left(\frac{1}{r} + \frac{1}{r^2}\right) + \frac{|\Theta^0 - T|}{r^2}\right) f^0 \in \mathrm{L}^1(\Pi_0).$$

Assumption 2.2. *$u^0 \in \mathrm{L}^2(\Omega_0)$ and $\mathrm{div}_x\, u^0 = 0$.*

Assumption 2.3. *$Y^0 \in \mathrm{L}^2(\Omega_0)$ and $0 \le Y^0 \le 1$ in Ω_0.*

Assumption 2.4. *$\Theta^0 \in \mathrm{L}^2(\Omega_0)$ and $\Theta^0 > 0$ in Ω_0.*

Let us now define the notion of weak solution of the problem.

Definition 2.2.2. *We say that a 4-tuple (u, Y, Θ, f) is a weak solution of the system (2.2.4)–(2.2.10) with initial data $(u^0, Y^0, \Theta^0, f^0)$ if the following conditions are satisfied. The distribution function must verify :*

- $\overline{f} \in \mathrm{L}^\infty((0,\tau) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \mathscr{C}^0([0,\tau];\mathrm{L}^p(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*))$ *for all* $p \in [1,\infty)$,
- $(1 + r^3 + r^3|v|^2) f \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^1(\Pi_t))$,
- $(Y - Y_{v,\mathrm{surf}})\left(\frac{1}{r} + \frac{1}{r^2}\right) f \in \mathrm{L}^1_{\mathrm{loc}}(\widehat{\Pi})$,
- $\frac{\Theta - T}{r^2} f \in \mathrm{L}^1_{\mathrm{loc}}(\widehat{\Pi})$.

The fluid quantities must satisfy :

- $u \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$,
- $u - w \in \mathcal{V}_0$,
- $Y \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$,
- $\Theta \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$.

Finally, the following weak formulations must hold for any $\psi \in \mathscr{W}$, $\varphi \in \mathscr{V}$, and $\zeta \in \mathscr{X}$:

$$\iiint_{\widehat{\Pi}} f \left(\partial_t \psi + v \cdot \nabla_x \psi + \frac{u - v}{r^2 + \frac{r_{\mathrm{drug}}^3}{r}} \cdot \nabla_v \psi \right) + \iiint_{\widehat{\Pi}} f \left(\frac{Y - Y_{v,\mathrm{surf}}}{r} \left(\partial_r \psi + \frac{1}{r} \partial_T \psi \right) + \frac{\Theta - T}{r^2} \partial_T \psi \right) = - \iint_{\Pi_0} f^0 \psi(0, \cdot), \quad (2.2.18)$$

$$\iint_{\widehat{\Omega}} (u \cdot \partial_t \varphi + (u \otimes u) : \nabla_x \varphi - \nabla_x u : \nabla_x \varphi) = - \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r(v - u) f \right) \cdot \varphi - \int_{\Omega_0} u^0 \cdot \varphi(0, \cdot), \quad (2.2.19)$$

$$\iint_{\widehat{\Omega}} (-Y \partial_t \zeta + \zeta u \cdot \nabla_x Y + \nabla_x Y \cdot \nabla_x \zeta) = \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r(Y_{v,\mathrm{surf}} - Y) f \right) \zeta + \int_{\Omega_0} Y^0 \, \zeta(0, \cdot), \quad (2.2.20)$$

$$\iint_{\widehat{\Omega}} (-\Theta \partial_t \zeta + \zeta u \cdot \nabla_x \Theta + \nabla_x \Theta \cdot \nabla_x \zeta) = \iint_{\widehat{\Omega}} \left(\int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r(T - \Theta) f \right) \zeta + \int_{\Omega_0} \Theta^0 \, \zeta(0, \cdot). \quad (2.2.21)$$

The main result of this paper can now be properly stated.

Theorem 2.2.3. *Under Assumptions 2.2–2.1, there exists a weak solution to Problem (2.2.4)–(2.2.10) in the sense of Definition 2.2.2.*

We follow the same strategy as in [BGM17]. Namely, we solve an approximated system by a fixed-point procedure and recover a solution to the initial problem through compactness properties.

In order to obtain similar estimates as the ones we formally derived above, we need to ensure that f vanishes for radii below some positive value. Therefore, we introduce a function $\eta \in \mathscr{C}^\infty(\mathbb{R}_+)$ such that $\eta \equiv 0$ on $[0, r_{\mathrm{drug}}/2]$ and $\eta \equiv 1$ on $[r_{\mathrm{drug}}, +\infty)$, with $\eta' \geq 0$.

To solve the Navier-Stokes equations in a cylindrical domain, we follow a penalization procedure. We then use the Friedrichs approximation to solve the equations. Let $\pi_{\mathrm{L}} : \mathrm{L}^2(\mathbb{R}^3) \to \{u \in \mathrm{L}^2(\mathbb{R}^3),\, \mathrm{div}_x\, u = 0\}$ be the Leray projection. We choose $N \in \mathbb{N}$ and let P_N be the orthogonal projection

$$P_N : \mathrm{L}^2(\mathbb{R}^3) \to \mathscr{F}_N = \{u \in \mathrm{L}^2(\mathbb{R}^3),\, \widehat{u}(\xi) = 0,\, \forall |\xi| \geq N\},$$

where $\widehat{u}$ denotes the Fourier transform of u. We also need to truncate the right-hand side of (2.2.4), as in [BGM17], to apply standard results for the Friedrichs approximation. To preserve the energy estimate (2.2.17), we perform the same truncation in (2.2.2). Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be an odd, increasing, bounded function, with $0 \leq \chi(v) \leq v$ for all $v \geq 0$. We write $\chi(v) = (\chi(v_1), \chi(v_2), \chi(v_3))$ for any $v \in \mathbb{R}^3$. Furthermore, we also need to truncate the variation of temperature that appears in (2.2.2) in order to solve this equation.

Unfortunately, the penalization strategy used for solving the Navier-Stokes equation does not preserve the boundary condition $u = w$. Therefore, we have to modify the velocity field for the convection-diffusion equations solved by Y and Θ. For $t \in [0, \tau]$, define the projection

$$\pi_{\Omega_t} : \mathrm{L}^2(\Omega_t) \to \left\{ u \in \mathrm{H}^1(\Omega_t),\, \mathrm{div}_x\, u = 0,\, u_{|\partial\Omega_t} = 0 \right\}.$$

We also need the sets

$$K_N = \left\{ (t,x) \in \widehat{\Omega},\, d((t,x), \widehat{\Gamma}) < 1/N \right\},$$

where $d(\cdot, \widehat{\Gamma})$ denotes the Euclidean distance to the boundary $\widehat{\Gamma}$, defined on $\mathbb{R}_+ \times \mathbb{R}^3$.

We can then consider the following approximated problem :

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,(G_1 f) + \partial_r\,(G_2 f) + \partial_T\,(G_3 f) = 0 \qquad \text{in } \widehat{\Pi}, \tag{2.2.22}$$

$$\partial_t u + P_N \pi_{\mathrm{L}} (\mathrm{div}_x (u \otimes u)) - \Delta_x u + N P_N \pi_{\mathrm{L}} \left((u - w) \mathbf{1}_{\widehat{\Omega}^c} \right) = P_N \pi_{\mathrm{L}} F \qquad \text{in } (0,\tau) \times \mathbb{R}^3, \tag{2.2.23}$$

$$\mathrm{div}_x\, u = 0 \qquad \text{in } (0,\tau) \times \mathbb{R}^3, \tag{2.2.24}$$

$$\partial_t Y + z_N \cdot \nabla_x Y - \Delta_x Y + cY = S_Y \qquad \text{in } \widehat{\Omega}, \tag{2.2.25}$$

$$\partial_t \Theta + z_N \cdot \nabla_x \Theta - \Delta_x \Theta + c\Theta = S_T \qquad \text{in } \widehat{\Omega}, \tag{2.2.26}$$

where

$$F = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} \chi(v-u) r\eta \overline{f}, \qquad z_N = w + \pi_{\Omega_t} \left((u-w) \mathbf{1}_{K_N} \right),$$

$$c = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r\eta f, \qquad S_Y = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} Y_{v,\mathrm{surf}}\, r\eta f$$

$$S_T = \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} rT\eta f, \qquad G_1 = \chi(u-v) \frac{\eta}{r^2 + \frac{r_{\mathrm{drug}}^3}{r}},$$

$$G_2 = \frac{\overline{Y} - Y_{v,\mathrm{surf}}}{r} \eta, \qquad G_3 = \frac{1}{r} G_2 + \frac{\chi(\overline{\Theta} - T)}{r^2} \eta.$$

This system is completed with the boundary conditions (2.2.3) for f, (2.2.8) for Y, and (2.2.10) for Θ. Furthermore, f, Y, Θ and u satisfy, respectively, the initial conditions (2.2.11), (2.2.13), (2.2.14), and $u(0) = P_N \overline{u^0}$.

We make the same Assumption 2.2 on u^0 but need stronger assumptions on the other initial data.

Assumption 2.5. $f^0 \in \mathrm{L}^\infty(\Pi_0)$ *is nonnegative and compactly supported and, for almost every* $r < r_{\mathrm{drug}}$*,* $f^0(\cdot,\cdot,r,\cdot) \equiv 0$*, and*

$$\left(1 + r^3 + r^3|v|^2 + \left| Y^0 - Y_{v,\mathrm{surf}} \right| \left(\frac{1}{r} + \frac{1}{r^2} \right) + \frac{|\Theta^0 - T|}{r^2} \right) f^0 \in \mathrm{L}^1(\Pi_0).$$

Assumption 2.6. $Y^0 \in \mathrm{H}^1(\Omega_0)$ *and* $0 \leq Y^0 \leq 1$ *in* Ω_0*.*

Assumption 2.7. $\Theta^0 \in \mathrm{H}^1(\Omega_0)$ *and* $\Theta^0 \geq 0$ *in* Ω_0*.*

Existence of a solution for (2.2.22)–(2.2.26) with conditions (2.2.3), (2.2.8) and (2.2.10), and initial data satisfying Assumptions 2.2 and 2.5–2.7 is obtained thanks to the Schauder fixed-point theorem that we recall here in an appropriate setting (see [Bon62]).

Theorem 2.2.4. *Let E be a real normed vector space and $\mathscr{C}$ a closed convex nonempty subset of E. If $\Lambda : \mathscr{C} \to \mathscr{C}$ is a continuous map such that $\Lambda(\mathscr{C})$ is a precompact subset of E, then Λ has a fixed point.*

In the following, we use

$$E = \big\{(u,Y,\Theta) \mid u \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\mathbb{R}^3) \cap \mathscr{C}^0([0,\tau];\mathrm{L}^2(\mathbb{R}^3)),\\ Y \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t) \cap \mathscr{C}^0([0,\tau];\mathrm{L}^2(\Omega_t))),\ \Theta \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t) \cap \mathscr{C}^0([0,\tau];\mathrm{L}^2(\Omega_t)))\big\}$$

and we first set

$$\mathscr{C} = \{(u,Y,\Theta) \in E \mid 0 \leq Y \leq 1\},$$

the real convex to which we apply the theorem being in fact a subset of $\mathscr{C}$, as described below.

The vector-space E is endowed with the following norm. Let $\gamma \in \mathrm{L}^2(\mathbb{R}^3)$. We define the norm $\|.\|_E$ by, for all $(u,Y,\Theta) \in E$,

$$\|(u,Y,\Theta)\|_E = \|Y\|_{\mathrm{L}^2(\widehat{\Omega})} + \|\Theta\|_{\mathrm{L}^2(\widehat{\Omega})} + \sup_{t\in[0,\tau]} \|u(t,\cdot)\gamma\|_{\mathrm{L}^1(\mathbb{R}^3)}\,.$$

Note that thanks to γ, when dealing with the convergence of (u_n, Y_n, Θ_n) in E, we will only have to prove convergence of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(B))$ for a ball B, as long as the sequence is bounded in $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))$.

Let us apply Theorem 2.2.4 to the map $\Lambda : \mathscr{C} \to \mathscr{C}$, the image of a triplet $(u,Y,\Theta) \in \mathscr{C}$ being the only triplet $(\widetilde{u}, \widetilde{Y}, \widetilde{\Theta})$ satisfying the following systems of equations :

$$\partial_t \widetilde{u} + P_N \pi_{\mathrm{L}}(\mathrm{div}_x(\widetilde{u} \otimes \widetilde{u})) - \Delta_x \widetilde{u} + N P_N \pi_{\mathrm{L}}\left((\widetilde{u} - w)\mathbf{1}_{\widehat{\Omega}^c}\right) = P_N \pi_{\mathrm{L}} F_{u,Y,\Theta}, \tag{2.2.27}$$
$$\mathrm{div}_x\, \widetilde{u} = 0, \tag{2.2.28}$$
$$u_{|t=0} = P_N u^0 \tag{2.2.29}$$

in $(0,\tau) \times \mathbb{R}^3$,

$$\partial_t \widetilde{Y} + \widetilde{z}_N \cdot \nabla_x \widetilde{Y} - \Delta_x \widetilde{Y} + c_{f_{u,Y,\Theta}} \widetilde{Y} = S_{Y, f_{u,Y,\Theta}}, \tag{2.2.30}$$
$$\nabla_x \widetilde{Y} \cdot n_t = 0, \tag{2.2.31}$$
$$Y_{|t=0} = Y^0, \tag{2.2.32}$$

and

$$\partial_t \widetilde{\Theta} + \widetilde{z}_N \cdot \nabla_x \widetilde{\Theta} - \Delta_x \widetilde{\Theta} + c_{f_{u,Y,\Theta}} \widetilde{\Theta} = S_{T, f_{u,Y,\Theta}}, \tag{2.2.33}$$
$$\nabla_x \widetilde{\Theta} \cdot n_t = 0, \tag{2.2.34}$$
$$\Theta_{|t=0} = \Theta^0, \tag{2.2.35}$$

in $\widehat{\Omega}$, where $f_{u,Y,\Theta}$ is the unique weak solution to the problem

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\,(G_1 f) + \partial_r\,(G_2 f) + \partial_T\,(G_3 f) = 0 \quad \text{on } \widehat{\Pi}, \tag{2.2.36}$$
$$f = 0, \quad \text{on } \widehat{\Sigma}^-, \tag{2.2.37}$$
$$f_{|t=0} = f^0, \tag{2.2.38}$$

with

$$F_{u,Y,\Theta} = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} \chi(v-u) r\eta \overline{f_{u,Y,\Theta}},$$

$$\widetilde{z}_N = w - \pi_{\Omega_t}\left((\widetilde{u}-w)\mathbf{1}_{K_N}\right),$$

$$c_{f_{u,Y,\Theta}} = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r\eta \overline{f_{u,Y,\Theta}},$$

$$S_{T,f_{u,Y,\Theta}} = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} rT\eta \overline{f_{u,Y,\Theta}}, \qquad S_{Y,f_{u,Y,\Theta}} = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} Y_{v,\mathrm{surf}} r\eta \overline{f_{u,Y,\Theta}},$$

$$G_1 = \chi(u-v)\frac{r\eta}{r^3+r_{\mathrm{drug}}^3}, \quad G_2 = \frac{\overline{Y}-Y_{v,\mathrm{surf}}}{r}\eta, \quad G_3 = \frac{1}{r}G_2 + \frac{\chi(\overline{\Theta}-T)}{r^2}\eta.$$

The rest of the first part of this chapter is dedicated to the proof of Theorem 2.2.3. First, we recall the results obtained in [BGM17] regarding the Vlasov equation in a time-dependent domain with boundary condition (2.2.37) and its consequences. Then we briefly recall that Problem (2.2.27)–(2.2.29) has a unique strong solution and show that this solution is also the unique solution in a less regular setting. We continue by tackling both convection-diffusion problems (2.2.30)–(2.2.32) and (2.2.33)–(2.2.35). Our approach relies on a change of variable in order to work over a fixed domain and we use a standard approximation procedure to prove the existence and uniqueness of a solution to the problem over a fixed domain. We can then apply Theorem 2.2.4 to Λ thanks to compactness results of Aubin-Lions type from [Mou16] in a time-dependent framework. Finally, we use compactness again to prove the existence of a solution for a slightly modified version of (2.2.2)–(2.2.14), where η still appears, and the DiPerna-Lions theory to get rid of this truncation.

2.3 Existence of a solution to the approximated problem

2.3.1 Study of the Vlasov equation

Let $(u, Y, \Theta) \in \mathscr{C}$. The following result is an immediate consequence of Theorem 3.1, Proposition 3.2 and Remark 3.3 of [BGM17].

Theorem 2.3.1. *Problem (2.2.36)–(2.2.38) has a unique weak solution $f_{u,Y,\Theta} \in \mathrm{L}^\infty(\widehat{\Pi})$ in the sense of Definition 2.2.2. Moreover, $\overline{f_{u,Y,\Theta}} \in \mathscr{C}^0([0,\tau];\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*))$ for all $1 \le p < \infty$. The trace of $f_{u,Y,\Theta}$ on $\widehat{\Sigma}$ is well-defined as the unique element $\gamma f \in \mathrm{L}^\infty(\widehat{\Sigma})$ such that, for all test functions $\psi \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}\times\mathbb{R})$, all $\beta \in \mathscr{C}^1(\mathbb{R})$, and all $0 \le t_0 \le t_1 \le \tau$,*

$$\begin{aligned}\int_{t_0}^{t_1}\iint_{\Pi_t} &\beta(f)\left(\partial_t\psi + v\cdot\nabla_x\psi + G_1\cdot\nabla_v\psi + G_2\partial_r\psi + G_3\partial_T\psi\right)\\ &- \int_{t_0}^{t_1}\iint_{\Pi_t}\left(f\beta'(f)-\beta(f)\right)\psi\left(\mathrm{div}_v\, G_1 + \partial_r G_2 + \partial_T G_3\right)\\ &= \iint_{\Pi_{t_1}}\beta(f(t_1))\psi(t_1) - \iint_{\Pi_{t_0}}\beta(f(t_0))\psi(t_0) + \int_{t_0}^{t_1}\iint_{\Sigma_t}\beta(\gamma f)\psi\, v\cdot n_t.\end{aligned}$$

Furthermore, f is nonnegative, so is γf, f is compactly supported, and

$$\left\|\overline{f_{u,Y,\Theta}}\right\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*))} \lesssim \left\|f^0\right\|_{\mathrm{L}^p(\Pi_0)}. \tag{2.3.1}$$

We also have $S_{T,f_{u,Y,\Theta}}, S_{Y,f_{u,Y,\Theta}}, c_{f_{u,Y,\Theta}} \in \mathscr{C}^0([0,\tau];\mathrm{L}^p(\Omega_t))$ *for all* $p \in [1,+\infty)$ *and, for all* $p \in [1,+\infty]$,

$$\left\|S_{T,f_{u,Y,\Theta}}\right\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\Omega_t))} \lesssim \left\|f^0\right\|_{\mathrm{L}^p(\Pi_0)}, \tag{2.3.2}$$

$$\left\|S_{Y,f_{u,Y,\Theta}}\right\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\Omega_t))} \lesssim \left\|f^0\right\|_{\mathrm{L}^p(\Pi_0)}, \tag{2.3.3}$$

and

$$\left\|c_{f_{u,Y,\Theta}}\right\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^p(\Omega_t))} \lesssim \left\|f^0\right\|_{\mathrm{L}^p(\Pi_0)}. \tag{2.3.4}$$

Finally, $F_{u,Y,\Theta} \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(\mathbb{R}^3))$ *and*

$$\|F_{u,Y,\Theta}\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))} \lesssim \left\|f^0\right\|_{\mathrm{L}^2(\Pi_0)}. \tag{2.3.5}$$

Remark 2.3.2. *Let us be more accurate with respect to the multiplicative constant appearing in (2.3.1). It depends on* p, τ, r_{drug}, $\|\eta_r'\|_{\mathrm{L}^\infty(\mathbb{R})}$, $\|\nabla Y_{v,\mathrm{surf}}\|_{\mathrm{L}^\infty(\mathbb{R}_+^*\times\mathbb{R}_+^*)}$ *and* $\|\chi'\|_{\mathrm{L}^\infty(\mathbb{R})}$. *Therefore, we need to be vigilant when choosing a sequence of truncations* $(\chi_n)_{n\in\mathbb{N}}$ *in Section 2.4.*

Remark 2.3.3. *The fact that the solution* f *remains compactly supported comes from the fact that* $G_1, G_2, G_3 \in \mathrm{L}^\infty((0,\tau)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$. *It is a standard result that the size of the support only depends on the initial datum and the* $\mathrm{L}^1(0,\tau;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*))$ *norm of* G_1, G_2, G_3. *The only parameter involved in the bounds (2.3.2)–(2.3.5) that is of interest is the size of the support of* f, *and therefore the* L^∞ *norms of* $|G_1|+|G_2|+|G_3|$.

2.3.2 Study of the approximated Navier-Stokes equations

Let $u^0 \in \mathrm{L}^2(\mathbb{R}^3)$ such that $\mathrm{div}_x\, u^0 = 0$ and let $F \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(\mathbb{R}^3))$. Our focus in this section is the following problem on $(0,\tau)\times\mathbb{R}^3$:

$$\partial_t u + P_N\pi_{\mathrm{L}}(\mathrm{div}_x(u\otimes u)) - \Delta_x u + NP_N\pi_{\mathrm{L}}\left((u-w)\mathbf{1}_{\widehat{\Omega}^c}\right) = P_N\pi_{\mathrm{L}}F, \tag{2.3.6}$$

$$\mathrm{div}_x\, u = 0, \tag{2.3.7}$$

$$u_{|t=0} = P_N u^0. \tag{2.3.8}$$

In this section, $\lesssim_N$ will denote an inequality up to a multiplicative constant which can depend on τ, w and N. Applying the Cauchy-Lipschitz theorem and the standard procedure to obtain the energy estimates leads to the following result.

Theorem 2.3.4. *Problem (2.3.6)–(2.3.8) has a unique solution* $u \in \mathscr{C}^1([0,\tau];\mathscr{F}_N)$. *Furthermore, for all* $t \in [0,\tau]$,

$$\|u(t)\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \int_0^t \|\nabla_x u\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,\mathrm{d}s \lesssim_N 1 + \left\|u^0\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \|F\|^2_{\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)} \tag{2.3.9}$$

and

$$\|\partial_t u\|^2_{\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)} + \|\nabla_x u\|^2_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^2))} + \|u\|^2_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))} \lesssim_N 1 + \left\|u^0\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \|F\|^2_{\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)}. \tag{2.3.10}$$

We also use a uniqueness result for weaker solutions of (2.3.6)–(2.3.8) with respect to the time variable.

Definition 2.3.5. *We say that* $u_* \in \mathscr{C}([0,\tau];\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)) \cap \mathrm{L}^2(0,\tau;\mathscr{F}_N)$ *is a weak-in-time solution of (2.3.6)–(2.3.8) if, for all* $\varphi \in \mathscr{D}((0,\tau)\times\mathbb{R}^3)$ *such that* $\operatorname{div}_x \varphi = 0$,

$$\int_0^\tau \int_{\mathbb{R}^3} \left(-u_* \cdot \partial_t \varphi + P_N \pi_L \left(\operatorname{div}_x(u_* \otimes u_*)\right) \cdot \varphi + \nabla_x u_* : \nabla_x \varphi\right) + \int_0^\tau \int_{\mathbb{R}^3} N P_N \pi_L \left((u_* - w)\mathbf{1}_{\widehat{\Omega}^c}\right) \cdot \varphi = \int_0^\tau \int_{\mathbb{R}^3} P_N \pi_L(F) \cdot \varphi, \quad (2.3.11)$$

with $u_*(0) = P_N u^0$ *and* $\operatorname{div}_x u_* = 0$.

Theorem 2.3.6. *The solution* $u \in \mathscr{C}^1([0,\tau];\mathscr{F}_N)$ *given by Theorem 2.3.4 is the only weak-in-time solution of (2.3.6)–(2.3.8) in the sense of Definition 2.3.5.*

This result stems from Definition 2.3.5 and the derivation of a standard energy estimate.

Remark 2.3.7. *The unique weak-in-time solution of (2.3.6)–(2.3.8) given by Theorem 2.3.6 is in fact a Leray solution. It is the spatial regularization that allows us to recover uniqueness.*

2.3.3 Study of the convection-diffusion equations

Note that the equations on Y and Θ display the same structure. Therefore, we consider a more general framework covering both cases. Let $V^0 \in \mathrm{H}^1(\Omega_0)$, $z \in \mathrm{L}^\infty(0,\tau;\mathrm{H}^1(\Omega_t))$ such that $\operatorname{div}_x z = 0$ and $z_{|\partial\Omega_t} = w_{|\partial\Omega_t}$ for almost every $t \in (0,\tau)$. Let $c \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^4(\Omega_t))$ and $S \in \mathrm{L}^2(\widehat{\Omega})$ such that c and S are nonnegative.

This section is dedicated to solving the problem

$$\partial_t V + z \cdot \nabla_x V - \Delta_x V + cV = S \text{ in } \widehat{\Omega}, \quad (2.3.12)$$

$$\nabla_x V \cdot n_t = 0 \text{ on } \widehat{\Gamma}, \quad (2.3.13)$$

$$V(0,\cdot) = V^0 \text{ in } \Omega_0. \quad (2.3.14)$$

The notion of weak solution of this problem is defined as follows.

Definition 2.3.8. *We say that* $V \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(\Omega_t)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$ *is a weak solution of (2.3.12)–(2.3.14) if* $V(0) = V^0$ *and, for all* $\zeta \in \mathscr{C}^1(\overline{\widehat{\Omega}})$, *for all* $t \in (0,\tau)$,

$$\int_{\Omega_t} V(t)\zeta(t) - \int_{\Omega_0} V^0\zeta(0) - \iint_{\widehat{\Omega}_t} V\partial_s\zeta + \int_0^t B(V(s),\zeta(s);s) = \iint_{\widehat{\Omega}_t} S\zeta, \quad (2.3.15)$$

where B *is the time-dependent bilinear form defined, for almost every* $t \in (0,\tau)$ *and all* $(v_1,v_2) \in \mathrm{H}^1(\Omega_t)^2$, *by*

$$B(v_1,v_2;t) = \int_{\Omega_t} \left(\nabla_x v_1 \cdot \nabla_x v_2 + v_2 z(t) \cdot \nabla_x v_1 + c(t) v_1 v_2\right).$$

We will prove the following result.

Theorem 2.3.9. *Problem (2.3.12)–(2.3.14) has a unique weak solution* V *in the sense of Definition 2.3.8. Furthermore,* V *satisfies the weak maximum principle : if* V^0 *is nonnegative on* Ω_0, *then* V *is nonnegative on* $\widehat{\Omega}$. *Moreover, for all* $t \in (0,\tau)$,

$$\|V(t)\|^2_{\mathrm{L}^2(\Omega_t)} + 2\int_0^t \|\nabla_x V\|^2_{\mathrm{L}^2(\Omega_s)} + 2\iint_{\widehat{\Omega}_t} cV \lesssim \left\|V^0\right\|^2_{\mathrm{L}^2(\Omega_0)} + \|S\|^2_{\mathrm{L}^2(\widehat{\Omega})}. \quad (2.3.16)$$

Finally, for all $\zeta \in \mathscr{D}(\widehat{\Omega})$,

$$|\langle \partial_t V, \zeta \rangle| \lesssim \left(\|V^0\|^2_{\mathrm{L}^2(\Omega_0)} + \|S\|^2_{\mathrm{L}^2(\widehat{\Omega})} \right) \\ \times \left(1 + \|z\|^2_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t))} + \|c\|^2_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^4(\Omega_t))} \right) \|\zeta\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}, \quad (2.3.17)$$

where $\langle \cdot, \cdot \rangle$ *denotes the duality bracket* $\mathscr{D}'(\widehat{\Omega})$–$\mathscr{D}(\widehat{\Omega})$.

In order to prove this theorem, we use a change of variable so as to work in a fixed domain. For $t \in (0,\tau)$ and $y \in \Omega_0$, let

$$V_0(t,y) = V(t, \mathcal{A}_t(y)), \qquad z_0(t,y) = z(t, \mathcal{A}_t(y)),$$
$$w_0(t,y) = w(t, \mathcal{A}_t(y)), \qquad c_0(t,y) = c(t, \mathcal{A}_t(y)),$$
$$S_0(t,y) = S(t, \mathcal{A}_t(y)), \quad J(t,y) = \mathrm{Cof}(\nabla_y \mathcal{A}_t(y)), \quad D(t,y) = J(t,y)^\intercal J(t,y),$$

and consider the problem on $(0,\tau) \times \Omega_0$

$$\partial_t V_0 + (z_0 - w_0) \cdot J \nabla_y V_0 - \mathrm{div}_y\,(D \nabla_y V_0) + c_0 V_0 = S_0, \quad (2.3.18)$$
$$D \nabla_y V_0 \cdot n_0 = 0, \quad (2.3.19)$$
$$V_0(0,\cdot) = V^0. \quad (2.3.20)$$

Definition 2.3.10. *We say that* $V_0 \in \mathscr{C}^0([0,\tau]; \mathrm{L}^2(\Omega_0)) \cap \mathrm{L}^2(0,\tau; \mathrm{H}^1(\Omega_0))$ *such that* $\partial_t V_0 \in \mathrm{L}^2(0,\tau; \mathrm{H}^1(\Omega_0)')$ *is a weak solution to Problem (2.3.18)–(2.3.20) if* $V_0(0) = V^0$ *and, for all* $\zeta \in \mathscr{C}^1(\overline{\Omega_0})$, *for any* $t \in (0,\tau)$,

$$\int_0^t \langle \partial_t V_0, \zeta \rangle_1 + \int_0^t \int_{\Omega_0} B_0(V_0(s), \zeta(s); s) = \int_0^t \int_{\Omega_0} S_0 \zeta, \quad (2.3.21)$$

where $\langle \cdot, \cdot \rangle_1$ *denotes the duality bracket* $\mathrm{H}^1(\Omega_0)'$–$\mathrm{H}^1(\Omega_0)$ *and* B_0 *is the time-dependent bilinear form defined, for all* $(v_1, v_2) \in \mathrm{H}^1(\Omega_0)^2$ *and almost every* $t \in (0,\tau)$, *by*

$$B_0(v_1, v_2; t) = \int_{\Omega_0} [D(t) \nabla_y v_1 \cdot \nabla_y v_2 + v_2 (z_0 - w_0)(t) \cdot J(t) \nabla_y v_1 + c_0(t) v_1 v_2].$$

Note that $\mathrm{div}_y\, J = 0$ and that $\mathrm{div}_x\, w = 0$ implies that $\det J = 1$ as well as $\mathrm{div}_y(J^\intercal w_0) = 0$. Therefore, thanks to the fact that for every $t \in (0,\tau)$, $\mathcal{A}_t$ is a $\mathscr{C}^1$-diffeomorphism, by a change of variable, one can easily check that V is a weak solution of Problem (2.3.12)–(2.3.14) in the sense of Definition 2.3.8 if and only if V_0 is a weak solution of Problem (2.3.18)–(2.3.20) in the sense of Definition 2.3.10.

Let us now focus on Problem (2.3.18)–(2.3.20). In order to prove the existence of a weak solution, we follow the steps of [Eva10] – which deals with the Dirichlet boundary condition rather than the Neumann one as well as different regularity hypotheses on the coefficients – and use a Galerkin approximation. Let $(v_n)_{n \in \mathbb{N}}$ be an orthogonal basis of $\mathrm{H}^1(\Omega_0)$ such that $(v_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathrm{L}^2(\Omega_0)$ and, for all $n \in \mathbb{N}$, let $\mathcal{H}_n = \mathrm{Span}(v_0, \ldots, v_n)$. Let $n \in \mathbb{N}$. By the linear Cauchy-Lipschitz theorem, there exists a unique solution $V_{0,n} \in W^{1,\infty}(0,\tau; \mathcal{H}_n)$ to the problem

$$\int_{\Omega_0} \partial_t V_{0,n} v_k + B_0(V_{0,n}(t), v_k; t) = \int_{\Omega_0} S_0 v_k \qquad \text{a.e. } t \in (0,\tau), \forall k \in [\![0,n]\!], \tag{2.3.22}$$

$$V_{0,n\,|t=0} = P_{\mathcal{H}_n} V^0. \tag{2.3.23}$$

We write

$$V_{0,n}(t,y) = \sum_{k=0}^{n} q_k^n(t) v_k(y).$$

Then, if we multiply (2.3.22) by q_k^n, sum over k for each k and integrate on $(0,t)$, for $t \in (0,\tau)$, we get

$$\int_0^t \int_{\Omega_0} \left(V_{0,n}\partial_s V_{0,n} + V_{0,n}(z_0 - w_0)\cdot J\nabla_y V_{0,n} + D\nabla_y V_{0,n}\cdot\nabla_y V_{0,n} + c_0|V_{0,n}|^2\right) = \int_0^t \int_{\Omega_0} S_0 V_{0,n}.$$

First note that

$$\int_0^t \int_{\Omega_0} V_{0,n}\partial_s V_{0,n} = \frac{1}{2}\left\|V_{0,n}(t)\right\|^2_{\mathrm{L}^2(\Omega_0)} - \frac{1}{2}\left\|V_{0,n}(0)\right\|^2_{\mathrm{L}^2(\Omega_0)}.$$

Furthermore, for almost every $t \in (0,\tau)$,

$$\begin{aligned}\int_{\Omega_0} V_{0,n}(z_0 - w_0)\cdot J\nabla_y V_{0,n} &= \int_{\partial\Omega_0} |V_{0,n}|^2 (z_0 - w_0)\cdot J n_0 \\ &\quad - \int_{\Omega_0} V_{0,n}(z_0 - w_0)\cdot J\nabla_y V_{0,n} - \int_{\Omega_0} |V_{0,n}|^2 (\operatorname{div}_y J)\cdot(z_0 - w_0) \\ &\quad - \int_{\Omega_0} |V_{0,n}|^2 \operatorname{Tr}(J^\intercal \nabla_y (z_0 - w_0)).\end{aligned} \tag{2.3.24}$$

Since $\operatorname{div}_y J = 0$ and

$$\begin{aligned}\operatorname{Tr}(J^\intercal\nabla_y(z_0 - w_0)) &= \operatorname{Tr}[\operatorname{Cof}(\nabla_y \mathcal{A}_t(y))^\intercal \nabla_x (z-w)(t,\mathcal{A}_t(y))\nabla_y\mathcal{A}_t(y)] \\ &= \frac{1}{\det(\nabla_y\mathcal{A}_t(y))}\operatorname{div}_x(z-w)(t,\mathcal{A}_t(y)) = 0,\end{aligned}$$

we deduce from (2.3.24) that

$$\int_{\Omega_0} V_{0,n}(z_0 - w_0)\cdot J\nabla_y V_{0,n} = 0.$$

Finally, noticing that $(t,y) \mapsto \sigma(J(t,y))$ is a continuous map from $[0,\tau]\times\overline{\Omega_0}$ to $\mathbb{R}$, and since $[0,\tau]\times\overline{\Omega_0}$ is compact and $\det J = 1$, there exists $\alpha > 0$ such that

$$\int_0^t \int_{\Omega_0} D\nabla_y V_{0,n}\cdot\nabla_y V_{0,n} \geq \alpha \int_0^t\int_{\Omega_0} |\nabla_y V_{0,n}|^2.$$

Therefore,

$$\|V_{0,n}(t)\|^2_{\mathrm{L}^2(\Omega_0)} + 2\alpha\int_0^t\int_{\Omega_0}|\nabla_y V_{0,n}|^2 + 2\int_0^t\int_{\Omega_0} c_0|V_{0,n}|^2 \\ \leq \left\|V^0\right\|^2_{\mathrm{L}^2(\Omega_0)} + \|S_0\|^2_{\mathrm{L}^2((0,\tau)\times\Omega_0)} + \int_0^t \|V_{0,n}(s)\|^2_{\mathrm{L}^2(\Omega_0)}$$

and by Grönwall's lemma,

$$\|V_{0,n}(t)\|^2_{\mathrm{L}^2(\Omega_0)} + 2\alpha\int_0^t \|\nabla_y V_{0,n}\|^2_{\mathrm{L}^2(\Omega_0)} + 2\int_0^t\int_{\Omega_0} c_0|V_{0,n}|^2 \lesssim 1, \tag{2.3.25}$$

where the underlying constant depends on V^0 and S. Thus, there exists $V_0 \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_0)) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_0))$ such that, up to a subsequence, $V_{0,n} \rightharpoonup V_0$ and $\nabla_y V_{0,n} \rightharpoonup \nabla_y V_0$ weakly in $\mathrm{L}^2((0,\tau)\times\Omega_0)$ and $V_{0,n} \rightharpoonup V_0$ weakly-$*$ in $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_0))$.

In order to take the limit $n \to +\infty$ in (2.3.22)–(2.3.23), we prove that $(\partial_t V_{0,n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_0)')$. Let $v \in \mathrm{H}^1(\Omega_0)$. Applying (2.3.22) for $n \in \mathbb{N}$ and almost every $t \in (0,\tau)$ yields

$$\left|\langle\partial_t V_{0,n}, v\rangle_1\right| = \left|\int_{\Omega_0}\partial_t V_{0,n} P_{\mathcal{H}_n} v\right| \\ \leq \left|\int_{\Omega_0}(P_{\mathcal{H}_n}v)J^\intercal(z_0-w_0)\cdot\nabla_y V_{0,n}\right| + \left|\int_{\Omega_0} D\nabla_y V_{0,n}\cdot\nabla_y(P_{\mathcal{H}_n}v)\right| + \left|\int_{\Omega_0} c_0 V_{0,n}(P_{\mathcal{H}_n}v)\right| \\ + \left|\int_{\Omega_0} S_0(P_{\mathcal{H}_n}v)\right|.$$

Since

$$\left|\int_{\Omega_0}(P_{\mathcal{H}_n}v)J^\intercal(z_0-w_0)\cdot\nabla_y V_{0,n}\right| \lesssim \|V_{0,n}\|_{\mathrm{H}^1(\Omega_0)}\|z_0-w_0\|_{\mathrm{H}^1(\Omega_0)}\|v\|_{\mathrm{H}^1(\Omega_0)},$$

we get, thanks to (2.3.25) and the fact that $z_0, w_0 \in \mathrm{L}^\infty(0,\tau;\mathrm{H}^1(\Omega_0))$ and $c_0 \in \mathrm{L}^\infty(0,\tau;\mathrm{L}^4(\Omega_0))$,

$$\int_0^\tau \|\partial_t V_{0,n}\|^2_{\mathrm{H}^1(\Omega_0)'} \lesssim \int_0^\tau \|V_{0,n}\|^2_{\mathrm{H}^1(\Omega_0)}\|z_0-w_0\|^2_{\mathrm{H}^1(\Omega_0)} \\ + \int_0^\tau \|D\nabla_y V_{0,n}\|^2_{\mathrm{L}^2(\Omega_0)} + \|c_0 V_{0,n}\|^2_{\mathrm{L}^2(\Omega_0)} + \|S_0\|^2_{\mathrm{L}^2(\Omega_0)} \lesssim 1.$$

Therefore, up to a subsequence, $(\partial_t V_{0,n})_{n\in\mathbb{N}}$ converges weakly in $\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_0)')$ to $\partial_t V_0$. We deduce from this that $V_0 \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(\Omega_0))$.

We can now pass to the limit in (2.3.22)–(2.3.23) in a similar way as in [Eva10] and prove the existence of a solution to Problem (2.3.18)–(2.3.20). To obtain its uniqueness, we begin by showing that V_0 satisfies the maximum principle. After an approximation procedure, we can choose ζ_0 as the nonpositive part of V_0, which we write $(V_0)^-$ in (2.3.21). For all $t \in (0,\tau)$, we have

$$\left\|(V_0)^-(t)\right\|^2_{\mathrm{L}^2(\Omega_0)} + 2\int_0^t\int_{\Omega_0} \mathbf{1}_{V_0<0}\,|J\nabla_y V_0|^2 + 2\int_0^t\int_{\Omega_0} \mathbf{1}_{V_0<0}\, c_0|V_0|^2 \leq \left\|(V^0)^-\right\|^2_{\mathrm{L}^2(\Omega_0)} - \int_0^t\int_{\Omega_0} S_0(V_0)^-,$$

where $\mathbf{1}_A$ denotes the characteristic function of a subset $A \subset \Omega_0$. Since, here, V^0, c_0 and S_0 are nonnegative, this yields $V_0 \geq 0$. The uniqueness of the weak solution to (2.3.18)–(2.3.20) follows.

Estimate (2.3.16) can be obtained by taking $\zeta = V$ in (2.3.15) and applying Grönwall's lemma. For (2.3.17), we can apply (2.3.15) to $\zeta \in \mathscr{D}(\widehat{\Omega})$ and reason similarly to what we have done for $\partial_t V_0$ above, except for the following term, for which we write

$$\left|\iint_{\widehat{\Omega}} \zeta z\cdot\nabla_x V\right| \leq \|z\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega_t))}\,\|V\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))}\,\|\zeta\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}\,.$$

This concludes the proof of Theorem 2.3.9.

2.3.4 Application of the Schauder theorem

Thanks to the results stated in the previous sections, we can check that the map Λ defined by (2.2.27)–(2.2.35) satisfies the hypotheses of the Schauder fixed-point theorem. From Theorems 2.3.1, 2.3.4, and 2.3.9, we deduce that Λ is well-defined and that there exists a constant $C > 0$ depending only on w, N, χ and the initial data, such that

$$\forall (u,Y,\Theta) \in E, \qquad \|\Lambda(u,Y,\Theta)\|_E \leq C.$$

Step 1 – Λ is compact

Let $(u_n, Y_n, \Theta_n)_{n\in\mathbb{N}}$ be a bounded sequence of $(E, \|.\|_E)$. Let

$$(\widetilde{u_n}, \widetilde{Y_n}, \widetilde{\Theta_n})_{n\in\mathbb{N}} = (\Lambda(u_n, Y_n, \Theta_n))_{n\in\mathbb{N}}$$

and let $(f_n)_{n\in\mathbb{N}}$ be the sequence of solutions to the corresponding Vlasov equations. Thanks to Remark 2.3.3, there exists a compact of $\overline{\widehat{\Omega}} \times \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}_+$ that contains all the supports of the functions f_n. The bounds (2.3.2)–(2.3.5) are therefore uniform with respect to $n \in \mathbb{N}$.

The convergence, up to a subsequence, of $(\widetilde{u_n})_{n\in\mathbb{N}}$ is a consequence of the Arzelà-Ascoli theorem, which we apply in $\mathscr{C}^0([0,\tau];\mathrm{L}^2(B))$ for any closed ball B of $\mathbb{R}^3$. The sequence is indeed uniformly equicontinuous thanks to the bound (2.3.10) and the fact that, for all $\varphi \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(B))$ such that $\partial_t\varphi \in \mathrm{L}^2((0,\tau)\times\mathbb{R}^3)$,

$$\forall t_1, t_2 \in (0,\tau), \qquad \|\varphi(t_2) - \varphi(t_1)\|^2_{\mathrm{L}^2(B)} \leq |t_2 - t_1|\,\|\partial_t\varphi\|^2_{\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)}\,.$$

Moreover, for all $t \in [0,\tau]$, $(\widetilde{u_n}(t))_{n\in\mathbb{N}}$ is precompact in $\mathrm{L}^2(B)$ thanks to the Rellich-Kondrachov theorem. Then there exists $u \in \mathscr{C}^0([0,\tau];\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3))$ such that the sequence $(\widetilde{u_n})_{n\in\mathbb{N}}$ converges to u in $\mathscr{C}^0([0,\tau];\mathrm{L}^2(B))$ for any closed ball B of $\mathbb{R}^3$. Thanks to Estimates (2.3.5) and (2.3.9) and the monotone convergence theorem, $\widetilde{u}$ belongs to $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))$. Then the equicontinuity of the sequence $(\widetilde{u_n})_{n\in\mathbb{N}}$ ensures that $u \in \mathscr{C}^0([0,\tau];\mathrm{L}^2(\mathbb{R}^3))$. Furthermore, Estimate (2.3.9) shows that $(\nabla_x\widetilde{u_n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)$ and therefore, up to a subsequence, it converges weakly to $\nabla_x\widetilde{u}$ in $\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)$. The weak lower semi-continuity of $\|.\|_{\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)}$ then implies that $u \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\mathbb{R}^3))$. Finally, we prove that $(\widetilde{u}_n)_{n\in\mathbb{N}}$ converges to u in E. For any $R>0$, if we denote by B_R the ball of $\mathbb{R}^3$ centered at 0 and of radius R, we have, for all $t\in[0,\tau]$,

$$\|(\widetilde{u}(t)-\widetilde{u_n}(t))\gamma\|_{\mathrm{L}^1(\mathbb{R}^3)} \le \|\widetilde{u}(t)-\widetilde{u_n}(t)\|_{\mathrm{L}^2(B_R)}\|\gamma\|_{\mathrm{L}^2(\mathbb{R}^3)} + \|\widetilde{u}(t)-\widetilde{u_n}(t)\|_{\mathrm{L}^2(\mathbb{R}^3)}\left\|\gamma\mathbf{1}_{|x|>R}\right\|_{\mathrm{L}^2(\mathbb{R}^3)}$$

Thanks to Estimate (2.3.9) and the fact that $\gamma\in\mathrm{L}^2(\mathbb{R}^3)$, we deduce from this that

$$\lim_{n\to+\infty}\sup_{t\in[0,\tau]}\|(\widetilde{u}(t)-\widetilde{u_n}(t))\gamma\|_{\mathrm{L}^1(\mathbb{R}^3)}=0.$$

We now deal with the convergence of $(\widetilde{Y_n})_{n\in\mathbb{N}}$, the case of $(\widetilde{\Theta_n})_{n\in\mathbb{N}}$ being completely similar. We begin by noticing that thanks to Theorem 2.3.9, for all $n\in\mathbb{N}$, $\widetilde{Y_n}$ is nonnegative and, using the the same argument, since $Y_{v,\mathrm{surf}}\le 1$, we can prove that $\widetilde{Y_n}\le 1$. In order to get compactness, we apply an Aubin-Lions-type result : Corollary 1 from [Mou16]. Indeed, thanks to the bounds (2.3.3) and (2.3.16), $(\widetilde{Y_n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$. And with (2.3.4), (2.3.5), (2.3.9) and (2.3.17), for all $\zeta\in\mathscr{D}(\widehat{\Omega})$, we obtain

$$\left|\left\langle\partial_t\widetilde{Y_n},\zeta\right\rangle\right| \lesssim_N \|\zeta\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}.$$

This allows to state that, up to a subsequence, $(\widetilde{Y_n})_{n\in\mathbb{N}}$ converges in $\mathrm{L}^2(\widehat{\Omega})$.

In conclusion, we have obtained that, up to a subsequence, $(\widetilde{u_n},\widetilde{Y_n},\widetilde{\Theta_n})_{n\in\mathbb{N}}$ converges in $(E,\|.\|_E)$.

Step 2 – Λ is continuous

Let $(u_n,Y_n,\Theta_n)_{n\in\mathbb{N}}$ be a sequence of E converging towards (u,Y,Θ). It is enough to prove that, from any subsequence of

$$(\widetilde{u_n},\widetilde{Y_n},\widetilde{\Theta_n})_{n\in\mathbb{N}} = (\Lambda(u_n,Y_n,\Theta_n))_{n\in\mathbb{N}},$$

we can extract a subsequence which converges to $(\widetilde{u},\widetilde{Y},\widetilde{\Theta})=\Lambda(u,Y,\Theta)$ in E. For the sake of clarity, we will not modify the indices when dealing with subsequences. The compactness of Λ ensures that $(\widetilde{u_n},\widetilde{Y_n},\widetilde{\Theta_n})_{n\in\mathbb{N}}$ converges to (U,H,Z) in E up to a subsequence.

The convergences of $(\widetilde{u_n})_{n\in\mathbb{N}}$ in $\mathscr{C}^0([0,\tau]\times\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3))$ and in w-$\mathrm{L}^2((0,\tau)\times\mathbb{R}^3)$ provide that, with standard arguments, we can take the limit $n\to+\infty$ in all the terms appearing in (2.3.11), except the one on the right-hand side. For this term, Estimate (2.3.1) ensures that, up to a subsequence, $(\widetilde{f_n})_{n\in\mathbb{N}}$ converges weakly-$*$ in $\mathrm{L}^\infty((0,\tau)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ and weakly in $\mathrm{L}^p((0,\tau)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ for all $p\in(1,\infty)$. Furthermore, the convergence of (u_n,Y_n,Θ_n) in E provides the convergences of $(Y_n)_{n\in\mathbb{N}}$ and $(\Theta_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^2(\widehat{\Omega})$. With this, we can take the limit $n\to+\infty$ in (2.2.18). The uniqueness of the weak solution, given by Theorem 2.3.1, yields that the limit of $(f_n)_{n\in\mathbb{N}}$ is $f_{u,Y,\Theta}$. In turn, this results in the convergence of the right-hand side term in (2.3.11) and, thanks to the uniqueness of the weak-in-time solution proved in Theorem 2.3.6, we obtain that the whole sequence $(\widetilde{u_n})_{n\in\mathbb{N}}$ converges to $U=\widetilde{u}$ in $\mathscr{C}^0([0,\tau];\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3))$.

We get the convergence in $\mathrm{L}^2(\widehat{\Omega})$ of $(\widetilde{Y_n})_{n\in\mathbb{N}}$ and $(\widetilde{\Theta_n})_{n\in\mathbb{N}}$ towards, respectively, $Z=\widetilde{Y}$ and $H=\widetilde{\Theta}$ in a similar way. As a conclusion, Λ is continuous.

Step 3 – Conclusion

Finally, we can apply Theorem 2.2.4 to the map Λ restricted to the set

$$\mathscr{C}' = \{(u, Y, \Theta) \in \mathscr{C},\ \|(u, \Theta, Y)\|_E \leq C\},$$

where $C > 0$ has been defined at the beginning of this section. We obtain the existence of $(u, Y, \Theta) \in \mathscr{C}'$ such that $\Lambda(u, Y, \Theta) = (u, Y, \Theta)$.

2.4 Conclusion of the proof of Theorem 2.2.3

Let f^0, u^0, Y^0, Θ^0 satisfy Assumptions 2.1–2.4.

Let $(Y_n^0)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{H}^1(\Omega_0)$ that converges to Y^0 in $\mathrm{L}^2(\Omega_0)$ and such that $0 \leq Y_n^0 \leq 1$ and $\|Y_n^0\|_{\mathrm{L}^2(\Omega_0)} \leq \|Y^0\|_{\mathrm{L}^2(\Omega_0)}$ for all $n \in \mathbb{N}$. Similarly, let $(\Theta_n^0)_{n\in\mathbb{N}}$ be a nonnegative sequence of $\mathrm{H}^1(\Omega_0)$ that converges to Θ^0 and such that $\|\Theta_n^0\|_{\mathrm{L}^2(\Omega_0)} \leq \|\Theta^0\|_{\mathrm{L}^2(\Omega_0)}$ for all $n \in \mathbb{N}$. Finally, let $(f_n^0)_{n\in\mathbb{N}}$ be a nonnegative sequence of $\mathrm{L}^\infty(\Pi_0)$ such that, for all $n \in \mathbb{N}$, f_n is compactly supported and $f_n^0 \in \mathrm{L}^\infty(0, \tau; \mathrm{L}^p(\Pi_t))$ and such that $(f_n)_{n\in\mathbb{N}}$ converges to f^0 in $\mathrm{L}^p(\Pi_0)$ for all $p \in [1, \infty)$ and w^*-$\mathrm{L}^\infty(\Pi_0)$. We also assume that, for all $n \in \mathbb{N}$, $f_n^0(\cdot, \cdot, r, \cdot) \equiv 0$ for almost every $r \in [0, r_{\mathrm{drug}}]$. Lastly, we choose f_n^0 so that

$$\|f_n^0\|_{\mathrm{L}^\infty(\Pi_0)} \leq \|f^0\|_{\mathrm{L}^\infty(\Pi_0)}, \tag{2.4.1}$$

$$\iint_{\Pi_0} r^3 f_n^0 \leq \iint_{\Pi_0} r^3 f^0, \tag{2.4.2}$$

$$\iint_{\Pi_0} r^3 |v|^2 f_n^0 \leq \iint_{\Pi_0} r^3 |v|^2 f^0, \tag{2.4.3}$$

$$\iint_{\Pi_0} |v|^2 f_n^0 \leq \iint_{\Pi_0} |v|^2 f^0, \tag{2.4.4}$$

$$\iint_{\Pi_0} r^3 T^2 f_n^0 \leq \iint_{\Pi_0} r^3 T^2 f^0. \tag{2.4.5}$$

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of truncation functions satisfying the same hypotheses as χ in Section 2.2. Additionally, we suppose that, for all $n \in \mathbb{N}$, $\|\chi_n\|_{\mathrm{L}^\infty(\mathbb{R})} \leq n$, $0 \leq \chi_n' \leq 1$, and for all $v \in \mathbb{R}_+$, $|\chi_n(v)| \leq v$. Finally, we assume that $\chi_{n\,|[-n+1/n, n-1/n]} = \mathrm{Id}_{[-n+1/n, n-1/n]}$ so that $(\chi_n)_{n\in\mathbb{N}}$ converges uniformly to $\mathrm{Id}_\mathbb{R}$.

Using the results of Section 2.3.4, for all $n \in \mathbb{N}$ there exist u_n, Y_n, Θ_n, f_n weakly solving the system

$$\partial_t f_n + v \cdot \nabla_x f_n + \mathrm{div}_v(G_{1,n} f_n) + \partial_r(G_{2,n} f_n) + \partial_T(G_{3,n} f_n) = 0 \qquad \text{on } \widehat{\Pi}, \tag{2.4.6}$$

$$f_n = 0 \qquad \text{on } \widehat{\Sigma}^-, \tag{2.4.7}$$

$$\begin{aligned} \partial_t u_n + P_n \pi_\mathrm{L}(\mathrm{div}_x(u_n \otimes u_n)) - \Delta_x u_n + n P_n \pi_\mathrm{L}\left((u_n - w)\mathbf{1}_{\widehat{\Omega}^c}\right) \\ = P_n \pi_\mathrm{L} \int_{\mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*} r\eta \overline{f_n} \chi_n(v - u_n) \qquad \text{in } (0, \tau) \times \mathbb{R}^3, \end{aligned} \tag{2.4.8}$$

$$\mathrm{div}_x\, u_n = 0 \qquad \text{in } (0, \tau) \times \mathbb{R}^3, \tag{2.4.9}$$

$$\partial_t Y_n + z_n \cdot \nabla_x Y_n - \Delta_x Y_n = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r\eta(Y_{v,\mathrm{surf}} - Y_n) f_n \qquad \text{in } \widehat{\Omega}, \tag{2.4.10}$$

$$\nabla_x Y_n \cdot n_t = 0 \qquad \text{on } \widehat{\Gamma}, \tag{2.4.11}$$

$$\partial_t \Theta_n + z_n \cdot \nabla_x \Theta_n - \Delta_x \Theta_n = \int_{\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r\eta(T - \Theta_n) f_n \qquad \text{in } \widehat{\Omega}, \tag{2.4.12}$$

$$\nabla_x \Theta_n \cdot n_t = 0 \qquad \text{on } \widehat{\Gamma}, \tag{2.4.13}$$

where

$$z_n = w - \pi_{\Omega_t}\left((u_n - w)\mathbf{1}_{K_n}\right),$$

$$G_{1,n} = \chi_n(u_n - v)\frac{\eta}{r^2 + \frac{r_{\mathrm{drug}}^3}{r}}, \qquad G_{2,n} = \frac{\overline{Y_n} - Y_{v,\mathrm{surf}}}{r}\eta, \qquad G_{3,n} = G_{2,n} + \frac{\overline{\Theta_n} - T}{r^2}\eta.$$

Moreover, these solutions satisfy the initial conditions

$$u_n(0) = P_n u^0, \quad Y_n(0) = Y_n^0, \quad \Theta_n(0) = \Theta_n^0, \quad f_n(0) = f_n^0.$$

Remark 2.4.1. *Note that, thanks to Theorem 2.3.6, $u_n \in \mathscr{C}^1([0,\tau];\mathscr{F}_n)$ is a strong solution of the regularized Navier-Stokes equations.*

Thanks to Theorem 2.3.1, Remark 2.3.2 and Assumption 2.4.1, we have

$$\|f_n\|_{\mathrm{L}^\infty(\widehat{\Pi})} \lesssim \left\|f_n^0\right\|_{\mathrm{L}^\infty(\Pi_0)} \lesssim 1.$$

Therefore, there exists $f \in \mathrm{L}^\infty(\widehat{\Pi})$ such that, up to a subsequence, $(f_n)_{n\in\mathbb{N}}$ converges to f weakly-$*$ in $\mathrm{L}^\infty(\widehat{\Pi})$. Another consequence of (2.3.1) is the convergence of $(f_n)_{n\in\mathbb{N}}$ weakly in $\mathrm{L}^q(0,\tau;\mathrm{L}^p(\Pi_t))$ for all $1 < p, q < \infty$. In particular, we have $f \geq 0$.

Estimate (2.3.9) on u_n is no longer satisfactory since it depends on n. We are thus led to write an energy estimate such as (2.2.17). Multiplying (2.4.8) by u_n and integrating over $(0,t)\times\mathbb{R}^3$ for $t \in (0,\tau)$ yields

$$\begin{aligned}\frac{1}{2}\|u_n(t)\|_{\mathrm{L}^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla_x u_n\|_{\mathrm{L}^2(\mathbb{R}^3)}^2 + n\int_0^t\int_{\mathbb{R}^3} \mathbf{1}_{\widehat{\Omega}^c}(u_n - w)\cdot u_n \\ = \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*} r\eta\overline{f_n}\chi_n(v - u_n)\cdot u_n + \frac{1}{2}\left\|P_n u^0\right\|_{\mathrm{L}^2(\mathbb{R}^3)}^2.\end{aligned}$$

Furthermore, thanks to the stability of the renormalized solutions of the Vlasov equation, due to the DiPerna-Lions theory (see [DL89]), we can deal with the Vlasov equation as if its coefficients and f_n were smooth. For more details, the reader can refer to Remark 3.1 of [HMM20]. Then, multiplying (2.4.6) by $(r^3 + r_{\mathrm{drug}}^3)|v|^2/2$ and integrating over $\widehat{\Pi}_t$ leads to

$$\frac{1}{2}\iint_{\Pi_t}(r^3+r_{\mathrm{drug}}^3)|v|^2 f_n \leq \iiint_{\widehat{\Pi}_t} r\eta f_n\chi_n(u_n - v)\cdot v + 3\iiint_{\widehat{\Pi}_t} r|v|^2\eta f_n + \frac{1}{2}\iint_{\Pi_0}(r^3+r_{\mathrm{drug}}^3)|v|^2 f_n^0,$$

because f_n, Y_n and $Y_{v,\mathrm{surf}}$ are nonnegative, $Y_n, Y_{v,\mathrm{surf}} \leq 1$ and $f_{n\,|\widehat{\Sigma}^-} = 0$. Summing these inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\left\|u_n(t)\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \frac{1}{2}\iint_{\Pi_t}(r^3+r^3_{\mathrm{drug}})|v|^2 f_n + \int_0^t \left\|\nabla_x u_n\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} \\ + n\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\widehat{\Omega}^c}(u_n-w)\cdot u_n + \iiint_{\widehat{\Pi}_t} r\eta f_n\chi_n(v-u_n)\cdot(v-u_n) \\ \leq 3\iiint_{\widehat{\Pi}_t} r|v|^2\eta f_n + \frac{1}{2}\left\|u^0\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \frac{1}{2}\iint_{\Pi_0}(r^3+r^3_{\mathrm{drug}})|v|^2 f^0 \quad (2.4.14)\end{aligned}$$

Since $\chi_n(v)\cdot v \geq 0$ for all v, we only need to replace the unsigned term

$$n\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\widehat{\Omega}^c}(u_n-w)\cdot u_n \qquad \text{by} \qquad n\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\widehat{\Omega}^c}|u_n-w|^2.$$

We proceed in the same way as [BGM17]. By multiplying (2.4.8) by $P_n w$ and integrating over $(0,t)\times\mathbb{R}^3$, we get

$$\begin{aligned}-n\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\widehat{\Omega}^c}(u_n-w)\cdot w = \int_{\mathbb{R}^3} u_n(t)\cdot w(t) - \int_{\mathbb{R}^3} u_n^0\cdot w(0) - \int_0^t\int_{\mathbb{R}^3} u_n\cdot\partial_t w \\ + \int_0^t\int_{\mathbb{R}^3}\nabla_x u_n : P_n\nabla_x w + \int_0^t\int_{\mathbb{R}^3}(\mathrm{div}_x(u_n\otimes u_n))\cdot w \\ + \int_0^t\iiiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^*_+\times\mathbb{R}^*_+} r\eta_r\theta_n\overline{f_n}\chi_n(v-u_n)\cdot P_n w. \quad (2.4.15)\end{aligned}$$

We then use Cauchy-Schwarz's and Young's inequalities, as well as the fact that $|\chi_n(z)|^2 \leq \chi_n(z)\cdot z$ for all $z\in\mathbb{R}^3$, to find, by summing (2.4.14) and (2.4.15),

$$\begin{aligned}\frac{1}{4}\left\|u_n(t)\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \frac{1}{2}\iint_{\Pi_t}(r^3+r^3_{\mathrm{drug}})|v|^2 f_n + \frac{1}{8}\int_0^t \left\|\nabla_x u_n\right\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + n\int_0^t\int_{\mathbb{R}^3}\mathbf{1}_{\widehat{\Omega}^c}|u_n-w|^2 \\ + \frac{1}{2}\iiint_{\widehat{\Pi}_t} r\eta f_n\chi_n(v-u_n)\cdot(v-u_n) \lesssim 1. \quad (2.4.16)\end{aligned}$$

Therefore, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))\cap\mathrm{L}^2(0,\tau;\mathrm{H}^1(\mathbb{R}^3))$. As a consequence, there exists $u\in\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))\cap\mathrm{L}^2(0,\tau;\mathrm{H}^1(\mathbb{R}^3))$ such that, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges to u in w-$\mathrm{L}^2(0,\tau;\mathrm{H}^1(\mathbb{R}^3))$ and w^*-$\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\mathbb{R}^3))$.

We also need a new estimate for $(\Theta_n)_{n\in\mathbb{N}}$. After an approximation procedure, we can take $\zeta = \Theta_n\mathbf{1}_{[0,t]}$ in (2.3.15) and obtain

$$\left\|\Theta_n(t)\right\|^2_{\mathrm{L}^2(\Omega_t)} + 2\int_0^t\left\|\nabla_x\Theta_n\right\|^2_{\mathrm{L}^2(\Omega_s)} + 2\iiint_{\widehat{\Pi}_t} r(\Theta_n-T)\Theta_n\eta f_n = \left\|T_n^0\right\|^2_{\mathrm{L}^2(\Omega_0)}.$$

Furthermore, multiplying (2.4.6) by r^3T^2 and integrating over $\widehat{\Pi}_t$ yields, thanks to the boundary condition (2.4.7) and the fact that $0\leq Y_n, Y_{v,\mathrm{surf}}\leq 1$ and $f_n\geq 0$,

$$\iint_{\Pi_t} r^3T^2 f_n(t) \leq 6\iiint_{\widehat{\Pi}_t} rT^2\eta f_n + 4\iiint_{\widehat{\Pi}_t} rT\eta f_n + 2\iiint_{\widehat{\Pi}_t} r\chi_n(\Theta_n-T)T\eta f_n + \int_{\Pi_0} r^3T^2 f_n^0.$$

Summing these inequalities leads to

$$\begin{aligned}\|\Theta_n(t)\|^2_{\mathrm{L}^2(\Omega_t)} + \iint_{\Pi_t} r^3T^2 f_n(t) + 2\int_0^t \|\nabla_x\Theta_n\|^2_{\mathrm{L}^2(\Omega_s)} + 2\iiint_{\widehat{\Pi}_t} r|\Theta_n - T|^2\eta f_n \\ \leq 6\iiint_{\widehat{\Pi}_t} rT^2\eta f_n + 4\iiint_{\widehat{\Pi}_t} rT\eta f_n + \iint_{\Pi_0} r^3T^2 f_n^0 \\ + 2\iiint_{\widehat{\Pi}_t} rT\left[\chi_n(\Theta_n - T) - (\Theta_n - T)\right]\eta f_n. \quad (2.4.17)\end{aligned}$$

As in the introduction of this chapter, we deal with the first two terms of the right-hand side by ensuring, thanks to the Grönwall lemma again, that

$$\iint_{\Pi_t} r^3 f_n(t) \lesssim 1. \quad (2.4.18)$$

Since $|\chi_n(\Theta_n - T) - (\Theta_n - T)| \leq 2|\Theta_n - T|\mathbf{1}_{|\Theta_n - T| > n-1/n}$ and thanks to the Young inequality and the definition of the truncation η, we also have

$$\begin{aligned}\iiint_{\widehat{\Pi}_t} rT\left[\chi_n(\Theta_n - T) - (\Theta_n - T)\right]\eta f_n \leq 2\iiint_{\widehat{\Pi}_t} rT|\Theta_n - T|\eta f_n \\ \leq \frac{4}{r_{\mathrm{drug}}}\iiint_{\widehat{\Pi}_t} r^3T^2 f_n + \frac{1}{2}\iiint_{\widehat{\Pi}_t} r|\Theta_n - T|^2\eta f_n. \quad (2.4.19)\end{aligned}$$

Then, (2.4.17) leads to

$$\iint_{\Pi_t} r^3T^2 f_n \lesssim 1, \quad (2.4.20)$$

thanks to the Grönwall lemma again. Note that by a straightforward application of Fatou's lemma, (2.4.18) and (2.4.20) yield

$$\iint_{\Pi_t} r^3T^2 f < \infty, \qquad \iint_{\Pi_t} r^3 f < \infty.$$

As a conclusion, we can write

$$\|\Theta_n(t)\|^2_{\mathrm{L}^2(\Omega_t)} + \iint_{\Pi_t} r^3T^2 f_n(t) + 2\int_0^t \|\nabla_x\Theta_n\|^2_{\mathrm{L}^2(\Omega_s)} + \iiint_{\widehat{\Pi}_t} r|\Theta_n - T|^2\eta f_n \lesssim 1. \quad (2.4.21)$$

Therefore $(\Theta_n)_{n\in\mathbb{N}}$ and $(\nabla_x\Theta_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{L}^2(\widehat{\Omega})$. As a consequence, there exists $\Theta, H \in \mathrm{L}^2(\widehat{\Omega})$ such that, up to a subsequence, $(\Theta_n)_{n\in\mathbb{N}}$ and $(\nabla_x\Theta_n)_{n\in\mathbb{N}}$ converge to Θ and H, respectively, in w-$\mathrm{L}^2(\widehat{\Omega})$. Considering this in the distribution sense, we recover that $H = \nabla_x\Theta$ and therefore $\Theta \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$.

Let $\mathcal{B}$ be an open ball such that, for every $t \in [0,\tau]$, $\overline{\Omega_t} \subset \mathcal{B}$. We define $(\overline{\Theta_n})_{n\in\mathbb{N}}$ as the sequence of continuations of $(\Theta_n)_{n\in\mathbb{N}}$ by 0 on $(\mathbb{R}_+^* \times \mathcal{B})\backslash\widehat{\Omega}$. From Estimate (2.4.21), we infer that $(\overline{\Theta_n})_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}_+^*;\mathrm{L}^2(\mathcal{B}))$. Therefore, up to a subsequence, $(\overline{\Theta_n})_{n\in\mathbb{N}}$ converges in w^*-$\mathrm{L}^\infty(\mathbb{R}_+^*;\mathrm{L}^2(\mathcal{B}))$ and, by a similar argument as above, its limit equals Θ on $\widehat{\Omega}$.

In the same way, we obtain an estimate on $(Y_n)_{n\in\mathbb{N}}$. Choosing $\zeta = Y_n\mathbf{1}_{[0,t]}$ in (2.3.8), we get, since $0 \leq Y_n, Y_{v,\mathrm{surf}} \leq 1$,

$$\frac{1}{2}\|Y_n(t)\|^2_{\mathrm{L}^2(\Omega_t)} + \int_0^t \|\nabla_x Y_n\|^2_{\mathrm{L}^2(\Omega_s)} \leq \iiint_{\widehat{\Pi}_t} r\eta f_n + \frac{1}{2}\left\|Y_n^0\right\|^2_{\mathrm{L}^2(\Omega_0)},$$

hence, using (2.4.18),

$$\frac{1}{2}\|Y_n(t)\|^2_{\mathrm{L}^2(\Omega_t)} + \int_0^t \|\nabla_x Y_n\|^2_{\mathrm{L}^2(\Omega_s)} \lesssim 1. \tag{2.4.22}$$

As above, we infer that there exists $Y \in \mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^2(\mathcal{B})) \cap \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$ such that, up to a subsequence, $(Y_n)_{n\in\mathbb{N}}$ converges to Y in w-$\mathrm{L}^2(\widehat{\Omega})$ and w^*-$\mathrm{L}^\infty(\mathbb{R}_+^*;\mathrm{L}^2(\mathcal{B}))$.

The weak convergences we have obtained are enough to justify the asymptotics $n \to +\infty$ in the linear terms appearing in (2.2.18)–(2.2.21). For the other terms, we will need to prove strong compactness for $(u_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and $(\Theta_n)_{n\in\mathbb{N}}$.

Let us first consider the sequence $(u_n)_{n\in\mathbb{N}}$. We apply Theorem 4 of [Mou16] which is an Aubin-Lions type of result specifically engineered to deal with the Navier-Stokes equation in a time-dependent domain. Namely, under the additional hypotheses that the sequence of normal traces of u_n in $\mathrm{L}^2(0,\tau;\mathrm{H}^{-\frac{1}{2}}(\Omega_t))$ is compact and that we have a control of the action of $\partial_t u_n$ on test functions with divergence equal to zero, the sequence $(u_n)_{n\in\mathbb{N}}$ converges in $\mathrm{L}^2(\widehat{\Omega})$ up to a subsequence.

The first hypothesis is verified since, thanks to (2.4.16), $(u_n)_{n\in\mathbb{N}}$ converges to w in $\mathrm{L}^2(\widehat{\Omega}^c)$ and the normal trace operator on $\widehat{\Gamma}$ is continuous from $\{\varphi \in \mathrm{L}^2(\widehat{\Omega}^c), \operatorname{div}_x \varphi \in \mathrm{L}^2(\widehat{\Omega}^c)\}$ to $\mathrm{L}^2(0,\tau;\mathrm{H}^{-\frac{1}{2}}(\Omega_t))$. Note that this implies $u - w \in \mathcal{V}_0$.

For the second one, we can prove that, for any $\varphi \in \mathscr{D}(\widehat{\Omega})$ such that $\operatorname{div}_x \varphi = 0$, we have

$$\left|\int_0^t \int_{\Omega_t} \partial_t u_n \cdot \varphi\right| \lesssim \|\varphi\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}.$$

This is obtained thanks to the Hölder and Gagliardo-Nirenberg inequalities, the continuous injection $\mathrm{H}^2(\Omega_t) \hookrightarrow \mathrm{L}^\infty(\Omega_t)$, and Estimate (2.4.16). We only detail one computation : since $|\chi(v)|^2 \leq \chi(v)\cdot v$ for all $v \in \mathbb{R}^3$, the Galgiardo-Nireberg and Cauchy-Schwarz inequalities yield

$$\begin{aligned}\left|\iiint_{\widehat{\Pi}} r\eta f_n \chi_n(v-u_n)\cdot\varphi\right| &\leq \int_0^\tau \|\varphi\|_{\mathrm{H}^2(\Omega_t)} \iint_{\Pi_t} r\eta f_n|\chi_n(v-u_n)| \\ &\leq \int_0^\tau \|\varphi\|_{\mathrm{H}^2(\Omega_t)} \left(\iint_{\Pi_t} r\eta f_n\right)^{\frac{1}{2}} \left(\iint_{\Pi_t} r\eta f_n |\chi_n(v-u_n)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sup_{t\in(0,\tau)} \iint_{\Pi_t} r^3 f_n\right)^{\frac{1}{2}} \left(\iiint_{\widehat{\Pi}} r\eta f_n \chi_n(v-u_n)\cdot(v-u_n)\right)^{\frac{1}{2}} \|\varphi\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))} \\ &\lesssim \|\varphi\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}\end{aligned}$$

thanks to (2.4.16) and (2.4.18). Therefore, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathrm{L}^2(\widehat{\Omega})$.

To obtain compactness for the sequence $(\Theta_n)_{n\in\mathbb{N}}$, we apply Corollary 1 of [Mou16]. Since the sequence is bounded in $\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega_t))$, we just have to prove that, for all $\zeta \in \mathscr{D}(\widehat{\Omega})$,

$$\iint_{\widehat{\Omega}} \Theta_n \partial_t \zeta \lesssim \|\zeta\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^2(\Omega_t))}.$$

We do not detail the proof since the right-hand side term of the equation is dealt with in the same way as for $(u_n)_{n\in\mathbb{N}}$ by using of (2.4.18) and (2.4.21). Therefore, up to a subsequence, $(\Theta_n)_{n\in\mathbb{N}}$ converges to Θ in $\mathrm{L}^2(\widehat{\Omega})$. The same goes for $(Y_n)_{n\in\mathbb{N}}$, converging to Y in $\mathrm{L}^2(\widehat{\Omega})$ up to a subsequence.

This allows to take the limit $n \to +\infty$ in all the nonlinear terms that appear in Definition 2.2.2. We focus on the only difficulty arising in the asymptotics in the coupling terms from the right-hand sides of the equations. We tackle this case for the Navier-Stokes equation but a similar treatment provides the expected results for the convection-diffusion equations.

Thanks to the strong convergence of $(u_n)_{n\in\mathbb{N}}$, we obtain the L^2 convergence of $(r\eta\chi_n(v - u_n(t)))_{n\in\mathbb{N}}$ in a bounded subset of $\Omega_t \times \mathbb{R}^3 \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ for almost every t. Indeed, let $R, \theta > 0$ and K_v a compact of $\mathbb{R}^3$. Up to a subsequence, we can assume that $(u_n)_{n\in\mathbb{N}}$ converges almost everywhere to $u \in \mathrm{L}^2(\widehat{\Omega})$. Since $(u_n(t))_{n\in\mathbb{N}}$ converges to $u(t)$ in $\mathrm{L}^2(\Omega_t)$ for almost every t, the Vitali convergence theorem ensures that $(|u_n(t)|^2)_{n\in\mathbb{N}}$ is uniformly integrable. Since $|\chi_n(v - u_n(t))| \le |v| + |u_n(t)|$ and K_v is bounded, we deduce from this that $(|\chi_n(v-u_n(t))|^2)_{n\in\mathbb{N}}$ is uniformly integrable for almost every t. Therefore, since this sequence converges almost everywhere, the Vitali convergence theorem ensures the convergence of $(\chi_n(v - u_n(t)))_{n\in\mathbb{N}}$ to $v - u(t)$ in $\mathrm{L}^2(\Omega_t \times K_v)$ for almost every t. This leads to $(r\eta\chi_n(v - u_n(t)))_{n\in\mathbb{N}}$ converging to $(r\eta(v - u(t)))_{n\in\mathbb{N}}$ in $\mathrm{L}^2(\Omega_t \times K_v \times [0, R] \times [0, \theta])$ for almost every $t \in (0, \tau)$.

Let $\varphi \in \mathscr{D}(\widehat{\Omega})$. Thanks to the bounds (2.4.16) and (2.4.20), we can prove that the following convergence occurs uniformly with respect to $n \in \mathbb{N}$:

$$\iiint_{\widehat{\Pi}} \mathbf{1}_{|v|\le M}\mathbf{1}_{r\le M}\mathbf{1}_{T\le M} r\eta f_n\chi_n(v-u_n)\cdot\varphi \xrightarrow[M\to+\infty]{} \iiint_{\widehat{\Pi}} r\eta f_n\chi_n(v-u_n)\cdot\varphi.$$

Therefore,

$$\lim_{M\to+\infty}\lim_{n\to+\infty}\iiint_{\widehat{\Pi}} \mathbf{1}_{|v|\le M}\mathbf{1}_{r\le M} r\eta f_n\chi_n(v-u_n)\cdot\varphi \\ = \lim_{n\to+\infty}\lim_{M\to+\infty}\iiint_{\widehat{\Pi}} \mathbf{1}_{|v|\le M}\mathbf{1}_{r\le M}\mathbf{1}_{T\le M} r\eta f_n\chi_n(v-u_n)\cdot\varphi.$$

Then we deduce from the local convergence we have just obtained, the dominated convergence theorem and the weak convergence of $(f_n)_{n\in\mathbb{N}}$ that

$$\iiint_{\widehat{\Pi}} \mathbf{1}_{|v|\le M}\mathbf{1}_{r\le M}\mathbf{1}_{T\le M} r\eta f_n\chi_n(v-u_n)\cdot\varphi \xrightarrow[n\to+\infty]{} \iiint_{\widehat{\Pi}} \mathbf{1}_{|v|\le M}\mathbf{1}_{r\le M}\mathbf{1}_{T\le M} r\eta f(v-u)\cdot\varphi.$$

Besides, thanks to (2.4.16), (2.4.18) and Fatou's lemma, we can prove that $r\eta f(v - u) \in \mathrm{L}^1(\widehat{\Pi})$, which allows to take the limit $M \to +\infty$ and obtain the convergence

$$\iiint_{\widehat{\Pi}} r\eta f_n\chi_n(v-u_n)\cdot\varphi \xrightarrow[n\to+\infty]{} \iiint_{\widehat{\Pi}} r\eta f(v-u)\cdot\varphi.$$

As a conclusion, we have obtained that u, Y, Θ and f weakly solve a system of equations similar to Problem (2.2.2)–(2.2.14). We have yet to deal with the factor η still appearing in these equations. Namely, we finally prove that under the assumption that $f^0(\cdot, \cdot, r, \cdot) \equiv 0$ for almost every $r \in [0, r_{\mathrm{drug}}]$, we have $f(\cdot, \cdot, \cdot, r, \cdot) \equiv 0$ for $r \in [0, r_{\mathrm{drug}}]$, so that $\eta f = f$. We use again the DiPerna-Lions theory (see [DL89]) which enables us, thanks to the uniqueness of the solution of the Vlasov equation, to consider a sequence of Vlasov equations with smooth coefficients for which the corresponding sequence of solutions $(f_n)_{n\in\mathbb{N}}$ strongly converges to f. We can thus assume that the functions $(f_n^0)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and $(\Theta_n)_{n\in\mathbb{N}}$ are smooth and that $Y_n \geq 0$ for all $n \in \mathbb{N}$ and $f_n^0(\cdot,\cdot,r,\cdot) \equiv 0$ for every $r \in [0, r_{\text{drug}}]$. We also take a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of $(0,1)$ with limit 0 and replace Y_n by $Y_n + \varepsilon_n$ in the part of the Vlasov equation that describes radius evolution. With these assumptions, the characteristic curves are defined by

$$\begin{cases} \dot{X}_n(t;t_0,x,v,r,T) = V_n, \\ \dot{V}_n(t;t_0,x,v,r,T) = \frac{R_n\eta(R_n)}{R_n^3+r_{\text{drug}}^3}\chi_n(u_n(t,X_n)-V_n), \\ \dot{R}_n(t;t_0,x,v,r,T) = \frac{Y_n(t,X_n)+\varepsilon_n-Y_{v,\text{surf}}(R_n,H_n)}{R_n}\eta(R_n), \\ \dot{H}_n(t;t_0,x,v,r,T) = \frac{Y_n(t,X_n)-Y_{v,\text{surf}}(R_n,H_n)}{R_n^2}\eta(R_n) + \frac{\Theta_n-H_n}{R_n^2}\eta(R_n), \\ (X_n,V_n,R_n,H_n)(t_0;t_0,x,v,r,T) = (x,v,r,T), \end{cases}$$

where $(x,v,r,T) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+^*\times\mathbb{R}_+^*$ and $t_0 \in [0,\tau)$. Thanks to the Cauchy-Lipschitz theorem, the previous differential system has a unique global solution $Z_n = (X_n, V_n, R_n, H_n)$.

Let $n \in \mathbb{N}$. Assume that $r > r_{\text{drug}}$, let $t_* = \inf\{t \geq t_0,\ R_n(t;t_0,x,v,r,T) = r_{\text{drug}}\}$ and suppose that $t_* < \tau$. Since $Y_{v,\text{surf}}(r_{\text{drug}},\theta) = 0$ for all $\theta > 0$, $Y_n \geq 0$ and $\varepsilon_n > 0$, we have $\dot{R}_n(t_*;t_0,x,v,r,T) > 0$, which contradicts the definition of t_* since $R_n(\cdot;t_0,x,v,r,T)$ is continuous. Therefore, if $r > r_{\text{drug}}$, then $R_n(t;t_0,x,v,r,T) > r_{\text{drug}}$ for all $t \geq t_0$.

Let $(t,x,v,r,T) \in \widehat{\Pi}$, with $r \leq r_{\text{drug}}$. By the methods of characteristics, there exists a function J_n such that, for every $(t,x',v',r',T') \in \widehat{\Pi}$,

$$f_n(t, Z(t;0,x',v',r',T')) = f_n^0(x',v',r',T')J_n(t;0,x',v',r',T').$$

Therefore, with $(x',v',r',T') = Z_n(0;t,x,v,r,T)$, we get

$$f_n(t,x,v,r,T) = f_n^0(Z_n(0;t,x,v,r,T))J_n(t, Z_n(0;t,x,v,r,T)).$$

If we had $R_n(0;t,x,v,r,T) > r_{\text{drug}}$, then, as proved in the last paragraph, we would have $r = R_n(t;t,x,v,r,T) > r_{\text{drug}}$. By contradiction, we get $R_n(0;t,x,v,r,T) \leq r_{\text{drug}}$ and the hypothesis on f_n^0 yields

$$f_n(t,x,v,r,T) = 0.$$

Consequently, for all $n \in \mathbb{N}$ and $r \in [0, r_{\text{drug}}]$, $f_n(\cdot,\cdot,\cdot,r,\cdot) \equiv 0$. The weak convergence of $(f_n)_{n\in\mathbb{N}}$ yields the same property for f. Therefore, $\eta f = f$, which concludes the proof of Theorem 2.2.3.

2.5 Numerical approach

After proving the existence of a weak solution, the rest of this chapter is dedicated to the numerical resolution in dimension 3 of the hygroscopic model presented in [Bou+20]. Therefore, we come back to the real, more physically relevant, model presented in Chapter 1 (page 8) and consider the system (1.3.1)–(1.3.4). We work in a fixed spatial domain Ω, whose boundary is divided into three subdomains $\Gamma = \Gamma^{\text{in}} \cup \Gamma^{\text{wall}} \cup \Gamma^{\text{out}}$ to which we shall refer as the inlet, the wall, and the exit, respectively. The boundary conditions are then given by (1.3.5)–(1.3.8).

Following [Bou+15 ; Bou+20], we only consider *very thin sprays*, for which the Brinkman force can be neglected in the Navier-Stokes equations (1.3.2) (F is set to 0). They are therefore

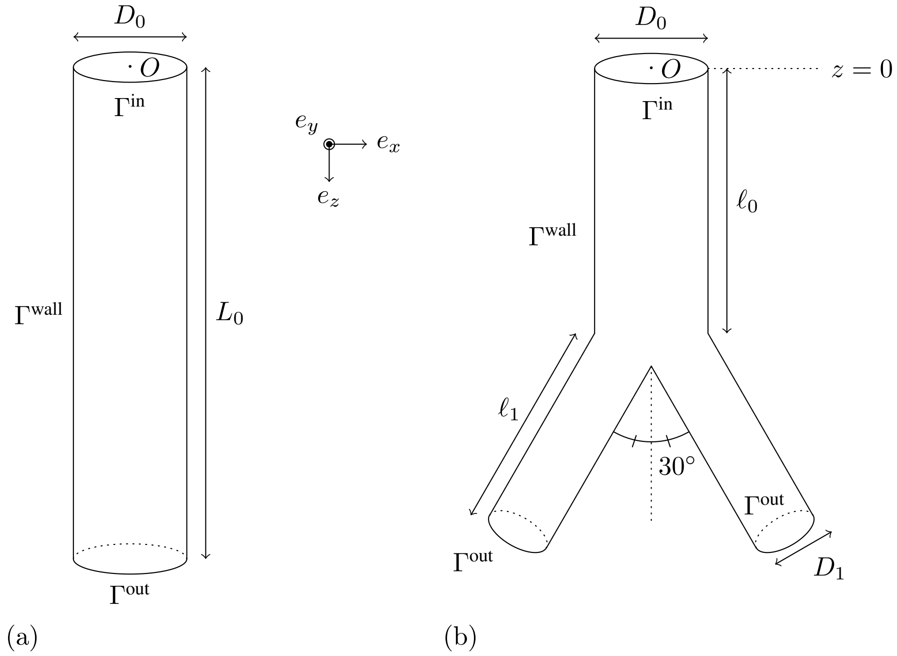


FIGURE 2.1 – Schematic view of (a) the cylindrical domain, (b) the branched structure.

decoupled from the remaining equations and can be solved independently. This hypothesis is of significant importance as we are led to execute lengthy three-dimensional calculations only once.

We implement the same numerical scheme as in [Bou+20] but in three dimensions. We invite the reader to refer to this article for the full presentation of the model, where the physical constants are not normalized as in the previous analysis, as well as for details about the scheme itself. In the following, we only recall the working assumptions and provide some remarks on the implementation. All the computations are performed with `FreeFem++` [Hec12] [1].

Our computations are performed on two distinct domains. The first one is a cylinder of diameter $D_0 = 0.8$ cm and length $L_0 = 9.4$ cm, see Figure 2.1a. The second one is a branched structure modelling an idealized trachea and the first generation of the airways. The length of the trachea is $\ell_0 = 2.9$ cm and its diameter is D_0. Both bronchi form an angle of 30° with the vertical axis, their length is set to $\ell_1 = 1.8$ cm, and their diameter is $D_1 = 0.6$ cm, see Figure 2.1b. In both cases, the center of the boundary Γ^{in} is the origin of the space coordinate system.

The cylindrical domain serves as a tool to validate our three-dimensional code. Since we neglect the aerosol retroaction term in (1.3.2), a Poiseuille profile at the inlet leads to a stationary Poiseuille flow solving the Navier-Stokes equations in Ω. We also checked that if the initial distribution of the particles has a cylindrical symmetry, then it is preserved through time. Finally, we use the cylindrical domain in Section 2.6 to test model simplifications in a simpler geometrical and computational setting than the branched structure.

As stated above, we neglect the Brinkman force in the Navier-Stokes equation (1.3.2), so that (1.3.2) form an uncoupled system of equations. We therefore solve (1.3.2) with initial condition $u^0 = 0$ and, at the domain inlet, a boundary condition u^{in} following a Poiseuille law, oriented

1. Note that we had to use version 3.43 as opposed to version 4.4.2 which was available at the time of because the function `convect` did not seem to properly behave in three dimensions.

vertically downwards and its modulus given, for $(x, y, 0) \in \Gamma^{\text{in}}$, by

$$|u^{\text{in}}(x, y, 0)| = u_0 \left(1 - \frac{x^2 + y^2}{(D_0/2)^2}\right),$$

with $u_0 = 50.0$ cm.s^{-1}. When we solve the equations, the velocity seems to reach a stationary state, at least in the domains we studied. We use this solution in the rest of the computations.

Recall that the Equation (1.3.3) describing the evolution of the water vapor mass fraction $Y_{\text{v,air}}$ reads

$$\rho_{\text{air}}(\partial_t Y_{\text{v,air}} + u \cdot \nabla_x Y_{\text{v,air}}) - \text{div}_x(D_{\text{v}}(T_{\text{air}})\nabla_x Y_{\text{v,air}}) = S_Y, \tag{2.5.1}$$

where ρ_{air} is the air mass density, $D_{\text{v}}(T_{\text{air}})$ is the binary diffusion coefficient of water vapor in the air, and S_Y accounts for the water mass exchanges between the air and the particles. In the rest of this numerical study, we consider the following conditions on the air temperature :

$$T^0_{\text{air}} = 310 \text{ K}, \qquad T^{\text{in}}_{\text{air}} = 297 \text{ K}, \qquad T_{\text{wall}} = 310 \text{ K}.$$

Then, a comparison of the orders of magnitude of the coefficients in (2.5.1) shows that the leading term is the diffusion one and therefore $Y_{\text{v,air}}$ should rapidly reach an equilibrium and the aerosol should not have any influence, which is confirmed by numerical simulations. For example, in the case of the branched structure, we have

$$\frac{\rho_{\text{air}}}{\Delta t} \sim 7 \times 10^{-1}\,\text{g.cm}^{-3}.\text{s}^{-1}, \qquad \frac{\rho_{\text{air}}|u|}{\Delta x} \sim 5 \times 10^{-1}\,\text{g.cm}^{-3}.\text{s}^{-1},$$

$$\frac{D_{\text{v}}(T^{\text{in}}_{\text{air}})}{(\Delta x)^2} \sim 11\,\text{g.cm}^{-3}.\text{s}^{-1}, \qquad |S_Y| \sim 5 \times 10^{-5}\,\text{g.cm}^{-3}.\text{s}^{-1},$$

where Δt is the time step and Δx is the average diameter of a cell of the three-dimensional mesh. Moreover, it appears that the variation of $D_{\text{v}}(T_{\text{air}})$ as T_{air} changes does not influence $Y_{\text{v,air}}$. Therefore, we solve Equation (1.3.3) independently with $D_{\text{v}}(T_{\text{air}})$ considered equal to $D_{\text{v}}(T^{\text{in}}_{\text{air}})$ and we set $Y^0_{\text{v,air}} = Y^{\text{in}}_{\text{v,air}} = 1.81\%$ and $Y^{\text{wall}}_{\text{v,air}} = 3.70\%$. These remarks lead to a decreasing computation time, but less significant than in the case of the Navier-Stokes equations.

Following [Bou+15, Remark 7], we solve the Vlasov equation (1.3.1) thanks to a time subcycling strategy. At each time step, we solve as many systems of ODEs of the following form as there are numerical particles p, i. e.

$$\dot{x}_p(t) = v_p(t), \tag{2.5.2}$$
$$\dot{v}_p(t) = \alpha(r_p(t))(u(t, x_p(t)) - v_p(t)) + g, \tag{2.5.3}$$
$$\dot{r}_p(t) = a(r_p(t), T_p(t), Y_{\text{v,air}}(t, x_p(t))), \tag{2.5.4}$$
$$\dot{T}_p(t) = b(r_p(t), T_p(t), Y_{\text{v,air}}(t, x_p(t)), T_{\text{air}}(t, x_p(t))). \tag{2.5.5}$$

Note that these ODEs for a given particle p are independent from the ODEs for all the other ones and it is therefore natural to perform a parallel computation to solve them and thus greatly reduce the computation time. We must emphasize that the number of numerical particles remains very small with respect to the number of physical ones, which implies a slender computational cost.

Finally, in order to avoid needing a precise parametrization of the boundary when dealing with the deposition condition, i.e. to be able to simply load a mesh file and solve the equations and verify the deposition criterion below without more precise knowledge of the domain, we define a standard notion of distance to the considered subset Σ of Γ by solving

$$\begin{cases} -\Delta_x g_\Sigma &= 1 \quad \text{on } \Omega, \\ g_\Sigma &= 0 \quad \text{on } \Sigma, \\ \nabla_x g_\Sigma \cdot n &= 0 \quad \text{on } \Gamma\backslash\Sigma, \end{cases}$$

and setting

$$\forall x \in \Omega, \qquad d_\Sigma(x) = \begin{cases} \frac{g_\Sigma(x)}{\|\nabla_x g_\Sigma(x)\|} & \text{if } \frac{g_\Sigma(x)}{\nabla_x g_\Sigma(x)} < \eta, \\ \eta & \text{else,} \end{cases}$$

and $d_\Sigma(x) = g_\Sigma(x) = 0$ if $x \in \Sigma$, for some $\eta > 0$. The functions $d_\Sigma \geq 0$ satisfy $d_\Sigma(x) = 0 \Leftrightarrow x \in \Sigma$ and enable us to discriminate between particles near the boundary under consideration.

The deposition/exit test is therefore performed as follows at each time step of the subcycle solving the ODEs for the particles.

- If at some time step, the droplet is outside Ω, we go back to the previous position (still inside the domain) and compare the distances of the particle to the wall, and both exits, and consider that the particle was deposited or exited accordingly.
- If the droplet remains inside Ω, we assume that it is deposited if both the following conditions are met :

  $$d_{\Gamma^{\text{wall}}}(x_p) \leq \alpha\, h_{\min} \qquad \text{and} \qquad v_p\, \Delta t \leq \alpha\, h_{\min},$$

  where $h_{\min}$ is the minimal diameter of a mesh cell and $\alpha > 0$. Let us emphasize the fact that Δt is the global time step, as opposed to the one used to solve the ODEs. These conditions mean that if the particle is close enough to the wall and its velocity is small enough, then we consider the particle to be deposited. This is physically justified by the fact that under these assumptions, the mucus on the wall tends to force the adhesion of the particle to the wall, see [KE85].

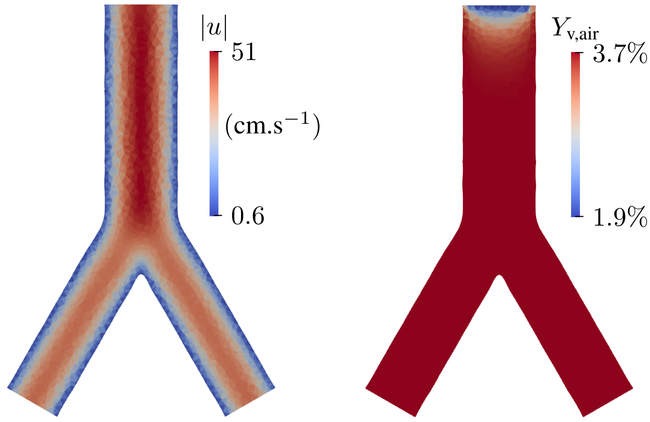
Note that the definition of $d_{\Gamma^{\text{wall}}}$ implies to take η greater than $\alpha h_{\min}$ for the deposition test to make sense. Furthermore, we have investigated the influence of the choice of α for values between 0.5 and 2 and have found the deposition rate to have a *stable* behavior, in the sense that a small variation of the parameter α does not induce great chances to the deposition rate. For instance, when a set of initial positions of the particles leads to 3 deposited numerical particles for $\alpha = 0.5$, it leads to 19 deposited particles for $\alpha = 2$. Finally, working under the initial conditions presented above, we find that the limiting factor for deposition is the distance to the wall rather than the velocity.

An intuitive choice would have been to say that a particle with radius r_p at position x_p is deposited if $d_{\Gamma^{\text{wall}}}(x_p) \leq r_p$. This is implemented in [Bou+20] and the numerical simulations are run on a very fine mesh. On the contrary, for the sake of simplicity, we choose to use a mesh sufficiently fine *fluid-wise* but not *particle-wise*. Therefore, we need to adapt the deposition condition as described above.

We also considered the following deposition criterion

$$d_{\Gamma^{\text{wall}}}(x_p) \leq \ell_{\text{lash}},$$

where $\ell_{\text{lash}} = 0.1$ cm is the average length of a bronchial lash. It led to a deposition rate of 20%, a different order of magnitude from the 8% obtained in [Bou+20] and was not further investigated but should be taken into account in further research.



(a) (b)

FIGURE 2.2 – (a) Modulus of the velocity $|u|$ and (b) water vapor mass fraction $Y_{\mathrm{v,air}}$ in the cross-section $y = 0$.

2.6 Numerical results

2.6.1 Full model

As stated above, all the tests discussed below were investigated both in the cylindrical domain and the branched structure presented in Figure 2.1 and led to the same kind of comments and conclusions. Therefore, we only present our results in the latter case.

Our experiment consists in releasing five waves of 100 numerical particles, each representing 10^4 physical particles, between the times $\Delta t = 1.53454\ 10^{-3}$ s and $101\Delta t$. All the particles initially have the same velocity $v_p(0) = 50\ e_z$ (in cm.s^{-1}), the same radius $r_p(0) = r_{\mathrm{drug}} = 22.5\ \mu$m, and the same temperature $T_p(0) = T_{\mathrm{air}}^{\mathrm{in}}$. Their position is defined randomly in the disk of radius $D_0/4$ centered on the origin and lying in the plane $z = 0$.

As mentioned in the previous section, the air velocity u and the water vapor mass fraction $Y_{\mathrm{v,air}}$ are assumed to be stationary. Figure 2.2 displays the profile of $|u|$ and $Y_{\mathrm{v,air}}$ in the cross-section $y = 0$ of the bifurcation.

In Figure 2.3, we show the evolution of the particles in the domain. At first, in the trachea, the bundle of droplets assumes the shape of a paraboloid, which is consistent with the fact that the air flow is almost Poiseuille in this region.

The evolution of the air temperature as well as the particles is represented in Figure 2.4. Again, as expected, the temperature in the trachea evolves as if the air propagated as a Poiseuille flow, the *cold wave* assuming the shape of a parabola in the cross section $y = 0$. We notice that the particles move a little faster than the wave, because of the gravitational effects.

Figure 2.5 allows to see the local effects of the particles on the air temperature as it displays the temperature at a given time (a) with and (b) without the particles. As noticed in the two-dimensional case in [Bou+20], the droplet temperature is higher than the cold wave and they heat the air surrounding them as they go through the branched structure. This effect is not negligible as the air temperature around the particles can go up to about 302 K.

FIGURE 2.3 – Dynamics of the particles at times $t = (10 + 25i)\Delta t$, $0 \leq i \leq 7$, i.e. between times $t_0 = 1.53454\ 10^{-3}$ s and $t_7 = 2.8389\ 10^{-2}$ s.

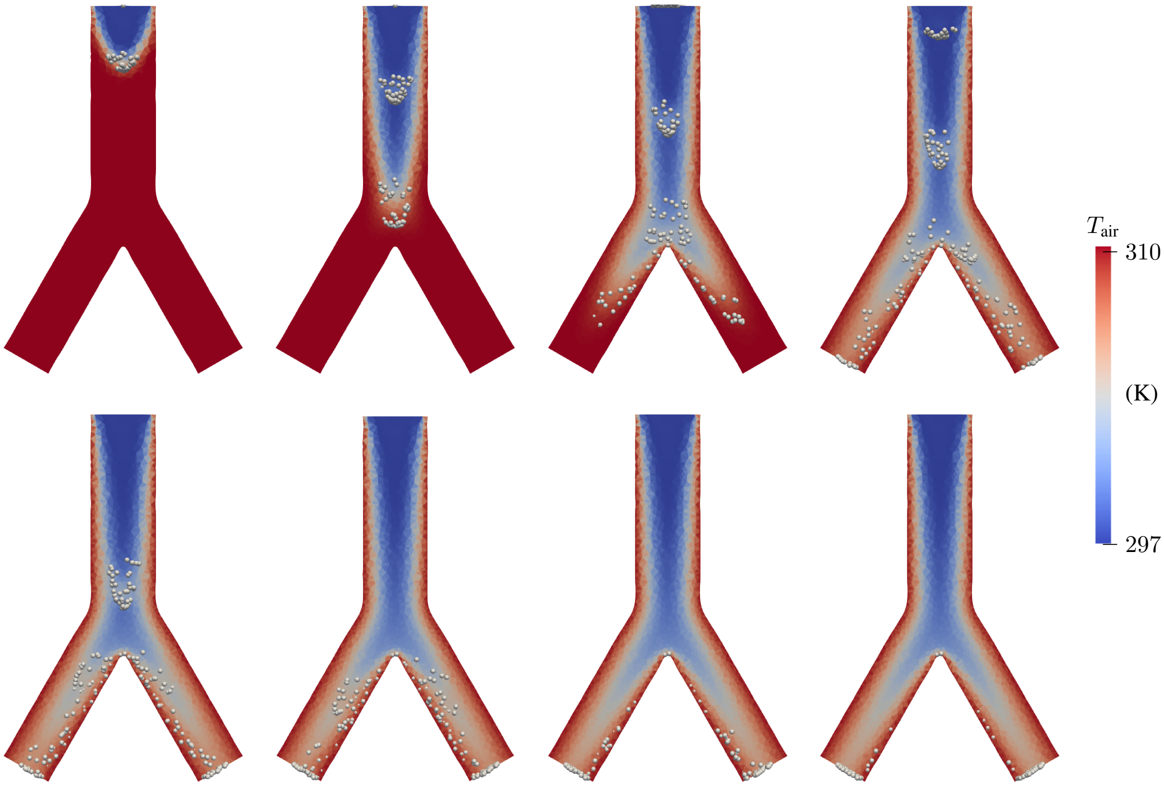


FIGURE 2.4 – Dynamics of the particles and the air temperature, at times $t = (10 + 25 \times i)\Delta t$, $0 \leq i \leq 7$. The temperature is displayed on the domain $y = 0$ while the particles with a positive y coordinate are projected onto the plane $y = 0$.

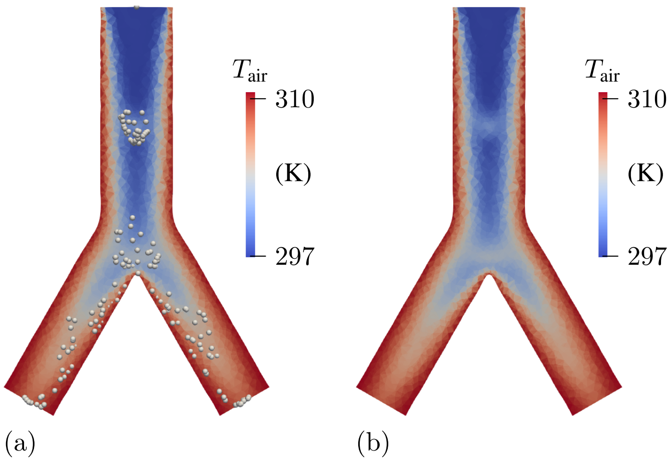
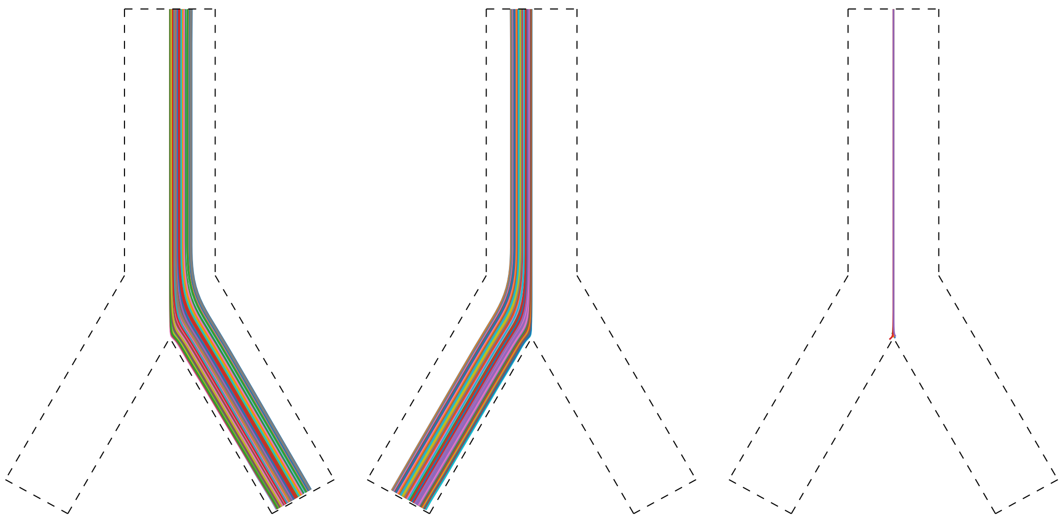


FIGURE 2.5 – Local effects of the aerosol on the air temperature, at time $t = 0.12123$ s



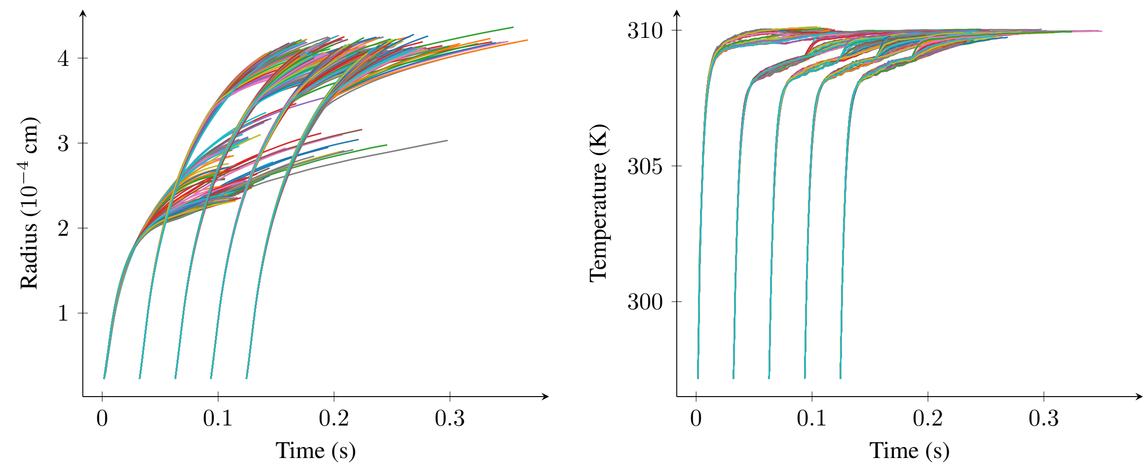
(a) (b) (c)

FIGURE 2.6 – Particle trajectories in the full model, the plane $y = 0$ (a) towards the left branch, (b) towards the right branch, (c) until deposition.

Let us now present the results for the particles. The following plots are the results of one computation. Yet we find similar behaviors, from a statistical point of view, regardless of the initial uniform distribution of the particles in the disk of radius $D_0/4$. In our test case, 5 particles are deposited, 239 exit the domain through the left branch (we use the patient's point of view) and 256 go through the right branch. Figure 2.6 displays their trajectories. We can see that only particles at the center of the inlet are deposited at the cusp of the branched structure and there is no deposition on the walls of the branches. The domain being asymmetrical in the case study of [Bou+20], we cannot draw much conclusions for the comparison of the results. Note nonetheless that the deposition of droplets, as expected, is lower for our three-dimensional domain.

On Figure 2.7, we represent the evolution of the radius and temperature of the particles until their exit from the domain or their deposition. Similarly to what is described in two dimensions in [Bou+20], the first injection of particles behaves quite differently from the other four. The explanation lies in the fact that they encounter different air temperatures. Indeed, the first particles are transported through hot air, the cold wave being just behind them, while the other injections flow through cold air. Note that the entry of the particles into each branch coincides with the spreading of radii and temperatures among the injections.

Since we run computations with a random initial distribution of particles, we need to repeat the experiment 10 times and average to draw conclusions on the data we obtain. In Table 2.1 (Full Model column, Model B being described below), we provide the mean percentage of deposited particles and of particles passing through the left or right branch and the corresponding mean times, as well as the mean radius and temperature of the droplets after their exit or deposition. To determine the accuracy, we compute the standard deviation with two significant figures and approximate the mean value accordingly. For example, we find a standard deviation of $0.52\ 10^{-4}$ cm for the radius of the deposited particles and, consequently, the mean value is given as $3.26\ 10^{-4}$ cm. Yet, for the temperature, we obtain a very low standard deviation of 0.082 K, which is irrelevant with respect to the precision of the numerical scheme and we therefore only provide 3 significant figures for these data.



(a) (b)

FIGURE 2.7 – (a) Radius and (b) temperature evolution of the particles.

Mean	**Full Model**	**Model B**
Deposition rate	1.62 %	1.62 %
Left exit rate	49.1 %	49.1 %
Right exit rate	49.28 %	49.28 %
Radius depos. (10^{-4} cm)	3.26	3.37
Radius left exit (10^{-4} cm)	3.82	3.91
Radius right exit (10^{-4} cm)	3.81	3.91
Temperature depos. (K)	310	310
Temperature left exit (K)	310	310
Temperature right exit (K)	310	310
Depos. time (s)	0.1423	0.1424
Left exit time (s)	0.1935	0.1934
Right exit time (s)	0.1938	0.1937

TABLEAU 2.1 – Statistics for the particles in the branched structure.

2.6.2 Potential simplifications of the model

Let us now consider simplifications that could be made to the model. Our main focus is the effect of such modifications on the eventual deposition and the location of the deposited particles, as well as the evolution of the size of the droplets. The first simplification is tested in both cylindrical and branched structure cases while the other ones are only tested in the cylinder.

2.6.2.1 Can we assume that the water vapor mass fraction is constant in Ω ?

As explained in the previous section, we consider the water vapor mass fraction $Y_{\mathrm{v,air}}$ to be constant in time. As we can see on Figure 2.2b, $Y_{\mathrm{v,air}}$ does not vary much in the branched structure and only differs from Y_{wall} close to the inlet. We now assume that $Y_{\mathrm{v,air}} = Y_{\mathrm{wall}}$ is constant throughout the domain. The remaining equations are (1.3.1),(1.3.2) and (1.3.4) and constitute Model B. For our experiments, we set the particles in the same initial conditions as above, and compare the results. We find that the global behavior of a droplet is not changed. If a particle exits (left or right) or is deposited with a space-dependent $Y_{\mathrm{v,air}}$, it does so with a constant $Y_{\mathrm{v,air}}$ too. Statistics on the mean radii, temperatures and deposition/exit times are provided in Table 2.1 (Model B). In order to compare more accurately the differences between the models, Table 2.2 provides statistics on the relative errors that we compute for each particle of our 10 experiments. We do not provide the values for the temperature as the mean relative error is less than the precision that can be expected from the approximation scheme.

	Mean relative error	**Standard deviation**
Radius depos.	3.5 %	1.9 %
Radius left exit	2.7 %	1.3 %
Radius right exit	2.7 %	1.3 %
Depos. time	0.02 %	0.03 %
Left exit time	0.05 %	0.09 %
Right exit time	0.05 %	0.1 %

Tableau 2.2 – Statistics on the relative errors at end time between the Full Model and Model B.

2.6.2.2 Can we further assume that water vapor mass fraction at the droplet surface is constant ?

As we explained above, we have validated our code in the case of a cylindrical domain, where all the computations can be done by hand, and the outcomes do not differ from the Reference case of the branched structure (except that there is no deposition in the cylinder). In particular, the assumption $Y_{\mathrm{v,air}} = Y_{\mathrm{wall}}$ leads to results which are similar to those of the previous paragraph. To answer the question under review in the following, we only consider the case of a cylindrical domain.

As shown in Figure 2.8, the water vapor mass fraction at the droplet surface seems to quickly reach an equilibrium. Though, for the particles in the first injection, this value seems a little greater than for the other injections, we now consider that $Y_{\mathrm{v,surf}}$ is a constant. This simplifies the differential equation (2.5.4) satisfied by the radius as it becomes :

$$\dot{r} = \frac{cD_{\mathrm{v}}(T_{\mathrm{air}})}{r}$$

for some constant $c > 0$. If $D_{\mathrm{v}}(T_{\mathrm{air}})$ were constant, the radii would then be expressed analytically thanks to a square root, which is in fact reminiscent of the profiles of Figure 2.7a. Yet the computations result in a droplet temperature behavior that is completely different from the reference case, as shown in Figure 2.9, and that does not seem physically relevant.

2.6.2.3 Can we assume the air and particle temperatures to be constant ?

In [Bou+20], the authors show that considering the air and particle temperatures as constants results in a 367 % increase in the deposition rate and a 155 % increase in the final mean radius. We have considered this hypothesis in the cylinder case and have found an increase of 30 % in the final mean radius. Such an increase, in coherence with the two-dimensional results, is enough to discard the hypothesis and we do not investigate further the case of a branched structure.

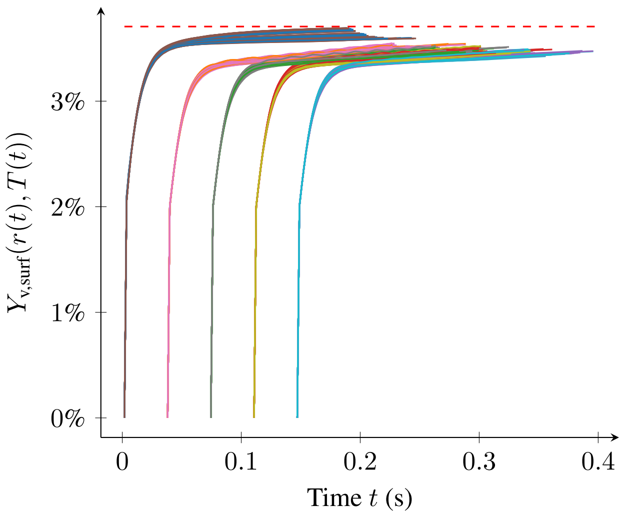
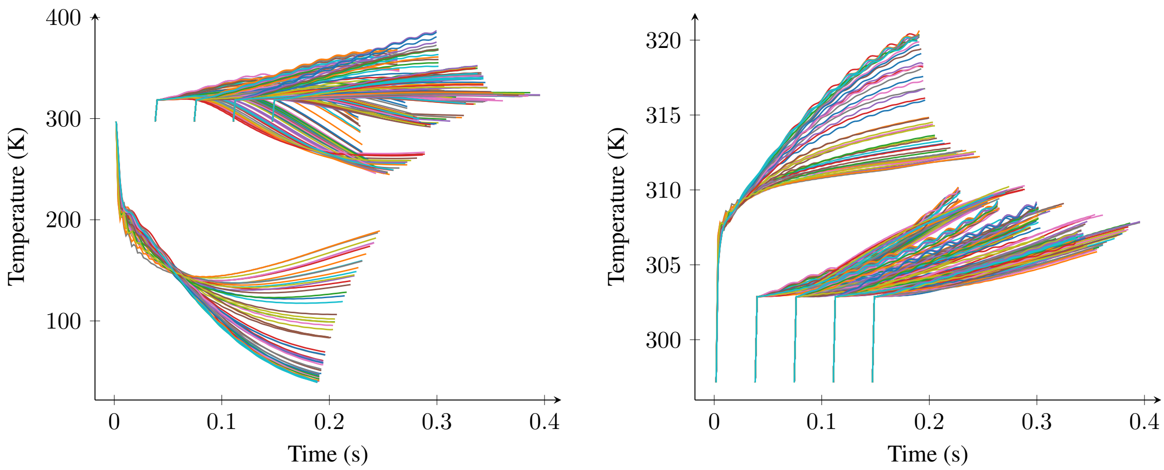


FIGURE 2.8 – Water vapor mass fraction at the surface of the droplet in the cylinder. The dotted line represents the value $Y_{\text{v,air}} = Y_{\text{wall}}$.



(a) (b)

FIGURE 2.9 – Temperature of the particles in the cylinder for $Y_{\text{v,air}} = Y_{\text{wall}}$ and (a) $Y_{\text{v,surf}} = 0.03$ and (b) $Y_{\text{v,surf}} = 0.035$.

Note that if we also consider $Y_{\text{v,surf}}$ to be constant, equal to 3.5%, then we obtain an increase of 18 % in the final mean radius.

2.6.3 Conclusions

We can compare our results with those obtained in two dimensions in [Bou+20]. We find similar evolutions for the radius and temperature of the particles in the reference case. The particles flowing through the trachea also have a local influence on the air temperature, heating it by up to 1.6 %. Yet, the air temperature reaches a stationary state that does not depend on the fact that particles have been present in the domain.

As expected from the symmetry of the domain, the left and right exit rate are almost the same. The deposition phenomenon for small-sized particles seems to be mainly governed by the geometry of the domain and the initial position of the particles. An increase of around 3% in the particle radius between the Full Model and Model B does not lead to a change in the deposition of the droplets. We cannot really comment on the deposition rate as the domain considered in [Bou+20] was more realistic and asymmetrical. To provide a meaningful comparison, we should compare our three-dimensional results to two-dimensional results in a symmetrical bifurcation. We could also create a three-dimensional mesh resembling the two-dimensional one from [Bou+20]. We should expect a lower deposition rate in dimension three since the particles have more space when they come close to the walls of the branch, see [Bou+15].

When studying simplifications of the model, we find that the water vapor mass fraction in the air does not influence the behavior of the particles. Indeed, when we assume it is constant, it almost does not effect the temperature or trajectory of the droplets. Futhermore, this assumption induces a mean relative error on the radii of the deposited particles of around 3.5%, which can be neglected in this context. On the contrary, considering the water vapor mass fraction at the surface of the droplets to be constant leads to irrelevant results. Similarly if the water vapor mass fraction in the air and the particle and air temperatures are taken as constants, the changes on the radii are too significant for this simplification to be adopted.

As was already stated in [Bou+20], further investigations should focus on the effect of the excipient on the aerosol behavior, the effects of the geometry (since we only considered domains with a vertical main axis), or the influence of the initial radius on the deposition of the particles. The presence of mucus moving towards the inlet under the influence of the motion of the bronchial lashes should be modelled more precisely. Finally, taking into account the variation of physical parameters or the presence of an obstruction in the airways would improve the modelling of respiratory diseases.

Chapitre 3

Limites hydrodynamiques pour le système de Vlasov–Navier–Stokes

En collaboration avec Daniel Han-Kwan (Cmls, École polytechnique, Cnrs), nous présentons dans [HM21] un cadre permettant de traiter des limites hydrodynamiques du système de Vlasov–Navier–Stokes. Plus précisément, nous étudions des régimes de haute friction prenant en compte la légèreté (respectivement la petitesse) des particules et ayant pour limite le système appelé Transport-Navier–Stokes (respectivement Transport-Navier–Stokes inhomogène).

Sommaire du présent chapitre

3.1 Introduction

This work is concerned with the mathematical description of the motion of a dispersed phase of small particles (*e.g.* a spray or an aerosol) flowing in a surrounding incompressible homogeneous fluid. In the class of the so-called *fluid-kinetic* models (see [Wil85], [ORo81]), the cloud of particles (resp. the fluid) is described by its distribution function f in the phase space (resp. its velocity u and its pressure p). We assume that the particles do not directly act on one another and that their interaction with the fluid can be represented by a drag acceleration of the particles and, conversely, a drag force applied to the fluid, called the Brinkman force. This leads to the Vlasov–Navier–Stokes system :

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \operatorname{div}_v [f(u - v)] = 0, \end{cases} \tag{3.1.1}$$

where

$$\rho_f(t,x) := \int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v,$$
$$j_f(t,x) := \int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v$$

stand respectively for the density and momentum of the particles.

In this work, we study (3.1.1) for $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$, where $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3$. The flat torus $\mathbb{T}^3$ is equipped with the normalized Lebesgue measure, so that $\mathrm{Leb}(\mathbb{T}^3) = 1$. The equations (3.1.1) are endowed with the initial conditions

$$u|_{t=0}(x) = u^0(x),$$
$$f|_{t=0}(x,v) = f^0(x,v).$$

The Vlasov–Navier–Stokes system can be (at least formally) derived as a *mean-field* limit of an N-particle system interacting with a fluid. For some (partial) rigorous results, we refer to [All90 ; DGR08 ; Hil18 ; HMS19 ; CH20 ; GH19 ; Höf18a ; Mec19], though the full justification is still an outstanding open problem.

In another direction, [Ber+17; Ber+18] proposed a program (similar to the Bardos-Golse-Levermore program [BGL91] for the hydrodynamic limits of the Boltzmann equation) to derive (3.1.1) from multiphase Boltzmann equations describing a gas mixture. Again, the complete rigorous justification is an open problem.

We also refer the reader to [Bou+15] for a discussion of several physical extensions and applications of the model.

The Vlasov–Navier–Stokes system is endowed with a remarkable energy–dissipation structure, on which most of the recent mathematical theories build. Introducing the energy

$$\mathrm{E}(t) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3} |u(t,x)|^2\mathrm{d}x,$$

and the dissipation

$$\mathrm{D}(t) = \int_{\mathbb{T}^3} |\nabla_x u(t,x)|^2\mathrm{d}x + \int_{\mathbb{T}^3\times\mathbb{R}^3} |v-u(t,x)|^2 f(t,x,v)\mathrm{d}x\mathrm{d}v,$$

we have (at least formally) the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E} + \mathrm{D} = 0. \tag{3.1.2}$$

In this work we specifically aim at studying **high friction** regimes of the Vlasov–Navier–Stokes system and at justifying its approximation by reduced purely hydrodynamic equations of Navier–Stokes type.

As a concrete illustration, one regime we consider corresponds to

$$\begin{cases} \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p = \dfrac{1}{\varepsilon}\left(j_f - \rho_f u\right), \\ \mathrm{div}_x\, u = 0, \\ \partial_t f + v\cdot\nabla_x f + \dfrac{1}{\varepsilon}\,\mathrm{div}_v\left[f(u-v)\right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)\mathrm{d}v, \quad j_f(t,x) = \int_{\mathbb{R}^3} vf(t,x,v)\mathrm{d}v, \end{cases}$$

for $\varepsilon \ll 1$, which, at least formally, leads to the **Inhomogeneous Incompressible Navier–Stokes** equations

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \nabla_x((1+\rho)u\otimes u) - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0. \end{cases}$$

We also study other high friction regimes, which we introduce in the next section. A goal of this work is to develop robust methods to deal with all limits in a general framework.

3.1.1 High friction scalings of the Vlasov–Navier–Stokes system

In this work we study two classes of high friction limits for the Vlasov–Navier–Stokes system, referred to as the **light** particle and the **fine** particle regimes. To explain their physical relevancy, following [GJV04a] (see also the work of Caflisch and Papanicolaou [CP83] in which several scalings are discussed in view of experimental parameter values), we write (3.1.1) in the following dimensionless form :

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \mathrm{K}\Delta_x u + \nabla_x p = \mathrm{C}\left(j_f - \rho_f u\right), \\ \operatorname{div}_x u = 0, \\ \partial_t f + \mathrm{A} v \cdot \nabla_x f + \mathrm{B}\operatorname{div}_v \left[f\left(\frac{1}{\mathrm{A}} u - v\right)\right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)\mathrm{d}v, \quad j_f(t,x) = \mathrm{A}\int_{\mathbb{R}^3} v f(t,x,v)\mathrm{d}v, \end{cases}$$

with

- $\mathrm{A} = \frac{\sqrt{\theta}}{U}$, where $\sqrt{\theta}$ (resp. U) is the typical thermal velocity of the particles (resp. velocity of the fluid), and $U = \frac{L}{T}$, where L (resp. T) stands for the characteristic observation length (resp. observation time) ;
- $\mathrm{B} = \frac{T}{\tau}$, where τ is the so-called Stokes relaxation time, given by the formula

$$\tau = \frac{2a^2 \rho_{\text{part}}}{9\mu},$$

  in which a (resp. ρ_{part}) stands for the renormalized [1] radius (resp. the mass density) of a particle and μ is the diffusivity constant of the fluid ;
- $\mathrm{C} = \frac{T}{\tau}\frac{\rho_{\text{part}}}{\rho_{\text{fluid}}} = \mathrm{B}\frac{\rho_{\text{part}}}{\rho_{\text{fluid}}}$, where ρ_{fluid} is for the density of the fluid ;
- $\mathrm{K} = \frac{2}{9}\left(\frac{a}{L}\right)^2 \frac{T}{\tau}\frac{\rho_{\text{part}}}{\rho_{\text{fluid}}} = \frac{2}{9}\left(\frac{a}{L}\right)^2 \mathrm{C}$.

We shall study in this work the following classes of regimes.

3.1.1.1 *Light* and *light and fast* particle regimes

The **light particle** regime corresponds to

$$\mathrm{A} = 1, \quad \mathrm{B} = \frac{1}{\varepsilon}, \quad \mathrm{C} = 1, \quad \mathrm{K} = 1,$$

leading to the system

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \frac{1}{\varepsilon}\operatorname{div}_v \left[f(u - v)\right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)\mathrm{d}v, \quad j_f(t,x) = \int_{\mathbb{R}^3} v f(t,x,v)\mathrm{d}v. \end{cases}$$

This corresponds to the physical situation where :

- $U \sim \sqrt{\theta}$, *i.e.* the particles and fluid have comparable velocities,
- $\tau \ll T$, *i.e.* the Stokes relaxation time is small compared to the observation time,
- $\rho_{\text{part}} \ll \rho_{\text{fluid}}$, which means that the particles are light compared to the fluid. Asymptotically they become inertialess.

1. Very loosely speaking, the Vlasov–Navier–Stokes system comes from the $N \to +\infty$ limit of an N-particle system, in which the radius r of a particle also tends to 0 as N$\to +\infty$, namely $r \sim \frac{1}{N} a$.

A variant of this scaling corresponds to the **light and fast particle** regime, for which

$$\mathrm{A} = \frac{1}{\varepsilon^\alpha}, \quad \mathrm{B} = \frac{1}{\varepsilon}, \quad \mathrm{C} = 1, \quad \mathrm{K} = 1,$$

where $\alpha > 0$, which leads to the system

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = j_f - \rho_f u, \\ \operatorname{div}_x u = 0, \\ \partial_t f + \dfrac{1}{\varepsilon^\alpha} v \cdot \nabla_x f + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f(\varepsilon^\alpha u - v) \right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \quad j_f(t,x) = \dfrac{1}{\varepsilon^\alpha} \displaystyle\int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v. \end{cases}$$

Compared to the previous regime, the only difference is that $U \ll \sqrt{\theta}$, *i.e.* the velocity of the particles is large compared to that of the fluid. Otherwise, as before, the Stokes relaxation time is small compared to the observation time and the particles are light compared to the fluid.

3.1.1.2 Fine particle regime

The **fine particle** regime corresponds to

$$\mathrm{A} = 1, \quad \mathrm{B} = \frac{1}{\varepsilon}, \quad \mathrm{C} = \frac{1}{\varepsilon}, \quad \mathrm{K} = 1,$$

leading to the system

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = \dfrac{1}{\varepsilon} \left(j_f - \rho_f u \right), \\ \operatorname{div}_x u = 0, \\ \partial_t f + v \cdot \nabla_x f + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f(u - v) \right] = 0, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \quad j_f(t,x) = \displaystyle\int_{\mathbb{R}^3} v f(t,x,v) \mathrm{d}v. \end{cases}$$

This corresponds to the physical situation where :

- $U \sim \sqrt{\theta}$, *i.e.* the particles and fluid have comparable velocities,
- $\tau \ll T$, *i.e.* the Stokes settling time is small compared to the observation time,
- $\rho_{\text{part}} \sim \rho_{\text{fluid}}$, which means that the particles and fluid have comparable mass densities,
- $a \ll L$, *i.e.* the size of the particles is small compared to the observation length.

3.1.1.3 General scalings for the equations

In summary, we shall study the following scalings for the Vlasov–Navier–Stokes system

$$\begin{cases} \partial_t u_{\varepsilon,\gamma,\sigma} + (u_{\varepsilon,\gamma,\sigma} \cdot \nabla_x) u_{\varepsilon,\gamma,\sigma} - \Delta_x u_{\varepsilon,\gamma,\sigma} + \nabla_x p_{\varepsilon,\gamma,\sigma} = \dfrac{1}{\gamma}\left(j_{f_{\varepsilon,\gamma,\sigma}} - \rho_{f_{\varepsilon,\gamma,\sigma}} u_{\varepsilon,\gamma,\sigma}\right), \\ \operatorname{div}_x u_{\varepsilon,\gamma,\sigma} = 0, \\ \partial_t f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\sigma} v \cdot \nabla_x f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f_{\varepsilon,\gamma,\sigma}(\sigma u_{\varepsilon,\gamma,\sigma} - v)\right] = 0, \\ \rho_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \displaystyle\int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}v, \quad j_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \dfrac{1}{\sigma}\displaystyle\int_{\mathbb{R}^3} v f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}v, \end{cases} \tag{3.1.3}$$

in which we consider

- $\sigma = 1, \gamma = 1$, for the light particle regime,
- $\sigma = \varepsilon^\alpha, \gamma = 1$ with $\alpha > 0$, for the light and fast particle regime,
- $\sigma = 1, \gamma = \varepsilon$, for the fine particle regime.

The scaled energy and dissipation functionals read :

$$\mathrm{E}_{\varepsilon,\gamma,\sigma}(t) = \frac{\varepsilon}{\sigma^2\gamma}\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t,x)|^2 \mathrm{d}x, \tag{3.1.4}$$

$$\mathrm{D}_{\varepsilon,\gamma,\sigma}(t) = \int_{\mathbb{T}^3} |\nabla_x u_{\varepsilon,\gamma,\sigma}(t,x)|^2\mathrm{d}x + \frac{1}{\gamma}\int_{\mathbb{T}^3\times\mathbb{R}^3} \left|\frac{v}{\sigma} - u_{\varepsilon,\gamma,\sigma}\right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v. \tag{3.1.5}$$

The Brinkman force is given by

$$F_{\varepsilon,\gamma,\sigma} := \frac{1}{\gamma}(j_{f_{\varepsilon,\gamma,\sigma}} - \rho_{f_{\varepsilon,\gamma,\sigma}} u_{\varepsilon,\gamma,\sigma}).$$

In the light and fast particle regime, we shall restrict to the range of parameters $\alpha \in (0, 1/2]$. Dealing with $\alpha > 1/2$ would be possible, but in view of the scaled energy $\mathrm{E}_{\varepsilon,\gamma,\sigma}$, it would systematically require a well-preparedness assumption in order to ensure that it is initially bounded. Remark that the parameter $\alpha = 1/2$ seems particularly natural as the energies of the kinetic and the fluid part have then the same order in $\mathrm{E}_{\varepsilon,\gamma,\sigma}$.

3.1.2 Formal derivation of the limits

We explain in this section how to obtain the formal high friction limits for the previous regimes.

3.1.2.1 *Light* and *light and fast* particle regimes

We start with the light particle limit of (3.1.3), that corresponds to $\gamma = 1, \sigma = 1$. The conservation of mass and momentum read

$$\begin{cases} \partial_t \rho_{f_\varepsilon} + \operatorname{div}_x j_{f_\varepsilon} = 0, \\ \partial_t j_{f_\varepsilon} + \operatorname{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \,\mathrm{d}v \right) = \dfrac{1}{\varepsilon}\left(\rho_{f_\varepsilon} u_\varepsilon - j_{f_\varepsilon}\right). \end{cases}$$

We deduce that we must have

$$\rho_{f_\varepsilon} u_\varepsilon - j_{f_\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0.$$

Assuming the following convergences

$$(\rho_{f_\varepsilon}, j_{f_\varepsilon}, u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\rho, j, u),$$

we thus formally [2] get

$$j = \rho u,$$

and (ρ, u) has to satisfy the following system :

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t u + u \cdot \nabla_x u - \Delta_x u + \nabla_x p = 0, \\ \operatorname{div}_x u = 0, \end{cases} \tag{3.1.6}$$

which corresponds to a transport equation driven by a velocity field satisfying the incompressible Navier–Stokes equation, and which we refer to as **Transport–Navier–Stokes**.

Remark 3.1.1. *Contrary to the other regimes studied in this paper, the form of the scaled dissipation* (3.1.5) *does not straightforwardly imply that the distribution function f_ε weakly converges to a Dirac distribution in velocity. However, it will be a consequence of the upcoming analysis that*

$$f_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \rho \otimes \delta_{v=u}.$$

The light and fast particle limit of (3.1.3), that corresponds to $\gamma = 1$, $\sigma = \varepsilon^\alpha$, can be formally analyzed in a similar fashion, except that the expressions of the scaled energy and dissipation (3.1.4)–(3.1.5) coupled with the energy–dissipation identity (3.1.2) yield more information in this regime. Assume the following convergences

$$(\rho_{f_\varepsilon}, u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (\rho, u).$$

By the scaled energy–dissipation identity (3.1.2), (3.1.4)–(3.1.5), we have

$$\int_0^{+\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon |v - \varepsilon^\alpha u_\varepsilon|^2 \, \mathrm{d}x\mathrm{d}v\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} 0$$

at rate $\varepsilon^{2\alpha}$, and therefore we deduce that formally

$$f_\varepsilon(t, x, v) \underset{\varepsilon\to 0}{\longrightarrow} \rho(t, x) \otimes \delta_{v=0}. \tag{3.1.7}$$

The conservation of mass and momentum read in this regime

$$\begin{cases} \partial_t \rho_{f_\varepsilon} + \operatorname{div}_x j_{f_\varepsilon} = 0, \\ \partial_t j_{f_\varepsilon} + \dfrac{1}{\varepsilon^{2\alpha}} \operatorname{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \, \mathrm{d}v \right) = \dfrac{1}{\varepsilon} \left(\rho_{f_\varepsilon} u_\varepsilon - j_{f_\varepsilon} \right). \end{cases}$$

By (3.1.7), we expect that

$$\left(\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \, \mathrm{d}v \right) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

2. assuming in particular that products also pass to the limit

so that
$$\rho_{f_\varepsilon} u_\varepsilon - j_{f_\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0.$$
As a conclusion, we again formally obtain that (ρ, u) has to satisfy the Transport–Navier–Stokes system (3.1.6).

3.1.2.2 Fine particle regime

Finally, we consider the fine particle regime for (3.1.3) that is $\gamma = \varepsilon$, $\sigma = 1$.

Assume the following convergences
$$(\rho_{f_\varepsilon}, u_\varepsilon) \xrightarrow[\varepsilon\to 0]{} (\rho, u).$$
In view of the scaled dissipation (3.1.5) in this regime, we expect
$$\int_0^{+\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon |v - u_\varepsilon|^2 \mathrm{d}x\mathrm{d}v\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} 0$$
and thus,
$$f_\varepsilon \xrightarrow[\varepsilon\to 0]{} \rho \otimes \delta_{v=u}. \tag{3.1.8}$$
The conservation of mass and momentum read in this regime
$$\begin{cases} \partial_t \rho_{f_\varepsilon} + \mathrm{div}_x\, j_{f_\varepsilon} = 0, \\ \partial_t j_{f_\varepsilon} + \mathrm{div}_x \left(\displaystyle\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \, \mathrm{d}v \right) = -F_\varepsilon. \end{cases}$$
where we have set
$$F_\varepsilon = \frac{1}{\varepsilon}(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon).$$
By (3.1.8), we get
$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$
As we also formally have, thanks to (3.1.8)
$$\int_{\mathbb{R}^3} v \otimes v f_\varepsilon \, \mathrm{d}v \xrightarrow[\varepsilon\to 0]{} \rho u \otimes u,$$
if we assume that
$$F_\varepsilon \xrightarrow[\varepsilon\to 0]{} F,$$
we must formally have
$$\partial_t(\rho u) + \nabla_x(\rho u \otimes u) = -F.$$
Passing to the limit in the Navier–Stokes equations, we obtain
$$\partial_t u + \nabla_x(\rho u \otimes u) + \nabla_x p = F, \qquad \mathrm{div}_x\, u = 0,$$

hence (ρ, u) must satisfy the system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t((1+\rho)u) + \nabla_x((1+\rho)u \otimes u) - \Delta_x u + \nabla_x p = 0, \\ \operatorname{div}_x u = 0, \end{cases} \tag{3.1.9}$$

which corresponds to the classical **Inhomogeneous Incompressible Navier–Stokes** equations (in short, referred to as the Inhomogeneous Navier–Stokes system in the rest of the paper). The Cauchy problem for these equations, for global weak solutions *à la Leray* or strong solutions *à la Fujita–Kato* has been the object of many research works. We may refer, among many others, to the monographs of Antontsev, Kazhikhov and Monakhov [AKM90] and Lions [Lio96] and [Sim90 ; Dan04 ; AGZ11 ; AGZ12 ; PZZ13 ; Pou15 ; DM19] (and references therein).

Remark 3.1.2. *Note that while modelling additional fragmentation phenomena, [BDM14] also formally derived a similar system of equations. We invite the reader to refer to [HM21, Section 7] for the rigorous justification of the limit in this context.*

3.1.3 Main results of this work

To conclude this introduction, let us state the main results of this work. The system (3.1.3) is endowed with initial conditions

$$u_{\varepsilon,\gamma,\sigma}|_{t=0} = u^0_{\varepsilon,\gamma,\sigma}, \quad f_{\varepsilon,\gamma,\sigma}|_{t=0} = f^0_{\varepsilon,\gamma,\sigma}.$$

that may depend on ε. We start by presenting the assumptions about the initial data that we will use throughout this paper.

Assumption 3.1. *Let $(u^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0} \subset \mathrm{H}^1(\mathbb{T}^3)$ and $(f^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0} \subset \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$ such that*

- *there exists $q > 4$ such that for any $\varepsilon > 0$,*

$$(1+|v|^q) f^0_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{T}^3)) \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3);$$

- *there exist $r \in (2,3)$ and $p \in \left(3, \frac{3(2+r)}{4}\right]$ such that $u^0_{\varepsilon,\gamma,\sigma} \in \mathrm{B}^{1,3}_2(\mathbb{T}^3) \cap \mathrm{B}^{2-2/r,3}_r(\mathbb{T}^3) \cap \mathrm{B}^{s,p}_p(\mathbb{T}^3)$, where $s = 2 - 2/p$;*
- *there exists $M > 1$ such that, for any $\varepsilon > 0$,*

$$\left\| u^0_{\varepsilon,\gamma,\sigma} \right\|_{\mathrm{H}^1 \cap \mathrm{B}^{s,p}_p(\mathbb{T}^3)} + \left\| f^0_{\varepsilon,\gamma,\sigma} |v|^q \right\|_{\mathrm{L}^1(\mathbb{R}^3; \mathrm{L}^\infty(\mathbb{T}^3))} + \sup_{(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3} (1+|v|^q) f^0_{\varepsilon,\gamma,\sigma}(x,v) \le M.$$

The definition of the Besov spaces $\mathrm{B}^{s,q}_r(\mathbb{T}^3)$ is recalled in Definition A.2 of the Appendix.

Remark 3.1.3. *In the* light *and* light and fast *particle regimes, it is actually possible to lower down the required regularity on the initial fluid velocity, namely by only asking $u^0_{\varepsilon,\gamma,\sigma} \in \mathrm{B}^{\widetilde{s},p}_{\frac{p}{p-1}}(\mathbb{T}^3)$, with $\widetilde{s} = 2 - 2\frac{p-1}{p} = \frac{2}{p}$ instead of $\mathrm{B}^{s,p}_p(\mathbb{T}^3)$. Nevertheless, we shall not provide such details in this manuscript and refer the reader to [HM21].*

Remark 3.1.4. *Note that the parameters p, q in Assumption 3.1 verify $p < q$.*

We will also need the standard Fujita–Kato [FK64] type of smallness assumption for the initial data of the Navier–Stokes equations :

Assumption 3.2. *The initial data* $(u^0_{\varepsilon,\gamma,\sigma})_{\varepsilon>0}$ *satisfy*

$$\forall \varepsilon > 0, \qquad \|u^0_{\varepsilon,\gamma,\sigma}\|_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)} \leq C^*/2,$$

where C^* *is the universal constant given by Theorem C.2.*

The definition of the homogeneous Sobolev space $\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)$ is recalled in Definition A.1 of the Appendix.

To simplify the upcoming statements, we shall assume here that there exist $(u^0, \rho^0) \in \mathrm{L}^2(\mathbb{T}^3) \times \mathrm{L}^\infty(\mathbb{T}^3)$ such that

$$\|u^0_{\varepsilon,\gamma,\sigma} - u^0\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \quad W_1\left(\rho_{f^0_{\varepsilon,\gamma,\sigma}}, \rho^0\right) \xrightarrow[\varepsilon\to 0]{} 0,$$

which can always be ensured up to an extraction by standard compactness arguments. Here W_1 stands for the Wasserstein-1 distance whose definition is recalled in Definition A.3 of the Appendix. Also, we shall use the notation $\langle \cdot \rangle$ for the mean value on $\mathbb{T}^3$.

To state the convergence results, let us distinguish between the light, light and fast, and fine particle regimes.

3.1.3.1 Light particle regime

In the light particle regime ($(\gamma, \sigma) = (1, 1)$ in (3.1.3)), we derive the Transport–Navier–Stokes system (3.1.6). This is achieved within three sets of assumptions :

- Under the sole Assumption 3.1, a situation we refer to as the *general* case, one can find a time $T > 0$ below which $u_{\varepsilon,1,1}(t)$ converges towards $u(t)$ in $\mathrm{L}^2(\mathbb{T}^3)$, while $f_{\varepsilon,1,1} \xrightarrow[\varepsilon\to 0]{} \rho \otimes \delta_{v=u}$ *up to an integration in time*, where (ρ, u) satisfies (3.1.6) with initial conditions (ρ^0, u^0). The fact that the convergence of $f_{\varepsilon,1,1}$ is not pointwise and holds only after integration in time comes from the fact that no such convergence is assumed for $t = 0$. It thus formally means that there is an initial layer which turns out to be integrable and negligible when $\varepsilon \to 0$.
- With additional assumptions, the so-called *mildly well-prepared* case, which loosely speaking consists in assuming that the initial fluid velocity is close enough to its average, we are able to show that the aforementioned convergence results hold for all times $T > 0$.
- Finally, under *well-prepared* assumptions, which in particular ensure the initial convergence $f^0_{\varepsilon,1,1} \xrightarrow[\varepsilon\to 0]{} \rho^0 \otimes \delta_{v=u^0}$, we also obtain pointwise convergence for $f_{\varepsilon,1,1}$.

Theorem 3.1.5 (Light particle regimes)**.** *Let* $(u_{\varepsilon,1,1}, f_{\varepsilon,1,1})$ *be a global weak solution associated to the initial condition* $(u^0_{\varepsilon,1,1}, f^0_{\varepsilon,1,1})$*. We have the following convergence results.*

1. General case. *Under Assumption 3.1, there exists* $T > 0$ *such that*

$$\int_0^T W_1(f_{\varepsilon,1,1}(t), \rho(t) \otimes \delta_{v=u(t)})\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0 \tag{3.1.10}$$

and for all $t \in [0, T]$*,*

$$\|u_{\varepsilon,1,1}(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \tag{3.1.11}$$

where (ρ, u) *satisfies the Transport–Navier–Stokes system* (3.1.6)*.*

2. Mildly well-prepared case. *Under the additional Assumption 3.2, there exists* $\eta > 0$ *small enough such that if*

$$\|u^0_{\varepsilon,1,1} - \langle u^0_{\varepsilon,1,1}\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

the convergences (3.1.10) *and* (3.1.11) *hold for any* $T > 0$.

3. Well-prepared case. *Finally, if we further assume that*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} |v - u^0_{\varepsilon,1,1}(x)| f^0_{\varepsilon,1,1}(x,v)\,\mathrm{d}x\mathrm{d}v \xrightarrow[\varepsilon\to 0]{} 0,$$

then, for all $t > 0$,

$$W_1(f_{\varepsilon,1,1}(t), \rho(t)\otimes\delta_{v=u(t)}) \xrightarrow[\varepsilon\to 0]{} 0. \tag{3.1.12}$$

All convergence results are quantitative with respect to T *and* ε.

As a matter of fact, we shall provide more general and more precise statements. We refer to Section 3.4.5 for case **1** and to Section 3.4.3 for cases **2** and **3**.

3.1.3.2 Light and fast particle regime

In the light and fast particle regime ($(\gamma,\sigma) = (1,\varepsilon^\alpha)$ in (3.1.3)), we also derive the Transport–Navier–Stokes system (3.1.6).

We obtain similar results to the light particle regime, and the previous formal discussion mostly still prevails here. The main difference with the light particle regime is as follows. As its denomination suggests, this regime allows for the description of *fast* particles, which means in particular that at initial time, the Brinkman force may not be uniformly bounded with respect to ε (mind the $1/\sigma$ in the definition of $j_{\varepsilon,\gamma,\sigma}$). However our results show that because of the high friction, the order of magnitude of the particle velocities becomes instantaneously of order 1. Again, there is therefore in general an initial layer whose contribution vanishes when integrated in time as $\varepsilon \to 0$.

It turns out that we can deal with the *light* and *light and fast* particle regimes in a completely unified manner, under the additional well-preparedness assumption

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^p f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v)\mathrm{d}x\mathrm{d}v \lesssim \varepsilon^{\alpha p - 1+\kappa}, \tag{3.1.13}$$

for some $\kappa \in (0,1)$, where p is the regularity index introduced in Assumption 3.1. To preserve the readability of this manuscript, we restrict ourselves to this framework and invite the reader to refer to [HM21] for the proof that this additional assumption can be in fact dispensed with. We can nonetheless already note that (3.1.13) is significant only for $\alpha > 1/p$.

In the following statement, we keep the distinction between the general, mildly well-prepared, and well-prepared cases in order to suggest to the reader the kind of result we achieve in [HM21] thanks to a slight refinement of the method we present in this chapter.

Theorem 3.1.6 (Light and fast particle regimes)**.** *Let* $(u_{\varepsilon,1,\varepsilon^\alpha}, f_{\varepsilon,1,\varepsilon^\alpha})$ *be a global weak solution associated to the initial condition* $(u^0_{\varepsilon,1,\varepsilon^\alpha}, f^0_{\varepsilon,1,\varepsilon^\alpha})$. *Assume that* (3.1.13) *is satisfied. Then the following convergence results hold.*

A. *The case* $\alpha < 1/2$.

1. General case. *Under Assumption 3.1, there exists* $T > 0$ *such that*

$$\int_0^T W_1(f_{\varepsilon,1,\varepsilon^\alpha}(t), \rho(t)\otimes\delta_{v=0})\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0 \tag{3.1.14}$$

and for all $t \in [0,T]$,

$$\|u_{\varepsilon,1,\varepsilon^\alpha}(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0, \tag{3.1.15}$$

where (ρ, u) *satisfies the Transport–Navier–Stokes system* (3.1.6).
2. Mildly well-prepared case. *Under the additional Assumption 3.2, there exists* $\eta > 0$ *small enough such that if*

$$\left\| u^0_{\varepsilon,1,\varepsilon^\alpha} - \langle u^0_{\varepsilon,1,\varepsilon^\alpha} \rangle \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

then the convergences (3.1.14) *and* (3.1.15) *hold for any* $T > 0$.
3. Well-prepared case. *Finally, if we further assume that*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} |v| f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v)\,\mathrm{d}x\mathrm{d}v \xrightarrow[\varepsilon\to 0]{} 0,$$

then, for all $t > 0$,

$$W_1(f_{\varepsilon,1,\varepsilon^\alpha}(t), \rho(t)\otimes\delta_{v=0}) \xrightarrow[\varepsilon\to 0]{} 0. \tag{3.1.16}$$

Furthermore all convergence results are quantitative with respect to T *and* ε.

B. *The case* $\alpha = 1/2$.

The previous results still hold, with the following additional assumptions, for the general case and the mildly well-prepared case :

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f^0_{\varepsilon,1,\varepsilon^{\frac{1}{2}}}(x,v)\mathrm{d}x\mathrm{d}v \leq \eta,$$

for $\eta > 0$ *small enough. Moreover the results are quantitative under the assumption*

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,1,\varepsilon^{\frac{1}{2}}}|v|\mathrm{d}x\mathrm{d}v \xrightarrow[\varepsilon\to 0]{} 0.$$

As in the light particle regime, we also refer to Section 3.4.5 for case **1** and to Section 3.4.3 for cases **2** and **3**.

3.1.3.3 Fine particle regime

In the fine particle regime ($(\gamma,\sigma) = (\varepsilon, 1)$ in (3.1.3)) we derive the Inhomogeneous Navier–Stokes system (3.1.9). This proves to be much more singular than the previous two regimes. As a consequence, more assumptions on the initial data are required to justify the convergence results. In particular, we are no longer able to state any result with the sole Assumption 3.1 and need to always consider well-prepared initial data. Moreover, a smallness assumption on the kinetic initial distribution function is required.

Theorem 3.1.7 (Fine particle regimes)**.** *Let* $(u_{\varepsilon,\varepsilon,1}, f_{\varepsilon,\varepsilon,1})$ *be a global weak solution associated to the initial condition* $(u^0_{\varepsilon,\varepsilon,1}, f^0_{\varepsilon,\varepsilon,1})$. *We have the following convergence results.*
1. Mildly well-prepared case. *Under Assumption 3.1, there exist* ε_0, $\eta > 0$, $M' > 0$, *such that, if for all* $\varepsilon \in (0, \varepsilon_0)$,

$$\left\| f^0_{\varepsilon,\varepsilon,1} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \eta$$

and

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v - u^0_{\varepsilon,\varepsilon,1}(x)|^p}{\varepsilon^{p-1}} f^0_{\varepsilon,\varepsilon,1}(x,v)\mathrm{d}x\mathrm{d}v \leq M',$$

where p *is the regularity index introduced in Assumption 3.1, then there exists* $T > 0$ *such that for all* $t \in [0, T]$,

$$W_1(f_{\varepsilon,\varepsilon,1}(t), \rho(t)\otimes\delta_{v=u(t)}) \xrightarrow[\varepsilon\to 0]{} 0, \tag{3.1.17}$$

and for all $t \in [0, T]$,

$$\|u_{\varepsilon,1,1}(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon \to 0]{} 0, \tag{3.1.18}$$

where (ρ, u) *satisfies the Inhomogeneous Navier–Stokes system* (3.1.9).
2. Well-prepared case. *Under the additional Assumption 3.2 there exists* $\eta' > 0$ *small enough such that if*

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} \left| v - \frac{\langle j^0_{\varepsilon,\varepsilon,1} \rangle}{\langle \rho^0_{\varepsilon,\varepsilon,1} \rangle} \right|^2 f^0_{\varepsilon,\varepsilon,1} \, \mathrm{d}x\mathrm{d}v + \int_{\mathbb{T}^3} |u^0_{\varepsilon,\varepsilon,1} - \langle u^0_{\varepsilon,\varepsilon,1} \rangle|^2 \mathrm{d}x + \left| \frac{\langle j^0_{\varepsilon,\varepsilon,1} \rangle}{\langle \rho^0_{\varepsilon,\varepsilon,1} \rangle} - \langle u^0_{\varepsilon,\varepsilon,1} \rangle \right|^2 \leq \eta,$$

then the convergences (3.1.17) *and* (3.1.18) *hold for any* $T > 0$.

If we further assume that the solution (ρ, u) *to* (3.1.9) *is smooth enough, then the convergences* (3.1.17) *and* (3.1.18) *are quantitative with respect to* T *and* ε.

We have used the following notations for the two first initial moments in the statement of the theorem :

$$\rho^0_{\varepsilon,\varepsilon,1} = \int_{\mathbb{R}^3} f^0_{\varepsilon,\varepsilon,1}(\cdot, v)\mathrm{d}v, \quad j^0_{\varepsilon,\varepsilon,1} = \int_{\mathbb{R}^3} v f^0_{\varepsilon,\varepsilon,1}(\cdot, v)\mathrm{d}v.$$

Like for the two other regimes, we shall provide more complete statements. Let us refer to Section 3.5.5 for case **1** and to Sections 3.5.3 and 3.5.4 for case **2**.

We end the statement of the results with a remark that concerns all regimes.

Remark 3.1.8. *The assumptions in Theorems 3.1.5, 3.1.6 and 3.1.7 are somewhat comparable to those of [HMM20] for the long time behavior of the Vlasov–Navier–Stokes system* (3.1.1)*, except that we require in Assumption 3.1 extra bounds in Besov spaces. The reason is as follows. The analysis of* (3.1.1) *relies on the instantaneous smoothing effect of the heat semi-group on* $\mathbb{T}^3$ *and uses rough bounds (similar to those of Section 3.3.4) close to* $t = 0$*; we cannot argue similarly for the study of high friction regimes as these rough bounds blow up when* $\varepsilon \to 0$*, see Section 3.3.4.*

3.2 State of the art and methodology of this work

3.2.1 State of the art

Let us start by reviewing the mathematical literature that is relevant to this work.
Cauchy problem for Vlasov–Navier–Stokes. The Cauchy problem was first studied for related reduced models in the pioneering works of Anoshchenko and Boutet de Monvel-Berthier [AB97] and Hamdache [Ham98]. Existence of global weak solutions to the Vlasov–Navier–Stokes system on $\mathbb{T}^3$ was then established in the seminal work of Boudin, Desvillettes, Grandmont and Moussa [Bou+09]. This was later extended to the case of bounded domains of $\mathbb{R}^3$ in [WY15] and to time-dependent bounded domains of $\mathbb{R}^3$ in [BGM17]. Lately, in [BMM20], the authors obtain similar results for an augmented model in which hygroscopic effects are taken into account and lead to considering the size and temperature variation of the particles in the Vlasov equation. In dimension two, the uniqueness of weak solutions in the torus or in the whole space is obtained in [Han+20]. The existence of smooth solutions has also been studied, for instance [CK15] provides a local well-posedness result in dimension three.

Large time behavior for Vlasov–Navier–Stokes. Large time behavior for a related reduced model was first tackled in the pioneering work of Jabin [Jab00a]. Concerning the Vlasov–Navier–Stokes system on $\mathbb{T}^3$, a conditional result, describing the large time behavior of solutions, was obtained by Choi and Kwon in [CK15] ; loosely speaking a global bound on the moment ρ_f needs to be *a priori* assumed. More recently, this restriction has been removed in [HMM20] for initial conditions close to equilibrium, in a framework *à la* Fujita–Kato. This was recently extended to the whole space case in [Han20] and to bounded domains with absorption boundary conditions in [EHM21]. As we shall soon see, this question is closely linked to the limits we consider in this paper. In the Vlasov–Navier–Stokes system, concentration effects in velocity are at play, which eventually lead to a *monokinetic* behavior for the kinetic distribution function, that is to say, convergence to a Dirac mass in velocity. This is precisely this behavior that is described in [HMM20 ; Han20].

In a somewhat different direction, [GHM18] provides the existence and stability of regular equilibria for the system in a domain with partly absorbing boundary conditions and injection of particles and fluid.

High friction limits for Vlasov–(Fokker–Planck)–(Navier)–Stokes. High friction limits for fluid-kinetic systems have been the topic of several recent works, most of the time with an additional Fokker–Planck operator in the kinetic equation (that accounts for the effect of Brownian motion), resulting in the Vlasov–Fokker–Planck equation

$$\partial_t f_{\varepsilon,\gamma,\sigma} + \frac{1}{\sigma} v \cdot \nabla_x f_{\varepsilon,\gamma,\sigma} + \frac{1}{\varepsilon} \operatorname{div}_v \left[f_{\varepsilon,\gamma,\sigma}(\sigma u_{\varepsilon,\gamma,\sigma} - v) + \nabla_v f_{\varepsilon,\gamma,\sigma} \right] = 0,$$

instead of the Vlasov equation in (3.1.3).

Light particle regimes. Light particle regimes have been studied by Jabin in [Jab00b] for a Vlasov equation where the fluid velocity in 3.1.3 is computed by means of a convolution operator with a smooth kernel applied to a moment of the distribution function ; a general framework has been put forward but it does not apply to the full Vlasov–Navier–Stokes system. A similar problem is dealt with in [GP04] where the fluid velocity is a given random vector field. A toy model for (3.1.3) is tackled in dimension $d = 1$ in [Gou01] ; the analysis strongly relies on the one-dimensional framework (notably, Sobolev embeddings are much more favorable) and cannot be directly adapted to the three-dimensional case. More recently, the light particle regime was studied by Höfer in [Höf18b] for the Vlasov-Stokes system on the whole space in the presence of a gravity force (a limit referred to as the inertialess limit there).

The light and fast particle regime was studied in a seminal work by Goudon, Jabin and Vasseur [GJV04a] in the Vlasov–Fokker–Planck case for the critical parameter $\alpha = 1/2$. They were able to justify the asymptotics in dimension $d = 2$, leading to the Kramer-Smoluchowski equations, thanks to global entropy bounds related to the Fokker–Planck operator.

Fine particle regimes. In another seminal work [GJV04b], Goudon, Jabin and Vasseur consider the fine particle regime in the whole space, also for the Vlasov–Fokker–Planck–Navier–Stokes system. They relied on a *relative entropy* method (see Section 3.2.2.1 below for a short review) to derive the Inhomogeneous Navier–Stokes equations, see also [MV08].

Other hydrodynamic limits with monokinetic behavior. It is important to note that in the Vlasov–Fokker–Planck cases studied in [GJV04a ; GJV04b], the distribution function converges (at least formally) to a *local Maxwellian*, precisely because of the Fokker–Planck operator. Without it, as for the large time asymptotics, we expect a monokinetic behavior at the limit, that is to say convergence towards a Dirac mass in velocity. To conclude this review of the literature, let us briefly discuss other hydrodynamic limits for Vlasov-type equations in which Dirac distributions in velocity also show up. In a very influential work [Bre00], Brenier studies the quasineutral limit (*i.e.* the small Debye length regime) of the Vlasov-Poisson system. He shows that if the sequence of initial data converges in a certain sense to a Dirac distribution in velocity (with mass one) – an assumption one can refer to as *well-prepared* initial data – then the solution also displays a monokinetic behavior with a velocity that satisfies the incompressible Euler equations. More recently, Kang and Vasseur [KV15] and Kang and Figalli [FK19] have studied high friction limits for the so-called kinetic Cucker-Smale equation (see also the recent work [CC20]). They show that for well-prepared initial data, the solution asymptotically converges towards a Dirac mass in velocity, with a velocity satisfying pressureless Euler-type equations. See also the related recent works on hydrodynamic limits of Fritz-Nagumo kinetic equations [CFF19 ; Cre20].

3.2.2 Strategy

Before describing the strategy at stake in this work, we provide an overview of the classical methods that are available to tackle hydrodynamic limits for kinetic equations.

3.2.2.1 A review of classical methods

There are mainly three classes of methods that have been devised to study such questions.

Weak compactness methods. The broad principle of this class of methods consists in finding uniform (with respect to the small parameter ε) bounds to obtain the weak compactness of certain relevant quantities in the equations. Such bounds often come from conservation laws of the system (*e.g.* conservation of mass, energy, etc.). From the mathematical point of view, this usually also involves finding some (strong) compactness in order to overcome possible oscillations and eventually pass to the limit in nonlinear terms.

With regard to the Navier–Stokes limit of the Boltzmann equation, a program based on this method was set up by Bardos, Golse and Levermore [BGL91 ; BGL93] and a complete justification has been obtained, see [GS04 ; GS09 ; Ars12] and references therein. Let us also refer to the recent monograph of Arsénio and Saint-Raymond [AS19] for generalizations in several directions.

With regard to high friction limits, this is the strategy applied in the 1D toy model of [Gou01], and in [GJV04a] in the Vlasov–Fokker–Planck case for the light and fast particle regime for $\alpha = 1/2$.

Modulated energy/relative entropy methods. Following the pioneering works of Dafermos [Daf79] and Yau [Yau91] and more specifically Brenier [Bre00] and Golse [BGP00] for kinetic equations, this class of methods consists in *modulating* a well chosen functional (often the energy or entropy) with solutions to the target limit equations and showing that this new functional is vanishing as $\varepsilon \to 0$. One must also ensure that the modulated functional indeed allows to quantify the convergence to the limit.

This method was notably used to treat the incompressible Euler limit of the Boltzmann equation, see [Sai03 ; Sai09]. We can also again refer to the recent [AS19].

For what concerns high friction limits, this is the strategy applied in [GJV04b] in the Vlasov–Fokker–Planck case for the fine particle regime. This is also the main strategy at play for the aforementioned hydrodynamic limits for Vlasov-Poisson [Bre00] and Cucker-Smale [KV15 ; FK19 ; CC20].

Higher regularity/Strong compactness methods. This class of methods consists in building strong solutions enjoying uniform regularity with respect to the small parameter ε. This has proven efficient for various hydrodynamic limits for the Boltzmann equation, whether with an approach based on an (Hilbert or Chapman-Enskog) expansion with respect to ε, see *e.g.* [Nis78 ; DEL89], or in a regime close to equilibrium [BU91]. The latter, which most often requires a fine understanding of the spectral properties of the linearized Boltzmann operator, has witnessed several recent developments, see *e.g.* [Bri15 ; BMM19 ; JXZ18 ; GT20 ; ALT20] and references therein.

3.2.2.2 Strategy of this work

The strategy that we implement can be seen as a mix between all the aforementioned approaches. It is important to note that an argument solely based on propagation of higher regularity for all quantities is likely to fail, as we expect to obtain Dirac masses in velocity in the limit for the distribution function : as a result, higher regularity with respect to the velocity variable cannot be expected.

As usual in singular limits problems, the key is to obtain appropriate uniform estimates. Here, we aim at controlling the local density ρ_{f_ε} in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))$. Such a control does not straightforwardly follow from the natural conservation laws of the system. Inspired by the analysis of [HMM20] (which, we recall, concerns the long time behavior of solutions to (3.1.1)), such bounds for ρ_{f_ε} actually follow from a control on the fluid velocity field of the form

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \ll 1, \tag{3.2.1}$$

thanks to a natural straightening change of variables in velocity in the style of [BD85]. This paves the way to the short time results of Theorems 3.1.5, 3.1.6 and 3.1.7, using weak compactness techniques to pass to the limit. For the large time results, thanks to the work of Choi and Kwon [CK15], it turns out that a uniform control of ρ_{f_ε} in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))$ entails a uniform exponential decay in time for the so-called *modulated energy*, a functional encoding the large time monokinetic behavior of the distribution function. We emphasize that this is not sufficient to directly show any convergence as $\varepsilon \to 0$. However it provides some control, namely integrability in time and smallness, to eventually deduce uniform bounds such as (3.2.1) for arbitrarily large times.

We shall set up a bootstrap argument to prove (3.2.1). To this end, we use higher order estimates for u_ε which are obtained thanks to maximal regularity estimates for the Stokes equation. These estimates are then interpolated with the pointwise in time $\mathrm{L}^2(\mathbb{T}^3)$ bounds stemming from the modulated energy dissipation. The main difficulty lies in obtaining appropriate uniform in ε estimates of the Brinkman force

$$F_{\varepsilon,\gamma,\sigma} = \frac{1}{\gamma}\left(j_{f_{\varepsilon,\gamma,\sigma}} - \rho_{f_{\varepsilon,\gamma,\sigma}} u_{\varepsilon,\gamma,\sigma}\right)$$

in $\mathrm{L}^p(0,T;\mathrm{L}^p(\mathbb{T}^3))$ for large enough values of p. To this end, a process of *desingularization* (with respect to ε) of the Brinkman force is required.

As we shall see below, the *fine particle regime* is significantly more singular than the light particle regimes. To tackle the former, we rely on some ideas from [Han20], which deals with the large time behavior of small data solutions to the Vlasov–Navier–Stokes system posed on $\mathbb{R}^3 \times \mathbb{R}^3$. The general strategy follows that of the torus case [HMM20], but the slow decay (polynomial on the whole space, vs. exponential on the torus) of the solution to the Stokes equation makes (among other things) a finer understanding of the structure of the Brinkman force compulsory. This has led to a family of identities for a notion of *higher* dissipation (see [Han20, Lemma 4.2]) allowing for a better decay, on which we shall also rely.

Let us finally mention that in the fine particle regime, we also take advantage of a *relative entropy* functional, which is useful to obtain quantitative convergence estimates.

Remark 3.2.1. *The difficulties in order to apply the weak compactness strategy to study the fine particle regime are thoroughly discussed in [Mou18, Chapitre 2, Section 2.3.3].*

3.2.3 Outline of the proofs and organization of the chapter

The aim of this section is to provide a detailed overview of the proofs, give a flavor of the analysis and explain, at the same time, the organization of this chapter. As we already stated, the study of high friction limits is closely related to the long time behavior problem, and the general strategy we follow is inspired from [HMM20]. In this presentation, we shall mostly focus on *large time* results (*i.e.* for arbitrary times $T > 0$) which require some well-prepared initial conditions. We recall that such assumptions can also be dispensed with, at the expense of restricting to *short time* results.

Preliminaries. The beginning of the analysis of all three regimes is common and consists, loosely speaking, in a careful extension of the preliminaries of [HMM20] (plus some additional technical results), which tracks down the dependence with respect to the small parameter ε. This step is performed in Section 3.3.

Let us introduce the main objects of interest in this work. We begin by the the characteristic curves associated to the Vlasov equation

$$\begin{cases} \dot{X}_{\varepsilon,\gamma,\sigma}(s;t,x,v) = \dfrac{1}{\sigma} V_{\varepsilon,\gamma,\sigma}(s;t,x,v), \\ \dot{V}_{\varepsilon,\gamma,\sigma}(s;t,x,v) = \dfrac{1}{\varepsilon}\left(\sigma u_{\varepsilon,\gamma,\sigma}\left(s, X_{\varepsilon,\gamma,\sigma}(s;t,x,v)\right) - V_{\varepsilon,\gamma,\sigma}(s;t,x,v)\right), \\ X_{\varepsilon,\gamma,\sigma}(t;t,x,v) = x, \\ V_{\varepsilon,\gamma,\sigma}(t;t,x,v) = v, \end{cases} \tag{3.2.2}$$

which are particularly useful since we have the classical representation formula

$$f_{\varepsilon,\gamma,\sigma}(t,x,v) = e^{\frac{3t}{\varepsilon}} f^0\left(X_{\varepsilon,\gamma,\sigma}(0;t,x,v), V_{\varepsilon,\gamma,\sigma}(0;t,x,v)\right). \tag{3.2.3}$$

We claim that the key is to understand how to obtain *uniform* (with respect to ε) estimates for the moments $\rho_{\varepsilon,\gamma,\sigma} = \int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma} \mathrm{d}v$ and $j_{\varepsilon,\gamma,\sigma} = \frac{1}{\sigma}\int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma} v \, \mathrm{d}v$ of the form

$$\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} + \|j_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \lesssim 1.$$

Controlling $\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))}$ proves interesting because of the following facts, that stem from the study of the long time behavior for the Vlasov–Navier–Stokes system.

Building on the scaled energy (3.1.4) and following Choi and Kwon [CK15], we define the *modulated energy* as the following functional

$$\begin{aligned} \mathscr{E}_{\varepsilon,\gamma,\sigma}(t) =& \frac{\varepsilon}{2\gamma}\int_{\mathbb{T}^3\times\mathbb{R}^3} \left|\frac{v}{\sigma} - \frac{\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle}{\langle \rho_{\varepsilon,\gamma,\sigma}(t)\rangle}\right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\,\mathrm{d}x\mathrm{d}v \\ &+ \frac{1}{2}\int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t,x) - \langle u_{\varepsilon,\gamma,\sigma}(t)\rangle|^2 \mathrm{d}x + \frac{\varepsilon\langle \rho_{\varepsilon,\gamma,\sigma}(t)\rangle}{2(\gamma + \varepsilon\langle \rho_{\varepsilon,\gamma,\sigma}(t)\rangle)}\left|\frac{\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle}{\langle \rho_{\varepsilon,\gamma,\sigma}(t)\rangle} - \langle u_{\varepsilon,\gamma,\sigma}(t)\rangle\right|^2, \end{aligned}$$

where we recall that $\langle g\rangle$ stands for the mean value of g on the torus $\mathbb{T}^3$. Using the energy–dissipation identity (3.1.2) and the conservation laws of the system, we obtain for almost all

$t \geq s \geq 0$ (including $s = 0$),

$$\mathscr{E}_{\varepsilon,\gamma,\sigma}(t) + \int_s^t \mathrm{D}_{\varepsilon,\gamma,\sigma}(\tau)\mathrm{d}\tau \leq \mathscr{E}_{\varepsilon,\gamma,\sigma}(s). \tag{3.2.4}$$

As in [CK15], a remarkable consequence of this identity is a conditional exponential decay of the modulated energy $\mathscr{E}_{\varepsilon,\gamma,\sigma}$. Let $T \in \mathbb{R}^+ \cup \{+\infty\}$. *Assuming* a control on the density $\rho_{\varepsilon,\gamma,\sigma}$ of the form

$$\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \mathrm{C}_0, \quad \forall \varepsilon \in (0,1), \tag{3.2.5}$$

one can prove (see Lemma 3.3.20)

$$\forall t \in [0,T], \qquad \mathscr{E}_{\varepsilon,\gamma,\sigma}(t) \leq Ce^{-\lambda_{\varepsilon,\gamma,\sigma} t}\mathscr{E}_{\varepsilon,\gamma,\sigma}(0), \tag{3.2.6}$$

where $\lambda_{\varepsilon,\gamma,\sigma} > 0$ is bounded from below by a positive constant and $C > 0$ depends only on $\lambda_{\varepsilon,\gamma,\sigma}$, *independently* of ε. As a result, ensuring the global control

$$\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \mathrm{C}_0, \quad \forall \varepsilon \in (0,1)$$

is the key to understanding the long time behavior of $\mathscr{E}_{\varepsilon,\gamma,\sigma}$; it turns out that this is then sufficient to describe the long time behavior of the solutions to the Vlasov–Navier–Stokes system. This fact is at the heart of the strategy of [HMM20]. In the context of high friction limits, such a control of $\mathscr{E}_{\varepsilon,\gamma,\sigma}$ will provide *uniform* integrability in time properties, as well as *smallness* since $\mathscr{E}_{\varepsilon,\gamma,\sigma}(0)$ can be chosen as small as desired.

The strategy to control the moments directly comes from [HMM20]. *Assuming* a control on the fluid velocity $u_{\varepsilon,\gamma,\sigma}$ of the form

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq 1/30, \tag{3.2.7}$$

one proves (see Lemma 3.3.22) that for all $x \in \mathbb{T}^3$, the map

$$\Gamma^{t,x}_{\varepsilon,\gamma,\sigma} : v \mapsto V_{\varepsilon,\gamma,\sigma}(0;t,x,v) \tag{3.2.8}$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself and satisfies

$$\forall v \in \mathbb{R}^3, \qquad \det \mathrm{D}_v \Gamma^{t,x}_{\varepsilon,\gamma,\sigma}(v) \geq \frac{e^{\frac{3t}{\varepsilon}}}{2}.$$

We can then use the change of variables in velocity $w := \Gamma^{t,x}_{\varepsilon,\gamma,\sigma}(v)$ in the formula for the density $\rho_{\varepsilon,\gamma,\sigma}$:

$$\begin{aligned}
\rho_{\varepsilon,\gamma,\sigma}(t,x) &= e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0(X_{\varepsilon,\gamma,\sigma}(0;t,x,v), V_{\varepsilon,\gamma,\sigma}(0;t,x,v))\mathrm{d}v \\
&= e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0(\widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(0), w)|\det \mathrm{D}_v \Gamma^{t,x}_{\varepsilon,\gamma,\sigma}|^{-1}(w)\mathrm{d}w,
\end{aligned}$$

with

$$\widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(s) := X_{\varepsilon,\gamma,\sigma}(s;t,x,[\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(w)),$$

which, using Assumption 3.1, yields a control of the form (3.2.5). Under the same assumption (3.2.7), one also proves that the map $x \mapsto \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(s)$ is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{T}^3$ to itself with Jacobian bounded from below by 1/2.

Therefore, the goal is now to obtain the bound (3.2.7). To this end, one looks for higher order estimates for $u_{\varepsilon,\gamma,\sigma}$, which rely on the parabolic nature of the Navier–Stokes equation. We shall use $\mathrm{L}^p\mathrm{L}^p$ (for $p > 3$) maximal estimates for the Stokes operator on $\mathbb{T}^3$ to get

$$\begin{aligned}\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} &+ \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\lesssim \|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|(u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x)u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{B}^{s,p}_p(\mathbb{T}^3)},\end{aligned}$$

in which $F_{\varepsilon,\gamma,\sigma} = (j_{\varepsilon,\gamma,\sigma} - \rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma})/\gamma$ stands for the Brinkman force and $s = 2 - 2/p$. To control the term due to the nonlinearity $u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x u_{\varepsilon,\gamma,\sigma}$, one has to deal with strong enough solutions to the Navier–Stokes equation. Namely, we require a $\mathrm{L}^\infty\mathrm{H}^1$–$\mathrm{L}^2\dot{\mathrm{H}}^2$ control of the form

$$\|u_{\varepsilon,\gamma,\sigma}(t)\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \int_0^t \|\Delta_x u_{\varepsilon,\gamma,\sigma}(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \lesssim \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \|\mathrm{F}_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)}. \quad (3.2.9)$$

To enforce this property, we consider a Fujita–Kato type of framework, which consists in ensuring that

$$\int_0^t \left\|e^{s\Delta}u^0_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \|F_{\varepsilon,\gamma,\sigma}(s)\|^2_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)}\,\mathrm{d}s \le C^*, \quad (3.2.10)$$

for some universal constant $C^* \in (0,1)$. We can achieve this with

- a smallness assumption in $\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)$ for the initial fluid velocity,
- or by considering only small times.

In either case, we will still need to make sure that $\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} \ll 1$. The idea is to set up a bootstrap argument. For a given $\varepsilon > 0$, we begin by saying that $T > 0$ is a *strong existence time* if

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \le 1/30 \text{ and (3.2.10) holds.} \quad (3.2.11)$$

Because of the previous considerations, for a given strong existence time $T > 0$, on the interval $[0,T]$, one has a uniform control on $\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))}$ which in turn allows to prove the uniform decay for $\mathscr{E}_{\varepsilon,\gamma,\sigma}$.

After proving the existence of a strong existence time (that may at this stage depend on ε), we finally define

$$T^*_\varepsilon = \sup\{T > 0,\ T \text{ is a strong existence time}\}.$$

Following these preliminaries, the main part of the analysis is to prove that $T^*_\varepsilon = +\infty$ under the smallness assumption for the initial fluid velocity. We also show that we can dispense with this assumption and still find a strong existence time that does not depend on ε.

The *light* and *light and fast* particle regimes can be treated, at first, in a unified manner, based on this $\mathrm{L}^p\mathrm{L}^p$ strategy. It however requires a well-posedness assumption for treating the cases $\alpha > 1/p$. To remove it, a refined approach based on $\mathrm{L}^{\frac{p}{p-1}}\mathrm{L}^p$ parabolic estimates is introduced in [HM21].

A similar $\mathrm{L}^p\mathrm{L}^p$ strategy could be adapted to close the bootstrap argument in the fine particle regime. Nevertheless, we choose to develop a different argument which has two advantages :

- it directly yields pointwise in time convergence for the full distribution function towards the limit Dirac mass ;
- it could provide some insight as to how to deal with this limit in the whole space $\mathbb{R}^3$ instead of $\mathbb{T}^3$.

The bootstrap argument in the *light* and *light and fast* particle regimes. Section 3.4 is dedicated to both the light particle and light and fast particle regimes. To ease readability, we drop the parameter $\gamma = 1$ in the following formulas. We shall focus on the proof that $T_\varepsilon^\star = +\infty$ under suitable assumptions on the initial data. We assume by contradiction that $T_\varepsilon^\star < +\infty$ and aim at proving that for all $T < T_\varepsilon^\star$, the condition (3.2.11) can be improved, which by a continuity argument, will lead to a contradiction with the definition of $T_\varepsilon^\star$.

Let us first discuss the L^2 estimate for the Brinkman force $F_{\varepsilon,\sigma}$. The smallness property one needs to enforce follows from the dissipation $\mathrm{D}_{\varepsilon,\sigma}$. By the Cauchy-Schwarz inequality, we indeed have

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^3))} \le \|\rho_{\varepsilon,\sigma}\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \left(\int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,\sigma}|v-u_{\varepsilon,\sigma}|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}},$$

and we use that since $T < T_\varepsilon^\star$, we do have $\|\rho_{\varepsilon,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \lesssim 1$. By the modulated energy–dissipation identity (3.2.4), for all $T > 0$,

$$\int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,\sigma}|v-u_{\varepsilon,\sigma}|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \le \mathscr{E}_{\varepsilon,\sigma}(0),$$

which yields the required bound for $\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{T}^3))}$, choosing $\mathscr{E}_{\varepsilon,\sigma}(0)$ small enough.

The heart of the proof is to obtain a refined estimate for the Brinkman force, which stems from a kind of desingularization of its expression with respect to the small parameter ε. Writing

$$F_{\varepsilon,\sigma}(t,x) = \int_{\mathbb{R}^3} f_{\varepsilon,\sigma}(t,x,v)\left(\frac{v}{\sigma} - u_{\varepsilon,\sigma}(t,x)\right)\mathrm{d}v,$$

the representation formula (3.2.3) and the change of variables in velocity $w := \Gamma^{t,x}_{\varepsilon,\sigma}(v)$ (defined in (3.2.8)) yield

$$F_{\varepsilon,\sigma}(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left(\frac{1}{\sigma}[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) - u_{\varepsilon,\sigma}(t,x)\right)\left|\det\nabla_w[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right| \mathrm{d}w.$$

The key is the following identity, obtained from the equation of characteristics (3.2.2) and an integration by parts in time, taking advantage of the fast effect of the friction term, embodied by the presence of a factor $1/\varepsilon$ in the exponential functions in time :

$$\begin{aligned}\frac{1}{\sigma}[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) - u_{\varepsilon,\sigma}(t,x) = e^{-\frac{t}{\varepsilon}}&\left(\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right)\\ &- \int_0^t e^{\frac{s-t}{\varepsilon}} \partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\mathrm{d}s\\ &- \int_0^t e^{\frac{s-t}{\varepsilon}} V_{\varepsilon,\sigma}\left(s;t,x,[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right)\cdot\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\mathrm{d}s.\end{aligned}$$

Thanks to an estimate on the Jacobian of the change of variable (see Lemma 3.3.22), we end up with

$$|F_{\varepsilon,\sigma}(t,x)| \le F^0_{\varepsilon,\sigma} + F^{dt}_{\varepsilon,\sigma} + F^{dx}_{\varepsilon,\sigma}, \tag{3.2.12}$$

where

$$
\begin{aligned}
F^0_{\varepsilon,\sigma} &= e^{\frac{-t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right| \mathrm{d}w, \\
F^{dt}_{\varepsilon,\sigma} &= \int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}w, \\
F^{dx}_{\varepsilon,\sigma} &= \int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|V_{\varepsilon,\sigma}\left(s;t,x,[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right)\cdot \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}w.
\end{aligned}
$$

Let $p>3$ be the regularity index of Assumption 3.1. Building on this decomposition, one proves (see Lemmas 3.4.2, 3.4.4 and 3.4.5) that

$$
\begin{aligned}
\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}\| \lesssim \frac{\varepsilon^{\frac{1}{p}}}{\sigma}\left\||v|^p f^0_{\varepsilon,\sigma}\right\|^{\frac{1}{p}}_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)} + \varepsilon + \varepsilon\mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1}{2}} \\
+ \varepsilon\left\|\partial_t u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \varepsilon\left\|\Delta_x u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.
\end{aligned} \tag{3.2.13}
$$

Combining this with the aforementioned L^p maximal estimate and taking ε small enough to absorb some terms of the right-hand side, one finally obtains

$$
\left\|\partial_t u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|\Delta_x u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim 1 + \frac{\varepsilon^{\frac{1}{p}}}{\sigma}\left\||v|^p f^0_{\varepsilon,\sigma}\right\|^{\frac{1}{p}}_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)}. \tag{3.2.14}
$$

We will therefore need an additional *well-preparedness* assumption on the initial distribution function $f^0_{\varepsilon,\sigma}$ depending on $\sigma=\varepsilon^\alpha$ in the light and fast regime, namely

$$
\frac{\varepsilon^{\frac{1}{p}}}{\sigma}\left\||v|^p f^0_{\varepsilon,\sigma}\right\|^{\frac{1}{p}}_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)} \lesssim 1, \tag{3.2.15}
$$

in order to make sure that the second term is bounded with respect to ε. As p may be taken arbitrarily close to 3, this is significant only for the parameter range $\alpha>1/3$. In particular, note that this is not necessary in the light regime (that corresponds to $\alpha=0$). As explained above, a refined strategy, based on anisotropic $\mathrm{L}^{\frac{p}{p-1}}\mathrm{L}^p$ parabolic estimates would allow us to dispense with the additional assumption (3.2.15) (*cf.* [HM21] for the details).

To conclude, one may interpolate this higher order estimate with the L^2 norm of $u_{\varepsilon,\sigma}-\langle u_{\varepsilon,\sigma}\rangle$ thanks to the Gagliardo-Nirenberg inequality :

$$
\left\|\nabla_x u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \left\|u_{\varepsilon,\sigma}-\langle u_{\varepsilon,\sigma}\rangle\right\|^{1-\beta_p}_{\mathrm{L}^2(\mathbb{T}^3)}\left\|\Delta_x u_{\varepsilon,\sigma}\right\|^{\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)},
$$

for some $\beta_p\in(0,1)$. Using the exponential decay of $\mathscr{E}_{\varepsilon,\sigma}(t)$ (recall (3.2.6)), we therefore get

$$
\int_0^T \left\|\nabla_x u_{\varepsilon,\sigma}(t)\right\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathrm{d}t \lesssim \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}}\left\|\Delta_x u_{\varepsilon,\sigma}\right\|^{\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}},
$$

which can be made as small as desired by picking $\mathscr{E}_{\varepsilon,\sigma}(0)$ small enough. This allows to conclude the bootstrap argument and conclude that necessarily, $T^\star_\varepsilon=+\infty$.

Section 3.4.3 is then dedicated to the proofs of convergence. We first assume that $(\rho^0_{\varepsilon,\sigma}, u^0_{\varepsilon,\sigma})$ weakly converges to (ρ^0,u^0). Now that we have proven that $T^\star_\varepsilon=+\infty$ for any small $\varepsilon>0$, we can combine the energy–dissipation identity (3.1.2) and the bound (3.2.14) to deduce from the Aubin-Lions lemma the strong convergence of $(u_{\varepsilon,\sigma})$ in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$, for any $T>0$. The control (3.2.14) can also be used to prove the vanishing of $F_{\varepsilon,\sigma}$ in $\mathrm{L}^p((0,T)\times\mathbb{T}^3)$ thanks to (3.2.13), up to a slightly stronger additional assumption in the light and fast regime. This suffices to take the limit in the Navier–Stokes equations and the conservation of mass, and thus obtain the convergence of $(\rho_{\varepsilon,\sigma}, u_{\varepsilon,\sigma})$ to the solution to the Transport–Navier–Stokes system (3.1.6).

If we further assume the strong convergence of $(u^0_{\varepsilon,\sigma})$ to u^0, then the structure of the Navier–Stokes (resp. transport) equations satisfied by $(u_{\varepsilon,\sigma})$ and u (resp. $(\rho_{\varepsilon,\sigma})$ and ρ) provide a quantitative result for the pointwise convergence of $(u_{\varepsilon,\sigma})$ in $\mathrm{L}^2(\mathbb{T}^3)$ (resp. $(\rho_{\varepsilon,\sigma})$ in the Wasserstein-1 metric). The quantitative convergence result for $(f_{\varepsilon,\sigma})$, up to an integration in time, is then straightforward.

In order to obtain pointwise convergence for $(f_{\varepsilon,\sigma})$, we note that for any test function ψ, and almost every $t \in (0,T)$, we have

$$\left|\langle f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi\rangle\right| \leq \|\nabla_v \psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,\sigma}(t)|v - \sigma u_{\varepsilon,\sigma}(t)|\mathrm{d}x\mathrm{d}v,$$

which is reminiscent of the expression of the Brinkman force. Indeed, using the decomposition described above, we derive a quantitative convergence result in the case of well-prepared initial data.

In Section 3.4.5, we explain how to dispense with the assumptions of smallness for the initial fluid velocity and modulated energy and prove short time convergence results. The initial bounds on the initial data in Assumption 3.1 are in fact sufficient to ensure (3.2.10) holds for every $\varepsilon \in (0,1)$, provided the time t is less than some small time $T_M > 0$ that is independent of ε. We then apply the same bootstrap argument as above and prove that, up to reducing the value of T_M (regardless of ε), every time $T \in [0, T_M]$ is a strong existence time. The proof of convergence is then identical to that exposed in the previous paragraphs.

The bootstrap argument in the fine particle regimes. In Section 3.5, we tackle the most singular high friction regime studied in this work, that is the fine particle regime. To ease readability, we drop the parameters $\sigma = 1$ and $\gamma = \varepsilon$ in this presentation. As already explained, the difficulty lies in the fact that the Brinkman force in this regime, namely

$$F_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)(v - u_\varepsilon(t,x))\mathrm{d}v,$$

is singular with respect to ε. By opposition to the light particle case, thanks to the modulated energy–dissipation identity and assuming the global control

$$\|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,+\infty\times\mathbb{T}^3)} \leq \mathrm{C}_0,$$

one only obtains the bound

$$\|F_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}_+\times\mathbb{T}^3)} \lesssim \frac{1}{\sqrt{\varepsilon}}\mathscr{E}_\varepsilon(0),$$

which in general blows up as $\varepsilon \to 0$, and is as such not satisfactory in view of the $\mathrm{L}^\infty\mathrm{H}^1$–$\mathrm{L}^2\dot{\mathrm{H}}^2$ estimate for the fluid velocity that we aim at establishing. Therefore, the approach developed for the light particle regimes has to be modified.

We propose another desingularization procedure of the Brinkman force, that somehow reflects the fine algebraic structure of the Vlasov–Navier–Stokes system. Inspired by [Han20], the idea is to consider the *higher dissipation* functional

$$\mathrm{D}_\varepsilon^{(r)}(t) := \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon \frac{|v-u_\varepsilon(t,x)|^r}{\varepsilon^r}\mathrm{d}x\mathrm{d}v.$$

for $r \geq 2$. Such an object is useful in view of studying the Brinkman force, because of the estimate

$$\|F_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)} \leq \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{r}} \left(\int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\,\mathrm{d}t\right)^{\frac{1}{r}},$$

see Lemma 3.5.3. The key is an identity satisfied by $\int_0^t \mathrm{D}_\varepsilon^{(r)}(t)\mathrm{d}t$, similar to the ones introduced in [Han20], which is provided in Lemma 3.5.4. This results in the following estimate :

$$\begin{aligned}\int_0^t \mathrm{D}_\varepsilon^{(r)}(s)\mathrm{d}t &+ \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t)\frac{|v-u_\varepsilon(t)|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v \\ &\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r + \left\||\nabla_x u_\varepsilon| m_r^{\frac{1}{r}}\right\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r \\ &\qquad + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0 \frac{|v-u_\varepsilon^0|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v,\end{aligned}$$

that we use for $r = p$ (where p is the regularity index of Assumption 3.1) to get, assuming the smallness condition

$$\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \ll 1,$$

the well-preparedness assumption

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0 \frac{|v-u_\varepsilon^0|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v \lesssim 1,$$

and imposing ε small enough,

$$\|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim 1.$$

We point out that the smallness condition on f_ε^0 is used only in this argument. To prove the bound (3.2.7), we also require the smallness of the initial modulated energy. Finally, by a similar analysis for $r = 2$, we obtain a relevant control of the Brinkman force in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$, which allows to conclude the bootstrap argument in this case.

Section 3.5.3 is dedicated to a non-quantitative convergence result. We prove the strong convergence of (u_ε) and the weak convergence of (ρ_ε) as in the *light* and *light and fast* particle regimes. Furthermore, the energy–dissipation inequality yields the weak convergence of (j_ε). Because of the singularity of the Brinkman force, this does not imply that (F_ε) vanishes at $\varepsilon \to 0$. But thanks to the bounds derived for the bootstrap argument, we still know that (F_ε) converges weakly. This suffices to take the limit $\varepsilon \to 0$ in the conservations of mass and momentum as well as the Navier–Stokes equations and thus obtain the convergence of $(\rho_\varepsilon, u_\varepsilon)$ to the solution to the Inhomogeneous incompressible Navier–Stokes system. As in the *light* and *light and fast* particle regimes, a convergence result for (f_ε), up to an integration in time, then follows.

If we further assume the strong convergence of (u_ε^0) to u^0 and some regularity on the limit solution u, then we can obtain a pointwise convergence result for (ρ_ε) and, thanks to the well-preparedness assumption and the key estimate on the higher order dissipation, a pointwise convergence result for (f_ε). Note that, at this stage, we do not have quantitative estimates on $u_\varepsilon - u$, which prevents us from getting a quantitative result for (ρ_ε) and (f_ε) as well.

This indeed appears more difficult and is the object of Section 3.5.4. We introduce yet another functional, the so-called *relative entropy*

$$\mathscr{H}_\varepsilon(t) := \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v-u(t,x)|^2\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3} |u_\varepsilon(t,x)-u(t,x)|^2\mathrm{d}x.$$

As stated above, such functionals have already proven useful in many related high friction limits, see *e.g.* [GJV04b ; FK19]. A side goal of this section is to explain why it cannot be *directly* used in the fine particle regime that we study. Concretely, it turns out that a bound on the density

$$\|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T\times\mathbb{T}^3)} \leq \mathrm{C}_0, \quad \forall\varepsilon\in(0,1), \tag{3.2.16}$$

which is exactly the same as that required in the other parts of the analysis, is needed to complete the analysis. It appears that such a bound cannot be enforced with a proof solely based on relative entropy. But obtaining this bound has been precisely the object of the previous analysis.

By explicit computations (see Lemma 3.5.17) one gets

$$\begin{aligned}\mathscr{H}_\varepsilon(t) + \int_0^t\int_{\mathbb{T}^3} |\nabla_x(u_\varepsilon-u)|^2\mathrm{d}x\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3} |v-u_\varepsilon|^2 f_\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}s \\ \leq \mathscr{H}_{\varepsilon,\sigma}(0) + \int_0^t \sum_{j=1}^4 I_j(s)\mathrm{d}s.\end{aligned}$$

with

$$\begin{aligned}
I_1 &:= -\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(v-u)\otimes(v-u):\nabla_x u\,\mathrm{d}x\mathrm{d}v,\\
I_2 &:= -\int_{\mathbb{R}^3} (u_\varepsilon-u)\otimes(u_\varepsilon-u):\nabla_x u\,\mathrm{d}x,\\
I_3 &:= \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(v-u_\varepsilon)\cdot G\,\mathrm{d}x\mathrm{d}v,\\
I_4 &:= \int_{\mathbb{T}^3} (\rho_\varepsilon-\rho)(u_\varepsilon-u)\cdot G\,\mathrm{d}x,
\end{aligned}$$

where $G = \frac{\nabla_x p - \Delta_x u}{1+\rho}$. The terms I_1 and I_2 are the easiest to handle. To study I_3 one may use either the energy–dissipation inequality or, better, the dissipation $\mathrm{D}_\varepsilon^{(2)}$ to obtain the refined inequality

$$\int_0^T |I_3(t)|\mathrm{d}t \leq \|\rho_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \left(\varepsilon^2\int_0^T \mathrm{D}_\varepsilon^{(2)}(t)\mathrm{d}t\right)^{\frac{1}{2}} \|G\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon.$$

The control of I_4 relies on the following bound, which is valid under a regularity assumption on

(ρ, u) and on the condition that (3.2.16) holds :

$$\|\rho_\varepsilon(t) - \rho(t)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} \lesssim \|\rho_\varepsilon^0 - \rho^0\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon \int_0^t \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s + \int_0^t \|u_\varepsilon(s) - u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s. \quad (3.2.17)$$

Gathering all pieces together, we finally obtain the estimate

$$\mathscr{H}_\varepsilon(t) \lesssim \|u_\varepsilon^0 - u^0\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \|\rho_\varepsilon^0 - \rho^0\|^2_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon,$$

from which quantitative convergence bounds follow.

As in the *light* and *light and fast* particle regimes, we eventually explain in Section 3.5.5 how to dispense with the smallness assumptions on the initial fluid velocity and modulated energy, at the expense of a short time constraint.

3.3 Preliminary results

Let us start the analysis by introducing in a unified manner the first common arguments for all the regimes studied in this work. We recall that we aim at studying

$$\begin{cases} \partial_t u_{\varepsilon,\gamma,\sigma} + (u_{\varepsilon,\gamma,\sigma} \cdot \nabla_x) u_{\varepsilon,\gamma,\sigma} - \Delta_x u_{\varepsilon,\gamma,\sigma} + \nabla_x p_{\varepsilon,\gamma,\sigma} = \dfrac{1}{\gamma}\left(j_{f_{\varepsilon,\gamma,\sigma}} - \rho_{f_{\varepsilon,\gamma,\sigma}} u_{\varepsilon,\gamma,\sigma}\right), \\ \mathrm{div}_x\, u_{\varepsilon,\gamma,\sigma} = 0, \\ \partial_t f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\sigma} v \cdot \nabla_x f_{\varepsilon,\gamma,\sigma} + \dfrac{1}{\varepsilon}\mathrm{div}_v\left[f_{\varepsilon,\gamma,\sigma}(\sigma u_{\varepsilon,\gamma,\sigma} - v)\right] = 0, \\ \rho_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \displaystyle\int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}v, \quad j_{f_{\varepsilon,\gamma,\sigma}}(t,x) = \dfrac{1}{\sigma}\displaystyle\int_{\mathbb{R}^3} v f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}v. \end{cases} \quad (3.3.1)$$

Throughout this section, we fix $\varepsilon > 0$ and consider

$$(\gamma, \sigma) \in \{(1, \varepsilon^\alpha), (\varepsilon, 1)\},$$

for the light ($\alpha = 0$), light and fast ($\alpha \in (0, 1/2]$), and fine particle regimes, respectively. This part of the paper is organized as follows :

- Section 3.3.1 is a short reminder about the notion of weak solutions to the Vlasov–Navier–Stokes system.
- In Sections 3.3.2, 3.3.3 and 3.3.4, we gather several rather standard estimates for the Vlasov equation with friction and Navier–Stokes equations.
- Section 3.3.5 introduces a key object, the *modulated energy* and proves its exponential decay under the assumption of an L^∞ bound for the kinetic density.
- In Section 3.3.6, we study straightening changes of variables in velocity (resp. in space) which will be used at multiple times to obtain uniform bounds for moments and the Brinkman force. They are in particular the key to the aforementioned L^∞ bound for the kinetic density. We also introduce the notion of *strong existence times* which are, loosely speaking, times for which the changes of variables are admissible.
- In Section 3.3.7, we gather some estimates for the convective term in the Navier–Stokes equations.
- Finally, we initialize the bootstrap argument in Section 3.3.8.

3.3.1 Weak solutions

We begin by presenting the notion of weak solutions of the Vlasov–Navier–Stokes system and introducing useful notations.

Definition 3.3.1. *We define the kinetic energy of the system* (3.1.3)*, for every* $t \geq 0$*, by*

$$\mathrm{E}_{\varepsilon,\gamma,\sigma}(t) = \frac{1}{2}\left\|u_{\varepsilon,\gamma,\sigma}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{\varepsilon}{2\gamma\sigma^2}\int_{\mathbb{T}^3\times\mathbb{R}^3}|v|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v$$

and the dissipation by

$$\mathrm{D}_{\varepsilon,\gamma,\sigma}(t) = \left\|\nabla_x u_{\varepsilon,\gamma,\sigma}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{1}{\gamma}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma} - u_{\varepsilon,\gamma,\sigma}(t,x)\right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v.$$

As already said in the introduction, one can formally check that, for every $t \geq 0$, the following identity holds :

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{\varepsilon,\gamma,\sigma}(t) + \mathrm{D}_{\varepsilon,\gamma,\sigma}(t) = 0.$$

This identity plays a crucial role in the analysis of the Vlasov–Navier–Stokes system, be it for the proof of the existence of solutions ([Bou+09], [BGM17], [BMM20]), the long time behavior ([CK15], [Han20], [HMM20]) or the asymptotic analysis we conduct in this paper. As such, we shall only consider solutions that verify an inequality version of this energy–dissipation identity.

Definition 3.3.2. *Assume*

1. $u^0_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^3) = \{U \in \mathrm{L}^2(\mathbb{T}^3),\ \mathrm{div}_x U = 0\}$,
2. $0 \leq f^0_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$,
3. $(x,v) \mapsto f^0_{\varepsilon,\gamma,\sigma}(x,v)|v|^2 \in \mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)$,

then a global weak solution of the Vlasov–Navier–Stokes system (3.1.3) *with initial condition* $(u^0_{\varepsilon,\gamma,\sigma}, f^0_{\varepsilon,\gamma,\sigma})$ *is a pair* $(u_{\varepsilon,\gamma,\sigma}, f_{\varepsilon,\gamma,\sigma})$ *such that*

- *the distribution function* $f_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$ *is a renormalized solution of the Vlasov equation,*
- *the fluid velocity* $u_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^3)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^1(\mathbb{T}^3))$ *is a Leray solution of the Navier–Stokes equations,*
- $j_{\varepsilon,\gamma,\sigma} - \rho_{\varepsilon,\gamma,\sigma}u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^{-1}(\mathbb{T}^3))$,
- *for almost all* $t \geq s \geq 0$ *(including* $s = 0$*), the energy–dissipation inequality holds :*

$$\mathrm{E}_{\varepsilon,\gamma,\sigma}(t) + \int_s^t \mathrm{D}_{\varepsilon,\gamma,\sigma}(\tau)\mathrm{d}\tau \leq \mathrm{E}_{\varepsilon,\gamma,\sigma}(s). \tag{3.3.2}$$

Such global weak solutions are built in [Bou+09]. Note that Assumption 3.1 ensures that the requirements of Definition 3.3.2 are met. In the rest of this work, we consider a weak solution $(u_{\varepsilon,\gamma,\sigma}, f_{\varepsilon,\gamma,\sigma})$ in the sense of Definition 3.3.2.

The energy–dissipation estimate (3.3.2) has a possible consequence on the estimate of the Brinkman force which we recall is defined as

$$F_{\varepsilon,\gamma,\sigma} := \frac{1}{\gamma}\left(j_{\varepsilon,\gamma,\sigma} - \rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma}\right).$$

This is the object of the next remark.

Remark 3.3.3. *Let us assume the global uniform bound*

$$\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))} < +\infty,$$

Then by the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
&\|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2(\mathbb{R}_+;\mathrm{L}^2(\mathbb{T}^3))} \\
&\leq \|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))} \frac{1}{\gamma^2} \int_0^{+\infty} \int_{\mathbb{T}^3\times\mathbb{R}^3} \left| \frac{v}{\sigma} - u_{\varepsilon,\gamma,\sigma}(t,x) \right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v\mathrm{d}t \\
&\leq \frac{1}{\gamma} \|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))} \int_0^{+\infty} \mathrm{D}_{\varepsilon,\gamma,\sigma}(t)\mathrm{d}t \\
&\leq \frac{1}{\gamma} \|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))} \mathrm{E}_{\varepsilon,\gamma,\sigma}(0),
\end{aligned}$$

thanks to the energy–dissipation estimate (3.3.2)*. In the* light *and* light and fast *particle regimes, this provides a uniform bound for $F_{\varepsilon,\gamma,\sigma}$ in* $\mathrm{L}^2(\mathbb{R}_+\times\mathbb{T}^3)$ *; in sharp contrast, in the fine particle regime it only yields*

$$\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(\mathbb{R}_+\times\mathbb{T}^3)} \lesssim \frac{1}{\sqrt{\varepsilon}},$$

which may blow up as $\varepsilon \to 0$ and is thus not satisfactory. This is a first indication that the fine particle regime is more singular.

3.3.2 Conservation laws for the Vlasov equation

In this paragraph, we focus on the Vlasov equation and state basic conservation laws and estimates that can be justified thanks to the DiPerna-Lions theory [DL89], as described in the following remark.

Remark 3.3.4. *As in [HMM20, Remark 3.1], we shall use the DiPerna-Lions theory [DL89] repeatedly without writing down the argument explicitly. The* implicit *argument will always be :*

- *consider a sequence of regularized initial data $(f_n^0)_{n\in\mathbb{N}}$, an approximating sequence of fluid velocities $(u_n)_{n\in\mathbb{N}}$, and the associated smooth solutions $(f_n)_{n\in\mathbb{N}}$ of the Vlasov equation ;*
- *prove the desired estimate for f_n ;*
- *pass to the limit using the strong stability property of renormalized solutions [DL89].*

Eventually, we will prove that the velocity field $u_{\varepsilon,\gamma,\sigma}$ is smooth enough to apply the classical Cauchy-Lipschitz theorem, and therefore the DiPerna-Lions theory will not actually be needed.

Let us first define the notations associated to the characteristics of the equation.

Definition 3.3.5. *Let $t \geq 0$, $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$. We denote by*

$$(X_{\varepsilon,\gamma,\sigma}(\cdot;t,x,v), V_{\varepsilon,\gamma,\sigma}(\cdot;t,x,v))$$

the solution to the system of differential equations

$$\begin{cases} \dot{X}_{\varepsilon,\gamma,\sigma}(s;t,x,v) = \dfrac{1}{\sigma} V_{\varepsilon,\gamma,\sigma}(s;t,x,v), \\ \dot{V}_{\varepsilon,\gamma,\sigma}(s;t,x,v) = \dfrac{1}{\varepsilon}\left(\sigma u_{\varepsilon,\gamma,\sigma}(s, X_{\varepsilon,\gamma,\sigma}(s;t,x,v)) - V_{\varepsilon,\gamma,\sigma}(s;t,x,v)\right), \\ X_{\varepsilon,\gamma,\sigma}(t;t,x,v) = x, \\ V_{\varepsilon,\gamma,\sigma}(t;t,x,v) = v. \end{cases} \tag{3.3.3}$$

We can now state the following conservation laws of the Vlasov equation.

Lemma 3.3.6. *Under Assumption 3.1, for all $\varepsilon > 0$, for almost any $t \geq 0$,*

$$f_{\varepsilon,\gamma,\sigma}(t,x,v) \geq 0 \qquad a.e.\, (x,v) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

and

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma}(t,x,v)\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\gamma,\sigma}(x,v)\mathrm{d}x\mathrm{d}v.$$

Furthermore, the equations for local conservation of mass and momentum read

$$\partial_t \rho_{\varepsilon,\gamma,\sigma} + \mathrm{div}_x\, j_{\varepsilon,\gamma,\sigma} = 0, \tag{3.3.4}$$

and

$$\partial_t j_{\varepsilon,\gamma,\sigma} + \frac{1}{\sigma^2}\mathrm{div}_x \left(\int_{\mathbb{R}^3} v\otimes v\, f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}v\right) = -\frac{\gamma}{\varepsilon} F_{\varepsilon,\gamma,\sigma}. \tag{3.3.5}$$

$\triangleright$ Let us assume that both $u_{\varepsilon,\gamma,\sigma}$ and $f_{\varepsilon,\gamma,\sigma}$ are smooth functions, the general case follows by the DiPerna-Lions theory (recall Remark 3.3.4). Then, thanks to the method of characteristics, we have, for $t \geq 0$ and $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$,

$$f_{\varepsilon,\gamma,\sigma}(t,x,v) = e^{\frac{3t}{\varepsilon}} f^0_{\varepsilon,\gamma,\sigma}(X_{\varepsilon,\gamma,\sigma}(0;t,x,v), V_{\varepsilon,\gamma,\sigma}(0;t,x,v)),$$

hence the nonnegativity of $f_{\varepsilon,\gamma,\sigma}$.

The second statement stems from the integration of the Vlasov equation

$$\partial_t f_{\varepsilon,\gamma,\sigma} + \frac{1}{\sigma} v\cdot\nabla_x f_{\varepsilon,\gamma,\sigma} + \frac{1}{\varepsilon}\mathrm{div}_v\left[f_{\varepsilon,\gamma,\sigma}(\sigma u_{\varepsilon,\gamma,\sigma} - v)\right] = 0 \tag{3.3.6}$$

over $(0,t)\times\mathbb{T}^3\times\mathbb{R}^3$, for any $t \geq 0$, while the conservation of mass results from integrating only over $\mathbb{R}^3$ and the conservation of momentum comes from multiplying (3.3.6) by v/σ and integrating over $\mathbb{R}^3$. □

The maximum principle for Vlasov equations yields the following inequality.

Lemma 3.3.7. *Under Assumption 3.1, we have*

$$\forall t \geq 0, \qquad \|f_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leq e^{\frac{3t}{\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}.$$

$\triangleright$ This is a straightforward consequence of the method of characteristics. □

Remark 3.3.8. *The blow-up as ε goes to 0 in the right-hand side of this inequality is consistent with the expected convergence of $(f_{\varepsilon,\gamma,\sigma})$ to a Dirac mass, as presented in Section 3.1.2.*

Finally, we shall need the following conservation property of the total momentum.

Lemma 3.3.9. *Under Assumption 3.1, for almost any $t \geq 0$,*

$$\left\langle \frac{\varepsilon}{\gamma} j_{\varepsilon,\gamma,\sigma}(t) + u_{\varepsilon,\gamma,\sigma}(t) \right\rangle = \left\langle \frac{\varepsilon}{\gamma} j^0_{\varepsilon,\gamma,\sigma} + u^0_{\varepsilon,\gamma,\sigma} \right\rangle.$$

$\triangleright$ By integrating the Navier–Stokes equation and (3.3.5) on $(0,t) \times \mathbb{T}^3$, we obtain, for almost any $t \geq 0$,

$$\langle u_{\varepsilon,\gamma,\sigma}(t) \rangle = \langle F_{\varepsilon,\gamma,\sigma}(t) \rangle \qquad \text{and} \qquad \langle j_{\varepsilon,\gamma,\sigma}(t) \rangle = \frac{-\gamma}{\varepsilon} \langle F_{\varepsilon,\gamma,\sigma}(t) \rangle,$$

hence the result. $\square$

3.3.3 Higher regularity estimate for the Navier–Stokes equations

In this section, we present a higher regularity estimate for the Navier–Stokes equations. We also introduce some notations that we use to abbreviate the results given in Appendix C.

According to Theorem C.2, the estimate

$$\|u_{\varepsilon,\gamma,\sigma}(t)\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \int_0^t \|\Delta_x u_{\varepsilon,\gamma,\sigma}(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \, \mathrm{d}s \lesssim \Psi_{\varepsilon,\gamma,\sigma,0}, \tag{3.3.7}$$

where

$$\Psi_{\varepsilon,\gamma,\sigma,0} = \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)}.$$

holds for $t \geq 0$ if, for some universal constant $C^* > 0$,

$$\int_0^t \left\|e^{s\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \mathrm{d}s + \int_0^t \|F_{\varepsilon,\gamma,\sigma}(s)\|^2_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)} \mathrm{d}s \leq C^* \tag{3.3.8}$$

is satisfied.

Remark 3.3.10. *The subscript* 0 *in* $\Psi_{\varepsilon,\gamma,\sigma,0}$ *may be misleading as in the definition of* $\Psi_{\varepsilon,\gamma,\sigma,0}$*, the term* $\|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)}$ *depends on time* t*. However we shall always consider so-called strong existence times (see Definition 3.3.27) for which* $\|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} \leq C$*, for some fixed* $C > 0$*, so that the notation makes sense.*

Thanks to the Sobolev embeddings $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^r(\mathbb{T}^3)$ $(r \leq 6)$, we draw the following first consequence of the higher regularity estimate (3.3.7).

Corollary 3.3.11. *Under Assumption 3.1 and if* $T > 0$ *is such that* (3.3.7) *holds on* $[0,T]$*, then, for every* $2 \leq r \leq 6$*,*

$$\|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^r(\mathbb{T}^3))} \lesssim \Psi^{\frac{1}{2}}_{\varepsilon,\gamma,\sigma,0}.$$

Now we investigate when (3.3.8) is satisfied. To this end, we start with the following preliminary result which corresponds to [HMM20, Lemma 4.3].

Lemma 3.3.12. *Under Assumption 3.1, for all* $T > 0$*,* $F_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2(0,T;\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3))$*.*

$\triangleright$ We follow [HMM20, Lemmas 4.2 and 4.3] and only provide the proof for the sake of completeness. First, note that, for any $T > 0$, thanks to Hölder's inequality and the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$, we have

$$\|\rho_{\varepsilon,\gamma,\sigma} u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3))} \lesssim \|\rho_{\varepsilon,\gamma,\sigma}\|^{\frac{1}{2}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))} \|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^3))},$$

so we only need to prove that

$$\rho_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3)) \text{ and } j_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty(0,T;\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3)).$$

Recall the interpolation estimate : for any $0 \leq \ell \leq k$ and any nonnegative $g \in \mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$,

$$\|m_\ell g\|_{\mathrm{L}^{\frac{k+3}{\ell+3}}(\mathbb{T}^3)} \lesssim (M_k g)^{\frac{\ell+3}{k+3}} \|g\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{k-\ell}{k+3}}, \tag{3.3.9}$$

where

$$m_\ell g = \int_{\mathbb{R}^3} |v|^\ell g(\cdot,v)\mathrm{d}v \qquad \text{and} \qquad M_k g = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^k g(x,v)\mathrm{d}x\mathrm{d}v.$$

Applying (3.3.9) for $f_{\varepsilon,\gamma,\sigma}$ with $(\ell,k) = (0,3)$ and $(\ell,k) = (1,3)$ yields, for almost every $t \in (0,T)$,

$$\|\rho_{\varepsilon,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} = \|m_0 f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim (M_3 f_{\varepsilon,\gamma,\sigma})^{\frac{1}{2}} \|f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{2}},$$

and

$$\begin{aligned}\|j_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3)} &\leq \sigma^{-1} \|m_1 f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3)} \\ &\lesssim \sigma^{-1}(M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{2}{3}} \|f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{3}}.\end{aligned}$$

Thanks to Lemma 3.3.7, we know that $f_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3))$ so that we shall focus on proving that $M_3 f_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty((0,T))$.

Multiplying the Vlasov equation by $|v|^3$ and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} M_3 f_{\varepsilon,\gamma,\sigma}(t) + \frac{3}{\varepsilon} M_3 f_{\varepsilon,\gamma,\sigma}(t) \leq \frac{3\sigma}{\varepsilon} \int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t)|\, |m_2 f_{\varepsilon,\gamma,\sigma}(t)| \mathrm{d}x. \tag{3.3.10}$$

We apply (3.3.9) with $(\ell,k) = (2,3)$ and Lemma 3.3.7 to obtain, for almost every $t \geq 0$,

$$\begin{aligned}\|m_2 f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^{\frac{6}{5}}(\mathbb{T}^3)} &\lesssim (M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{5}{6}} \|f_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{6}} \\ &\lesssim e^{\frac{t}{2\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{6}} (M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{5}{6}}.\end{aligned}$$

Therefore, thanks to Hölder's inequality

$$\int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t)|\, |m_2 f_{\varepsilon,\gamma,\sigma}(t)| \mathrm{d}x \lesssim e^{\frac{t}{2\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{6}} (M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{5}{6}} \|u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^6(\mathbb{T}^3)}.$$

Injecting this in (3.3.10) yields, using once again the embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} (M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{1}{6}} + \frac{1}{2\varepsilon} (M_3 f_{\varepsilon,\gamma,\sigma}(t))^{\frac{1}{6}} \lesssim \frac{\sigma}{\varepsilon} e^{\frac{t}{2\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{6}} \|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{H}^1(\mathbb{T}^3)},$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(e^{\frac{t}{2\varepsilon}} (M_3 f_{\varepsilon,\gamma,\sigma})^{\frac{1}{6}} \right) \lesssim \frac{\sigma}{\varepsilon} e^{\frac{t}{\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \|u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{H}^1(\mathbb{T}^3)}.$$

Assumption 3.1 implies that $M_3 f^0_{\varepsilon,\gamma,\sigma} < \infty$, so that, thanks to the Cauchy-Schwarz inequality and the energy–dissipation estimate (3.3.2),

$$M_3 f_{\varepsilon,\gamma,\sigma}(t) \lesssim M_3 f^0_{\varepsilon,\gamma,\sigma} + \left(\frac{\sigma}{\varepsilon}\right)^6 (t^3+t^6) e^{\frac{3t}{\varepsilon}} \left\| f^0_{\varepsilon,\gamma,\sigma} \right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \mathrm{E}_{\varepsilon,\gamma,\sigma}(0)^3,$$

which concludes the proof. □

In the following lemma, we highlight two cases for which (3.3.8) is verified.

Lemma 3.3.13. *Under Assumption 3.1,*

1. *there exists $T_{M,\varepsilon} > 0$, depending on M and ε, such that* (3.3.8) *holds for almost every $t \in [0, T_{M,\varepsilon}]$.*
2. *under Assumption 3.2 and if $T > 0$ verifies*

$$\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq C^*/2,$$

then, (3.3.8) *holds for almost every $t \in [0,T]$.*

▷ Thanks to Sobolev interpolation and Lemma C.1,

$$\begin{aligned}\int_0^T \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \mathrm{d}t &\lesssim \left(\sup_{t\in(0,T)} \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)} \right) \int_0^T \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3)} \mathrm{d}t \\ &\lesssim \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)} \int_0^T \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3)} \mathrm{d}t,\end{aligned}$$

and

$$\int_0^T \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3)} \mathrm{d}t \leq \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|^2_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)}.$$

- Under Assumption 3.2, since $\int_0^t \|F_{\varepsilon,\gamma,\sigma}(s)\|^2_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)} \mathrm{d}s \leq \|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}$, the conclusion is straightforward.
- If we only consider Assumption 3.1, then $(\left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)})_{\varepsilon>0}$ is uniformly bounded and therefore, for every $t \in \mathbb{R}_+$,

$$\left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \leq M,$$

  from which we derive the existence of $T_M > 0$ such that

$$\int_0^{T_M} \left\|e^{t\Delta} u^0_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \leq \frac{C^*}{2}.$$

  Similarly, thanks to Sobolev's embeddings and Lemma 3.3.12, there exists $T_{M,\varepsilon} > 0$ such that

$$\int_0^{T_{M,\varepsilon}} \|F_{\varepsilon,\gamma,\sigma}(t)\|^2_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)} \mathrm{d}t \lesssim \int_0^{T_{M,\varepsilon}} \|F_{\varepsilon,\gamma,\sigma}(t)\|^2_{\mathrm{L}^{\frac{3}{2}}(\mathbb{T}^3)} \mathrm{d}t \leq \frac{C^*}{2},$$

  which concludes the proof of the lemma.

□

The purpose of the next lemma is to ensure that there exists a time $T_{M,\varepsilon} > 0$ such that $F_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2((0,T_{M,\varepsilon})\times\mathbb{T}^3)$, in order to be able to apply (3.3.7) at least for short times.

Lemma 3.3.14. *Under Assumption 3.1, there exists $T_{M,\varepsilon} > 0$ such that*

$$F_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2((0,T_{M,\varepsilon})\times\mathbb{T}^3).$$

$\triangleright$ Following, once again, the proof of [HMM20, Lemma 4.2], we obtain, for almost every $t \in \mathbb{R}_+$,

$$M_5 f_{\varepsilon,\gamma,\sigma}(t) \lesssim M_5 f^0_{\varepsilon,\gamma,\sigma} + \left(\frac{\sigma}{\varepsilon}\right)^8 e^{\frac{3t}{\varepsilon}} \left\|f^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \left(\int_0^t \|u_{\varepsilon,\gamma,\sigma}(s)\|_{\mathrm{L}^8(\mathbb{T}^3)}\,\mathrm{d}s\right)^8.$$

Thanks to Lemma 3.3.13 and Theorem C.2, we have $u_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2(0,T_{M,\varepsilon};\mathrm{H}^{\frac{3}{2}}(\mathbb{T}^3))$ for some $T_{M,\varepsilon} > 0$. This implies $u_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2(0,T_{M,\varepsilon};\mathrm{L}^r(\mathbb{T}^3))$ for any $r \in [1,\infty)$. Then, thanks to the interpolation estimate (3.3.9) with $(\ell,k) = (1,5)$,

$$\|j_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim \sigma^{-1}\left(M_5 f_{\varepsilon,\gamma,\sigma}(t)\right)^{\frac{1}{2}} \|f_{\varepsilon,\gamma,\sigma}(t)\|^{\frac{1}{2}}_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} < \infty,$$

and $j_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2(0,T_{M,\varepsilon};\mathrm{L}^2(\mathbb{T}^3))$. Similarly, using (3.3.9) with $(\ell,k) = (0,5)$,

$$\|\rho_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^{\frac{8}{3}}(\mathbb{T}^3)} \lesssim \left(M_5 f_{\varepsilon,\gamma,\sigma}(t)\right)^{\frac{3}{8}} \|f_{\varepsilon,\gamma,\sigma}(t)\|^{\frac{5}{8}}_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} < \infty$$

and we conclude with Hölder's inequality that $\rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^2((0,T_{M,\varepsilon})\times\mathbb{T}^3)$. $\square$

3.3.4 Rough bounds on the first moments

In order to have a first rough estimate on the Brinkman force, we will use the following bounds on the moments $\rho_{\varepsilon,\gamma,\sigma}$ and $j_{\varepsilon,\gamma,\sigma}$.

Lemma 3.3.15. *Under Assumption 3.1, there exist a time $T_{\varepsilon,\gamma,\sigma} > 0$ and a continuous function $\varphi^{\mathrm{force}}_{\varepsilon,\gamma,\sigma}$, increasing with respect to both its variables, and such that for every $T \in [0,T_{\varepsilon,\gamma,\sigma}]$,*

$$\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \leq \varphi^{\mathrm{force}}_{\varepsilon,\gamma,\sigma}(T,M),$$

$$\|j_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \leq \varphi^{\mathrm{force}}_{\varepsilon,\gamma,\sigma}(T,M),$$

and

$$\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \leq \varphi^{\mathrm{force}}_{\varepsilon,\gamma,\sigma}(T,M)$$

Remark 3.3.16. *Note that the functions $\varphi^{\mathrm{force}}_{\varepsilon,\gamma,\sigma,0}$ obtained in the proof blow up as $\varepsilon \to 0$.*

$\triangleright$ By the method of characteristics, for any $t \geq 0$ and $x \in \mathbb{T}^3$, we have

$$\rho_{\varepsilon,\gamma,\sigma}(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon,\gamma,\sigma}(X_{\varepsilon,\gamma,\sigma}(0;t,x,v), V_{\varepsilon,\gamma,\sigma}(0;t,x,v)\mathrm{d}v.$$

Using Assumption 3.1, we get

$$|\rho_{\varepsilon,\gamma,\sigma}(t,x)| \leq M e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} \frac{\mathrm{d}v}{1+|V_{\varepsilon,\gamma,\sigma}(0;t,x,v)|^q}, \tag{3.3.11}$$

with $q > 3$. Therefore, we need a lower bound for $|V_{\varepsilon,\gamma,\sigma}(0;t,x,v)|$. For $t \geq 0$ and $(x,v) \in \mathbb{T}^3\times\mathbb{R}^3$, we can integrate the equation of characteristics (3.3.3) to find that

$$V_{\varepsilon,\gamma,\sigma}(0;t,x,v) = e^{\frac{t}{\varepsilon}} v - \frac{\sigma}{\varepsilon}\int_0^t e^{\frac{\tau}{\varepsilon}} u_{\varepsilon,\gamma,\sigma}(\tau, X_{\varepsilon,\gamma,\sigma}(\tau;t,x,v))\mathrm{d}\tau.$$

Furthermore, thanks to Lemma 3.3.13, there exists $T_{\varepsilon,\gamma,\sigma} > 0$ such that (3.3.8) and therefore (3.3.7) hold, as well as Lemma 3.3.14. This yields

$$\|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T_{\varepsilon,\gamma,\sigma})\times\mathbb{T}^3)} \lesssim \Psi_{\varepsilon,\gamma,\sigma,0}^{\frac{1}{2}} \lesssim 1.$$

Therefore, thanks to the Gagliardo-Nirenberg inequality (Theorem B.1),

$$\|u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \|u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}^{\frac{1}{4}} \|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}^{\frac{3}{4}} + \|u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^2(\mathbb{T}^3)},$$

and using Hölder's inequality and the energy–dissipation estimate (3.3.2), we get, for every $T \in [0, T_{\varepsilon,\gamma,\sigma}]$,

$$\begin{aligned}
&\|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \\
&\quad\lesssim T^{\frac{5}{8}} \|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))}^{\frac{1}{4}} \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}^{\frac{3}{4}} + T\,\|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))} \\
&\quad\lesssim T^{\frac{5}{8}} \mathrm{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1}{2}} + T\,\mathrm{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we infer from Assumption 3.1 that there exists a continuous function $\psi_{\varepsilon,\gamma,\sigma}$ that is increasing with respect to both its variables and such that

$$|V(0;t,x,v)| \geq |v| - \frac{\sigma e^{\frac{T}{\varepsilon}}}{\varepsilon} \|u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \geq |v| - \psi_{\varepsilon,\gamma,\sigma}(T,M).$$

Then, thanks to (3.3.11), since $q > 3$,

$$\rho_{\varepsilon,\gamma,\sigma}(t,x) \lesssim M e^{\frac{3t}{\varepsilon}} \left(\int_{|v|\leq 2\psi_{\varepsilon,\gamma,\sigma}(T,M)} \mathrm{d}v + \int_{|v|\geq 2\psi_{\varepsilon,\gamma,\sigma}(T,M)} \frac{\mathrm{d}v}{1+|v|^q} \right) \lesssim \varphi_{\varepsilon,\gamma,\sigma}(T,M)$$

for some continuous function $\varphi_{\varepsilon,\gamma,\sigma}$ that is increasing with respect to both its variables.

Similarly, by the method of characteristics,

$$|j_{\varepsilon,\gamma,\sigma}(t,x)| \leq \sigma^{-1} M e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} \frac{|v|\mathrm{d}v}{1+|V(0;t,x,v)|^q}$$

and we can proceed as above to obtain the stated result since $q > 4$ in Assumption 3.1.

We can then conclude, since $F_{\varepsilon,\gamma,\sigma} = (j_{\varepsilon,\gamma,\sigma} - \rho_{\varepsilon,\gamma,\sigma} u_{\varepsilon,\gamma,\sigma})/\gamma$, by applying Corollary 3.3.11. □

3.3.5 Modulated energy dissipation

Following [CK15] and [HMM20], we introduce a *modulated* energy for which, under certain conditions, we can prove an exponential decay.

Definition 3.3.17. *We define the modulated energy, for every $t \geq 0$, by*

$$\begin{aligned}
\mathscr{E}_{\varepsilon,\gamma,\sigma}(t) = \frac{\varepsilon}{2\gamma} \int_{\mathbb{T}^3\times\mathbb{R}^3} \left| \frac{v}{\sigma} - \frac{\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle}{\langle \rho_{\varepsilon,\gamma,\sigma}\rangle} \right|^2 f_{\varepsilon,\gamma,\sigma}(t,x,v)\,\mathrm{d}x\mathrm{d}v& \\
+ \frac{1}{2}\int_{\mathbb{T}^3} |u_{\varepsilon,\gamma,\sigma}(t,x) - \langle u_{\varepsilon,\gamma,\sigma}(t)\rangle|^2 \mathrm{d}x& \\
+ \frac{\varepsilon\langle \rho_{\varepsilon,\gamma,\sigma}\rangle}{2(\gamma + \varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle)} \left| \frac{\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle}{\langle \rho_{\varepsilon,\gamma,\sigma}\rangle} - \langle u_{\varepsilon,\gamma,\sigma}(t)\rangle \right|^2&.
\end{aligned}$$

Remark 3.3.18. *Recall that, thanks to Lemma 3.3.6, $\langle \rho_{\varepsilon,\gamma,\sigma}(t)\rangle$ does not depend on time, which justifies the notation in the functional above.*

The modulated energy satisfies the same type of energy–dissipation estimate as the energy.

Lemma 3.3.19. *Under Assumption 3.1, for every $\varepsilon > 0$, for almost every $0 \le s \le t$ (including $s = 0$),*

$$\mathscr{E}_{\varepsilon,\gamma,\sigma}(t) + \int_s^t \mathrm{D}_{\varepsilon,\gamma,\sigma}(\tau)\mathrm{d}\tau \le \mathscr{E}_{\varepsilon,\gamma,\sigma}(s).$$

$\triangleright$ We expand the quadratic terms and find that for all $t \ge 0$

$$\begin{aligned}\mathscr{E}_{\varepsilon,\gamma,\sigma}(t) &= \mathrm{E}_{\varepsilon,\gamma,\sigma}(t) - \frac{\varepsilon}{2\gamma}\frac{|\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle|^2}{\langle \rho_{\varepsilon,\gamma,\sigma}\rangle} - \frac{|u_{\varepsilon,\gamma,\sigma}(t)|^2}{2} \\ &\quad + \frac{\varepsilon}{2(\gamma+\varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle)}\frac{|\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle|^2}{\langle \rho_{\varepsilon,\gamma,\sigma}\rangle} + \frac{\varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}{2(\gamma+\varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle)}|\langle u_{\varepsilon,\gamma,\sigma}(t)\rangle|^2 \\ &\quad - \frac{\varepsilon}{\gamma+\varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle\cdot\langle u_{\varepsilon,\gamma,\sigma}(t)\rangle \\ &= \mathrm{E}_{\varepsilon,\gamma,\sigma}(t) - \frac{\gamma}{2(\gamma+\varepsilon\langle\rho_{\varepsilon,\gamma,\sigma}\rangle)}\left|\frac{\varepsilon}{\gamma}\langle j_{\varepsilon,\gamma,\sigma}(t)\rangle + \langle u_{\varepsilon,\gamma,\sigma}(t)\rangle\right|^2.\end{aligned}$$

We conclude thanks to the energy–dissipation estimate (3.3.2) and Lemma 3.3.9.

□

As shown in [CK15] and [HMM20], under an assumption on the first moment of $f_{\varepsilon,\gamma,\sigma}$, we have an exponential decay of the modulated energy.

Lemma 3.3.20. *Under Assumption 3.1, if $T > 0$ is such that $\rho_{\varepsilon,\gamma,\sigma} \in \mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$, then*

$$\forall t \in [0,T], \qquad \mathscr{E}_{\varepsilon,\gamma,\sigma}(t) \le Ce^{-\lambda_{\varepsilon,\gamma,\sigma}t}\mathscr{E}_{\varepsilon,\gamma,\sigma}(0), \tag{3.3.12}$$

with

$$\lambda_{\varepsilon,\gamma,\sigma} = \min\left(\frac{2\gamma}{\varepsilon\left(\gamma + 2\,\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}\right)}, 1\right),$$

where $C > 0$ depends only on $\lambda_{\varepsilon,\gamma,\sigma}$.

$\triangleright$ We adapt the proof of [HMM20, Lemma 3.4]. All we need to prove is that

$$\forall t \in [0,T], \qquad \mathrm{D}_{\varepsilon,\gamma,\sigma}(t) \ge \lambda_{\varepsilon,\gamma,\sigma}\mathscr{E}_{\varepsilon,\gamma,\sigma}(t),$$

since the exponential decay (3.3.12) then follows from Lemma 3.3.19 and [HMM20, Lemma 9.3]. For the sake of readability, we shall not write the time, position or velocity variables in the rest of the proof.

By the Poincaré-Wirtinger inequality, we get

$$\mathrm{D}_{\varepsilon,\gamma,\sigma} \ge \frac{1}{\gamma}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma} - u_{\varepsilon,\gamma,\sigma}\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v + \|u_{\varepsilon,\gamma,\sigma} - \langle u_{\varepsilon,\gamma,\sigma}\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^3)}. \tag{3.3.13}$$

Furthermore, expanding the quadratic factor yields

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3}&\left|\frac{v}{\sigma}-u_{\varepsilon,\gamma,\sigma}\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v\\ &=\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v+\int_{\mathbb{T}^3}\rho_{\varepsilon,\gamma,\sigma}|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle|^2\,\mathrm{d}x\\ &\qquad+2\int_{\mathbb{T}^3\times\mathbb{R}^3}\left(\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right)\cdot\left(\langle u_{\varepsilon,\gamma,\sigma}\rangle-u_{\varepsilon,\gamma,\sigma}\right)f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v.\end{aligned}$$

For any $a\in(0,1)$, we have

$$\begin{aligned}2\int_{\mathbb{T}^3\times\mathbb{R}^3}&\left(\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right)\cdot\left(\langle u_{\varepsilon,\gamma,\sigma}\rangle-u_{\varepsilon,\gamma,\sigma}\right)f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v\\ &\geq -a\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v-\frac{1}{a}\int_{\mathbb{T}^3}\rho_{\varepsilon,\gamma,\sigma}|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle|^2\,\mathrm{d}x,\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-u_{\varepsilon,\gamma,\sigma}\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v&\geq(1-a)\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v\\ &\quad-\left(\frac{1}{a}-1\right)\int_{\mathbb{T}^3}\rho_{\varepsilon,\gamma,\sigma}|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle|^2\,\mathrm{d}x.\end{aligned}\tag{3.3.14}$$

In order to deal with the first term, we write

$$\begin{aligned}\left|\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2&=\left|\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2\\ &\quad+2\left(\frac{v}{\sigma}-\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}\right)\cdot\left(\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right)+\left|\frac{v}{\sigma}-\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}\right|^2\end{aligned}$$

and obtain, after integrating against $f_{\varepsilon,\gamma,\sigma}$,

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3}&\left|\frac{v}{\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v\\ &=\langle\rho_{\varepsilon,\gamma,\sigma}\rangle\left|\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right|^2+\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-\frac{\langle j_{\varepsilon,\gamma,\sigma}\rangle}{\langle\rho_{\varepsilon,\gamma,\sigma}\rangle}\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v\\ &\geq\frac{2\gamma}{\varepsilon}\left(\mathscr{E}_{\varepsilon,\gamma,\sigma}-\frac{1}{2}\left\|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\right).\end{aligned}$$

Therefore, the inequality (3.3.14) becomes

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma}-u_{\varepsilon,\gamma,\sigma}\right|^2 f_{\varepsilon,\gamma,\sigma}\,\mathrm{d}x\mathrm{d}v &\\ \geq \frac{2\gamma(1-a)}{\varepsilon}&\left(\mathscr{E}_{\varepsilon,\gamma,\sigma}-\frac{1}{2}\left\|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\right)\\ &-\left(\frac{1}{a}-1\right)\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}\left\|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)}.\end{aligned}$$

We set

$$a=\frac{2\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3}}{\gamma+2\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}}\in(0,1)$$

so that

$$\frac{1}{\gamma}\left(\frac{1}{a}-1\right)\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}=\frac{1}{2}.$$

Injecting this into (3.3.13) yields

$$\begin{aligned}\mathrm{D}_{\varepsilon,\gamma,\sigma}&\geq\frac{2\gamma}{\varepsilon\left(\gamma+2\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}\right)}\left(\mathscr{E}_{\varepsilon,\gamma,\sigma}-\frac{1}{2}\left\|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\right)\\ &\quad+\frac{1}{2}\left\|u_{\varepsilon,\gamma,\sigma}-\langle u_{\varepsilon,\gamma,\sigma}\rangle\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\\ &\geq\min\left(\frac{2\gamma}{\varepsilon\left(\gamma+2\|\rho_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}\right)},1\right)\mathscr{E}_{\varepsilon,\gamma,\sigma},\end{aligned}$$

hence the result. □

We will heavily rely on this exponential decay of the modulated energy, through the following estimates.

Corollary 3.3.21. *Under Assumption 3.1, for every $r>3$, if $T>0$ is such that $\rho_{\varepsilon,\gamma,\sigma}\in\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$ and $\Delta_x u_{\varepsilon,\gamma,\sigma}\in\mathrm{L}^r((0,T)\times\mathbb{T}^3)$, then for any $t\in[0,T]$,*

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^r(\mathbb{T}^3)}\lesssim\mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1-\alpha_r}{2}}e^{-\frac{(1-\alpha_r)\lambda_{\varepsilon,\gamma,\sigma}}{2}t}\|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|^{\alpha_r}_{\mathrm{L}^r(\mathbb{T}^3)},$$

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\lesssim\mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1-\beta_r}{2}}e^{-\frac{(1-\beta_r)\lambda_{\varepsilon,\gamma,\sigma}}{2}t}\|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|^{\beta_r}_{\mathrm{L}^r(\mathbb{T}^3)},$$

where α_r,β_r are defined in Corollary B.2.

▷ This is a direct consequence of the Gagliardo-Nirenberg inequality (Corollary B.2) and Lemma 3.3.20 since $r>3$. □

3.3.6 Changes of variable in velocity or in space

We present in this section two changes of variables that will allow us to derive crucial estimates on the moments and the Brinkman force.

First, we adapt what [HMM20] calls the *straightening* change of variables. Under a smallness condition on $\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}$, we define a diffeomorphism in velocity as follows.

Lemma 3.3.22. *Fix $c_* > 0$ such that $c_* e^{c_*} < 1/9$. Then, for any $t \in \mathbb{R}_+$ satisfying*

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))} \leq c_*, \tag{3.3.15}$$

and any $x \in \mathbb{T}^3$, the map

$$\Gamma^{t,x}_{\varepsilon,\gamma,\sigma} : v \mapsto V_{\varepsilon,\gamma,\sigma}(0; t, x, v)$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{R}^3$ to itself and satisfies

$$\forall v \in \mathbb{R}^3, \qquad \det \mathrm{D}_v \Gamma^{t,x}_{\varepsilon,\gamma,\sigma}(v) \geq \frac{e^{\frac{3t}{\varepsilon}}}{2}.$$

$\triangleright$ Defining $Y_{\varepsilon,\gamma,\sigma} = \frac{\sigma}{\varepsilon} X_{\varepsilon,\gamma,\sigma}$, the equations of characteristics in Definition 3.3.5 can be written as follows : for any $t \geq 0$ and $(y, v) \in \frac{\sigma}{\varepsilon}\mathbb{T}^3 \times \mathbb{R}^3$,

$$\begin{cases} \dot{Y}_{\varepsilon,\gamma,\sigma}(s; t, y, v) = \dfrac{1}{\varepsilon} V_{\varepsilon,\gamma,\sigma}(s; t, x, v) \\ \dot{V}_{\varepsilon,\gamma,\sigma}(s; t, y, v) = \dfrac{1}{\varepsilon} \left(\sigma u_{\varepsilon,\gamma,\sigma}\left(s, \dfrac{\varepsilon}{\sigma} Y_{\varepsilon,\gamma,\sigma}(s; t, y, v)\right) - V_{\varepsilon,\gamma,\sigma}(s; t, y, v)\right) \\ Y_{\varepsilon,\gamma,\sigma}(t; t, x, v) = y \\ V_{\varepsilon,\gamma,\sigma}(t; t, y, v) = v. \end{cases}$$

Following [HMM20], we define, for any $s \in \mathbb{R}_+$ and $z = (y, v) \in \frac{\sigma}{\varepsilon}\mathbb{T}^3 \times \mathbb{R}^3$,

$$w_{\varepsilon,\gamma,\sigma}(s, z) = \left(\frac{v}{\varepsilon}, \frac{1}{\varepsilon}\left(\sigma u_{\varepsilon,\gamma,\sigma}\left(s, \frac{\varepsilon}{\sigma} y\right) - v\right)\right)$$

and $Z_{\varepsilon,\gamma,\sigma} = (Y_{\varepsilon,\gamma,\sigma}, V_{\varepsilon,\gamma,\sigma})$, so that

$$\partial_s Z_{\varepsilon,\gamma,\sigma}(s; t, z) = w_{\varepsilon,\gamma,\sigma}(s, Z_{\varepsilon,\gamma,\sigma}(s; t, y, v)),$$

which after differentiation with respect to z yields

$$\partial_s \mathrm{D}_z Z_{\varepsilon,\gamma,\sigma}(s; t, z) = \mathrm{D}_z w_{\varepsilon,\gamma,\sigma}(s, Z(s; t, z)) \mathrm{D}_z Z_{\varepsilon,\gamma,\sigma}(s; t, z).$$

Since for any $s \in \mathbb{R}_+$,

$$\|\mathrm{D}_z w_{\varepsilon,\gamma,\sigma}(s, \cdot)\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \frac{1}{\varepsilon} + \|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(\mathbb{T}^3)},$$

Grönwall's lemma yields, for any $0 \leq s \leq t$,

$$\|\mathrm{D}_z Z_{\varepsilon,\gamma,\sigma}(s; t, \cdot)\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{\frac{t-s}{\varepsilon}} e^{\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}}.$$

In particular,

$$\|\mathrm{D}_v Y_{\varepsilon,\gamma,\sigma}(s; t, \cdot)\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{\frac{t-s}{\varepsilon}} e^{\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}}.$$

Using

$$V_{\varepsilon,\gamma,\sigma}(0; t, y, v) = e^{\frac{t}{\varepsilon}} v - \frac{\sigma}{\varepsilon} \int_0^t e^{\frac{s}{\varepsilon}} u_{\varepsilon,\gamma,\sigma}(s, \varepsilon Y_{\varepsilon,\gamma,\sigma}(s; t, y, v)/\sigma) \mathrm{d}s,$$

we get

$$\begin{aligned}
&\left\| e^{\frac{-t}{\varepsilon}} \mathrm{D}_v V_{\varepsilon,\gamma,\sigma}(0;t,\cdot) - \mathrm{Id} \right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \\
&\qquad \leq \int_0^t e^{\frac{s-t}{\varepsilon}} \left\| \nabla_x u_{\varepsilon,\gamma,\sigma}(s) \right\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \left\| \mathrm{D}_v Y_{\varepsilon,\gamma,\sigma}(s;t,\cdot) \right\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \mathrm{d}s \\
&\qquad \leq \left\| \nabla_x u_{\varepsilon,\gamma,\sigma} \right\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))} e^{\| \nabla_x u_{\varepsilon,\gamma,\sigma} \|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}} \\
&\qquad \leq c_* e^{c_*} < \frac{1}{9}
\end{aligned}$$

and we conclude using Theorem D.1. □

Remark 3.3.23. *We set in the following $c_* = 1/30$ and the inequality $c_* e^{c_*} < 1/9$ holds.*

This change of variables in velocity leads to the following crucial $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{T}^3)$ bound.

Corollary 3.3.24. *Under Assumption 3.1, if* (3.3.15) *holds, then*

$$\| \rho_{\varepsilon,\gamma,\sigma} \|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \lesssim \left\| f^0_{\varepsilon,\gamma,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \leq M.$$

▷ Following the method of characteristics, we write, for any $(t,x) \in [0,T] \times \mathbb{T}^3$,

$$\rho_{\varepsilon,\gamma,\sigma}(t,x) = \int_{\mathbb{R}^3} f_{\varepsilon,\gamma,\sigma}(t,x,v) \mathrm{d}v = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_\varepsilon(X_{\varepsilon,\gamma,\sigma}(0;t,x,v), V_{\varepsilon,\gamma,\sigma}(0;t,x,v)) \mathrm{d}v.$$

We perform the change of variable $w = V_{\varepsilon,\gamma,\sigma}(0;t,x,v)$ thanks to Lemma 3.3.22 and get

$$\begin{aligned}
|\rho_{\varepsilon,\gamma,\sigma}(t,x)| &\lesssim \int_{\mathbb{R}^3} f^0_\varepsilon \left(X_{\varepsilon,\gamma,\sigma} \left(0; t, x, [\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(w) \right), w \right) \mathrm{d}w \\
&\lesssim \int_{\mathbb{R}^3} \left\| f^0_{\varepsilon,\gamma,\sigma}(\cdot, w) \right\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \mathrm{d}w = \left\| f^0_{\varepsilon,\gamma,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}.
\end{aligned}$$

□

Remark 3.3.25. *Thanks to this result, we can apply Lemma 3.3.20. In particular, $\lambda_{\varepsilon,\gamma,\sigma}$ is bounded from above and below by positive constants that are independent of ε.*

In the proof of the previous corollary, the uniform bound

$$\left| f^0_\varepsilon \left(X_{\varepsilon,\gamma,\sigma} \left(0; t, x, [\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(w) \right), w \right) \right| \leq \left\| f^0_{\varepsilon,\gamma,\sigma}(\cdot, w) \right\|_{\mathrm{L}^\infty(\mathbb{T}^3)}$$

was sufficient, but we will encounter cases in Sections 3.4 and 3.5 where we will need more precision. For this purpose, we will apply a change of variables relative to the position variable after the above change of velocity variables has been performed.

Lemma 3.3.26. *For any $t \in \mathbb{R}_+$ satisfying*

$$\| \nabla_x u_{\varepsilon,\gamma,\sigma} \|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \frac{1}{30},$$

for any $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$, we define

$$\widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma} : s \mapsto X_{\varepsilon,\gamma,\sigma}(s; t, x, [\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v)). \tag{3.3.16}$$

Then for any $0 \leq s \leq t$ and $v \in \mathbb{R}^3$, the map

$$\Phi^{s;t,v}_{\varepsilon,\gamma,\sigma} : x \mapsto \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(s)$$

is a $\mathscr{C}^1$-diffeomorphism from $\mathbb{T}^3$ to itself and satisfies

$$\forall x \in \mathbb{T}^3, \qquad \det(\nabla_x \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(s)) \geq \frac{1}{2}.$$

$\triangleright$ The assumptions of Lemma 3.3.22 are satisfied thus $[\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v)$ is well-defined for any $(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3$. The equations of characteristics yield, for any $0 \leq \tau \leq t$,

$$V_{\varepsilon,\gamma,\sigma}(\tau; t, x, [\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v)) = e^{\frac{t-\tau}{\varepsilon}}[\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v) - \frac{\sigma}{\varepsilon}\int_\tau^t e^{\frac{\zeta-\tau}{\varepsilon}} u_{\varepsilon,\gamma,\sigma}\left(\zeta, \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(\zeta)\right) \mathrm{d}\zeta,$$

where

$$[\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v) = e^{-\frac{t}{\varepsilon}}v + \frac{\sigma}{\varepsilon}\int_0^t e^{\frac{\tau-t}{\varepsilon}} u_{\varepsilon,\gamma,\sigma}\left(\tau, \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(\tau)\right) \mathrm{d}\tau.$$

Therefore, for any $s \in [0,t]$,

$$\begin{aligned}\widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(s) - x &= -\frac{1}{\sigma}\int_s^t V_{\varepsilon,\gamma,\sigma}\left(\tau; t, x, [\Gamma^{t,x}_{\varepsilon,\gamma,\sigma}]^{-1}(v)\right) \mathrm{d}\tau \\ &= \varepsilon(e^{-\frac{t}{\varepsilon}} - e^{-\frac{s}{\varepsilon}})v + \int_0^\infty \left[e^{\frac{\tau-t}{\varepsilon}}\mathbf{1}_{\tau\leq t} - e^{\frac{\tau-s}{\varepsilon}}\mathbf{1}_{\tau\leq s} - \mathbf{1}_{s\leq\tau\leq t}\right] u_{\varepsilon,\gamma,\sigma}\left(\tau, \widetilde{X}^{t,x,v}_{\varepsilon,\gamma,\sigma}(\tau)\right) \mathrm{d}\tau.\end{aligned}$$

By differentiating, we get

$$\begin{aligned}\left\|\mathrm{D}_x \widetilde{X}^{t,\cdot,\cdot}_{\varepsilon,\gamma,\sigma} - \mathrm{Id}\right\|_{\mathrm{L}^\infty((0,t)\times\mathbb{T}^3\times\mathbb{R}^3)} &\leq 3\left\|\nabla_x u_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))} \\ &\quad + 3\left\|\mathrm{D}_x \widetilde{X}^{t,\cdot,\cdot}_{\varepsilon,\gamma,\sigma} - \mathrm{Id}\right\|_{\mathrm{L}^\infty((0,t)\times\mathbb{T}^3\times\mathbb{R}^3)} \left\|\nabla_x u_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))},\end{aligned}$$

hence, thanks to the assumption on $\left\|\nabla_x u_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^3))}$,

$$\left\|\mathrm{D}_x \widetilde{X}^{t,\cdot,\cdot}_{\varepsilon,\gamma,\sigma} - \mathrm{Id}\right\|_{\mathrm{L}^\infty((0,t)\times\mathbb{T}^3\times\mathbb{R}^3)} \leq \frac{1}{9}$$

and we conclude with Theorem D.1. $\square$

We now define *strong existence times*, so that if T is a strong existence time, then all the assumptions required in Lemmas 3.3.13, 3.3.22, and 3.3.26 are satisfied.

Definition 3.3.27. *We say $T > 0$ is a strong existence time if*

$$\left\|\nabla_x u_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \frac{1}{30}$$

and

$$\max\left(\int_0^T \left\|e^{t\Delta}u^0_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \mathrm{d}t, \int_0^T \left\|F_{\varepsilon,\gamma,\sigma}(t)\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \mathrm{d}t\right) \leq \frac{C^*}{2}.$$

3.3.7 Estimates on the convection term

We will need several estimates on the convective term in the Navier–Stokes equation. First, we show that this term is in $\mathrm{L}^p((0,T)\times\mathbb{T}^3)$, where p is the regularity index of Assumption 3.1, for appropriate values of $T>0$. We provide a local in time and non-uniform bound that we will use to initialize the bootstrap argument. We conclude this section by a uniform bound that is only valid for strong existence times.

Lemma 3.3.28. *Under Assumption 3.1, the following holds. Let p be the regularity index appearing in the statement. Let $T>0$ such that (3.3.7) holds on $[0,T]$ and $\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}<+\infty$. Then $(u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x)u_{\varepsilon,\gamma,\sigma}\in\mathrm{L}^p((0,T)\times\mathbb{T}^3)$.*

In particular there exists $T_{\varepsilon,\gamma,\sigma}>0$ such that $(u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x)u_{\varepsilon,\gamma,\sigma}\in\mathrm{L}^p((0,T_{\varepsilon,\gamma,\sigma})\times\mathbb{T}^3)$.

$\triangleright$ Let $T>0$ such that (3.3.7) holds on $[0,T]$ and $\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}<+\infty$. Following the proof of [HMM20, Lemma 6.2] and thanks to (3.3.7), we obtain

$$\begin{aligned}
&\|u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^a(0,T;\mathrm{L}^b(\mathbb{T}^3))}\\
&\quad\lesssim \|u_{\varepsilon,\gamma,\sigma}\|^a_{\mathrm{L}^\infty(0,T;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|^{\frac{r_1}{a}}_{\mathrm{L}^{r_1}(0,T;\mathrm{L}^{r_2}(\mathbb{T}^3))}\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|^{1-\frac{r_1}{a}}_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))}\\
&\quad\lesssim_M \|\nabla_x u_{\varepsilon,\gamma,\sigma}\|^{\frac{r_1}{a}}_{\mathrm{L}^{r_1}(0,T;\mathrm{L}^{r_2}(\mathbb{T}^3))},
\end{aligned}\tag{3.3.17}$$

for any $(a,b,r_1,r_2)\in(1,\infty)$ such that $r_1\le a$, $2\le b\le r_2$, and

$$\frac{1}{b}=\frac{1}{6}+\frac{r_1}{a}\frac{1}{r_2}+\frac{1}{2}\left(1-\frac{r_1}{a}\right).\tag{3.3.18}$$

We begin by taking $(a,b,r_1,r_2)=(2,3,2,6)$ in (3.3.17) and get, thanks to (3.3.7),

$$\|u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_M 1.$$

Recall that thanks to Corollary 3.3.11 and Lemma 3.3.15, we also have

$$\|j_{\varepsilon,\gamma,\sigma}-\rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1.$$

Therefore, Theorem E.2 yields

$$\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^3(\mathbb{T}^3))}+\|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1.$$

Then, by Sobolev embedding, we infer that, for any $r_2\in[2,+\infty)$,

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2(0,T;\mathrm{L}^{r_2}(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1.$$

With $a=r\in[2,3)$, $b=3$ and $r_1=2$, (3.3.17) yields,

$$\|u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^r(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1.$$

Again, we have

$$\|j_{\varepsilon,\gamma,\sigma}-\rho_{\varepsilon,\gamma,\sigma}u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^r(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1$$

and thus

$$\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^r(0,T;\mathrm{L}^3(\mathbb{T}^3))}+\|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^r(0,T;\mathrm{L}^3(\mathbb{T}^3))}\lesssim_{M,T,\varepsilon} 1,$$

which yields, by Sobolev embedding, for any $r \in [2,3)$ and $r_2 \in [2,+\infty)$,

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^r(0,T;\mathrm{L}^{r_2}(\mathbb{T}^3))} \lesssim_{M,T,\varepsilon} 1.$$

To conclude, we aim to apply (3.3.17) with $a = b = p > 3$ and $r_1 \in [2,3)$. The condition (3.3.18) implies

$$r_2 = \frac{6r_1}{-4p + 3r_1 + 6}$$

so that the inequality $p \leq r_2$ becomes

$$p < \frac{3(r_1+2)}{4}.$$

Finally the second part of the statement follows thanks to Lemmas 3.3.13 and 3.3.14: there exists $T = T_{\varepsilon,\gamma,\sigma} > 0$ such that (3.3.7) and $\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} < +\infty$ hold. □

Lemma 3.3.29. *Under Assumption 3.1, if $T > 0$ is a strong existence time, then*

$$\|(u_{\varepsilon,\gamma,\sigma} \cdot \nabla_x)\, u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \Psi^{\frac{1}{2}}_{\varepsilon,\gamma,\sigma,0} \mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|^{\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)},$$

where p is the regularity index appearing in the assumption and β_p is given by Corollary B.2.

▷ Since T is a strong existence time, thanks to Corollaries 3.3.11 and 3.3.21, we have

$$\|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathbb{T}^3))} \lesssim \Psi^{\frac{1}{2}}_{\varepsilon,\gamma,\sigma,0}$$

and, for every $t \in (0,T)$,

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1-\beta_p}{2}} e^{-\frac{(1-\beta_p)\lambda_{\varepsilon,\gamma,\sigma}}{2}t} \|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|_{\mathrm{L}^p(\mathbb{T}^3)}.$$

Therefore, using Hölder's inequality, we get

$$\begin{aligned}
&\|(u_{\varepsilon,\gamma,\sigma} \cdot \nabla_x) u_{\varepsilon,\gamma,\sigma}\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\qquad \lesssim \Psi^{\frac{p}{2}}_{\varepsilon,\gamma,\sigma,0} \mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{(1-\beta_p)p}{2}} \int_0^T e^{-\frac{(1-\beta_p)p\lambda_{\varepsilon,\gamma,\sigma}}{2}t} \|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|^{\beta_p p}_{\mathrm{L}^p(\mathbb{T}^3)}\,\mathrm{d}t \\
&\qquad \lesssim \Psi^{\frac{p}{2}}_{\varepsilon,\gamma,\sigma,0} \mathscr{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{(1-\beta_p)p}{2}} \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|^{\beta_p p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}
\end{aligned}$$

and the lemma follows. □

3.3.8 Initialization of the bootstrap argument

In this section, the last one that is common to all the regimes, we initiate the bootstrap argument that we will use to determine strong existence times, under several sets of assumptions.

First, we prove a non-uniform existence result.

Lemma 3.3.30. *Under Assumption 3.1, for every $\varepsilon > 0$, there exists a strong existence time.*

▷ Let p be the regularity index of Assumption 3.1. We apply the parabolic regularity estimates from Theorem E.2 to obtain, for any $T > 0$,

$$\begin{aligned}\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} &+ \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\lesssim \|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|(u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x)u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{B}^{s,p}_p(\mathbb{T}^3)}.\end{aligned}$$

To control the first term, we remember that thanks to Lemma 3.3.13, there exists $T_{\varepsilon,\gamma,\sigma} > 0$ such that (3.3.7) and Lemma 3.3.14 hold. Then, thanks to Lemma 3.3.15, there exists a continuous function $\varphi^{\text{force}}_{\varepsilon,\gamma,\sigma,0}$ such that

$$\|F_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T_{\varepsilon,\gamma,\sigma})\times\mathbb{T}^3)} \le \varphi^{\text{force}}_{\varepsilon,\gamma,\sigma,0}(T_{\varepsilon,\gamma,\sigma}). \tag{3.3.19}$$

Moreover, Lemma 3.3.28 shows that there exists a continuous function $\varphi^{\text{conv}}_{\varepsilon,\gamma,\sigma,0}$ that vanishes at 0 and such that

$$\|(u_{\varepsilon,\gamma,\sigma}\cdot\nabla_x)u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T_{\varepsilon,\gamma,\sigma})\times\mathbb{T}^3)} \le \varphi^{\text{conv}}_{\varepsilon,\gamma,\sigma,0}(T_{\varepsilon,\gamma,\sigma}).$$

Therefore, we can take $T = T_{\varepsilon,\gamma,\sigma}$ so that

$$\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \varphi_{\varepsilon,\gamma,\sigma,0}(T) + \left\|u^0_{\varepsilon,\gamma,\sigma}\right\|_{\mathrm{B}^{s,p}_p(\mathbb{T}^3)}, \tag{3.3.20}$$

for some continuous function $\varphi_{\varepsilon,\gamma,\sigma,0}$.

Using the energy–dissipation estimate, the Gagliardo-Nirenberg inequality (see Corollary B.2) and Hölder's inequality, we get

$$\begin{aligned}&\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \\ &\lesssim \|u_{\varepsilon,\gamma,\sigma}\|^{1-\beta_p}_{\mathrm{L}^\infty((0,T);\mathrm{L}^2(\mathbb{T}^3))}\int_0^T \|\Delta_x u_{\varepsilon,\gamma,\sigma}(t)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)}\,\mathrm{d}t + T\,\|u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^\infty((0,T);\mathrm{L}^2(\mathbb{T}^3))} \\ &\lesssim T^{1-\frac{\beta_p}{p}}\,\mathrm{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac{1-\beta_p}{2}}\,\|\Delta_x u_{\varepsilon,\gamma,\sigma}\|^{\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + T\,\mathrm{E}_{\varepsilon,\gamma,\sigma}(0)^{\frac12}.\end{aligned}$$

Thanks to (3.3.20), we conclude that we can take T small enough such that

$$\|\nabla_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \le \frac{1}{30}.$$

Finally, as in the proof of Lemma 3.3.13, we show that we can reduce $T > 0$ to ensure that

$$\int_0^T \left\|e^{t\Delta}u_{\varepsilon,\gamma,\sigma}\right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}\,\mathrm{d}t \le \frac{C^*}{2}.$$

Furthermore, thanks to (3.3.19) and Hölder's inequality, we have

$$\|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \le T^{\frac{p}{(p-1)}}\,\varphi^{\text{force}}_{\varepsilon,\gamma,\sigma,0}(T),$$

so that, up to reducing $T > 0$, we have

$$\|F_{\varepsilon,\gamma,\sigma}\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \le \frac{C^*}{2}.$$

Gathering all the pieces together, this leads to the existence of a strong existence time. □

Arguing as in the proof of Lemma 3.3.30, we also obtain the following result, which is useful to justify that the computations of the following sections make sense.

Lemma 3.3.31. *Let $T > 0$ be a strong existence time. Then*

$$\|\partial_t u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\gamma,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} < +\infty.$$

We can now define, for every $\varepsilon > 0$,

$$T_\varepsilon^* = \sup\{T > 0,\, T \text{ is a strong existence time}\}.$$

Our goal is twofold. First, we aim to prove that under Assumption 3.1 and additional smallness assumptions on the initial data depending on the regime, we have $T_\varepsilon^* = +\infty$, at least for small values of ε.

Then, we shall take a step back and prove that we can dispense with some of these additional smallness assumptions on the initial data at the cost of constraining the time horizon. The issue will be to prove that we can find a minimal time horizon that does not depend on ε (and therefore does not vanish as ε goes to 0).

To achieve these goals, we cannot apply the same strategy as in Lemma 3.3.30, since the estimates we used do not provide a uniform bound with respect to ε. We will thus need a new approach to control the Brinkman force that will provide an adequate desingularization, which we will achieve thanks to the changes of variables presented in Section 3.3.6. In the *light* and *light and fast* particle regimes, we can use the method of characteristics and suitable identities relying on integration by parts to obtain satisfactory uniform bounds. In the fine particle regime, though the previous argument could be refined and applied, we choose to present another strategy based on another functional, which we refer to as higher dissipation, following [Han20]. These different approaches lead to different arguments to prove the expected convergences. In Section 3.4, we present the strategy used for the *light* and *light and fast* particle regimes. Section 3.5 is then dedicated to the fine particle regime.

3.4 *Light* and *light and fast* particle regimes

In this section, we study in a unified manner the *light* and *light and fast* particle regimes, that correspond to the choices $(\gamma, \sigma) = (1, \varepsilon^\alpha)$, for some $\alpha \in [0, 1/2]$. For the sake of readability, we systematically drop the parameter γ: for instance, $u_{\varepsilon,1,\sigma}$ will be referred to as $u_{\varepsilon,\sigma}$. In short we study the behavior as $\varepsilon \to 0$ of solutions to the systems

$$\begin{cases} \partial_t u_{\varepsilon,\sigma} + (u_{\varepsilon,\sigma} \cdot \nabla_x) u_{\varepsilon,\sigma} - \Delta_x u_{\varepsilon,\sigma} + \nabla_x p_{\varepsilon,\sigma} = j_{f_{\varepsilon,\sigma}} - \rho_{f_{\varepsilon,\sigma}} u_{\varepsilon,\sigma}, \\ \operatorname{div}_x u = 0, \\ \partial_t f_{\varepsilon,\sigma} + \dfrac{1}{\sigma} v \cdot \nabla_x f_{\varepsilon,\sigma} + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f_{\varepsilon,\sigma}(\sigma u_{\varepsilon,\sigma} - v) \right] = 0, \\ \rho_{f_{\varepsilon,\sigma}}(t,x) = \displaystyle\int_{\mathbb{R}^3} f_{\varepsilon,\sigma}(t,x,v) \mathrm{d}v, \quad j_{f_{\varepsilon,\sigma}}(t,x) = \dfrac{1}{\sigma} \displaystyle\int_{\mathbb{R}^3} v f_{\varepsilon,\sigma}(t,x,v) \mathrm{d}v, \end{cases}$$

with $\sigma = 1$ for the light particle regime and $\sigma = \varepsilon^\alpha$ for the light and fast particle regime.

The main part of the proof is to complete the bootstrap argument set up in Section 3.3. To this purpose, we shall establish precise estimates on the Brinkman force

$$F_{\varepsilon,\sigma} = j_{f_{\varepsilon,\sigma}} - \rho_{f_{\varepsilon,\sigma}} u_{\varepsilon,\sigma}$$

thanks to a desingularization with respect to ε, based on the Lagrangian structure of the Vlasov equation. We then study the limit $\varepsilon \to 0$ and prove convergence (and convergence rates) to the solutions of the Transport–Navier–Stokes equations (3.1.6). This eventually leads to the proof of Theorems 3.1.5 and 3.1.6, but with an additional assumption on the initial data :

$$\left\| |v|^p f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)} \lesssim \frac{\sigma^p}{\varepsilon^{1-\kappa}}, \tag{3.4.1}$$

for some $\kappa > 0$. Note that this is a true restriction only for $\alpha > 1/3$. Treating the cases $\alpha \in (1/3, 1/2]$ without (3.4.1) requires a refined strategy presented in [HM21].

This section is organized as follows :

- The desingularization of the Brinkman force is achieved in Section 3.4.1, leading to uniform L^p (in time and space) bounds.
- Thanks to these crucial bounds, in Section 3.4.2, the bootstrap argument set up in Section 3.3 is concluded under the assumption of a small initial modulated energy. We rely on an L^p parabolic maximal estimate combined with the exponential decay of the modulated energy through an interpolation argument. We therefore obtain that all positive times are *strong existence times* in the sense of Definition 3.3.27.
- In Section 3.4.3, we perform the proof of convergence for the mildly well-prepared and well-prepared cases of Theorems 3.1.5 and 3.1.6, mainly using an $\mathrm{L}^\infty\mathrm{L}^2$–$\mathrm{L}^2\dot{\mathrm{H}}^1$ energy estimate for the Navier–Stokes part and Wasserstein stability estimates for the Vlasov part.
- Finally the general case of Theorems 3.1.5 and 3.1.6, in which we obtain a short-time convergence result, is discussed in Section 3.4.5.

3.4.1 Uniform estimates on the Brinkman force in L^p

Our aim is to use the method of characteristics and the changes of variables described in Subsection 3.3.6 to get a precise bound for the Brinkman force. For a strong existence time $T > 0$, we shall rely on the exponential decay in time of the modulated energy on $[0, T]$ as proven in Lemma 3.3.20.

Consider $\varepsilon \in (0, 1)$ and let $T > 0$ be a strong existence time. For almost any $t \in [0, T]$ and $x \in \mathbb{T}^3$, we have, by the method of characteristics,

$$F_{\varepsilon,\sigma}(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}(X_{\varepsilon,\sigma}(0;t,x,v), V_{\varepsilon,\sigma}(0;t,x,v)) \left(\frac{v}{\sigma} - u_{\varepsilon,\sigma}(t,x)\right) \mathrm{d}v,$$

where we recall $(X_{\varepsilon,\sigma}(0;t,x,v), V_{\varepsilon,\sigma}(0;t,x,v))$ is defined in (3.3.3). Thanks to Lemma 3.3.22, we can apply the change of variables in velocity $v = \Gamma^{t,x}_{\varepsilon,\sigma}(w)$ so that

$$F_{\varepsilon,\sigma}(t,x) = e^{\frac{3t}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon} \left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left(\frac{1}{\sigma}[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) - u_{\varepsilon,\sigma}(t,x)\right) \left|\det \nabla_w [\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right| \mathrm{d}w,$$

where we recall $\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)$ is defined in (3.3.16) and therefore, still by Lemma 3.3.22, this yields

$$|F_{\varepsilon,\sigma}(t,x)| \lesssim \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma} \left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\frac{1}{\sigma}[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) - u_{\varepsilon,\sigma}(t,x)\right| \mathrm{d}w. \tag{3.4.2}$$

By integrating the equation of characteristics (3.3.3) relative to the velocity, we get

$$[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) = e^{-\frac{t}{\varepsilon}} w + \frac{\sigma}{\varepsilon} \int_0^t e^{\frac{s-t}{\varepsilon}} u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right) \mathrm{d}s.$$

In order to desingularize this expression with respect to ε, we perform an integration by parts in time and obtain

$$\begin{aligned}[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) &= \sigma u_{\varepsilon,\sigma}(t,x) + e^{-\frac{t}{\varepsilon}}\left(w - \sigma u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right)\\
&\quad - \sigma\int_0^t e^{\frac{s-t}{\varepsilon}}\partial_s u_{\varepsilon,\sigma}\left(s,\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\mathrm{d}s\\
&\quad - \sigma\int_0^t e^{\frac{s-t}{\varepsilon}} V_{\varepsilon,\sigma}\left(s;t,x,[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right)\cdot\nabla_x u_{\varepsilon,\sigma}\left(s,\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\mathrm{d}s.\end{aligned}$$

Therefore, (3.4.2) can be written as follows.

Lemma 3.4.1. *For every $\varepsilon > 0$, if $T > 0$ is a strong existence time then for almost any $(t,x) \in [0,T]\times\mathbb{T}^3$,*

$$|F_{\varepsilon,\sigma}(t,x)| \lesssim F^0_{\varepsilon,\sigma} + F^{dt}_{\varepsilon,\sigma} + F^{dx}_{\varepsilon,\sigma},$$

where

$$\begin{aligned}F^0_{\varepsilon,\sigma}(t,x) &= e^{\frac{-t}{\varepsilon}}\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\left|\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right|\mathrm{d}w,\\
F^{dt}_{\varepsilon,\sigma}(t,x) &= \int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\left|\partial_s u_{\varepsilon,\sigma}\left(s,\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|\mathrm{d}s\mathrm{d}w,\\
F^{dx}_{\varepsilon,\sigma}(t,x) &= \int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\\
&\qquad\times\left|V_{\varepsilon,\sigma}\left(s;t,x,[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right)\cdot\nabla_x u_{\varepsilon,\sigma}\left(s,\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|\mathrm{d}s\mathrm{d}w.\end{aligned}$$

Let us study these three terms separately. In what follows, p stands for the regularity index of Assumption 3.1.

Lemma 3.4.2. *Under Assumption 3.1, there exists $\mu_p > 0$ such that for every $\varepsilon \in (0,1)$, if $T > 0$ is a strong existence time and if there exists $\kappa \in (0,1)$ such that*

$$\left\||v|^p f^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)} \lesssim \frac{\sigma^p}{\varepsilon^{1-\kappa}}, \tag{3.4.3}$$

then

$$\left\|F^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim M^{\mu_p}\varepsilon^{\frac{\kappa}{p}}.$$

Remark 3.4.3. *This is the only term for which the extra assumption* (3.4.3) *is useful. As already mentioned, it is actually restrictive only for $\alpha > 1/3$, since $p > 3$ can be taken arbitrarily close to* 3.

$\triangleright$ Thanks to Hölder's inequality, we have

$$\begin{aligned}&\left\|F^0_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}\\
&\quad\lesssim \int_0^T e^{-\frac{pt}{\varepsilon}}\int_{\mathbb{T}^3}\left|\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\left|\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right|\mathrm{d}w\right|^p\mathrm{d}x\mathrm{d}t\\
&\quad\lesssim \int_0^T e^{-\frac{pt}{\varepsilon}}\int_{\mathbb{T}^3}\left(\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\mathrm{d}w\right)^{p-1}\\
&\qquad\times\left(\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0),w\right)\left|\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right|^p\mathrm{d}w\right)\mathrm{d}x.\end{aligned}$$

We apply Lemma 3.3.26 and the change of variables in space $x' = \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)$ so that

$$\begin{aligned}
&\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left|\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right|^p \mathrm{d}x\mathrm{d}w \\
&\qquad \lesssim \sigma^{-p}\int_{\mathbb{T}^3\times\mathbb{R}^3} |w|^p f^0_{\varepsilon,\sigma}(x,w)\mathrm{d}x\mathrm{d}w + \left\|f^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\left\|u^0_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^p(\mathbb{T}^3)} \\
&\qquad \lesssim \varepsilon^{\kappa-1} + M^{\mu_p},
\end{aligned}$$

for some $\mu_p > 0$, hence

$$\left\|F^0_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon^\kappa M^{\mu_p p},$$

up to modifying the value of $\mu_p > 0$. □

We deal with the second term in a similar fashion.

Lemma 3.4.4. *Under Assumption 3.1, for every* $\varepsilon \in (0,1)$, *if* $T > 0$ *is a strong existence time,*

$$\left\|F^{dt}_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon M \left\|\partial_t u_{\varepsilon,\sigma}\right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.$$

▷ Thanks to Hölder's inequality, for almost all $(t,x) \in (0,T)\times\mathbb{T}^3$,

$$\begin{aligned}
|F^{dt}_{\varepsilon,\sigma}(t,x)|^p &\le \left(\int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\mathrm{d}s\mathrm{d}w\right)^{p-1} \\
&\quad \times\left(\int_{\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}s\mathrm{d}w\right) \\
&\lesssim \left\|f^0_{\varepsilon,\sigma}\right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\varepsilon^{p-1} \\
&\quad \times \int_0^t e^{\frac{s-t}{\varepsilon}}\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}w\mathrm{d}s.
\end{aligned}$$

Applying Lemma 3.3.26, we can now use the change of variables in space $x' = \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s) = \Phi^{s;t,w}_{\varepsilon,\sigma}(x)$ in the following integral to obtain

$$\begin{aligned}
&\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}x\mathrm{d}w \\
&\quad \lesssim \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,[\Phi^{s;t,w}_{\varepsilon,\sigma}]^{-1}(x),w}_{\varepsilon,\sigma}(0), w\right)|\partial_s u_{\varepsilon,\sigma}(s,x)|^p\mathrm{d}x\mathrm{d}w \\
&\quad \lesssim \left\|f^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\left\|\partial_s u_{\varepsilon,\sigma}(s)\right\|^p_{\mathrm{L}^p(\mathbb{T}^3)}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\left\|F^{dt}_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} &\lesssim \left\|f^0_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\varepsilon^{p-1}\int_0^T \left\|\partial_s u_{\varepsilon,\sigma}(s)\right\|^p_{\mathrm{L}^p(\mathbb{T}^3)}\left(\int_s^T e^{\frac{s-t}{\varepsilon}}\mathrm{d}t\right)\mathrm{d}s \\
&\lesssim \varepsilon^p\left\|f^0_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\left\|\partial_t u_{\varepsilon,\sigma}\right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)},
\end{aligned}$$

which concludes the proof of the lemma. □

The control of the last term in the Brinkman force is more intricate. We shall rely on the exponential decay of the modulated energy.

Lemma 3.4.5. *Under Assumption 3.1, there exists $\mu_p \geq 0$ such that for every $\varepsilon \in (0,1)$, if $T > 0$ is a strong existence time,*

$$\left\| F^{dx}_{\varepsilon,\sigma} \right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon \left\| \Delta_x u_{\varepsilon,\sigma} \right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \varepsilon M^{\mu_p} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1}{2}}.$$

$\triangleright$ By integrating the equation of characteristics, we get, for all $0 \leq s \leq t \leq T$,

$$V_{\varepsilon,\sigma}(s;t,x,[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)) = e^{-\frac{s}{\varepsilon}} w + \frac{\sigma}{\varepsilon} \int_0^s e^{\frac{\tau - s}{\varepsilon}} u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right) \mathrm{d}\tau,$$

so that

$$\begin{aligned}
|F^{dx}_{\varepsilon,\sigma}(t,x)| \leq & \int_{\mathbb{R}^3} e^{-\frac{t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) |w| \int_0^t \left| \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}w \\
& + \frac{\sigma}{\varepsilon} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \times \\
& \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau - s}{\varepsilon}} \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right| \left| \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}\tau\mathrm{d}s\mathrm{d}w \\
& =: I_1(t,x) + I_2(t,x).
\end{aligned}$$

Thanks to Hölder's inequality, we have

$$\begin{aligned}
|I_1(t,x)|^p \leq & \, e^{-\frac{pt}{\varepsilon}} \left[\int_0^t \left(\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) |w|^{\frac{p}{p-1}} \mathrm{d}w \right)^{\frac{p-1}{p}} \right. \\
& \left. \times \left(\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left| \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}w \right)^{\frac{1}{p}} \mathrm{d}s \right]^p \\
\leq & \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} e^{-\frac{pt}{\varepsilon}} \left[\int_0^t e^{\frac{s}{\varepsilon}} \varepsilon (1 - e^{-\frac{t}{\varepsilon}}) \right. \\
& \left. \times \left(\int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w) |\nabla_x u_{\varepsilon,\sigma}(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s))|^p \mathrm{d}w \right)^{\frac{1}{p}} \frac{e^{-\frac{s}{\varepsilon}} \mathrm{d}s}{\varepsilon(1 - e^{\frac{-t}{\varepsilon}})} \right]^p,
\end{aligned}$$

and thanks to Jensen's inequality applied on the probability space

$$\left((0,t), \frac{e^{-\frac{s}{\varepsilon}} \mathrm{d}s}{\varepsilon(1 - e^{-\frac{t}{\varepsilon}})} \right),$$

we obtain

$$\begin{aligned}
|I_1(t,x)|^p \leq \varepsilon^{p-1} & \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \\
& \times \int_0^t e^{\frac{p(s-t)}{\varepsilon}} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left| \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}w\mathrm{d}s.
\end{aligned}$$

We apply Lemma 3.3.26 and use the change of variables $x' = \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)$. Therefore, as in the proof of Lemma 3.4.4, we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^3} |I_1(t,x)|^p \mathrm{d}x\mathrm{d}t \leq \varepsilon^{p-1} \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \\
\times \int_0^T \int_0^t e^{\frac{p(s-t)}{\varepsilon}} \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^p \mathrm{d}x\mathrm{d}w \right) \mathrm{d}s\mathrm{d}t \\
\lesssim \varepsilon^{p-1} \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \\
\times \int_0^T \int_0^t e^{\frac{p(s-t)}{\varepsilon}} \left\|\nabla_x u_{\varepsilon,\sigma}(s)\right\|^p_{\mathrm{L}^p(\mathbb{T}^3)} \mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Since T is a strong existence time, we can apply Lemmas 3.3.20–3.3.21. Then, thanks to Hölder's and Young's inequality as well as the value of $\lambda_{\varepsilon,\sigma}$ given by Lemma 3.3.20 and Corollary 3.3.24, we conclude that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^3} |I_1(t,x)|^p \mathrm{d}x\mathrm{d}t \\
&\quad \lesssim \varepsilon^{p-1} \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{p(1-\alpha_p)}{2}} \\
&\qquad \times \int_0^T \left\|\Delta_x u_{\varepsilon,\sigma}(s)\right\|^{p\alpha_p}_{\mathrm{L}^p(\mathbb{T}^3)} e^{-\frac{p(1-\alpha_p)\lambda_\varepsilon}{2} s} \left(\int_s^T e^{\frac{p(s-t)}{\varepsilon}} \mathrm{d}t \right) \mathrm{d}s \\
&\quad \lesssim \varepsilon^{p} \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{p(1-\alpha_p)}{2}} \frac{1}{\lambda_{\varepsilon,\sigma}^{1-\alpha_p}} \left\|\Delta_x u_{\varepsilon,\sigma}\right\|^{p\alpha_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\quad \lesssim \varepsilon^{p} \left\| |v|^{\frac{p}{p-1}} f^0_{\varepsilon,\sigma} \right\|^{\frac{p-1}{1-\alpha_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| f^0_{\varepsilon,\sigma} \right\|^{\frac{1}{1-\alpha_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{p}{2}} + \varepsilon^p \left\|\Delta_x u_{\varepsilon,\sigma}\right\|^{p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.
\end{aligned}$$

The control of I_2 is a little more technical. We apply Hölder's inequality twice and obtain

$$\begin{aligned}
|I_2(t,x)|^p \leq \Bigg[\varepsilon^{-1} \int_{\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \int_0^t \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| e^{\frac{s-t}{\varepsilon}} \\
\times \left(\int_0^s \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right|^p e^{\frac{p(\tau-s)}{2\varepsilon}} \mathrm{d}\tau \right)^{\frac{1}{p}} \left(\int_0^s e^{\frac{p(\tau-s)}{2(p-1)\varepsilon}} \mathrm{d}\tau \right)^{\frac{p-1}{p}} \mathrm{d}s\mathrm{d}w \Bigg]^p \\
\lesssim \varepsilon^{-1} \Bigg[\int_{\mathbb{R}^3} \int_0^t f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| e^{\frac{s-t}{2\varepsilon}} \\
\times e^{\frac{s-t}{2\varepsilon}} \left(\int_0^s \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right|^p e^{\frac{p(\tau-s)}{2\varepsilon}} \mathrm{d}\tau \right)^{\frac{1}{p}} \mathrm{d}s\mathrm{d}w \Bigg]^p \\
\lesssim \varepsilon^{-1} \left(\int_{\mathbb{R}^3} \int_0^t f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) e^{\frac{p(s-t)}{2\varepsilon}} \int_0^s \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right|^p e^{\frac{p(\tau-s)}{2\varepsilon}} \mathrm{d}\tau\mathrm{d}s\mathrm{d}w \right) \\
\times \left(\int_{\mathbb{R}^3} \int_0^t f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right|^{\frac{p}{p-1}} e^{\frac{p(s-t)}{2(p-1)\varepsilon}} \mathrm{d}s\mathrm{d}w \right)^{p-1}.
\end{aligned}$$

Therefore,

$$\int_0^T \int_{\mathbb{T}^3} |I_2(t,x)|^p \mathrm{d}x\mathrm{d}t \lesssim \varepsilon^{-1} \left\| f^0_{\varepsilon,\sigma} \right\|^{p-1}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left(\sup_{t\in(0,T)} I_3(t) \right) \times I_4,$$

where

$$I_3(t) := \int_0^t e^{\frac{p(s-t)}{2\varepsilon}} \int_0^s e^{\frac{p(\tau-s)}{2\varepsilon}} \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right|^p \mathrm{d}x\mathrm{d}w\mathrm{d}\tau\mathrm{d}s,$$

and

$$I_4 := \int_0^T \left(\int_0^t \|\nabla_x u_{\varepsilon,\sigma}(s)\|^{\frac{p}{p-1}}_{\mathrm{L}^\infty(\mathbb{T}^3)} e^{\frac{p(s-t)}{2(p-1)\varepsilon}} \mathrm{d}s \right)^{p-1} \mathrm{d}t.$$

On the one hand, we can apply Corollary 3.3.11 and Lemma 3.3.26 with the change of variable $x' = \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)$ as before and obtain

$$\sup_{t\in(0,T)} I_3(t) \lesssim \varepsilon^2 \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi^{\frac{p}{2}}_{\varepsilon,0}.$$

On the other hand, using Jensen's inequality, we have

$$I_4 \lesssim \varepsilon^{p-2} \int_0^T \int_0^t \|\nabla_x u_{\varepsilon,\sigma}(s)\|^p_{\mathrm{L}^\infty(\mathbb{T}^3)} e^{\frac{p(s-t)}{2(p-1)\varepsilon}} \mathrm{d}s\mathrm{d}t.$$

Thanks to Corollary 3.3.21 and Hölder's inequality, we find that

$$I_4 \lesssim \varepsilon^{p-1} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{(1-\beta_p)p}{2}} \|\Delta_x u_{\varepsilon,\sigma}\|^{\beta_p p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.$$

We can conclude that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^3} |I_2(t,x)|^p \mathrm{d}x\mathrm{d}t &\lesssim \varepsilon^p \left\| f^0_{\varepsilon,\sigma} \right\|^p_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi^{\frac{p}{2}}_{\varepsilon,\sigma,0} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{(1-\beta_p)p}{2}} \|\Delta_x u_{\varepsilon,\sigma}\|^{\beta_p p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3))} \\ &\lesssim \varepsilon^p \left\| f^0_{\varepsilon,\sigma} \right\|^{\frac{p}{1-\beta_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi^{\frac{p}{2(1-\beta_p)}}_{\varepsilon,\sigma,0} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{p}{2}} + \varepsilon^p \|\Delta_x u_{\varepsilon,\sigma}\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3))}.\end{aligned}$$

Gathering all pieces together, this allows to conclude the proof. □

3.4.2 Conclusion of the bootstrap argument in the mildly well-prepared case

Recall that, as stated in Section 3.3.8, we consider, under Assumption 3.1, for every $\varepsilon > 0$,

$$T^*_\varepsilon = \sup\{T > 0,\, T \text{ is a strong existence time}\}.$$

The aim of this section is to prove that under the additional Assumption 3.2, (3.4.3) and the smallness of the initial modulated energy, we have $T^*_\varepsilon = +\infty$, at least for ε close to 0. We only need to prove that, for every $T < T^*_\varepsilon$,

$$\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \le \frac{1}{40} \qquad \text{and} \qquad \|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \le \frac{C^*}{4}.$$

Indeed, as we have seen in the proof of Lemma 3.3.13, Assumption 3.2 already ensures that, for all $T \geq 0$,

$$\int_0^T \left\| e^{t\Delta} u_{\varepsilon,\sigma} \right\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}^4 \mathrm{d}t \leq \frac{C^*}{4}.$$

Then, thanks to the timewise continuity of the norms that are involved in the definition of a strong existence time, we deduce that $T_\varepsilon^* = +\infty$.

We begin by proving the following lemma.

Lemma 3.4.6. *Under Assumption 3.1, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ if* (3.4.3) *holds and $T < T_\varepsilon^*$, then*

$$\|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_p},$$

for some $\omega_p > 0$.

▷ Thanks to Theorem E.2, we have

$$\begin{aligned}
&\|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\qquad\qquad \lesssim \|F_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|(u_{\varepsilon,\sigma}\cdot\nabla_x)u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|u_{\varepsilon,\sigma}^0\right\|_{\mathrm{B}_p^{s,p}(\mathbb{T}^3)}.
\end{aligned}$$

We can now apply the results from Lemmas 3.3.29 and 3.4.2–3.4.5. We find that

$$\begin{aligned}
&\|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\quad \lesssim \varepsilon M \|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \varepsilon \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \varepsilon^\kappa M^{\mu_p} + \varepsilon M^{\mu_p} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} \\
&\qquad\qquad + \Psi_{\varepsilon,\sigma,0}^{\frac{1}{2}} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^{\beta_p} + M
\end{aligned}$$

and we conclude thanks to Young's inequality and taking ε small enough. □

We have the following immediate consequence.

Corollary 3.4.7. *Under Assumption 3.1, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, if* (3.4.3) *holds and*

$$\mathscr{E}_{\varepsilon,\sigma}(0) \leq \eta, \tag{3.4.4}$$

then for any $T < T_\varepsilon^*$,

$$\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \frac{1}{40}.$$

▷ Since $p > 3$, we can apply Corollary 3.3.21 and Lemma 3.4.6 and find that there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, for some $\omega_p > 0$, we have

$$\int_0^T \|\nabla_x u_{\varepsilon,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \mathrm{d}t \lesssim \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^{\beta_p} \lesssim \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} M^{\omega_p \beta_p},$$

which can be made as small as necessary by reducing the value of $\eta > 0$ in (3.4.4). □

There remains to verify that

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}. \tag{3.4.5}$$

We shall provide two proofs of this fact. The first one uses (3.4.4) and is straightforward, while the second one does not and is more technical, being based on the methods developed earlier to prove $\mathrm{L}^p\mathrm{L}^p$ estimates for the Brinkman force. It will however come in handy in order to

- obtain convergence rates for the fluid velocity (Section 3.4.3) ;
- treat the general case (Section 3.4.5).

Lemma 3.4.8. *Under Assumption 3.1 and 3.4.4, if T is a strong existence time,*

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2(\mathbb{R}_+;\mathrm{L}^2(\mathbb{T}^3))} \leq \left\|f^0_{\varepsilon,\sigma}\right\|^{\frac{1}{2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1}{2}}.$$

$\triangleright$ The argument is almost the same as in Remark 3.3.3, except that we use the modulated energy–dissipation inequality of Lemma 3.3.19. This yields

$$\|F_{\varepsilon,\sigma}\|^2_{\mathrm{L}^2(\mathbb{R}_+;\mathrm{L}^2(\mathbb{T}^3))} \leq \|\rho_{\varepsilon,\sigma}\|_{\mathrm{L}^\infty(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0),$$

and we conclude by Corollary 3.3.24. $\square$

We deduce that (3.4.5) holds, taking η small enough in (3.4.4). Let us now present the second approach leading to (3.4.5). The idea is to mimic Lemmas 3.4.2–3.4.5, which results in the following lemma.

Lemma 3.4.9. *Under Assumption 3.1, there exists $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, if* (3.4.3)–(3.4.4) *hold, then for any $T < T^*_\varepsilon$,*

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon M \|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \varepsilon \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}} M^{\mu_2}, \tag{3.4.6}$$

for some $\mu_2 > 0$.

$\triangleright$ Note that we can reproduce the proofs of Lemmas 3.4.2–3.4.4 and obtain, for every $\varepsilon \in (0, 1)$,

$$\begin{aligned} \left\|F^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &\lesssim M^{\mu_2}\varepsilon^{\frac{\kappa}{2}}, \\ \left\|F^{dt}_{\varepsilon,\sigma}\right\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &\lesssim \varepsilon M \|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}. \end{aligned}$$

Nevertheless, the Gagliardo-Nirenberg inequality cannot be applied exactly in the same way and some adaptation of the proof of Lemma 3.4.5 is required. We have

$$|F^{dx}_{\varepsilon,\sigma}| \leq I_1 + I_2,$$

where, for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$,

$$I_1(t, x) = \int_{\mathbb{R}^3} e^{-\frac{t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) |w| \int_0^t \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}w,$$

and

$$\begin{aligned} I_2(t, x) = \frac{\sigma}{\varepsilon} \int_{\mathbb{R}^3} & f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \\ & \times \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau-s}{\varepsilon}} \left|u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right| \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}\tau\mathrm{d}s\mathrm{d}w. \end{aligned}$$

We can still follow the proof of Lemma 3.4.5 and obtain

$$\|I_1\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon M + \varepsilon \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}.$$

For I_2, we get

$$\int_0^T \int_{\mathbb{T}^3} |I_2(t,x)|^2 \mathrm{d}x\mathrm{d}t \lesssim \varepsilon^{-1} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left(\sup_{t\in(0,T)} I_3(t) \right) \times I_4,$$

where

$$I_3(t) = \int_0^t e^{\frac{s-t}{\varepsilon}} \int_0^s e^{\frac{\tau-s}{\varepsilon}} \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(\tau)\right)\right|^2 \mathrm{d}x\mathrm{d}w\mathrm{d}\tau\mathrm{d}s,$$

and

$$I_4(t) = \int_0^T \left(\int_0^t \|\nabla_x u_{\varepsilon,\sigma}(s)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)} e^{\frac{s-t}{\varepsilon}} \mathrm{d}s \right) \mathrm{d}t.$$

We still have

$$\sup_{t\in(0,T)} I_3(t) \lesssim \varepsilon^2 \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi_{\varepsilon,\sigma,0} \lesssim \varepsilon^2 M^{\mu_2}.$$

Finally, for what concerns I_4, thanks to Jensen's inequality and Corollary 3.3.21 and Lemma 3.4.6, for ε small enough, we have

$$I_4 \lesssim \varepsilon \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_{\varepsilon,\sigma}\|^{\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim \varepsilon M^{\nu_2}.$$

The proof of the result is complete. □

Therefore, the L^2 parabolic estimate can be written as follows.

Lemma 3.4.10. *Under Assumption 3.1, there exists $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, if* (3.4.3)–(3.4.4) *hold, then for any $T < T^*_\varepsilon$,*

$$\|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_2},$$

for some $\omega_2 > 0$.

▷ Theorem E.2 ensures that

$$\begin{aligned}\|\partial_t u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &+ \|\Delta_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \\ &\lesssim \|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \|(u_{\varepsilon,\sigma}\cdot\nabla_x)u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \left\|u^0_{\varepsilon,\sigma}\right\|_{\mathrm{H}^1(\mathbb{T}^3)}.\end{aligned}$$

Lemma 3.4.9 enables us to control the first right-hand side term. On the other hand, thanks to Hölder's inequality and Lemmas 3.3.20–3.3.21 and 3.4.6, we have, for ε small enough,

$$\begin{aligned}\|(u_{\varepsilon,\sigma}\cdot\nabla)u_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &\lesssim \|u_{\varepsilon,\sigma}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))} \int_0^T \|\nabla_x u_{\varepsilon,\sigma}(t)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)} \mathrm{d}t \\ &\lesssim \Psi_{\varepsilon,\sigma,0}\mathscr{E}_{\varepsilon,\sigma}(0)^{1-\beta_p} \int_0^T \|\Delta_x u_{\varepsilon,\sigma}(t)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)} e^{-(1-\beta_p)\lambda_\varepsilon t}\mathrm{d}t \\ &\lesssim \Psi_{\varepsilon,\sigma,0}\mathscr{E}_{\varepsilon,\sigma}(0)^{1-\beta_p} \|\Delta_x u_{\varepsilon,\sigma}\|^{2\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_p}\end{aligned}$$

for some $\omega_p > 0$, and the result follows. □

Injecting this in (3.4.6), we obtain, by taking ε small enough, that

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}.$$

The bootstrap argument is finally complete.

Remark 3.4.11. *Let us make the small modulated energy assumption from* (3.4.4) *more explicit. This will justify the assumptions stated in Theorems 3.1.5–3.1.6 (in the so-called mildly-well-prepared case).*

- *In the light particle regime, we have*

$$\mathscr{E}_{\varepsilon,1,1}(0) = \frac{\varepsilon}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|v - \frac{\langle j^0_{\varepsilon,1,1}\rangle}{\langle\rho^0_{\varepsilon,1,1}\rangle}\right|^2 f^0_{\varepsilon,1,1}(x,v)\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3}|u^0_{\varepsilon,1,1}(x) - \langle u^0_{\varepsilon,1,1}\rangle|^2\mathrm{d}x + \frac{\varepsilon\langle\rho^0_{\varepsilon,1,1}\rangle}{2(1+\varepsilon\langle\rho^0_{\varepsilon,1,1}\rangle)}|\langle j^0_{\varepsilon,1,1}\rangle - \langle u^0_{\varepsilon,1,1}\rangle|^2.$$

Assumption 3.1 ensures that the first and last term can be made as small as needed by reducing ε. There only remains the middle term, and we can thus replace (3.4.4), *up to taking $\varepsilon > 0$ small enough, by the assumption*

$$\left\|u^0_{\varepsilon,1,1} - \langle u^0_{\varepsilon,1,1}\rangle\right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta, \tag{3.4.7}$$

up to reducing η.

- *In the light and fast particle regime, we have*

$$\mathscr{E}_{\varepsilon,1,\varepsilon^\alpha}(0) = \frac{\varepsilon}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\varepsilon^\alpha} - \frac{\langle j^0_{\varepsilon,1,\varepsilon^\alpha}\rangle}{\langle\rho^0_{\varepsilon,1,\varepsilon^\alpha}\rangle}\right|^2 f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v)\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^3}|u^0_{\varepsilon,1,\varepsilon^\alpha}(x) - \langle u^0_{\varepsilon,1,\varepsilon^\alpha}\rangle|^2\mathrm{d}x + \frac{\varepsilon\langle\rho^0_{\varepsilon,1,\varepsilon^\alpha}\rangle}{2(1+\varepsilon\langle\rho^0_{\varepsilon,1,\varepsilon^\alpha}\rangle)}|\langle j^0_{\varepsilon,1,\varepsilon^\alpha}\rangle - \langle u^0_{\varepsilon,1,\varepsilon^\alpha}\rangle|^2,$$

so that, thanks to Assumption 3.1 and (3.4.3), *to ensure that* (3.4.4) *holds, we need only assume*

$$\left\|u^0_{\varepsilon,1,\varepsilon^\alpha} - \langle u^0_{\varepsilon,1,\varepsilon^\alpha}\rangle\right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta. \tag{3.4.8}$$

Without the extra assumption (3.4.3), *we need to impose that*

$$\varepsilon^{1-2\alpha}\int_{\mathbb{T}^3\times\mathbb{R}^3}|v|^2 f^0_{\varepsilon,1,\varepsilon^\alpha}(x,v)\mathrm{d}x\mathrm{d}v \leq \eta,$$

which is only restrictive for $\alpha = 1/2$, and accounts for the statement of Theorem 3.1.6.

3.4.3 Convergence in the mildly well-prepared and well-prepared cases

We study the convergence of the sequences $(f_{\varepsilon,\sigma})_{\varepsilon>0}$, $(\rho_{\varepsilon,\sigma})_{\varepsilon>0}$ and $(u_{\varepsilon,\sigma})_{\varepsilon>0}$ under several sets of assumptions for the initial data. We are able to provide rates of convergences in some of the cases. For the sake of readability, we do not keep track very precisely of the dependence on the initial data and express these rates in terms of the global initial bound $M > 1$ defined in Assumption 3.1.

Our first result only requires the weak convergence of the fluid velocity and the particle density.

Theorem 3.4.12. *Under Assumptions 3.1–3.2, there exists $\eta > 0$ such that if* (3.4.3)–(3.4.4) *hold and if*

$$u_{\varepsilon,\sigma} \xrightarrow[\varepsilon\to 0]{} u^0 \text{ in } w\text{-}\mathrm{L}^2(\mathbb{T}^3) \qquad \text{and} \qquad \rho^0_{\varepsilon,\sigma} \xrightarrow[\varepsilon\to 0]{} \rho^0 \text{ in } w^*\text{-}\mathrm{L}^\infty(\mathbb{T}^3),$$

then, for any $T > 0$, $(u_\varepsilon)_{\varepsilon>0}$ converges to u in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$, $(\rho_\varepsilon)_{\varepsilon>0}$ converges weakly-$*$ to ρ in $\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$, where (ρ, u) is a strong solution of

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(\rho u) = 0, \\ \rho|_{t=0} = \rho^0, \\ \partial_t u + (u\cdot\nabla_x)u - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0, \\ u|_{t=0} = u^0. \end{cases} \tag{3.4.9}$$

$\triangleright$ The energy–dissipation estimate (3.3.2) shows that the sequence $(u_{\varepsilon,\sigma})$ is uniformly bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^3))$. Therefore, up to a subsequence that we shall not write explicitly, $(u_{\varepsilon,\sigma})$ converges to some u in w-$\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^3))$. Furthermore, by the results of Section 3.4.2, for ε small enough, all times are strong existence times. Thanks to Lemma 3.4.6, we can apply the Aubin-Lions lemma (see for example [BF13]) and obtain the strong convergence of $(u_{\varepsilon,\sigma})$ to u in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

Moreover, we can now use Corollary 3.3.24, which ensures that $(\rho_{\varepsilon,\sigma})$ is bounded in $\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$, so that it converges, up to a subsequence, to some ρ in w^*-$\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$. Therefore, $(\rho_{\varepsilon,\sigma}u_{\varepsilon,\sigma})$ converges to ρu in w-$\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

Furthermore, we can also apply Lemmas 3.4.2–3.4.6 to find that $(F_{\varepsilon,\sigma})$ converges to 0 in $\mathrm{L}^p((0,T)\times\mathbb{T}^3)$, which implies the same convergence in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$. As a consequence, $(j_{\varepsilon,\sigma})$ converges to ρu in w-$\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

We can finally take the limit $\varepsilon\to 0$ in the distributional sense in the Navier–Stokes equation

$$\partial_t u_{\varepsilon,\sigma} + u_{\varepsilon,\sigma}\cdot\nabla_x u_{\varepsilon,\sigma} - \Delta_x u_{\varepsilon,\sigma} + \nabla_x u_{\varepsilon,\sigma} = F_{\varepsilon,\sigma}$$

and in the conservation law

$$\partial_t\rho_{\varepsilon,\gamma,\sigma} + \mathrm{div}_x\, j_{\varepsilon,\gamma,\sigma} = 0$$

and obtain the derivation of (3.4.9).

Let us note that thanks to the higher order energy estimate (3.3.7) and another weak compactness argument, we have $u \in \mathrm{L}^\infty(0,+\infty;\mathrm{H}^1(\mathbb{T}^3)) \cap \mathrm{L}^2(0,+\infty;\mathrm{H}^2(\mathbb{T}^3))$.

To conclude, we remark that the limit system has a unique such smooth solution by standard uniqueness results in the Fujita–Kato framework (see *e.g.* [Che+06, Theorem 3.3]). Consequently the convergence of $(u_{\varepsilon,\sigma})$ and $(\rho_{\varepsilon,\sigma})$ holds without taking a subsequence. $\square$

If we assume a stronger form of convergence of the initial fluid velocity, then we have a stronger result for $(u_{\varepsilon,\sigma})$ and $(\rho_{\varepsilon,\sigma})$ as well. We can also state the convergence of $(f_{\varepsilon,\sigma})$ and provide explicit rates in term of ε. Recall that W_1 denotes the Wasserstein-1 distance (see Definition A.3).

Theorem 3.4.13. *Under Assumptions 3.1–3.2, there exists $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, if* (3.4.3)–(3.4.4) *hold, there exists $\omega_p > 0$ and $C > 0$, independent from ε, such that,* *if $T>1$, for any $t\in[0,T]$,*

$$\|u_{\varepsilon,\sigma}(t)-u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}\lesssim e^{CM^2T}M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)}+\varepsilon^{\frac{\kappa}{2}}\right),$$

$$W_1(\rho_{\varepsilon,\sigma}(t),\rho(t))\lesssim W_1(\rho^0_{\varepsilon,\sigma},\rho^0)+T^{\frac12}e^{CM^2T}M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)}+\varepsilon^{\frac{\kappa}{2}}\right).$$

and

$$\int_0^T W_1(f_{\varepsilon,\sigma}(t),\rho(t)\otimes\delta_{v=\sigma u(t)})\mathrm{d}t\lesssim TW_1(\rho^0_{\varepsilon,\sigma},\rho^0)+T^{\frac32}e^{CM^2T}M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)}+\varepsilon^{\frac{\kappa}{2}}\right),$$

where (ρ,u) is the unique solution of (3.4.9)*.*

Remark 3.4.14. *We only assume $T>1$ in order to simplify the time-dependence of the estimates above. In particular, the following result holds for any $T>0$ with the same convergence rates in terms of ε.*

As a straightforward consequence we deduce

Corollary 3.4.15. *Under Assumptions 3.1–3.2, there exists $\eta>0$ such that if*

- (3.4.3)–(3.4.4) *hold,*
- *$(u^0_{\varepsilon,\sigma})_{\varepsilon>0}$ converges to u^0 in $\mathrm{L}^2(\mathbb{T}^3)$,*
- *$(\rho^0_{\varepsilon,\sigma})_{\varepsilon>0}$ converges to ρ^0 in $(\mathcal{P}_1(\mathbb{T}^3),W_1)$,*

then, for any $T>0$,

- *$(u_{\varepsilon,\sigma})_{\varepsilon>0}$ converges to u in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$,*
- *$(\rho_{\varepsilon,\sigma})_{\varepsilon>0}$ weakly-$*$ converges to ρ in $\mathrm{L}^\infty(0,T;\mathcal{P}_1(\mathbb{T}^3))$,*
- *$(f_{\varepsilon,\sigma}-\rho\otimes\delta_{v=\sigma u})_{\varepsilon>0}$ weakly-$*$ converges to 0 in $\mathrm{L}^1(0,T;\mathcal{P}_1(\mathbb{T}^3))$.*

Let us prove Theorem 3.4.13.

$\triangleright$ Let (ρ,u) be the unique strong solution to the system (3.4.9). We set

$$w_{\varepsilon,\sigma}=u_{\varepsilon,\sigma}-u$$

which satisfies the equation

$$\partial_t w_{\varepsilon,\sigma}+(u\cdot\nabla_x)w_{\varepsilon,\sigma}-\Delta_x w_{\varepsilon,\sigma}+\nabla_x(p_\varepsilon-p)=F_{\varepsilon,\sigma}-(w_{\varepsilon,\sigma}\cdot\nabla_x)u_{\varepsilon,\sigma}.$$

We multiply this by $w_{\varepsilon,\sigma}$ and integrate over $(0,t)\times\mathbb{T}^3$ for any $t\in(0,T)$ to get

$$\begin{aligned}\frac12\|w_{\varepsilon,\sigma}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}+\int_0^t\int_{\mathbb{T}^3}w_{\varepsilon,\sigma}(s,x)\cdot(u(s)\cdot\nabla_x)w_{\varepsilon,\sigma}(s,x)\mathrm{d}s\mathrm{d}x+\int_0^t\|\nabla_x w_{\varepsilon,\sigma}(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\mathrm{d}s\\ =\frac12\|w_{\varepsilon,\sigma}(0)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}+\int_0^t\int_{\mathbb{T}^3}F_{\varepsilon,\sigma}(s,x)\cdot w_{\varepsilon,\sigma}(s,x)\mathrm{d}s\mathrm{d}x\\ -\int_0^t\int_{\mathbb{T}^3}(w_{\varepsilon,\sigma}(s,x)\cdot\nabla_x)u_{\varepsilon,\sigma}(s,x)\cdot w_{\varepsilon,\sigma}(s,x)\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Since $\mathrm{div}_x\,u=0$, we have

$$\int_0^t\int_{\mathbb{T}^3}w_{\varepsilon,\sigma}(s,x)\cdot(u(s,x)\cdot\nabla_x)w_{\varepsilon,\sigma}(s,x)=0.$$

Furthermore, thanks to the Gagliardo-Nirenberg inequality (Theorem B.1), we can write

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{T}^3} (w_{\varepsilon,\sigma}(s,x)\cdot\nabla_x)u_{\varepsilon,\sigma}(s,x)\cdot w_{\varepsilon,\sigma}(s,x)\mathrm{d}s\mathrm{d}x\right| \\ &\leq \int_0^t \|w_{\varepsilon,\sigma}(s)\|^2_{\mathrm{L}^4(\mathbb{T}^3)} \|\nabla_x u_{\varepsilon,\sigma}(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\ &\lesssim \int_0^t \|w_{\varepsilon,\sigma}(s)\|^{\frac12}_{\mathrm{L}^2(\mathbb{T}^3)} \|\nabla_x w_{\varepsilon,\sigma}(s)\|^{\frac32}_{\mathrm{L}^2(\mathbb{T}^3)} \|\nabla_x u_{\varepsilon,\sigma}(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\ &\qquad + \int_0^t \|w_{\varepsilon,\sigma}(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \|\nabla_x u_{\varepsilon,\sigma}(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s.\end{aligned}$$

Since T is a strong existence time, we can apply Lemma 3.3.13 and Young's inequality to find that for any $a>0$,

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{T}^3} (w_{\varepsilon,\sigma}(s,x)\cdot\nabla_x)u_{\varepsilon,\sigma}(s,x)\cdot w_{\varepsilon,\sigma}(s,x)\mathrm{d}s\mathrm{d}x\right| \\ &\lesssim a^{\frac43}\int_0^t \|\nabla_x w_{\varepsilon,\sigma}(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s + \frac{1}{a^4}\left(\Psi_{\varepsilon,\sigma,0} + \Psi^{\frac12}_{\varepsilon,\sigma,0}\right)\int_0^t \|w_{\varepsilon,\sigma}(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s.\end{aligned}$$

We can take $a>0$ small enough so that Grönwall's inequality and Lemma 3.3.13 show that there exists a universal constant $C>0$ such that for any $t\in[0,T]$,

$$\|w_{\varepsilon,\sigma}(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim e^{CM^2T}\left(\|w_{\varepsilon,\sigma}(0)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \|F_{\varepsilon,\sigma}\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\right),$$

hence, by Lemmas 3.4.9–3.4.10, since T is a strong existence time, we obtain

$$\|u_{\varepsilon,\sigma}-u\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim e^{CM^2T}M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u(0)\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^\kappa\right), \tag{3.4.10}$$

for some $\omega_p>0$.

As we have seen previously, $(\rho_{\varepsilon,\sigma})$ converges weakly-$*$ to ρ in $\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$ such that

$$\partial_t\rho + \mathrm{div}_x(\rho u) = 0,$$

and $\rho|_{t=0}=\rho^0$. Let us now compute a rate of convergence. As before, we study the equation satisfied by the difference $\chi_{\varepsilon,\sigma}=\rho_{\varepsilon,\sigma}-\rho$. We have

$$\partial_t\chi_{\varepsilon,\sigma} + \mathrm{div}_x(\chi_{\varepsilon,\sigma}u_{\varepsilon,\sigma}) = -\,\mathrm{div}_x\, G_{\varepsilon,\sigma},$$

where $G_{\varepsilon,\sigma}=F_{\varepsilon,\sigma}+\rho(u_{\varepsilon,\sigma}-u)$. We consider the characteristic curves associated to this transport equation, defined, for every $s,t\geq 0$, by

$$\frac{\mathrm{d}}{\mathrm{d}s}Y(s;t,x) = u_{\varepsilon,\sigma}(s,Y(s;t,x)) \qquad \text{and} \qquad Y(t;t,x)=x.$$

Let $s,t\geq 0$. Since $\mathrm{div}_x\, u_{\varepsilon,\sigma}=0$, we have $\det\nabla_x Y(s;t,x)=1$. Furthermore, thanks to Grönwall's inequality, since T is a strong existence time,

$$\|\nabla_x Y(s;t,x)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \leq e^{\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))}} \lesssim 1.$$

Therefore, for any $\psi \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $\|\nabla_x \psi\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \leq 1$, the method of characteristics and the change of variable $x' = Y(s;t,x)$ yield, for any $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^3} \chi_{\varepsilon,\sigma}(t)\psi \mathrm{d}x &= \int_{\mathbb{T}^3} \chi_{\varepsilon,\sigma}(0)\psi \mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} (\mathrm{div}_x\, G_\varepsilon)(s, Y(s;t,x))\psi(x)\mathrm{d}x\mathrm{d}s \\ &= \int_{\mathbb{T}^3} (\rho^0_{\varepsilon,\sigma} - \rho^0)\psi \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} G_\varepsilon(s,x)\nabla_x \left(\psi(Y(t;s,x))\right) \mathrm{d}x\mathrm{d}s.\end{aligned}$$

Moreover, for every $s, t \geq 0$,

$$\|\nabla_x \left(\psi(Y(t;s,x))\right)\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \|\nabla_x Y(t;s,x)\nabla_x \psi(Y(t;s,x))\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim 1,$$

so that, thanks to the previous estimate (3.4.10) for $u_{\varepsilon,\sigma} - u$, we have, for any $t \in [0,T]$,

$$\begin{aligned}&\Big|\int_0^t \int_{\mathbb{T}^3} G_{\varepsilon,\sigma}(s,x)\nabla_x \left(\psi(Y(t;s,x))\right) \mathrm{d}x\mathrm{d}s\Big| \\ &\qquad \leq \|G_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \|\nabla_x \left(\psi(Y(t;s,x))\right)\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \\ &\qquad \lesssim T^{\frac{1}{2}} \left(\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + M\, \|u_{\varepsilon,\sigma} - u\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\right) \\ &\qquad \lesssim T^{\frac{1}{2}} e^{CM^2T} M^{\omega_p} \left(\left\|u^0_{\varepsilon,\sigma} - u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}}\right).\end{aligned}$$

for some $\omega_p > 0$. Therefore, for every $t \in [0,T]$,

$$W_1(\rho_{\varepsilon,\sigma}(t), \rho(t)) \lesssim W_1(\rho^0_{\varepsilon,\sigma}, \rho^0) + T^{\frac{1}{2}} e^{CM^2T} M^{\omega_p} \left(\left\|u^0_{\varepsilon,\sigma} - u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}}\right). \tag{3.4.11}$$

We can now prove the estimate on $(f_{\varepsilon,\sigma})_{\varepsilon>0}$. We write, for almost all $t \in (0,T)$,

$$\begin{aligned}f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)} &= \left(f_{\varepsilon,\sigma}(t) - \rho_{\varepsilon,\sigma}(t) \otimes \delta_{v=\sigma u_{\varepsilon,\sigma}(t)}\right) \\ &\qquad + \left(\rho_{\varepsilon,\sigma}(t) \otimes \delta_{v=\sigma u_{\varepsilon,\sigma}(t)} - \rho(t) \otimes \delta_{v=\sigma u(t)}\right).\end{aligned}$$

For $\psi \in \mathscr{C}^\infty_c(\mathbb{T}^3 \times \mathbb{R}^3)$ such that $\|\nabla_{x,v}\psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leq 1$, we have

$$\begin{aligned}&|\langle \rho_{\varepsilon,\sigma}(t) \otimes \delta_{v=\sigma u_{\varepsilon,\sigma}(t)} - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi\rangle| \\ &\qquad \leq \left|\int_{\mathbb{T}^3} (\rho_{\varepsilon,\sigma} - \rho)(t)\psi(x, \sigma u_{\varepsilon,\sigma}(t,x))\mathrm{d}x\right| + \int_{\mathbb{T}^3} \rho(t,x)|\psi(x,\sigma u_{\varepsilon,\sigma}(t,x)) - \psi(x,\sigma u)|\mathrm{d}x.\end{aligned}$$

On the one hand, for almost every $t \in (0,T)$,

$$\left|\int_{\mathbb{T}^3} (\rho_{\varepsilon,\sigma} - \rho)(t)\psi(x, \sigma u_{\varepsilon,\sigma}(t,x))\mathrm{d}x\right| \leq W_1(\rho_{\varepsilon,\sigma}(t), \rho(t)) \sup_{x\in\mathbb{T}^3} |\nabla_x \left(\psi(x, \sigma u_{\varepsilon,\sigma}(t,x))\right)|$$

and, for every $x \in \mathbb{T}^3$,

$$\begin{aligned}|\nabla_x \left(\psi(x, \sigma u_{\varepsilon,\sigma}(t,x))\right)| &\leq \|\nabla_x \psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} + \sigma \|\nabla_x u_{\varepsilon,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \|\nabla_v \psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \\ &\leq 1 + \sigma \|\nabla_x u_{\varepsilon,\sigma}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}.\end{aligned}$$

We use the fact that since T is a strong existence time,

$$\|\nabla_x u_{\varepsilon,\sigma}(t)\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \lesssim 1,$$

so that

$$\int_0^T W_1(\rho_{\varepsilon,\sigma}(t),\rho(t)) \sup_{x\in\mathbb{T}^3} |\nabla_x (\psi(x,\sigma u_{\varepsilon,\sigma}(t,x)))| \,\mathrm{d}t \\ \lesssim (T+\sigma)\left(\sup_{s\in(0,T)} W_1(\rho_{\varepsilon,\sigma}(s),\rho(s))\right) \lesssim T\left(\sup_{s\in(0,T)} W_1(\rho_{\varepsilon,\sigma}(s),\rho(s))\right).$$

On the other hand

$$\int_{\mathbb{T}^3} \rho(t,x)|\psi(x,\sigma u_{\varepsilon,\sigma}(t,x)) - \psi(x,\sigma u)|\mathrm{d}x \\ \leq \sigma \|\rho\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \|\nabla_v\psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \|u_{\varepsilon,\sigma}(t)-u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}.$$

Hence, recalling that

$$W_1(\rho_{\varepsilon,\sigma}(t)\otimes\delta_{v=\sigma u_{\varepsilon,\sigma}(t)}, \rho(t)\otimes\delta_{v=\sigma u(t)}) = \sup_{\|\nabla_{x,v}\psi\|=1} |\langle \rho_{\varepsilon,\sigma}(t)\otimes\delta_{v=\sigma u_{\varepsilon,\sigma}(t)} - \rho(t)\otimes\delta_{v=\sigma u(t)}, \psi\rangle|,$$

we obtain, since T is a strong existence time and thanks to the previous estimates (3.4.10)–(3.4.11),

$$\int_0^T W_1(\rho_{\varepsilon,\sigma}(t)\otimes\delta_{v=\sigma u_{\varepsilon,\sigma}(t)}, \rho(t)\otimes\delta_{v=\sigma u(t)})\mathrm{d}t \\ \lesssim TW_1(\rho^0_{\varepsilon,\sigma},\rho^0) + T^{\frac{3}{2}} e^{CM^2T} M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}}\right),$$

for some $\omega_p > 0$. Furthermore, we also obtain

$$\int_0^T W_1(f_{\varepsilon,\sigma}(t),\rho_{\varepsilon,\sigma}(t)\otimes\delta_{v=\sigma u_{\varepsilon,\sigma}(t)})\mathrm{d}t \\ \leq \int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_{\varepsilon,\sigma}(t,x,v)|v-\sigma u_{\varepsilon,\sigma}(t,x)|\mathrm{d}x\mathrm{d}v\mathrm{d}t \\ \leq \sigma T^{1-\frac{1}{p}}\left(\int_0^T\int_{\mathbb{T}^3}\left(\int_{\mathbb{R}^3} f_{\varepsilon,\sigma}(t,x,v)\left|\frac{v}{\sigma}-u_{\varepsilon,\sigma}(t,x)\right|\mathrm{d}v\right)^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}},$$

and thanks to Lemmas 3.4.2–3.4.6, we get

$$\int_0^T W_1(f_{\varepsilon,\sigma}(t),\rho_{\varepsilon,\sigma}(t)\otimes\delta_{v=\sigma u_{\varepsilon,\sigma}(t)})\mathrm{d}t \lesssim \sigma T^{1-\frac{1}{p}} M^{\omega_p}\varepsilon^{\frac{\kappa}{p}}.$$

We can now conclude that

$$\int_0^T W_1(f_{\varepsilon,\sigma}(t),\rho(t)\otimes\delta_{v=\sigma u(t)})\mathrm{d}t \lesssim TW_1(\rho^0_{\varepsilon,\sigma},\rho^0) + T^{\frac{3}{2}} e^{CM^2T} M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma}-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}}\right).$$

All claimed estimates are finally proved. □

Under an additional well-preparedness assumption, we can even prove pointwise convergence in time for the distribution function.

Theorem 3.4.16. *Under Assumptions 3.1–3.2, there exists $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, if (3.4.3)–(3.4.4) hold there exists $\omega_p > 0$ and $C > 0$, independent from ε such that, if $T > 1$, then for almost every $t \in (0, T)$,*

$$W_1(f_{\varepsilon,\sigma}(t), \rho(t) \otimes \delta_{v=\sigma u(t)}) \lesssim \sigma e^{CM^2 T} M^{\omega_p} \left(\left\| u^0_{\varepsilon,\sigma} - u^0 \right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}} \right) + \sigma \left\| \left| \frac{v}{\sigma} - u^0_{\varepsilon,\sigma} \right| f_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)} + \sigma M^{\omega_p} \varepsilon^{1-\frac{1}{p}}.$$

As stated in Remark 3.4.14, we only work under the assumption $T > 1$ in order to have a simpler time-dependence in the convergence rates. In particular, the following convergence results hold for any $T > 0$ with the same rate in terms of the parameter ε.

Corollary 3.4.17. *In the light particle regime, under Assumptions 3.1–3.2, there exists $\varepsilon_0 > 0$ and $\eta > 0$, such that if*

$$\forall \varepsilon \in (0, \varepsilon_0), \qquad \left\| u^0_{\varepsilon,1,1} - \langle u^0_{\varepsilon,1,1} \rangle \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

and if $(u^0_{\varepsilon,1,1})_{\varepsilon>0}$ converges to u^0 in $\mathrm{L}^2(\mathbb{T}^3)$, and

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v - u^0_{\varepsilon,1,1}(x)| f^0_{\varepsilon,1,1}(x, v) \, \mathrm{d}x\mathrm{d}v \xrightarrow[\varepsilon \to 0]{} 0,$$

then $(f_{\varepsilon,1,1})_{\varepsilon>0}$ weakly-$*$ converges to $\rho \otimes \delta_{v=u}$ in $\mathrm{L}^\infty(0, T; \mathcal{P}_1(\mathbb{T}^3))$.

Corollary 3.4.18. *In the light and fast particle regime, under Assumption 3.1–3.2, there exists $\varepsilon_0 > 0$ and $\eta > 0$, such that if there exists $\kappa \in (0, 1)$ for which*

$$\forall \varepsilon \in (0, \varepsilon_0), \qquad \left\| |v|^p f^0_{\varepsilon,1,\varepsilon^\alpha} \right\|_{\mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)} \lesssim \varepsilon^{\alpha p + \kappa - 1},$$

and

$$\forall \varepsilon \in (0, \varepsilon_0), \qquad \left\| u^0_{\varepsilon,1,\varepsilon^\alpha} - \langle u^0_{\varepsilon,1,\varepsilon^\alpha} \rangle \right\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta$$

then $(f_{\varepsilon,1,\varepsilon^\alpha})_{\varepsilon>0}$ weakly-$*$ converges to $\rho \otimes \delta_{v=0}$ in $\mathrm{L}^\infty(0, T; \mathcal{P}_1(\mathbb{T}^3))$.

The proof of Theorem 3.4.16 is similar to the derivation of uniform estimates for the Brinkman force in Section 3.4.1.
$\triangleright$ For $\psi \in \mathscr{C}^\infty_c(\mathbb{T}^3 \times \mathbb{R}^3)$ with $\|\nabla_{x,v}\psi\|_{\mathrm{L}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq 1$, we have, thanks to the method of characteristics and the change of variable $w = \Gamma^{t,x}_{\varepsilon,\sigma}(v)$, for almost every $t \in (0, T)$,

$$\begin{aligned}
&\langle f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi \rangle \\
&\quad = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\varepsilon,\sigma}(t, x, v) [\psi(x, v) - \psi(x, \sigma u(t, x))] \mathrm{d}x\mathrm{d}v \\
&\quad = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f^0_{\varepsilon,\sigma} \left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w \right) \\
&\qquad \times \left[\psi\left(x, [\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right) - \psi(x, \sigma u(t, x)) \right] \left| \det \nabla_w [\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) \right| e^{\frac{3t}{\varepsilon}} \mathrm{d}x\mathrm{d}w,
\end{aligned}$$

and Lemma 3.3.22 yields

$$|\langle f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi\rangle| \lesssim \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \left|[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) - \sigma u(t,x)\right| \mathrm{d}x\mathrm{d}w.$$

As in the proof of Lemma 3.4.2 we integrate the characteristics equation and perform an integration by parts to get, for every $w \in \mathbb{R}^3$,

$$\begin{aligned}
[\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w) = \sigma u_{\varepsilon,\sigma}(t,x) + e^{-\frac{t}{\varepsilon}}\left(w - \sigma u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right) - \sigma \int_0^t e^{\frac{s-t}{\varepsilon}} \partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right) \\
- \sigma \int_0^t e^{\frac{s-t}{\varepsilon}} V_{\varepsilon,\sigma}\left(s; t, x, [\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right) \cdot \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right) \mathrm{d}s.
\end{aligned}$$

Then, we separate the integral into four parts :

$$\begin{aligned}
&\frac{1}{\sigma}|\langle f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi\rangle| \\
&\quad\lesssim \int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) |u_{\varepsilon,\sigma}(t,x) - u(t,x)| \mathrm{d}x\mathrm{d}w \\
&\qquad + \left|\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left(\frac{w}{\sigma} - u^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0)\right)\right) \mathrm{d}x\mathrm{d}w\right| \\
&\qquad + \int_{\mathbb{T}^3\times\mathbb{R}^3}\int_0^s e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right)\left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}x\mathrm{d}w \\
&\qquad + \int_{\mathbb{T}^3\times\mathbb{R}^3}\int_0^s e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(0), w\right) \\
&\qquad\qquad \times \left|V\left(s; t, x, [\Gamma^{t,x}_{\varepsilon,\sigma}]^{-1}(w)\right) \cdot \nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}^{t,x,w}_{\varepsilon,\sigma}(s)\right)\right| \mathrm{d}s\mathrm{d}x\mathrm{d}w \\
&\quad\le I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us provide, for each of these integrals, a rate of convergence (to 0). Using the Cauchy-Schwarz inequality and the previously stated convergence of $(u_{\varepsilon,\sigma})_{\varepsilon>0}$, we get

$$|I_1| \le \left\|f^0_{\varepsilon,\sigma}\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|u_{\varepsilon,\sigma}(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim e^{CM^2T} M^{\omega_p}\left(\left\|u^0_{\varepsilon,\sigma} - u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}}\right).$$

Furthermore, thanks to Lemma 3.3.26 and the corresponding change of variables,

$$I_2 \lesssim \int_{\mathbb{T}^3\times\mathbb{R}^3}\left|\frac{v}{\sigma} - u^0_{\varepsilon,\sigma}(x)\right| f^0_{\varepsilon,\sigma}(x,v)\, \mathrm{d}x\mathrm{d}v.$$

We deal with I_3 and I_4 as in Section 3.4.1, but here the computations are simpler. Thanks to

Hölder's inequality and Lemmas 3.3.26 and 3.4.6, we have

$$\begin{aligned}
I_3 &\leq \int_0^t e^{\frac{s-t}{\varepsilon}} \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}_{t,x,w}(0), w\right) \mathrm{d}x\mathrm{d}w \right)^{1-\frac{1}{p}} \\
&\qquad\qquad \times \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}\left(\widetilde{X}_{t,x,w}(0), w\right) \left|\partial_s u_{\varepsilon,\sigma}\left(s, \widetilde{X}_{t,x,w}(s)\right)\right|^p \mathrm{d}x\mathrm{d}w \right)^{\frac{1}{p}} \mathrm{d}s \\
&\leq \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \int_0^t e^{\frac{s-t}{\varepsilon}} \left\| \partial_s u_{\varepsilon,\sigma}(s) \right\|_{\mathrm{L}^p(\mathbb{T}^3)} \mathrm{d}t \\
&\leq \varepsilon^{1-\frac{1}{p}} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| \partial_s u_{\varepsilon,\sigma} \right\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_p} \varepsilon^{1-\frac{1}{p}},
\end{aligned}$$

for some $\omega_p > 0$. Finally, we also have

$$\begin{aligned}
I_4 &\leq \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} e^{-\frac{t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}_{t,x,w}(0), w\right) |w| \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}_{t,x,w}(s)\right)\right| \mathrm{d}x\mathrm{d}w\mathrm{d}s \\
&\quad + \frac{\sigma}{\varepsilon} \int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} e^{\frac{s-t}{\varepsilon}} f^0_{\varepsilon,\sigma}\left(\widetilde{X}_{t,x,w}(0), w\right) \int_0^s e^{\frac{\tau-s}{\varepsilon}} \left| u_{\varepsilon,\sigma}\left(\tau, \widetilde{X}_{t,x,w}(\tau)\right)\right| \\
&\qquad\quad \times \left|\nabla_x u_{\varepsilon,\sigma}\left(s, \widetilde{X}_{t,x,w}(s)\right)\right| \mathrm{d}\tau\mathrm{d}x\mathrm{d}w\mathrm{d}s \\
&\leq I_5 + I_6.
\end{aligned}$$

We conclude thanks to Hölder's inequality and Corollary 3.3.21 and Lemma 3.4.6 that

$$\begin{aligned}
I_5 &\leq \left\| f^0_{\varepsilon,\sigma} |v|^{\frac{p}{p-1}} \right\|^{1-\frac{1}{p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\| f^0_{\varepsilon,\sigma} \right\|^{\frac{1}{p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \int_0^t e^{-\frac{t}{\varepsilon}} \left\| \nabla_x u_{\varepsilon,\sigma}(s) \right\|_{\mathrm{L}^p(\mathbb{T}^3)} \mathrm{d}s \\
&\lesssim M^{\omega_p} \varepsilon^{1-\frac{\alpha_p}{p}},
\end{aligned}$$

and

$$\begin{aligned}
I_6 &\leq \frac{\sigma}{\varepsilon} \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau-s}{\varepsilon}} \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}(\widetilde{X}_{t,x,w}(0), w) |u_{\varepsilon,\sigma}(\tau, \widetilde{X}_{t,x,w}(\tau))|^p \mathrm{d}x\mathrm{d}w \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f^0_{\varepsilon,\sigma}(\widetilde{X}_{t,x,w}(0), w) |\nabla_x u_{\varepsilon,\sigma}(s, \widetilde{X}_{t,x,w}(s))|^{\frac{p}{p-1}} \mathrm{d}x\mathrm{d}w \right)^{1-\frac{1}{p}} \mathrm{d}\tau\mathrm{d}s \\
&\leq \frac{\sigma}{\varepsilon} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \int_0^t \int_0^s e^{\frac{s-t}{\varepsilon}} e^{\frac{\tau-s}{\varepsilon}} \left\| u_{\varepsilon,\sigma}(\tau) \right\|_{\mathrm{L}^p(\mathbb{T}^3)} \left\| \nabla_x u_{\varepsilon,\sigma}(s) \right\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \mathrm{d}\tau\mathrm{d}s \\
&\lesssim \frac{\sigma}{\varepsilon} \left\| f^0_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_{\varepsilon,\sigma}(0)^{\frac{1-\beta_p}{2}} \Psi^{\frac{1}{2}}_{\varepsilon,0} \\
&\quad \times \int_0^t e^{\frac{\tau-s}{\varepsilon}} \left\| \Delta_x u_{\varepsilon,\sigma}(s) \right\|^{\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)} \left(\int_\tau^t e^{\frac{s-\tau}{\varepsilon}} \mathrm{d}s \right) \mathrm{d}\tau \\
&\lesssim M^{\omega_p} \sigma \varepsilon^{1-\frac{\beta_p}{p}}.
\end{aligned}$$

Therefore, there exists $\omega_p > 0$ such that

$$|\langle f_{\varepsilon,\sigma}(t) - \rho(t) \otimes \delta_{v=\sigma u(t)}, \psi \rangle| \lesssim \sigma e^{CM^2 T} M^{\omega_p} \left(\|u^0_{\varepsilon,\sigma} - u^0\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{\kappa}{2}} \right) \\ + \sigma \left\| \left| \frac{v}{\sigma} - u^0_{\varepsilon,\sigma} \right| f_{\varepsilon,\sigma} \right\|_{\mathrm{L}^1(\mathbb{T}^3 \times \mathbb{R}^3)} + \sigma M^{\omega_p} \varepsilon^{1-\frac{1}{p}},$$

hence the theorem. □

In [HM21, Section 6], we apply the tools developed to deal with the fine particle regime to the *light* and *light and fast* particle regimes and obtain alternative proofs of the above results under slightly different sets of assumptions.

3.4.4 Application to long time behavior

As a side consequence of the analysis, we obtain the description of the long time behavior of solutions to (3.1.3) in the *light* and *light and fast* particle regimes. What we specifically have proved are variants of the main result of [HMM20] with assumptions which are somewhat tailored to the particular regimes we consider.

Loosely speaking, [HMM20, Theorem 2.1] asserts, forgetting for a while about the small parameter ε, that if the initial data (u^0, f^0) is close to equilibrium, then any solution remains close and the dynamics ultimately resembles that of a Dirac mass in velocity.

We observe that as corollaries of the previous analysis, we get similar results but with assumptions bearing solely on the velocity field in the *light* and *light and fast* particle regime for $\alpha \in (0, 1/3]$:

Corollary 3.4.19. *In the* light *and* light and fast *particle regimes, for $\alpha \in [0, 1/3]$, under Assumptions 3.1 and 3.2, for ε small enough, there is $\eta > 0$ such that, if*

$$\|u^0_{\varepsilon,\varepsilon^\alpha} - \langle u^0_{\varepsilon,\varepsilon^\alpha} \rangle\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq \eta,$$

then there exist $C, \lambda > 0$ independent of ε such that, for all $t \geq 0$,

$$\mathscr{E}_{\varepsilon,\varepsilon^\alpha}(t) \leq C \mathscr{E}_{\varepsilon,\varepsilon^\alpha}(0) e^{-\lambda t}.$$

The fact that the decay of the modulated energy entails a monokinetic behavior as $t \to +\infty$ is for instance proved in [HMM20, Theorem 1.1]

▷ This is a consequence of Lemmas 3.3.19 and 3.3.20, of the bootstrap analysis led in Section 3.4.2, of Remark 3.4.11 and the observation that (3.4.3) is restrictive only for $\alpha > 1/3$. □

3.4.5 Convergence in the general case

In this paragraph we aim to prove similar convergence results as the previous theorems under the sole Assumption 3.1.

In turn, the time horizon under which we are able to prove convergence results shall be constrained.

Recall that we consider, under Assumption 3.1, for every $\varepsilon > 0$,

$$T^*_\varepsilon = \sup\{T > 0,\, T \text{ is a strong existence time}\}.$$

Our goal is to find a time T_M, independent of ε, such that $T_M < T^*_\varepsilon$. Let us first note that in view of the proof of Lemma 3.3.13, under Assumption 3.1, there exists a time T_0, independent of ε, such that

$$\int_0^{T_0} \left\| e^{t\Delta} u_{\varepsilon,\sigma} \right\|^4_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \mathrm{d}t \leq \frac{C^*}{4}.$$

Thus, it suffices to show that there exists $T_M \leq \min(T^*_\varepsilon, T_0)$ independent of ε, such that

$$\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \leq \frac{1}{40} \qquad \text{and} \qquad \|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}.$$

Let $T \in (0, \min(T^*_\varepsilon, T_0))$. We can apply Lemmas 3.4.9 and 3.4.10 which ensure that for ε small enough,

$$\|F_{\varepsilon,\sigma}\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}.$$

To conclude, by a variant of the proof of Corollary 3.4.7, we get

$$\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \lesssim T^{\omega_p} M^{\omega'_p} (1 + \varepsilon^{\frac{\kappa}{p}}),$$

so that, imposing T small enough instead of relying on the smallness of the initial modulated energy $\mathscr{E}_{\varepsilon,\sigma}(0)$, this yields

$$\|\nabla_x u_{\varepsilon,\sigma}\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \frac{1}{40}.$$

Therefore, we have proved that there exists a time $T_M > 0$, independent of ε, that is a strong existence time.

The convergence results in Theorems 3.4.12–3.4.13 and Corollary 3.4.16 still hold when replacing the smallness assumptions on the initial condition by the smallness of the time horizon. Their proof are indeed almost identical.

3.5 Fine particle regime

We study in this section the fine particle regime, which corresponds to $(\gamma, \sigma) = (\varepsilon, 1)$ in (3.1.3). For the sake of readability, we shall no longer write the parameters (γ, σ): for instance, $u_{\varepsilon,\varepsilon,1}$ will be referred to as u_ε. For the reader's convenience, let us write down the system corresponding to this regime :

$$\begin{cases} \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x) u_\varepsilon - \Delta_x u_\varepsilon + \nabla_x p = \dfrac{1}{\varepsilon} \left(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon \right), \\ \operatorname{div}_x u_\varepsilon = 0, \\ \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \dfrac{1}{\varepsilon} \operatorname{div}_v \left[f_\varepsilon (u_\varepsilon - v) \right] = 0, \\ \rho_{f_\varepsilon}(t,x) = \displaystyle\int_{\mathbb{R}^3} f_\varepsilon(t,x,v) \mathrm{d}v, \quad j_{f_\varepsilon}(t,x) = \displaystyle\int_{\mathbb{R}^3} v f_\varepsilon(t,x,v) \mathrm{d}v. \end{cases}$$

As explained in the introduction, the fine particle regime is more singular than the *light* and *light and fast* ones. Yet, in the case of well-prepared initial data, the same desingularization (as in Section 3.4.1) of the Brinkman force

$$F_\varepsilon = \frac{1}{\varepsilon} \left(j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon \right)$$

could be applied and would allow to prove the expected convergence. Nevertheless, we choose to present a different strategy to desingularize F_ε which yields stronger results. For this purpose, we introduce and study what [Han20] calls *higher* fluid-kinetic dissipation. This allows to prove qualitative convergence results for f_ε and $(\rho_\varepsilon, u_\varepsilon)$. However this approach is not sufficient to provide convergence rates. To that purpose, we shall rely on a relative entropy method. Let us emphasize the fact that this method *by itself does not* allow to conclude, but rather has to be used in conjunction with the previous analysis.

Remark 3.5.1. *The higher fluid-kinetic dissipation allows, in [Han20], to extend the study of the long-time-behavior of the Vlasov–Navier–Stokes system from* $\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$ *to the domain* $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$. *It might also be the right approach to study high friction limits in the domain* $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, *which we do not consider here.*

This section is organized as follows :

- In Section 3.5.1, we introduce the relevant higher dissipation functionals and prove a key identity, leading to the desingularization of the Brinkman force and eventually to uniform L^p (in time and space) bounds. This process requires to consider mildly well-prepared initial data as defined in the statement of Theorem 3.1.7.
- Thanks to these crucial bounds, for well-prepared initial data, in Section 3.5.2, we conclude the bootstrap argument, obtaining that all positive times are *strong existence times.* Whereas the general strategy is the same as for the *light* and *light and fast* particle regimes (recall Section 3.4.2), a special care is required here to handle the L^2 norm of the Brinkman force (recall Remark 3.3.3).
- Section 3.5.3 concludes the convergence proof in the well-prepared case.
- In Section 3.5.4, we introduce a *relative entropy* method, allowing to prove quantitative convergence results in the well-prepared case of Theorem 3.1.7, when one assumes that the solutions of the limiting system are smooth enough. This analysis can be performed only because we have already proved that all positive times are strong existence times.
- Finally Section 3.5.5 explains how to get a small time convergence result in the mildly well-prepared case.

3.5.1 Uniform estimates on the Brinkman force in L^p

To conclude the bootstrap argument initiated in Section 3.3.8, we need to control the Brinkman force in various L^r spaces. We will rely on the exponential decay of the modulated energy on $[0, T]$, when T is a strong existence time (recall Lemma 3.3.20). Notice, in this regime, the factor ε^{-1} in the expression of F_ε which is therefore more singular than in Section 3.4. To overcome this difficulty, we follow [Han20] and introduce the *higher* fluid-kinetic dissipation. This section is an adaptation of [Han20, Section 4] which tracks down the dependence with respect to ε and which is detailed for the sake of completeness.

Definition 3.5.2. *Let* $r \geq 2$. *The higher fluid-kinetic dissipation (of order* r*) is defined, for almost every* $t \geq 0$, *by*

$$\mathrm{D}_\varepsilon^{(r)}(t) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon \frac{|v - u_\varepsilon(t,x)|^r}{\varepsilon^r} \mathrm{d}x\mathrm{d}v.$$

In the rest of this section, we work under Assumption 3.1, and consider a strong existence time $T > 0$.

The following lemma ensures that a control of $\mathrm{D}_\varepsilon^{(r)}$ implies estimates for F_ε in L^r.

Lemma 3.5.3. *For $r \geq 2$, we have*

$$\|F_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)} \leq \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{r}} \left(\int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\,\mathrm{d}t\right)^{\frac{1}{r}}.$$

▷ This is a direct consequence of Hölder's inequality and Corollary 3.3.24, since T is a strong existence time :

$$\begin{aligned}\|F_\varepsilon(t)\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^r &\leq \int_0^T \int_{\mathbb{T}^3} \rho_\varepsilon(t,x)^{r-1} \left(\int_{\mathbb{R}^3} \frac{|v-u_\varepsilon|^r}{\varepsilon^r} f_\varepsilon(t,x,v)\mathrm{d}v\right)\mathrm{d}x\mathrm{d}t \\ &\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{r-1} \int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\,\mathrm{d}t,\end{aligned}$$

which corresponds to the desired estimate. □

The key to desingularize $\mathrm{D}_\varepsilon^{(r)}$ is the following identity, obtained by using the method of characteristics and integration by parts.

Lemma 3.5.4. *For all $r \geq 2$, the following identity holds.*

$$\begin{aligned}\int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\mathrm{d}t = -\frac{1}{r}\left[\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\frac{|v-u_\varepsilon(t,x)|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v\right]_0^T \\ -\int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)(\partial_t u_\varepsilon + (\nabla_x u_\varepsilon)v)\cdot\frac{(v-u_\varepsilon(t,x))|v-u_\varepsilon(t,x)|^{r-2}}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v\mathrm{d}t.\end{aligned}$$

▷ By the method of characteristics and a change of variables,

$$\begin{aligned}\mathrm{D}_\varepsilon^{(r)}(t) &= \varepsilon^{-r}e^{\frac{3t}{\varepsilon}}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(X_\varepsilon(0;t,x,v),V_\varepsilon(0;t,x,v))|v-u_\varepsilon(t,x)|^r\mathrm{d}x\mathrm{d}v \\ &= \varepsilon^{-r}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v)|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t,0,x,v))|^r\mathrm{d}x\mathrm{d}v.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^r \\ &\quad= r\frac{\mathrm{d}}{\mathrm{d}t}\left(V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right)\cdot\left(V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right) \\ &\qquad\times|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^{r-2} \\ &\quad= r\left(\frac{u_\varepsilon(t,X_\varepsilon(t;0,x,v))-V_\varepsilon(t;0,x,v)}{\varepsilon}-\frac{\mathrm{d}}{\mathrm{d}t}\left(u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right)\right) \\ &\qquad\cdot\left(V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right)|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^{r-2}.\end{aligned}$$

This yields

$$
\begin{aligned}
&\frac{1}{\varepsilon}|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^r \\
&\qquad = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}|V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^r \\
&\qquad\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left(u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right)\cdot\left(V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))\right) \\
&\qquad\qquad \times |V_\varepsilon(t;0,x,v)-u_\varepsilon(t,X_\varepsilon(t;0,x,v))|^{r-2}
\end{aligned}
$$

and the result follows after integrating by parts. □

Once in this form, the control of $\mathrm{D}_\varepsilon^{(r)}$ is a consequence of the following estimates, which are straightforward applications of Hölder's inequality.

Lemma 3.5.5. *For all $r \geq 2$, we have the following estimates*

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\partial_t u_\varepsilon\cdot\frac{(v-u_\varepsilon(t,x))|v-u_\varepsilon(t,x)|^{r-2}}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v\mathrm{d}t\right| \\
&\qquad\qquad \lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{\frac{1}{r}}\|\partial_t u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}\left(\int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\mathrm{d}t\right)^{1-\frac{1}{r}}, \\
&\left|\int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)(\nabla_x u_\varepsilon)v)\cdot\frac{(v-u_\varepsilon(t,x))|v-u_\varepsilon(t,x)|^{r-2}}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v\mathrm{d}t\right| \\
&\qquad\qquad \lesssim \left\||\nabla_x u_\varepsilon|m_{r,\varepsilon}^{\frac{1}{r}}\right\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}\left(\int_0^T \mathrm{D}_\varepsilon^{(r)}(t)\mathrm{d}t\right)^{1-\frac{1}{r}},
\end{aligned}
$$

where

$$
m_{r,\varepsilon} := \int_{\mathbb{R}^3} f_\varepsilon|v|^r\mathrm{d}v.
$$

A combination of the two results leads, thanks to Young's inequality, to the following lemma.

Lemma 3.5.6. *The higher fluid-kinetic dissipation of order $r \geq 2$ satisfies, for almost every $t \in (0,T)$,*

$$
\begin{aligned}
&\int_0^t \mathrm{D}_\varepsilon^{(r)}(s)\mathrm{d}s + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\frac{|v-u_\varepsilon(t,x)|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v \\
&\qquad\qquad \lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\|\partial_t u_\varepsilon\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r \\
&\qquad\qquad\qquad + \left\||\nabla_x u_\varepsilon|m_{r,\varepsilon}^{\frac{1}{r}}\right\|_{\mathrm{L}^r((0,T)\times\mathbb{T}^3)}^r + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0\frac{|v-u_\varepsilon^0|^r}{\varepsilon^{r-1}}\mathrm{d}x\mathrm{d}v.
\end{aligned}
$$

We now apply this estimate with $r = p$, the regularity index appearing in Assumption 3.1. The first term will be absorbed in the parabolic estimate, under a smallness condition on f_ε^0 and the last term will lead us to assume well-preparedness of the initial data. We still have to deal with the middle one.

Lemma 3.5.7. *Let $T>0$ be a strong existence time. The following estimate holds*

$$\begin{aligned}\left\| |\nabla_x u_\varepsilon| m_{p,\varepsilon}^{\frac{1}{p}} \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} &\lesssim \varepsilon^{1-\alpha_p} \left\| f_\varepsilon^0 |v|^p \right\|^{\frac{1}{1-\alpha_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3))} \mathscr{E}_\varepsilon(0)^{\frac{p}{2}} \\ &+ \left\| f_\varepsilon^0 \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}} \Psi_{\varepsilon,0}^{\frac{p}{2(1-\beta_p)}} \\ &+ \left(\varepsilon^{1-\alpha_p} + \left\| f_\varepsilon^0 \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}} \right) \left\| \Delta_x u_\varepsilon \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.\end{aligned}$$

$\triangleright$ The method of characteristics yields

$$\begin{aligned}&\left\| |\nabla_x u_\varepsilon| m_{p,\varepsilon}^{\frac{1}{p}} \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\qquad = \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^p e^{\frac{3t}{\varepsilon}} f_\varepsilon^0(X_\varepsilon(0;t,x,v), V_\varepsilon(0;t,x,v)) |v|^p \mathrm{d}x\mathrm{d}v\mathrm{d}t\end{aligned}$$

so that by the change of variables in velocity of Lemma 3.3.22, we get

$$\begin{aligned}&\left\| |\nabla_x u_\varepsilon| m_{p,\varepsilon}^{\frac{1}{p}} \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\qquad = \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^p e^{\frac{3t}{\varepsilon}} f_\varepsilon^0(\widetilde{X}_{t,x,w}(0), w) \left| [\Gamma_\varepsilon^{t,x}]^{-1}(w) \right|^p \\ &\qquad\qquad \times |\det \nabla_w \Gamma_\varepsilon^{t,x}(w)|^{-1} \mathrm{d}x\mathrm{d}w\mathrm{d}t \\ &\qquad \lesssim \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^p f_\varepsilon^0(\widetilde{X}_\varepsilon^{t,x,w}(0), w) \left| [\Gamma_\varepsilon^{t,x}]^{-1}(w) \right|^p \mathrm{d}x\mathrm{d}w\mathrm{d}t.\end{aligned}$$

By integration of the characteristics equation on velocity (3.3.3), we have

$$[\Gamma_\varepsilon^{t,x}]^{-1}(w) = e^{-\frac{t}{\varepsilon}} w + \frac{1}{\varepsilon} \int_0^t e^{\frac{s-t}{\varepsilon}} u_\varepsilon(s, \widetilde{X}_\varepsilon^{t,x,w}(s)) \mathrm{d}s.$$

Therefore,

$$\begin{aligned}\left\| |\nabla_x u_\varepsilon| m_{p,\varepsilon}^{\frac{1}{p}} \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} &\lesssim \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^p f_\varepsilon^0(\widetilde{X}_\varepsilon^{t,x,w}(0), w) e^{-\frac{pt}{\varepsilon}} |w|^p \mathrm{d}x\mathrm{d}v\mathrm{d}t \\ + \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^p f_\varepsilon^0(\widetilde{X}_\varepsilon^{t,x,w}(0), w) &\left(\frac{1}{\varepsilon} \int_0^t e^{\frac{s-t}{\varepsilon}} \left| u_\varepsilon(s, \widetilde{X}_\varepsilon^{t,x,w}(s)) \right| \mathrm{d}s \right)^p \mathrm{d}x\mathrm{d}w\mathrm{d}t \\ &=: I_1 + I_2.\end{aligned}$$

On the one hand, thanks to Corollary 3.3.21 and Hölder's and Young's inequalities,

$$\begin{aligned}I_1 &\leq \left\| f_\varepsilon^0 |v|^p \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \int_0^T e^{-\frac{pt}{\varepsilon}} \left\| \nabla_x u_\varepsilon(t) \right\|^p_{\mathrm{L}^p(\mathbb{T}^3)} \mathrm{d}t \\ &\lesssim \left\| f_\varepsilon^0 |v|^p \right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_\varepsilon(0)^{\frac{(1-\alpha_p)p}{2}} \int_0^T e^{-\frac{pt}{\varepsilon}} \left\| \Delta_x u_\varepsilon(t) \right\|^{\alpha_p p}_{\mathrm{L}^p(\mathbb{T}^3)} \mathrm{d}t \\ &\lesssim \varepsilon^{1-\alpha_p} \left\| f_\varepsilon^0 |v|^p \right\|^{\frac{1}{1-\alpha_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3))} \mathscr{E}_\varepsilon(0)^{\frac{p}{2}} + \varepsilon^{1-\alpha_p} \left\| \Delta_x u_\varepsilon \right\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.\end{aligned}$$

On the other hand, applying Jensen's inequality and Lemma 3.3.26,

$$\begin{aligned}
I_2 &\lesssim \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3}\int_0^t |\nabla_x u_\varepsilon(t,x)|^p \left|u_\varepsilon\left(s,\widetilde{X}_\varepsilon^{t,x,w}(s)\right)\right|^p f_\varepsilon^0\left(\widetilde{X}_\varepsilon^{t,x,w}(0),w\right) e^{\frac{s-t}{\varepsilon}}\mathrm{d}s\mathrm{d}x\mathrm{d}w\mathrm{d}t \\
&\lesssim \frac{1}{\varepsilon}\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\int_0^T\int_0^t \|\nabla_x u_\varepsilon(t)\|^p_{\mathrm{L}^\infty(\mathbb{T}^3)}\|u_\varepsilon(s)\|^p_{\mathrm{L}^p(\mathbb{T}^3)} e^{\frac{s-t}{\varepsilon}}\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Recall that Corollaries 3.3.11 and 3.3.21 ensure that for every $s\in(0,T)$,

$$\|u_\varepsilon(s)\|_{\mathrm{L}^p(\mathbb{T}^3)}\lesssim \Psi_{\varepsilon,0}^{\frac{1}{2}},$$

and for every $t\in(0,T)$,

$$\|\nabla_x u_\varepsilon(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\lesssim \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} e^{-\frac{(1-\beta_p)\lambda_\varepsilon}{2}t}\|\Delta_x u_\varepsilon(t)\|^{\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)}.$$

Therefore, Hölder's and Young's inequalities yield

$$\begin{aligned}
I_2 &\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\Psi_{\varepsilon,0}^{\frac{p}{2}}\mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}}\|\Delta_x u_\varepsilon\|^{\beta_p p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}}\Psi_{\varepsilon,0}^{\frac{p}{2(1-\beta_p)}} + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3\mathrm{L}^\infty(\mathbb{T}^3))}\mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}}\|\Delta_x u_\varepsilon\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)},
\end{aligned}$$

which concludes the proof. □

In conclusion, we have obtained the following estimate.

Lemma 3.5.8. *Under Assumption 3.1, if $T>0$ is a strong existence time, then*

$$\begin{aligned}
\int_0^T \mathrm{D}_\varepsilon^{(p)}(t)\mathrm{d}t &+ \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\frac{|v-u_\varepsilon(t,x)|^p}{\varepsilon^{p-1}}\mathrm{d}x\mathrm{d}v \\
&\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\|\partial_t u_\varepsilon\|^p_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\quad + \left(\varepsilon^{1-\alpha_p} + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3))}\mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}}\right)\|\Delta_x u_\varepsilon\|^p_{\mathrm{L}^p(0,T)\times\mathbb{T}^3)} \\
&\quad + \varepsilon^{1-\alpha_p}\left\|f_\varepsilon^0|v|^p\right\|^{\frac{1}{1-\alpha_p}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\mathscr{E}_\varepsilon(0)^{\frac{p}{2}} \\
&\quad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)p}{2}}\Psi_{\varepsilon,0}^{\frac{p}{2(1-\beta_p)}} \\
&\quad + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v)\frac{|v-u_\varepsilon^0(x)|^p}{\varepsilon^{p-1}}\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

3.5.2 Conclusion of the bootstrap argument in the well-prepared case

This section is the analogue of Section 3.4.2 for the *light* and *light and fast* particle regimes. Recall that, as is now usual, we consider, under Assumption 3.1, for every $\varepsilon>0$,

$$T_\varepsilon^* = \sup\{T>0,\ T \text{ is a strong existence time}\}.$$

The aim of this Subsection is to prove that under Assumption 3.2 and additional well-preparedness assumptions on the initial data, we have $T_\varepsilon^* = +\infty$, at least for ε small enough.

We need only prove that, for every $T < T_\varepsilon^*$,

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \le \frac{1}{40} \qquad \text{and} \qquad \|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \le \frac{C^*}{4}.$$

Indeed, as seen from the proof of Lemma 3.3.13, Assumption 3.2 ensures that for all $T \ge 0$,

$$\int_0^T \left\|e^{t\Delta} u_\varepsilon\right\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}^4 \mathrm{d}t \le \frac{C^*}{4}.$$

Then, thanks to the timewise continuity of the norms involved, we deduce that $T_\varepsilon^* = +\infty$.

We begin by proving the following lemma.

Lemma 3.5.9. *Under Assumption 3.1, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, if*

$$\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \le \eta \tag{3.5.1}$$

and

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} \frac{|v - u_\varepsilon^0(x)|^p}{\varepsilon^{p-1}} f_\varepsilon^0(x,v)\mathrm{d}x\mathrm{d}v \le M, \tag{3.5.2}$$

then for any $T < T_\varepsilon^*$,

$$\|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_p}$$

for some $\omega_p > 0$.

$\triangleright$ Thanks to Theorem E.2, we have

$$\begin{aligned}&\|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\qquad\qquad \lesssim \|F_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|(u_\varepsilon\cdot\nabla_x)u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|u_\varepsilon^0\right\|_{\mathrm{B}_p^{s,p}(\mathbb{T}^3)}.\end{aligned}$$

Combining the estimates in Lemmas 3.3.29, 3.5.3, and 3.5.8, we get

$$\begin{aligned}
&\|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\quad\lesssim \|F_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|(u_\varepsilon\cdot\nabla_x)u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \left\|u_\varepsilon^0\right\|_{\mathrm{B}_p^{s,p}(\mathbb{T}^3)} \\
&\quad\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\qquad + \left(\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{p}} \varepsilon^{\frac{1-\alpha_p}{p}} + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{R}^3))} \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}}\right) \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p(0,T)\times\mathbb{T}^3)} \\
&\qquad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{p}} \varepsilon^{\frac{1-\alpha_p}{p}} \left\|f_\varepsilon^0|v|^p\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{\frac{1}{p(1-\alpha_p)}} \mathscr{E}_\varepsilon(0)^{\frac{1}{2}} \\
&\qquad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} \Psi_{\varepsilon,0}^{\frac{1}{2(1-\beta_p)}} \\
&\qquad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{p}} \left\|f_\varepsilon^0(x,v)\frac{|v-u_\varepsilon^0(x)|^p}{\varepsilon^{p-1}}\right\|_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)}^{\frac{1}{p}} \\
&\qquad + \Psi_{\varepsilon,0}^{\frac{1}{2}} \mathscr{E}_\varepsilon(0)^{\frac{(1-\beta_p)}{2}} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^{\beta_p} + \left\|u_\varepsilon^0\right\|_{\mathrm{B}_p^{s,p}(\mathbb{T}^3)}.
\end{aligned}$$

We can use the smallness condition (3.5.1), apply Young's inequality and, thanks to Assumption 3.1, we find that there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, if (3.5.2) holds, then

$$\begin{aligned}
&\|\partial_t u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\
&\qquad \lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{p}} \varepsilon^{\frac{1-\alpha_p}{p}} M^{\omega_p} + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} M^{\omega_p} \\
&\qquad\quad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{1-\frac{1}{p}} M^{\omega_p} + M^{\omega_p}
\end{aligned}$$

for some $\omega_p > 0$, hence the result. □

Remark 3.5.10. *The well-preparedness assumption* (3.5.2) *implies the following convergence rate*

$$W_1(f_\varepsilon^0, \rho_\varepsilon^0 \otimes \delta_{v=u_\varepsilon^0}) \lesssim M\varepsilon^{1-\frac{1}{p}}.$$

We have the following immediate consequence.

Corollary 3.5.11. *Under Assumption 3.1, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, if* (3.5.1)–(3.5.2) *hold and if*

$$\mathscr{E}_\varepsilon(0) \leq \eta, \tag{3.5.3}$$

then for any $T < T_\varepsilon^*$,

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \frac{1}{40}.$$

▷ Since $p > 3$, we can apply Corollary 3.3.21 and Lemma 3.5.9 and find that there exist ε_0 and $\eta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, if (3.5.1)–(3.5.2) hold then, for some $\omega_p > 0$,

$$\int_0^T \|\nabla_x u_\varepsilon(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \,\mathrm{d}t \lesssim \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^{\beta_p} \lesssim \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} M^{\omega_p},$$

which can be made as small as necessary by reducing the value of $\eta > 0$ in (3.5.3). □

There remains to verify that

$$\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}.$$

We cannot use the exact same strategy as in Lemma 3.5.8 because the Gagliardo-Nirenberg inequality does not apply exactly the same way, but we are able to prove the following result.

Lemma 3.5.12. *Under Assumption 3.1, for every $T < T_\varepsilon^*$,*

$$\begin{aligned}
\int_0^T \mathrm{D}_\varepsilon^{(2)}(t)\mathrm{d}t &\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}^2 + \varepsilon^{1-\alpha_2} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}^2 \\
&\quad + \varepsilon^{1-\alpha_2} \left\|f_\varepsilon^0 |v|^2\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{\frac{1}{1-\alpha_2}} \mathscr{E}_\varepsilon(0) \\
&\quad + \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi_{\varepsilon,0} \mathscr{E}_\varepsilon(0)^{1-\beta_p} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}^{2\beta_p} \\
&\quad + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v) \frac{|v - u_\varepsilon^0(x)|^2}{\varepsilon} \mathrm{d}x\mathrm{d}v.
\end{aligned}$$

▷ Applying Lemma 3.5.6 with $r = 2$, we obtain

$$\int_0^T \mathrm{D}_\varepsilon^{(2)}(t)\mathrm{d}t \lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\partial_t u_\varepsilon\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \left\||\nabla_x u_\varepsilon| m_2^{\frac{1}{2}}\right\|^2_{\mathrm{L}^2((0,T)\times\mathbb{R}^3)} + \left\|f_\varepsilon^0 \frac{|v-u_\varepsilon^0|^2}{\varepsilon}\right\|_{\mathrm{L}^1(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Following the proof of Lemma 3.5.7, we have

$$\begin{aligned}
&\left\||\nabla_x u_\varepsilon| m_2^{\frac{1}{2}}\right\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\\
&\lesssim \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^2 f_\varepsilon^0\left(\widetilde{X}_\varepsilon^{t,x,w}(0),w\right) e^{-\frac{t}{\varepsilon}}|w|^2 \mathrm{d}x\mathrm{d}w\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3} |\nabla_x u_\varepsilon(t,x)|^2 f_\varepsilon^0\left(\widetilde{X}_\varepsilon^{t,x,w}(0),w\right)\left(\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}\left|u_\varepsilon\left(s,\widetilde{X}_\varepsilon^{t,x,w}(s)\right)\right|\mathrm{d}s\right)^2 \mathrm{d}x\mathrm{d}w\mathrm{d}t\\
&\lesssim I_1 + I_2,
\end{aligned}$$

with

$$I_1 \lesssim \varepsilon^{1-\alpha_2}\left\|f_\varepsilon^0|v|^2\right\|^{\frac{1}{1-\alpha_2}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\mathscr{E}_\varepsilon(0) + \varepsilon^{1-\alpha_2}\|\Delta_x u_\varepsilon\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}.$$

To control I_2, we need to rely on Lemma 3.5.9. Thanks to Jensen's inequality the Gagliardo-Nirenberg theorem, we have

$$\begin{aligned}
I_2 &\lesssim \varepsilon^{-1}\int_0^T \int_{\mathbb{T}^3\times\mathbb{R}^3}\int_0^t e^{\frac{s-t}{\varepsilon}}|\nabla_x u_\varepsilon(t,x)|^2 f_\varepsilon^0\left(\widetilde{X}_\varepsilon^{t,x,w}(0),w\right)\\
&\qquad\qquad \times\left|u_\varepsilon\left(s,\widetilde{X}_{x,t,w}(s)\right)\right|^2\mathrm{d}s\mathrm{d}x\mathrm{d}w\mathrm{d}t\\
&\lesssim \varepsilon^{-1}\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\int_0^T\int_0^t e^{\frac{s-t}{\varepsilon}}\|\nabla_x u_\varepsilon(t)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)}\|u_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\mathrm{d}s\mathrm{d}t\\
&\lesssim \varepsilon^{-1}\left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\Psi_{\varepsilon,0}\,\mathscr{E}_\varepsilon(0)^{1-\beta_p}\\
&\qquad\qquad \times\int_0^T\int_0^t e^{\frac{s-t}{\varepsilon}}e^{(1-\beta_p)\lambda_\varepsilon t}\|\Delta_x u_\varepsilon(t)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)}\mathrm{d}s\mathrm{d}t\\
&\lesssim \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}\Psi_{\varepsilon,0}\,\mathscr{E}_\varepsilon(0)^{1-\beta_p}\|\Delta_x u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)}.
\end{aligned}$$

Gathering all pieces together, we obtain the claimed estimate. □

Therefore, the L^2 parabolic estimate can be written as follows.

Lemma 3.5.13. *Under Assumption 3.1, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, if* (3.5.1)–(3.5.2) *are satisfied, then for any $T < T^*$,*

$$\|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \|\Delta_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim M^{\omega_2},$$

for some $\omega_2 > 0$.

▷ Thanks to Theorem E.2, we have

$$\begin{aligned}\|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &+ \|\Delta_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \\ &\lesssim \|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \|(u_\varepsilon\cdot\nabla_x)u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \left\|u_\varepsilon^0\right\|_{\mathrm{H}^1(\mathbb{T}^3)}. \quad (3.5.4)\end{aligned}$$

On the one hand, combining Lemmas 3.5.3 and 3.5.12, we get

$$\begin{aligned}\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim\ & \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \\ &+ \varepsilon^{\frac{1-\alpha_2}{2}} \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{R}^3)} \\ &+ \varepsilon^{\frac{1-\alpha_2}{2}} \left\|f_\varepsilon^0\right\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left\|f_\varepsilon^0|v|^2\right\|^{\frac{1}{2(1-\alpha_2)}}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \mathscr{E}_\varepsilon(0)^{\frac12} \\ &+ \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \Psi_{\varepsilon,0}^{\frac12}\mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_\varepsilon\|^{\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &+ \left\|f_\varepsilon^0\right\|^{\frac12}_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v)\frac{|v-u_\varepsilon^0(x)|^2}{\varepsilon}\mathrm{d}x\mathrm{d}v\right)^{\frac12}.\end{aligned}$$

Since

$$\begin{aligned}\left\|f_\varepsilon^0|v|^2\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))} &\le \int_{|v|\le1}\left\|f_\varepsilon^0(v)\right\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathrm{d}v + \int_{|v|\ge1}\left\|f_\varepsilon^0(v)\right\|_{\mathrm{L}^\infty(\mathbb{T}^3)}|v|^p\mathrm{d}v \\ &\lesssim \eta + M,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v)\frac{|v-u_\varepsilon^0(x,v)|^2}{\varepsilon}\mathrm{d}x\mathrm{d}v &\le \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0(x,v)\mathrm{d}x\mathrm{d}v\right)^{1-\frac2p}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0\frac{|v-u_\varepsilon^0|^p}{\varepsilon^{\frac p2}}\mathrm{d}x\mathrm{d}v\right)^{\frac2p} \\ &\le \eta^{1-\frac2p}\varepsilon^{1-\frac2p}\left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0\frac{|v-u_\varepsilon^0|^p}{\varepsilon^{p-1}}\mathrm{d}x\mathrm{d}v\right)^{\frac2p} \lesssim \eta^{1-\frac2p}M^{\frac2p}\varepsilon^{1-\frac2p},\end{aligned}$$

we have, thanks to Lemma 3.5.9, for ε and η small enough,

$$\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim \eta\,\|\partial_t u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \eta\varepsilon^{\frac{1-\alpha_2}{2}}\|\Delta_x u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} + \varepsilon^{\omega_2}\eta^{\omega_2'}M^{\omega_2''}, \quad (3.5.5)$$

for some $\omega_2, \omega_2', \omega_2'' > 0$.

On the other hand, thanks to Hölder's inequality and Lemmas 3.3.20–3.3.21 and 3.5.9, we have, for ε and η small enough,

$$\begin{aligned}\|(u_\varepsilon\cdot\nabla)u_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} &\lesssim \|u_\varepsilon\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))}\int_0^T\|\nabla_x u_\varepsilon(t)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathrm{d}t \\ &\lesssim \Psi_{\varepsilon,0}\mathscr{E}_\varepsilon(0)^{1-\beta_p}\int_0^T\|\Delta_x u_\varepsilon(t)\|^{2\beta_p}_{\mathrm{L}^p(\mathbb{T}^3)}e^{-(1-\beta_p)\lambda_\varepsilon t}\mathrm{d}t \\ &\lesssim \Psi_{\varepsilon,0}\mathscr{E}_\varepsilon(0)^{1-\beta_p}\|\Delta_x u_\varepsilon\|^{2\beta_p}_{\mathrm{L}^p((0,T)\times\mathbb{T}^3)} \\ &\lesssim \eta^{1-\beta_p}M^{\omega_p},\end{aligned}$$

for some $\omega_p > 0$. Hence the result. □

We inject the estimate of Lemma 3.5.13 into the estimate (3.5.5) that was obtained in the course the proof to find that if η and ε are small enough, then

$$\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4},$$

and the bootstrap argument is finally complete.

3.5.3 Non quantitative convergence in the well-prepared case

In the following paragraphs, we will consider a fixed time horizon $T > 0$ and study the convergence of the sequences $(f_\varepsilon)_{\varepsilon>0}$, $(\rho_\varepsilon)_{\varepsilon>0}$ and $(u_\varepsilon)_{\varepsilon>0}$ under several sets of convergence assumptions for the initial data.

Our first result only requires the weak convergence of the fluid velocity and the particle density, but the greater singularity of the Brinkman force prevents us from finding explicit convergence rates as in Section 3.4.

Theorem 3.5.14. *Under Assumptions 3.1–3.2, there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that if, for all $\varepsilon \in (0, \varepsilon_0)$, (3.5.1)–(3.5.3) hold and if*

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0 \text{ in } w\text{-}\mathrm{L}^2(\mathbb{T}^3) \qquad \text{and} \qquad \rho_\varepsilon^0 \xrightarrow[\varepsilon\to 0]{} \rho^0 \text{ in } w^*\text{-}\mathrm{L}^\infty(\mathbb{T}^3),$$

then $(u_\varepsilon)_{\varepsilon>0}$ converges to u in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$, $(\rho_\varepsilon)_{\varepsilon>0}$ converges weakly-$*$ to ρ in $\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$ where (ρ, u) satisfies, with $\widetilde{\rho} = 1 + \rho$,

$$\begin{cases} \partial_t\widetilde{\rho} + \mathrm{div}_x(\widetilde{\rho}u) = 0, \\ \widetilde{\rho}|_{t=0} = 1 + \rho^0, \\ \partial_t(\widetilde{\rho}u) + \mathrm{div}_x(\widetilde{\rho}u \otimes u) - \Delta_x u + \nabla_x p = 0, \\ \mathrm{div}_x\, u = 0, \\ u|_{t=0} = u^0. \end{cases}$$

Furthermore, for almost all $t \in (0,T)$,

$$W_1(\rho_\varepsilon(t), \rho(t)) \lesssim W_1(\rho_\varepsilon^0, \rho^0) + \|u_\varepsilon - u\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} + \varepsilon \xrightarrow[\varepsilon\to 0]{} 0, \tag{3.5.6}$$

and $(f_\varepsilon)_{\varepsilon>0}$ converges to $\rho \otimes \delta_{v=u}$ in the sense that

$$\int_0^T W_1(f_\varepsilon(t), \rho(t) \otimes \delta_{v=u(t)})\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0.$$

$\triangleright$ The energy dissipation estimate (3.3.2) first shows that the sequence (u_ε) is uniformly bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^3))$. Therefore, up to a subsequence that we shall not write, (u_ε) converges to some u in w-$\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^3))$. Furthermore, since we have proven in the previous section that all times are strong existence times and thanks to Lemma 3.5.13, we can apply the Aubin-Lions lemma and obtain the strong convergence of (u_ε) to u in $\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

Moreover, the fact that T is a strong existence time enables us to apply Corollary 3.3.24, which ensures that (ρ_ε) is bounded in $\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$, so that it converges, up to a subsequence, to some ρ in w^*-$\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)$. Therefore, $(\rho_\varepsilon u_\varepsilon)$ converges to ρu in w-$\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

Contrary to the *light* and *light and fast* regimes, we cannot state that the Brinkman force F_ε converges to 0. Yet, thanks to Lemmas 3.5.3, 3.5.8 and 3.5.9, (F_ε) is bounded and therefore converges, up to a subsequence, to some F in w-$\mathrm{L}^2((0,T)\times\mathbb{T}^3)$.

Finally, (j_ε) also converges to ρu in w-$\mathrm{L}^2((0,T)\times\mathbb{T}^3)$. Indeed, for any $\varphi\in\mathrm{L}^2((0,T)\times\mathbb{T}^3)$, thanks to Hölder's inequality and the modulated energy–dissipation estimate from Lemma 3.3.20, we have, for $\varepsilon\in(0,\varepsilon_0)$,

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}^3}(j_\varepsilon(t,x)-\rho_\varepsilon(t,x)u_\varepsilon(t,x))\varphi(t,x)\mathrm{d}x\mathrm{d}t\right| &\leq \int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v-u_\varepsilon(t,x)||\varphi(t,x)|\mathrm{d}x\mathrm{d}v\mathrm{d}t\\
&\leq \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{\frac12}\|\varphi\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\left(\int_0^T\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v-u_\varepsilon(t,x)|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t\right)^{\frac12}\\
&\leq \left\|f_\varepsilon^0\right\|_{\mathrm{L}^1(\mathbb{R}^3;\mathrm{L}^\infty(\mathbb{T}^3))}^{\frac12}\|\varphi\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\,\varepsilon^{\frac12}\mathscr{E}_\varepsilon(0)^{\frac12}\\
&\lesssim_M \sqrt{\varepsilon}\,\eta\,\|\varphi\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}\,.\end{aligned}$$

We can take the limit $\varepsilon\to 0$ in the conservation of mass (3.3.4) and obtain

$$\partial_t\rho+\mathrm{div}_x(\rho u)=0.$$

Let us now take the limit $\varepsilon\to 0$ in the conservation of momentum (3.3.5)

$$\partial_t j_\varepsilon+\mathrm{div}_x\left(\int_{\mathbb{R}^3} f_\varepsilon\, v\otimes v\,\mathrm{d}v\right)=-F_\varepsilon.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^3} f_\varepsilon v\otimes f\mathrm{d}v &= \int_{\mathbb{R}^3} f_\varepsilon(v-u_\varepsilon)\otimes(v-u_\varepsilon)\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^3} f_\varepsilon(v\otimes u_\varepsilon+u_\varepsilon\otimes v)\mathrm{d}v-\int_{\mathbb{R}^3} f_\varepsilon u_\varepsilon\otimes u_\varepsilon\mathrm{d}v\\
&:= I_{\varepsilon,1}+I_{\varepsilon,2}+I_{\varepsilon,3}.\end{aligned}$$

Thanks to the modulated energy estimate from Lemma 3.3.20,

$$\int_0^T\int_{\mathbb{T}^3}|I_{\varepsilon,1}|\lesssim\int_0^T\int_{\mathbb{T}^3\mathbb{R}^3} f_\varepsilon|v-u_\varepsilon|^2\mathrm{d}v\xrightarrow[\varepsilon\to0]{}0.$$

Furthermore, we can write

$$\begin{aligned}I_{\varepsilon,2}-2\rho u\otimes u &= j_\varepsilon\otimes u_\varepsilon-\rho u\otimes u+u_\varepsilon\otimes j_\varepsilon-\rho u\otimes u\\
&=(j_\varepsilon-\rho u)\otimes u+j_\varepsilon\otimes(u_\varepsilon-u)+u_\varepsilon\otimes(j_\varepsilon-\rho u)+\rho(u_\varepsilon-u)\otimes u,\end{aligned}$$

so that we can prove that $(I_{\varepsilon,2})_{\varepsilon>0}$ converges to $2\rho\, u\otimes u$ in the distribution sense thanks to the previous bounds, weak convergences and strong convergence of $(u_\varepsilon)_{\varepsilon>0}$. Finally

$$I_{\varepsilon,3}+\rho u\otimes u=(\rho-\rho_\varepsilon)\,u\otimes u+\rho_\varepsilon(u_\varepsilon-u)\otimes u,$$

so $(I_{\varepsilon,3})_{\varepsilon>0}$ converges to $-\rho\, u\otimes u$ in the distribution sense. Therefore, we have proven, in the distribution sense,

$$\partial_t(\rho u)+\mathrm{div}_x(\rho\, u\otimes u)=-F. \tag{3.5.7}$$

Thanks to the strong convergence of $(u_\varepsilon)_{\varepsilon>0}$ and the weak convergence of $(F_\varepsilon)_{\varepsilon>0}$ to F, we can also take the limit in the Navier–Stokes equations to obtain that u satisfies

$$\partial_t u + \operatorname{div}_x(u \otimes u) - \Delta_x u + \nabla_x p = F. \tag{3.5.8}$$

Summing (3.5.7) and (3.5.8) leads to

$$\partial_t(\widetilde{\rho} u) + \operatorname{div}_x(\widetilde{\rho} u \otimes u) - \Delta_x u + \nabla_x p = 0,$$

where $\widetilde{\rho} = 1 + \rho$.

We conclude that no subsequence is required by the same weak compactness and uniqueness argument as at the end of of the proof Theorem 3.4.12. The fact that the limit system has a unique solution follows from [PZZ13].

Let us now prove the desired convergence for (ρ_ε). Consider

$$\chi_\varepsilon = \rho - \rho_\varepsilon.$$

We have

$$\partial_t \chi_\varepsilon = -\operatorname{div}_x(\rho u) + \operatorname{div}_x j_\varepsilon = -\operatorname{div}_x((\rho - \rho_\varepsilon)u) - \operatorname{div}_x(\rho_\varepsilon u) + \operatorname{div}_x j_\varepsilon$$

so that χ_ε is a solution to the transport equation with source

$$\partial \chi_\varepsilon + \operatorname{div}_x(\chi_\varepsilon u) = -\operatorname{div}_x J_\varepsilon,$$

where $J_\varepsilon = \rho_\varepsilon(u - u_\varepsilon) - \varepsilon F_\varepsilon$. Following the proof of Theorem 3.4.13, we obtain that, for almost every $t \in (0, T)$,

$$\begin{aligned} W_1(\rho_\varepsilon(t), \rho(t)) &\lesssim W_1(\rho_\varepsilon^0, \rho^0) + \|u_\varepsilon - u\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} + \varepsilon \|F_\varepsilon\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} \\ &\lesssim W_1(\rho_\varepsilon^0, \rho^0) + \|u_\varepsilon - u\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} + \varepsilon, \end{aligned}$$

since T is a strong existence time.

To prove the convergence of $(f_\varepsilon)_{\varepsilon>0}$, we first note that, for any $\varphi \in \mathscr{C}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ such that $\|\nabla_{x,v}\varphi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leq 1$, thanks to Lemma 3.3.20, for almost every $t \in (0, T)$,

$$\begin{aligned} |\langle f_\varepsilon(t) - \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)}, \varphi\rangle| &\leq \|\nabla_{x,v}\varphi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v - u_\varepsilon(t,x)|\mathrm{d}x \\ &\leq \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v - u_\varepsilon(t,x)|\mathrm{d}x\mathrm{d}x \end{aligned}$$

so that, for almost every $t \in (0, T)$,

$$W_1(f_\varepsilon(t), \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)}) \leq \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v - u_\varepsilon(t,x)|\mathrm{d}x,$$

and thus by the modulated energy–dissipation inequality this yields

$$\int_0^T W_1(f_\varepsilon(t), \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)})\mathrm{d}t \leq \sqrt{\varepsilon}\mathscr{E}_\varepsilon(0)^{\frac{1}{2}}\sqrt{T}.$$

Furthermore, for almost every $t \in (0,T)$,

$$|\langle \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)} - \rho(t) \otimes \delta_{v=u(t)}, \varphi\rangle| \\ \lesssim \left| \int_{\mathbb{T}^3} (\rho_\varepsilon(t,x) - \rho(t,x))\varphi(x, u_\varepsilon(t,x))\mathrm{d}x \right| + \|\rho(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}.$$

For almost every $t \in (0,T)$, we have the bound

$$\left| \int_{\mathbb{T}^3} (\rho_\varepsilon(t,x) - \rho(t,x))\varphi(x, u_\varepsilon(t,x))\mathrm{d}x \right| \leq \left(1 + \|\nabla_x u_\varepsilon(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right) W_1(\rho_\varepsilon(t), \rho(t)).$$

and thus it will be treated thanks to (3.5.6). We eventually obtain

$$\int_0^T W_1(\rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)}, \rho(t) \otimes \delta_{v=u(t)})\mathrm{d}t \lesssim_M \int_0^T W_1(\rho_\varepsilon(t), \rho(t))\mathrm{d}t + \|u_\varepsilon(t) - u(t)\|^2_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}.$$

This leads to the expected convergence for (f_ε).

□

Assume now that

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0 \text{ in } \mathrm{L}^2(\mathbb{T}^3). \tag{3.5.9}$$

(As already explained in the introduction, thanks to Assumption 3.1, this can always be ensured up to taking a subsequence.) By the Aubin-Lions lemma, up to a subsequence, we then have

$$\sup_{[0,T]} \|u_\varepsilon - u\|_{\mathrm{L}^2(\mathbb{T}^3)} \xrightarrow[\varepsilon\to 0]{} 0.$$

As opposed to the *light* and *light and fast* regimes, it is not straightforward to ensure that this holds without taking a subsequence, and if it is the case, to provide a rate of convergence. This will be the object of the next section. We can still provide quantitative pointwise convergence for (f_ε), that will depend on this yet unclear convergence.

Corollary 3.5.15. *Under the assumptions of Theorem 3.5.14 and* (3.5.9)*, for almost every* $t \in (0,T)$*,*

$$W_1(f_\varepsilon(t), \rho(t) \otimes \delta_{v=u(t)}) \lesssim W_1(\rho_\varepsilon^0, \rho^0) + \|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \|u_\varepsilon - u\|_{\mathrm{L}^2((0,t)\times\mathbb{T}^3)} + \varepsilon.$$

▷ To prove the pointwise convergence of $(f_\varepsilon)_{\varepsilon>0}$, we first note that, for any $\psi \in \mathscr{C}^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ such that $\|\nabla_{x,v}\psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \leq 1$, thanks Hölder's inequality and Lemmas 3.5.8 and 3.5.9, for almost every $t \in (0,T)$,

$$\left|\langle f_\varepsilon(t) - \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)}, \psi\rangle\right| \leq \|\nabla_{x,v}\psi\|_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)|v - u_\varepsilon(t,x)|\mathrm{d}x\mathrm{d}v \\ \lesssim M^{\omega_p}\varepsilon^{p-1} \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(t,x,v)\frac{|v - u_\varepsilon(t,x)|^p}{\varepsilon^{p-1}}\mathrm{d}x\mathrm{d}v \lesssim M^{\omega'_p}\varepsilon^{p-1},$$

for some $\omega_p, \omega_p' > 0$. Furthermore, for almost every $t \in (0,T)$,

$$|\langle \rho_\varepsilon(t) \otimes \delta_{v=u_\varepsilon(t)} - \rho(t) \otimes \delta_{v=u(t)}, \psi \rangle| \leq \left| \int_{\mathbb{T}^3} (\rho_\varepsilon(t,x) - \rho(t,x)) \psi(t,x,u_\varepsilon(t,x)) \mathrm{d}x \right| + \left| \int_{\mathbb{T}^3} \rho(t,x) (\psi(t,x,u_\varepsilon(t,x) - \psi(t,x,u(t,x))) \mathrm{d}x \right|.$$

On the one hand,

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (\rho_\varepsilon(t,x) - \rho(t,x)) \psi(t,x,u_\varepsilon(t,x)) \mathrm{d}x \right| &\leq W_1(\rho_\varepsilon(t), \rho(t)) \sup_{x \in \mathbb{T}^3} |\nabla_x (\psi(x, u_\varepsilon(t,x)))| \\ &\leq W_1(\rho_\varepsilon(t), \rho(t)) \left(1 + \|\nabla_x u_\varepsilon(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right) \\ &\lesssim W_1(\rho_\varepsilon(t), \rho(t)). \end{aligned}$$

On the other hand, since T is a strong existence time, thanks to Corollary 3.3.24, for almost every $t \in (0,T)$,

$$\begin{aligned} \left| \int_{\mathbb{T}^3} \rho_\varepsilon(t,x) (\psi(t,x,u_\varepsilon(t,x) - \psi(t,x,u(t,x))) \mathrm{d}x \right| &\lesssim \int_{\mathbb{T}^3} |u_\varepsilon(t,x) - u(t,x)| \mathrm{d}x \\ &\leq \|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}, \end{aligned}$$

which completes the proof of the theorem.

□

Remark 3.5.16. *As for the* light *and* light and fast *particle regime, we obtain as a straightforward side consequence of the analysis, the description of the long time behavior of solutions to the Vlasov–Navier–Stokes system in the fine particle regime. In this regime, it turns out that the required assumptions are essentially the same as in [HMM20], as opposed to the results of Section 3.4.4. More interestingly, by uniformity with respect to ε of the exponential decay of the modulated energy, we also recover as a corollary a description of the long time behavior of solutions to the Inhomogeneous Navier–Stokes equations* (3.1.9). *This is of course a well-known result, see* e.g. *[Pou15, Theorem 1.4].*

3.5.4 Relative entropy estimates

We introduce in this section a relative entropy method for the fine particle regime with the aim to

- justify the pointwise in time convergence of (u_ε) towards u (without taking a subsequence),
- obtain full quantitative (with respect to ε) estimates.

This will be achieved after assuming that the solution to the limit system is smooth enough. A side objective is to justify why the related entropy method by itself does not allow to prove any convergence result but rather has to be used in conjunction with the previous bootstrap analysis. Namely, the relative entropy method requires exactly the same uniform bound for ρ_ε in L^∞ as for the other parts of this work.

We introduce the relative entropy

$$\mathscr{H}_\varepsilon(t) := \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon(t,x,v) |v - u(t,x)|^2 \mathrm{d}x \mathrm{d}v + \frac{1}{2} \int_{\mathbb{T}^3} |u_\varepsilon(t,x) - u(t,x)|^2 \mathrm{d}x,$$

where (ρ, u) is a smooth solution to the Inhomogeneous incompressible Navier–Stokes equations (note that ρ does not appear explicitly in $\mathscr{H}_\varepsilon$, though). The precise required smoothness shall be clarified in the upcoming statements.

The following is the analogue of [GJV04b, Lemma 3] in the case without diffusion for the Vlasov equation.

Lemma 3.5.17. *Under Assumption 3.1, the relative entropy satisfies for all $t \geq 0$*

$$\mathscr{H}_\varepsilon(t) + \int_0^t \int_{\mathbb{T}^3} |\nabla_x(u_\varepsilon - u)|^2 \mathrm{d}x\mathrm{d}s + \frac{1}{\varepsilon}\int_0^t \int_{\mathbb{T}^3\times\mathbb{R}^3} |v - u_\varepsilon|^2 f_\varepsilon \mathrm{d}x\mathrm{d}v\mathrm{d}s \leq \mathscr{H}_\varepsilon(0) + \int_0^t \sum_{j=1}^4 I_j(s)\mathrm{d}s. \tag{3.5.10}$$

with

$$\begin{aligned}
I_1 &:= -\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\otimes(v-u) : \nabla_x u \mathrm{d}x\mathrm{d}v,\\
I_2 &:= -\int_{\mathbb{R}^3} (u_\varepsilon - u)\otimes(u_\varepsilon - u) : \nabla_x u \mathrm{d}x,\\
I_3 &:= \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v - u_\varepsilon)\cdot G \mathrm{d}x\mathrm{d}v,\\
I_4 &:= \int_{\mathbb{T}^3} (\rho_\varepsilon - \rho)(u_\varepsilon - u)\cdot G\mathrm{d}x,
\end{aligned}$$

where $G = \frac{\nabla_x p - \Delta_x u}{1+\rho}$.

$\triangleright$ Let us argue as if f_ε were a smooth function. We first compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon |v-u|^2 \mathrm{d}x\mathrm{d}v\\
&\quad= \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} |v-u|^2 \partial_t f_\varepsilon \mathrm{d}x\mathrm{d}v - \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot\partial_t u \mathrm{d}x\mathrm{d}v\\
&\quad= -\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon ((v-u)\cdot\nabla_x u)\cdot v \mathrm{d}x\mathrm{d}v + \frac{1}{\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot(u_\varepsilon - v)\mathrm{d}x\mathrm{d}v\\
&\qquad+ \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot(u\cdot\nabla_x u)\mathrm{d}x\mathrm{d}v \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot G\mathrm{d}x\mathrm{d}v\\
&\quad= I_1 + \frac{1}{\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot(u_\varepsilon - v)\mathrm{d}x\mathrm{d}v + \int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (v-u)\cdot G\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^3} |u_\varepsilon - u|^2 \mathrm{d}x &= \int_{\mathbb{T}^3} (u_\varepsilon - u)\cdot\partial_t(u_\varepsilon - u)\mathrm{d}x\\
&= \int_{\mathbb{T}^3} (u_\varepsilon - u)\cdot(u\cdot\nabla_x u - u_\varepsilon\cdot\nabla_x u_\varepsilon)\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon (u_\varepsilon - u)\cdot(v - u_\varepsilon)\mathrm{d}x\mathrm{d}v\\
&\quad+ \int_{\mathbb{T}^3} (u_\varepsilon - u)\cdot G\mathrm{d}x + \int_{\mathbb{T}^3} (u_\varepsilon - u)\cdot\Delta_x u_\varepsilon \mathrm{d}x.
\end{aligned}$$

Since $\operatorname{div} u_\varepsilon = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{T}^3} (u_\varepsilon - u) \cdot (u \cdot \nabla_x u - u_\varepsilon \cdot \nabla_x u_\varepsilon) \mathrm{d}x \\
&\qquad = -\int_{\mathbb{T}^3} (u_\varepsilon - u) \otimes (u_\varepsilon - u) : \nabla_x u \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} u_\varepsilon \cdot \nabla_x |u_\varepsilon - u|^2 \mathrm{d}x \\
&\qquad = I_2.
\end{aligned}$$

Since $\operatorname{div} u = 0$ and by definition of G,

$$\Delta_x u = -(1+\rho) G + \nabla_x p,$$

we can also rewrite

$$\begin{aligned}
\int_{\mathbb{T}^3} (u_\varepsilon - u) \cdot \Delta_x u_\varepsilon \mathrm{d}x &= \int_{\mathbb{T}^3} (u_\varepsilon - u) \cdot \Delta_x (u_\varepsilon - u) \mathrm{d}x + \int_{\mathbb{T}^3} (u_\varepsilon - u) \cdot \Delta_x u \mathrm{d}x \\
&= -\int_{\mathbb{T}^3} |\nabla_x (u_\varepsilon - u)|^2 \mathrm{d}x - \int_{\mathbb{T}^3} (1+\rho)(u_\varepsilon - u) \cdot G \mathrm{d}x
\end{aligned}$$

All in all, we have obtained

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\mathbb{T}^3} |u_\varepsilon - u|^2 \mathrm{d}x = I_2 - \int_{\mathbb{T}^3} |\nabla_x (u_\varepsilon - u)|^2 \mathrm{d}x \\
+ \frac{1}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (u_\varepsilon - u) \cdot (v - u_\varepsilon) \mathrm{d}x \mathrm{d}v - \int_{\mathbb{T}^3} \rho (u_\varepsilon - u) \cdot G \mathrm{d}x.
\end{aligned}$$

We finally write

$$\frac{1}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (u_\varepsilon - u) \cdot (v - u_\varepsilon) \mathrm{d}x \mathrm{d}v + \frac{1}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (u_\varepsilon - u) \cdot (v - u_\varepsilon) \mathrm{d}x \mathrm{d}v = -\frac{1}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v - u_\varepsilon|^2 f_\varepsilon \mathrm{d}x \mathrm{d}v$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (v - u) \cdot G \mathrm{d}x \mathrm{d}v - \int_{\mathbb{T}^3} \rho (u_\varepsilon - u) \cdot G \mathrm{d}x \\
&\qquad = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_\varepsilon (v - u_\varepsilon) \cdot G \mathrm{d}x \mathrm{d}v + \int_{\mathbb{T}^3} (\rho_\varepsilon - \rho)(u_\varepsilon - u) \cdot G \mathrm{d}x \\
&\qquad = I_3 + I_4.
\end{aligned}$$

Adding everything up, this yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}_\varepsilon + \int_{\mathbb{T}^3} |\nabla_x (u_\varepsilon - u)|^2 \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v - u_\varepsilon|^2 f_\varepsilon \mathrm{d}x \mathrm{d}v = \sum_{j=1}^{4} I_j,$$

which concludes the computation. To obtain (3.5.10) in the general case, we notice that we can take $|v - u(t,x)|^2$ as a test function in the weak formulation of the Vlasov equation (by a standard density argument). □

The control of I_4 requires a pointwise estimate on $(\rho_\varepsilon - \rho)$ in $\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)$, as opposed to the Wasserstein control obtained in Theorem 3.5.14. Nevertheless, the same strategy leads to the following result.

Lemma 3.5.18. *Let $T > 0$. Assume there exist $C > 0$ and $\varepsilon_0 \in (0, 1)$ such that*

$$\forall \varepsilon \in (0, \varepsilon_0), \qquad \|\rho_\varepsilon\|_{\mathrm{L}^\infty(0,T\times\mathbb{T}^3)} \leq C. \tag{3.5.11}$$

Then for all $t \in (0, T)$,

$$\|\rho_\varepsilon(t) - \rho(t)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} \lesssim \left\|\rho_\varepsilon^0 - \rho^0\right\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon \int_0^t \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \int_0^t \|u_\varepsilon(s) - u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)}.$$

$\triangleright$ As in the proof of Theorem 3.5.14, $\chi_\varepsilon = \rho - \rho_\varepsilon$ satisfies the following transport equation

$$\partial\chi_\varepsilon + \mathrm{div}_x(\chi_\varepsilon u) = -\,\mathrm{div}_x\, J_\varepsilon,$$

where $J_\varepsilon = \rho_\varepsilon(u - u_\varepsilon) - \varepsilon F_\varepsilon$. Recall the characteristics of the equation : for $t \geq 0$ and $x \in \mathbb{T}^3$, we consider the solution $Y(\cdot; t, x)$ to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}s} Y(s; t, x) = u(s, Y(s; t, x)) \qquad \text{and} \qquad Y(t; t, x) = x.$$

Since $\mathrm{div}_x\, u = 0$, we have $\det \nabla_x Y(s; t, x) = 1$ for every $s, t \geq 0$ and $x \in \mathbb{T}^3$. Therefore, for every $\varphi \in \mathscr{C}^\infty(\mathbb{T}^3)$ such that $\|\nabla_x\varphi\|_{\mathrm{L}^2(\mathbb{T}^3)} \leq 1$, for every $t \in [0, T]$, thanks to the method of characteristics,

$$\begin{aligned}
\int_{\mathbb{T}^3} \chi_\varepsilon(t, x)\varphi(x)\mathrm{d}x &= \int_{\mathbb{T}^3} \chi_\varepsilon(0, x)\varphi(x)\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} (\mathrm{div}_x\, J_\varepsilon)(s, Y(s, t, x))\varphi(x)\mathrm{d}x\mathrm{d}s \\
&= \int_{\mathbb{T}^3} (\rho_\varepsilon^0 - \rho^0)\varphi + \int_0^t \int_{\mathbb{T}^3} J_\varepsilon(s, Y(s, t, x)) \cdot \nabla_x\varphi(x)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Applying the change of variable $x' = Y(s; t, x)$ yields

$$\left(\int_{\mathbb{T}^3} |J_\varepsilon(s, Y(s, t, x))|^2 \mathrm{d}x\right)^{\frac{1}{2}} \lesssim \varepsilon \|F_\varepsilon\|_{\mathrm{L}^2(\mathbb{T}^3)} + \|\rho_\varepsilon\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \|u_\varepsilon - u\|_{\mathrm{L}^2(\mathbb{T}^3)}$$

so that, thanks to the Cauchy-Schwarz inequality,

$$\left|\int_{\mathbb{T}^3} \chi_\varepsilon(t, x)\varphi(x)\mathrm{d}x\right| \lesssim \left\|\rho_\varepsilon^0 - \rho^0\right\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon \int_0^t \|F_\varepsilon(s)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \int_0^t \|u_\varepsilon(s) - u(s)\|_{\mathrm{L}^2(\mathbb{T}^3)},$$

which leads to the expected result. $\square$

Let us now state the estimate on $\mathscr{H}_\varepsilon$ we have obtained thanks to the previous results.

Proposition 3.5.19. *Let $T > 0$. Assume that*

$$\|G\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} + \|\nabla_x G\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \lesssim 1. \tag{3.5.12}$$

There exist $\varepsilon_0 > 0$ and $C_T > 0$ such that, if (3.5.11) *holds, then for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in (0, T)$,*

$$\mathscr{H}_\varepsilon(t) \leq C_T \left(\mathscr{H}_\varepsilon(0) + \|\rho_\varepsilon(0) - \rho(0)\|^2_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \int_0^T |I_3|\mathrm{d}t\right).$$

▷ Clearly, we have

$$|I_1(t)| \lesssim \|\nabla_x u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \int_{\mathbb{T}^3\times\mathbb{R}^3} |v-u(t,x)|^2 f_\varepsilon(t,x,v)\mathrm{d}x\mathrm{d}v \lesssim \|\nabla_x u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathscr{H}_\varepsilon(t),$$
$$|I_2(t)| \lesssim \|\nabla_x u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \int_{\mathbb{T}^3} |u_\varepsilon-u(t,x)|^2 \mathrm{d}x \lesssim \|\nabla_x u(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\mathscr{H}_\varepsilon(t).$$

On the other hand, applying Lemma 3.5.18 and Young's inequality, we obtain for all $t \in (0,T)$, for any $a>0$,

$$\begin{aligned}\int_0^t |I_4|(s)\mathrm{d}s &\le \int_0^t \|\rho_\varepsilon(s)-\rho(s)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} \|\nabla_x((u_\varepsilon(s)-u(s))\cdot G(s))\|_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s\\
&\le 2aT\|\rho_\varepsilon(0)-\rho(0)\|^2_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + 2aT^2\varepsilon^2\int_0^t \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s\\
&\quad + \left(2aT^2 + \frac{\|\nabla_x G\|^2_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}}{4a}\right)\int_0^t \|u_\varepsilon(s)-u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s\\
&\quad + \frac{\|G\|^2_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}}{4a}\int_0^t \|\nabla_x(u_\varepsilon(s)-u(s))\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\varepsilon^2\int_0^t \|F_\varepsilon(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s &\le \|\rho_\varepsilon\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)}\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(s,x,v)|v-u_\varepsilon(s,x)|^2\mathrm{d}x\mathrm{d}v\mathrm{d}s\\
&\le C\int_0^t\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon(s,x,v)|v-u_\varepsilon(s,x)|^2\mathrm{d}x\mathrm{d}v\mathrm{d}s.\end{aligned}$$

The conclusion follows from Lemma 3.5.17 and Gronwall's lemma, by taking $a = \|G\|^2_{\mathrm{L}^\infty(\mathbb{T}^3\times\mathbb{R}^3)}$ and ε_0 sufficiently small.

□

We are now ready to state quantitative convergence results. We only need to verify that the bound (3.5.11) holds. To do so, we rely on the analysis of the previous sections.

Theorem 3.5.20. *Let $T>0$. Under Assumptions 3.1–3.2, if we assume that*

$$\|G\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} + \|\nabla_x G\|_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \lesssim 1,$$

there exists $\varepsilon_0>0$ and $\eta>0$ such that if, for all $\varepsilon\in(0,\varepsilon_0)$, (3.5.1)–(3.5.3) *hold and if*

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u^0 \text{ in } \mathrm{L}^2(\mathbb{T}^3) \qquad \text{and} \qquad \rho^0_\varepsilon \xrightarrow[\varepsilon\to 0]{} \rho^0 \text{ in } \dot{\mathrm{H}}^{-1}(\mathbb{T}^3)$$

then $(u_\varepsilon)_{\varepsilon>0}$ converges to u in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^3))$ and $(\rho_\varepsilon)_{\varepsilon>0}$ converges to ρ in $\mathrm{L}^\infty(0,T;\dot{\mathrm{H}}^{-1}(\mathbb{T}^3))$. Furthermore, for every $t\in(0,T)$

$$\begin{aligned}&\|u_\varepsilon(t)-u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \|\rho_\varepsilon(t)-\rho(t)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)}\\
&\qquad\qquad \lesssim_{T,M,G} \left\|u^0_\varepsilon-u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \left\|\rho^0_\varepsilon-\rho^0\right\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon^{\frac12}. \quad (3.5.13)\end{aligned}$$

▷ Under this set of assumptions, we have proven all times are strong existence times, see

Section 3.5.2. Therefore, Lemma 3.3.24 applies and ensures that the bound (3.5.11) holds.

We use the Cauchy-Schwarz inequality and Lemmas 3.3.24, 3.5.12 and 3.5.13 to find that

$$\int_0^T |I_3(t)|\mathrm{d}t \leq \|\rho_\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^\infty((0,T)\times\mathbb{T}^3)} \left(\varepsilon^2 \int_0^T \mathrm{D}_\varepsilon^{(2)}(t)\mathrm{d}t\right)^{\frac{1}{2}} \|G\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \lesssim_{T,M,G} \varepsilon.$$

Furthermore, thanks to Hölder's inequality

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0 |v-u_\varepsilon^0|^2 \lesssim_M \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} f_\varepsilon^0 |v-u_\varepsilon^0|^p\right)^{\frac{2}{p}} \lesssim_M \varepsilon^{2-\frac{2}{p}}.$$

Thus, applying Proposition 3.5.19, for every $t \in [0,T]$,

$$\mathscr{H}_\varepsilon(t) \lesssim_{T,M,G} \left\|u_\varepsilon^0 - u^0\right\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \left\|\rho_\varepsilon^0 - \rho^0\right\|^2_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon,$$

hence the result for $\|u_\varepsilon(t) - u(t)\|_{\mathrm{L}^2(\mathbb{T}^3)}$.

Recall that since T is a strong existence time,

$$\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{2}.$$

Therefore, Lemma 3.5.18 yields

$$\begin{aligned}\|\rho_\varepsilon(t) - \rho(t)\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} &\lesssim_T \left\|\rho_\varepsilon^0 - \rho^0\right\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \varepsilon \|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \\ &\quad + \|u_\varepsilon - u\|_{\mathrm{L}^1(0,T;\mathrm{L}^2(\mathbb{T}^3))} \\ &\lesssim_{T,M,G} \left\|\rho_\varepsilon^0 - \rho^0\right\|_{\dot{\mathrm{H}}^{-1}(\mathbb{T}^3)} + \left\|u_\varepsilon^0 - u^0\right\|_{\mathrm{L}^2(\mathbb{T}^3)} + \varepsilon^{\frac{1}{2}},\end{aligned}$$

hence the theorem. □

To conclude, Theorem 3.5.20 combined with Corollary 3.5.15 allows us to derive the quantitative convergence of (f_ε).

Remark 3.5.21. *Another benefit of the introduction of higher dissipation functionals appears here. To estimate the contribution of I_3, one can also use the energy–dissipation estimate. However this would yield $\varepsilon^{\frac{1}{4}}$ instead of $\varepsilon^{\frac{1}{2}}$ as in* (3.5.13).

Remark 3.5.22. *Note that we could have kept track of the time dependence and provided explicit quantitative results, as we have done in Section 3.4.*

3.5.5 Convergence in the mildly well-prepared case

In the previous subsections, we derived convergence results for any time horizon under

- a smallness assumption for the initial fluid velocity in the sense of Assumption 3.2,
- a smallness assumption for the initial modulated energy in the sense of (3.5.3),
- a smallness assumption for the initial distribution function in the sense of (3.5.1),
- well-preparedness for the initial data in the sense of (3.5.2).

In this section, we show how to obtain short-time convergence results when dispensing with Assumptions 3.2 and (3.5.3), which corresponds to the mildly well-prepared case of Theorem 3.1.7.

According to Lemma 3.3.30, there exists a strong existence time $T_{M,\varepsilon} > 0$. Let us prove that we can find a strong existence time that does not depend on ε. We will do so using a bootstrap argument similar to the one used in the previous sections. We define

$$T_\varepsilon^* = \sup\{T > 0,\, T \text{ is a strong existence time}\}.$$

Our goal is to find a time T_M, independent of ε, such that $T_M < T_\varepsilon^*$. As we have proven in Section 3.4.5 in the *light* and *light and fast* particle regimes, there exists $T_M > 0$, independent of ε, such that for every $\varepsilon \in (0,1)$,

$$\int_0^{T_M} \|e^{t\Delta} u_\varepsilon^0\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}^4 \,\mathrm{d}t \leq \frac{C^*}{4}.$$

For what concerns $\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)}$, we can apply Lemma 3.5.13 and its proof to ensure that there exists $\varepsilon_0 > 0$, $\eta > 0$ and $T_M > 0$ (independent of ε) such that if (3.5.1)–(3.5.2) hold, then for every $T < \min(T_\varepsilon^*, T_M)$,

$$\|F_\varepsilon\|_{\mathrm{L}^2((0,T)\times\mathbb{T}^3)} \leq \frac{C^*}{4}.$$

There only remains to obtain a satisfactory bound for $\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^3))}$. To achieve this, we apply Corollary 3.3.21 and Lemma 3.5.9, which yield

$$\|\nabla_x u_\varepsilon\|_{\mathrm{L}^1(0,T;\mathrm{L}^\infty(T^3))} \lesssim T^{1-\beta_p} \mathscr{E}_\varepsilon(0)^{\frac{1-\beta_p}{2}} \|\Delta_x u_\varepsilon\|_{\mathrm{L}^p((0,T)\times T^3)}^{\beta_p} \lesssim T^{1-\beta_p} M^{\omega_p},$$

for some $\omega_p > 0$. The right-hand side can be made as mall as necessary (*e.g.* $<1/40$) by reducing the value of $T_M > 0$, still independently from ε.

This concludes the bootstrap argument, and we can now perform the same analysis as in Subsections 3.5.3 and 3.5.4 and obtain the same convergence results when we replace Assumptions 3.2 and (3.5.3) by the time horizon constraint $T < T_M$.

A Sobolev and Besov (semi-)norms and Wasserstein-1 distance

Let $k \in \mathbb{Z}^3$. The k-th Fourier coefficient of $f \in \mathrm{L}^1(\mathbb{T}^3)$ is given by

$$c_k(f) := \int_{[0,2\pi]^3} f(x) e^{-ik\cdot x} \mathrm{d}x.$$

As usual, this definition is extended by duality to tempered distributions $f \in \mathcal{S}'(\mathbb{T}^3)$. Let us first recall the very classical definition of inhomogeneous and homogeneous Sobolev spaces.

Definition A.1. *Let $s \in \mathbb{R}$. The (inhomogeneous) Sobolev space $\mathrm{H}^s(\mathbb{T}^3)$ is defined by*

$$\mathrm{H}^s(\mathbb{T}^3) := \left\{ f \in \mathcal{S}'(\mathbb{T}^3), \|u\|_{\mathrm{H}^s(\mathbb{T}^3)}^2 := \frac{1}{(2\pi)^3} \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |c_k(f)|^2 < +\infty \right\}.$$

The homogeneous Sobolev space $\dot{\mathrm{H}}^s(\mathbb{T}^3)$ *is defined by*

$$\dot{\mathrm{H}}^s(\mathbb{T}^3) := \left\{ f \in \mathcal{S}'(\mathbb{T}^3), \|u\|^2_{\dot{\mathrm{H}}^s(\mathbb{T}^3)} := \frac{1}{(2\pi)^3} \sum_{k\in\mathbb{Z}^3} |k|^{2s} |c_k(f)|^2 < +\infty \right\}.$$

We now provide a definition of Besov spaces on the torus $\mathbb{T}^3$. We refer to [BCD11, Chapter 2] for a reference on the topic. The existence of the functions χ and φ below is provided by [BCD11, Proposition 2.10]

Definition A.2. *Let* $\mathcal{C} = \{\xi \in \mathbb{R}^3,\, 3/4 \leq |\xi| \leq 8/3\}$. *Let* $\chi \in \mathcal{D}(B(0, 4/3)), \varphi \in \mathcal{D}(\mathcal{C})$ *be radial functions, with values in* $[0,1]$ *such that*

$$\begin{aligned} \forall \xi \in \mathbb{R}^3,\, \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \\ |j - j'| \geq 2 \implies \textit{Supp } \varphi(2^{-j}\cdot) \cap \textit{Supp } \varphi(2^{-j'}\cdot) = \emptyset, \\ j \geq 1 \implies \textit{Supp } \chi \cap \textit{Supp } \varphi(2^{-j}\cdot) = \emptyset. \end{aligned}$$

Consider the operators Δ_j, $j \in \mathbb{Z}$, *acting on* $\mathcal{S}'(\mathbb{T}^3)$ *defined by*

- $\Delta_j := 0$ *for* $j \leq -2$,
- $\Delta_{-1} f := \chi(D) f$, *with* $c_k(\chi(D)f) = \chi(k) c_k(f)$,
- $\Delta_j f := \varphi(2^{-j}D) f$, *with* $c_k(\varphi(2^{-j}D)f) = \varphi(2^{-j}k) c_k(f)$.

Let $s \in \mathbb{R}, q, r \in [1, +\infty]$. *The Besov space* $\mathrm{B}^{s,q}_r(\mathbb{T}^3)$ *is defined as*

$$\mathrm{B}^{s,q}_r(\mathbb{T}^3) := \left\{ f \in \mathcal{S}'(\mathbb{T}^3),\, \|u\|_{\mathrm{B}^{s,q}_r(\mathbb{T}^3)} := \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \|\Delta_j f\|^r_{\mathrm{L}^q(\mathbb{T}^3)} \right)^{1/r} \right\}.$$

To conclude this short section of functional analysis reminders, let us give the definition of the Wasserstein-1 distance W_1 that is intensively used in this work.

Definition A.3. *Let* X *be either* $\mathbb{T}^3$ *or* $\mathbb{T}^3 \times \mathbb{R}^3$. *Let* μ, ν *be two probability measures on* X. *The Wasserstein-1 distance between* μ *and* ν *is defined as*

$$W_1(\mu, \nu) = \sup_{\|\nabla\psi\|_\infty \leq 1} \int_X \psi \, (\mathrm{d}\mu - \mathrm{d}\nu) \,.$$

We recall that W_1 allows to metricize the weak-$\star$ convergence on $\mathcal{P}_1(X)$, the set of probability measures on X with finite first moment.

B Gagliardo-Nirenberg interpolation estimates

Theorem B.1. *Consider* $1 \leq p, q, r \leq \infty$ *and* $m \in \mathbb{N}$. *Assume that* $j \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$ *satisfy*

$$\frac{1}{p} = \frac{j}{3} + \left(\frac{1}{r} - \frac{m}{3}\right)\alpha + \frac{1-\alpha}{q},$$

$$\frac{j}{m} \leq \alpha \leq 1,$$

with the exception $\alpha < 1$ if $m - j - 3/r \in \mathbb{N}$. Then, the following holds. For any $g \in \mathrm{L}^q(\mathbb{T}^3)$, if $\mathrm{D}^m g \in \mathrm{L}^r(\mathbb{T}^3)$, then $\mathrm{D}^j g \in \mathrm{L}^p(\mathbb{T}^3)$ and we have the estimate

$$\left\|\mathrm{D}^j g\right\|_{\mathrm{L}^p(\mathbb{T}^3)} \lesssim \|\mathrm{D}^m g\|_{\mathrm{L}^r(\mathbb{T}^3)}^{\alpha} \|g\|_{\mathrm{L}^q(\mathbb{T}^3)}^{1-\alpha} + \|g\|_{\mathrm{L}^q(\mathbb{T}^3)},$$

where the constant behind $\lesssim$ does not depend on g. If $\langle \mathrm{D}^j g\rangle = 0$, then the term $\|g\|_{\mathrm{L}^q(\mathbb{T}^3)}$ in the right-hand side can be dispensed with.

We introduce in the following corollary some notations we use in this paper.

Corollary B.2. *For any $p \geq 2$, we set*

$$\alpha_p = \frac{5p-6}{7p-6}, \qquad \beta_p = \frac{5p}{7p-6}.$$

Let $g \in L^2(\mathbb{T}^3)$. We have

$$\begin{aligned}
\forall p \geq 2, \mathrm{D}^2 g \in \mathrm{L}^p(\mathbb{T}^3) \quad &\Longrightarrow \quad \|\nabla g\|_{\mathrm{L}^p(\mathbb{T}^3)} \lesssim \|\Delta_x g\|_{\mathrm{L}^p(\mathbb{T}^3)}^{\alpha_p} \|g - \langle g\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)}^{1-\alpha_p},\\
\forall p > 3, \mathrm{D}^2 g \in \mathrm{L}^p(\mathbb{T}^3) \quad &\Longrightarrow \quad \|\nabla g\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \|\Delta_x g\|_{\mathrm{L}^p(\mathbb{T}^3)}^{\beta_p} \|g - \langle g\rangle\|_{\mathrm{L}^2(\mathbb{T}^3)}^{1-\beta_p}.
\end{aligned}$$

C Estimates on the incompressible Navier–Stokes equations

This brief section is dedicated to the presentation of some higher order energy estimates for the Navier–Stokes system with a source term :

$$\begin{cases}
\partial_t u + (u \cdot \nabla_x)u - \Delta_x u + \nabla_x p = F,\\
\mathrm{div}_x\, u = 0,\\
u|_{t=0} = u^0.
\end{cases} \tag{C.1}$$

Let us first state an estimate on the solution of the heat equation.

Lemma C.1. *Let $u^0 \in \dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)$. For every $t \geq 0$,*

$$\frac{1}{2}\left\|e^{t\Delta}u^0\right\|_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)}^2 + \int_0^t \left\|e^{s\Delta}u^0\right\|_{\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3)}^2 \mathrm{d}s \leq \frac{1}{2}\left\|u^0\right\|_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)}^2.$$

This result is one of the main ingredients in proving the following property of smooth solutions to the Navier–Stokes equations.

Theorem C.2. *There exists a universal constant $C^* \in (0,1)$ such that the following holds. Consider $u^0 \in \mathrm{H}^1_{\mathrm{div}}(\mathbb{T}^3)$, $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^{-\frac{1}{2}}(\mathbb{T}^3))$ and $T > 0$ such that*

$$\int_0^T \left\|e^{t\Delta}u^0\right\|_{\dot{\mathrm{H}}^1(\mathbb{T}^3)}^4 \mathrm{d}t + \int_0^T \|F(t)\|_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)}^2 \mathrm{d}t \leq C^*. \tag{C.2}$$

Then there exists on $(0,T)$ a unique Leray solution of (C.1). It belongs to $\mathrm{L}^\infty(0,T;\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)) \cap \mathrm{L}^2(0,T;\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3))$ and satisfies, for a.e. $0 \leq t \leq T$,

$$\|u(t)\|_{\dot{\mathrm{H}}^{\frac{1}{2}}(\mathbb{T}^3)}^2 + \int_0^t \|u(s)\|_{\dot{\mathrm{H}}^{\frac{3}{2}}(\mathbb{T}^3)}^2 \mathrm{d}s \lesssim \left\|u^0\right\|_{\dot{\mathrm{H}}^{\frac{1}{2}}}^2 + \int_0^t \|F(s)\|_{\dot{\mathrm{H}}^{-\frac{1}{2}}(\mathbb{T}^3)}^2 \mathrm{d}s.$$

Furthermore, if $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^3))$*, the solution belongs to* $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbb{T}^3)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbb{T}^3))$ *and satisfies for a.e.* $0 \le t \le T$

$$\|u(t)\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \int_0^t \|\Delta_x u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \lesssim \left\|u^0\right\|^2_{\mathrm{H}^1(\mathbb{T}^3)} + \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s. \tag{C.3}$$

$\triangleright$ We refer the reader to [RRS16, Theorem 10.1] for what concerns the estimate for u in $\mathrm{L}^\infty(0,T;\dot{\mathrm{H}}^{\frac12}(\mathbb{T}^3)) \cap \mathrm{L}^2(0,T;\dot{\mathrm{H}}^{\frac32}(\mathbb{T}^3))$. The additional assumption $u^0 \in \mathrm{H}^1(\mathbb{T}^3)$ allows to follow the proof of [HMM20, Proposition 9.10] on $[0,T]$ and this yields the second part of the statement. $\square$

D Perturbation of the identity map

We make use of the following version of the inverse function theorem.

Theorem D.1. *For* $\Omega = \mathbb{T}^3$ *or* $\Omega = \mathbb{R}^3$*, if* $\phi : \Omega \to \Omega$ *is* $\mathscr{C}^1$ *and satisfies* $\|\nabla\phi\|_\infty < 1$*, then* $f = \mathrm{Id} + \phi$ *is a* $\mathscr{C}^1$*-diffeomorphism of* Ω *to itself satisfying* $\|\nabla f\|_\infty \le (1 - \|\nabla\phi\|_\infty)^{-1}$*. If furthermore* $\|\nabla\phi\|_\infty \le 1/9$*, then* $\det \nabla f \ge 1/2$.

E Maximal parabolic regularity

Let us state the maximal parabolic regularity estimates for the Stokes equation on which we rely heavily in this work. The following follows from the classical [GS91, Theorem 2.7] (see also the expository notes [Sal16]) and the transference argument for instance described in [HMM20, Corollary 9.8] .

Theorem E.1. *Let* $q, r > 1$*. Let* $u^0 \in \mathscr{C}^\infty(\mathbb{T}^3)$*. Let* $T > 0$ *and* $F \in \mathrm{L}^q(\mathbb{R}^*_+, \mathrm{L}^r(\mathbb{T}^3))$*, and suppose* u *is the unique tempered solution of the inhomogeneous Stokes equation*

$$\begin{cases} \partial_t u - \Delta_x u = F, \\ \mathrm{div}_x\, u = 0, \\ u|_{t=0} = u^0. \end{cases}$$

Then

$$\|\partial_t u\|_{\mathrm{L}^q(\mathbb{R}^*_+,\mathrm{L}^r(\mathbb{T}^3))} + \|\Delta_x u\|_{\mathrm{L}^q(\mathbb{R}^*_+,\mathrm{L}^r(\mathbb{T}^3))} \lesssim_{q,r} \|F\|_{\mathrm{L}^q(\mathbb{R}^*_+,\mathrm{L}^r(\mathbb{T}^3))} + \left\|u^0\right\|_{\mathrm{B}^{s,r}_q(\mathbb{T}^3)},$$

with $s = 2 - \frac{2}{q}$.

We can now state the resulting maximal parabolic estimate for the Navier–Stokes equation, which follows from Theorem E.1.

Theorem E.2. *Let* $q, r > 1$*. Let* $u^0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^3) \cap \mathrm{B}^{s,r}_q(\mathbb{T}^3)$ *for* $s = 2 - 2/q$ *and* $F \in \mathrm{L}^q(\mathbb{R}^*_+, \mathrm{L}^r(\mathbb{T}^3))$*. Suppose* u *is a Leray solution of the Navier–Stokes equations*

$$\begin{cases} \partial_t u + (u \cdot \nabla_x) u - \Delta_x u + \nabla_x p = F, \\ \mathrm{div}_x\, u = 0, \\ u|_{t=0} = u^0, \end{cases}$$

then

$$\|\partial_t u\|_{\mathrm{L}^q(\mathbb{R}_+^*,\mathrm{L}^r(\mathbb{T}^3))} + \|\Delta_x u\|_{\mathrm{L}^q(\mathbb{R}_+^*,\mathrm{L}^r(\mathbb{T}^3))} \lesssim_{q,r} \|(u\cdot\nabla_x)u\|_{\mathrm{L}^q(\mathbb{R}_+^*,\mathrm{L}^r(\mathbb{T}^3))} + \|F\|_{\mathrm{L}^q(\mathbb{R}_+^*,\mathrm{L}^r(\mathbb{T}^3))} + \left\|u^0\right\|_{\mathrm{B}_q^{s,r}(\mathbb{T}^3)},$$

with $s = 2 - \frac{2}{q}$.

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Analyse mathématique et asymptotique de modèles couplés fluide-cinétique issus de la mécanique des fluides et des sciences du vivant

Résumé

Dans ce manuscrit, nous nous intéressons à des modèles, dits fluide-cinétique, décrivant l'évolution de particules en suspension dans un fluide porteur. Un tel système physique est représenté mathématiquement par le couplage d'équations aux dérivées partielles multi-échelles issues de la mécanique des fluides. Plus précisément, nous faisons l'hypothèse que le fluide porteur peut être décrit par des quantités macroscopiques, sa vitesse et sa pression, grâce aux équations de Navier-Stokes incompressible. Le spray de particules est quant à lui décrit à l'échelle mésoscopique, comme classiquement en théorie cinétique des gaz, par sa fonction de densité dans l'espace des phases, régie par une équation de type Vlasov. La prise en compte de l'accélération de traînée fournie par le fluide aux particules et, en retour, la force de rétroaction subie par le fluide entraîne un couplage fort du système d'équations étudié.

Dans un premier temps, nous portons notre attention sur un modèle fluide-cinétique récemment proposé pour décrire le mouvement d'un aérosol thérapeutique au sein des voies respiratoires supérieures. En plus des interactions présentées ci-dessus, les effets de l'humidité de l'air ambiant sur la taille et la température des particules sont pris en compte par l'introduction d'équations de convection-diffusion décrivant la fraction massique de vapeur d'eau dans l'air et sa température, ainsi que l'intégration de la variation du rayon et de la température des particules dans l'équation de transport régissant l'aérosol. Nous démontrons l'existence de solutions faibles globales dans un domaine borné dépendant du temps pour ce système d'équations puis nous présentons quelques résultats d'expérimentations numériques.

Enfin, nous étudions plusieurs régimes de haute friction pour le système de Vlasov-Navier-Stokes présenté précédemment. Nous définissons un cadre permettant de traiter rigoureusement ces limites hydrodynamiques lorsque les particules sont légères ou petites par rapport au fluide. Nous obtenons à la limite les systèmes Transport-Navier-Stokes ou Navier-Stokes inhomogène, respectivement.

Mots clés : fluide-cinétique, Vlasov-Navier-Stokes, solutions faibles, simulation numérique, limite hydrodynamique

Abstract

In this manuscript, we consider so-called fluid-kinetic models that describe the evolution of particles flowing through a fluid. Such a physical system is represented mathematically by the coupling of multi-scale partial differential equations stemming from fluid mechanics. More precisely, we assume that the sustaining fluid can be described by macroscopic quantities, its velocity and pressure, thanks to the incompressible Navier-Stokes equations. As for the particle spray, it is described at the mesoscopic scale, as usual in the kinetic theory of gases, by its density function in the phase space, which obeys a Vlasov-type equation. Taking into account the drag acceleration exerted by the fluid on the particles and, conversely, the drag force applied on the fluid by the spray leads to a strong coupling of the system of equations under study.

First, we focus on a fluid-kinetic model recently proposed to describe the motion of a therapeutic aerosol in the superior regions of the airways. In addition to the interactions presented above, the effects of the airway humidity on the particle size and temperature are taken into account by introducing convection-diffusion equations describing the water vapor mass fraction and the temperature of the air, as well as integrating the size and temperature variations into the transport equation satisfied by the aerosol density function. We prove the existence of global weak solutions in a time-dependent domain for this system of equations and present some results of numerical experiments.

Finally, we study several high-friction regimes for the Vlasov-Navier-Stokes system presented above. We define a framework allowing to properly justify these hydrodynamic limits in the case where the particles are light (resp. small) with respect to the fluid. At the limit, we derive the Transport-Navier-Stokes (resp. Inhomogeneous Navier-Stokes) systems.

Keywords: fluid-kinetic, Vlasov-Navier-Stokes, weak solutions, numerical simulation, hydrodynamic limit

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