Symplect'hearing the contact horizon and contacting the symplectic inner beauty

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par

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“Not all who wander are lost.”

Cheshire Cat
in *L.Caroll’s “Alice’s Adventures in Wonderland*".

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Foreword

This memoir, appearing as a wander through symplectic topology and contact geometry presents most of the results I obtained since the end of my PhD, and puts them in context. It reviews the main questions I am interested in, explains my approach to those, and asks further questions.

A problem that has always fascinated me in the study of manifolds with boundary can be phrased as:

How much structure do you need on a domain for the boundary to carry relevant information on the interior? Reciprocally, how much does the interior of a domain "know" about its boundary?

This is inspired by Marc Kac's paper [Kac66] "*Can you hear the shape of a drum?*". This paper generalises a question of H. A. Lorentz[1] which was answered by Weyl[2]. Weyl proved that one can recover the area of a domain by examining how rapidly the Dirichlet eigenvalues of the Laplace operator grow [Wey11].

Symplectic and contact geometry originated in a mathematical formulation of the classical mechanics of dynamical systems with finitely many degrees of freedom. The objects studied are smooth manifolds with an additional structure, *symplectic* in the even-dimensional case and *contact* in the odd-dimensional one. One of the most prominent features of symplectic and contact geometry is that rigidity and flexibility phenomena coexist. Flexibility is illustrated by Darboux's

[1]Zum Schluss soll ein mathematisches Problem Erwähnung finden, as vielleicht bei den anwesenden Mathematikern Interesse erwecken wird. Es stammt aus der Strahlungstheorie von Jeans. In einer vollkommen spiegelnden Hülle können sich stehende elektromagnetische Schwingungen ausbilden, ähnlich den Tönen einer Orgelpfeife; wir wollen nur auf die sehr hohen Obertöne das Augenmerk richten. Jeans fragt nach der auf ein Frequenzintervall dn fallenden Energie. Dazu berechnet er zuerst die Anzahl der zwischen den Frequenzen n und $n + dn$ liegenden Obertöne und multipliziert die Zahl dann mit der zu jeder Frequenz gehörigen Energie, die nach einem Satze der statistischen Mechanik für alle Frequenzen gleich ist. Auf diese Weise bekommt er in der Tat das richtige Gesetz der Strahlung für langwellige Wärmestrahlen. Hierbei entsteht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n + dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist. Für mehrere einfache Formen der Hülle, wo sich die Rechnung durchführen lässt, wird der Satz in einer Leidener Dissertation bestätigt werden. Es ist nicht zu zweifeln, dass er allgemein, auch für mehrfach zusammenhängende Räume, gültig ist. Analoge Sätze werden auch bei andern schwingenden Gebüden, wie elastischen Membranen und Luftmassen etc., bestehen
To conclude, there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in Jeans' theory of radiation. In an enclosure with a perfectly reflecting surface, standing electromagnetic waves can form, similar to tones of an organ pipe; we shall focus only on very high overtones. Jeans asks for the energy falling on a frequency interval dn. To do this, he first calculates the number of overtones lying between the frequencies n and $n + dn$ and then multiplies this number by the energy belonging to each frequency, which according to a theorem of statistical mechanics is the same for all frequencies. In this way, he indeed gets the right law of radiation for long-wave radiation. Here arises the mathematical problem of proving that the number of sufficiently high overtones between n and $n + dn$ is independent of the shape of the enclosure and proportional only to its volume. For several simple shapes on which the calculations can be carried out, this theorem has been confirmed in a Leiden dissertation. There is no doubt that it holds in general, even for multiply connected regions. Analogous results for other vibrating structures, such as elastic membranes, air masses, etc. should also hold.

[2]who also introduced the term *symplectic* in [Wey39]

theorem (locally all symplectic, respectively all contact, manifolds are “the same”) and by various h-principles. Rigidity is illustrated by Gromov’s non-squeezing theorem (which is at the origin of symplectic topology); it states that one can symplectically embedd a ball in a cylinder if and only if the radius of the ball is less than that of the cylinder.

Understanding rigidity and flexibility is one of my goals. I approach this goal by considering a symplectic manifold whose boundary is a contact manifold. The central question, almost formulated as such in [CFHW96], is then:

“*How much does the contact boundary know about the symplectic interior, and, reciprocally, how much does the symplectic interior know about its contact boundary?*”

Two classical conjectures are directly related to this question. Weinstein conjecture (conjecture 1.1.1) concerns the existence of periodic orbits in the *Reeb dynamics* on a compact contact manifold and Viterbo’s conjecture (conjecture 4.0.1) concerns *symplectic embeddings* and obstructions (*symplectic capacities*) thereof.

This memoir is divided in two part; the First part is devoted to the *how much does the symplectic interior know about its contact boundary?* viewpoint. Contact manifolds come with (a lot of) dynamical systems (Reeb vector fields) and their study, in particular the periodic orbits, is the main focus of this first part. The Second part is about the *how much does the contact boundary know about the symplectic interior* viewpoint. The focus is on obstructions to symplectic embeddings of a symplectic manifold in another one coming from the dynamics (periods of orbits) on the boundaries.

List of publications (since PhD)

[GK16] Gutt, Jean; Kang, Jungsoo : *On the minimal number of periodic orbits on some hypersurfaces in* $\mathbb{R}^{2n}$, Annales de l'institut Fourier, 66 no. 6 (2016), p. 2485–2505.

[Gut17] Gutt, Jean : *The positive equivariant symplectic homology as an invariant for some contact manifolds*, Journal of Symplectic Geometry, Vol. 15, No. 4 (2017), pp. 1019–1069.

[AGH18] Albers, Peter; Gutt, Jean; Hein, Doris : *Periodic Reeb orbits on prequantization bundles*, Journal of Modern Dynamics, Vol 12, 2018, pp. 123–150.

[GH18] Gutt, Jean; Hutchings, Michael : *Symplectic capacities from positive* S^1*-equivariant symplectic homology*, Algebraic & Geometric Topology, Vol 18, Issue 6 (2018), 3537–3600.

[GU19] Gutt, Jean; Usher, Mike : *Symplectically knotted codimension-zero embeddings of domains in* $\mathbb{R}^4$, Duke Mathematical Journal, 168 (2019), no. 12, 2299–2363.

[AGKM20] Abreu, Miguel; Gutt, Jean; Kang, Jungsoo; Macarini, Leonardo : *Two closed orbits for non-degenerate Reeb flows*, Mathematical Proceedings of the Cambridge Philosophical Society, pages 1–36, 2020.

[GHR20] Gutt, Jean; Hutchings, Michael; Ramos, Vinicius : *Examples around the strong Viterbo conjecture*, accepted for publication in Journal of Fixed Point Theory and Applications, Preprint arXiv:2003.10854, 21 pages.

Contents

Part I.

Reeb dynamics

1. Introduction to Part I

1.1. General context

A contact manifold is an odd-dimensional manifold M^{2n-1} endowed with a contact structure, i.e. a codimension one distribution ξ having a maximal non-integrability property. If we write locally the distribution as the kernel of a 1-form, $\xi = \ker \alpha$, the condition is that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing; such a 1-form α is called a contact form. Throughout this memoir we shall always assume that a contact structure is co-oriented, that is, α is defined globally.

One of the simplest examples of closed contact manifolds is the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ with the standard contact form $\alpha_0 \in \Omega^1(S^{2n-1})$ which is given by the restriction to the sphere of the Liouville 1-form $\lambda_0 \in \Omega^1(\mathbb{R}^{2n})$.

$$\alpha_0 := \lambda_0\big|_{S^{2n-1}} := \frac{1}{2}\sum_{j=1}^{n}(x^j dy^j - y^j dx^j)\big|_{S^{2n-1}}. \tag{1.1.1}$$

where x_j, y_j are the standard coordinates on $\mathbb{R}^{2n}$.

To a contact form α on M corresponds a unique vector field R_α (the Reeb vector field) characterized by the equations $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$. The Reeb vector field never vanishes; hence its flow does not have any fixed point. Periodic orbits are thus the most noticeable objects thereof. In his “traité de la mécanique céleste", Poincaré pointed out the interest of periodic orbits:

> Ce qui nous rend ces solutions périodiques si précieuses, c’est qu’elles sont, pour ainsi dire, la seule brêche par où nous puissions essayer de pénétrer dans une place jusqu’ici réputée inabordable.

If α is a contact form on M, $f\alpha$ is also a contact form for any non-vanishing function $f \in C^\infty(M,\mathbb{R})$. There are thus as many Reeb vector fields on a contact manifold M as there are non-vanishing smooth functions on M. Nonetheless there is, conjecturally, a very strong rigidity phenomenon.

Conjecture 1.1.1 (Weinstein, [Wei79]). *Every contact form on a compact contact manifold carries at least one periodic Reeb orbit.*

The Weinstein conjecture was proven in dimension three by Taubes in 2007 [Tau07]. Taubes’ result was later improved independently by Cristofaro-Gardiner and Hutchings [CGH16] and by Ginzburg, Hein, Hryniewicz and Macarini [GHHM13] who proved that every contact form on a compact contact manifold of dimension three carries at least two geometrically distinct periodic Reeb orbits. Recently, Cristofaro-Gardiner, Hutchings and Pomerleano [CGHP19] have proven that, modulo assumptions[1], every contact form on a compact contact manifold of dimension three carries either two or infinitely many geometrically distinct periodic Reeb orbits. This last result does not generalize to higher dimensions since Albers, Geiges and Zehmisch [AGZ18] constructed

[1]The assumptions are that the contact form is non-degenerate and the first Chern class of the contact structure is torsion.

examples, in all odd dimensions greater than three, of contact forms on compact connected contact manifolds carrying an arbitrarily large (but finite) number of geometrically distinct periodic Reeb orbits.

Those results motivate:

Question 1.1.2. *Given a contact manifold, what is the lower bound on the number of geometrically distinct periodic Reeb orbits and what is the topological (or analytic) significance of that bound?*

Note that, at the time of writing, except for a few manifolds, we do not have any idea what this bound should be.

1.1.1. degenerate vs non-degenerate

The bound in question 1.1.2 might depend on whether the contact form α is **degenerate** or **non-degenerate**. Similarly to the Morse condition for smooth function, the non-degeneracy condition is to ensure isolation of the periodic Reeb orbits.

Definition 1.1.3. *A contact form is* ***non-degenerate*** *if all periodic Reeb orbits are non-degenerate. A periodic Reeb orbit γ of period T is* non degenerate *if* 1 *is not an eigenvalue of the Poincaré return map; i.e.* 1 *is not an eigenvalue of the differential of the flow restricted to the contact structure ξ $\phi_{\star}^{R_\alpha,T} : \xi_{\gamma(0)} \to \xi_{\gamma(T)}$.*

For smooth functions on a compact manifold $f : M \to \mathbb{R}$, we have lower bounds on the minimal number of critical points of f.
If f **is Morse**, then we have the Morse inequalities

$$\# \operatorname{Crit}(f) \geqslant \sum_{i=1}^{\dim(M)} b_i(M)$$

where the $b_i(M)$ are the Betti numbers of M.
If f **is not Morse**, then

$$\# \operatorname{Crit}(f) \geqslant \operatorname{cuplength}(M) + 1$$

where the cuplength of M is defined as follows

Definition 1.1.4. *Let M be a manifold. The* ***cuplength*** *of M is defined as*

$$\operatorname{cuplength}(M) := \max\left\{k \in \mathbb{N} \mid \exists \beta_1, \ldots, \beta_k \in H^{\geqslant 1}(M) \text{ such that } \beta_1 \cup \ldots \cup \beta_k \neq 0\right\}.$$

For instance, looking at the 2-torus T^2, every Morse function $f : T^2 \to \mathbb{R}$ must have at least 4 critical points, but there exists smooth functions $g : T^2 \to \mathbb{R}$ with only 3 critical points (which is minimal). Figure 1.1 represent an immersion [Cur92] of T^2 in $\mathbb{R}^3$ where the height function has only 3 critical points.

1.1.2. Smallest orbit

Another natural question is

Question 1.1.5. *What is the minimal period among all periodic Reeb orbits?*

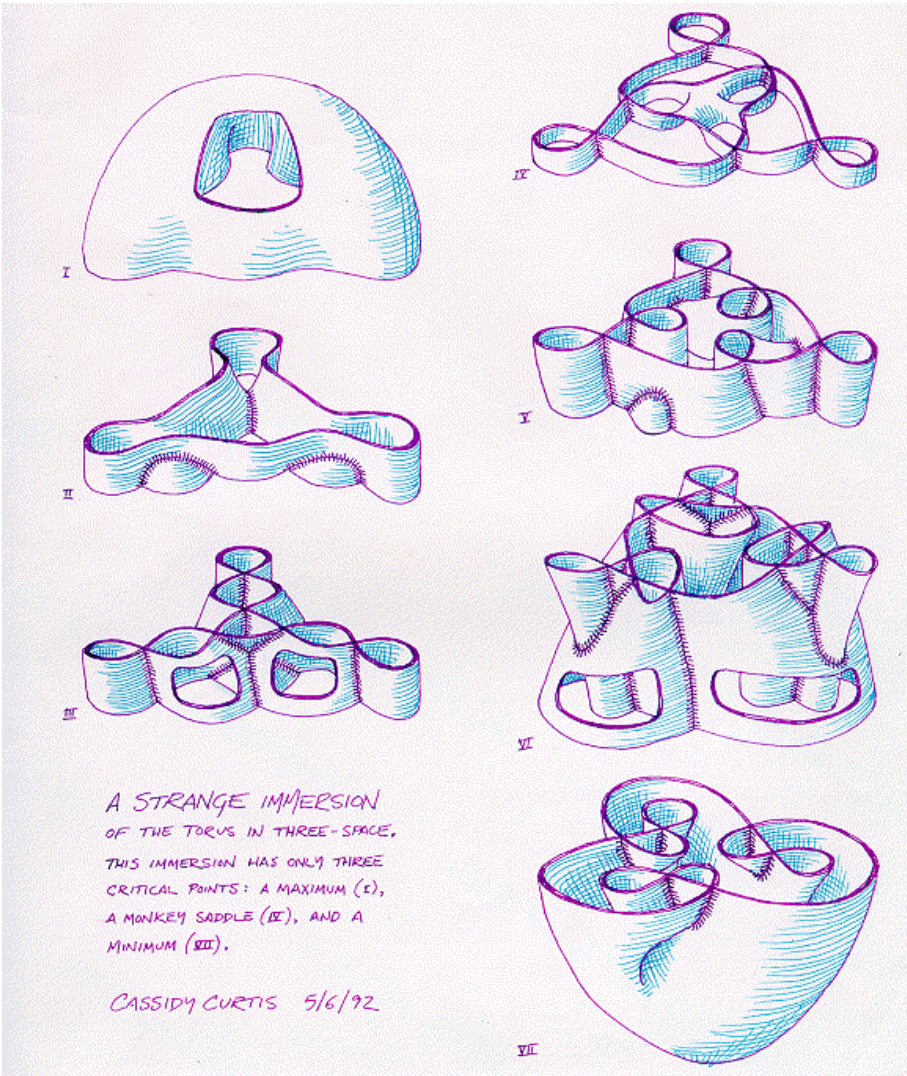


Figure 1.1.: Immersion of T^2, from [Cur92], for which the height function only have 3 critical points

Note that, as stated, question 1.1.5 is not well-posed. Indeed, when multiplying the contact form α by a constant k, the Reeb vector field is multiplied by $\frac{1}{k}$. A more "reasonable" quantity (since invariant by rescaling) to look at is called the **systolic ratio** of the contact form α and is defined as

$$\frac{(T_{\min,\alpha})^n}{\mathrm{Vol}(M,\alpha)}$$

where $T_{\min,\alpha}$ denote the smallest period of a periodic orbit of R_α and $\mathrm{Vol}(M,\alpha) = \int_M \alpha \wedge (d\alpha)^{n-1}$.

Question 1.1.5 would then become

Question 1.1.6. *Given a contact manifold (M,ξ), is there a bound (upper and/or lower) for the systolic ratios of all contact form defining the contact structure ξ?*

1.2. Star-shaped hypersurfaces

A distinguished class of contact manifolds consists of the boundaries of some star-shaped[2] domains with respect to the origin in $\mathbb{R}^{2n}$. It appears naturally in many dynamical problems. For instance, hypersurfaces bounding strictly convex domains (called strictly convex hypersurfaces) arise as regularized energy hypersurfaces in the planar restricted three-body problem. The boundary Σ of a star-shaped domain X is called a star-shaped hypersurface in $\mathbb{R}^{2n}$. The 1-form $\alpha_0 := \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)\big|_\Sigma$ is a contact form on Σ. The 2-form $\omega_0 := d\alpha_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is a symplectic form on X

Lemma-Remark 1.2.1. *The study of the Reeb field on all star-shaped hypersurfaces is equivalent to the study of the Reeb field for all contact forms defining the standard contact structure[3] on the sphere S^{2n-1}.*

Proof. Let $X \subset \mathbb{R}^{2n}$ be a star-shaped bounded domain with smooth boundary Σ. Then

$$\lambda_0 = \frac{1}{2} := \sum_{i=1}^n x^i dy^i - y^i dx^i$$

restricts to a contact form on Σ. Let $h_\Sigma : S^{2n-1} \to \mathbb{R}$ a smooth positive function such that, $X = \{rz \,|\, z \in S^{2n-1}, 0 \leqslant r \leqslant h_\Sigma(z)\}$. We now look at the diffeomorphism $\varphi : S^{2n-1} \to \Sigma : z \mapsto h_\Sigma(z)z$ and we can easily check that

$$\varphi^\star(\lambda_0|_\Sigma) = h_\Sigma^2 \lambda_0|_{S^{2n-1}}.$$

□

Star-shaped hypersurfaces are one of the few manifolds for which we have a candidate for the bound in Question 1.1.2.

Conjecture 1.2.2. *Every star-shaped hypersurface in $\mathbb{R}^{2n}$ carries at least n geometrically distinct periodic Reeb orbits.*

1.2.1. $n = 2$

A crucial ingredient to study the planar restricted three-body problem is a **global surface of section**, a notion which goes back to Poincaré. A global surface of section (of disk type) is an embedded 2-disk in an energy hypersurface of dimension 3. It is equipped with the Poincaré return map encoding the qualitative properties of the dynamics. In addition, the boundary is required to be a periodic orbit, called a binding orbit. A global surface of section reduces the complexity by one dimension. Finding this fascinating object is, in general, a nontrivial problem. However a beautiful theorem due to Hofer, Wysocki, and Zehnder [HWZ98] asserts that every dynamically

[2] by star-shaped, I mean that the radial vector field is everywhere transverse to the boundary.
[3] Defined as $\operatorname{Ker} \alpha_0$.

convex[4] hypersurface in $\mathbb{R}^4$ has a global surface of section. In contrast to perturbation methods, [HWZ98] uses holomorphic curve theory and dynamical convexity is essential for a compactness result of holomorphic planes.

Both methods have their own merits. In perturbation theory, one begins with a well-known dynamical system (e.g. Kepler problem) where we know which embedded disk is a global surface of section. The disk survives under small perturbations but it is difficult to estimate how long this lasts. For example perturbation methods show that in the planar restricted three-body problem the most famous orbit, the retrograde orbit, is a binding orbit of a global surface of section if the situation is close enough to the Kepler problem. On the other hand, the theorem in [HWZ98] enables us to easily know the existence of global surfaces of section in a given dynamical system because strict convexity of a dynamical system is a property that can be checked a priori. For example in [AFF+11] the authors proved strict convexity and hence the existence of a global surface of section in the planar restricted three-body problem close to the Hill's lunar problem where perturbation methods do not apply. But the method in [HWZ98] does not tell where the resulting global surface of section is located. This was already pointed out by Hofer, Wysocki, and Zehnder and led them to raise the question whether a periodic orbit with the smallest action always is a binding orbit of a global surface of section. In the following we call a periodic orbit with the smallest action a smallest periodic orbit.

1.2.2. $n \geqslant 2$

The problem of finding periodic Reeb orbits on a contact manifold which is embedded in a symplectic manifold can often be translated into the problem of finding periodic orbits of a Hamiltonian vector field on a prescribed energy level. For instance, if X is a star-shaped domain in $\mathbb{R}^{2n}$ such that $0 \in \operatorname{Int} X$, finding periodic orbits of the Reeb vector field on the boundary Σ of X (for the standard contact form α_0) amounts to finding periodic orbits of the Hamiltonian vector field defined by a power of the gauge function, on the boundary of X which is a level set of this Hamiltonian. Indeed, the **gauge function** of X, $j_X : \mathbb{R}^{2n} \to [0, \infty)$ is defined by

$$j_X(x) := \min\left\{h \middle| \tfrac{x}{h} \in C\right\}$$

and the Hamiltonian vector field associated to $H_\beta = j_X(x)^\beta$ is

$$X_{H_\beta} = \frac{\beta}{2} R_{\alpha_0}.$$

Finding periodic Reeb orbits on Σ thus translates into finding $T \in \mathbb{R}^{>0}$ and a smooth curve $x : [0, T] \to \mathbb{R}^{2n}$ such that

$$\begin{cases} H_\beta\big(x(t)\big) = 1 & \forall t \\ \dot{x}(t) = X_{H_\beta} \\ x(0) = x(T) \end{cases} \tag{1.2.1}$$

Solutions of (1.2.1) are usually called **closed characteristics**.

In this context, a beautiful idea was developped for strictly convex Hamiltonians: the Clarke-Ekeland dual principle [CE80]. A lot of foundational research in Hamiltonian dynamics and symplectic geometry is based on it. Nevertheless after holomorphic curve theory has become one of the main tools of symplectic geometry, this elegant idea has received little attention. A reason

[4] Dynamical convexity is a generalization of strict convexity, see Definition 1.3.2

is that the Clarke-Ekeland dual principle is only valid under the condition of strict convexity. In exchange however this tells stories that methods in modern symplectic geometry have not seen so far. The main reason is that, in contrast to the classical action functional, the Clarke-Ekeland dual action functional attains a minimizer. This minimizer yields a smallest periodic orbit. Thus one of the things the dual principle tells is that in strictly convex Hamiltonian systems a periodic orbit with the smallest action has minimal index. This is precisely the index for being a binding orbit in dimension 3.

Another formulation (which generalizes to the case where Σ is only continuous) uses the exterior normal vector, ν_Σ and the complex structure J on $\mathbb{R}^{2n}$. The Reeb vector field, R_α is proportional to $J\nu_\Sigma$ since $\iota(J\nu_\Sigma)d\alpha = 0$ because $\iota(J\nu_\Sigma)d\alpha(Y) = \omega(J\nu_\Sigma, Y) = -\langle \nu_\Sigma, Y\rangle = 0$ for all $Y \in T\Sigma$. Thus

$$R_\alpha = cJ\nu_\Sigma$$

with $|c| = \|R_\alpha\|$.

Given in $\mathbb{R}^{2n}$ a star-shaped domain X with boundary Σ, one can define for $x \in \Sigma$, the **normal cone** to X at x, $N_\Sigma(x)$.

$$N_\Sigma(x) := \{y \in \mathbb{R}^{2n} \,|\, \langle x' - x, y\rangle \leqslant 0, \forall x' \in X\}.$$

Problem (1.2.1) then becomes finding $T > 0$ and an absolutely continuous curve $x : [0, T] \to \mathbb{R}^{2n}$ such that:

$$\begin{cases} x(t) \in \Sigma \quad \forall t \\ \dot{x}(t) \in JN_\Sigma(x) \\ x(0) = x(T) \end{cases}$$

This later formulation allows to look at periodic Reeb orbits on polytopes, for which algorithmic methods were recently implemented, [CH20].

1.3. Convexity

Regarding question 1.1.6, It is shown in [HZ11, ABHS17] that the systolic ratio of (S^3, ξ_0) is unbounded. [ABHS17] build examples of star-shaped hypersurface in $\mathbb{R}^4$ with arbitrarily large systolic ratio. On the other hand, it is believed that *convex domains* carry special rigidity phenomena which general starshaped domains do not have. In particular, Viterbo [Vit00] conjectured a systolic inequality for convex domains

Conjecture 1.3.1 (Weak Viterbo conjecture). *If $X \subset \mathbb{R}^{2n}$ is a convex set, then*

$$\left(T_{\min}\right)^n \leqslant n!\,\mathrm{Vol}(X).$$

Moreover, equality holds if and only if X is symplectomorphic to the ball.

Note that here $\mathrm{Vol}(X)$ denotes the Euclidean volume of X. The Euclidean volume of X and the contact volume of the boundary Σ of X $\mathrm{Vol}(\Sigma, \alpha_0)$ are related by

$$n!\,\mathrm{Vol}(X) = \mathrm{Vol}(\Sigma, \alpha_0).$$

Convexity is not a symplectically invariant property. This was already pointed out a long time ago but only a few symplectic substitutions have been suggested. The most prominent one is **dynamical convexity**, introduced in [HWZ98], where they show that strict convexity guarantees dynamical convexity. A natural question is whether these two notions agree.

Definition 1.3.2. *A contact form α on S^{2n-1} is* ***dynamically convex*** *if all closed Reeb orbits have Conley-Zenhder index at least $n+1$.*

Question 1.3.3. *Is every dynamically convex domain symplectomorphic to a convex domain?*

1.4. Structure of Part I

The rest of this first part is divided in two chapters. Chapter 2 presents the various results I obtained in the direction of Question 1.1.2, first on star-shaped hypersurfaces in $\mathbb{R}^{2n}$ then on more general manifolds. Chapter 3 is an exposition of the tools (and their properties) used in the proofs of the results in Chapter 2. In particular, §3.2 consists of an exposition of equivariant symplectic homology which is also relevant to Part II.

2. Results

2.1. Star-shaped hypersurfaces in $\mathbb{R}^{2n}$

2.1.1. In any dimension

The first result in the direction of Conjecture 1.2.2 is the proof by Rabinowitz [Rab79] of the existence of one periodic orbit on every star-shaped hypersurface; this was extended to all hypersurfaces in $\mathbb{R}^{2n}$, of contact type by Viterbo [Vit87]. Conjecture 1.2.2 was proven by Ekeland and Lasry [EL80] and by Beresticky, Lasry, Mancini and Ruf [BLMR85] for convex hypersurfaces which are "pinched" between two spheres whose ratio of radii is strictly less than $\sqrt{2}$.

Theorem 2.1.1 ([EL80, BLMR85]). *Let Σ be a star-shaped hypersurface in $\mathbb{R}^{2n}$. Assume there exists a point $x_0 \in \mathbb{R}^{2n}$ and numbers $0 < R_1 \leqslant R_2$ such that:*

$$\forall x \in \Sigma, \quad R_1 \leqslant \|x - x_0\| \leqslant R_2 \qquad \textit{with } \frac{R_2}{R_1} < \sqrt{2}$$

Assume also that $\forall x \in \Sigma, \quad T_x\Sigma \cap B_{R_1}(x_0) = \emptyset$. Then Σ carries at least n geometrically distinct periodic Reeb orbits.

Long, Zhu, Hu et Wang [LZ02, WHL07] managed to remove the pinching assumption and showed that every strictly convex hypersurface carries at least $\lfloor \frac{n}{2} \rfloor + 1$ geometrically distinct periodic Reeb orbits. They proved moreover that if all periodic Reeb orbits are non-degenerate, then there are at least n of them. Those results rely on variational methods: the action functional and its dual in the sense of Clarke-Ekeland, for which the convexity of the hypersurface is crucial. The second ingredient in those proofs is a detailed study of the Conley-Zehnder index; it is an integer number (or half-integer in the degenerate cases) associated to every periodic Reeb orbit.

My approach was to replace the variational tools by tools of a more symplectic nature. I developped properties of the positive S^1-equivariant symplectic homology (denoted by CH from now on) and obtained the following results. First, [Gut17] an elementary proof of theorem 2.1.1 of Ekeland and Lasry and Beresticky, Lasry, Mancini and Ruf (with a non-degeneracy assumption). Then, with Jungsoo Kang [GK16], we considerably weakened the convexity assumption, keeping a non-degeneracy assumption.

Theorem 2.1.2 ([GK16]). *Let (Σ, α_0) be a non-degenerate star-shaped hypersurface in $\mathbb{R}^{2n}$ such that all periodic orbits have Conley-Zehnder index at least $n-1$. Then (Σ, α_0) carries at least n simple periodic Reeb orbits.*

Replacing the convexity assumption by something weaker than dynamical convexity was perceived as an interesting step. The main idea is to combine the homology CH (to find many periodic Reeb orbits) and a "translation" of the common index jump theorem [LZ02] (to distinguish which orbits are geometrically distinct).

Theorem 2.1.3 (common index jump theorem). *Let $\gamma_1, \ldots, \gamma_k$ be simple periodic orbits on a given contact manifold of dimension $2n-1$. Assume that all the iterates of the periodic orbits are nondegenerate and that all the mean indices*[1] *of the periodic orbits are positive; $\widehat{CZ}(\gamma_i) > 0$ for all $i \in [0, k]$. Then, for any given $M \in \mathbb{N}$, there exist infinitely many $N \in \mathbb{N}$ and $(m_1, \ldots, m_k) \in \mathbb{N}^k$ such that for any $m \in \{1, \ldots, M\}$*

$$\mathrm{CZ}(\gamma_i^{2m_i - m}) = 2N - \mathrm{CZ}(\gamma_i^m) \quad \textit{and} \quad \mathrm{CZ}(\gamma_i^{2m_i + m}) = 2N + \mathrm{CZ}(\gamma_i^m)$$

and

$$2N - (n-1) \leqslant \mathrm{CZ}(\gamma_i^{2m_i}) \leqslant 2N + (n-1).$$

Since then, our techniques have been generalized to other manifolds and pushed further by other authors, [GG16, GGM18, AM16, DLW16, DLLW16]. To the best of my knowledge, the current optimal statement (compilation of the results by the aforementioned authors) is

Theorem 2.1.4. *Let (Σ, α_0) be a compact star-shaped hypersurface in $\mathbb{R}^{2n}$.*

- *If Σ is dynamically convex (possibly degenerate) then there is at least $\lceil \frac{n}{2} \rceil + 1$ simple periodic Reeb orbits.*
- *If Σ is non-degenerate, all periodic Reeb orbits have positive mean Conley-Zehnder index, and there are no orbits of CZ-index 0 (if n is even) or no orbits of CZ-index -1, 0 or 1 (if n is odd), then there are at least n simple periodic Reeb orbits.*
- *If $\Sigma \subset \mathbb{R}^8$ is convex, then Conjecture 1.2.2 holds.*

Removing completely the assumption on the indices, we showed [GK16] that there are generally "a lot" of periodic orbits, unless the quantities $\frac{\mathscr{A}(\gamma)}{\widehat{\mathrm{CZ}}(\gamma)}$ (where $\widehat{\mathrm{CZ}}$ and $\mathscr{A}$ respectively denote the mean Conley-Zehnder index and the action (i.e. the period)) are all equal.

Proposition 2.1.5 ([GK16]). *Let (Σ, α) be a nondegenerate starshaped hypersurface in $\mathbb{R}^{2n}$, for n odd, with two simple periodic orbits γ and δ. Then Σ carries another simple periodic orbit unless*

$$\frac{\mathscr{A}(\gamma)}{\widehat{CZ}(\gamma)} = \frac{\mathscr{A}(\delta)}{\widehat{CZ}(\delta)} \tag{2.1.1}$$

This last Proposition shows the existence of rigidity and raised the question

Question 2.1.6. *What is the topological significance of the quantities $\frac{\mathscr{A}(\gamma)}{\widehat{CZ}(\gamma)}$?*

Those quantities already appeared in [EH87, Eke90, Gür15]. A non-degenerate contact form α is called *perfect* if the number of good periodic nondegenerate orbits with Conley Zehnder index k is equal to the dimension of the k-th positive S^1-equivariant symplectic homology group. Gürel [Gür15] proved that if a non-degenerate contact form on the sphere is perfect, then all the quantities $\frac{\mathscr{A}(\gamma)}{\widehat{\mathrm{CZ}}(\gamma)}$ are equal. With Jungsoo [GK16], we proved that if a non-degenerate contact form on the sphere is perfect, then there are precisely n even simple periodic orbits of the Reeb vector field. If the contact form is moreover dynamically convex, the converse is also true.

The first guess is that the quantities $\frac{\mathscr{A}(\gamma)}{\widehat{\mathrm{CZ}}(\gamma)}$ exhibit some kind of symmetry of the hypersurface. A diffeomorphism $f : (\Sigma, \alpha) \to (\Sigma, \alpha)$ is called a *strict (anti)-contactomorphism* if $f^\star \alpha = \alpha$ or if $f^\star \alpha = -\alpha$, respectively.

[1] The mean index of a periodic orbit γ is defined as $\widehat{\mathrm{CZ}}(\gamma) = \lim_{m\to\infty} \frac{\mathrm{CZ}(\gamma^m)}{m}$

Question 2.1.7. *If Σ is a star-shaped hypersurface in $\mathbb{R}^{2n}$ and $f : (\Sigma, \alpha_0) \to (\Sigma, \alpha_0)$ is a strict (anti)-contactomorphism, are all periodic Reeb orbits invariant under f?*

We showed [GK16] that if a non-degenerate star-shaped hypersurface (Σ, α_0) in $\mathbb{R}^{2n}$ is dynamically convex and has precisely n geometrically distinct periodic Reeb orbits, and if there exists a strict (anti)-contactomorphism of (Σ, α_0), then all periodic orbits are invariant under it.

"Symmetric" hypersurfaces (in particular under the involution; i.e. $\Sigma = -\Sigma$) have been studied in [Wan09, LWZ19, LW18, JKS18, JKS20, GM19].

2.1.2. In dimension 3

The most notable result in dimension three is due to Hofer Wysocki and Zehnder [HWZ98]

Theorem 2.1.8. *Any dynamically convex star-shaped hypersurface in $\mathbb{R}^4$ carries either 2 or infinitely many simple periodic Reeb orbits.*

The key idea is to find for every dynamically convex star-shaped hypersurface in $\mathbb{R}^4$ a disk-like global surface of section and then use a result by Franks [Fra92].

Definition 2.1.9. *Let φ^t be a smooth flow on a closed manifold M of dimension 3. An embedded surface $\Sigma \hookrightarrow M$ is a **global surface of section** for φ^t if:*

1. *Each component of the boundary $\partial\Sigma$ of Σ is a periodic orbit of φ^t.*
2. *The flow φ^t is transverse to $\Sigma \backslash \partial\Sigma$.*
3. *For every $p \in \Sigma\backslash\partial\Sigma$, there exist $t^+ \in \mathbb{R}_{>0}$ and $t^- \in \mathbb{R}_{<0}$ such that both $\varphi^{t^+}(p)$ and $\varphi^{t^-}(p)$ belong to $\Sigma\backslash\partial\Sigma$.*

*If Σ is diffeomorphic to a disk, then Σ is called **disk-like**.*

Theorem 2.1.10 ([HWZ98]). *Any dynamically convex Reeb flow on (S^3, ξ_0) admits a disk-like global surface of section.*

Theorem 2.1.11 ([Hry12, Hry14]). *Let γ be a periodic orbit of a dynamically convex Reeb flow on (S^3, ξ_0). Then γ bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number -1. Moreover, such an orbit binds an open book decomposition whose pages are disk-like global surfaces of section.*

Question 2.1.12 ([HWZ98]). *Does a periodic orbit with the smallest action in a (dynamically) convex hypersurface in $\mathbb{R}^4$ always bound a global surface of section?*

Question 2.1.13. *On a (dynamically) convex hypersurface in $\mathbb{R}^4$, is the smallest periodic orbit unknotted and has self-linking number -1?*

One can also do the "reverse process" and build a contact form on S^3 starting with a compactly supported Hamiltonian diffeomorphism on the disk (viewed as a global surface of section) (see [Bra08, Ush20]). Using this, Abbondandolo, Bramham, Hryniewicz and Salomao build a dynamicaly convex contact form on S^3 with a systolic ratio of almost 2.

Theorem 2.1.14 ([ABHS19]). *For every $\epsilon > 0$ there is a dynamically convex contact form α on S^3 such that*

$$2 - \epsilon < \frac{T_{\min}^2}{\mathrm{Vol}(S^3, \alpha \wedge d\alpha)} < 2$$

In particular, the supremum of the systolic ratio over all dynamically convex contact forms on S^3 is at least 2.

2.2. Other manifolds

Concerning the minimal number of periodic Reeb orbits on contact manifolds (of dimension $\geqslant 5$) other than the sphere, very little is known and I would like to point out that nothing is known outside some restricted class of **prequantization bundles**. That is, E is a $\mathbb{C}$-bundle over a symplectic manifold (W, ω) with $c_1(E) = -[\omega] \in H^2(W; \mathbb{Z})$. In particular, the cohomology class $[\omega]$ of the symplectic form must admit an integral lift. A Hermitian connection on E gives rise to a connection 1-form α_0 on the corresponding S^1-bundle Σ over W. The 1-form α_0 is naturally a contact form. Its Reeb vector field is the infinitesimal generator of the S^1-action on Σ, see [Gei08, Section 7.2] for more details. Moreover, the Hermitian structure defines circle resp. disk bundles S_R resp. D_R of radius $R > 0$. We extend α_0 to $E \backslash M$ by pullback.

We call a hypersurface $\Sigma_f \subset E$ **graphical** if it can be written as the graph of a function $f : \Sigma \to \mathbb{R}_{>0}$ inside E

$$\Sigma_f = \{f(x)x \mid x \in \Sigma\} . \tag{2.2.1}$$

Then $\alpha_f := f\alpha_0$ is a contact form on Σ_f.

Conjecture 2.2.1. *Assuming (W, ω) is a closed connected symplectic manifold with integral symplectic form $[\omega] \in \mathrm{H}^2(M, \mathbb{Z})$ in the construction above, then the graphical hypersurface Σ_f carries at least k simple periodic Reeb orbits with*

$$k = \begin{cases} \sum_{i=1}^{\dim W} b_i(W) & \text{if } \alpha_f \text{ is non-degenerate} \\ \mathrm{cuplength}(W) + 1 & \text{if } \alpha_f \text{ is degenerate} \end{cases}$$

I gave the first results, in this context, answering partially Conjecture 2.2.1, thanks to the use of the homology CH.

Proposition 2.2.2 ([Gut14c]). *Let Σ_f be a graphical hypersurface in E such that the intersection of Σ_f with each fiber is a circle. Then Σ_f carries at least $\sum_{i=0}^{\dim W} b_i$ geometrically distinct periodic Reeb orbits, where b_i denote the Betti numbers of W.*

With Peter Albers and Doris Hein [AGH18], we gave one of the first geometrical explanation[2] of the minimal number of geometrically distinct periodic Reeb orbits for some hypersurfaces in prequantization bundles. This lower bound is given in terms of the cuplength of the base. In particular for star-shaped hypersurfaces in $\mathbb{R}^{2n}$, the minimal number originates in the cuplength of $\mathbb{C}P^{n-1}$.

Theorem 2.2.3 ([AGH18]). *Let E be prequantization bundle over the symplectic manifold (W^{2n}, ω). Assume that the graphical hypersurface $\Sigma_f \subset E$ is pinched between S_{R_1} and S_{R_2} with $\frac{R_2}{R_1} < \sqrt{2}$. Then there exist either infinitely many periodic Reeb orbits of R_{α_f} or there are periodic orbits $\gamma_1, \ldots, \gamma_c$ of R_{α_f} with $c = \mathrm{cuplength}(W) + 1$ such that*

$$\pi R_1^2 < \mathscr{A}_{\alpha_f}(\gamma_1) < \ldots < \mathscr{A}_{\alpha_f}(\gamma_c) < \pi R_2^2$$

where $\mathscr{A}_{\alpha_f}(\gamma) := \int_\gamma \alpha_f$ is the action or period of a Reeb orbit γ.

Note that the two previous results do not assume non-degeneracy of the contact form.

[2] See also [Mos76]

Corollary 2.2.4. *In the context of Theorem 2.2.3, either the minimal period of periodic Reeb orbits of R_{α_f} is less than πR_1^2 or α_f carries at least* $\mathrm{cuplength}(W)+1$ *simple periodic Reeb orbits.*

In short, there is either a short periodic orbit or $\mathrm{cuplength}(M)+1$ simple periodic Reeb orbits. As a particular case of Corollary 2.2.4, we have Theorem 2.1.1. We recall that S^{2n-1} is the S^1-bundle corresponding to a prequantization bundle over $\mathbb{C}P^{n-1}$ and that $\mathrm{cuplength}(\mathbb{C}P^{n-1}) = n-1$.

Removing the pinching condition (but adding a non-degeneracy assumption), we proved, with Miguel Abreu, Jungsoo Kang and Leonardo Macarini [AGKM20] that under a mild growing condition of the homology CH, there are always at least two periodic Reeb orbits.

Theorem 2.2.5 ([AGKM20]). *Let (M^{2n+1},ξ) be a closed contact manifold admitting a strong symplectic filling W such that $c_1(TW)=0$. Let Γ be a set of free homotopy classes of loops in W closed under iterations and assume that there exist $K\in\mathbb{N}$ and a non-vanishing section σ of the determinant line bundle $\Lambda^{n+1}_{\mathbb{C}}TW$ such that*

$$\dim CH_n(W,\Gamma) < \dim CH_{n+jK}(W,\Gamma)$$

or

$$\dim CH_{-n}(W,\Gamma) < \dim CH_{-n-jK}(W,\Gamma)$$

for every $j\in\mathbb{N}$, where the grading in $CH_*(W,\Gamma)$ is taken with respect to the homotopy class of σ. Then every non-degenerate Reeb flow on M carries either infinitely many geometrically distinct closed Reeb orbits or at least two geometrically distinct closed Reeb orbits γ_1 and γ_2 such that their Conley-Zehnder indices satisfy $\mu(\gamma_1^k)\neq\mu(\gamma_2^k)$ for some $k\in\mathbb{N}$. Moreover, all these orbits have free homotopy class in Γ.

We then showed that Theorem 2.2.5 applies to many manifolds: to displaceable contact manifolds exactly embedded in an exact symplectic manifold, to unit cotangent bundle of closed, spin, oriented manifolds of dimension bigger than one (with an assumption on the π_1), to good toric contact manifolds, to prequantization bundles, to connected sums of Liouville domains,...

2.2.1. Displaceable contact manifolds

Given a contact manifold (M,ξ) and an exact symplectic manifold $(X,d\lambda)$, an embedding $M\hookrightarrow X$ is called an exact contact embedding if it is bounding and if there exists a contact form α supporting ξ such that $\alpha-\lambda|_M$ is exact. Here bounding means that M separates X into two connected components, with one of them relatively compact. This embedding is displaceable if M can be displaced from itself by a Hamiltonian diffeomorphism with compact support on X. We say that X is convex at infinity if there exists an exhaustion $X=\cup_k X_k$ by compact subsets $X_k\subset X_{k+1}$ with smooth boundaries such that $\lambda|_{\partial X_k}$ is a contact form for every k. A big class of contact manifolds admitting displaceable exact contact embeddings in exact symplectic manifolds that are convex at infinity is given in [BC02]: the boundary of every subcritical Stein manifold.

Let (M,ξ) be a contact manifold admitting a displaceable exact contact embedding into a convex at infinity exact symplectic manifold X such that $c_1(TX)|_{\pi_2(X)}=0$ and denote by W the compact region in X bounded by M.We showed that W satisfies the hypothesis of Theorem 2.2.5 for $\Gamma=\{0\}$. Hence, we get the following result.

Corollary 2.2.6 ([AGKM20]). *Let (M,ξ) be a contact manifold admitting a displaceable exact contact embedding into a convex at infinity exact symplectic manifold X with $c_1(TX)|_{\pi_2(X)}=0$ and* *denote by W the compact region in X bounded by M. Then every non-degenerate Reeb flow on M has at least two simple closed orbits contractible in W. If $c_1(TX) = 0$ and $H^1(M,\mathbb{R}) = 0$ then every Reeb flow on a contact finite quotient of M carries at least two simple closed orbits. Moreover, the closed lifts of iterates of these orbits to M are contractible in W.*

2.2.2. Cosphere bundles and closed geodesics

Let N be a closed Riemannian manifold and ΛN its free loop space. There is an isomorphism between the (non-equivariant) symplectic homology of T^*N and the homology of ΛN twisted by a local system of coefficients. For the S^1-equivariant version, if N is orientable and spin, we have the isomorphism

$$\mathrm{CH}_*(D^*N) \cong H_*(\Lambda N/S^1, N; \mathbb{Q}), \tag{2.2.2}$$

where $N \subset \Lambda N/S^1$ indicates the subset of constant loops, D^*N is the obvious filling of the cosphere bundle S^*N given by the unit disk bundle. The grading of $\mathrm{CH}_*(D^*N)$ is given by a non-vanishing section of $\Lambda^{n+1}_{\mathbb{C}}TD^*N$ induced from the choice of a volume form in the base so that the Conley-Zehnder index of a non-degenerate closed geodesic coincides with its Morse index, see e.g. [BO16]. This isomorphism respects the filtration given by the free homotopy classes, that is,

$$\mathrm{CH}^\Gamma_*(D^*N) \cong H_*(\Lambda^\Gamma N/S^1, N; \mathbb{Q}) \tag{2.2.3}$$

for every set Γ of free homotopy classes in D^*N, where $\Lambda^\Gamma N$ denotes the set of loops in N with free homotopy class in Γ. (Note that, since $\pi_1(D^*N) \cong \pi_1(N)$, the set of free homotopy classes in D^*N and N are naturally identified. Moreover, $N \subset \Lambda^0 N/S^1$ and therefore if Γ does not contain the trivial free homotopy class then the right hand side of the isomorphism (2.2.3) has to be understood as $H_*(\Lambda^\Gamma N/S^1; \mathbb{Q})$.) For general N, it is expected that the same isomorphism holds with a local system of coefficients as in the non-equivariant case but a rigorous proof has not been written in the literature yet.

It turns out that if N is simply connected and $H_*(\Lambda N/S^1, N; \mathbb{Q})$ is not asymptotically unbounded then it satisfies the assumption in Theorem 2.2.5. Using this, we can prove the following result. Before we state it, let us recall a definition and introduce a notation. A topological space X is k-simple if $\pi_1(X)$ acts trivially on $\pi_k(X)$. If a closed manifold N has dimension bigger than one and $\pi_1(N) \cong \mathbb{Z}$ then N is not rationally aspherical, that is, there exists $j > 1$ such that $\pi_j(N) \otimes \mathbb{Q} \neq 0$. Let k be the smallest such j. In what follows, ξ_{can} denotes the canonical contact structure on S^*N.

Corollary 2.2.7 ([AGKM20]). *Let N be a closed oriented spin manifold with dimension bigger than one. Suppose that N satisfies one of the following conditions:*

(i) $\pi_1(N)$ *is finite;*

(ii) $\pi_1(N) \cong \mathbb{Z}$, $\pi_2(N) = 0$ *and N is k-simple, with k as discussed above;*

(iii) $\pi_1(N)$ *is infinite and there is no $a \in \pi_1(N)$ such that every non-zero $b \in \pi_1(N)$ is conjugate to some power of a.*

In case (i), we have that every non-degenerate contact form on (S^*N, ξ_{can}) has at least two simple closed orbits. Under hypothesis (ii) or (iii), we have two simple closed orbits for any contact form on (S^*N, ξ_{can}), without assuming that it is non-degenerate.

Remark 2.2.8. *The hypothesis that N is oriented spin is used only to have the isomorphism* (2.2.3). *Possibly, it can be relaxed once we have this isomorphism with the relative homology of $(\Lambda N/S^1, N)$ twisted by a local system of coefficients.*

Remark 2.2.9. *In case (ii), the hypothesis that N is k-simple can be relaxed in the following way: let a be a generator of $\pi_1(N)$ and denote by A the linear map corresponding to the action of a on $\pi_k(N) \otimes \mathbb{Q}$. Then it is enough that $\ker(A - Id) \neq 0$. This hypothesis and the assumption that $\pi_2(N) = 0$ are probably just technical but we do not know how to drop them so far.*

Theorem 2.2.5 is used to prove Corollary 2.2.7 only under hypothesis (i). For hypotheses (ii) and (iii), we show the existence of two periodic orbits γ_1 and γ_2 such that no iterate of γ_1 is freely homotopic to γ_2. This is easy in case (iii) using (2.2.3) but highly non-trivial in case (ii) where we show that $\mathrm{CH}_*^0(D^*N) \neq 0$ and $\mathrm{CH}_*^a(D^*N) \neq 0$ for some non-trivial homotopy class a. The proof in case (ii) actually shows the following result. It can be considered as a sort of Lusternik-Fet theorem for Reeb flows; see e.g. [AB16].

Theorem 2.2.10 ([AGKM20])**.** *Let N be a closed not rationally aspherical manifold. Suppose that N is oriented spin, $\pi_1(N)$ is abelian, $\pi_2(N) = 0$ and N is k-simple, with k as discussed above. Then every (possibly degenerate) Reeb flow on S^*N carries a* contractible *closed orbit. As a consequence, if, furthermore, $\pi_1(N)$ is infinite, then every Reeb flow on S^*N has at least two simple closed orbits.*

The hypothesis that N is oriented spin and the second and third conditions in item (ii) can be dropped when we restrict ourselves to Reeb flows given by geodesic flows of Finsler metrics as follows. The proof of item (i) in Theorem 2.2.5 uses only the fact that, given a non-degenerate contact form α on M, $\mathrm{CH}_*^\Gamma(W)$ is the homology of a chain complex generated by the good periodic orbits of α with free homotopy class in Γ graded by the Conley-Zehnder index; the nature of the differential is absolutely unessential. Let F be a Finsler metric on N. It is well known that the closed geodesics of F are the critical points of the corresponding energy functional defined on the free loop space. We will say that F *has only one prime closed geodesic* if either the corresponding geodesic flow has only one simple closed orbit or F is reversible (i.e. $F(x,v) = F(x,-v)$ for every $(x,v) \in TN$) and its geodesic flow has only two simple periodic orbits (given by the lifts of a closed geodesic $\gamma(t)$ and its reversed geodesic $\gamma(-t)$).

It turns out that if F is bumpy (i.e. its geodesic flow is non-degenerate) and has only one prime closed geodesic then $H_*(\Lambda N/S^1, N; \mathbb{Q})$ is the homology of the chain complex generated by the good periodic orbits of the geodesic flow of F with trivial differential. Using this fact we can prove the following result. In what follows, we say that F *has at least two prime closed geodesics* if it does not have only one prime closed geodesic in the sense above. (Note that every Finsler metric has at least one prime closed geodesic.)

Corollary 2.2.11 ([AGKM20])**.** *Let N be a closed manifold with dimension bigger than one. Suppose that N satisfies one of the following conditions:*

(i) $\pi_1(N)$ *is finite;*

(ii) $\pi_1(N) \cong \mathbb{Z}$*;*

(iii) $\pi_1(N)$ *is infinite and there is no $a \in \pi_1(N)$ such that every non-zero $b \in \pi_1(N)$ is conjugate to some power of a.*

In case (i), we have that every bumpy Finsler metric F on N has at least two prime closed geodesics. Under hypothesis (ii) or (iii), we have two prime closed geodesics for any Finsler metric F on N, without assuming that it is bumpy.

Remark 2.2.12. *Our contribution in this corollary is that we find two closed geodesics when N has finite fundamental group and F is bumpy. The remaining cases can be covered by classical minimax methods. This is in contrast with Corollary 2.2.7 for which these minimax methods are not available, making the proof of item (ii) much harder than in the case of geodesic flows.*

Remark 2.2.13. *We are not aware of any example of N which is excluded in the statement, see [Tai10, Section 5]. For instance, if $\pi_1(N)$ is abelian, N meets the hypothesis in Corollary 2.2.11.*

2.2.3. Good toric contact manifolds

Toric contact manifolds are the odd dimensional analogues of toric symplectic manifolds. They can be defined as contact manifolds of dimension $2n+1$ equipped with an effective Hamiltonian action of a torus of dimension $n+1$. Good toric contact manifolds of dimension three are (S^3, ξ_{st}) and its finite quotients. Good toric contact manifolds of dimension greater than three are compact toric contact manifolds whose torus action is not free. These form the most important class of compact toric contact manifolds and can be classified by the associated moment cones, in the same way that Delzant's theorem classifies compact toric symplectic manifolds by the associated moment polytopes. We refer to [Ler03] for details.

In [AM12] the authors show that on any good toric contact manifold (M^{2n+1}, ξ) such that $c_1(\xi) = 0$, any non-degenerate toric contact form is even, that is, all contractible closed orbits of its Reeb flow have even contact homology degree, where the contact homology degree of a closed orbit γ is given by $\mu(\gamma) + n - 2$. (As proved in [AM20], this is also true for the non-contractible closed Reeb orbits.) Suppose that M admits a symplectic filling W with vanishing first Chern class. Then, as showed in [AM12, AM20], $\mathrm{CH}^0_*(W)$ can be computed in a purely combinatorial way in terms of the associated momentum cone. Using this computation, we showed that W satisfies the hypothesis of Theorem 2.2.5 for $\Gamma = \{0\}$ and consequently we get the following result. Note that the fundamental group of every good toric contact manifold M is finite and consequently $H^1(M, \mathbb{R}) = 0$.

Corollary 2.2.14 ([AGKM20]). *Let (M, ξ) be a good toric contact manifold admitting a strong symplectic filling W such that $c_1(TW) = 0$. Then every non-degenerate contact form on a contact finite quotient of M carries at least two geometrically distinct contractible closed orbits.*

Remark 2.2.15. *It turns out that every good toric contact manifold (M, ξ) in dimensions three and five such that $c_1(\xi) = 0$ admits a (toric) filling with vanishing first Chern class [AM20].*

2.2.4. Prequantization circle bundles over symplectic manifolds

Let (B^{2n}, ω) be a closed integral symplectic manifold. Consider the prequantization circle bundle (M, ξ) of (B, ω), that is, the contact manifold given by the total space of a principal circle bundle over B whose first Chern class is $-[\omega]$ and with contact structure given by the kernel of a connection form. Suppose that M admits a symplectic filling W with vanishing first Chern class. Then, under some assumptions on B, we can show that W satisfies the hypothesis of Theorem 2.2.5 with $\Gamma = \{0\}$. More precisely, we have the following result. In what follows,

$$c_B := \inf\{k \in \mathbb{N} \mid \exists S \in \pi_2(B) \text{ with } \langle c_1(TB), S\rangle = k\}$$

denotes the minimal Chern number of B.

Corollary 2.2.16 ([AGKM20]). *Let (M,ξ) be a prequantization circle bundle of a closed integral symplectic manifold (B,ω) such that $\omega|_{\pi_2(B)} \neq 0$, $c_1(TB)|_{\pi_2(B)} \neq 0$ and, furthermore, $H_k(B;\mathbb{Q}) = 0$ for every odd k or $c_B > n$. Suppose that M admits a strong symplectic filling W such that $c_1(TW) = 0$. Then every non-degenerate Reeb flow on M carries at least two geometrically distinct closed orbits contractible in W. If, additionally, $H^1(M,\mathbb{R}) = 0$ then every contact form on a contact finite quotient of M carries at least two geometrically distinct closed orbits. Moreover, the closed lifts of iterates of these orbits to M are contractible in W.*

Remark 2.2.17. *It follows from the Gysin exact sequence that $H^1(M,\mathbb{R}) = 0$ whenever $H_1(B;\mathbb{Q}) = 0$.*

Remark 2.2.18. *When $\omega|_{\pi_2(B)} = 0$ and B satisfies some extra conditions (for instance, when $\pi_i(B) = 0$ for every $i \geqslant 2$) it is proved in [GGM15] (c.f. [GS18]) that every Reeb flow on M (possibly degenerate) carries infinitely many simple closed orbits.*

Remark 2.2.19. *We have that $H_*(B;\mathbb{Q})$ vanishes in odd degrees and $c_1(TB)|_{\pi_2(B)} \neq 0$ whenever B admits a Hamiltonian circle action with isolated fixed points.*

Note that the prequantization bundle M has a natural symplectic filling W given by the corresponding disk bundle in the complex line bundle $L \xrightarrow{\pi} B$ whose first Chern class is $-[\omega]$. Suppose that B is monotone, that is, $[\omega] = \lambda c_1(TB)$ for some $\lambda \in \mathbb{R}$. (We say that B is positive monotone if $\lambda > 0$.) One can check that

$$c_1(TW) = (1-\lambda)\pi^* c_1(TB).$$

Consequently, when $\lambda = 1$ we have that $c_1(TW) = 0$. Now, suppose that λ is an integer bigger than one and let $\widetilde{M}$ be the prequantization bundle of $(B, \frac{1}{\lambda}\omega)$. It is easy to see that M is the finite quotient of $\widetilde{M}$ by the $\mathbb{Z}_\lambda$-action induced by the obvious S^1-action on $\widetilde{M}$. Thus, we have the following corollary; see Remark 2.2.17.

Corollary 2.2.20 ([AGKM20]). *Let (M,ξ) be the prequantization circle bundle of a closed integral symplectic manifold (B,ω) such that $\omega|_{\pi_2(B)} \neq 0$, $c_1(TB)|_{\pi_2(B)} \neq 0$ and, furthermore, $H_k(B;\mathbb{Q}) = 0$ for every odd k or $c_B > n$. Suppose that $[\omega] = \lambda c_1(TB)$ for some $\lambda \in \mathbb{N}$ and that $H_1(B;\mathbb{Q}) = 0$. Then every contact form on a contact finite quotient of M carries at least two geometrically distinct closed orbits. Moreover, the closed lifts of iterates of these orbits to M have contractible projections to B.*

2.2.5. Brieskorn spheres

Given $a = (a_0, \dots, a_{n+1}) \in \mathbb{N}^{n+2}$ define Σ_a as the intersection of the hypersurface

$$z_0^{a_0} + \cdots + z_{n+1}^{a_{n+1}} = 0$$

in $\mathbb{C}^{n+2}$ with the unit sphere $S^{2n+3} \subset \mathbb{C}^{n+2}$. It is well known that $\alpha_a = \frac{i}{8}\sum_{j=0}^{n+1} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j)$ defines a contact form on Σ_a and $(\Sigma_a, \xi_a := \ker \alpha_a)$ is called a Brieskorn manifold. When n is even, $a_0 = \pm 1 \bmod 8$ and $a_1 = \cdots = a_{n+1} = 2$ we have that Σ_a is diffeomorphic to the sphere S^{2n+1} and called a Brieskorn sphere. Brieskorn spheres admit strong symplectic fillings given by Liouville domains W satisfying $c_1(TW) = 0$ and it turns out that W satisfies the hypothesis of Theorem 2.2.5 with $\Gamma = \{0\}$. Therefore, we obtain the following result which is a generalization of [Kan13, Theorem C].

Corollary 2.2.21 ([AGKM20]). *Let M be a contact finite quotient of a Brieskorn sphere. Then every non-degenerate Reeb flow on M carries at least two geometrically distinct closed orbits.*

2.2.6. Connected sums

Let (W_1, λ_1) and (W_2, λ_2) be two Liouville domains of dimension $2n+2$. The boundary connected sum of them is again a Liouville domain $(W_1 \# W_2, \lambda_1 \# \lambda_2)$ and the contact connected sum $(\partial W_1 \# \partial W_2, \xi_1 \# \xi_2)$ is the boundary of it. The following result establishes that the main hypothesis of Theorem 2.2.5 is preserved by boundary connected sums of Liouville domains, furnishing many other examples of contact manifolds satisfying the assumptions of Theorem 2.2.5.

Theorem 2.2.22 ([AGKM20]). *Let (W_1, λ_1) and (W_2, λ_2) be Liouville domains of dimension $2n+2$ with vanishing first Chern class. Assume that there are non-vanishing sections σ_1 and σ_2 of $\Lambda^{n+1}_{\mathbb{C}} TW_1$ and $\Lambda^{n+1}_{\mathbb{C}} TW_2$ respectively satisfying the hypothesis of Theorem 2.2.5 with Γ given by the set of all free homotopy classes. Suppose that $c_1(T(W_1 \# W_2)) = 0$ and let σ be a non-vanishing section of $\Lambda^{n+1}_{\mathbb{C}} T(W_1 \# W_2)$ extending σ_1 and σ_2. Then $W_1 \# W_2$ satisfies the hypothesis of Theorem 2.2.5 with the grading of $\mathrm{CH}_*(W_1 \# W_2)$ induced by σ.*

2.2.7. A question

A question which emerged from all those examples is

Question 2.2.23. *If a $2n-1$ dimensional manifold M admits a contact form α such that all periodic Reeb orbits have Conley-Zehnder index at least $n+1$ (dynamically convex), is M diffeomorphic to a sphere?*

3. The methods

3.1. Action functional

The problem (1.2.1) of finding periodic orbits on a fixed energy level can be transformed in finding periodic orbits with fixed period in the whole space. Indeed, solutions (T, γ) of

$$\begin{cases} \dot{x}(t) = X_{H_\beta} \\ x(0) = x(S) \end{cases}$$

come in continuous families parametrized by the energy level E. More precisely, if $\widetilde{\gamma}$ is a solution of

$$\begin{cases} \dot{x}(t) = X_{H_\beta} \\ x(0) = x(1) \end{cases} \tag{3.1.1}$$

then, $\gamma : [0, T] \to \Sigma$ is a closed characteristic with

$$T := E^{\frac{2-\beta}{\beta}}$$
$$\gamma(t) := E^{-\frac{1}{\beta}} \widetilde{\gamma}\left(E^{\frac{2-\beta}{\beta}} t\right).$$

This reduces the fixed energy problem to the fixed period problem.

It is known, since Lagrange, that solutions of problem (3.1.1) correspond to critical points of the **action functional**.

$$\mathscr{A}_{H_\beta} : C^\infty\left(S^1, \mathbb{R}^{2n}\right) \to \mathbb{R}$$

$$\mathscr{A}_{H_\beta}(\gamma) = -\tfrac{1}{2} \int_0^1 J\dot{\gamma}(t) \cdot \gamma(t) dt - \int_0^1 H_\beta\left(\gamma(t)\right) dt.$$

3.1.1. Strategy to find periodic orbits

In view of Conjecture 1.2.2, the goal is to find critical points of the action functional corresponding to geometrically distinct periodic orbits. So far, all the results are proved in two steps. The first one is to find critical points. This is done almost always using some type of Morse theoretic argument (recently, [GG16], also introduce the use of Lusternik–Schnirelmann theory). The second step (for which most of the assumptions in the statements are for) is to distinguish, among the critical points found in step 1, which one originate from iterate of the same orbit and which one correspond to geometrically distinct periodic orbits. Arguments for this second step use mostly combinatorial properties of the Conley-Zehnder index. We won't recall the definition of the Conley-Zehnder index and its properties; we refer to [Gut14a, CZ84, RS93, Lon00] and references therein.

3.1.2. Modifying the Hamiltonian

Since H_β is autonomous, every 1-periodic orbit, γ_{H_β} of X_{H_β}, corresponding to the periodic Reeb orbit γ, gives birth to a S^1-family of 1-periodic orbits of X_{H_β} which is denoted by S_γ. For Morse theoretic arguments, it is easier to have isolated critical points.

We can modify the Hamiltonian H_β, as in [CFHW96], to deform this autonomous Hamiltonian into a time-dependent Hamiltonian H_δ with only non degenerate 1-periodic orbits. The Hamiltonian $H_\delta(\theta,p)$ will coincide with $H_\beta(p)$ outside a neighbourhood of the image of the non-constant 1-periodic orbits of X_H. We proceed as follows:
We choose a perfect Morse function on the circle, $\check{f}: S^1 \to \mathbb{R}$.
For each 1-periodic orbit γ_{H_β} of X_{H_β}, we consider the integer $l_{\gamma_{H_\beta}}$ so that γ_{H_β} is a $l_{\gamma_{H_\beta}}$-fold cover of a simple periodic orbit:

$$l_{\gamma_{H_\beta}} := \max\{k \in \mathbb{N} \,|\, \gamma_{H_\beta}(\theta + \tfrac{1}{k}) = \gamma_{H_\beta}(\theta) \ \ \forall \theta \in S^1\}.$$

This number $l_{\gamma_{H_\beta}}$ is constant on the S^1-family of 1-periodic orbits of X_H corresponding to the periodic Reeb orbit γ. We set $l_\gamma = l_{\gamma_{H_\beta}} = \frac{1}{T}$ where T is the period of γ.
We choose a symplectic trivialization $\psi := (\psi_1,\psi_2) : U_\gamma \to V \subset S^1 \times \mathbb{R}^{2n-1}$ between open neighborhoods $U_\gamma \subset \partial W \times \mathbb{R}^+ \subset \widehat{W}$ of the image of γ_{H_β} and V of $S^1 \times \{0\}$ such that $\psi_1(\gamma_{H_\beta}(\theta)) = l_\gamma \theta$. Here $S^1 \times \mathbb{R}^{2n-1}$ is endowed with the standard symplectic form. Let $\check{g}: S^1 \times \mathbb{R}^{2n-1} \to [0,1]$ be a smooth cutoff function supported in a small neighborhood of $S^1 \times \{0\}$ such that $\check{g}_{|_{S^1\times\{0\}}} \equiv 1$.
We denote by $\check{f}_\gamma$ the function defined on S_γ by $\check{f} \circ \psi_{1|_{S_\gamma}}$.

For $\delta > 0$ and $(\theta, p, \rho) \in S^1 \times U_\gamma$, we define

$$H_\delta(\theta,p,\rho) := h(\rho) + \delta \check{g}\big(\psi(p,\rho)\big)\check{f}\big(\psi_1(p,\rho) - l_\gamma\theta\big). \tag{3.1.2}$$

The Hamiltonian H_δ coincides with H_β outside the open sets $S^1 \times U_\gamma$.

Lemma 3.1.1 ([CFHW96, BO09]). *The 1-periodic obits of H_δ, for δ small enough, are either constant orbits (the same as those of H_β) or nonconstant orbits which are non degenerate and form pairs $(\hat{\gamma},\check{\gamma})$ which coincide with the orbits in S_γ starting at the minimum and the maximum of $\check{f}_\gamma$ respectively, for each Reeb orbit γ such that S_γ appears in the 1-periodic orbits of H_β. Their Conley-Zehnder index is given by $\mu_{CZ}(\hat{\gamma}) = \mu_{CZ}(\gamma) - 1$ and $\mu_{CZ}(\check{\gamma}) = \mu_{CZ}(\gamma)$.*

3.1.3. Reformulation of the functional

We can reformulate problem (3.1.1) as finding 1-periodic orbits of H_δ. More generally, Let $H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth time-periodic Hamiltonian on $\mathbb{R}^{2n}$. The 1-periodic orbits of X_H are the critical points of the **action functional**

$$\mathscr{A}_H : C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$$

$$\mathscr{A}_H(\gamma) := -\tfrac{1}{2}\int_0^1 J\dot{\gamma}(t)\cdot\gamma(t)dt - \int_0^1 H\big(t,\gamma(t)\big)dt \tag{3.1.3}$$

$$= -\int_0^1 \gamma^\star\lambda_0 - \int_0^1 H\big(t,\gamma(t)\big)dt. \tag{3.1.4}$$

The idea to approach Question 1.1.2 is to have a homology for this action functional and mimicking the Morse inequalities. The action functional is not bounded from below nor from above. To build a homology of this functional, three ideas have been developped.

1. Build an infinite dimensional version of Morse homology (Floer / symplectic homology) (§3.2).

2. Modify the functional to a new functional (Clarke-Ekeland dual) where one can apply finite dimensional Morse homology (§3.3).

3. Do a relative homology so that all intersections of stable and unstable manifolds are finite dimensional (§3.4)

3.2. Equivariant symplectic homology

(Positive) symplectic homology was developed by Viterbo [Vit99], using works of Cieliebak, Floer, and Hofer [FH94, CFH95]. The S^1-equivariant version of (positive) symplectic homology was originally defined by Viterbo [Vit99], and an alternate definition using family Floer homology was given by Bourgeois-Oancea [BO16, §2.2], following a suggestion of Seidel [Sei08]. We will use the family Floer homology definition here which is often amenable to computations. We follow the treatment in [Gut17], with some minor tweaks which do not affect the results.

Let (X, λ) be a Liouville domain, so that X is a compact smooth manifold with boundary and $\lambda \in \Omega^1(X)$ has the properties that $d\lambda$ is non-degenerate and that $\lambda|_{\partial X}$ is a contact form. We say that (X, λ) is *non-degenerate* if the linearized return map of the Reeb flow at each closed Reeb orbit on ∂X, acting on the contact hyperplane $\ker \lambda$, does not have 1 as an eigenvalue. We will also assume that the first Chern class of TX vanishes on $\pi_2(X)$.

In this situation, for each $L \in \mathbb{R}$ we have an L-filtered positive S^1-equivariant symplectic homology, $SH^{S^1,+,L}(X, \lambda)$, which will be defined properly in §3.2.3.3. To simplify notation, we often denote $SH^{S^1,+,L}(X, \lambda)$ by $CH^L(X, \lambda)$ below[1]. These are $\mathbb{Q}$-vector[2] spaces that come equipped with maps $\iota_{L_1,L_2} \colon CH^{L_1}(X, \lambda) \to CH^{L_2}(X, \lambda)$ for $L_1 \leqslant L_2$ such that $\iota_{L,L}$ is the identity and $\iota_{L_2,L_3} \circ \iota_{L_1,L_2} = \iota_{L_1,L_3}$.[3] The assumption on $c_1(TX)$ implies that the $CH^L(X, \lambda)$ are $\mathbb{Z}$-graded. The (unfiltered) positive S^1-equivariant symplectic homology of (X, λ) is $CH(X, \lambda) = \varinjlim_L CH^L(X, \lambda)$ where the direct limit is constructed using the maps ι_{L_1,L_2}.

Proposition 3.2.1 ([Gut17, GH18, GU19]). *The positive S^1-equivariant symplectic homology $CH(X, \lambda)$ has the following properties:*

(Free homotopy classes) *$CH(X, \lambda)$ has a direct sum decomposition*

$$CH(X, \lambda) = \bigoplus_{\Gamma} CH(X, \lambda, \Gamma)$$

where Γ ranges over free homotopy classes of loops in X. We let $CH(X, \lambda, 0)$ denote the summand corresponding to contractible loops in X.

[1] The reason for this notation is that positive S^1-equivariant symplectic homology can be regarded as a substitute for linearized contact homology, which can be defined without transversality difficulties [BO16, §3.2].

[2] It is also possible to define positive S^1-equivariant symplectic homology with integer coefficients. However the torsion in the latter is not relevant to the applications explained here, and it will simplify our discussion to discard it.

[3] Warning: In [GH18] the map that we denote by ι_{L_1,L_2} is denoted by ι_{L_2,L_1}.

(Action filtration) *For each $L \in \mathbb{R}$, there is a $\mathbb{Q}$-module $CH^L(X,\lambda,\Gamma)$ which is an invariant of (X,λ,Γ). If $L_1 < L_2$, then there is a well-defined map*

$$\iota_{L_1,L_2} : CH^{L_1}(X,\lambda,\Gamma) \longrightarrow CH^{L_2}(X,\lambda,\Gamma). \tag{3.2.1}$$

These maps form a directed system, and we have the direct limit

$$\lim_{L\to\infty} CH^L(X,\lambda,\Gamma) = CH(X,\lambda,\Gamma).$$

We denote the resulting map $CH^L(X,\lambda,\Gamma) \to CH(X,\lambda,\Gamma)$ by ι_L. We write $CH^L(X,\lambda) = \bigoplus_\Gamma CH^L(X,\lambda,\Gamma)$.

(U map) *There is a distinguished map*

$$U : CH(X,\lambda,\Gamma) \longrightarrow CH(X,\lambda,\Gamma),$$

which respects the action filtration in the following sense: For each $L \in \mathbb{R}$ there is a map

$$U_L : CH^L(X,\lambda,\Gamma) \longrightarrow CH^L(X,\lambda,\Gamma).$$

If $L_1 < L_2$ then $U_{L_2} \circ \iota_{L_1,L_2} = \iota_{L_1,L_2} \circ U_{L_1}$. The map U is the direct limit of the maps U_L, i.e.

$$\iota_L \circ U_L = U \circ \iota_L. \tag{3.2.2}$$

(Reeb Orbits) *Assume as above that (X,λ) is a non-degenerate Liouville domain with $c_1(TX)|_{\pi_2(X)} = 0$. There is an $\mathbb{R}$-filtered chain complex $\big(CC_*(X,\lambda),\partial\big)$, freely generated over $\mathbb{Q}$ by the good[4] Reeb orbits of $\lambda|_{\partial X}$ with the generator corresponding to a Reeb orbit γ having filtration level equal to the action $\int_\gamma \lambda$ and grading equal to the Conley-Zehnder index of γ, such that for each $k \in \mathbb{Z}$ and $L \in \mathbb{R}$ the space $CH^L_k(X,\lambda)$ is the kth homology of the subcomplex $CC^L_*(X,\lambda)$ of $CC_*(X,\lambda)$ consisting of elements with filtration level at most L, and such that for $L_1 \leqslant L_2$ the image of the map $\iota_{L_1,L_2}\colon CH^{L_1}_k(X,\lambda) \to CH^{L_2}_k(X,\lambda)$ is isomorphic to the image of the inclusion-induced map $H_k\big(CC^{L_1}_*(X,\lambda)\big) \to H_k\big(CC^{L_2}_*(X,\lambda)\big)$.*

Moreover, the boundary operator ∂ on $CC_*(X,\lambda)$ strictly decreases filtration, in the sense that if $x \in CC^L_*(X,\lambda)$ then there is $\epsilon > 0$ such that $\partial x \in CC^{L-\epsilon}_*(X,\lambda)$.

(δ map) *There is a distinguished map*

$$\delta : CH(X,\lambda,\Gamma) \longrightarrow H_*(X,\partial X;\mathbb{Q}) \otimes H_*(BS^1;\mathbb{Q})$$

which vanishes whenever $\Gamma \neq 0$.

(Scaling) *If r is a positive real number, then there are canonical isomorphisms*

$$CH(X,\lambda,\Gamma) \xrightarrow{\simeq} CH(X,r\lambda,\Gamma),$$
$$CH^L(X,\lambda,\Gamma) \xrightarrow{\simeq} CH^{rL}(X,r\lambda,\Gamma)$$

which commute with all of the above maps.

[4]Recall that a Reeb orbit γ is bad if it is an even degree multiple cover of another Reeb orbit γ' such that the Conley-Zehnder indices of γ and γ' have opposite parity. Otherwise, γ is good.

(Star-Shaped Domains) *If X is a nice star-shaped domain in $\mathbb{R}^{2n}$ and λ_0 is the restriction of the standard Liouville form $\lambda_0 = \frac{1}{2}\sum_{i=1}^{n}(x_i\, dy_i - y_i\, dx_i)$, then:*

(i) *$CH(X, \lambda_0)$ and $CH^L(X, \lambda_0)$ have canonical $\mathbb{Z}$ gradings. With respect to this grading, we have*

$$CH_*(X, \lambda_0) \simeq \begin{cases} \mathbb{Q}, & \text{if } * \in n+1+2\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.3}$$

(ii) *The map δ sends a generator of $CH_{n-1+2k}(X, \lambda_0)$ to a generator of $H_{2n}(X, \partial X; \mathbb{Q})$ tensor a generator of $H_{2k-2}(BS^1; \mathbb{Q})$.*

(iii) *The U map has degree -2 and is an isomorphism*

$$CH_*(X, \lambda_0) \xrightarrow{\simeq} CH_{*-2}(X, \lambda_0),$$

except when $* = n+1$.

(iv) *If $\lambda_0|_{\partial X}$ is nondegenerate and has no Reeb orbit γ with $\mathcal{A}(\gamma) \in (L_1, L_2]$ and $\mathrm{CZ}(\gamma) = n-1+2k$, then the map*

$$\imath_{L_2,L_1} : CH^{L_1}_{n-1+2k}(X, \lambda_0) \to CH^{L_2}_{n-1+2k}(X, \lambda_0)$$

is surjective.

Now suppose that (X', λ') is another nondegenerate Liouville domain and $\varphi : (X, \lambda) \to (X', \lambda')$ is a generalized Liouville embedding (see Definition 6.3.4) with $\varphi(X) \subset \mathrm{int}(X')$. One can then define a *transfer morphism*

$$\Phi : CH(X', \lambda') \longrightarrow CH(X, \lambda),$$

Proposition 3.2.2 ([Gut17, GH18]). *The transfer morphism Φ has the following properties:*

(Action) *Φ respects the action filtration in the following sense: For each $L \in \mathbb{R}$ there are distinguished maps*

$$\Phi^L : CH^L(X', \lambda') \longrightarrow CH^L(X, \lambda)$$

such that if $L_1 < L_2$ then

$$\Phi^{L_2} \circ \imath_{L_2,L_1} = \imath_{L_2,L_1} \circ \Phi^{L_1}, \tag{3.2.4}$$

and Φ is the direct limit of the maps Φ^L, i.e.

$$\imath_L \circ \Phi^L = \Phi \circ \imath_L. \tag{3.2.5}$$

(Functoriality) *The transfer map is functorial in the sense that if (X_1, λ_1), (X_2, λ_2), and (X_3, λ_3) are Liouville domains domains and if $\phi : X_1 \hookrightarrow X_2$ and $\psi : X_2 \hookrightarrow X_3$ are either generalized Liouville embeddings or isomorphisms of Liouville domains, then the following diagram is commutative:*

$$\begin{tikzcd} CH^L(X_3, \lambda_3) \arrow[r, "\Phi^L_\psi"] \arrow[rr, bend right, "\Phi^L_{\psi\circ\phi}"'] & CH^L(X_2, \lambda_2) \arrow[r, "\Phi^L_\phi"] & CH^L(X_1, \lambda_1). \end{tikzcd} \tag{3.2.6}$$

(Commutativity with U) *For each $L \in \mathbb{R}$, the diagram*

$$\begin{array}{ccc} CH^L(X',\lambda') & \xrightarrow{\Phi^L} & CH^L(X,\lambda) \\ \big\downarrow U^L & & \big\downarrow U^L \\ CH^L(X',\lambda') & \xrightarrow{\Phi^L} & CH^L(X,\lambda) \end{array} \tag{3.2.7}$$

commutes.

(Commutativity with δ) *The diagram*

$$\begin{array}{ccc} CH(X',\lambda') & \xrightarrow{\Phi} & CH(X,\lambda) \\ \big\downarrow \delta & & \big\downarrow \delta \\ H_*(X',\partial X';\mathbb{Q})\otimes H_*(BS^1;\mathbb{Q}) & \xrightarrow{\rho\otimes 1} & H_*(X,\partial X;\mathbb{Q})\otimes H_*(BS^1;\mathbb{Q}) \end{array} \tag{3.2.8}$$

commutes. Here $\rho : H_*(X',\partial X';\mathbb{Q}) \to H_*(X,\partial X;\mathbb{Q})$ denotes the composition

$$H_*(X',\partial X';\mathbb{Q}) \longrightarrow H_*(X',X'\backslash\varphi(\mathrm{int}(X));\mathbb{Q}) \xrightarrow{\simeq} H_*(\varphi(X),\varphi(\partial X);\mathbb{Q}) = H_*(X,\partial X;\mathbb{Q})$$

where the first map is the map on relative homology induced by the triple $(X',X'\backslash\varphi(\mathrm{int}(X)),\partial X')$, and the second map is excision.

3.2.1. Symplectic homology

Let (X,λ) be a Liouville domain with boundary Y. Let R_λ denote the Reeb vector field associated to λ on Y. Below, let $\mathrm{Spec}(Y,\lambda)$ denote the set of periods of Reeb orbits, and let $\epsilon = \frac{1}{2}\min \mathrm{Spec}(Y,\lambda)$.

Recall that the completion $(\widehat{X},\widehat{\lambda})$ of (X,λ) is defined by

$$\widehat{X} := X \cup \big([0,\infty)\times Y\big) \quad \text{and} \quad \widehat{\lambda} := \begin{cases} \lambda & \text{on } X, \\ e^\rho \lambda|_Y & \text{on } [0,\infty)\times Y \end{cases}$$

where ρ denotes the $[0,\infty)$ coordinate. Write $\widehat{\omega} = d\widehat{\lambda}$. Consider a 1-periodic Hamiltonian on $\widehat{X}$, i.e. a smooth function

$$H : S^1 \times \widehat{X} \longrightarrow \mathbb{R}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Such a function H determines a vector field X_H^θ on $\widehat{X}$ for each $\theta \in S^1$, defined by $\widehat{\omega}(X_H^\theta,\cdot) = dH(\theta,\cdot)$. Let $\mathscr{P}(H)$ denote the set of 1-periodic orbits of X_H, i.e. smooth maps $\gamma : S^1 \to \widehat{X}$ satisfying the equation $\gamma'(\theta) = X_H^\theta\big(\gamma(\theta)\big)$.

Definition 3.2.3. *An* ***admissible Hamiltonian*** *is a smooth function $H : S^1 \times \widehat{X} \to \mathbb{R}$ satisfying the following conditions:*

(1) *The restriction of H to $S^1 \times X$ is negative, autonomous (i.e. S^1-independent), and C^2-small (so that there are no non-constant 1-periodic orbits). Furthermore,*

$$H > -\epsilon \tag{3.2.9}$$

on $S^1 \times X$.

(2) *There exists* $\rho_0 \geqslant 0$ *such that on* $S^1 \times [\rho_0, \infty) \times Y$ *we have*

$$H(\theta, \rho, y) = \beta e^{\rho} + \beta' \tag{3.2.10}$$

with $0 < \beta \notin \operatorname{Spec}(Y, \lambda)$ *and* $\beta' \in \mathbb{R}$*. The constant* β *is called the* ***limiting slope*** *of* H*.*

(3) *There exists a small, strictly convex, increasing function* $h : [1, e^{\rho_0}] \to \mathbb{R}$ *such that on* $S^1 \times [0, \rho_0] \times Y$*, the function* H *is* C^2*-close to the function sending* $(\theta, \rho, x) \mapsto h(e^{\rho})$*. The precise sense of "small" and "close" that we need here is explained in Remarks 3.2.4 and 3.2.8.*

(4) *The Hamiltonian* H *is nondegenerate, i.e. all* 1*-periodic orbits of* X_H *are nondegenerate.*

We denote the set of admissible Hamiltonians by $\mathscr{H}_{std}$*.*

Remark 3.2.4. *Condition (1) implies that the only* 1*-periodic orbits of* X_H *in* X *are constants; they correspond to critical points of* H*.*

The significance of condition (2) is as follows. On $S^1 \times [0, \infty) \times Y$*, for a Hamiltonian of the form* $H_1(\theta, \rho, y) = h_1(e^{\rho})$*, we have*

$$X^{\theta}_{H_1}(\rho, y) = -h_1'(e^{\rho}) R_{\lambda}(y).$$

Hence for such a Hamitonian H_1 *with* h_1 *increasing, a* 1*-periodic orbit of* X_{H_1} *maps to a level* $\{\rho\} \times Y$*, and the image of its projection to* Y *is the image of a (not necessarily simple) periodic Reeb orbit of period* $h_1'(e^{\rho})$*. In particular, condition (2) implies that there is no* 1*-periodic orbit of* X_H *in* $[\rho_0, \infty) \times Y$*.*

Condition (3) ensures that for any non-constant 1*-periodic orbit* γ_H *of* X_H*, there exists a (not necessarily simple) periodic Reeb orbit* γ *of period* $T < \beta$ *such that the image of* γ_H *is close to the image of* γ *in* $\{\rho\} \times Y$ *where* $T = h'(e^{\rho})$*.*

Definition 3.2.5. *An* S^1*-family of almost complex structures* $J : S^1 \to End(T\widehat{X})$ *is* ***admissible*** *if it satisfies the following conditions:*

- J^{θ} *is* $\widehat{\omega}$*-compatible for each* $\theta \in S^1$*.*
- *There exists* $\rho_1 \geqslant 0$ *such that on* $[\rho_1, \infty) \times Y$*, the almost complex structure* J^{θ} *does not depend on* θ*, is invariant under translation of* ρ*, sends* ξ *to itself compatibly with* $d\lambda$*, and satisfies*

$$J^{\theta}(\partial_{\rho}) = R_{\lambda}. \tag{3.2.11}$$

We denote the set of all admissible J *by* $\mathscr{J}$*.*

Given $J \in \mathscr{J}$, and $\gamma_-, \gamma_+ \in \mathscr{P}(H)$, let $\widetilde{\widehat{M}}(\gamma_-, \gamma_+; J)$ denote the set of maps

$$u : \mathbb{R} \times S^1 \longrightarrow \widehat{X}$$

satisfying Floer's equation

$$\frac{\partial u}{\partial s}(s, \theta) + J^{\theta}\big(u(s, \theta)\big)\left(\frac{\partial u}{\partial \theta}(s, \theta) - X^{\theta}_H\big(u(s, \theta)\big)\right) = 0 \tag{3.2.12}$$

as well as the asymptotic conditions

$$\lim_{s \to \pm\infty} u(s, \cdot) = \gamma_{\pm}.$$

If J is generic and $u \in \widetilde{\widetilde{M}}(\gamma_-, \gamma_+; J)$, then $\widetilde{\widetilde{M}}(\gamma_-, \gamma_+; J)$ is a manifold near u whose dimension is the Fredholm index of u defined by

$$\operatorname{ind}(u) = \mathrm{CZ}_\tau(\gamma_+) - \mathrm{CZ}_\tau(\gamma_-).$$

Here CZ_τ denotes the Conley-Zehnder index computed using trivializations τ of $\gamma_\pm^\star T\widehat{X}$ that extend to a trivialization of $u^\star T\widehat{X}$. Note that $\mathbb{R}$ acts on $\widetilde{\widetilde{M}}(\gamma_-, \gamma_+; J)$ by translation of the domain; we denote the quotient by $\widetilde{M}(\gamma_-, \gamma_+; J)$.

Definition 3.2.6. *Let $H \in \mathscr{H}_{\mathrm{std}}$, and let $J \in \mathscr{J}$ be generic. Define the Floer chain complex $(CF(H,J), \partial)$ as follows. The chain module $CF(H,J)$ is the free $\mathbb{Q}$-module*[5] *generated by the set of 1-periodic orbits $\mathscr{P}(H)$. If $\gamma_-, \gamma_+ \in \mathscr{P}(H)$, then the coefficient of γ_+ in $\partial\gamma_-$ is obtained by counting Fredholm index 1 points in $\widetilde{M}(\gamma_-, \gamma_+; J)$ with signs determined by a system of coherent orientations as in [FH93]. (The chain complexes for different choices of coherent orientations are canonically isomorphic.)*

Let $HF(H,J)$ denote the homology of the chain complex $(CF(H,J), \partial)$. Given H, the homologies for different choices of generic J are canonically isomorphic to each other, so we can denote this homology simply by $HF(H)$.

The construction of the above canonical isomorphisms is a special case of the following more general construction. Given two admissible Hamiltonians $H_1, H_2 \in \mathscr{H}_{\mathrm{std}}$, write $H_1 \leqslant H_2$ if $H_1(\theta, x) \leqslant H_2(\theta, x)$ for all $(\theta, x) \in S^1 \times \widehat{X}$. In this situation, one defines a *continuation morphism* $HF(H_1) \to HF(H_2)$ as follows; cf. [Gut17, Thm. 4.5] and the references therein. Choose generic $J_1, J_2 \in \mathscr{J}$ so that the chain complexes $CF(H_i, J_i)$ are defined for $i = 1, 2$. Choose a generic homotopy $\{(H_s, J_s)\}_{s \in \mathbb{R}}$ such that H_s satisfies equation (3.2.10) for some β, β' depending on s; $J_s \in \mathscr{J}$ for each $s \in \mathbb{R}$; $\partial_s H_s \geqslant 0$; $(H_s, J_s) = (H_1, J_1)$ for $s << 0$; and $(H_s, J_s) = (H_2, J_2)$ for $s >> 0$. One then defines a chain map $CF(H_1, J_1) \to CF(H_2, J_2)$ as a signed count of Fredholm index 0 maps $u : \mathbb{R} \times S^1 \to \widehat{X}$ satisfying the equation

$$\frac{\partial u}{\partial s} + J_s^\theta \circ u \left(\frac{\partial u}{\partial \theta} - X_{H_s}^\theta \circ u \right) = 0 \tag{3.2.13}$$

and the asymptotic conditions $\lim_{s \to -\infty} u(s, \cdot) = \gamma_1$ and $\lim_{s \to \infty} u(s, \cdot) = \gamma_2$. The induced map on homology gives a well-defined map $HF(H_1) \to HF(H_2)$. If $H_2 \leqslant H_3$, then the continuation map $HF(H_1) \to HF(H_3)$ is the composition of the continuation maps $HF(H_1) \to HF(H_2)$ and $HF(H_2) \to HF(H_3)$.

Definition 3.2.7. *We define the* symplectic homology *of (X, λ) to be the direct limit*

$$SH(X, \lambda) := \varinjlim_{H \in \mathscr{H}_{\mathrm{adm}}} HF(H)$$

with respect to the partial order $\leqslant$ and continuation maps defined above.

[5] It is also possible to use $\mathbb{Z}$ coefficients here, but we will use $\mathbb{Q}$ coefficients in order to later establish the Reeb Orbits property in Proposition 3.2.1, which leads to the Reeb Orbits property of the capacities c_k. In special cases when the Conley-Zehnder index of a 1-periodic orbit is unambiguously defined, for example when all 1-periodic orbits are contractible and $c_1(TX)|_{\pi_2(X)} = 0$, the chain complex is graded by minus the Conley-Zehnder index.

3.2.2. Positive symplectic homology

Positive symplectic homology is a modification of symplectic homology in which constant 1-periodic orbits are discarded.

To explain this, let $H : S^1 \times \widehat{X} \to \mathbb{R}$ be a Hamiltonian in $\mathscr{H}_{\mathrm{std}}$. The *Hamiltonian action* functional $\mathscr{A}_H : C^\infty(S^1, \widehat{X}) \to \mathbb{R}$ is defined by

$$\mathscr{A}_H(\gamma) := -\int_{S^1} \gamma^\star \widehat{\lambda} - \int_{S^1} H\big(\theta, \gamma(\theta)\big)\, d\theta.$$

If $J \in \mathscr{J}$, then the differential on the chain complex $(CF(H,J), \partial)$ decreases the Hamiltonian action $\mathscr{A}_H$. As a result, for any $L \in \mathbb{R}$, we have a subcomplex $CF^{\leqslant L}(H,J)$ of $CF(H,J)$, generated by the 1-periodic orbits with Hamiltonian action less than or equal to L.

To see what this subcomplex can look like, note that the 1-periodic orbits of $H \in \mathscr{H}_{\mathrm{std}}$ fall into two classes: (i) constant orbits corresponding to critical points in X, and (ii) non-constant orbits contained in $[0, \rho_0] \times Y$.

If x is a critical point of H on X, then the action of the corresponding constant orbit is equal to $-H(x)$. By (3.2.9), this is less than ϵ.

By Remark 3.2.4, a non-constant 1-periodic orbit of X_H is close to a 1-periodic orbit of $-h'(e^\rho)R_\lambda$ located in $\{\rho\} \times Y$ for $\rho \in [0, \rho_0]$ with $h'(e^\rho) \in \mathrm{Spec}(Y, \lambda)$. The Hamiltonian action of the latter loop is given by

$$-\int_{S^1} e^\rho \lambda(-h'(e^\rho)R_\lambda)\, d\theta - \int_{S^1} h(e^\rho)\, d\theta = e^\rho h'(e^\rho) - h(e^\rho). \tag{3.2.14}$$

Since h is strictly convex, the right hand side is a strictly increasing function of ρ.

Remark 3.2.8. *In Definition 3.2.3, we assume that h is sufficiently small so that the right hand side of (3.2.14) is close to the period $h'(e^\rho)$, and in particular greater than ϵ. We also assume that H is sufficiently close to $h(e^\rho)$ on $S^1 \times [0, \rho_0] \times Y$ so that the Hamiltonian actions of the 1-periodic orbits are well approximated by the right hand side of (3.2.14), so that:*

(i) *The Hamiltonian action of every 1-periodic orbit of X_H corresponding to a critical point on X is less than ϵ; and the Hamiltonian action of every other 1-periodic orbit is greater than ϵ.*

(ii) *If γ is a Reeb orbit of period $T < \beta$, and if γ' is a 1-periodic orbit of X_H in $[0, \rho_0] \times Y$ associated to γ, then*

$$|\mathscr{A}_H(\gamma') - T| < \min\left\{\beta^{-1}, \tfrac{1}{3}\,\mathrm{gap}(\beta)\right\}.$$

Here $\mathrm{gap}(\beta)$ denotes the minimum difference between two elements of $\mathrm{Spec}(Y, \lambda)$ that are less than β.

We can now define positive symplectic homology.

Definition 3.2.9. *Let (X, λ) be a Liouville domain, let H be a Hamiltonian in $\mathscr{H}_{\mathrm{std}}$, and let $J \in \mathscr{J}$. Consider the quotient complex*

$$CF^+(H,J) := \frac{CF(H,J)}{CF^{\leqslant \epsilon}(H,J)}.$$

The homology of the quotient complex is independent of J, so we can denote this homology by $HF^+(H)$. More generally, if $H_1 \leqslant H_2$, then the chain map used to define the continuation map

$HF(H_1) \to HF(H_2)$ descends to the quotient, since the Hamiltonian action is nonincreasing along a solution of (3.2.13) when the homotopy is nondecreasing. Thus we obtain a well-defined continuation map $HF^+(H_1) \to HF^+(H_2)$ satisfying the same properties as before.

We now define the positive symplectic homology *of (X, λ) to be the direct limit*

$$SH^+(X, \lambda) := \varinjlim_{H \in \mathscr{H}_{std}} HF^+(H).$$

Positive symplectic homology can sometimes be better understood using certain special admissible Hamiltonians obtained as follows.

Definition 3.2.10. *[BO09] Let (X, λ) be a Liouville domain. An* admissible Morse-Bott Hamiltonian *is an autonomous Hamiltonian $H : \widehat{X} \to \mathbb{R}$ such that:*

(1) *The restriction of H to X is a Morse function which is negative and C^2-small (so that the Hamiltonian vector field has no non-constant 1-periodic orbits).*

(2) *There exists $\rho_0 \geqslant 0$ such that on $[\rho_0, \infty) \times Y$ we have*

$$H(\rho, x) = \beta e^{\rho} + \beta'$$

with $0 < \beta \notin \mathrm{Spec}(Y, \lambda)$ and $\beta' \in \mathbb{R}$.

(3) *On $[0, \rho_0) \times Y$ we have*

$$H(\rho, x) = h(e^{\rho})$$

where h is as in Definition 3.2.3, and moreover $h'' - h' > 0$.

We denote the set of admissible Morse-Bott Hamiltonians by $\mathscr{H}_{\mathrm{MB}}$.

Given $H \in \mathscr{H}_{\mathrm{MB}}$, each 1-periodic orbit of X_H is either: (i) a constant orbit corresponding to a critical point of H in X, or (ii) a non-constant 1-periodic orbit, with image in $\{\rho\} \times Y$ for $\rho \in (0, \rho_0)$, whose projection to Y has the same image as a Reeb orbit of period $e^{\rho} h'(\rho)$. Since H is autonomous, every Reeb orbit γ with period less than β gives rise to an S^1 family of 1-periodic orbits of X_H, which we denote by S_γ.

An admissible Morse-Bott Hamiltonian as in Definition 3.2.10 can be deformed into an admissible Hamiltonian as in Definition 3.2.3, which will be time-dependent and have nondegenerate 1-periodic orbits:

Lemma 3.2.11. *([CFHW96, Prop. 2.2] and [BO09, Lem. 3.4]) An admissible Morse-Bott Hamiltonian H can be perturbed to an admissible Hamiltonian H' whose 1-periodic orbits consist of the following:*

(i) *Constant orbits at the critical points of H.*

(ii) *For each Reeb orbit γ with period less than β, two nondegenerate orbits $\hat{\gamma}$ and $\check{\gamma}$. Given a trivialization τ of $\xi|\gamma$, their Conley-Zehnder indices are given by $-\mathrm{CZ}_\tau(\hat{\gamma}) = \mathrm{CZ}_\tau(\gamma) + 1$ and $-\mathrm{CZ}_\tau(\check{\gamma}) = \mathrm{CZ}_\tau(\gamma)$.*

Remark 3.2.12. *The references [CFHW96] and [BO09] use the notation $\gamma_{\min}$ instead of $\hat{\gamma}$, and γ_{Max} instead of $\check{\gamma}$. The motivation is that these orbits are distinguished in their S^1-family as critical points of a perfect Morse function on S^1.*

3.2.3. S^1-equivariant symplectic homology

3.2.3.1. S^1-equivariant homology

Let X be a topological space endowed with an S^1-action. If the S^1-action is free, $X/_{S^1}$ is a topological space. The aim of S^1-equivariant homology is to build on the space X a homology which coincides, when the action is free, with the singular homology of the quotient. One considers the universal principal S^1-bundle $ES^1 \to BS^1$.The diagonal action on $X \times ES^1$ is free and one denotes by $X \times_{S^1} ES^1$ the quotient $(X \times ES^1)/_{S^1}$.

Definition 3.2.13 (Borel)**.** *Let X be a topological space endowed with an S^1-action. The S^1*-equivariant homology of X *with $\mathbb{Z}$-coefficients is*

$$H_*^{S^1}(X) := H_*(X \times_{S^1} ES^1, \mathbb{Z}).$$

An axiomatic definition of equivariant homology was stated later by Basu, [Bas], based on the following Proposition:

Proposition 3.2.14. *The S^1-equivariant homology with $\mathbb{Z}$-coefficients is a functor $H_*^{S^1}$ from the category of S^1-spaces and S^1-maps to the category of abelian groups and homomorphisms. Let X be a topological space endowed with a S^1-action, $H_*^{S^1}$ associates to X a sequence of abelian groups: $H_i^{S^1}(X,\mathbb{Z}), i \geqslant 0$. Let $f : X \to Y$ be an S^1-equivariant map between topological spaces endowed with an S^1-action. It induces homomorphisms $f_i^{S^1} : H_i^{S^1}(X,\mathbb{Z}) \to H_i^{S^1}(Y,\mathbb{Z})$. The functor $H_*^{S^1}$ satisfy the two following conditions:*

1. *If the S^1-action on X is free, then $H_*^{S^1}(X,\mathbb{Z}) = H_*(X/_{S^1},\mathbb{Z})$ (the singular homology of $X/_{S^1}$).*
2. *If $f : X \to Y$ induces an isomorphism $f_* : H_*(X,\mathbb{Z}) \to H_*(Y,\mathbb{Z})$, then it also induces an isomorphism $f_*^{S^1} : H_*^{S^1}(X,\mathbb{Z}) \to H_*^{S^1}(Y,\mathbb{Z})$.*

Any functor satisfying the two conditions of Proposition 3.2.14 is given by Definition 3.2.13. Indeed, the projection $pr_1 : X \times ES^1 \to X : (x,e) \mapsto x$ is an S^1-equivariant map which induces an isomorphism

$$pr_{1*} : H_*(X \times ES^1, \mathbb{Z}) \to H_*(X,\mathbb{Z})$$

since ES^1 is contractible. By 2, pr_{1*} induces an isomorphism

$$pr_{1*}^{S^1} : H_*^{S^1}(X \times ES^1, \mathbb{Z}) \to H_*^{S^1}(X,\mathbb{Z}).$$

Condition 1 then implies

$$H_*^{S^1}(X,\mathbb{Z}) \cong H_*(X \times_{S^1} ES^1, \mathbb{Z}).$$

3.2.3.2. S^1-equivariant symplectic homology

Let (X,λ) be a Liouville domain with boundary Y. We now review how to define the S^1-equivariant symplectic homology $SH^{S^1}(X,\lambda)$.

The S^1-equivariant symplectic homology $SH^{S^1}(X,\lambda)$ is defined as a limit as $N \to \infty$ of homologies $SH^{S^1,N}(X,\lambda)$, where N is a nonnegative integer. To define the latter, fix the perfect Morse function $f_N : \mathbb{C}P^N \to \mathbb{R}$ defined by

$$f_N\left([w^0 : \ldots : w^n]\right) = \frac{\sum_{j=0}^N j|w^j|^2}{\sum_{j=0}^N |w^j|^2}.$$

Let $\widetilde{f_N} : S^{2N+1} \to \mathbb{R}$ denote the pullback of f_N to S^{2N+1}. We will consider gradient flow lines of $\widetilde{f_N}$ and f_N with respect to the standard metric on S^{2N+1} and the metric that this induces on $\mathbb{C}P^N$.

Remark 3.2.15. *The family of functions f_N has the following two properties which are needed below. We have two isometric inclusions $i_0, i_1 : \mathbb{C}P^N \to \mathbb{C}P^{N+1}$ defined by $i_0([z_0 : \ldots : z_N]) = [z_0 : \ldots : z_N : 0]$ and $i_1([z_0 : \ldots : z_N]) = [0 : z_0 : \ldots : z_N]$. Then:*

(1) The images of i_0 and i_1 are invariant under the gradient flow of f_{N+1}.

(2) We have $f_N = f_{N+1} \circ i_0 = f_{N+1} \circ i_1 + \text{constant}$, so that the gradient flow of f_{N+1} pulls back via i_0 or i_1 to the gradient flow of f_N.

Now choose a "parametrized Hamiltonian"

$$H : S^1 \times \widehat{X} \times S^{2N+1} \longrightarrow \mathbb{R} \tag{3.2.15}$$

which is S^1-invariant in the sense that

$$H(\theta + \varphi, x, \varphi z) = H(\theta, x, z) \qquad \forall \theta, \varphi \in S^1 = \mathbb{R}/\mathbb{Z},\ x \in \widehat{X},\ z \in S^{2N+1}.$$

Here the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $S^{2N+1} \subset \mathbb{C}^{N+1}$ is defined by $\varphi \cdot z = e^{2\pi i \varphi} z$.

Definition 3.2.16. *A parametrized Hamiltonian H as above is* admissible *if:*

(i) For each $z \in S^{2N+1}$, the Hamiltonian

$$H_z = H(\cdot, \cdot, z) : S^1 \times \widehat{X} \longrightarrow \mathbb{R}$$

satisfies conditions (1), (2), and (3) in Definition 3.2.3, with β and β' independent of z.

(ii) If z is a critical point of $\widetilde{f}_N$, then the 1-periodic orbits of H_z are nondegenerate.

(iii) H is nondecreasing along downward gradient flow lines of $\widetilde{f}_N$.

Let $\mathscr{P}^{S^1}(\tilde{f}_N, H)$ denote the set of pairs (z, γ), where $z \in S^{2N+1}$ is a critical point of $\tilde{f}_N$, and γ is a 1-periodic orbit of the Hamitonian H_z. Note that S^1 acts freely on the set $\mathscr{P}^{S^1}(\tilde{f}_N, H)$ by

$$\varphi \cdot (z, \gamma) = \big(\varphi \cdot z, \gamma(\cdot - \varphi)\big).$$

If $p = (z, \gamma) \in \mathscr{P}^{S^1}(\tilde{f}_N, H)$, let S_p denote the orbit of (z, γ) under this S^1 action.

Next, choose a generic map

$$J : S^1 \times S^{2N+1} \to \mathscr{J}, \quad (\theta, z) \mapsto J_z^\theta, \tag{3.2.16}$$

which is S^1-invariant in the sense that

$$J_{\varphi \cdot z}^{\theta + \varphi} = J_z^\theta$$

for all $\varphi, \theta \in S^1$ and $z \in S^{2N+1}$.

Let $p^- = (z^-, \gamma^-)$ and $p^+ = (z^+, \gamma^+)$ be distinct elements of $\mathscr{P}^{S^1}(\tilde{f}_N, H)$. Define $\widehat{\mathscr{M}}(S_{p^-}, S_{p^+}; J)$ to be the set of pairs (η, u), where $\eta : \mathbb{R} \to S^{2N+1}$ and $u : \mathbb{R} \times S^1 \to \widehat{X}$, satisfying the following equations:

$$\begin{cases} \dot{\eta} + \vec{\nabla} \tilde{f}_N(\eta) = 0, \\ \partial_s u + J_{\eta(s)}^\theta \circ u \big(\partial_\theta u - X_{H_{\eta(s)}^\theta} \circ u\big) = 0, \\ \lim_{s \to \pm\infty} \big(\eta(s), u(s, \cdot)\big) \in S_{p^\pm}. \end{cases} \tag{3.2.17}$$

Here the middle equation is a modification of Floer's equation (3.2.12) which is "parametrized by η". Note that $\mathbb{R}$ acts on the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ by reparametrization: if $\sigma \in \mathbb{R}$, then

$$\sigma \cdot (\eta, u) = \big(\eta(\cdot - \sigma), u(\cdot - \sigma, \cdot)\big).$$

In addition, S^1 acts on the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ as follows: if $\tau \in S^1$, then

$$\tau \cdot (\eta, u) := \big(\tau \cdot \eta, u(\cdot, \cdot - \tau)\big).$$

Let $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ denote the quotient of the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ by these actions of $\mathbb{R}$ and S^1.

If J is generic, then $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ is a manifold near (η, u) of dimension

$$\operatorname{ind}(\eta, u) = (\operatorname{ind}(f_N, z^-) - \mathrm{CZ}_\tau(\gamma^-)) - (\operatorname{ind}(f_N, z^+) - \mathrm{CZ}_\tau(\gamma^+)) - 1.$$

Here $\operatorname{ind}(f_N, z^\pm)$ denotes the Morse index of the critical point $z^\pm$ of f_N, and CZ_τ denotes the Conley-Zehnder index with respect to a trivialization τ of $(\gamma^\pm)^\star T\widehat{X}$ that extends over $u^\star T\widehat{X}$.

Definition 3.2.17. *[BO16, §2.2] Define a chain complex $\left(CF^{S^1,N}(H,J), \partial^{S^1}\right)$ as follows. The chain module $CF^{S^1,N}(H,J)$ is the free $\mathbb{Q}$ module[6] generated by the orbits S_p. If S_{p^-}, S_{p^+} are two such orbits, then the coefficient of S_{p^+} in $\partial^{S^1} S_{p^-}$ is a signed count of elements (η, u) of $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ with* $\operatorname{ind}(\eta, u) = 1$.

We denote the homology of this chain complex by $HF^{S^1,N}(H)$. This does not depend on the choice of J, by the usual continuation argument; one defines continuation chain maps using a modification of (3.2.17) in which the second line is replaced by an "η-parametrized" version of Floer's continuation equation (3.2.13).

We now define a partial order on the set of pairs (N, H), where N is a nonnegative integer and H is an admissible parametrized Hamiltonian (3.2.15), as follows. Let $\widetilde{i}_0 : S^{2N+1} \to S^{2N+3}$ denote the inclusion sending $z \mapsto (z, 0)$. (This lifts the inclusion i_0 defined in Remark 3.2.15.) Then $(N_1, H_1) \leqslant (N_2, H_2)$ if and only if:

- $N_1 \leqslant N_2$, and
- $H_1 \leqslant (\widetilde{i}_0^{\,\star})^{N_2 - N_1} H_2$ pointwise on $S^1 \times \widehat{X} \times S^{2N_1+1}$.

In this case we can define a continuation map $HF^{S^1,N_1}(H_1) \to HF^{S^1,N_2}(H_2)$ using an increasing homotopy from H_1 to $(\widetilde{i}_0^{\,\star})^{N_2 - N_1} H_2$ on $S^1 \times \widehat{X} \times S^{2N_1+1}$.

Definition 3.2.18. *Define the* S^1-equivariant symplectic homology

$$SH_*^{S^1}(X, \lambda) := \varinjlim_{N,H} HF_*^{S^1,N}(H).$$

It is sometimes useful to describe S^1-equivariant symplectic homology in terms of individual Hamiltonians on $S^1 \times \widehat{X}$, rather than S^{2N+1}-families of them, by the following procedure.

Remark 3.2.19. *[Gut14c, §2.1.1] Fix an admissible Hamiltonian $H' : S^1 \times \widehat{X} \to \mathbb{R}$ and a nonnegative integer N. Consider a sequence of admissible parametrized Hamiltonians $\{H_k\}_{k=0,\dots,N}$ as in (3.2.15), where H_k is defined on $S^1 \times \widehat{X} \times S^{2k+1}$, with the following properties:*

[6] It is also possible to define $SH^{S^1,+}$, using $\mathbb{Z}$ coefficients, as with SH.

- *For each $k = 0, \dots, N-1$, the pullbacks $\widetilde{i}_0^\star H_{k+1}$ and $\widetilde{i}_1^\star H_{k+1}$ agree with H_k up to a constant. Here $\widetilde{i}_1 : S^{2k+1} \to S^{2k+3}$ denotes the lift of i_1 sending $z \mapsto (0, z)$.*
- *For each $k = 0, \dots, N$ and each $z \in \mathrm{Crit}(\tilde{f}_k)$, we have*
$$H_k(\theta, x, z) = H'\big(\theta - \phi(z), x\big) + c. \tag{3.2.18}$$
Here c is a constant depending on k and z; and the map $\phi : \mathrm{Crit}(\tilde{f}_k) \to S^1$ sends a critical point $(0, \dots, 0, e^{2\pi i \psi}, 0, \dots, 0) \mapsto \psi$.

Next, choose a sequence of families of almost complex structures $J_k : S^1 \times S^{2k+1} \to \mathscr{J}(\widehat{X})$ for $k = 0, \dots, N$ such that:

- *J_k is generic so that the chain complex $\left(CF^{S^1,k}(H_k, J_k), \partial^{S^1}\right)$ is defined.*
- *$\widetilde{i}_0^\star J_{k+1} = \widetilde{i}_1^\star J_{k+1} = J_k$.*

The chain complex $\left(CF^{S^1,N}(H_N, J_N), \partial^{S^1}\right)$ can now be described as follows. By (3.2.18), we can identify the chain module as

$$CF^{S^1,N}(H_N, J_N) = \mathbb{Q}\{1, u, \dots, u^N\} \otimes_{\mathbb{Q}} CF(H', J_0). \tag{3.2.19}$$

This identification sends a pair (z, γ), where $z \in \mathrm{Crit}(\widetilde{f}_N)$ is a lift of an index $2k$ critical point of f_N and γ is a reparametrization of a 1-periodic orbit γ' of H', to $u^k \otimes \gamma'$.

Since the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\widetilde{i}_1$, the differential has the form

$$\partial^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_i(\gamma) \tag{3.2.20}$$

where the operator φ_i on $CF(H', J_0)$ does not depend on k. In particular, φ_0 is the differential on $CF(H', J_0)$. We can also formally write

$$\partial^{S^1} = \sum_{i=0}^{N} u^{-i} \otimes \varphi_i$$

where it is understood that u^{-i} annihilates terms of the form $u^j \otimes \gamma$ with $i > j$.

The usual continuation arguments show that the homology of this chain complex does not depend on the choice of sequences $\{H_k\}$ and $\{J_k\}$ satisfying the above assumptions. We denote this homology by $HF^{S^1,N}(H')$.

Since in the above construction we assume that the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\widetilde{i}_0$, it follows that when $N_1 \leqslant N_2$ we have a well-defined map $HF^{S^1,N_1}(H') \to HF^{S^1,N_2}(H')$ induced by inclusion of chain complexes.

As before, if $H_1' \leqslant H_2'$, then there is a continuation map $HF^{S^1,N}(H_1') \to HF^{S^1,N}(H_2')$ satisfying the usual properties.

As in [BO16, §2.3], we now have:

Proposition 3.2.20. *The S^1-equivariant homology of (X, λ) is given by*

$$SH_*^{S^1}(X, \lambda) = \varinjlim_{N \in \mathbb{N},\, H' \in \mathscr{H}_{std}} HF^{S^1,N}(H').$$

3.2.3.3. Positive S^1-equivariant symplectic homology

As for symplectic homology, S^1-equivariant symplectic homology also has a positive version in which constant 1-periodic orbits are discarded.

Definition 3.2.21. *Let $H : S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ be an admissible parametrized Hamiltonian. The* parametrized action functional *$\mathscr{A}_H : S^{2N+1} \times C^\infty(S^1, \widehat{X}) \longrightarrow \mathbb{R}$ is defined by*

$$\mathscr{A}_H(z,\gamma) := -\int_\gamma \widehat{\lambda} - \int_{S^1} H(\theta, \gamma(\theta), z)\, d\theta. \tag{3.2.21}$$

Lemma 3.2.22. *If H is an admissible parametrized Hamiltonian, and if J is a generic S^1-invariant family of almost complex structures as in (3.2.16), then the differential ∂^{S^1} on $CF^{S^1,N}(H,J)$ does not increase the parametrized action (3.2.21).*

Proof. Given a solution (η, u) to the equations (3.2.17), one can think of η as fixed and regard u as a solution to an instance of equation (3.2.13), where J_s and H_s in (3.2.13) are determined by η. By condition (iii) in Definition 3.2.16, this instance of (3.2.13) corresponds to a nondecreasing homotopy of Hamiltonians. Consequently, the action is nonincreasing along this solution of (3.2.13) as before. □

It follows from Lemma 3.2.22 that for any $L \in \mathbb{R}$, we have a subcomplex $CF^{S^1,N,\leqslant L}(H,J)$ of $CF^{S^1,N}(H,J)$, spanned by S^1-orbits of pairs (z,γ) where $z \in \mathrm{Crit}(\tilde{f}_N)$ and γ is a 1-periodic orbit of H_z with $\mathscr{A}_H(z,\gamma) \leqslant L$.

As in §3.2.2, if the S^1-orbit of (z,γ) is a generator of $CF^{S^1,N}(H,J)$, then there are two possibilities: (i) γ is a constant orbit corresponding to a critical point of H_z on X, and $\mathscr{A}_H(z,\gamma) < \epsilon$; or (ii) γ is close to a Reeb orbit in $\{\rho\} \times Y$ with period $-h'(e^\rho)$, and $\mathscr{A}_H(z,\gamma)$ is close to this period; in particular $\mathscr{A}_H(z,\gamma) > \epsilon$.

Definition 3.2.23. *Consider the quotient complex*

$$CF^{S^1,N,+}(H,J) := \frac{CF^{S^1,N}(H,J)}{CF^{S^1,N,\leqslant\epsilon}(H,J)}. \tag{3.2.22}$$

As in Definition 3.2.9, the homology of the quotient complex is independent of J, so we can denote this homology by $HF^{S^1,N,+}(H)$; and we have continuation maps $HF^{S^1,N_1,+}(H_1) \to HF^{S^1,N_2,+}(H_2)$ when $(N_1,H_1) \leqslant (N_2,H_2)$. We now define the positive S^1-equivariant symplectic homology *by*

$$SH^{S^1,+}(X,\lambda) := \varinjlim_{N,H} HF^{S^1,N,+}(H). \tag{3.2.23}$$

Returning to the situation of Remark 3.2.19, define $HF^{S^1,N,+}(H')$ to be the homology of the quotient of the chain complex (3.2.19) by the subcomplex spanned by $u^k \otimes \gamma$ where γ is a critical point of H' in X. We then have the following analogue of Proposition 3.2.20:

Proposition 3.2.24. *The positive S^1-equivariant homology of (X,λ) is given by*

$$SH^{S^1,+}(X,\lambda) = \varinjlim_{N \in \mathbb{N},\, H' \in \mathscr{H}_{std}} HF^{S^1,N,+}(H').$$

3.3. Clarke duality

3.3.1. Origin of Clarke duality

The *Legendre tranform* of a function $F \in C^1(\mathbb{R}^N, \mathbb{R})$ is defined by the implicit formula

$$F^*(v) = (v,u) - F(u)$$

$$v = \nabla F(u)$$

when ∇F is invertible. It has the remarkable property that

$$(\nabla F)^{-1} = \nabla F^*.$$

Its geometrical meaning is the following: the tangent hyperplane to the graph of F with normal $[v,1]$ is given by $\{[w,s] \in \mathbb{R}^{N+1} \,|\, s = (w,v) - F^*(v)\}$. Thus the graph of F can be described in a dual way, either as a set of points or as an envelope of tangent hyperplanes.

The *Fenchel transform* extends the Legendre transform to not necessarily smooth convex functions by using affine minorants instead of tangent hyperplanes. To motivate, notice that when F is convex, the function $\tilde{F}_v : u \mapsto (v,u) - F(u)$ is concave and the definition of the Legendre transform just expresses that u is a critical point of $\tilde{F}_v$, and hence the global maximum of $\tilde{F}_v$ is achieved at u. Consequently,

$$F^*(v) = \sup_{w \in \mathbb{R}^n} [(v,w) - F(w)]$$

and the right-hand member of this equality, which is defined as an element of $]-\infty, \infty]$ without the smoothness and invertibility conditions required by the Legendre transforms is, by definition, the Fenchel transform of the convex function F.

In classical Hamiltonian mechanics, if the Lagrangian $L(t,q,r)$ is given, the corresponding Hamiltonian $H = H(t,q,p)$ is the Legendre transform of $L(t,q,\cdot)$, namely

$$H(t,q,p) = (p,q) - L(t,q,r)$$

where r is expressed in terms of (t,q,p) through the relation $p = \partial_r L(t,q,r)$.

Besides this Hamiltonian duality, there is, in the study of Hamiltonian systems, another duality based on the Legendre transform of $H(t,\cdot,\cdot)$. If we write $u = (q,p)$, the Hamiltonian equations can be written in the compact form

$$-J\dot{u}(t) = \nabla H(t,u(t)).$$

Setting $\dot{v} = -J\dot{u}$, so that $u = Jv - c$ where c is a constant, we obtain

$$\dot{v} = \nabla H(t,u) \quad \text{or equivalently} \quad u = \nabla H^*(t,\dot{v})$$

if the Legendre transform $H^*(t,.)$ of $H(t,.)$ exists. Therefore, the Hamiltonian equations expressed in terms of v become $Jv - \nabla H^*(t,\dot{v}) = c$. The integrated Euler-Lagrange equations corresponding to the critical points of the functions χ defined on a suitable space of T-periodic functions is

$$\chi(v) = \int_0^T \tfrac{1}{2}(J\dot{v}(t), v(t)) + H^*(t,\dot{v}(t))\, dt$$

This dual action χ can therefore be used as well as the Hamiltonian action to prove the existence of T-periodic solutions of the Hamiltonian system.

3.3.2. Clarke's dual action functional

Definition 3.3.1. ***Clarke's dual action functional*** *is defined by the formula*

$$\mathscr{A}_H^*(\gamma) := -\tfrac{1}{2}\int_0^1 J\dot{\gamma}(t)\cdot\gamma(t)dt - \int_0^1 H^*\big(t, J\dot{\gamma}(t)\big)dt.$$

The functional $\mathscr{A}_H^*$ is continuously differentiable on the Hilbert space

$$\mathbb{H}_1 := H_1(S^1, \mathbb{R}^{2n})/\mathbb{R}^{2n},$$

where the action of $\mathbb{R}^{2n}$ onto the Sobolev space $H_1(S^1, \mathbb{R}^{2n})$ is given by translations. Rather than working with equivalence classes of curves modulo translations, it is convenient to work with genuine curves by identifying $\mathbb{H}_1$ with the space of closed curves with zero mean:

$$\mathbb{H}_1 = \left\{ x \in H_1(S^1, \mathbb{R}^{2n}) \,\middle|\, \int_{S^1} x(t)dt = 0 \right\}.$$

Let $\Pi : H_1(S^1, \mathbb{R}^{2n}) \to \mathbb{H}_1$ be the quotient projection.

There is a one-to-one correspondence between the critical points of $\mathscr{A}_H$ and $\mathscr{A}_H^*$. More precisely, we have the following result

Lemma 3.3.2 ([Eke90, AK19])**.** *If x is a critical point of $\mathscr{A}_H$, then $\Pi(x)$ is a critical point of $\mathscr{A}_H^*$. Conversely, every critical point x of $\mathscr{A}_H^*$ is smooth and there exists a unique vector $v_0 \in \mathbb{R}^{2n}$ such that $x + v_0$ is a critical point of $\mathscr{A}_H$. In this case, we have*

$$\mathscr{A}_H(x + v_0) = \mathscr{A}_H^*(x).$$

Question 3.3.3. *Is it possible to extend the Clarke duality, first to star-shaped domains and thence to all symplectic manifold?*

3.3.3. Morse complex of the dual action functional

Proposition 3.3.4. *Assume that the Hamiltonian $H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is smooth and satisfies the conditions 3.2.3.(2) (Asymptotics) and 3.2.3.(3) (Convexity). Then the dual action functional $\mathscr{A}_H^* : \mathbb{H}_1 \to \mathbb{R}$ satisfies the Palais-Smale condition.*

If we assume that the smooth Hamiltonian H satisfies 3.2.3.(2), 3.2.3.(3), and 3.2.3.(4) (Non-degeneracy), the functional $\mathscr{A}_H^*$ is Morse, meaning that the (Gateaux) second differential of $\mathscr{A}_H^*$ at each critical point is non-degenerate. However, the functional $\mathscr{A}_H^*$ is in general not of class C^2 (it is not even twice differentiable), so some care is needed in order to associate a Morse complex to it. The strategy from [AK19] is to use the fact that $\mathscr{A}_H^*$ is smooth when restricted to a suitable finite dimensional smooth submanifold of $\mathbb{H}_1$, which contains all the critical points of $\mathscr{A}_H^*$ and is defined by a saddle-point reduction.

Given a natural number $N \in \mathbb{N}$, consider the splitting

$$\mathbb{H}_1 = \mathbb{H}_1^{N,+} \oplus \widehat{\mathbb{H}}_1^{N,+}$$

with

$$\mathbb{H}_1^{N,+} := \left\{ x \in \mathbb{H}_1 \,\middle|\, x(t) = \sum_{k=1}^{N} x_k e^{2\pi i k t},\ x_k \in \mathbb{R}^{2n} \right\},$$

$$\widehat{\mathbb{H}}_1^{N,+} := \left\{ x \in \mathbb{H}_1 \,\middle|\, x(t) = \sum_{k\leqslant -1} x_k e^{2\pi i k t} + \sum_{k\geqslant N+1} x_k e^{2\pi i k t},\ x_k \in \mathbb{R}^{2n} \right\}.$$

This splitting is orthogonal with respect to the $\mathbb{H}_1$ and to the L^2 inner products. We identify $\mathbb{H}_1$ with the product space $\mathbb{H}_1^{N,+} \times \widehat{\mathbb{H}}_1^{N,+}$. The following proposition summarizes the main properties of the saddle point reduction.

Proposition 3.3.5 ([AK19]). *Assume that the Hamiltonian $H \in C^\infty(S^1, \mathbb{R}^{2n})$ satisfies 3.2.3.(2) and 3.2.3.(3). If $N \in \mathbb{N}$ is large enough, then the following facts hold:*

1. *For every $x \in \mathbb{H}_1^{N,+}$ the restriction of $\mathscr{A}_H^*$ to $\{x\} \times \widehat{\mathbb{H}}_1^{N,+}$ has a unique critical point $\big(x, Y(x)\big)$, which is a non-degenerate global minimizer of this restriction.*

2. *The map $Y : \mathbb{H}_1^{N,+} \to \widehat{\mathbb{H}}_1^{N,+}$ takes values in $C^\infty(S^1, \mathbb{R}^{2n})$ and is smooth with respect to the C^k-norm for any $k \in \mathbb{N}$ on the target. In particular, its graph*

$$M := \Big\{(x, y) \in \mathbb{H}_1^{N,+} \times \widehat{\mathbb{H}}_1^{N,+} \,|\, y = Y(x)\Big\}$$

   is a smooth $2nN$-dimensional submanifold of $\mathbb{H}_1$.

3. *The restriction of $\mathscr{A}_H^*$ to M, which is denoted by $\psi_H^* : M \to \mathbb{R}$ is smooth.*

4. *A point $z \in \mathbb{H}_1$ is a critical point of $\mathscr{A}_H^*$ if and only if it belongs to M and is a critical point of ψ_H^*. In this case, the Morse index and the nullity with respect to the two functionals coincide:*

$$\operatorname{ind}(z; \mathscr{A}_H^*) = \operatorname{ind}(z; \psi_H^*) \qquad null(z; \mathscr{A}_H^*) = null(z; \psi_H^*).$$

5. *If M is endowed with the Riemannian metric induced by the inclusion into $\mathbb{H}_1$, the functional ψ_H^* satisfies the Palais-Smale condition.*

If we further assume that the Hamiltonian H satisfies 3.2.3.(4), we obtain that ψ_H^* is a smooth Morse function with finitely many critical points and satisfying the Palais-Smale condition on the finite-dimensional manifold M. As such, it has a Morse complex, uniquely defined up to chain isomorphisms, denoted by

$$\{CM_*(\psi_H^*), \partial^M\}.$$

The space $CM_*(\psi_H^*)$ is the $\mathbb{Q}$-vector space generated by the critical points of ψ_H^*, graded by the Morse index. The boundary operator $\partial^M : CM_*(\psi_H^*) \to CM_{*-1}(\psi_H^*)$ is defined for all $x \in Crit(\psi_H^*)$ by the formula

$$\partial^M(x) = \sum_y \#\mathscr{M}(x, y) y$$

where y ranges over all critical points with Morse index equal to the index of x minus 1 and $\#\mathscr{M}(x, y)$ is the number of negative gradient flow lines of ψ_H^* going from x to y. Here, the negative gradient vector field of ψ_H^* is induced by a generic Riemannian metric on M, uniformly equivalent to the standard one and such that the negative gradient flow is Morse-Smale, meaning that stable and unstable manifolds of pairs of critical points meet transversally. Changing the generic metric changes the Morse complex by a chain isomorphism. The homology of the Morse complex $CM_*(\psi_H^*)$ is isomorphic to the singular homology of the pair $\big(M, \{\psi_H^* < a\}\big)$, where a is any number which is smaller than the smallest critical level of ψ_H^*:

$$HM_k(\psi_H^*) \simeq H_k\big(M, \{\psi_H^* < a\}\big).$$

3.4. Morse Homology for the action functional

3.4.1. Morse homology for Hilbert spaces

Abbondandolo and Majer [AM05, AM03, AM01, AM04], have defined a relative Morse homology on Hilbert spaces for some functionals. This applies, in particular, to the action functional but, we start by recalling the general definition, following the aforementionned references. Let $\mathscr{H}$ be a real Hilbert space and L a linear, invertible, self-adjoint operator on $\mathscr{H}$, one considers the class of functionals $f : \mathscr{H} \to \mathbb{R}$ of the form

$$f(x) = \frac{1}{2}(Lx, x) + b(x)$$

where b is C^2 and $\nabla b : \mathscr{H} \to \mathscr{H}$ is a compact map. Denote this class of functionals by $\mathscr{F}(L)$. The main idea is that under suitable assumptions, even so the Morse indices and coindices of the critical points are infinite, the intersections of of stable and unstable manifolds, $W^u(x) \cap W^s(y)$ are finite dimensional. To prove such a result (and to define a relative Morse index) requires a orthogonal decomposition of the Hilbert space $\mathscr{H}$ in two subspaces.

Given a bounded self-adjoint operator $S : \mathscr{H} \to \mathscr{H}$, denote by $V^+(S)$ (respectively $V^-(S)$) the maximal S-invariant subspace on which S is strictly positive (respectively strictly negative). The spaces $V^+(S)$ and $V^-(S)$ are called the **positive eigenspace of** S and the **negative eigenspace of** S respectively. Since the operator L has been fixed, we denote by $\mathscr{H}^+$ and $\mathscr{H}^-$ the positive and negative eigenspaces of L

$$\mathscr{H}^+ := V^+(L), \qquad \mathscr{H}^- := V^-(L).$$

Note that we have $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$.

The Hessian of a functional $f \in \mathscr{F}(L)$ at x is given by

$$D^2 f(x) = L + D^2 b(x).$$

Note that $D^2 f(x)$ is a Fredholm operator since $D^2 b(x)$ is a compact linear operator (because ∇b is compact).

We now recall the notion of "relative Morse index" for the critical points of f.

Definition 3.4.1. *Let V and W be closed linear subspaces of a Hilbert space $\mathscr{H}$. They form a **Fredholm pair** if $\dim(V \cap W) < \infty$, $V + W$ is closed and $\dim \frac{\mathscr{H}}{V+W} = \dim(V+W)^\perp = \dim(V^\perp \cap W^\perp) < \infty$.*

Remark 3.4.2. *An operator $A : \mathscr{H}_1 \to \mathscr{H}_2$ is Fredholm if and only if $\left(\mathscr{H}_1 \times \{0\}, Graph(A)\right)$ is a Fredholm pair in $\mathscr{H}_1 \times \mathscr{H}_2$. The **index of a Fredholm pair** (V, W) is defined as*

$$ind(V, W) = \dim(V \cap W) - \operatorname{codim}(V + W) \in \mathbb{Z}.$$

Let V and W be closed linear subspaces of a Hilbert space $\mathscr{H}$. W is a **compact perturbation of** V if $P_W - P_V$ is compact, where P is the orthogonal projection. In particular $(V, W^\perp)$ is a Fredholm pair. The relative dimension of V with respect to W is defined as $\dim(V, W) := ind(V, W^\perp) = \dim(V \cap W^\perp) - \dim(V^\perp \cap W)$.

If A is a self-adjoint Fredholm operator and K is a compact operator, $V^-(A)$ is a compact perturbation of $V^-(A + K)$.

Going back to the functional f, we have $D^2 f(x) = L + D^2 b(x)$ where $D^2 b(x)$ is a compact operator. We have that $V^-(D^2 f(x))$ is a compact perturbation of $\mathscr{H}^-$ and we can define the **relative Morse index** of x as

$$ind_{\mathscr{H}^-}(x) = \dim\Big(V^-\big(D^2 f(x)\big), \mathscr{H}^-\Big).$$

Remark that when $\mathscr{H}^- = \{0\}$, this index is the usual Morse index. We denote by $\mathrm{crit}_k(f)$ the set of critical points of f of relative Morse index k.

Now, let x and y be critical points of f, we look at $W^u(x) \cap W^s(y)$ to define moduli spaces of gradient trajectories $u' = \nabla f(u)$.

Let $I \subset \mathbb{R} \cup \{-\infty, +\infty\}$ be an interval.

Definition 3.4.3. *A functional $f \in C^2(\mathscr{H})$ is called* Morse *on I if the Hessian $D^2 f(x)$ is invertible for every critical point x such that $f(x) \in I$.*

Assuming that the functional f is Morse, we have the two following facts $\forall p \in W^u(x)$:

1. $T_p W^u(x)$ is a compact perturbation of $\mathscr{H}^-$ with relative dimension $ind_{\mathscr{H}^-}(x)$
2. $\big(T_p W^s(x), \mathscr{H}^-\big)$ are Fredholm pairs

If $p \in W^u(x) \cap W^s(y)$, $\big(T_p W^u(x), T_p W^s(y)\big)$ is a Fredholm pair of index $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y)$.

In our case, the gradient trajectories are of the form:

$$u'(t) = -\nabla f(u) = -Lu - \nabla b$$

So $u' + Lu = -\nabla b$, multiplying by e^{tL}, we have

$$\frac{d}{dt} e^{tL} u = e^{tL}(u' + Lu) = -e^{tL} \nabla b$$

and thus

$$u(t) = e^{-tL}\left(u(0) - \int_0^t e^{sL} \nabla b\big(u(s)\big) ds\right).$$

Definition 3.4.4. *A functional $f \in C^1(\mathscr{H})$ satisfies the* Palais-Smale condition *on I if every sequence $(x_n) \subset \mathscr{H}$ such that $\lim_{n\to\infty} f(x_n) = c \in I$ and $\lim_{n\to\infty} \nabla f(x_n) = 0$ is relatively compact.*

Lemma 3.4.5. *The functional f satisfies the Palais-Smale condition (PS) if and only if all PS sequences are bounded.*

Proof. Indeed, $\nabla f(x) = Lx + \nabla b(x)$. Take a PS sequence x_n, so $\nabla f(x_n) \to 0$ and, since ∇b is compact, $\nabla b(x_n) \to z$. Therefore $Lx_n \to -z$. Since L is invertible, $x_n \to -L^{-1}z$. □

Definition 3.4.6. *A functional $f \in C^2(\mathscr{H})$ has the* Morse-Smale property *on I up to order k if it is a Morse function on I and the unstable and stable manifolds of every pair of critical points $x, y \in f^{-1}(I)$ such that $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) \leqslant k$, meet transversally*

Theorem 3.4.7. *Assume that the functional $f \in \mathscr{F}(L)$ satisfies PS and the Morse-Smale property up to order k on the interval I. Let $x, y \in f^{-1}(I)$ be two critical points of f such that $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) \leqslant k$. Then $W^u(x) \cap W^s(y)$, if nonempty, is an embedded C^1-submanifold of $\mathscr{H}$ of dimension*

$$\dim\big(W^u(x) \cap W^s(y)\big) = ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y).$$

Moreover, we have the following:

- *When $k \geqslant 0$, $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) \leqslant 0$, and $x \neq y$, we have $W^u(x) \cap W^s(y) = \varnothing$.*
- *When $k \geqslant 0$, and $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) = 1$, $W^u(x) \cap W^s(y) \cup \{x, y\}$ is compact.*

This theorem implies, in particular, that when $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) = 1$, there is a finite number of trajectories from x to y. The manifolds $W^u(x) \cap W^s(y)$ admit an orientation [AM05, §3.5] and thus, when $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) = 1$, all the trajectories from x to y come with a sign.

The idea of orientation is the following. Let $F_p(\mathscr{H})$ denote the set of Fredholm pairs in $\mathscr{H}$. We have the non-trivial line bundle

$$\begin{array}{ccc} \Lambda^{\max}(V \cap W) \otimes \Lambda^{\max}\left(\left(\frac{\mathscr{H}}{V+W}\right)^*\right) & \hookrightarrow & \operatorname{Det}\left(F_p(\mathscr{H})\right) \\ & & \downarrow \\ & & F_p(\mathscr{H}) \end{array}$$

If x is a critical point of the functional f, the pair $\left(T_x W^u(x), \mathscr{H}^+\right)$ is in $F_p(\mathscr{H})$. We choose an orientation of the determinanl line bundle over this pairs and we do the same at every (critical) point. This induces an orientation over $\left(T_x W^s(x), \mathscr{H}^-\right)$.
Thus, $\left(T_p W^u(x), \mathscr{H}^+\right)$ and $\left(T_p W^s(x), \mathscr{H}^-\right)$ are oriented for all $p \in W^u(x)$. This induces a canonical orientation of $\left(T_p W^u(x), T_p W^s(x)\right)$. If the functional is Morse-Smale, we are done.

When $ind_{\mathscr{H}^-}(x) - ind_{\mathscr{H}^-}(y) = 1$, let $\#\mathscr{N}(x, y)$ denote the count, with signs, of trajectories from x to y.

Given an interval I of the extended real line and a functional f satisfying the following conditions

(M.1) $f \in \mathscr{F}(L)$;

(M.2) f satisfies the PS condition on I;

(M.3) f is a Morse function on I;

(M.4) f has the Morse-Smale property on I up to order 2;

(M.5) for every $a \in I$ and every $k \in \mathbb{Z}$, the set $\operatorname{crit}_k\left(f, I \cap (-\infty, a]\right)$ is finite;

we can define a Morse homology of the pair (f, I). The Morse complex in degree k is defined as

$$CM_k(f, I) := \oplus_{x \in \operatorname{crit}_k(f,I)} \mathbb{Q}\langle x \rangle.$$

and the boundary operator $\partial_k^{f,I} : CM_k(f, I) \to CM_{k-1}(f, I)$ is defined, for $x \in \operatorname{crit}_k(f, I)$, as

$$\partial_k^{f,I}(x) = \sum_{y \in \operatorname{crit}_{k-1}(f,I)} \#\mathscr{N}(x, y) y.$$

Theorem 3.4.8. *Assuming the functional f satisfies (M.1)–(M.5), the boundary operator $\partial_k^{f,I}$ is an actual boundary homomorphism, i.e.*

$$\partial_k^{f,I} \circ \partial_k^{f,I} = 0.$$

Therefore the pair $\left(CM(f, I), \partial_k^{f,I}\right)$ is a chain complex called the *Morse complex* of (f, I) and its homology is called the *Morse homology* of (f, I).

3.4.2. The case of the action functional for star-shaped domains

Let $\Lambda(\mathbb{R}^{2n}) := C^\infty(S^1, \mathbb{R}^{2n})$ be the free loop space of $\mathbb{R}^{2n}$. The Hamiltonian action functional $\mathscr{A}_H$ on $\Lambda(\mathbb{R}^{2n})$ is defined as

$$\mathscr{A}_H(\gamma) := -\int_\gamma \lambda_0 - \int_{S^1} H\big(\theta, \gamma(\theta)\big) d\theta. \tag{3.4.1}$$

To ensure, we have a Morse theory of this $\mathscr{A}_H$, we have to complete $\Lambda(\mathbb{R}^{2n})$ in a Hilbert manifold; its structure will be induced by $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$. Then we shall extend the functional $\mathscr{A}_H$ and check that it satisfies the 5 conditions (M.1)–(M.5) listed above.

3.4.2.1. The Hilbert manifold

Since $\Lambda(\mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n})$, every element $x \in \Lambda(\mathbb{R}^{2n})$ can be written as a Fourrier series with coefficients in $\mathbb{R}^{2n}$.

$$x(t) = \sum_{k\in\mathbb{Z}} x_k e^{2\pi i k t}.$$

Using this Fourrier decomposition, $\Lambda(\mathbb{R}^{2n})$ can be completed in the Sobolev space: $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$ (which carries a Hilbert structure).

$$H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) := \left\{ x \in L^2(S^1, \mathbb{R}^{2n}) \mid \sum_{k\in\mathbb{Z}} |k| \|x_k\|^2 < \infty \right\}.$$

We have the orthogonal decomposition

$$H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) = E^+ \oplus E^0 \oplus E^-$$

with respect to the inner product $\langle x, y \rangle := \langle x_0, y_0 \rangle + 2\pi \sum_{0\neq k\in\mathbb{Z}} |k| \langle x_k, y_k \rangle$ and where

$$\begin{aligned} E^- &= \{x \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \mid x_k = 0 \text{ for } k \geqslant 0\} \\ E^0 &= \{x \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \mid x_k = 0 \text{ for } k \neq 0\} \cong \mathbb{R}^{2n} \\ E^+ &= \{x \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}) \mid x_k = 0 \text{ for } k \leqslant 0\}. \end{aligned}$$

Let P_{E^+}, P_{E^-} and P_{E^0} denote the orthogonal projections on E^+, E^- and E^0 respectively.

3.4.2.2. The functional

Recall the class $\mathscr{F}(L)$ of functionals for which the Morse homology is defined. Let $\mathscr{H}$ be a real Hilbert space and let L be a linear, invertible, self-adjoint operator on $\mathscr{H}$. We are looking at the functional $f : \mathscr{H} \to \mathbb{R}$

$$f(x) = \frac{1}{2}(Lx, x) + b(x)$$

where b is C^2 and $\nabla b : \mathscr{H} \to \mathscr{H}$ is a compact map. In the case of a nice star-shaped domain in $\mathbb{R}^{2n}$, the Hilbert space is $\mathscr{H} = H^{\frac{1}{2}}$ and the functional is given by

$$\mathscr{A}_H(x) = -\tfrac{1}{2}\int J\dot{x}\cdot x dt - \int_0^1 H\big(t, x(t)\big) dt. \tag{3.4.2}$$

The fact that this functional coincides with the one from equation (3.4.1) is a direct computation.

Fixing L, we denote by $\mathscr{H}^+$ the maximal L-invariant subspace on which L is positive and by $\mathscr{H}^-$ the maximal L-invariant subspace on which L is negative. We have $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$. Here $\mathscr{H}$ decomposes as $\mathscr{H} = E^+ \oplus E^0 \oplus E^-$ where $E^0 \cong \mathbb{R}^{2n}$ is the set of constant loops. We split E^0 arbitrarily in $E^0 = E^0_+ \oplus E^0_-$ where $E^0_+ \cong \mathbb{R}^n \cong E^0_-$. In the previous notation, we take $\mathscr{H}^+ = E^+ \oplus E^0_+$ and $\mathscr{H}^- = E^- \oplus E^0_-$ by extending L with the matrix $\begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$.

Remark 3.4.9. *By taking the splitting of E^0 to be given by $E^0_- = \langle x_1, \ldots, x_n \rangle$, we have that the CZ-index is equal to the relative Morse index, see [Abb01].*

The functional then writes as

$$\mathscr{A}_H = -\tfrac{1}{2}\int_0^1 J\dot{x}\cdot x\,dt - \int_0^1 H\big(t,x(t)\big)dt \tag{3.4.3}$$

$$= \tfrac{1}{2}\big(\|P_{E^+}(x)\|^2_{H^{\frac{1}{2}}} - \|P_{E^-}(x)\|^2_{H^{\frac{1}{2}}}\big) - \int_0^1 H\big(t,x(t)\big)dt \tag{3.4.4}$$

$$= \tfrac{1}{2}(Lx\,,\,x)_{\frac{1}{2}} \underbrace{-\tfrac{1}{2}\|P_{E^0_+}x\|^2 + \tfrac{1}{2}\|P_{E^0_-}x\|^2 - \int_0^1 H\big(t,x(t)\big)dt}_{b}\,. \tag{3.4.5}$$

Proposition 3.4.10. *[HZ11, Lemma 3.4] The map $b : \mathscr{H} \to \mathbb{R}$ from equation (3.4.5) is differentiable. Its gradient $\nabla b : \mathscr{H} \to \mathscr{H}$ is continuous and compact.*

We need a better understanding of the Sobolev spaces H^s before going on. Indeed not all element of $H^{\frac{1}{2}}$ can be represented by a continuous function.

Proposition 3.4.11. *[HZ11, Proposition 3.4] Let $s > \frac{1}{2}$. If $x \in H^s(S^1,\mathbb{R}^{2n})$, then $x \in C^0(S^1,\mathbb{R}^{2n})$. Moreover, there is a constant c, depending on s, such that*

$$\|x\|_{C^0} \leqslant c\|x\|_{H^s}, \qquad \forall x \in H^s(S^1,\mathbb{R}^{2n}).$$

Recall that from [HZ11, Proposition 3.3], for $t > s \geqslant 0$, the inclusion map $I : H^t(S^1,\mathbb{R}^{2n}) \to H^s(S^1,\mathbb{R}^{2n})$ is compact.

The following inclusion j, and its adjoint j^*, will play a key role in the following.

$$j : H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n}) \to H^0(S^1,\mathbb{R}^{2n}) = L^2(S^1,\mathbb{R}^{2n})$$
$$j^* : L^2(S^1,\mathbb{R}^{2n}) \to H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n})$$

Proposition 3.4.12. *[HZ11, Proposition 3.5]*

$$j^*\big(L^2(S^1,\mathbb{R}^{2n})\big) \subset H^1(S^1,\mathbb{R}^{2n}) \quad \text{and} \quad \|j^*(y)\|_{H^1} \leqslant \|y\|_{L^2}.$$

Proposition 3.4.13. *[HZ11] The Hamiltonian action functional $\mathscr{A}_H : \mathscr{H} \to \mathbb{R}$ is a smooth functional. Its gradient, with respect to the inner product on $H^{\frac{1}{2}}$ is given by*

$$\nabla_{\frac{1}{2}}\mathscr{A}_H(x) = -P_{E^+}(x) + P_{E^-}(x) + j^*\nabla H\big(\cdot,x(\cdot)\big).$$

Moreover, $\nabla_{\frac{1}{2}}\mathscr{A}_H$ is Lipschitz continuous on $\mathscr{H}$ with uniform Lipschitz constant. Its Jacobian is given by

$$\nabla^2_{\frac{1}{2}}\mathscr{A}_H(x) = -P_{E^+} + P_{E^-} + j^*\nabla^2 H\big(\cdot,x(\cdot)\big).$$

We denote by $X := -\nabla_{\frac{1}{2}}\mathscr{A}_H$ the gradient vector field of the Hamiltonian action functional. Assume that $x \in \mathscr{H}$ is a critical point of the action functional; i.e. $\nabla_{\frac{1}{2}}\mathscr{A}_H(x) = 0$. Then $x \in C^\infty(S^1, \mathbb{R}^{2n})$. Moreover it solves the Hamiltonian equation

$$\dot{x}(t) = J\nabla H\big(x(t)\big).$$

Lemma 3.4.14. *[HZ11, Lemma 3.7] The flow of $\dot{x} = X(x)$ is globally defined, maps bounded sets to bounded sets and admits the representation*

$$x \cdot t = e^t x^- + x^0 + e^{-t}x^+ + K(t,x)$$

where $K : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ is continuous and maps bounded sets in precompact sets.

3.4.2.3. The conditions (M.1)–(M.5) are satisfied by $\mathscr{A}_H$

Condition (M.1) is satisfied and the decomposition $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$ is as in §3.4.1. To ensure the Morse property, we have to pick a generic Hamiltonian

Proposition 3.4.15 ([AM01]). *There is a residual set (in the sense of Baire) $\mathscr{H}_{reg} \subset C^\infty(S^1 \times \mathbb{R}^{2n}, \mathbb{R})$ of Hamiltonians such that the negative $H^{\frac{1}{2}}$-gradient X of $\mathscr{A}_H$ is a Morse vector field for evey $H \in \mathscr{H}_{reg}$. In particular, the set of critical points of $\mathscr{A}_H$ is a finite set.*

To ensure transversality (condition (M.4)), we need to perturb the vector field $X = -\nabla_{\frac{1}{2}}\mathscr{A}_H$ by adding a small compactly supported vector field $\overline{X}$. We do it this way rather than following [AM05] in preparation for transversality for hybrid-type curves §3.5. Let $\mathscr{K}(\mathscr{H}) \subset C^3_b(\mathscr{H})$ be the closed subspace of all C^3-vector fields which are compact and bounded on $\mathscr{H}$. We choose a C^1-function $g : \mathscr{H} \to \mathbb{R}^+$ satisfying

1. $g(p) > 0$ everywhere else; i.e. for all $p \in \mathscr{H} \setminus \operatorname{Crit}\mathscr{A}_H$,
2. $g(p) \leqslant \frac{1}{2}\|\nabla_{\frac{1}{2}}\mathscr{A}_H(p)\|_{H^{\frac{1}{2}}}$ for all $p \in \mathscr{H}$.

In particular, we have $g(x) = 0$ for all $x \in \operatorname{Crit}\mathscr{A}_H$. We consider the subset of vector fields

$$\mathscr{K}_g := \Big\{\overline{X} \in \mathscr{K}(\mathscr{H}) \,|\, \exists c > 0 \text{ such that } \|\overline{X}_p\|_{H^{\frac{1}{2}}} \leqslant c g(p) \quad \forall p \in \mathscr{H}\Big\}.$$

This set is a Banach space when equipped with the following norm:

$$\|\overline{X}\|_{\mathscr{K}_g} := \sup_{p \in \mathscr{H} \setminus \operatorname{Crit}(\mathscr{A}_H)} \frac{\|\overline{X}_p\|_{H^{\frac{1}{2}}}}{g(p)} + \|\nabla \overline{X}\|_{C^2}.$$

We denote the open unit ball in $\mathscr{K}_g$, with respect to the above norm, by $\mathscr{K}_{g,1}$. It is a Banach manifold with trivial tangent bundle.

Lemma 3.4.16. *Let $\overline{X} \in \mathscr{K}_{g,1}$ and let $\widetilde{X} := -\nabla_{\frac{1}{2}}\mathscr{A}_H + \overline{X}$. Then*

1. The singular points of $\widetilde{X}$ are the critical points of the action functional

$$\operatorname{sing}(\widetilde{X}) = \operatorname{Crit}(\mathscr{A}_H).$$

2. *For all $x \in \mathrm{Crit}(\mathscr{A}_H)$, we have*
$$D\widetilde{X}(x) = -D^2\mathscr{A}_H(x).$$

3. *The action functional is a Lyapunov function for $\widetilde{X}$; i.e.*
$$D\mathscr{A}_H(p)\big(\widetilde{X}(p)\big) < 0 \qquad \text{for all } p \in \mathscr{H}\setminus \mathrm{Crit}(\mathscr{A}_H).$$

Theorem 3.4.17. *There is a residual subset $\mathscr{K}_{reg} \subset \mathscr{K}_{g,1}$ of compact vector fields $\overline{X}$ such that the perturbed vector field $\widetilde{X} := -\nabla_{\frac{1}{2}}\mathscr{A}_H + \overline{X}$ fulfills the Morse-Smale condition up to order 2.*

We are therefore in a situation where we can define the Morse homology as in Section 3.4.1.

3.4.3. Continuations

In view of an isomorphism with symplectic homology, we need to be able to change the Hamiltonian. Let f_0 and f_1 be two functionals and let f_s be a homotopy interpolating between the two; $f_s = f_0$ for $s \leqslant \epsilon$ and $f_s = f_1$ for $s \geqslant 1-\epsilon$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ a smooth function with two critical points: a maximum at 0, with $\varphi(0) = 1$ and a minimum at 1 with $\varphi(1) = 0$. Let $\widetilde{f} : \mathbb{R} \times \mathscr{H} \to \mathbb{R}$ be the functional defined by $\widetilde{f}(s,x) = \varphi(s) + f_s(x)$. The critical points of $\widetilde{f}$ of index k are
$$\mathrm{crit}_k \widetilde{f} = \{0\} \times \mathrm{crit}_{k-1} f_0 \bigcup \{1\} \times \mathrm{crit}_k f_1.$$

The associated differential $\partial_{\widetilde{f}}$ writes as
$$\partial_{\widetilde{f}} = \begin{pmatrix} \partial_{f_0} & \phi \\ 0 & \partial_{f_1} \end{pmatrix}.$$

This ϕ is precisely the continuation map (as in finite-dimensional Morse homology)

3.5. All three homologies coincide

This section describes some known results and ongoing work (joint with V. Ramos). The main (ongoing) statement is that given an admissible Hamiltonian, there are chain complexes isomorphisms between the three aforementionned constructions which commute with continuations. One of the isomorphism was proved by Abbondandolo and Kang. To prove the other one is under progress.

Theorem 3.5.1 ([AK19]). *Let $H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamiltonian function satisfying the conditions 3.2.3.(2), 3.2.3.(3), and 3.2.3.(4). Then there exists a chain complex isomorphism*
$$\Theta : \big(CM_{*-n}(\psi_H^*), \partial^M\big) \to (CF_*(H,J), \partial)$$

The isomorphism Θ is defined as a count of "hybrid trajectories". Let x and y be 1-periodic orbits of X_H. We shall see $\Pi(x) \in \mathbb{H}_1$ as a critical point of $\mathscr{A}_H^*$ (and hence of ψ_H^*) and $y \in H^{\frac{1}{2}}$ as a critical point of $\mathscr{A}_H$. Let J be a family of uniformly bounded ω_0-compatible almost complex structures on $\mathbb{R}^{2n}$ parametrized by $[0,\infty) \times S^1$ such that $J = J_0$ on $[0,1] \times S^1$. Denote by
$$\mathscr{M}(x,y) = \mathscr{M}(x,y,H,J)$$

the space of smooth maps $u : [0,\infty) \times S^1 \to \mathbb{R}^{2n}$ which solve the Floer equation

$$\partial_s u + J(s,t,u)(\partial_t u - X_{H_t}(u)) = 0 \text{ on } [0,\infty) \times S^1$$

with the asymptotic condition

$$\lim_{s\to\infty} u(s,\cdot) = y \text{ in } C^\infty(S^1, \mathbb{R}^{2n}),$$

and the boundary condition

$$u(0,\cdot) \in \Pi^{-1} W^u\big((\Pi(x); -\nabla\psi_H^*) + \mathbb{H}^-_{\frac{1}{2}}.$$

where $W^u\big((\Pi(x); -\nabla\psi_H^*)$ is the unstable manifold of the negative gradient vector field of ψ_H^* at $\Pi(x)$ in a finite dimensional submanifold M of $\mathbb{H}_1$, which is used to construct the Morse complex of ψ_H^* in §3.3.

Abbondandolo and Kang proved that, generically, $\mathcal{M}(x,y)$ is a smooth manifold of dimension $\mathrm{CZ}(x) - \mathrm{CZ}(y)$; moreover if $\mathrm{CZ}(x) = \mathrm{CZ}(y)$ then the manifold is compact and thus consists of finitely many points. They then define for all $k \in \mathbb{Z}$ the isomorphism $\Theta_k : \big(CM_{k-n}(\psi_H^*), \partial^M\big) \to (CF_k(H,J), \partial)$ by

$$\Theta_k\big(\Pi(x)\big) = \sum_y \#\mathcal{M}(x,y)y$$

where the sum runs over all 1-periodic orbits y of X_H of Conley-Zehnder index k.

Note that the aforementionned isomorphism is defined with $\mathbb{Z}_2$ coefficients. It should extend to $\mathbb{Q}$ coefficients after an orientation have been added.

For the other isomorphism, the statement I am trying to prove with V. Ramos is

Statement 3.5.2. *Let $H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamiltonian function satisfying the conditions 3.2.3.(2), 3.2.3.(3), and 3.2.3.(4). Then there exists a chain complex isomorphism*

$$\Phi : \big(CM_*(H), \partial^M\big) \to (CF_*(H,J), \partial)$$

We define this chain map $\Phi : CM(H) \to CF(H)$ by counting hybrid curves in a similar manner as [AK19]. Let $Z = [0,\infty) \times S^1$ and let $x, y \in \mathcal{P}(H)$. We also let H and X be generic as explained in Section 3.2.1. We define

$$\mathcal{M}_{hyb}(x,y,X) = \Big\{u \in H^1_{loc}(Z, \mathbb{R}^{2n}) \mid \bar\partial_{J_0,H}(u) = 0, u(0,\cdot) \in W^u_X(x), \lim_{s\to\infty} u(s,\cdot) = y\Big\}.$$

As before, generically, $\mathcal{M}_{hyb}(x,y)$ is a smooth manifold of dimension $\mathrm{CZ}(x) - \mathrm{CZ}(y)$; moreover if $\mathrm{CZ}(x) = \mathrm{CZ}(y)$ then the manifold is compact and thus consists of finitely many points. So we let

$$\Phi(x) = \sum_{\mu(y)=\mu(x)} \#\mathcal{M}_{hyb}(x,y)\cdot y.$$

We still have to prove that Θ and Φ commute with continuation maps and that we can "extend" them to the S^1-equivariant setup.

Part II.

Symplectic embeddings

4. Introduction to Part II

If X and X' are domains[1] in $\mathbb{R}^{2n} = \mathbb{C}^n$, a **symplectic embedding** from X to X' is a smooth embedding $\varphi : X \hookrightarrow X'$ such that $\varphi^\star\omega = \omega$, where ω denotes the standard symplectic form on $\mathbb{R}^{2n}$. If there exists a symplectic embedding from X to X', we write $X \underset{s}{\hookrightarrow} X'$.

An important problem in symplectic topology is to determine when symplectic embeddings exist, and more generally to classify the symplectic embeddings between two given domains. Modern work on this topic began with the Gromov nonsqueezing theorem [Gro85], which asserts that the ball

$$B^{2n}(r) = \left\{z \in \mathbb{C}^n \;\middle|\; \pi|z|^2 \leqslant r\right\}$$

symplectically embeds into the cylinder

$$Z^{2n}(R) = \left\{z \in \mathbb{C}^n \;\middle|\; \pi|z_1|^2 \leqslant R\right\}$$

if and only if $r \leqslant R$. Many questions about symplectic embeddings remain open, even for simple examples such as ellipsoids and polydisks.

If there exists a symplectic embedding $X \underset{s}{\hookrightarrow} X'$, then we have the volume constraint $\mathrm{vol}(X) \leqslant \mathrm{vol}(X')$. To obtain more nontrivial obstructions to the existence of symplectic embeddings, one often uses various symplectic capacities. Definitions of the latter term vary; here we define a **symplectic capacity** to be a function c which assigns to each domain in $\mathbb{R}^{2n}$, possibly in some restricted class, a number $c(X) \in [0, \infty]$, satisfying the following axioms:

(Monotonicity) If X and X' are domains in $\mathbb{R}^{2n}$, and if there exists a symplectic embedding $X \underset{s}{\hookrightarrow} X'$, then $c(X) \leqslant c(X')$.

(Conformality) If r is a positive real number then $c(rX) = r^2 c(X)$.

We say that a symplectic capacity c is **normalized** if it is defined at least for convex domains and satisfies

$$c\left(B^{2n}(1)\right) = c\left(Z^{2n}(1)\right) = 1.$$

The first example of a normalized symplectic capacity is the **Gromov width** defined by

$$c_{\mathrm{Gr}}(X) = \sup\left\{r \;\middle|\; B^{2n}(r) \underset{s}{\hookrightarrow} X\right\}.$$

This trivially satisfies all of the axioms except for the normalization requirement $c_{\mathrm{Gr}}(Z^{2n}(1))$, which holds by Gromov non-squeezing theorem. A similar example is the **cylindrical capacity** defined by

$$c_Z(X) = \inf\left\{R \;\middle|\; X \underset{s}{\hookrightarrow} Z^{2n}(R)\right\}.$$

[1] In this memoir, a “domain” is the closure of an open set. One can of course also consider domains in other symplectic manifolds, but we will not do so here.

Additional examples of normalized symplectic capacities are the Hofer-Zehnder capacity c_{HZ} defined in [HZ11] and the Viterbo capacity c_{SH} defined in [Vit99]. There are also useful families of symplectic capacities parametrized by a positive integer k including the Ekeland-Hofer capacities c_k^{EH} defined in [EH89, EH90] using calculus of variations; the "equivariant capacities" c_k^{CH} defined in [GH18] using positive equivariant symplectic homology; and in the four-dimensional case, the ECH capacities c_k^{ECH} defined in [Hut11] using embedded contact homology. For each of these families, the $k = 1$ capacities c_1^{EH}, c_1^{CH}, and c_1^{ECH} are normalized. For more about symplectic capacities in general we refer to [CHLS07, Sch18] and the references therein.

The goal of this second part is to present some results and examples related to the following conjecture, which apparently has been folkore since the 1990s.

Conjecture 4.0.1 (strong Viterbo conjecture). *If X is a convex domain in $\mathbb{R}^{2n}$, then all normalized symplectic capacities of X are equal.*

Viterbo originally conjectured the following statement[2] in [Vit00]:

Conjecture 4.0.2 (Viterbo conjecture). *If X is a convex domain in $\mathbb{R}^{2n}$ and if c is a normalized symplectic capacity, then*

$$c(X) \leqslant (n!\,\mathrm{Vol}(X))^{1/n}. \tag{4.0.1}$$

The inequality (4.0.1) is true when c is the Gromov width c_{Gr}, by the volume constraint. Thus Conjecture 4.0.1 implies Conjecture 4.0.2. The Viterbo conjecture recently gained even more attention as it was shown in [AAKO14] that it implies the Mahler conjecture[3] in convex geometry.

Lemma 4.0.3. *If X is a domain in $\mathbb{R}^{2n}$, then $c_{\mathrm{Gr}}(X) \leqslant c_Z(X)$, with equality if and only if all normalized symplectic capacities of X agree (when they are defined for X).*

Proof. It follows from the definitions that if c is a normalized symplectic capacity defined for X, then $c_{\mathrm{Gr}}(X) \leqslant c(X) \leqslant c_Z(X)$. □

Thus the strong Viterbo conjecture is equivalent to the statement that every convex domain X satisfies $c_{\mathrm{Gr}}(X) = c_Z(X)$. We now discuss some examples where it is known that $c_{\mathrm{Gr}} = c_Z$. Hermann [Her98] showed that all T^n-invariant convex domains do satisfy $c_{\mathrm{Gr}} = c_Z$. This generalizes to S^1-invariant convex domains by the following elementary argument:

Proposition 4.0.4 (Y. Ostrover, private communication). *Let X be a compact convex domain in $\mathbb{C}^n$ which is invariant under the S^1 action by $e^{i\theta} \cdot z = (e^{i\theta}z_1, \dots, e^{i\theta}z_n)$. Then $c_{\mathrm{Gr}}(X) = c_Z(X)$.*

Proof. By compactness, there exists $z_0 \in \partial X$ minimizing the distance to the origin. Let $r > 0$ denote this minimal distance. Then the ball $(|z| \leqslant r)$ is contained in X, so by definition $c_{\mathrm{Gr}}(X) \geqslant \pi r^2$.

By applying an element of $U(n)$, we may assume without loss of generality that $z_0 = (r, 0, \dots, 0)$. By a continuity argument, we can assume without loss of generality that ∂X is a smooth hypersurface in $\mathbb{R}^{2n}$. By the distance minimizing property, the tangent plane to ∂X at z_0 is given by $(z \cdot (1, 0, \dots, 0) = r)$ where $\cdot$ denotes the real inner product. By convexity, X is contained in

[2] Viterbo also conjectured that equality holds in (4.0.1) only if $\mathrm{int}(X)$ is symplectomorphic to an open ball.

[3] The Mahler conjecture [Mah39] states that for any n-dimensional normed space V, we have

$$\mathrm{Vol}(B_V)\,\mathrm{Vol}(B_{V*}) \geqslant \frac{4^n}{n!},$$

where B_V denotes the unit ball of V, and B_{V*} denotes the unit ball of the dual space V^*. For some examples of Conjectures 4.0.1 and 4.0.2 related to the Mahler conjecture see [SL20].

the half-space $(z \cdot (1, 0, \ldots, 0) \leqslant r)$. By the S^1 symmetry, X is also contained in the half-space $(z \cdot (e^{i\theta}, 0, \ldots, 0) \leqslant r)$ for each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus X is contained in the intersection of all these half-spaces, which is the cylinder $|z_1| \leqslant r$. Then $c_Z(X) \leqslant \pi r^2$ by definition. □

Remark 4.0.5. *A similar argument shows that if $k \geqslant 3$ is an integer and if $X \subset \mathbb{C}^n$ is a convex domain invariant under the $\mathbb{Z}/k$ action by $j \cdot z = (e^{2\pi i j/k} z_1, \ldots, e^{2\pi i j/k} z_n)$, then*

$$\frac{c_Z(X)}{c_{\mathrm{Gr}}(X)} \leqslant \frac{k}{\pi} \tan(\pi/k).$$

The role of the convexity hypothesis in Conjecture 4.0.1 is somewhat mysterious. We shall explore to what extent non-convex domains satisfy $c_{\mathrm{Gr}} = c_Z$.

4.1. Structure of Part 2

Part 2 is structured as follows: Chapter 5 is devoted to toric domains which will provide the framework for all results concerning symplectic embeddings. Chapter 6 presents the ECH capacities, the Ekeland-Hofer capacities and the new capacities from positive S^1-equivariant symplectic homology as well as computations and applications. Chapter 7 consists of known and new results around Conjecture 4.0.1. Chapter 8 presents a new notion of inequivalence of symplectic embeddings and examples thereof. The Last Chapter, about symplectic convexity, consists essentially of a list of questions and open problems which I intend to work upon.

5. Toric domains

This chapter introduces toric domains and one of the main result is Proposition 5.1.4 which gives a necessary and sufficient geometric condition for a toric domain to be dynamically convex.

5.1. Definition and examples

To describe an important family of examples of symplectic manifolds, let $\mathbb{R}^n_{\geqslant 0}$ denote the set of $x \in \mathbb{R}^n$ such that $x_i \geqslant 0$ for all $i = 1, \dots, n$. Define the moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$ by

$$\mu(z_1, \dots, z_n) = \pi(|z_1|^2, \dots, |z_n|^2).$$

If Ω is a domain in $\mathbb{R}^n_{\geqslant 0}$, define the **toric domain**

$$X_\Omega = \mu^{-1}(\Omega) \subset \mathbb{C}^n.$$

The factors of π ensure that

$$\mathrm{vol}(X_\Omega) = \mathrm{vol}(\Omega). \tag{5.1.1}$$

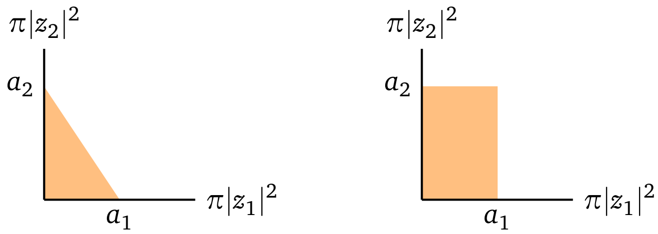
Example 5.1.1. *If $a_1, \dots, a_n > 0$, define the **ellipsoid***

$$E(a_1, \dots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leqslant 1 \right\} \tag{5.1.2}$$

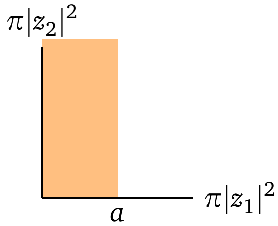
*We will occasionally find it convenient to extend this to the case that some $a_i = 0$ by taking $E(\dots, 0, \dots) = \varnothing$. The **polydisk** is defined as*

$$P(a_1, \dots, a_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \pi |z_i|^2 \leqslant a_i, \;\; \forall i = 1, \dots, n \right\}. \tag{5.1.3}$$

*Also, define the **ball** $B(a) = E(a, \dots, a)$.*



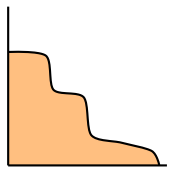
Example 5.1.2. *The four dimensional cylinder, $Z(a) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \pi |z_1|^2 \leqslant a \right\}$ is a (limit of) toric domain whose underlying domain in $\mathbb{R}^2$ is an "infinite strip".*



Let $\partial_+\Omega$ denote the set of $\mu \in \partial\Omega$ such that $\mu_j > 0$ for all $j = 1, \ldots, n$.

Definition 5.1.3 ([GHR20]). *A **monotone toric domain** is a compact toric domain X_Ω with smooth boundary such that if $\mu \in \partial_+\Omega$ and if v is an outward normal vector at μ, then $v_j \geqslant 0$ for all $j = 1, \ldots, n$.*

*A **strictly monotone toric domain** is a compact toric domain X_Ω with smooth boundary such that if $\mu \in \overline{\partial_+\Omega}$ and if v is a nonzero outward normal vector at μ, then $v_j > 0$ for all $j = 1, \ldots, n$.*



Note that monotone toric domains do not have to be convex; see §5.2.4 for details on conditions for toric domains to be convex. (Toric domains that are convex are already covered by Proposition 4.0.4.)

To clarify the hypothesis, let X be a compact domain in $\mathbb{R}^{2n}$ with smooth boundary, and suppose that X is "star-shaped", meaning that the radial vector field on $\mathbb{R}^{2n}$ is transverse to ∂X. Then there is a well-defined Reeb vector field R on ∂X. We say that X is **dynamically convex** if, in addition to the above hypotheses, every Reeb orbit γ has Conley-Zehnder index $\mathrm{CZ}(\gamma) \geqslant n + 1$ if nondegenerate, or in general has minimal Conley-Zehnder index[1] at least $n + 1$. It was shown by Hofer-Wysocki-Zehnder [HWZ95] that if X is strictly convex, then X is dynamically convex. However the Viterbo conjecture would imply that not every dynamically convex domain is symplectomorphic to a convex domain; see Remark 7.0.2 below.

Proposition 5.1.4 ([GHR20]). *(proved in §5.2.4) Let X_Ω be a compact star-shaped toric domain in $\mathbb{R}^4$ with smooth boundary. Then X_Ω is dynamically convex if and only if X_Ω is a strictly monotone toric domain.*

Two special type of monotone toric domains are defined as follows. Given $\Omega \subset \mathbb{R}^n_{\geqslant 0}$, define

$$\widehat{\Omega} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; (|x_1|, \ldots, |x_n|) \in \Omega \right\}.$$

Definition 5.1.5. *[GH18] A **convex toric domain** is a toric domain X_Ω such that $\widehat{\Omega}$ is compact and convex.*

[1]If γ is nondegenerate then the Conley-Zehnder index $\mathrm{CZ}(\gamma) \in \mathbb{Z}$ is well defined. If γ is degenerate then there is an interval of possible Conley-Zehnder indices of nondegenerate Reeb orbits near γ after a perturbation, and for dynamical convexity we require the minimum number in this interval to be at least $n+1$. In the 4-dimensional case ($n = 2$), this means that the dynamical rotation number of the linearized Reeb flow around γ, which we denote by $\rho(\gamma) \in \mathbb{R}$, is greater than 1.

This terminology may be misleading because a "convex toric domain" is not the same thing as a compact toric domain that is convex in $\mathbb{R}^{2n}$; see Proposition 5.1.7 below.

Definition 5.1.6. *[GH18] A **concave toric domain** is a toric domain X_Ω such that Ω is compact and $\mathbb{R}^n_{\geqslant 0} \backslash \Omega$ is convex.*

We remark that if X_Ω is a convex toric domain or concave toric domain and if X_Ω has smooth boundary, then it is a monotone toric domain.

Proposition 5.1.7. *A toric domain X_Ω is a convex subset of $\mathbb{R}^{2n}$ if and only if the set*

$$\widetilde{\Omega} = \left\{ \mu \in \mathbb{R}^n \;\middle|\; \pi\left(|\mu_1|^2, \ldots, |\mu|^2\right) \in \Omega \right\} \tag{5.1.4}$$

is convex in $\mathbb{R}^n$.

Proof. ($\Rightarrow$) The set $\widetilde{\Omega}$ is just the intersection of the toric domain X_Ω with the subspace $\mathbb{R}^n \subset \mathbb{C}^n$. If X_Ω is convex, then its intersection with any linear subspace is also convex.

($\Leftarrow$) Suppose that the set $\widetilde{\Omega}$ is convex. Let $z, z' \in X_\Omega$ and let $t \in [0,1]$. We need to show that

$$(1-t)z + tz' \in X_\Omega.$$

That is, we need to show that

$$\left(\left|(1-t)z_1 + tz'_1\right|, \ldots, \left|(1-t)z_n + z'_n\right|\right) \in \widetilde{\Omega}. \tag{5.1.5}$$

We know that the 2^n points $(\pm|z_1|, \ldots, \pm|z_n|)$ are all in $\widetilde{\Omega}$, as are the 2^n points $(\pm|z'_1|, \ldots, \pm|z'_n|)$. By the triangle inequality we have

$$|(1-t)z_j + tz'_j| \leqslant (1-t)|z_j| + t|z'_j|$$

for each $j = 1, \ldots, n$. It follows that the point in (5.1.5) can be expressed as $(1-t)$ times a convex combination of the points $(\pm|z_1|, \ldots, \pm|z_n|)$, plus t times a convex combination of the points $(\pm|z'_1|, \ldots, \pm|z'_n|)$. Since $\widetilde{\Omega}$ is convex, it follows that (5.1.5) holds. $\square$

Example 5.1.8. If X_Ω is a convex toric domain, then X_Ω is a convex subset of $\mathbb{R}^{2n}$.

Proof. Similarly to the above argument, this boils down to showing that if $w, w' \in \mathbb{C}$ and $0 \leqslant t \leqslant 1$ then

$$|(1-t)w + tw'|^2 \leqslant (1-t)|w|^2 + t|w'|^2.$$

The above inequality follows by expanding the left hand side and using the triangle inequality. $\square$

However the converse is not true:

Example 5.1.9. Let $p > 0$, and let Ω be the positive quadrant of the L^p unit ball,

$$\Omega = \left\{ \mu \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \sum_{j=1}^n \mu_j^p \leqslant 1 \right\}.$$

Then X_Ω is a concave toric domain iff $p \leqslant 1$, and a convex toric domain iff $p \geqslant 1$. By Proposition 5.1.7, the domain X_Ω is convex in $\mathbb{R}^{2n}$ if and only if $p \geqslant 1/2$.

5.2. Dynamics on the boundary

We perturb Ω to have some additional properties that will be useful. We may assume the following, where Σ denotes the closure of the set $\partial\Omega \cap \mathbb{R}^n_{>0}$:

(i) Σ is a smooth hypersurface in $\mathbb{R}^n$.

(ii) The Gauss map $G : \Sigma \to S^{n-1}$ is a smooth embedding, and ∂X_Ω is a smooth hypersurface in $\mathbb{R}^{2n}$. In particular, X_Ω is a nice star-shaped domain.

(iii) If $w \in \Sigma$ and if $w_i = 0$ for some i, then the i^{th} component of $G(w)$ is positive and small.

5.2.1. Reeb vector field

We first compute the Reeb vector field on $\partial X_\Omega = \mu^{-1}(\Sigma)$.

Let $w \in \Sigma$ and let $z \in \mu^{-1}(w)$. Also, write $G(w) = (\nu_1, \ldots, \nu_n)$. Observe that

$$\sum_i \nu_i w_i = \|G(w)\|^*_\Omega.$$

We now define local coordinates on a neighborhood of z in $\mathbb{C}^n$ as follows. For $i = 1, \ldots, n$, let $\mathbb{C}_i$ denote the i^{th} summand in $\mathbb{C}^n$. If $z_i = 0$, then we use the standard coordinates x_i and y_i on $\mathbb{C}_i$. If $z_i \neq 0$, then on $\mathbb{C}_i$ we use local coordinates μ_i and θ_i, where $\mu_i = \pi(x_i^2 + y_i^2)$, and θ_i is the angular polar coordinate.

In these coordinates, the standard Liouville form (1.1.1) is given by

$$\lambda_0 = \frac{1}{2} \sum_{w_i=0} (x_i\, dy_i - y_i\, dx_i) + \frac{1}{2\pi} \sum_{w_i \neq 0} \mu_i\, d\theta_i.$$

Also, the tangent space to ∂X_Ω at z is described by

$$T_z \partial X_\Omega = \bigoplus_{w_i=0} \mathbb{C}_i \oplus \left\{ \sum_{w_i \neq 0} \left(a_i \frac{\partial}{\partial \mu_i} + b_i \frac{\partial}{\partial \theta_i} \right) \,\middle|\, \sum_{w_i \neq 0} \nu_i a_i = 0 \right\}.$$

It follows from the above three equations that the Reeb vector field at z is given by

$$R = \frac{2\pi}{\|G(w)\|^*_\Omega} \sum_{w_i \neq 0} \nu_i \frac{\partial}{\partial \theta_i}. \tag{5.2.1}$$

For future reference, we also note that the contact structure ξ at z is given by

$$\xi_z = \bigoplus_{w_i=0} \mathbb{C}_i \oplus \left\{ \sum_{w_i \neq 0} \left(a_i \frac{\partial}{\partial \mu_i} + b_i \frac{\partial}{\partial \theta_i} \right) \,\middle|\, \sum_{w_i \neq 0} \nu_i a_i = 0,\ \sum_{w_i \neq 0} w_i b_i = 0 \right\}. \tag{5.2.2}$$

5.2.2. Reeb orbits

We now compute the Reeb orbits and their basic properties.

It is convenient here to define a (discontinuous) modification $\widetilde{G} : \Sigma \to \mathbb{R}^n$ of the Gauss map G by setting a component of the output to zero whenever the corresponding component of the input is zero. That is, for $i = 1, \ldots, n$ we define

$$\widetilde{G}(w)_i = \begin{cases} G(w)_i, & w_i \neq 0, \\ 0, & w_i = 0. \end{cases} \tag{5.2.3}$$

Observe from (5.2.1) that the Reeb vector field R is tangent to $\mu^{-1}(w)$. Let $Z(w)$ denote the number of components of w that are equal to zero; then $\mu^{-1}(w)$ is a torus of dimension $n - Z(w)$. It follows from (5.2.1) that if $\widetilde{G}(w)$ is a scalar multiple of an integer vector, then $\mu^{-1}(w)$ is foliated by an $(n - Z(w) - 1)$-dimensional Morse-Bott family of Reeb orbits; otherwise $\mu^{-1}(w)$ contains no Reeb orbits.

Let V denote the set of nonnegative integer vectors v such that v is a scalar multiple of an element $\tilde{v}$ of the image of the modified Gauss map $\widetilde{G}$. Given $v \in V$, let $d(v)$ denote the greatest common divisor of the components of v. Let $\mathscr{P}(v)$ denote the set of $d(v)$-fold covers of simple Reeb orbits in the torus $\mu^{-1}\left(\widetilde{G}^{-1}(\tilde{v})\right)$. Then it follows from the above discussion that the set of Reeb orbits on ∂X_Ω equals $\sqcup_{v \in V} \mathscr{P}(v)$. Moreover, condition (iii) above implies that $v \in V$ whenever $\sum_i v_i \leqslant k$.

Equation (5.2.1) implies that each Reeb orbit $\gamma \in \mathscr{P}(v)$ has symplectic action

$$\mathscr{A}(\gamma) = \|v\|_\Omega^*.$$

Also, we can define a trivialization τ of $\xi|_\gamma$ from (5.2.2), identifying ξ_z for each $z \in \gamma$ with a codimension two subspace of $\mathbb{R}^{2n}$ with coordinates x_i, y_i for each i with $w_i = 0$, and coordinates a_i, b_i for each i with $w_i \neq 0$. Then, we have

$$c_1(\gamma, \tau) = \sum_{i=1}^{n} v_i. \tag{5.2.4}$$

5.2.3. Nondegeneracy

We now approximate the convex toric domain X_Ω by a nice star-shaped domain X' such that $\lambda_0|_{\partial X'}$ is nondegenerate.

Given $v \in V$ with $d(v) = 1$, one can perturb ∂X_Ω in a neighborhood of the $n - Z(v)$ dimensional torus swept out by the Reeb orbits in $\mathscr{P}(v)$, using a Morse function f on the $n - Z(v) - 1$ dimensional torus $\mathscr{P}(v)$, to resolve the Morse-Bott family $\mathscr{P}(v)$ into a finite set of nondegenerate Reeb orbits corresponding to the critical points of f (possibly together with some additional Reeb orbits of much larger symplectic action). Owing to the strict convexity of Σ, each such nondegenerate Reeb orbit γ will have Conley-Zehnder index with respect to the above trivialization τ in the range

$$Z(v) \leqslant \mathrm{CZ}_\tau(\gamma) \leqslant n - 1. \tag{5.2.5}$$

It then follows from (5.2.4) that

$$Z(v) + 2\sum_{i=1}^{n} v_i \leqslant \mathrm{CZ}(\gamma) \leqslant n - 1 + 2\sum_{i=1}^{n} v_i. \tag{5.2.6}$$

In particular,

$$\mathrm{CZ}(\gamma) = 2k + n - 1 \Longrightarrow k \leqslant \sum_{i=1}^{n} v_i \leqslant k + \frac{n - 1 - Z(v)}{2}. \tag{5.2.7}$$

Moreover, even if we drop the assumption that $d(v) = 1$, then after perturbing the orbits in $\mathscr{P}(v/d(v))$ as above, the family $\mathscr{P}(v)$ will still be replaced by nondegenerate orbits each satisfying (5.2.6) (possibly together with additional Reeb orbits of much larger symplectic action), as long as $d(v)$ is not too large with respect to the perturbation.

Now choose $\epsilon > 0$ small and choose

$$R > \max\left\{ \|v\|_{\Omega}^{*} \;\middle|\; v \in \mathbb{N}^{n},\ \sum_{i} v_i \leqslant k + \frac{n-1}{2} \right\}.$$

We can then perturb X_Ω to a nice star-shaped domain X' with $\lambda_0|_{\partial X'}$ nondegenerate such that for each $v \in V$ with $\|v\|_{\Omega}^{*} < R$, the Morse-Bott family $\mathscr{P}(v)$ is perturbed as above; each nondegenerate orbit γ arising from each such $\mathscr{P}(v)$ has symplectic action satisfying

$$\mathscr{A}(\gamma) \geqslant \|v\|_{\Omega}^{*} - \epsilon; \tag{5.2.8}$$

and there are no other Reeb orbits of symplectic action less than R.

5.2.4. Proof of Proposition 5.1.4.

Proof. As a preliminary remark, note that if a Reeb orbit has rotation number $\rho > 1$, then so does every iterate of the Reeb orbit. Thus X_Ω is dynamically convex if and only if every **simple** Reeb orbit has rotation number $\rho > 1$.

Since X_Ω is star-shaped, Ω itself is also star-shaped. Since X_Ω is compact with smooth boundary, $\overline{\partial_+\Omega}$ is a smooth arc from some point $(0,b)$ with $b > 0$ to some point $(a,0)$ with $a > 0$.

We can find the simple Reeb orbits and their rotation numbers by the calculations in [CCGF⁺14, §3.2] and [GH18, §2.2]. The conclusion is the following. There are three types of simple Reeb orbits on ∂X_Ω:

(i) There is a simple Reeb orbit corresponding to $(a,0)$, whose image is the circle in ∂X_Ω with $\pi|z_1|^2 = a$ and $z_2 = 0$.

(ii) Likewise, there is a simple Reeb orbit corresponding to $(0,b)$, whose image is the circle in ∂X_Ω with $z_1 = 0$ and $\pi|z_2|^2 = b$.

(iii) For each point $\mu \in \partial_+\Omega$ where $\partial_+\Omega$ has rational slope, there is an S^1 family of simple Reeb orbits whose images sweep out the torus in ∂X_Ω where $\pi(|z_1|^2, |z_2|^2) = \mu$.

Let s_1 denote the slope of $\overline{\partial_+\Omega}$ at $(a,0)$, and let s_2 denote the slope of $\overline{\partial_+\Omega}$ at $(0,b)$. Then the Reeb orbit in (i) has rotation number $\rho = 1 - s_1^{-1}$, and the Reeb orbit in (ii) has rotation number $\rho = 1 - s_2$. For a Reeb orbit in (iii), let $v = (v_1, v_2)$ be the outward normal vector to $\partial_+\Omega$ at μ, scaled so that v_1, v_2 are relatively prime integers. Then each Reeb orbit in this family has rotation number $\rho = v_1 + v_2$.

If X_Ω is strictly monotone, then $s_1, s_2 < 0$, and for each Reeb orbit of type (iii) we have $v_1, v_2 \geqslant 1$. It follows that every simple Reeb orbit has rotation number $\rho > 1$.

Conversely, suppose that every simple Reeb orbit has rotation number $\rho > 1$. Applying this to the Reeb orbits (i) and (ii), we obtain that $s_1, s_2 < 0$. Thus $\partial_+\Omega$ has negative slope near its endpoints. The arc $\partial_+\Omega$ can never go horizontal or vertical in its interior, because otherwise there would be a Reeb orbit of type (iii) with $v = (1,0)$ or $v = (0,1)$, so that $\rho = 1$. Thus X_Ω is strictly monotone. □

6. Symplectic capacities

This chapter presents some known capacities (ECH and Ekeland-Hofer) and the capacities I introduced with Michael Hutchings. The latter are defined using positive S^1-equivariant symplectic homology. A nice feature is that they can be computed explicitely for all convex or concave toric domains. A nice application is to prove (Theorem 6.3.18) that the inclusion gives the "optimal" symplectic embedding of a cube in any concave or convex toric domain.

6.1. ECH capacities

Let (Y, λ) be a non-degenerate 3-dimensional contact manifold. The embedded contact homology (ECH) of Y is the homology of a chain complex (over $\mathbb{Z}$) which is generated by the ECH generators. We refer to [Hut14] and the reference therein for a complete presentation.

Definition 6.1.1. *An ECH generator is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the α_i are distinct periodic Reeb orbits, the m_i are positive integers and if α_i is hyperbolic, then $m_i = 1$.*
The symplectic action of an ECH generator is defined as

$$I(\alpha) := \sum_i m_i \mathscr{A}(\alpha_i)$$

The differential counts certain embedded pseudo-holomorphic curves in $\mathbb{R} \times Y$. In general the ECH is a topological invariant of compact three-manifolds, related to Seiberg-Witten Floer homology (see [Tau10, Hut14]).

The ECH spectrum of Y is a sequence of real numbers

$$0 < c_1^{ECH}(Y) \leqslant c_2^{ECH}(Y) \leqslant \cdots \leqslant \infty$$

such that $c_k^{ECH}(Y)$ is the minimal L such that the grading $2k$ class in ECH can be represented in the ECH chain complex by a linear combination of ECH generators each having symplectic action $\leqslant L$. When Y is the boundary of a symplectic manifold X, $c_k^{ECH}(Y)$ is called the k^{th} ECH capacity of X.

We now recall some facts about ECH capacities which we will use to prove Theorem 7.0.1.

Definition 6.1.2. *A* weakly convex toric domain *in $\mathbb{R}^4$ is a compact toric domain $X_\Omega \subset \mathbb{R}^4$ such that Ω is convex, and $\partial_+\Omega$ is an arc with one endpoint on the positive μ_1 axis and one endpoint on the positive μ_2 axis.*

Theorem 6.1.3 (Cristofaro-Gardiner [CG14])**.** *In four dimensions, let X_Ω be a concave toric domain, and let $X_{\Omega'}$ be a weakly convex toric domain. Then there exists a symplectic embedding $\operatorname{int}(X_\Omega) \underset{s}{\hookrightarrow} X_{\Omega'}$ if and only if $c_k^{\mathrm{ECH}}(X_\Omega) \leqslant c_k^{\mathrm{ECH}}(X_{\Omega'})$ for all $k \geqslant 0$.*

To make use of this theorem, we need some formulas to compute the ECH capacities c_k^{ECH}. To start, let us consider a 4-dimensional concave toric domain X_Ω. Associated to X_Ω is a "weight sequence" $W(X_\Omega)$, which is a finite or countable multiset of positive real numbers defined in [CCGF+14], see also [Ram17], as follows. Let r be the largest positive real number such that the triangle $\Delta^2(r) \subset \Omega$. We can write $\Omega \backslash \Delta^2(r) = \widetilde{\Omega}_1 \sqcup \widetilde{\Omega}_2$, where $\widetilde{\Omega}_1$ does not intersect the μ_2-axis and $\widetilde{\Omega}_2$ does not intersect the μ_1-axis. It is possible that $\widetilde{\Omega}_1$ and/or $\widetilde{\Omega}_2$ is empty. After translating the closures of $\widetilde{\Omega}_1$ or $\widetilde{\Omega}_2$ by $(-r,0)$ and $(0,-r)$ and multiplying them by the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, respectively, we obtain two new domains Ω_1 and Ω_2 in $\mathbb{R}^2_{\geqslant 0}$ such that X_{Ω_1} and X_{Ω_2} are concave toric domains. We then inductively define

$$W(X_\Omega) = (r) \cup W(X_{\Omega_1}) \cup W(X_{\Omega_2}), \tag{6.1.1}$$

where '$\cup$' denotes the union of multisets, and the term $W(X_{\Omega_i})$ is omitted if Ω_i is empty.

Let us call two subsets of $\mathbb{R}^2$ "affine equivalent" if one can be obtained from the other by the composition of a translation and an element of $\mathrm{GL}(2,\mathbb{Z})$. If $W(X_\Omega) = (a_1, a_2, \dots)$, then the domain Ω is canonically decomposed into triangles, which are affine equivalent to the triangles $\Delta^2(a_1), \Delta^2(a_2), \dots$ and which meet only along their edges; the first of these triangles is $\Delta^2(r)$. See [Hut19, §3.1] for more details. We now recall the "Traynor trick":

Proposition 6.1.4. *[Tra95] If $T \subset \mathbb{R}^2_{\geqslant 0}$ is a triangle affine equivalent to $\Delta^2(a)$, then there is a symplectic embedding $\mathrm{int}(B^4(a)) \underset{s}{\hookrightarrow} X_{\mathrm{int}(T)}$.*

As a result, there is a symplectic embedding

$$\coprod_i \mathrm{int}(B^4(a_i)) \subset X_\Omega.$$

Consequently, by the monotonicity property of ECH capacities, we have

$$c_k^{\mathrm{ECH}}\left(\coprod_i \mathrm{int}(B^4(a_i))\right) \leqslant c_k^{\mathrm{ECH}}(X_\Omega). \tag{6.1.2}$$

Theorem 6.1.5 ([CCGF+14]). *If X_Ω is a four-dimensional concave toric domain with weight expansion $W(X_\Omega) = (a_1, a_2, \dots)$, then equality holds in (6.1.2).*

To make this more explicit, we know from [Hut11] that[1]

$$c_k^{\mathrm{ECH}}\left(\coprod_i \mathrm{int}(B^4(a_i))\right) = \sup_{k_1+\cdots=k} \sum_i c_{k_i}^{\mathrm{ECH}}(\mathrm{int}(B^4(a_i))) \tag{6.1.3}$$

and

$$c_k^{\mathrm{ECH}}(\mathrm{int}(B^4(a))) = c_k^{\mathrm{ECH}}(B^4(a)) = da, \tag{6.1.4}$$

where d is the unique nonnegative integer such that

$$d^2 + d \leqslant 2k \leqslant d^2 + 3d.$$

To state the next lemma, given $a_1, a_2 > 0$, define the polydisk

$$P(a_1,a_2) = \left\{ z \in \mathbb{C}^2 \;\middle|\; \pi|z_1|^2 \leqslant a_1,\ \pi|z_2|^2 \leqslant a_2 \right\}.$$

This is a convex toric domain $X_{\Omega'}$ where Ω' is a rectangle of side lengths a_1 and a_2.

[1] For the sequence of numbers a_i coming from a weight expansion, or for any finite sequence, the supremum in (6.1.3) is achieved, so we can write 'max' instead of 'sup'.

Lemma 6.1.6. *Let X_Ω be a four-dimensional concave toric domain. Let $(a,0)$ and $(0,b)$ be the points where $\partial_+\Omega$ intersects the axes. Let μ be a point on $\overline{\partial_+\Omega}$ minimizing $\mu_1+\mu_2$, and write $r=\mu_1+\mu_2$. Then there exists a symplectic embedding*

$$\operatorname{int}(X_\Omega) \underset{s}{\hookrightarrow} P(r, \max(b, a-\mu_1)).$$

Proof. One might hope for a direct construction using some version of "symplectic folding" [Sch99], but we will instead use the above ECH machinery. By Theorem 6.1.3, it is enough to show that

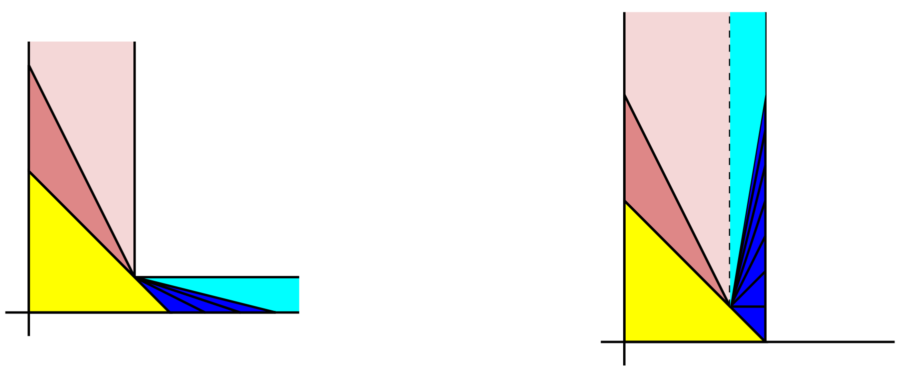
$$c_k^{\mathrm{ECH}}(X_\Omega) \leqslant c_k^{\mathrm{ECH}}(P(r, \max(b, a-\mu_1)) \tag{6.1.5}$$

for each nonnegative integer k.

Consider the weight expansion $W(X_\Omega)=(a_1,a_2,\dots)$ where $a_1=r$. The decomposition of Ω into triangles corresponding to the weight expansion consists of the triangle $\Delta^2(r)$, plus some additional triangles in the quadrilateral with corners $(0,r),(\mu_1,\mu_2),(\mu_1,b),(0,b)$, plus some additional triangles in the quadrilateral with corners $(\mu_1,\mu_2),(r,0),(a,0),(a,\mu_2)$; see Figure 6.1a. The latter quadrilateral is affine equivalent to the quadrilateral with corners $(\mu_1,\mu_2),(r,0),(r,a-\mu_1),(\mu_1,a-\mu_1)$; see Figure 6.1b. This allows us to pack triangles affine equivalent to $\Delta^2(a_1),\Delta^2(a_2),\dots$ into the rectangle with horizontal side length r and vertical side length $\max(b,a-\mu_1)$. Thus by the Traynor trick, we have a symplectic embedding

$$\coprod_i \operatorname{int}(B(a_i)) \underset{s}{\hookrightarrow} P(r, \max(b, a-\mu_1)).$$

Then Theorem 6.3.14 and the monotonicity of ECH capacities imply (6.1.5). $\square$



(a) Weights of X_Ω (b) Ball packing into a polydisk

Figure 6.1.: Embedding a concave toric domain into a polydisk

6.2. Ekeland-Hofer capacities

Let $\mathscr{H} = H^{1/2}(S^1, \mathbb{R}^{2n})$. We can decompose $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^0 \oplus \mathscr{H}^-$ where $\mathscr{H}^+$ and $\mathscr{H}^-$ are the subsets of the functions that only contain positive and negative Fourier coefficients, respectively, and $\mathscr{H}^0$ is the subset of constant functions. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ a Hamiltonian which is quadratic outside of a compact set and whose support contains Ω. For $x \in \mathscr{H}$, we define

$$\mathscr{A}_H(x) = -\frac{1}{2}\int_0^1 J\dot{x}(t) \cdot x(t) dt - \int_0^1 H\big(t, x(t)\big) dt.$$

For an S^1 invariant subset $X \subset \mathscr{H}$, one can define an index $\alpha(X) \in \mathbb{Z}$ as follows. Consider the classifying map $f : X \times_{S^1} ES^1 \to BS^1 = \mathbb{C}P^\infty$. So f induces a map $f^* : H^*(\mathbb{C}P^\infty) \to H^*_{S^1}(X) = H^*(X \times_{S^1} ES^1)$. Let u be a generator of the ring $H^*(\mathbb{C}P^\infty)$ and define

$$\alpha(X) = \max\{k \in \mathbb{Z} | f^* u^k \neq 0\}.$$

Now we give an alternative description of the index α. Let $U : H^{S^1}_*(X) \to H^{S^1}_*(X)$ be the map defined by $U(\gamma) = f^\star u \cap \gamma$.

Proposition 6.2.1. *Let X be an S^1-space. Then*

$$\alpha(X) = \max\{k \in \mathbb{Z} | U^k \neq 0\}.$$

Proof. We first remark that there is a map ${}^\tau U$ on the dual. We have $H^*_{S^1}(X) \simeq \mathrm{Hom}\left(H^{S^1}_*(X); \mathbb{Q}\right)$.

$$\begin{tikzcd}
\mathrm{Hom}\left(H^{S^1}_*(X); \mathbb{Q}\right) & \mathrm{Hom}\left(H^{S^1}_*(X); \mathbb{Q}\right) \arrow[l, "{}^\tau U"'] \arrow[d, "\simeq"'] \\
H^*_{S^1}(X) \arrow[u, "\simeq"'] & H^*_{S^1}(X) \arrow[l, "\cup f^\star(u)"']
\end{tikzcd}$$

$${}^\tau U \simeq \cup f^\star(u)$$

Fact 6.2.2. $U = 0$ if and only if ${}^\tau U = 0$.

The fact follows from the "duality" between U and ${}^\tau U$; i.e. $\langle {}^\tau U a, b\rangle = \langle a, Ub\rangle$.

Thus,

$$\sup\{k | U^k \neq 0\} = \sup\{k | ({}^\tau U)^k \neq 0\} = \sup\{k | f^\star u^k \neq 0\}$$

Indeed, to see the last equality, note first that $\geqslant$ is obvious and $\leqslant$ is because $f^\star u^k \cup 1 = f^\star u^k$. □

Ekeland and Hofer defined a subgroup Γ of the group of homeomorphisms of $\mathscr{H}$ with compact support (which we recall later) and denoted by S^+ the unit sphere in $\mathscr{H}^+$. For an S^1-invariant subspace $\xi \subset \mathscr{H}$, they also defined

$$\mathrm{ind}(\xi) = \min_{h \in \Gamma} \alpha(X \cap h(S^+))$$

Finally, they defined

$$c_k^{EH}(H) = \inf\left\{\sup \mathscr{A}_H(\xi) \middle| \mathrm{ind}(\xi) \geqslant k\right\}.$$

Let us now recall the definition of the group Γ: a homeomorphism h belongs to Γ if h is of the following form:

$$h(x) = e^{\gamma^+(x)} x^+ + x^0 + e^{\gamma^-(x)} x^- + K(x)$$

where γ^+ and γ^+ are maps $\mathscr{H} \to \mathbb{R}$ which are required to be continuous, S^1-invariant and mapping bounded sets into bounded sets while $K : \mathscr{H} \to \mathscr{H}$ is continuous, S^1-equivariant, mapping bounded sets to pre-compact sets. Additionally, there must exists a number $\rho > 0$ such that, either $\mathscr{A}_H(x) \leqslant 0$ or $\|x\| \geqslant \rho$ implies $\gamma^+(x) = \gamma^-(x) = 0$ and $K(x) = 0$.

Computations of these capacities are known in a few examples. To state these, if $a_1, \dots, a_n > 0$, let $(M_k(a_1, \dots, a_n))_{k=1,2,\dots}$ denote the sequence of positive integer multiples of $a_1, \dots, a_n$, arranged in nondecreasing order with repetitions. We then have:

- [EH90, Prop. 4] The Ekeland-Hofer capacities of an ellipsoid are given by

$$c_k^{\mathrm{EH}}(E(a_1, \dots, a_n)) = M_k(a_1, \dots, a_n). \tag{6.2.1}$$

- [EH90, Prop. 5] The Ekeland-Hofer capacities of a polydisk are given by

$$c_k^{\mathrm{EH}}(P(a_1, \dots, a_n)) = k \cdot \min(a_1, \dots, a_n). \tag{6.2.2}$$

- Generalizing (6.2.2), it is asserted in [CHLS07, Eq. (3.8)] that if $X \subset \mathbb{R}^{2n}$ and $X' \subset \mathbb{R}^{2n'}$ are compact star-shaped domains, then for the (symplectic) Cartesian product $X \times X' \subset \mathbb{R}^{2(n+n')}$, we have

$$c_k^{\mathrm{EH}}(X \times X') = \min_{i+j=k} \{c_i^{\mathrm{EH}}(X) + c_j^{\mathrm{EH}}(X')\}, \tag{6.2.3}$$

  where i and j are nonnegative integers and we interpret $c_0^{\mathrm{EH}} = 0$.

6.3. Capacities from positive S^1-equivariant symplectic homology

6.3.1. Nondegenerate Liouville domains

We first define the capacities c_k for nondegenerate Liouville domains, imitating the definition of ECH capacities in [Hut11, Def. 4.3].

Definition 6.3.1. *Let (X, λ) be a nondegenerate Liouville domain and let k be a positive integer. Define*

$$c_k(X, \lambda) \in (0, \infty]$$

to be the infimum over L such that there exists $\alpha \in CH^L(X, \lambda)$ satisfying

$$\delta U^{k-1} \iota_L \alpha = [X] \otimes [\mathrm{pt}] \in H_*(X, \partial X) \otimes H_*(BS^1). \tag{6.3.1}$$

6.3.2. Arbitrary Liouville domains

We now extend the definition of c_k to an arbitrary Liouville domain (X, λ). To do so, we use the following procedure to perturb a possibly degenerate Liouville domain to a nondegenerate one.

First recall that there is a distinguished Liouville vector field V on X characterized by $\iota_V d\lambda = \lambda$. Write $Y = \partial X$. The flow of V then defines a smooth embedding

$$(-\infty, 0] \times Y \longrightarrow X, \tag{6.3.2}$$

sending $\{0\} \times Y$ to Y in the obvious way, such that if ρ denotes the $(-\infty, 0]$ coordinate, then ∂_ρ is mapped to the vector field V. This embedding pulls back the Liouville form λ on X to the 1-form $e^\rho(\lambda|_Y)$ on $(-\infty, 0] \times Y$. The **completion** of (X, λ) is the pair $(\widehat{X}, \widehat{\lambda})$ defined as follows. First,

$$\widehat{X} = X \cup_Y ([0, \infty) \times Y),$$

glued using the identification (6.3.2). Observe that $\widehat{X}$ has a subset which is identified with $\mathbb{R} \times Y$, and we denote the $\mathbb{R}$ coordinate on this subset by ρ. The 1-form λ on X then extends to a unique 1-form $\widehat{\lambda}$ on $\widehat{X}$ which agrees with $e^\rho(\lambda|_Y)$ on $\mathbb{R} \times Y$.

Now if $f : Y \to \mathbb{R}$ is any smooth function, define a new Liouville domain (X_f, λ_f), where

$$X_f = \widehat{X} \setminus \{(\rho, y) \in \mathbb{R} \times Y \mid \rho > f(y)\},$$

and λ_f is the restriction of $\widehat{\lambda}$ to X_f. For example, if $f \equiv 0$, then $(X_f, \lambda_f) = (X, \lambda)$. In general, there is a canonical identification

$$\begin{aligned} Y &\longrightarrow \partial X_f, \\ y &\longmapsto (f(y), y) \in \mathbb{R} \times Y. \end{aligned}$$

Under this identification,

$$\lambda_f|_{\partial X_f} = e^f \lambda|_Y.$$

We now consider c_k of nondegenerate perturbations of a possibly degenerate Liouville domain.

Lemma 6.3.2. *(cf. [Hut11, Lem. 3.5])*

(a) If (X, λ) is any Liouville domain, then

$$\sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}) = \inf_{f_+ > 0} c_k(X_{f_+}, \lambda_{f_+}). \tag{6.3.3}$$

Here the supremum and infimum are taken over functions $f_- : Y \to (-\infty, 0)$ and $f_+ : Y \to (0, \infty)$ respectively such that the contact form $e^{f_\pm}(\lambda|_Y)$ is nondegenerate.

(b) If (X, λ) is nondegenerate, then the supremum and infimum in (6.3.3) agree with $c_k(X, \lambda)$.

As a result of Lemma 6.3.2, it makes sense to extend Definition 6.3.1 as follows:

Definition 6.3.3. *If (X, λ) is any Liouville domain, let us define $c_k(X, \lambda)$ to be the supremum and infimum in (6.3.3).*

Definition 6.3.4. *Let (X, λ) and (X', λ') be Liouville domains of the same dimension. A* ***generalized Liouville embedding*** *$(X, \lambda) \to (X', \lambda')$ is a symplectic embedding $\varphi : (X, d\lambda) \to (X', d\lambda')$ such that*

$$\left[(\varphi^\star \lambda' - \lambda)|_{\partial X} \right] = 0 \in H^1(\partial X; \mathbb{R}).$$

Of course, if $H^1(\partial X; \mathbb{R}) = 0$, for example if X is a nice star-shaped domain in $\mathbb{R}^{2n}$, then every symplectic embedding is a generalized Liouville embedding.

Theorem 6.3.5. *The functions c_k of Liouville domains satisfy the following axioms:*

(Conformality) If (X, λ) is a Liouville domain and r is a positive real number, then $c(X, r\lambda) = rc(X, \lambda)$.

(Increasing) $c_1(X,\lambda) \leqslant c_2(X,\lambda) \leqslant \cdots \leqslant \infty$.

(Restricted Monotonicity) If there exists a generalized Liouville embedding $(X,\lambda) \to (X',\lambda')$*, then* $c_k(X,\lambda) \leqslant c_k(X',\lambda')$.

(Contractible Reeb Orbits) If $c_k(X,\lambda) < \infty$*, then* $c_k(X,\lambda) = \mathscr{A}(\gamma)$ *for some Reeb orbit* γ *of* $\lambda|_{\partial X}$ *which is contractible[2] in* X.

Remark 6.3.6. *In the case where* X *is a star-shaped domain in* $\mathbb{R}^{2n}$ *and if* $\lambda_0|_{\partial X}$ *is nondegenerate, then* $c_k(X) = \mathscr{A}(\gamma)$ *for some Reeb orbit* γ *of* $\lambda_0|_{\partial X}$ *with* $\mathrm{CZ}(\gamma) = 2k+n-1$.

Remark 6.3.7. *Monotonicity does not extend from generalized Liouville embeddings to arbitrary symplectic embeddings: in some cases there exists a symplectic embedding* $(X,d\lambda) \to (X',d\lambda')$ *even though* $c_k(X,\lambda) > c_k(X',\lambda')$*. For example, suppose that* $T \subset X'$ *is a Lagrangian torus. Let* λ_T *denote the standard Liouville form on the cotangent bundle* T^*T*. By the Weinstein Lagrangian tubular neighborhood theorem, there is a symplectic embedding* $(X,d\lambda) \to (X',d\lambda')$*, where* $X \subset T^*T$ *is the unit disk bundle for some flat metric on* T*, and* $\lambda = \lambda_T|_X$*. Then* (X,λ) *is a Liouville domain. But* $\lambda|_{\partial X}$ *has no Reeb orbits which are contractible in* X*, so by the Contractible Reeb Orbits axiom,* $c_k(X,\lambda) = \infty$ *for all* k.

Note that the symplectic embedding $(X,d\lambda) \to (X',d\lambda')$ *is a generalized Liouville embedding if and only if* T *is an* ***exact*** *Lagrangian torus in* (X',λ')*, that is* $\lambda'|_T$ *is exact. The Restricted Monotonicity axiom then tells us that* if (X',λ') is a Liouville domain with $c_1(X',\lambda') < \infty$, then (X',λ') does not contain any exact Lagrangian torus.

Remark 6.3.8. *The functions* c_k *are defined for disconnected Liouville domains. However, it follows from the definition that*

$$c_k\left(\coprod_{i=1}^{m}(X_i,\lambda_i)\right) = \max_{i=1,\ldots,m} c_k(X_i,\lambda_i).$$

As a result, Restricted Monotonicity for embeddings of disconnected Liouville domains does not tell us anything more than it already does for their connected components.

Remark 6.3.9. *One can ask whether, by analogy with ECH capacities [Hut11, Prop. 1.5], the existence of a generalized Liouville embedding* $\coprod_{i=1}^{m}(X_i,\lambda_i) \to (X',\lambda')$ *implies that*

$$\sum_{i=1}^{m} c_{k_i}(X_i,\lambda_i) \leqslant c_{k_1+\cdots+k_m}(X',\lambda') \tag{6.3.4}$$

for all positive integers $k_1,\cdots,k_m$*. We have heuristic reasons to expect this when the* k_i *are all multiples of* $n-1$*. However it is false more generally.*

For example, in $2n$ *dimensions, the Traynor trick [Tra95] can be used to symplectically embed the disjoint union of* n^2 *copies of the ball* $B(1/2-\varepsilon)$ *into the ball* $B(1)$*, for any* $\varepsilon > 0$*. If* (6.3.4) *is true with all* $k_i = 1$*, then we obtain*

$$n^2(1/2-\varepsilon) \leqslant n.$$

But this is false when $n > 2$ *and* $\varepsilon > 0$ *is small enough.*

[2] Here $\mathscr{A}(\gamma)$ denotes the symplectic action of γ, which is defined by $\mathscr{A}(\gamma) = \int_\gamma \lambda$.

6.3.3. Computations

One can compute the capacities c_k for many examples of star-shaped domains in $\mathbb{R}^{2n}$, using only the axioms in Theorem 6.3.5.

We now compute the capacities c_k of a convex toric domain X_Ω in $\mathbb{R}^{2n}$. If $v \in \mathbb{R}^n_{\geqslant 0}$ is a vector with all components nonnegative, define[3]

$$\|v\|^*_\Omega = \max\{\langle v, w\rangle \mid w \in \Omega\} \tag{6.3.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Let $\mathbb{N}$ denote the set of nonnegative integers.

Theorem 6.3.10 ([GH18]). *Suppose that X_Ω is a convex toric domain in $\mathbb{R}^{2n}$. Then*

$$c_k(X_\Omega) = \min\left\{ \|v\|^*_\Omega \;\middle|\; v = (v_1, \dots, v_n) \in \mathbb{N}^n,\ \sum_{i=1}^n v_i = k \right\}. \tag{6.3.6}$$

In fact, (6.3.6) holds for any function c_k defined on nice star-shaped domains in $\mathbb{R}^{2n}$ and satisfying the axioms in Theorem 6.3.5.

Example 6.3.11. *The polydisk $P(a_1, \dots, a_n)$ is a convex toric domain X_Ω, where Ω is the rectangle*

$$\Omega = \{x \in \mathbb{R}^n_{\geqslant 0} \mid x_i \leqslant a_i,\ \forall i = 1, \dots, n\}.$$

In this case

$$\|v\|^*_\Omega = \sum_{i=1}^n a_i v_i.$$

It then follows from (6.3.6) that

$$c_k(P(a_1, \dots, a_n)) = k \cdot \min\{a_1, \dots, a_n\}.$$

Example 6.3.12. *The ellipsoid $E(a_1, \dots, a_n)$ is a convex toric domain X_Ω, where Ω is the simplex*

$$\Omega = \left\{ x \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \sum_{i=1}^n \frac{x_i}{a_i} \leqslant 1 \right\}.$$

In this case

$$\|v\|^*_\Omega = \max_{i=1,\dots,n} a_i v_i.$$

Then (6.3.6) gives

$$c_k(E(a_1, \dots, a_n)) = \min_{\sum_i v_i = k} \max_{i=1,\dots,n} a_i v_i.$$

It is a combinatorial exercise[4] to check that

$$\min_{\sum_i v_i = k} \max_{i=1,\dots,n} a_i v_i = M_k(a_1, \dots, a_n). \tag{6.3.7}$$

We conclude that

$$c_k(E(a_1, \dots, a_n) = M_k(a_1, \dots, a_n). \tag{6.3.8}$$

[3] The reason for this notation is as follows. Let $\|\cdot\|_\Omega$ denote the norm on $\mathbb{R}^n$ whose unit ball is $\widehat{\Omega}$. Then in equation (6.3.5), $\|\cdot\|^*_\Omega$ denotes the dual norm on $(\mathbb{R}^n)^*$, where the latter is identified with $\mathbb{R}^n$ using the Euclidean inner product.

[4] To do the exercise, by a continuity argument we may assume that a_i/a_j is irrational when $i \neq j$, so that the positive integer multiples of the numbers a_i are distinct. If $v \in \mathbb{N}^n$ and $\sum_i v_i = k$, then the k numbers ma_i where $1 \leqslant i \leqslant n$ and $1 \leqslant m \leqslant v_i$ are distinct, which implies that the left hand side of (6.3.7) is greater than or equal to the right hand side. To prove the reverse inequality, if $L = M_k(a_1, \dots, a_n)$, then the numbers $v_i = \lfloor L/a_i \rfloor$ satisfy $\sum_i v_i = k$ and $\max_{i=1,\dots,n} a_i v_i = L$.

Comparing the above two examples with equations (6.2.1) and (6.2.2) suggests that our capacities c_k may agree with the Ekeland-Hofer capacities c_k^{EH}:

Conjecture 6.3.13. *Let X be a compact star-shaped domain in $\mathbb{R}^{2n}$. Then*

$$c_k(X) = c_k^{\mathrm{EH}}(X)$$

for every positive integer k.

We can also compute the capacities c_k of another family of examples: concave toric domains. Suppose that X_Ω is a concave toric domain. Let Σ denote the closure of the set $\partial\Omega \cap \mathbb{R}^n_{>0}$. Similarly to (6.3.5), if $v \in \mathbb{R}^n_{\geqslant 0}$, define[5]

$$[v]_\Omega = \min\left\{\langle v, w\rangle \,\middle|\, w \in \Sigma\right\}. \tag{6.3.9}$$

Theorem 6.3.14 ([GH18])**.** *If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then*

$$c_k(X_\Omega) = \max\left\{[v]_\Omega \,\middle|\, v \in \mathbb{N}^n_{>0},\ \sum_{i=1}^n v_i = k+n-1\right\}. \tag{6.3.10}$$

Note that in (6.3.10), all components of v are required to be positive, while in (6.3.6), we only required that all components of v be nonnegative.

Example 6.3.15. *Let us check that (6.3.10) gives the correct answer when X_Ω is an ellipsoid $E(a_1, \ldots, a_n)$. Similarly to Example 6.3.12, we have*

$$[v]_\Omega = \min_{i=1,\ldots,n} a_i v_i.$$

Thus, we need to check that

$$\max_{\sum_i v_i = k+n-1}\ \min_{i=1,\ldots,n} a_i v_i = M_k(a_1, \ldots, a_n) \tag{6.3.11}$$

where, unlike Example 6.3.12, now all components of v must be positive integers. This can be proved similarly to (6.3.7).

A quick application of Theorem 6.3.14, pointed out by Schlenk [Sch18, Cor. 11.5], is to compute the Gromov width of any concave toric domain[6]:

Corollary 6.3.16 ([GH18])**.** *If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then*

$$c_{\mathrm{Gr}}(X_\Omega) = \max\{a \mid B(a) \subset X_\Omega\}.$$

Proof. Let $a_{\max}$ denote the largest real number a such that $B(a) \subset X_\Omega$. By the definition of the Gromov width c_{Gr}, we have $c_{\mathrm{Gr}}(X_\Omega) \geqslant a_{\max}$. To prove the reverse inequality $c_{\mathrm{Gr}}(X_\Omega) \leqslant a_{\max}$, suppose that there exists a symplectic embedding $B(a) \to X_\Omega$; we need to show that $a \leqslant a_{\max}$. By equation (6.3.8), the monotonicity property of c_1, and Theorem 6.3.14, we have

$$\begin{aligned}
a &= c_1(B(a)) \\
&\leqslant c_1(X_\Omega) \\
&= [(1, \ldots, 1)]_\Omega \\
&= \min\left\{\sum_{i=1}^n w_i \,\middle|\, w \in \Sigma\right\} \\
&= a_{\max}.
\end{aligned}$$

□

[5] Unlike (6.3.5), the function $[\cdot]_\Omega$ is not a norm; instead it satisfies the reverse inequality $[v+v']_\Omega \geqslant [v]_\Omega + [v']_\Omega$.

[6] The four-dimensional case of this was shown using ECH capacities in [CCGF+14, Cor. 1.10].

6.3.4. Application to cube capacities

We now use the above results to solve some symplectic embedding problems where the domain is a cube.

Given $\delta > 0$, define the **cube**

$$\square_n(\delta) = P(\delta, \dots, \delta) \subset \mathbb{C}^n.$$

Equivalently,

$$\square_n(\delta) = \left\{ z \in \mathbb{C}^n \;\middle|\; \max_{i=1,\dots,n} \left\{ \pi |z_i|^2 \right\} \leqslant \delta \right\}.$$

Definition 6.3.17 ([GH18]). *Given a $2n$-dimensional symplectic manifold (X, ω), define the* ***cube capacity***

$$c_\square(X, \omega) = \sup \left\{ \delta > 0 \mid \text{there exists a symplectic embedding } \square_n(\delta) \longrightarrow (X, \omega) \right\}.$$

It is immediate from the definition that $c_\square$ is a symplectic capacity.

Theorem 6.3.18 ([GH18]). *Let $X_\Omega \subset \mathbb{C}^n$ be a convex toric domain or a concave toric domain. Then*

$$c_\square(X_\Omega) = \max\{\delta \mid (\delta, \dots, \delta) \in \Omega\}.$$

That is, $c_\square(X_\Omega)$ is the largest δ such that $\square_n(\delta)$ is a subset of X_Ω; one cannot do better than this obvious symplectic embedding by inclusion.

Since the proof of Theorem 6.3.18 is short, we will give it now. We need to consider the non-disjoint union of n symplectic cylinders,

$$L_n(\delta) = \left\{ z \in \mathbb{C}^n \;\middle|\; \min_{i=1,\dots,n} \left\{ \pi |z_i|^2 \right\} \leqslant \delta \right\}.$$

Lemma 6.3.19 ([GH18]). $c_k(L_n(\delta)) = \delta(k + n - 1)$.

Proof. Observe that $L_n(\delta) = X_{\Omega_\delta}$ where

$$\Omega_\delta = \left\{ x \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \min_{i=1,\dots,n} x_i \leqslant \delta \right\}.$$

As such, Ω_δ is the union of a nested sequence of concave toric domains. By an exhaustion argument, the statement of Theorem 6.3.14 is valid for X_{Ω_δ}. Similarly to Example 6.3.11, we have

$$[v]_{\Omega_\delta} = \delta \sum_{i=1}^{n} v_i.$$

The lemma then follows from equation (6.3.10). $\square$

Proposition 6.3.20 ([GH18]). $c_\square(L_n(\delta)) = \delta$.

Proof. We have $\square_n(\delta) \subset L_n(\delta)$, so by the definition of $c_\square$, it follows that $c_\square(L_n(\delta)) \geqslant \delta$.

To prove the reverse inequality $c_\square(L_n(\delta)) \leqslant \delta$, suppose that there exists a symplectic embedding $\square_n(\delta') \to L_n(\delta)$; we need to show that $\delta' \leqslant \delta$. By the Monotonicity property of the capacities c_k, we know that

$$c_k(\square_n(\delta')) \leqslant c_k(L_n(\delta))$$

for each positive integer k. By Example 6.3.11 and Lemma 6.3.19, this means that

$$k\delta' \leqslant \delta(k+n-1).$$

Since this holds for arbitrarily large k, it follows that $\delta' \leqslant \delta$ as desired. ☐

Proof of Theorem 6.3.18. Let $\delta > 0$ be the largest real number such that $(\delta, \ldots, \delta) \in \Omega$. It follows from the definitions of convex and concave toric domain that

$$\square_n(\delta) \subset X_\Omega \subset L_n(\delta).$$

The first inclusion implies that $\delta \leqslant c_\square(X_\Omega)$ by the definition of $c_\square$, while the second inclusion implies that $c_\square(X_\Omega) \leqslant \delta$ by Proposition 6.3.20. Thus $c_\square(X_\Omega) = \delta$. ☐

Remark 6.3.21. *The proof of Theorem 6.3.18 shows more generally that any star-shaped domain $X \subset \mathbb{C}^n$ such that*

$$\square_n(\delta) \subset X \subset L_n(\delta) \tag{6.3.12}$$

satisfies $c_\square(X) = \delta$.

7. Results towards the strong Viterbo conjecture

In this chapter, we prove the strong Viterbo conjecture for all dynamically convex toric domains in $\mathbb{R}^4$. We then study non convex domains and check whether they satisfy the equality of all capacities. The last section of this chapter is devoted to higher dimensions.

Theorem 7.0.1 ([GHR20]). *If X_Ω is a monotone toric domain in $\mathbb{R}^4$, then $c_{\mathrm{Gr}(X)} = c_Z(X)$.*

Proof. Let r be the largest positive real number such that $\Delta^2(r) \subset \Omega$. We have $B^4(r) \subset X_\Omega$, so $r \leqslant c_{\mathrm{Gr}}(X_\Omega)$, and we just need to show that $c_Z(X_\Omega) \leqslant r$.

Let μ be a point on $\partial\Omega_+$ such that $\mu_1 + \mu_2 = r$. By an approximation argument, we can assume that X_Ω is strictly monotone, so that the tangent line to $\partial_+\Omega$ at μ is not horizontal or vertical. Then we can find $a, b > r$ such that Ω is contained in the quadrilateral with vertices $(0,0)$, $(a,0)$, (μ_1, μ_2), and $(0,b)$. It then follows from Lemma 6.1.6 that there exists a symplectic embedding $\mathrm{int}(X_\Omega) \underset{s}{\hookrightarrow} P(r,R)$ for some $R > 0$. Since $P(r,R) \subset Z^4(r)$, it follows that $c_Z(X_\Omega) \leqslant r$. $\square$

By proposition 5.1.4, Theorem 7.0.1 implies that all dynamically convex toric domains in $\mathbb{R}^4$ have $c_{\mathrm{Gr}} = c_Z$.

If X is a star-shaped domain with smooth boundary, let $A_{\min}(X)$ denote the minimal period of a Reeb orbit on ∂X.

Remark 7.0.2. *Without the toric hypothesis, not all dynamically convex domains in $\mathbb{R}^4$ have $c_{\mathrm{Gr}} = c_Z$. In particular, it is shown in [ABHS17] that for $\epsilon > 0$ small, there exists a dynamically convex domain X in $\mathbb{R}^4$ such that $A_{\min}(X)^2/(2\operatorname{vol}(X)) \geqslant 2 - \epsilon$. One has $c_1^{\mathrm{CH}}(X) \geqslant A_{\min}(X)$ by [GH18, Thm. 1.1], but $c_{\mathrm{Gr}}(X)^2 \leqslant 2\operatorname{vol}(X)$ by the volume constraint. Thus*

$$\frac{c_Z(X)}{c_{\mathrm{Gr}}(X)} \geqslant \sqrt{2-\epsilon}.$$

Remark 7.0.3. *It is also not true that all star-shaped toric domains have $c_{\mathrm{Gr}} = c_Z$. Counterexamples have been known for a long time, see e.g. [Her98], and in §7.1 we consider a new family of examples where we can explicitly compute both c_{Gr} and c_Z.*

For monotone toric domains in higher dimensions, we do not know how to prove that all normalized symplectic capacities agree, but we can at least prove the following:

Theorem 7.0.4 ([GHR20]). *(proved in §7.3) If X_Ω is a monotone toric domain in $\mathbb{R}^{2n}$, then*

$$c_{\mathrm{Gr}}(X_\Omega) = c_1^{\mathrm{CH}}(X_\Omega). \tag{7.0.1}$$

Returning to convex domains, some normalized symplectic capacities are known to agree (not the Gromov width or cylindrical capacity however), as we review in the following theorem:

Theorem 7.0.5 (Ekeland, Hofer, Zehnder, Abbondandolo-Kang, Irie). *If X is a convex domain in $\mathbb{R}^{2n}$, then:*

(a) $c_1^{\mathrm{EH}}(X) = c_{\mathrm{HZ}}(X) = c_{\mathrm{SH}}(X) = c_1^{\mathrm{CH}}(X)$.

(b) If in addition ∂X is smooth[1]*, then all of the capacities in (a) agree with $A_{\min}(X)$.*

Proof. Part (b) implies part (a) by a continuity argument.

Part (b) was shown for $c_{\mathrm{HZ}}(X)$ by Hofer-Zehnder in [HZ11] and for $c_{\mathrm{SH}}(X)$ by Irie [Iri19] and Abbondandolo-Kang [AK19]. The agreement of these two capacities with $c_1^{\mathrm{CH}}(X)$ for convex domains now follows from the combination of [GH18, Theorem 1.24] and [GS18, Lemma 3.2], as explained by Irie in [Iri19, Remark 2.15]. Finally, part (b) for $c_1^{\mathrm{EH}}(X)$ has been claimed and understood for a long time, but since we could not find a complete proof in the literature we give one here in §7.2. □

7.1. A family of non-monotone toric examples

We now study a family of examples of non-monotone toric domains, and we determine when they satisfy the conclusions of Conjecture 4.0.1 or Conjecture 4.0.2.

For $0 < a < 1/2$, let Ω_a be the convex polygon with corners $(0,0)$, $(1-2a,0)$, $(1-a,a)$, $(a,1-a)$ and $(0,1-2a)$, and write $X_a = X_{\Omega_a}$. Then X_a is a weakly convex (but not monotone) toric domain.

Proposition 7.1.1 ([GHR20]). *Let $0 < a < 1/2$. Then the Gromov width and cylindrical capacity of X_a are*

$$c_{\mathrm{Gr}}(X_a) = \min(1-a, 2-4a), \tag{7.1.1}$$
$$c_Z(X_a) = 1-a. \tag{7.1.2}$$

Corollary 7.1.2. *Let $0 < a < 1/2$ and let X_a be as above. Then:*

(a) The conclusion of Conjecture 4.0.1 holds for X_a, i.e. all normalized symplectic capacities defined for X_a agree, if and only if $a \leqslant 1/3$.

(b) The conclusion of Conjecture 4.0.2 holds for X_a, i.e. every normalized symplectic capacity c defined for X_a satisfies $c(X_a) \leqslant \sqrt{2\,\mathrm{Vol}(X_a)}$, if and only if $a \leqslant 2/5$.

Proof of Corollary 7.1.2. (a) By Lemma 4.0.3, we need to check that $c_{\mathrm{Gr}}(X_a) = c_Z(X_a)$ if and only if $a \leqslant 1/3$. This follows directly from (7.1.1) and (7.1.2).

(b) Since c_Z is the largest normalized symplectic capacity, the conclusion of Conjecture 4.0.2 holds for X_a if and only if

$$c_Z(X_a) \leqslant \sqrt{2\,\mathrm{Vol}(X_a)}. \tag{7.1.3}$$

By equation (5.1.1), we have

$$\mathrm{Vol}(X_{\Omega_a}) = \frac{1-4a^2}{2}.$$

It follows from this and (7.1.2) that (7.1.3) holds if and only if $a \leqslant 2/5$. □

[1] Without the smoothness assumption, it is shown in [AAO14, Prop. 2.7] that $c_{\mathrm{HZ}}(X)$ agrees with the minimum action of a "generalized closed characteristic" on ∂X.

To prove Proposition 7.1.1, we will use the following formula for the ECH capacities of a weakly convex toric domain X_Ω. Let r be the smallest positive real number such that $\Omega \subset \Delta^2(r)$. Then $\Delta^2(r)\backslash\Omega = \widetilde{\Omega}_1 \sqcup \widetilde{\Omega}_2$ where $\widetilde{\Omega}_1$ does not intersect the μ_2-axis and $\widetilde{\Omega}_2$ does not intersect the μ_1-axis. It is possible that $\widetilde{\Omega}_1$ and/or $\widetilde{\Omega}_2$ is empty. As in the discussion preceding (6.1.1), the closures of $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are affine equivalent to domains Ω_1 and Ω_2 such that X_{Ω_1} and X_{Ω_2} are concave toric domains. Denote the union (as multisets) of their weight sequences by

$$W(X_{\Omega_1}) \cup W(X_{\Omega_2}) = (a_1, \dots).$$

We then have:

Theorem 7.1.3 (Choi-Cristofaro-Gardiner [CG14]). *If X_Ω is a four-dimensional weakly convex toric domain as above, then*

$$c_k^{\mathrm{ECH}}(X_\Omega) = \inf_{l \geqslant 0} \left\{ c_{k+l}^{\mathrm{ECH}}\left(B^4(r)\right) - c_l^{\mathrm{ECH}}\left(\coprod_i B^4(a_i)\right) \right\}. \tag{7.1.4}$$

We need one more lemma, which follows from [LMS13, Cor. 4.2]:

Lemma 7.1.4. *Let $\mu_1, \mu_2 \geqslant a > 0$. Let Ω be the "diamond" in $\mathbb{R}^2_{\geqslant 0}$ given by the convex hull of the points $(\mu_1 \pm a, \mu_2)$ and $(\mu_1, \mu_2 \pm a)$. Then there is a symplectic embedding*

$$\mathrm{int}(B^4(2a)) \underset{s}{\hookrightarrow} X_\Omega.$$

Proof of Proposition 7.1.1. To prove (7.1.1), we first describe the ECH capacities of X_a. In the formula (7.1.4) for X_a, we have $r = 1$, while the weight expansions of Ω_1 and Ω_2 are both (a, a); the corresponding triangles are shown in Figure 7.1(b). Thus by Theorem 7.1.3 and equation (6.1.3), we have

$$c_k^{\mathrm{ECH}}(X_a) = \inf_{l_1, \dots, l_4 \geqslant 0} \left\{ c_{k+l_1+l_2+l_3+l_4}^{\mathrm{ECH}}\left(B^4(1)\right) - \sum_{i=1}^{4} c_{l_i}^{\mathrm{ECH}}\left(B^4(a)\right) \right\}. \tag{7.1.5}$$

We also note from (6.1.4) that

$$c_1^{\mathrm{ECH}}(B^4(r)) = c_2^{\mathrm{ECH}}(B^4(r)) = r, \qquad c_5^{\mathrm{ECH}}(B^4(r)) = 2r.$$

Taking $k = 1$ and $(l_1, \dots, l_4) = (1, 0, 0, 0)$ in equation (7.1.5), we get

$$c_1^{\mathrm{ECH}}(X_{\Omega_a}) \leqslant 1 - a. \tag{7.1.6}$$

Taking $k = 5$ and $(l_1, \dots, l_4) = (1, 1, 1, 1)$ in equation (7.1.5), we get

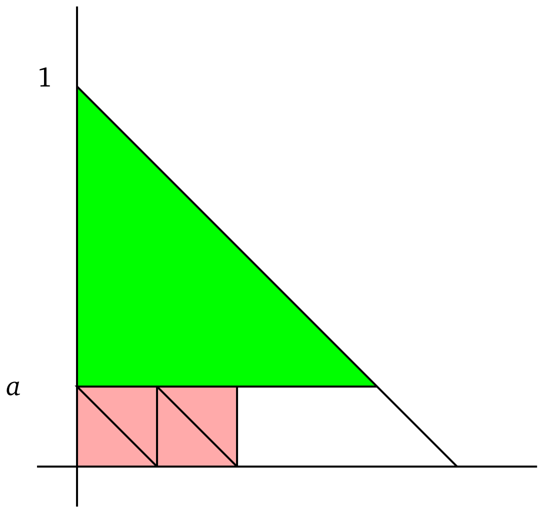
$$c_1^{\mathrm{ECH}}(X_{\Omega_a}) \leqslant 2 - 4a. \tag{7.1.7}$$

By (7.1.6) and (7.1.7) and the fact that c_1^{ECH} is a normalized symplectic capacity, we conclude that

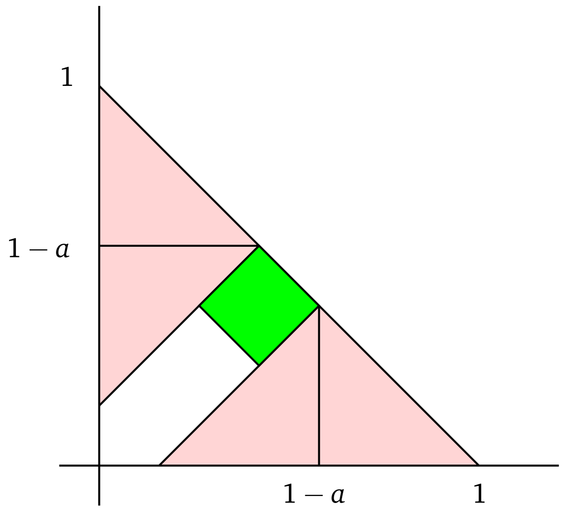
$$c_{\mathrm{Gr}}(X_{\Omega_a}) \leqslant \min(1 - a, 2 - 4a). \tag{7.1.8}$$

To prove the reverse inequality to (7.1.8), suppose first that $0 < a \leqslant 1/3$. It is enough to prove that there exists a symplectic embedding $\mathrm{int}(B^4(1-a)) \underset{s}{\hookrightarrow} X_{\Omega_a}$. By Theorem 6.1.3, it is enough to show that

$$c_k^{\mathrm{ECH}}(B^4(1-a)) \leqslant c_k^{\mathrm{ECH}}(X_{\Omega_a})$$



(a) $0 < a \leqslant 1/3$



(b) $1/3 \leqslant a < 1/2$

Figure 7.1.: Ball packings

for all nonnegative integers k. By equation (7.1.5), the above inequality is equivalent to

$$c_k^{\mathrm{ECH}}(B^4(1-a)) + \sum_{i=1}^{4} c_{l_i}^{\mathrm{ECH}}(B^4(a)) \leqslant c_{k+l_1+l_2+l_3+l_4}^{\mathrm{ECH}}(B^4(1)) \tag{7.1.9}$$

for all nonnegative integers $k, l_1, \ldots, l_4 \geqslant 0$. To prove (7.1.9), by the monotonicity of ECH capacities and the disjoint union formula (6.1.3), it suffices to find a symplectic embedding

$$\operatorname{int}(B^4(1-a) \sqcup B^4(a) \sqcup B^4(a) \sqcup (B^4(a) \sqcup B^4(a)) \underset{s}{\hookrightarrow} B^4(1).$$

This embedding exists by the Traynor trick (Proposition 6.1.4) using the triangles shown in Figure 7.1(a).

Finally, when $1/3 \leqslant a < 1/2$, it is enough to show that there exists a symplectic embedding $\operatorname{int}(B^4(2-4a)) \underset{s}{\hookrightarrow} X_{\Omega_a}$. This exists by Lemma 7.1.4 using the diamond shown in Figure 7.1(b).

This completes the proof of (7.1.1). Equation (7.1.2) follows from Theorem 7.1.5 below. □

Theorem 7.1.5 ([GHR20]). *Let $X_\Omega \subset \mathbb{R}^4$ be a weakly convex toric domain, see Definition 6.1.2. For $j = 1, 2$, let*

$$M_j = \max\{\mu_j \mid \mu \in \Omega\}.$$

Assume that there exists $(M_1, \mu_2) \in \overline{\partial_+ \Omega}$ with $\mu_2 \leqslant M_1$, and that there exists $(\mu_1, M_2) \in \overline{\partial_+ \Omega}$ with $\mu_1 \leqslant M_2$. Then

$$c_Z(X_\Omega) = \min(M_1, M_2).$$

That is, under the hypotheses of the theorem, the optimal symplectic embedding of X_Ω into a cylinder is the inclusion of X_Ω into either $(\pi|z_1|^2 \leqslant M_1)$ or $(\pi|z_2|^2 \leqslant M_2)$.

Proof. From the above inclusions we have $c_Z(X_\Omega) \leqslant \min(M_1, M_2)$. To prove the reverse inequality, suppose that there exists a symplectic embedding

$$X_\Omega \underset{s}{\hookrightarrow} Z^4(R). \tag{7.1.10}$$

We need to show that $R \geqslant \min(M_1, M_2)$. To do so, we will use ideas[2] from [Hut16].

Let $\epsilon > 0$ be small. Let $(A, 0)$ and $(0, B)$ denote the endpoints of $\overline{\partial_+\Omega}$. By an approximation argument, we can assume that $\overline{\partial_+\Omega}$ is smooth, and that $\partial_+\Omega$ has positive slope less than ϵ near $(A, 0)$ and slope greater than ϵ^{-1} near $(0, B)$. As in the proof of Proposition 5.1.4, there are then three types of Reeb orbits on ∂X_Ω:

(i) There is a simple Reeb orbit whose image is the circle with $\pi|z_1|^2 = A$ and $z_2 = 0$. This Reeb orbit has symplectic action (period) equal to A, and rotation number $1 - \epsilon^{-1}$.

(ii) There is a simple Reeb orbit whose image is the circle with $z_1 = 0$ and $\pi|z_2|^2 = B$. This Reeb orbit has symplectic action B and rotation number $1 - \epsilon^{-1}$.

(iii) For each point $\mu \in \partial_+\Omega$ where $\partial_+\Omega$ has rational slope, there is an S^1 family of simple Reeb orbits in the torus where $\pi(|z_1|^2, |z_2|^2) = \mu$. If $\nu = (\nu_1, \nu_2)$ is the outward normal vector to $\partial_+\Omega$ at μ, scaled so that ν_1, ν_2 are relatively prime integers, then these Reeb orbits have rotation number $\nu_1 + \nu_2$ and symplectic action $\mu \cdot \nu$, see [GH18, §2.2].

We claim now that

(*) Any Reeb orbit on ∂X_Ω with positive rotation number has symplectic action at least $\min(M_1, M_2)$.

To prove this claim, we only need to check the type (iii) simple Reeb orbits where $\nu_1 + \nu_2 \geqslant 1$. We must have $\nu_1 \geqslant 1$ or $\nu_2 \geqslant 1$. If $\nu_1 \geqslant 1$, then by the hypotheses of the theorem there exists μ_2' such that $(M_1, \mu_2') \in \overline{\partial_+\Omega}$ and $M_1 \geqslant \mu_2'$. Since Ω is convex and ν is an outward normal at μ, the symplectic action

$$\mu \cdot \nu \geqslant (M_1, \mu_2') \cdot \nu = M_1 + (\nu_1 - 1)(M_1 - \mu_2') + (\nu_1 + \nu_2 - 1)\mu_2' \geqslant M_1.$$

Likewise, if $\nu_2 \geqslant 1$, then the symplectic action $\mu \cdot \nu \geqslant M_2$.

Now starting from the symplectic embedding (7.1.10), by replacing X_Ω with an appropriate subset and replacing $Z^4(R)$ with an appropriate superset, we obtain a symplectic embedding $X' \underset{s}{\hookrightarrow} \operatorname{int}(Z')$, where:

- Z' is an ellipsoid whose boundary has one simple Reeb orbit γ_+ with symplectic action $\mathscr{A}(\gamma_+) = R + \epsilon$ and Conley-Zehnder index $\mathrm{CZ}(\gamma_+) = 3$, another simple Reeb orbit with very large symplectic action, and no other simple Reeb orbits.
- X' is a (non-toric) star-shaped domain with smooth boundary, all of whose Reeb orbits are nondegenerate. Every Reeb orbit on $\partial X'$ with rotation number greater than or equal to 1 has action at least $\min(M_1, M_2) - \epsilon$.

[2] The main theorem in [Hut16] gives a general obstruction to a symplectic embedding of one four-dimensional convex toric domain into another, which sometimes goes beyond the obstruction coming from ECH capacities. This theorem can be generalized to weakly convex toric domains; but rather than carry out the full generalization, we will just explain the simple case of this that we need.

The symplectic embedding gives rise to a strong symplectic cobordism W whose positive boundary is $\partial Z'$ and whose negative boundary is $\partial X'$. The argument in [Hut16, §6] shows that for a generic "cobordism-admissible" almost complex structure J on the "completion" of W, there exists an embedded J-holomorphic curve u with one positive end asymptotic to the Reeb orbit γ_+ in $\partial Z'$, negative ends asymptotic to some Reeb orbits $\gamma_1, \ldots, \gamma_m$ in $\partial X'$, and Fredholm index $\operatorname{ind}(u) = 0$. The Fredholm index is computed by the formula

$$\operatorname{ind}(u) = 2g + \left[\mathrm{CZ}(\gamma_+) - 1\right] - \sum_{i=1}^{m} \left[\mathrm{CZ}(\gamma_i) - 1\right] \tag{7.1.11}$$

where g denotes the genus of u. Furthermore, since J-holomorphic curves decrease symplectic action, we have

$$\mathscr{A}(\gamma_+) \geqslant \sum_{i=1}^{m} \mathscr{A}(\gamma_i). \tag{7.1.12}$$

We claim now that at least one of the Reeb orbits γ_i has action at least $\min(M_1, M_2) - \epsilon$. Then the inequality (7.1.12) gives

$$R + \epsilon \geqslant \min(M_1, M_2) - \epsilon,$$

and since $\epsilon > 0$ was arbitrarily small, we are done.

To prove the above claim, suppose to the contrary that all of the Reeb orbits γ_i have action less than $\min(M_1, M_2) - \epsilon$. Then all of the Reeb orbits γ_i have rotation number $\rho(\gamma_i) < 1$, which means that they all have Conley-Zehnder index $\mathrm{CZ}(\gamma_i) \leqslant 1$. It now follows from (7.1.11) that $\operatorname{ind}(u) \geqslant 2$, which is a contradiction[3]. □

7.2. The first Ekeland-Hofer capacity

The goal of this section is to (re)prove the following theorem. This is well-known in the community and is attributed to Ekeland, Hofer and Zehnder [EH90, HZ87]. It was first mentioned by Viterbo in [Vit89, Proposition 3.10].

Theorem 7.2.1 (Ekeland-Hofer-Zehnder). *Let $W \subset \mathbb{R}^{2n}$ be a compact convex domain with smooth boundary. Then*

$$c_1^{\mathrm{EH}}(W) = A_{\min}(W).$$

Proof. Since W is star-shaped, there is a unique differentiable function $r : \mathbb{R}^{2n} \to \mathbb{R}$ which is C^∞ in $\mathbb{R}^{2n} \backslash \{0\}$ satisfying $r(cz) = c^2 r(z)$ for $c \geqslant 0$ such that

$$W = \{z \in \mathbb{R}^{2n} \mid r(z) \leqslant 1\},$$
$$\partial W = \{z \in \mathbb{R}^{2n} \mid r(z) = 1\}.$$

Let $\alpha = A_{\min}(W)$ and fix $\epsilon > 0$. Let $f \in C^\infty_{\geqslant 0}(\mathbb{R})$ be a convex function such that $f(r) = 0$ for $r \leqslant 1$ and $f(r) = (\alpha + \epsilon)(r - 1)$ for $r \geqslant 2$. In particular,

$$f(r) \geqslant (\alpha + \epsilon)(r - 1), \quad \text{for all } r. \tag{7.2.1}$$

[3] One way to think about the information that we are getting out of (7.1.11), as well as the general symplectic embedding obstruction in [Hut16], is that we are making essential use of the fact that every holomorphic curve has nonnegative genus.

We now choose a convex function $H \in C^\infty(\mathbb{R}^{2n})$ such that

$$\begin{array}{rcll} H(z) & = & f(r(z)), & \text{if } r(z) \leqslant 2, \\ H(z) & \geqslant & f(r(z)), & \text{for all } z \in \mathbb{R}^{2n}, \\ H(z) & = & c\,|z|^2, & \text{if } z >> 0 \text{ for some } c \in \mathbb{R}_{>0} \backslash \pi\mathbb{Z}. \end{array} \tag{7.2.2}$$

Let $x_0 \in E$ be an action-minimizing Reeb orbit on ∂W, reparametrized as a map $x_0 : \mathbb{R}/\mathbb{Z} = S^1 \to \mathbb{R}^{2n}$ of speed α, so that $A(x_0) = \alpha$ and $r(x_0) \equiv 1$ and $\dot{x}_0 = \alpha J \nabla r(x_0)$. From a simple calculation we deduce that x_0 is a critical point of the functional $\Psi : E \to \mathbb{R}$ defined by

$$\Psi(x) = A(x) - \alpha \int_0^1 r(x(t))\,dt. \tag{7.2.3}$$

Observe that $\Psi(cx) = c^2\Psi(x)$ for $c \geqslant 0$. So sx_0 is a critical point of Ψ for all $s \geqslant 0$. Let $\xi = [0,\infty) \cdot P^+ x_0 \oplus E^0 \oplus E^-$.

We now claim that $\Psi(x) \leqslant 0$ for all $x \in \xi$. To prove this, let $\xi_s = sP^+x_0 \oplus E^0 \oplus E^-$. Observe that Ψ_{ξ_s} is a concave function. Since sx_0 is a critical point of Ψ_{ξ_s} it follows that $\max \Psi(\xi_s) = \Psi(sx_0) = s^2\Psi(x_0) = 0.$.

From (7.2.1), (7.2.2) and (7.2.3) we obtain

$$\mathscr{A}_H(x) \leqslant \Psi(x) + \alpha + \epsilon - \epsilon \int_0^1 r(x(t))\,dt \leqslant \alpha + \epsilon.$$

Note that ξ is S^1-invariant. Moreover it is proven in [EH89] that $h(\xi) \cap S^+ \neq \varnothing$ for all $h \in \Gamma$. So $c_{H,1} \leqslant \alpha + \epsilon$. Hence $c_1^{\mathrm{EH}}(W) \leqslant \alpha + \epsilon$ for all $\epsilon > 0$. Therefore

$$c_1^{\mathrm{EH}}(W) \leqslant \alpha.$$

To prove the reverse inequality, recall from [EH90, Prop. 2] that $c_1^{\mathrm{EH}}(W)$ is the symplectic action of some Reeb orbit on ∂W. Thus

$$c_1^{\mathrm{EH}}(W) \geqslant \alpha.$$

□

7.3. High dimensions

We start by proving Theorem 7.0.4. (Some related arguments appeared in [GH18, Lem. 1.19].) If $a_1, \ldots, a_n > 0$, define the "L-shaped domain"

$$L(a_1, \ldots, a_n) = \left\{ \mu \in \mathbb{R}^n_{\geqslant 0} \,\middle|\, \mu_j \leqslant a_j \text{ for some } j \right\}.$$

Lemma 7.3.1. *If $a_1, \ldots, a_n > 0$, then*

$$c_1^{\mathrm{CH}}\left(X_{L(a_1,\ldots,a_n)}\right) = \sum_{j=1}^n a_j.$$

Proof. Observe that

$$\mathbb{R}^n_{\geqslant 0} \backslash L(a_1, \ldots, a_n) = (a_1, \infty) \times \cdots \times (a_n, \infty).$$

is convex. Thus $X_{L(a_1,\dots,a_n)}$ satisfies all the conditions in the definition of "concave toric domain", except that it is not compact.

A formula for c_k^{CH} of a concave toric domain is given in [GH18, Thm. 1.14]. The $k = 1$ case of this formula asserts that if X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then

$$c_1^{\mathrm{CH}}(X_\Omega) = \min\left\{ \sum_{i=1}^{n} \mu_i \;\middle|\; \mu \in \overline{\partial_+ \Omega} \right\}. \tag{7.3.1}$$

By an exhaustion argument (see [GH18, Rmk. 1.3]), this result also applies to $X_{L(a_1,\dots,a_n)}$. For $\Omega = L(a_1,\dots,a_n)$, the minimum in (7.3.1) is realized by $\mu = (a_1,\dots,a_n)$. □

Lemma 7.3.2. *If X_Ω is a monotone toric domain in $\mathbb{R}^{2n}$ and if $\mu \in \partial_+\Omega$, then $\Omega \subset L(\mu_1,\dots,\mu_n)$.*

Proof. By an approximation argument we can assume without loss of generality that X_Ω is strictly monotone. Then $\partial_+\Omega$ is the graph of a positive function f over an open set $U \subset \mathbb{R}^{n-1}_{\geqslant 0}$ with $\partial_j f < 0$ for $j = 1,\dots,n-1$. It follows that if $(\mu'_1,\dots,\mu'_{n-1}) \in U$ and $\mu'_j > \mu_j$ for all $j = 1,\dots,n-1$, then $f(\mu'_1,\dots,\mu'_{n-1}) < f(\mu_1,\dots,\mu_{n-1})$. Consequently Ω does not contain any point μ' with $\mu'_j > \mu_j$ for all $j = 1,\dots,n$. This means that $\Omega \subset L(\mu_1,\dots,\mu_n)$. Figure 7.2 illustrates this inclusion for $n = 2$. □

Proof of Theorem 7.0.4. For $a > 0$, consider the simplex

$$\Delta^n(a) = \left\{ \mu \in \mathbb{R}^n_{\geqslant 0} \;\middle|\; \sum_{j=1}^{n} \mu_i \leqslant a \right\}.$$

Observe that the toric domain $X_{\Delta^n(a)}$ is the ball $B^{2n}(a)$. Now let $a > 0$ be the largest real number such that $\Delta^n(a) \subset \Omega$; see Figure 7.2.

We have $B^{2n}(a) \subset X_\Omega$, so by definition $a \leqslant c_{\mathrm{Gr}}(X_\Omega)$. Since c_1^{CH} is a normalized symplectic capacity, $c_{\mathrm{Gr}}(X_\Omega) \leqslant c_1^{\mathrm{CH}}(X_\Omega)$. By the maximality property of a, there exists a point $\mu \in \overline{\partial_+\Omega}$ with $\sum_{j=1}^n \mu_j = a$. By an approximation argument we can assume that $\mu \in \partial_+\Omega$. By Lemma 7.3.2, $X_\Omega \subset X_{L(\mu_1,\dots,\mu_n)}$. By the monotonicity of c_1^{CH} and Lemma 7.3.1, we then have

$$c_1^{\mathrm{CH}}(X_\Omega) \leqslant c_1^{\mathrm{CH}}\left(X_{L(\mu_1,\dots,\mu_n)}\right) = \sum_{j=1}^{n} \mu_j = a.$$

Combining the above inequalities gives $c_{\mathrm{Gr}}(X_\Omega) = c_1^{\mathrm{CH}}(X_\Omega) = a$. □

We conclude this section by a quick sketch on how to prove Theorem 7.0.5.

Proof of Theorem 7.0.5. We assume that ∂X is smooth. By monotonicity of the capacities the result follows for all convex domains.

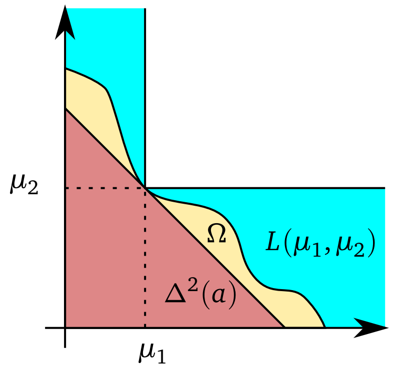


Figure 7.2.: The inclusions $\Delta^n(a) \subset \Omega \subset L(\mu_1, \dots, \mu_n)$ for $n = 2$

We will use the following commutative diagram [BO13, Theorem 1.2]

$$
\begin{tikzcd}
SH_k^{\epsilon}(X) \arrow[rr] \arrow[dd, "a"] & & SH_k^{L}(X) \arrow[dl] \arrow[dd, "b"] \\
& SH_k^{+,L}(X) \arrow[ul, "{[-1]}"] \arrow[dd, "c"] & \\
SH_k^{S^1,\epsilon}(X) \arrow[rr] \arrow[dd] & & SH_k^{S^1,L}(X) \arrow[dl] \arrow[dd] \\
& CH_k^{L}(X) \arrow[ul, "{[-1]}"] \arrow[dd] & \\
SH_{k-2}^{S^1,\epsilon}(X) \arrow[rr] \arrow[uuuu, bend left, "{[-1]}"] & & SH_{k-2}^{S^1,L}(X) \arrow[dl] \arrow[uuuu, bend right, "{[-1]}"'] \\
& CH^{L}(X)_{k-2} \arrow[ul, "{[-1]}"] \arrow[uuuu, bend left, "{[-1]}"] &
\end{tikzcd}
\tag{7.3.2}
$$

By [GH18, Lemma 4.7], the first equivariant capacity is given by

$$
c_1^{CH}(X) = \inf\{L \mid CH_{n+1}^{L}(X) \neq 0\}. \tag{7.3.3}
$$

Since X is convex, ∂X is dynamically convex, which implies that the three elements of the lower triangle in Equation (7.3.2) vanish in degrees $n-1$ and $n-2$.

$$
\begin{gathered}
SH_{n-2}^{S^1,\epsilon}(X) = 0 = SH_{n-1}^{S^1,\epsilon}(X) \\
SH_{n-2}^{S^1,\epsilon}(X) = 0 = SH_{n-1}^{S^1,\epsilon}(X) \\
CH_{n-2}^{L}(X) = 0 = CH_{n-1}^{L}(X)
\end{gathered}
$$

Thus, in degree n, the maps a, b and c are isomorphisms. Therefore

$$
\inf\{L \mid CH_{n+1}^{L}(X) \neq 0\} = \inf\{L \mid SH_{n+1}^{+,L}(X) \neq 0\}. \tag{7.3.4}
$$

From Viterbo's isomorphism ([Vit99, Proposition 1.4]) we know that $SH^{\epsilon}_{n+1}(X) = 0$ and $SH^{\epsilon}_{n}(X) = \mathbb{Q}$. So from the upper triangle in Equation (7.3.2)we obtain the following exact sequence:

$$0 \longrightarrow SH^{L}_{n+1}(X) \longrightarrow SH^{+,L}_{n+1}(X) \xrightarrow{\delta} SH^{\epsilon}_{n}(X) \longrightarrow SH^{L}_{n}(X) \longrightarrow 0$$

Now recall from [AK19, Main Corolary] that $c_{SH}(X) = l_{min}(\partial X)$. If the map $SH^{<\epsilon}_{n}(X) \to SH^{<L}_{n}(X)$ is zero, then the map δ is surjective, in particular $SH^{+,L}_{n+1}(X) \neq 0$. So

$$l_{min}(\partial X) = c_{SH}(X) = \inf\{L > \epsilon \,|\, SH^{<\epsilon}_{n}(X) \to SH^{<L}_{n}(X) \text{ is zero}\} \geqslant \inf\{L \,|\, SH^{+,L}_{n+1}(X) \neq 0\}. \tag{7.3.5}$$

It now follows from Theorem 7.2.1, (7.3.3), (7.3.4) and (7.3.5) that

$$c^{EH}_{1}(X) = l_{min}(\partial X) = c_{SH}(X) \geqslant c^{CH}_{1}(X) \geqslant l_{min}(\partial X).$$

Therefore

$$c^{EH}_{1}(X) = c^{CH}_{1}(X) = c_{SH}(X).$$

□

8. Knotted embeddings

Recent years have seen a significant improvement in our understanding of when one region in $\mathbb{R}^4$ symplectically embeds into another, see *e.g.* [McD09], [MS12], [CG19]. Complementing this existence question, one can ask whether embeddings are unique up to an appropriate notion of equivalence; in particular, if $A \subset U \subset \mathbb{R}^4$ this entails asking whether every symplectic embedding $A \hookrightarrow U$ is equivalent to the inclusion. Somewhat less is known about this uniqueness question, though there are positive results in [McD09],[CG19] and negative results in [FHW94], [Hin13]. We showed with M. Usher [GU19] that modern techniques of constructing symplectic embeddings $B \hookrightarrow U$ often give rise, when restricted to certain subsets $A \subset B \cap U$, to embeddings $A \hookrightarrow U$ that are distinct from the inclusion in a strong sense.

The subsets of $\mathbb{R}^4$ (and in some cases more generally in $\mathbb{R}^{2n} \cong \mathbb{C}^n$) that we consider are toric domains, see §5.

We use the following standard notational convention:

Definition 8.0.1. If $A \subset \mathbb{C}^n$ and $\alpha > 0$, we define $\alpha A = \{\sqrt{\alpha}a | a \in A\}$.

(The square root ensures that any capacity c will obey $c(\alpha A) = \alpha c(A)$, and also that we have $E(\alpha a_1, \ldots, \alpha a_n) = \alpha E(a_1, \ldots, a_n)$ and similarly for polydisks.)

For any subset $B \subset \mathbb{C}^n$ let B° denote the interior of B. We were mostly concerned with symplectic embeddings $X \hookrightarrow \alpha X^\circ$ where X is a concave or convex toric domain and $\alpha > 1$. The definitions imply that concave or convex toric domains X always satisfy $X \subset \alpha X^\circ$ for all $\alpha > 1$, so one such embedding is given by the inclusion of X into αX°. However we will find that in many cases there are other such embeddings that are inequivalent to the inclusion in the following sense:

Definition 8.0.2. *Let A and U be symplectic manifolds, and let φ_1 and φ_2 be symplectic embeddings $A \hookrightarrow U$. We say φ_1 and φ_2 are* ***equivalent*** *if there exists a symplectomorphism $\Psi : U \to U$ such that $\Psi \circ \varphi_1(A) = \varphi_2(A)$. Otherwise they are called* ***inequivalent****.*

In the particular case of nested domains in $\mathbb{C}^n$, we introduced the notion of *knottedness* as follows.

Definition 8.0.3. *Let $A \subset U \subset \mathbb{C}^n$, with A closed and U open, and let $\phi \colon A \to U$ be a symplectic embedding.*[1] *We say that ϕ is* ***unknotted*** *if there is a symplectomorphism $\Psi \colon U \to U$ such that $\Psi(A) = \phi(A)$. We say that ϕ is* ***knotted*** *if it is not unknotted.*

Note that we do not require the map Ψ to be compactly supported, or Hamiltonian isotopic to the identity, or even to extend continuously to the closure of U; accordingly our definition of knottedness is in principle more restrictive than others that one might use.

In [GU19] we have proven the existence of knotted embeddings from X to αX° for many toric domains $X \subset \mathbb{C}^2$ and suitable $\alpha > 1$.

[1] Since A may not be a manifold or even a manifold with boundary we should say what it means for $\phi \colon A \to U$ to be a symplectic embedding; our convention will be that it means that there is an open neighborhood of A to which ϕ extends as a symplectic embedding. When A is a manifold with boundary it is not hard to see using a relative Moser argument that this is equivalent to the statement that $\phi \colon A \to U$ is a smooth embedding of manifolds with boundary which preserves the symplectic form.

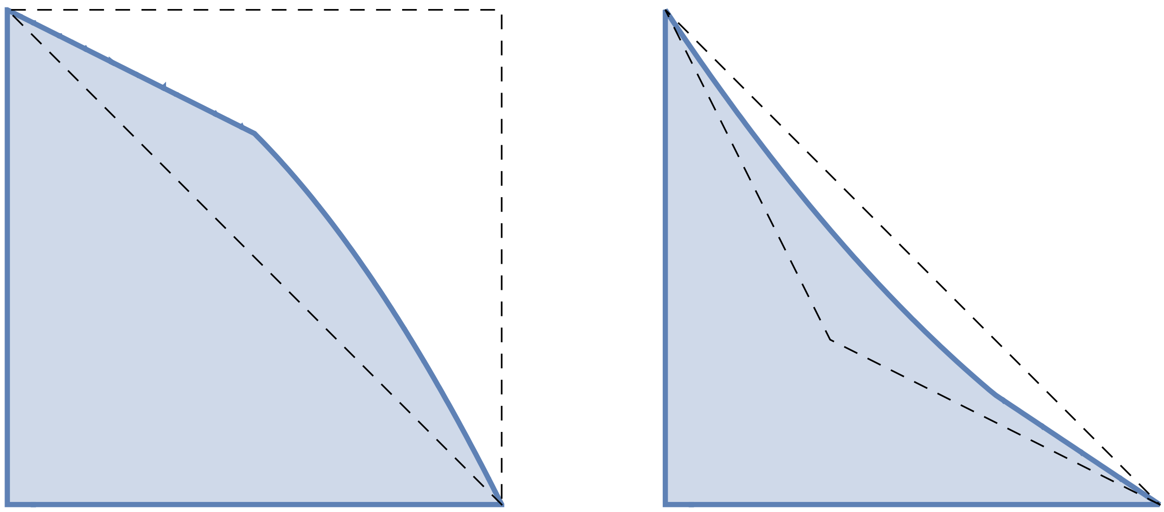


Figure 8.1.: The shaded regions are examples of choices of Ω such that Theorem 8.0.4 gives knotted embeddings $X_\Omega \to \alpha X_\Omega^\circ$ for suitable $\alpha > 1$. The dashed lines delimit the regions which are assumed to contain $(\partial\Omega) \cap (0,\infty)^n$ in, respectively, Cases (i) and (ii) of the theorem.

Theorem 8.0.4. *Let $X \subset \mathbb{C}^2$ belong to any of the following classes of domains:*

(i) All convex toric domains X such that, for some $c > 0$, $B^4(c) \subsetneq X \subset P(c,c)$.

(ii) All concave toric domains X_Ω such that, for some $c > 0$,

$$\{(x,y) \in [0,\infty)^2 | \min\{2x+y, x+2y\} \leqslant c\} \subset \Omega \subsetneq \{(x,y) \in [0,\infty)^2 | x+y \leqslant c\}.$$

(iii) All complex ℓ^p balls $\{(w,z) \in \mathbb{C}^2 || w|^p + |z|^p \leqslant r^p\}$ for $p > \frac{\log 9}{\log 6} \approx 1.23$, except for $p = 2$.

(iv) All polydisks $P(a,b)$ for $a \leqslant b < 2a$.

Then there exist $\alpha > 1$ and a knotted embedding $\phi\colon X \to \alpha X^\circ$.

For context, recall that McDuff showed in [McD91] that the space of symplectic embeddings from one four-dimensional ball to another is always connected; by the symplectic isotopy extension theorem this implies that symplectic embeddings $B^4(c) \to \alpha B^4(c)^\circ$ can never be knotted. (In particular the exclusion of $B^4(c)$ from each of the classes (i),(ii),(iii) above is necessary.) McDuff's result was later extended to establish the connectedness of the space of embeddings of one four-dimensional ellipsoid into another [McD09] or of a four-dimensional concave toric domain into a convex toric domain [CG19]. So Theorem 8.0.4 reflects that embeddings from concave toric domains into concave ones, or convex toric domains into convex ones, can behave differently than embeddings from concave toric domains into convex ones.

We do not know whether the bound $b < 2a$ in part (iv) of Theorem 8.0.4 is sharp. The bound $p > \frac{\log 9}{\log 6}$ in part (iii) is not sharp; we are aware of extensions of our methods that lower this bound slightly, though in the interest of brevity we do not include them. Note that the domains in part (iii) are concave when $p < 2$ and convex when $p > 2$ (in the latter case the result follows directly from part (i)).

While our primary focus in this paper is on domains in $\mathbb{R}^4$, we show in Theorem 8.0.5 that the embeddings from Cases (i) and (iv) of Theorem 8.0.4 remain knotted after being trivially extended to the product of X_Ω with an ellipsoid of sufficiently large Gromov width. It remains an interesting problem to find knotted embeddings involving broader classes of high-dimensional domains that do not arise from lower-dimensional constructions.

Theorem 8.0.5. *Let $X \subset \mathbb{C}^2$ belong to any of the following classes of domains:*

(i) All convex toric domains X such that, for some $c > 0$, $B^4(c) \subsetneq X \subset P(c,c)$.

(ii) All polydisks $P(a,b)$ for $a \leqslant b < 2a$.

Then there exist numbers $\alpha > 1$ and $R > 0$ and a knotted embedding $\phi\colon X \times E(b_1,\ldots,b_{n-2}) \to \alpha\big(X \times E(b_1,\ldots,b_{n-2})\big)^\circ$ for any $b_1,\ldots,b_{n-2}$ with each $b_i \geqslant R$.

By the way, embeddings such as those in Theorem 8.0.4 can only be knotted for a limited range of α, since the extension-after-restriction principle [Sch05, Proposition A.1] implies that for any compact set $X \subset \mathbb{C}^n$ which is star-shaped with respect to the origin and contains the origin in its interior and any symplectic embedding $\phi\colon X \to \mathbb{C}^n$, there is $\alpha_0 > 1$ such that $\phi(X) \subset \alpha_0 X^\circ$ and such that ϕ is unknotted when considered as a map to αX° for all $\alpha \geqslant \alpha_0$. The values for α that we find in the proof of Theorem 8.0.4 vary from case to case, but in each instance lie between 1 and 2. This suggests the:

Question 8.0.6. *Do there exist a domain $X \subset \mathbb{R}^{2n}$, a number $\alpha > 2$, and a knotted symplectic embedding $\phi\colon X \to \alpha X^\circ$?*

Theorem 8.0.4 concerns embeddings of a domain X into the interior of a dilate αX° of X; of course it is also natural to consider embeddings in which the source and target are not simply related by a dilation. Our methods in principle allow for this, though the proofs that the embeddings are knotted become more subtle. We carried this out for embeddings of four-dimensional polydisks into other polydisks, and in particular we proved the following:

Theorem 8.0.7. *Given any $y \geqslant 1$, there exist polydisks $P(a,b)$ and $P(c,d)$ and knotted embeddings of $P(a,b)$ into $P(1,y)^\circ$ and of $P(1,y)$ into $P(c,d)^\circ$.*

Theorem 8.0.7 and Case (iv) of Theorem 8.0.4 should be compared to [FHW94, Section 3.3], in which it is shown that, if $a \leqslant b < c$ but $a+b > c$, then the embeddings $\phi_1,\phi_2\colon P(a,b) \to P(c,c)^\circ$ given by $\phi_1(w,z) = (w,z)$ and $\phi_2(w,z) = (z,w)$ are not isotopic through compactly supported symplectomorphisms of $P(c,c)^\circ$. However our embeddings are different than these; in fact the embeddings from [FHW94] are not even knotted in our (rather strong) sense since there is a symplectomorphism of the open polydisk $P(c,c)^\circ$ mapping $P(a,b)$ to $P(b,a)$. If one instead considers embeddings into $P(c,d)$ with $c < d$ chosen such that $P(c,d)^\circ$ contains both $P(a,b)$ and $P(b,a)$ and $a+b > d$, then $P(a,b)$ and $P(b,a)$ are inequivalent to each other under the symplectomorphism group of $P(c,d)^\circ$. However in situations where this construction and the construction underlying Theorem 8.0.4 (iv) and Theorem 8.0.7 both apply, our knotted embeddings represent different knot types than both $P(a,b)$ and $P(b,a)$.

Let us be a bit more specific about how we prove Theorem 8.0.4; the proof of Theorem 8.0.7 is conceptually similar. The knotted embeddings $\phi\colon X \to \alpha X^\circ$ described in Theorem 8.0.4 are obtained as compositions of embeddings $X \to E \to \alpha X^\circ$ where E is an ellipsoid. In the cases that X is convex, the first map $X \to E$ is just an inclusion, while the second map $E \to \alpha X^\circ$ is

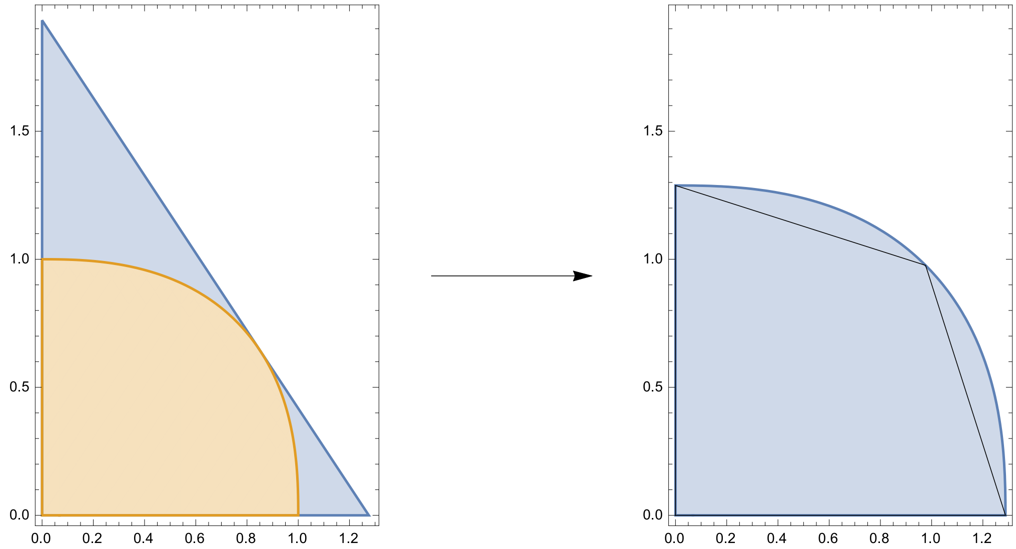


Figure 8.2.: The strategy underlying our knotted embedding in the case that X is the ℓ^5 ball of capacity 1, as in Case (i) or (iii) of Theorem 8.0.4. X is the toric domain associated to the smaller region on the left; the toric domain associated to the triangle on the left is the ellipsoid $E = E((3/2)^{3/5}, 3^{3/5})$, which in particular contains X. The larger region at right is obtained by dilating X by $\alpha = (1+\epsilon)(3/2)^{3/5}$ for a small $\epsilon > 0$, and we showed that there is a symplectic embedding $\phi\colon E \to \alpha X^\circ$ (in fact, ϕ has image contained in the preimage under μ of the inscribed quadrilateral on the right). Our knotted embedding is $\phi|_X$; Theorem 8.0.9(a) implies that any unknotted embedding $X \to \alpha X^\circ$ that extends to a symplectic embedding $E \to \alpha X^\circ$ would have $\alpha \geqslant 2^{3/5}$, whereas in this construction α can be taken arbitrarily close to $(3/2)^{3/5}$.

obtained by using recent developments from [McD09],[CG19] that ultimately have their roots in Taubes-Seiberg-Witten theory. (For a limited class of convex toric domains X that are close to a cube $P(c,c)$, we provide a much more elementary and explicit construction in Section 8.1.) In the cases that X is concave the reverse is true: $E \to \alpha X^\circ$ is an inclusion while $X \to E$ is obtained from these more recent methods. Meanwhile, we use the properties of transfer maps in filtered S^1-equivariant symplectic homology to obtain a lower bound on possible values α such that there can exist any unknotted embedding $X \to \alpha X^\circ$ which factors through an ellipsoid E. In each case in Theorem 8.0.4, we will find compositions $X \to E \to \alpha X^\circ$ arising from the constructions for which α is less than this symplectic-homology-derived lower bound, leading to the conclusion that the composition must be knotted. Figure 8.2 and its caption explain this more concretely in a representative special case.

To carry this out systematically, let us introduce the following two quantities associated to a star-shaped domain $X \subset \mathbb{C}^n$, where the symbol $\hookrightarrow$ always denotes a symplectic embedding:

$$\delta_{\mathrm{ell}}(X) = \inf\{\alpha \geqslant 1 | (\exists a_1, \ldots, a_n)(X \hookrightarrow E(a_1, \ldots, a_n) \hookrightarrow \alpha X^\circ)\} \tag{8.0.1}$$

and

$$\delta^{\mathrm{u}}_{\mathrm{ell}}(X) = \inf\left\{\alpha \geqslant 1 \;\middle|\; \begin{array}{c}(\exists a_1, \ldots, a_n, f : X \hookrightarrow E(a_1, \ldots, a_n),\\ g\colon E(a_1, \ldots, a_n) \hookrightarrow \alpha X^\circ)(g \circ f \text{ is unknotted.})\end{array}\right\} \tag{8.0.2}$$

(The u in $\delta^{\mathrm{u}}_{\mathrm{ell}}$ stands for "unknotted.") To put this into a different context, as was suggested to us by Y. Ostrover and L. Polterovich, one can define a pseudometric on the space of star-shaped domains in $\mathbb{C}^n$ by declaring the distance between two domains X and Y to be the logarithm of the infimal $\alpha \in \mathbb{R}$ such that there is a sequence of symplectic embeddings $\alpha^{-1/2}X \hookrightarrow Y \hookrightarrow \alpha^{1/2}X^{\circ}$; a more refined version of this pseudometric would additionally ask that neither of the resulting compositions $X \to \alpha X^{\circ}$ and $Y \to \alpha Y^{\circ}$ be knotted. Then (at least if $n = 2$) the logarithm of $\delta_{\mathrm{ell}}(X)$ or of $\delta^{\mathrm{u}}_{\mathrm{ell}}(X)$ is the distance from X to the set of ellipsoids with respect to such a pseudometric. (In the case of $\delta^{\mathrm{u}}_{\mathrm{ell}}$ this statement depends partly on the result from [McD09] that when E is an ellipsoid in $\mathbb{R}^4$ a symplectic embedding $E \hookrightarrow \alpha E^{\circ}$ is never knotted.)

We will prove Theorem 8.0.4 by proving, for each X as in the statement, a strict inequality $\delta_{\mathrm{ell}}(X) < \delta^{\mathrm{u}}_{\mathrm{ell}}(X)$. This entails finding upper bounds for $\delta_{\mathrm{ell}}(X)$ by exhibiting particular compositions of embeddings $X \hookrightarrow E \hookrightarrow \alpha X^{\circ}$, and finding lower bounds for $\delta^{\mathrm{u}}_{\mathrm{ell}}(X)$ using filtered positive S^1-equivariant symplectic homology. As it happens, for convex or concave toric domains both our upper bounds and our lower bounds can be conveniently expressed in terms of the following notation:

Notation 8.0.8. *For a domain $\Omega \subset [0,\infty)^n$ we define functions $\|\cdot\|^*_{\Omega}$ and $[\,\cdot\,]_{\Omega}$ from $\mathbb{R}^n$ to $\mathbb{R}$ as follows:*

- *For $\vec{\alpha} \in \mathbb{R}^n$, $\|\vec{\alpha}\|^*_{\Omega} = \sup\{\vec{\alpha} \cdot \vec{v} \mid \vec{v} \in \Omega\}$.*
- *For $\vec{\alpha} \in \mathbb{R}^n$, $[\vec{\alpha}]_{\Omega} = \inf\{\vec{\alpha} \cdot \vec{v} \mid \vec{v} \in [0,\infty)^n \setminus \Omega\}$.*

The estimates for $\delta^{\mathrm{u}}_{\mathrm{ell}}$ that are relevant to Theorem 8.0.4 are given by the following result:

Theorem 8.0.9. *(a) If $X_{\Omega} \subset \mathbb{C}^2$ is a convex toric domain, then*

$$\delta^{\mathrm{u}}_{\mathrm{ell}}(X_{\Omega}) \geqslant \frac{\|(1,1)\|^*_{\Omega}}{\max\{\|(1,0)\|^*_{\Omega}, \|(0,1)\|^*_{\Omega}\}}.$$

(b) If $X_{\Omega} \subset \mathbb{C}^2$ is a concave toric domain, then

$$\delta^{\mathrm{u}}_{\mathrm{ell}}(X_{\Omega}) \geqslant \frac{\min\{[(2,1)]_{\Omega}, [(1,2)]_{\Omega}\}}{[(1,1)]_{\Omega}}.$$

As for upper bounds on δ_{ell}, we proved the following:

Theorem 8.0.10. *(a) Suppose that $\Omega \subset [0,\infty)^2$ is a domain such that $\hat{\Omega}$ is convex and such that Ω contains points $(a,0), (0,b), (x,y)$ with $0 < x \leqslant a \leqslant b \leqslant x+y$. Then*

$$\delta_{\mathrm{ell}}(X_{\Omega}) \leqslant \left\| \left(\frac{1}{a}, \frac{1}{x+y} \right) \right\|^*_{\Omega}.$$

(b) Suppose that $\Omega \subset [0,\infty)^2$ is a domain that contains $(0,0)$ in its interior and whose complement in $\overline{[0,\infty)^2}$ is convex, and such that points $(a,0), (0,b), (x,y)$ with $0 < x+y \leqslant a \leqslant b$ all belong to $\overline{[0,\infty)^2 \setminus \Omega}$. Then

$$\delta_{\mathrm{ell}}(X_{\Omega}) \leqslant \frac{1}{\left[\left(\frac{1}{b}, \frac{1}{x+y} \right) \right]_{\Omega}}.$$

(c) For a polydisk $P(a,b)$ with $a \leqslant b \leqslant 2a$ we have

$$\delta_{\mathrm{ell}}(P(a,b)) \leqslant \left\| \left(\frac{3}{a+b}, \frac{1}{2a+b} \right) \right\|^*_{[0,a]\times[0,b]}.$$

8.1. An explicit construction

The embeddings that underlie Theorem 8.0.10 are obtained by very indirect methods and are difficult to understand concretely. We will now explain a more direct construction that, for instance, leads to an explicit formula for a knotted embedding $P(1,1) \to \alpha P(1,1)^\circ$ for any $\alpha \in \left(\frac{1}{2-\sqrt{2}}, 2\right)$.

The key ingredient is a toric structure on the complement of the antidiagonal in $S^2 \times S^2$ that appears (at least implicitly) in [EP09, Example 1.22], [FOOO12], [OU16, Section 2]. View S^2 as the unit sphere in $\mathbb{R}^3$ and let $A = \{(v,w) \in S^2 \times S^2 \,|\, w = -v\}$ be the antidiagonal. Define functions $F_1, F_2 \colon S^2 \times S^2 \to \mathbb{R}$ by

$$F_1(v,w) = v_3 + w_3 \qquad F_2(v,w) = \|v+w\|.$$

Now F_2 fails to be smooth along $A = F_2^{-1}(\{0\})$, but on $S^2 \times S^2 \backslash A$ the Hamiltonian flows of the functions F_1 and F_2 induce S^1-actions that commute with each other and are rather simple to understand: F_1 induces simultaneous rotation of the factors about the z-axis, and F_2 induces the flow which rotates the pair $(v,w) \in S^2 \times S^2 \backslash A$ about an axis in the direction of $v+w$. In formulas:

$$\begin{aligned}\phi^t_{F_1}&((v_1,v_2,v_3),(w_1,w_2,w_3)) \qquad (8.1.1)\\ &= \Big(\big((\cos t)v_1 - (\sin t)v_2, (\sin t)v_1 + (\cos t)v_2, v_3\big), \big((\cos t)w_1 - (\sin t)w_2, (\sin t)w_1 + (\cos t)w_2, w_3\big)\Big)\end{aligned}$$

and

$$\phi^t_{F_2}(v,w) = \left(\frac{v+w}{2} + (\cos t)\frac{v-w}{2} + (\sin t)\frac{w \times v}{\|v+w\|}, \frac{v+w}{2} + (\cos t)\frac{w-v}{2} + (\sin t)\frac{v \times w}{\|v+w\|}\right). \qquad (8.1.2)$$

Define

$$J \colon S^2 \times S^2 \to \mathbb{R}^2 \quad \text{by} \quad J(v,w) = (2 - \|v+w\|, \|v+w\| - v_3 - w_3),$$

i.e. $J = (2 - F_2, -F_1 + F_2)$. Then J is smooth away from A, and its restriction to $S^2 \times S^2 \backslash A$ is the moment map for a Hamiltonian T^2-action.[2] It is not hard to see that J has image equal to $\Delta := \{(x,y) \in [0,\infty)^2 | x/2 + y/4 \leqslant 1\}$, and that the preimage of $\{x/2 + y/4 = 1\}$ is equal to $Q := \{(v,w) \in S^2 \times S^2 | v_3 + w_3 = -\|v+w\|\}$. (In other words, Q is the locus of pairs $(v,w) \in S^2 \times S^2$ such that $v+w$ is on the nonpositive z axis.)

Proposition 8.1.1. *Let $\Delta^\circ = \left\{(x,y) \in [0,\infty)^2 \,\middle|\, \frac{x}{2} + \frac{y}{4} < 1\right\}$ and define $s \colon \Delta^\circ \to S^2 \times S^2$ by*

$$\begin{aligned}s(x,y) = \Bigg(&\left(\sqrt{x\left(1 - \frac{x}{4}\right)}, \sqrt{y\left(1 - \frac{x}{2} - \frac{y}{4}\right)}, 1 - \frac{x+y}{2}\right),\\ &\left(-\sqrt{x\left(1 - \frac{x}{4}\right)}, \sqrt{y\left(1 - \frac{x}{2} - \frac{y}{4}\right)}, 1 - \frac{x+y}{2}\right)\Bigg).\end{aligned}$$

Then, writing $E(4\pi, 8\pi)^\circ = \left\{(w,z) \in \mathbb{C}^2 \,\middle|\, \frac{|w|^2}{4} + \frac{|z|^2}{8} < 1\right\}$, the map

$$\Phi(|z_1|e^{i\theta}, |z_2|e^{i\varphi}) = \phi^{\varphi}_{F_1}\left(\phi^{\theta-\varphi}_{F_2}\left(s\left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}\right)\right)\right)$$

defines a symplectomorphism $\Phi \colon E(4\pi, 8\pi)^\circ \to S^2 \times S^2 \backslash Q$ which satisfies $J \circ \Phi(z_1, z_2) = \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}\right)$.

[2] Here we view T^2 as $(\mathbb{R}/2\pi\mathbb{Z})^2$. On the other hand the map $\mu(w,z) = (\pi|w|^2, \pi|z|^2)$ that we have considered elsewhere is the moment map for a Hamiltonian $(\mathbb{R}/\mathbb{Z})^2$-action; to get a $(\mathbb{R}/2\pi\mathbb{Z})^2$-action one would take $\frac{\mu}{2\pi}$.

Proof. First we observe that s indeed takes values in $S^2\times S^2\subset\mathbb{R}^3\times\mathbb{R}^3$, which follows by computing

$$\begin{aligned} & x\left(1-\frac{x}{4}\right)+y\left(1-\frac{x}{2}-\frac{y}{4}\right)+\left(1-\frac{x+y}{2}\right)^2 \\ &= x+y-\frac{x^2+y^2}{4}-\frac{xy}{2}+1-x-y+\frac{x^2+2xy+y^2}{4}=1. \end{aligned}$$

Given $(x,y)\in\Delta^\circ$, if we write $(v,w)=s(x,y)$, then

$$\|v+w\|^2=4y\left(1-\frac{x}{2}-\frac{y}{4}\right)+(2-x-y)^2=x^2-4x+4=(2-x)^2,$$

so (since $x<2$)

$$J\big(s(x,y)\big)=\big(2-\|v+w\|,-v_3-w_3+\|v+w\|\big)=(x,x+y-2+2-x)=(x,y).$$

In particular, the image of s is contained in $S^2\times S^2\backslash Q=J^{-1}(\Delta^\circ)$, and it intersects each fiber of $J|_{J^{-1}(\Delta^\circ)}$ just once.

Moreover, since the image of s is contained in $\{(v,Rv)\,|\,v\in S^2\}$ where R is the reflection through the v_2v_3-plane and hence is antisymplectic, we see that $s^*\Omega=0$ where Ω is the standard product symplectic form on $S^2\times S^2$. Thus $s\colon\Delta^\circ\to J^{-1}(\Delta^\circ)$ is a Lagrangian right inverse to the moment map J.

Write $\psi^1_{(\theta,\varphi)}(z_1,z_2)=(e^{-i\theta}z_1,e^{-i\varphi}z_2)$ for the standard T^2-action on $E(4\pi,8\pi)^\circ$ (with moment map $\frac{\mu}{2\pi}$ having image equal to Δ°; the negative signs in front of θ and φ arise because our convention for Hamiltonian vector fields is $\omega_0(X_H,\cdot)=dH$). Likewise write $\psi^2_{(\theta,\varphi)}=\phi^{-\varphi}_{F_1}\circ\phi^{\varphi-\theta}_{F_2}$ for the T^2-action on $S^2\times S^2\backslash Q$ induced by the moment map J. Our map Φ maps the Lagrangian section of $\frac{\mu}{2\pi}$ given by the nonnegative real locus of $E(4\pi,8\pi)^\circ$ to the Lagrangian section of $J|_{S^2\times S^2\backslash Q}$ given by the image of s, and Φ obeys $J\circ\Phi=\frac{\mu}{2\pi}$ and, for all $(\theta,\varphi)\in T^2$, $\Phi\circ\psi^1_{(\theta,\varphi)}=\psi^2_{(\theta,\varphi)}\circ\Phi$. These facts are easily seen to imply that Φ is a symplectomorphism, as it identifies action-angle coordinates on $E(4\pi,8\pi)^\circ$ with action-angle coordinates on $S^2\times S^2\backslash Q$. The last statement is immediate from the formula for Φ and the facts that $J\circ s$ is the identity and that J is preserved under the Hamiltonian flows of F_1 and F_2. ☐

Remark 8.1.2. *With sufficient effort, one can derive the following equivalent formula for the map* $\Phi\colon E(4\pi,8\pi)^\circ\to S^2\times S^2$ *from Proposition 8.1.1: regarding S^2 as the unit sphere in $\mathbb{C}\times\mathbb{R}$, we have*

$$\begin{aligned} \Phi(w,z) &= \big(\Gamma(w,z),\Gamma(-w,z)\big) \quad \textit{where} \\ \Gamma(w,z) &= \left(\frac{\sqrt{8-|w|^2}\,\big((8-2|w|^2-|z|^2)w+\bar{w}z^2\big)}{8(4-|w|^2)}+\frac{iz}{4}\sqrt{8-2|w|^2-|z|^2},\right. \\ &\qquad \left. 1-\frac{|w|^2+|z|^2}{4}-\frac{\sqrt{(8-|w|^2)(8-2|w|^2-|z|^2)}}{4(4-|w|^2)}Im(w\bar{z})\right). \end{aligned} \tag{8.1.3}$$

Since $E(4\pi,8\pi)^\circ$ is precisely the locus where $2|w|^2+|z|^2<8$, this makes clear that Φ is a smooth (indeed even real-analytic) map despite the appearance of square roots in the formula for s in Proposition 8.1.1.

Now if $D(4\pi)$ denotes the open disk of area 4π (so radius 2) in $\mathbb{C}$, there is a symplectomorphism $\sigma\colon S^2\backslash\{(0,-1)\} \to D(4\pi)$ defined by

$$\sigma(z, v_3) = \sqrt{\frac{2}{1+v_3}}z \tag{8.1.4}$$

where as in Remark 8.1.2 we regard S^2 as the unit sphere in $\mathbb{C}\times\mathbb{R}$.

So if we let $\mathscr{I} = \big(\{(0,-1)\}\times S^2\big)\cup\big(S^2\times\{(0,-1)\}\big)$ then $\sigma\times\sigma$ defines a symplectomorphism $S^2\times S^2\backslash\mathscr{I} \cong P(4\pi,4\pi)^\circ = D(4\pi)\times D(4\pi)$.

For $v = (z, v_3) \in S^2 \subset \mathbb{C}\times\mathbb{R}$, we have

$$\|v + (0,-1)\|^2 = |z|^2 + v_3^2 - 2v_3 + 1 = 2 - 2v_3$$

and hence

$$J\big(v,(0,-1)\big) = J\big((0,-1),v\big) = \Big(2-\sqrt{2-2v_3}, \sqrt{2-2v_3} + (1-v_3)\Big).$$

Thus

$$J(\mathscr{I}) \subset \big\{(x,y)\in\mathbb{R}^2\,|\,(2-x)^2 = 2(x+y)-4\big\} = \big\{(x,y)\in\mathbb{R}^2\,|\,y = \frac{x^2}{2} - 3x + 4\big\}.$$

Since $\frac{\mu}{2\pi} = J\circ\Phi$, we have $\frac{\mu}{2\pi}(\Phi^{-1}(\mathscr{I})) = J(\mathscr{I})$. From this we obtain the following:

Proposition 8.1.3. *Suppose that X_Ω is a convex toric domain where $\Omega \subset \big\{(2\pi x, 2\pi y)\in[0,\infty)^2 | y < \frac{x^2}{2} - 3x + 4\big\}$. Then there is an ellipsoid E such that $X_\Omega \subset E^\circ$ and such that the map Φ from Proposition 8.1.1 maps E to a subset of $S^2\times S^2\backslash\mathscr{I}$. Hence $(\sigma\times\sigma)\circ\Phi|_E$ is a symplectic embedding from E to $P(4\pi,4\pi)^\circ$.*

Proof. The sets $\frac{1}{2\pi}\Omega$ and $S := \big\{(x,y)\in[0,\infty)^2\,|\,y \geqslant \frac{x^2}{2} - 3x + 4\big\}$ are disjoint, closed, convex subsets of $\mathbb{R}^2$, and the first of these sets is compact, so the hyperplane separation theorem shows that they must be separated by a line ℓ, which passes through the first quadrant since both sets are contained in the first quadrant. This line ℓ must have negative slope, since S intersects all lines with positive slope and also intersects all horizontal or vertical lines that pass through the first quadrant. So we can write the separating line as $\ell = \big\{(x,y)\in\mathbb{R}^2\,|\,\frac{x}{a}+\frac{y}{b} = 1\big\}$ with $a, b > 0$, and then it will hold that $\frac{1}{2\pi}\Omega \subset \big\{\frac{x}{a}+\frac{y}{b} < 1\big\}$ and $S \subset \{\frac{x}{a}+\frac{y}{b} > 1\}$. The first inclusion shows that $X_\Omega \subset E(2\pi a, 2\pi b)^\circ$. Meanwhile since $(2,0),(0,4)\in S\subset\big\{\frac{x}{a}+\frac{y}{b} > 1\big\}$, we have $a < 2$ and $b < 4$. So $E(2\pi a, 2\pi b)$ is contained in the domain of the map Φ from Proposition 8.1.1, and by the discussion before the proposition the fact that $\ell\cap S = \varnothing$ implies that $E(2\pi a, 2\pi b)\cap\Phi^{-1}(\mathscr{I}) = \varnothing$. Hence the proposition holds with $E = E(2\pi a, 2\pi b)$. □

Corollary 8.1.4. *Suppose that X_Ω is a convex toric domain with $\Omega\subset\{(2\pi x, 2\pi y)\in[0,\infty)^2|y < \frac{x^2}{2} - 3x + 4\}$, and that we have $P(4\pi,4\pi)\subset\alpha X_\Omega$ for some $\alpha < \delta_{\mathrm{ell}}^{\mathrm{u}}(X_\Omega)$. Then $(\sigma\times\sigma)\circ\Phi|_{X_\Omega}\colon X_\Omega \hookrightarrow P(4\pi,4\pi)^\circ \subset \alpha X_\Omega^\circ$ defines a knotted embedding $X_\Omega \hookrightarrow \alpha X_\Omega^\circ$.*

Proof. By Proposition 8.1.3 we have an ellipsoid E and a sequence $X_\Omega \hookrightarrow E^\circ \hookrightarrow P(4\pi,4\pi)^\circ \subset \alpha X_\Omega^\circ$ where the first map is the inclusion and the second map is $(\sigma\times\sigma)\circ\Phi|_E$. So the corollary follows directly from the assumption that $\alpha < \delta_{\mathrm{ell}}^{\mathrm{u}}(X_\Omega)$ and the definition of $\delta_{\mathrm{ell}}^{\mathrm{u}}$. □

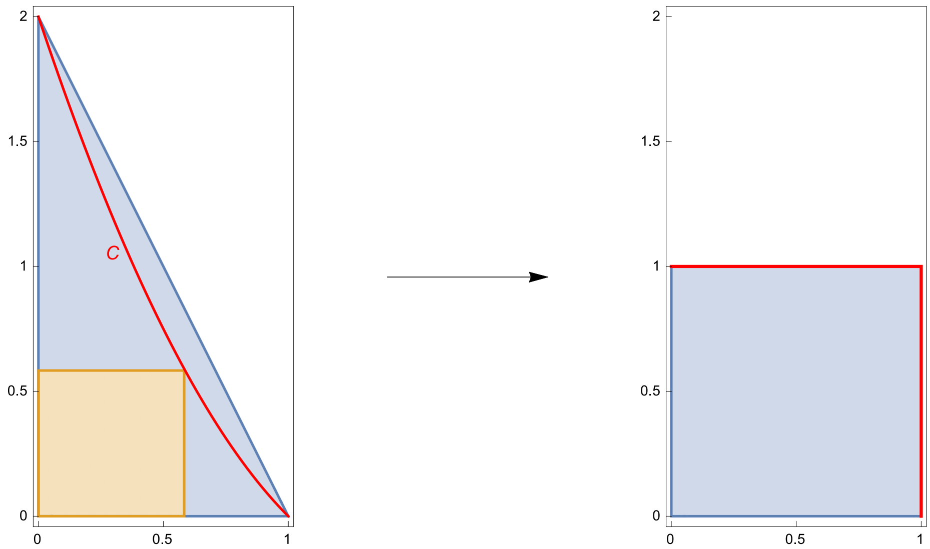


Figure 8.3.: After appropriate rescalings, the map Φ from Proposition 8.1.1 sends the interior of the ellipsoid $E(1,2)$ to a product of spheres of area 1, with the preimage of $(S^2 \times \{(0,0,-1)\}) \cup (\{(0,0,-1)\} \times S^2)$ contained in $\mu^{-1}(C)$ where C is the red curve at left. Consequently the preimage under μ of any domain lying below C, such as the small square at left, is embedded into the polydisk $P(1,1)^\circ$ by a rescaling of $(\sigma\times\sigma)\circ\Phi$. This gives an explicit knotted embedding $P(c,c) \hookrightarrow P(1,1)^\circ$ for $1/2 < c < 2-\sqrt{2}$.

We emphasize that this embedding $(\sigma\times\sigma)\circ\Phi$ is completely explicit: σ is defined in (8.1.4) and Φ is defined in Proposition 8.1.1 based partly on the formulas (8.1.1) and (8.1.2), or even more explicitly is given by (8.1.3).

Example 8.1.5. *For instance, Ω could be taken to be a square $[0,2\pi c]^2$ with c smaller than the smallest root of the polynomial $\frac{x^2}{2} - 4x + 4$, namely $4-2\sqrt{2}$ (see Figure 8.3). So we obtain an embedding $(\sigma\times\sigma)\circ\Phi\colon P(2\pi c, 2\pi c) \hookrightarrow P(4\pi,4\pi)^\circ = \frac{2}{c}P(2\pi c,2\pi c)^\circ$, which is knotted provided that $\frac{2}{c} < \delta^{\mathrm{u}}_{\mathrm{ell}}(P(2\pi c,2\pi c))$. By Theorem 8.0.9 we have $\delta^{\mathrm{u}}_{\mathrm{ell}}(P(a,a)) \geqslant 2$ for any a, so our embedding is knotted provided that $1 < c < 4-2\sqrt{2}$. So after conjugating by appropriate rescalings our explicit embedding $(\sigma\times\sigma)\circ\Phi$ defines a knotted embedding $P(a,a) \hookrightarrow \alpha P(a,a)^\circ$ provided that $2 > \alpha > \frac{1}{2-\sqrt{2}} \approx 1.71$. For comparison, our less explicit construction (leading to the bound $\delta_{\mathrm{ell}}(P(a,a)) \leqslant 3/2$ from Theorem 8.0.10) gives knotted embeddings $P(a,a) \hookrightarrow \alpha P(a,a)^\circ$ whenever $2 > \alpha > 1.5$.*

Choosing the scaling so that the codomain is $P(4\pi,4\pi)^\circ$, the image of this embedding $\alpha^{-1}P(4\pi,4\pi) \hookrightarrow P(4\pi,4\pi)^\circ$ is not hard to describe explicitly as a subset of $P(4\pi,4\pi)^\circ$: it is given as the region

$$\left\{(z_1,z_2) \in P(4\pi,4\pi)^\circ \middle| G_2(z_1,z_2) \geqslant 2-2/\alpha,\ -G_1(z_1,z_2)+G_2(z_1,z_2) \leqslant 2/\alpha\right\},$$

where $G_i = F_i \circ (\sigma\times\sigma)^{-1}$, i.e.,

$$G_1(z_1,z_2) = 2 - \frac{|z_1|^2+|z_2|^2}{2}$$

and

$$G_2(z_1,z_2)^2 = \left(\sqrt{1-\frac{|z_1|^2}{4}}\,\mathrm{Re}(z_1) + \sqrt{1-\frac{|z_2|^2}{4}}\,\mathrm{Re}(z_2)\right)^2 + \left(\sqrt{1-\frac{|z_1|^2}{4}}\,\mathrm{Im}(z_1) + \sqrt{1-\frac{|z_2|^2}{4}}\,\mathrm{Im}(z_2)\right)^2 + \left(2-\frac{|z_1|^2+|z_2|^2}{2}\right)^2.$$

Corollary 8.1.4 also applies to some other convex toric domains besides the cube $P(a,a)$, though it as not as broadly applicable as Theorem 8.0.4. For example the reader may check that, in Corollary 8.1.4, for appropriate α one can take X_Ω equal to a polydisk $P(1,a)$ with $1 \leqslant a \leqslant 1.2$, or to an appropriately rescaled ℓ^p ball as in Theorem 8.0.4 for $p \geqslant 10$.

Remark 8.1.6. *By construction, the embedding Φ from Proposition 8.1.1 maps the torus $T_{\sqrt{2}} := \left\{(w,z)\in\mathbb{C}^2 \,\middle|\, |w| = |z| = \sqrt{2}\right\}$ to the Lagrangian torus in $S^2\times S^2$ that is denoted K in [EP09, Example 1.22], and which can be identified with the Chekanov-Schlenk twist torus Θ, see [CS10],[OU16]. Since, as shown in [EP09], there is no symplectomorphism mapping K to the Clifford torus in $S^2\times S^2$ (i.e., to the image of $T_{\sqrt{2}}$ under the standard embedding $(\sigma\times\sigma)^{-1}$ of $P(4\pi,4\pi)^\circ$ into $S^2\times S^2$), one easily infers independently of our other results that $(\sigma\times\sigma)\circ\Phi\colon P(2\pi c,2\pi c)\hookrightarrow P(4\pi,4\pi)^\circ$ must not be isotopic to the inclusion by a compactly supported Hamiltonian isotopy for $1 < c < 4-2\sqrt{2}$ (for such a Hamiltonian isotopy could be extended to $S^2\times S^2$, giving a symplectomorphism that would send K to the Clifford torus). However this argument based on Lagrangian tori does not seem to adapt to yield the full result that $(\sigma\times\sigma)\circ\Phi$ is knotted in the stronger sense of Definition 8.0.3.*

By the way, if $c < 1$, our embedding $(\sigma\times\sigma)\circ\Phi\colon P(2\pi c,2\pi c)\hookrightarrow P(4\pi,4\pi)^\circ$ is unknotted. Indeed in this case the ball $B^4(4\pi c)$ is contained both in $P(4\pi,4\pi)^\circ$ and in $E(4\pi,8\pi)\setminus\Phi^{-1}(\mathscr{I})$, and so both $(\sigma\times\sigma)\circ\Phi|_{P(2\pi c,2\pi c)}$ and the inclusion $P(2\pi c,2\pi c)\hookrightarrow P(4\pi,4\pi)^\circ$ extend to embeddings $B^4(4\pi c,4\pi c)\hookrightarrow P(4\pi,4\pi)^\circ$; these two embeddings of the ball are symplectically isotopic by [CG19, Proposition 1.5]. Thus a transition between knottedness and unknottedness occurs at the value $c=1$, which is precisely the first value for which $P(2\pi c,2\pi c)$ contains the torus $T_{\sqrt{2}}$ mentioned at the start of the remark.

Remark 8.1.7. *A similar construction to that in Proposition 8.1.1, using results from [OU16, Section 3], allows one to construct a symplectic embedding of $E(3\pi,12\pi)^\circ$ into $\mathbb{C}P^2$ where the symplectic form on $\mathbb{C}P^2$ is normalized to give area 6π to a complex projective line, such that the torus $T_{\sqrt{2}}$ is sent to the $\mathbb{C}P^2$ version of the Chekanov-Schlenk twist torus Θ. Combining this with a symplectomorphism from the complement of a line in $\mathbb{C}P^2$ to a ball and restricting to $P(2\pi c,2\pi c)$ for c slightly larger than 1, we obtain a symplectic embedding $P(2\pi c,2\pi c)\hookrightarrow B^4(6\pi)^\circ$ which cannot be Hamiltonian isotopic to the inclusion because Θ is not Hamiltonian isotopic to the Clifford torus. It is less clear whether this embedding $P(2\pi c,2\pi c)\hookrightarrow B^4(6\pi)^\circ$ is knotted in the sense of Definition 8.0.3; the symplectic-homology-based methods in the present paper seem ill-equipped to address this because the filtered positive S^1-equivariant symplectic homology of $B^4(6\pi)$ does not have as rich a structure as that of the domains X that appear in Theorem 8.0.4.*

9. Symplectic convexity

As we saw throughout this memoir, convexity plays a big role in symplectic geometry, and is often synonymous of rigidity but is not invariant under symplectomorphism. The main question is

Question 9.0.1. *What is the symplectic analogue of convexity?*

In $\mathbb{R}^{2n}$, defining symplectic convexity as being symplectomorphic to a convex domain is a hardly verifiable condition and therefore not ideal. The main alternative is dynamical convexity.

Question 1.3.3. *Is every dynamically convex domain in $\mathbb{R}^{2n}$ symplectomorphic to a convex domain in $\mathbb{R}^{2n}$?*

The answer is NO. Recently, Chaidez and Edtmair [CE20, Theorem 1.8] constructed a dynamically convex domain which is not symplectomorphic to a convex one. The main new ingredient in Chaidez and Edtmair is the **Ruelle invariant** (see [Rue85, CE20, Hut19]).

The "Ruelle invariant" is defined for a contact form on a homology three-sphere, which, roughly speaking, is a measure of the average rotation rate of the Reeb flow. We follow the exposition from [Hut19] in what follows.

Let $\widetilde{\mathrm{Sp}}(\mathbb{R}^2,\Omega_0)$ denote the universal cover of the group $\mathrm{Sp}(\mathbb{R}^2,\Omega_0)$ of 2×2 symplectic matrices. There is a standard "rotation number" function

$$\mathrm{rot} : \widetilde{\mathrm{Sp}}(\mathbb{R}^2,\Omega_0) \longrightarrow \mathbb{R}$$

defined as follows. Let $A \in \mathrm{Sp}(\mathbb{R}^2,\Omega_0)$, and let $\widetilde{A} \in \widetilde{\mathrm{Sp}}(\mathbb{R}^2,\Omega_0)$ be a lift of A. This lift $\widetilde{A}$ can be represented by a path $\{A_t\}_{t\in[0,1]}$ in $\mathrm{Sp}(\mathbb{R}^2,\Omega_0)$ with $A_0 = Id$ and $A_1 = A$. If v is a nonzero vector in $\mathbb{R}^2$, then the path of vectors $\{A_t v\}_{t\in[0,1]}$ rotates by some angle which we denote by $2\pi\rho(v) \in \mathbb{R}$. We then define

$$\mathrm{rot}\left(\widetilde{A}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho\left(A^{k-1}v\right).$$

This does not depend on the choice of nonzero vector v. For example, if A is conjugate to rotation by angle $2\pi\theta$, then $\mathrm{rot}\left(\widetilde{A}\right)$ is a lift of θ from $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$. The rotation number is a quasimorphism: if $\widetilde{B}$ is another element of $\widetilde{\mathrm{Sp}}(\mathbb{R}^2,\Omega_0)$, then

$$\left|\mathrm{rot}\left(\widetilde{A}\widetilde{B}\right) - \mathrm{rot}\left(\widetilde{A}\right) - \mathrm{rot}\left(\widetilde{B}\right)\right| < 1.$$

Now let Y be a homology three-sphere, and let λ be a contact form on Y with associated contact structure ξ and Reeb vector field R. For $t \in \mathbb{R}$, let $\phi_t : Y \to Y$ denote the diffeomorphism given by the time t Reeb flow. For each $y \in Y$, the derivative of ϕ_t restricts to a linear map

$$d\phi_t : \xi_y \longrightarrow \xi_{\phi_t(y)}$$

which is symplectic with respect to $d\lambda$. Now fix a symplectic trivialization of ξ, consisting of a symplectic linear map $\tau : \xi_y \to \mathbb{R}^2$ for each $y \in Y$. Then for $y \in Y$ and $t \in \mathbb{R}$, the composition

$$\mathbb{R}^2 \xrightarrow{\tau^{-1}} \xi_y \xrightarrow{d\phi_t} \xi_{\phi_t(y)} \xrightarrow{\tau} \mathbb{R}^2$$

is a symplectic matrix which we denote by $A^{\tau}_{y,t}$. In particular, if $y \in Y$ and $T \geqslant 0$, then the path of symplectic matrices $\{A^{\tau}_{y,t}\}_{t\in[0,T]}$ defines an element of $\widetilde{\mathrm{Sp}}(2)$. We denote its rotation number by

$$\mathrm{rot}_{\tau}(y,T) = \mathrm{rot}\left(\{A^{\tau}_{y,t}\}_{t\in[0,T]}\right) \in \mathbb{R}.$$

One can use the quasimorphism property to show [CGS20, Rue85] that for almost all $y \in Y$, the limit

$$\rho(y) = \lim_{T\to\infty} \frac{1}{T}\mathrm{rot}_{\tau}(y,T)$$

is well defined and independent of τ, and the function ρ is integrable.

Definition 9.0.2. *If Y is a homology three-sphere and λ is a contact form on Y, define the* ***Ruelle invariant***

$$\mathrm{Ru}(Y,\lambda) = \int_Y \rho\,\lambda \wedge d\lambda.$$

Definition 9.0.3. *If X is a star-shaped domain in $\mathbb{R}^4$ with smooth boundary, then we define*

$$\mathrm{Ru}(X) = \mathrm{Ru}\left(\partial X, \lambda_0|_{\partial X}\right).$$

For example of computations, we have the following proposition:

Proposition 9.0.4 (Gutt-Zhang). *Let X_{Ω} be any 4-dimensional toric star-shaped domain*[1]*. Then its Ruelle invariant is*

$$\mathrm{Ru}(X_{\Omega}) = a(\Omega) + b(\Omega)$$

where $a(\Omega)$ and $b(\Omega)$ are the w_1-intercept and w_2-intercept, respectively, of the moment image Ω in $[0,\infty)^2$ (in (w_1,w_2)-coordinate.

The question of being symplectomorphic to a convex domain still remains for the particular example of Theorem 2.1.14. If this example were to be symplectomorphic to a convex domain, this would imply that the weak Viterbo conjecture is false.

Another approach to Question 1.3.3, is by using some interleaving distance, see [Ush20, SZ]. The notion of knotted embedding (Definition 8.0.3) allows one to define a Banach-Mazur type distance on the set of star-shaped domains in $\mathbb{R}^{2n}$. Given two star-shaped domains X and Y in $\mathbb{R}^{2n}$, their distance $\rho(X,Y)$ is defined as

$$\inf\left\{ \log\lambda \;\middle|\; \begin{cases} \text{there exists symplectic embeddings } \lambda^{-1}X \hookrightarrow Y \hookrightarrow \lambda X; \\ \text{there exists symplectic embeddings } \lambda^{-1}Y \hookrightarrow X \hookrightarrow \lambda Y; \\ \text{the two compositions above are unknotted.} \end{cases} \right\}.$$

This distance is invariant under symplectomorphism. By [Joh48, Theorem III], any convex body $K \subset \mathbb{R}^n$ has an associated John ellipsoid E_K which obeys $E_K \subset K \subset nE_K$. Thus any dynamically convex domain which is "far" from the set of ellipsoids would not be symplectomorphic to any convex domain. This criterion fails on the example from Theorem 2.1.14. This prompted the question

[1] Here, a toric star-shaped domain means ∂X_{Ω} is smooth and the radial vector field of $\mathbb{R}^4$ is transversal to ∂X_{Ω}. In particular, $\partial\Omega$ is smooth and the radial vector field of $\mathbb{R}^2$ is transversal to $\partial\Omega$.

Question 9.0.5. *Are all dynamically convex domain close (at interleaving distance less than $\sqrt{n}$) to the set of ellipsoids?*

Another argument in favor of Viterbo's conjecture is

Theorem 9.0.6 (Hryniewicz, private communication)**.** *In dimension 4, the strong Viterbo Conjecture implies a positive answer to Question 2.1.12*

An alternative notion of symplectic convexity, in dimension 4, is to be monotone toric (see Definition 5.1.3).

Question 9.0.7. *Is every convex set in $\mathbb{R}^4$ symplectomorphic to a monotone toric domain?*

An affirmative answer would provide a proof of the strong Viterbo Conjecture in dimension 4.

Perspectives

In the text, I have mentioned questions and developments directly related to the results presented in this memoir and which I intend to study. This last section will be devoted to present a longer term perspective for my research.

Coming back to the motivation of uncovering the link between the geometry of a symplectic manifold and the contact geometry of its boundary, the long term dream is to try to mimick what has been done in the Riemannian case and relate those links to operator theory.

Does there exist an operator (or a family of operators) defined on an "appropriate space" (depending on the given symplectic manifold) such that its spectrum is related to symplectic capacities?

This is in the spirit of Lorentz question (see Foreword) about the Dirichlet spectrum of the Laplacian. Recall that if (M,g) is a Riemannian manifold with boundary, then M has a **Laplace operator** $\triangle$, defined by $\triangle(f) := -\operatorname{div}(\operatorname{grad} f)$, that acts on smooth functions on M. The **spectrum** of M is the sequence of eigenvalues of $\triangle$. The **Dirichlet spectrum** is the spectrum of $\triangle$ acting on smooth functions that vanish on the boundary and the **Neumann spectrum** is that of $\triangle$ acting on functions with vanishing normal derivative at the boundary. The **spectral gap** is the smallest positive eigenvalue of $\triangle$.

The dreamy idea is to first try to build such an operator for star-shaped domains in $\mathbb{R}^{2n}$, starting with $n=2$, then to extend it to cotangent bundles and prequantization bundles and see if it can be generalised to other symplectic manifold. The definition of such an operator would require additional geometric structures (other than the symplectic form), for instance a compatible almost complex structure or an appropriate symplectic connexion.

In the near future, the first step is to study the desired spectrum (i.e. symplectic capacities). The various directions which are described in what follows are: the asymptotics of the capacities, their behaviour under symplectic products, the smallest capacity, and applications to Reeb dynamics. The next step, consisting in the combined approaches of building an operator from its spectrum and computing the spectrum of known operators carrying some symplectic data, being at this point wildly speculative, will not be developed here.

As seen in Chapter 6, there are, in dimension 4, two distinct sequences of capacities, the ECH capacities (§6.1) and the capacities from positive S^1-equivariant symplectic homology (§6.3) / Ekeland-Hofer capacities (§6.2). The latter being well-defined in higher dimensions. In the following, we shall distinguish the case of dimension 4, where we are going to focus on ECH capacities, and the higher dimensional case, where we shall consider the capacities from positive S^1-equivariant symplectic homology.

Dimension 4

There is the following fundamental result about the ECH spectrum

Theorem 0.1 ([CGHR14]). *Let (X, λ) be a four dimensional Liouville domain such that $c_k^{ECH}(X, \lambda) < \infty$ for all k. Then*

$$\lim_{k\to\infty} \frac{\left(c_k^{ECH}(X, \lambda)\right)^2}{k} = 4\,\mathrm{Vol}(X, \lambda).$$

This Theorem is to be compared to the following Theorem, named Weyl's law (which generalises Weyl's answer to Lorentz question). We refer to [Ivr16] and references therein for a nice history of Weyl's law and later work.

Theorem 0.2 (Weyl's law in dimension 4, [Wey11, Wey50]). *Let X be a star-shaped domain in $\mathbb{R}^4$ with smooth boundary $Y := \partial X$. Denote by $\{\lambda_k \,|\, k \in \mathbb{N}\}$ the Dirichlet spectrum (associated to X). Then*

$$\lim_{k\to\infty} \frac{\lambda_k^2}{k} = \frac{32\pi^2}{\mathrm{Vol}(X)}$$

For the higher order terms, there is the Weyl conjecture which is proven under the assumption that the set of all periodic geodesic billiards has measure 0.

On the ECH side the higher order terms have been studied in [Hut19, CGS20]. Given a Liouville domain (X, λ), one define the error term as

$$e_k(X) := c_k^{ECH}(X) - 2\sqrt{k\,\mathrm{Vol}(X)}$$

Conjecture 0.3 ([Hut19]). *If X is a star-shaped domain in $\mathbb{R}^4$, then*

$$e_k(X) = O(1)$$

So far, [Sun18, CGS20, Hut19], the current statement is that $e_k(X) = O(k^{\frac{1}{4}})$. In fact, Hutchings proved that

Theorem 0.4 ([Hut19]). *If $X \subset \mathbb{R}^4$ is a strictly convex or strictly concave toric domain, then*

$$\lim_{k\to\infty} e_k(X) = -\tfrac{1}{2}\,\mathrm{Ru}(X) \tag{0.1}$$

where $\mathrm{Ru}(X)$ is the Ruelle invariant (see Definition 9.0.3). Hutchings conjectured Equation (0.1) to hold for all **generic** star-shaped domains. The term "generic" is crucial since (0.1) is false for the ball. Also,

Counterexample 0.5 (Gutt-Zhang). *The ellipsoid $E(1, 2)$ and the polydisk $P(1, 1)$ have the same ECH capacities and the same volume (and thus same e_k). But, by Proposition 9.0.4, their Ruelle invariant is different.*

Remark 0.6. *The ellipsoid $E(1, 2)$ and the polydisk $P(1, 1)$ where already distinguished as non-symplectomorphic by the third Ekeland-Hofer capacity or the third capacity from positive S^1-equivariant symplectic homology.*

$$c_3^{EH}\big(E(1,2)\big) = c_3\big(E(1,2)\big) = 3 \qquad \textit{and} \qquad c_3^{EH}\big(P(1,1)\big) = c_3\big(P(1,1)\big) = 2.$$

This led Jun Zhang and I to the three following questions / future directions:

Question 0.7. *Can one generalise the Ruelle invariant to contact flows (i.e. flows preserving the contact structure but not necessarily a contact form) and extract dynamical information, in particular on periodic orbits, on those flows?*

Question 0.8. *Can we compute the Ruelle invariant (for Reeb flows) for other manifolds than the sphere. For instance for prequantized spaces (e,g, unit cosphere bundles)?*

Question 0.9. *Can we generalise the Ruelle invariant to higher dimensions?*

At the moment, we focus our attention on the latter question where the starting point is to use the map ρ from [CZ84] (and constructed in details in [Gut14b, Gut14a]) to generalize the rotation number.

Dimension $\geqslant 4$

The main sequences of symplectic capacities defined in any dimension are the Ekeland-Hofer capacities (for star-shaped domains in $\mathbb{R}^{2n}$, §6.2) and for all Liouville domains the capacities from positive S^1-equivariant symplectic homology (§6.3). The conjecture is that those coincides:

Conjecture 6.3.13 ([GH18])**.** *For all star-shaped domain X in $\mathbb{R}^{2n}$, we have*

$$c_k^{EH}(X) = c_k(X).$$

Remark 0.10. *More evidence for this conjecture: Theorem 6.3.10 implies that our capacities c_k satisfy the Cartesian product property (6.2.3) in the special case when X and X' are convex toric domains. We do not know whether the capacities c_k satisfy this property in general.*

The proof of this conjecture is ongoing work with V. Ramos. Our strategy is the following:

1. Define an S^1-equivariant Morse theory in infinite dimension for the Hamiltonian action functional, for a fixed Hamiltonian. This was done in the non-equivariant case for star-shaped domains by Abbondandolo and Majer [AM05]. We shall adapt this construction to the S^1-equivariant setting, define $HM(\mathscr{A})$ as the direct limit of this S^1-equivariant Morse homology in infinte dimensions over a family of admissible Hamiltonians (in the sense of the homology CH).

2. Show that $HM(\mathscr{A})$ is isomorphic to the homology CH. In fact, we would like to show that the two chain complexes (for the same fixed Hamiltonian) of those two homologies are chain-complex isomorphic and that this isomorphism "commutes" with the direct limit operation. M. Hecht [Hec13] showed such an isomorphism on tori in the non-equivariant case and for a fixed Hamiltonian. See §3.5 for more details about those two first points.

3. Show that the Ekeland-Hofer capacities, c^{EH}, are spectral invariants of the homology $HM(\mathscr{A})$. The major problem here is to understand the Fadell-Rabinowitz index in the context of symplectic homology.

4. The fact that the Ekeland-Hofer capacities are the same as the equivariant capacities should result from the three previous points.

For the asymptotics of those capacities, little is known and it will be different from the ECH case. Indeed, the example of the Polydisk (see Example 6.3.11) in particular show that the capacities from positive S^1-equivariant symplectic homology do not detect the volume, not even asymptotically.

The proof of Theorem 6.3.18 shows that if $X \subset \mathbb{C}^n$ is a star-shaped domain satisfying (6.3.12), then

$$\lim_{k \to \infty} \frac{c_k(X)}{k} = c_\square(X). \tag{0.2}$$

This is related to the following question of Cieliebak-Mohnke [CM14].

Given a domain $X \subset \mathbb{R}^{2n}$, define the **Lagrangian capacity** $c_L(X)$ to be the supremum over A such that there exists an embedded Lagrangian torus $T \subset X$ such that the symplectic area of every map $(D^2, \partial D^2) \to (X, T)$ is an integer multiple of A. It is asked in [CM14] whether if $X \subset \mathbb{R}^{2n}$ is a convex domain then

$$\lim_{k \to \infty} \frac{c_k^{EH}(X)}{k} = c_L(X). \tag{0.3}$$

It is confirmed by [CM14, Cor. 1.3] that (0.3) holds when X is a ball.

Observe that if X is any domain in $\mathbb{C}^n$, then the Lagrangian capacity is related to the cube capacity by

$$c_\square(X) \leqslant c_L(X),$$

because if $\square_n(\delta)$ symplectically embeds into X, then the restriction of this embedding maps the "corner"

$$\mu^{-1}(\delta, \dots, \delta) \subset \square_n(\delta)$$

to a Lagrangian torus T in X such that the symplectic area of every disk with boundary on T is an integer multiple of δ. Thus the asymptotic result (0.2) implies that if $X \subset \mathbb{C}^n$ is a domain satisfying (6.3.12), then

$$\lim_{k \to \infty} \frac{c_k(X)}{k} \leqslant c_L(X).$$

Assuming Conjecture 6.3.13, this proves one inequality in (0.3) for these examples.

Question 0.11. *What about the higher order asymptotics?*

In order to construct an operator whose spectrum is (or at least contains) the capacites from positive S^1-equivariant symplectic homology, the more information we have on the behavior of c_k the merrier. This imply finding new algebraic structures on the homology CH. For those, we would like to start by exploring the two following directions: the pair-of-pants product and the Künneth formula. The pair-of-pant product is defined in non-equivariant symplectic homology. The lift of this product to the homology CH is not a product anymore but becomes a bracket. We plan to use Floer trajectories, with an additional constraint on the angle, to build a product structure on the homology CH. To construct a Künneth-type long exact sequence for the homology CH, we plan to use techniques such as those in the construction of the long exact sequence for non-equivariant symplectic homology by Oancea [Oan08]. This would in principle lead to the Cartesian product property (6.2.3).

$$c_k(X \times X') = \min_{i+j=k} \{c_i(X) + c_j(X')\},$$

where i and j are positive integers and $X \subset \mathbb{R}^{2n}$ and $X' \subset \mathbb{R}^{2n'}$ are star-shaped domains.

Another approach to build algebraic structures on the homology CH of the unit disk bundle $(DT^\star N, \lambda_{can})$ (whose boundary is the contact manifold $ST^\star N$), for N a closed spin oriented manifold, is to prove the following isomorphism:

$$CH_*(DT^\star N, \lambda_{can}) \cong H_*(\Lambda N / S^1, N; \mathbb{Q}), \tag{0.4}$$

where ΛN is the free loop space of N and $N \subset \Lambda N$ indicates the subset of constant loops. Then we would "push" the operations on $H_*(\Lambda N/S^1, N; \mathbb{Q})$ onto CH via this isomorphism. This isomorphism is mentioned without any proof in [Vit88] and [BO16]. We would like to point out that there are several approaches to the non-equivariant version of this isomorphism [AS06, Abo15, Vit88, SW06] and we expect that both the methods of Abbondandolo and Schwarz [AS06] and of Abouzaid [Abo15] should adapt to the S^1-equivariant setting that we consider.

Another sequence of symplectic capacities called the **higher symplectic capacities** was introduced by K. Siegel [Sie19b, Sie19a]. Those are defined using Rational Symplectic Field Theory [EGH00] and are conjectured to wield similar properties as ECH capacities. They make use of the $\mathscr{L}_\infty$-structure. The higher symplectic capacities differ from the capacities from positive S^1-equivariant symplectic homology, as shown by computation in [Sie19a]. These capacities are particularly suited for obstructions of stabilized symplectic embeddings (i.e. of the form 4-dimensional manifold $\times \mathbb{R}^2 n$).

Question 0.12. *What is the asymptotics of the higher symplectic capacities?*

Application to Dynamics

The symplectic capacities from positive S^1-equivariant symplectic homology carry relevant information on the dynamics of the Reeb vector field on the boundary for all contact form defining the contact structure

Theorem 0.13 ([GG16]). *Let Y be a star-shaped hypersurface in $\mathbb{R}^{2n}$. If Y carries only finitely many simple periodic Reeb orbits, then, for all $i \geqslant 1$*

$$c_i(Y) < c_{i+1}(Y)$$

The same statement holds true in $\mathbb{R}^4$ with the ECH capacities. The actual statement from [GG16] is a bit stronger. It states that the inequality remains valid for "capacities" defined in a similar way as in §6.3 but taking the "inverse image by powers of U of any class in $H_*(X, \partial X) \otimes H_*(BS^1)$ (with correct degree)". It is therefore very tempting to ask for a lower bound on the minimal number of geometrically distinct periodic Reeb orbits in a prescribed homotopy class. Together, with J. Kang, we consider prequantization bundles i.e. complex line bundles E over a symplectic manifold (M, ω) sucht that $c_1(E) = -[\omega] \in H^2(M; \mathbb{Z})$. The circle bundle in E is naturally a contact manifold. Without restriction on the homotopy class, the minimal number of geometrically distinct periodic Reeb orbits is bounded below (in some case) by the sum of the Betti numbers of the base (in the non-degenerate case) and by the cuplength of the base (in the degenerate case).

We plan to check whether these lower bounds remain valid if we consider only periodic Reeb orbits homotopic to a fiber (in particular non-contractible). The restriction to this particular free homotopy class of loops comes from the fact that in the case of $\mathbb{R}P^{2n-1}$, finding periodic Reeb orbits homotopic to a fibre is equivalent to finding periodic orbits on a star-shaped hypersurface in $\mathbb{R}^{2n}$ which are invariant by antipodal reflection.

The positive S^1-equivariant symplectic homology CH decomposes as a direct sum of homologies corresponding to orbits in different homotopy classes. It is difficult to obtain information on this decomposition from the global homology. We plan to use the fact that periodic orbits are homotopic to a fibre if and only if they are the boundary of a disk and have winding number equal to 1. We try to construct a variant of the homology CH, using positivity of intersection in a similar manner as what is done in [AK19] to detect the orbits in a given homotopy class. Another approach would consist in defining an S^1-equivariant version of the Rabinowitz-Floer homology [AF12].

Smallest capacity

In this last section, we describe some questions and research related to the first capacity (or related spectral invariant[1]) which we call spectral gap in what follows. The first question would be

Question 0.14. *What is the significance of the spectral gap?*

Can the spectral gap be read on the barcode of the homology CH? Also, if all eigenvalues are distinct, can we extract a lower bound on the minimal number of simple periodic Reeb orbits?

The hope is that the spectral gap will shed some light on Question 2.2.23. So far, Jean-François Barraud and I try to apply the "crocodile walk" techniques [Bar18]; it generate the fundamental group from the Floer moduli spaces. We try to extend this to equivariant symplectic homology. Assuming the contact manifold is fillable by a cotangent bundle and assuming dynamical convexity we try to extract information about its π_1 from CH.

Question 2.2.23 is related to questions stemming from algebraic geometry. We refer to [McL16] and references therein for the algebraic geometrical interpretation.

Definition 0.15. *Given a compact contact manifold (M,ξ), define the* ***minimal log discrepancy*** *as*

$$\mathrm{min.\,log.\,discr.}(M,\xi) := \sup_{\alpha} \min_{\gamma\in\mathscr{P}(\alpha)} \{\left(\mathrm{CZ}(\gamma)+n-3\right)\tfrac{1}{2}+1\}$$

where the supremum is taken over all contact forms α such that $\ker\alpha=\xi$ and $\mathscr{P}(\alpha)$ is the set of all periodic Reeb orbits of R_α.

Remark that $\mathrm{min.\,log.\,discr.}(S^{2n-1},\xi_{std})=n$. Question 2.2.23 reformulates in this context as

Question 0.16. *Given a 2n-1-dimensional compact contact manifold (M,ξ), is it true that*

$$\mathrm{min.\,log.\,discr.}(M,\xi)\leqslant n$$

with equality if and only if M is diffeomorphic to the sphere S^{2n-1}?

Note that this question englobes Shokurov's conjecture [Sho02] and a positive answer to the first part would give a disproval of a conjecture (expected to be false) of a conjecture by Thurston. Recall that a c-symplectic manifold is a triple (X,J,c) such that

1. X is a $2n$-dimensional compact manifold,
2. J is an almost complex structure on X,
3. $c\in H^2(X,\mathbb{R})$ such that $c^n := c\cup\cdots\cup c\neq 0$.

Conjecture 0.17 (Thurston)**.** *c-symplectic $\Rightarrow$ symplectic.*

[1] One direction is to impose condition on the linking number or on the index of the orbit whose action represent the symplectic capacity

It is expected that for a generic compact contact manifold, $\text{min.log.discr.}(M, \xi) = -\infty$. The approach to this conjecture is to test it on link of affine variety (i.e. the intersection of an affine variety and a large ball) where the homology CH and the spectral gap, together with the additional ambiant structure might generalise [McL16]. The other approach is by looking the case $n = 2$ and using ECH.

There are some links [EMY03] (also McLean, private communications) between minimal log discrepancy and arc space (i.e. space of holomorphic disks). This prompted the question whether the two-systole can be detected from symplectic capacities?

Defining the two-systole of the 4-dimensional torus endowed with a Riemmannian metric to be

$$\text{sys}_2(T^4, g) := \inf A(\mathscr{C})$$

where the infimum is taken over all non-trivial cycle $\mathscr{C}$ of $H_2(T^4, \mathbb{Z}_2)$ and $A(\mathscr{C})$ is the area of $\mathscr{C}$. We have the following question:

Conjecture 0.18 (Balacheff-Gutt). *Let (T^4, ω_0) be the standard symplectic 4-torus. There exists a constant $K > 0$ such that for all Riemmannian metrics which are ω_0-compatible, we have*

$$\left(\text{sys}_2(T^4, g)\right)^2 \leqslant K^2 \,\text{Vol}(T^4, g).$$

Remark 0.19. *This statement is known to be false for a non ω_0-compatible metric but true for all flat metrics which are ω_0-compatible [BB15].*

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Résumé

Ce mémoire se présente comme une promenade dans le domaine de la topologie symplectique et de la géométrie de contact, présentant dans leur contexte les résultats que j'ai obtenus depuis la fin de ma thèse.

Le fil conducteur de mes recherches est la question suivante: *Dans le cadre d'une variété symplectique à bord de type contact, quelle information l'intérieur possède t-il sur le bord et réciproquement, quelle information le bord possède t-il sur l'intérieur?*

Ce mémoire est divisé en deux parties correspondant aux deux volets de la question ci-dessus.

La première partie est consacrée à l'étude du nombre minimal d'orbites périodiques du champ de Reeb d'abord sur des hypersurfaces étoilées dans $\mathbb{R}^{2n}$ ensuite sur des variétés plus générales. Un des outils principaux est l'homologie symplectique S^1-équivariante positive; elle est construite à partir d'orbites périodiques de champs de vecteurs hamiltoniens sur une variété symplectique dont le bord est la variété de contact considérée.

La deuxième partie est consacrée aux plongements symplectiques d'une variété symplectique dans une autre et plus précisément à leur obstructions (capacités symplectiques). Nous présentons une nouvelle construction d'une suite de capacités symplectiques ainsi que quelques applications et calculs. La conjecture forte de Viterbo énonce que toutes les capacités normalisées coïncident sur les domaines convexes de $\mathbb{R}^{2n}$. Nous en donnons une démonstration en dimension 4 dans le cadre des domaines toriques monotones (que nous introduisons). Nous définissons une nouvelle notion d'équivalence de plongements symplectiques et donnons des exemples de plongements non-équivalents.

Le dernier chapitre présente certaines perspectives envisagées pour mes recherches futures.