Comportements asymptotiques et transition de phase pour des marches aléatoires en milieux aléatoires et des marches renforcées

Rémy Poudevigne-Auboiron

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Rémy Poudevigne--Auboiron

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Devant le jury composé de :

M. Roland Bauerschmidt	University of Cambridge	Rapporteur
Mme Oriane Blondel	Université Lyon 1	Examinatrice
Mme Codina Cotar	University College	Examinatrice
Mme Margherita Disertori	Universität Bonn	Examinatrice
M. Nathanaël Enriquez	Université Paris-Sud	Examinateur
M. Christophe Garban	Université Lyon 1	Examinateur
M. Pierre Mathieu	Université d'Aix-Marseille	Examinateur
M. Christophe Sabot	Université Lyon 1	Directeur de thèse

Suite aux rapports de :

M. Alejandro Ramírez	Universidad Católica de Chile
M. Roland Bauerschmidt	University of Cambridge

Institut Camille Jordan
UMR 5208 du CNRS

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Thèse de doctorat

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Comportements asymptotiques et transition de phase pour des marches aléatoires en milieux aléatoires et des marches renforcées

Cette thèse a pour but d'étudier certains comportements de marches aléatoires en milieux aléatoires et de marches renforcées. Nous regardons d'une part les marches aléatoires en milieu de Dirichlet et d'autres part deux modèles de marches renforcées : la marche aléatoire renforcée linéairement par arête et le processus de saut renforcé par sommet.
Les marches aléatoires en milieux de Dirichlet sont un cas particulier de marches aléatoires en milieux aléatoires présentant une importante propriété simplifiant leur étude: l'invariance statistique par retournement du temps. Dans une première partie nous utilisons cette propriété pour caractériser le comportement limite de ces marches en dimensions 3 et supérieures dans le cas où elles sont transitoires à vitesse nulle. Dans ce cas nous montrons que leur comportement est caractérisé par un processus stable. Dans une seconde partie nous montrons que la propriété d'invariance statistique par retournement du temps est caractéristique des marches aléatoires en milieu de Dirichlet.
La marche aléatoire renforcée linéairement par arête et le processus de saut renforcé par sommet sont deux modèles de processus renforcés intimement liés. Dans ces deux modèles la marche a tendance a revenir vers les zones déjà visitées. Nous montrons que certaines quantités caractéristiques de ces deux modèles présentent une certaine monotonie en leurs paramètres. Cela induit un certain nombre de conséquences notamment une unicité de la transition de phase entre récurrence et transitivité, la récurrence en dimension 2 et une loi du $0-1$ pour la récurrence. Dans un second temps on s'intéresse également à une version biaisée du modèle de marche aléatoire renforcée linéairement par arête pour lequel on montre qu'il conserve un comportement similaire pour certains types de graphes.

Asymptotic behaviour and phase transition for random walks in random environments and reinforced random walks

In this thesis, we study some behaviours of random walks in random environemnts and reinforced random walks. We will first look at random walks in Dirichlet environment and then at two models of reinforced walks: the linealry edge-reinforced random walk and the vertex reinforced jump process.
Random walks in Dirichlet environment are a special case of random walk in random environments that exhibit an important property simplifiant leur étude: the statistical invariance by time reversal. In chapter 2 we will use this property to characterize the asymptotic behaviour of these walks in dimensions 3 and higher when they are transient with zero speed. In this case we show that their behaviour is characterized by a stable process. In chapter 3 we show that this property of statistical invariance by time reversal is actually characteristic of random walks in Dirichlet environments.
The linearly edge-reinforced random walk and the vertex reinforced jump process are two closely linked models of reinforced processes. In both models the walk tends to come back to areas it has already visited. In chapter 4, we will show that some characteristic quantities exhibit some monotonicity in their parameters. This induces some consequences: unicity for the phase transition between recurrence and transience, recurrence in dimension 2, and a $0-1$ law for recurrence. Then, in chapter 5 we will look at a biased version of the linearly edge-reinforced random walk for which we show that its behaviour stays similar to the original model on some infinite graphs.

Chapter 0

Résumé en Français

Dans cette thèse nous allons étudier certains comportements de marches aléatoires en milieux aléatoires et de marches renforcées. Nous regardons d'une part les marches aléatoires en milieu de Dirichlet et d'autres part deux modèles de marches renforcées: la marche aléatoire renforcée linéairement par arête et le processus de saut renforcé par sommet.
Les marches aléatoires en milieu de Dirichlet sont un cas particulier de marches aléatoires en milieux aléatoires. Ces dernières sont très bien comprises sur $\mathbf{Z}$, lorsque les probabilités de transitions en chaque site sont iid [90], car la marche est alors nécessairement réversible ce qui permet l'introduction d'un potentiel qui simplifie grandement l'étude de la marche. Ce potentiel permet de déterminer si la marche est récurrente ou transitoire ainsi que de déterminer, dans le cas où elle est transitoire, si elle est balistique ou non. Cette réversibilité et donc la notion de potentiel disparaissent en dimensions 2 et supérieures. Le comportement général est donc beaucoup moins bien compris en dimension plus grande que 2 même si certains résultats importants ont été prouvés, comme un critère sous lequel il y a balisticité et même un TCL [83]. Cependant, il reste de nombreuses questions ouvertes. On ne sait pas encore montrer que les marches en dimensions 3 et supérieures sont transitoires. On ne connaît pas de critères simples permettant de déterminer s'il y a balisticité, notamment on ne sait pas prouver que dans un environnement iid avec une loi uniformément elliptique la transience directionnelle implique la balisticité. Cependant, l'étude de ces problèmes reste accessible sur certains modèles comme le modèle de marche aléatoire en milieu de Dirichlet.
La marche aléatoire renforcée linéairement, introduite par Diaconis et Coppersmith [24] est un modèle de marche aléatoire biaisée vers les arêtes déjà visitées. Ce modèle est partiellement échangeable ce qui permet de montrer que c'est un mélange de marches aléatoires parmi des conductances [30] ce qui en fait une marche aléatoire dans un milieu aléatoire. C'est en voyant ce modèle comme une marche aléatoire en milieu aléatoire que la plupart des résultats connus sur ce modèle sont démontrés. De même, le processus de saut renforcé par sommet introduit par Davis et Volkov dans [26] est également biaisé vers les zones déjà visitées. Ce processus est également partiellement échangeable et il est en fait très proche de la marche aléatoire renforcée linéairement [72].

0.1 La marche aléatoire en milieu de Dirichlet

Le modèle de marche aléatoire en milieu de Dirichlet possède une propriété fondamentale facilitant son étude : la marche renversée est aussi une marche aléatoire dans un environnement de Dirichlet. La marche renversée n'est en général définie que sur des graphes finis puisque sa définition fait intervenir la loi invariante de la marche. Cela dit, cette propriété d'invariance des marches dans un environnement de Dirichlet par inversion du temps est en fait vraie pour tout graphe fini. Cette propriété permet de démontrer le caractère transitoire de la marche en dimensions 3 et supérieures [69] ainsi qu'en dimension 2 si les probabilités de transitions ne sont pas symétriques [74] mais dans le cas symétrique on ne sait pas démontrer la récurrence. L'invariance par renversement du temps permet aussi de démontrer, dans le cas où les probabilités de transitions ne sont pas symétriques, le caractère transitoire directionnel [88] ainsi que l'existence de temps de renouvellement (la marche est en-deçà d'un certain niveau avant ces temps et au-delà après) et ainsi de montrer une loi des grands nombres.

Le cas balistique, c'est-à-dire le cas où $\frac{X_n}{n}$ converge vers une vitesse v non nulle est plutôt bien compris, un théorème central limite a même été prouvé [20]. Même dans le cas général, il existe des critères permettant de montrer un théorème central limite dans le cas balistique [83]. Cependant, il n'existe pas encore de critères simples permettant de déterminer si la marche est balistique dans le cas général mais il existe tout de même plusieurs critères de balisticité ([83], [41]). Pour les marches dans un environnement de Dirichlet, le cas sous-

balistique est moins bien compris : en effet il a été prouvé [18] que $\frac{\log X_n}{n} \to$ converge vers un certain κ qui dépend des paramètres de la loi de Dirichlet mais il n'a pas encore été prouvé que $\frac{X_n}{n^\kappa}$ converge vers une loi limite non dégénérée. Un problème similaire, celui d'une marche parmi des conductances aléatoires biaisées (on prend des conductances iid et on biaise en multipliant par une exponentielle en la distance selon une direction) a été étudié et il a été montré [42] que dans ce cas $\frac{X_n}{n^\kappa}$ converge vers une loi limite non dégénérée. Dans le chapitre 2 on montre un résultat similaire dans le cas des marches aléatoires en milieu de Dirichlet. Plus précisément, on montre que la comportement de la marche en dimension 3 et plus, dans le cas transitoire sous-balistique est caractérisé par un processus κ-stable pour un κ explicite. Plus précisément, le processus renormalisé par $n^{-\kappa}$ converge vers l'inverse d'un processus κ-stable. Il est aussi possible de caractérisé les temps d'atteintes de niveau par le processus κ-stable.
Dans le chapitre 3 on s'intéresse à la propriété d'invariance statistique par retournement du temps. On montre que sur tout graphe vérifiant quelques hypothèses techniques cette propriété caractérise les marches aléatoires en milieu de Dirichlet. Plus précisément, on montre que sur tout graphe vérifiant ces quelques hypothèses techniques si l'environnement et l'environnement renversé ont des probabilités de transitions indépendantes à chaque site alors l'environnement est un environnement de Dirichlet ou un environnement déterministe.

0.2 La marche aléatoire renforcée linéairement

La marche aléatoire renforcée linéairement est une marche biaisée vers les arêtes déjà visitées. Plus précisément la probabilité d'emprunter une arête est proportionnelle au nombre de fois que cette arête a été visitée plus un poids initial a. Quelle que soit la dimension de l'espace $\mathbf{Z}^d$ sur lequel on regarde la marche, pour un renforcement suffisamment fort, c'est-à-dire un poids initial petit, la marche devient récurrente et même positivement récurrente. On a même que le nombre moyen de fois que l'on visite un point décroît exponentiellement avec la distance à l'origine ([72], [2]). Inversement, en dimension 3 et plus il a été montrer que cette marche est transitoire pour un poids initial suffisamment grand [32]. En dimension 2 il a été démontré que la marche est récurrente quelque soit le poids initial ([60],[76]). Des résultats ont été démontrés pour le processus de saut renforcé par sommet.
Dans le chapitre 4 nous montrons que diminuer le poids initial rend les deux modèles «plus récurrents». Cela implique qu'il existe une unique transition de phase entre les comportements récurrent et transitoire pour les deux modèle. Ce résultat a d'autre conséquence : nous étendons les résultats de récurrence à tous les graphes récurrents (au sens des réseaux éléctriques) et nous montrons que les deux modèles satisfont une loi du 0-1 pour la récurrence ce qui étend un résultat de [76].
Enfin, dans le chapitre 5 nous étudions une modification de la marche aléatoire renforcée linéairement. Cette nouvelle version a un biais sur les arêtes dans une certaine direction. Cela a pour conséquence de briser l'échangeabilité partielle. Par conséquent il n'est plus possible de voir ce modèle comme un mélange de marches réversibles ce qui nous prive d'un outil puissant. Nous montrons cependant que pour un biais suffisamment faible, pour une certaine famille de graphes, ce modèle biaisé est également récurrent.

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Chapter 1

Introduction

1.1 Simple random walk and electrical networks

1.1.1 Simple random walk

In this section we will start by describing the simplest example of a random walk: the aptly named simple random walk. Then we will discuss a first generalization of the random walk: the random walk among conductances. This model has links with the physical concept of electrical network which are useful for its study. It will be useful for the next section where we will start by looking at the behaviour of a random version of its model but more importantly in section 1.4 where the models we study can be seen as random walks among random conductances.
But first, let us start by defining the simple random walk. It is a Markov chain on $\mathbf{Z}^d$ (for any dimension d), started at 0 where at each step the walker choose a neighbouring position uniformly at random (with probability $1/2d$). This means that if S_n is the position at time n of the random walk then the random variables $(S_{n+1} - S_n)_{n \in \mathbf{N}}$ are iid.
This simple description as a sum of iid random variables makes it possible to do explicit calculations and get incredibly precise results on the behaviour of this walk. First by simply applying the law of large number we get:

$$\frac{S_n}{n} \to 0 \text{ almost surely,}$$

and then, by applying the central limit theorem we get:

$$\frac{S_n}{\sqrt{n}} \to \mathscr{N}(0, I_d) \text{ in law.}$$

An important question for random walks is whether they are **recurrent** or **transient**, depending on the dimension.

Definition 1.

- *The simple random walk is said to be* ***recurrent*** *if it comes back to the origin infinitely often almost surely.*
- *The simple random walk is said to be* ***transient*** *if it comes back to the origin finitely many times almost surely.*

In dimension $d = 1$, the probability that the walk comes back to the origin at time $2n$ is equal to:

$$\mathbb{P}(S_{2n} = 0) = 2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{n\pi}}.$$

From this it is possible to show that in dimension d, the probability that the walk comes back to the origin at time $2n$ is of order $n^{-d/2}$. This means that on average, the walk comes back infinitely often to the origin in dimensions 1 and 2 and only finitely many times in dimension $d \geq 3$. From this we can deduce the following:

Theorem 1. *In dimensions $d = 1$ and $d = 2$, the simple random walk is recurrent while in dimensions $d \geq 3$, the simple random walk is transient.*

A simple generalization of the simple random walk is to still look at it as a sum of iid random variables but changing the law of these variables. We could authorize steps of length more than one but we will restrict ourselves to jumps to the nearest neighbours. The walk is said to be balanced if $\mathbb{E}(S_1) = 0$ and biased otherwise. If the walk is balanced then the previous results hold true (if you can go in all directions with positive probability). If the walk is biased then the walk is transient in any dimension. More precisely we still have a law of large number and a CLT which yield:

$$\frac{S_n}{n} \to \mathbb{E}(S_1) \text{ and } \frac{S_n - n\mathbb{E}(S_1)}{\sqrt{n}} \to \mathrm{Var}(S_1)\mathscr{N}(0,1).$$

This is not the only possible generalization of the simple random walk. One interesting such generalization is the electrical network. Representing the simple random walk as an electrical network creates a probabilistic interpretation of physical concepts like resistances, conductances, current, potential, and energy.

1.1.2 Electrical network

Before explaining the link between electrical networks and the simple random walk, we must first give a precise definition of an electrical network.

Definition 2. *An electrical network is a non-directed graph (V, E) to which we add positive weights $(W_e)_{e\in E}$ on the edges. The weights are called* ***conductances*** *and their inverse* ***resistances****.*

Now, how do we link electrical networks to random walks ? First we choose an arbitrary starting point $x_0 \in V$ (it can even be random). Then, when the walk S is at a point x, it chooses a neighbour proportionally to the conductances, i.e:

$$\mathbb{P}_{x_0}(S_{n+1} = y | S_n = x) = \frac{W_{\{x,y\}}}{\sum\limits_{z\sim x} W_{\{x,z\}}}.$$

To get the simple random walk in dimension d, we just need to take the graph $\mathbf{Z}^d$ and set all the conductances to be equal. For now, this definition gives a generalization of the simple random walk but we loose the notion of sum of iid random variables in this general setting. Fortunately, it gives rise to new tools.

Definition 3. *Let (V, E, W) be an electrical network and $In, Out \in V$ two distinct vertices of the graph. A* ***unitary flow*** *f from In to Out is a function from the* ***directed edges*** *to $\mathbf{R}$ such that:*

- *for every vertices $x \sim y$, $f((x,y)) = -f((y,x))$,*
- *for every vertices $x \in V\backslash\{In, Out\}$, $\sum\limits_{y\sim x} f((x,y)) = 0$,*
- $\sum\limits_{y\sim In} f((In,y)) = 1.$

It follows from this definition that we also have $\sum\limits_{y\sim Out} f((y, Out)) = 1$.

Definition 4. *Let (V, E, W) be an electrical network and $In, Out \in V$ two distinct vertices of the graph. A* ***unitary potential*** *U from In to Out is a function from U to $\mathbf{R}$ such that $U(In) = 1$ and $U(Out) = 0$.*

From these two definitions we can define the notion of **energy**.

Definition 5. *Let (V, E, W) be an electrical network, U a unitary potential on this network, and j a unitary flow on this network. Their respective energies $\mathscr{E}$ are defined by:*

$$\mathscr{E}(j) = \frac{1}{2} \sum_{\{x,y\}\in E} \frac{1}{W_{\{x,y\}}} (j((x,y)))^2 \text{ and}$$
$$\mathscr{E}(U) = \frac{1}{2} \sum_{\{x,y\}\in E} W_{\{x,y\}} (U(x) - U(y))^2.$$

In both sums, each non-directed edge is counted only once.

In these definitions of the energy we just sum on every edge the energy contained within the edge. The energy contained within an edge is given by the formulae $\frac{1}{2}Rj^2$ and $\frac{1}{2}C(\nabla U)^2$ (where R is the resistance and C the conductance). As often in physics, we will try to minimize the energy.

Theorem 2. *Let (V, E, W) be a finite, connected electrical network and $In, Out \in V$ two distinct vertices of the graph. There exists a unique unitary potential U_{min} from In to Out and a unique unitary flow j_{min} from In to Out that minimize the energy. The potential U_{min} is harmonic on $V \backslash \{In, Out\}$, that is to say:*

$$\forall x \in V \backslash \{In, Out\}, \ \sum_{y \sim x} W_{\{x,y\}} U_{min}(y) = U_{min}(x) \sum_{y \sim x} W_{\{x,y\}}.$$

As for the minimal unitary flow j_{min}, it derives from the minimal unitary potential U_{min}, that is to say there exists a constant C such that:

$$\forall x \sim y, \ C j_{min}((x, y)) = W_{\{x,y\}} \left(U_{min}(y) - U_{min}(x) \right).$$

Remark 1. *The minimal potential from Out to In is equal to 1 minus the minimal potential from In to Out. Similarly, the minimal flow from Out to In is equal to minus the minimal flow from In to Out.*

The potential $U_{\min}$ has a simple probabilistic interpretation. It represents, for any vertex, whether it is easier to go to In or Out starting from that point. More precisely if τ_x is the first time such that $S_{\tau_x} = x$. We then have:

$$\mathbb{P}_x \left(\tau_{In} < \tau_{Out} \right) = U_{\min}(x).$$

This interpretation leads to an extremely useful result. Let U be unitary the potential from In to Out that minimizes the energy. We have:

$$\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right) = \frac{2\mathscr{E}(U)}{\sum\limits_{y \sim In} W_{\{In,y\}}},$$

where τ_{In}^{+} is the **return time**: the first time strictly after 0 such that $S_{\tau_{In}^{+}} = In$. The proof is just a simple calculation that only uses that U_{min} is harmonic outside of $\{x, y\}$:

$$\begin{aligned}
2\mathscr{E}(U) &= \sum_{\{x,y\} \in E} W_{\{x,y\}} \left(U(x) - U(y) \right)^2 \\
&= \sum_{x \in V} U(x) \sum_{y \sim x} W_{\{x,y\}} \left(U(x) - U(y) \right) \\
&= \sum_{x \in V \backslash \{In, Out\}} U(x) \times 0 + 0 \times \sum_{y \sim Out} W_{\{Out,y\}} \left(0 - U(y) \right) + 1 \times \sum_{y \sim In} W_{\{In,y\}} (1 - U(y)) \\
&= \left(\sum_{y \sim In} W_{\{In,y\}} \right) \mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right).
\end{aligned}$$

Similarly, let j be the unitary flow from In to Out that minimizes the energy. We have:

$$\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right) = \frac{1}{2\mathscr{E}(j) \sum\limits_{y \sim In} W_{\{In,y\}}}.$$

Definition 6. *The quantity $\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right) \sum\limits_{y \sim In} W_{\{In,y\}}$ is called the **effective conductance** between In and Out. Its inverse is called the **effective resistance** between In and Out.*

Remark 2. *The effective conductance and resistance between In and Out are the same as the effective conductance and resistance between Out and In.*

This means that we can easily bound the value of $\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right)$ by exhibiting explicit unitary flows and potential. Indeed, if U is a unitary potential from In to Out, j a unitary flow from In to Out and $U^{\min}$ and $j^{\min}$ the unitary potential and flow from In to Out that minimize the energy we have:

$$\begin{aligned}
\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right) &= \frac{2\mathscr{E}\left(U^{\min} \right)}{\sum\limits_{y \sim In} W_{In,y}} \leq \frac{2\mathscr{E}(U)}{\sum\limits_{y \sim In} W_{In,y}} \text{ and} \\
\mathbb{P}_{In} \left(\tau_{In}^{+} > \tau_{Out} \right) &= \frac{1}{2\mathscr{E}\left(j^{\min} \right) \sum\limits_{y \sim In} W_{In,y}} \geq \frac{1}{2\mathscr{E}(j) \sum\limits_{y \sim In} W_{In,y}}.
\end{aligned}$$

It is actually possible to use this method to get the following property on effective conductances and resistances that considerably simplify their study:

Theorem 3 (Rayleigh monotonicity)**.** *Let $G = (V, E)$ be a finite graph and In and Out two vertices of that graph. Let W^-, W^+ be weights on that graph such that for any $e \in E, W_e^- \leq W_e^+$. Let C_{eff}^- and C_{eff}^+ be the effective conductances between In and Out on (G, W^-) and (G, W^+) respectively. Let R_{eff}^- and R_{eff}^+ be the effective resistances between In and Out on (G, W^-) and (G, W^+) respectively. We have:*

$$C_{\mathit{eff}}^- \leq C_{\mathit{eff}}^+, \text{ and } R_{\mathit{eff}}^- \geq R_{\mathit{eff}}^+.$$

Now, it is possible to use these results to gain information on transience and recurrence on infinite graphs. If we think about it in a non-rigorous way, by taking Out to be infinity then $\mathbb{P}_{In}\left(\tau_{In}^+ > \tau_{Out}\right)$ becomes $\mathbb{P}_{In}\left(\tau_{In}^+ = \infty\right)$ (the walk goes to infinity before coming back to In means that the walk never comes back to In). This means that it is equal to 0 if the walk is recurrent and is strictly positive if it is transient. Then we know we can bound this value by using the energy so we can use potentials and flows to show whether the walk starting from In is recurrent or transient. This can be done rigorously and we get the following results:

Theorem 4. *Let (V, E, W) be an electrical network and $In \in V$ a vertex. The walk starting from In is transient if and only if there is a unitary flow starting from In (and with no Out) with finite energy.*

Theorem 5. *Let (V, E, W) be an electrical network and $In \in V$ a vertex. The walk starting from In is recurrent if and only if there is a sequence of unitary potential starting from In (and with no Out) such that their energy goes to 0.*

From these two results it is possible to get back the result from theorem 1 that the simple random walk is recurrent in dimension 1 and 2 but transient in dimensions 3 and above. One advantage of this method compared to explicit calculations is that it is not reliant on all the weights being the same. Indeed, by modifying slightly the weights but looking at the same potentials and flows, the energies are not modified by much and transience and recurrence are not changed. More precisely:

Theorem 6. *Let $G = (V, E)$ be an infinite graph and In a vertex of that graph. Let W^-, W^+ be weights on that graph such that for any $e \in E, W_e^- \leq W_e^+$. If the walk on (G, W^-) starting from In is transient then so is the walk on (G, W^+) starting from In. Similarly if the walk on (G, W^+) starting from In is recurrent then so is the walk on (G, W^-) starting from In.*

In particular, we get by comparing an electrical network to the simple random walk:

Theorem 7. *In dimension $d \in \{1, 2\}$, an electrical network on $\mathbf{Z}^d$ with weights W for which there exists a constant C such that for any edge e, $W_e \leq C$ is recurrent.*
In dimension $d \geq 3$, an electrical network on $\mathbf{Z}^d$ with weights W for which there exists a constant C such that for any edge e, $W_e \geq C$ is transient.

Remark 3. *For the recurrence in dimension 1 and 2, weights equal to 0 are allowed which means that in dimension 2 the walk on the infinite percolation cluster is recurrent for instance.*

Another interesting aspect of the electrical network is its **invariant measure**. The invariant measure π is extremely simple to compute in the case of electrical networks. Indeed, for a finite electrical network (V, E, W), the invariant measure π is given by:

$$\forall x \in V, \ \pi_x := \sum_{y \sim x} W_{\{x,y\}}.$$

Indeed, we have:

$$\forall x \in V, \ \sum_{y \sim x} \frac{W_{\{x,y\}}}{\sum\limits_{z \sim y} W_{\{y,z\}}} \pi_y = \sum_{y \sim x} W_{\{x,y\}} = \pi_x.$$

This results also extends to infinite graphs. The measure π defined as above is an invariant measure in the case of infinite graphs.
This result is important because of its link to another important property of electrical networks: they are **reversible**. This means that for any path $x_0, x_1, \ldots, x_n$, we have:

$$\mathbb{P}\left(X_n = x_n, \ldots, X_1 = x_1 | X_0 = 0\right) \pi_{x_0} = \mathbb{P}\left(X_n = x_0, X_{n-1} = x_1, \ldots, X_1 = x_{n-1} | X_0 = x_n\right) \pi_{x_n}.$$

This has several consequences. First this gives bounds on the probability to be at a specific vertex at a specific time. For instance:

$$\mathbb{P}_0(S_n = x) = \mathbb{P}_x(S_n = 0) \frac{\pi_x}{\pi_0} \leq \frac{\pi_x}{\pi_0}. \tag{1.1}$$

This can be used if π_x is small compared to π_0. Another example is the following bound:

$$\mathbb{P}_0(S_{2n} = x) \leq \sqrt{\mathbb{P}_0(S_{2n} = 0) \mathbb{P}_x(S_{2n} = x) \frac{\pi_x}{\pi_0}}.$$

The proof is quite short:

$$\begin{aligned}
\mathbb{P}_0(S_{2n} = x) &= \sum_y \mathbb{P}_0(S_n = y)\mathbb{P}_y(S_n = x) \\
&= \sum_y \mathbb{P}_0(S_n = y)\mathbb{P}_x(S_n = y)\frac{\pi_x}{\pi_y} \\
&\leq \sqrt{\frac{\pi_x}{\pi_0} \sum_y \mathbb{P}_0(S_n = y)^2 \frac{\pi_0}{\pi_y} \sum_y \mathbb{P}_x(S_n = y)^2 \frac{\pi_x}{\pi_y}} \\
&= \sqrt{\mathbb{P}_0(S_{2n} = 0)\mathbb{P}_x(S_{2n} = x)\frac{\pi_x}{\pi_0}}
\end{aligned}$$

This can be used to bound transition probabilities when the probability to return to the starting points are well understood. It is actually possible to get good estimates of transition probabilities when the conductances are uniformly elliptic. The conductances are said to be uniformly elliptic when there exist two constants $c_1 \leq c_2$ such that for every edge e, $c_1 \leq w_e \leq c_2$. When the graph is $\mathbf{Z}^d$ and the conductances are uniformly elliptic then we can get precise bounds on the transition probabilities. In [28], Delmotte proved the following bounds:

Theorem 8. *For any dimension d, for any uniformly elliptic conductances on $\mathbf{Z}^d$, there exists constants c_l, C_l, c_r, C_r such that for any $n \geq 1$, for any vertices x, y such that $d(x, y) \leq 2n$ and such that the graph distance between x and y is even, we have:*

$$\frac{c_l}{n^{d/2}} e^{-C_l d(x,y)^2} \leq \mathbb{P}_x(X_{2n} = y) \leq \frac{c_r}{n^{d/2}} e^{-C_r d(x,y)^2}.$$

The reason we look at X_{2n} is because otherwise there would be some issues with the parity of the distance to the starting point. This issue only concerns the lower bound. The result by Delmotte is actually much more general than this since it characterizes exactly which graph satisfy the above property and for which conductances.

1.2 Random walks in random environments

If we want to introduce an extra layer of randomness on the simple random walk, there are multiple ways to do it. We could look at random graphs but here we will restrict ourselves to the study of $\mathbf{Z}^d$ with random transition probabilities. We will only talk about two specific cases:

- random walks among random conductances, and
- random walks in iid environments.

We will start by random walks among random conductances which are a generalization of the model studied in the previous section where we add some randomness on the conductances. As discuss previously, in section 1.4 we will discuss a specific kind of random walk in random conductances. Moreover this will serve as an introduction to random walk in iid environments because both model share some similar behaviours and some tools and concept apply for both concept. Furthermore because the random conductance model allows for several simplification compared to the iid environment model it is better understood. Finally, the next section is devoted to the study of a specific kind of random walk in iid environments.
Before we start discussing the two models of random walks, we will explain what we mean by **environment**.

Definition 7. *The set Ω of environments on a graph $\mathscr{G} = (V, E)$ is the set of all possible transition probabilities on this graph:*

$$\Omega := \left\{ \omega \in [0,1]^E,\ \forall x \in V, \sum_{y \sim x} \omega(x, y) = 1 \right\}.$$

*An **environment** is an element of the set Ω.*
In the case of random walks in conductances the environment is usually described in terms of conductances instead of transition probabilities.

The random walk in iid environments consist of having random transition probabilities at each site (vertex) instead of deterministic ones. These transition probabilities are iid at each site. The random walk among conductances consists of looking at the simple random walk as a random walk in an electrical network. Except that instead of looking at deterministic conductances we look at random ones. Once again they are chosen to be iid.

1.2.1 Random walks among random conductances

We will start by the case of random walk among random conductances. In this model, the environment is always reversible, this means that we restrict ourselves to a subset of all possible environments but all the results and techniques that come from electrical networks can be applied to this model.
Before going into the details of what we know of this model we will first see what difference can arise with the simple random walk when we authorise random conductances. First, when the conductances are of order 1, the behaviour of the walk is quite similar to that of the simple random walk. If we think of effective conductances then if all the conductances are between $1/c$ and c for some constant c then the effective conductance between two points will not differ by a factor more than c from the effective conductance between those same two points for the simple random walk.
If the conductances can be allowed to be 0 then we effectively change the geometry of the underlying graph. For instance we can go from a walk on $\mathbf{Z}^d$ to a walk on a percolation cluster or from a walk on $\mathbf{Z}^3$ to a walk on $\mathbf{Z}^2$ if all the conductances in a specific direction are equal to 0.
If the conductances can be arbitrarily large then we can see traps appear. Traps in this case will be portions of the graph where the conductances are large compared to the surrounding area (think of an edge with large conductance surrounded by edges with conductance of order 1). It is extremely easy for the walk to enter such an area because the walk tends to favour large conductances. On the other hand, it is quite hard for the walk to exit such a region, this means that the walk stays trapped in this region for a large amount of time.
If the conductances can be arbitrarily small then we can see another kind of trap appear. Traps in this case will be portions of the graph where the conductances are of order 1 surrounded by edges with small conductances (think of one edge with conductance of order 1 surrounded by edges with small conductances). If the walk enters such a trap then it will stay there a long time. However, contrarily to the previous kind of trap it is extremely hard to enter such a trap. In fact, the longer the walk stays in a trap, the harder it is to enter such a trap. This mean that these trap do not have much impact on average but can have an impact on the likelihood of unlikely events.

Uniform ellipticity

We first want to look at the case where the conductances are bounded from above and from below. Let μ be a probability measure on $\mathbf{R}^+$ such that there exists a constant $c > 1$ such that $\mu\left(\frac{1}{c}, c\right) = 1$. Set a dimension d. Let E_d be the set of edges of $\mathbf{Z}^d$. Let $\Omega := (\mathbf{R}^+)^{E_d}$ be the set of environments on $\mathbf{Z}^d$. For any environment $\omega \in \Omega$ and any vertex $x \in \mathbf{Z}^d$, let P_x^ω be the probability measure associated with the reversible Markov chain $(X_n)_{n\in\mathbf{N}}$ started at x and with conductances given by the environment ω. Now let $\mathbb{P}_\mu$ be the probability measure on Ω where all the conductances are iid of law μ. Let $\mathbb{E}_\mu$ be the expectation with respect to $\mathbb{P}_\mu$ and $\mathbb{E}_x^\omega$ the expectation with respect to P_x^ω. Finally, let $\mathbb{P}_x^\mu$ be the law of the walk starting in x, averaged on the environments ω: $\mathbb{P}_x^\mu(\cdot) := \mathbb{E}_\mu\left(P_x^\omega(\cdot)\right)$ and $\mathbb{E}_x^\mu$ the associated expectation.
An important distinction when we look at random walks in random environment is whether a results holds for almost every environment or whether it only holds true on average. If the result is true for almost every environment then we say that it is a **quenched** result and if it is only true on average then we say that it is an **annealed** or **averaged** result. For instance if we look at the position of the walk X after time n, it is obvious that $\mathbb{E}_\mu\left(\mathbb{E}_0^\omega\left(X_n\right)\right) = 0$ because of symmetries. This is an annealed result and the corresponding quenched result would be that for almost every environment, $\mathbb{E}_0^\omega\left(X_n\right) = 0$ which is obviously false.
One example of quenched result in the case of bounded conductances is the result by Delmotte. It tells us that there exist constants c_l, C_l, c_r, C_r such that for any $n \geq 1$, for any vertices x, y such that the graph distance $d(x,y)$ between x and y is even, $d(x,y) \leq 2n$ and such that the graph distance between x and y, and for $\mathbb{P}_\mu$ almost every environment ω, we have:

$$\frac{c_l}{n^{d/2}} e^{-C_l d(x,y)^2} \leq P_x^\omega(X_{2n} = y) \leq \frac{c_r}{n^{d/2}} e^{-C_r d(x,y)^2}.$$

This in turn leads to the same annealed statement. There exist constants c_l, C_l, c_r, C_r such that for any $n \geq 1$, for any vertices x, y such that $d(x,y) \leq 2n$ and such that the graph distance between x and y is even, we have:

$$\frac{c_l}{n^{d/2}} e^{-C_l d(x,y)^2} \leq \mathbb{P}_x^\mu(X_{2n} = y) \leq \frac{c_r}{n^{d/2}} e^{-C_r d(x,y)^2}.$$

These results strongly suggest that the walk behaves like a Brownian motion. To prove this, we actually need the randomness of the environment. Indeed in one dimension, we could imagine the case where all the conductances to the left of the origin are equal to 2 while all the conductances to the right of the origin are equal to 1. In such a case the walk would spend twice as much time to the right of the origin as it spends on the left and thus it could not converge to a brownian motion. With the randomness of the environment such imbalances cannot exist on a large scale and therefore it was shown in [84] that the walk satisfies a quenched

CLT where the variance does not depend on the environment. That is to say that there exists $\sigma > 0$ such that for $\mathbb{P}_\mu$ almost every environment the processes $t \to \frac{1}{\sqrt{n}} X_{nt}$ converge to a Brownian motion of covariance matrix $\sigma^2 I_d$. This also implies an annealed CLT with the same covariance matrix.

What happens on the percolation cluster ?

If we only authorize the conductances to have value 0 or 1 then we get a walk on a percolation cluster. If the probability to have the value 0 is too high then there is no infinite percolation cluster and the behaviour of the walks is less interesting, for instance it cannot be transient. So we will restrict ourselves to the supercritical case where there is a unique infinite cluster. In particular we will not consider the one dimensional case. And we will also condition on the fact that the origin is in the infinite percolation cluster.
The geometry of the percolation cluster is similar to that of $\mathbf{Z}^d$ on a large scale. For instance the distances are the same up to a multiplicative constant, and balls have similar volumes. However on small scales, every pattern can appear on the percolation cluster. This means that on a small scale the percolation cluster and $\mathbf{Z}^d$ can be extremely different. For instance the percolation cluster on $\mathbf{Z}^d$ can locally look like $\mathbf{Z}$. This means that over large amount of time the walk on $\mathbf{Z}^d$ and on the percolation cluster should be similar but they can be extremely different at the beginning. This is illustrated by the following result:

Theorem 9 (Theorem 1 of [3])**.** *Let μ be the probability measure defined by $\mu(\{1\}) = p = 1 - \mu(\{0\})$ such that p is larger than the critical parameter for the percolation. There exist constants c_l, C_l, c_r, C_r such that for $\mathbb{P}_\mu$ almost every environment ω there exists a constant S_ω such that $\forall t \geq S_\omega,\ \forall x \in \mathbf{Z}^d$:*

$$\frac{c_l}{n^{d/2}} e^{-C_l d(0,x)^2} \leq P_0^\omega(X_t = x) \leq \frac{c_r}{n^{d/2}} e^{-C_r d(0,x)^2} \text{ if } 0 \text{ and } x \text{ are in the percolation cluster and } d(0,x) \leq t,$$

where $d(\cdot,\cdot)$ is the graph distance on the percolation cluster.

In this case the walk is slightly different than the one we defined previously. Here the time is continuous instead of discrete. The time between the consecutive jumps is not 1 any more but iid exponential random variables, independent of everything else. The proof can be extended to the discrete time case.
As can be expected from the previous result, a quenched CLT, similar to that of the previous section, also holds for the random walk on the percolation cluster as was proved simultaneously in [10] and [57].

With only an upper bound

Now what happens if we only impose an upper bound on the conductances ? We still allow the value 0, so we impose that the probability that an edge has a positive conductance is larger than the critical threshold for percolation. We call $\mathscr{C}(\omega)$ the infinite cluster for the environment ω (an edge is closed if its conductance is 0), which is almost surely well-defined. The main difference with the previous cases is that now the walk can get trapped in a finite area. This happens if the walk enters an area with conductances of order 1 surrounded by small conductances. It is hard for the walk to enter or exit such an area. This means that it is unlikely for the walk to enter such a place but once it enters it, it can stay in it for a really long time. Because of this the gaussian bounds may not hold any more. For instance if we look at the probability of going back to 0 after time n, the walk can wander a bit, enter a trap surrounded by conductances of order $1/n$ (which has a probability of happening of order $1/n$) then exit it (still a probability of order $1/n$ to happen at a specific time) and then go back to 0. If the trap is close enough to the origin then finding the trap and going back to the origin at time n is greater than something of order $1/n^2$ instead of $n^{-d/2}$ for gaussian bounds. More precisely, it was shown in [11] that for almost every environments ω (such that 0 is in the infinite cluster) there exists a constant $C(\omega)$ such that for every $n \geq 1$:

$$P_0^\omega(X_n = 0) \leq C(\omega) \begin{cases} n^{-d/2}, & d = 2,3, \\ n^{-2}\log(n) & d = 4, \\ n^{-2} & d \geq 5. \end{cases}$$

Conversely, it was also shown in [11] that you can get arbitrarily close to this bound in dimension 5 and more. In [16] it was shown that it is possible to get arbitrarily close to the bound in dimension 4. This means that in dimension 4 and above we can observe an anomaly for the quenched probability of coming back to the origin at time n. If we instead look at the annealed probability to come back to the origin, the anomaly can exist in all dimensions (because of the environments where the origin is in a trap).
We could imagine that this implies that the CLT fails in this case but this is not what happens. Indeed it was shown in [27] that the walk satisfies an annealed CLT. This result is actually more general because it does not require the conductances to be independent, only weaker assumptions. This was later improved in [56] where it was proved that the walk satisfies a quenched CLT where the variance is deterministic. That it to say that there exists $\sigma > 0$ such that for $\mathbb{P}_\mu$ almost every environment the processes $t \to \frac{1}{\sqrt{n}} X_{nt}$ converge to a brownian

motion of covariance matrix $\sigma^2 I_d$.
To understand why the CLT holds we introduce an important tool for random walks among conductances: **the point of view of the particle**. Instead of looking at a trajectory $(X_n)_{n\in\mathbf{N}}$ within an environment ω, we look at a sequence of environments $(\omega_n)_{n\in\mathbf{N}}$. Let θ_u be the shift of vector u for environments $(\theta_u(\omega)(x,y) := \omega(x+u, y+u))$. The trajectory from the point of view of the particle $(\omega_n)_{n\in\mathbf{N}}$ is defined by $\omega_n := \theta_{-X_n}(\omega)$. Instead of following the particle within the environment we shift the environment so that the particle remains at the origin. This has an important consequence: the sequence $(\omega_n)_{n\in\mathbf{N}}$ is a Markov chain while the sequence $(X_n)_{n\in\mathbf{N}}$ is not (for it to be one we need to know the environment). Furthermore it is a reversible Markov chain. This means that we know an invariant measure for it, the measure:

$$\mathbb{Q}_\mu(\mathrm{d}\omega) := \left(\sum_{x\sim 0} \omega(\{0,x\})\right) 1_{0\in\mathscr{C}(\omega)} \mathbb{P}_\mu(\mathrm{d}\omega).$$

The quantity $\left(\sum_{x\sim 0} \omega(\{0,x\})\right) 1_{0\in\mathscr{C}(\omega)}$ is bounded so the measure $\mathbb{Q}_\mu$ has a finite mass and by multiplying it by a constant we can obtain a probability measure $\mathbb{Q}^1_\mu$. Let $\tilde{\mathbb{P}}_\mu$ be the measure $\mathbb{P}_\mu$ conditioned on the origin being part of the infinite cluster. Clearly, the measures $\mathbb{Q}^1_\mu$ and $\tilde{\mathbb{P}}_\mu$ are absolutely continuous with respect to one another. This means that events that are almost sure for one are also almost sure for the other. Moreover we can apply ergodic theorems to $\mathbb{Q}^1_\mu$. For instance let A be a measurable event, we have $\tilde{\mathbb{P}}_\mu$ almost surely:

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\omega_i \in A} \to \mathbb{Q}^1_\mu(A).$$

This invariant measure was used in [56] to show a quenched CLT (invariant measures had already been used previously in [52] to show an annealed CLT for bounded conductances). The idea of the proof is to set a threshold ϵ, consider the percolation cluster $\mathscr{C}^\epsilon(\omega)$ where an edge is closed if its conductance is lower than ϵ, and look at the walk X only at time when it is on the percolation cluster $\mathscr{C}^\epsilon(\omega)$. This gives a new walk X^ϵ. For this new walk it is possible to apply the same methods as for the percolation cluster which gives a quenched CLT. Then we use the invariant measure to bound the proportion of the time the walk spends outside $\mathscr{C}^\epsilon(\omega)$. This is used to show that the processes X^ϵ and X are always "close". Then by taking ϵ to 0 it is possible to show that the original walk X also satisfies a quenched CLT.
The reason why the CLT holds but the gaussian bounds do not can be seen in the invariant measure. The failure of the gaussian bounds comes from traps, that is to say areas surrounded by small conductances. The walk is unlikely to enter them but it can be sufficient for the gaussian bounds to fail around the origin. However, the invariant measure and the ergodic principle tells us that the walk cannot spend too much time in traps which means that the traps cannot have too important an impact on the global trajectory.

General measure

The general case is more complicated. When the walk stumbles upon an edge with large conductance, it is likely to cross it many time before visiting other parts of the graph. If the conductances have no expectation then we can expect that the walk will spend most of its time trapped on edges with larger and larger conductances. This is reflected by the fact that the invariant probability introduced earlier does not exist any more when the conductances do not have finite expectation. Therefore we cannot expect to have a CLT under this condition. For instance, if μ has a heavy enough polynomial tail, it has been shown in [4] that the process does not converge to a brownian motion but to a **fractional-kinetics process**.

Definition 8. *The fractional-kinetics process $Z_{d,\alpha}$ is defined as follows: let B be a brownian motion in dimension d with covariance I_n and let S_α be a completely asymmetric (positive) α-stable Levy process, then for all $s \in \mathbf{R}$:*

$$Z_{d,\alpha}(s) := B(S_\alpha^{-1}(s)).$$

However, when the conductances have finite expectation then it was proved in [1] that the walk converges to a Brownian motion.
It is important to note that the reason why the CLT fails is because the walk can be trapped locally. This does not mean that on a global scale the behaviour is much different from a slowed down Brownian motion (i.e with a time change). To investigate this, we can introduce a time changed version of our process, the variable speed random walk (VSRW). Instead of waiting a time 1 or a random exponential time of expectation 1 between each jump, when it is on a vertex x, the VSRW wait a random exponential time of expectation $\sum_{y\sim x} \omega(\{x,y\})$. The idea is that if the VSRW encounters an edge with a large conductance W, it will cross it a number of time of order W before visiting other edges but will only wait a time of order $1/W$ between each crossing. This means

that it will only spend a time of order 1 on this edge before visiting an other one. Therefore the VSRW should not be trapped on edges with large conductances. A clue in that direction is that for the VSRW, the invariant probability measure exists. It was actually shown in [1] that as long as there is an infinite percolation cluster with probability one, the VSRW conditioned on the origin being part of the percolation cluster converges to a non-degenerate Brownian motion.

1.2.2 Random walks in iid environments

We will only consider nearest neighbours walks on $\mathbf{Z}^d$ and call E_d the set of oriented vertices (the vertices are between nearest neighbours). We set a dimension d, let S_d be the set defined by $S_d := \{x \in (0,\infty)^d, \sum x_i = 1\}$ and Ω the set of environments on $\mathbf{Z}^d$ defined by:

$$\Omega := \left\{ \omega \in (0,\infty)^{E_d},\ \forall x \in \mathbf{Z}^d,\ \sum_{y \sim x} \omega(x,y) = 1 \right\}.$$

For any law μ on S_d, the law $\mathbb{P}_\mu$ on the set of environments is the law such that the transition probabilities at each vertex are i.i.d of law μ.
The case of the dimension $d = 1$ and $d \geq 2$ are quite different. The reason for this is that random walks in random environments are still reversible in dimension $d = 1$ (and more generally on trees) but not in dimension $d \geq 2$. This means that all tools used for the random walk among random conductances can also be used in dimension 1 while new tools need to be developed for dimension 2 and above. For this reason dimension 1 is better understood than higher dimensions.

Dimension 1

The one dimensional case has been studied since the 70's. It was first studied by Solomon [80] who in 1975 identified 3 different regimes through explicit calculations:

Theorem 10. *Let* $\rho_0 := \frac{\omega(0,1)}{\omega(0,-1)} = \frac{\omega(0,1)}{1-\omega(0,1)}$.
If $\mathbb{E}_\mu(\log(\rho_0)) = 0$ *then the random walk* X *is recurrent.*
If $\mathbb{E}_\mu(\log(\rho_0)) > 0$ *(respectively* $\mathbb{E}_\mu(\log(\rho_0)) < 0$*) then* $X_n \to +\infty$ *(respectively* $-\infty$*) almost surely.*
In the case $\mathbb{E}_\mu(\log(\rho_0)) > 0$, *if* $\mathbb{E}_\mu(1/\rho_0) < 1$ *then*

$$\frac{X_n}{n} \to \frac{1 - \mathbb{E}_\mu(1/\rho_0)}{1 + \mathbb{E}_\mu(1/\rho_0)} \text{ almost surely,}$$

and if $\mathbb{E}_\mu(1/\rho_0) \geq 1$ *then*

$$\frac{X_n}{n} \to 0 \text{ almost surely.}$$

Remark 4. *There is still a characterization of recurrence/transience (and of the direction of transience) if* $\mathbb{E}_\mu(\log(\rho_0))$ *is ill-defined.*

The original proof is based on explicit calculations but to better understand what happens it easier to adopt the point of view of the random potential introduced by Sinai in [78]. In this case, we view the walk as a random walk among random conductances. For any $N \in \mathbf{Z}$, let $\rho_n := \frac{\omega(n,n+1)}{\omega(n,n-1)}$. We define the potential $U : \mathbf{Z} \mapsto \mathbf{R}$, for all $x \in \mathbf{Z}$ by:

$$U(x) := \begin{cases} \sum_{i=1}^{x} -\log(\rho_i) & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ \sum_{i=x}^{0} \log(\rho_i) & \text{if } x \leq -1. \end{cases}$$

In the formalism of the previous section, the conductance for the edge $\{x, x+1\}$ would be given by $\exp^{-U(x)}$. Now that the conductances are defined we first need to understand the behaviour of the conductances before we can look at the walk within those conductances. The potential that we have defined is a sum of independent random variables. If $\mathbb{E}_\mu(\log(\rho_0))$ is well defined then we have that:

$$\begin{aligned} &\lim_{n\to+\infty} U(n) = -\infty \text{ and } \lim_{n\to-\infty} U(n) = +\infty && \text{if } \mathbb{E}_\mu(\log(\rho_0)) > 0 \\ &\liminf_{n\to+\infty} U(n) = \liminf_{n\to-\infty} U(n) = -\infty \text{ and } \limsup_{n\to+\infty} U(n) = \limsup_{n\to-\infty} U(n) = +\infty && \text{if } \mathbb{E}_\mu(\log(\rho_0)) = 0 \\ &\lim_{n\to+\infty} U(n) = +\infty \text{ and } \lim_{n\to-\infty} U(n) = -\infty && \text{if } \mathbb{E}_\mu(\log(\rho_0)) < 0 \end{aligned}$$

Now, in the first and third cases, the potential is biased which means that the conductances go to 0 in one direction and $+\infty$ in the other. The walk tends to favour higher conductances and therefore the walk will be transient in the direction where the conductances go to infinity (the direction where the potential goes to $-\infty$). This is where the transient case comes from. If $\mathbb{E}_\mu(\log(\rho(0))) = 0$ the potential is not biased and as a consequence the walk will not favour any direction and will be recurrent. This explains why the value of $\mathbb{E}_\mu(\log(\rho(0)))$ determines the recurrent or transient case.
In the transient case, the walk can have positive speed (ballistic) or zero-speed (sub-ballistic). This last regime does not exists for the simple random walk. The zero speed is due to a trapping phenomenon that is slightly different than the one that appears for random walks among random conductances in the unbounded case. In both cases the slowing-down is due to finite regions of the graph where the walk stays trapped a long time. However the exact nature of the traps differs between the two models.

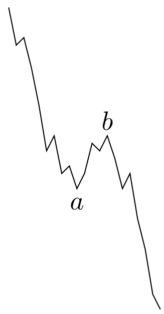


Figure 1.1: Potential with a trap between a and b

In the figure above, we see that the potential goes to $-\infty$ which means that the walk tends to go to the right. However, when it reaches a, it needs to climb the potential to reach b. This is complicated and it is not to hard to show that it will at least take a time of order $\exp(U(b) - U(a))$. This means that the walk is trapped in the valley around a for a time of order $\exp(U(b) - U(a))$. The difference between the zero-speed and the positive speed regime depends on those valleys. If there are too many valleys where the walk is trapped for too long then it will have zero-speed. If it is not the case then it will have positive speed.
Now that we know what the three regimes are, we can study more precisely the behaviour of the walk in each of these regimes. First, if the walk is recurrent, it was shown by Sinai in [78] that in time n the walk only travels a distance of order $\log(n)^2$. Then, Golosov [45] and Kesten [50] determined independently the limit of the rescaled random walk. This gives the following theorem:

Theorem 11. *Assume that $\mathbb{E}_\mu(\log(\rho_0)) = 0$, there exists $\epsilon > 0$ such that $\mu(\epsilon, 1-\epsilon) = 1$, and $0 < \sigma^2 := \mathbb{E}_\mu\left((\log(\rho_0))^2\right) < \infty$. We have under $\mathbb{P}_0^\mu$:*

$$\frac{\sigma^2}{(\log(n))^2} X_n \to Z \text{ in law,}$$
$$\frac{\sigma^2}{(\log(n))^2} \max_{0\le k\le n} X_k \to \overline{Z} \text{ in law,}$$

where Z is symmetric, $\overline{Z}$ is positive and their laws are characterized by the Laplace transform:

$$\mathbb{E}\left(\exp(-\lambda|Z|)\right) = \frac{\cosh(\sqrt{2\lambda}) - 1}{\lambda\cosh(\sqrt{2\lambda})},$$
$$\mathbb{E}\left(\exp(-\lambda\overline{Z})\right) = \frac{\tanh(\sqrt{2\lambda})}{\sqrt{2\lambda}}.$$

To understand why the walk only goes at a distance of order $\log(n)^2$ in time n instead of $\sqrt{n}$ for the simple random walk, we must once again understand what traps look like in this case.
If we look at the figure above, we see that there is a valley in the potential between b and c where the minimum is attained in a. When the walk is in such a valley, with high probability it goes to the bottom of the valley (a in our example). Then, the time the walk spends in the valley is of order $\exp(U_{max} - U_{min})$ where U_{min} is the potential at the bottom of the valley ($U(a)$ in our example) and U_{max} the minimum of the two potentials at the top of the valley ($U(c)$ in our example). If, before time n, the walks encounters a valley where the minimum and the maximum of the potential differ by $\log(n)$ then the walk will spend a time of order n in it and should therefore still be in it at time n. The question now becomes, where can we find such a valley? The

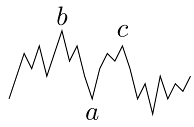


Figure 1.2: Potential with a trap between b and c with a minimum in a

increments of the potential are i.i.d, with zero expectation and finite variance. This means that we can expect the potential to behave like a Brownian motion. We therefore should see a valley of depth $\log(n)$ at a distance of order $\log(n)^2$. More precisely, the valley should be of length $\log(n)^2$ and contain the origin. This is where the $\log(n)^2$ rescaling comes from. For a fixed environment, at time n the walk is with high probability at the bottom of this valley of size $\log(n)$. Then if we average on the environment the position of this minimum we find the law of the previous theorem.
Now for the transient case, the limiting behaviour was first studied in 1975 in [51]. It was found that under some assumptions on the law of the environment, the position of the walk at time n could be described by stable laws. More precisely:

Theorem 12. *Assume that $\mathbb{E}_\mu(\log(\rho_0)) = \mathbb{E}(-U(1)) > 0$, there exists $\kappa > 0$ such that*

$$\mathbb{E}_\mu\left(\left(\frac{1}{\rho_0}\right)^\kappa\right) = 1 \text{ and } \mathbb{E}_\mu\left(\left(\frac{1}{\rho_0}\right)^\kappa \log\left(\frac{1}{\rho_0}\right) 1_{\rho_0 \leq 1}\right) < \infty,$$

and the law of $\log(\rho_0)$ is non-arithmetic. Then, depending on κ we have different limit theorems.
If $0 < \kappa < 1$, there exists a positive random variable Z with a κ-stable law such that under $\mathbb{P}_0^\mu$:

$$n^{-\kappa} X_n \to \left(\frac{1}{Z}\right)^\kappa \text{ in law.}$$

If $\kappa = 1$, there exists a function $\delta : [0, \infty) \mapsto \mathbf{R}$, a constant $c > 0$ and a random variable Z with a 1-stable law such that under $\mathbb{P}_0^\mu$:

$$\delta(t) \sim c\frac{t}{\log(t)} \text{ and } \frac{\log^2(t)}{t}(X_n - \delta(n)) \to Z \text{ in law.}$$

If $1 < \kappa < 2$, there exists a constant $C > 0$ and a random variable Z with a κ-stable law and zero mean such that under $\mathbb{P}_0^\mu$:

$$n^{-\frac{1}{\kappa}}(X_n - Cn) \to Z \text{ in law.}$$

If $\kappa = 2$, there exists a constant $C > 0$ and a centered gaussian Z such that under $\mathbb{P}_0^\mu$:

$$\frac{1}{\sqrt{n\log(n)}}(X_n - Cn) \to Z \text{ in law.}$$

If $\kappa > 2$, there exists a constant $C > 0$ and a centered gaussian Z such that under $\mathbb{P}_0^\mu$:

$$\frac{1}{\sqrt{n}}(X_n - Cn) \to Z \text{ in law.}$$

For the case where $\kappa < 1$, the precise κ-stable law was identified in [39]. In the particular case where the transition probabilities have a Beta law (which is the same as the Dirichlet law of the next chapter in dimension 1) of parameters (α, β) with $\kappa = \alpha - \beta \in (0, 1)$ the expression is quite simple. Let S^κ a positive random variable with Laplace transform $\mathbb{E}(e^{-\lambda S^\kappa}) = e^{-\lambda^\kappa}$, the stable random variable of the previous theorem is given by:

$$\frac{\sin(\pi\kappa)}{2^\kappa \pi} \frac{B(\alpha, \beta)^2}{\psi(\alpha) - \psi(\beta)} \left(\frac{1}{S^\kappa}\right)^\kappa.$$

When $\kappa > 1$, which implies $\mathbb{E}_\mu(\frac{1}{\rho_0}) < 1$, we see that the walk has a positive speed whose expression was already found in [80].

Higher dimensions

In higher dimensions, the walk is no longer reversible. This means that we loose the notion of potential. For this reason random walks in iid environments in dimension 2 and higher are not as well understood as in dimension 1 even though progress has been made. We will first start by stating some general results and presenting some useful tools.
First we need to define ellipticity which is a common assumption in random walks in iid random environments.

Definition 9. *An environment ω on a graph (V, E) is elliptic if for every edge $\{x, y\} \in E$, $\omega(x, y) > 0$. It is uniformly elliptic if there exists an $\epsilon > 0$ such that for every edge $\{x, y\} \in E$, $\omega(x, y) > \epsilon$.*

If we think of the random conductance model, ellipticity would be that there are no edges with conductance zero and uniform ellipticity that the conductance are bounded. We will always assume that the environments are almost surely elliptic and sometimes we will also assume that they are uniformly elliptic. One of the first things we would want to prove is that under only an assumption of ellipticity, the walk satisfies a law of large number i.e almost surely $\frac{X_n}{n} \to v$ where v is deterministic. This is actually quite hard to prove with no extra assumptions. The first result in that direction was a zero-one law regarding directional transience by Kalikow in [47]. This was then improved by Merkl and Zerner in [61] by only assuming that the environments are elliptic instead of uniformly elliptic.

Theorem 13. *Assume that almost surely the environment is elliptic. For any $l \in \mathbf{R}^d \backslash \{0\}$ let A_l be the event $\{X_n.l \to \infty\}$. For any $l \in \mathbf{R}^d \backslash \{0\}$:*

$$\mathbb{P}_0^\mu(A_l \cup A_{-l}) \in \{0, 1\}.$$

Furthermore, still in [61], it was shown in dimension 2 (and conjectured to be true in all dimensions) that there was actually a more satisfying $0-1$ law.

Theorem 14. *Assume that almost surely the environment is elliptic and set the dimension $d = 2$. For any $l \in \mathbf{R}^2 \backslash \{0\}$ let A_l be the event $\{X_n.l \to \infty\}$. For any $l \in \mathbf{R}^2 \backslash \{0\}$:*

$$\mathbb{P}_0^\mu(A_l) \in \{0, 1\}.$$

The idea is to start two walks in $\mathbf{Z}^2$, one starting from the origin and going in the direction l and one starting far from the origin, in the direction l and going in the direction $-l$. By carefully choosing the starting point of the second walk it is possible to make them both intersect. Making the two walks intersect heavily relies on the graph being $\mathbf{Z}^2$. Now make a third walk start at the intersection point, by Kalikow's $0-1$ law, this walk will either go in the direction l or in the direction $-l$. Since the first walk has already gone a long way in the direction l it will continue in his direction with high probability and therefore the third walk is strongly drifted in the direction l. Similarly by comparing with the second walk it is also strongly drifted in the direction $-l$. This leads to a contradiction which means that we cannot have both $\mathbb{P}_0^\mu(A_l) > 0$ and $\mathbb{P}_0^\mu(A_{-l}) > 0$.
To understand the behaviour of the walk when it is transient in a direction, Sznitman and Zerner introduced a useful tool in [86]. We will only state a simpler but equivalent definition in a specific case.

Definition 10. *Set a dimension $d \in \mathbf{N}^*$. Let $(e_1, \dots, e_d)$ be the canonical basis of $\mathbf{Z}^d$ and for any $i \in [\![1, d]\!]$, set $e_{i+d} := -e_i$. For any direction $j \in [\![1, 2d]\!]$, the sequence of* ***renewal times*** *$(\tau_i^{e_j})_{i \in \mathbf{N}^*}$ are defined by:*

$$\begin{aligned} \tau_1^{e_j} &= \inf\{n \in \mathbf{N},\ \forall m \geq n, X_m.e_j \geq X_n.e_j F \text{ and } \forall m < n, X_m.l < X_n.l\} \text{ and} \\ \tau_{k+1}^{e_j} &= \inf\{n > \tau_k,\ \forall m \geq n, X_m.l \geq X_n.e_j \text{ and } \forall m < n, X_m.l < X_n.l\}. \end{aligned}$$

These are times after which the walk does not backtrack. This can be generalized to any direction $l \in \mathbf{R}^d \backslash \{0\}$ but for technical reasons the definition is more complicated in this case. For instance we need to impose that for some constant $a > 0$, $\left(X_{\tau_{i+1}^l} - X_{\tau_{i+1}^l}\right).l > a$. The idea is that since before and after those times the walk is in different part of the graph and the environment is iid, the behaviour of the walk before and after those time should be independent.

Proposition 1.2.2.1. *For all $k \in \mathbf{N}^*$, let $\mathscr{G}_k$ be the σ-field defined by:*

$$\mathscr{G}_k := \sigma(\tau_1^l, \dots, \tau_k^l, (X_n)_{0 \leq n \leq \tau_k^l}, (\omega(x, \cdot))_{x.l < X_{\tau_k^l}.l}).$$

We have, for all $k \geq 1$:

$$\mathbb{P}_0^\mu\left((X_{\tau_k^l + n})_{n \geq 0} \in \cdot, (\omega(X_{\tau_k^l} + x, \cdot))_{x.l \geq 0} \in \cdot | \mathscr{G}_k\right) = \mathbb{P}_0^\mu\left((X_n)_{n \geq 0} \in \cdot, (\omega(x, \cdot))_{x.l \geq 0} \in \cdot | \tau_1^l = 0\right).$$

This means that the trajectories and the transition probabilities inside slabs between two consecutive renewal times (after the first one) are i.i.d random variables. Now that we have i.i.d random variables we can expect to get a law of large numbers. There is still a small issue: it is not completely obvious that such renewal times exists even if the walk is transient in the direction l. It was actually proved, still in [61] that if the walk is transient in a dimension l then all the random variables τ_i^l are finite almost surely and furthermore:

$$\mathbb{E}_0^\mu\left(X_{\tau_{i+1}^l}.l - X_{\tau_i^l}.l\right) < +\infty.$$

This is because every time the walk reaches a new level, it has a probability p of never coming back to this level independently of what happened before. Therefore when $X.l = n$, there should already have been around np renewal times on average. Then using that the renewal slabs are iid it is possible to conclude. This leads to a law of large number.

Theorem 15 ([86])**.** *If $\mathbb{E}_0^\mu(\tau_2^l|\tau_1^l = 0) < +\infty$ then conditioned on $X_n.l \to \infty$, we have almost surely:*

$$\frac{X_n}{n} \to \frac{\mathbb{E}_0^\mu\left(X_{\tau_2^l}|\tau_1^l = 0\right)}{\mathbb{E}_0^\mu\left(\tau_2^l|\tau_1^l = 0\right)}.$$

If $\mathbb{E}_0^\mu(\tau_2^l|\tau_1^l = 0) = +\infty$ then conditioned on $X_n.l \to \infty$, we have almost surely:

$$\frac{X_n.l}{n} \to 0.$$

If the renewal times are not well defined, it is still possible to get a partial law of large number.

Theorem 16 ([91])**.** *Set $l \in \mathbf{R}^d\backslash\{0\}$. If $\mathbb{P}_0^\mu(A_l \cup A_{-l}) = 1$, there exists $v_l^-, v_l^+ \in [0,\infty)$ such that $\mathbb{P}_0^\mu$ almost surely:*

$$\frac{X_n.l}{n} \to -v_l^- 1_{A_{-l}} + v_l^+ 1_{A_l}.$$

If $\mathbb{P}_0^\mu(A_l \cup A_{-l}) = 0$, $\mathbb{P}_0^\mu$ almost surely:

$$\frac{X_n.l}{n} \to 0.$$

Only the second part comes from [91], the first part is a simple consequence of $\mathbb{E}_0^\mu\left(X_{\tau_2^l}.l|\tau_1^l = 0\right) < +\infty$. In dimension 2, making use of the full $0-1$ law, by taking $e_1 = (1,0)$ and $e_2 = (0,1)$, there exists constants $v_1, v_2 \in \mathbf{R}$ such that $\frac{X_n.e_1}{n} \to v_1$ and $\frac{X_n.e_2}{n} \to v_2$ which means that $\mathbb{P}_0^\mu$ almost surely:

$$\frac{X_n}{n} \to V_1 e_1 + v_2 e_2 =: v.$$

This means that we have a law of large number in dimension 2. In dimension 3 and higher it is not hard to see that there can be at most two different value of the limit speed and that they are in opposite directions. In dimension 5 and above, it was shown in [9] that if there are two possible limit speeds then one of them is zero. The picture is thus as follows: under ellipticity, almost surely $\frac{X_n}{n}$ converges to a (random) limit speed. In dimensions 1 and 2 this speed is unique, in dimensions 3 and 4 there are at most two limit speeds and they are in opposite directions and in dimension 5 and above there is at most one non-zero limit speed.

1.2.3 A priori conditions

In order to go further, Sznitman introduced some a priori conditions under which ballisticity or quenched CLTs could be proved. Other a piori conditions have been introduced since but all have been proven to be equivalent under uniform ellipticity. Let us start with those introduced by Sznitman in [83] and [81].

Definition 11. *Set $\gamma \in (0,1)$ and $l \in \mathbf{R}^d\backslash\{0\}$. Let T_L^l and $\tilde{T}_L^l$ be the stopping times defined by:*

$$T_L^l := \inf\{n, X_n.l \geq L\} \text{ and } \tilde{T}_L^l := \inf\{n, X_n.l \leq -L\}.$$

The condition $(T)_\gamma$ in direction l is verified if for every l' in a neighbourhood of l:

$$\limsup_{L\to+\infty} L^{-\gamma} \log\left(\mathbb{P}_0^\mu\left(T_L^{l'} > \tilde{T}_L^{l'}\right)\right) < 0.$$

The condition (T) in direction l is said to be verified if the condition $(T)_1$ in direction l is verified.
The condition (T') in direction l is said to be verified if the condition $(T)_\gamma$ in direction l is verified for all $\gamma \in (0,1)$.

We also need the condition $(P)_M$ introduced in [13].

Definition 12. *Set $l \in R_d$ such that $||l||_2 = 1$. Let R be a rotation of $\mathbf{R}^d$ such that $R((1,0,\ldots,0)) = l$. For any $L, \tilde{L} \in \mathbf{R}^+$, let $B_{l,L,\tilde{L}}$ be the box defined by:*

$$B_{l,L,\tilde{L}} := R\left((-L,L)\times(-\tilde{L},\tilde{L})^{d-1}\right) \cap \mathbf{Z}^d.$$

Let $T_{B_{l,L,\tilde{L}}}$ be the first time at which the walk exits $B_{l,L,\tilde{L}}$. A random walk in iid uniformly elliptic environment is said to satisfy condition $(P)_M$ is direction l if there exists $\tilde{L} \leq 70L^3$ and $L > c_0$ (c_0 only depends on the dimension and the uniform ellipticity constant) such that:

$$\mathbb{P}_0^\mu\left(X_{T_{B_{l,L,\tilde{L}}}}.l < L\right) \leq \frac{1}{L^M}.$$

All these conditions are of similar spirit, they all mean that the walk is much more likely to go in direction l over long distances than it is to go in direction $-l$. It was actually shown that under some assumptions these conditions are equivalent.

Theorem 17 ([35],[13],[46]). *Under uniform ellipticity, in dimension $d \geq 2$, conditions T, $T^{'}$, T_γ for any $\gamma \in (0,1)$ and $(P)_M$ for $M > 15d+5$ are all equivalent.*

Now we can look at what those conditions entail. In cite [81] it is shown that under uniform ellipticity those conditions imply ballisticity and an annealed CLT.

Theorem 18. *Assume that any of the above conditions hold in the direction l and that the environment is uniformly elliptic in dimension $d \geq 2$. Then there exists $v \in \mathbf{R}^d \backslash \{0\}$ such that $v.l > 0$ and*

$$\frac{X_n}{n} \to v, \ \mathbb{P}_0^\mu \text{ almost surely.}$$

Furthermore, under $\mathbb{P}_0^\mu$, the law of the processes $t \to \frac{X_{nt}-ntv}{\sqrt{n}}$ converges in $D(\mathbf{R}_+, \mathbf{R}^d)$ to the law of a non-degenerate Brownian motion.

This was then improved to a quenched CLT under moment conditions on the renewal times which are implied by those conditions in [14] (in dimension 4 and higher by using the non-intersection of random walks in the same environment in those dimensions) and [66] (in all dimensions but under much higher moment conditions).
An other important result assuming the conditions $(T), (T')$ or $(P)_M$ was proved in [12].

Theorem 19. *In dimensions 4 and higher, under uniform ellipticity and any of the conditions $(T), (T')$ or $(P)_M$, there is an invariant measure for the point of view of the particle $\mathbb{Q}$, absolutely continuous with respect to $\mathbb{P}_\mu$ and such that for any $k \in \mathbf{N}$,*

$$\mathbb{E}_\mu\left(\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_\mu}\right)^k\right) < +\infty.$$

The assumption uniform ellipticity is not necessary to prove ballisticity under the various conditions $(T), (T)', (P)_M$. This assumption has been weakened in [22],[19],[41].

1.3 Random walks in Dirichlet environment

1.3.1 The model

The random walk in Dirichlet environment is the special case of random walk in iid environment where the law μ is a Dirichlet law.

Definition 13. *Given a family of positive weights $(\alpha_1, \ldots, \alpha_n)$, the Dirichlet law of parameter $\alpha := (\alpha_1, \ldots, \alpha_n)$ has density:*

$$\frac{\Gamma\left(\sum\limits_{i=1}^n \alpha_i\right)}{\prod\limits_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^n x_i^{\alpha_i - 1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}$$

on the simplex

$$\{(x_1, \ldots, x_n) \in (0,1]^n, \sum_{i=1}^n x_i = 1\}.$$

This is a generalization of the β-law for higher dimensions. An other way of seeing it is as a renormalization of independent gamma random variables.

Proposition 1.3.1.1. *Let $(\alpha_1, \ldots, \alpha_n)$ be a family of positive weights and $(Y_1, \ldots, Y_n)$ be independent gamma random variables of respective parameter $(\alpha_1, \ldots, \alpha_n)$. Let $Y := \sum Y_i$, the vector $\left(\frac{Y_1}{Y}, \ldots, \frac{Y_n}{Y}\right)$ is distributed according to a Dirichlet law of parameter $(\alpha_1, \ldots, \alpha_n)$. Furthermore, the random variable Z and the random vector $\left(\frac{Y_1}{Y}, \ldots, \frac{Y_n}{Y}\right)$ are independent.*

This property has a couple of consequences.

Proposition 1.3.1.2. *Let $(\alpha_1, \ldots, \alpha_{n+m})$ be a family of positive weights and $(Z_1, \ldots, Z_{n+m})$ be a random vector following a Dirichlet law of parameter $(\alpha_1, \ldots, \alpha_{n+m})$. The vectors $\left(\frac{Z_1}{\sum\limits_{i=1}^n Z_i}, \ldots \frac{Z_n}{\sum\limits_{i=1}^n Z_i}\right)$ and $\left(\frac{Z_{n+1}}{\sum\limits_{i=1}^m Z_{n+i}}, \ldots \frac{Z_{n+m}}{\sum\limits_{i=1}^m Z_{n+i}}\right)$ are independent and their distribution are Dirichlet of respective parameters $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha_{n+1}, \ldots, \alpha_{n+m})$.*

This essentially means that the transition probabilities are as independent as they can be. The only information you can get from a subset of the transition probabilities is that their sum plus the sum of the others are equal to 1. An other interesting property is that the moments can be easily computed which will be useful later on.

Proposition 1.3.1.3. *Let $(\alpha_1, \ldots, \alpha_n)$ be a family of positive weights and $(Z_1, \ldots, Z_n)$ be a random vector following a Dirichlet law of parameter $(\alpha_1, \ldots, \alpha_n)$. Let $(\beta_1, \ldots, \beta_n) \in [0, \infty)^n$ be non-negative reals. We have:*

$$\mathbb{E}\left(\prod_{i=1}^{n} Z_i^{\beta_i}\right) = \frac{\Gamma(\sum \alpha_i)}{\Gamma(\sum \alpha_i + \beta_i)} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)}.$$

From this we can deduce another representation of the random walk in Dirichlet environment by computing the probability of paths.

Proposition 1.3.1.4. *Let $\mathscr{G} = (V, E)$ be a finite directed graph. Let $(\alpha_e)_{e \in E}$ be a family of positive weights and $x_0 \in V$ a starting point. Let $(\tilde{X}_n)_{n \in \mathbf{N}}$ be the reinforced random walk given by $\tilde{X}_0 = x_0$ almost surely and:*

$$\mathbb{P}(\tilde{X}_{n+1} = y | \tilde{X}_0, \ldots, \tilde{X}_n) = \frac{N_{(\tilde{X}_n, y)}(n)}{\sum\limits_{z \in V, (\tilde{X}_n, z) \in E} N_{(\tilde{X}_n, z)}(n)},$$

where $N_e(n) := \alpha_e + \sum\limits_{i=0}^{n-1} 1_{(\tilde{X}_i, \tilde{X}_{i+1}) = e}$.
The random walk $(\tilde{X}_n)_{n \in \mathbf{N}}$ is called the directed edge linearly reinforced random walk on $\mathscr{G}$ with initial weights $(\alpha_e)_{e \in E}$ and has the same law as the annealed law of a random walk in random environment where the transition probabilities are independent at each site and given by Dirichlet random variables of parameter (α).

We will use slightly different notations in this part for the law of the environment and the annealed law. If $(\alpha_e)_{e \in E}$ is a family of weights we will write $\mathbb{P}^{(\alpha)}$ for the law on the set of environments such that the transition probabilities at each site are independent and given by α (this corresponds to $\mathbb{P}_\mu$ in the previous chapters). Notice that if the α are not the same at each site then the environment is not iid. Then the corresponding annealed law will be $\mathbb{P}_0^{(\alpha)}$ (instead of $\mathbb{P}_0^\mu$). When we work on $\mathbf{Z}^d$ with iid transition probabilities we will only write $(\alpha_1, \ldots, \alpha_{2d})$ instead of the α on all the edges.
For any $i \in [\![1, d]\!]$ let $e_i \in \mathbf{R}^d$ be the vector with all coefficients equal the 0 except the i^{th} which has value 1 and let e_{i+d} be defined by $e_{i+d} := -e_i$. For any set of positive weights $\alpha = (\alpha_1, \ldots, \alpha_{2d})$ let $d_\alpha := \sum\limits_{i=1}^{2d} \alpha_i e_i$.

1.3.2 The fundamental property

The fundamental property of the random walk in Dirichlet environment is its **statistical invariance by time reversal**. Before we can explain what this precisely means, we first need to introduce a few notions. We start with the notion of null-divergence.

Definition 14. *Let $\mathscr{G} = (V, E)$ be a finite directed graph. Let $(\alpha_e)_{e \in E}$ be a family of positive weights. We say that the weights have null divergence if for every $x \in V$:*

$$\sum_{y \in V, (x,y) \in E} \alpha_{(x,y)} = \sum_{y \in V, (y,x) \in E} \alpha_{(y,x)}.$$

Notice that on a torus or on $\mathbf{Z}^d$ if the weights are the same at each site, then the weights have null divergence. We now define what we mean by time reversal.

Definition 15. *For any finite graph $\mathscr{G} = (V, E)$, its reversed graph $(\tilde{\mathscr{G}} = \tilde{V}, \tilde{E})$ is obtained by keeping all the vertices and flipping all the edges ie: $\tilde{V} = V$ and $\tilde{E} = \{(x, y), (y, x) \in E)\}$.*
Let ω be an environment on $\mathscr{G} = (V, E)$. The reversed environment $\tilde{\omega}$ on the reversed graph $(\tilde{V}, \tilde{E})$ is defined by $\tilde{\omega}(x, y) = \omega(y, x)\frac{\pi_y}{\pi_x}$ where π is the stationary distribution (i.e for any vertex x, $\pi_x = \sum \omega(y, x)\pi_y$).

A way of seeing the time-reversed environment is the following. If the law of the starting point of the walk $(X_n)_{n \in \mathbf{N}}$ in the environment is distributed as the invariant probability then the reversed trajectory $(X_n, X_{n-1}, \ldots, X_0)$ behaves like a Markov chain in the reversed environment. The reversed environment is in a way the environment for the walk if we look backward in time instead of forward. Notice that if an environment is reversible (i.e given by conductances) then it is its own reversed environment. The fundamental property of the Dirichlet environment is the following:

Theorem 20 ([69])**.** *Let $\mathscr{G} = (V, E)$ be a finite directed graph. Let $(\alpha_e)_{e\in E}$ be a family of positive weights on the edges of $\mathscr{G}$. Let $(\tilde{\alpha}_e)_{e\in\tilde{E}}$ be the family of positive weights on the edges of the reversed graph $\tilde{\mathscr{G}}$ such that:*

$$\forall (x, y) \in E,\ \tilde{\alpha}_{(y,x)} := \alpha_{(x,y)}.$$

If α is of null divergence then the law of the reversed environment under $\mathbb{P}^{(\alpha)}$ is the same as the law of the environment under $\mathbb{P}^{(\tilde{\alpha})}$.

The idea of the proof is to notice that the probabilities of a cycle in an environment and the reversed cycle in the reversed environment are the same. Then we see that the probabilities of the reversed cycles under $\mathbb{P}^{(\alpha)}$ are the same as the cycles under $\mathbb{P}^{(\tilde{\alpha})}$. From this it is possible to extend the result to any path which yields the desired result.
This property of the Dirichlet environment of statistical stability by time reversal allows for some explicit computations which yields a surprising number of results. For instance, we have the following information on the asymptotic direction of the walk:

Theorem 21 ([74],[88])**.** *In any dimension $d \geq 1$, for any direction $l \in \mathbf{R}^d\backslash\{0\}$ such that $||l||_2 = 1$ and $l.d_\alpha > 0$:*

$$\mathbb{P}_0^{(\alpha)}\left(X_n.l \to +\infty\right) > 0$$

Notice that because of the $0-1$ law on $\mathbf{Z}^2$ (14) this means that the probability in the theorem is equal to 1 in dimension 2. From this it is possible to show that the asymptotic direction in $\mathbf{Z}^2$ is equal to the expectation of the first step if this expectation is non-zero. For general random walks in $\mathbf{Z}^2$, we do not know the asymptotic direction, we do not even have a general criterion to know when there is an asymptotic direction.
The idea to prove this result is to look at a finite approximation of $\mathbf{Z}^d$ and use the statistical invariance by time reversal on it. For instance if we assume that $\alpha_1 > \alpha_3$ in $\mathbf{Z}^2$ ($d_\alpha.e_1 > 0$), we can look at the following graph.

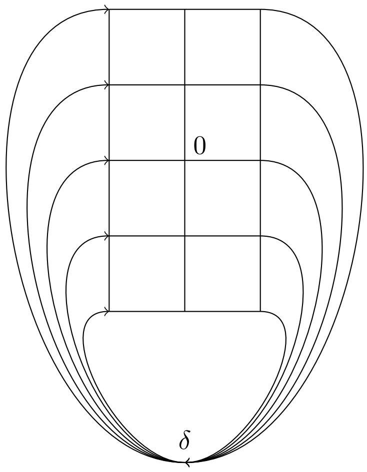


Figure 1.3: Finite approximation of $\mathbf{Z}^2$

The weights within the grid are the same as on $\mathbf{Z}^2$ but the weights going to δ are equal to $\alpha_1 - \alpha_3$ and those going out of δ are also equal to $\alpha_1 - \alpha_3$. These weights have null divergence. The probability to go from a point on the left to a point on the right without going back to the left is larger or equal to the probability to go from δ to δ and only visit the points on the left once. If we look at the reversed environment, this is the same as starting in δ waiting to reach a point on the left and immediately crossing to δ once a point on the left is reached. This has a positive probability that does not depend on the size of the graph chosen. Then by making the size of the graph go to infinity we conclude that the walk has a positive probability of going to infinity to the left. This can be done in any dimension and with vectors in $\mathbf{Q}^d$ instead of just e_1.
In dimension 3 and above, the previous result plus theorem 13 implies that the walk is transient if the weights are not symmetric. If the weight are symmetric however, the previous result does not give any insight on the behaviour of the walk. It was found in [69] that in this case, the walk is transient (in dimension $d \geq 3$). This implies the following result.

Theorem 22 ([69])**.** *On $\mathbf{Z}^d$ with $d \geq 3$, random walks in Dirichlet environment are transient.*

Unfortunately, in dimension $d = 2$, it is not known whether the walk is recurrent if the weights are symmetric.

1.3.3 Point of view of the particle and traps

An important tool in the random conductance model was the point of view of the particle. For general random walk in random environment in dimension $d \geq 2$ it is not known under which conditions an absolutely continuous invariant measure for the point of view of the particle exists. We know it exists in dimension $d \geq 4$, under uniform ellipticity and any of the conditions (T), (T') or $(P)_M$ but we do not know whether those assumptions are necessary or if weaker ones suffice. The problem is better understood in the Dirichlet case and can be reduced to a simple quantity κ that we will now define.

Definition 16. *Set positive weights $(\alpha_1, \dots, \alpha_{2d})$. The constant κ is defined by:*

$$\kappa := 2\sum_{i=1}^{2d} \alpha_i - \max_{i \in [\![1,d]\!]} (\alpha_i + \alpha_{i+d}).$$

The simplest way to understand κ is to look at the time you can stay trapped on an edge. If you are in 0, the probability to go through the edge $(0, e_i)$ and then through the edge $(e_i, 0)$ is equal to $\omega(0, e_i)\omega(e_i, 0)$ this means that the number of time you cross this edge is a geometric random variable of expectation $\frac{\omega(0,e_i)\omega(e_i,0)}{1-\omega(0,e_i)\omega(e_i,0)}$. It is simple to show that

$$\mathbb{P}^{(\alpha)}\left(\frac{\omega(0, e_i)\omega(e_i, 0)}{1 - \omega(0, e_i)\omega(e_i, 0)} \geq t\right) \sim ct^{\alpha_i + \alpha_{i+d} - 2\sum_{i=1}^{2d} \alpha_i}.$$

So the probability of never being at a distance more than 1 of the origin before time t has a polynomial tail of exponent κ. This means that if $\kappa \leq 1$ the annealed average time spent on edges before leaving them has infinite expectation. On the other hand if $\kappa > 1$ the annealed average time spent on edges before leaving them has finite expectation.
Intuitively, if $\kappa \leq 1$ there is no typical environment for the point of view of the particle. As time passes, the particle gets stuck on edges where the time needed to escape grows larger and larger. That is to say at time n the walk is most likely stuck on an edge where you need a time of order n to escape. Therefore there can be no invariant measure for the point of view of the particle absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$. On the other hand, if $\kappa > 1$ the walks waits a time of order n^κ before being stuck a time n in an edge so it should not slow down the walk too much. This is indeed what was observed in [70]

Theorem 23. *If $d \geq 3$ and $\kappa > 1$ then there exists a probability measure $\mathbb{Q}^{(\alpha)}$ invariant for the walk from the point of view of the particle and absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$. Furthermore, for every $s < \kappa$:*

$$\mathbb{E}_{\mathbb{P}^{(\alpha)}}\left(\left(\frac{\mathrm{d}\mathbb{Q}^{(\alpha)}}{\mathrm{d}\mathbb{P}^{(\alpha)}}\right)^s\right) < +\infty.$$

On the other hand, if $d \geq 3$ and $\kappa \leq 1$ there is no invariant probability measure for the walk from the point of view of the particle, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$.

The reason why there is no invariant probability measure when $\kappa \leq 1$ is that the walk stays trapped for too long in traps of finite size (edges when κ is close to 1 and larger subset of the graph when κ goes to 0). To circumvent this issue we look at an accelerated walk where the time spent on a vertex depends on the environment around this edge. For instance if the problematic traps are at most of size m then we accelerate the walk so that the time spent in boxes of size m before leaving them is of order 1 or less. This way we might expect that this accelerated walk has no problematic traps. This was done in [18]. Before stating the precise result, we need a definition that will be useful afterwards.

Definition 17. *For any environment ω, any integer $m \geq 1$ and any vertex $x \in \mathbf{Z}^d$, let $\gamma_m^\omega(x)$ be the probability that starting from x the walk reaches the border of the box of size m centered on x ($x + [-m, m]^d$) before visiting a vertex twice.*
The accelerated walk $(Y_t^m)_{t \in (0,\infty)}$ is the continuous time random walk that has the same trajectory as $(X_n)_{n \in \mathbf{N}}$ but waits a random exponential time of parameter $1/\gamma_m^\omega(x)$ before jumping when it is in x instead of waiting a time 1.

Theorem 24. *Let $d \geq 3$ be the dimension and let $\alpha = (\alpha_1, \dots, \alpha_{2d})$ be a family of positive weights. If m is large enough then the random walk from the point of view of the particle associated with Y^m has a stationary distribution $\mathbb{Q}^{m,\alpha}$ absolutely continuous with respect to $\mathbb{P}^\alpha$. For any $\beta > 1$, there exists an m such that $\frac{\mathrm{d}\mathbb{Q}^{m,\alpha}}{\mathrm{d}\mathbb{P}^\alpha}$ is in L^β.*

The existence of these invariant measures has an important consequence: in dimension $d \geq 3$ the walk either has a deterministic asymptotic direction or it is not transient in any direction. Furthermore if $\kappa > 1$ the walk has an asymptotic speed (which is 0 iff the walk is not directionally transient). This coupled to theorem 21 and the result in dimension 2 gives us the following result.

Theorem 25 ([18],[88]). *In all dimensions $d \geq 1$, if $d_\alpha \neq 0$ then*

$$\frac{X_n}{||X_n||_2} \to \frac{d_\alpha}{||d_\alpha||_2} \textit{almost surely.}$$

However if $d_\alpha = 0$ then for any $l \in \mathbf{R}^d \backslash \{0\}$,

$$-\infty = \liminf X_n.l < \limsup X_n.l = +\infty.$$

Furthermore, in dimension $d \geq 3$ if $\kappa > 1$ and $d_\alpha \neq 0$ then there exists $c > 0$ such that

$$\frac{X_n}{n} \to cd_\alpha \textit{ almost surely.}$$

In dimension $d \geq 3$ if $\kappa \leq 1$ or $d_\alpha = 0$ then :

$$\frac{X_n}{n} \to 0 \textit{ almost surely.}$$

This gives a complete picture for directional transience for the Dirichlet environment. In dimension 1 and in dimension $d \geq 3$ we have an explicit criterion for ballisticity. We even have a quenched CLT under some assumptions on the weights

Theorem 26 ([20]). *In any dimension $d \geq 1$, if $\kappa > 2$ and $\sum_{i=1}^{d} |\alpha_i - \alpha_{i+d}| > 1$ then for some $c > 0$, for $\mathbb{P}^\alpha$ almost every environment, the sequence of processes $t \to \frac{X_{\lfloor nt \rfloor} - cntd_\alpha}{\sqrt{n}}$ converges in distribution to a non-degenerate brownian motion.*

When $d_\alpha \neq 0$, we know that the walk is transient in direction d_α. When $\kappa > 1$ in dimension $d \geq 3$, the walk has positive speed so we know how fast the walk goes in the asymptotic direction, but what happens when $\kappa \leq 1$? In this case the walk is slowed down because of finite subsets of $\mathbf{Z}^d$ where it stays trapped for a long time. The main cause of this slowing down are edges where the walk can stay trapped a long time (other traps are less likely). The time the walk spends trapped on edges is governed by the coefficient κ. Thus it was shown in [18] that X_n was of order n^κ. More precisely:

$$\frac{\log(X_n.d_\alpha)}{\log(n)} \to \kappa \text{ in probability for } \mathbb{P}_0^{(\alpha)}.$$

This kind of slowing down appears in multiple models for instance in dimension 1 for generic random walk in random environment ([51],[39]), for random walks among unbounded conductances ([4]), for random walks on Galton-Watson trees ([8]), for random walks among biased conductances on a percolation cluster ([40]) and for random walks among unbounded biased conductances ([43]). In three of those cases (dimension 1, random walks among unbounded conductances and random walks among unbounded biased conductances) the behaviour of the walk is described by a stable subordinator. In the first one, the hitting times of the levels converge after rescaling to a stable subordinator and for the last two, the trajectory of the walk converges (after rescaling) to a brownian motion with a time change given by a stable subordinator (the fractional kinetics introduced in section 1.2.1). In the remaining two cases (Galton-Watson trees and biased percolation cluster) the slow-down does not come from the randomness of the conductances but the randomness of the geometry of the underlying graph. For technical reasons, this implies that there can be no convergence to a stable subordinator (this is for the same reasons that in theorem 12, the support of the measure has to be non-arithmetic).

1.3.4 Our results regarding random walks in Dirichlet environment

In the case of random walks in Dirichlet environment, the randomness comes from the transition probabilities, not the geometry of the graph. Therefore, we can expect that in the sub-ballistic case, the walk can be described by a stable subordinator. This is indeed what we will show in chapter 2. Before we can properly state our result, we will first define precisely what the stable subordinator is.

Definition 18. *For any $\kappa \in (0, 1)$ let $\mathscr{S}^\kappa$ be the Lévy process where the increments are completely asymmetric κ-stable random variables. The increment have the following characterizations:*

$$\forall \lambda \in \mathbf{R}, \forall s \in \mathbf{R}^+, \mathbb{E}\left(\exp\left(i\lambda \mathscr{S}_s^\kappa\right)\right) = \exp\left(-s|\lambda|^\kappa \left(1 - isgn(\lambda) \tan\left(\frac{\pi\kappa}{2}\right)\right)\right)$$

and for any $s \in \mathbf{R}^+$, $\mathscr{S}_s^\kappa$ and $s^{\frac{1}{\kappa}} \mathscr{S}_1^\kappa$ have the same law.
Since this process is non-decreasing and càdlàg we can define the càdlàg inverse $\tilde{\mathscr{S}}^\kappa$ by:

$$\tilde{\mathscr{S}}_t^\kappa := \inf\{s, \mathscr{S}_s^\kappa \geq t\}.$$

We can now state our theorems:

Theorem. *Set $d \geq 3$ and $\alpha \in (0,\infty)^{2d}$. Let $(\tau_n^{e_1})_{n\in\mathbf{N}}$ be the sequence defined in 10. Let $X^n(t)$ be defined by:*

$$X^n(t) = n^{-\kappa} X_{\lfloor nt \rfloor}.$$

If $\kappa < 1$ and $d_\alpha \neq 0$, there exists positive constants c_1, c_2, c_3 such that for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(t \to n^{-\frac{1}{\kappa}} \tau^{e_1}_{\lfloor nt \rfloor}\right) \to c_1 \mathscr{S}^\kappa,$$

for the M_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$(x \to \inf\{t \geq 0, X^n(t).e_1 \geq x\}) \to c_2 \mathscr{S}^\kappa$$

and for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$X^n \to c_3 \tilde{\mathscr{S}}^\kappa d_\alpha.$$

Remark 5. *We will give a quick explanation on what the M_1 and J_1 topologies are, for a precise definition see [79],[89]. They were both introduced as a generalization of the infinite norm for càdlàg functions. In the J_1 topology, a sequence of càdlàg functions f_n converges to f if there exists a sequence of increasing homomorphisms $\lambda_n : [0,1] \mapsto [0,1]$ such that*

$$\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0,$$

and

$$\sup_{t\in[0,1]} |f_n(\lambda_n(t)) - f(t)| \to 0.$$

It is essentially the same as the infinite norm except that you can "wiggle" the function time-wise. The M_1 topology is a topology on the graphs of the functions where we add vertical segments every time there is a jump. The main difference between the M_1 and J_1 topology is that there is almost no difference between one jump and small consecutive jumps in the M_1 topology while the difference is significant in the J_1 topology. The reason why we only have a convergence in M_1 for the hitting times $n^{-\frac{1}{\kappa}} \inf\{t \geq 0, Y(t).e_1 \geq nx\}$ is because there are consecutive jumps. Indeed, if there is a large jump between $\inf\{t \geq 0, X(t).e_1 \geq n\}$ and $\inf\{t \geq 0, Y(t).e_1 \geq n+1\}$ it is likely that there is a trap with high strength close-by which means that it is likely that there also is a large jump between $\inf\{t \geq 0, Y(t).e_1 \geq n+1\}$ and $\inf\{t \geq 0, X(t).e_1 \geq n+2\}$.

Theorem. *Set $d \geq 3$ and $\alpha \in (0,\infty)^{2d}$. Let $(\tau_n^{e_1})_{n\in\mathbf{N}}$ be the sequence defined in 10. If $d_{\alpha\neq 0}$ and $\kappa = 1$, there exists positive constants c_1, c_2, c_3 such that we have the following convergences in probability (for the annealed law):*

$$\frac{1}{n\log(n)} \tau_n \to c_1,$$

$$\frac{1}{n\log(n)} \inf\{i, X_i.e_1 \geq n\} \to c_2,$$

$$\frac{\log(n)}{n}(X_n) \to c_3 d_\alpha.$$

Remark 6. *We cannot replace the convergence in probability by an almost sure convergence. This is because if we look at a sum of iid random variables Z_i with a heavy tail $\mathbb{P}(Z_i \geq t) \sim ct^{-1}$ then we do not have an almost sure convergence. In fact, there are infinitely many i such that:*

$$Z_i \geq i\log(i)\log(\log(i)).$$

These two results rely on the existence of an invariant measure from the point of view of the particle which in turn depends on the statistical invariance by time reversal. This property actually characterizes the Dirichlet environment. Indeed we will see in chapter 3 that the following theorem holds true.

Theorem 27 (chapter 3)**.** *Let (V,E) be a finite directed graph and ω transition probabilities on this graph that satisfy the following properties:*

- *the graph has no multiple edges,*
- *the graph and the reversed graph are 2-connected,*

- *the transition probabilities are of positive expectation,*
- *the transition probabilities are independent at each site.*

If the transition probabilities of the reversed environment are also independent, then the transition probabilities are independent Dirichlet random variables with null divergence or are deterministic.

In the proof, we use that the transition probabilities of the environment and the reversed environment are both independent at each site to get informations on the probability of some carefully chosen cycles. Then we use these probabilities to get equalities regarding the moments of transition probabilities on different vertices in both the environment and the reversed environment. Then by induction we show that the moments we get are those of some Dirichlet environment with null divergence.

1.4 Linearly edge reinforced random walk and Vertex reinforced jump process

We will look at three different models that are closely linked to each other. The first one, the linearly edge-reinforced random walk (ERRW) is a discrete-time reinforced random walk that was first introduced by Coppersmith and Diaconis in 1986 [24]. The second one, the vertex reinforced jump process is a continuous-time reinforced random walk introduced by Davis and Volkov in [26]. The last one, the $H^{2|2}$ model (see section 1.4.5 for a brief introduction) is a spin-model introduced in the context of random band matrices by Zirnbauer, Disertori and Spencer (in [92],[34]). The link between the three models was found in 2015 by Sabot and Tarrès in [72].

1.4.1 Linearly edge-reinforced random walk and partial exchangeability

First we begin by defining the first model, the linearly edge-reinforced random walk. It is defined on any weighted graph with locally finite degree and positive weights.

Definition 19. *Let $\mathscr{G} = (V, E)$ be a locally finite, non-directed graph. To every edge $e \in E$ we associate a positive weight a_e. Let $x_0 \in V$ be a vertex of $\mathscr{G}$. The ERRW $(Y_n)_{n\in\mathbf{N}}$ starting from x_0 is the random process which takes its values in V defined by:*

$$Y_0 = x_0 \text{ a.s, and}$$

$$\mathbb{P}\left(Y_{n+1} = y | Y_0, \ldots, Y_n\right) = 1_{y\sim Y_n} \frac{a_{\{Y_n,y\}} + Z_n(\{Y_n, y\})}{\sum\limits_{z\sim Y_n} a_{\{Y_n,z\}} + Z_n(\{Y_n, z\})},$$

where the random variables $(Z_n)_{n\in\mathbf{N}}$ are defined by:

$$\forall e \in E,\ Z_n(e) = \sum_{i=0}^{n-1} 1_{\{Y_i,Y_{i+1}\}=e}.$$

As we can see from the definition, this random walk prefers going back to edges it has already visited. Furthermore, the smaller the initial weights are, the stronger this attraction is. On the other hand, when the initial weights are large, the walk does not feel the reinforcement in the beginning and the walk looks like a walk in an electrical network (at least in the beginning).
On general graphs, the link with random walks in random environment is not clear from this definition. There is however one type of graphs on which the link is more obvious: trees. On trees when the walk is on a vertex x, and it leaves through the edge $\{x, y\}$, it will necessarily come back through the edge $\{x, y\}$ (if it comes back). This means that for the vertex x, the behaviour of the walk is the same as choosing an adjacent vertex y proportionally to its weight $a_{\{x,y\}}$, increasing the weight $a_{\{x,y\}}$ by two and then starting over. This is the same behaviour as a Pólya urn which is well understood.

Theorem 28. *Set $n \in \mathbf{N}\backslash\{0, 1\}$. Let $(a_1, \ldots, a_n) \in (0, \infty)^n$ be positive weights. Let $(X_i)_{i\in\mathbf{N}}$ be random variables defined as follows:*

$$\forall m \in \mathbf{N},\ \forall i \in [\![1, n]\!],\ \mathbb{P}\left(X_m = i | X_0, \ldots, X_{m-1}\right) = \frac{a_i + 2\sum\limits_{j=0}^{m-1} 1_{X_j=i}}{2m + \sum\limits_{k=1}^{n} a_i}.$$

There exists a random vector $(p_1, \ldots, p_n) \in [0, 1]^n$ such that $\sum p_i = 1$ and almost surely:

$$\forall i \in [\![1, n]\!], \frac{1}{m}\sum_{j=0}^{m-1} 1_{X_j=i} \to p_i.$$

The distribution of the vector $(p_1, \ldots, p_n)$ is a Dirichlet of parameter $\left(\frac{a_1}{2}, \ldots, \frac{a_n}{2}\right)$ and knowing $(p_1, \ldots, p_n)$, the random variables $(X_i)_{i\in\mathbf{N}}$ are iid and:

$$\mathbb{P}(X_i = j | p_1, \ldots, p_n) = p_j.$$

This means that if we only look at the behaviour of the ERRW around a vertex, it is the same as a random walk in Dirichlet environment. Furthermore, the behaviour on each vertex is independent from what happens on the other vertices (if the walk leaves by the edge $\{x, y\}$, it will come back through the edge $\{x, y\}$ independently of what goes on for the other vertices). This means that on a tree, the ERRW has the same law as a random walk in Dirichlet environment.
Now we look at the parameters of the Dirichlet environment: for any vertex x that is not the starting point, let x^- be its parent vertex (the only neighbour of x that is closer to the origin than x). The parameters of the Dirichlet at x are $\frac{a_{\{x,x^-\}}+1}{2}$ for the parent and $\frac{a_{\{x,y\}}}{2}$ for the other neighbours y. There is a $+1$ for the parent vertex because the first time the walk reaches x, it has to come from x^- which means that $\{x, x^-\}$ has been visited exactly once and all the other neighbouring edges $\{x, y\}$ have never been visited before. Therefore, there is a small bias towards the root of the tree. If the initial weights are small enough then on average the walk is drifted towards the root, if they are not small enough (and vertices have more than one child) then the walk is drifted towards infinity. This was shown more rigorously by Pemantle in [62] where he showed that there was a phase transition. For a d-ary tree ($d \geq 2$) there exists a constant $a_d > 0$ such that if all the initial weights are the same and smaller than a_d then almost surely the environment is positive recurrent, and if all the initial weights are the same and larger than a_d then almost surely the environment is positive recurrent.
On general graphs, things are more complicated. The arguments used for trees do not work any more but we can use that the ERRW is **partially exchangeable**.

Definition 20. *A random walk $(X_i)_{i\in\mathbf{N}}$ on a non-directed graph $\mathscr{G} = (V, E)$ starting at $x_0 \in V$ is partially exchangeable if the probability of a path $x_0, \ldots, x_n$ only depends on the number of crossing of edges.*

It was shown in [30] that a partially exchangeable random walk on a finite graph is a random walk among random conductances (the conductances are not necessarily independent or identically distributed). By approximating an infinite graph by an increasing sequence of finite graph and showing that the resulting law on environments were tight, Merkl and Rolles where able to prove in [59] that this result can be extended to locally finite graph.

Theorem 29. *Let $\mathscr{G} = (V, E)$ be a locally finite non-directed graph. Let $(a_e)_{e\in E}$ be positive weights on the edges and $x_0 \in V$ the starting point. There exists a measure $\mathbb{Q}^{a,x_0}$ on $(0, +\infty)^E$ such that the ERRW with initial weights W and started at x_0 has the same law has a random walk in random conductances started at x_0 where the law of the conductances are given by $\mathbb{Q}^{a,x_0}$.*

On finite graphs this measure actually has an explicit expression that was found by Coppersmith and Diaconis in [24] and proved by Keane and Rolles in [48] (the article [24] was never published and for that reason the proof was written in [48]).

Theorem 30. *Let $\mathscr{G} = (V, E)$ be a finite non-directed graph. Let $(a_e)_{e\in E}$ be positive weights on the edges and $x_0 \in V$ the starting point. Set e_0 an arbitrary edge that contains x_0. The measure $\mathbb{Q}^{W,x_0}$ on $(0, +\infty)^E$ has density:*

$$C \prod_{\{x,y\}\in E} \left(\frac{\omega_{\{x,y\}}}{\sqrt{\pi_x(\omega)\pi_y(\omega)}}\right)^{a_{\{x,y\}}} \prod_{x\in V} \pi_x(\omega)^{-1/2} \sqrt{Detmin(H(\omega))} \prod_{e\in E\setminus\{e_0\}} \frac{\mathrm{d}\omega_e}{\omega_e},$$

where $\pi_x(\omega) = \sum\limits_{y\sim x} \omega_{\{x,y\}}$, $Detmin(H(\omega))$ is the determinant of any minor of $H(\omega)$ and $H(\omega)$ is the matrix defined by:

$$\begin{aligned} H(x, y) &= -\,\omega_{\{x,y\}} \text{ if } x \neq y \\ H(x, x) &= \sum_{y\sim x} \omega(x, y). \end{aligned}$$

This expression is explicit but not easy to analyse. For instance, there is no longer independence between the conductances and there is no simple way of extending the measure for infinite graphs. It was however used in [58] to show recurrence on $\mathbf{Z} \times \{0, 1\}$ for large enough initial weights and then in [67] to show recurrence on $\mathbf{Z} \times \mathscr{T}$ where $\mathscr{T}$ is a finite tree for large enough initial weights. The measure was also used in [60] to show that the ERRW on a modification of $\mathbf{Z}^2$ is positive recurrent for small enough initial weights. The modification of $\mathbf{Z}^2$ is simply $\mathbf{Z}^2$ where every edge is replaced by at least 130 edges in series. The reason why this modification is necessary is quite simple, the method used shows that the conductances decay at least polynomially from the origin, to show positive recurrence the decay needs to be high enough and to increase this decay, they need

to decrease the initial weights and replace the edges by a series of edges.
This was later improved simultaneously and independently by Sabot, Tarrès [72] and Angel, Crawford, Kozma [2].

Theorem 31 ([72],[2]). *For any $d \geq 1$, there exists $a_d > 0$ such that the edge-reinforced random walk on $\mathbf{Z}^d$ with initial weights all equal to $a < a_d$ is positive recurrent.*

They actually show that the conductances decrease exponentially from the origin. Sabot and Tarrès used the link between ERRW, VRRJP and the supersymmetric hyperbolic sigma (SUSY) model to adapt results already known for the SUSY model. Angel, Crawford and Kozma used techniques similar to those used for the SUSY model, directly for the ERRW. The proof uses that the ERRW is a random walk among random conductances but surprisingly does not use the associated measure found in [24] and [48]. The idea is as follows: when the initial weights are small enough, whenever the walk visits a vertex for the first time the edge it just came through has a weight larger than 1 while the weights of the other edges are small. This means that with high probability the walk will go back through the edge it just came through. This means that for this vertex, with high probability, the conductances of the edge the walk came from is much larger than the others. Then, when you look at the path the walk used to go from the origin to a point x, the conductances get smaller and smaller the further along the path you go. From this you can get an exponential decay of the conductances.

1.4.2 Vertex reinforced jump process

The second model we will look at is the vertex reinforced jump process.

Definition 21. *The VRJP on a locally finite graph $\mathscr{G} = (V, E)$ with weights $(W_e)_{e\in E}$ is the continuous-time process $(\tilde{Y}_t)_{t\in\mathbf{R}^+}$ that starts at some vertex x_0 and that, conditionally on the past at time t, if $\tilde{Y}_t = x$, jumps to a neighbour y of x at rate*

$$W_{\{x,y\}}\tilde{\ell}_x(t),$$

where

$$\tilde{\ell}_x(t) := \int_0^t 1_{\tilde{Y}_s = x} \mathrm{d}s.$$

This was how the VRJP was first introduced in [26]. However, by a simple time change we can find equivalent definitions, one of which will be extremely useful.

Proposition 1.4.2.1. *Let $\mathscr{G} = (V, E)$ be a locally finite graph, $(W_e)_{e\in E}$ positive weights on that graph and x_0 a vertex of that graph. Let $(Y_t)_{t\in\mathbf{R}^+}$ be the continuous jump process that starts at some vertex x_0 and that, conditionally on the past at time t, if $Y_t = x$, jumps to a neighbour y of x at rate*

$$W_{\{x,y\}}e^{\ell_x(t)+\ell_y(t)},$$

where

$$\ell_x(t) := \int_0^t 1_{Y_s = x} \mathrm{d}s.$$

Let $A : \mathbf{R}^+ \mapsto \mathbf{R}^+$ be the increasing continuous random process defined by:

$$A(t) = \sum_{x\in V} e^{\ell_x(t)} - 1.$$

The time-changed process $\left(Y_{A^{-1}(t)}\right)_{t\in\mathbf{R}^+}$ has the same law has the VRJP on $\mathscr{G}$ with weights $(W_e)_{e\in E}$ and starting point x_0.

As we can see in both definitions of the VRJP, it has some similarities with the ERRW. First, both are reinforced processes, which are biased towards the parts of the graph they have already visited (edges for the ERRW and vertices for VRRW). Furthermore, the higher the initial weights, the less time the process spends between each jump and the less it feels the reinforcement. The behaviour should then be similar to the ERRW, for small initial weights the walk spends most of its times close to the origin and for large initial weights its trajectories are similar to a random walk in an electrical network (at least in the beginning).
These similarities come from a deep link between the EERW and the VRJP. To highlight this link, we must first introduce a time modification of the ERRW. This modification was introduced in [25] and [77]. On each

edge we define a time process independent from the other edges. Let $(t_k^e)_{(e,k)\in E\times \mathbf{N}}$ be independent random variables of parameter 1 and for every edge e define:

$$V_n^e := \sum_{i=0}^{n-1} \frac{1}{a_e + i} t_i^e.$$

For every vertex $x \in V$ let $\ell_x(t)$ be the local time on x at time t. For every edge $\{x, y\} \in E$ whenever the process is on x or y and there exist n such that $V_n^e = \ell_x(t) + \ell_y(t)$ the process crosses the edge $\{x, y\}$.
Put in another way, you put a clock on every edge that waits a random exponential time of parameter $a_e + n$ (where n is the number of times the edge has been crossed). This clock only runs when the walk is on either end of the edge. Whenever the clock rings the walk crosses the edge. Because of the memoryless properties of the exponential, whenever the walks is on a vertex x, the time before a neighbouring clock rings is an exponential and it is easy to see that the probability that the walk crosses a specific edge is the same as for the ERRW. Therefore, both processes have the same trajectories.
Just like the VRJP this process jumps faster and faster as time goes on. The precise link between the two comes from a property of the clock ringing times.

Theorem 32 ([49]). *Set $a > 0$. Let $(t_k)_{k\in\mathbf{N}}$ be independent random variables of parameter 1 and define the counting process $N : \mathbf{R}^+ \mapsto \mathbf{N}$ by:*

$$N(t) := \sup \left\{ n \in \mathbf{N},\ \sum_{i=0}^{n-1} \frac{1}{a_e + i} t_i^e \leq t \right\}.$$

There exists a random variable X such that almost surely:

$$N(t)e^{-t} \to X.$$

The variable X is a Gamma random variable of parameter a and knowing X, N has the law of the counting process of a Poisson point process of intensity $\mathrm{d}\mu(t) = Xe^t \mathrm{d}\lambda(t)$ (where λ is the Lebesgue measure on $\mathbf{R}^+$).

This implies the following result proved in [72].

Theorem 33. *Let $(\tilde{X}_t)_{t\in\mathbf{R}^+}$ be a continuous time ERRW with initial weights $(a_e)_{e\in E}$ and starting point x_0. There exists independent random variables $(W_e)_{e\in E}$ where W_e is a Gamma random variable of parameter a_e such that conditioned on $(W_e)_{e\in E}$, the continuous-time ERRW is a VRJP with initial weights W_e.*
In particular, the ERRW $(X_n)_{n\in\mathbf{N}}$ is equal in law to the discrete time process associated with a VRJP with random independent Gamma initial weights $(W_e)_{e\in E}$ of respective parameter $(a_e)_{e\in E}$.

This means that if we get results on the VRJP we should get similar results for the ERRW. The first result we get is that the VRJP is partially exchangeable (for a time-changed version of it that does not speed up over time) which implies, just like for the ERRW that the Trajectories of the VRJP have the same law as a mixture of random walks among random conductances. Unlike for the ERRW, not all environments are possible. More precisely, we have the following result from [72]

Theorem 34. *For any $n \in \mathbf{N}^*$, let $\mathscr{H}_0 := \{u \in \mathbf{R}^n, u_1 + \dots + u_n = 0\}$. Let $\mathscr{G} = (V, E)$ be a finite graph with m vertices, $x_0 \in V$ a starting point and $(W_e)_{e\in E}$ a family of positive weights. There exists a random vector U living in $\mathscr{H}_0$ such that the trajectories of the VRJP on $\mathscr{G}$, starting at x_0 with weights W have the same law as a mixture of random walk among random conductances where the conductances ω_e are given by:*

$$\forall \{x, y\} \in E,\ \omega_{\{x,y\}} = W_{\{x,y\}} e^{U_x} e^{U_y}.$$

As we can see, the environment is given by a random potential which limits the possible environments (note that on a tree all environments can be obtained with such a potential). The law of this random potential is actually known and is somewhat simpler than that of the ERRW. Surprisingly, this measure first appeared in the supersymmetric hyperbolic sigma (also called $H^{2|2}$) model in the context of random band matrices (the model was introduced in [92] and the measure first appeared in[34]).

Theorem 35. *The distribution of the random vector $(U_1, \dots, U_n)$ of theorem 34 for the VRJP on a finite graph $\mathscr{G}$ with n vertices, with weights W and starting point i_0 is given by the following measure (which is a probability measure):*

$$\mu_n^{W,i_0}(\mathrm{d}u) := \left(\frac{1}{2\pi}\right)^{\frac{n-1}{2}} e^{u_{i_0}} e^{-\frac{1}{2}\left(\sum\limits_{i\sim j} W_{\{i,j\}}(e^{u_i - u_j} + e^{u_j - u_i} - 2)\right)} \sqrt{DetMin(H_{W,u})} \mathrm{d}u_1 \dots \mathrm{d}u_{n-1},$$

where Detmin$(H_{W,u})$ is the determinant of any minor of $H_{W,u}$ and $H_{W,u}$ is the matrix defined by:

$$\begin{aligned} H_{W,u}(x,y) &= -\,\omega_{\{x,y\}}e^{u_x+u_y} \text{ if } x \neq y \\ H_{W,u}(x,x) &= \sum_{y\sim x} W_{x,y}e^{u_x+u_y}. \end{aligned}$$

The fact that this is indeed a probability measure is not at all obvious. It is even used to deduce various non-trivial equalities for this measure. For instance changing the starting point from i to j is the same as multiplying the density by $e^{u_j-u_i}$ so

$$\mathbb{E}_{\mu_n^{W,i}}\left(e^{U_j-U_i}\right) = \mathbb{E}_{\mu_n^{W,j}}(1) = 1.$$

Similarly if we multiply all the weights by $(1+\lambda) > 0$, we multiply the square root of the determinant by $(1+\lambda)^{\frac{n-1}{2}}$ and the term in the exponential by $(1+\lambda)$ so we get:

$$\mathbb{E}_{\mu_n^{W,i}}\left(e^{-\frac{\lambda}{2}\left(\sum\limits_{i\sim j} W_{\{i,j\}}(e^{u_i-u_j}+e^{u_j-u_i}-2)\right)}\right) = \mathbb{E}_{\mu_n^{(1+\lambda)W,i}}\left(\frac{1}{(1+\lambda)^{\frac{n-1}{2}}}\right) = \frac{1}{(1+\lambda)^{\frac{n-1}{2}}}.$$

This means that under $\mu_n^{W,i}$, $\frac{1}{2}\left(\sum\limits_{i\sim j} W_{\{i,j\}}(e^{u_i-u_j}+e^{u_j-u_i}-2)\right)$ is a gamma random variable of parameter $\frac{n-1}{2}$. It is also possible to get other identities by differentiating the densities with respect to the parameters $(W_e)_{e\in E}$.
This link between the VRJP and the supersymmetric hyperbolic sigma model made it possible to use results from the latter model (or adapt them) to deduce results on the former. The results for the $H^{2|2}$ model were obtained for weights all equal to one another while we have to consider random weights when looking at the ERRW. For instance, we have the following result for the VRJP with equal weights:

Theorem 36 (Theorem 2 of [33])**.** *Set $d \in \mathbf{N}$. There exists $w_r^d > 0$ such that for any $0 < w < w_r^d$, the VRJP on any finite subset Λ of $\mathbf{Z}^d$ started at 0 with weights $(W_e)_{e\in E}$ all equal to w satisfies the following inequality:*

$$\exists C > 0, \alpha \in (0,1), \forall x \in \mathbf{Z}^d,\ \mathbb{E}_{\mu_{|\Lambda|}^{W,0}}\left(e^{\frac{U_x-U_0}{2}}\right) \leq C\alpha^{|x|_1}.$$

The constants C, α do not depend on the choice of Λ.

From this it is easy to see through Borel Cantelli that almost surely the conductances decay exponentially in the distance to the origin. From this it is then possible to show that the walk is positive recurrent. Indeed, by using 1.1 we see that the probability that the walk gets to a distance n of the origin before coming back to the origin decays exponentially fast in n. This leads to a simple corollary.

Corollary 1. *Set $d \in \mathbf{N}$. There exists $w_r^d > 0$ such that for any $0 < w < w_r^d$, the VRJP on $\mathbf{Z}^d$ started at 0 with weights $(W_e)_{e\in E}$ all equal to w is a mixture of positive recurrent random walk among random conductances.*

There are also results regarding transience. For instance, it was shown in [34] that there is no decay of the conductances in dimension 3 and above for large enough initial weights:

Theorem 37. *Set $d \geq 3$. There exists $w_t^d > 0$ such that for any $w > w_t^d$, the VRJP on any finite subset Λ of $\mathbf{Z}^d$ started at 0 with weights $(W_e)_{e\in E}$ all equal to w satisfies the following inequality:*

$$\forall m \leq w^{\frac{1}{8}}, \forall x \in \mathbf{Z}^d,\ \mathbb{E}_{\mu_{|\Lambda|}^{W,0}}\left(\cosh(U_x-U_0)^m\right) \leq 2.$$

From this it is possible to prove that the VRJP is transient when the initial weights are larger than w_t^d.

Corollary 2 ([72])**.** *Set $d \in \mathbf{N}$. There exists $w_t^d > 0$ such that for any $w > w_t^d$, the VRJP on $\mathbf{Z}^d$ started at 0 with weights $(W_e)_{e\in E}$ all equal to w is transient with probability 1.*

Similarly this can be adapted to show that the ERRW is transient when the initial weights are larger enough.

Corollary 3 ([32])**.** *Set $d \in \mathbf{N}$. There exists $a_t^d > 0$ such that for any $a > w_t^d$, the ERRW on $\mathbf{Z}^d$ started at 0 with weights $(a_e)_{e\in E}$ all equal to a is transient with probability 1.*

Based on these results we see that in dimension 3 or more there is a phase transition between positive recurrent and transient. However, we need additional tools to study the VRJP with initial weights that are neither very small nor very large.

1.4.3 The beta-field

To simplify the study of the VRJP, another representation of its random environment was introduced in [73] and [55] (it was introduced in [73] with the second parameter η equal to 0 and then generalized to any η in [55]).

Proposition 1.4.3.1. *Let n be an integer, $(\eta_i)_{1\leq i\leq n}$ a family of non-negative parameters and $W \in M_n(\mathbf{R})$ a symmetric matrix with non-negative coefficients. Let $1_n \in \mathbf{R}^n$ be the vector $(1,\dots,1)$. The measure $\nu_n^{W,\eta}$ on $(0,\infty)^n$ is defined by the following density:*

$$\nu_n^{W,\eta}(\mathrm{d}\beta_1\dots\mathrm{d}\beta_n) := e^{-\frac{1}{2}\left(1_n H_\beta\, {}^t 1_n \;+\; \eta H_\beta^{-1}\, {}^t\eta \;-\; 2\sum\limits_{1\leq i\leq n}\eta_i\right)} \frac{1}{\sqrt{det(H_\beta)}} 1_{H_\beta>0}\mathrm{d}\beta_1\dots\mathrm{d}\beta_n,$$

where $\forall i,j \in [\![1,n]\!]$,

$$\begin{aligned} H_\beta(i,i) =& 2\beta_i - W(i,i),\\ H_\beta(i,j) =& -W(i,j) \text{ if } i\neq j \end{aligned}$$

and $H_\beta > 0$ means that H_β is positive definite. Then, $\nu_n^{W,\eta}$ is a ***probability*** *measure.*
We call $\tilde{\nu}_n^{W,\eta}$ the distribution of H_β when $(\beta_i)_{1\leq i\leq n}$ is distributed according to $\nu_n^{W,\eta}$.

Once again, it is not obvious that this family of measures is actually a family of probability measures. It was proved in [73] for $\eta = 0$ and in [55] for general η.
Unlike the measure μ, the family of measure ν does not depend on any specific starting point. The link with the VRJP is the following:

Theorem 38 (Theorem 3 of [73])**.** *Let $\mathscr{G} = (V,E)$ be a finite graph with n vertices, $(W_e)_{e\in E}$ positive weights and $i_0 \in V$ a starting point.*
Let $(U_x)_{x\in V}$ be distributed according to μ_n^{W,i_0} and γ a gamma random variable of parameter $(1/2,1/2)$ independent of U. We define the random vector $(\beta_x)_{x\in V}$ by:

$$\beta_x := \gamma 1_{x=i_0} + \frac{1}{2}\sum_{y\sim x} W_{\{x,y\}} e^{U_y-U_x}.$$

The random vector $(\beta_x)_{x\in V}$ is distributed according to $\nu_n^{W,0}$ (there is a slight abuse of notations, to make it rigorous, to every vertex of V we must associate a unique integer in $[\![1,n]\!]$).
Conversely, let H_β be distributed according to $\tilde{\nu}_n^{W,0}$ and let G_β be its inverse. The random vector $(U_x)_{x\in V}$ defined by:

$$U_x := \log\left(\frac{G_\beta(i_0,x)}{G_\beta(i_0,i_0)}\right) - \frac{1}{n}\sum_{y\in V}\log\left(\frac{G_\beta(i_0,y)}{G_\beta(i_0,i_0)}\right)$$

is distributed according to μ_n^{W,i_0}. The random variable $\frac{1}{G_\beta(i_0,i_0)}$ is a gamma random variable of parameter $(1/2,1/2)$ independent of U.

This β-field gives another description of the environment of the VRJP. Surprisingly, the starting point does not appear in the measure which gives a coupling of the VRJP for all possible starting points.
Even though this representation might seem more complicated than the previous one it has some nice properties that make it easier to use in some circonstances. First, it is possible to get the Laplace transform of the β-field ([73],[55]):

$$\mathbb{E}_{\nu_n^{W,0}}\left(e^{-\sum\lambda_x\beta_x}\right) = e^{-\sum\limits_{x\sim y} W_{\{x,y\}}\left(\sqrt{1+\lambda_x}\sqrt{1+\lambda_y}-1\right)} e^{-\sum_{x\in V}\eta_i(\sqrt{1+\lambda_i}-1)} \prod_{x\in V}\frac{1}{\sqrt{1+\lambda_x}}.$$

This Laplace transform tells us two important properties of the β-field. First the field is 1-dependent (the fields inside two subsets of V at distance two or more of each other are independent). Also the law of the β-field in a subset U of V only depends on the weights of the edges which have at least one endpoint in V. This was used in [76] to extend the β-field to infinite graphs. However on infinite graphs there is a small subtlety: there might be more than one representation as a random walk among random conductances. If the VRJP is almost surely recurrent then the representation is unique up to multiplicative constants (because of the law of large number, the number of times the walk crosses an edge $\{x,y\}$ divided by the number of times it is in x converges to $2\omega(x,y)/\pi_x$). If the VRJP is transient, there can be multiple representations. This is the case on trees where it was shown that there are multiple representations in [23]. On $\mathbf{Z}^d$ it was shown in [44] that for large enough initial weights there is only one representation for the VRJP.
The family of laws ν as a final interesting property: it is stable under marginals and conditioning.

Theorem 39 ([55],[76])**.** *Let n_1, n_2 be two integers, and $n := n_1 + n_2$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative coefficients and $(\eta_i)_{i \in [\![1, n_1+n_2]\!]}$ a family of non-negative coefficients. Let $(\beta_i)_{i \in [\![1,n_1+n_2]\!]}$ be random variables with a $\nu_n^{W,\eta}$ distribution and $H_\beta \in M_n(\mathbf{R})$ the matrix defined by:*

$$\forall i,j \in [\![1,n]\!], H_\beta(i,j) := \begin{cases} 2\beta_i - W(i,i) \text{ if } i = j, \\ -W(i,j) \text{ if } i \neq j. \end{cases}$$

We make the following bloc decomposition:

$$W = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}, H_\beta = \begin{pmatrix} H_\beta^{11} & H_\beta^{12} \\ H_\beta^{21} & H_\beta^{22} \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix},$$

where $W^{11}, H_\beta^{11} \in M_{n_1}(\mathbf{R})$, $W^{12}, H_\beta^{12} \in M_{n_1,n_2}(\mathbf{R})$, $W^{21}, H_\beta^{21} \in M_{n_2,n_1}(\mathbf{R})$, $W^{22}, H_\beta^{22} \in M_{n_2}(\mathbf{R})$, $\eta^1 \in \mathbf{R}^{n_1}$ and $\eta^2 \in \mathbf{R}^{n_2}$. Then the family $(\beta_i)_{1 \le i \le n_1}$ is distributed according to $\nu_{n_1}^{W^{11},\hat{\eta}}$ where

$$\hat{\eta} \in \mathbf{R}^{n_1} \text{ and } \forall i \in [\![1,n_1]\!],\ \hat{\eta}_i := \eta_i + \sum_{k=1}^{n_2} W^{12}(i,k).$$

Conditionally on $(\beta_i)_{1 \le i \le n_1}$, the family $(\beta_i)_{n_1+1 \le i \le n_1+n_2}$ is distributed according to $\nu_{n_2}^{\check{W},\check{\eta}}$ where

$$\check{W} = W^{22} + W^{21} \left(H_\beta^{11}\right)^{-1} W^{12},$$

and

$$\check{\eta} \in \mathbf{R}^{n_2} \text{ and } \check{\eta} = \eta^2 + W^{21} \left(H_\beta^{11}\right)^{-1} \eta^1.$$

Finally there is one important tool that needs to be discussed: the ψ-field. It should be understood as something similar to the effective conductance to infinity for electrical networks. Set an infinite, connected, locally finite graph $\mathscr{G} = (V,E)$ and initial weights $(W_e)_{e \in E}$ we define the sequence of graphs $\mathscr{G}_n = (V_n, E_n)$ obtained by keeping a finite subset of $\mathscr{G}$ and collapsing all other vertices into one vertex δ_n.

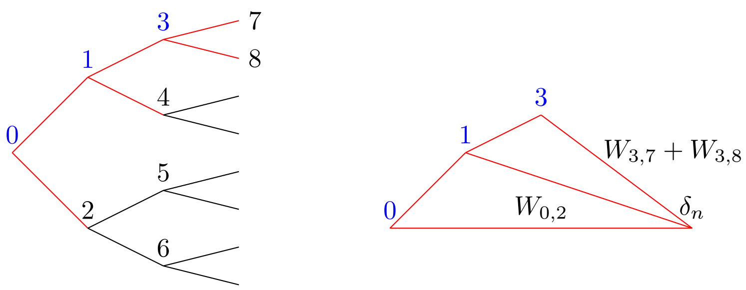


Figure 1.4: Finite approximation of a graph $\mathscr{G}$

We define $\psi_n(x) := \frac{G_{\beta,n}(x,\delta_n)}{G_{\beta,n}(\delta_n,\delta_n)}$ where $G_{\beta,n}$ is the inverse of H_β on $\mathscr{G}_n$.

Theorem 40 ([76])**.** *For some choice of coupling of the β-fields on a sequence of graphs $\mathscr{G}_n$ that is increasing, for any $x \in V$ and for n large enough:*

$$\mathbb{E}\left(\psi_{n+1}(x) | \psi_n(x)\right) = \psi_n(x).$$

It was later shown in [31] that other quantities possessed the martingale property when going from a graph to a larger graph. However, ψ is of special interest. First, since $\psi_n(x) \ge 0$, there exists a random variable $\psi_\infty(x)$ such that a.s

$$\psi_n(x) \to \psi_\infty(x).$$

The quantity $\psi_\infty(0)$ has the following interpretation in terms of recurrence/transience.

Theorem 41 ([76])**.** *If $\psi_\infty(0) = 0$ the VRJP starting at 0 is recurrent, otherwise it is transient.*

This is important for the transience and recurrence of the VRJP and the ERRW. First, if the law of the weights are invariant by translation and the weights are ergodic then $(\psi_\infty(x))_{x \in \mathbf{Z}^d}$ is ergodic and its law is invariant by translation. This means we have a $0-1$ law for recurrence/transience of the VRJP and the ERRW.

Theorem 42. *For any $d \ge 1$, the VRJP with all initial weights equal to $w > 0$ is recurrent with probability 0 or 1.*
Similarly, the ERRW with all initial weights equal to $a > 0$ is recurrent with probability 0 or 1.

Furthermore, instead of having to show that the conductances decay exponentially fast or faster than some polynomial to prove recurrence, we now only need any kind of decay for the martingale $(\psi_n(0))_{n\in\mathbf{N}}$. This means that the polynomial decay in dimension 2 for the ERRW found in [60] is now enough to prove that the ERRW on $\mathbf{Z}^2$ is recurrent if all the initial weights are equal. A similar decay was found simultaneously and independently in [71] and [53] for the VRJP on $\mathbf{Z}^2$. It was previously found in [6] that the VRJP was recurrent in $\mathbf{Z}^2$ in a weaker sense: the expected time the process spends at the origin is infinite. This leads to the following picture:

Theorem 43. *In dimension 1 and 2, the ERRW and the VRJP with weights all equal to $a > 0$ are both recurrent.*
In dimension $d \geq 3$ there exists weights $0 < a_r^d \leq a_t^d < +\infty$ and $0 < w_r^d \leq w_t^d < +\infty$ such that the ERRW with weights all equal to $a > 0$ is recurrent if $a < a_r^d$ and transient if $a > a_t^d$ and the VRJP with weights all equal to $w > 0$ is recurrent if $w < w_r^d$ and transient if $w > w_t^d$.

1.4.4 Our results regarding ERRW and VRJP

In Chapter 4 we show that the VRJP and the ERRW exhibit some monotonicity in the initial weights. This means that $a_r^d = a_t^d$ and $w_r^d = w_t^d$. We then get the following picture:

Theorem 44 (Chapter 4)**.** *In dimension 1 and 2, the ERRW and the VRJP with weights all equal to $a > 0$ are both recurrent.*
In dimension $d \geq 3$ there exists weights $0 < a_c^d < +\infty$ and $0 < w_x^d < +\infty$ such that the ERRW with weights all equal to $a > 0$ is recurrent if $a < a_c^d$ and transient if $a > a_c^d$ and the VRJP with weights all equal to $w > 0$ is recurrent if $w < w_c^d$ and transient if $w > w_c^d$.

This is done by showing that the martingale property of the ψ-field is actually much more general, by an appropriate coupling of the β-fields. There are other quantities that are martingales and this martingale property appears just by lowering any weights, it is not necessary to increase the size of a graph (collapsing two points into one is the same as increasing the weight between them to infinity similarly splitting δ_n into multiple point is the same as decreasing the weight between those point from infinity to something finite). This plus the $0-1$ law 42 allows us to conclude.
What we show is that all the quantities $\frac{G_\beta(i,j)}{G_\beta(i,i)}$ have a martingale property when we decrease the weights of the VRJP.

Theorem 45 (Chapter 4)**.** *Let $n \geq 2$ be an integer, let $W^-, W^+ \in M_n(\mathbf{R})$ be two symmetric matrices with null diagonal coefficients and non-negative off-diagonal coefficients such that for any $i,j \in [\![1,n]\!]$, $W^-(i,j) \leq W^+(i,j)$ and i and j are W^--connected. Let H^- and H^+ be two matrices distributed according to $\tilde{\nu}_n^{W^-,0}$ and $\tilde{\nu}_n^{W^+,0}$ respectively, and let their inverse be G^- and G^+ respectively. For any convex function f, any integer $i \in [\![1,n]\!]$ and any deterministic vector $X \in [0,\infty)^n$:*

$$\mathbb{E}\left(f\left(\frac{\sum_{j=1}^{n} X_j G^-(i,j)}{G^-(i,i)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{\sum_{j=1}^{n} X_j G^+(i,j)}{G^+(i,i)}\right)\right).$$

This may not look like a martingale property but by Theorem 4.1 of [37] we know that this is the same as saying that there is a coupling between G^+ and G^- such that

$$\mathbb{E}\left(\sum_{j=1}^{n} X_i \frac{G^-(i,j)}{G^-(i,i)} | G^+\right) = \sum_{j=1}^{n} X_i \frac{G^+(i,j)}{G^+(i,i)}.$$

This is similar in a way to the Rayleigh monotonicity for electrical networks (theorem 3); here the increase of $\mathbb{E}(f(G(i,j)/G(i,i)))$ plays the role of the decrease effective conductance between i and j when the weights decrease. This result can be applied to $\psi_n(0)$ to show the following property.

Theorem 46. *Let $\mathscr{G} = (V,E)$ be an infinite, non-directed, connected graph without loops or multiple edges and $0 \in V$ a vertex in this graph. Let $(W_e^-)_{e\in E}$ and $(W_e^+)_{e\in E}$ be two families of positive weights such that for any $e \in E$, $0 < W_e^- \leq W_e^+$. The probability that the VRJP with initial weights W^- is recurrent is greater or equal than the probability that the VRJP with initial weights W^+ is recurrent.*

To prove this, we increase the weights one by one and show that the result is true if we increase only one weight. The β-field on $V\backslash\{i,j\}$ does not depend on $W_{i,j}$ by the restriction property 39. This means that when we increase a weight $W_{\{i,j\}}$, we can fix the value of the β-field outside $\{i,j\}$ and through careful use of the restriction property we can reduce the problem to the study of the model on the graph $\{i,j\}$. Then we only have to show through a coupling on all the laws $\nu_2^{W,0}$ that decreasing the weight gives rise to a martingale property. This gives the more technical result below.

Theorem 47. *Set an integer $n \in \mathbf{N}$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative off diagonal coefficients and null diagonal coefficients. Let $W^1, W^2 \in M_{n,1}(\mathbf{R})$ be two matrices with non-negative coefficients and let $W^3 \in M_{n,1}(\mathbf{R})$ be the matrix defined by $W^3 := W^1 + W^2$. Let $w^-, w^+ \in [0,\infty)$ be two positive real with $w^- < w^+$. We define the matrices W^-, W^+ and W^∞ by:*

$$W^- := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^- \\ {}^tW^2 & w^- & 0 \end{pmatrix}, W^+ := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^+ \\ {}^tW^2 & w^+ & 0 \end{pmatrix} \text{ and } W^\infty := \begin{pmatrix} W & W^3 \\ {}^tW^3 & 0 \end{pmatrix}.$$

If $n = 0$, we just have:

$$W^- := \begin{pmatrix} 0 & w^- \\ w^- & 0 \end{pmatrix}, W^+ := \begin{pmatrix} 0 & w^+ \\ w^+ & 0 \end{pmatrix} \text{ and } W^\infty := (0).$$

For any vector $X \in \mathbf{R}^{n+2}$ we define the vector $\overline{X} \in \mathbf{R}^{n+1}$ by:

$$\forall i \in [\![1,n]\!],\ \overline{X}_i := X_i \text{ and}$$
$$\overline{X}_{n+1} := X_{n+1} + X_{n+2}.$$

For any vector $X^1 \in [0,\infty)^{n+2}$ there exists random matrices H^-, H^+ and H^∞ (with inverse G^-, G^+ and G^∞ respectively) that are distributed according to $\tilde{\nu}_{n+2}^{W^-,0}, \tilde{\nu}_{n+2}^{W^+,0}$ and $\tilde{\nu}_{n+1}^{W^\infty,0}$ respectively such that

$${}^tX^1G^-X^1 = {}^tX^1G^+X^1 = {}^t\overline{X^1}G^\infty\overline{X^1} \text{ almost surely,}$$

for all $i \in [\![1,n]\!]$, $H^-(i,i) = H^+(i,i) = H^\infty(i,i)$ and for any vector $X^2 \in [0,\infty)^{n+2}$ we have:

$$\mathbb{E}\left({}^tX^1G^+X^2|H^\infty\right) = {}^t\overline{X^1}G^\infty\overline{X^2}, \text{ and}$$
$$\mathbb{E}\left({}^tX^1G^-X^2|H^+\right) = {}^tX^1G^+X^2 \text{ if } n+1 \text{ and } n+2 \text{ are } H^-\text{-connected.}$$

In this theorem, X can be seen as the vertex i from the previous theorems (if $X_i = 1$ and all the other coordinates are zero, we find the previous theorem). The matrix H^∞ corresponds to the case where the weight $W_{i,j}$ we change is made to go to infinity which is the same as fusing together the vertices i and j.
This result has other applications. First, we can use it to show that both the VRJP and the ERRW satisfy a $0-1$ law for recurrence/transience for any positive initial weights, improving theorem 42 by Sabot and Zeng [76].

Theorem 48. *For any locally finite graph $\mathscr{G} = (V,E)$ and any vertex $x_0 \in V$, the VRJP on $\mathscr{G} = (V,E)$ starting at 0 and with independent positive random weights $(W_e)_{e\in E}$ is recurrent with probability 0 or 1. In particular, the ERRW on $\mathscr{G}$, starting at x_0 and with initial deterministic positive weights $(a_e)_{e\in E}$ is recurrent with probability 0 or 1.*

The idea is for any n to use the graph $\mathscr{G}_n$ which is the graph $\mathscr{G}$ where all the vertices at distance n are fused into one. For this graph it is possible to show that $\phi_\infty(0)/\phi_n(0)$ is independent of $\phi_n(0)$. Then we can use our graph to compare $\phi_\infty(0)$ on the two graphs (there is one direction that is given directly by the theorem, the other one is more subtle but essentially our result allows us to identify a worst case scenario that is easier to study than the general case). From this it is possible to show that on our original graph $\mathbb{P}(\phi_\infty(0) = 0|\phi_n(0))$ does not depend on $\phi_n(0)$. From this it is then possible to show a $0-1$ law.
We also provide an alternative proof of this decay in chapter 4 that works for all recurrent graphs:

Theorem 49 (Chapter 4)**.** *Let $\mathscr{G} = (V,E)$ be an infinite, locally finite graph and $x_0 \in V$ a vertex. Let $(W_e)_{e\in E}$ be a family of positive weights. If the random walk on $\mathscr{G}$ starting at x_0 with deterministic conductances $(c_e)_{e\in E} = (W_e)_{e\in E}$ is recurrent then so are the ERRW and the VRJP starting at x_0 and with initial weights $(W_e)_{e\in E}$.*

It is possible to use our theorem 59 to compare the VRJP to a VRJP with "infinite" weights which corresponds to the simple random walk. However, in chapter 4, we will prove this result using a simpler method than does not use our result, only the representation of the VRJP with the β-field. The idea is to use that the β-field behaves nicely under conditioning. The behaviour is similar to that of electrical networks. If you take a

random walk on an electrical network (V, E, c) but only look at it when it is in a subset V' of V it behaves like a random walk on an electrical network (V', E', c') where the conductances c' are equal to the conductances c plus a term that corresponds to paths in $V \backslash V'$. Similarly if you take a β-field on (V, E, W) conditioned on its value on a subset $V \backslash V'$ of V it has the same law as a β-field on (V', E', W') where the weights W' are equal to the weights W plus a term that corresponds to paths in $V \backslash (V'$. It is possible to show by induction that when V' consists of only two points, the weight W' is smaller or equal in expectation to the conductance c' (if $c = W$). If the graph is recurrent this means that the effective weight between 0 and δ_n goes to 0 which in turns means that $\psi_n(0)$ goes to 0.
Even though it is not stated this way, we believe that the proof in [71] also works for recurrent graphs (with maybe an additional assumption that the weights must be bounded from below by a positive constant).
Finally in chapter 5 we study a biased version of the ERRW for which we show that if $\mathscr{G}$ is a finite graph then this biased ERRW is recurrent on $\mathscr{G} \times \mathbf{Z}$ if the bias is small enough.

1.4.5 The supersymmetric hyperbolic sigma model

In this section we will briefly introduce the supersymmetric hyperbolic sigma model (also called $H^{2|2}$ for short). It was introduced by Zirnbauer in [92] as a toy model for some model with supersymmetries and that describes quantum transport and localization. The model is a spin model on a graph where the spins have 2 commuting components (reals), two anticommuting components (Grassmann variables which live in a more complicated space) and a fifth component such that the norm of the spin is equal to -1. The space in which the Grassmann variables leave is a real algebra with a dimension that depends on the number of vertices of the graph. To avoid going into too much details, think of Grassmann variables as formal variables with the following properties: for any real s, and any Grassmann variables ψ_1, ψ_2 we have:

$$s\psi_1 = \psi_1 s \text{ and } \psi_1\psi_2 = -\psi_2\psi_1.$$

In particular this means that if ψ is a Grassmann variable then $\psi^2 = 0$. Now, to each vertex i of a (finite) graph we give a spin $\sigma_i := (z_i, x_i, y_i, \xi_i, \eta_i)$ where x_i, y_i are reals, ξ_i, η_i are Grassmann variables and z_i is defined by:

$$z_i := \sqrt{1 + x_i^2 + y_i^2} + \frac{1}{\sqrt{1 + x_i^2 + y_i^2}} \xi_i \eta_i.$$

This is the first order expansion of $\sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i}$ and we can easily show that $z_i^2 = 1 + x_i^2 + y_i^2 + 2\xi_i\eta_i$ because $(\xi_i\eta_i)^2 = 0$. This means that we can see z_i as $\sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i}$. We define the scalar products $< \sigma_i.\sigma_j >$ on spins by:

$$< \sigma_i.\sigma_j >:= -z_i z_j + x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j.$$

We have:

$$\begin{aligned}
< \sigma_i.\sigma_i >= & -\left(\sqrt{1 + x_i^2 + y_i^2} + \frac{1}{\sqrt{1 + x_i^2 + y_i^2}} \xi_i \eta_i\right)^2 + x_i^2 + y_i^2 + 2\xi_i\eta_i \\
= & -(1 + x_i^2 + y_i^2) - 2\xi_i\eta_i - \frac{1}{1 + x_i^2 + y_i^2} \xi_i\eta_i\xi_i\eta_i + x_i^2 + y_i^2 + 2\xi_i\eta_i \\
= & -1 \qquad \text{because } \xi_i\eta_i\xi_i\eta_i = -\xi_i(\eta_i\eta_i)\xi_i = 0.
\end{aligned}$$

The spins $\sigma_i := (z_i, x_i, y_i, \xi_i, \eta_i)$ live in a space that we call $H^{2|2}$ because there are 2 commuting components, 2 anticommuting components and the norm of the elements is -1 (which make it a hyperbolic space). Now we need to define a notion of integration over the Grassmann variables to be able to define measures on the spins. We look at functions F of the form:

$$F(z, x, y, \xi, \eta) = \sum_{I_1, I_2 \subset V} f_{I_1, I_2}(x, y) \prod_{i_1 \in I_1} \xi_{i_1} \prod_{i_2 \in I_2} \eta_{i_2}. \tag{1.2}$$

The order of the product is important because it can change the sign of the product depending on the order of the product. We must therefore set the order once and for all. This set of functions might seem limited but because of the properties of the Grassmann variables it is includes most usual functions. For instance the function $\sqrt{1 + \xi_1}$ can be written as $1 + \frac{1}{2}\xi_1$ (we do have $(1 + \frac{1}{2}\xi_1)^2 = 1 + \xi_1$). Similarly the function $e^{\xi_1 + \xi_2\eta_2}$ can be written as $1 + \xi_1 + \xi_2\eta_2 + \xi_1\xi_2\eta_2$.
Now for each of the monomials of such functions we define:

$$\int f_{I_1, I_2}(x, y) \prod_{i_1 \in I_1} \xi_{i_1} \prod_{i_2 \in I_2} \eta_{i_2} \mathrm{d}\xi_j := \begin{cases} 0 & \text{if } j \notin I_1 \\ \varepsilon(I_1, I_2, i) f_{I_1, I_2}(x, y) \prod\limits_{i_1 \in I_1 \backslash \{j\}} \xi_{i_1} \prod\limits_{i_2 \in I_2} \eta_{i_2} \mathrm{d}\xi_j & \text{if } j \in I_1, \end{cases}$$

where $\varepsilon(I_1, I_2, j)$ depends on the position of the variable ξ_j in the product: if it is at an odd position (first,third... starting from the left) then it is equal to 1, otherwise it is equal to -1. We can define something similar when integrating with respect to η. This is similar to a derivation. This means that if we integrate over η and ξ we are left with only the term $f_{V,V}(x,y)$. Then we can define the full integral by:

$$\int\limits_{x,y,\xi,\eta} F = \int\limits_{x,y\in(\mathbf{R}^2)^V} f_{V,V}(x,y) \prod_{i\in V} \mathrm{d}x_i \mathrm{d}y_i.$$

We then have the following proposition regarding supersymmetry:

Theorem 50 (lemma 16 of [34]). *If a function F is of the form 1.2 we say that it is supersymmetric if:*

$$\sum_{i\in V}(\xi_i\partial_{x_i} + \eta_i\partial_{y_i} + x\partial_{\eta_i} - y\partial_{\xi_i})F = 0.$$

If a function is supersymmetric, smooth and has sufficient decay then:

$$\int\limits_{x,y,\xi,\eta} F = (2\pi)^{|V|} f_{\emptyset,\emptyset}(0,0).$$

It is possible to show that the function $(x,y,\xi,\eta) \to z_i$ and $(x,y,\xi,\eta) \to < \sigma_i.\sigma_j >$ are supersymmetric. If we look at the functions $F^{W,h}$ defined by:

$$e^{-\sum\limits_{\{i,j\}\in E} W_{i,j}(-<\sigma_i.\sigma_j>-1)+h\sum\limits_{i\in V}(z_i-1)} \prod_{i\in V}\frac{1}{z_i},$$

it is also possible to show that there are supersymmetric for any choice of W, h. This means that:

$$\int F^{W,h} = (2\pi)^{|V|}.$$

Then it is also possible to make a change of variables and integrate over the Grassmann variables which leads to:

$$\int F^{W,h} = \int\limits_{(\mathbf{R}^2)^V} e^{-\sum\limits_{\{i,j\}\in E} W_{i,j}(\cosh(t_i-t_j)-1+\frac{1}{2}(s_i-s_j)^2e^{t_i+t_j})+h\sum\limits_{i\in V}\left(\cosh(t_i)-1+\frac{1}{2}s_i^2e^{t_i}\right)} \mathrm{Det}(D_{W,h}) \prod_{i\in V} e_i^t \mathrm{d}t_i \mathrm{d}s_i,$$

where

$$D_{W,h}(i,j) = \begin{cases} -W_{i,j}e^{t_i+t_j} & \text{if } i\neq j \\ he^{t_i} + \sum\limits_{k\sim i} W_{i,k}e^{t_i+t_k} & \text{if } i=j \end{cases}$$

Then by integrating over s (it is just a gaussian) you are left with:

$$\int\limits_{\mathbf{R}^V} \left(\frac{1}{2\pi}\right)^{|V|/2} e^{-\sum\limits_{\{i,j\}\in E} W_{i,j}(\cosh(t_i-t_j)-1)+h\sum\limits_{i\in V}(\cosh(t_i)-1)} \sqrt{\mathrm{Det}(D_{W,h})} \prod_{i\in V} e^{t_i}\mathrm{d}t_i = 1.$$

This is essentially the same measure as the one we use for the VRJP and it is possible to prove that the VRJP measure is a probability measure from this equality.
It is easy to construct similar $H^{n|m}$ model for different number of commuting and anti-commuting variables. Surprisingly, some of these models seem to be connected to other probabilistic models. For instance it was shown in [5] that the $H^{0|2}$ and $H^{2|4}$ are linked to Bernoulli bond percolation conditioned on not containing open loops.

Chapter 2

Limit theorem for sub-ballistic Random Walks in Dirichlet Environment in dimension $d \geq 3$

Up to minor modifcations, this chapter is a reproduction of the article [63] available on Arxiv.

Abstract

We look at random walks in Dirichlet environment. It was known that in dimension $d \geq 3$, if the walk is sub-ballistic, the displacement of the walk is polynomial of order κ for some explicit κ. We show that the walk, after renormalization, actually converges to a κ-stable completely asymmetric Levy Process.

2.1 Introduction and results

2.1.1 Introduction

Random walks in random environments (RWRE) have been studied for several decades and are now rather well understood in the one dimensional case (see Solomon [80],Kesten, Kozlov, Spitzer [51] and Sinaï [78]). Important progress has been made in higher dimension, mainly in 3 directions: under a ballisticity condition, for small perturbation of the simple random walk ([21], [85], [17], [68], [54]) and in Dirichlet environment.
The most studied ballisticity conditions come from the conditions (T) and (T') introduced by Sznitman in [83], [81]. They have been shown to be equivalent in [46] and also to be equivalent to an effective polynomial condition [13], [22]. By assuming any of these, in the ballistic regime, directional transience, ballisticity, and a CLT have been proved. Quenched CLTs have also been proved in various cases, either by assuming an annealed CLT, uniform ellipticity and a condition introduced by Kalikow [82], or by assuming the existence of high enough moments for the renewal times (see [86] for a definition of the renewal times) and uniform ellipticity of the environment [66] and [14] in dimension $d \geq 4$.
All these results show limit theorems in the ballistic case, that is to say that the walk has a positive speed. In dimension 2 and higher no complete limit theorems are known for the RWRE in the sub-ballistic case. However in dimension 1 we know that a sub-ballistic regime exists, where the walk can behave like the inverse of a stable subordinator [51] [39]. This sub-ballistic regime is caused by the existence of traps where the walk spends most of its time. This trapping phenomenon appears in other models closely related to the RWRE for instance the Bouchaud Trap Model (see [7] for a precise definition and an overview of the results). The model of random walks in random conductances also exhibits a similar trapping phenomenon. Indeed an annealed limit theorem (the limit is the inverse of a stable subordinator) and an equivalent to the CLT [43] have been proved for the biased random walk in random conductances. Similar results have been obtained for the biased walk in the percolation cluster and in Galton-Watson trees, but in both cases there is no convergence to a limit law [8], [40]. In the special case of iid RWRE a trapping phenomenon that leads to sub-ballistic behaviour has been identified in [18], [19] and [41] but no limit theorem has been proved.
The random walk in Dirichlet environment (RWDE) is a model where the transition probabilities are iid Dirichlet random variables (see [75] for an overview). It was first introduced because of its link to the linearly directed-edge reinforced random walk ([62],[38]). It also has a property of invariance by time reversal that allows explicit calculations (see [69]). In particular, it gives a simple criterion for existence of absolutely continuous

invariant distribution from the point of view of the particle, directional transience and ballisticity in dimension $d \geq 3$ ([87], [18], [88], [70]). In the non-ballistic case the walk is directionally transient but the limit law was still unknown ([18]), it was only known that for some explicit $\kappa \in (0,1]$, $\frac{\log(|X_n|)}{\log(n)} \to \kappa$.
In this paper we give the annealed limit law for the sub-ballistic regime ($\kappa \leq 1$) in dimension $d \geq 3$. In the case $\kappa = 1$ we have the limit law of $\frac{1}{n\log(n)} Y_n$ (where Y is the random walk) while for $\kappa < 1$ we have the limit law of the process. To the best of our knowledge, this is the first stable limit theorem for non reversible RWRE in iid environment, in dimension $d \geq 2$.

2.1.2 Definitions and statement of the results

In all the paper we set $d \geq 3$. Let $(e_1, \ldots, e_d)$ be the canonical base of $\mathbf{Z}^d$ and for any $j \in [\![d+1, 2d]\!]$, set $e_j = -e_{j-d}$. For any $z \in \mathbf{Z}^d$, let $||z|| := \sum\limits_{i=1}^{d} |z_i|$ be the L_1-norm of z. For any $x, y \in \mathbf{Z}^d$ we will write $x \sim y$ if $||y - x|| = 1$. Let $E = \{(x,y) \in (\mathbf{Z}^d)^2, x \sim y\}$ be the set of directed edges of Z^d and let $\tilde{E} = \{\{x,y\}, (x,y) \in (\mathbf{Z}^d)^2, x \sim y\}$ be the set of non-directed edges. Let Ω be the set of environments on $\mathbf{Z}^d$:

$$\Omega = \{\omega = (\omega(x,y))_{x\sim y} \in (0,1]^E \text{ such that } \forall x \in \mathbf{Z}^d, \sum_{i=1}^{2d} \omega(x, x+e_i) = 1\}.$$

For each $\omega \in \Omega$, let $(Y_n)_{n\in\mathbf{N}}$ be the Markov chain on $\mathbf{Z}^d$ defined by $Y_0 = 0$ almost surely and the following transition probabilities:

$$\forall y \in \mathbf{Z}^d, \forall i \in [\![1, 2d]\!],\ \mathrm{P}_0^\omega(Y_{n+1} = y + e_i | Y_n = y) = \omega(y, y + e_1).$$

Let $\mathrm{E}_{\mathrm{P}_0^\omega}$ be the expectation with respect to P_0^ω.
Given a family of positive weights $(\alpha_1, \ldots, \alpha_{2d})$, we consider the case where the transition probabilities at each site are iid Dirichlet random variables of parameter $\alpha := (\alpha_1, \ldots, \alpha_{2d})$, that is with density:

$$\frac{\Gamma\left(\sum\limits_{i=1}^{2d} \alpha_i\right)}{\prod\limits_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i - 1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{2d-1}$$

on the simplex

$$\{(x_1, \ldots, x_{2d}) \in (0,1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}.$$

Let $\mathbb{P}^{(\alpha)}$ be the law obtained on Ω this way. Let $\mathbb{E}_{\mathbb{P}^{(\alpha)}}$ be the expectation with respect to $\mathbb{P}^{(\alpha)}$ and let $\mathbb{P}_0^{(\alpha)}[.] := \mathbb{E}_{\mathbb{P}^{(\alpha)}}[\mathrm{P}_0^\omega(.)]$ be the annealed law of the process starting at 0. Let $(\tau_i)_{i\in\mathbf{N}^*}$ be the renewal times, in the direction e_1, introduced in [86]:

Definition 22. *We define $(\tau_i)_{i\in\mathbf{N}^*}$, the renewal times in the direction e_1, by:*

$$\tau_1 = \inf\{n \in \mathbf{N}, \forall i < n, Y_i.e_1 < Y_n.e_1 \textit{ and } \forall i > n, Y_i.e_1 > Y_n.e_1\}$$

and for all $i > 1$:

$$\tau_{i+1} = \inf\{n > \tau_i, \forall i < n, Y_i.e_1 < Y_n.e_1 \textit{ and } \forall i > n, Y_i.e_1 > Y_n.e_1\}.$$

The renewal times are used to create independence thanks to the following theorem (Theorem 1.4 of [86]).

Proposition 2.1.2.1. *For all $k \in \mathbf{N}^*$, let $\mathscr{G}_k$ be the σ-field defined by:*

$$\mathscr{G}_k := \sigma(\tau_1, \ldots, \tau_k, (Y_n)_{0\leq n\leq\tau_k}, (\omega(x,\cdot))_{x.e_1 < Y_{\tau_k}.e_1}).$$

We have, for all $k \geq 1$:

$$\mathbb{P}_0^{(\alpha)}\left((Y_{\tau_k+n})_{n\geq 0} \in \cdot, (\omega(Y_{\tau_k} + x, \cdot))_{x.e_1\geq 0} \in \cdot | \mathscr{G}_k\right) = \mathbb{P}_0^{(\alpha)}\left((Y_n)_{n\geq 0} \in \cdot, (\omega(x,\cdot))_{x.e_1\geq 0} \in \cdot | \tau_1 = 0\right).$$

This means that the trajectories and the transition probabilities inside slabs between two consecutive renewal times (after the first one) are i.i.d random variables.

Definition 23. *We define the drift d_α by:*

$$d_\alpha := \sum \alpha_i e_i.$$

If $d_\alpha \neq 0$, we will assume, without loss of generality, that $\alpha_1 > \alpha_{1+d}$.

Definition 24. *We define the two parameters κ and κ' by:*

$$\kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\dots,d}(\alpha_i + \alpha_{i+d})$$

and

$$\kappa' = 3\left(\sum_{i=1}^{2d} \alpha_i\right) - 2\max_{i=1,\dots,d}(\alpha_i + \alpha_{i+d}).$$

For any direction $j \in [\![1, d]\!]$ we also define the parameter κ_j by:

$$\kappa_j := 2\left(\sum_{i=1}^{2d} \alpha_i\right) - (\alpha_j + \alpha_{j+d})$$

In [70], it was proved that, for $d \geq 3$, when $\kappa > 1$, there exists an invariant probability measure $\mathbb{Q}^{(\alpha)}$ for the environment from the point of view of the particle, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$. From that it is possible to show that directional transience and ballisticity are equivalent when $\kappa > 1$. Furthermore, we know for which parameter the walk is directionally transient.

Theorem 1 (Corollary 1 of [88]). *If $d \geq 3$ and $d_\alpha \neq 0$, then for $\mathbb{P}^{(\alpha)}$ almost every environment, the walk is directionally transient with asymptotic direction d_α, that is to say:*

$$\frac{Y_n}{||Y_n||} \to \frac{d_\alpha}{||d_\alpha||},\ P_0^\omega \text{ almost surely.}$$

However, when $\kappa \leq 1$, such an invariant probability does not exist because of traps. But, in [18], it was proved that, by accelerating the walk, we can get an invariant probability for this accelerated walk, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$.
This lead to the following limit theorem in [18]:

Proposition 2.1.2.2. *If $\kappa \leq 1$, $d \geq 3$ and $d_\alpha \neq 0$. Let $l \in \{e_1, \dots, e_{2d}\}$ be such that $d_\alpha.l > 0$. Then we have the following convergence in probability (for the annealed law):*

$$\frac{\log(Y_n.l)}{\log(n)} \to \kappa.$$

We will now give a precise definition of the accelerated walk. We call directed path a sequence of vertices $\sigma = (x_0, \dots, x_n)$ such that $(x_i, x_{i+1}) \in E$ for all i. To simplify notations, we will write $\omega_\sigma := \prod_{i=0}^{n-1} \omega(x_i, x_{i+1})$. For any positive integer m, we define the accelerating function $\gamma_\omega^m(x)$ by:

$$\gamma_\omega^m(x) := \frac{1}{\sum \omega_\sigma},$$

where the sum is on all finite simple (each vertex is visited at most once) paths σ in $x + [\![-m, m]\!]^d$, starting from x, going to the border of $x + [\![-m, m]\!]^d$ and stopped the first time they reach this border. We will call X_t^m the continuous-time Markov chain whose jump rate from x to y is $\gamma_\omega^m(x)\omega(x, y)$, with $X_0^m = 0$. This means that $Y_n = X_{t_n^m}^m$ and $X_t^m = \sum_k Y_k 1_{t_k^m \leq t < t_{k+1}^m}$, for $t_n^m = \sum_{k=1}^{n} \frac{1}{\gamma_\omega^m(Y_k)} \mathscr{E}_k$, where the $\mathscr{E}_i$ are iid exponential random variables of parameter 1. The walk X_t^m can be viewed as an accelerated version of the walk Y_n.
Now, we need to introduce an other object: the walk seen from the point of view of the particle. First, let $(\theta_x)_{x \in \mathbf{Z}^d}$ be the shift on the environment defined by: $\theta_x \omega(y, z) := \omega(x + y, x + z)$. We call process seen from the point of view of the particle the process defined by $\overline{\omega_t^m} = \theta_{X_t^m} \omega$. Unlike the walk Y, under $\mathbb{P}_0^{(\alpha)}$, $\overline{\omega_t^m}$ is a Markov process on Ω. Its generator R is given by:

$$Rf(\omega) = \sum_{i=1}^{2d} \gamma_\omega^m(0)\omega(0, e_i) f(\theta_{e_i}\omega),$$

for all bounded measurable functions f on Ω.

Theorem 2. *(Theorem 2.1 of [18])*
In dimension $d \geq 3$, if m is large enough then the process $\left(\overline{\omega_t^m}\right)_{t \in \mathbf{R}^+}$ has a stationary distribution $\mathbb{Q}^{m,\alpha}$. For any $\beta > 1$ there exists an m such that $\frac{d\mathbb{Q}^{m,\alpha}}{d\mathbb{P}^\alpha}$ is in L^β.

We will write $\mathbb{Q}_0^{m,\alpha}(\cdot)$ for $\mathbb{Q}^{m,\alpha}\left(\mathrm{P}_0^{\omega}(\cdot)\right)$ To simplify the notations, we will drop the (α) from $\mathbb{P}^{(\alpha)}, \mathbb{P}_0^{(\alpha)}, \mathbb{Q}^{m,\alpha}$ and $\mathbb{Q}_0^{m,\alpha}$ when there is no ambiguity. We will also write X_t, $\mathbb{Q}$ and $\mathbb{Q}_0$ instead of X_t^m, $\mathbb{Q}^m$ and $\mathbb{Q}_0^m$ when there is no ambiguity on m.
We need a last definition to be able to state the limit theorems.

Definition 25. *For any $\kappa \in (0,1)$ let $\mathscr{S}^\kappa$ be the Lévy process where the increments are completely asymmetric κ-stable random variables. The increment have the following characterizations:*

$$\forall \lambda \in \mathbf{R}, \forall s \in \mathbf{R}^+, \mathbb{E}\left(\exp\left(i\lambda \mathscr{S}_s^\kappa\right)\right) = \exp\left(-s|\lambda|^\kappa \left(1 - isgn(\lambda)\tan\left(\frac{\pi\kappa}{2}\right)\right)\right)$$

and for any $s \in \mathbf{R}^+$, $\mathscr{S}_s^\kappa$ and $s^{\frac{1}{\kappa}}\mathscr{S}_1^\kappa$ have the same law.
Since this process is non-decreasing and càdlàg we can define the càdlàg inverse $\tilde{\mathscr{S}^\kappa}$ by:

$$\tilde{\mathscr{S}_t^\kappa} := \inf\{s, \mathscr{S}_s^\kappa \geq t\}.$$

The following two theorems, which are the main results of this paper, give a full annealed limit theorem:

Theorem. *Set $d \geq 3$ and $\alpha \in (0,\infty)^{2d}$. Let $Y^n(t)$ be defined by:*

$$Y^n(t) = n^{-\kappa} Y_{\lfloor nt \rfloor}.$$

If $\kappa < 1$ and $d_\alpha \neq 0$, there exists positive constants c_1, c_2, c_3 such that for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(t \to n^{-\frac{1}{\kappa}} \tau_{\lfloor nt \rfloor}\right) \to c_1 \mathscr{S}^\kappa,$$

for the M_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(t \to n^{-\frac{1}{\kappa}} \inf\{t \geq 0, Y(t).e_1 \geq nx\}\right) \to c_2 \mathscr{S}^\kappa$$

and for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$Y^n \to c_3 \tilde{\mathscr{S}^\kappa} d_\alpha.$$

Remark 7. *We will give a quick explanation on what the M_1 and J_1 topologies are, for a precise definition see [79],[89]. They were both introduced as a generalization of the infinite norm for càdlàg functions. In the J_1 topology, a sequence of càdlàg functions f_n converges to f if there exists a sequence of increasing homomorphisms $\lambda_n : [0,1] \mapsto [0,1]$ such that*

$$\sup_{t\in[0,1]} |\lambda_n(t) - t| \to 0,$$

and

$$\sup_{t\in[0,1]} |f_n(\lambda_n(t)) - f(t)| \to 0.$$

It is essentially the same as the infinite norm except that you can "wiggle" the function time-wise. The M_1 topology is a topology on the graphs of the functions where we add vertical segments every time there is a jump. The main difference between the M_1 and J_1 topology is that there is almost no difference between one jump and small consecutive jumps in the M_1 topology while the difference is significant in the J_1 topology. The reason why we only have a convergence in M_1 for the hitting times $n^{-\frac{1}{\kappa}}\inf\{t \geq 0, Y(t).e_1 \geq nx\}$ is because there are consecutive jumps. Indeed, if there is a large jump between $\inf\{t \geq 0, Y(t).e_1 \geq n\}$ and $\inf\{t \geq 0, Y(t).e_1 \geq n+1\}$ it is likely that there is a trap with high strength close-by which means that it is likely that there also is a large jump between $\inf\{t \geq 0, Y(t).e_1 \geq n+1\}$ and $\inf\{t \geq 0, Y(t).e_1 \geq n+2\}$.

Theorem. *If $d \geq 3$ and $\kappa = 1$, there exists positive constants c_1, c_2, c_3 such that we have the following convergences in probability (for the annealed law):*

$$\frac{1}{n\log(n)}\tau_n \to c_1,$$

$$\frac{1}{n\log(n)}\inf\{i, Y_i.e_1 \geq n\} \to c_2,$$

$$\frac{\log(n)}{n}(Y_n) \to c_3 d_\alpha.$$

Remark 8. *We cannot replace the convergence in probability by an almost sure convergence. This is because if we look at a sum of iid random variables Z_i with a heavy tail $\mathbb{P}(Z_i \geq t) \sim ct^{-1}$ then we do not have an almost sure convergence. In fact, there are infinitely many i such that:*

$$Z_i \geq i \log(i) \log(\log(i)).$$

A tool that will be central in the proof is the study of traps. We now give a precise definition of traps.

Definition 26. *A trap is any undirected edge $\{x, y\}$ such that $\omega(x, y) + \omega(y, x) > \frac{3}{2}$.*
The strength of a trap is the quantity $\frac{1}{(1-\omega(x,y))+(1-\omega(y,x))}$.

Remark 9. *$\frac{3}{2}$ has been chosen because it ensures that $\omega(x, y), \omega(y, x) > \frac{1}{2}$ which in turn means that for every point x, there is at most one point y such that (x, y) is a trap.*

2.1.3 Sketch of the proof

The proofs for $\kappa < 1$ and $\kappa = 1$ are mostly the same and therefore we will explain both at the same time.

Only the renewal times matter

We first show that the number of points visited between two renewal times has a finite expectation (lemma 2.2.1.2). This means that the walk does not "wander far" between two renewal times. So we only have to know the renewal times and the position of the walk at the renewal times to prove both theorems (lemma 2.2.1.3). By proposition 2.1.2.1, the random variables $(\tau_{i+1} - \tau_i)$ are iid which simplifies the study of the process $i \to \tau_i$

The time between renewal times only depends on the strength of the traps

Then we use the stationary law of the accelerated walk to get two results: firstly, the time spent outside of traps is negligible (lemma 2.2.5.4); secondly, the number of time N the walk enters a trap has a finite moment of order $\kappa + \varepsilon$ for some $\varepsilon > 0$ if $\kappa < 1$. If $\kappa = 1$, then N has a finite expectation (lemma 2.2.3.3). This means the time spent in a trap mostly depends on its strength.
Now we want to show that the number of times the walk enters a trap and the time it stays in the trap each time are approximately independent.
We get two different results in this direction:

The strength of the traps are essentially independent

The first result (lemma 2.2.3.1) is that in a way the time spent in traps are independent random variables. These random variables have a tail in $Ct^{-\kappa}$ where the constant C depends on where the walk enters and exits the trap and how many times it does. More precisely, we first set an environment and a path in this environment. Then we forget all the transition probabilities in the traps, this means that if $\{x, y\}$ is a trap, then we only remember the "renormalized" transition probabilities:

$$\left(\frac{\omega(x,z)}{1-\omega(x,y)}\right)_{z\sim x, z\neq y} \text{ and } \left(\frac{\omega(y,z)}{1-\omega(y,x)}\right)_{z\sim y, z\neq x}.$$

Then every time the path visits a trap we only remember where it enters the trap and where it exits the trap, we forget the number of back and forths inside the trap. Then, only knowing these information, the strength of the traps are independent.

The number of times a trap is visited and its strength are essentially independent

The second result (lemma 2.2.3.4) allows us to bound the probability that both the number of times the walk enters a trap and the strength of the trap are high. We use the fact that for an edge (x, y) if we know all the transition probabilities outside of x, y and we know the $\left(\frac{\omega(x,z)}{1-\omega(x,y)}\right)_{z\sim x}$ and the $\left(\frac{\omega(y,z)}{1-\omega(y,x)}\right)_{z\sim y}$ then the number of times the walk enters the trap is essentially independent of the strength of the trap (it depends mostly on $\frac{1-\omega(x,y)}{1-\omega(y,x)}$ and hardly on the strength of the trap). This means that it is unlikely that the traps with a high strength are visited many times.

Conclusion

Thanks to these results we get that if we set an integer A and we only look at traps that are entered less than A times then we have a good approximation of the total time spent in traps (lemma 2.2.4.2). The higher A is, the better the approximation gets. Now if we only look at the traps the walk enters less than A times, we get a finite sum of sums of iid random variables by lemma 2.2.3.1. This means that, after renormalization, the time spent in traps entered less than A times converges to a stable distribution if $\kappa < 1$. It converges to a constant if $\kappa = 1$ (lemma 2.2.4.3). Then the only thing left is to make A go to infinity and we get the first two results of both theorems.
Finally to prove the last part of both theorems we just use basic inversion arguments.

2.2 The proof

2.2.1 Number of points visited between renewal times

In this section we show that the expectation of the number of point visited between two renewal times is finite. This means that only knowing the values of the renewal times will be enough to prove theorem 1 and 2.

Lemma 2.2.1.1. *For m such that $\mathbb{Q}^m$ exists, let $(T_i^m)_{i\in\mathbf{N}^*}$ be the renewal times for the walk X^m i.e $T_i^m := t_{\tau_i}^m$ or to put it another way $X^m_{T_i^m} = Y_{\tau_i}$. There exists a constant C_m such that for all $i \in \mathbf{N}^*$, $\mathbb{E}_{\mathbb{P}_0^{(\alpha)}}(T_{i+1}^m - T_i^m) = C_m$ and $\mathbb{P}_0^{(\alpha)}$ almost surely:*

$$\frac{1}{n}T_n^m \to C_m.$$

Proof. Let D be the random distance defined by $D = Y_{\tau_2} - Y_{\tau_1}$. First we will show that $\mathbb{E}_{\mathbb{P}_0}(D) < \infty$.
Let $(\tau_i)_{i\in\mathbf{N}^*}$ be the different renewal times along the direction e_1. Now let $(d_i)_{i\in\mathbf{N}^*}$ be the sequence defined by:

$$\forall i \in \mathbf{N}^*, d_i = Y_{\tau_i}.e_1.$$

Let $\tilde{L}^\tau(i)$ be the number of renewal times before the walks travels a distance i in the direction e_1 ie:

$$\forall i \in \mathbf{N}^*, \tilde{L}^\tau(i) = \inf\{n, d_n \geq i\}.$$

The sequence of random variables $(d_{i+1} - d_i)_{i\in\mathbf{N}^*}$ is iid by lemma 2.1.2.1. Therefore, if the expectation of $D = d_2 - d_1$ is infinite then $\frac{d_n}{n} \to \infty$, $\mathbb{P}_0$ almost surely. Now, for every $i \in \mathbf{N}^*$, we have $d_{\tilde{L}^\tau(i)} \geq i$ and therefore $\frac{\tilde{L}^\tau(i)}{i} \leq \frac{\tilde{L}^\tau(i)}{d_{\tilde{L}^\tau(i)}}$. If $\mathbb{P}_0$ almost surely $\frac{n}{d_n} \to 0$ we would have $\frac{\tilde{L}^\tau(i)}{i} \to 0$ $\mathbb{P}_0$ almost surely. Since $\frac{\tilde{L}^\tau(i)}{i+1} \leq 1$ we would get that $\mathbb{E}_{\mathbb{P}_0}\left(\frac{\tilde{L}^\tau(i)}{i}\right) \to 0$. However, there is a constant $C > 0$ such that every time the walk reaches a new height along e_1, it is a renewal time with probability C (independent of the walk up to that time) so $\mathbb{E}_{\mathbb{P}_0}\left(\frac{\tilde{L}^\tau(i)}{i}\right) \geq C$. Therefore we get that the expectation of the distance the walk travels in the direction e_1 between two renewal times is finite.

Now we can look at the accelerated walk X^m. We would like the sequence $(T_{i+1}^m - T_i^m)_{i\in\mathbf{N}^*}$ to be a sequence of iid random variables. Unfortunately, the definition of the accelerated random walk uses vertices in a box of size m around the vertex on which the walk currently is, so we need to wait at least $2m+3$ renewal times to be sure to be at a distance at least $2m+1$ of all the vertices visited before time $T_{i+1}^m - 1$. So we only have that for any $j \in \mathbf{N}$, the sequence $\left(T^m_{(2m+3)i+j+1} - T^m_{(2m+3)i+j}\right)_{i\in\mathbf{N}^*}$ is a sequence of iid random variables. Furthermore the sequence $(T_{i+1}^m - T_i^m)_{i\geq m+2}$ is identically distributed.

We know that there exists a constant $c > 0$ such that $\mathbb{P}_0$ almost surely $\frac{X_t^m.e_1}{t} \to c > 0$. If the expectation of the time the accelerated walk spends between two renewal times is infinite then $\frac{T_i^m}{i} \to \infty$, $\mathbb{P}_0$ almost surely since the random variables $\left(T^m_{(2m+3)i+1} - T^m_{(2m+3)i}\right)_{i\in\mathbf{N}^*}$ are iid. Therefore we would have $\frac{X^m_{T_i^m}.e_1}{T_i^m}\frac{T_i^m}{i} \to \infty$ so $\frac{Y_{\tau_i}.e_1}{i} \to \infty$ which is absurd because: $\frac{Y_{\tau_i}.e_1}{i} = \frac{d_i}{i}$ and $\frac{d_i}{i}$ satisfies a law of large number. Therefore the expectation of time the accelerated walk spends between two renewal times is finite and there exists a constant $C > 0$ such that:

$$\forall i \geq m+2,\ \mathbb{E}_{\mathbb{P}_0}(T_{i+1}^m - T_i^m) = C.$$

And by the law of large number, $\mathbb{P}_0$ almost surely:

$$\frac{1}{i}T_i^m \to C.$$

□

Lemma 2.2.1.2. *The number of different points the walk visits between two renewal times has a finite expectation (Note that the number of different points visited between two renewal times is the same for the walk Y and the accelerated walks X^m).*

Proof. We choose m large enough such that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ for some $\gamma > 1$. In the following we will write T_i instead of T_i^m to simplify the notations. Let β be such that $\frac{1}{\gamma} + \frac{1}{\beta} = 1$. Let c_∞ be the constant such that $\mathbb{P}_0$ almost surely: $\frac{1}{i}T_i \to c_\infty$, it exists by lemma 2.2.1.1. Let $(R_i)_{i\in\mathbf{N}^*}$ be the sequence defined by: $\forall i \in \mathbf{N}^*, R_i = \#\{x, \exists j \leq \tau_i, Y_j = x\}$. The random variables $(R_{i+1} - R_i)_{i\geq 1}$ are iid by proposition 2.1.2.1. Thus if the number of different points the walk visits between two renewal times has an infinite expectation (for $\mathbb{P}_0$) then $\frac{R_i}{i} \to \infty$, $\mathbb{P}_0$ almost surely and therefore $\mathbb{Q}_0^m$ almost surely. However we have for any $C > 0$:

$$\begin{aligned}
\mathbb{Q}_0^m(R_n \geq Cn) &\leq \mathbb{Q}_0^m(T_n \geq 2c_\infty n) + \mathbb{Q}(R_n \geq Cn \text{ and } T_n < 2c_\infty n) \\
&= o(1) + \mathbb{Q}_0^m(R_n \geq Cn \text{ and } T_n < 2c_\infty n) \\
&\leq o(1) + \mathbb{Q}_0^m\left(\sum_{0\leq i\leq 2c_\infty n} \#\{x, \exists t \in [i, i+1), X_t = x\} \geq Cn\right) \\
&\leq o(1) + \frac{1}{Cn}\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{0\leq i\leq 2c_\infty n} \#\{x, \exists t \in [i, i+1), X_t = x\}\right) \\
&\leq o(1) + \frac{4c_\infty}{C}\mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0,1), X_t = x\}\right) \text{ for } n \text{ large enough.}
\end{aligned}$$

Now we just have to prove that $\mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0,1), X_t = x\}\right)$ is finite. We use the fact that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ and therefore $\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}$ is also in L^γ.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_0^m}\left(\#\{x, \exists t \in [0,1), X_t = x\}\right) &= \mathbb{E}_{\mathbb{P}_0}\left(\#\{x, \exists t \in [0,1), X_t = x\}\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right) \\
&\leq \mathbb{E}_{\mathbb{P}_0}\left(\#\{x, \exists t \in [0,1), X_t = x\}^\beta\right)^{\frac{1}{\beta}}\left(\mathbb{E}_{\mathbb{P}_0}\left(\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right)^\gamma\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

So we just need to prove that $\mathbb{E}_{\mathbb{P}}\left(\#\{x, \exists t \in [0,1), X_t = x\}^\beta\right)$ is finite. This is an immediate consequence of lemma 4 of [18]. Therefore, for C large enough, we get:

$$\mathbb{Q}_0^m(R_n \geq Cn) \leq o(1) + \frac{1}{2}.$$

Therefore, the number of different points the walk visits between two renewal times has a finite expectation. □

Now, we show that the trajectory of the walk cannot deviate too much from a straight line.

Lemma 2.2.1.3. *Let $L^\tau(n) = \min\{i, \tau_i \geq n\}$. There exists $D \in \mathbf{R}^d$ such that $\mathbb{P}_0$ almost surely:*

$$\frac{Y_n}{L^\tau(n)} \to D.$$

Proof. By proposition 2.1.2.1, $(Y_{\tau_{i+1}} - Y_{\tau_i})_{i\geq 1}$ is a sequence of iid random variables (for $\mathbb{P}_0$). Let $R_i := \#\left\{x \in \mathbf{Z}^d, \exists j < \tau_i,\ Y_j = x\right\}$ be the number of different points visited before time τ_i. By lemma 2.2.1.2 , $R_i - R_{i-1}$ has a finite expectation and since $|Y_{\tau_{i+1}} - Y_{\tau_i}|_1 \leq R_{i+1} - R_i$, we get that $|Y_{\tau_{i+1}} - Y_{\tau_i}|_1$ also has a finite expectation. So there exists $D \in \mathbf{Z}^d$ such that $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_n}}{n} \to D.$$

Now we want to show that $\frac{\left|Y_n - Y_{\tau(L_n^\tau)}\right|_1}{L^\tau(n)} \to 0$, $\mathbb{P}_0$ almost surely. We clearly have:

$$\frac{\left|Y_n - Y_{\tau(L^\tau(n))}\right|_1}{L^\tau(n)} \leq \frac{R_{L^\tau(n)} - R_{L^\tau(n)-1}}{L^\tau(n)}$$

but since $\mathbb{E}_{\mathbb{P}_0}(R_i - R_{i-1})$ is finite, $\frac{R_i - R_{i-1}}{i} \to 0$, $\mathbb{P}_0$ almost surely, so:

$$\frac{\left|Y_n - Y_{\tau(L^\tau(n))}\right|_1}{L^\tau(n)} \to 0,\ \mathbb{P}_0 \text{ almost surely .}$$

So we get that $\mathbb{P}_0$ almost surely: $\frac{Y_n}{L^\tau(n)} \to D$. □

2.2.2 Number of visits of traps

This section is devoted to refining some results of [18] to get an upper bound on the number of visits of traps. First we must get some results on finite graphs and then we will extend these results on $\mathbf{Z}^d$.

Definition 27. *Let $G = (V, E)$ be a finite, directed graph. A vertex $\delta \in V$ is a cemetery vertex if*

- *no edge exits δ, ie $\forall x \in V, (\delta, x) \not\in E$,*
- *for every vertex $x \in V$ there exists a directed path from x to δ.*

In this section we will only consider graphs with no multiple edges, no elementary loops (one edge starting and ending at the same point), and such that for every $x, y \in V \backslash \{\delta\}$, $(x, y) \in E$ if and only if $(y, x \in E)$.
We will first extend the definition of $\gamma_\omega^m(x)$ for those graphs. Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph, $(\alpha(e))_{e \in E}$ be a family of real numbers, and $\mathbb{P}^\alpha$ be the corresponding Dirichlet distribution (independent at each site).

Definition 28. *For $x \in G$ and $\Lambda \subset V \cup \{\delta\}$, we define the following generalization of γ_ω^m:*

$$\gamma_{G,\omega}^\Lambda(x) := \frac{1}{\sum\limits_\sigma \omega_\sigma},$$

where we sum on simple paths from x to the border of Λ (i.e $\{y \in \Lambda, \exists z \notin \Lambda, \{x, y\} \in V\}$) that stay in Λ.

Remark 10. *We notice that, in $\mathbf{Z}^d$, for any $m \in \mathbf{N}^*$:*

$$\forall x \in \mathbf{Z}^d,\ \gamma_\omega^m(x) = \gamma_{\mathbf{Z}^d,\omega}^{x+[\![-m,m]\!]^d}(x).$$

We will also use the following acceleration function.

Definition 29. *For any graph G and any environment ω on G we define the partial acceleration function γ_G^ω by:*

$$\gamma_G^\omega(x) = \max_{y \sim x} \left(\frac{1}{1 - \omega(x, y) + 1 - \omega(y, x)} \right).$$

When there is no ambiguity we will write $\gamma^\omega(x)$ instead of $\gamma_G^\omega(x)$

Remark 11. *Let x be a vertex in $\mathbf{Z}^d$. If it is in a trap then $\gamma^\omega(x)$ is equal to the strength of the trap. Otherwise $\gamma^\omega(x) \leq 2$.*

We have the following result, in the case of finite graphs:

Lemma 2.2.2.1. *(Proposition A.2 of [18])*
Let $n \in \mathbf{N}^*$. Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph possessing at most n edges and such that every vertex is connected to δ by a directed path. We furthermore suppose that G has no multiple edges, no elementary loop, and that if $(x, y) \in E$ and $y \neq \delta$, then $(y, x) \in E$. Let $(a(e))_{e \in E}$ be positive real numbers. Then, for every vertex $x \in V$, there exist real numbers $C, r > 0$ such that, for small $\varepsilon > 0$,

$$\mathbb{P}^{(a)} \left(\gamma_{G,\omega}^{\{\delta\}}(x) \geq \frac{1}{\varepsilon} \right) \leq C \varepsilon^\beta (-\ln \varepsilon)^r$$

where the value of β is explicit and given in [18] but to simplify the notations we will only use the fact that it is bigger than or equal to κ' in the case we will look at.

Lemma 2.2.2.2. *(Lemma 8 of [87])*
Let $(p_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (p_i^{(r)})_{1 \leq i \leq n_r}$ be independent Dirichlet random variables with respective parameters $(\alpha_i^{(1)})_{1 \leq i \leq n_1}, \ldots, (\alpha_i^{(r)}$
Let $m_1, \ldots, m_r$ be integers such that $\forall i \leq r, 1 \leq m_i < n_i$, and let $\Sigma = \sum\limits_{j=1}^r \sum\limits_{i=1}^{m_j} p_i^{(j)}$ and $\beta = \sum\limits_{j=1}^r \sum\limits_{i=1}^{m_j} \alpha_i^{(j)}$. There exists positive constants C, C' such that, for any positive measurable function $f : \mathbf{R} \times \mathbf{R}^{\sum_j m_j} \mapsto \mathbf{R}$,

$$\mathbb{E}\left[f\left(\frac{p_1^{(1)}}{\Sigma}, \ldots, \frac{p_{m_1}^{(1)}}{\Sigma}, \ldots, \frac{p_1^{(r)}}{\Sigma}, \ldots, \frac{p_{m_r}^{(r)}}{\Sigma} \right) \right] \leq C \tilde{\mathbb{E}}\left[f\left(\tilde{p}_1^{(1)}, \ldots, \tilde{p}_{m_1}^{(1)}, \ldots, \tilde{p}_1^{(r)}, \ldots, \tilde{p}_{m_r}^{(r)} \right) \right],$$

where, under the probability $\tilde{\mathbb{P}}$, $(\tilde{p}_1^{(1)}, \ldots, \tilde{p}_{m_1}^{(1)}, \ldots, \tilde{p}_1^{(r)}, \ldots, \tilde{p}_{m_r}^{(r)})$ is sampled from a Dirichlet distribution of parameter $(\tilde{\alpha}_1^{(1)}, \ldots, \tilde{\alpha}_{m_1}^{(1)}, \ldots, \tilde{\alpha}_1^{(r)}, \ldots, \tilde{\alpha}_{m_r}^{(r)})$.

The following lemma shows that the value of the acceleration function $\gamma_\omega^m(x)$ depends mostly on the strength of the trap that contains x (if there is one). This means that the number of visits to a vertex depends mostly on the strength of the trap containing this vertex.

Lemma 2.2.2.3. *Set $\alpha \in (0,\infty)^{2d}$. In $\mathbf{Z}^d$, for any $\beta \in \left[\kappa, \frac{\kappa+\kappa'}{2}\right)$, for any $m \geq 2$:*

$$\mathbb{E}_{\mathbb{P}_0^{(\alpha)}}\left(\left(\frac{\gamma_\omega^m(0)}{\gamma_{\mathbf{Z}^d}^\omega(0)}\right)^\beta\right) < \infty.$$

Proof. Let $m \geq 2$ be an integer. We will use the results we have on finite graphs for this lemma. First we notice that the value of $\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)}\right)^\beta$ only depends on a finite amount of edges and vertices around 0. This means that we can look at this quantity on a finite graph and have the same law. The finite graph $G^m = (V^m, E^m)$ we want is obtained by contracting all the points $x \in \mathbf{Z}^d$ such that $||x||_1 \geq m$ in a single point δ (the cemetery vertex) and deleting all the edges going from this vertex to the rest of the environment. For any environment ω on $\mathbf{Z}^d$ we have an equivalent environment ω^m on G^m: if $(x,y) \in E$ and $(x,y) \in E^m$ then $\omega(x,y) = \tilde{\omega}(x,y)$ and for any $x \in V^m \backslash \{\delta\}$, $\tilde{\omega}(x,\delta) = \sum_{y \in \mathbf{Z}^d, ||y||_1 = m} \omega(x,y)$. Now we have:

$$\gamma_\omega^m(0) = \gamma_{G^m,\omega^m}^{\{\delta\}}(0)$$

and

$$\gamma_{\mathbf{Z}^d}^\omega(0) = \gamma_{G^m}^{\omega^m}(0).$$

So we just have to show that

$$\mathbb{E}_{\mathbb{P}^{(\alpha)}}\left(\left(\frac{\gamma_{G^m,\omega^m}^{\{\delta\}}(0)}{\gamma_{G^m}^{\omega^m}(0)}\right)^\beta\right) < \infty.$$

For any point $y \sim 0$ and any environment ω we define Σ_y^ω by:

$$\Sigma_y^\omega = 2 - \omega(0,y) - \omega(y,0).$$

For any point $x \in G^m$ such that $x \sim 0$, we define $G_x^m = (V_x^m, E_x^m)$ by contracting the vertices 0 and x into a single vertex 0 and deleting the edges $(0,x)$ and $(x,0)$. The edges $(0,y)$ and $(y,0)$ stay the same for any $y \sim 0$ such that $x \neq y$. However, the edges (x,y) and (y,x) become $(0,y)$ and $(y,0)$ respectively, for any $y \sim x$ such that $0 \neq y$. We can also define ω_x^m by:

$$\forall (y,z) \in E^m, y \notin \{0,x\},\ \omega_x^m(y,z) := \omega^m(y,z)$$
$$\forall (y,z) \in E^m, y \in \{0,x\}, (y,z) \in E_x^m,\ \omega_x^m(y,z) := \frac{\omega^m(y,z)}{\Sigma_y^\omega}$$

Let $x \sim 0$ be a vertex of G^m. If we think of $\frac{1}{\gamma_{G^m,\omega^m}^{\{\delta\}}}$ as a sum on simple paths, we have:

$$\frac{1}{\gamma_{G^m,\omega^m}^{\{\delta\}}} \geq \Sigma_x^{\omega^m} \omega^m(0,x) \frac{1}{\gamma_{G^m,\omega^m}^{\{\delta\}}}$$

Indeed, if we look at $\frac{1}{\gamma_{G_x^m,\omega_x^m}^{\{\delta\}}}$ as a sum on simple paths σ from 0 to δ ($\sigma_0 = 0$), either the first vertex σ_1 visited by the path is such that $(0,\sigma_1) \in E^m$ or $(x,\sigma_1) \in E^m$. We define $\tilde{\sigma}$ by: if $(0,\sigma_1) \in E^m$ then $\tilde{\sigma} := \sigma$ and we have:

$$\omega^m(\tilde{\sigma}) = \Sigma_x^{\omega^m} \omega_x^m(\sigma) \geq \Sigma_x^{\omega^m} \omega^m(0,x)\omega_x^m(\sigma),$$

and if $(x,\sigma_1) \in E^m$ then $\tilde{\sigma}_i := \sigma_{i-1}$ for $i \geq 2$ and $\tilde{\sigma}_0 := 0$ and $\tilde{\sigma}_1 := x$ and we get:

$$\omega^m(\tilde{\sigma}) = \Sigma_x^{\omega^m} \omega^m(0,x)\omega_x^m(\sigma).$$

For any environment ω, let $x(\omega^m)$ be the point that maximises $(y \to \omega^m(0,y))$. We have $\tilde{\omega}(0,y) \geq \frac{1}{2d}$ and therefore:

$$\frac{1}{\gamma_{G^m,\omega^m}^{\{\delta\}}} \geq \frac{1}{2d} \Sigma_{x(\omega^m)}^{\omega^m} \frac{1}{\gamma_{G_{x(\omega^m)}^m,\omega_{x(\omega^m)}^m}^{\{\delta\}}}.$$

So we get, for any $\varepsilon > 0$:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma^{\{\delta\}}_{G^m,\omega^m}(0)}{\gamma^{\omega^m}_{G^m}(0)}\right) \geq \frac{1}{\varepsilon}\right) \leq \mathbb{P}^{(\alpha)}\left(\frac{2d\varepsilon}{\gamma^{\omega^m}_{G^m}(0)} \geq \Sigma^{\omega^m}_{x(\omega^m)}\frac{1}{\gamma^{\{\delta\}}_{G^m_{x(\omega^m)},\omega^m_{x(\omega^m)}}}\right)$$

$$= \sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(y = x(\omega^m) \text{ and } \frac{2d\varepsilon}{\gamma^{\omega^m}_{G^m}(0)} \geq \Sigma^{\omega^m}_{y}\frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right)$$

$$\leq \sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(\frac{2d\varepsilon}{\gamma^{\omega^m}_{G^m}(0)} \geq \Sigma^{\omega^m}_{y}\frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right).$$

by definition of $\gamma^{\omega^m}_{G^m}(0)$:

$$\forall y \sim 0,\ \gamma^{\omega^m}(0)_{G^m}\Sigma^{\omega^m}_y \geq 1.$$

Therefore:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma^{\{\delta\}}_{G^m,\omega^m}(0)}{\gamma^{\omega^m}_{G^m}(0)}\right) \geq \frac{1}{\varepsilon}\right) \leq \sum_{y\sim 0}\mathbb{P}^{(\alpha)}\left(2d\varepsilon \geq \frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right).$$

Now we can apply lemma 2.2.2.2 which gives, for any $y \sim 0$:

$$\mathbb{P}^{(\alpha)}\left(2d\varepsilon \geq \frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right) \leq C\tilde{\mathbb{P}}\left(2d\varepsilon \geq \frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right),$$

where under $\tilde{\mathbb{P}}$, ω^m_y are independent Dirichlet random variables (on the graph G^m_y and the parameters of the Dirichlet are the same as in $\mathbf{Z}^d$). Now, according to lemma 2.2.2.1 there exists two constants C', r such that:

$$\forall\varepsilon \quad \text{small enough} \quad, \tilde{\mathbb{P}}\left(2d\varepsilon \geq \frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right) \leq C'\varepsilon^{\kappa'}\left(-\log(\varepsilon)\right)^r.$$

This means that by changing the constant C', we get:

$$\forall\varepsilon \geq 0, \tilde{\mathbb{P}}\left(2d\varepsilon \geq \frac{1}{\gamma^{\{\delta\}}_{G^m_y,\omega^m_y}}\right) \leq C'\varepsilon^{\frac{\kappa+\kappa'}{2}}.$$

So there exists a constant D that does not depend on ε such that:

$$\mathbb{P}^{(\alpha)}\left(\left(\frac{\gamma^{\{\delta\}}_{G^m,\omega^m}(0)}{\gamma^{\omega^m}_{G^m}(0)}\right) \geq \frac{1}{\varepsilon}\right) \leq D\varepsilon^{\frac{\kappa+\kappa'}{2}}.$$

We have the result we want. □

Unfortunately this statement cannot be efficiently used with the invariant distribution $\mathbb{Q}^m$ because we can visit multiple points between times 0 and 1 since the time is continuous. So we need a version of the previous lemma that takes this continuity into account.

Lemma 2.2.2.4. *Set $\alpha \in (0,\infty)^{2d}$. For every $\beta < \frac{\kappa+\kappa'}{2}$, there exists an integer m such that:*

$$\mathbb{E}_{\mathbb{Q}^m_0}\left(\sum_{x\in\mathbf{Z}^d}\left(\int_{t=0}^{1}\frac{\gamma^m_\omega(x)}{\gamma^\omega_{\mathbf{Z}^d}(x)}1_{X^m_t=x}\mathrm{d}t\right)^\beta\right) < \infty.$$

Proof. Let $p \in (1,\infty)$ be a constant such that $\beta p^2 < \frac{\kappa+\kappa'}{2}$ and let γ be such that $\frac{1}{p}+\frac{1}{\gamma} = 1$. Now let m be an integer such that $\frac{\mathrm{d}\mathbb{Q}^m}{\mathrm{d}\mathbb{P}}$ is in L^γ. This means that $\frac{\mathrm{d}\mathbb{Q}^m_0}{\mathrm{d}\mathbb{P}_0}$ is also in L^γ. We will only work in $\mathbf{Z}^d$ so we will write

γ^ω instead of $\gamma^\omega_{\mathbf{Z}^d}$.

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbf{Z}^d}\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\right)\\
&=\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right)\\
&\le\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^{p\beta}\right)^{\frac{1}{p}}\mathbb{E}_{\mathbb{P}_0}\left(\left(\frac{\mathrm{d}\mathbb{Q}_0^m}{\mathrm{d}\mathbb{P}_0}\right)^\gamma\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

This means we just need to show that $\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^{p\beta}\right)^{\frac{1}{p}}$ is finite. Let D_1^m be the random variable defined by:

$$D_1^m:=\sum_{i=1}^{d}\max_{t\in[0,1]}|X_t^m.e_i|.$$

We have:

$$\begin{aligned}
\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^{p\beta}\right)^{\frac{1}{p}}&\le\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}\right)^{p\beta}1_{\exists t\in[0,1],X_t^m=x}\right)^{\frac{1}{p}}\\
&\le\sum_{x\in\mathbf{Z}^d}\left(\mathbb{E}_{\mathbb{P}_0}\left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}\right)^{p\beta}1_{D_1^m\ge||x||_\infty}\right)\right)^{\frac{1}{p}}\\
&\le\sum_{x\in\mathbf{Z}^d}\left(\mathbb{E}_{\mathbb{P}_0}\left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}\right)^{p^2\beta}\right)\right)^{\frac{1}{p^2}}\left(\mathbb{E}_{\mathbb{P}_0}\left(1_{D_1^m\ge||x||_\infty}\right)\right)^{\frac{1}{\alpha}}\\
&=\sum_{x\in\mathbf{Z}^d}\left(\mathbb{E}_{\mathbb{P}}\left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}\right)^{p^2\beta}\right)\right)^{\frac{1}{p^2}}\left(\mathbb{E}_{\mathbb{P}_0}\left(1_{D_1^m\ge||x||_\infty}\right)\right)^{\frac{1}{\alpha}}
\end{aligned}$$

Now since the environment for $\mathbb{P}$ is iid, $\mathbb{E}_{\mathbb{P}}\left(\left(\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}\right)^{p^2\beta}\right)$ does not depend on x and we get:

$$\sum_{x\in\mathbf{Z}^d}\mathbb{E}_{\mathbb{P}_0}\left(\left(\int_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^{p\beta}\right)^{\frac{1}{p}}\le\left(\mathbb{E}_{\mathbb{P}}\left(\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)}\right)^{p^2\beta}\right)\right)^{\frac{1}{p^2}}\sum_{x\in\mathbf{Z}^d}\left(\mathbb{E}_{\mathbb{P}_0}\left(1_{D_1^m\ge||x||_\infty}\right)\right)^{\frac{1}{\gamma}}.$$

And since there exists a constant C such that for every $i\ge1$ there are at most Ci^{d-1} points x such that $||x||_\infty=i$, we get:

$$\sum_{x\in\mathbf{Z}^d}\left(\mathbb{E}_{\mathbb{P}_0}\left(1_{D_1\ge||x||_\infty}\right)\right)^{\frac{1}{\alpha}}\le1+C\sum_{i\ge1}i^{d-1}\left(\mathbb{E}_{\mathbb{P}_0}\left(1_{D_1\ge i}\right)\right)^{\frac{1}{\alpha}}$$

which is finite by lemma 4 of [18]. And by lemma 2.2.2.3 we get:

$$\mathbb{E}_{\mathbb{P}}\left(\left(\frac{\gamma_\omega^m(0)}{\gamma^\omega(0)}\right)^{p^2\beta}\right)<\infty.$$

So we get the result we want. □

2.2.3 Independence of the traps

This section will be devoted to the precise study of traps. The notion of trap was defined in the introduction in definition26. In the previous section we have essentially shown that the total amount of time spent on a trap mostly depends on its strength. Now, we need a way to create independence between the times spent in the different traps. We will do it in two steps. First we will show that the strength of the traps are essentially independent and then we will show that the strength of a trap and the number of times it is visited are essentially independent. However, we first need to introduce a few objects to characterize this independence precisely.

Definition 30. *Let* $\mathscr{T}^\omega$ *be the set of traps* $\{x,y\} \in \tilde{E}$ *for the environment* ω.
$\tilde{\mathscr{T}}^\omega$ *is the set of vertices* $x \in \mathbf{Z}^d$ *such that there exist* y *such that* $\{x,y\} \in \mathscr{T}^\omega$.
For any subset J *of* $[\![1,d]\!]$ *we define* $\mathscr{T}^\omega_J$, *the traps with direction in* J *by:*

$$\mathscr{T}^\omega_J = \{\{x,y\} \in \mathscr{T},\ \exists j \in J, y = x + e_j \text{ or } y = x - e_j\}.$$

For any subset J *of* $[\![1,d]\!]$, $\tilde{\mathscr{T}}^\omega_J$ *is the set of vertices* $x \in \mathbf{Z}^d$ *such that there exist* y *such that* $\{x,y\} \in \mathscr{T}^\omega_J$.
In the following we will omit the ω *when there is no ambiguity.*

Definition 31. *We say that two environments* ω_1 *and* ω_2 *are trap-equivalent if:*
- they have the same traps:

$$\mathscr{T}^{\omega_1} = \mathscr{T}^{\omega_2},$$

- at each vertex not in a trap, the transition probabilities are the same for both environment:

$$\forall x \notin \tilde{\mathscr{T}}^{\omega_1},\ \forall y \sim x,\ \omega_1(x,y) = \omega_2(x,y),$$

- at each vertex x *in a trap* $\{x,y\}$, *the transition probabilities conditioned on not crossing the trap are the same:*

$$\forall (x,y) \in E,\ \{x,y\} \in \mathscr{T}^{\omega_1},\ \forall z \sim x, z \neq y, \frac{\omega_1(x,z)}{1-\omega_1(x,y)} = \frac{\omega_2(x,y)}{1-\omega_2(x,y)}.$$

We will denote by $\tilde{\Omega}$ *the set of all equivalence classes for the trap-equivalence relation.*

Definition 32. *Set* $\tilde{\omega} \in \tilde{\Omega}$. *Let* $\mathscr{T}$ *be its set of trap and* σ *a path starting at* 0 *that only stays a finite amount of time every time it enters a trap. We want to define a path, with the same trajectory as* σ *outside the traps, which does not keep information regarding the time spent in the traps. We essentially want to erase all the back and forths inside traps. To that extent we define the sequences of integer times* $(t_i), (s_i)$ *by:*

$$\begin{aligned}
&t_0 = 0,\\
&s_i = \inf\{n \geq t_i, (\sigma_n = \sigma_{t_i} \text{ or } \{\sigma_n, \sigma_{t_i}\} \in \mathscr{T}) \text{ and } (\sigma_{n+1} \neq \sigma_{t_i} \text{ and } \{\sigma_{n+1}, \sigma_{t_i}\} \notin \mathscr{T})\},\\
&t_{i+1} = \begin{cases} s_i + 1 & \text{if } \sigma_{s_i} = \sigma_{t_i} \\ s_i & \text{otherwise.} \end{cases}
\end{aligned}$$

If σ_{t_i} *is in a trap then* $[t_i, s_i]$ *is the interval of time spent in this trap before leaving it.*
The partially forgotten path $\tilde{\sigma}$ *associated with* σ *in the environment* $\tilde{\omega}$ *is defined by:*

$$\tilde{\sigma}_i := \sigma_{t_i}.$$

Similarly we can define the partially-forgotten walk $(\tilde{Y}_n)_{n\in\mathbf{N}}$ *associated with* $(Y_n)_{n\in\mathbf{N}}$

Definition 33. *For all* $i \in \mathbf{N}^*$, *let* I_i *be the set defined by:*

$$I_i = [\![1,d]\!] \times \{a,b,c,d \in \mathbf{N}, a \geq 1, a+b+c+d = i\}.$$

And I^n *be defined by:*

$$I^n = \bigcup_{1 \leq i \leq n} I_i.$$

Let σ *be a path starting at* 0 *and* $\tilde{e} \in \tilde{E}$ *be an undirected edge. We define the sequences* (t_i^{in}) *(the times when the path enters* $\tilde{e}$*) and* (t_i^{out}) *(the times when the path exits* $\tilde{e}$*) by:*

$$\begin{aligned}
t_1^{in} &= \inf\{n, \tilde{Y}_n \in \tilde{e}\},\\
t_{i+1}^{in} &= \inf\{n > t_i^{in}, \tilde{Y}_n \in \tilde{e} \text{ and } \tilde{Y}_{n-1} \notin \tilde{e}\},\\
t_i^{out} &= \inf\{n \geq t_i^{in}, \tilde{Y}_n \in \tilde{e} \text{ and } \tilde{Y}_{n+1} \notin \tilde{e}\}\}.
\end{aligned}$$

Since the walk is almost surely transient by theorem 1, we have that for i *large enough* $t_i^{in} = t_i^{out} = \infty$ *almost surely.*
Now let $x := \sigma_{t_1^{in}}$ *and* y *be such that* $\{x,y\} = \tilde{e}$. *Let* $j \in [\![1,d]\!]$ *be such that either* $x = y + e_j$ *or* $x = y - e_j$ *(*j *is the direction of the edge) and* n *be such that* $t_n^{in} < \infty$ *and* $t_{n+1}^{in} = \infty$. *Now we can define* $N_{x\to x}, N_{x\to y}, N_{y\to x}, N_{y\to y}$ *by:*

$$\begin{aligned}
N_{x\to x} &= \#\{i \leq n, t_i^{in} = x \text{ and } t_i^{out} = x\},\\
N_{x\to y} &= \#\{i \leq n, t_i^{in} = x \text{ and } t_i^{out} = y\},\\
N_{y\to x} &= \#\{i \leq n, t_i^{in} = y \text{ and } t_i^{out} = x\},\\
N_{y\to y} &= \#\{i \leq n, t_i^{in} = y \text{ and } t_i^{out} = y\}.
\end{aligned}$$

The configuration p *of the edge* $\tilde{e}$, *for the path* σ, *is the element of* I_n *defined by:*

$$p^\sigma_{\{x,y\}} := (j, N_{x\to x}, N_{x\to y}, N_{y\to x}, N_{y\to y}).$$

Remark 12. *Set $\tilde{\omega} \in \tilde{\Omega}$. Let σ_1, σ_2 be two paths starting at 0 with the same partially forgotten path in $\tilde{\omega}$. For any undirected edge $\tilde{e}$, the configuration of $\tilde{e}$ is the same for σ_1 and σ_2. Therefore we only need to know the partially forgotten path to know the configuration of an edge.*

Now we can say in what way the strength of the traps are independent.

Lemma 2.2.3.1. *For any environment $\omega \in \Omega$, let $\tilde{\omega} \in \tilde{\Omega}$ be its equivalence class for the trap-equivalent relation. Now let $(\tilde{Y}_i)$ be the partially forgotten walk. We will write $\overline{\alpha} := \sum\limits_{1\le i\le 2d} \alpha_i$ and for any vertex z and integer i we will use the notation $\alpha(z, z+e_i) := \alpha_i$. Knowing $\tilde{\omega}$ and $(\tilde{Y}_i)$, the strength of the various traps are independent. Furthermore, let $\{x, y\}$ be a trap and $p = (j, N_{x\to x}, N_{x\to y}, N_{y\to x}, N_{y\to y})$ its configuration. To simplify notations we will write $N_x := N_{x\to x} + N_{y\to x}$, $N_y := N_{x\to y} + N_{y\to y}$ and $N := N_x + N_y$. Let (r, k) be defined by $(1-\omega(x,y), 1-\omega(y,x)) = ((1+k)r, (1-k)r)$. The density of law of (r,k) (with respect to the Lebesgue measure) knowing $\tilde{\omega}$ and $\tilde{Y}$ is:*

$$C_p r^{\kappa_j - 1}(1+k)^{N_x + \overline{\alpha} - \alpha(x,y) - 1}(1-k)^{N_y + \overline{\alpha} - \alpha(y,x) - 1} h_p(r(1+k), r(1-k)) 1_{0 \le r \le \frac{1}{4}} 1_{-1\le k \le 1},$$

where C_p is a constant that only depends on p and α, and h_p is a function that only depends on p and α and that satisfies the following bound:

$$\forall r \le \frac{1}{4}, \quad |\log(h_p(r(1+k), r(1-k)))| \le 5(N + 2\overline{\alpha})r.$$

And for the law of the strength s of the trap, there exists a constant D that only depends on the configuration of the trap such that for any $A \ge 2$:

$$DA^{-\kappa_j} \exp\left(-\frac{5(N+2\overline{\alpha})}{A}\right) \le \mathbb{P}_0\left(s \ge A | \tilde{\omega}, \tilde{Y}\right) \le DA^{-\kappa_j} \exp\left(\frac{5(N+2\overline{\alpha})}{A}\right).$$

Proof. In the following, we will write $\overline{\alpha} := \sum\limits_{i=1}^{2d} \alpha_i$ and if $y = x + e_i$ we will write $\alpha(x,y) := \alpha_i$. First we need to show that the strength of the traps is approximately independent of the trajectory of the walk. We will take an environment ω and let $\tilde{\omega}$ be the set of all environments that are trap-equivalent to ω. Now for any path σ starting at 0, let $\tilde{\sigma}^{\tilde{\omega}}$ be the set of all path that starts at 0 and that have the same partially-forgotten path as σ. We want to see how the law of the environment is changed knowing the partially-forgotten path and the equivalence class of the environment. We get that the density of the environment (we look at an environment of finite size, large enough to contain the path we look at) (for $\mathbb{P}^{(\alpha)}$) knowing the equivalence class of the environment is equal to:

$$C \prod_{\{x,y\}\in\mathscr{T}} (\varepsilon_x)^{\overline{\alpha} - \alpha(x,y) - 1}(1-\varepsilon_x)^{\alpha(x,y)-1}(\varepsilon_y)^{\overline{\alpha} - \alpha(y,x) - 1}(1-\varepsilon_y)^{\alpha(y,x)-1} 1_{\varepsilon_x + \varepsilon_y < \frac{1}{2}} \mathrm{d}\varepsilon_x \mathrm{d}\varepsilon_y, \tag{2.1}$$

where $\varepsilon_x = 1 - \omega(x,y)$ and $\varepsilon_y = 1 - \omega(y,x)$. Now, knowing the environment, the probability of having the given partially-forgotten walk is the same in parts of the environment where there is no trap. The only thing that depends on the specific environment is the times when the walk crosses the traps. Let $\{x, y\}$ be a trap, and for any $z_1, z_2 \in \{x, y\}$ let $\tilde{p}(z_1, z_2)$ be the probability to exit the path by z_2, starting at z_1, we get:

$$\tilde{p}(x,x) = \frac{\varepsilon_x}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}, \qquad \tilde{p}(y,y) = \frac{\varepsilon_y}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y},$$
$$\tilde{p}(x,y) = \frac{\varepsilon_y(1-\varepsilon_x)}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}, \qquad \tilde{p}(y,x) = \frac{\varepsilon_x(1-\varepsilon_y)}{\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y}.$$

So for any environment ω, we get that the probability of a partially-forgotten path (for $\mathrm{P}_0^{(\alpha)}$), is equal to:

$$\begin{aligned}
&C \prod_{\{x,y\}\in\mathscr{T}} \tilde{p}(x,x)^{N_{x\to x}} \tilde{p}(x,y)^{N_{x\to y}} \tilde{p}(y,x)^{N_{y\to x}} \tilde{p}(y,y)^{N_{y\to y}} \\
=&C \prod_{\{x,y\}\in\mathscr{T}} \frac{\varepsilon_x^{N_{x\to x}}(\varepsilon_y(1-\varepsilon_x))^{N_{x\to y}}(\varepsilon_x(1-\varepsilon_y))^{N_{y\to x}}\varepsilon_y^{N_{y\to y}}}{(\varepsilon_x + \varepsilon_y - \varepsilon_x\varepsilon_y)^{N_{x\to x}+N_{x\to y}+N_{y\to x}+N_{y\to y}}} \\
=&C \prod_{\{x,y\}\in\mathscr{T}} \frac{\varepsilon_x^{N_{x\to x}+N_{y\to x}}\varepsilon_y^{N_{x\to y}+N_{y\to y}}}{(\varepsilon_x+\varepsilon_y)^{N_{x\to x}+N_{x\to y}+N_{y\to x}+N_{y\to y}}} \frac{(1-\varepsilon_x)^{N_{x\to y}}(1-\varepsilon_y)^{N_{y\to x}}}{\left(1 - \frac{\varepsilon_x\varepsilon_y}{\varepsilon_x+\varepsilon_y}\right)^{N_{x\to x}+N_{x\to y}+N_{y\to x}+N_{y\to y}}}.
\end{aligned} \tag{2.2}$$

We define $h_{\{x,y\}}$ by:

$$h_{\{x,y\}}(\varepsilon_x,\varepsilon_y)=\frac{(1-\varepsilon_x)^{N_{x\to y}}(1-\varepsilon_y)^{N_{y\to x}}}{\left(1-\frac{\varepsilon_x\varepsilon_y}{\varepsilon_x+\varepsilon_y}\right)^{N_{x\to x}+N_{x\to y}+N_{y\to x}+N_{y\to y}}}(1-\varepsilon_x)^{\alpha(x,y)-1}(1-\varepsilon_y)^{\alpha(y,x)-1}.$$

Now we get that the density probability of having a given environment knowing the equivalence class of the environment and the partially forgotten path is equal to the product of 2.1 and 2.2 up to a multiplicative constant C that depends on the partially-forgotten path:

$$C\prod_{\{x,y\}\in\mathscr{T}}\frac{\varepsilon_x^{N_{x\to x}+N_{y\to x}+\overline{\alpha}-\alpha(x,y)-1}\varepsilon_y^{N_{x\to y}+N_{y\to y}+\overline{\alpha}-\alpha(y,x)-1}}{(\varepsilon_x+\varepsilon_y)^{N_{x\to x}+N_{x\to y}+N_{y\to x}+N_{y\to y}}}h_{\{x,y\}}(\varepsilon_x,\varepsilon_y)1_{\varepsilon_x+\varepsilon_y<\frac{1}{2}}\mathrm{d}\varepsilon_x\mathrm{d}\varepsilon_y.$$

This means that for $\mathbb{P}_0^{(\alpha)}$, knowing the equivalence class of the environment and the partially forgotten path, the transition probabilities for each trap are independent, so we will look at each trap independently. Let's fix a trap $\{x,y\}$ and to simplify notations, we will write $N_x=N_{x\to x}+N_{y\to x}$, $N_y=N_{x\to y}+N_{y\to y}$ and $N=N_x+N_y$. We define r and k by $r=\frac{\varepsilon_x+\varepsilon_y}{2}$ and $k=\frac{\varepsilon_x-\varepsilon_y}{\varepsilon_x+\varepsilon_y}$ which gives $\varepsilon_x=r(1+k)$ and $\varepsilon_y=r(1-k)$ the law of the transition probabilities becomes:

$$\begin{aligned}&C\frac{r}{2}\frac{(r(1+k))^{N_x+\overline{\alpha}-\alpha(x,y)-1}(r(1-k))^{N_y+\overline{\alpha}-\alpha(y,x)-1}}{(2r)^{N_x+N_y+2}}h_{\{x,y\}}(r(1+k),r(1-k))1_{r<\frac{1}{2}}\mathrm{d}r\mathrm{d}k\\ =&C'r^{\kappa_j-1}(1+k)^{N_x+\overline{\alpha}-\alpha(x,y)-1}(1-k)^{N_y+\overline{\alpha}-\alpha(y,x)-1}h_{\{x,y\}}(r(1+k),r(1-k))1_{r<\frac{1}{2}}\mathrm{d}r\mathrm{d}k.\end{aligned}$$

Now we want to give bounds on $h_{\{x,y\}}$. Since for all $r\le\frac{1}{2}$, $|\log(1-r)|\le 2r$, we get:

$$\begin{aligned}&|\log(h_{\{x,y\}}(r(1+k),r(1-k)))|\\ \le&|(N(x,y)+\alpha(x,y)-1)\log(1-r(1+k))|+|(N(y,x)+\alpha(y,x)-1)\log(1-r(1-k))|\\ &+|N\log(1-\frac{r(1-k^2)}{2})|\\ \le&(N(x,y)+\alpha(x,y))4r+(N(y,x)+\alpha(y,x))4r+Nr\\ \le&5(N_x+N_y+\alpha_x+\alpha_y)r.\end{aligned}$$

Let $D=\int\limits_{k=-1}^{1}C'(1+k)^{N_x+\overline{\alpha}-\alpha(x,y)-1}(1-k)^{N_y+\overline{\alpha}-\alpha(y,x)-1}$, for any $A\ge 2$, we have:

$$DA^{-\kappa_j}\exp\left(-\frac{5(N+2\alpha)}{A}\right)\le\mathbb{P}_0\left(s\ge A|\tilde{\omega},\tilde{Y}\right)\le DA^{-\kappa_j}\exp\left(\frac{5(N+2\alpha)}{A}\right).$$

□

Now we want to show that there cannot be too many traps that are visited many times.

Lemma 2.2.3.2. *Set $\alpha\in(0,\infty)^{2d}$. For any $\beta\in\left[\kappa,\frac{\kappa+\kappa'}{2}\right)$ with $\beta\le 1$ there exists a finite constant $C>0$ such that for every $i\in\mathbf{N}\setminus\{0,1\}$:*

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}}\#\{j\in[\![\tau_i,\tau_{i+1}-1]\!],Y_j\in\{x,y\}\text{ and }Y_{j+1}\notin\{x,y\}\}^\beta\right)=C.$$

Proof. We want to show that

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}}\#\{j\in[\![\tau_i,\tau_{i+1}-1]\!],Y_j\in\{x,y\}\text{ and }Y_{j+1}\notin\{x,y\}\}^\beta\right)$$

can be bounded away from infinity by using the inequality from lemma 2.2.2.4:

$$\mathbb{E}_{\mathbb{Q}_0^m}\left(\sum_{x\in\mathbf{Z}^d}\left(\int\limits_{t=0}^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t=x}\mathrm{d}t\right)^\beta\right)<\infty,$$

which is true for any $\beta \in \left[\kappa, \frac{\kappa+\kappa'}{2}\right)$, and for any integer m such that $\mathbb{Q}_0^m$ exists.
To that end we need to introduce the intermediate quantity S_n^m:

$$S_n^m := \sum_{i=0}^{n} \sum_{\{x,y\}\in\mathscr{T}} \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)^\beta + \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t \right)^\beta,$$

where (T_i^m) are the renewal times for the walk (X_t^m), with the convention that $T_0^m := 0$. By definition of X^m, the time the walk X^m spends in a vertex x is a sum of ℓ_x iid exponential random variables of expectation $\frac{1}{\gamma_\omega^m(x)}$, where ℓ_x is the number of times the walk Y visits the point x. Therefore the quantity

$$\int_0^\infty \gamma_\omega^m(x) 1_{X_t=x} \mathrm{d}t$$

should be close to ℓ_x. Then, every time the walk Y enters the trap $\{x,y\}$ is stays a time of order $\gamma^\omega(x)$. This means that $\frac{\ell_x}{\gamma^\omega(x)}$ should be almost equal to the number of times the trap is entered. Finally, we get that for every trap the quantities

$$\sum_{\{x,y\}\in\mathscr{T}} \#\{j \in [\![\tau_i, \tau_{i+1}-1]\!], Y_j \in \{x,y\} \text{ and } Y_{j+1} \notin \{x,y\}\}^\beta$$

and

$$\sum_{\{x,y\}\in\mathscr{T}} \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)^\beta + \left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t \right)^\beta$$

should be of the same order. Then we just need to bound the second quantity with lemma 2.2.2.4 and a law of large number.
For any $k \in [\![0, 2m+3]\!]$ the random variables $(S_{(2m+3)i+k+1}^m - S_{(2m+3)i+k}^m)_{i\geq 1}$ are iid (the definition of $\gamma_\omega^m(x)$ depends on a box f size m around x and traps span over 2 vertices that's why we cannot consider the sequence $(S_{i+1}^m - S_i^m)_{i\geq 1}$). This means that there is a positive constant C_0 that can be infinite such that $\mathbb{E}_{\mathbb{P}_0}\left(S_{2m+3}^m - S_{2m+2}^m\right) = C_0$ and

$$\frac{1}{n} S_n^m \to C_0 \ \mathbb{P}_0 \text{ a.s and therefore } \mathbb{Q}_0 \text{ a.s.}$$

For any $x \in \mathbf{Z}^d$ there is at most one integer i such that $\left(\int_{T_i^m}^{T_{i+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)$ is non-zero and therefore:

$$S_n^m = \sum_{\{x,y\}\in\mathscr{T}} \left(\int_0^{T_{n+1}^m} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)^\beta + \left(\int_0^{T_{n+1}^m} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t \right)^\beta.$$

By lemma 2.2.1.1 there is a finite constant D^m such that $\frac{1}{n} T_n^m \to D^m$ $\mathbb{P}_0$ and $\mathbb{Q}_0$ almost surely. We get:

$$\frac{1}{n} \sum_{\{x,y\}\in\mathscr{T}} \left(\int_0^{D^m n} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)^\beta + \left(\int_0^{D^m n} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t \right)^\beta \to C_0 \ \mathbb{Q}_0 \text{ a.s.}$$

Therefore,

$$\liminf \frac{1}{n} \mathbb{E}_{\mathbb{Q}_0} \left(\sum_{\{x,y\}\in\mathscr{T}} \left(\int_0^{D^m n} \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)} 1_{X_t^m=x} \mathrm{d}t \right)^\beta + \left(\int_0^{D^m n} \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)} 1_{X_t^m=y} \mathrm{d}t \right)^\beta \right) \geq C_0.$$

Since $\beta \leq 1$ we have:

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}_{\mathbb{Q}_0}\left(\sum_{\{x,y\}\in\mathscr{T}}\left(\int_0^{D^m n}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_0^{D^m n}\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta\right)\\
\leq&\frac{1}{n}\sum_{i=0}^{\lfloor D^m n\rfloor}\mathbb{E}_{\mathbb{Q}_0}\left(\sum_{\{x,y\}\in\mathscr{T}}\left(\int_i^{i+1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_i^{i+1}\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta\right)\\
=&\frac{\lfloor D^m n\rfloor+1}{n}\mathbb{E}_{\mathbb{Q}_0}\left(\sum_{\{x,y\}\in\mathscr{T}}\left(\int_0^{1}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_0^{1}\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta\right)\\
&<\infty \text{ by lemma 2.2.2.4.}
\end{aligned}$$

So C_0 is finite.
Now we want to get a bound on Y from a bound on X^m. For any trap $\{x,y\}\in\mathscr{T}$ let $N_{\{x,y\}}$ be the number of times the trap $\{x,y\}$ is entered. Let $\mathscr{T}^{\omega,n}$ be the subset of $\mathscr{T}^\omega$ defined by:

$$\mathscr{T}^{\omega,n}:=\left\{\{x,y\}\in\mathscr{T}^\omega,\ Y_{\tau_1}.e_1\leq x.e_1\leq Y_{\tau_1}.e_1+n \text{ and } Y_{\tau_1}.e_1\leq y.e_1\leq Y_{\tau_1}.e_1+n\right\}.$$

We chose a partially-forgotten path σ and we look at the law of the total time the walk X spends in a trap $\{x,y\}\in\mathscr{T}^\omega$ knowing Y_{τ_1} and $\tilde{Y}=\sigma$, where $\tilde{Y}$ is the partially forgotten walk. We now have two sources of randomness: the number of back and forth the walk does every time it visits a trap and the time the continuous speed-walk X^m spends for every step.
Knowing the partially-forgotten walk, $N_{\{x,y\}}$ is deterministic. Let $t^j_{\{x,y\}}$ be the j^{th} time the walk Y enters the trap $\{x,y\}$ and $\tilde{t}^j_{\{x,y\}}$ be the j^{th} time the walk Y exits the trap $\{x,y\}$. We define $H^j_{\{x,y\}}$ by $H^j_{\{x,y\}}:=\left\lfloor\frac{\tilde{t}^j_{\{x,y\}}-t^j_{\{x,y\}}}{2}\right\rfloor$, the number of back and forths in the trap $\{x,y\}$ during the j^{th} visit to the trap. For any integer n and for any trap $\{x,y\}\in\mathscr{T}^{\omega,n}$ we have that knowing the environment, Y_{τ_1} and the partially forgotten walk, $\left(H^j_{\{x,y\}}\right)_{j\in\mathbf{N},\{x,y\}\in\mathscr{T}}$ is a sequence of independent geometric random variables of parameter $(1-\omega(x,y))(1-\omega(y,x))$. Finally, for every $x\in\tilde{\mathscr{T}}$, let ℓ^j_x be the number of time x is visited between times $t^j_{\{x,y\}}$ and $\tilde{t}^j_{\{x,y\}}$. We define ε^j_x by $\varepsilon^j_x:=\ell^j_x-H^j_{\{x,y\}}$. Knowing the partially forgotten walk, ε^j_x is deterministic (it is equal to 0 iff the walk enters and leaves the trap by y during the j^{th} visit) and $\varepsilon^j_x\in\{0,1\}$. We have:

$$\int_0^\infty\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t=\sum_{j=1}^{N_{\{x,y\}}}\sum_{k=1}^{\varepsilon^j_x+H^j_{\{x,y\}}}\mathscr{E}^{k,j}_{m,x}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)},$$

where the $(\mathscr{E}^{k,j}_{m,x})_{x\in\mathbf{Z}^d,k,j\in\mathbf{N}}$ are independent exponential random variables of parameter $\gamma_\omega^m(x)$, they correspond to the time the accelerated walk spends on each vertex. By technical lemma 2.3.0.4 (the proof of which is in the annex) we get that there exists a constant $C_1>0$ such that for any integer n and any trap $\{x,y\}\in\mathscr{T}^{\omega,n}$:

$$C_1(N_{\{x,y\}})^\beta\leq\mathbb{E}_{\mathrm{P}_0^\omega}\left(\left(\int_0^\infty\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_0^\infty\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta|\tilde{Y},Y_{\tau_1}\right).\tag{2.3}$$

Unfortunately, we cannot directly use this inequality to conclude because it does not behave nicely with the renewal times. Indeed if you know that a trap spans over two renewal blocks, it means that you cannot do any back and forth inside the trap and the previous inequality becomes false. Instead we will have to first consider traps in $\mathscr{T}^{\omega,n}$. First, by definition of the renewal times, no trap in $\mathscr{T}^{\omega,n}$ can be visited before time τ_1 or after time τ_{n+2} since $Y_{\tau_{n+2}}.e_1\geq Y_{\tau_1}.e_1+n+1$. Therefore:

$$\begin{aligned}
&\sum_{\{x,y\}\in\mathscr{T}^{\omega,n}}\left(\int_0^\infty\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_0^\infty\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta\\
\leq&\sum_{\{x,y\}\in\mathscr{T}^{\omega}}\left(\int_{T_1^m}^{T_{n+2}^m}\frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_{T_1^m}^{T_{n+2}^m}\frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta
\end{aligned}$$

Therefore we get:

$$\frac{1}{n+1}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}^{\omega,n}}\left(\int_0^\infty \frac{\gamma_\omega^m(x)}{\gamma^\omega(x)}1_{X_t^m=x}\mathrm{d}t\right)^\beta+\left(\int_0^\infty \frac{\gamma_\omega^m(y)}{\gamma^\omega(y)}1_{X_t^m=y}\mathrm{d}t\right)^\beta\right)\leq C_0<\infty$$

This in turns gives:

$$\frac{1}{n+1}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}^{\omega,n}}\left(N_{\{x,y\}}\right)^\beta\right)\leq\frac{C_0}{C_1}<\infty$$

Now let $C_2:=\mathbb{E}_{\mathbb{P}_0}\left(\sum\limits_{\{x,y\}\in\mathscr{T}}\#\{j\in[\![\tau_i,\tau_{i+1}-1]\!],Y_j\in\{x,y\}\text{ and }Y_{j+1}\notin\{x,y\}\}^\beta\right)$ be the quantity we want to bound. By the law of large number, we have that $\mathbb{P}_0$ a.s and therefore $\mathbb{Q}_0$ a.s:

$$\frac{1}{n}\sum_{i=1}^n\sum_{\{x,y\}\in\mathscr{T}}\#\{j\in[\![\tau_i,\tau_{i+1}-1]\!],Y_j\in\{x,y\}\text{ and }Y_{j+1}\notin\{x,y\}\}^\beta\to C_2$$

Now, as a consequence of lemma 2.2.1.2 and the law of large number, there exists a finite constant $D>0$ such that $\mathbb{P}_0$ a.s and therefore $\mathbb{Q}_0$ a.s, $\frac{1}{n}Y_{\tau_n}.e_1\to D$. Furthermore, a trap spans over at most two renewal blocks so for any trap $\{x,y\}$:

$$\sum_{i\geq1}\#\{j\in[\![\tau_i,\tau_{i+1}-1]\!],Y_j\in\{x,y\}\text{ and }Y_{j+1}\notin\{x,y\}\}^\beta\leq 2(N_{\{x,y\}})^\beta.$$

As a consequence, $\mathbb{P}_0$ a.s:

$$\liminf\frac{1}{n}\frac{1}{n+1}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}^{\omega,Dn}}\left(N_{\{x,y\}}\right)^\beta\right)\geq\frac{C_2}{2}.$$

Finally we get:

$$\frac{C_2}{2}\leq D\frac{C_0}{C_1}$$

so C_2 is finite. □

The next lemma is just a variation of the previous one, with the difference that the sum has a deterministic number of terms instead of a random one which makes it simpler to use.

Lemma 2.2.3.3. *For any $j\in[\![1,d]\!]$ let (x_i^j,y_i^j) be the i^{th} trap in the direction j the walk encounters after τ_2. Let N_i^j be the number of times the walk enters this trap.*
If $\kappa\leq1$, for any $\beta\in[\kappa,\frac{\kappa+\kappa'}{2})$ with $\beta\leq1$ there is a constant C such that for any $j\in[\![1,d]\!]$:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n(N_i^j)^\beta\right)\leq Cn.$$

If $\kappa=1$ there exists a positive concave function ϕ defined on $[0,\infty)$ such that $\phi(t)$ goes to infinity when t goes to infinity. And such that if $\Phi(t)=\int\limits_{x=0}^t\phi(x)\mathrm{d}x$ then there exists a constant C such that for any $n\in\mathbf{N}$:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^n\Phi(N_i^j)\right)\leq Cn.$$

Those results are also true if (x_i^j,y_i^j) is the i^{th} trap in the direction j the walk encounters after τ_2 such that $x_i.e_1,y_i.e_1\geq Y_{\tau_2}.e_1$.

Proof. Let $p>0$ be the probability, for $\mathbb{P}_0$, that there is at least one trap in the direction j between times τ_2 and τ_3-1. Let $\mathscr{T}_j$ be the set of traps in the direction j. Now let the sequence (n_i) be defined by:

$$\begin{aligned}n_0&=1,\\n_{i+1}&=\min\{k>n_i,\exists\{x,y\}\in\mathscr{T}_j,\exists n\in[\![\tau_k,\tau_{k+1}-1]\!],Y_n\in\{x,y\}\}.\end{aligned}$$

Now, if $\kappa \leq 1$, let $Z_i^j = \sum\limits_{\{x,y\}\in\mathscr{T}_j} \#\{m \in [\![\tau_{n_i}, \tau_{n_i+1}-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta$. The $(Z_i^j)_{i\geq 1}$ are clearly identically distributed and we have:

$$\mathbb{E}_{\mathbb{P}_0}(Z_i^j) = \frac{1}{p}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}_j} \#\{m \in [\![\tau_i, \tau_{i+1}-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right).$$

So let $C_j = \mathbb{E}_{\mathbb{P}_0}(Z_i^j)$ which is finite by lemma 2.2.3.2. We clearly have:

$$\sum_{i=1}^{m} (N_i^j)^\beta \leq \sum_{i=1}^{2m} Z_i^j.$$

The sum has to go up to $2m$ because in the second sum some traps can appear twice if they are in between two renewal slabs. Indeed, in this case they can be visited before and after the renewal time (if they are in the direction e_1). We now have:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{m}(N_i^j)^\beta\right) \leq 2C_j m.$$

Similarly, if $\{\overline{x}_i, \overline{y}_i\}$ is the i^{th} trap in the direction j the walk encounters after τ_2 such that $x_i.e_1, y_i.e_1 \geq Y_{\tau_2}.e_1$ and $\overline{N}_i^j$ the number of times the walk enters this trap then we have:

$$\sum_{i=1}^{m}(\overline{N}_i^j)^\beta \leq \sum_{i=1}^{2m+1} Z_i.$$

If $\kappa = 1$, by lemma 2.2.3.2,

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}_j} \#\{m \in [\![\tau_2, \tau_3-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right) < \infty.$$

Therefore, by forthcoming technical lemma 2.3.0.1 there exists a positive, concave function ϕ defined on $[0,\infty)$ such that $\phi(t)$ goes to infinity when t goes to infinity and such that, if $\Phi(t) := \int\limits_{x=0}^{t} \phi(x)\mathrm{d}x$ then:

$$\mathbb{E}_{\mathbb{P}_0}\left(\Phi\left(2\sum_{\{x,y\}\in\mathscr{T}_j} \#\{m \in [\![\tau_2, \tau_3-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right)\right) < \infty,$$

where $\Phi(t) := \int\limits_{x=0}^{t} \phi(x)\mathrm{d}x$. We have that $x \to \frac{\Phi(x)}{x}$ is increasing and therefore, by writing $g(x) = \frac{\Phi(x)}{x}$, for any non-negative sequence $(a_i)_{1\leq i\leq n}$:

$$\begin{aligned}\sum_{1\leq i\leq n} \Phi(a_i) &= \sum_{1\leq i\leq n} a_i g(a_i)\\ &\leq \sum_{1\leq i\leq n} a_i g\left(\sum_{1\leq j\leq n} a_j\right)\\ &= \left(\sum_{1\leq i\leq n} a_i\right) g\left(\sum_{1\leq i\leq n} a_i\right)\\ &= \Phi\left(\sum_{1\leq i\leq n} a_i\right).\end{aligned}$$

So we get:

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_0}\left(\sum_{\{x,y\}\in\mathscr{T}_j} \Phi\left(2\#\{m \in [\![\tau_2, \tau_3-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right)\right)\\ \leq&\mathbb{E}_{\mathbb{P}_0}\left(\Phi\left(2\sum_{\{x,y\}\in\mathscr{T}_j} \#\{m \in [\![\tau_2, \tau_3-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right)\right) < \infty.\end{aligned}$$

Let $Z_i^j := \sum\limits_{\{x,y\}\in\mathscr{T}_j} \Phi\left(\#\{m \in [\![\tau_{n_i}, \tau_{n_i+1}-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right)$. The $(Z_i^j)_{i\geq 1}$ are clearly identically distributed and we have:

$$\mathbb{E}_{\mathbb{P}_0}(Z_i) = \frac{1}{p}\mathbb{E}_{\mathbb{P}_0}\left(2\sum_{\{x,y\}\in\mathscr{T}_j} \Phi\left(\#\{m \in [\![\tau_2, \tau_3-1]\!], Y_m \in \{x,y\} \text{ and } Y_{m+1} \notin \{x,y\}\}^\beta\right)\right) < \infty.$$

So let $C_j = \mathbb{E}_{\mathbb{P}_0}(Z_i^j)$, which is finite. We clearly have:

$$\sum_{i=1}^{m} \Phi(N_i^j) \leq \sum_{i=1}^{2m} Z_i.$$

Once again, the sum has to go up to $2m$ because in the second sum some traps can appear twice if they are in between two renewal slabs. Indeed, in this case they can be visited before and after the renewal time (if they are in the direction e_1). so:

$$\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{m} \Phi(N_i^j)\right) \leq 2C_j m.$$

Similarly, if $\{\overline{x}_i, \overline{y}_i\}$ is the i^{th} trap in the direction j the walk encounters after τ_2 such that $x_i.e_1, y_i.e_1 \geq Y_{\tau_2}.e_1$ and $\overline{N}_i^j$ the number of times the walk enters this trap then we have:

$$\sum_{i=1}^{m} \Phi(\overline{N}_i^j) \leq \sum_{i=1}^{2m+1} Z_i^j.$$

and we get the result we want. $\square$

The following lemma gives us some independence between the strength of a trap and the number of times the walk enters this trap.

Lemma 2.2.3.4. *Let $j \in [\![1, d]\!]$ be an integer that represents the direction of the traps we will consider. Let $\{x_i^j, y_i^j\}$ be the i^{th} trap in the direction j (ie $x_i^j - y_i^j \in \{e_j, -e_j\}$) to be visited after time τ_2 and such that $x_i^j.e_1 \geq Y_{\tau_2}.e_1$ and $y_i^j.e_1 \geq Y_{\tau_2}.e_1$. Now let s_i^j be the strength of the trap. Let N_i^j be the number of times the trap $\{x_i^j, y_i^j\}$ is exited. Let $\kappa_j = 2\sum\limits_{i=1}^{2d} \alpha_i - \alpha_j - \alpha_{j+d}$. For any $\gamma \in [0,1]$, there exists a constant C that does not depend on i such that:*

$$\forall A \geq 2,\ \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^\gamma 1_{s_i^j \geq A}\right) \leq \frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}((N_i^j)^\gamma).$$

We also have that for any positive concave function ϕ such that $\phi(0) = 1$ with $\Phi(t) = \int\limits_{x=0}^{t} \phi(x)\mathrm{d}x$ we get:

$$\forall A \geq 2,\ \mathbb{E}_{\mathbb{P}_0}\left(\Phi(N_i^j) 1_{s_i^j \geq A}\right) \leq \frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)).$$

Proof. First if H is a geometric random variable of parameter p then for any $\gamma \in [0,1]$ we have the following three inequalities:

$$\mathbb{E}((1+H)^\gamma) \geq 1 = p^\gamma \frac{1}{p^\gamma}, \tag{2.4}$$

$$\mathbb{E}((1+H)^\gamma) \geq \mathbb{P}\left(Z \geq \frac{1}{p}\right)\frac{1}{p^\gamma} \geq (1-p)^{\frac{1}{p}-1}\frac{1}{p^\gamma}, \tag{2.5}$$

$$\mathbb{E}((1+H)^\gamma) \leq \mathbb{E}((1+H))^\gamma = \frac{1}{p^\gamma}. \tag{2.6}$$

Inequalities 2.4 and 2.5 give us that there is a constant C_γ such that $\mathbb{E}((1+H)^\gamma) \geq C_\gamma \frac{1}{p^\gamma}$, inequality 2.4 gives us the result for $p \geq \frac{1}{2}$ and since $(1-p)^{\frac{1}{p}-1}$ converges to $\exp(-1)$ when p goes to 0, inequality 2.5 gives us the result for $p \leq \frac{1}{2}$.
By lemma 2.3.0.2 we get that there is a constant C_ϕ such that:

$$\frac{1}{2}\frac{1}{p}\phi\left(\frac{1}{p}\right) \leq \mathbb{E}(\Phi(1+H)) \leq C_\phi \frac{1}{p}\phi\left(\frac{1}{p}\right). \tag{2.7}$$

Let $t \in \mathbf{N}$ be an integer. In the following we will call renewal hyperplan the set of vertices $\{x, x.e_1 = Y_t.e_1\}$. We look at the n^{th} time, after time t, that the walk encounters a vertex that touches a trap $\{x,y\}$ in the direction j that has never been visited before and such that $x.e_1, y.e_1 \geq Y_t.e_1$. We want to show that the strength of the trap is basically independent from the number of times the walk leaves the trap and from the random variable $1_{\tau_2=t}$. Let x, y be the corresponding trap with x being the first vertex visited.

Now we look at the trap $\{x,y\}$. Let i be such that $y = x+e_i$, we will write $\alpha_x := \alpha_i$, $\alpha_y := \alpha_{i+d}$ and $\overline{\alpha} := \sum\limits_{k=1}^{2d} \alpha_k$. The density probability (for $\mathbb{P}^{(\alpha)}$) for the transition probabilities $\omega(x,y)$ and $\omega(y,x)$, knowing all the transition probabilities $(\omega(z_1,z_2))_{z_1 \in \mathbf{Z}^d \setminus \{x,y\}}$, the renormalized transition probabilities $(\frac{\omega(x,z)}{1-\omega(x,y)})_{z\neq y}, (\frac{\omega(y,z)}{1-\omega(y,x)})_{z\neq x}$ and that $\{x,y\}$ is a trap is:

$$C\omega(x,y)^{\alpha_x-1}(1-\omega(x,y))^{\overline{\alpha}-\alpha_x-1}\omega(y,x)^{\alpha_y-1}(1-\omega(y,x))^{\overline{\alpha}-\alpha_y-1}1_{\omega(x,y)+\omega(y,x)\geq\frac{3}{2}}.$$

Now we make the change of variables:

$$1-\omega(y,x) = r(1-k),\ 1-\omega(x,y) = r(1+k),$$

which gives a density probability of:

$$2rCr^{\kappa_j-2}(1-k)^{\overline{\alpha}-\alpha_y-1}(1+k)^{\overline{\alpha}-\alpha_x-1}(1-r(1+k))^{\alpha_x-1}(1-r(1-k))^{\alpha_y-1}1_{r\leq\frac{1}{4}}\mathrm{d}r\mathrm{d}k.$$

Let $h(r,k)$ be defined by:

$$h(r,k) = (1-r(1+k))^{\alpha_x-1}(1-r(1-k))^{\alpha_y-1}.$$

For $0 \leq r \leq \frac{1}{4}$ and $-1 \leq k \leq 1$ we have:

$$\log(h(r,k)) \leq |\alpha_x - 1|\left|\log\left(\frac{1}{2}\right)\right| + |\alpha_y-1|\left|\log\left(\frac{1}{2}\right)\right| \leq (\alpha_x+\alpha_y+2)\log(2).$$

So for $0 \leq r \leq \frac{1}{4}$ and $-1 \leq k \leq 1$ we have:

$$2^{-(\alpha_x+\alpha_y+2)} \leq h(r,k) \leq 2^{\alpha_x+\alpha_y+2}.$$

Now the density probability is:

$$2Ch(r,k)r^{\kappa_j-1}(1-k)^{\overline{\alpha}-\alpha_y-1}(1+k)^{\overline{\alpha}-\alpha_x-1}1_{r\leq\frac{1}{4}}\mathrm{d}r\mathrm{d}k.$$

Now we look at a specific environment ω and an edge $\{x',y'\}$ in that environment. To simplify the notation we will write $\varepsilon_{x'} = 1-\omega(x',y')$ and $\varepsilon_{y'} = 1-\omega(y',x')$. When the walk leaves the trap there are three possibilities:
-the walk goes to infinity before going back to the trap or the renewal hyperplan
-the walk goes to the renewal hyperplan before it goes back to the trap (this does not necessarily mean that the walk will go back to the trap after going to the renewal hyperplan)
-the walk goes back to the trap before it goes to the renewal hyperplan (this does not necessarily mean that the walk will eventually go to the renewal hyperplan).

If the walk is in x' let $\beta_{x'}^{\infty}$ be the probability, knowing that the next step isn't crossing the trap, that the walk goes to infinity without going to the renewal hyperplan or the trap. Similarly, let $\beta_{x'}^{0}$ be the probability, knowing that the next step isn't crossing the trap, that the walk goes to the renewal hyperplan before it goes back to the trap (this does not mean that the walk necessarily goes back to the trap). We will also define $\beta_{x'}$ by $\beta_{x'} := \beta_{x'}^{\infty} + \beta_{x'}^{0}$. Similarly we will define $\beta_{y'}, \beta_{y'}^{\infty}, \beta_{y'}^{0}$.

Now, if the walk is in x', the probability that when the walk leaves the trap it either never comes back to the trap or goes to the renewal hyperplan before it goes back to the trap is:

$$\frac{\varepsilon_{x'}}{\varepsilon_{x'}+\varepsilon_{y'}-\varepsilon_{x'}\varepsilon_{y'}}\beta_{x'} + \frac{\varepsilon_{y'}(1-\varepsilon_{x'})}{\varepsilon_{x'}+\varepsilon_{y'}-\varepsilon_{x'}\varepsilon_{y'}}\beta_{y'} = \frac{\varepsilon_{x'}\beta_{x'}+\varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}}{\varepsilon_{x'}+\varepsilon_{y'}-\varepsilon_{x'}\varepsilon_{y'}}.$$

Similarly, if the walk is in y', this probability is:

$$\frac{\varepsilon_{x'}(1-\varepsilon_{y'})\beta_{x'}+\varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'}+\varepsilon_{y'}-\varepsilon_{x'}\varepsilon_{y'}}.$$

Now we want to show that that both these quantities are almost equal to:

$$\frac{\varepsilon_{x'}\beta_{x'}+\varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'}+\varepsilon_{y'}}.$$

We will only show it for the first quantity, the proof is the same for the second one. We recall that $\varepsilon_{x'}, \varepsilon_{y'} \leq \frac{1}{2}$, therefore:

$$0 \leq \varepsilon_{x'}\varepsilon_{y'} \leq \frac{1}{2}(\varepsilon_{x'} + \varepsilon_{y'})$$

and

$$0 \leq \varepsilon_{x'}\varepsilon_{y'}\beta_{y'} \leq \frac{1}{2}(\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}).$$

So we get:

$$\frac{1}{2}\frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'}} \leq \frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}} \leq 2\frac{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}{\varepsilon_{x'} + \varepsilon_{y'}}.$$

Similarly, if the walk is in x', the probability that the walk goes to infinity knowing that the walk either goes to infinity or the renewal hyperplan before coming to the trap is:

$$\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}^{\infty}}{\varepsilon_{x'} + \varepsilon_{y'} - \varepsilon_{x'}\varepsilon_{y'}} \frac{\varepsilon_{x'} + \varepsilon_{yx'} - \varepsilon_{x'}\varepsilon_{y'}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}} = \frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}}.$$

And if it is in y' this probability is:

$$\frac{\varepsilon_{x'}(1-\varepsilon_{y'})\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}(1-\varepsilon_{y'})\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.$$

We want to show that both these probabilities are almost equal to $\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}$. We will only show it for the first one:

$$\begin{aligned}\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}} \leq & \frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}} \\ \leq & \frac{1}{(1-\varepsilon_x)}\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}} \\ \leq & 2\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.\end{aligned}$$

And we also get, the same way:

$$\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}(1-\varepsilon_{x'})\beta_{y'}} \geq \frac{1}{2}\frac{\varepsilon_{x'}\beta_{x'}^{\infty} + \varepsilon_{y'}\beta_{y'}^{\infty}}{\varepsilon_{x'}\beta_{x'} + \varepsilon_{y'}\beta_{y'}}.$$

Now we get back to the trap $\{x, y\}$. Let N be the number of times the walks leaves the trap $\{x, y\}$ before going to the renewal hyperplan (so if the walk never goes to the renewal hyperplan, N is just the number of times the walk leaves the trap $\{x, y\}$). We get that knowing $\varepsilon_x, \varepsilon_y$ and N, the probability (for P_0^{ω}) that the walk never goes to the renewal hyperplan is between $\frac{1}{2}\frac{\varepsilon_x\beta_x^{\infty} + \varepsilon_y\beta_y^{\infty}}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}$ and $2\frac{\varepsilon_x\beta_x^{\infty} + \varepsilon_y\beta_y^{\infty}}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}$.
We also have that there exists two geometric random variables N^- and N^+ respectively of parameter $\frac{1}{2}\frac{\varepsilon_x\beta_x + \varepsilon_y\beta_y}{\varepsilon_x + \varepsilon_y}$ and $2\frac{\varepsilon_x\beta_x + \varepsilon_y\beta_y}{\varepsilon_x + \varepsilon_y}$ such that P_0^{ω} almost surely:

$$1 + N^- \leq N \leq 1 + N^+.$$

Therefore, by equations 2.4, 2.5, 2.6 and 2.7 there exists two positive constants C_1 and C_2 (that depend on γ and Φ) such that for f equal to either $x \to x^{\gamma}$ or Φ:

$$C_1 f\left(\frac{\varepsilon_x + \varepsilon_y}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\right) \leq \mathbb{E}_{\mathrm{P}_0^{\omega}}(f(N)) \leq C_2 f\left(\frac{\varepsilon_x + \varepsilon_y}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\right). \tag{2.8}$$

Now let f be either $x \to x^{\gamma}$ or Φ. We need to show that N is almost independent from $1_{\tau_2 = t}$. Let t_{xy} be the first time the walk is in x or y and let B be the event that "τ_2 can be equal to t" ie there exists $t' < t$ (t' plays the role of τ_1) such that:
- $\forall i < t',\ X_i.e_1 < X_{t'}.e_1$,
- $\forall i \in [\![t', t-1]\!],\ X_{t'}.e_1 \leq X_i.e_1 < X_t.e_1$,
- $\forall i \in [\![t, t_{xy}]\!],\ X_i.e_1 \geq X_t.e_1$,
- $\forall i \in [\![0, t'-1]\!] \cup [\![t'+1, t-1]\!], (\exists j < i,\ X_j.e_1 \geq X_i.e_1)$ or $(\exists j \in [\![i+1, t-1]\!] X_j.e_1 < X_i.e_1)$.

We have that if B isn't true then τ_2 cannot be equal to t. If B is true the $\tau_2 = t$ iff the walk never crosses the renewal hyperplan after time t_{xy}. So, for any environment ω:

$$\frac{1}{2}\frac{\varepsilon_x\beta_x^{\infty} + \varepsilon_y\beta_y^{\infty}}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\mathrm{P}_0^{\omega}(B) \leq \mathrm{P}_0^{\omega}(\tau_2 = t|N) \leq 2\frac{\varepsilon_x\beta_x^{\infty} + \varepsilon_y\beta_y^{\infty}}{\varepsilon_x\beta_x + \varepsilon_y\beta_y}\mathrm{P}_0^{\omega}(B) \tag{2.9}$$

To simplify notations we will write

$$\overline{h}(k) := \frac{(1+k)\beta_x^\infty + (1-k)\beta_y^\infty}{(1+k)\beta_x + (1-k)\beta_y}.$$

We have (in the following, the constant C will depend on the line):

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x+\varepsilon_y\le\frac{1}{A}}1_{\tau_2=t}\right)\\
\le&2\mathbb{E}_{\mathbb{P}_0}\left(f(N)\frac{\varepsilon_x\beta_x^\infty+\varepsilon_y\beta_y^\infty}{\varepsilon_x\beta_x+\varepsilon_y\beta_y}1_{\varepsilon_x+\varepsilon_y\le\frac{1}{A}}\right) \text{ by 2.9}\\
\le&C\mathbb{E}_{\mathbb{P}_0}\left(f\left(\frac{\varepsilon_x+\varepsilon_y}{\varepsilon_x\beta_x+\varepsilon_y\beta_y}\right)1_{\varepsilon_x+\varepsilon_y\le\frac{1}{A}}\frac{\varepsilon_x\beta_x^\infty+\varepsilon_y\beta_y^\infty}{\varepsilon_x\beta_x+\varepsilon_y\beta_y}\right) \text{ by 2.8.}
\end{aligned}$$

Now we use the fact that the various β only depend on the trajectory of the walk up to the time it encounters the n^{th} trap in the direction j after time t, the transition probabilities $(\omega(z_1,z_2))_{z_1\in\mathbf{Z}^d\setminus\{x,y\}}$, the renormalized transition probabilities $(\frac{\omega(x,z)}{1-\omega(x,y)})_{z\neq y}$, $(\frac{\omega(y,z)}{1-\omega(y,x)})_{z\neq x}$ and that $\{x,y\}$ is a trap. But the law of $(\omega(x,y),\omega(x,y))$ is independent of this so we get:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x+\varepsilon_y\le\frac{1}{A}}1_{\tau_2=t}\right)\\
\le&C\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{2A}}\int_{k=-1}^{1} f\left(\frac{r}{r(1+k)\beta_x+r(1-k)\beta_y}\right)2Ch(r,k)r^{\kappa_j-1}(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\mathrm{d}r\right)\\
\le&C\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{2A}} r^{\kappa_j-1}\mathrm{d}r\int_{k=-1}^{1} f\left(\frac{1}{(1+k)\beta_x+(1-k)\beta_y}\right)(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\right)\\
=&C\left(\frac{2}{A}\right)^{\kappa_j}\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{4}} r^{\kappa_j-1}\mathrm{d}r\int_{k=-1}^{1} f\left(\frac{1}{(1+k)\beta_x+(1-k)\beta_y}\right)(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\right)\\
\le&\frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(\int_{r=0}^{\frac{1}{4}}\int_{k=-1}^{1} f\left(\frac{r}{r(1+k)\beta_x+r(1-k)\beta_y}\right)2Ch(r,k)r^{\kappa_j-1}(1-k)^{\alpha_y}(1+k)^{\alpha_x}\overline{h}(k)\mathrm{d}k\mathrm{d}r\right)\\
\le&\frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x+\varepsilon_y\le\frac{1}{2}}\frac{\varepsilon_x\beta_x^\infty+\varepsilon_y\beta_y^\infty}{\varepsilon_x\beta_x+\varepsilon_y\beta_y}\right)\\
\le&\frac{C}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(f(N)1_{\varepsilon_x+\varepsilon_y<\frac{1}{2}}1_{\tau_2=t}\right).
\end{aligned}$$

Then, by summing on all t we get the result. □

2.2.4 The time the walk spends in trap

Now that we have some independence, we can start to look at the precise behaviour of the time spent in the traps. First we want to show that the number of times the walk enters a trap times the strength of said trap is a good approximation of the total time spent in this trap.

Lemma 2.2.4.1. *Let $j\in[\![1,d]\!]$ be a direction. Now let $\{x_i^j,y_i^j\}$ be the i^{th} trap in the direction j entered after time τ_2 and such that $x_i^j.e_1, y_i^j.e_1\ge Y_{\tau_2}.e_1$. Let s_i^j be the strength of this trap, N_i^j the number of times the walk enters this trap and $\ell_i^j=\#\{n,Y_n\in\{x_i^j,y_i^j\}\}$ the time spent in the trap. We have for any environment ω, for any $A,B\ge0$, for any integer m and for any $C\in\mathbf{R}^+\cup\{\infty\}$:*

$$P_0^\omega\left(\sum_{i=1}^n \ell_i^j 1_{N_i^j\ge m}1_{s_i^j\le C}\ge A \text{ and } \sum N_i^j 1_{N_i^j\ge m}s_i^j 1_{s_i^j\le C}\le B\right)\le\frac{5B}{A}.$$

Proof. Let ω be an environment, $(\tilde{Y}_i)_{i\in\mathbf{N}}$ be the partially forgotten walk on this environment. Let $p_i^j=\omega(x_i^j,y_i^j)\omega(y_i^j,x_i^j)$. Now the number of back and forths inside the trap (x_i^j,y_i^j) during its k^{th} visit is equal to $H_{i,k}^j$ where $H_{i,k}^j$ is a geometric random variable of parameter p_i^j. Knowing the partially-forgotten walk and p_i^j, the $H^j_{i,k}$ are independent and we get for any j:

$$\mathbb{E}_{\mathrm{P}^\omega_0}\left(\sum_{i=1}^n 1_{N^j_i\geq m}1_{s^j_i\leq C}\sum_{k=1}^{N^j_i}2H^j_{i,k}|\tilde{Y}\right) =\sum_{i=1}^n 1_{N^j_i\geq m}1_{s^j_i\leq C}\sum_{k=1}^{N^j_i}2\frac{p^j_i}{1-p^j_i}$$
$$\leq 2\sum_{i=1}^n 1_{N^j_i\geq m}1_{s^j_i\leq C}N^j_i\frac{1}{1-p^j_i}.$$

Now we use the fact that $\omega(x^j_i,y^j_i)\geq\frac{1}{2}$ to show that $1-p^j_i\geq\frac{1}{2s^j_i}$:

$$\begin{aligned}1-p^j_i =&1-(1-(1-\omega(x^j_i,y^j_i)))(1-(1-\omega(y^j_i,x^j_i))\\
=&(1-\omega(x^j_i,y^j_i))+(1-\omega(y^j_i,x^j_i))-(1-\omega(x^j_i,y^j_i))(1-\omega(y^j_i,x^j_i))\\
\geq&(1-\omega(x^j_i,y^j_i))+(1-\omega(y^j_i,x^j_i))-\frac{1}{2}(1-\omega(y^j_i,x^j_i))\\
\geq&\frac{(1-\omega(x^j_i,y^j_i))+(1-\omega(y^j_i,x^j_i))}{2}\\
=&\frac{1}{2s^j_i}.\end{aligned}$$

So we get:

$$\mathbb{E}_{\mathrm{P}^\omega_0}\left(\sum_{i=1}^n 1_{N^j_i\geq m}1_{s^j_i\leq C}\sum_{k=1}^{N^j_i}2H^j_{i,k}|\tilde{Y}\right)\leq 4\sum_{i=1}^n N^j_i 1_{N^j_i\geq m}s^j_i 1_{s^j_i\leq C}.$$

The actual value of ℓ^j_i can be slightly larger than $\sum\limits_{k=1}^{N^j_i}2H^j_{i,k}$ because this only counts the back-and-forths, so we miss the correct amount by 1 every time the walks crosses the trap an even number of times and by 2 every time the walks crosses the trap an odd number of times. So we get that the time ℓ^j_i the walk spends in the i^{th} trap is smaller than $2N^j_i+\sum\limits_{j=1}^{N^j_i}2H^j_{i,k}$. For any positive constants $A,B>0$, let $E^n(B)$ be the event $\sum\limits_{i=1}^n N^j_i 1_{N^j_i\geq m}1_{s^j_i\leq C}s^j_i\leq B$, we have:

$$\begin{aligned}&\mathrm{P}^\omega_0\left(\sum_{i=1}^n \ell^j_i 1_{N^j_i\geq m}1_{s^j_i\leq C}\geq A \text{ and } \sum N^j_i 1_{s^j_i\leq C}1_{N^j_i\geq m}s^j_i\leq B\right)\\
=&\mathbb{E}_{\mathrm{P}^\omega_0}\left(\mathrm{P}^\omega_0\left(\sum_{i=1}^n \ell^j_i 1_{s^j_i\leq C}1_{N^j_i\geq m}\geq A|\tilde{Y}\right)1_{E^n(B)}\right)\\
\leq&\mathbb{E}_{\mathrm{P}^\omega_0}\left(\frac{\sum\limits_{i=1}^n N^j_i 1_{N^j_i\geq m}1_{s^j_i\leq C}(4s^j_i+2)}{A}1_{E^n(B)}\right)\\
\leq&\mathbb{E}_{\mathrm{P}^\omega_0}\left(\frac{\sum\limits_{i=1}^n 5N^j_i 1_{N^j_i\geq m}s^j_i 1_{s^j_i\leq C}}{A}1_{E^n(B)}\right) \text{ since } s^j_i>2\\
\leq&\mathbb{E}_{\mathrm{P}^\omega_0}\left(\frac{5B}{A}1_{E^n(B)}\right)\leq\frac{5B}{A}.\end{aligned}$$

□

Now we want to show that we can neglect the time spent in traps in directions such that $\kappa_j\neq\kappa$ and in traps that are visited a lot of times. This will allow us to have traps that are rather similar so that the time spent in those traps are almost identically distributed.

Lemma 2.2.4.2. *Let $j\in[\![1,d]\!]$ be an integer that represents the direction of the trap we will consider. Let $\{x_i,y_i\}$ be the i^{th} trap in the direction j visited by the walk after time τ_2 and such that $x_i.e_1\geq Y_{\tau_2}.e_1$ and*

$y_i.e_1 \geq Y_{\tau_2}.e_1$. Let $\kappa_j = 2\alpha - \alpha_j - \alpha_{j+d} \geq \kappa$.
If $\kappa < 1$ there are two cases: If $\kappa_j = \kappa$, for any $\varepsilon > 0$ there exists an integer m_ε such that for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq \varepsilon.$$

If $\kappa_j > \kappa$, for any $\varepsilon > 0$ there exists an integer n_ε such that for $n \geq n_\varepsilon$:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq \varepsilon.$$

If $\kappa = 1$ there are two cases: If $\kappa_j = \kappa$, for any $\varepsilon > 0$ there exists an integer m_ε such that for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n \log(n)\right) \leq \varepsilon.$$

If $\kappa_j > 1$, for any $\varepsilon > 0$ there exists an integer n_ε such that for $n \geq n_\varepsilon$:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n \log(n)\right) \leq \varepsilon.$$

Proof. For all $i \geq 0$ let t_i be the time at which the walk Y enters its i^{th} trap ($\{x_i, y_i\}$) in the direction j after τ_2 and such that $x_i.e_1 \geq Y_{\tau_2}.e_1$ and $y_i.e_1 \geq Y_{\tau_2}.e_1$. We will write x_i the vertex such that $x_i = Y_{t_i}$. Let s_i^j be the strength of the trap $\{x_i^j, y_i^j\}$. For any $A, B > 0$:

$$\begin{aligned}
\mathbb{P}_0(\exists i \leq n, s_i^j \geq A \text{ and } N_i^j \geq B) &\leq \mathbb{P}_0\left(\left(\sum_{i=1}^{n} N_i^j 1_{s_i^j \geq A}\right)^{\kappa} \geq B^\kappa\right) \\
&\leq \mathbb{P}_0\left(\sum_{i=1}^{n} (N_i^j)^\kappa 1_{s_i^j \geq A} \geq B^\kappa\right) \\
&\leq \frac{1}{B^\kappa}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n} (N_i^j)^\kappa 1_{s_i^j \geq A}\right) \\
&\leq \frac{c}{B^\kappa}\frac{1}{A^{\kappa_j}}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n} (N_i^j)^\kappa\right) \quad \text{by lemma 2.2.3.4} \\
&\leq \frac{c}{B^\kappa}\frac{1}{A^{\kappa_j}} Cn \quad \text{by lemma 2.2.3.3.}
\end{aligned} \tag{2.10}$$

We will first look at the case $\kappa < 1$.
Now, we want to show that we can neglect traps with a high N_i^j or a low s_i^j. We get that for any positive

integer M, any real $A \geq 2$ and any $\beta \in [\kappa, 1]$ and $\eta > 0$ such that $\beta + \eta \leq \min\left(\frac{\kappa+\kappa'}{2}, 1\right)$:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<A} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}}\right)\\
\leq&\mathbb{P}_0\left(\sum_{i=1}^{n}(N_i^j)^{\beta}(s_i^j)^{\beta} 1_{s_i^j<A} 1_{N_i^j \geq M} \geq (an)^{\frac{\beta}{\kappa}}\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n}(N_i^j)^{\beta}(s_i^j)^{\beta} 1_{s_i^j<A} 1_{N_i^j \geq M}\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n}(N_i^j)^{\beta+\eta}(s_i^j)^{\beta} 1_{s_i^j<A}\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\mathbb{E}_{\mathbb{P}_0}\left(\int_{t=0}^{A^{\beta}}\sum_{i=1}^{n}(N_i^j)^{\beta+\eta} 1_{(s_i^j)^{\beta}\geq t}\mathrm{d}t\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\sum_{i=1}^{n}\int_{t=0}^{A^{\beta}}\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{(s_i^j)^{\beta}\geq t}\right)\mathrm{d}t\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\sum_{i=1}^{n}\left(2\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right)+\int_{t=2^{\beta}}^{A^{\beta}}\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{s_i\geq t^{\frac{1}{\beta}}}\right)\mathrm{d}t\right).
\end{aligned}$$

By lemma 2.2.3.4, there exists a constant c such that $\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta} 1_{s_i^j\geq t^{\frac{1}{\beta}}}\right) \leq \mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right) c t^{-\frac{\kappa}{\beta}}$, for $t \geq 2^{\beta}$ so:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<A} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}}\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\sum_{i=1}^{n}\left(2\mathbb{E}_{\mathbb{P}_0}((N_i^j)^{\beta+\eta})+\mathbb{E}_{\mathbb{P}_0}\left((N_i^j)^{\beta+\eta}\right)\int_{t=2^{\beta}}^{A^{\beta}} c t^{-\frac{\kappa}{\beta}}\mathrm{d}t\right)\\
\leq&(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\sum_{i=1}^{n}\left(2+c\int_{t=2^{\beta}}^{A^{\beta}} t^{-\frac{\kappa}{\beta}}\mathrm{d}t\right)E_{\mathbb{P}_0}((N_i^j)^{\beta+\eta})\\
\leq&dn(an)^{-\frac{\beta}{\kappa}}M^{-\eta}\left(2+c\int_{t=2^{\beta}}^{A^{\beta}} t^{-\frac{\kappa}{\beta}}\mathrm{d}t\right) \text{ by lemma 2.2.3.3.}
\end{aligned} \tag{2.11}$$

Now for $\kappa_j = \kappa$ if we take $\beta \in (\kappa, 1]$ such that $\beta < \frac{\kappa+\kappa'}{2}$, $\eta = 0$ and $A = bn^{\frac{1}{\kappa}}$ we get:

$$\begin{aligned}
\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<bn^{\frac{1}{\kappa}}} 1_{N_i^j \geq M} \geq (an)^{\frac{1}{\kappa}}\right) \leq&\frac{d}{a}n^{1-\frac{\beta}{\kappa}}\left(2+\frac{\beta c}{\beta-\kappa}\left(bn^{\frac{1}{\kappa}}\right)^{\beta-\kappa}\right)\\
\leq&\frac{d}{a}n^{1-\frac{\beta}{\kappa}}\left(2+\frac{\beta c}{\beta-\kappa}b^{\beta-\kappa}n^{\frac{\beta-\kappa}{\kappa}}\right)\\
=&2\frac{d}{a}n^{1-\frac{\beta}{\kappa}}+\frac{d}{a}\frac{\beta c}{\beta-\kappa}b^{\beta-\kappa}.
\end{aligned} \tag{2.12}$$

Now, we get by lemma 2.2.4.1 that for any positive constants A, B and any positive integer m:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{n}\ell_i^j 1_{N_i^j\geq m} \geq A\right)\\
\leq&\mathbb{P}_0\left(\sum_{i=1}^{n}\ell_i^j 1_{N_i^j\geq m} \geq A \text{ and } \sum_{i=1}^{n} N_i^j 1_{N_i^j\geq m} s_i^j \leq B\right)+\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j\geq m} s_i^j \geq B\right)\\
\leq&\frac{5B}{A}+\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j\geq m} s_i^j \geq B\right).
\end{aligned} \tag{2.13}$$

So for any $\varepsilon > 0$, for any $a > 0$, by taking $B = \varepsilon^2 n^{\frac{1}{\kappa}}$ and $A = \varepsilon n^{\frac{1}{\kappa}}$ in 2.13, we have for any positive integer m:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m} \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq 5\varepsilon + \mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq \varepsilon^2 n^{\frac{1}{\kappa}}\right).$$

And we have for any $b > 0$:

$$\begin{aligned}&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j \geq \varepsilon^2 n^{\frac{1}{\kappa}}\right)\\ \leq&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq b n^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}}\right) + \mathbb{P}_0\left(\exists i \leq n, N_i^j \geq m \text{ and } s_i^j \geq b n^{\frac{1}{\kappa}}\right).\end{aligned}$$

We have by 2.10:

$$\mathbb{P}_0\left(\exists i \leq n, N_i^j \geq m \text{ and } s_i^j \geq b n^{\frac{1}{\kappa}}\right) \leq \frac{cdn}{(mb)^{\kappa} n} = \frac{cd}{(mb)^{\kappa}}.$$

And by 2.12, taking $b = \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}}$:

$$\begin{aligned}\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} n^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}}\right) \leq&\frac{d}{\varepsilon^{2\kappa}}\left(2n^{1-\frac{\beta}{\kappa}} + \frac{\beta c}{\beta-\kappa}\varepsilon^{\frac{2\kappa+1}{\beta-\kappa}(\beta-\kappa)}\right)\\ =&\frac{d}{\varepsilon^{2\kappa}}\left(2n^{1-\frac{\beta}{\kappa}} + \frac{\beta c}{\beta-\kappa}\varepsilon^{2\kappa+1}\right).\end{aligned}$$

So for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j 1_{N_i^j \geq m} s_i^j 1_{s_i^j \leq \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} n^{\frac{1}{\kappa}}} \geq \varepsilon^2 n^{\frac{1}{\kappa}}\right) \leq \frac{2d\beta c}{\beta-\kappa}\varepsilon$$

which means that for n large enough and m_ε such that $m_\varepsilon \varepsilon^{\frac{2\kappa+1}{\beta-\kappa}} \geq \varepsilon^{-\frac{1}{\kappa}}$ we have:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n^{\frac{1}{\kappa}}\right) \leq 5\varepsilon + cd\varepsilon + \frac{2d\beta c}{\beta-\kappa}\varepsilon.$$

And we have the result we want.
If $\kappa_j > \kappa$ there exists $\beta \in (\kappa, \kappa_j)$ such that $\beta \leq 1$ and $\beta \leq \frac{\kappa+\kappa'}{2}$ we get by taking $M = 1$ and $A = \infty$ in 2.11:

$$\begin{aligned}\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j \geq (an)^{\frac{1}{\kappa}}\right) \leq& d a^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}}\left(2 + \int_{t=2}^{\infty} t^{-\frac{\kappa_j}{\beta}}\,\mathrm{d}t\right)\\ =& d a^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}}\left(2 + \frac{\beta}{\kappa_j-\beta}2^{1-\frac{\kappa_j}{\beta}}\right)\\ =& C a^{-\frac{\beta}{\kappa}} n^{1-\frac{\beta}{\kappa}} \text{ for some constant } C.\end{aligned}$$

And then lemma 2.2.4.1 gives us the result we want.
Now we can look at the case $\kappa = 1$.
Let ϕ be a positive concave function such that $\phi(t)$ goes to infinity when t goes to infinity. We define Φ by $\Phi(x) := \int_{t=0}^{x} \phi(t)\,\mathrm{d}t$. Let f be defined by $f(0) := \phi(0) > 0$ and $\forall x > 0,\ f(x) := \frac{\Phi(x)}{x}$, we clearly have that $f(x) \geq f(0)$ and we have for any $y > x > 0$:

$$\begin{aligned}f(y) =&\frac{1}{y}\int_{t=0}^{y} \phi(t)\mathrm{d}t\\ =&\frac{1}{y}\frac{y}{x}\int_{t=0}^{x} \phi\left(\frac{y}{x}t\right)\mathrm{d}t\\ \geq&\frac{1}{x}\int_{t=0}^{x} \phi(t)\,\mathrm{d}t\\ =&f(x).\end{aligned}$$

We get that for any positive integer M and any real $A \geq 2$:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<A} 1_{N_i^j \geq M} \geq an\log(n)\right) \\
\leq& \frac{1}{an\log(n)} \mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<A} 1_{N_i^j \geq M}\right) \\
\leq& \frac{1}{an\log(n)f(M)} \mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{n} N_i^j f(N_i^j) s_i^j 1_{s_i^j<A}\right) \\
\leq& \frac{1}{an\log(n)f(M)} \sum_{i=1}^{n} \int_{t=0}^{A} \mathbb{E}_{\mathbb{P}_0}\left(\Phi(N_i^j) 1_{s_i^j \geq t}\right) \mathrm{d}t \\
\leq& \frac{1}{an\log(n)f(M)} \sum_{i=1}^{n} \left(2\mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)) + \int_{t=2}^{A} \mathbb{E}_{\mathbb{P}_0}\left(\Phi(N_i^j) 1_{s_i^j \geq t}\right) \mathrm{d}t\right).
\end{aligned}$$

Now, by lemma 2.2.3.4 we get:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j<A} 1_{N_i^j \geq M} \geq an\log(n)\right) \\
\leq& \frac{1}{an\log(n)f(M)} \sum_{i=1}^{n} \left(2\mathbb{E}_{\mathbb{P}_0}(\Phi(N_i^j)) + \mathbb{E}_{\mathbb{P}_0}\left(\Phi(N_i^j)\right) \int_{t=2}^{A} ct^{-\kappa_j} \mathrm{d}t\right) \\
\leq& \frac{1}{an\log(n)f(M)} \sum_{i=1}^{n} \left(2 + c\int_{t=2}^{A} t^{-\kappa_j} \mathrm{d}t\right) E_{\mathbb{P}_0}(\Phi(N_i^j)) \\
\leq& \frac{dn}{an\log(n)f(M)} \left(2 + c\int_{t=2}^{A} t^{-\kappa_j} \mathrm{d}t\right) \text{ by lemma 2.2.3.3.}
\end{aligned} \tag{2.14}$$

If $\kappa_j = 1$, we get, by taking $A = n^2$ (for $n \geq 2$) in 2.14:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{s_i^j \leq A} 1_{N_i^j \geq M} \geq an\log(n)\right) \leq \frac{d}{a\log(n)f(M)}(2 + 2c\log(n)) \leq \frac{C}{af(M)}.$$

And by taking $A = n^2$ and $B = 1$ in equation 2.10 we have for some constant c:

$$\mathbb{P}_0\left(\exists i \leq n, s_i^j \geq n^2\right) \leq \frac{c}{n}.$$

So for any $\varepsilon > 0$ we get, by taking m_ε such that $f(m_\varepsilon) \geq \frac{1}{\varepsilon^3}$ and using lemma 2.2.4.1:

$$\begin{aligned}
\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{N_i^j \geq m_\varepsilon} \geq \varepsilon n\log(n)\right) \leq& 5\varepsilon + \mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j 1_{N_i^j \geq M} \geq \varepsilon^2 n\log(n)\right) \\
\leq& 5\varepsilon + \frac{c}{n} + C\varepsilon.
\end{aligned}$$

So there exists a constant C such that for any $\varepsilon > 0$ there exists m_ε such that:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j 1_{\ell_i^j \geq m_\varepsilon} \geq \varepsilon n\log(n)\right) \leq C\varepsilon.$$

If $\kappa_j > 1$, we take $M = 0$ and $A = \infty$ in 2.14 we get for some constant C:

$$\mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j \geq an\log(n)\right) \leq \frac{d}{a\log(n)f(0)} \left(2 + c\int_{t=2}^{\infty} t^{-\kappa_j} \mathrm{d}t\right) = \frac{C}{a\log(n)}.$$

And therefore by lemma 2.2.4.1, for any $\varepsilon > 0$

$$\begin{aligned}\mathbb{P}_0\left(\sum_{i=1}^{n} \ell_i^j \geq \varepsilon n \log(n)\right) \leq& 5\varepsilon + \mathbb{P}_0\left(\sum_{i=1}^{n} N_i^j s_i^j \geq \varepsilon^2 n \log(n)\right)\\ \leq& 5\varepsilon + \frac{C}{\varepsilon^2 \log(n)}.\end{aligned}$$

So we have the result we want $\square$

Now we have all the tools to get a first limit theorem on the time spent in traps.

Lemma 2.2.4.3. *Set $\alpha \in (0,\infty)^{2d}$ and let $\overline{\alpha} := \sum_{i=1}^{2d} \alpha_i$. Let $J = \{j \in [\![1,d]\!], 2\overline{\alpha} - \alpha_j - \alpha_{j+d} = \kappa\}$ and $\tilde{\mathscr{T}_j}$ be the set of vertices x such that there exists $j \in J$ such that either $(x, x+e_j) \in \mathscr{T}$ or $(x, x-e_j) \in \mathscr{T}$. Let $\{x_i^j, y_i^j\}$ be the i^{th} trap in the direction j encountered after time τ_2.*
For $\kappa < 1$, for any m there exists a constant C_m such that:

$$n^{-\frac{1}{\kappa}} \sum_{j \in J} \sum_{i \geq 0} \ell_i^j 1_{N_i^j \leq m} 1_{\exists k \leq \tau_{n+1}-1, Y_k \in \{x_i^j, y_i^j\}} \to C_m \mathscr{S}_1^{\kappa} \text{ in law for } \mathbb{P}_0.$$

For $\kappa = 1$, for any m there exists a constant $C_m > 0$ such that:

$$\frac{1}{n\log(n)} \sum_{j \in J} \sum_{i \geq 0} \ell_i^j 1_{N_i^j \leq m} 1_{\exists k \leq \tau_{n+1}-1, Y_k \in \{x_i^j, y_i^j\}} \to C_m \text{ in probability for } \mathbb{P}_0.$$

Proof. For every configuration $p \in \bigcup I_n$ let C_p be the expectation of the number of traps of configuration p encountered between times τ_2 and $\tau_3 - 1$ (it is also the expectation of the number of traps of configuration p encountered between times τ_i and $\tau_{i+1} - 1$ for any $i \geq 2$). We clearly have:

$$C_p \leq \mathbb{E}_{\mathbb{P}_0}\left(\sum_{x \in \mathbf{Z}^d} 1_{\exists i \in [\tau_2, \tau_3 - 1], Y_i = x}\right) < \infty.$$

Once we know that a trap is in a direction $j \in J$ and has a configuration p for some partially forgotten random walk, the exact number of back and forth the walk does in this trap is still random, because the exact number of back and forths knowing the transition probabilities of the trap is random and because the transition probabilities of the trap are still random, following the law (cf lemma 2.2.3.1):

$$C \frac{\varepsilon_x^{p_x} \varepsilon_y^{p_y}}{(\varepsilon_x + \varepsilon_y)^{p^s}} h(\varepsilon_x, \varepsilon_y) 1_{\varepsilon_x + \varepsilon_y \leq \frac{1}{2}},$$

where $\varepsilon_x := 1 - \omega(x,y)$, $\varepsilon_y := 1 - \omega(y,x)$ and the value of p_x, p_y, p^s are explicit but irrelevant, except for the fact that $p_x + p_y - p^s = \kappa - 2$. Let N be such that $p \in I_N$ (ie the walks exits the trap N times) we also have that there exists a constant C_α that only depends on α such that:

$$|\log(h(\varepsilon_x, \varepsilon_y))| \leq C_\alpha N(\varepsilon_x + \varepsilon_y).$$

Now if we make the change of variable $2r = \varepsilon_x + \varepsilon_y$, $k = \frac{\varepsilon_x - \varepsilon_y}{\varepsilon_x + \varepsilon_y}$, we get that the law of the transition probabilities becomes:

$$\begin{aligned}&2rCr^{p_x+p_y-p^s} \frac{(1+k)^{p_x}(1-k)^{p_y}}{(2r)^{p^s}} h(r(1+k), r(1-k)) 1_{r \leq \frac{1}{4}} \mathrm{d}r\mathrm{d}k\\ =&2^{1-p^s} C r^{\kappa-1} (1+k)^{p_x}(1-k)^{p_y} h(r(1+k), r(1-k)) 1_{r \leq \frac{1}{4}} \mathrm{d}r\mathrm{d}k.\end{aligned}$$

The number of back and forths is the sum of N iid geometric random variable $(H_1, \ldots, H_N)$ of parameter $q = 1 - \varepsilon_x - \varepsilon_y + \varepsilon_x\varepsilon_y = 1 - 2r + r^2(1-k^2)$. This gives us the following bound:

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{N} H_i \geq a|q\right) &\leq N\mathbb{P}\left(H_1 \geq \frac{a}{N}|q\right)\\ &\leq N(1-q)q^{\frac{a}{N}}\\ &\leq N\exp\left(\log(1-2r+r^2(1-k^2))\frac{a}{N}\right)\\ &\leq N\exp\left((-2r+r^2)\frac{a}{N}\right).\end{aligned}$$

For $r \in \left[\frac{2\kappa n \log(a)}{a}, \frac{1}{2}\right]$ we have $-2r + r^2 \leq -r$ and

$$\begin{aligned} N \exp\left((-2r + r^2)\frac{a}{N}\right) &\leq N \exp\left(-r\frac{a}{N}\right) \\ &\leq N \exp\left(-\frac{2\kappa N \log(a)}{a}\frac{a}{N}\right) \\ &= N a^{-2\kappa}. \end{aligned}$$

Now let ℓ^- be equal to twice the number of back-and-forths: $\ell^- := 2\sum_{i=1}^{N} H_i$. Now we look at $\mathbb{P}\left(\ell^- \geq a \text{ and } r \leq \frac{2\kappa N \log(a)}{a}\right)$, we want to show that it is equivalent to $Ca^{-\kappa}$ for some constant C. First we want to have a good approximation of $\mathbb{P}\left(2\sum_{i=1}^{N} H_i \geq a | q\right)$ for large q. Now let $\tilde{H}_1, \ldots, \tilde{H}_n$ be iid exponential random variables of parameter $-\log(q)$ such that for every i, $H_i = \lfloor \tilde{H}_i \rfloor$. And we define $\tilde{\ell}^- = 2\sum_{i=1}^{n} \tilde{H}_i$. Now it is easy to show by induction on n that:

$$\mathbb{P}_0\left(\tilde{\ell}^- \geq 2a | q\right) = \sum_{j=0}^{N-1} \frac{(-a\log(q))^j}{j!} \exp(\log(q)a).$$

Now we clearly have:

$$\ell^- \leq \tilde{\ell}^- \leq \ell^- + 2N$$

so

$$\mathbb{P}_0\left(\ell^- \geq a | q\right) \leq \mathbb{P}_0\left(\tilde{\ell}^- \geq a | q\right)$$

and

$$\mathbb{P}_0\left(\ell^- \geq a | q\right) \geq \mathbb{P}_0\left(\tilde{\ell}^- \geq a - 2N | q\right).$$

We want to show that $\mathbb{P}\left(\tilde{\ell}^- \geq a | q\right)$ and $\mathbb{P}\left(\tilde{\ell}^- \geq a - 2N | q\right)$ are more or less equal. We clearly have:

$$\mathbb{P}_0\left(\tilde{\ell}^- \geq a - 2N | q\right) \leq \mathbb{P}_0\left(\tilde{\ell}^- \geq a | q\right)$$

and we also have:

$$\begin{aligned} \mathbb{P}_0\left(\tilde{\ell}^- \geq 2a - 2N | q\right) &= \sum_{j=0}^{N-1} \frac{(-a\log(p))^j}{j!}\left(1 - \frac{N}{a}\right)^j \exp(\log(q)a)\exp(-\log(p)N) \\ &\geq \exp(-\log(q)N)\left(1 - \frac{N}{a}\right)^N \sum_{j=0}^{N-1} \frac{(-a\log(q))^j}{j!} \exp(\log(q)a). \end{aligned}$$

First we want to show that we can replace $\log(q)$ by $-2r$. We clearly have $\log(q) \leq -2r + r^2$. We also have $\log(q) \geq \log(1-2r)$ and for $r \in [0, \frac{1}{4}]$, there exists a constant C that does not depend on r such that $\log(1-2r) \geq -2r - Cr^2$. So we get:

$$2r - r^2 \leq -\log(q) \leq 2r + Cr^2.$$

So

$$\exp(-2ar)\exp(-Car^2) \leq \exp(a\log(q)) \leq \exp(-2ar)\exp(ar^2).$$

So we get:

$$\forall j, \quad \frac{(-a\log(q))^j}{j!} \exp(\log(q)a) \leq \frac{(2ar)^j}{j!} \exp(-2ar)\left(1 + \frac{Cr}{2}\right)^j \exp(ar^2)$$

and

$$\frac{(-a\log(q))^j}{j!} \exp(\log(q)a) \geq \frac{(2ar)^j}{j!} \exp(-2ar)\left(1 - \frac{r}{2}\right)^j \exp(-Car^2).$$

Now we will define $g^+(a,r)$ and $g^-(a,r)$ by:

$$\begin{aligned} g^+(a,r) &= \left(1 + \frac{Cr}{2}\right)^j \exp(ar^2)\exp\left(2rC_\alpha N\right) \\ g^-(a,r) &= \left(1 - \frac{r}{2}\right)^j \exp(-Car^2)\exp\left(-2rC_\alpha N\right)\exp(\left(2r - r^2\right)N)\left(1 - \frac{N}{a}\right)^N, \end{aligned}$$

where C is the same constant as in the previous inequality and C_α is the same as in 2.2.4. And for every $r \le \frac{1}{4}, k \in [-1,1]$ we have:

$$\frac{(-a\log(q))^j}{j!}\exp(\log(q)a)h(r(1-k),r(1+k)) \le \frac{(2ar)^j}{j!}\exp(-2ar)g^+(a,r)$$

and

$$\frac{(-a\log(q))^j}{j!}h(r(1-k),r(1+k))\exp(\log(q)a) \ge \frac{(2ar)^j}{j!}\exp(-2ar)g^-(a,r).$$

We clearly have that $g^+(a,r)$ is increasing in r while $g^-(a,r)$ is decreasing in r and $g^+(a,0)=1$ and $g^-(a,0) = \left(1-\frac{N}{a}\right)^N$.

So, for any $c>0$, we have the following 2 inequalities:

$$\begin{aligned}
&\mathbb{P}_0(\ell^- \ge 2a \text{ and } 1-q \le c)\\
\le&\mathbb{P}_0(\tilde{\ell}^- \ge 2a \text{ and } 1-q \le c)\\
\le&\mathbb{P}_0(\tilde{\ell}^- \ge 2a \text{ and } r \le c) \qquad \text{since } 1-q \ge 2r-r^2 \ge r\\
=&\int\limits_{r=0}^{c}\int\limits_{k=-1}^{1} 2^{1-p^s}Cr^{\kappa-1}(1+k)^{p_x}(1-k)^{p_y}h(r(1+k),r(1-k))\mathbb{P}_0(\tilde{\ell}^- \ge 2a|q)\mathrm{d}k\mathrm{d}r\\
\le&\int\limits_{r=0}^{c}\int\limits_{k=-1}^{1} 2^{1-p^s}Cr^{\kappa-1}(1+k)^{p_x}(1-k)^{p_y}\sum_{j=0}^{N-1}\frac{(2ar)^j}{j!}\exp(-2ar)g^+(a,r)\mathrm{d}k\mathrm{d}r\\
\le&g^+(a,c)\int\limits_{k=-1}^{1}(1+k)^{p_x}(1-k)^{p_y}\mathrm{d}k\int\limits_{r=0}^{c}2^{1-p^s}Cr^{\kappa-1}\sum_{j=0}^{N-1}\frac{(2ar)^j}{j!}\exp(-2ar)\mathrm{d}r,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}_0(\ell^- \ge 2a \text{ and } 1-q \le c)\\
\ge&\mathbb{P}_0(\tilde{\ell}^- \ge 2a-2N \text{ and } 1-q \le c)\\
\ge&\mathbb{P}_0(\tilde{\ell}^- \ge 2a-2N \text{ and } 2r \le c) \qquad \text{since } 1-q \le 2r\\
=&\int\limits_{r=0}^{\frac{c}{2}}\int\limits_{k=-1}^{1} 2^{1-p^s}Cr^{\kappa-1}(1+k)^{p_x}(1-k)^{p_y}h(r(1+k),r(1-k))\mathbb{P}_0(\tilde{\ell}^- \ge 2a-2N|q)\mathrm{d}k\mathrm{d}r\\
\ge&\int\limits_{r=0}^{\frac{c}{2}}\int\limits_{k=-1}^{1} 2^{1-p^s}Cr^{\kappa-1}(1+k)^{p_x}(1-k)^{p_y}\sum_{j=0}^{N-1}\frac{(a2r)^j}{j!}\exp(-2ar)g^-(a,r)\mathrm{d}k\mathrm{d}r\\
\ge&g^-\left(a,\frac{c}{2}\right)\int\limits_{k=-1}^{1}(1+k)^{p_x}(1-k)^{p_y}\mathrm{d}k\int\limits_{r=0}^{\frac{c}{2}}2^{1-p^s}Cr^{\kappa-1}\sum_{j=0}^{N-1}\frac{(a2r)^j}{j!}\exp(-2ar)\mathrm{d}r.
\end{aligned}$$

If we take $c=a^{-\frac{3}{4}}$ we clearly get when $a\to\infty$, $g^-(a,a^{-\frac{3}{4}})\to 1$ and $g^+(a,a^{-\frac{3}{4}})\to 1$. Furthermore, for any constant c':

$$\begin{aligned}
&\int\limits_{r=0}^{c'a^{-\frac{3}{4}}}2^{1-p^s}Cr^{\kappa-1}\sum_{j=0}^{N-1}\frac{(a2r)^j}{j!}\exp(-2ar)\mathrm{d}r\\
=&(2a)^{-\kappa}\int\limits_{r=0}^{2c'a^{\frac{1}{4}}}2^{1-p^s}Cr^{\kappa-1}\sum_{j=0}^{N-1}\frac{r^j}{j!}\exp(-r)\mathrm{d}r\\
\sim&(2a)^{-\kappa}\sum_{j=0}^{N-1}\frac{\Gamma(j+1)}{j!}\\
=&(2a)^{-\kappa}N.
\end{aligned}$$

Therefore we get:

$$\mathbb{P}_0(\ell^- \ge 2a \text{ and } 1-q \le a^{\frac{3}{4}}) \sim N\left(\int\limits_{k=-1}^{1}(1+k)^{p_x}(1-k)^{p_y}\mathrm{d}k\right)2^{1-p^s}C(2a)^{-\kappa}.$$

So there exist a constant C that only depends on α such that:

$$\mathbb{P}_0(\ell^- \geq 2a \text{ and } 1-q \leq a^{-\frac{3}{4}}) \sim CNa^{-\kappa}.$$

So we get for some constant C':

$$\mathbb{P}_0(\ell^- \geq a) \sim C'Na^{-\kappa}.$$

Now let ℓ be the total time spent in the trap. It is equal to ℓ^- plus the number of time the walk enters and exits the trap by the same vertex plus twice the number of times the walk enters and exits the trap by different vertices. This means there exists a constant δ_p that only depends on the configuration such that $\ell = \ell^- + \delta_p$. This, in turn, means that we have also the asymptotic equality:

$$\mathbb{P}_0(\ell \geq a) \sim C'Na^{-\kappa}.$$

Now, let ℓ_i^p be the time spent in the i^{th} trap with configuration p.
First, if $\kappa < 1$, by Theorem 3.7.2 of [36] we get that for some constant c_p:

$$n^{-\frac{1}{\kappa}} \sum_{i=1}^{n} \ell_i^p \to c_p \mathscr{S}_1^\kappa \text{ in law for } \mathbb{P}_0.$$

Now we use the fact that the number of trap of configuration p between two renewal times has a finite expectation C_p to show that we have the convergence we want. Let $M_{n,p}$ be the number of traps of configuration p the walk has entered before the n^{th} renewal time. For any $\varepsilon > 0$ and any p we have:

$$\mathbb{P}_0(M_{n,p} \in [(C_p-\varepsilon)n, (C_p+\varepsilon)n]) \to 1.$$

Therefore for any configuration p:

$$n^{-\frac{1}{\kappa}} \sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n} \ell_i^p \to (2\varepsilon)^{\frac{1}{\kappa}} c_p S_\kappa \text{ in law for } \mathbb{P}_0.$$

And for any $m \in \mathbf{N}$:

$$n^{-\frac{1}{\kappa}} \sum_{p\in I^m} \sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n} \ell_i^p \to (2\varepsilon)^{\frac{1}{\kappa}} \left(\sum_{p\in I^m} (c_p)^\kappa \right)^{\frac{1}{\kappa}} S_\kappa \text{ in law for } \mathbb{P}_0.$$

We write $I^m(J)$ all the configuration of I^m that are in a direction $j \in J$. Now, using the fact that the ℓ_p^i are non negative, for any $n \in \mathbf{N}$ and any $\varepsilon > 0$ small enough, we have:

$$\begin{aligned}
&\mathbb{P}_0\left(n^{-\frac{1}{\kappa}} \left| \sum_{p\in I^m(J)} \sum_{i=1}^{M_{n,p}} \ell_i^p - \sum_p \sum_{i=1}^{C_p n} \ell_p^i \right| \geq \eta \right) \\
\leq &\mathbb{P}_0(\exists p \in I^m(J), N_{n,p} \notin [(C_p-\varepsilon)n, (C_p+\varepsilon)n]) + \mathbb{P}\left(n^{-\frac{1}{\kappa}} \sum_{p\in I^m(J)} \sum_{i=(C_p-\varepsilon)n}^{(C_p+\varepsilon)n} \ell_i^p \geq \eta \right) \\
= &o(1) + \mathbb{P}_0\left((2\varepsilon)^{\frac{1}{\kappa}} \left(\sum_{p\in I^m(J)} (c_p)^\kappa \right)^{\frac{1}{\kappa}} \mathscr{S}_1^\kappa \geq \eta \right).
\end{aligned}$$

Since it is true for all ε, we get that

$$n^{-\frac{1}{\kappa}} \left| \sum_{p\in I^m(J)} \sum_{i=1}^{M_{n,p}} \ell_i^p - \sum_{p\in I^m(J)} \sum_{i=1}^{C_p n} \ell_p^i \right| \to 0 \text{ in probability for } \mathbb{P}_0.$$

And since

$$n^{-\frac{1}{\kappa}} \sum_{p\in I^m(J)} \sum_{i=1}^{C_p n} \ell_i^p \to \left(\sum_{p\in I^m(J)} (c_p)^\kappa \right)^{\frac{1}{\kappa}} \mathscr{S}_1^\kappa \text{ in probability for } \mathbb{P}_0,$$

we get:

$$n^{-\frac{1}{\kappa}} \sum_{p\in I^m(J)} \sum_{i=1}^{M_{n,p}} \ell_i^p \to \left(\sum_{p\in I^m(J)} (c_p)^\kappa \right)^{\frac{1}{\kappa}} \mathscr{S}_1^\kappa \text{ in law for } \mathbb{P}_0$$

Now if $\kappa = 1$, we first want to show that we can neglect the values larger than $n\log(n)$. Let p be a configuration, ℓ_i^p the total time spent in the i^{th} trap in the configuration p encountered, C_p the constant such that the number of trap encountered before time $\tau_{n+1}-1$ is equivalent to $C_p n$, $M_{n,p}$ the number of traps in the configuration p encountered before the time $\tau_{n+1}-1$ and c_p the constant such that $\mathbb{P}_0(\ell_i^p \geq t) \sim c_p n^{-1}$. We get:

$$\begin{aligned}\mathbb{P}_0(\exists i \leq M_{n,p}, \ell_i^p \geq n\log(n)) \leq& \mathbb{P}_0(\exists i \leq 2C_p n, \ell_i^p \geq n\log(n)) + \mathbb{P}_0(M_{n,p} \geq 2C_p n)\\ \leq& 2C_p n \frac{c_p}{n\log(n)} + o(1).\\ =& o(1)\end{aligned}$$

Now we can compute the expectation and variance of $\ell_i^p \wedge n\log(n)$:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_0}(\ell_i^p \wedge n\log(n)) \sim& \int_{t=1}^{n\log(n)} \frac{c_p}{t}\mathrm{d}t\\ \sim& c_p\log(n).\end{aligned}$$

Now for the variance we get:

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_0}(\ell_i^p \wedge n\log(n)) \leq& \mathbb{E}_{\mathbb{P}_0}((\ell_i^p \wedge n\log(n))^2)\\ \sim& \int_{t=1}^{n\log(n)} 2t\frac{c_p}{t}\mathrm{d}t\\ \sim& 2c_p n\log(n).\end{aligned}$$

So for n large enough:

$$\mathrm{Var}_{\mathbb{P}_0}(\ell_i^p \wedge n\log(n)) \leq 4c_p n\log(n).$$

First, for any constant c, for n big enough:

$$\begin{aligned}&\mathbb{P}_0\left(\left|\sum_{i=1}^{cn} \ell_i^p \wedge n\log(n) - cnc_p\log(n)\right| \geq \varepsilon n\log(n)\right)\\ \leq& \mathbb{P}_0\left(\left|\sum_{i=1}^{cn} \ell_i^p \wedge n\log(n) - cn\mathbb{E}\left(\ell_1^p \wedge n\log(n)\right)\right| \geq \frac{1}{2}\varepsilon n\log(n)\right) \text{ for } n \text{ big enough, by 2.2.4}\\ \leq& cn\frac{4\mathrm{Var}_{\mathbb{P}_0}\left(\ell_1^p \wedge n\log(n)\right)}{(\varepsilon n\log(n))^2}\\ \leq& cn\frac{16c_p n\log(n)}{(\varepsilon n\log(n))^2}\\ =& \frac{16cc_p}{\log(n)} = o(1)\end{aligned}$$

This means that we have the following results:

$$\mathbb{P}_0\left(\sum_{i=1}^{(C_p+\varepsilon)n} \ell_i^p \wedge n\log(n) - (C_p+\varepsilon)nc_p\log(n) \geq \varepsilon n\log(n)\right) \to 0$$

and

$$\mathbb{P}_0\left(\sum_{i=1}^{(C_p-\varepsilon)n} \ell_i^p \wedge n\log(n) - (C_p+\varepsilon)nc_p\log(n) \leq -\varepsilon n\log(n)\right) \to 0$$

Then, by definition of C_p we get, for any $\varepsilon \geq 0$:

$$\mathbb{P}_0\left(|M(n,p) - C_p n| \geq \varepsilon n\right) \to 0.$$

Then, using the fact that $\sum_{i=1}^{n} \ell_i^p \wedge a$ is increasing in n for any a, we get:

$$\begin{aligned}&\mathbb{P}_0\left(\sum_{i=1}^{M(n,p)} \ell_i^p \geq (C_p+\varepsilon)(c_p+\varepsilon)n\log(n)\right)\\ \leq& \mathbb{P}_0(M(n,p) \geq (C_p+\varepsilon)n) + \mathbb{P}_0\left(\sum_{i=1}^{(C_p+\varepsilon)n} \ell_i^p \geq (C_p+\varepsilon)(c_p+\varepsilon)n\log(n)\right)\\ =& o(1).\end{aligned}$$

Similarly, we have:

$$\begin{aligned}&\mathbb{P}_0\left(\sum_{i=1}^{M(n,p)}\ell_i^p \le (C_p-\varepsilon)(c_p-\varepsilon)n\log(n)\right)\\ \le&\mathbb{P}_0(M(n,p)\ge (C_p-\varepsilon)n)+\mathbb{P}_0\left(\sum_{i=1}^{(C_p-\varepsilon)n}\ell_i^p \ge (C_p-\varepsilon)(c_p-\varepsilon)n\log(n)\right)\\ =&o(1).\end{aligned}$$

Therefore,

$$\frac{1}{n\log(n)}\sum_{i=1}^{M(n,p)}\ell_i^p \to C_p c_p \text{ in probability for } \mathbb{P}_0.$$

Now we just have to sum on all configurations $p\in I^m$ that are in a direction $j\in J$ to get the result we want.

□

2.2.5 Only the time spent in traps matter

Now to properly show the result we want, we have to show that some quantities and some events are negligible, this is what this section is devoted to.

Lemma 2.2.5.1. *Let j be in $[\![1,d]\!]$. Let $\{x_i^j,y_i^j\}$ be the i^{th} trap visited by the walk in the direction j after time τ_2, s_i^j its strength, ℓ_i^j the time spent in this trap and N_i^j the number of times the walk enters the trap:*

$$\begin{aligned}\ell_i^j&=\sum_{k\ge 0}1_{Y_k\in\{x_i^j,y_i^j\}},\\ N_i^j&=\sum_{k\ge 0}1_{Y_k\in\{x_i,y_i\} \text{ and } Y_{k+1}\notin\{x_i^j,y_i^j\}}.\end{aligned}$$

Let $\kappa_j=2\sum_{i=1}^{2d}\alpha_i-\alpha_j-\alpha_{j+d}\ge\kappa$. Let $M(n,j)$ be the number of traps in the direction j encountered between times τ_2 and τ_n-1.
If $\kappa<1$ and $\kappa_j=\kappa$, for any $\varepsilon>0$ there exists $\varepsilon'>0$ such that for n large enough:

$$\mathbb{P}_0\left(\sum_{i=1}^{M(n,j)}\ell_i^j 1_{s_i^j\le\varepsilon' n^{\frac{1}{\kappa}}}\ge\varepsilon n^{\frac{1}{\kappa}}\right)\le\varepsilon.$$

Proof. Let $\gamma\in\left(\kappa,\frac{\kappa+\kappa'}{2}\right)$ be such that $\gamma\le 1$. Let β be a positive real. Let $\{\overline{x}_i^j,\overline{y}_i^j\}$ be the i^{th} trap visited by the walk in the direction j after time τ_2 such that $\{\overline{x}_i^j.e_1,\overline{y}_i^j.e_1\ge Y_{\tau_2}.e_1$. Let $\overline{s}_i^j$ be its strength $\overline{\ell}_i^j$ the time spent in this trap and $\overline{N}_i^j$ the number of times the trap is visited.
By lemma 2.2.1.2 the number of traps encountered between 2 renewal times has a finite expectation and since the $(M(2i+1,j)-M(2i,j))_{i\in\mathbf{N}^*}$ are iid and so are the $(M(2i+2,j)-M(2i+1,j))_{i\in\mathbf{N}^*}$, there exists a constant C_j such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}M(n,j)\to C_j.$$

So for any $\varepsilon>0$, for n large enough:

$$\mathbb{P}_0(M(n,j)\ge 2C_jn)\le\frac{\varepsilon}{4}.$$

We have for n large enough:

$$\begin{aligned}
&\mathbb{P}_0\left(\sum_{i=1}^{M(n,j)} \ell_i^j 1_{s_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \varepsilon n^{\frac{1}{\kappa}}\right)\\
\le&\mathbb{P}_0\left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)+\mathbb{P}_0\left(\sum_{i=1}^{M(3,j)} \ell_i^j 1_{s_i\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)\\
\le&\mathbb{P}_0\left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)+\mathbb{P}_0\left(\tau_3 \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)\\
\le&\mathbb{P}_0\left(\sum_{i=1}^{M(n,j)} \overline{\ell}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)+\frac{\varepsilon}{4} \text{ for } n \text{ large enough}\\
\le&\mathbb{P}_0\left(\sum_{i=1}^{2C_jn} \overline{\ell}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)+2\frac{\varepsilon}{4} \text{ for } n \text{ large enough .}
\end{aligned}$$

Then by lemma 2.2.4.1 we have:

$$\mathbb{P}_0\left(\sum_{i=1}^{2C_jn} \overline{\ell}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{1}{2}\varepsilon n^{\frac{1}{\kappa}}\right)\le \frac{\varepsilon}{4}+\mathbb{P}_0\left(\sum_{i=1}^{2C_jn} \overline{N}_i^j\overline{s}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right).$$

And finally we have:

$$\begin{aligned}
\mathbb{P}_0\left(\sum_{i=1}^{2C_jn} \overline{N}_i^j\overline{s}_i^j 1_{\overline{s}_i^j\le \varepsilon' n^{\frac{1}{\kappa}}} \ge \frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right) \le&\mathbb{P}_0\left(\sum_{i=1}^{2C_jn} (\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j\le \beta n^{\frac{1}{\kappa}}} \ge \left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right)^\gamma\right)\\
\le&\left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right)^{-\gamma}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{i=1}^{2C_jn} (\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j\le \beta n^{\frac{1}{\kappa}}}\right)\\
=&\left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right)^{-\gamma}\sum_{i=1}^{2C_jn}\mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i^j\le \beta n^{\frac{1}{\kappa}}}\right).
\end{aligned}$$

Then by lemma 2.2.3.4 we get, for some constant c that does not depend on β:

$$\begin{aligned}
\left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right)^{-\gamma}\sum_{i=1}^{2C_jn}\mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma(\overline{s}_i^j)^\gamma 1_{\overline{s}_i\le \beta n^{\frac{1}{\kappa}}}\right) \le& c\left(\frac{\varepsilon^2}{40} n^{\frac{1}{\kappa}}\right)^{-\gamma}\sum_{i=1}^{2C_jn}\mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right)\left(\varepsilon' n^{\frac{1}{\kappa}}\right)^{\gamma-\kappa}\\
=&c\left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa} n^{-1}\sum_{i=1}^{2C_jn}\mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right).
\end{aligned}$$

And by lemma 2.2.3.3 there exists a constant c that does not depend on β such that:

$$\left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa} n^{-1}\sum_{i=1}^{2C_jn}\mathbb{E}_{\mathbb{P}_0}\left((\overline{N}_i^j)^\gamma\right)\le c\left(\frac{\varepsilon^2}{40}\right)^{-\gamma}(\varepsilon')^{\gamma-\kappa}.$$

So by taking β small enough we get the result we wanted. □

Lemma 2.2.5.2. *Let $J=\{j\in [\![1,d]\!], \kappa_j>\kappa\}$.*
If $\kappa=1$ there exists a constant C such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{i=0}^{\tau_n-1} 1_{Y_i\in\tilde{\mathscr{T}}_J}\to C.$$

If $\kappa<1$ there exists a constant $C>0$ and a constant $\gamma\in(\kappa,1]$ such that $\mathbb{P}_0$ almost surely:

$$\limsup n^{-\frac{1}{\gamma}}\sum_{k=0}^{\tau_n-1} 1_{Y_k\in\tilde{\mathscr{T}}_J}\le C.$$

Proof. For any $j \in J$ we define $\kappa_j = 2\sum_{i=1}^{2d} \alpha_i - \alpha_j - \alpha_{j+d} > \kappa$. Let $\{x_i^j, y_i^j\}$ be the i^{th} trap in the direction j the walk enters after time τ_2 and such that $x_i^j.e_1, y_i^j.e_1 \geq Y_{\tau_2}.e_1$. Let N_i^j be the number of times the walk exits $\{x_i^j, y_i^j\}$ and ℓ_i^j the time the walk spends in this trap. Let $M(i,j)$ be the number of traps in the direction j entered before time τ_i. The $(M(2i+2,j)-M(2i+1,j))_{i\in\mathbf{N}^*}$ are iid and so are the $(M(2i+1,j)-M(2i,j))_{i\in\mathbf{N}^*}$, they also all have the same law (the only issue is that since a trap span over two vertices, there might be a slight overlap between traps of two different 'renewal slabs'). Now, since the number of different vertices the walk encounters between two renewal times has a finite expectation, the $(M(i+1,j)-M(i,j))$ have a finite expectation and therefore there exists a constant C_j such that $\mathbb{P}_0$ almost surely:

$$M(n,j) - C_j n \to -\infty.$$

Now let $\tilde{Y}$ be the partially forgotten walk associated with Y. We get that knowing the environment, the partially forgotten walk and the renewal position Y_{τ_2} the time spend in the $\{x_i^j, y_i^j\}$, the k^{th} time the walk enters this trap is equal to $\varepsilon_{i,k}^j + 2H_{i,k}^j$ where $\varepsilon_{i,k}^j$ is 1 if the walk enters the trap by the same vertex it leaves it and 2 otherwise and $H_{i,k}^j$ is a geometric random variable that counts the number of back and forths. The parameter of $H_{i,k}^j$ is $p_i^j := \omega(x_i^j, y_i^j)\omega(y_i^j, x_i^j)$.

First, lets look at the case $\kappa = 1$. Since the $\left(\sum_{j=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i\in\tilde{\mathcal{T}}_J}\right)_{i\in\mathbf{N}^*}$ are iid and so are the $\left(\sum_{j=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i\in\tilde{\mathcal{T}}_J}\right)_{i\in\mathbf{N}^*}$, we just have to prove that their expectation is not infinite to have the result we want. If their expectation were infinite, then we would have that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{M(n,j)} \ell_i^j \to \infty.$$

Therefore we would have $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n} \ell_i^j \to \infty.$$

But

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}_0^\omega}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n} \ell_i^j|\tilde{Y}\right) =& \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j} \mathbb{E}_{\mathrm{P}_0^\omega}\left(\varepsilon_{i,k}^j + 2H_{i,k}^j|\tilde{Y}\right) \\
=& \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j}\left(\varepsilon_{i,k}^j + 2\frac{p_i^j}{1-p_i^j}\right) \\
\leq& 2\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\sum_{k=1}^{N_i^j}\frac{1}{1-p_i^j} \\
\leq& C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j} N_i^j s_i^j,
\end{aligned}$$

where s_i^j is the strength of the trap $\{x_i^j, y_i^j\}$. Now we get:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_0}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n} \ell_i^j\right) \leq& \frac{1}{n}\mathbb{E}_{\mathbb{P}_0}\left(C\sum_{j\in J}\sum_{i=1}^{nC_j} N_i^j s_i^j\right) \\
=& C\frac{1}{n}\mathbb{E}_{\mathbb{P}_0}\left(\sum_{j\in J}\sum_{i=1}^{nC_j} N_i^j \int_{t=0}^{\infty} 1_{(s_i^j)\geq t}\mathrm{d}t\right) \\
\leq& C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\mathbb{E}_{\mathbb{P}_0}\left(N_i^j\left(2 + \int_{t=2}^{\infty} 1_{s_i^j\geq t}\mathrm{d}t\right)\right) \\
\leq& C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\left(2\mathbb{E}_{\mathbb{P}_0}(N_i^j) + C\int_{t=2}^{\infty}\mathbb{E}_{\mathbb{P}_0}(N_i^j 1_{s_i^j\geq t})\mathrm{d}t\right).
\end{aligned}$$

Now by lemma 2.2.3.4 we know that there exists a constant C such that for any $t \geq 2$:

$$\mathbb{E}_{\mathbb{P}_0}(N_i^j 1_{s_i^j \geq t}) \leq C t^{-\kappa_j} \mathbb{E}_{\mathbb{P}_0}(N_i^j).$$

So there exists a constant C' (the value of this constant will change depending on the line) such that:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_0}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\ell_i^j\right) \leq & C'\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{nC_j}\mathbb{E}_{\mathbb{P}_0}(N_i^j) \\
\leq & C'\sum_{j\in J} C_j \quad \text{by lemma 2.2.3.3} \\
\leq & C'.
\end{aligned}$$

This means that we cannot have $\frac{1}{n}\sum\limits_{j\in J}\sum\limits_{i=1}^{C_j n} N_i^j \to \infty$ $\mathbb{P}_0$ almost surely. Therefore the random variables $\left(\sum\limits_{j=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i\in\tilde{\mathcal{T}}_J}\right)_{i\in\mathbf{N}^*}$ have finite expectation and so have the random variables $\left(\sum\limits_{j=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i\in\tilde{\mathcal{T}}_J}\right)_{i\in\mathbf{N}^*}$. So we have the result we want.

If $\kappa < 1$, we will basically use the same method. First there exists $\gamma \in (\kappa, 1]$ such that $\gamma < \frac{\kappa+\kappa'}{2}$ and for every $j \in J$, $\gamma < \kappa_j$.

We have that:

$$\limsup n^{-\frac{1}{\gamma}}\sum_{k=0}^{\tau_n-1} 1_{Y_k\in\tilde{\mathcal{T}}_J} = \limsup n^{-\frac{1}{\gamma}}\sum_{i=2}^{n-1}\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}.$$

And since:

$$\left(n^{-\frac{1}{\gamma}}\sum_{i=2}^{n}\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma} \leq \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{n}\left(\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma}$$

we also have:

$$\limsup n^{-\frac{1}{\gamma}}\sum_{i=2}^{n}\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J} \leq \left(\limsup \frac{1}{n}\sum_{i=2}^{n}\left(\sum_{k=\tau_i}^{\tau_{i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma}\right)^{\frac{1}{\gamma}}.$$

Now, since the random variables $\left(\left(\sum\limits_{k=\tau_{2i}}^{\tau_{2i}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma}\right)_{i\in\mathbf{N}^*}$ are iid and so are the random variables $\left(\left(\sum\limits_{k=\tau_{2i+1}}^{\tau_{2i+1}-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma}\right)_{i\in\mathbf{N}^*}$ we have that there exists a constant $C_\infty \in [0,\infty]$ such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{i=2}^{n}\left(\sum_{k=\tau_i}^{\tau_i-1} 1_{Y_k\in\tilde{\mathcal{T}}_J}\right)^{\gamma} \to C_\infty.$$

Now, by definition of the C_j and since $(a+b)^\gamma \leq a^\gamma + b^\gamma$ we have that if $C_\infty = \infty$ then $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\left(N_i^j\right)^{\gamma} \to \infty.$$

However we have (using the same techniques and notations as in the case $\kappa = 1$):

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}_0^\omega}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}(\ell_i^j)^\gamma|\tilde{Y}\right) = & \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\mathbb{E}_{\mathrm{P}_0^\omega}\left(\left(\sum_{k=1}^{N_i^j}\varepsilon_i^j + 2H_{i,k}^j\right)^{\gamma}|\tilde{Y}\right) \\
\leq & \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\mathbb{E}_{\mathrm{P}_0^\omega}\left(\sum_{k=1}^{N_i^j}\varepsilon_i^j + 2H_{i,k}^j|\tilde{Y}\right)^{\gamma} \\
\leq & \frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\left(N_i^j\frac{2}{p_i^j}\right)^{\gamma} \\
\leq & C\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}\left(N_i^j s_i^j\right)^{\gamma}.
\end{aligned}$$

Now by the same method as the one for $\kappa = 1$, by using lemma 2.2.3.4 and lemma 2.2.3.3 we get:

$$\mathbb{E}_{\mathbb{P}_0}\left(\frac{1}{n}\sum_{j\in J}\sum_{i=1}^{C_j n}(\ell_i^j)^\gamma\right) \le C.$$

This means that $C_\infty < \infty$ and therefore:

$$\limsup n^{-\frac{1}{\gamma}} \sum_{k=0}^{\tau_n - 1} 1_{Y_k \in \tilde{\mathscr{T}}_J} \le (C_\infty)^{\frac{1}{\gamma}} < \infty.$$

□

Lemma 2.2.5.3. *Let $A_{\varepsilon,n}^{i_1,i_2}(i)$ be the event that the walk visits at least two trap of strength at least $\varepsilon n^{\frac{1}{\kappa}}$ between times τ_i and $\tau_{i+i_1} - 1$ and that it enters these traps at most i_2 times. We have that for any $i_1 \ge 1$:*

$$\mathbb{P}_0\left(\bigcup_{2\le i\le n} A_{\varepsilon,n}^{i_1,i_2}(i)\right) \to 0.$$

Proof. Let $\overline{\alpha} := \sum_{i=1}^{2d} \alpha_i$. Let $M(i)$ be the number of traps visited before time τ_i. We know by lemma 2.2.1.2 that the number $M(i+i_1) - M(i)$ of traps visited between times τ_i and $\tau_{i+i_1} - 1$ has a finite expectation (for $\mathbb{P}_0$) and by proposition 2.1.2.1 the $((M(2i+2) - M(2i+1))_{i\ge 1}$ are iid and so are the $(M(2i+1) - M(2i))_{i\ge 1}$. This means that there is a positive constant C such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}M(n) \to C.$$

Now let $M^{i_2}(i)$ be the number of traps visited at most i_2 times before time τ_i. We know that:

$$\mathbb{P}_0(M^{i_2}(n+i_1) \ge 2Cn) \to 0.$$

Now, for any $\eta > 0$ we have:

$$\begin{aligned}
\mathbb{P}_0\left(\exists i \le n, M(i+i_1) - M(i) \ge \eta n\right) \le & \sum_{i\le n} \mathbb{P}_0\left(M(i+i_1) - M(i) \ge \eta n\right) \\
= & o(1) + \sum_{2\le i\le n} \mathbb{P}_0\left(M(i+i_1) - M(i) \ge \eta n\right) \\
= & o(1) + (n-1)\mathbb{P}_0\left(M(i+i_1) - M(i) \ge \eta n\right) \\
= & o(1) \text{ since } M(i+i_1) - M(i) \text{ has a finite expectation.}
\end{aligned}$$

Now let A_i be the event "the i^{th} trap visited by the walk is of strength at least $\varepsilon n^{\frac{1}{\kappa}}$ and that the walk enters this trap at most i_2 times". We have:

$$\begin{aligned}
& \mathbb{P}_0\left(\exists i \le 2Cn, \exists j \le \eta n, A_i \text{ and } A_{i+j}\right) \\
\le & \mathbb{P}_0\left(\exists i \le \frac{2C}{\eta}, \exists j_1, j_2 \in [\![i\eta n, i\eta n + 2\eta n]\!], j_1 \ne j_2 \text{ and } A_{j_1} \text{ and } A_{j_2}\right) \\
\le & \sum_{i=0}^{\frac{2C}{\eta}} \mathbb{P}_0\left(\exists j_1, j_2 \in [\![i\eta n, i\eta n + 2\eta n]\!], j_1 \ne j_2 \text{ and } A_{j_1} \text{ and } A_{j_2}\right) \\
\le & \sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1 = i\eta n}^{i\eta n + 2\eta n} \sum_{j_2 = i\eta n}^{i\eta n + 2\eta n} \mathbb{P}_0\left(A_{j_1} \text{ and } A_{j_2}\right) 1_{j_1 \ne j_2}.
\end{aligned}$$

Now let $(\tilde{Y}_n)_{n\in\mathbf{N}}$ be the partially forgotten walk, by lemma 2.2.3.1 if s_j is the strength of the j^{th} trap visited and N_j is the number of times the walk enters the j^{th} trap, there exists a constant D_j that only depends on its configuration such that for any $B > 2$,

$$\mathbb{P}_0\left(s_j \ge B | \tilde{Y}, \tilde{\omega}\right) \le D_j B^{-\kappa} \exp\left(\frac{5(N_i + 2\overline{\alpha})}{B}\right).$$

Let D^{i_2} be the maximum value of $D_j \exp\left(\frac{5(Z_i+2\overline{\alpha})}{2}\right)$ we can get for configuration of traps entered at most i_2 times. We get that for any j:

$$\mathbb{P}_0(s_j \geq B \text{ and } N_j \leq i_2|\tilde{Y},\tilde{\omega}) \leq D^{i_2}B^{-\kappa}.$$

We also know that the strength of the traps are independent, knowing the partially forgotten walk and the equivalence class of the environment for the trap-equivalent relation. Therefore we have, for any $\eta > 0$:

$$\begin{aligned}
&\sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1=i\eta n}^{i\eta n+2\eta n} \sum_{j_2=i\eta n}^{i\eta n+2\eta n} \mathbb{P}_0\left(A_{j_1} \text{ and } A_{j_2}\right) 1_{j_1\neq j_2}\\
\leq&\sum_{i=0}^{\frac{2C}{\eta}} \sum_{j_1=i\eta n}^{i\eta n+2\eta n} \sum_{j_2=i\eta n}^{i\eta n+2\eta n} (D^{i_2})^2\left(\varepsilon n^{\frac{1}{\kappa}}\right)^{-2\kappa}\\
\leq&2\frac{2C}{\eta}(\eta n)^2(D^{i_2})^2\varepsilon^{-2\kappa}n^{-2} \text{ for } \eta \text{ small enough}\\
=&4C\eta(D^{i_2})^2\varepsilon^{-2\kappa}.
\end{aligned}$$

Now, by taking a sequence $(\eta_n)_{n\in\mathbf{N}^*}$ of positive reals such that $\eta_n \to 0$ and such that:

$$\mathbb{P}_0\left(\exists i \leq n, M(i+i_1) - M(i) \geq \eta_n n\right) \to 0,$$

we get:

$$\begin{aligned}
\mathbb{P}_0\left(\bigcup_{2\leq i\leq n} A_{\varepsilon,n}^{i_1,i_2}(i)\right) \leq&\mathbb{P}_0\left(M(n+i_1) \leq 2Cn\right) \text{ or } \left(\exists i \leq n, M(i+i_1) - M(i) \geq \eta_n n\right)\\
&+\mathbb{P}_0\left(\exists i \leq 2Cn, \exists j \leq \eta_n n, A_i + \mathbb{P}_0 A_{i+j}\right).
\end{aligned}$$

Therefore:

$$\mathbb{P}_0\left(\bigcup_{2\leq i\leq n} A_{\varepsilon,n}^{i_1,i_2}(i)\right) \to 0$$

□

Lemma 2.2.5.4. *If $\kappa = 1$ there exists a constant C such that $\mathbb{P}_0$ almost surely:*

$$\frac{1}{n}\sum_{i=0}^{\tau_n-1} 1_{Y_i\notin\tilde{\mathscr{T}}} \to C.$$

If $\kappa < 1$, there exists a constant $C > 0$ and a constant $\beta < \frac{1}{\kappa}$ such that $\mathbb{P}_0$ almost surely, for n large enough:

$$\sum_{x\in\mathbf{Z}^d}\sum_{i=0}^{\tau_n-1} 1_{Y_i=x}1_{x\notin\tilde{\mathscr{T}}} \leq Cn^{\beta}.$$

Proof. Let m be such that $\mathbb{Q}^m$ is well defined. Let $(t_i^m)_{i\in\mathbf{N}}$ be the times at which X^m changes position, with $t_0 := 0$. We have $X^m_{t_i^m} = Y_i$ for all $i \in \mathbf{N}$. Let $(\mathscr{E}_i)_{i\in\mathbf{N}}$ be a sequence of random variables defined by $\mathscr{E}_i = (t_{i+1}^m - t_i^m)\gamma_\omega^m(Y_i)$. By definition of X and Y, $(\mathscr{E}_i)_{i\in\mathbf{N}}$ is a sequence of iid exponential random variables of parameter 1, independent of the walk and the environment.

We will first look at the case $\kappa = 1$.
If $\sum_{i=\tau_2}^{\tau_3-1} 1_{Y_i\notin\tilde{\mathscr{T}}}$ has a finite expectation for $\mathbb{P}_0$, since the $\left(\sum_{i=\tau_{2i}}^{\tau_{2i+1}-1} 1_{Y_i\notin\tilde{\mathscr{T}}}\right)_{i\in\mathbf{N}^*}$ are iid and so are the $\left(\sum_{i=\tau_{2i+1}}^{\tau_{2i+2}-1} 1_{Y_i\notin\tilde{\mathscr{T}}}\right)_{i\in\mathbf{N}^*}$ then we have the result we want. On the other hand, if $\sum_{i=\tau_1}^{\tau_2-1} 1_{Y_i\notin\tilde{\mathscr{T}}}$ has an infinite expectation then, since the random variables $\left(\sum_{i=\tau_i}^{\tau_{i+1}-1} 1_{Y_i\notin\tilde{\mathscr{T}}}\right)_{i\geq 2}$ are non negative, $n^{-1}\sum_{i=\tau_1}^{\tau_n-1} 1_{Y_i\notin\tilde{\mathscr{T}}} \to \infty$ $\mathbb{P}_0$ almost surely.
By the law of large number, we get that $\mathbb{P}_0$ almost surely:

$$\exists k \in \mathbf{N}, \forall n \geq k, \sum_{i=0}^{\tau_n-1} \mathscr{E}_i 1_{Y_i\notin\tilde{\mathscr{T}}} \geq \frac{1}{2}\sum_{i=0}^{\tau_n-1} 1_{Y_i\notin\tilde{\mathscr{T}}}.$$

For any point x, if x is not in a trap then, by definition of traps:

$$\frac{1}{\gamma^{\omega}(x)} \geq \frac{1}{2}.$$

This yields:

$$\sum_{i=0}^{\tau_n-1} \mathscr{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \leq 2 \sum_{i=0}^{\tau_n-1} \mathscr{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \frac{1}{\gamma^{\omega}(Y_i)}.$$

And by writing $T_n^m = t_{\tau_n}^m$ we have:

$$\sum_{i=0}^{\tau_n-1} \mathscr{E}_i 1_{Y_i \notin \tilde{\mathcal{T}}} \leq 2 \int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t.$$

We know by lemma 2.2.1.1 that there exists a constant d_m such that $\mathbb{P}_0$ almost surely:

$$T_n^m - d_m n \to -\infty.$$

We get:

$$\exists k \in \mathbf{N}, \forall n \geq k, \int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t \leq \int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t.$$

Finally, if $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{i=0}^{\tau_n} 1_{Y_i \notin \tilde{\mathcal{T}}} \to \infty.$$

Then $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t \to \infty.$$

And therefore, since $\mathbb{Q}_0^m$ is absolutely continuous with respect to $\mathbb{P}_0$ we get that $\mathbb{Q}_0^m$ almost surely:

$$\frac{1}{n} \int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t \to \infty.$$

So we would have, since $\frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)}$ is positive:

$$\frac{1}{n} \mathbb{E}_{\mathbb{Q}_0^m} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t \right) \to \infty$$

which would mean, since $\mathbb{Q}_0^m$ is a stationary law:

$$\mathbb{E}_{\mathbb{Q}^m} \left(\int_0^1 \frac{\gamma_\omega^m(X_t^m)}{\gamma^{\omega}(X_t^m)} \mathrm{d}t \right) = \infty.$$

Which is false by lemma 2.2.2.4 so we get the result we want.
Now for the case $\kappa < 1$.
Let $\beta \in \left(\kappa, \frac{\kappa+\kappa'}{2}\right)$ be a real such that $\beta \leq 1$. If $\sum_{x \in \mathbf{Z}^d} \left(\sum_{i=\tau_2}^{\tau_3-1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}}$ has an infinite expectation (for $\mathbb{P}_0$), since the $\left(\sum_{x \in \mathbf{Z}^d} \left(\sum_{i=\tau_{2j}}^{\tau_{2j+1}-1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \right)_{j \in \mathbf{N}^*}$ are iid, we would have that $\mathbb{P}_0$ almost surely:

$$n^{-1} \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_n-1} 1_{Y_i = x} \right)^{\beta} 1_{x \notin \tilde{\mathcal{T}}} \to \infty.$$

By lemma 2.3.0.5 we get that there exists a constant $C > 0$ such that $\mathbb{P}_0$ almost surely:

$$\exists m \in \mathbf{N}, \forall n \geq m, \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_{n+1}-1} \mathscr{E}_i 1_{Y_i = x} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}} \geq C \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_{n+1}-1} 1_{Y_i = x} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}}.$$

We also have, by writing $T_n^m = t_{\tau_n}^m$:

$$\begin{aligned}
\sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_{n+1}-1} \mathscr{E}_i 1_{Y_i = x} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}} \leq & 4^\beta \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_{n+1}-1} \mathscr{E}_i 1_{Y_i = x} \frac{1}{\gamma^\omega(x)} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}} \\
\leq & 4^\beta \sum_{x \in \mathbf{Z}^d} \left(\int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta .
\end{aligned}$$

We know by lemma 2.2.1.1 that there exists a constant d_m such that $\mathbb{P}_0$ almost surely:

$$T_n^m - d_m n \to -\infty.$$

We get:

$$\exists m \in \mathbf{N}, \forall n \geq m, \sum_{x \in \mathbf{Z}^d} \left(\int_0^{T_n^m} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \leq \sum_{x \in \mathbf{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta .$$

Finally, if $\mathbb{P}_0$ almost surely

$$\frac{1}{n} \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_n - 1} 1_{Y_i = x} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}} \to \infty$$

then $\mathbb{P}_0$ almost surely

$$\frac{1}{n} \sum_{x \in \mathbf{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \to \infty.$$

And therefore, since $\mathbb{Q}_0^m$ is absolutely continuous with respect to $\mathbb{P}_0$ we get that $\mathbb{Q}_0^m$ almost surely:

$$\frac{1}{n} \sum_{x \in \mathbf{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \to \infty.$$

So we would have:

$$\frac{1}{n} \mathbb{E}_{\mathbb{Q}_0^m} \left(\sum_{x \in \mathbf{Z}^d} \left(\int_0^{d_m n} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \right) \to \infty.$$

And therefore:

$$\frac{1}{n} \sum_{i=0}^{d_m n} \mathbb{E}_{\mathbb{Q}_0^m} \left(\sum_{x \in \mathbf{Z}^d} \left(\int_i^{i+1} \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \right) \to \infty.$$

This would mean, since $\mathbb{Q}_0^m$ is a stationary law that

$$\mathbb{E}_{\mathbb{Q}_0^m} \left(\sum_{x \in \mathbf{Z}^d} \left(\int_0^1 \frac{\gamma_\omega^m(X_t^m)}{\gamma^\omega(X_t^m)} 1_{X_t^m = x} \mathrm{d}t \right)^\beta \right) = \infty$$

which is false by lemma 2.2.2.4. Therefore there exists a constant $C > 0$ such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n} \sum_{x \in \mathbf{Z}^d} \left(\sum_{i=0}^{\tau_n - 1} 1_{Y_i = x} \right)^\beta 1_{x \notin \tilde{\mathscr{T}}} \to C.$$

So $\mathbb{P}_0$ almost surely for n large enough:

$$\frac{1}{n}\left(\sum_{x\in\mathbf{Z}^d}\sum_{i=0}^{\tau_n-1} 1_{Y_i=x}1_{x\notin\tilde{\mathcal{T}}}\right)^{\beta} \leq \frac{1}{n}\sum_{x\in\mathbf{Z}^d}\left(\sum_{i=0}^{\tau_n-1} 1_{Y_i=x}\right)^{\beta} 1_{x\notin\tilde{\mathcal{T}}} \leq 2C.$$

And therefore:

$$\sum_{x\in\mathbf{Z}^d}\sum_{i=0}^{\tau_n-1} 1_{Y_i=x}1_{x\notin\tilde{\mathcal{T}}} \leq (2Cn)^{\frac{1}{\beta}}.$$

□

2.2.6 Proof of the theorems

Now we can finally prove both theorems.

Theorem 51. *Set $d \geq 3$ and $\alpha \in (0,\infty)^{2d}$. Let $Y^n(t)$ be defined by:*

$$Y^n(t) = n^{-\kappa} Y_{\lfloor nt\rfloor}.$$

If $\kappa < 1$ and $d_\alpha \neq 0$, there exists positive constants c_1, c_2, c_3 such that for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$\left(t \to n^{-\frac{1}{\kappa}}\tau_{\lfloor nt\rfloor}\right) \to c_1 \mathscr{S}^{\kappa},$$

for the M_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$(x \to \inf\{t \geq 0, Y^n(t).e_1 \geq x\}) \to c_2 \mathscr{S}^{\kappa}$$

and for the J_1 topology and for $\mathbb{P}_0^{(\alpha)}$:

$$Y^n \to c_3 \tilde{\mathscr{S}^{\kappa}} d_\alpha.$$

Proof. The proof will be divided in three parts, one for each result. The second part and the third one rely on the first part. However, the second part and the third part are independent from one another.

First Part

First we will prove that there exists a constant c such that for any $t \in \mathbf{R}^+$ and any increasing sequence (x_n) such that $x_n \to \infty$, we have the following convergence in law, for $\mathbb{P}_0$:

$$x_n^{-\frac{1}{\kappa}}\tau_{\lfloor x_n t\rfloor} \to c t^{\frac{1}{\kappa}} \mathscr{S}_1^{\kappa}.$$

The result is obvious for $t = 0$. For $t > 0$, lemmas 2.2.5.4 and 2.2.5.2 tell us that we only have to consider the time spent in traps in directions j such that $\kappa_j = \kappa$. Then lemma 2.2.4.2 tells us that with probability larger than $1-\varepsilon$ the time spent in such traps is not more than the time spent in traps where the walks come back at most m_ε times (for some m_ε) plus at most $\varepsilon x_n^{\frac{1}{\kappa}}$. We also know by lemma 2.2.4.3 that for any m_ε there exists a constant c_ε such that the time spent in traps where the walks come back at most m_ε times renormalized by $x_n^{-\frac{1}{\kappa}}$ converges in law (for $\mathbb{P}_0$) to $c_\varepsilon t^{\frac{1}{\kappa}} S^\kappa$ so we get the result we want by having ε go to 0 since c_ε is increasing and cannot go to infinity. Since the $(\tau_{i+1}-\tau_i)_{i\geq 1}$ are iid (for $\mathbb{P}_0$) by proposition 2.1.2.1, we also get that for any sequence $(n_i)_{i\in\mathbf{N}^*}$ with $n_i \geq 1$, $\left(i^{-\frac{1}{\kappa}}(\tau_{n_i+it}-\tau_{n_i})\right)_{i\geq 1}$ converges in law (for $\mathbb{P}_0$) to $c_1 t^{\frac{1}{\kappa}}\mathscr{S}_1^{\kappa}$.

Now we want to show that the family of process $\left(t \to x_n^{-\frac{1}{\kappa}}\tau_{\lfloor x_n t\rfloor}\right)_{n\in\mathbf{N}}$ is tight. We will only look at the convergence and tightness for the processes on an interval $[0, A]$. We use the characterisation given in Theorem 15.3 of [15]:

(i) for each positive ε there exists c such that:

$$\mathbb{P}\left(\sup_{t\in[0,T]} |f(t)| > c\right) \leq \varepsilon,$$

(ii) for each $\varepsilon > 0$ and $\eta > 0$, there exist a δ, $0 < \delta < T$, and an integer n_0 such that:

$\forall n \geq n_0,\ \mathbb{P}(w_{f_n}(\delta) \geq \eta) \leq \varepsilon$

and

$\forall n \geq n_0,\ \mathbb{P}(v_{f_n}(0,\delta) \geq \eta) \leq \varepsilon$ and $\mathbb{P}(v_{f_n}(T,\delta) \geq \eta) \leq \varepsilon$, where w_f and v_f are defined by:

$$\begin{aligned} w_f(\delta) &= \sup\{\min\left(|f(t)-f(t_1)|, |f(t_2)-f(t)|\right) t_1 \le t \le t_2 \le T, t_2 - t_1 \le \delta\}, \\ v_f(t,\delta) &= \sup\{|f(t_1)-f(t_2)| : t_1, t_2 \in [0,T] \cap (t-\delta, t+\delta)\}. \end{aligned}$$

For a sequence of non-decreasing processes (W_n) defined on $[0,T]$, this characterization is implied by the following:

$$\begin{aligned} &(i) \text{ for each positive } \varepsilon \text{ there exist } C \text{ such that} \\ &\quad \mathbb{P}(W_n(T) \ge C) \le \varepsilon, \text{ for } n \ge 1, \\ &(ii) \text{ for each } \varepsilon > 0 \text{ there exist a } \delta \in (0,T), \text{ such that for } n \ge 1 \\ &(a) \ \ \forall x \in [\delta, T-\delta], \ \mathbb{P}(W_n(x+\delta) - W_n(x) \ge \varepsilon \text{ and } W_n(x) - W_n(x-\delta) \ge \varepsilon) \le \varepsilon \\ &\quad \text{and} \\ &(b) \ \ \mathbb{P}(W_n(\delta) - W_n(0) \ge \varepsilon) \le \varepsilon \\ &\quad \text{and} \\ &(c) \ \ \mathbb{P}(W_n(T) - W_n(T-\delta) \ge \varepsilon) \le \varepsilon. \end{aligned}$$

For the first property, since we know that the sequence $\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor}\right)_{n\in\mathbf{N}}$ converges in law for $\mathbb{P}_0$, the family $\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor}\right)_{n\in\mathbf{N}}$ is tight and therefore for any $\varepsilon > 0$ there exists B_ε such that:

$$\forall n \in \mathbf{N}, \ \mathbb{P}_0\left(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor} \in [0, B_\varepsilon]\right) \ge 1 - \varepsilon.$$

So:

$$\forall \varepsilon > 0, \exists B_\varepsilon, \forall n \in \mathbf{N}, \mathbb{P}_0\left(\forall t \in [0,A], \ x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor} \in [0, B_\varepsilon]\right) \ge 1 - \varepsilon.$$

Now we will prove the two side conditions ($ii.b$ and $ii.c$). For ($ii.b$), we first choose δ such that $\mathbb{P}_0\left(c_1 \delta^{\frac{1}{\kappa}} \mathscr{S}_1^\kappa \ge \varepsilon\right) \le \frac{\varepsilon}{2}$. This proves the result for n large enough and then, since the processes we consider are càdlàg, we decrease δ up to the point where we have the result for n small and we get the result we want.

For ($ii.c$), the proof will be essentially the same. Since the increments are iid (except for the first one of which we do not know the law) the law of $x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n A \rfloor} - x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n (A-\delta) \rfloor}$ converges to $c_1 \delta^{\frac{1}{\kappa}} S_\kappa$. So we get that for some δ, for n large enough we have the result we want. For small n we only use the fact that the processes are càdlàg so we get the result we want by decreasing δ.

Now we can prove ($ii.a$). Let $J = \{j \in [\![1,d]\!], \kappa_j = \kappa\}$. First we have, by lemmas 2.2.5.4 and 2.2.5.2, that for n large enough, the time spent in vertices that are not part of a trap in a direction $j \in J$ before time $\tau_{\lfloor x_n t \rfloor}$ is smaller than $\frac{1}{3}\varepsilon x_n^{\frac{1}{\kappa}}$ with probability at least $1 - \frac{1}{3}\varepsilon$. Similarly by lemma 2.2.4.3 there exists m_ε such that for n large enough the time spent in traps in direction $j \in J$ such that the walk enters at least m_ε times the trap is lower than $\frac{1}{3}\varepsilon x_n^{\frac{1}{\kappa}}$ with probability at least $1 - \frac{1}{3}\varepsilon$. And finally, there exists β_ε such that for n large enough, by lemma 2.2.5.1, with probability at least $1 - \frac{1}{3}\varepsilon$ the time spent in traps in direction $j \in J$ such that their strength is at most $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ is lower than $\frac{1}{3}\varepsilon x_n^{\frac{1}{\kappa}}$. Condition ($ii.c$) is not verified if either of the previous three events are not verified which happens with probability at most $1 - \varepsilon$. However if the previous events are verified and there is no i such that there are at least two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+2\delta x_n} - 1$ then the main condition is true.

So now we just have to prove that for δ small enough, with high probability there is no i such that there are at least two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+2\delta x_n} - 1$. By lemma 2.2.5.3 we have that for any $m \in \mathbf{N}$ the probability that there exists $i \le x_n$ such that there are two traps of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ between times τ_i and $\tau_{i+m} - 1$ goes to 0 when n goes to infinity. So let B_i be the event: "there exists a trap of strength at least $\beta_\varepsilon x_n^{\frac{1}{\kappa}}$ visited at most m_ε times between times τ_i and $\tau_{i+1} - 1$". We define the finite sequence (n_i) by:

$$\begin{aligned} n_1 &= \inf\{j \ge 1, B_j\}, \\ n_{i+1} &= \inf\{j \ge n_i + m, B_j\}. \end{aligned}$$

We also define $\tilde{n}_i$ by $\tilde{n}_i = \sup\{j, n_j \le x_i\}$. First we want to prove that $\tilde{n}_i$ cannot be too large. We know that there exists a constant C such that if $M(n)$ is the number of different traps in a direction j visited before time τ_n then for n large enough: $\mathbb{P}_0(M(x_n) \geq Cx_n) \leq \varepsilon$ and by lemma 2.2.3.4 we clearly have that $\mathbb{E}(\tilde{n}_n 1_{M(x_n)\leq Cx_n}) \leq \frac{cC}{\beta^\kappa}$. Therefore if we take $B \geq \frac{cC}{\varepsilon\beta^\kappa}$ we get that for n large enough, $\mathbb{P}_0(\tilde{n}_n \geq B) \leq 2\varepsilon$. Now we want to show that for $\delta > 0$ small enough, $\mathbb{P}_0(\exists i \leq B, n_{i+1} - n_i \leq 2\delta x_n) \leq \varepsilon$ which would yields the desired result. For any i, we have, by proposition 2.1.2.1:

$$\mathbb{P}_0(n_{i+1} - n_i \leq 2\delta x_n) \leq \mathbb{P}_0(n_1 \leq 2\delta x_n).$$

And therefore:

$$\mathbb{P}_0(\exists i \leq \tilde{n}_n,\ n_{i+1} - n_i \leq 2\delta x_n) \leq \mathbb{P}_0(\tilde{n}_n > B) + B\mathbb{P}_0(n_1 \leq 2\delta x_n).$$

We have that there is a constant C such that for n large enough, $\mathbb{P}_0(M(2\delta x_n) \geq 2C\delta x_n) \leq \frac{\varepsilon}{B}$. And then by lemma 2.2.3.4 we have that the expectation of the number of traps of strength at least $\beta x_n^{\frac{1}{\kappa}}$ among the first $2\delta x_n$ traps is lower than $2\delta x_n \frac{c}{\beta^\kappa x_n}$ and therefore for δ small enough, $\mathbb{P}_0(\exists i \leq \tilde{n}_n,\ n_{i+1} - n_i \leq 2\delta x_n) \leq \varepsilon$. So we have that the sequence of processes is tight.
Now we want to show that its limit is $c_1\mathscr{S}^\kappa$. Let m be an integer and $(x_i)_{0\leq i\leq n}$ be reals such that $0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1$. We have, since the $(\tau_{i+1} - \tau_i)_{i\geq 1}$ are iid and independent from τ_1:

$$(x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n y_i \rfloor})_{0\leq i\leq m} \to (\mathscr{S}^\kappa(y_i))_{0\leq i\leq m}.$$

So we have convergence in the J_1 topology for any increasing sequence x_i that goes to infinity.

Second Part

Let L be defined by:

$$L(t) := \inf\{i, Y_i.e_1 \geq t\}.$$

And let L_n be the renormalized L:

$$L_n(t) := n^{-\frac{1}{\kappa}} L(nt)$$

Notice that:

$$L_{n^\kappa}(t) = \inf\{i, Y^n(i).e_1 \geq t\}.$$

We have, by definition of τ and L:

$$\forall n \in \mathbf{N}^*,\ L(Y_{\tau_n}.e_1) = \tau_n.$$

We first want to show that the sequence L_n is tight in the M_1 topology. We use the characterisation given in Theorem 12.12.3 of [89]:

(i) for each positive ε there exists c such that:

$$\mathbb{P}\left(\sup_{t\in[0,T]} |f(t)| > c\right) \leq \varepsilon,$$

(ii) for each $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta,\ 0 < \delta < T$, and an integer n_0 such that:

$\forall n \geq n_0,\ \mathbb{P}(w_{f_n}(\delta) \geq \eta) \leq \varepsilon$

and

$\forall n \geq n_0,\ \mathbb{P}(v_{f_n}(0, \delta) \geq \eta) \leq \varepsilon$ and $\mathbb{P}(v_{f_n}(T, \delta) \geq \eta) \leq \varepsilon$.

Where w_f and v_f are defined by:

$$w_f(\delta) = \sup\{\inf_{\alpha\in[0,1]} |f(t) - (\alpha f(t_1) + (1-\alpha) f(t_2))|, t_1 \leq t \leq t_2 \leq T, t_2 - t_1 \leq \delta\},$$
$$v_f(t, \delta) = \sup\{|f(t_1) - f(t_2)| : t_1, t_2 \in [0, T] \cap (t - \delta, t + \delta)\}.$$

First we have:

$$\begin{aligned}\mathbb{P}_0\left(\sup_{t\in[0,T]} |L_n(t)| > c\right) &= \mathbb{P}_0\left(L(nT) > cn^{\frac{1}{\kappa}}\right)\\ &\leq \mathbb{P}_0\left(\tau_{nT} > cn^{\frac{1}{\kappa}}\right),\end{aligned}$$

which is smaller than ε for all n, for c large enough.
Next, since H_n is non-decreasing, we have:

$$\mathbb{P}_0(w_{L_n}(\delta) = 0) = 1.$$

Then, we first use the fact that:

$$v_{L_n}(0,\delta) \le n^{-\frac{1}{\kappa}} \tau_{n\delta}$$

to get that for δ small enough:

$$\forall n \ge n_0,\ \mathbb{P}_0(v_{L_n}(0,\delta) \ge \eta) \le \varepsilon.$$

The bound for $v_{L_n}(T,\delta)$ is similar but slightly trickier. We know that for $c = (\mathbb{E}(Y_{\tau_2} - Y_{\tau_1}).e_1)^{-1}$, $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}(Y_{\tau_{cn(T-2\delta)}}.e_1, Y_{\tau_{cn(T+\delta)}}.e_1) \to (T-2\delta, T+\delta).$$

Therefore, using the fact that L_n is increasing, with probability going to 1:

$$L_n(T) - L_n(T-\delta) \le n^{-\frac{1}{\kappa}}(\tau_{cn(T+\delta)} - \tau_{cn(T-2\delta)}).$$

And we have the result we want for δ small enough and n large enough. So we have that the sequence $(L_n)_{n\in\mathbf{N}^*}$ is tight. Now we just have to show that its limit is $C\mathscr{S}^\kappa$ for some constant C. Set $c = (\mathbb{E}(Y_{\tau_2} - Y_{\tau_1}).e_1)^{-1}$. We will show that $L_n(x)$ is almost equal to $\tau_n(cx)$ which will yield the result. Set $\varepsilon > 0$ and $x \in [0,\infty)$. We want to show that $\mathbb{P}_0(|L_n(x) - \tau_n(cx)| \ge \varepsilon) \to 0$. We will use the following inequality:

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \ge \varepsilon) \le \inf_{\delta>0} \mathbb{P}_0(L_n(x) \ge \tau_n(cx+\delta)) + \mathbb{P}_0(\tau_n(cx+\delta) - \tau_n(cx) \ge \varepsilon).$$

We clearly have, for any $\delta > 0$

$$\limsup_{n\to\infty} \mathbb{P}_0(L_n(x) \ge \tau_n(cx+\delta)) = 0.$$

And for some constant $\tilde{C}$ that does not depend on x or c

$$\mathbb{P}_0(\tau_n(cx+\delta) - \tau_n(cx) \ge \varepsilon) \to \mathbb{P}_0(\tilde{C}\mathscr{S}^\kappa(\delta) \ge \varepsilon).$$

Therefore

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \ge \varepsilon) \to 0.$$

Similarly we get:

$$\mathbb{P}_0(L_n(x) - \tau_n(cx) \le -\varepsilon) \to 0.$$

Therefore the limit of L^n is $t \to \tilde{C}\mathscr{S}^\kappa(ct)$ which is equal to $C\mathscr{S}^\kappa$ for some constant C.

Third Part

We will look at a sequence of processes $t \to \overline{\tau}_n(t)$ such that the law of $\overline{\tau}_n$ is the same as that of $t \to x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}$ and such that almost surely $\overline{\tau}_n \to \overline{\tau}$ in the J_1 topology with the law of $\overline{\tau}$ being that of $\mathscr{S}^\kappa$. We want to show that the law of the inverse of $\overline{\tau}_n$ converges to that of the inverse of $\mathscr{S}^\kappa$. This is a direct consequence of lemmas 2.3.0.6 and 2.3.0.7. Now if we define $L^\tau(t)$ by $L^\tau(t) = \min\{n \in \mathbf{N}, \tau_n \ge t\}$, we have that in J_1 topology:

$$\frac{1}{x_n} L^\tau\left(x_n^{\frac{1}{\kappa}} t\right) \to \tilde{\mathscr{S}^\kappa}(t)$$

for any increasing sequence x_n such that $x_n \to \infty$. Therefore, for any increasing sequence x_n such that $x_n \to \infty$:

$$\frac{1}{x_n^\kappa} L^\tau(x_n t) \to \tilde{\mathscr{S}^\kappa}(t).$$

Now by lemma 2.2.1.3 there exists $v \in \mathbf{R}^d$ such that $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_{\lfloor t \rfloor}}}{t} \to v.$$

This means that in the J_1 topology, we have the following convergence (in law):

$$\left(t \to \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}\right) \to (t \to tv).$$

And therefore, in the J_1 topology,

$$\left(t \to x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}, t \to \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}\right) \to (c_1\mathscr{S}^\kappa, t \to tv).$$

Now we will look at $(\overline{\tau}_n, d_n)$ where for any n the law of $(\overline{\tau}_n, d_n)$ is the same as the law of $t \to x_n^{-\frac{1}{\kappa}} \tau_{\lfloor x_n t \rfloor}, t \to \frac{Y_{\tau_{\lfloor x_n t \rfloor}}}{x_n}$ and such that almost surely:

$$(\overline{\tau}_n, d_n) \to (c_1 \mathscr{S}^\kappa, t \to tv) .$$

Let $\overline{\tau}$ be such that almost surely $\overline{\tau}_n \to \overline{\tau}$. Let $\Delta_{[0,A]}$ be the distance associated with the infinite norm on [0,A]. If we look at $d_{\overline{\tau}_n^{-1}(t)}$ where $\overline{\tau}_n^{-1}(t) = \inf\{x, \overline{\tau}_n(x) \geq t\}$ we get:

$$\begin{aligned}\Delta_{[0,A]} \left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}^{-1}(t)v\right) &\leq \Delta_{[0,A]} \left(d_n(\overline{\tau}_n^{-1}(t), \overline{\tau}_n^{-1}(t)v\right) + \Delta_{[0,A]} \left(\overline{\tau}_n^{-1}(t)v, \overline{\tau}^{-1}(t)v\right) \\ &= \Delta_{[0,A]} \left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}_n^{-1}(t)v\right) + ||v|| \Delta_{[0,A]} \left(\overline{\tau}_n^{-1}(t), \overline{\tau}^{-1}(t)\right) .\end{aligned}$$

So for any $B, \varepsilon > 0$:

$$\begin{aligned}&\mathbb{P}_0(\Delta_{[0,A]} \left(d_n(\overline{\tau}_n^{-1}(t)), \overline{\tau}^{-1}(t)v\right) \geq \varepsilon) \\ \leq &\mathbb{P}_0(\overline{\tau}_n^{-1}(A) > B) + \mathbb{P}_0 \left(\exists t \in [0, B], ||d_n(t) - tv|| \geq \frac{\varepsilon}{2}\right) + \mathbb{P}_0 \left(\Delta_{[0,A]} \left(\overline{\tau}_n^{-1}(t), \overline{\tau}^{-1}(t)\right) \geq \frac{\varepsilon}{2}\right) \\ = &\mathbb{P}_0(\overline{\tau}_n^{-1}(A) > B) + o(1) \\ = &\mathbb{P}_0(\overline{\tau}_n(B) < A) + o(1) \\ = &\mathbb{P}_0(\overline{\tau}(B) < A) + o(1).\end{aligned}$$

We clearly have that when B goes to infinity, $\mathbb{P}_0(\overline{\tau}(B) < A)$ goes to 0 so we have that in the J_1 topology:

$$d_n(\overline{\tau}_n^{-1}(t)) \to \overline{\tau}^{-1}(t)v.$$

Since we have that in law (in the following we will write $\tau(x)$ instead of τ_x for the formulas to stay readable):

$$d_n(\overline{\tau}_n^{-1}(t)) = \frac{1}{x_n} Y_{\tau\left(\lfloor x_n (x_n^{-1} L^\tau ((x_n)^{\frac{1}{\kappa}} t))\rfloor\right)} = \frac{1}{x_n} Y_{\tau\left(\lfloor \overline{\tau}((x_n)^{\frac{1}{\kappa}} t)\rfloor\right)}$$

we get that in the J_1 topology for any increasing sequence x_n:

$$x_n^{-\kappa} Y_{\tau(\lfloor L^\tau (x_n t)\rfloor)} \to c_1^{-\kappa} \tilde{\mathscr{S}}^\kappa(t)v.$$

Now we only have to show that $Y_{\tau(\lfloor L^\tau(x_n t)\rfloor)}$ and Y_t are almost equal. For every $i > 0$ let R_i be the number of different points visited between times τ_i and $\tau_{i+1} - 1$ and let R_0 be the number of different points visited before time $\tau_i - 1$ (0 if $\tau_i = 0$). The $(R_i)_{i \in \mathbf{N}}$ are independent and the $(R_i)_{i \in \mathbf{N}^*}$ are iid with finite expectation by lemma 2.2.1.2. Let $\varepsilon > 0$ be a constant and let $B > 0$ be such that for x large enough, $\mathbb{P}_0(x^{-\kappa} L^\tau(xA) \geq B) \leq \frac{\varepsilon}{2}$ (taking B such that $\mathbb{P}_0(c_1^{-\kappa} \tilde{\mathscr{S}}^\kappa(A) \geq B) \leq \frac{\varepsilon}{4}$ works). We get that for x large enough:

$$\begin{aligned}\mathbb{P}_0(\exists t \leq xA, x^{-\kappa} ||Y_{\tau(\lfloor L^\tau(t)\rfloor)} - Y_t|| \geq \varepsilon) \leq &\frac{\varepsilon}{2} + \mathbb{P}_0(\exists i \leq Bx^\kappa, R_i \geq \varepsilon x^\kappa) \\ \leq &\frac{\varepsilon}{2} + \mathbb{P}_0(R_0 \geq \varepsilon x^\kappa) + \mathbb{P}_0(\exists i \in [\![1, Bx^\kappa]\!], R_i \geq \varepsilon x^\kappa) \\ \leq &\frac{\varepsilon}{2} + o(1) + Bx^\kappa \mathbb{P}_0(R_1 \geq \varepsilon x^\kappa) \\ = &\frac{\varepsilon}{2} + o(1).\end{aligned}$$

So for any $\varepsilon > 0$ we have that for x large enough:

$$\mathbb{P}_0(\exists t \leq xA, x^{-\kappa} ||Y_{\tau(\lfloor L^\tau(xt)\rfloor)} - Y_t|| \geq \varepsilon) \leq \varepsilon.$$

So we get that in the J_1 topology:

$$x^{-\kappa} Y_{\lfloor xt \rfloor} \to \tilde{\mathscr{S}}^\kappa(t)v.$$

Since v and d_α are collinear, we get the result we want. $\square$

Theorem 52. *If $d \geq 3$ and $\kappa = 1$, there exists positive constants c_1, c_2, c_3 such that we have the following convergences in probability (for $\mathbb{P}_0$):*

$$\frac{1}{n \log(n)} \tau_n \to c_1,$$

$$\frac{1}{n \log(n)} \inf\{i, Y_i.e_1 \geq n\} \to c_2,$$

$$\frac{\log(n)}{n} (Y_n) \to c_3 d_\alpha.$$

Proof. Let $J = \{j \in [\![1,d]\!], \kappa_j = \kappa\}$.
By lemma 2.2.5.4 we get that there exists a constant C such that $\mathbb{P}_0$ almost surely:

$$\frac{1}{n}\sum_{i=0}^{\tau_n} 1_{Y_i \notin \tilde{\mathscr{T}}} \to C.$$

So we only have to look at the time spent in the traps. By lemma 2.2.5.2 we get that for any $\varepsilon > 0$, for n large enough:

$$\mathbb{P}_0\left(\frac{1}{n\log(n)}\sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathscr{T}}} 1_{Y_i \notin \tilde{\mathscr{T}}_J} \geq \varepsilon\right) \leq \varepsilon.$$

Therefore we only have to look at the time spent in traps in a direction $j \in J$. For any trap $\{x, y\}$ let $\tilde{N}_x$ be the number of times the walks exits the trap $\{x, y\}$, we have $\tilde{N}_w = \tilde{N}_y$. Let $\varepsilon > 0$ be a positive constant. By lemma 2.2.4.2 there exists a m_ε such that:

$$\mathbb{P}_0\left(\frac{1}{n\log(n)}\sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathscr{T}}_J} 1_{\tilde{N}_{Y_i} \geq m_\varepsilon} \geq \varepsilon\right) \leq \varepsilon.$$

And by lemma 2.2.4.3 we get that there is a constant C_{m_ε} such that:

$$\frac{1}{n\log(n)}\sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathscr{T}}_J} 1_{\tilde{N}_{Y_i} \leq m_\varepsilon} \to C_{m_\varepsilon} \text{ in probability.}$$

So for n large enough:

$$\mathbb{P}_0\left(\frac{1}{n\log(n)}\sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathscr{T}}} \in [C_{m_\varepsilon} - 2\varepsilon, C_{m_\varepsilon} + 2\varepsilon]\right) \geq 1 - 2\varepsilon.$$

This means that there exists a constant C_∞ such that:

$$\frac{1}{n\log(n)}\sum_{i=1}^{\tau_{n+1}-1} 1_{Y_i \in \tilde{\mathscr{T}}} \to C_\infty \text{ in probability.}$$

And therefore:

$$\frac{1}{n\log(n)}\tau_{n+1} \to C_\infty \text{ in probability.}$$

So we have proved the first part of the theorem.
Now, by lemma 2.2.1.3 we have for some $C > 0$, $\mathbb{P}_0$ almost surely:

$$\frac{Y_{\tau_n}.e_1}{n} \to C.$$

So for any $\varepsilon > 0$, by writing $L(n) := \min\{i, Y_i.e_1 \geq n\}$ and $C^+ = \frac{1}{C(1-\varepsilon)}$:

$$\begin{aligned}
&\mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n)] \\
\leq &\mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n) \text{ and } \tau_{C^+n} \leq C_\infty C^+(1+\varepsilon)n\log(n)] \\
&\quad + \mathbb{P}_0[\tau_{C^+n} > C_\infty C^+(1+\varepsilon)n\log(n)] \\
= &\mathbb{P}_0[L(n) \geq C_\infty C^+(1+\varepsilon)n\log(n) \text{ and } \tau_{C^+n} \leq C_\infty C^+(1+\varepsilon)n\log(n)] + o(1) \\
\leq &\mathbb{P}_0[L(n) \geq \tau_{C^+n}] + o(1) \\
= &\mathbb{P}_0[Y_{\tau_{C^+n}}.e_1 \leq n] + o(1) \\
= &\mathbb{P}_0\left[\frac{Y_{\tau_{C^+n}}.e_1}{C^+n} \leq C(1-\varepsilon)\right] + o(1) \\
= &o(1).
\end{aligned}$$

The same way we get, by taking $C^- = \frac{1}{C(1+\varepsilon)}$:

$$\begin{aligned}
&\mathbb{P}_0(L(n) \le C_\infty C^-(1-\varepsilon) n \log(n)) \\
\le &\mathbb{P}_0(L(n) \le C_\infty C^-(1-\varepsilon) n \log(n) \text{ and } \tau_{C^- n} \ge C_\infty C^-(1-\varepsilon) n \log(n)) \\
&\quad + \mathbb{P}_0(\tau_{C^- n} < C_\infty C^-(1-\varepsilon) n \log(n)) \\
= &\mathbb{P}_0(L(n) \le C_\infty C^-(1-\varepsilon) n \log(n) \text{ and } \tau_{C^- n} \ge C_\infty C^-(1-\varepsilon) n \log(n)) + o(1) \\
\le &\mathbb{P}_0(L(n) \le \tau_{C^- n}) + o(1) \\
= &\mathbb{P}_0(Y_{\tau_{C^- n}}.e_1 \ge n) + o(1) \\
= &\mathbb{P}_0\left(\frac{Y_{\tau_{C^- n}}.e_1}{C^- n} \ge C(1+\varepsilon)\right) + o(1) \\
= &o(1).
\end{aligned}$$

So we get the second result. Now for the last result, we define $L^\tau(n) = \min\{i, \tau_i \ge n\}$ so $\tau_{L^\tau(n)-1} < n \le \tau_{L^\tau(n)}$. We get, for n big enough:

$$\mathbb{P}_0\left(L^\tau(n) \ge C_\infty^{-1}(1+2\varepsilon)\frac{n}{\log(n)}\right) \le \mathbb{P}_0\left(\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}} \le n\right).$$

And we have:

$$C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)} \log\left(C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}\right) = C_\infty^{-1}(1+\varepsilon)n(1+o(1)).$$

And therefore, using the result of part one:

$$\frac{\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}}}{n} \to C_\infty C_\infty^{-1}(1+\varepsilon) = (1+\varepsilon).$$

So we get that:

$$\mathbb{P}_0\left(\tau_{C_\infty^{-1}(1+\varepsilon)\frac{n}{\log(n)}} \le n\right) \to 0.$$

And therefore:

$$\mathbb{P}_0\left(L^\tau(n) \ge C_\infty^{-1}(1+2\varepsilon)\frac{n}{\log(n)}\right) \to 0.$$

The proof of the lower bound is exactly the same:

$$\mathbb{P}_0\left(L^\tau(n) \le C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}\right) \le \mathbb{P}_0\left(\tau_{C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}} \ge n\right).$$

But we have:

$$n^{-1}\tau_{C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}} \to (1-\varepsilon).$$

So

$$\mathbb{P}_0\left(L^\tau(n) \le C_\infty^{-1}(1-\varepsilon)\frac{n}{\log(n)}\right) \to 0.$$

And therefore:

$$\frac{\log(n)}{n} L^\tau(n) \to C_\infty^{-1}.$$

Now, by lemma 2.2.1.3 $\frac{Y_i}{L^\tau(i)} \to D$, $\mathbb{P}_0$ almost surely so we get:

$$\frac{\log(n)}{n} Y_n \to C_\infty^{-1} D.$$

□

2.3 Annex

Lemma 2.3.0.1. *Let X be a non-negative random variable such that $\mathbb{E}(X) < \infty$. There exists an increasing, positive, concave function ϕ such that $\phi(t)$ goes to infinity when t goes to infinity and:*

$$\mathbb{E}(\Phi(X)) < \infty,$$

where $\Phi(t) = \int\limits_{x=0}^{t} \phi(x)\mathrm{d}x$.

Proof. First we show that there exists a non-decreasing, positive function $f : \mathbf{R}^+ \to \mathbf{R}^+$ such that $f(t)$ goes to infinity when t goes to infinity and:

$$\mathbb{E}(Xf(X)) < \infty.$$

To do that we first define the sequence (t_i) by:

$$\begin{aligned} t_0 =& 0 \\ t_{i+1} =& 1 + \inf \left\{ x \geq t_i, \mathbb{E}(X 1_{X>x}) \leq 2^{-(i+1)} \mathbb{E}(X) \right\}. \end{aligned}$$

Now we define f by:

$$f(x) = 1 + \sum_{i \geq 0} 1_{x \geq t_i}.$$

We clearly have that f is non-decreasing, positive ($f(t) \geq 2$) and that $f(t)$ goes to infinity when t goes to infinity. As for the expectation we have:

$$\begin{aligned} \mathbb{E}(Xf(X)) =& \mathbb{E}\left(\sum_{i \geq 0} X 1_{X \geq t_i} \right) + \mathbb{E}(X) \\ =& \sum_{i \geq 0} \mathbb{E}\left(X 1_{X \geq t_i} \right) + \mathbb{E}(X) \\ \leq& \sum_{i \geq 0} 2^{-i} \mathbb{E}(X) + \mathbb{E}(X) \\ \leq& 3\mathbb{E}(X) < \infty. \end{aligned}$$

Now we want to find an increasing concave function ϕ lower than f such that $\phi(t)$ goes to infinity when t goes to infinity. To that effect we will define the sequences (a_i) and (b_i) by:

$$\begin{aligned} a_0 =& 1, \\ b_0 =& \frac{1}{t_1}, \\ \forall i \in \mathbf{N},\ a_{i+1} =& a_i + b_i(t_{i+1} - t_i), \\ \forall i \in \mathbf{N},\ \min(b_{i+1} =& b_i, \frac{(i+2) - a_i}{t_{i+1} - t_i}) \end{aligned}$$

and we define ϕ by:

$$\forall i \in \mathbf{N}, \forall x \in [t_i, t_{i+1}), \phi(x) = a_i + b_i(x - t_i).$$

The function ϕ is continuous and its slope is decreasing so it is clearly concave.
We now have to prove that $\lim_{t \to \infty} \phi(t) = \infty$. First we want to show that for every $i \in \mathbf{N}$, $a_i \leq i + 1$. It is obvious for $i \in \{0, 1\}$ and for $i > 0$ we have:

$$a_i \leq a_{i-1} + \frac{(i+1) - a_{i-1}}{t_i - t_{i-1}}(t_i - t_{i-1}) = i + 1.$$

Now we want to show that there can be no i such that $b_i \leq 0$. If there was, we could define j by $j = \min\{i, b_i \leq 0\}$, we would have $j \geq 1$ and:

$$\frac{(j+1) - a_{j-1}}{t_j - t_{j-1}} \leq 0.$$

But since $a_{j-1} \leq j$ it is impossible so all the b_i are positive and therefore ϕ is increasing. Now we will prove that $\lim_{i \to \infty} a_i = \infty$. First we notice that if $b_{i+1} < b_i$ then $b_{i+1} = \frac{(i+2) - a_i}{t_{i+1} - t_i}$ so $a_{i+1} = i + 2$. Therefore, either the b_i are stationary and ϕ is larger than some affine function with positive slope which implies the result we want or the sequence b_i is not stationary and there are infinitely many i such that $a_{i+1} = i + 2$ and therefore we have the result we want.
We still have to show that $\phi \leq f$. We know that ϕ is increasing and we have:

$$\forall i \in \mathbf{N}, \forall x \in [t_i, t_{i+1}), f(x) - \phi(x) = i + 2 - \phi(x) \geq i + 2 - \phi(t_{i+1}) = i + 2 - a_{i+1} \geq 0.$$

So we have the desired result. $\square$

Lemma 2.3.0.2. *Let ϕ be a non-decreasing, positive concave function and $\Phi(x) := \int_{t=0}^{x} \phi(t)\mathrm{d}t$. There exists a constant C_ϕ such that if X is a geometric random variable with success probability p:*

$$\frac{1}{2}\Phi\left(\frac{1}{p}\right) \le \frac{1}{2}\frac{1}{p}\phi\left(\frac{1}{p}\right) \le \mathbb{E}(\Phi(1+X)) \le C_\phi \frac{1}{p}\phi\left(\frac{1}{p}\right) \le 2C_\phi \Phi\left(\frac{1}{p}\right).$$

Proof. Φ is convex so if X is a geometric random variable with success probability p:

$$\begin{aligned}
\mathbb{E}(\Phi(1+X)) \ge& \Phi(\mathbb{E}(1+X)) \\
=& \Phi\left(\frac{1}{p}\right) \\
=& \int_{t=0}^{\frac{1}{p}} \phi(t)\mathrm{d}t \\
=& \frac{1}{p}\int_{t=0}^{1} \phi\left(t\frac{1}{p}\right)\mathrm{d}t \\
\ge& \frac{1}{p}\int_{t=0}^{1} t\phi\left(\frac{1}{p}\right) + (1-t)\phi\left(0\right)\mathrm{d}t \\
\ge& \frac{1}{p}\int_{t=0}^{1} t\phi\left(\frac{1}{p}\right)\mathrm{d}t \\
=& \frac{1}{2}\frac{1}{p}\phi\left(\frac{1}{p}\right).
\end{aligned}$$

Now for the upper bound, we will first look at the case where $p \le \frac{1}{2}$:

$$\begin{aligned}
\mathbb{E}(\Phi(1+X)) =& \mathbb{E}\left(\int_{t=0}^{\infty} \phi(t)1_{1+X\ge t}\mathrm{d}t\right) \\
=& \int_{t=0}^{\infty} \phi(t)\mathbb{P}(X \ge t-1)\mathrm{d}t \\
\le& \int_{t=0}^{\infty} \phi(t)(1-p)^{t-1}\mathrm{d}t \\
\le& 2\int_{t=0}^{\infty} \phi(t)\exp(t\log(1-p))\mathrm{d}t \\
=& 2\frac{-1}{\log(1-p)}\int_{t=0}^{\infty} \phi\left(-\frac{t}{\log(1-p)}\right)\exp(-t)\mathrm{d}t \\
\le& 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right) + \int_{t=1}^{\infty} \phi\left(-\frac{t}{\log(1-p)}\right)\exp(-t)\mathrm{d}t\right).
\end{aligned}$$

Now we use the fact that ϕ is concave, this gives us, for $t \ge 1$:

$$\frac{1}{t}\phi\left(-\frac{t}{\log(1-p)}\right) + \left(1-\frac{1}{t}\right)\phi(0) \le \phi\left(-\frac{1}{\log(1-p)}\right).$$

Since ϕ is positive, we get:

$$\phi\left(-\frac{t}{\log(1-p)}\right) \le t\phi\left(-\frac{1}{\log(1-p)}\right).$$

So we get:

$$\begin{aligned}
\mathbb{E}(\Phi(1+X)) \leq & 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right)+\int_{t=1}^{\infty} t\phi\left(-\frac{1}{\log(1-p)}\right)\exp(-t)\mathrm{d}t\right)\\
\leq & 2\frac{-1}{\log(1-p)}\left(\phi\left(-\frac{1}{\log(1-p)}\right)+\phi\left(-\frac{1}{\log(1-p)}\right)\right)\\
= & 4\frac{-1}{\log(1-p)}\phi\left(-\frac{1}{\log(1-p)}\right).
\end{aligned}$$

Since $-\frac{p}{\log(1-p)} \leq 1$ and ϕ is increasing, we get:

$$-\frac{1}{\log(1-p)}\phi\left(-\frac{1}{\log(1-p)}\right) \leq \frac{1}{p}\phi\left(\frac{1}{p}\right).$$

And therefore, if $p \leq \frac{1}{2}$:

$$\mathbb{E}(\Phi(1+X)) \leq 4\frac{1}{p}\phi\left(\frac{1}{p}\right).$$

If $p \geq \frac{1}{2}$ we can couple X with a geometric random variable Y of parameter $\frac{1}{2}$ such that almost surely $Y \geq X$ and since Φ is increasing:

$$\mathbb{E}(\Phi(1+X)) \leq \mathbb{E}(\Phi(1+Y)) \leq 8\phi(2) \leq 8\phi(2)\frac{1}{p}\frac{\phi\left(\frac{1}{p}\right)}{\phi(1)} = 8\frac{\phi(2)}{\phi(1)}\frac{1}{p}\phi\left(\frac{1}{p}\right) \leq 16\frac{1}{p}\phi\left(\frac{1}{p}\right).$$

We get the upper bound we wanted.
Now we just have to prove that for any $x \geq 0$, $\frac{1}{2}x\phi(x) \leq \Phi(x) \leq x\phi(x)$. For the upper bound we have:

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t \leq \int_0^x \phi(x)\mathrm{d}t = x\phi(x).$$

And for the lower bound we have:

$$\Phi(x) = \int_0^x \phi(t)\mathrm{d}t = \int_0^x \phi\left(\frac{t}{x}x\right)\mathrm{d}t \geq \int_0^x \frac{t}{x}\phi(x)\,\mathrm{d}t = \frac{1}{2}x\phi(x).$$

□

Lemma 2.3.0.3. *Let X be a positive random variable, and let $a = \mathbb{E}(X)$ and $\tilde{X} = X - a$. If $Var(X) \leq a^2$ then:*

$$\forall\gamma \in [0,1],\ \ Var(X^\gamma) \leq 2a^{2\gamma}\left(\frac{Var(X)}{a^2}\right).$$

Proof. For any $x \in [-1,\infty)$, let $f_x : [0,1] \mapsto \mathbf{R}$ the function defined by

$$f_x(\gamma) := \gamma \to (1+x)^\gamma.$$

This function is convex and $f_x(1) = 1+x$ and $f_x'(1) = (1+x)\log(1+x)$ so:

$$\forall\gamma \in [0,1],\ \ f_x(\gamma) \geq 1+x+(\gamma-1)(1+x)\log(1+x) \geq 1+x-(1-\gamma)(1+x)x \geq 1+\gamma x-(1-\gamma)x^2.$$

By Jensen inequality, we have:

$$\mathbb{E}(X^\gamma) \leq a^\gamma.$$

Since $\mathbb{E}(X^\gamma) = a^\gamma\mathbb{E}\left(\left(1+\frac{\tilde{X}}{a}\right)^\gamma\right)$, we also get:

$$\mathbb{E}(X^\gamma) \geq a^\gamma\left(1-(1-\gamma)\frac{\mathrm{Var}(X)}{a^2}\right).$$

So if $\mathrm{Var}(X) \leq a^2$, then

$$-\mathbb{E}(X^\gamma)^2 \leq -a^{2\gamma}\left(1-(1-\gamma)\frac{\mathrm{Var}(X)}{a^2}\right)^2 \leq -a^{2\gamma}\left(1-2(1-\gamma)\frac{\mathrm{Var}(X)}{a^2}\right).$$

We also have:

$$\mathbb{E}(X^{2\gamma}) \leq \mathbb{E}(X^2)^\gamma = \left(a^2 + \mathrm{Var}(X)\right)^\gamma \leq a^{2\gamma}\left(1+\gamma\frac{\mathrm{Var}(X)}{a^2}\right).$$

Finally we get:

$$\mathrm{Var}\left(X^\gamma\right) \leq a^{2\gamma}\left(1+\gamma\frac{\mathrm{Var}(X)}{a^2} - 1 + 2(1-\gamma)\frac{\mathrm{Var}(X)}{a^2}\right) = a^{2\gamma}(2-\gamma)\frac{\mathrm{Var}(X)}{a^2}.$$

□

Lemma 2.3.0.4. *Let $p \in (0,\infty)$ be a positive real, $N \geq 1$ an integer, $h \in (\frac{1}{4},1)$ and $q \in (0,\infty)$ with $1 \geq q(1-h) \geq \frac{1}{2}$. Let (ε_i) be a sequence of integer in $\{0,1\}$. Let $(H_i)_{i\in\mathbf{N}}$ be a sequence of iid random variables following a geometric law of parameter h (here h is the probability of success). Let $(\mathscr{E}_{i,j})_{i,j\in\mathbf{N}}$ be a sequence of iid random variables , independent of (H_i) and following an exponential law of parameter p. Now let Z be defined by:*

$$Z = \sum_{i=1}^{N}\sum_{j=1}^{\varepsilon_i+H_i}\mathscr{E}_{i,j}\frac{p}{q}.$$

There exists a constant C such that if $N \geq 1$:

$$\forall\gamma\in[0,1],\ \ Var(Z^\gamma) \leq CN^{2\gamma-1} \leq CN^\gamma.$$

We also have that there are two constant $c_1, c_2 > 0$ that do not depend on γ such that:

$$c_1N^\gamma \leq \mathbb{E}(Z^\gamma) \leq c_2N^\gamma.$$

Proof. First we look at the expectation of Z, we get:

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=1}^{N}\mathbb{E}\left(\sum_{j=1}^{\varepsilon_i+H_i}\frac{1}{p}\frac{p}{q}\right)\\
&= \sum_{i=1}^{N}\frac{1}{q}\left(\varepsilon_i + \frac{h}{1-h}\right)\\
&= \frac{1}{q(1-h)}\sum_{i=1}^{N}\varepsilon_i(1-h)+h.
\end{aligned}$$

Now we will look at the variance but first we need a small result to simplify the notations, for this result, M will be a non negative random variable and $(X_i)_{i\in\mathbf{N}}$ a sequence of iid real random variables, independent of M. We get:

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{M}X_i\right) &= \mathbb{E}\left(\left(\sum_{i=1}^{M}X_i\right)^2\right) - \left(\mathbb{E}\left(\sum_{i=1}^{M}X_i\right)\right)^2\\
&= \mathbb{E}\left(M\mathbb{E}(X_1^2) + M(M-1)\mathbb{E}(X_1)^2\right) - \mathbb{E}(M)^2\mathbb{E}(X_1)^2\\
&= \mathbb{E}(M)\mathrm{Var}(X_1) + \mathrm{Var}(M)\mathbb{E}(X_1)^2.
\end{aligned}$$

Now we can compute the variance of Z. First we have:

$$\mathrm{Var}\left(Z\right) = \sum_{i=1}^{N}\mathrm{Var}\left(\sum_{j=1}^{\varepsilon_i+H_i}\mathscr{E}_{i,j}\frac{p}{q}\right) = \frac{p^2}{q^2}\sum_{i=1}^{N}\mathrm{Var}\left(\sum_{j=1}^{\varepsilon_i+H_i}\mathscr{E}_{i,j}\right).$$

Then we have:

$$\begin{aligned}
\frac{p^2}{q^2}\sum_{i=1}^{N}\mathbb{E}(\varepsilon_i+H_i)\mathrm{Var}(\mathscr{E}_{i,1}) &= \frac{p^2}{q^2}\sum_{i=1}^{N}\left(\varepsilon_i+\frac{h}{1-h}\right)\frac{1}{p^2}\\
&= \frac{1}{q^2(1-h)^2}\sum_{i=1}^{N}\varepsilon_i(1-h)^2 + h(1-h),
\end{aligned}$$

$$\begin{aligned}
\frac{p^2}{q^2}\sum_{i=1}^{N}\mathrm{Var}((\varepsilon_i+H_i)^2)\mathbb{E}(\mathscr{E}_{i,1})^2 &= \frac{p^2}{q^2}\sum_{i=1}^{N}\frac{h}{1-h}\frac{1}{p^2}\\
&= \frac{1}{q^2(1-h)^2}\sum_{i=1}^{N}h(1-h),
\end{aligned}$$

So we get, by summing these two equalities:

$$\mathrm{Var}\,(Z) = \frac{1}{q^2(1-h)^2}\sum_{i=1}^{N}\varepsilon_i(1-h)^2 + 2h(1-h).$$

We have assumed that $h \geq \frac{1}{4}$ and $\frac{1}{2} \leq q(1-h) \leq 1$ therefore we have:

$$\frac{1}{4}N \leq \mathbb{E}(Z) \leq 4N,$$

$$\mathrm{Var}(Z) \leq 20N.$$

Therefore we have:

$$\frac{\mathrm{Var}(Z)}{(\mathbb{E}(Z))^2} \leq 320\frac{1}{N}.$$

So by lemma 2.3.0.3, for $N \geq 320$ we have:

$$\forall \gamma \in [0,1],\ \mathrm{Var}\,(Z^\gamma) \leq 2\mathbb{E}(Z)^{2\gamma}\left(\frac{\mathrm{Var}(Z)}{\mathbb{E}(Z)^2}\right) \leq 4^{2\gamma}N^{2\gamma}\frac{640}{N}.$$

And if $N \leq 320$ we have:

$$\forall \gamma \in [0,1],\ \mathrm{Var}\,(Z^\gamma) \leq \mathbb{E}(Z^{2\gamma}) \leq \mathbb{E}(Z^2) \leq (20N + 16N^2).$$

So there exist a constant C such that if $1 \leq N \leq 320$:

$$\mathrm{Var}\,(Z^\gamma) \leq C\frac{1}{N}.$$

So

$$\forall \gamma \in [0,1],\ \mathrm{Var}\,(Z^\gamma) \leq CN^{2\gamma-1}.$$

So we have that there exists a constant C such that if $N \geq 1$:

$$\forall \gamma \in [0,1],\ \mathrm{Var}\,(Z^\gamma) \leq CN^{2\gamma-1} \leq CN^\gamma.$$

Now for the expectation, we first have the upper bound:

$$\mathbb{E}(Z^\gamma) \leq \mathbb{E}(Z)^\gamma \leq (4N)^\gamma.$$

For the lower bound, we will use Holder inequality:

$$\mathbb{E}(Z) = \mathbb{E}\left(Z^{\frac{\gamma}{2-\gamma}}Z^{2\frac{1-\gamma}{2-\gamma}}\right) \leq \mathbb{E}\left(Z^{\frac{\gamma}{2-\gamma}(2-\gamma)}\right)^{\frac{1}{2-\gamma}}\mathbb{E}\left(Z^{2\frac{1-\gamma}{2-\gamma}\frac{2-\gamma}{1-\gamma}}\right)^{\frac{1-\gamma}{2-\gamma}}.$$

This yields:

$$\mathbb{E}(Z)^{2-\gamma} \leq \mathbb{E}(Z^\gamma)\mathbb{E}(Z^2)^{1-\gamma}$$

ie:

$$E(Z^\gamma) \geq \frac{\mathbb{E}(Z)^{2-\gamma}}{\mathbb{E}(Z^2)^{1-\gamma}}.$$

Now we have $E(Z^2) = \mathrm{Var}(Z) + \mathbb{E}(Z)^2$ since $\mathrm{Var}(Z) \leq 80\mathbb{E}(Z)$ and $\mathbb{E}(Z) \geq \frac{1}{4}$ we have $\mathrm{Var}(Z) \leq 320\mathbb{E}(Z)^2$ and therefore: $E(Z^2) \leq 321E(Z)^2$ which yields:

$$E(Z^\gamma) \geq \frac{\mathbb{E}(Z)^{2-\gamma}}{(321\mathbb{E}(Z)^2)^{1-\gamma}} \geq \frac{\mathbb{E}(Z)^\gamma}{321^{1-\gamma}} \geq \frac{\mathbb{E}(Z)^\gamma}{321}.$$

□

Lemma 2.3.0.5. *Let $\beta \in [0,1]$. Let $(N_i)_{i\in\mathbf{N}^*}$ be a sequence of random positive integers and $(A_i)_{i\in\mathbf{N}}$ be a sequence of random finite subsets of $\mathbf{N}$ with the following two properties:*

$$\forall i \geq 0, A_i \subset A_{i+1},$$

$$\#A_i \to \infty.$$

Let $(Z_i)_{i\in\mathbf{N}}$ be independent exponential random variables of parameter 1 independent of $(A_i), (N_i)$.
Then there exists a constant $C > 0$ such that almost surely:

$$\exists m \in \mathbf{N}, \forall n \geq m, \sum_{i\in A_n}\left(\sum_{j=1}^{N_i}Z_i\right)^\beta \geq C\sum_{i\in A_n}(N_i)^\beta.$$

Proof. Let C be such that $2C - 2^{1-\beta} > 0$ Let $(n_i)_{i\in\mathbf{N}}$ be the sequence defined by:

$$n_i = \min\left\{i, \# \sum_{i\in A_n} (N_i)^\beta \geq 2^i\right\}.$$

We have that if

$$\exists m \in \mathbf{N}, \forall j \geq m, \sum_{i\in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta \geq 2C \sum_{i\in A_{n_j}} (N_i)^\beta$$

and M is such an m then for every $n \geq n_M$, if j is the integer that satisfies $n_j \leq n < n_{j+1}$, we have:

$$\begin{aligned}\sum_{i\in A_n} (N_i)^\beta \leq & 2^{j+1} \\ \leq & 2 \sum_{i\in A_{n_j}} (N_i)^\beta \\ \leq & 2C \sum_{i\in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta \\ \leq & 2C \sum_{i\in A_n} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta.\end{aligned}$$

By lemma 2.3.0.3, for any $i \in \mathbf{N}^*$:

$$\mathrm{Var}\left(\left(\sum_{j=1}^{N_i} \mathscr{E}_{i,j}\right)^\beta |(A_k),(N_k)\right) \leq 2(N_i)^{2\beta-1} \leq 2(N_i)^\beta.$$

And by Hölder:

$$\begin{aligned}\mathbb{E}\left(\left(\sum_{j=1}^{N_i} \mathscr{E}_{i,j}\right)^\beta |(A_k),(N_k)\right) \geq & \mathbb{E}\left(\sum_{j=1}^{N_i} \mathscr{E}_{i,j}|(A_k),(N_k)\right)^{2-\beta} \mathbb{E}\left(\left(\sum_{j=1}^{N_i} \mathscr{E}_{i,j}\right)^2 |(A_k),(N_k)\right)^{-(1-\beta)} \\ = & (N_i)^{2-\beta}(N_i^2 + N_i)^{-(1-\beta)} \\ \geq & (N_i)^{2-\beta}(2N_i^2)^{-(1-\beta)} \\ = & 2^{\beta-1}(N_i)^\beta.\end{aligned}$$

Now we get:

$$\begin{aligned}& \sum_{j\geq 0} \mathbb{P}\left(\sum_{i\in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta \leq 2C \sum_{i\in A_{n_j}} (N_i)^\beta\right) \\ \leq & \sum_{j\geq 0} \mathbb{E}\left(\frac{\mathrm{Var}\left(\sum_{i\in A_{n_j}} \left(\sum_{k=1}^{N_i} Z_i\right)^\beta |(A_k),(N_k)\right)}{\left((2C - 2^{1-\beta}) \sum_{i\in A_{n_j}} (N_i)^\beta\right)^2}\right) \\ \leq & \sum_{j\geq 0} \mathbb{E}\left(\frac{2}{(2C - 2^{1-\beta})^2 \sum_{i\in A_{n_j}} (N_i)^\beta}\right) \\ \leq & \frac{2}{(2C - 2^{1-\beta})^2} \sum_{j\geq 0} 2^{-j} < \infty.\end{aligned}$$

So by Borell-Cantelli we get the result we want □

Lemma 2.3.0.6. *Let f, g be two non-decreasing positive càdlàg functions with $f(0) = g(0) = 0$. Let $A, B > 0$ be constants such that $f(A) \geq B$ and $g(A) \geq B$. Let $\varepsilon, \delta > 0$ be such that:*

$$\forall t \in [0, A+\varepsilon], g(t+\varepsilon) \geq g(t) + \delta$$

and

$$\sup\{|f(t)-g(t)|, t\in[0,A+2\varepsilon]\} \leq \frac{\delta}{2}.$$

Then:

$$\sup\{|f^{-1}(x)-g^{-1}(t)|, t\in[0,B]\} \leq 2\varepsilon.$$

Proof. Let t be in $[0,B]$. First we have:

$$f\left(g^{-1}(t)+2\varepsilon\right) \geq g\left(g^{-1}(t)+2\varepsilon\right) - \frac{\delta}{2} \geq g(g^{-1}(t)+\varepsilon)+\delta-\frac{\delta}{2} \geq t.$$

Therefore $f^{-1}(t) \leq g^{-1}(t)+\varepsilon$. Similarly we have:

$$f\left(g^{-1}(t)-\varepsilon\right) \leq g\left(g^{-1}(t)-\varepsilon\right) + \frac{\delta}{2} \leq g(g^{-1}(t))-\delta+\frac{\delta}{2} < t.$$

Therefore $f^{-1}(t) \geq g^{-1}(t)-\varepsilon$. So we have the result we want. □

Lemma 2.3.0.7. *Let $t \to \mathscr{S}^\kappa(t)$ be the jump process where $\mathscr{S}^\kappa(1)$ is a completely asymmetric, positive stable law of parameter κ. For any $\varepsilon > 0$ and any $B > 0$ there exists $A > 0$ and $\delta > 0$ such that:*

$$\mathbb{P}(\mathscr{S}^\kappa(A) \geq B) \geq 1-\varepsilon,$$

$$\mathbb{P}(\exists t \leq A-\varepsilon, \mathscr{S}^\kappa(t+\varepsilon)-\mathscr{S}^\kappa(t) < \delta) \leq \varepsilon.$$

Proof. There clearly exists an A that satisfies the first property. Now we need to find a δ that satisfies the second inequality for this A. We will look at a slightly different property:

$$\exists i \leq \frac{2A}{\varepsilon}, \mathscr{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathscr{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta.$$

Since for every $t \leq A-\varepsilon$ there exists $i \leq \frac{2A}{\varepsilon}$ such that: $[i\frac{\varepsilon}{2}, (i+1)\frac{\varepsilon}{2}] \subset [t, t+\varepsilon]$, we have that for any $\delta > 0$:

$$\mathbb{P}(\exists t \leq A-\varepsilon, \mathscr{S}^\kappa(t+\varepsilon)-\mathscr{S}^\kappa(t+\varepsilon) \leq \delta) \leq \mathbb{P}\left(\exists i \leq \frac{2A}{\varepsilon}, \mathscr{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathscr{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta\right).$$

And there clearly exists δ such that

$$\mathbb{P}\left(\exists i \leq \frac{2A}{\varepsilon}, \mathscr{S}^\kappa\left(i\frac{\varepsilon}{2}\right) - \mathscr{S}^\kappa\left((i+1)\frac{\varepsilon}{2}\right) \leq \delta\right) \leq \varepsilon.$$

So we get the result we want. □

Chapter 3

Random walk in random environment and their time-reversed counterpart

Up to minor modifications, this chapter is the same as the article [65] available on Arxiv.

Abstract

The random walk in Dirichlet environment is a random walk in random environment where the transition probabilities are independent Dirichlet random variables. This random walk exhibits a property of statistical invariance by time-reversal which leads to several results. More precisely, a time-reversed random walk in Dirichlet environment (with null divergence) is also a random walk in random environment where the transition probabilities are independent Dirichlet random variables with different parameters. We show that on all graphs that satisfy a few weak assumptions, a random walk in random environment with independent transition probabilities and such that the transition probabilities of the time-reversed random walk in random environment are also independent is a random walk in Dirichlet environment.

3.1 Introduction and results

3.1.1 Introduction

In this paper we deal with random walks in random environments (RWRE) with independent transition probability. This model has been studied for several decades [90]: since the 80s in dimension 1 and for two decades in higher dimensions. In one dimension RWRE exhibit a key property: reversibility. Thanks to this property, the one dimensional case is now well understood (see Solomon [80],Kesten, Kozlov, Spitzer [51] and Sinaï [78]). Unfortunately, in higher dimensions RWRE with iid transition are no longer reversible. For this reason, RWRE in higher dimension is not as well understood as the one dimensional case but important progress has been made. For instance, under some assumptions (see [83],[81],[46],[13],[22]) regarding the directional transience of the walk and ellipticity, ballisticity and annealed CLT have been proved. Some quenched CLTs have also been proved under stronger assumptions ([82],[86],[66],[14]). Another direction taken was to look at small perturbation of the simple walk ([21],[85],[17],[68],[54]).

In this paper we look at a specific case of RWRE: random walks in Dirichlet environment. That is to say random walk where the transition probabilities at each site are iid and have a Dirichlet distribution. It was first introduced because of its link to the linearly directed-edge reinforced random walk ([62]). This model (under an additional property of null-divergence) exhibits a property of invariance after time-reversal, that is to say that the time reversed random walk is also a random walk in Dirichlet environment ([69],[74]). This property makes some calculations explicit which allows to find some non-trivial results. For instance, it was shown that in dimension $d \geq 3$, the walk is transient ([69]) and there is an invariant distribution for the process seen from the point of view of the particle ([70]). It is also known for which parameters the walk is directionaly transient ([74],[88],[18]) and in this case the walk is either ballistic or converges to a stable Levy process ([63]). These are still open questions in the general case.

A natural question is whether the random walk in Dirichlet environment is the only RWRE with independent transition probability that exhibits the time-reversal property. We show that, indeed, under some weak conditions on the graph considered, if both the environment and the time-reversed environment have transition probabilities independent at each site, then the transition probabilities follow a Dirichlet distribution (with null-divergence).

3.1.2 Definition and statement of the results

We will first need a few definitions before we can properly state the result. We will look at random environments on directed graphs.

Definition 34. *An environment ω on an oriented graph $G = (V, E)$ is a function from the set of edges E to $[0, 1]$ such that for any vertex $x \in E$ we have:*

$$\sum_{y,(x,y)\in E} \omega(x, y) = 1.$$

For any oriented graph G let Ω^G be the set of all environments on G.

The goal of this paper is to study the time-reversed walk. It is obtained by reversing the graph and the environment on this graph. To any oriented graph and any environment on this graph the associated reversed graph and environment are defined as follows.

Definition 35. *For any graph (V, E), its reversed graph $(\tilde{V}, \tilde{E})$ is obtained by keeping all the vertices and flipping all the edges ie: $\tilde{V} = V$ and $\tilde{E} = \{(x, y), (y, x) \in E)\}$*

Definition 36. *Let (V, E) be a graph and ω an environment on this graph. The reversed environment $\tilde{\omega}$ on the reversed graph $(\tilde{V}, \tilde{E})$ is defined by $\tilde{\omega}(x, y) = \omega(y, x)\frac{\pi_y}{\pi_x}$ where π is the stationary distribution (i.e for any vertex x, $\pi_x = \sum \omega(y, x)\pi_y$).*

For a given environment, it is not easy to compute its reversed environment. As a consequence, for a given graph and a given law on its environments, it is in general hard to compute the law of the reversed environment. However, in the specific case of independent sites with Dirichlet distribution and null divergence, the computation becomes easy. We will now explain what the Dirichlet distribution is. For any family $(\alpha_1, \ldots, \alpha_n)$ of positive weights, Dirichlet random variables of parameter $\alpha := (\alpha_1, \ldots, \alpha_{2d})$ have the following density:

$$\frac{\Gamma\left(\sum_{i=1}^{2d} \alpha_i\right)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i - 1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_{2d-1}$$

on the simplex

$$\{(x_1, \ldots, x_{2d}) \in (0, 1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}.$$

Let $G = (V, E)$ be a directed graph. Let $\alpha := (\alpha_e)_{e\in E}$ be a family of positive weights. Now, let $\mathbb{P}^{G,\alpha}$ be the law on Ω where the transition probabilities at each vertex are independent. For any vertex $x \in V$, the law of the family $(\omega(x, y))_{y,(x,y)\in E}$ under $\mathbb{P}^{G,\alpha}$ is a Dirichlet law of parameters $(\alpha_{(x,y)})_{y,(x,y)\in E}$. It was proved in [69] that if the divergence of α is null:

$$\forall x \in V, \quad \sum_{y,(x,y)\in E} \alpha_{(x,y)} = \sum_{y,(y,x)\in E} \alpha_{(y,x)}$$

then the law of the reversed environment is $\mathbb{P}^{\tilde{G},\check{\alpha}}$ with $\check{\alpha}_{(x,y)} = \alpha_{(y,x)}$. This property makes many calculations explicit. We want to see if it is possible to find other law on the set of environments where the transition probabilities are iid at each site and the law of the reversed environment is also iid on each site. We show that under weak assumptions on the graph considered, no such other non-deterministic law exists. To precisely state this, we first need a couple of definitions regarding graphs.

Definition 37. *A directed graph (V, E) is strongly connected if for any pair of vertices $(x, y) \in V^2$ there exists a path along the oriented edges of the graph that goes from x to y.*

Definition 38. *A directed graph (V, E) is 2-connected if it is strongly connected and if for any vertex $x \in V$, the graph obtained by removing x and all the edges coming from x or going to x is still strongly connected.*

Now we can finally state the theorem.

Theorem 53. *Let (V, E) be a finite directed graph and ω transition probabilities on this graph that satisfy the following properties:*

- *the graph has no multiple edges,*
- *the graph and the reversed graph are 2-connected,*
- *the transition probabilities are of positive expectation,*
- *the transition probabilities are independent.*

If the transition probabilities of the reversed environment are also independent then, the transition probabilities are independent Dirichlet random variables with null divergence or are deterministic.

3.2 The proof

3.2.1 A few general results on graphs

We will use the following notations for simplification.

Definition 39. *Let $G = (V, E)$ be a directed graph, for every vertex $x \in V$ we define E_x as the set of all edges starting at x and E^x as the set of all edges ending at x. And we also define: $V_x = \{y \in V, (x, y) \in E_x\}$ and $V^x = \{y \in V, (y, x) \in E^x\}$.*

In almost all the proofs we will use paths and cycle on graphs. They are defined as follows.

Definition 40. *A path γ on a graph (V, E) is a sequence of vertices $(\gamma_1, \ldots, \gamma_n)$ such that for any $i < n, (\gamma_i, \gamma_{i+1}) \in E$.*
The length of $|\gamma|$ of the path γ is equal to n.
We will write $(x, y) \in \gamma$ to say that there exists an integer $i \in [\![1, n-1]\!]$ such that $(x, y) = (\gamma_i, \gamma_{i+1})$.
A cycle σ is a path such that $\sigma_1 = \sigma_{|\sigma|}$.

We will now prove a couple of results on oriented graphs that will help us for the proof of the main result.

Lemma 3.2.1.1. *If (V,E) be a 2-connected graph, with no multiple edges and at least three vertices, then for every $x \in V$, E_x contains at least two edges.*

Proof. Let x be a vertex in V, then E_x contains at least one edge because otherwise there would be no non-trivial path starting at x and the graph would not be strongly connected. Now, if E_x had only one edge (x, y) then for the same reasons, the graph obtained by removing the vertex y would not be strongly connected. Therefore, E_x contains at least two elements □

Lemma 3.2.1.2. *Let (V,E) be a 2-connected graph, with no multiple edges. Let x_1,x_2 and y be three distinct vertices in V. There exists two simple paths γ^1 and γ^2 going from x_1 to y and from x_2 to y respectively such that the only point at which they intersect is y (ie $\gamma^1_i = \gamma^2_j$ iff $\gamma^1_i = y$).*

Proof. First, for any $z_1, z_2 \in V$, let $\Gamma(z_1, z_2)$ be the set of simple paths that go from z_1 to z_2 along the directed edges of E. Now for any three distinct points $x_1, x_2, z \in V$, we look at the set of paths:

$$\tilde{\Gamma}'(x_1, x_2, y) := \bigcup_{z \in V} \Gamma(x_1, z) \times \Gamma(x_2, z) \times \Gamma(z, y).$$

Let $\tilde{\Gamma}$ be the subset of $\tilde{\Gamma}'$ that only contains triplets $\gamma^1, \gamma^2, \tilde{\gamma}$ such that the only point at which at least two of the paths intersect is the endpoint of γ^1 and γ^2 which is also the starting point of $\tilde{\gamma}$.
We first want to show that $\tilde{\Gamma}$ is not empty. We take two simple paths γ^1 and γ^2 going from x_1 to y and from x_2 to y respectively. If they do not intersect except at y then we have proved the lemma, otherwise we define:

$$\tau_2 := \inf\left\{i \geq 0, \exists j \geq 0, \gamma^1_j = \gamma^2_i\right\} < \infty,$$

and

$$\tau_1 := \inf\left\{i \geq 0, \gamma^1_i = \gamma^2_{\tau_2}\right\} < \infty.$$

The triplet $((\gamma^1_0, \ldots, \gamma^1_{\tau_1}), (\gamma^2_0, \ldots, \gamma^2_{\tau_2}), (\gamma^1_{\tau_1}, \ldots, \gamma^1_{|\gamma^1|}))$ is in $\tilde{\Gamma}$ by definition of τ_1 and τ_2. Therefore $\tilde{\Gamma}$ is not empty. Now we look at a triplet $(\gamma^1, \gamma^2, \tilde{\gamma}) \in \tilde{\Gamma}$ that minimizes the length $|\tilde{\gamma}|$. If $\tilde{\gamma}$ is just a point then we have the result we want. Otherwise we show that we can shorten the length of $\tilde{\gamma}$. First, let z be the point at which the three paths of the triplet intersect. Now let p be a path that goes from x_1 to y without going through z.

Let $\tau_1 = \sup\{i, \exists j \in \mathbf{N}, p_i = \gamma^1_j \text{ or } p_i = \gamma^2_j\}$ and $\tau_2 = \inf\{i > \tau_1, \exists j \in \mathbf{N}, p_i = \tilde{\gamma}_j\}$. We will assume that p_{τ_1} is a point of γ_1 (the proof is exactly the same if it is a point of γ_2). We also define $\tilde{\tau}_1 = \inf\{i \in \mathbf{N}, \gamma^1_i = p_{\tau_1}\}$ and $\tilde{\tau}_2 = \sup\{i \in \mathbf{N}, \tilde{\gamma}_i = p_{\tau_2}\}$. The following triplet is an element of $\tilde{\Gamma}$ with a smaller length for the third element compared to $(\gamma^1, \gamma^2, \tilde{\gamma})$:

$$((\gamma^1_0, \dots, \gamma^1_{\tilde{\tau}_1}, p_{\tau_1+1}, \dots, p_{\tau_2}), (\gamma^2_0, \dots, \gamma^2_{|\gamma^2|}, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{\tilde{\tau}_2}), (\tilde{\gamma}_{\tilde{\tau}_2}, \dots, \tilde{\gamma}_{\tilde{\gamma}}))$$

and therefore we have the desired result. □

3.2.2 Moment functions

Definition 41. *Let $G = (V, E)$ be a directed graph. A function $N : E \mapsto \mathbf{R}$ is said to be of null divergence if:*

$$\forall x \in V, \sum_{e \in E_x} N(e) = \sum_{e \in E^x} N(e).$$

Definition 42. *A moment function f of a graph (V, E) will be a set of functions $\{f_x, x \in V\}$ such that for every vertex $x \in V$, f_x is a function from $\mathbf{N}^{E_x}$ to $(0, \infty)$. To simplify notation, instead of writing:*

$$f_x(n_{xy_1}, n_{xy_2}, \dots, n_{xy_r})$$

we will write:

$$f_x\left(\sum_{e \in E_x} n_e \vec{e}\right).$$

Definition 43. *Let $G = (V, E)$ be a graph, f a moment function of G and $\check{f}$ be a moment function of the reversed graph $\tilde{G} = (v, \tilde{E})$. f and $\check{f}$ are compatible if for all functions $N : E \mapsto \mathbf{N}$ of null divergence:*

$$\prod_{x \in V} f_x\left(\sum_{y \in V_x} N((x,y))\vec{xy}\right) = \prod_{x \in V} \check{f}_x\left(\sum_{y \in V^x} N((y,x))\overrightarrow{xy}\right).$$

Let (V, E) be a graph and ω a random environment on this graph that both satisfy the conditions of the theorem. To prove our theorem, we first want to show that the moment of our transition probabilities are of the form :

$$\mathbb{E}\left(\prod_{y \in V^x} \omega(x,y)^{n_{(x,y)}}\right) = \frac{\prod_{y \in V^x} h_{(x,y)}(n_{(x,y)})}{\tilde{h}_x\left(\sum_{y \in V^x} n_{(x,y)}\right)},$$

for some functions $(h_e)_{e \in E}$ ad $(\tilde{h}_x)_{x \in V}$. In order to do that, we first need to find suitable candidates for the functions $(h_e)_{e \in E}$ and $(\tilde{h}_x)_{x \in V}$. The following lemma will help us do that (in the proof of the theorem where we prove that the moments of the transition probabilities of the environment ω and its reversed environment are compatible moment functions).

Lemma 3.2.2.1. *Let $G = (V, E)$ be a 2-connected directed graph. Let f be a moment function of G and $\check{f}$ be a moment function of the reversed graph $\tilde{G}$ such that f and $\check{f}$ are compatible. We also assume that:*

$$\forall x \in V, f_x(0) = \check{f}_x(0) = 1.$$

Then for every vertex $x \in V$ there exists a function $\tilde{h}_x : \mathbf{N} \mapsto (0, \infty)$ and for every edge $e \in E$ there exists a function $h_e : \mathbf{N} \mapsto (0, \infty)$ such that:

$$\forall x \in V, \forall y \in V_x, \forall n \in \mathbf{N}, f_x(n\overrightarrow{xy}) = \frac{h_{(x,y)}(n)}{\tilde{h}_x(n)} \text{ and}$$
$$\forall x \in V, \forall y \in V^x, \forall n \in \mathbf{N}, \check{f}_x(n\overrightarrow{xy}) = \frac{h_{(y,x)}(n)}{\tilde{h}_x(n)}.$$

Proof. For $n = 0$, the result is obvious, we just need to take $h_e(0) = 1$ for all $e \in E$ and $\tilde{h}_x(0) = 1$ for all $x \in V$. Now we choose $n \geq 1$. For every edge $(x, y) \in E$ we write:

$$g_{(x,y)}(n) := \frac{f_x(n\overrightarrow{xy})}{\check{f}_y(n\overrightarrow{yx})}.$$

For a simple cycle (i.e a cycle that never visits a point more than once, except for the first point because it is also the last one) σ, the compatibility of f and $\check{f}$ tells us that:

$$\prod_i f_{\sigma(i)}(n\overrightarrow{\sigma(i)\sigma(i+1)}) = \prod_i \check{f}_{\sigma(i)}(n\overrightarrow{\sigma(i)\sigma(i-1)}),$$

which means:

$$\prod_i g_{(\sigma(i),\sigma(i+1))}(n) = 1.$$

Since any cycle is the union of simple cycles, the above property is true for any cycle. Now, we choose a vertex $x \in V$ and we set $\tilde{h}_x(n) := 1$. Let y be a vertex in V, γ^1 and γ^2 two paths from x to y and $\tilde{\gamma}$ a path from y to x. We have:

$$\prod_i g_{(\gamma_i^1,\gamma_{i+1}^1)}(n) \prod_i g_{(\tilde{\gamma}_i,\tilde{\gamma}_{i+1})}(n) = 1 \text{ and } \prod_i g_{(\gamma_i^2,\gamma_{i+1}^2)}(n) \prod_i g_{(\tilde{\gamma}_i,\tilde{\gamma}_{i+1})}(n) = 1.$$

Therefore,

$$\prod_i g_{(\gamma_i^1,\gamma_{i+1}^1)}(n) = \prod_i g_{(\gamma_i^2,\gamma_{i+1}^2)}(n).$$

This equality allows us to define $\tilde{h}_y(n)$ by:

$$\tilde{h}_y(n) := \prod_i g_{(\gamma_i^1,\gamma_{i+1}^1)}(n),$$

because it does not depend on the path we chose to get to y. Now let y_1 and y_2 be two vertices such that (y_1, y_2) is an edge. Let γ be a path that goes from x to y_1 and $\tilde{\gamma}$ be the same path to which we add the edge (y_1, y_2) at the end, we have:

$$g_{(y_1,y_2)}(n) = \frac{g_{(y_1,y_2)}(n) \prod g_{(\gamma(i),\gamma(i+1))}(n)}{\prod g_{(\gamma(i),\gamma(i+1))}(n)} = \frac{\prod g_{(\tilde{\gamma}(i),\tilde{\gamma}(i+1))}(n)}{\prod g_{(\gamma(i),\gamma(i+1))}(n)} = \frac{\tilde{h}_{y_2}(n)}{\tilde{h}_{y_1}(n)}.$$

And now, if we choose $h_{(y_1,y_2)}(n)$ such that:

$$f_{y_1}(n\overrightarrow{y_1y_2}) = \frac{h_{(y_1,y_2)}(n)}{\tilde{h}_{y_1}(n)},$$

then we also have:

$$\check{f}_{y_2}(n\overrightarrow{y_2y_1}) = \frac{h_{(y_1,y_2)}(n)}{\tilde{h}_{y_2}(n)}.$$

□

Unfortunately the functions we have found are not uniquely defined. Indeed, for any $n \in \mathbf{N}^*$ we can multiply all values $(h_e(n))_{e\in E}$ and $(\tilde{h}_x)_{x\in V}$ by a constant $\Delta(n) \neq 0$ and all the properties of the previous lemma would stay true. The goal of the next lemma is to show that it is the only change we can make to the previous functions. This will be used, in addition to the previous lemma, to show (in the proof of the theorem) that the moments of transition probabilities ω that satisfy the conditions of the theorem are of the form:

$$\mathbb{E}\left(\prod_{y\in V^x} \omega(x,y)^{n_{(x,y)}}\right) = \frac{\prod_{y\in V^x} h_{(x,y)}(n_{(x,y)})}{\tilde{h}_x\left(\sum_{y\in V^x} n_{(x,y)}\right)},$$

for some functions $(\tilde{h}_x)_{x\in V}$ and $(h_e)_{e\in E}$.

Lemma 3.2.2.2. *Let $G = (V, E)$ be a 2-connected graph and f and $\check{f}$ two compatible moment functions such that:*

$$\forall (x,y) \in E, \forall n \in N, f_x(n\overrightarrow{xy}) = \check{f}_y(n\overrightarrow{yx}) = 1.$$

There exists $\Delta : \mathbf{N} \mapsto (0,\infty)$ with $\Delta(0) = \Delta(1) = 1$ such that:

$$\forall x \in V, f_x\left(\sum_{y\in V_x} N((x,y))\overrightarrow{xy}\right) = \frac{\prod_{y\in V_x} \Delta(N((x,y)))}{\Delta\left(\sum_{y\in V_x} N((x,y))\right)}$$

and,

$$\forall x \in V, \check{f}_x\left(\sum_{y\in V^x} N((y,x))\overrightarrow{xy}\right) = \frac{\prod_{y\in V^x} \Delta(N((y,x)))}{\Delta\left(\sum_{y\in V^x} N((y,x))\right)}.$$

Proof. We will construct Δ by induction. There will be two parts to the proof, first the existence of $\Delta(2)$ and then the existence of $\Delta(i)$ for $i \geq 3$.
First we prove the existence of $\Delta(2)$. Let $x \in V$ be a vertex, we want to show that there exists $\Delta_x(2)$ such that:

$$\forall y_1, y_2 \in V_x, y_1 \neq y_2 \implies f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = \frac{\Delta(1)^2}{\Delta_x(2)} = \frac{1}{\Delta_x(2)}$$
$$\text{and } \forall y_1, y_2 \in V^x, y_1 \neq y_2 \implies \check{f}_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = \frac{1}{\Delta_x(2)}.$$

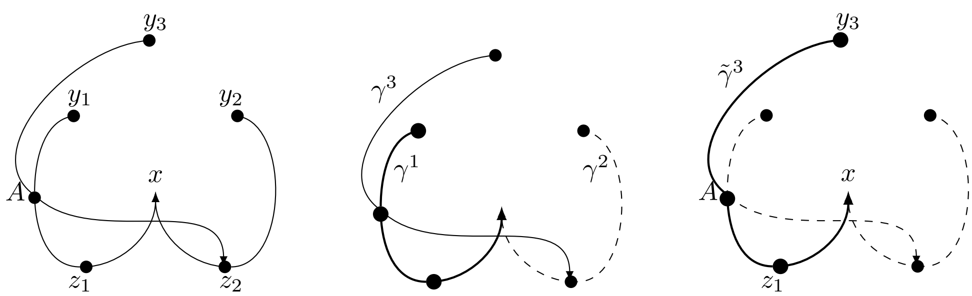
If both V_x and V^x have only two elements then we call the two elements of V_x: y_1 and y_2. By lemma 3.2.1.2 there exists two paths γ^1 and γ^2 that go respectively from y_1 to x and from y_2 to x and that only intersect in x. We call z_1 and z_2 the vertices such that the last edges crossed by γ^1 and γ^2 are (z_1, x) and (z_2, x) respectively. We call σ^1 and σ^2 the simple cycles such that σ^1 goes through (x, y_1) and then follows the path γ^1 and σ^2 goes through (x, y_2) and then follows the path γ^2. By definition of γ^1 and γ^2, the cycles σ^1 and σ^2 only intersect in x, so every vertex other than x is visited by at most only one of the two cycles (and only once because they are simple cycles), in particular $z_1 \neq z_2$. We have that:

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) \prod_{v\in V\setminus\{x\}} f_v\left(\sum_{u\in V_v} (1_{(v,u)\in\sigma^1} + 1_{(v,u)\in\sigma^2})\overrightarrow{vu}\right)$$
$$= \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2}) \prod_{v\in V\setminus\{x\}} \check{f}_v\left(\sum_{u\in V^v} (1_{(u,v)\in\sigma^1} + 1_{(u,v)\in\sigma^2})\overrightarrow{vu}\right).$$

This means that

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2}).$$

Therefore $\Delta_x(2)$ exists.
Otherwise we can assume, without loss of generality, that V_x has at least three elements (if it is not the case, then V^x has at least three elements by lemma 3.2.1.1 and we just have to consider the reversed graph). For the sake of clarity, the figure below describes what is written in this paragraph. Let y_1, y_2 and $y_3 \in V_x$ be three distinct vertices. Let γ^1 and γ^2 be two simple paths that go from y_1 to x and from y_2 to x respectively and that only intersect in x. We call z_1 the vertex such that (z_1, x) is the last edge crossed by γ^1 and z_2 the vertex such that (z_2, x) is the last edge crossed by γ^2. Now let γ^3 be a path from y_3 to z_1 or z_2 that does not go through x. Let τ_1 be the first time such that $\gamma^3(\tau_1)$ is either a vertex of γ^1 or γ^2 and let A be the corresponding vertex, $A := \gamma^3(\tau_1)$. We can assume that A is a vertex of γ^1 (if it is not the case we can just exchange the role of y_1 and y_2). Let τ_2 be the time such that $\gamma^1_{\tau_2} = \gamma^3_{\tau_1}$. Let $\tilde{\gamma}^3$ be the path that starts at y_3 then follows γ^3 up to $\gamma^3_{\tau_1}$ and then follows γ^1 from $\gamma^1_{\tau_2}$ to x. In particular, the last edge crossed by $\tilde{\gamma}^3$ is (z_1, x).



Now we have:

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) \prod_{v\in V\setminus\{x\}} f_v\left(\sum_{u\in V_v} (1_{(v,u)\in\gamma^1} + 1_{(v,u)\in\gamma^2})\overrightarrow{vu}\right)$$
$$= \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2}) \prod_{v\in V\setminus\{x\}} \check{f}_v\left(\sum_{u\in V^v} (1_{(u,v)\in\gamma^1} + 1_{(u,v)\in\gamma^2})\overrightarrow{vu}\right)$$

Now since every vertex besides x is visited at most once by γ^1 and γ^2 (since these two simple paths only intersect in x) we get:

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2})$$

by the same arguments, using γ^2 and $\tilde{\gamma}^3$ we get

$$f_x(\overrightarrow{xy_3} + \overrightarrow{xy_2}) = \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2}).$$

So

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = f_x(\overrightarrow{xy_3} + \overrightarrow{xy_2})$$

Now, if we do exactly the same thing except we switch y_2 and y_3. We get that either

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_3}) = f_x(\overrightarrow{xy_2} + \overrightarrow{xy_1})$$

or

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_3}) = f_x(\overrightarrow{xy_2} + \overrightarrow{xy_3}).$$

Either way, we get:

$$f_x(\overrightarrow{xy_1} + \overrightarrow{xy_3}) = f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = f_x(\overrightarrow{xy_2} + \overrightarrow{xy_3}) = \check{f}_x(\overrightarrow{xz_1} + \overrightarrow{xz_2}).$$

Now either V^x has three elements or more in which case the same arguments hold for $\check{f}$ or it has only two elements and we only have to consider $\check{f}_x(\overrightarrow{xz_2} + \overrightarrow{xz_1})$, in both case we know that $f_x(\overrightarrow{xy_1} + \overrightarrow{xy_2}) = \check{f}_x(\overrightarrow{xz_2} + \overrightarrow{xz_1})$ so $\Delta_x(2)$ exists.

Now we have to prove that $\Delta_x(2)$ does not depend on x. Let x and y be two points such that $(x, y) \in E$. We want to prove that there exists two simple cycles that both contain the edge (x, y) and that only intersect in x and y. It is clearly equivalent to prove that there exists two simple paths that begin at y and end at x and that only intersect at x and y. To prove that, let $z \in V^x$ be a vertex, we know that there exists two simple paths p^1 and p^2 that go respectively from y to x and from z to x and that only intersect at x, now we look at the two simple cycles defined as follows: σ^1 is the cycle that starts at x then goes through (x, y) and then follows p^1 back to x and σ^2 is the cycle that starts at x then goes through (x, y) and (y, z) and then follows p^2 back to x. This gives us:

$$\prod_{v \in V} f_v \left(\sum_{u \in V_v} \left(1_{(v,u) \in \sigma^1} + 1_{(v,u) \in \sigma^2} \right) \overrightarrow{vu} \right) = \prod_{v \in V} \check{f}_v \left(\sum_{u \in V^v} \left(1_{(u,v) \in \sigma_1} + 1_{(u,v) \in \sigma_2} \right) \overrightarrow{vu} \right).$$

Since for every $v \in V \backslash \{x, y\}$, v cannot be visited by both σ^1 and σ^2, we get that for every $v \in V \backslash \{x, y\}$:

$$f_v \left(\sum_{u \in V_v} \left(1_{(v,u) \in \sigma^1} + 1_{(v,u) \in \sigma^2} \right) \overrightarrow{vu} \right) = 1 \text{ and } \check{f}_v \left(\sum_{u \in V^v} \left(1_{(u,v) \in \sigma^1} + 1_{(u,v) \in \sigma^2} \right) \overrightarrow{vu} \right) = 1.$$

Thus:

$$f_x(2\overrightarrow{xy}) f_y \left(\sum_{u \in V_y} \left(1_{(y,u) \in \sigma^1} + 1_{(y,u) \in \sigma^2} \right) \overrightarrow{yu} \right) = \check{f}_x \left(\sum_{u \in V^x} \left(1_{(u,x) \in \sigma^1} + 1_{(u,x) \in \sigma^2} \right) \overrightarrow{xu} \right) \check{f}_y(2\overrightarrow{yx}).$$

This in turns means that

$$f_y \left(\sum_{u \in V_y} \left(1_{(y,u) \in \sigma^1} + 1_{(y,u) \in \sigma^2} \right) \overrightarrow{yu} \right) = \check{f}_x \left(\sum_{u \in V^x} \left(1_{(u,x) \in \sigma^1} + 1_{(u,x) \in \sigma^2} \right) \overrightarrow{xu} \right).$$

Therefore

$$\frac{\Delta(1)^2}{\Delta_x(2)} = \frac{\Delta(1)^2}{\Delta_y(2)}.$$

So we get $\Delta_x(2) = \Delta_y(2)$ and since the graph is connected, $\Delta(2)$ exists.

Now we can prove the existence of $\Delta(i)$ for $i \geq 3$ by induction.

First we assume that $\Delta(i)$ exists for $i \leq n$ for some $n \geq 2$. We want to prove that $\Delta(n+1)$ exists. First let $x \in V$ be a point of the graph and set two vertices $y_1, y_2 \in V_x$. We know that there exists two simple paths $\gamma^1 : y_1 \to x$ and $\gamma^2 : y_2 \to x$ that only intersect in x by lemma 3.2.1.2. We will call z_1 and z_2 the points such that the last edge through which γ^1 and γ^2 go are (z_1, x) and (z_2, x) respectively. We now consider two sequences of points $(y_3 \dots y_{n+1}) \in V_x$ and $(z_3 \dots z_{n+1}) \in V^x$ and a sequence of simple paths $(\gamma^3 \dots \gamma^{n+1})$ such that for every i, γ^i goes from y_i to z_i without passing through x and then goes through the edge (z_i, x). Now, for every $i \leq n+1$ we look at the simple cycle σ^i which starts at x then goes along the edge (x, y_i), then follows the path γ^i. By construction of the cycles we have:

$$\forall v \in V \backslash \{x\}, \# \left\{i, v \in \sigma^i\right\} \leq n.$$

Therefore we get:

$$\begin{aligned}
&f_x\left(\sum_{i \leq n+1} \overrightarrow{xy_i}\right) \prod_{v \in V \backslash \{x\}} \frac{\prod\limits_{e \in E_v} \Delta(\# \left\{i, e \in \sigma^i\right\})}{\Delta(\# \left\{i, v \in \sigma^i\right\})} \\
=&\check{f}_x\left(\sum_{i \leq n+1} \overrightarrow{xz_i}\right) \prod_{v \in V \backslash \{x\}} \frac{\prod\limits_{e \in E^v} \Delta(\# \left\{i, e \in \sigma^i\right\})}{\Delta(\# \left\{i, v \in \sigma^i\right\})}.
\end{aligned}$$

So we get:

$$\frac{f_x\left(\sum\limits_{i \leq n+1} \overrightarrow{xy_i}\right)}{\prod\limits_{e \in E_x} \Delta(\# \left\{i, (x, y_i) = e\right\})} = \frac{\check{f}_x\left(\sum\limits_{i \leq n+1} \overrightarrow{xz_i}\right)}{\prod\limits_{e \in E^x} \Delta(\# \left\{i, (z_i, x) = e\right\})}.$$

Now we just need to use this equality to prove that $\frac{f_x\left(\sum\limits_{i \leq n+1} \overrightarrow{xy_i}\right)}{\prod\limits_{e \in E_x} \Delta(\#\{i,(x,y_i)=e\})}$ does not depend on the sequence (y_i).

Since the value of $(z_3, \ldots, z_{n+1})$ does not depend on $(y_3, \ldots, y_{n+1})$ we have that $\frac{f_x\left(\sum\limits_{i \leq n+1} \overrightarrow{xy_i}\right)}{\prod\limits_{e \in E_x} \Delta(\#\{i,(x,y_i)=e\})}$ does not depend on $(y_3, \ldots, y_{n+1})$. To simplify notations we will write:

$$g(y_1, \ldots, y_{n+1}) = \frac{f_x\left(\sum\limits_{i \leq n+1} \overrightarrow{xy_i}\right)}{\prod\limits_{e \in E_x} \Delta(\# \left\{i, (x, y_i) = e\right\})}$$

Now let $(y^1_1, \ldots, y^1_{n+1})$ and $(y^2_1, \ldots, y^2_{n+1})$ be two sequences of vertices in V_x, we have:

$$\begin{aligned}
g(y^1_1, \ldots, y^1_{n+1}) &= g(y^1_1, y^1_2, y^2_1, \ldots, y^2_{n-1}) \\
&= g(y^2_1, y^1_2, y^1_1, y^2_2, \ldots, y^2_{n-1}) \\
&= g(y^2_1, y^1_2, y^2_2, \ldots, y^2_n) \\
&= g(y^2_1, y^2_2, y^1_2, y^2_3, \ldots, y^2_n) \\
&= g(y^2_1, y^2_2, y^2_3, \ldots, y^2_{n+1}).
\end{aligned}$$

So we have that for every $x \in V$ there is a $\Delta_x(n+1)$ such that:

$$f_x\left(\sum_{i \leq n+1} \overrightarrow{xy_i}\right) = \frac{\prod\limits_{e \in E_x} \Delta(\# \left\{i, (x, y_i) = e\right\})}{\Delta_x(n+1)}$$

and

$$\check{f}_x\left(\sum_{i \leq n+1} \overrightarrow{xz_i}\right) = \frac{\prod\limits_{e \in E^x} \Delta(\# \left\{i, (z_i, x) = e\right\})}{\Delta_x(n+1)}.$$

Now we just need to prove that this $\Delta_x(n+1)$ does not depend on x. Let $x, y \in V$ be two vertices such that (x, y) is an edge, we want to prove that $\Delta_x(n+1) = \Delta_y(n+1)$. This will yield the result we want since the graph is connected. Let z be a point in $V_y \backslash \{x\}$ which is not empty by lemma 3.2.1.1. Let p^1 and p^2 be two simple paths that go respectively from y to x and from z to x and that only intersect in x. We will not look at those paths but at the simple cycles σ^1 and σ^2 defined as follows: σ^1 starts at x, then goes along the edge (x, y) and finally follows the path p^1 back to x, σ^2 starts at x, then goes along the edges (x, y) and (y, z) and finally follows the path p^2 back to x. Those two simple cycles only intersect in x and y. This gives us:

$$\prod_{v \in V} f_v\left(\sum_{u \in V_v} \left(n 1_{(v,u) \in \sigma^1} + 1_{(v,u) \in \sigma^2}\right) \overrightarrow{vu}\right) = \prod_{v \in V} \check{f}_v\left(\sum_{u \in V^v} \left(n 1_{(u,v) \in \sigma_1} + 1_{(u,v) \in \sigma_2}\right) \overrightarrow{vu}\right).$$

Since for every $v \in V\backslash\{x,y\}$, v cannot be visited by both σ^1 and σ^2, we get that for every $v \in V\backslash\{x,y\}$:

$$f_v\left(\sum_{u\in V_v}\left(n1_{(v,u)\in\sigma^1}+1_{(v,u)\in\sigma^2}\right)\overrightarrow{vu}\right)=1,$$
$$\text{and } \check{f}_v\left(\sum_{u\in V^v}\left(n1_{(u,v)\in\sigma^1}+1_{(u,v)\in\sigma^2}\right)\overrightarrow{vu}\right)=1,$$

Now, we only have to look at x and y. We get:

$$f_x((n+1)\overrightarrow{xy})f_y\left(\sum_{u\in V_y}\left(n1_{(y,u)\in\sigma^1}+1_{(y,u)\in\sigma^2}\right)\overrightarrow{yu}\right)=\check{f}_x\left(\sum_{u\in V^x}\left(n1_{(u,x)\in\sigma^1}+1_{(u,x)\in\sigma^2}\right)\overrightarrow{xu}\right)\check{f}_y((n+1)\overrightarrow{yx}).$$

This in turns means that

$$f_y\left(\sum_{u\in V_y}\left(n1_{(y,u)\in\sigma^1}+1_{(y,u)\in\sigma^2}\right)\overrightarrow{yu}\right)=\check{f}_x\left(\sum_{u\in V^x}\left(n1_{(u,x)\in\sigma^1}+1_{(u,x)\in\sigma^2}\right)\overrightarrow{xu}\right).$$

Therefore
$$\frac{\Delta(n)\Delta(1)}{\Delta_x(n+1)}=\frac{\Delta(n)\Delta(1)}{\Delta_y(n+1)}.$$
So we get $\Delta_x(n+1)=\Delta_y(n+1)$ and since the graph is connected, $\Delta(n+1)$ exists. Therefore Δ exists by induction and we have proved the lemma. □

Now, the last thing we need to do is show that if the moments of transition probabilities are of the form

$$\mathbb{E}\left(\prod_{y\in V^x}\omega(x,y)^{n_{(x,y)}}\right)=\frac{\prod_{y\in V^x}h_(x,y)(n_{(x,y)})}{\tilde{h}_x\left(\sum_{y\in V^x}n_{(x,y)}\right)},$$

for some functions $(h_e)_{e\in E}$ and $(\tilde{h}_x)_{x\in V}$, then they follow a Dirichlet law or are deterministic. This can be done by using the following lemma and that for transition probabilities ω, any vertex $x\in V$ and any integers $(n_{(x,y)})_{y\in V_x}$, we have:

$$\mathbb{E}\left(\prod_{y\in V_x}\omega(x,y)^{n_{(x,y)}}\right)=\sum_{z\in V_x}\mathbb{E}\left(\prod_{y\in V_x}\omega(x,y)^{n_{(x,y)}+1_{y=z}}\right).$$

This equality is a direct consequence of this other equality: $\sum_{y\in V_x}\omega(x,y)=1$.

Lemma 3.2.2.3. *In this lemma, for any function $g:\mathbf{N}^d\mapsto\mathbf{R}$ we will write $g\left(\sum_i n_i\overrightarrow{e_i}\right)$ instead of $g(n_1,\dots,n_d)$. Let $f:\mathbf{N}^d\mapsto\mathbf{R}$ be a function that satisfies:*

$$f(0)=1 \text{ and } \forall(n_1,\dots,n_d)\in\mathbf{N}^d,\ f\left(\sum_{1\le i\le d}n_i\overrightarrow{e_i}\right)=\sum_{1\le j\le d}f\left(\overrightarrow{e_j}+\sum_{1\le i\le d}n_i\overrightarrow{e_i}\right),$$

and such that there exists functions $(h_i)_{1\le i\le d}$ from $\mathbf{N}$ to $\mathbf{R}$ and $\tilde{h}:\mathbf{N}\mapsto\mathbf{R}^*$ that satisfy:

$$f\left(\sum_{1\le i\le d}n_i\overrightarrow{e_i}\right)=\frac{\prod_{1\le i\le d}h_i(n_i)}{\tilde{h}\left(\sum_{1\le i\le d}n_i\right)},$$
$$\text{and } \forall i,\ h_i(0)=1 \text{ and } h_i(1)\neq 0,$$
$$\text{and } \tilde{h}(0)=1.$$

Then, either $\tilde{h}(2) \neq \tilde{h}(1)^2$ and there exists constants $(\beta_i)_{1\leq i\leq d} \in \mathbf{R}$ and $\gamma \neq 0$ such that:

$$f\left(\sum_{1\leq i\leq d} n_i \overrightarrow{e_i}\right) = \frac{\Gamma\left(\sum\limits_{1\leq i\leq d} \beta_i\right)}{\Gamma\left(\sum\limits_{1\leq i\leq d} n_i + \beta_i\right)} \prod_{1\leq i\leq d} \frac{\Gamma(n_i+\beta_i)}{\Gamma(\beta_i)},$$
$$\text{and } \forall i \in [|1,\dots,d|], \forall n \in \mathbf{N}, h_i(n) = \gamma^n H(\beta_i, n),$$
$$\text{and } \forall n \in \mathbf{N}, \tilde{h}(n) = \gamma^n H\left(\sum_{1\leq i\leq d} \beta_i, n\right),$$

where $H(a,n) = \prod\limits_{0\leq i\leq n-1} (a+i)$.
Or $\tilde{h}(2) = \tilde{h}(1)^2$ and there exists constants $(c_i)_{1\leq i\leq d}$ and $\gamma \neq 0$ such that:

$$f\left(\sum_{1\leq i\leq d} n_i \overrightarrow{e_i}\right) = \prod_{1\leq i\leq d} (c_i)^{n_i},$$
$$\text{and } \forall i \in [|1,\dots,d|], \forall n \in \mathbf{N}, h_i(n) = (\gamma c_i)^n,$$
$$\text{and } \forall n \in \mathbf{N}, \tilde{h}(n) = \gamma^n.$$

Proof. First, we want to show that proving the result when $\tilde{h}(1) = 1$ is enough. Indeed if we look at the functions:

$$g_i(n) = \frac{h_i(n)}{\tilde{h}(1)^n} \text{ and } \tilde{g}(n) = \frac{\tilde{h}(n)}{\tilde{h}(1)^n},$$

We still have:

$$f\left(\sum_{1\leq i\leq d} n_i \overrightarrow{e_i}\right) = \frac{\prod\limits_{1\leq i\leq d} g_i(n_i)}{\tilde{g}\left(\sum\limits_{1\leq i\leq d} n_i\right)}.$$

We also have that $\tilde{h}(2) = \tilde{h}(1)^2$ if and only if $\tilde{g}(2) = \tilde{g}(1)^2$. We also have $\tilde{g}(1) = 1$. Therefore looking at the case $\tilde{h}(1) = 1$ is enough so we will only look at that case.
-If $\tilde{h}(2) \neq \tilde{h}(1)^2$. Let $\beta := \frac{\tilde{h}(1)^2}{\tilde{h}(2)-\tilde{h}(1)^2}$ so that $\frac{\beta}{1+\beta} = \frac{\tilde{h}(1)^2}{\tilde{h}(2)}$. In particular, $\beta \neq 0$. We will choose the following values for the β_i and prove that they yield the desired result:

$$\forall i, \beta_i := f(\overrightarrow{e_i})\beta.$$

Now we will prove by induction on $\sum\limits_{1\leq i\leq d} n_i$ that these values of β_i are correct. To avoid problems of definition of the gamma function, we will use the following function $H : \mathbf{R} \times \mathbf{N} \mapsto \mathbf{N}$ defined by:

$$H(a,0) = 1,$$
$$\text{and } H(a,n) = \prod_{0\leq i\leq n-1} (a+i).$$

When the gamma function is well-defined, we have the equality:

$$H(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Now, we want to show by induction that:

$$h_i(n) = \frac{1}{\beta^n} H(\beta_i, n) \text{ and } \tilde{h}(n) = \frac{1}{\beta^n} H(\beta, n).$$

The result is obviously true for $h_i(0)$, $\tilde{h}(0)$ and $\tilde{h}(1)$. We also have, by definition of $(\beta_i)_{1\leq i\leq d}$:

$$h_i(1) = \frac{h_i(1)}{\tilde{h}(1)} = f(\overrightarrow{e_i}) = \frac{\beta_i}{\beta} = \frac{H(\beta_i, 1)}{\beta}.$$

We also have, by definition of β:

$$\tilde{h}(2) = \frac{1+\beta}{\beta} = \frac{\beta(1+\beta)}{\beta^2},$$

so we have the desired result. Now we have to look at $f(\overrightarrow{2e_i})$:

$$\begin{aligned}
f(\overrightarrow{2e_i}) &= f(\overrightarrow{e_i}) - \sum_{j \neq i} f(\overrightarrow{e_i} + \overrightarrow{e_j}) \\
&= \frac{\beta_i}{\beta} - \sum_{j \neq i} \frac{\beta_i \beta_j}{\beta(1+\beta)} \\
&= \frac{\beta_i}{\beta} - \frac{\beta_i}{(1+\beta)} \sum_{j \neq i} \frac{\beta_j}{\beta} \\
&= \frac{\beta_i}{\beta} - \frac{\beta_i}{(1+\beta)} \frac{\beta - \beta_i}{\beta} \\
&= \frac{\beta_i(1+\beta_i)}{\beta(1+\beta)}.
\end{aligned}$$

So:

$$h_i(2) = \tilde{h}(2) f(\overrightarrow{2e_i}) = \frac{1+\beta}{\beta} \frac{\beta_i(1+\beta_i)}{\beta(1+\beta)} = \frac{\beta_i(1+\beta_i)}{\beta^2}.$$

We have the desired result, now we can prove the result by induction on n. We have already proved it if $n \leq 2$. Now we assume it is proved for $n \leq N$ and we will prove it for $N+1$. First we use the equality:

$$f((N-1)\overrightarrow{e_1} + \overrightarrow{e_2}) = \sum_{1 \leq i \leq d} f((N-1)\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_i}).$$

So:

$$\begin{aligned}
\frac{h_1(N-1)h_2(1)}{\tilde{h}(N)} &= \frac{1}{\tilde{h}(N+1)} \left(h_1(N)h_2(1) + h_1(N-1)h_2(2) + \sum_{3 \leq i \leq d} h_1(N-1)h_2(1)h_i(1) \right) \\
&= \frac{1}{\beta^{N+1}\tilde{h}(N+1)} H(\beta_1, N-1)\beta_2 \left(\beta_1 + N - 1 + \beta_2 + 1 + \sum_{3 \leq i \leq d} \beta_i \right) \\
&= \frac{1}{\beta^{N+1}\tilde{h}(N+1)} H(\beta_1, N-1)\beta_2(\beta + N).
\end{aligned}$$

So we get:

$$\frac{H(\beta_1, N-1)}{\beta^{N-1}\tilde{h}(N)} = \frac{H(\beta_1, N-1)(\beta+N)}{\beta^N \tilde{h}(N+1)}.$$

Therefore:

$$\tilde{h}(N+1)H(\beta_1, N-1) = \frac{1}{\beta}\tilde{h}(n)H(\beta_1, N-1)(\beta + N).$$

if we replace β_1 by any β_i. So either there exists i such that $H(\beta_i, N-1) \neq 0$ and we get:

$$\tilde{h}(N+1) = \frac{H(\beta, N)}{\beta^{N+1}}$$

or $\forall 1 \leq i \leq d, H(\beta_i, N-1) = 0$, which means that:

$$\forall i, -\beta_i \in \mathbf{N} \text{ and } \beta_i \geq -(N-2).$$

In the latter case, we have:

$$f\left(\sum_{1 \leq i \leq d} -\beta_i \overrightarrow{e_i} \right) = \frac{\prod_{1 \leq i \leq d} H(\beta_i, -\beta_i)}{\beta^{-\beta}\tilde{h}(-\beta)} \neq 0.$$

However, we also have:

$$f\left(\sum_{1 \leq i \leq d} -\beta_i \overrightarrow{e_i} \right) = \sum_{1 \leq j \leq d} f\left(\overrightarrow{e_j} + \sum_{1 \leq i \leq d} -\beta_i \overrightarrow{e_i} \right) = 0$$

so it is not possible.
Finally:

$$\begin{aligned}
h_i(N+1) &= f((N+1)\overrightarrow{e_i})\tilde{h}(N+1) \\
&= \tilde{h}(N+1)\left(f(N\overrightarrow{e_i}) - \sum_{j\neq i} f(N\overrightarrow{e_i} + \overrightarrow{e_j})\right) \\
&= \frac{\tilde{h}(N+1)}{\tilde{h}(N)} h_i(N) - \sum_{j\neq i} h_i(N)h_j(1) \\
&= \frac{\tilde{h}(N+1)}{\tilde{h}(N)} h_i(N) - \sum_{j\neq i} h_i(N)\frac{\beta_j}{\beta} \\
&= h_i(N)\left(\frac{\beta+N}{\beta} - \sum_{j\neq i}\frac{\beta_j}{\beta}\right) \\
&= h_i(N)\frac{\beta_i+N}{\beta} \\
&= \frac{H(\beta_i, N+1)}{\beta^{N+1}}.
\end{aligned}$$

So we have the result we want.
-If $\tilde{h}(2) = \tilde{h}(1)^2 = 1$, we want to show that $\tilde{h}(n) = 1$ and $h_i(n) = f(\overrightarrow{e_i})^n$. We note $C_i := f(\overrightarrow{e_i})$ to simplify notations. We have the following equality:

$$\sum_{1\leq i\leq d} C_i = \sum_{1\leq i\leq d} f(\overrightarrow{e_i}) = f(0) = 1$$

We want to show the result by induction on n, it is obvious for $n = 0$ and $n = 1$. Now, we assume the result is proved for $n \leq N$, we want to prove it for $N + 1$. First we use the equality:

$$f((N-1)\overrightarrow{e_1} + \overrightarrow{e_2}) = \sum_{1\leq i\leq d} f((N-1)\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_i}).$$

We get:

$$\begin{aligned}
C_1^{N-1}C_2 &= \frac{1}{\tilde{h}(N+1)}\left(C_1^N C_2 + C_1^{N-1}C_2^2 + \sum_{3\leq i\leq d} C_1^{N-1}C_2C_i\right) \\
&= \frac{C_1^{N-1}C_2}{\tilde{h}(N+1)}\left(C_1 + C_2 + \sum_{3\leq i\leq d} C_i\right) \\
&= \frac{C_1^{N-1}C_2}{\tilde{h}(N+1)},
\end{aligned}$$

since $C_1^{N-1}C_2 \neq 0$, we get $\tilde{h}(N+1) = 1$.
Finally:

$$\begin{aligned}
h_i(N+1) &= f((N+1)\overrightarrow{e_i}) \\
&= f(N\overrightarrow{e_i}) - \sum_{j\neq i} f(N\overrightarrow{e_i} + \overrightarrow{e_j}) \\
&= C_i^N - \sum_{j\neq i} C_i^N C_j \\
&= C_i^N\left(1 - \sum_{j\neq i} C_j\right) \\
&= C_i^{N+1}.
\end{aligned}$$

So, we have the result we wanted. □

Now we have all we need to prove the theorem.

Theorem. *Let (V,E) be a finite directed graph, with no multiple edges, 2-connected and such that its reversed graph is also 2-connected. Then the only RWREs with a non-deterministic environment on this graph with independent transition probability such that for every edge (x,y), we have $\mathbb{E}\left(p_\omega(x,y)\right)>0$ and the reversed walk also has independent transition probabilities is a RWRE where the transition probabilities are independent Dirichlet random variables with null divergence.*

Proof. Let $(w(x,y))_{(x,y)\in E}$ be random transition probabilities and $(\check{w}(x,y))_{(x,y)\in \tilde{E}}$ the transition probabilities of the reversed environment such that the transition probabilities are independent at each site both for the environment and the reversed environment. We define the moment functions f and $\check{f}$ by:

$$\forall x\in V, f_x\left(\sum_{y\in V_x} n_{xy}\overrightarrow{xy}\right)=\mathbb{E}\left(\prod_{y\in E_x}(w(x,y))^{n_{xy}}\right) \text{ and}$$
$$\forall x\in V, \check{f}_x\left(\sum_{y\in V^x} n_{yx}\overrightarrow{xy}\right)=\mathbb{E}\left(\prod_{y\in E^x}(\check{w}(x,y))^{n_{yx}}\right).$$

These two moments functions are compatible because the transition-probabilities of the time-reversed random walk are defined by $\check{w}(y,x)\pi_y=w(x,y)\pi_x$ where $(\pi_x)_{x\in V}$ is the stationary law. Therefore, if $N:E\mapsto\mathbf{N}$ is of null divergence then:

$$\begin{aligned}
\prod_{v\in V}\check{f}_v\left(\sum_{u\in V^v}N((u,v))\overrightarrow{vu}\right)&=\prod_{v\in V}\mathbb{E}\left(\prod_{u\in V^v}(\check{w}(v,u))^{N((u,v))}\right)\\
&=\mathbb{E}\left(\prod_{v\in V}\prod_{u\in V^v}(\check{w}(v,u))^{N((u,v))}\right)\\
&=\mathbb{E}\left(\prod_{v\in V}\prod_{u\in V^v}\left(w(u,v)\frac{\pi_u}{\pi_v}\right)^{N((u,v))}\right)\\
&=\mathbb{E}\left(\prod_{v\in V}\prod_{u\in V^v}(w(u,v))^{N((u,v))}\prod_{v\in V}(\pi_v)^{\sum\limits_{u\in E_v}N((v,u))-\sum\limits_{u\in V^v}N((u,v))}\right)\\
&=\mathbb{E}\left(\prod_{v\in V}\prod_{u\in V^v}(w(u,v))^{N((u,v))}\right)\\
&=\prod_{v\in V}\mathbb{E}\left(\prod_{u\in V_v}(w(v,u))^{N((v,u))}\right)\\
&=\prod_{v\in V}f_v\left(\sum_{u\in V_v}N((v,u))\overrightarrow{vu}\right).
\end{aligned}$$

Now we can apply the result of lemma 3.2.2.1 which gives the existence of functions $\tilde{h}_x:\mathbf{N}\mapsto(0,\infty)$ for every $x\in V$ and functions $h_e:\mathbf{N}\mapsto(0,\infty)$ for every $e\in E$ such that:

$$\forall x\in V,\forall y\in V_x,\forall n\in\mathbf{N}, f_x(n\overrightarrow{xy})=\frac{h_{(x,y)}(n)}{\tilde{h}_x(n)} \text{ and}$$
$$\forall x\in V,\forall y\in V^x,\forall n\in\mathbf{N}, \check{f}_x(n\overrightarrow{xy})=\frac{h_{(y,x)}(n)}{\tilde{h}_x(n)}.$$

Now we can consider the moment functions g and $\check{g}$ defined by:

$$\forall x\in V, g_x\left(\sum_{y\in V_x}N((x,y))\overrightarrow{xy}\right)=f_x\left(\sum_{y\in V_x}N((x,y))\overrightarrow{xy}\right)\frac{\tilde{h}_x\left(\sum\limits_{y\in V_x}N((x,y))\right)}{\prod\limits_{y\in V_x}h_{(x,y)}(N((x,y)))} \text{ and}$$

$$\forall x\in V, \check{g}_x\left(\sum_{y\in V^x}N((y,x))\overrightarrow{xy}\right)=\check{f}_x\left(\sum_{y\in V^x}N((y,x))\overrightarrow{xy}\right)\frac{\tilde{h}_x\left(\sum\limits_{y\in V^x}N((y,x))\right)}{\prod\limits_{y\in V^x}h_{(y,x)}(N((y,x)))}.$$

The moment functions g and $\check{g}$ are compatible that satisfy the hypotheses of lemma 3.2.2.2 so there exists a function $\Delta : \mathbf{N} \mapsto (0, \infty)$ such that:

$$\forall x \in V, g_x\left(\sum_{y\in V_x} N((x,y))\overrightarrow{xy}\right) = \frac{\prod\limits_{y\in V_x} \Delta(N((x,y)))}{\Delta\left(\sum\limits_{y\in V_x} N((x,y))\right)} \text{ and}$$

$$\forall x \in V, \check{g}_x\left(\sum_{y\in V^x} N((y,x))\overrightarrow{xy}\right) = \frac{\prod\limits_{y\in V^x} \Delta(N((y,x)))}{\Delta\left(\sum\limits_{y\in V^x} N((y,x))\right)}.$$

We define the following functions:

$$\forall e \in E, h'_e(n) := \Delta(n)h_e(n)$$
$$\text{and } \forall x \in V, \tilde{h}'_x(n) := \Delta(n)\tilde{h}_x(n).$$

We have:

$$\forall x \in V, f_x\left(\sum_{y\in V_x} N((x,y))\overrightarrow{xy}\right) = \frac{\prod\limits_{y\in V_x} h'_{(x,y)}(N((x,y)))}{\tilde{h}'_x\left(\sum\limits_{y\in V_x} N((x,y)) \text{ and}\right)}$$

$$\forall x \in V, \check{f}_x\left(\sum_{y\in V^x} N((y,x))\overrightarrow{xy}\right) = \frac{\prod\limits_{y\in V^x} h'_{(y,x)}(N((y,x)))}{\tilde{h}'_x\left(\sum\limits_{y\in V_x} N((y,x))\right)}.$$

Now, according to lemma 3.2.2.3, we get that for every $x \in V$, f_x is either the moments of a Dirichlet distribution or the moments of a deterministic distribution, the same is true for $\check{f}_x$. Either all of the f_x and $\check{f}_x$ are moments of Dirichlet distribution, or at least one of them is deterministic in which case we can assume that it is f_x for some $x \in V$ that will be fixed for the rest of the proof. We want to prove that in the later case, all the probability transitions are deterministic. For any $x \in V$, if either f_x or $\check{f}_x$ is deterministic then according to lemma 3.2.2.3 there is a $\gamma_x \neq 0$ such that $\tilde{h}'_x(n) = \gamma_x^n$ and therefore, still according to lemma 3.2.2.3 if $\tilde{h}'_x(n) = \gamma_x^n$ then we are in the case where $\tilde{h}'_x(2) = \tilde{h}'_x(1)^2$ and both f_x and $\check{f}_x$ are the moments of deterministic transitions probabilities. Now let $(x, y) \in E$ be an edge, if f_x is the moment of deterministic transitions probabilities then, once again by lemma 3.2.2.3, there is a $\gamma_{(x,y)} \neq 0$ such that $h'_{(x,y)}(n) = \gamma^n_{(x,y)}$. Now we get:

$$\check{f}_y\left(n\overrightarrow{yx}\right) = \frac{h'_{(x,y)}(n)}{\tilde{h}'_y(n)}.$$

According to lemma 3.2.2.3 the only way to have $h'_{(x,y)}(n) = \gamma^n_{(x,y)}$ is that $\check{f}_y$ and f_y are the moments of deterministic transition probabilities. Now since the graph is connected, we get that for all vertices $z \in V$, the function f_s is the moments of deterministic transition probabilities.
If for every $x \in V$, f_x is the moments of a Dirichlet distribution then we want to prove that we have null divergence. According to lemma 3.2.2.3 there exists $(\beta_e)_{e\in E}$ such that for every $x \in V$, f_x is the moments of a Dirichlet distribution of parameters $(\beta_e)_{e\in E_x}$ and $\check{f}_x$ is the moments of a Dirichlet distribution of parameters $(\beta_e)_{e\in E^x}$. There exists $(\gamma_x)_{x\in V}, (\beta_x)_{x\in V}$ such that:

$$\forall x \in V, \forall n \in \mathbf{N}, \tilde{h}'_x(n) = \gamma_x^n \frac{\Gamma(\beta_x + n)}{\Gamma(\beta_x)}.$$

Since for every $x \in V$, $\sum\limits_{y\in V_x} \omega(x, y) = 1$ almost surely, we have:

$$\forall x \in V \sum_{y\in V_x} \frac{h'_{(x,y)}(1)}{\tilde{h}'_x(1)} = 1.$$

This means, according to lemma 3.2.2.3 that:

$$\forall x \in V, \sum_{y\in V_x} \beta_{(x,y)} = \beta_x.$$

The same way we get, by looking at the reversed walk:

$$\forall x \in V, \sum_{y \in V^x} \beta_{(y,x)} = \beta_x.$$

This means that the parameters of the Dirichlet distributions have null divergence. $\square$

Chapter 4

Monotonicity and phase transition for the VRJP and the ERRW

This chapter is based on the article [64] available on Arxiv. There is however one major addition: the last section is devoted to proving a general $0-1$ law for recurrence/transience for the VRJP and the ERRW.

Abstract

The vertex-reinforced jump process (VRJP), introduced by Davis and Volkov in [26], is a continuous-time process that tends to come-back to already visited vertices. It is closely linked to the edge-reinforced random walk (ERRW) introduced by Coppersmith and Diaconis in 1986 ([24]) which is more likely to cross edges it has already crossed. On $\mathbf{Z}^d$ for $d \geq 3$, both models where shown to be recurrent for small enough initial weights ([72],[2]) and transient for large enough initial weights ([32],[72]). We show through a coupling of the VRJP for different weights that the VRJP (and the ERRW) exhibits some monotonicity. In particular, we show that increasing the initial weights of the VRJP and the ERRW makes them more transient which means that the recurrence/transience phase transition is necessarily unique. Furthermore, by making the weights go to infinity, we show that the recurrence of the ERRW and the VRJP is implied by the recurrence of a random walk in deterministic electrical network.

4.1 Introduction and results

4.1.1 Introduction

The edge-reinforced random walk (ERRW) was first introduced by Coppersmith and Diaconis in 1986 [24]. In this model, the more the walk crosses an edge, the likelier it is to cross it again in the future. This model was shown to be a random walk in random reversible environments ([29], [59]). This representation lead to several results on this model, first recurrence and transience on trees depending on the reinforcement [62] then recurrence on the ladder [58] and $\mathbf{Z}\times G$ [67] for large enough reinforcement and on a modification of $\mathbf{Z}^2$ for large enough reinforcement [60]. It was then shown by two different techniques that the ERRW on $\mathbf{Z}^d$ is recurrent for large enough reinforcement (in [2] by Angel, Crawford and Kozma and in [72] by Sabot and Tarrès). The technique used in [72] was based on a link between the ERRW, the vertex-reinforced jump process (VRJP, introduced by Davis and Volkov in [26]) and the super-symmetric hyperbolic sigma model (introduced in the context of random band matrices in [92],[34] by Zirnbauer, Disertori and Spencer). This relation led to several other results for both the ERRW and the VRJP: the transience and a CLT in dimension 3 and higher for small enough reinforcements ([32],[72],[76]), a $0-1$ law for recurrence on $\mathbf{Z}^d$ [76] and the recurrence in dimension 2 ([76],[60],[71]). This means that on the one hand, for $d \in \{1,2\}$ the ERRW and the VRJP are recurrent for any reinforcement. On the other hand, for $d \geq 3$ both the ERRW and the VRJP are recurrent for large enough reinforcement and transient for small enough reinforcements. We know that in-between, the VRJP and the ERRW are recurrent or transient but it was not known whether there is a unique phase transition. In this paper we show that we can couple the VRJP for different weights (more precisely, we couple the β-field associated to the VRJP that was introduced in [73]). This coupling leads to a monotonicity for the VRJP similar to the Rayleigh monotonicity for electrical networks. This gives us the uniqueness of the recurrence/transience phase transition for the VRJP and the ERRW in dimension 3 and higher. This monotonicity can also be used to show that the VRJP and the ERRW with constant weights are recurrent on recurrent graphs by seeing random walks in electrical networks as VRJPs with infinite weights.

4.1.2 Statement of the results

Let $\mathscr{G} = (V, E)$ be a locally finite, non-directed graph. To every edge $e \in E$ we associate a positive weight a_e. Let $x_0 \in V$ be a vertex of $\mathscr{G}$. The edge-reinforced random walk Y starting from x_0 is the random process which takes its values in V defined by:

$$Y_0 = x_0 \text{ a.s, and}$$

$$\mathbb{P}\left(Y_{n+1} = y | Y_0, \dots, Y_n\right) = 1_{y \sim Y_n} \frac{a_{\{Y_n, y\}} + Z_n(\{Y_n, y\})}{\sum\limits_{z \sim Y_n} a_{\{Y_n, z\}} + Z_n(\{Y_n, z\})},$$

where the random variables $(Z_n)_{n \in \mathbf{N}}$ are defined by:

$$\forall e \in E,\ Z_n(e) = \sum_{i=0}^{n-1} 1_{\{Y_i, Y_{i+1}\} = e}.$$

If the graph is $\mathbf{Z}^d$, this process can exhibit different behaviours depending on the initial weights. For small enough initial weights it is recurrent.

Theorem (Theorem 1 of [2] and corollary 2 of [72])**.** *For any K there exists $a_0 > 0$ such that if $\mathscr{G}$ is a graph with all degrees bounded by K, then the linearly edge reinforced random walk on $\mathscr{G}$ with initial weights $a \in (0, a_0)$ is positive recurrent.*

For large enough initial weights, the process is transient.

Theorem (Theorem 1 of [32])**.** *On $\mathbf{Z}^d$, $d \geq 3$, there exists $a_c(d) > 0$ such that, if $a_e > a_c(d)$ for all $e \in E$, then the ERRW with weights $(a_e)_{e \in E}$ is transient a.s.*

Note that the previous two theorems use results or ideas of [34] and [33]. The ERRW is linked to an other random process, the vertex-reinforced jump process (VRJP). The VRJP on a locally finite graph $\mathscr{G} = (V, E)$ is the continuous-time process $(\tilde{Y}_t)_{t \in \mathbf{R}^+}$ that starts at some vertex x_0 and that, conditionally on the past at time t, if $\tilde{Y}_t = x$, jumps to a neighbour y of x at rate

$$W_{\{x,y\}} \ell_x(t),$$

where

$$\ell_x(t) := \int_0^t 1_{\tilde{Y}_s = x} \mathrm{d}s.$$

The following link between the ERRW and the VRJP has been shown in [72].

Theorem (Theorem 1 of [72])**.** *The ERRW with weights $(a_e)_{e \in E}$ is equal in law to the discrete time process associated with a VRJP in random independent weights $W_e \sim Gamma(a_e, 1)$.*

In this article we show, through a coupling, that the VRJP has a property similar to Rayleigh's monotonicity for electrical network. This leads to several results for recurrence and transience. First, we show that the probability that the walk is recurrent is decreasing in the parameters of the VRJP. This is a corollary of our main theorem that will be stated at the end because it is technical and needs a few additional definitions.

Theorem 54. *Let $\mathscr{G} = (V, E)$ be an infinite, non-directed, connected graph without loops or multiple edges and $0 \in V$ a vertex in this graph. Let $(W_e^-)_{e \in E}$ and $(W_e^+)_{e \in E}$ be two families of positive weights such that for any $e \in E$, $0 < W_e^- \leq W_e^+$. The probability that the VRJP with initial weights W^- is recurrent is greater or equal than the probability that the VRJP with initial weights W^+ is recurrent.*

It was already proved that the VRJP on $\mathbf{Z}^d$ with constant weights or weights invariant by translation is recurrent with probability 0 or 1 in [76]. In addition to our theorem this means that the VRJP and the ERRW are recurrent for small enough weights and then transient for larger weights. This means that the VRJP and the ERRW exhibit a phase transition for recurrence/transience on $\mathbf{Z}^d$ when all the edges have the same weight.

Theorem 55. *Set $d \geq 3$ there exists $w_d \in (0, \infty)$ such that the VRJP on $\mathbf{Z}^d$ with initial weight $w \in (0, \infty)$ is recurrent if $w < w_d$ and transient if $w > w_d$.*

Theorem 56. *Set $d \geq 3$ there exists $a_d \in (0,\infty)$ such that the ERRW on $\mathbf{Z}^d$ with initial weight $a \in (0,\infty)$ is recurrent if $a < a_d$ and transient if $a > a_d$.*

We will also use our result to improve the $0-1$ law and extend it to all locally finite graphs and all positive random weights.

Theorem 57. *For any locally finite graph $\mathscr{G} = (V,E)$ and any vertex $x_0 \in V$, the VRJP on $\mathscr{G} = (V,E)$ starting at 0 and with independent positive random weights $(W_e)_{e\in E}$ is recurrent with probability 0 or 1. In particular, the ERRW on $\mathscr{G}$, starting at x_0 and with initial deterministic positive weights $(a_e)_{e\in E}$ is recurrent with probability 0 or 1.*

The link between the VRJP and electrical network goes beyond this monotonicity property. The following theorem shows that recurrence of electrical networks, VRJP and ERRW are also closely linked.

Theorem 58. *Let $\mathscr{G} = (V,E)$ be an infinite, locally finite graph and $x_0 \in V$ a vertex. Let $(W_e)_{e\in E}$ be a family of positive weights. If the random walk on $\mathscr{G}$ starting at x_0 with deterministic conductances $(c_e)_{e\in E} = (W_e)_{e\in E}$ is recurrent then so are the ERRW and the VRJP starting at x_0 and with initial weights $(W_e)_{e\in E}$.*

To state our technical main theorem, we need some extra definition and results related to the VRJP and the ERRW. First we need to introduce the β-field (introduced in [73] by Tarrès, Sabot and Zeng), a random vector defined for weighted graphs.

Definition 44. *Let n be an integer, $(\eta_i)_{1\leq i\leq n}$ a family of non-negative parameters and $W \in M_n(\mathbf{R})$ a symmetric matrix with non-negative coefficients. Let $1_n \in \mathbf{R}^n$ be the vector $(1,\dots,1)$. The measure $\nu_n^{W,\eta}$ on $(0,\infty)^n$ is defined by the following density:*

$$\nu_n^{W,\eta}(\mathrm{d}\beta_1\dots\mathrm{d}\beta_n) := e^{-\frac{1}{2}\left(1_n H_\beta\, {}^t1_n \,+\, \eta H_\beta^{-1}\, {}^t\eta \,-\, 2\sum\limits_{1\leq i\leq n}\eta_i\right)} \frac{1}{\sqrt{det(H_\beta)}} 1_{H_\beta>0}\mathrm{d}\beta_1\dots\mathrm{d}\beta_n,$$

where $\forall i,j \in [\![1,n]\!]$,

$$\begin{aligned} H_\beta(i,i) &= 2\beta_i - W(i,i), \\ H_\beta(i,j) &= -W(i,j) \text{ if } i \neq j \end{aligned}$$

and $H_\beta > 0$ means that H_β is positive definite.
This family of measures is actually a family of probability measures, as was proved in [73].
We call $\tilde{\nu}_n^{W,\eta}$ the distribution of H_β when $(\beta_i)_{1\leq i\leq n}$ is distributed according to $\nu_n^{W,\eta}$.

The link between the β-field and the VRJP is not obvious at first glance. It was shown in [73] (based on previous results in [72]) that the VRJP with weights W can be seen as a random walk in a random electrical network whose conductances are given by the weights W and the β-field. More precisely:

Theorem (Theorem 3 of [73]). *Let $\mathscr{G} = (V,E)$ be a non-directed graph and $(W_e)_{e\in E}$ weights on the edges. Let H_β be distributed according to $\tilde{\nu}_{|V|}^{W,0}$ and let G_β be the inverse of H_β. For any $x_0 \in V$ the discrete path of the VRJP (the sequence of vertices at each successive jump) on $\mathscr{G}$ with weights W, starting at x_0, is a random walk in random electrical network where the conductances $(c_e)_{e\in E}$ are given by:*

$$c_{\{x,y\}} = W_{\{x,y\}} G_\beta(x_0,x) G_\beta(x_0,y).$$

The reason we look at the β-field instead of the conductances is that the β-field has several interesting properties. First, the β-field does not depend on the starting point of the VRJP. Its Laplace transform has a simple expression and it is 1-dependent. But most importantly, the family of laws $\nu^{W,\eta}$ is stable by taking marginals or conditional distributions (lemma 5 of [76] and independently in [55]). More precisely:

Proposition 4.1.2.1. *Let n_1, n_2 be two integers, and $n := n_1 + n_2$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative coefficients and $(\eta_i)_{i\in[\![1,n_1+n_2]\!]}$ a family of non-negative coefficients. Let $(\beta_i)_{i\in[\![1,n_1+n_2]\!]}$ be random variables with a $\nu_n^{W,\eta}$ distribution and $H_\beta \in M_n(\mathbf{R})$ the matrix defined by:*

$$\forall i,j \in [\![1,n]\!], H_\beta(i,j) := \begin{cases} 2\beta_i - W(i,i) \text{ if } i = j, \\ -W(i,j) \text{ if } i \neq j. \end{cases}$$

We make the following bloc decomposition:

$$W = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}, H_\beta = \begin{pmatrix} H_\beta^{11} & H_\beta^{12} \\ H_\beta^{21} & H_\beta^{22} \end{pmatrix} \text{ and } \eta = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix},$$

where $W^{11}, H_\beta^{11} \in M_{n_1}(\mathbf{R})$, $W^{12}, H_\beta^{12} \in M_{n_1,n_2}(\mathbf{R})$, $W^{21}, H_\beta^{21} \in M_{n_2,n_1}(\mathbf{R})$, $W^{22}, H_\beta^{22} \in M_{n_2}(\mathbf{R})$, $\eta^1 \in \mathbf{R}^{n_1}$ *and* $\eta^2 \in \mathbf{R}^{n_2}$. *Then the family* $(\beta_i)_{1\leq i\leq n_1}$ *is distributed according to* $\nu_{n_1}^{W^{11},\hat{\eta}}$ *where*

$$\hat{\eta} \in \mathbf{R}^{n_1} \text{ and } \forall i \in [\![1, n_1]\!],\ \hat{\eta}_i := \eta_i + \sum_{k=1}^{n_2} W^{12}(i,k).$$

Conditionally on $(\beta_i)_{1\leq i\leq n_1}$, *the family* $(\beta_i)_{n_1+1\leq i\leq n_1+n_2}$ *is distributed according to* $\nu_{n_2}^{\check{W},\check{\eta}}$ *where*

$$\check{W} = W^{22} + W^{21}\left(H_\beta^{11}\right)^{-1} W^{12},$$

and

$$\check{\eta} \in \mathbf{R}^{n_2} \text{ and } \check{\eta} = \eta^2 + W^{21}\left(H_\beta^{11}\right)^{-1}\eta^1.$$

Definition 45. *Let n be an integer and let $H \in M_n(\mathbf{R})$ be a symmetric matrix. We say that two integers $1 \leq i, j \leq n$ are H-connected if there exists a finite sequence $(k_1, \ldots, k_m)$ such that $k_1 = i, k_m = j$ and for all $1 \leq a \leq m-1$, $H(k_a, k_{a+1}) \neq 0$.*

We can now state our (technical) main theorem which gives a coupling between VRJPs of different weights and a simpler corollary that is the equivalent of Rayleigh monotonicity for the VRJP.

Theorem 59. *Set an integer $n \in \mathbf{N}$. Let $W \in M_n(\mathbf{R})$ be a symmetric matrix with non-negative off diagonal coefficients and null diagonal coefficients. Let $W^1, W^2 \in M_{n,1}(\mathbf{R})$ be two matrices with non-negative coefficients and let $W^3 \in M_{n,1}(\mathbf{R})$ be the matrix defined by $W^3 := W^1 + W^2$. Let $w^-, w^+ \in [0,\infty)$ be two positive real with $w^- < w^+$. We define the matrices W^-, W^+ and W^∞ by:*

$$W^- := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^- \\ {}^tW^2 & w^- & 0 \end{pmatrix}, W^+ := \begin{pmatrix} W & W^1 & W^2 \\ {}^tW^1 & 0 & w^+ \\ {}^tW^2 & w^+ & 0 \end{pmatrix} \text{ and } W^\infty := \begin{pmatrix} W & W^3 \\ {}^tW^3 & 0 \end{pmatrix}.$$

If $n = 0$, we just have:

$$W^- := \begin{pmatrix} 0 & w^- \\ w^- & 0 \end{pmatrix}, W^+ := \begin{pmatrix} 0 & w^+ \\ w^+ & 0 \end{pmatrix} \text{ and } W^\infty := \begin{pmatrix} 0 \end{pmatrix}.$$

For any vector $X \in \mathbf{R}^{n+2}$ we define the vector $\overline{X} \in \mathbf{R}^{n+1}$ by:

$$\forall i \in [\![1, n]\!],\ \overline{X}_i := X_i \text{ and}$$
$$\overline{X}_{n+1} := X_{n+1} + X_{n+2}.$$

For any vector $X^1 \in [0,\infty)^{n+2}$ there exists random matrices H^-, H^+ and H^∞ (with inverse G^-, G^+ and G^∞ respectively) that are distributed according to $\tilde{\nu}_{n+2}^{W^-,0}, \tilde{\nu}_{n+2}^{W^+,0}$ and $\tilde{\nu}_{n+1}^{W^\infty,0}$ respectively such that

$${}^tX^1 G^- X^1 = {}^tX^1 G^+ X^1 = {}^t\overline{X^1} G^\infty \overline{X^1} \text{ almost surely,}$$

for all $i \in [\![1, n]\!]$, $H^-(i,i) = H^+(i,i) = H^\infty(i,i)$ and for any vector $X^2 \in [0,\infty)^{n+2}$ we have:

$$\begin{aligned} &\mathbb{E}\left({}^tX^1 G^+ X^2 | H^\infty\right) = {}^t\overline{X^1} G^\infty \overline{X^2}, \text{ and} \\ &\mathbb{E}\left({}^tX^1 G^- X^2 | H^+\right) = {}^tX^1 G^+ X^2 \text{ if } n+1 \text{ and } n+2 \text{ are } H^-\text{-connected.} \end{aligned}$$

It was already known that a special case of this theorem was true: the martingale property between H^+ and H^∞ under specific assumptions (the martingale property for ψ in [76]). However, the link between H^+ and H^- was not known.

Theorem 60. *Let $n \geq 2$ be an integer, let $W^-, W^+ \in M_n(\mathbf{R})$ be two symmetric matrices with null diagonal coefficients and non-negative off-diagonal coefficients such that for any $i, j \in [\![1, n]\!]$, $W^-(i,j) \leq W^+(i,j)$ and i and j are W^--connected. Let H^- and H^+ be two matrices distributed according to $\tilde{\nu}_n^{W^-,0}$ and $\tilde{\nu}_n^{W^+,0}$ respectively, and let their inverse be G^- and G^+ respectively. For any convex function f, any integer $i \in [\![1, n]\!]$ and any deterministic vector $X \in [0,\infty)^n$:*

$$\mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^-(i,j)}{G^-(i,i)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{\sum_{j=1}^n X_j G^+(i,j)}{G^+(i,i)}\right)\right).$$

For a specific choice of X and a specific choice of i, the random variable $\frac{\sum_{j=1}^{n} X_i G(i,j)}{G(i,i)}$ is equal to the random variable ψ defined in [76] (to be more precise, it is equal to an approximation of ψ on finite graphs). This random variable ψ is closely linked to the recurrence of the graph (it is equal to 0 iff the VRJP is recurrent). By using this theorem for ψ (to be more precise, on an approximation of ψ on finite graphs), it is then possible to deduce the uniqueness of the phase transition between recurrence and transience for the VRJP and the ERRW (on any graph).

4.2 A simplification

4.2.1 Schur's lemma

We will use Schur's decomposition several times in the paper. It is useful because it behaves nicely with the marginal and conditional laws of ν.

Lemma 4.2.1.1 (Schur decomposition)**.** *Let H be a symmetric, positive definite matrix. Let A, B, C be 3 matrices such that H can be decomposed in bloc as such:*

$$H = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}.$$

Its inverse is given by:

$$H^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(C - {}^tBA^{-1}B)^{-1}\, {}^tBA^{-1} & -A^{-1}B(C - {}^tBA^{-1}B)^{-1} \\ -(C - {}^tBA^{-1}B)^{-1}\, {}^tBA^{-1} & (C - {}^tBA^{-1}B)^{-1} \end{pmatrix}.$$

4.2.2 Reduction to 2 points

We want to show that we can reduce the problem to the study of ν_1 and ν_2, but first we need to prove a small lemma that will be useful in the following.

Lemma 4.2.2.1. *Let n be an integer, let $H \in M_n(\mathbf{R})$ be a symmetric, positive definite matrix with non-positive off-diagonal coefficients. For any integers $1 \leq i, j \leq n$, $H^{-1}(i,j) > 0$ iff i and j are H-connected.*

Proof. Since H is a symmetric, positive definite matrix, all its eigenvalues are positive reals. Let λ^- be the smallest eigenvalue of H and λ^+ its largest. Since H is symmetric, all its diagonal coefficients $H(i,i)$ satisfy the inequality $\lambda^- \leq H(i,i) \leq \lambda^+$. This means that all the coefficients of $I_n - \frac{1}{\lambda^+}H$ are non-negative and its eigenvalues are between 0 and $1 - \frac{\lambda^-}{\lambda^+} < 1$. This means that we have the following equality:

$$H^{-1} = \frac{1}{\lambda^+}\left(I_n - \left(I_n - \frac{1}{\lambda^+}H\right)\right)^{-1} = \frac{1}{\lambda^+}\sum_{k\geq 0}\left(I_n - \frac{1}{\lambda^+}H\right)^k.$$

For any integers i, j, i and j are H-connected iff there exists $m \geq 0$ such that $\left(I_n - \frac{1}{\lambda^+}H\right)^m > 0$ (since all the coefficients of $I_n - \frac{1}{\lambda^+}H$ are non-negative). This means that $H^{-1}(i,j) > 0$ iff i and j are H-connected. □

We will use the following lemma to reduce our problem to the study of ν_1 and ν_2.

Lemma 4.2.2.2. *Let $n \in \mathbf{N}^*$ be an integer. Let $H^{11} \in M_n(\mathbf{R})$ be a symmetric, positive definite matrix with non-positive off-diagonal coefficients. Let $H^{12} \in M_{n,2}(\mathbf{R})$ be a matrix with only non-positive coefficients. We also define the matrix $\overline{H}^{12} \in M_{n,1}(\mathbf{R})$ by:*

$$\overline{H}^{12} = H^{12}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now let $H \in M_{n+2}(\mathbf{R})$ and $\overline{H} \in M_{n+1}(\mathbf{R})$ be two symmetric, positive definite matrices with non-positive off-diagonal coefficients such that they have the following bloc decomposition:

$$H = \begin{pmatrix} H^{11} & H^{12} \\ {}^tH^{12} & H^{22} \end{pmatrix} \text{ and } \overline{H} = \begin{pmatrix} H^{11} & \overline{H}^{12} \\ {}^t\overline{H}^{12} & \overline{H}^{22} \end{pmatrix}.$$

Let G and $\overline{G}$ be the inverse of H and $\overline{H}$ respectively. We use the same bloc decomposition:

$$G =: \begin{pmatrix} G^{11} & G^{12} \\ {}^tG^{12} & G^{22} \end{pmatrix} \text{ and } \overline{G} := \begin{pmatrix} G^{11} & \overline{G}^{12} \\ {}^t\overline{G}^{12} & \overline{G}^{22} \end{pmatrix}.$$

For any vector $X \in \mathbf{R}^{n+2}$ we define the vector $\overline{X} \in \mathbf{R}^{n+1}$ by:

$$\forall i \in [\![1,n]\!],\ \overline{X}_i := X_i \text{ and}$$
$$\overline{X}_{n+1} := X_{n+1} + X_{n+2}.$$

For any vectors $X^1, X^2 \in [0,\infty)^{n+2}$ we can define:

- *$\alpha_1(X^1) \geq 0$ and $\alpha_2(X^1) \geq 0$ that only depend on X^1, H^{11} and H^{12},*
- *$\alpha_1(X^2) \geq 0$ and $\alpha_2(X^2) \geq 0$ that only depend on X^2, H^{11} and H^{12},*
- *$C(X^1, X^2) \geq 0$ that only depends on X^1, X^2, H^{11} and H^{12} (but not H^{22}),*

such that:

$$\begin{aligned} {}^tX^1GX^2 =& C(X^1,X^2) + \begin{pmatrix}\alpha_1(X^1) & \alpha_2(X^1)\end{pmatrix} G^{22} \begin{pmatrix}\alpha_1(X^2) \\ \alpha_2(X^2)\end{pmatrix} \\ {}^t\overline{X}^1\overline{GX}^2 =& C(X^1,X^2) + (\alpha_1(X^1)+\alpha_2(X^1))\overline{G}^{22}(\alpha_1(X^2)+\alpha_2(X^2)). \end{aligned}$$

The previous lemma allows us to transform the expression ${}^tX^1GX^2$ in the form $A + {}^tY^1G^{22}Y^2$. The properties of the family of law ν (4.1.2.1) tell us that the study of G^{22} knowing A, Y^1 and Y^2 is the same as the study of ν_2 for some parameters. This means that if we get some monotonicity for ν_2 we should be able to get it back for ν_n for any n.

proof of lemma 4.2.2.2. First we look at H. Let G be the inverse of H. We use the same bloc decomposition as for H:

$$G = \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix},$$

where $G^{11} \in M_n(\mathbf{R}),\ G^{12} \in M_{n,2}(\mathbf{R}),\ G^{21} \in M_{2,n}(\mathbf{R})$ and $G^{22} \in M_2(\mathbf{R})$. By Schur decomposition 4.2.1.1 we have:

$$\begin{aligned} G &= \begin{pmatrix} (H^{11})^{-1} + (H^{11})^{-1}H^{12}G^{22}\,{}^tH^{12}(H^{11})^{-1} & -(H^{11})^{-1}H^{12}G^{22} \\ -G^{22}\,{}^tH^{12}(H^{11})^{-1} & G^{22} \end{pmatrix} \\ &= \begin{pmatrix} I_n & -(H^{11})^{-1}H^{12} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -H^{21}(H^{11})^{-1} & I_2 \end{pmatrix} \end{aligned}$$

By definition of H, all the coefficients of $-H^{12}$ are non-negative and all the coefficients of $(H^{11})^{-1}$ are also non-negative since H^{11} is an M-matrix. This means that all the coefficients of $-(H^{11})^{-1}H^{12}$ are non-negative. Let $X^1, X^2 \in \mathbf{R}^{n+2}$ be two vectors with the following bloc decomposition:

$$X^1 := \begin{pmatrix} X^{11} \\ X^{12} \end{pmatrix} \text{ and } X^2 := \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix},$$

where $X^{11}, X^{21} \in \mathbf{R}^n$ and $X^{12}, X^{22} \in \mathbf{R}^2$. Let $M := -H^{21}(H^{11})^{-1}$. We have:

$$\begin{aligned} {}^tX^1GX^2 &= \begin{pmatrix} {}^tX^{11} & {}^tX^{12} \end{pmatrix} \begin{pmatrix} I_n & -(H^{11})^{-1}H^{12} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -H^{21}(H^{11})^{-1} & I_2 \end{pmatrix} \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix} \\ &= \begin{pmatrix} {}^tX^{11} & {}^tX^{12} \end{pmatrix} \begin{pmatrix} I_n & M \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ M & I_2 \end{pmatrix} \begin{pmatrix} X^{21} \\ X^{22} \end{pmatrix} \\ &= \begin{pmatrix} {}^tX^{11} & {}^tX^{11}\,{}^tM + {}^tX^{12} \end{pmatrix} \begin{pmatrix} (H^{11})^{-1} & 0 \\ 0 & G^{22} \end{pmatrix} \begin{pmatrix} X^{21} \\ MX^{21} + X^{22} \end{pmatrix} \\ &= {}^tX^{11}(H^{11})^{-1}X^{21} + ({}^tX^{11}\,{}^tM + {}^tX^{12})G^{22}(MX^{21} + X^{22}) \\ &= {}^tX^{11}(H^{11})^{-1}X^{21} + {}^t(MX^{11} + X^{12})G^{22}(MX^{21} + X^{22}). \end{aligned}$$

Now we can define $\alpha_1(X^1), \alpha_2(X^1), \alpha_1(X^2)$ and $\alpha_2(X^2)$ by:

$$\begin{pmatrix} \alpha_1(X^1) \\ \alpha_2(X^1) \end{pmatrix} := MX^{11} + X^{12} \text{ and } \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix} := MX^{21} + X^{22}.$$

We also define $C(X^1, X^2)$ by $C(X^1, X^2) := {}^tX^{11}(H^{11})^{-1}X^{21}$. We get:

$${}^tX^1GX^2 = C(X^1,X^2) + \begin{pmatrix}\alpha_1(X^1) & \alpha_2(X^1)\end{pmatrix} G^{22} \begin{pmatrix}\alpha_1(X^2) \\ \alpha_2(X^2)\end{pmatrix}.$$

Similarly, we get:

$$\begin{aligned} {}^t\overline{X}^1\overline{GX}^2 &= {}^tX^{11}(H^{11})^{-1}X^{21} + {}^t(-\overline{H}^{21}(H^{11})^{-1}X^{11} + \overline{X}^{12})\overline{G}^{22}(-\overline{H}^{21}(H^{11})^{-1}X^{21} + \overline{X}^{22}) \\ &= C(X^1,X^2) + \left(\alpha_1(X^1) + \alpha_2(X^1)\right)\overline{G}^{22}\left(\alpha_1(X^2) + \alpha_2(X^2)\right) \end{aligned}$$

□

4.3 The coupling

4.3.1 A change of variables

When we look at ν_2, instead of looking at the beta-field (β_1, β_2) we will look at two other variables that will make our coupling and various calculations more explicit. In the following lemma we state this change of variables and some relevant properties of the new variables.

Lemma 4.3.1.1. *We set a parameter $\lambda \in [0,1]$ and a parameter $w \geq 0$ such that if $w = 0$ then $\lambda \notin \{0,1\}$. Let $W := \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$. Let (β_1, β_2) be distributed according to $\nu_2^{W,0}$. We define the variables γ and Z by:*

$$\gamma := \frac{1}{\begin{pmatrix} \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}} = \frac{4\beta_1\beta_2 - w^2}{2w\lambda(1-\lambda) + 2\beta_2\lambda^2 + 2\beta_1(1-\lambda)^2},$$

$$Z := \frac{2\beta_1 - \lambda^2\gamma}{w + \lambda(1-\lambda)\gamma}.$$

We have that both Z and γ are positive and:

$$2\beta_1 = \lambda^2\gamma + (w + \lambda(1-\lambda)\gamma)Z,$$
$$2\beta_2 = (1-\lambda)^2\gamma + (w + \lambda(1-\lambda)\gamma)\frac{1}{Z}.$$

The random variable γ is the only random variable such that:

$$\begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix} - \gamma \begin{pmatrix} \lambda^2 & \lambda(1-\lambda) \\ \lambda(1-\lambda) & (1-\lambda)^2 \end{pmatrix}$$

is of rank one. The law of γ is that of a Gamma of parameter $(\frac{1}{2}, \frac{1}{2})$. The law of Z, knowing γ is given by:

$$\frac{\sqrt{W + \lambda(1-\lambda)\gamma}}{\sqrt{2\pi}} \exp\left(-(W + \lambda(1-\lambda)\gamma)\frac{(z-1)^2}{2z}\right) \frac{1}{z} \left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right) 1_{z>0} \mathrm{d}z.$$

This law is a mixture of an inverse gaussian law and its inverse.
If U is defined by $U := \sqrt{Z} - \frac{1}{\sqrt{Z}}$, its density, knowing γ, is given by:

$$\frac{\sqrt{w + \lambda(1-\lambda)\gamma}}{\sqrt{2\pi}} \exp\left(-(w + \lambda(1-\lambda)\gamma)\frac{u^2}{2}\right) \left(1 - (2\lambda - 1)\frac{u}{\sqrt{u^2+4}}\right) \mathrm{d}u.$$

This law is similar to a gaussian, in particular the law of $|U|$ is that of the absolute value of a gaussian.
We also have the following equality:

$$det \begin{pmatrix} 2\beta_1 & -w \\ -w & 2\beta_2 \end{pmatrix} = 4\beta_1\beta_2 - w^2 = (w + \lambda(1-\lambda)\gamma)\gamma \left((1-\lambda)\sqrt{Z} + \frac{\lambda}{\sqrt{Z}}\right)^2.$$

The random variable γ is a generalization of the random variable γ defined in [73], in which it is only defined for $\lambda \in \{0,1\}$. It is used to make a link between the β-field and the VRJP starting at a specific point.

Proof. Let $\mathscr{H} \subset (0,\infty)^2$ be the set defined by:

$$\mathscr{H} := \left\{(b_1, b_2) \in (0,\infty)^2, \begin{pmatrix} 2b_1 & -w \\ -w & 2b_2 \end{pmatrix} > 0\right\}.$$

Let $f : (0,\infty)^2 \mapsto \mathbf{R}^2$ be the function defined by:

$$f(c,z) := \left(\frac{\lambda^2 c + (w + \lambda(1-\lambda)c)z}{2}, \frac{(1-\lambda)^2 c + (w + \lambda(1-\lambda)c)\frac{1}{z}}{2}\right).$$

First we need to check that $f\left((0,\infty)^2\right) \subset \mathscr{H}$. First, $\frac{\lambda^2 c + (w+\lambda(1-\lambda)c)z}{2} > 0$ and $\frac{(1-\lambda)^2 c + (w+\lambda(1-\lambda)c)\frac{1}{z}}{2} > 0$. Then:

$$4\frac{\lambda^2 c + (w + \lambda(1-\lambda)c)z}{2} \frac{(1-\lambda)^2 c + (w + \lambda(1-\lambda)c)\frac{1}{z}}{2} - w^2 > wzw\frac{1}{z} - w^2 > 0.$$

This means that $f\left((0,\infty)^2\right) \subset \mathscr{H}$.

Now we need a small result on matrices that will make calculations on f simpler. Let $Y := \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}$. For any $(a_1, a_2) \in \mathscr{H}$ and $s \in \mathbf{R}$, we have:

$$\begin{aligned}
\det\left(\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix} - sY\,{}^tY\right) =& \det\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix} \det\left(I_2 - s\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix}^{-1} Y\,{}^tY\right) \\
=& \det\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix} \det\left(1 - s\,{}^tY\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix}^{-1} Y\right) \\
=& \det\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix} \left(1 - s\,{}^tY\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix}^{-1} Y\right).
\end{aligned}$$

This means that

$$\det\left(\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix} - sY\,{}^tY\right) = 0 \Leftrightarrow s = \frac{1}{{}^tY\begin{pmatrix} 2a_1 & -w \\ -w & 2a_2 \end{pmatrix}^{-1} Y}.$$

Now we notice that if $(b_1, b_2) := f(c,z)$ then

$$\begin{pmatrix} 2b_1 & -w \\ -w & 2b_2 \end{pmatrix} - cY\,{}^tY = \begin{pmatrix} (w+\lambda(1-\lambda)c)z & -(w+\lambda(1-\lambda)c) \\ -(w+\lambda(1-\lambda)c) & (w+\lambda(1-\lambda)c)\frac{1}{z} \end{pmatrix},$$

which is of rank one, and the eigenvector for the non-zero eigenvalue is $\begin{pmatrix} \sqrt{z} \\ \frac{1}{\sqrt{z}} \end{pmatrix}$.

Therefore if we know that $(b_1, b_2) = f(c,z)$ then

$$\begin{aligned}
c =& \frac{1}{{}^tY\begin{pmatrix} 2b_1 & -w \\ -w & 2b_2 \end{pmatrix}^{-1} Y} = \frac{4b_1b_2 - w^2}{2b_2\lambda^2 + 2b_1(1-\lambda)^2 + 2\lambda(1-\lambda)w}, \text{ and} \\
z =& \frac{2b_1 - \lambda^2 c}{w+\lambda(1-\lambda)\gamma} = \frac{w+\lambda(1-\lambda)\gamma}{2b_2 - (1-\lambda)^2 c}.
\end{aligned}$$

This means that f is injective and its inverse is the one we want. Conversely, f is surjective by using the same formula.

The Jacobian J_f of the change of variables f is equal to:

$$J_f(c,z) = \begin{pmatrix} \left(\lambda^2 + \lambda(1-\lambda)z\right)\frac{1}{2} & \left((1-\lambda)^2 + \lambda(1-\lambda)\frac{1}{z}\right)\frac{1}{2} \\ (w+\lambda(1-\lambda)c)\frac{1}{2} & -(w+\lambda(1-\lambda)c)\frac{1}{2z^2} \end{pmatrix}$$

and therefore the determinant D_f of the Jacobian is equal to :

$$\begin{aligned}
D_f(c,z) =& \frac{w+\lambda(1-\lambda)c}{4}\left((1-\lambda)^2 + \lambda(1-\lambda)\frac{1}{z} + \lambda^2\frac{1}{z^2} + \lambda(1-\lambda)\frac{1}{z}\right) \\
=& \frac{w+\lambda(1-\lambda)c}{4}\left(1-\lambda+\frac{\lambda}{z}\right)^2 \\
=& \frac{w+\lambda(1-\lambda)c}{4}\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)^2.
\end{aligned}$$

Now we can change variables (β_1, β_2) such that $H_\beta := \begin{pmatrix} 2b_1 & -w \\ -w & 2b_2 \end{pmatrix} > 0$ into variables (γ, z) defined by:

$$\begin{aligned}
\gamma :=& \frac{4\beta_1\beta_2 - w^2}{2w\lambda(1-\lambda) + 2\beta_2\lambda^2 + 2\beta_1(1-\lambda)^2}, \\
z :=& \frac{2\beta_1 - \lambda^2\gamma}{w+\lambda(1-\lambda)\gamma}.
\end{aligned}$$

We need to make a few calculations before we can express the law of (γ, Z). First we have, for any $(c,z) \in$

$(0,\infty)^2$, with $(b_1,b_2) := f(c,z)$:

$$\begin{aligned}
&4b_1b_2 - w^2\\
&= \left((w+\lambda(1-\lambda)c)z + \lambda^2 c\right)\left((w+\lambda(1-\lambda)c)\frac{1}{z} + (1-\lambda)^2 c\right) - w^2\\
&=(w+\lambda(1-\lambda)c)^2 + (w+\lambda(1-\lambda)c)\left(\lambda^2 c\frac{1}{z} + (1-\lambda)^2 cz\right) + \lambda^2(1-\lambda)^2c^2 - w^2\\
&=(\lambda(1-\lambda)c)^2 + 2w\lambda(1-\lambda)c + (w+\lambda(1-\lambda)c)\left(\lambda^2 c\frac{1}{x} + (1-\lambda)^2 cx\right) + \lambda^2(1-\lambda)^2c^2\\
&=2c(w+\lambda(1-\lambda)c) + (w+\lambda(1-\lambda)c)c\left(\lambda^2\frac{1}{z} + (1-\lambda)^2 z\right) + \lambda^2(1-\lambda)^2c^2\\
&=(w+\lambda(1-\lambda)c)c\left(\lambda^2\frac{1}{z} + (1-\lambda)^2 z + 2\right)\\
&=(w+\lambda(1-\lambda)c)c\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)^2.
\end{aligned}$$

Therefore we get:

$$\frac{D_f(c,z)}{\sqrt{4b_1b_2 - w^2}} = \frac{\sqrt{w+\lambda(1-\lambda)c}}{4\sqrt{c}}\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right).$$

We also have the following equality:

$$\begin{aligned}
b_1 + b_2 - w =& \lambda^2\frac{c}{2} + (w+\lambda(1-\lambda)c)\frac{z}{2} + (1-\lambda)^2\frac{c}{2} + (w+\lambda(1-\lambda)c)\frac{1}{2z}\\
&- ((w+\lambda(1-\lambda)c) - \lambda(1-\lambda)c)\\
=&(\lambda^2 + (1-\lambda)^2 + 2)\frac{c}{2} + \frac{1}{2}(w+\lambda(1-\lambda)c)\left(z + \frac{1}{z} - 2\right)\\
=&\frac{c}{2} + \frac{1}{2}(w+\lambda(1-\lambda)c)\frac{1}{z}(z-1)^2.
\end{aligned}$$

And therefore we get the following joint law for γ and Z (c represents γ and z represents Z):

$$\frac{2}{\pi}\frac{\sqrt{w+\lambda(1-\lambda)c}}{4\sqrt{c}}\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)\exp\left(-\frac{c}{2} - (w+\lambda(1-\lambda)c)\frac{(z-1)^2}{2z}\right)\mathrm{d}z\mathrm{d}c.$$

In particular, the law of Z, knowing γ, is given by

$$\frac{\sqrt{w+\lambda(1-\lambda)\gamma}}{\sqrt{2\pi}}\exp\left(-(w+\lambda(1-\lambda)\gamma)\frac{(z-1)^2}{2z}\right)\frac{1}{z}\left((1-\lambda)\sqrt{z} + \frac{\lambda}{\sqrt{z}}\right)\mathrm{d}z.$$

It is indeed a density since it is a mixture of an inverse gaussian and the inverse of an inverse gaussian. Now, we can look at the law of U. By definition, $U = \sqrt{Z} - \frac{1}{\sqrt{Z}}$. This means that $\sqrt{Z} = \frac{\sqrt{U^2+4}+U}{2}$ and $\frac{1}{\sqrt{Z}} = \frac{\sqrt{U^2+4}-U}{2}$. We therefore have $Z = \frac{U^2+2+U\sqrt{U^2+4}}{2}$. The density of U is thus:

$$\begin{aligned}
&\frac{1}{2}\left(2u + \sqrt{u^2+4} + \frac{u^2}{\sqrt{u^2+4}}\right)\frac{\sqrt{w+\lambda(1-\lambda)\gamma}}{\sqrt{2\pi}}\exp\left(-(w+\lambda(1-\lambda)\gamma)\frac{u^2}{2}\right)\\
&\quad\times\frac{2}{u^2+2+u\sqrt{u^2+4}}\left((1-\lambda)\frac{\sqrt{u^2+4}+u}{2} + \lambda\frac{\sqrt{u^2+4}-u}{2}\right)\mathrm{d}u\\
=&\frac{2u\sqrt{u^2+4}+2u^2+4}{2\sqrt{u^2+4}}\frac{\sqrt{w+\lambda(1-\lambda)\gamma}}{\sqrt{2\pi}}\exp\left(-(w+\lambda(1-\lambda)\gamma)\frac{u^2}{2}\right)\\
&\quad\times\frac{2}{u^2+2+u\sqrt{u^2+4}}\left((1-\lambda)\frac{\sqrt{u^2+4}+u}{2} + \lambda\frac{\sqrt{u^2+4}-u}{2}\right)\mathrm{d}u\\
=&\frac{\sqrt{w+\lambda(1-\lambda)\gamma}}{\sqrt{2\pi}}\exp\left(-(w+\lambda(1-\lambda)\gamma)\frac{u^2}{2}\right)\left(1 - (2\lambda-1)\frac{u}{\sqrt{u^2+4}}\right)\mathrm{d}u.
\end{aligned}$$

□

4.3.2 The tilted gaussian law

Definition 46. *For any $(K,\delta) \in (0,\infty) \times [-1,1]$ we define the tilted gaussian law $\tilde{\mathcal{N}}(K,\delta)$ by the following density:*

$$\sqrt{\frac{K}{2\pi}} \exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right) \mathrm{d}u.$$

It is indeed a density since it is the density of a gaussian plus an antisymmetric term that is smaller than the gaussian term.

Lemma 4.3.2.1. *Set $K > 0$ and $\delta, \delta' \in [-1,1]$. Let U be a random variable distributed according to $\tilde{\mathcal{N}}(K,\delta)$. We have the following equality:*

$$\mathbb{E}\left(\frac{1+\delta'\frac{U}{\sqrt{(U)^2+4}}}{1+\delta\frac{U}{\sqrt{(U)^2+4}}}\right) = 1.$$

Proof. We have:

$$\begin{aligned}
\mathbb{E}\left(\frac{1+\delta'\frac{U}{\sqrt{(U)^2+4}}}{1+\delta\frac{U}{\sqrt{(U)^2+4}}}\right) &= \int\limits_{u\in\mathbf{R}} \sqrt{\frac{K}{2\pi}} \exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1+\delta'\frac{u}{\sqrt{(u)^2+4}}}{1+\delta\frac{u}{\sqrt{(u)^2+4}}}\right) \mathrm{d}u \\
&= \int\limits_{u\in\mathbf{R}} \sqrt{\frac{K}{2\pi}} \exp\left(-\frac{Ku^2}{2}\right)\left(1+\delta'\frac{u}{\sqrt{(u)^2+4}}\right) \mathrm{d}u \\
&= 1.
\end{aligned}$$

□

Lemma 4.3.2.2. *Let $0 < K^- \leq K^+$. Set $\delta \in [-1,1]$. There exists two random variables U^- and U^+ distributed according to $\tilde{\mathcal{N}}(K^-,\delta)$ and $\tilde{\mathcal{N}}(K^+,\delta)$ respectively such that:*

$$\forall \delta' \in [-1,1],\ \mathbb{E}\left(\frac{1+\delta'\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}} | U^+\right) = \frac{1+\delta'\frac{U^+}{\sqrt{(U^+)^2+4}}}{1+\delta\frac{U^+}{\sqrt{(U^+)^2+4}}},$$

and

$$K^-(U^-)^2 = K^+(U^+)^2 \ a.s.$$

Proof. Let $K := \sqrt{\frac{K^+}{K^-}}$. Let U^+ be a random variable distributed according to $\tilde{\mathcal{N}}(K^+,\delta)$. First we define the random variables V^+ and V^- by:

$$\begin{aligned}
V^+ :=& \frac{U^+}{\sqrt{(U^+)^2+4}} \\
V^- :=& \frac{KU^+}{\sqrt{K^2(U^+)^2+4}}.
\end{aligned}$$

We notice that $0 \leq |V^+| \leq |V^-| < 1$. Let $p_1, p_2 \in \mathbf{R}$ be defined by:

$$p^+ := \frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+} \text{ and } p^- := \frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}.$$

Both p^+ and p^- are non-negative. We also have:

$$\begin{aligned}
p^+ + p^- =& \frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+} + \frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+} \\
=& \frac{1+\delta V^- + 1 - \delta V^- + \frac{V^+}{V^-}\left(1+\delta V^- - 1 + \delta V^-\right)}{2(1+\delta V^+)} \\
=& \frac{2+\frac{V^+}{V^-}2\delta V^-}{2(1+\delta V^+)} = 1.
\end{aligned}$$

Now, let U^- the random variable be such that knowing U^+:

$$U^- := \begin{cases} KU^+ \text{ with probability } p^+ \\ -KU^+ \text{ with probability } p^- \end{cases}.$$

Now we want to show that U^- is distributed according to $\tilde{\mathcal{N}}(K^-,\delta)$. We have, for any test function f:

$$\begin{aligned}\mathbb{E}\left(f(U^-)\right) =& \mathbb{E}\left(\mathbb{E}\left(f(U^-)|U^+\right)\right)\\ =& \mathbb{E}\left(\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}f\left(KU^+\right)+\frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}f\left(-KU^+\right)\right).\end{aligned}$$

First we get:

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}f\left(KU^+\right)\right)\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^+}{2\pi}}\exp\left(-\frac{K^+u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1}{2}\left(1+\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)\frac{1+\delta K\frac{u}{\sqrt{K^2u^2+4}}}{1+\delta\frac{u}{\sqrt{u^2+4}}}f\left(Ku\right)\right)\mathrm{d}u\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^+}{2\pi}}\exp\left(-\frac{K^+u^2}{2}\right)\left(1+\delta K\frac{u}{\sqrt{K^2u^2+4}}\right)\left(\frac{1}{2}\left(1+\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)f\left(Ku\right)\right)\mathrm{d}u\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^-}{2\pi}}\exp\left(-\frac{K^-u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1}{2}\left(1+\frac{\sqrt{u^2+4}}{\sqrt{u^2+4K}}\right)f\left(u\right)\right)\mathrm{d}u.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}f\left(-KU^+\right)\right)\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^+}{2\pi}}\exp\left(-\frac{K^+u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1}{2}\left(1-\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)\frac{1-\delta K\frac{u}{\sqrt{K^2u^2+4}}}{1+\delta\frac{u}{\sqrt{u^2+4}}}f\left(-Ku\right)\right)\mathrm{d}u\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^+}{2\pi}}\exp\left(-\frac{K^+u^2}{2}\right)\left(1-\delta K\frac{u}{\sqrt{K^2u^2+4}}\right)\left(\frac{1}{2}\left(1-\frac{\sqrt{K^2u^2+4}}{K\sqrt{u^2+4}}\right)f\left(-Ku\right)\right)\mathrm{d}u\\ =&\int_{u\in\mathbf{R}}\sqrt{\frac{K^-}{2\pi}}\exp\left(-\frac{K^-u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)\left(\frac{1}{2}\left(1-\frac{\sqrt{u^2+4}}{\sqrt{u^2+4K}}\right)f\left(u\right)\right)\mathrm{d}u.\end{aligned}$$

If we put both equalities together, we get for any test function f:

$$\mathbb{E}\left(f(U^-)\right)=\int_{u\in\mathbf{R}}\sqrt{\frac{K^-}{2\pi}}\exp\left(-\frac{K^-u^2}{2}\right)\left(1+\delta\frac{u}{\sqrt{u^2+4}}\right)f(u)\mathrm{d}u.$$

This means that U^- is indeed distributed according to $\tilde{\mathcal{N}}(K^-,\delta)$. Now we only need to show that U^+ and U^- satisfy the equality we want. First we notice that for any $x\in(-1,1)$:

$$\frac{1+\delta' x}{1+\delta x}=1+(\delta'-\delta)\frac{x}{1+\delta x}.$$

This means that we only need to show that:

$$\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}|U^+\right)=\frac{\frac{U^+}{\sqrt{(U^+)^2+4}}}{1+\delta\frac{U^+}{\sqrt{(U^+)^2+4}}}.$$

Which is the same as showing:

$$\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}|U^+\right)=\frac{V^+}{1+\delta V^+}.$$

By definition of U^-, V^- and V^+, we have:

$$\mathbb{E}\left(\frac{\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}|U^+\right)$$

$$\begin{aligned}
=&\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}\frac{\frac{KU^+}{\sqrt{(KU^+)^2+4}}}{1+\delta\frac{KU^+}{\sqrt{(KU^+)^2+4}}}+\frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}\frac{\frac{-KU^+}{\sqrt{(-KU^+)^2+4}}}{1+\delta\frac{-KU^+}{\sqrt{(-KU^+)^2+4}}}\\
=&\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{1+\delta V^-}{1+\delta V^+}\frac{V^-}{1+\delta V^-}+\frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{1-\delta V^-}{1+\delta V^+}\frac{-V^-}{1-\delta V^-}\\
=&\frac{1}{2}\left(1+\frac{V^+}{V^-}\right)\frac{V^-}{1+\delta V^+}+\frac{1}{2}\left(1-\frac{V^+}{V^-}\right)\frac{-V^-}{1+\delta V^+}\\
=&\frac{V^+}{1+\delta V^+}
\end{aligned}$$

□

Lemma 4.3.2.3. *Set $w>0$ and $W:=\begin{pmatrix}0 & w\\ w & 0\end{pmatrix}$. Now set 2 parameters $\lambda,\theta\in[0,1]$. Let (β_1,β_2) be distributed according to $\nu_2^{W,0}$. Let H_β be the random matrix defined by:*

$$H_\beta:=\begin{pmatrix}2\beta_1 & -w\\ -w & 2\beta_2\end{pmatrix}.$$

Let G_β be the inverse of H_β. We define the random variables γ and Z by:

$$\begin{aligned}
\gamma:=&\frac{4\beta_1\beta_2-w^2}{2w\lambda(1-\lambda)+2\beta_2\lambda^2+2\beta_1(1-\lambda)^2},\\
Z:=&\frac{2\beta_1-\lambda\gamma}{w+\lambda(1-\lambda)\gamma}.
\end{aligned}$$

We have:

$$\begin{pmatrix}\lambda & (1-\lambda)\end{pmatrix}G_\beta\begin{pmatrix}\theta\\ (1-\theta)\end{pmatrix}=\frac{\theta\frac{1}{\sqrt{Z}}+(1-\theta)\sqrt{Z}}{\gamma\left((1-\lambda)\sqrt{Z}+\lambda\frac{1}{\sqrt{Z}}\right)}.$$

Proof. First, by lemma 4.3.1.1 we have:

$$\begin{aligned}
2\beta_1&=(w+\lambda(1-\lambda)\gamma)\,Z+\lambda^2\gamma,\\
2\beta_2&=(w+\lambda(1-\lambda)\gamma)\,\frac{1}{Z}+(1-\lambda)^2\gamma,\\
w&=(w+\lambda(1-\lambda)\gamma)-\lambda(1-\lambda)\gamma.
\end{aligned}$$

To simplify notations, let $\tilde{w}$ be the random variable defined by $\tilde{w}:=w+\lambda(1-\lambda)\gamma$. A quantity that will be important in the following is the determinant of H_β: $4\beta_1\beta_2-w^2$. By lemma 4.3.1.1, we have:

$$4\beta_1\beta_2-w^2=\tilde{w}\gamma\left((1-\lambda)\sqrt{Z}+\lambda\frac{1}{\sqrt{Z}}\right)^2.$$

We know that :

$$G_\beta(1,1)=\frac{2\beta_2}{4\beta_1\beta-w^2},\ G_\beta(2,2)=\frac{2\beta_1}{4\beta_1\beta-w^2}\text{ and }G_\beta(1,2)=G_\beta(2,1)=\frac{w}{4\beta_1\beta-w^2}.$$

Therefore:

$$\begin{pmatrix}\lambda & 1-\lambda\end{pmatrix}G_\beta\begin{pmatrix}\theta\\ 1-\theta\end{pmatrix}=\frac{\lambda\theta2\beta_2+(\lambda(1-\theta)+(1-\lambda)\theta)w+(1-\lambda)(1-\theta)2\beta_1}{4\beta_1\beta_2-w^2}.$$

Now we also have:

$$\begin{aligned}
&\lambda\theta2\beta_2+(\lambda(1-\theta)+(1-\lambda)\theta)w+(1-\lambda)(1-\theta)2\beta_1\\
=&\lambda\theta\left(\tilde{w}\frac{1}{Z}+(1-\lambda)^2\gamma\right)+(\lambda(1-\theta)+(1-\lambda)\theta)(\tilde{w}-\lambda(1-\lambda)\gamma)+(1-\lambda)(1-\theta)\left(\tilde{w}Z+\lambda^2\gamma\right)\\
=&\lambda\theta\tilde{w}\frac{1}{Z}+(\lambda(1-\theta)+(1-\lambda)\theta)\tilde{W}+(1-\lambda)(1-\theta)\tilde{w}Z\\
=&\tilde{w}\left(\lambda\frac{1}{\sqrt{Z}}+(1-\lambda)\sqrt{Z}\right)\left(\theta\frac{1}{\sqrt{Z}}+(1-\theta)\sqrt{Z}\right).
\end{aligned}$$

We therefore get:

$$\begin{aligned}(\lambda \quad 1-\lambda)\, G_\beta \begin{pmatrix}\theta \\ 1-\theta\end{pmatrix} =& \frac{\tilde{w}\left(\lambda\frac{1}{\sqrt{Z}}+(1-\lambda)\sqrt{z}\right)\left(\theta\frac{1}{\sqrt{Z}}+(1-\theta)\sqrt{Z}\right)}{\tilde{w}\gamma\left((1-\lambda)\sqrt{Z}+\lambda\frac{1}{\sqrt{Z}}\right)^2} \\ =& \frac{\theta\frac{1}{\sqrt{Z}}+(1-\theta)\sqrt{Z}}{\gamma\left((1-\lambda)\sqrt{Z}+\lambda\frac{1}{\sqrt{Z}}\right)}.\end{aligned}$$

□

4.4 Main theorem

Some of the results are based on some manipulations on graph, mostly we will quotient graphs. We remind the reader of the definition of the quotient of a graph by one of its subset. We also add the notion of weight for these quotients.

Definition 47. *Let $\mathscr{G} = (V, E)$ be a locally finite, non-directed graph. Let $(W_e)_{e\in E}$ be a family of weights on the edges of $\mathscr{G} = (V, E)$. Let A be a subset of V. The quotient $(\tilde{V}^A, \tilde{E}^A), \tilde{W}^A$ of the weighted graph $\mathscr{G}, W$ by the subset of vertices A is defined by:*

$$\begin{aligned}&\tilde{V}^A := V\backslash A \cup \{x_A\} \\ &\tilde{E}^A := \{\{x,y\}\in E, x,y\in V\backslash A\} \cup \{\{x_A,y\}\in \left(\tilde{V}_A\right)^2, \exists x\in A, \{x,y\}\in E\} \\ &\forall \{x,y\}\in \tilde{E}^A, x,y\notin A, W^A_{\{x,y\}} := W_{\{x,y\}}, \\ &\forall x\in \tilde{V}^A\backslash\{x_A\} \text{ such that } \{x_A,x\}\in \tilde{E}^A, W^A_{\{x_A,a\}} := \sum_{y\in A} 1_{\{x,y\}\in E} W_{\{x,y\}}.\end{aligned}$$

We can now prove our main theorem.

proof of theorem 59. According to proposition 4.1.2.1, the marginal law of $(\beta_i)_{1\le i\le n}$ is the same under $\nu^{W^-,0}_{n+2}, \nu^{W^+,0}_{n+2}$ and $\nu^{W^\infty,0}_{n+1}$ and is equal to $\nu^{W,\eta}_n$ for some $\eta \in \mathbf{R}^n$. Let H be distributed according to $\tilde{\nu}^{W,\eta}_n$. Let $K \in [0,+\infty)$ be the random variable defined by

$$K := {}^t W^2 H^{-1} W^1,$$

and $\tilde{K}$ the random matrix defined by:

$$\tilde{K} = \begin{pmatrix}0 & K \\ K & 0\end{pmatrix}.$$

Set a vector $X^1 \in [0,\infty)^{n+2}$. Let $\alpha_1(X^1)$ and $\alpha_2(X^1)$ be the numbers defined in lemma 4.2.2.2 and $\alpha(X^1) := \alpha_1(X^1) + \alpha_2(X^1)$. Let $\lambda \in [0,1]$ be the random variable defined by

$$\lambda := \begin{cases}\frac{\alpha_1(X^1)}{\alpha_1(X^1)+\alpha_2(X^1)} & \text{if } \alpha_1(X^1)+\alpha_2(X^1) \neq 0 \\ 0 & \text{otherwise}\end{cases},$$

and $\delta \in [-1,1]$ the random variable defined by $\delta := 2\lambda - 1$.
If $n+1$ and $n+2$ are H^--connected then $K + w^- > 0$. Let γ be a random variable distributed according to a $\Gamma\left(\frac{1}{2}\right)$ distribution. Now let U^- and U^+ be two random variables distributed according to $\tilde{\mathscr{N}}(K+w^-,\delta)$ and $\tilde{\mathscr{N}}(K+w^+,\delta)$ respectively and such that

$$\forall \delta' \in [-1,1],\ \mathbb{E}\left(\frac{1+\delta'\frac{U^-}{\sqrt{(U^-)^2+4}}}{1+\delta\frac{U^-}{\sqrt{(U^-)^2+4}}}|U^+\right) = \frac{1+\delta'\frac{U^+}{\sqrt{(U^+)^2+4}}}{1+\delta\frac{U^+}{\sqrt{(U^+)^2+4}}}.$$

Such two random variables exist by lemma 4.3.2.2. We define the positive random variables Z^- and Z^+ by:

$$U^- = \sqrt{Z^-} - \frac{1}{\sqrt{Z^-}} \text{ and } U^+ = \sqrt{Z^+} - \frac{1}{\sqrt{Z^+}}.$$

Now, we define the random variables $\tilde{\beta}^-_{n+1}, \tilde{\beta}^-_{n+2}, \tilde{\beta}^+_{n+1}$ and $\tilde{\beta}^+_{n+2}$ by:

$$\begin{aligned}
2\tilde{\beta}^-_{n+1} &= \left(K + w^- + \lambda(1-\lambda)\gamma\right) Z^- + \lambda^2\gamma \\
2\tilde{\beta}^-_{n+2} &= \left(K + w^- + \lambda(1-\lambda)\gamma\right) Z^- + (1-\lambda)^2\gamma \\
2\tilde{\beta}^+_{n+1} &= \left(K + w^+ + \lambda(1-\lambda)\gamma\right) Z^+ + \lambda^2\gamma \\
2\tilde{\beta}^+_{n+2} &= \left(K + w^+ + \lambda(1-\lambda)\gamma\right) Z^+ + (1-\lambda)^2\gamma.
\end{aligned}$$

Let $\tilde{K}^-$ and $\tilde{K}^+$ be the matrices defined by:

$$\tilde{K}^- := \begin{pmatrix} 0 & w^- + K \\ w^- + K & 0 \end{pmatrix} \text{ and } \tilde{K}^+ := \begin{pmatrix} 0 & w^+ + K \\ w^+ + K & 0 \end{pmatrix}.$$

By lemma 4.3.1.1, knowing K and δ, $(\tilde{\beta}^-_{n+1}, \tilde{\beta}^-_{n+2})$ and $(\tilde{\beta}^-_{n+1}, \tilde{\beta}^-_{n+2})$are distributed according to $\nu_2^{\tilde{K}^-,0}$ and $\nu_2^{\tilde{K}^+,0}$ respectively. Now we can define the matrices H^-, H^+ and H^∞ by bloc:

$$H^- = \begin{pmatrix} H & -W^1 & -W^2 \\ -{}^tW^1 & 2\tilde{\beta}^-_{n+1} + {}^tW^1 H^{-1} W^1 & -w^- \\ -{}^tW^2 & -w^- & 2\tilde{\beta}^-_{n+2} + {}^tW^2 H^{-1} W^2 \end{pmatrix},$$

$$H^+ = \begin{pmatrix} H & -W^1 & -W^2 \\ -{}^tW^1 & 2\tilde{\beta}^-_{n+1} + {}^tW^1 H^{-1} W^1 & -w^+ \\ -{}^tW^2 & -w^+ & 2\tilde{\beta}^+_{n+2} + {}^tW^2 H^{-1} W^2 \end{pmatrix},$$

$$H^\infty = \begin{pmatrix} H & -W^1 - W^2 \\ -{}^tW^1 - {}^tW^2 & \gamma + \left({}^tW^1 + {}^tW^2\right) H^{-1} \left(W^1 + W^2\right) \end{pmatrix}.$$

By proposition 4.1.2.1 and lemma 4.2.1.1, H^-, H^+ and H^∞ are distributed according to $\tilde{\nu}^{W^-,0}_{n+2}, \tilde{\nu}^{W^+,0}_{n+2}$ and $\tilde{\nu}^{W^\infty,0}_{n+1}$ respectively. Let G^-, G^+ and G^∞ be the inverse of H^-, H^+ and H^∞ respectively. Let $G^{22,-}, G^{22,+}$ and $G^{22,\infty}$ be defined by:

$$\begin{aligned}
G^{22,-} &:= \begin{pmatrix} G^-(n+1,n+1) & G^-(n+1,n+2) \\ G^-(n+2,n+1) & G^-(n+2,n+2) \end{pmatrix}, \\
G^{22,+} &:= \begin{pmatrix} G^+(n+1,n+1) & G^+(n+1,n+2) \\ G^+(n+2,n+1) & G^+(n+2,n+2) \end{pmatrix} \text{ and} \\
G^{22,\infty} &:= \left(G^\infty(n+1,n+1)\right).
\end{aligned}$$

For any vector $X^2 \in [0,\infty)^{n+2}$, by lemma 4.2.2.2 there exists three non-negative random variables $C(X^1, X^2), \alpha_1(X^2)$ and $\alpha_2(X^2)$ that only depend on H, W^1 and W^2 such that:

$${}^tX^1 G^- X^2 = C(X^1, X^2) + \begin{pmatrix} \alpha_1(X^1) & \alpha_2(X^1) \end{pmatrix} G^{22,-} \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix},$$

$${}^tX^1 G^+ X^2 = C(X^1, X^2) + \begin{pmatrix} \alpha_1(X^1) & \alpha_2(X^1) \end{pmatrix} G^{22,+} \begin{pmatrix} \alpha_1(X^2) \\ \alpha_2(X^2) \end{pmatrix},$$

and

$${}^t\overline{X}^1 G^\infty \overline{X}^2 = C(X^1, X^2) + (\alpha_1(X^1) + \alpha_2(X^1)) G^{22,\infty} (\alpha_1(X^2) + \alpha_2(X^2)).$$

Let $\alpha(X^2) := \alpha_1(X^2) + \alpha_2(X^2)$ and let $\theta \in [-1,1]$ be defined by:

$$\theta := \begin{cases} \frac{\alpha_1(X^2)}{\alpha_1(X^2)+\alpha_2(X^2)} & \text{if } \alpha_1(X^2) + \alpha_2(X^2) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

We have:

$${}^tX^1 G^- X^2 = C(X^1, X^2) + \alpha(X^1)\alpha(X^2) \begin{pmatrix} \lambda & 1-\lambda \end{pmatrix} G^{22,-} \begin{pmatrix} \theta \\ 1-\theta \end{pmatrix},$$

$${}^tX^1 G^+ X^2 = C(X^1, X^2) + \alpha(X^1)\alpha(X^2) \begin{pmatrix} \lambda & 1-\lambda \end{pmatrix} G^{22,+} \begin{pmatrix} \theta \\ 1-\theta \end{pmatrix},$$

and

$${}^t\overline{X}^1 G^\infty \overline{X}^2 = C(X^1, X^2) + \alpha(X^1)\alpha(X^2) G^{22,\infty}.$$

By lemma 4.3.2.3 and by definition of U^- and U^+, we have:

$$\mathbb{E}\left({}^tX^1G^-X^2|H^+\right) = {}^tX^1G^+X^2,$$

and

$${}^tX^1G^-X^1 = C(X^1,X^1)\left(\alpha_1(X^1)+\alpha_2(X^1)\right)^2\frac{1}{\gamma} = {}^tX^1G^+X^1 = {}^t\overline{X^1}G^\infty\overline{X^1}.$$

By lemmas 4.3.2.3 and 4.3.2.1, we have:

$$\mathbb{E}\left({}^tX^1G^+X^2|H^\infty\right) = {}^t\overline{X^1}G^\infty\overline{X^2}.$$

□

proof of theorem 60. Set an integer $i \in [\![1,n]\!]$. We will only show this result when W^- and W^+ differ by only two symmetric coefficients (i.e one edge): (k,l) and (l,k). We can assume that $W^-(n-1,n) < W^+(n-1,n)$ because of the symmetries of the family of laws $\tilde{\nu}_n^{W,0}$. For any $j_1, j_2 \in [\![1,n]\!]$, j_1 and j_2 are W^--connected. This means that by the main theorem, there exists two matrices H^- and H^+ distributed according to $\tilde{\nu}_n^{W^-,0}$ and $\tilde{\nu}_n^{W^+,0}$ respectively, with inverse G^- and G^+ respectively and such that:

- $G^-(i,i) = G^+(i,i)$ almost surely,
- $\forall X \in [0,\infty)^n$, $\mathbb{E}\left(\sum\limits_{j=1}^n X_jG^-(i,j)|H^+\right) = \sum\limits_{j=1}^n X_jG^+(i,j)$.

This means that for any convex function f and any vector $X \in [0,\infty)^n$:

$$\mathbb{E}\left(f\left(\frac{\sum\limits_{j=1}^n X_jG^-(i,j)}{G^-(i,i)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{\sum\limits_{j=1}^n X_jG^+(i,j)}{G^+(i,i)}\right)\right).$$

□

4.5 Proofs of theorems 54,55 and 56

4.5.1 Proof of theorem 54

Proof. Let $d_{\mathscr{G}}(\cdot,\cdot)$ be the graph distance on $\mathscr{G}$. Let $\mathscr{G}_n$ be the graph obtained by fusing together all the vertices at a distance n or more from 0. This means that $\mathscr{G}_n = (V_n, E_n)$, with:

$$\begin{aligned} V_n &= \{x \in V, d_{\mathscr{G}}(0,x) < n\} \cup \{\delta_n\} \text{ and,}\\ E_n &= \left\{\{x,y\} \in E, (x,y) \in V_n^2\right\} \cup \left\{\{x,\delta_n\}, d_{\mathscr{G}}(0,x) = n-1, \exists y \in V\backslash V_n, d_{\mathscr{G}}(x,y) = 1\right\}.\end{aligned}$$

Let $|V_n|$ be the number of vertices in V_n. Let $W_n^- \in M_{|V_n|}(\mathbf{R})$ and $W_n^+ \in M_{|V_n|}(\mathbf{R})$ be the symmetric matrices defined by:

- for any $x,y \in V_n$ such that $\{x,y\} \notin E_n$, $W_n^-(x,y) = W_n^+(x,y) = 0$,
- for any $x,y \in V_n\backslash\{\delta\}$, $W_n^-(x,x) = W^-_{\{x,x\}}$ and $W_n^+(x,x) = W^+_{\{x,x\}}$,
- for any $x \in V_n\backslash\{\delta\}$, $W_n^-(x,\delta_n) = W_n^-(\delta_n,x) = \sum\limits_{y\in V,\{x,y\}\in E} W^-_{\{x,y\}}1_{y\notin V_n}$
- for any $x \in V_n\backslash\{\delta\}$, $W_n^+(x,\delta_n) = W_n^+(\delta_n,x) = \sum\limits_{y\in V,\{x,y\}\in E} W^+_{\{x,y\}}1_{y\notin V_n}$

This means that for any $x,y \in V_n$, $W_n^-(x,y) \leq W_n^+(x,y)$. Let H_n^- and H_n^+ be two random matrices distributed according to $\tilde{\nu}_{|V_n|}^{W_n^-,0}$ and $\tilde{\nu}_{|V_n|}^{W_n^+,0}$ respectively. Let G_n^- and G_n^+ be the inverse of H_n^- and H_n^+ respectively. By Theorem 1 of [76], there exists two non-negative random variables $\psi^-(0)$ and $\psi^+(0)$ such that:

$$\frac{G_n^-(0,\delta_n)}{G_n^-(\delta_n,\delta_n)} \xrightarrow[n\to\infty]{} \psi^-(0) \text{ in law, and}$$
$$\frac{G_n^+(0,\delta_n)}{G_n^+(\delta_n,\delta_n)} \xrightarrow[n\to\infty]{} \psi^+(0) \text{ in law.}$$

Furthermore, still by theorem 1 of [76], we have:

$$\begin{aligned}\mathbb{P}\left(\text{The VRJP with initial weights } w^- \text{ is recurrent}\right) &= \mathbb{P}\left(\psi^-(0)=0\right),\\ \mathbb{P}\left(\text{The VRJP with initial weights } w^+ \text{ is recurrent}\right) &= \mathbb{P}\left(\psi^+(0)=0\right).\end{aligned}$$

Let $f : [0,\infty) \mapsto \mathbf{R}$ be a continuous, bounded, convex function. By theorem 60, we have, for any $n \geq 1$:

$$\mathbb{E}\left(f\left(\frac{G_n^-(0,\delta_n)}{G_n^-(\delta_n,\delta_n)}\right)\right) \geq \mathbb{E}\left(f\left(\frac{G_n^+(0,\delta_n)}{G_n^+(\delta_n,\delta_n)}\right)\right).$$

This means that $\mathbb{E}\left(f(\psi^-(0))\right) \geq \mathbb{E}\left(f(\psi^+(0))\right)$. For any $n \geq 1$, let $f_n : [0,\infty) \mapsto \mathbf{R}$ be the function defined by:

$$f_n(x) = \begin{cases} 1 - nx \text{ if } 0 \leq x \leq \frac{1}{n} \\ 0 \text{ if } x > \frac{1}{n}. \end{cases}$$

For any $n \geq 1$, the function f_n is continuous, bounded and convex, so $\mathbb{E}\left(f_n(\psi^-(0))\right) \geq \mathbb{E}\left(f_n(\psi^+(0))\right)$. We notice that

$$\begin{aligned}\mathbb{E}\left(f_n(\psi^-(0))\right) &\xrightarrow[n\to\infty]{} \mathbb{P}\left(\psi^-(0)=0\right), \text{ and}\\ \mathbb{E}\left(f_n(\psi^+(0))\right) &\xrightarrow[n\to\infty]{} \mathbb{P}\left(\psi^+(0)=0\right).\end{aligned}$$

This means that $\mathbb{P}\left(\psi^-(0)=0\right) \geq \mathbb{P}\left(\psi^+(0)=0\right)$ and therefore the probability that the VRJP with initial weights w^- is recurrent is greater than the probability that the VRJP with initial weights w^+ is recurrent. □

4.5.2 Proof of theorem 55

Proof. Set a dimension $d \geq 3$. By proposition 3 of [76], for any $w \in (0,\infty)$, the VRJP on $\mathbf{Z}^d$ is either almost surely recurrent or almost surely transient. Furthermore, by theorem 60, the probability that the VRJP is recurrent is non-increasing in the initial weight. Therefore, there exists $w_d \in [0,\infty]$ such that the VRJP on $\mathbf{Z}^d$ with initial weight $w \in (0,\infty)$ is recurrent if $w < w_d$ and transient if $w > w_d$. Since the VRJP is recurrent in dimension 3 for small enough weights (corollary 3 of [72]), $w_d \neq 0$ and since it is transient for large enough weights (lemma 9 of [76]), $w_d \neq \infty$. □

4.5.3 Proof of theorem 56

Proof. Set a dimension $d \geq 3$. Let E^d be the set of vertices in $\mathbf{Z}^d$. Set $0 < a^- < a^+$. Let $(W_e^-)_{e\in E}$ be iid random Gamma variables with parameter a^- and let $(W_e')_{e\in E}$ be iid random Gamma variables with parameter $a^+ - a^-$. By theorem 1 of [72], the ERRW on $\mathbf{Z}^d$ with initial weight $a^- \in (0,\infty)$ is a mixture of VRJP on $\mathbf{Z}^d$ where the initial weights are $(W_e^-)_{e\in E}$ and the ERRW on $\mathbf{Z}^d$ with initial weight $a^+ \in (0,\infty)$ is a mixture of VRJP on $\mathbf{Z}^d$ where the initial weights are $(W_e^- + W_e')_{e\in E}$. Now, by theorem 60, the VRJP with initial weights $(W_e^-)_{e\in E}$ has a higher probability of being recurrent than the VRJP with initial weights $(W_e^- + W_e')_{e\in E}$. Therefore the probability that the ERRW with constant weight equal to a is recurrent is non-increasing in a. By proposition 5 of [76], the ERRW with initial weight a is either almost surely transient or almost surely recurrent. Therefore, there exists $a_d \in [0,\infty]$ such that the ERRW on $\mathbf{Z}^d$ with initial weight $a \in (0,\infty)$ is recurrent if $a < a_d$ and transient if $a > a_d$. Since the ERRW is recurrent in dimension 3 for small enough weights, $a_d \neq 0$ and since it is transient for large enough weights, $a_d \neq \infty$. □

4.6 Proof of theorem 58

4.6.1 Preliminaries

Definition 48. *Let $\mathscr{G} = (V,E)$ be a finite graph and $(W_e)_{e\in E}$ be positive weights. Let H_β be the random matrix distributed according to $\tilde{\nu}_n^{W,0}$ and G_β its inverse. Let $x, y \in V$ be two distinct vertices of $\mathscr{G}$. The effective weight between x and y, $w_{x,y}^{\mathit{eff}}$, is the random variable defined by:*

$$w_{x,y}^{\mathit{eff}} := \frac{G_\beta(x,y)}{G_\beta(x,x)G_\beta(y,y) - G_\beta(x,y)^2}.$$

Remark 13. *Let $\mathscr{G} = (V,E)$ be a finite graph and $(W_e)_{e\in E}$ be positive weights. Let $(\beta_i)_{i\in V}$ be random variables distributed according to $\nu_n^{W,0}$, H_β the corresponding matrix (distributed according to $\tilde{\nu}_n^{W,0}$) and G_β its inverse.*

Let $x, y \in V$ be two distinct vertices of $\mathscr{G}$ and w^{eff} the effective weight between x and y. Let $V_1 := \{x, y\}$ and $V_2 := V \backslash \{x, y\}$ be two subsets of V. The corresponding decomposition of H_β is given by:

$$H_\beta := \begin{pmatrix} H_\beta^{V_1} & -W^{V_1,V_2} \\ -{}^t W^{V_1,V_2} & H_\beta^{V_2} \end{pmatrix}.$$

By lemma 4.2.1.1,

$$W^{\text{eff}} = W_{x,y} + \left({}^t W^{V_1,V_2} \left(H_\beta^{V_2} \right)^{-1} W^{V_1,V_2} \right)(x,y). \tag{4.1}$$

Furthermore, by lemmas 4.1.2.1 and 4.2.1.1, the law of $\frac{G_\beta(x,y)}{G_\beta(y,y)}$ knowing the β-field on V_2 is the same as the law of $\frac{G_\beta(z_1,z_2)}{G_\beta(z_2,z_2)}$ on a two-vertices graph $\{z_1, z_2\}$ where $W_{z_1,z_2} = w^{\text{eff}}$.

Lemma 4.6.1.1. *Let $\mathscr{G} = (V, E)$ be a finite graph and $x_0, \delta \in V$ two distinct vertices. Let $(c_e)_{e \in E}$ be a family of random (not necessarily independent) positive conductances. Let c_{eff} be the (random) effective conductance between x_0 and δ for the electrical network with initial conductances $(c_e)_{e\in E}$. Let $\overline{c}_{\mathit{eff}}$ be the equivalent conductance between x_0 and δ if we set conductances $(\overline{c}_e)_{e\in E}$ defined by $\overline{c}_e := \mathbb{E}(c_e)$ on $\mathscr{G}$. We have the following inequality:*

$$\mathbb{E}(c_{\mathit{eff}}) \le \overline{c}_{\mathit{eff}}.$$

Proof. Let $(V_x)_{x\in V}$ be the (random) potential with $V_{x_0} = 1$ and $V_\delta = 0$ that minimizes the energy:

$$\mathscr{E} := \frac{1}{2} \sum_{\{x,y\}\in E} c_e (V_x - V_y)^2.$$

This potential is harmonic on $V \backslash \{x_0, \delta\}$ by the Dirichlet principle and therefore $(V_x - V_y)_{(x,y)\in E}$ is the flow that minimizes the energy and we get:

$$\mathscr{E} := \frac{1}{2} c_{\text{eff}}.$$

Now let $(\overline{V}_x)_{x\in V}$ be the potential with $\overline{V}_{x_0} = 1$ and $\overline{V}_\delta = 0$ that minimizes the energy:

$$\overline{\mathscr{E}} := \frac{1}{2} \sum_{\{x,y\}\in E} \overline{c}_e (\overline{V}_x - \overline{V}_y)^2.$$

We have:

$$\overline{\mathscr{E}} := \frac{1}{2} \overline{c}_{\text{eff}}.$$

Now since V minimizes $\mathscr{E}$, we have:

$$\mathscr{E} \le \frac{1}{2} \sum_{\{x,y\}\in E} c_e (\overline{V}_x - \overline{V}_y)^2.$$

Now, by taking the expectation we get:

$$\mathbb{E}(\mathscr{E}) \le \frac{1}{2} \sum_{\{x,y\}\in E} \mathbb{E}(c_e) (\overline{V}_x - \overline{V}_y)^2.$$

Therefore:

$$\frac{1}{2} \mathbb{E}(c_{\text{eff}}) \le \frac{1}{2} \sum_{\{x,y\}\in E_{n+1}} \mathbb{E}(c_e) (\overline{V}_x - \overline{V}_y)^2.$$

Then we get:

$$\frac{1}{2} \mathbb{E}(c_{\text{eff}}) \le \frac{1}{2} \overline{c}_{\text{eff}}.$$

And therefore:

$$\mathbb{E}(c_{\text{eff}}) \le \overline{c}_{\text{eff}}.$$

□

Proposition 4.6.1.2. *Let $\mathscr{G} = (V, E)$ be a finite graph and $x_0, \delta \in V$ two distinct vertices. Let $(W_e)_e$ be a family of random (not necessarily independent) positive weights. Let w^{eff} be the (random) effective weight between x_0 and δ for the VRJP with initial weights $(W_e)_{e\in E}$. Let c^{eff} be the effective conductance between x_0 and δ if we set conductances $(c_e)_{e\in E}$ defined by $c_e := \mathbb{E}(W_e)$ on $\mathscr{G}$. We have the following inequality:*

$$\mathbb{E}(w_{\mathit{eff}}) \le c_{\mathit{eff}}.$$

Proof. We will show the result by induction on the number of vertices of the graph. If the graph has two vertices $\{x_0, \delta\}$ (and therefore only one edge) the result is obvious.
Now we assume that the result is true for all graphs with n vertices or less, we will show it for any graph with $n+1$ vertices.
Let $\mathscr{G}_{n+1} = (V_{n+1}, E_{n+1})$ be a finite graph with exactly $n+1$ vertices, including x_0 and δ. Let $(W_e^{n+1})_{e\in E_{n+1}}$ be random weights on E_{n+1}. Let H_β be a random matrix distributed according to $\tilde{\nu}_n^{W,0}$. Let w_{n+1}^{eff} be the (random) effective weight between x_0 and δ. Let $(c_e^{n+1})_{e\in E_{n+1}}$ be deterministic conductances defined by $c_e^{n+1} = \mathbb{E}\left(W_e^{n+1}\right)$. We define two effective conductances between x_0 and δ on $\mathscr{G}_{n+1}$: one for random conductances W^{n+1} ($\overline{c}_{n+1}^{\text{eff}}$) and the other for deterministic conductances $(c_e)_{e\in E}$ (c_{n+1}^{eff}). By lemma 4.6.1.1:

$$\mathbb{E}\left(\overline{c}_{n+1}^{\text{eff}}\right) \le \mathbb{E}\left(\overline{c}_{n+1}^{\text{eff}}\right). \tag{4.2}$$

Now, let $y \in V_{n+1}$ be a vertex that is neither x_0 nor δ. Let $\mathscr{G}_n^y = (V_n^y, E_n^y)$ be the complete graph with n elements with $V_n^y = V_{n+1}\backslash\{y\}$. We can decompose V_{n+1} in V_n^y and $\{y\}$, the corresponding decomposition of H_β is given by:

$$H_\beta := \begin{pmatrix} H_\beta^{V_n} & -W^{V_1,y} \\ -{}^t\!W^{V_1,y} & 2\beta_y \end{pmatrix}.$$

By lemma 4.2.1.1, w_{n+1}^{eff} knowing H_β is equal to the effective weight w_n^{eff} on the graph $\mathscr{G}_n^y$ for weights and the β-field given by the matrix $H_\beta^{V_n^y} - \frac{1}{2\beta_y} W^{V_1^y,y}\,{}^t\!W^{V_1^y,y}$. This matrix, knowing β_y and W^{n+1} is distributed according to $\nu_n^{W'}$ with $W'_{x_1,x_2} = W_{x_1,x_2}^{n+1} + \frac{W_{x_1,y}^{n+1}W_{y,x_2}^{n+1}}{2\beta_y}$. By 4.1.2.1, if $K_y := \sum\limits_{x,\{x,y\}\in E_{n+1}} W_{y,x}^{n+1}$ the expectation of $\frac{1}{2\beta_y}$, knowing W^{n+1} is given by:

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{2\beta_y}\right) &= \int\limits_{b=0}^{\infty} \frac{1}{2b}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2b}} \exp\left(-\frac{1}{2}\left(2b + \frac{K_y^2}{2b} - 2K_y\right)\right) \mathrm{d}b \\
&= \int\limits_{b=0}^{\infty} \frac{b}{2}\sqrt{\frac{2}{\pi}}\sqrt{\frac{b}{2}} \exp\left(-\frac{1}{2}\left(\frac{2}{b} + \frac{K_y^2 b}{2} - 2K_y\right)\right) \frac{1}{b^2}\mathrm{d}b \\
&= \frac{1}{2}\int\limits_{b=0}^{\infty} \frac{1}{\sqrt{2b}}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\left(\frac{4}{2b} + \frac{K_y^2}{4}2b - 2K_y\right)\right) \mathrm{d}b \\
&= \frac{1}{K_y}\int\limits_{b=0}^{\infty} \frac{1}{\sqrt{2b}}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\left(\frac{K_y^2}{2b} + 2b - 2K_y\right)\right) \mathrm{d}b \\
&= \frac{1}{K_y} \text{by definition of } \nu_1^{0,K_y}.
\end{aligned}$$

Therefore for any $x_1, x_2 \in V_n^y$:

$$\mathbb{E}\left(W'_{x_1,x_2} | W^{n+1}\right) = W_{x_1,x_2}^{n+1} + \frac{W_{x_1,y}^{n+1}W_{y,x_2}^{n+1}}{\sum\limits_x W_{y,x}^{n+1}}.$$

Similarly the effective conductance $\overline{c}_{n+1}^{\text{eff}}$ between x_0 and δ on $\mathscr{G}_{n+1}$ with conductances W^{n+1} is equal to the effective conductance $\overline{c}_n^{\text{eff}}$ between x_0 and δ on $\mathscr{G}_n^y$ with conductances $\overline{c}'_{x_1,x_2} := W_{x_1,x_2}^{n+1} + \frac{W_{x_1,y}^{n+1}W_{y,x_2}^{n+1}}{\sum\limits_x W_{y,x}}$. This means that, for any $e \in E_n^y$:

$$\mathbb{E}\left(W'_e | W^{n+1}\right) = \overline{c}'_e,$$

so by the induction property:

$$\mathbb{E}\left(w_n^{\text{eff}}\right) \le \mathbb{E}\left(\overline{c}_n^{\text{eff}}\right),$$

which implies that

$$\mathbb{E}\left(w_{n+1}^{\text{eff}}\right) \le \mathbb{E}\left(c_{n+1}^{\text{eff}}\right).$$

$\square$

4.6.2 proof of theorem 58

Proof. Once we can compare the effective weight for the VRJP to effective conductance for an electrical network, the proof is quite straightforward. Let $\tilde{W}_e$ be weights and let $c_e := \mathbb{E}(\tilde{W}_e)$ be conductances. For any $n > 0$ we define $\overline{S}_n$ the vertices of V at distance n or more of x_0. Then $\mathscr{G}_n, \tilde{W}^n$ (with $\mathscr{G}_n := (V_n, E_n)$) is the quotient of the weighted graph $\mathscr{G}, \tilde{W}$ by $\overline{S}_n$ and δ_n is the point obtained by fusing all points of $\overline{S}_n$ into one. For any n, let H_n be distributed according to $\tilde{\nu}_{|V_n|}^{\tilde{W}^n,0}$ and let G_n be its inverse. By Theorem 1 of [76], to show that the VRJP with initial weights $\tilde{W}_e$ is recurrent, we only need to show that $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$. By remark 13, the law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ is entirely determined by the law of the effective weight. Since the effective conductive converges to 0, the effective weights converges to 0 in probability by lemma 4.6.1.2. Then, by remark 13, the law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ knowing the effective weight is the same as if the graph had only two points: x_0 and δ with a weight equal to the effective weight between them. Now let (β_1, β_2) be distributed according to $\nu_2^{w^{\text{eff}},0}$, the law of $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ is the same as the law of

$$\frac{\frac{w^{\text{eff}}}{4\beta_1\beta_2-(w^{\text{eff}})^2}}{\frac{2\beta_1}{4\beta_1\beta_2-(w^{\text{eff}})^2}} = \frac{w^{\text{eff}}}{2\beta_1}.$$

By taking $\lambda = 1$ in lemma 4.3.1.1, we get that

$$\frac{w^{\text{eff}}}{2\beta_1} = \frac{w^{\text{eff}}}{W^{\text{eff}\frac{1}{z}}} = Z,$$

where the law of Z (knowing w^{eff}) is given by:

$$\sqrt{\frac{w^{\text{eff}}}{2\pi}}\frac{1}{z\sqrt{z}}\exp\left(-\frac{w^{\text{eff}}}{2}\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^2\right)1_{z>0}\mathrm{d}z.$$

If w^{eff} goes to 0 then Z converges to 0 in probability and therefore $\frac{G_n(x_0,\delta_n)}{G_n(\delta_n,\delta_n)}$ converges to 0 in probability and we get the result we want. □

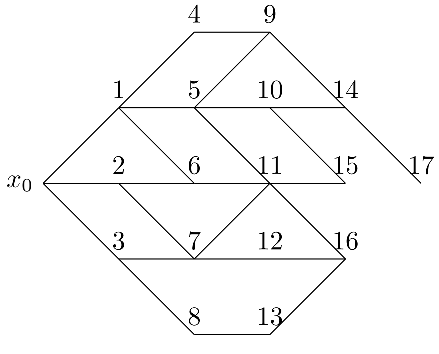
4.7 Proof of theorem 57

4.7.1 Preliminaries and definitions

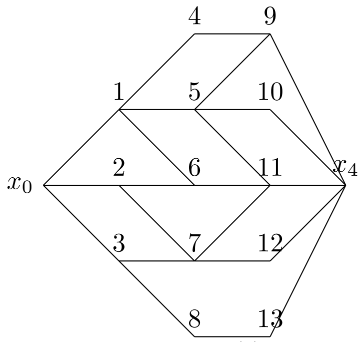
The proof uses the characterization of recurrence used by Sabot and Zeng in [76]: we use the positive martingale $(\psi_n)_{n\in\mathbf{N}}$ with its almost sure limit ψ_∞. The VRJP is recurrent iff $\psi_\infty = 0$. The goal is to show that the probability that $\psi_\infty = 0$ is independent of ψ_n for any n which implies the desired $0-1$ law. To do this, for any n we introduce a modification of the graph for which it is easy to show this property. Then we use our main theorem to show the recurrent/transient behaviour of the two graphs are the same. One direction is easy (the modified graph is more recurrent by our theorem), the other one is more subtle: we use our theorem to identify a worst case scenario and then we only have to study this worst case scenario.

First we define the modification of the graphs we will use in our proof and some notations that will be useful in the following.

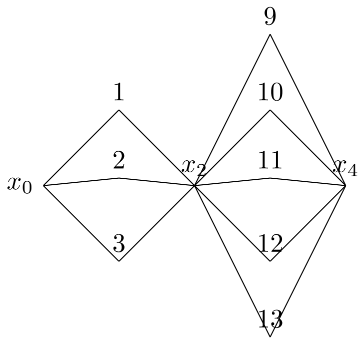
Definition 49. *Let $\mathscr{G} = (V, E)$ be a locally finite graph and $(W_e)_{e\in E}$ a family of positive weights on the edges. Let $d_{\mathscr{G}}(\cdot,\cdot)$ be the graph distance on $\mathscr{G}$. Set $0 < m < n$ and let S_m be the set of vertices at distance m of x_0 and $\tilde{S}_n$ the set of vertices at distance n or more of x_0. We define $(\mathscr{G}_n, W_n)$ as the quotient of $(\mathscr{G}, W)$ by $\tilde{S}_n$ and $(\mathscr{G}_n^m, W_n^m)$ as the quotient of $(\mathscr{G}_n, W_n)$ by S_m. We will write x_m and x_n for the points x_{S_m} and $x_{\tilde{S}_n}$ respectively. For instance, if we take $\mathscr{G}$ to be the graph below.*



The graph $\mathscr{G}_4$ will be given by:



Finally, the graph $\mathscr{G}_4^{(2)}$ is given by:



If we look at the above definitions the graphs we will look at will be naturally split into three subsets: the vertices at a distance less than m of the origin, the vertices at a distance m of the origin and the vertices at a distance larger than m of the origin. Because we split the graph in 3 instead of 2 as in the previous proofs, we will need a lemma that is a small variation on Schur's decomposition for 3 blocs of vertices instead of 2.

Lemma 4.7.1.1. *Let H be a symmetric, positive definite matrix. Let A^-, A^+, B^-, B^+, C be 5 matrices such that H can be decomposed in bloc as such:*

$$H = \begin{pmatrix} A^- & 0 & B^- \\ 0 & A^+ & B^+ \\ {}^tB^- & {}^tB^- & C \end{pmatrix}.$$

Then, its inverse G has the following bloc decomposition (the blocs are the same as for H):

$$G = H^{-1} = \begin{pmatrix} G^{1,1} & G^{1,2} & G^{1,3} \\ G^{2,1} & G^{2,2} & G^{2,3} \\ G^{3,1} & G^{3,2} & G^{3,3} \end{pmatrix},$$

with $G^{i,j} = {}^tG^{i,j}$. Furthermore, we have:

$$\begin{aligned} G^{1,2} =& (A^-)^{-1}B^-G^{3,3}B^+(A^+)^{-1}, \\ G^{1,3} =& -(A^-)^{-1}B^-G^{3,3}, \\ G^{2,3} =& -(A^+)^{-1}B^+G^{3,3}. \end{aligned}$$

Proof. By lemma 4.2.1.1 we have:

$$\begin{aligned} &\begin{pmatrix} G^{1,1} & G^{1,2} \\ G^{2,1} & G^{2,2} \end{pmatrix} \\ &= \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}^{-1} + \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}^{-1} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} G^{3,3} \begin{pmatrix} {}^tB^- & {}^tB^+ \end{pmatrix} \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (A^-)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} + \begin{pmatrix} (A^-)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} G^{3,3} \begin{pmatrix} {}^tB^- & {}^tB^+ \end{pmatrix} \begin{pmatrix} (A^-)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A^-)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} + \begin{pmatrix} (A^-)^{-1}B^- \\ (A^+)^{-1}B^+ \end{pmatrix} G^{3,3} \begin{pmatrix} {}^tB^-(A^-)^{-1} & {}^tB^+(A^+)^{-1} \end{pmatrix}. \end{aligned}$$

Therefore:

$$G^{1,2} = (A^-)^{-1}B^-G^{3,3}B^+(A^+)^{-1}.$$

Similarly, by lemma 4.2.1.1 we have:

$$\begin{aligned}\begin{pmatrix} G^{1,3} \\ G^{2,3} \end{pmatrix} &= \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}^{-1} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} G^{3,3} \\ &= \begin{pmatrix} (A^-)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \begin{pmatrix} B^- \\ B^+ \end{pmatrix} G^{3,3} \\ &= \begin{pmatrix} (A^-)^{-1} B^- G^{3,3} \\ (A^+)^{-1} B^+ G^{3,3} \end{pmatrix}.\end{aligned}$$

□

Finally we define a family of functions that will be useful in the following.

Definition 50. *For any $a > 0$ let $f_a : [0, \infty) \mapsto [0, 1]$ be the concave function defined by:*

$$\forall x \in [0, \infty),\ f_a(x) := \begin{cases} \frac{1}{a} x & \text{if } x \in [0, a], \\ 1 & \text{otherwise} \end{cases}$$

4.7.2 The proof

We have defined the graphs that we will look at. Now we will couple the various probability measures ν on those graphs. For the graphs $(\mathscr{G}_n)_{n\in\mathbf{N}}$ this is exactly what was done in [76]. For the graphs $(\mathscr{G}^m_{n+m})_{n\in\mathbf{N}}$ the idea is similar but the proof uses theorem 59.

Lemma 4.7.2.1. *Let $\mathscr{G} = (V, E)$ be a locally finite graph with positive weights $(W_e)_{e\in E}$ on the edges and $x_0 \in V$ a vertex of this graph. Set $m \in \mathbf{N}^*$ and let $V^-_m := \{x \in, d_{\mathscr{G}}(x_0, x) < m\}$ be the set of vertices at a distance less than m of x_0. There exists random matrices $(H_{m+i})_{i\in\mathbf{N}}$ and $(\overline{H}_{m+i})_{i\in\mathbf{N}}$ such that:*

$$\forall i \in \mathbf{N},\ H_{m+i} \text{ is distributed according to } \tilde{\nu}^{W_{m+i},0}_{|\mathscr{G}_{m+i}|} \text{ and } \overline{H}_{m+i} \text{ is distributed according to } \tilde{\nu}^{W^m_{m+i},0}_{|\mathscr{G}^m_{m+i}|},$$

$$\forall i \in \mathbf{N},\ \forall x \in V_m,\ H_{m+i}(x, x) = \overline{H}_{m+i}(x, x) = H_m(x, x),$$

$$\forall i \in \mathbf{N},\ \mathbb{E}\left(\frac{(H_{m+i+1})^{-1}(x_{m+i+1}, x_0)}{(H_{m+i+1})^{-1}(x_{m+i+1}, x_{m+i+1})} | H_{m+i}\right) = \frac{(H_{m+i})^{-1}(x_{m+i}, x_0)}{(H_i)^{-1}(x_{m+i}, x_{m+i})},$$

$$\forall i \in \mathbf{N},\ \mathbb{E}\left(\frac{(H_{m+i})^{-1}(x_{m+i}, x_0)}{(H_{m+i})^{-1}(x_{m+i}, x_{m+i})} | \overline{H}_{m+i}\right) = \frac{(\overline{H}_{m+i})^{-1}(x_{m+i}, x_0)}{(\overline{H}_{m+i})^{-1}(x_{m+i}, x_{m+i})} \text{ and}$$

$$H_m = \overline{H}_m.$$

Proof. This is a direct consequence of 59. □

Now we look more specificatly at the modified graphs $\mathscr{G}^m_n$ that we have defined. In particular we will look at the random variables $\frac{G(x_0,x_n)}{G(x_n,x_n)}$ on these graphs because they are linked to recurrence/transience according theorem 1 of [76].

Lemma 4.7.2.2. *Let $\mathscr{G} = (V, E)$ be a locally finite graph with positive weights $(W_e)_{e\in E}$ on the edges and $x_0 \in V$ a vertex of this graph. Set $m \in \mathbf{N}^*$ and for any $n \in \mathbf{N}$ let $\mathscr{G}^m_{m+n} = (V^m_{m+n}, E^m_{m+n})$ be the graph defined in 49 with weights W^m_{m+n}. Let $V^{m,-}_{m+n} := \{x \in V^n_{m+n},\ d_{\mathscr{G}^m_{m+n}}(x_0, x) < m\}$ be the set of vertices at a distance less than m of x_0 in $\mathscr{G}^m_{m+n}$. Let $V^{m,+}_{m+n} := \{x \in V^n_{m+n},\ d_{\mathscr{G}^m_{m+n}}(x_0, x) > m\}$ be the set of vertices at a distance more than m of x_0 in $\mathscr{G}^m_{m+n}$. Let $(H_{n+m})_{n\in\mathbf{N}}$ be a sequence of random matrices respectively distributed according to $\nu^{W^m_{m+n},0}_{|\mathscr{G}^m_{m+n}|}$ with respective inverse $(G_{n+m})_{n\in\mathbf{N}}$ such that:*

$$\forall n \in \mathbf{N}, \forall x \in V^{m,-}_{n+m},\ H_{n+m}(x, x) = H_m(x, x).$$

Then we have,

$$\forall n \in \mathbf{N},\ \frac{G_{n+m}(x_0, x_m)}{G_{n+m}(x_m, x_m)} = \frac{G_m(x_m, x_0)}{G_m(x_m, x_m)}.$$

Furthermore, for all $n \in \mathbf{N}$, $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ is independent of $\frac{G_{n+m}(x_0,x_m)}{G_{n+m}(x_m,x_m)}$, and $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ converges in law to a non-negative random variable $\overline{Y}_\infty$ when n goes to infinity. Finally, for any $n > 0$, the law of $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ does not depend on the weights of the edges of which at least one endpoint is in $V^{m,-}_{m+n}$.

Proof. Let $(H_{n+m})_{n\in\mathbf{N}}$ be a sequence of random matrices respectively distributed according to $\nu_{|\mathscr{G}_{m+n}^m|}^{W_{m+n}^m,0}$ with respective inverse $(G_{n+m})_{n\in\mathbf{N}}$ such that:

$$\forall n\in\mathbf{N},\forall x\in V_{n+m}^{m,-},\ H_{n+m}(x,x)=H_m(x,x).$$

We decompose V_{n+m}^m in $V_{n+m}^{m,-}$,$V_{n+m}^{m,+}$ and $\{x_m\}$. The corresponding decomposition of $(H_{n+m})_{n\in\mathbf{N}}$ and $(G_{n+m})_{n\in\mathbf{N}}$ are given by:

$$\forall n\in\mathbf{N},\ H_{n+m}=:\begin{pmatrix} H_{n+m}^- & 0 & W_{n+m}^{m,-}\\ 0 & H_{n+m}^+ & W_{n+m}^{m,+}\\ {}^t W_{n+m}^{m,-} & {}^t W_{n+m}^{m,+} & H(x_m,x_m)\end{pmatrix} \text{ and } G_{n+m}=:\begin{pmatrix} G_{n+m}^{1,1} & G_{n+m}^{1,2} & G_{n+m}^{1,3}\\ G_{n+m}^{2,1} & G_{n+m}^{2,2} & G_{n+m}^{2,3}\\ G_{n+m}^{3,1} & G_{n+m}^{3,2} & G_{n+m}^{3,3}\end{pmatrix}.$$

By lemma 4.7.1.1,

$$G_{n+m}^{1,2}=G_{n+m}^{1,3}\left(G_{n+m}^{3,3}\right)^{-1}\,{}^tG_{n+m}^{1,3}.$$

Then, by using that $G^{3,3}$ is a 1×1 matrix, we have that for all $(x^-,x^+)\in V_{n+m}^{m,-}\times V_{n+m}^{m,+}$:

$$G_{n+m}(x^-,x^+)=\frac{G_{n+m}(x^-,x_m)G_{n+m}(x^+,x_m)}{G_{n+m}(x_m,x_m)}.$$

We also have, as a consequence:

$$\begin{aligned}\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)} &=\frac{G_{n+m}(x_0,x_m)G_{n+m}(x_{n+m},x_m)}{G_{n+m}(x_m,x_m)G_{n+m}(x_0,x_m)}\\ &=\frac{G_{n+m}(x_{n+m},x_m)}{G_{n+m}(x_m,x_m)}\\ &=-\left((H_{n+m}^+)^{-1}W_{n+m}^{m,+}\right)(x_{n+m},x_m).\end{aligned}$$

Similarly,

$$\forall x\in V_{m+n}^{m,-},\ \frac{G_{n+m}(x,x_m)}{G_{n+m}(x_m,x_m)}=-\left((H_{n+m}^-)^{-1}W_{n+m}^{m,-}\right)(x,x_m). \tag{4.3}$$

We know that the law of H_{n+m}^+ does not depend on the weights of the edges of which at least one endpoint is in $V_{m+n}^{m,-}$ by definition of this set and proposition 4.1.2.1. Therefore, the law of $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ does not depend on the weights of the edges of which at least one endpoint is in $V_{m+n}^{m,-}$. Since the β-field is 1-dependent, H_{n+m}^- and H_{n+m}^+ are independent so $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ and $\frac{G_{n+m}(x_0,x_m)}{G_{n+m}(x_m,x_m)}$ are independent.
Furthermore, by construction of the matrices $(H_{n+m})_{n\in\mathbf{N}}$ and equality 4.3, for all $n\in\mathbf{N}$:

$$\forall x\in V_{m+n}^{m,-},\ \frac{G_{n+m}(x,x_m)}{G_{n+m}(x_m,x_m)}=\frac{G_m(x,x_m)}{G_m(x_m,x_m)}.$$

Now, by theorem 59, we can have a sequence of random matrices $(\overline{H}_{n+m})_{n\in\mathbf{N}}$ respectively distributed according to $\nu_{|\mathscr{G}_{m+n}^m|}^{W_{m+n}^m,0}$ with respective inverse $(\overline{G}_{n+m})_{n\in\mathbf{N}}$ such that:

$$\begin{aligned}&\forall n\in\mathbf{N},\forall x\in V_{n+m}^{m,-},\ \overline{H}_{n+m}(x,x)=\overline{H}_m(x,x),\\ &\forall n\in\mathbf{N},\forall x\in V_{n+m}^{m},\ \frac{\overline{G}_{n+m}(x,x_m)}{\overline{G}_{n+m}(x_m,x_m)}=\frac{\overline{G}_m(x,x_0)}{\overline{G}_m(x_m,x_m)},\\ &\forall n\in\mathbf{N},\forall x\in V_{n+m}^{m},\ \overline{G}_{n+m}(x_m,x_m)=\overline{G}_m(x_m,x_m).\end{aligned}$$

Since $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}=\frac{G_{n+m}(x_{n+m},x_m)}{G_{n+m}(x_m,x_m)}$, which is a positive martingale by construction of $(\overline{H}_{n+m})_{n\in\mathbf{N}}$, we have that $\frac{G_{n+m}(x_0,x_{n+m})}{G_{n+m}(x_0,x_m)}$ converges in law to a non-negative random variable when n goes to infinity. $\square$

Finally we will need this useful lemma to show that the behaviour of the VRJP on the graphs $\mathscr{G}_n$ and $\mathscr{G}_n^m$ are not too different.

Lemma 4.7.2.3. *Set $n\in\mathbf{N}^*$. Let $W,W'\in M_n(\mathbf{R})$ be two symmetric matrix with non-positive off-diagonal coefficients and non-negative diagonal coefficients. We assume that all integers in $[\![1,n]\!]$ are W-connected. Let G' be the inverse of a matrix distributed according to $\nu_n^{W+W',0}$. For any $\epsilon>0$ there exists a constant C_ϵ^W that only depends on W and such that*

$$\mathbb{P}\left(\exists (i,j)\in[\![1,n]\!]^2,\ G(i,j)\le C_\epsilon^W\right)\le\epsilon.$$

Proof. Set $\epsilon > 0$. Let G be the inverse of a matrix distributed according to $\nu_n^{W,0}$. Since for any $a > 0$, the function f_a is convex, we have by theorem 54:

$$\forall (i,j) \in [\![1,n]\!]^2,\ \mathbb{E}\left(f_a(G'(i,j))\right) \geq \mathbb{E}\left(f_a(G(i,j))\right)$$

Since all the coefficients of G are almost surely positive, there exists $a_\epsilon > 0$ such that:

$$\forall (i,j) \in [\![1,n]\!]^2,\ \mathbb{E}\left(f_{a_\epsilon}(G(i,j))\right) \geq 1 - \frac{\epsilon}{2n^2}.$$

This in turns means that

$$\forall (i,j) \in [\![1,n]\!]^2,\ \mathbb{E}\left(f_{a_\epsilon}(G'(i,j))\right) \geq 1 - \frac{\epsilon}{2n^2}.$$

This implies that:

$$\forall (i,j) \in [\![1,n]\!]^2,\ \mathbb{P}\left(G'(i,j) < \frac{a_\epsilon}{2}\right) \leq \frac{\epsilon}{n^2}.$$

Finally we get

$$\mathbb{P}\left(\exists (i,j) \in [\![1,n]\!]^2,\ G'(i,j) \leq \frac{a_\epsilon}{2}\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}\left(G'(i,j) \leq \frac{a_\epsilon}{2}\right) \leq \epsilon.$$

□

Now, we have all we need to prove the theorem.

proof of theorem 57. We will start by looking at the case of deterministic positive weights. The random case will then follow by an application of Kolmogorov's $0-1$ law.
Set $m \in \mathbf{N}^*$. Let $(H_{m+n})_{n\in\mathbf{N}}$ be random matrices of respective law $\left(\nu_{|\mathscr{G}_{n+m}|}^{W_{n+m},0}\right)_{n\in\mathbf{N}}$ and $(G_{m+n})_{n\in\mathbf{N}}$ their respective inverse such that

$$\forall n \in \mathbf{N},\ \mathbb{E}\left(\frac{G_{m+n+1}(x_0, x_{m+n+1})}{G_{m+n+1}(x_{m+n+1}, x_{m+n+1})}|H_{m+n}\right) = \frac{G_{m+n}(x_0, x_{m+n})}{G_{m+n}(x_{m+n}, x_{m+n})}.$$

Let $(H^m_{m+n})_{n\in\mathbf{N}}$ be random matrices of respective law $\left(\nu_{|\mathscr{G}^m_{n+m}|}^{W^m_{n+m},0}\right)_{n\in\mathbf{N}}$ and $(G^m_{m+n})_{n\in\mathbf{N}}$ their respective inverse such that

$$\forall n \in \mathbf{N},\ \mathbb{E}\left(\frac{G^m_{m+n}(x_0, x_{m+n+1})}{G^m_{m+n}(x_{m+n}, x_{m+n})}|H_{m+n}\right) = \frac{G_{m+n}(x_0, x_{m+n})}{G_{m+n}(x_{m+n}, x_{m+n})}.$$

Such matrices exist by theorem 59. Now, we want to compare G^m_{m+n} to G_{m+n}. The first step is to apply Schur lemma. Let $V^{m,-}_{m+n} := \{x \in V^n_{m+n},\ d_{\mathscr{G}^m_{m+n}}(x_0, x) < m\}$ be the set of vertices at a distance less than m of x_0 in $\mathscr{G}^m_{m+n}$. Let $V^{m,+}_{m+n} := \{x \in V^n_{m+n},\ d_{\mathscr{G}^m_{m+n}}(x_0, x) > m\}$ be the set of vertices at a distance more than m of x_0 in $\mathscr{G}^m_{m+n}$. We will first decompose V_{m+n} in $V^{m,-}_{m+n}$,$V^{m,+}_{m+n}$ and S_m. The corresponding bloc decomposition of H_{m+n} gives:

$$H_{m+n} =: \begin{pmatrix} H^{m,-}_{m+n} & 0 & \tilde{W}^-_{m+n} \\ 0 & H^{m,+}_{m+n} & \tilde{W}^+_{m+n} \\ {}^t\tilde{W}^-_{m+n} & {}^t\tilde{W}^+_{m+n} & H^{S_m}_{m+n} \end{pmatrix}.$$

Similarly the inverse G_{m+n} of H_{m+n} has the same bloc decomposition:

$$G_{m+n} =: \begin{pmatrix} G^{1,1}_{m+n} & G^{1,2}_{m+n} & G^{1,3}_{m+n} \\ {}^tG^{1,2}_{m+n} & G^{2,2}_{m+n} & G^{2,3}_{m+n} \\ {}^tG^{1,3}_{m+n} & {}^tG^{2,3}_{m+n} & G^{3,3}_{m+n} \end{pmatrix}.$$

Now we will decompose V^m_{m+n} in $V^{m,-}_{m+n}$,$V^{m,+}_{m+n}$ and $\{x_m\}$. The corresponding bloc decomposition of H^m_{m+n} gives:

$$H^m_{m+n} =: \begin{pmatrix} H^{m,-}_{m+n} & 0 & \tilde{W}^{m,-}_n \\ 0 & H^{m,+}_{m+n} & \tilde{W}^{m,-}_{m+n} \\ {}^t\tilde{W}^{m,-}_{m+n} & {}^t\tilde{W}^{m,+}_{m+n} & H^m_{m+n}(x_m, x_m) \end{pmatrix},$$

where $H^{m,-}_n$ and $H^{m,+}_n$ are the same as defined previously and if $1_{S_m} \in \mathbf{R}^{S_m}$ is the vector with only ones, then $\tilde{W}^{m,-}_{m+n} = \tilde{W}^-_{m+n}1_{S_m}$ and $\tilde{W}^{m,+}_n = \tilde{W}^+_n 1_{S_m}$. We have by lemma 4.7.1.1:

$$G^{1,2}_{m+n} = (H^{m,-}_{m+n})^{-1}\tilde{W}^-_{m+n}G^{3,3}_{m+n}\tilde{W}^+_{m+n}(H^{m,+}_{m+n})^{-1}.$$

We are interested in the law of $G^{3,3}_{m+n}$. Let W^0 be the matrix $-H^{S_m}_{m+n}$ with all the diagonal coefficients set to 0. Knowing $H^{m,-}_{m+n}$ and $H^{m,-}_{m+n}$ the law of $G^{3,3}_{m+n}$ is the same as that of the inverse of a matrix distributed according to $\nu^{W',0}_{|S_m|}$ where W' is defined by:

$$\begin{aligned}W' :=& W^0 + \begin{pmatrix} {}^t\tilde{W}^-_{m+n} & {}^t\tilde{W}^+_{m+n}\end{pmatrix}\begin{pmatrix} H^{m,-}_{m+n} & 0 \\ 0 & H^{m,+}_{m+n}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{W}^-_{m+n} \\ \tilde{W}^+_{m+n}\end{pmatrix}\\ =& W^0 + {}^t\tilde{W}^-_{m+n}(H^{m,-}_{m+n})^{-1}\tilde{W}^-_{m+n} + {}^t\tilde{W}^+_{m+n}(H^{m,+}_{m+n})^{-1}\tilde{W}^+_{m+n}.\end{aligned}$$

By lemma 4.7.2.3 there exists $C_\epsilon > 0$ that only depends on ${}^t\tilde{W}^-_{m+n}(H^{m,-}_{m+n})^{-1}\tilde{W}^-_{m+n}$ and such that with probably at least $1-\epsilon/2$, all the coefficients of $G^{3,3}$ are larger or equal to C_ϵ. By using that all weights are non-negative and all the coefficients of $(H^{m,-}_{m+n})^{-1}$ and $(H^{m,+}_{m+n})^{-1}$ are also non negative we have with probability at least $1-\epsilon/2$, knowing $H^{m,-}_{m+n}$ and $H^{m,+}_{m+n}$:

$$\begin{aligned}G^{1,2}_{m+n}(x_0,x_{m+n}) =& \left((H^{m,-}_{m+n})^{-1}\tilde{W}^-_{m+n}G^{3,3}_{m+n}\tilde{W}^+_{m+n}(H^{m,+}_{m+n})^{-1}\right)(x_0,x_{m+n})\\ \geq& C_\epsilon\left((H^{m,-}_{m+n})^{-1}\tilde{W}^-_{m+n}1_{S_m}\,{}^t1_{S_m}\tilde{W}^+_{m+n}(H^{m,+}_{m+n})^{-1}\right)(x_0,x_{m+n})\\ =& C_\epsilon\left((H^{m,-}_{m+n})^{-1}\tilde{W}^{m,-}_{m+n}\tilde{W}^{m,+}_{m+n}(H^{m,+}_{m+n})^{-1}\right)(x_0,x_{m+n})\\ =& \frac{C_\epsilon}{G^m_{m+n}(x_m,x_m)}G^m_{m+n}(x_0,x_{m+n})\end{aligned}$$

Knowing $H^{m,-}_{m+n}$ and $H^{m,+}_{m+n}$, the random variable $G^m_{m+n}(x_m,x_m)$ is the inverse of a Gamma random variable of parameter $1/2$. This means that knowing $H^{m,-}_{m+n}$ there exists a constant $\overline{C}_\epsilon$ such that with probability at least $1-\epsilon$:

$$G_{m+n}(x_0,x_{m+n}) \geq \overline{C}_\epsilon G^m_{m+n}(x_0,x_{m+n}).$$

Since $G_{m+n}(x_{m+n},x_{m+n}) = G^m_{m+n}(x_{m+n},x_{m+n})$ almost surely, we have that knowing $H^{m,-}_{m+n}$, with probability at least $1-\epsilon$:

$$\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})} \geq \overline{C}_\epsilon\frac{G^m_{m+n}(x_0,x_{m+n})}{G^m_{m+n}(x_{m+n},x_{m+n})}.$$

Therefore for any $a>0$,

$$\mathbb{E}\left(f_a\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right) \geq \mathbb{E}\left(f_{a/C_\epsilon}\left(\frac{G^m_{m+n}(x_0,x_{m+n})}{G^m_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right) - \epsilon.$$

We also know by construction of the matrix H^m_{n+m} that:

$$\mathbb{E}\left(f_a\left(\frac{G^m_{m+n}(x_0,x_{m+n})}{G^m_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right) \geq \mathbb{E}\left(f_a\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right).$$

By construction of the sequence of matrices:

$$\mathbb{E}\left(f_a\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right) = \mathbb{E}\left(f_a\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m}\right)\text{ almost surely}$$

and knowing $H^{m,-}_m$, the sequence $\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}$ is a positive martingale so it converges almost surely to a limit value ψ_∞. Furthermore, the function f_a is continuous and bounded so when n goes to infinity:

$$\mathbb{E}\left(f_a\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)|H^{m,-}_{m+n}\right) \to \mathbb{E}\left(f_a(\psi_\infty)|H^{m,-}_m\right).$$

By lemma 4.7.2.2, there exists a random variable ψ^m_∞ that does not depend on $H^{m,-}_m$ such that knowing $H^{m,-}_m$, $\left(\frac{G_{m+n}(x_0,x_{m+n})}{G_{m+n}(x_{m+n},x_{m+n})}\right)$ converges in law to ψ^m_∞ when n goes to infinity. This means that:

$$\mathbb{E}\left(f_a(\psi^m_\infty)|H^{m,-}_m\right) \geq \mathbb{E}\left(f_a(\psi_\infty)|H^{m,-}_m\right) \geq \mathbb{E}\left(f_{a/C_\epsilon}(\psi^m_\infty)|H^{m,-}_m\right) - \epsilon.$$

By making a go to 0 we get:

$$\mathbb{P}\left(\psi^m_\infty > 0|H^{m,-}_m\right) \geq \mathbb{P}\left(\psi_\infty > 0|H^{m,-}_m\right) \geq \mathbb{P}\left(\psi^m_\infty > 0|H^{m,-}_m\right) - \epsilon.$$

And finally by taking ϵ going to 0 we get:

$$\mathbb{P}\left(\psi_\infty = 0 | H_m^{m,-}\right) = \mathbb{P}\left(\psi_\infty^m = 0 | H_m^{m,-}\right).$$

This means that for any m there exists a constant p_m that does not depend on the weights on the edges with at least one endpoint within V_m^- such that:

$$\mathbb{P}\left(\psi_\infty = 0 | \psi_m\right) = p_m \text{ almost surely}.$$

Since ψ_m converges almost surely to ψ_∞ this means that $p_m \in \{0, 1\}$. From this we can conclude that the VRJP with deterministic initial positive weights W is recurrent with probability 0 or 1. Furthermore this probability does not depend on any finite subsets of weights so by Kolmogorov's $0-1$ law, this result is also true for random independent positive initial weights.

$\square$

Chapter 5

A biased ERRW

This chapter, unlike the previous ones, is not based on an article. In this chapter we will look at a modification of the ERRW. The study of this model led to the monotonicity behaviour of the previous chapter. This problem of monotonicity seemed more important than the one we will be discussing here and for this my research was more focused on the monotonicity problem. As a consequence, the problem we study here is not investigated has deeply as for the previous chapters.
The idea of the modification is to introduce a small bias. If it is small enough then we might still observe the exponential decay of some quantities. However by introducing this bias, we loose the partial exchangeability so the techniques developed for the ERRW do not apply in this case.

5.1 A modification of the ERRW

Let $\mathscr{G} = (V, E)$ be a locally finite non directed graph and $\overline{E} := \{(x, y) \in V^2,\ \{x, y\} \in E\}$ the directed edges. Let 0 be a vertex of this graph. To each non-directed edge $\{x, y\}$ we associate a positive weight $a_{\{x,y\}}$ and to each oriented edge (x, y) we associate a non-negative weight $\varepsilon_{(x,y)}$. For the sake of simplification we will use the following notation:

$$\forall x \in V,\ c_x^\varepsilon := \sum_{y\sim x} a_{\{x,y\}} + \varepsilon_{(x,y)}.$$

We define the biased edge reinforced random walk $(X_i)_{i\in\mathbf{N}}$ by:

$$X_0 = 0 \text{ a.s},$$

$$\mathbb{P}(X_{n+1} = y|X_0 = x_0, \dots, X_n = x_n) = \frac{a_{\{x_n,y\}} + \varepsilon_{(x_n,y)} + N_n(\{x_n, y\})}{c_{x_n}^\varepsilon + \sum\limits_{z\sim x_n} N_n(\{x_n, z\})},$$

where for all edge $\{x, y\} \in E$, for all $i \in \mathbf{N}$:

$$N_i(\{x, y\}) := \sum_{j=0}^{i-1} 1_{\{X_j,X_{j+1}\}=\{x,y\}}.$$

We want to show the following result.

Theorem 61. *Let $\tilde{\mathscr{G}}$ be a finite graph and set $a > 0$. To every non-directed edge e of $\tilde{\mathscr{G}} \times \mathbf{Z}$ we associate a weight $a_e := a$. There exists a constant c such that if to every directed edge of $\tilde{\mathscr{G}} \times \mathbf{Z}$ we associate a non-negative weight $\varepsilon \leq c$ then the biased edge reinforced random walk on $\tilde{\mathscr{G}} \times \mathbf{Z}$ is recurrent.*

Remark 14. *The link between this model and the result of the previous chapter is as follows: if the bias ε is the same in both direction of the edges (i.e $\varepsilon_{(x,y)} = \varepsilon_{(y,x)}$) then the biased VRJP is just the VRJP with weights $a + \varepsilon$. This means that comparing the biased VRJP to the VRJP as implications when comparing the VRJP with different weights.*

5.2 A martingale

We will start with a few definitions.

Definition 51. *Let $\mathbb{P}_{a,\varepsilon}$ be the probability measure associated with the biased edge reinforced random walk with weights $(a_e)_{e\in E}$ and $(\varepsilon_e)_{e\in\overline{E}}$. We will write $\mathbb{P}_a$ instead of $\mathbb{P}_{a,\varepsilon}$ when $\varepsilon = 0$. We define the following random variables:*

1. $p_a^\varepsilon(n) := \frac{a_{\{X_n,X_{n+1}\}}+\varepsilon_{(X_n,X_{n+1})}+N_n(\{X_n,X_{n+1}\})}{c^\varepsilon_{X_n}+\sum\limits_{z\sim X_n} N_n(\{X_n,z\})}$

2. $p_a(n) := \frac{a_{\{X_n,X_{n+1}\}}+N_n(\{X_n,X_{n+1}\})}{c^0_{X_n}+\sum\limits_{z\sim X_n} N_n(\{X_n,z\})}$

3. $M_n^{a,s} := \prod\limits_{i=0}^{n-1}\left(1+s\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)}\right)$

We will first show that $M^{a,s}$ is a martingale.

Lemma 5.2.0.1. *For any $s>0$, $M^{a,s}$ is a martingale under $\mathbb{P}_a$ for the filtration $\mathscr{F}_n := \sigma(X_0,\dots,X_n)$.*

Proof. We just have to do the calculations:

$$\begin{aligned}\mathbb{E}_a^{a,s}(M_{n+1}|\mathscr{F}_n) &= M_n^{a,s}\sum_{y\sim X_n}\mathbb{P}_a(X_{n+1}=y|\mathscr{F}_n)\left(1+s\frac{\mathbb{P}_a^\varepsilon(X_{n+1}=y|\mathscr{F}_n)-\mathbb{P}_a(X_{n+1}=y|\mathscr{F}_n)}{\mathbb{P}_a(X_{n+1}=y|\mathscr{F}_n)}\right)\\ &= M_n^{a,s}\sum_{y\sim X_n}\left(\mathbb{P}_a(X_{n+1}=y|\mathscr{F}_n)+s\mathbb{P}_a^\varepsilon(X_{n+1}=y|\mathscr{F}_n)-s\mathbb{P}_a(X_{n+1}=y|\mathscr{F}_n)\right)\\ &= M_n^{a,s}(1+s-s)\\ &= M_n^{a,s}.\end{aligned}$$

□

The martingale $M^{a,s}$ is a first order expansion of $(\left(\frac{\mathbb{P}_a^\varepsilon(X_0,..,X_n)}{\mathbb{P}_a(X_0,..,X_n)}\right)^s)$. To see how close both are, we introduce the process Δ defined by:

$$\forall n\in\mathbf{N}, \Delta_n := M_n^{a,s}\left(\frac{\mathbb{P}_a(X_0,..,X_n)}{\mathbb{P}_a^\varepsilon(X_0,..,X_n)}\right)^s.$$

We will also need to define $R_n := \#\{(X_i)_{i\le n}\}$ the range of X at time n.

Lemma 5.2.0.2. *Let $\mathscr{G}=(V,E)$ be a graph with bounded degree. If for every $(x,y)\in\overline{E}$, $\varepsilon_{(x,y)}\le\frac{1}{4}a_{\{x,y\}}$, then there exists a constant C that does not depend on either ε or a such that for all $s\in(0,2]$:*

$$\forall n\in\mathbf{N},\ |\log(\Delta_n^{a,\varepsilon,s})|\le CR_n\sup_{e\in\overline{E}}\max\left(\frac{\varepsilon_e}{a_e},\varepsilon_e\right)^2 \quad a.s.$$

Proof. We have:

$$\begin{aligned}\Delta_n^{a,\varepsilon,s} &= \prod_{i=0}^{n-1}\left(1+s\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)}\right)\left(\frac{p_a(i)}{p_a^\varepsilon(i)}\right)^s\\ &= \prod_{i=0}^{n-1}\left(1+s\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)}\right)\left(1+\frac{p_a(i)-p_a^\varepsilon(i)}{p_a^\varepsilon(i)}\right)^s.\end{aligned}$$

We see that at order 1, all the terms of the product are equal to 1. Let e_n be the random edge defined by $e_n := (X_n,X_{n+1})$ and $N_n(x) := \sum\limits_{y\sim x}N_n(\{x,y\})$. We have:

$$\begin{aligned}\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)} &= \left(\frac{a_{e_n}+\varepsilon_{e_n}+N_n(e_n)}{c^\varepsilon_{X_n}+N_n(X_n)}-\frac{a_{e_n}+N_n(e_n)}{c^0_{X_n}+N_n(X_n)}\right)\frac{c^0_{X_n}+N_n(X_n)}{a_{e_n}+N_n(e_n)}\\ &=\frac{a_{e_n}+\varepsilon_{e_n}+N_n(e_n)}{c^\varepsilon_{X_n}+N_n(X_n)}\frac{c^0_{X_n}+N_n(X_n)}{a_{e_n}+N_n(e_n)}-\frac{a_{e_n}+N_n(e_n)}{c^0_{X_n}+N_n(X_n)}\frac{c^0_{X_n}+N_n(X_n)}{a_{e_n}+N_n(e_n)}\\ &=\frac{a_{e_n}+\varepsilon_{e_n}+N_n(e_n)}{c^\varepsilon_{X_n}+N_n(X_n)}\frac{c^0_{X_n}+N_n(X_n)}{a_{e_n}+N_n(e_n)}-\frac{a_{e_n}+N_n(e_n)}{c^\varepsilon_{X_n}+N_n(X_n)}\frac{c^\varepsilon_{X_n}+N_n(X_n)}{a_{e_n}+N_n(e_n)}\\ &=\frac{\varepsilon_{e_n}\left(c^0_{X_n}+N_n(X_n)\right)-\left(c^\varepsilon_{X_n}-c^0_{X_n}\right)\left(a_{e_n}+N_n(e_n)\right)}{\left(c^\varepsilon_{X_n}+N_n(X_n)\right)\left(a_{e_n}+N_n(e_n)\right)}.\end{aligned}$$

Since ε_{e_n} and $\left(c^\varepsilon_{X_n}-c^0_{X_n}\right)$ are non-negative, we therefore have the following bounds:

$$-\frac{c^\varepsilon_{X_n}-c^0_{X_n}}{c^0_{X_n}+N_n(X_n)}\le\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)}\le\frac{\varepsilon_{e_n}}{a_{e_n}+N_n(e_n)}$$

For every edge $e \in E$, $\varepsilon_e \leq \frac{1}{4}a_e$, so $\left|\frac{p_a^\varepsilon(i)-p_a(i)}{p_a(i)}\right| \leq \frac{1}{4}$. Since $0 \leq s \leq 2$ then there is a constant C such that:

$$\left|\log\left(1 + s\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right) - s\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right| \leq C\left(\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right)^2$$

and

$$\begin{aligned}\left|s\log\left(1 + \frac{p_a(i) - p_a^\varepsilon(i)}{p_a^\varepsilon(i)}\right) + s\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right| &= \left|s\log\left(\frac{p_a(i)}{p_a^\varepsilon(i)}\right) + s\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right| \\ &= \left|-s\log\left(1 + \frac{p_a^\varepsilon - p_a(i)}{p_a(i)}\right) + s\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right| \\ &\leq C\left(\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right)^2.\end{aligned}$$

We therefore have the following bound on Δ:

$$\begin{aligned}&|\log(\Delta_n^{a,\varepsilon,s})| \\ \leq &\sum_{i=1}^{n-1} 2C\left(\frac{p_a^\varepsilon(i) - p_a(i)}{p_a(i)}\right)^2 \\ \leq &\sum_{i=1}^{n-1} 2C\left(\left(\frac{c_{X_n}^\varepsilon - c_{X_n}^0}{c_{X_n}^0 + N_n(X_n)}\right)^2 + \left(\frac{\varepsilon_{e_n}}{a_{e_n} + N_n(e_n)}\right)^2\right) \\ \leq &\sum_{x\in\{(X_i)_{i\leq n}\}} 2C\left(\frac{c_x^\varepsilon - c_x^0}{c_x^0}\right)^2 + 2C\left(c_x^\varepsilon - c_x^0\right)^2\frac{\pi^2}{6} + \sum_{e\in\{(X_i,X_{i+1})_{i\leq n}\}} 2C\left(\frac{\varepsilon_e}{a_e}\right)^2 + 2C\left(\varepsilon_e\right)^2\frac{\pi^2}{6} \\ \leq &\left(\sup_{e\in\overline{E}}\max\left(\frac{\varepsilon_e}{a_e}, \varepsilon_e\right)^2\right)\left(\sum_{x\in\{(X_i)_{i\leq n}\}} 2C\left(1 + \frac{\pi^2}{6}\right) + \sum_{e\in\{(X_i,X_{i+1})_{i\leq n}\}} 2C\left(1 + \frac{\pi^2}{6}\right)\right).\end{aligned}$$

Then by using that the graph is bounded we get the result we want. □

We can use this result to compare $\mathbb{P}_a^\varepsilon$ and $\mathbb{P}_a(A)$

Lemma 5.2.0.3. *Let $\mathscr{G} = (V,E)$ be a graph with bounded degree. Let η_ε be defined by $\eta_\varepsilon := \max\left(\frac{\varepsilon_e}{a_e}, \varepsilon_e\right)^2$. Let τ be a stopping time for the filtration $\mathscr{F}$, $\mathbb{P}_a$ a.s finite. Let c be a constant such that $R_\tau \leq c$ $\mathbb{P}_a$ a.s and let A be the event $A := \{X_\tau \neq 0\}$. If for every $(x,y) \in \overline{E}$, $\varepsilon_{(x,y)} \leq \frac{1}{4}a_{\{x,y\}}$, then there exists a constant C that only depends on the degree of the graph and such that for all $s \in (0,2]$:*

$$(\mathbb{P}_a^\varepsilon(A))^s \leq \mathbb{P}_a(A)^{s-1}\exp(C\eta_\varepsilon c).$$

Proof. We have:

$$\begin{aligned}(\mathbb{P}_a^\varepsilon(A))^s &= \left(\mathbb{E}_a^\varepsilon(1_A)\right)^s \\ &= \left(\mathbb{E}_a\left(1_A\left(M_\tau^{a,s}\Delta_\tau^{a,\varepsilon,s}\right)^{\frac{1}{s}}\right)\right)^s \\ &= \mathbb{P}_a(A)^s\left(\mathbb{E}_a\left(\left(M_\tau^{a,s}\Delta_\tau^{a,\varepsilon,s}\right)^{\frac{1}{s}}|A\right)\right)^s \\ &\leq \mathbb{P}_a(A)^s\mathbb{E}_a\left(M_\tau^{a,s}\Delta_\tau^{a,\varepsilon,s}|A\right) \text{ by Jensen inequality} \\ &\leq \mathbb{P}_a(A)^s\mathbb{E}_a\left(M_\tau^{a,s}|A\right)\exp(C\eta_\varepsilon B) \text{ by lemma 5.2.0.2} \\ &= \mathbb{P}_a(A)^{s-1}\mathbb{E}_a\left(M_\tau^{a,s}1_A\right)\exp(C\eta_\varepsilon B) \\ &\leq \mathbb{P}_a(A)^{s-1}\mathbb{E}_a\left(M_\tau^{a,s}\right)\exp(C\eta_\varepsilon B) \\ &\leq \mathbb{P}_a(A)^{s-1}\exp(C\eta_\varepsilon B).\end{aligned}$$

□

We can now prove the theorem.

proof of theorem 61. For any vertex x let τ_x be the first time at which the walk X attains x. Similarly let τ_0^+ let the first time after 0 at which the walk X returns to 0. To prove the recurrence we will use the exponential decay of the conductances under $\mathbb{P}_a$ on $\tilde{\mathscr{G}} \times \mathbf{Z}$:

$$\forall n \in \mathbf{N}, \sum_{x,d(0,x)=n} \mathbb{P}_a(\tau_x < \tau_0^+) \leq C\exp(-cn),$$

for some constants C, c that only depend on a and $\tilde{\mathscr{G}}$. Let τ_n be the first time such that $|X| = n$ or the walks returns to 0 for the $\left(\exp\left(\frac{c}{2}n\right)\right)^{\text{th}}$ times. We find:

$$\mathbb{P}_a\left(|X_{\tau_n}| = n\right) \le C \exp\left(-\frac{c}{2}n\right)$$

We also have $R_{\tau_n} \le 2|\tilde{\mathscr{G}}|n$. Now by lemma 5.2.0.3, by taking $s = 2$ we get:

$$\mathbb{P}_a^{\varepsilon}\left(|X_{\tau_n}| = n\right) \le C \exp\left(-\frac{c}{2}n\right) \exp\left(C'\eta_\varepsilon 2|\tilde{\mathscr{G}}|n\right),$$

where C' is the constant C of lemma 5.2.0.3. If η_ϵ is small enough, by Borell-Cantelli the walk is recurrent (or it stays on a finite subset of the graph but in this case it is easy to see that it is recurrent). $\square$

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Comportements asymptotiques et transition de phase pour des marches aléatoires en milieux aléatoires et des marches renforcées

Résumé. Cette thèse a pour but d'étudier certains comportements de marches aléatoires en milieux aléatoires et de marches renforcées. Nous regardons d'une part les marches aléatoires en milieu de Dirichlet et d'autres part deux modèles de marches renforcées : la marche aléatoire renforcée linéairement par arête et le processus de saut renforcé par sommet.
Les marches aléatoires en milieux de Dirichlet sont un cas particulier de marches aléatoires en milieux aléatoires présentant une importante propriété simplifiant leur étude: l'invariance statistique par retournement du temps. Dans une première partie nous utilisons cette propriété pour caractériser le comportement limite de ces marches en dimensions 3 et supérieures dans le cas où elles sont transitoires à vitesse nulle. Dans ce cas nous montrons que leur comportement est caractérisé par un processus stable. Dans une seconde partie nous montrons que la propriété d'invariance statistique par retournement du temps est caractéristique des marches aléatoires en milieu de Dirichlet.
La marche aléatoire renforcée linéairement par arête et le processus de saut renforcé par sommet sont deux modèles de processus renforcés intimement liés. Dans ces deux modèles la marche a tendance a revenir vers les zones déjà visitées. Nous montrons que certaines quantités caractéristiques de ces deux modèles présentent une certaine monotonie en leurs paramètres. Cela induit un certain nombre de conséquences notamment une unicité de la transition de phase entre récurrence et transitivité, la récurrence en dimension 2 et une loi du $0-1$ pour la récurrence. Dans un second temps on s'intéresse également à une version biaisée du modèle de marche aléatoire renforcée linéairement par arête pour lequel on montre qu'il conserve un comportement similaire pour certains types de graphes.

Asymptotic behaviour and phase transition for random walks in random environments and reinforced random walks

Abstract. In this thesis, we study some behaviours of random walks in random environemnts and reinforced random walks. We will first look at random walks in Dirichlet environment and then at two models of reinforced walks: the linealry edge-reinforced random walk and the vertex reinforced jump process.
Random walks in Dirichlet environment are a special case of random walk in random environments that exhibit an important property simplifiant leur étude: the statistical invariance by time reversal. In chapter 2 we will use this property to characterize the asymptotic behaviour of these walks in dimensions 3 and higher when they are transient with zero speed. In this case we show that their behaviour is characterized by a stable process. In chapter 3 we show that this property of statistical invariance by time reversal is actually characteristic of random walks in Dirichlet environments.
The linearly edge-reinforced random walk and the vertex reinforced jump process are two closely linked models of reinforced processes. In both models the walk tends to come back to areas it has already visited. In chapter 4, we will show that some characteristic quantities exhibit some monotonicity in their parameters. This induces some consequences: unicity for the phase transition between recurrence and transience, recurrence in dimension 2, and a $0-1$ law for recurrence. Then, in chapter 5 we will look at a biased version of the linearly edge-reinforced random walk for which we show that its behaviour stays similar to the original model on some infinite graphs.

Image de couverture: Mathilde Adorno