Backward stochastic differential equations and applications : optimal switching, stochastic games, partial differential equations and mean-field

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CONTENTS

CHAPTER 1

INTRODUCTION

The theory of Backward Stochastic Differential Equations (BSDEs in short) is much studied from the beginning of 1990. Motivated by the connection with different stochastic mathematical problems like mathematical finance problems, optimal control problems, differential games problems, PDEs, etc, the interest of BSDEs studies has broadly increased. BSDEs with a linear driver is firstly introduced by Bismut in 1973 [3] in studying the adjoint equations of stochastic optimal control problems. Later, in 1990, Pardoux and Peng [52] considered more general BSDEs where the driver verifies mainly a non-linear Lipschitz condition.

BSDEs may also arise in combined financial and insurance applications. For example, El Karoui et al. [24] introduced the connection between BSDEs and the theory of contingent claim valuation in a complete market. Dos Reis [21] studied the insurance related derivatives on the financial markets, which can be represented in terms of solutions of FBSDEs with quadratic growth. Delong [16] represented the linear BSDEs arising in life insurance and non-life insurance payment processes under systematic and unsystematic claims risk.

1.1 An overview of general results of BSDEs

Let $T > 0$ be a fixed real constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which carries a d-dimensional Brownian motion $B = (B_t)_{t\in[0,T]}$ whose natural filtration is $\mathcal{F}_t^0 := \sigma\{B_s, s \le t\}_{0\le t\le T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions, i.e., it is complete and right continuous. On the other hand, we define $\mathcal{P}$ as the σ-algebra on $[0, T] \times \Omega$ of the $\mathbb{F}$-progressively measurable sets. Next, we define the following spaces:

- $L^2 = \{\mathcal{F}_T$-measurable random variable ξ s.t. $\mathbb{E}(|\xi|^2) < \infty\}$;
- $\mathcal{S}^2 = \{\mathcal{P}$-measurable continuous processes $\phi = (\phi_t)_{t\in[0,T]}$ s.t. $\mathbb{E}(\sup_{t\in[0,T]} |\phi_t|^2) < \infty\}$;
- $\mathcal{A}^2 = \{$ Non-decreasing, continuous, $\mathcal{P}$-measurable processes $K = (K_t)_{t\le T}$ s.t. $K_0 = 0$ and $\mathbb{E}[K_T^2] < \infty$ $\}$;

- For $k \geq 1$, $\mathcal{H}^{2,k} = \{\mathcal{P}$-measurable, $\mathbb{R}^k$-valued processes $\phi = (\phi_t)_{t\in[0,T]}$ s.t. $\mathbb{E}(\int_0^T |\phi_t|_k^2 dt) < \infty\}$.

1.1.1 Classical results on standard BSDEs

Given an $\mathcal{F}_T$-measurable random variable ξ valued in $\mathbb{R}^p$ and a driver or generator $f(t,\omega,y,z)$: $[0,T]\times\Omega\times\mathbb{R}^p\times\mathbb{R}^{p\times d}\to\mathbb{R}^p$, $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{p+p\times d})/\mathcal{B}(\mathbb{R}^p)$-measurable. A solution of the BSDE associated with (f,ξ) is a pair $(Y_t,Z_t)_{t\leq T}$ of $\mathcal{P}-$measurable processes valued in $\mathbb{R}^{p+p\times d}$ such that:

$$\forall t \leq T,\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s. \tag{1.1}$$

Next let us consider the following assumptions.

Assumption 1.1.1.

1. *$(f(t,\omega,0,0))_{t\leq T}\in\mathcal{H}^{2,p}$ and ξ is square integrable;*

2. *The generator f satisfies the Lipschitz condition, i.e. there exists a constant C such that for any $t\in[0,T]$ and $(y_1,z_1),(y_2,z_2)\in\mathbb{R}^{p+p\times d}$ we have,*

$$\mathbb{P}-a.s.,\ |f(t,y_1,z_1)-f(t,y_2,z_2)| \leq C(|y_1-y_2|+|z_1-z_2|). \tag{1.2}$$

We then have the following result related to existence and uniqueness of the pair (Y,Z).

Theorem 1.1.2 (Pardoux-Peng [52]). *Under Assumption 1.1.1, the p-dimensional BSDE* (1.1) *has a unique solution $(Y_t,Z_t)_{t\leq T}$ such that:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2\right] < \infty. \tag{1.3}$$

Another useful result for solving different BSDEs with more general drivers is the comparison theorem. Indeed, one can compare the solutions of two BSDEs by comparing the drivers and the terminal conditions. This result is firstly introduced in one-dimensional case by El Karoui et al. [24].

Theorem 1.1.3 (El Karoui-Peng-Quenez [24]). *Assume $p=1$. Let $(Y^1,Z^1),(Y^2,Z^2)$ be two solutions of BSDEs associated respectively with (f_1,ξ_1) and (f_2,ξ_2) which satisfy Assumption 1.1.1. We also assume that for any $t\in[0,T]$,*

1. *$\xi_1\leq\xi_2, \mathbb{P}-a.s.$;*

2. *$f_1(t,Y_t^2,Z_t^2)\leq f_2(t,Y_t^2,Z_t^2), d\mathbb{P}\times dt-a.s.$*

Then $\mathbb{P}$-a.s., for any $t \in [0,T]$, $Y_t^1 \leq Y_t^2$.

Later, several works extend the classical results by relaxing the assumptions on the coefficients of BSDEs. By the construction of monotonic convergent sequences, the existence and eventually the uniqueness of the solution of BSDE are guaranteed:

- The coefficient f is locally Lipschitz and the terminal condition is bounded, Hamadène [26] proved the existence of the solution of one-dimensional BSDEs;
- f is of linear growth, continuous in (y,z) and the terminal condition is square integrable, Lepeltier and San Martin [47] proved the existence of a minimal solution of one-dimensional BSDEs;
- f is continuous in (y,z) and has a quadratic growth in Z and the terminal condition is also bounded, Kobylanski [44] proved the comparison result, as well as the existence and a stability results for one-dimensional BSDEs.

1.1.2 BSDEs in the markovian framework

One of the important settings of BSDEs is constructed under the markovian framework, i.e., the randomness of the coefficient and the terminal value of the BSDE comes from a diffusion process $(X_s^{t,x})_{s\in[t,T]}$ which is the solution of a standard SDE:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s, \quad s \in [t,T]; \\ X_t^{t,x} = x. \end{cases} \tag{1.4}$$

Next let the processes $(Y^{t,x}, Z^{t,x})$ solution of the following BSDE:

$$\forall s \leq T,\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dr. \tag{1.5}$$

The solution of (1.5) can be represented by a deterministic function $u(t,x)$ and is called the markovian solution. We now provide sufficient conditions on the data b, σ, f and h for which this markovian representation holds:

Assumption 1.1.4.

1. *The functions b and σ are continuous and uniformly Lipschitz with respect to x, i.e. there exists a constant C that for any $(t,x,x') \in [0,T] \times \mathbb{R}^{k+k}$,*

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'|.$$

As a result b and σ are of linear growth with respect to (w.r.t. for short) x, i.e.

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|).$$

2. *f is an $\mathbb{R}^p-$valued continuous function defined by:*

$$\begin{aligned} f :&[0,T] \times \mathbb{R}^k \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \to \mathbb{R}^p \\ &(t,x,y,z) \mapsto f(t,x,y,z). \end{aligned}$$

Moreover it is uniformly Lipschitz in (y,z), i.e., there exists a constant C such that

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \leq C(|y_1-y_2| + |z_1-z_2|);$$

3. *$f(t,x,0,0)$ and h are of polynomial growth, i.e. there exist constants c and p such that*

$$|f(t,x,0,0)| + |h(x)| \leq c(1+|x|^p).$$

Theorem 1.1.5 (El Karoui et al.[24]). *Under Assumption 1.1.4, for any $(t,x) \in [0,T]\times\mathbb{R}^k$ there exists two measurable deterministic functions $u(t,x)$ and $d(t,x)$ such that*

$$\mathbb{P}-a.s., \forall s \in [t,T], \quad Y_s^{t,x} = u(s,X_s^{t,x}), \quad Z_s^{t,x} = \sigma(s,X_s^{t,x})^\top d(s,X_s^{t,x}).$$

In addition, if the coefficients b,σ,f,h are globally Lipschitz w.r.t (x,y,z), uniformly in t to f, then u is locally Lipschitz in x and $1/2-$Hölder continuous in t. Moreover if b,σ,f,h are continuous differentiable with respect to (x,y,z) with bounded derivatives, then $\forall 0\leq t\leq s\leq T, x \in_R^k, Z_s^{t,x} = \sigma(s,X_s^{t,x})^\top \partial_x u(s,X_s^{t,x})\ d\mathbb{P}\times ds\ a.s.$ (see Corollary 4.1 in [24] for more details).

Now let us focus on the following quasilinear parabolic partial differential equation (PDE in short): $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\begin{cases} \partial_x u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma(t,x)\partial_x u(t,x)) = 0; \\ u(T,x) = h(x) \end{cases} \tag{1.6}$$

where $\mathcal{L}$ is the second order differential operator defined by

$$\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^\top)_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}\frac{\partial}{\partial x_i}.$$

The link between the solution of one-dimensional BSDE in the markovian framework (1.5) and the solution of the PDE (1.6) is the following:

Proposition 1.1.6 (El Karoui et al. [24]). *Under assumption 1.1.4, suppose that $u \in \mathcal{C}^{1,2}$ is the solution of the PDE* (1.6)*, then the following representation holds true:*

$$\forall s \in [t,T], u(s,X_s^{t,x}) = Y_s^{t,x}, \sigma(s,X_s^{t,x})^\top \partial_x u(s,X_s^{t,x}) = Z_s^{t,x},$$

where $(Y^{t,x}, Z^{t,x})$ is the unique solution of BSDE (1.5).

The solution of the one-dimensional BSDE in the markovian framework (1.5) is also related to the solution of the PDE (1.6) in viscosity sense which we recall the definition in the following.

Definition 1.1.7. *Let $p = 1$. Suppose that $u \in \mathcal{C}([0,T] \times \mathbb{R}^k)$ with $u(T,x) = h(x), x \in \mathbb{R}^k$. The function u is called a viscosity subsolution (resp. supersolution) of PDE 1.6 if for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ such that $\phi(t,x) = u(t,x)$ and $u(t,x)$ is a local maximum (resp. minimum) of $u - \phi$,*

$$\frac{\partial \phi}{\partial t} + \mathcal{L}\phi(t,x) + f(t,x,u(t,x),\sigma(t,x)^\top \partial_x \phi(t,x)) \leq 0 \ (resp. \geq 0).$$

Moreover, u is called a viscosity solution of PDE(1.6) *if it is both a viscosity subsolution and a viscosity supersolution of* (1.6).

Theorem 1.1.8 (Pardoux-Peng [53]). *Assume Assumption 1.1.4 fulfilled. Then $u(t,x) := Y_t^{t,x}$ is continuous and of polynomial growth, i.e.*

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k, \ |u(t,x)| \leq C(1 + |x|^p),$$

where C and p are two constants. Moreover it is the unique solution of PDE (1.6) *in the viscosity sense.*

1.1.3 BSDEs and zero-sum stochastic differential games

BSDEs are also connected to control and game problems. The solution of BSDEs associated to optimal control problems is firstly introduced by Bismut in 1973, then generalized by Pardoux and Peng [52]. Later on, the BSDEs theory was well developed in various directions. Hamadène and Lepeltier [32] introduced the connection of BSDEs with zero-sum stochastic differential games.

Definition 1.1.9.

1. *$\chi := C([0,T];\mathbb{R}^p)$ the set of continuous functions from $[0,T]$ into $\mathbb{R}^p$ and $\mathbb{P}$ the Wiener measure on χ;*
2. *We define U, V two sets of progressive measurable processes with values in compact sets $\bar{U}$ and $\bar{V}$ respectively; u is called an admissible control if $u \in U$; the same for $v \in V$;*

3. *For $1 \leq i,j \leq p, \sigma_{ij} : (t,x) \in [0,T] \times \chi \to \sigma_{ij}(t,x) \in \mathbb{R}$ is progressively measurable; we denote by $\sigma := (\sigma_{ij})_{i,j=1,p}$ and by $a := \sigma\sigma^\top$. We assume that σ is: (i) Lipschitz in x and of linear growth; (ii) is invertible and its inverse is bounded. Next let $(x_t)_{t\leq T}$ be the solution of the following SDE:*

$$x_t = x_0 + \int_0^t \sigma(s,x)dB_s, t \leq T \text{ and } x_0 \in \mathbb{R}^p,$$

where $(B_s)_{s\leq T}$ is a Brownian motion on $(\chi, \mathbb{P})$.

4. *$f(t,x,\bar{u},\bar{v})$ (resp. $c(t,x,\bar{u},\bar{v})$) is a measurable bounded function with values in $\mathbb{R}^p$ (resp. $\mathbb{R}$);*

5. *For any $u \in U, v \in V, P^{u,v}$ is a probability defined on $(\chi, \mathcal{F}_T)$ by:*

$$\frac{dP^{u,v}}{d\mathbb{P}} = exp\{\int_0^T \sigma^{-1}(s,x)f(s,x,u_s,v_s)dB_s - \frac{1}{2}\int_0^T |\sigma^{-1}(s,x)f(s,x,u_s,v_s)|^2 ds\}.$$

Then the process $(x_t)_{t\leq T}$ is, under $P^{u,v}$, a weak solution of $dx_t = f(t,x,u_t,v_t)dt + \sigma(t,x)dW_t^{u,v}$ where $(W_t^{u,v})_{t\leq T}$ is a Brownian motion under $P^{u,v}$;

6. *Let $(u_t)_{t\leq T}, (v_t)_{t\leq T}$ be two admissible controls. The payoff between two players, a minimizer (resp. maximizer) that acts with u (resp. v) is given by*

$$J(u,v) = \mathbb{E}^{u,v}\left[\xi + \int_0^T c(s,u_s,v_s)ds\right],$$

where $\mathbb{E}^{u,v}$ is the expectation w.r.t.$P^{u,v}$.

Next for any $(t,x,p,\bar{u},\bar{v}) \in [0,T] \times \chi \times \mathbb{R}^p \times \bar{U} \times \bar{V}$, we define the Hamiltonian of the game by $H(t,x,p,\bar{u},\bar{v}) := p\sigma^{-1}(t,x)f(t,x,\bar{u},\bar{v}) + c(t,x,\bar{u},\bar{v})$ and assume that the Isaacs condition is satisfied, i.e., for any $(t,x,p) \in [0,T] \times \chi \times \mathbb{R}^p$,

$$\max_{\bar{v}\in\bar{V}} \min_{\bar{u}\in\bar{U}} H(t,x,p,\bar{u},\bar{v}) = \min_{\bar{u}\in\bar{U}} \max_{\bar{v}\in\bar{V}} H(t,x,p,\bar{u},\bar{v}).$$

Then, by Benes selection Theorem (see [2]), there exists two $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^p)$-measurable functions $u^* : [0,T] \times \chi \times \mathbb{R}^p \to \bar{U}$ and $v^* : [0,T] \times \chi \times \mathbb{R}^p \to \bar{V}$, such that for any $t, x, \bar{u} \in \bar{U}, \bar{v} \in \bar{V}$,

$$\textbf{(H)} \qquad \begin{aligned} &H(t,x,p,u^*(t,x,p),v^*(t,x,p)) \leq H(t,x,p,\bar{u},v^*(t,x,p)), \\ &H(t,x,p,u^*(t,x,p),v^*(t,x,p)) \geq H(t,x,p,u^*(t,x,p),\bar{v}). \end{aligned}$$

Note that, conversely, (H) implies the Isaacs condition and then we have

$$H(t,x,p,u^*(t,x,p),v^*(t,x,p)) = \max_{\bar{v}\in\bar{V}} \min_{\bar{u}\in\bar{U}} H(t,x,p,\bar{u},\bar{v}).$$

Theorem 1.1.10 (Hamadène-Lepeltier [32]). *Assume that (H) is satisfied. Then the game has a saddle point $(u^*, v^*) \in U \times V$, i.e.,*

$$\forall u \in U, v \in V, J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*).$$

Moreover, $Y_0 = J(u^*, v^*)$, where (Y, z) is the unique solution of the following BSDE:

$$\begin{aligned} &-dY_t = H(t, x, z_t, u^*(t, x, z_t), v^*(t, x, z_t))dt - \tilde{z}_t dB_t,\ t \leq T; \\ &Y_T = \xi \end{aligned}$$

and $(u^*, v^*) := (u^*(t, x, z_t), v^*(t, x, z_t))_{t \leq T}$.

The zero-sum stochastic differential game problems and the associated BSDEs have been well documented in several works, see for example [14, 19, 30, 32, 33, 35, 41].

1.2 RBSDEs and DRBSDEs with interconnected barriers

In this section we recall some results on Reflected BSDEs (RBSDEs in short) and Doubly Reflected BSDEs (DRBSDEs in short), as well as the associated applications (see e.g.[11, 14, 18, 19, 20, 23, 27, 31, 37, 40, 41, 65]).

1.2.1 General results for reflected BSDEs in one-dimension

One-dimensional Reflected BSDE

Firstly let us recall some results of BSDEs with a reflecting obstacle. El karoui et al. [23] studied the reflected solution of BSDEs with a random lower obstacle. More precisely let $\xi \in L^2$, f is the generator defined from $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$ and L a continuous, progressively measurable process satisfying $\mathbb{E}\left[\sup_{0 \leq t \leq T}(L_t^+)^2\right] < \infty$. Then the triple (Y, Z, K) of processes is called a solution of the reflected BSDE associated with (f, ξ, L) if:

$$\begin{cases} Y \in \mathcal{S}^2, Z \in \mathcal{H}^{2,d}, K \in \mathcal{A}^2; \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t,\ t \in [0, T]; \\ Y_t \geq L_t,\ t \in [0, T]; \\ \int_0^T (Y_t - L_t)\, dK_t = 0. \end{cases} \tag{1.7}$$

Assumption 1.2.1.

1. $\xi \in L^2$;

2. $f(.,0,0) \in \mathcal{H}^{2,1}$, *moreover there exists a constant C such that* $\forall t \in [0,T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$,
$$|f(t,y,z) - f(t,y',z')| \leq C(|y-y'| + |z-z'|);$$
3. $\mathbb{E}\left[\sup_{t\in[0,T]}(L_t^+)^2\right] < \infty$;
4. $L_T \leq \xi$.

Theorem 1.2.2 (El Karoui et al. [23]). *Under Assumption 1.2.1, the following results hold true:*

1. *The RBSDE* (1.7) *associated with* (f,ξ,L) *has a unique solution* $(Y,Z,K) = (Y_t, Z_t, K_t)_{t\leq T}$ *in* $\mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{A}^2$.
2. *The following representation holds true: For any* $t \in [0,T]$,
$$Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\xi 1_{(\tau=T)} + \int_t^\tau f(s,Y_s,Z_s)ds + L_\tau 1_{(\tau<T)} \middle| \mathcal{F}_t\right]; \tag{1.8}$$
where $\mathcal{T}$ *is the set of stopping time dominated by T, and* $\mathcal{T}_t := \{\tau \in \mathcal{T};\, t \leq \tau \leq T\}$;
3. $\mathbb{E}\left(\sup_{t\in[0,T]} Y_t^2 + \int_0^T |Z_t|^2 dt + K_T^2\right) \leq C\mathbb{E}\left(\xi^2 + \int_0^T f^2(t,0,0)dt + \sup_{t\in[0,T]}(L_t^+)^2\right)$;
4. *Comparison result: Let* (Y,Z,K) *and* (Y',Z',K') *be two solutions of* (1.7) *respectively associated with* (ξ,f,L) *and* (ξ',f',L') *which satisfy Assumptions* (1.2.1). *If* $\xi \leq \xi', \forall (y,z) \in \mathbb{R}\times\mathbb{R}^d, f(t,y,z) \leq f'(t,y,z)$ $d\mathbb{P}\times dt$ *and* $\forall t \in [0,T], L_t \leq L'_t$ *a.s. then we have* $Y_t \leq Y'_t, \forall t \in [0,T]$ *a.s.;*

One-dimensional DRBSDE and related Dynkin games

Consider now two reflecting processes $(L_t)_{t\in[0,T]}$ and $(U_t)_{t\in[0,T]}$ under which the BSDE changes the direction once the solution *touches* either obstacle. This DRBSDEs is connected with the so-called zero-sum Dynkin game [14]. To be more precise, the game involves two players with antagonistic goals (one wishes to maximize his profit while another wishes to minimize his lost). Before the end of the game, each of whom implements a strategy and the first who decides to stop pays or earns a certain amount. When the game finishes, two players share the same payoff. The main objective of this game problem is to find an optimal strategy (if it exists), i.e., a strategy such that this game is fair for both players.

Definition 1.2.3. *We say that* (Y,Z,K^+,K^-) *is a solution of Doubly Reflected BSDE associated with two reflected obstacles L and U, terminal condition* ξ *and the generator f if the following system holds* *true:*

$$\begin{cases} Y \in \mathcal{S}^2, Z \in \mathcal{H}^{2,d}, K^+, K^- \in \mathcal{A}^2; \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T^+ - K_t^+ - (K_T^- - K_t^-); \ \forall t \in [0,T]; \\ L_t \leq Y_t \leq U_t; \\ \int_0^T (Y_t - L_t)\, dK_t^+ = 0 \text{ and } \int_0^T (Y_t - U_t)\, dK_t^- = 0. \end{cases} \tag{1.9}$$

Assumption 1.2.4.

1. *$\xi \in L^2$ and $(f(t,\omega,0,0))_{t\leq T} \in \mathcal{H}^{2,1}$;*
2. *There exists a constant C such that $\forall t \in [0,T], y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$,*
$$|f(t,y,z) - f(t,y',z')| \leq C(|y-y'| + |z-z'|);$$
3. *$L, U \in \mathcal{S}^2, \forall t \in [0,T], L_t < U_t$ and $L_T \leq \xi \leq U_T$;*
4. *(Mokobokzki's condition) Let $L_t^\xi := L_t 1_{(t<T)} + \xi 1_{(t=T)}, U_t^\xi := U_t 1_{(t<T)} + \xi 1_{(t=T)}$. There exists h and θ two continuous non-negative $\mathcal{F}-$supermartingales satisfying $\mathbb{E}[\sup_{t\in[0,T]} h_t^2] < \infty$ and $\mathbb{E}[\sup_{t\in[0,T]} \theta_t^2] < \infty$ such that: $\forall t \leq T$,*
$$L_t^\xi \leq h_t - \theta_t + \mathbb{E}[\xi | \mathcal{F}_t] \leq U_t^\xi.$$

Theorem 1.2.5 (Cvitanic-Karatzas [14])**.** *Suppose that Assumption 1.2.4 holds true, then the DRBSDE* (1.9) *has a unique solution $(Y, Z, K^+, K^-) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{A}^2 \times \mathcal{A}^2$.*

Next let us consider the connection between the solution (Y, Z, K^+, K^-) and the stochastic Dykin game. We denote by $\mathcal{M}_{t,T}$ the class of $\mathbb{F}-$stopping time, for σ, τ in class $\mathcal{M}_{t,T}$, we consider the following payoff:

$$R_t(\sigma,\tau) := \int_t^{\sigma\wedge\tau} f(r)dr + \xi 1_{(\sigma\wedge\tau=T)} + L_\tau 1_{(\tau<T,\tau\leq\sigma)} + U_\sigma 1_{(\sigma<\tau)}. \tag{1.10}$$

Proposition 1.2.6 (Cvitanic-Karatzas [14])**.** *Under Assumption* (1.2.4)*, the stochastic Dynkin game has a value, noted V_t, given by the unique solution of DRBSDEs* (1.9)*, i.e.*

$$V_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{M}_{t,T}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{M}_{t,T}} \mathbb{E}[R_t(\sigma,\tau) | \mathcal{F}_t] = \operatorname*{ess\,inf}_{\sigma\in\mathcal{M}_{t,T}} \operatorname*{ess\,sup}_{\tau\in\mathcal{M}_{t,T}} \mathbb{E}[R_t(\sigma,\tau) | \mathcal{F}_t] = Y_t.$$

Moreover there exists a saddle point of the game $(\tau^*, \sigma^*) \in (\mathcal{M}_{t,T})^2$ given by:

$$\sigma^* := \inf\{s \in [t,T]; Y_s = U_s\} \wedge T;$$

$$\tau^* := \inf\{s \in [t, T]; Y_s = L_s\} \wedge T.$$

Note that the Mokobodzki condition plays a crucial role when proving the unique solution of DRBSDEs. Since it is difficult to check the existence of a difference of non-negative supermartingales between the two barriers ([14, 30]), Cvitanic and Karatzas [14] provided another regularity condition on both of the obstacle processes which insures the existence and uniqueness of the solution of (1.9). Later Lepeltier, Hamadène and Matoussi [31] relaxed this latter condition by assuming it only on one of the obstacles:

Theorem 1.2.7 (Lepeltier et al. [31]). *Assume that Assumptions 1.2.4-(1-3) are satisfied. If there exists a sequence of process $(U_n)_{n\geq 0}$ such that for any $t \leq T, n \geq 0$,*

1. $U_t^n \geq U_t^{n+1}$ *and* $\lim_{n\to\infty} U_t^n = U_t$, $\mathbb{P} - a.s$;
2. $U_t^n = U_0^n + \int_0^t u_n(s)ds + \int_0^t v_n(s)dB_s$, *where* u_n, v_n *are* $\mathcal{F}_t-$*adapted processes such that* $\sup_{n\geq 0, t\in[0,T]} |u_t^n| \leq C^*$ *and* $\mathbb{E}\{\int_0^T |v_n(s)|^2 ds]^{1/2}\} < \infty$, *where* C^* *is a constant.*

Then DRBSDE (1.9) *has a unique solution.*

Hamadène and Hassani [27] showed existence and uniqueness of the solution of the DRBSDEs (1.9) by only assuming that the two obstacles are totally separated, i.e., $L < U$. However the processes K and Z are not necessarily integrable. Actually under this latter condition it is only shown that $\mathbb{P}$-a.s. $K_T(\omega) < \infty$ and $\int_0^T |Z_s(\omega)|^2 ds < \infty$.

1.2.2 Multidimensional RBSDEs with interconnected obstacle or oblique reflection

In recent years, the RBSDEs problems are also studied in higher dimensions. In connection with the switching problem, Hamadene and Jeanblanc [29] introduced those RBSDEs in dimension 2. Later, Hu and Tang [40] considered the multi-dimensional framework where the existence of the solution is obtained by penalization method, and the uniqueness of the solution derives from a verification theorem of the associated optimal switching problem. In the same year Hamadène and Zhang [37] studied a similar system of RBSDEs where the components of the driver depend on the solution. They showed that the unique solution of the multi-dimensional RBSDEs is also related to the value of an optimal switching problem. In this paper the monotonicity condition of the generator plays a key role while authors prove the existence of the solution by the penalization approach.

So let us introduce the following conditions:

Assumption 1.2.8. *For $m \geq 2$, $i = 1, ..., m$,*

1. $\mathbb{E}[\int_0^T |f^i(t, \omega, 0, 0)|^2 dt + |\xi^i|^2] < \infty$;

2. *$f^i(t,\vec{y},z) := f^i(t,y^1,y^2,...,y^m,z)$ is uniformly Lipschitz continuous in (y^i,z) and is continuous in y^j for $j \neq i$ and for $i,j = 1,...,m$ and $j \neq i$, $h^{ij}(t,y)$ is continuous in (t,y);*

3. *Let $A_i := \{1,...,m\} - \{i\}$, $f^i(t,\vec{y},z)$ is increasing in y^j for $j \neq i$, $h^{ij}(t,y)$ is increasing in y for $j \in A_i$;*

4. *For $i \in A_j, h_{ji}(t,y) \leq y$, moreover there does not exist a sequence $j_2 \in A_{j_1},...,j_k \in A_{j_{k-1}}, j_1 \in A_{j_k}$ such that $y^1 = h_{j_1 j_2}(t,y^2), y^2 = h_{j_2 j_3}(t,y^3),...,y^k = h_{j_k j_1}(t,y_1)$;*

5. *For $i = 1,...,m, \xi^i \geq \max_{j \in A_i} h_{ij}(T,\xi)$.*

Theorem 1.2.9 (Hamadène-Zhang [37]). *If Assumption 1.2.8 is fulfilled, then the following system of m-dimensional RBSDEs*

$$\begin{cases} Y^i \in \mathcal{S}^2, Z^i \in \mathcal{H}^{2,d}, K^i \in \mathcal{A}^2; \\ Y^i_t = \xi^i + \int_t^T f^i(s,\vec{Y}_s,Z^i_s)ds - \int_t^T Z^i_s dB_s + K^i_T - K^i_t; \\ Y^i_t \geq \max_{j \in A_i} h^{ij}(t,Y^j_t); \int_0^T \left[Y^i_t - \max_{j \in A_i} h^{ij}(t,Y^j_t) \right] dK^i_t = 0, \end{cases} \tag{1.11}$$

has a solution.

Uniqueness of the solution to the system of RBSDEs (1.11) is proved when for any $i = 1,\ldots,m$, f^i is Lipschitz w.r.t. $(\vec{y},z)$. Later this existence and uniqueness result is generalized by Chassagneux et al. [11] to the framework where f^i is no longer monotonic but only Lipschitz w.r.t. its components $(,z)$.

Besides, in the markovian framework, one can find the connection between the system of multi-dimensional RBSDEs and a specific system of PDEs. Indeed, let $(X^{t,x}_s)_{s\in[t,T]}$ be the diffusion process satisfying (1.4) and let us introduce the following items:

- Let $\Gamma := \{1,2,...,m\}$ the set of available switching modes;
- $\forall i,j \in \Gamma, i \neq j, g_{ij} : (t,x) \in [0,T] \times \mathbb{R}^k \mapsto g_{ij}(t,x) \in \mathbb{R}^+$ represents the switching cost function from mode i to mode j;
- $\forall i \in \Gamma, h^i : x \in \mathbb{R}^d \mapsto h^i(x) \in \mathbb{R}$ represents the terminal condition function;
- $f^i : (t,x,\vec{y},z) \in [0,T] \times \mathbb{R}^{k+m+d} \mapsto f^i(t,x,y^1,...,y^m,z) \in \mathbb{R}$ is the generator of the system of RBSDEs.

We denote by $\Gamma^{-i} := \Gamma - \{i\}$.

Definition 1.2.10. *Let $i \in \Gamma, t \in [0,T]$. The following triples $(Y^{i;t,x}, Z^{i;t,x}, K^{i;t,x}) \in \mathbb{R}^2 \times \mathcal{H}^{2,d} \times \mathcal{A}^2$, $i = 1,\ldots,m$, are called solution of the m-dimensional RBSDEs with interconnected obstacles associated with $(f^i, h^i, (g_{ij})_{j\in\Gamma^{-i}})$ if: For any $i = 1,\ldots,m$, $\forall s \in [t,T]$,*

$$\left\{\begin{array}{l} Y_s^{i;t,x} = h^i(X_T^{t,x}) + \int_s^T f^i\left(r, X_r^{t,x}, Y_r^{1;t,x}, ..., Y_r^{m;t,x}, Z_r^{i;t,x}\right) dr - \int_s^T Z_r^{i;t,x} dB_r + K_T^{i;t,x} - K_s^{i;t,x}; \\ Y_s^{i;t,x} \geq \max\limits_{j\in\Gamma^{-i}} \left[Y_s^{j;t,x} - g_{ij}(s, X_s^{t,x})\right]; \\ \int_t^T \left\{Y_s^{i;t,x} - \max\limits_{j\in\Gamma^{-i}} \left[Y_s^{j;t,x} - g_{ij}(s, X_s^{t,x})\right]\right\} dK_s^{i;t,x} = 0. \end{array}\right. \tag{1.12}$$

Assumption 1.2.11.

1. *b, σ are jointly continuous in (t,x) and Lipschitz w.r.t. x, i.e. there exists a constant $C \geq 0$ such that for any $t \in [0,T], x, x' \in \mathbb{R}^k$,*

   $$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'|.$$

   As a result b and σ are of linear growth, i.e.

   $$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|).$$

2. *For any $i \in \Gamma$, $f^i(t,x,\vec{y},z)$ is continuous in (t,x) uniformly w.r.t. $(\vec{y},z)$ and Lipschitz continuous w.r.t. $(\vec{y},z)$, i.e. for some $C \geq 0, \forall (t,x) \in [0,T] \times \mathbb{R}^k, \vec{y}, \vec{y'} \in \mathbb{R}^{|\Gamma|}, z, z' \in \mathbb{R}^d$,*

   $$|f^i(t,x,y^1,...,y^m,z) - f^i(t,x,y'^1,...,y'^m,z')| \leq C\left(|y^1 - y'^1| + ... + |y^m - y'^m| + |z - z'|\right);$$

3. *Monotonicity: $\forall j \in \Gamma^{-i}, y^j \in \mathbb{R} \mapsto f^i(t,x,\vec{y})$ is non-decreasing whenever the other components $(t,x,y^1,...,y^{j-1},y^{j+1},...,y^m)$ are fixed;*

4. *g_{ij} is jointly continuous in (t,x) and $\forall (t,x) \in [0,T] \times \mathbb{R}^k, g_{ij}(t,x) \geq 0$ and is of polynomial growth;*

5. *The non-free loop property: For any $(t,x) \in [0,T] \times \mathbb{R}^k$ and any sequence $i_1,...,i_p$ such that $i_1 \neq i_2, i_1 = i_p$ and **card**$\{i_1,...,i_p\} = p-1$, we have*

   $$g_{i_1 i_2}(t,x) + g_{i_2 i_3}(t,x) + ... + g_{i_{k-1} i_k}(t,x) > 0, \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

   By convention we set $g_{ii} = 0, \forall i \in \Gamma$;

6. *$\forall i \in \Gamma, h^i(x)$ is of polynomial growth and satisfies the consistency condition, i.e., $\forall x \in \mathbb{R}^k$,*

$$h^i(x) \geq \max_{j \in \Gamma^{-i}} \left(h^j(x) - g_{ij}(T,x) \right).$$

Theorem 1.2.12 (Hamadène-Morlais [34])**.** *Under Assumption 1.2.11, the system* (1.12) *has a unique solution $(Y^{i;t,x}, Z^{i;t,x}, K^{i;t,x})_{i\in\Gamma}$. Moreover, there exists deterministic continuous functions $(v^i(t,x))_{i\in\Gamma}$ of polynomial growth such that*

$$\forall i \in \Gamma, s \in [t,T],\ Y_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Moreover $(v^i(t,x))_{i\in\Gamma}$ are the unique solution in viscosity sense of the following system of PDEs with interconnected obstacles: $\forall i \in \Gamma, t \in [0,T]$

$$\begin{cases} \min\{v_i(t,x) - \max\limits_{j\in\Gamma^{-i}}(v^j(t,x) - g_{ij}(t,x)); -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) \\ \qquad -f^i(t,x,v^1(t,x),...,v^m(t,x),\sigma(t,x)^\top D_x v^i(t,x))\} = 0; \\ v^i(T,x) = h^i(x). \end{cases} \tag{1.13}$$

The multi-dimensional RBSDEs are connected to the multi-modes switching problems. In applications, the study of the optimal strategy related to the investment of multi-portfolio (e.g.[64, 34]), or even to find the optimal control in the natural resource industry (e.g.[20, 29]), one can apply the multidimensional RBSDEs model where its unique solution is nothing but the value function of the problem.

The stochastic switching problems

In accordance to the connection between multi-dimensional RBSDEs and the stochastic switching problems, Hamadène and Jeanblanc [29] introduced the two-mode switching problem by investigating into the real options problem, called *reversible investment problem*. By means of Snell envelop method, they showed that such a problem has an optimal strategy, under which the value of the problem is related to the 2-dimensional RBSDEs. Later this problem has been extended by a lot of researchers, for example, Hamadène and Hdhiri [28] studied the two-mode switching problem when the corresponding processes are driven by both a general Brownian filtration and an independent Poisson process; Porchet, Touzi and Warin [60] studied this problem by assuming that the payoff function is given by an exponential utility function.

Later the multiple switching problems are also studied. Djehiche, Hamadène and Popier [20] considered a real switching problem of a power plant. They show the existence of the optimal strategy and the link with the unique solution of the system of RBSDEs. In this work the driver f is path-independent.

Some more general cases of f, for example, the driver and the obstacle are interconnected to the solution of the RBSDEs, are also studied under different conditions (e.g.[40, 37, 34, 19]). Now we introduce some results on the multiple stochastic switching problems for later use. We stick to the markovian framework as the extension to the general one is immediate.

Definition 1.2.13.

- *We define a strategy* $(\delta, \xi) := ((\tau_n)_{n\geq 0}, (\xi_n)_{n\geq 0})$ *as*

  1. $(\tau_n)_{n\geq 1}$ *is a non-decreasing sequence of* $\mathbb{F}-$*stopping times; we set* $\tau_0 = 0$*;*
  2. $(\xi_n)_{n\geq 0}$ *the* $\Gamma-$*valued and* $\mathcal{F}_{\tau_n}-$*measurable random variable, and* ξ_0 *represents the initial state of the switching problem.*

- *we say that* $(\delta, \xi) := ((\tau_n)_{n\geq 0}, (\xi_n)_{n\geq 0})$ *is admissible if* $\mathbb{P}[\tau_n < T, \forall n \geq 0] = 0$*;*

- $(\alpha_t)_{t\leq T}$ *the indicator process defined by*

$$\forall t \in [0, T], \quad \alpha_t = 1_{[0,\tau_1)}(t) + \sum_{n\geq 1} \xi_n 1_{[\tau_{n-1},\tau_n)}(t);$$

- $(A^\alpha_s)_{s\leq T}$ *the cumulative cost function given by*

$$\forall s < T, \quad A^\alpha_s := \sum_{n\geq 1} g_{\xi_{n-1},\xi_n}(\tau_n, X^{t,x}_{\tau_n}) 1_{\tau_n \leq s}, \quad A^\alpha_T = \lim_{s\to T} A^\alpha_s;$$

- $\mathcal{D}^i_s := \{\alpha = ((\tau_n)_{n\geq 0}, (\xi_n)_{n\geq 0}) \in \mathcal{D}, \xi_0 = i, \tau_0 = s,$ *and* $\mathbb{E}[A^\alpha_T] < \infty\}$ *the set of admissible strategies.*

Theorem 1.2.14 (Hamadène-Morlais [34]). *Assume that Assumption 1.2.11-(1-2,4-6) is satisfied, then:*

(i) the switching problem has an optimal strategy, denoted $\alpha^* = (\delta^*, \xi^*)$*, i.e.,*

$$\sup_{\alpha\in\mathcal{D}^i_0} J^\alpha = J^{\alpha^*},$$

where $\alpha = (\delta, \xi)$ *and* $J^\alpha = \mathbb{E}[\int_0^T f^{\alpha_s}(s, X^{t,x}_s)ds - A^\alpha_T]$.

(ii) For any $i \in \Gamma, s \in [0, T]$

$$Y^i_s = \operatorname*{ess\,sup}_{s\in\mathcal{D}^i_s} (P^\alpha_s - A^\alpha_s),$$

where P^α is the solution of the following BSDE of non-standard type:

$$\begin{cases} P^\alpha \text{ RCLL and } \mathbb{E}[\sup_{s\le T}|P^\alpha_s|^2]<\infty, Q^\alpha\in\mathcal{H}^{2,d};\\ P^\alpha_s = h^\alpha(X^{t,x}_T)+\int_s^T f^\alpha(r,X^{t,x}_r,\vec{v}_r,Q^\alpha_r)dr-\int_s^T Q^\alpha_r dB_r-(A^\alpha_T-A^\alpha_s),\ \forall s\le T.\end{cases}$$

Note that similar results are also shown in Hu and Tang [40].

1.3 Systems of reflected BSDEs with interconnected bilateral obstacles: Existence, uniqueness and applications

Chapter 2 is a published co-work with Hamadène (ref.[35]).

1.3.1 Motivation

The main objective of this work is to study the system of multidimensional DRBSDEs with doubly interconnected barriers, then we connect the unique solution of DRBSDEs to the system of PDEs with doubly interconnected obstacles. The novelties of this work are: (i) firstly we obtain the existence of the solution via the penalization method in the general framework and not only the markovian one; (ii) secondly by relaxing the assumption of monotonicity on the driver f^{ij} and applying the results of the first part and the connection with switching game as well, we obtain the existence and the uniqueness of the solution of the system of DRBSDEs; (iii) thirdly we apply the unique solution of DRBSDEs in the second part to show the existence and the uniqueness of the system of PDEs in the viscosity sense.

Preliminaries

Let Γ^1,Γ^2 be the finite sets of the whole switching modes available for the controllers or players. Let $\Gamma:=\Gamma^1\times\Gamma^2$ and we denote by Λ its cardinal, i.e., $\Lambda:=|\Gamma|=|\Gamma^1|\times|\Gamma^2|$. On the other hand for $(i,j)\in\Gamma^1\times\Gamma^2$, we define $(\Gamma^1)^{-i}:=\Gamma^1-\{i\}$ and $(\Gamma^2)^{-j}:=\Gamma^2-\{j\}$.

A function $\Psi:(t,x)\in[0,T]\times\mathbb{R}^k\mapsto\Psi(t,x)\in\mathbb{R}^\ell$ is called of polynomial growth if there exists two non-negative real constants C and γ such that $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$|\Psi(t,x)|_\ell\le C(1+|x|_k^\gamma).$$

Hereafter this class of functions is denoted by Π_g.

Next let us denote by $\vec{y}$ the generic element $(y^{ij})_{(i,j)\in\Gamma}$ of $\mathbb{R}^\Lambda$ and let us introduce the following items: for any $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$,

i) f^{ij}: $(t,\omega,\vec{y},z) \in [0,T]\times\Omega\times\mathbb{R}^\Lambda\times\mathbb{R}^d \mapsto f^{ij}(t,\omega,\vec{y},z) \in \mathbb{R}$;

ii) $\underline{g}_{ik}$: $(t,\omega) \in [0,T]\times\Omega \mapsto \underline{g}_{ik}(t,\omega) \in \mathbb{R}^+$;

iii) $\overline{g}_{jl}$: $(t,\omega) \in [0,T]\times\Omega \mapsto \overline{g}_{jl}(t,\omega) \in \mathbb{R}^+$;

iv) ξ^{ij} is a r.v. valued in $\mathbb{R}$ and $\mathcal{F}_T$-measurable.

Finally let us introduce the following assumptions on $f^{ij}, \overline{g}_{ik}$ and $\underline{g}_{jl}$ for $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$.

Assumption 1.3.1.

1. *For any $(i,j) \in \Gamma^1\times\Gamma^2$,*

   a) *There exists a positive constant C and a non negative $\mathcal{P}$-measurable process $(\eta_t)_{t\le T}$ which satisfies $\mathbb{E}[\sup_{s\le T}|\eta_s|^2] < \infty$ such that: $\mathbb{P}$-a.s, $\forall(\vec{y},z) \in \mathbb{R}^{\Lambda+d}$, $t\in[0,T]$,*

   $$|f^{ij}(t,\vec{y},z)| \le C(1+\eta_t+|\vec{y}|),$$

   where $|\vec{y}|$ refers to the standard Euclidean norm of $\vec{y}$ in $\mathbb{R}^\Lambda$ (the same for $|z|$ below). Note that this implies that $\mathbb{E}[\int_0^T |f^{ij}(t,0,0)|^2 dt] < \infty$;

   b) *f^{ij} is Lipschitz continuous with respect to (w.r.t for short) $(\overrightarrow{y},z)$ uniformly in (t,ω), i.e. $\mathbb{P}$-a.s., for any $t\in[0,T]$, $(\overrightarrow{y_1},z_1)$ and $(\overrightarrow{y_2},z_2)$ elements of $\mathbb{R}^{\Lambda+d}$, we have*

   $$|f^{ij}(t,\overrightarrow{y_1},z_1) - f^{ij}(t,\overrightarrow{y_2},z_2)| \le C(|\overrightarrow{y_1}-\overrightarrow{y_2}|+|z_1-z_2|),$$

   where C is a fixed constant.

2. *For any $(i,j)\in\Gamma$,*

   a) *$\mathbb{E}(|\xi^{ij}|^2) < \infty$;*

   b) *ξ^{ij}, as the terminal condition at time T, satisfies the following consistency condition: $\mathbb{P}$-a.s.,*

   $$\max_{k\in(\Gamma^1)^{-i}}\left(\xi^{kj}-\underline{g}_{ik}(T)\right) \le \xi^{ij} \le \min_{l\in(\Gamma^2)^{-j}}\left(\xi^{jl}+\overline{g}_{jl}(T)\right).$$

3. *For all $i_1,i_2\in\Gamma^1$ (resp. $j_1,j_2\in\Gamma^2$) and $t\in[0,T]$, the process $\underline{g}_{i_1i_2}$(resp. $\overline{g}_{j_1j_2}$),*

   (i) is non-negative and continuous;

(ii) For any $k \in \Gamma^1$ (resp. $\ell \in \Gamma^2$) such that $|\{i_1, i_2, k\}| = 3$ (resp. $|\{j_1, j_2, \ell\}| = 3$), it holds:

$$\mathbb{P} - a.s.,\ \forall t \le T,\ \underline{g}_{i_1 i_2}(t) < \underline{g}_{i_1 k}(t) + \underline{g}_{k i_2}(t)\ \left(resp.\ \overline{g}_{j_1 j_2}(t) < \overline{g}_{j_1 \ell}(t) + \overline{g}_{\ell j_2}(t)\right); \tag{1.14}$$

4. *For any $(i,j),(k,\ell) \in \Gamma$, $\underline{g}_{ik}$(resp. $\overline{g}_{j\ell}$) is an Itô process, i.e.,*

$$\begin{cases} \underline{g}_{ik}(t) = \underline{g}_{ik}(0) + \int_0^t \underline{b}_{ik}(s)ds + \int_0^t \underline{\sigma}_{ik}(s)dB_s,\ t \le T, \\ \text{with } \underline{\sigma}_{ik} \in \mathcal{H}^{2,d} \text{ and } \underline{b}_{ik},\ \mathcal{P}\text{-measurable and } \mathbb{E}[\sup_{s\le T} |\underline{b}_{ik}(s)|^2] < \infty. \end{cases}$$

$$\left(resp. \begin{cases} \overline{g}_{j\ell}(t) = \overline{g}_{j\ell}(0) + \int_0^t \overline{b}_{j\ell}(s)ds + \int_0^t \overline{\sigma}_{j\ell}(s)dB_s,\ t \le T, \\ \text{with } \overline{\sigma}_{j\ell} \in \mathcal{H}^{2,d} \text{ and } \overline{b}_{j\ell},\ \mathcal{P}\text{-measurable and } \mathbb{E}[\sup_{s\le T} |\overline{b}_{j\ell}(s)|^2] < \infty. \end{cases}\right).$$

5. *Monotonicity:*
For any $(i,j) \in \Gamma$ and $(k,l) \in \Gamma^{-ij} := \Gamma - \{(i,j)\}$, the mapping $y^{kl} \mapsto f^{ij}(t, \overrightarrow{y}, z)$ is non-decreasing when the other components $(y^{pq})_{(p,q)\neq(k,l)}$ and z are fixed.

Definition 1.3.2. *A family $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is said to be a solution of the system of reflected BSDEs with doubly interconnected barriers associated with $((f^{ij})_{(i,j)\in\Gamma}, (\xi^{ij})_{(i,j)\in\Gamma}, (\underline{g}_{ik})_{i,k\in\Gamma^1}, (\overline{g}^{j,\ell})_{j,\ell\in\Gamma^2})$, if it satisfies the followings: $\forall (i,j) \in \Gamma$,*

$$\begin{cases} Y^{ij} \in \mathcal{S}^2, Z^{ij} \in \mathcal{H}^{2,d}, K^{ij,\pm} \in \mathcal{A}^2; \\ Y^{ij}_t = \xi^{ij} + \int_t^T f^{ij}(s,\omega,(Y^{kl}_s)_{(k,l)\in\Gamma^1\times\Gamma^2}, Z^{ij}_s)ds - \\ \qquad \int_t^T Z^{ij}_s dB_s + K^{ij,+}_T - K^{ij,+}_t - (K^{ij,-}_T - K^{ij,-}_t),\ \forall t \le T; \\ L^{ij}_t \le Y^{ij}_t \le U^{ij}_t,\ \forall t \in [0,T]; \\ \int_0^T (Y^{ij}_t - L^{ij}_t)dK^{ij,+}_t = 0 \quad and \quad \int_0^T (U^{ij}_t - Y^{ij}_t)dK^{ij,-}_t = 0, \end{cases} \tag{1.15}$$

where $L^{ij}_t := \max\limits_{k\in(\Gamma^1)^{-i}} \{Y^{kj}_t - \underline{g}_{ik}(t)\}$ and $U^{ij}_t := \min\limits_{l\in(\Gamma^2)^{-j}} \{Y^{il}_t + \overline{g}_{jl}(t)\}, \forall t \le T.$

1.3.2 Main results of this paper

Existence of solution under monotonicity condition

In the first place we prove the existence of a solution of (1.15) under Assumption 1.3.1. For this purpose we penalize both barriers in the following way: $\forall m, n \in \mathbb{N}$, $(i,j) \in \Gamma$,

$$\begin{cases} Y^{ij,m,n} \in \mathcal{S}^2, Z^{ij,m,n} \in \mathcal{H}^{2,d}; \\ Y_t^{ij,m,n} = \xi^{ij} + \int_t^T f^{ij,m,n}\left(s, (Y_s^{kl,m,n})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij,m,n}\right) ds - \int_t^T Z_s^{ij,m,n} dB_s,\ t \le T, \end{cases} \tag{1.16}$$

where

$$f^{ij,m,n}\left(t, (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z\right) = f^{ij}(t, \vec{y}, z) + n\left\{y_t^{ij} - \max_{k\in(\Gamma^1)^{-i}}\left[y_t^{kj} - \underline{g}_{ik}(t)\right]\right\}^- \\ -m\left\{y_t^{ij} - \min_{l\in(\Gamma^2)^{-j}}\left[y_t^{il} + \bar{g}_{jl}(t)\right]\right\}^+,$$

($x^+ = x \vee 0$ and $x^- = (-x)\vee 0$, $x \in \mathbb{R}$).

For all $(i,j)\in\Gamma$, the sequence $(Y^{ij,m,n})_{n\ge 0}$ increasingly converges to a process in $\mathcal{S}^2$, denoted $(\bar{Y}^{ij,m})_{m\ge 0}$. As the terminal condition of this sequence stays the same, the monotonic result is obtained by simply comparing the generators $(f^{ij,m,n})_{n\ge0}$. On the other hand, the process $(Z^{ij,m,n})_{n\ge0} \to_{n\to\infty} \bar{Z}^{ij,m}$ in $\mathcal{H}^{2,d}$, the penalized part of the lower obstacle also converges in $\mathcal{S}^2$, we denote by

$$\bar{K}_t^{ij,m,+} := \lim_{n\to\infty}\int_0^t n\left\{Y_s^{ij,m,n} - \max_{k\in\Gamma^{-i}}[Y_s^{kj,m,n} - \underline{g}_{ik}(s)]\right\}^- ds,\ t\le T,$$

thus the triple $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m}, \bar{K}^{ij,m})_{(i,j)\in\Gamma}$ is the unique solution of RBSDEs associated with $(\xi^{ij}, \bar{f}^{ij,m}, (\underline{g}_{ik})_{k\in\Gamma^1})_{(i,j)\in\Gamma}$ where

$$\bar{f}^{ij,m}(s, (y^{kl})_{(k,l)\in\Gamma}, z) := f^{ij}(s, (y^{kl})_{(k,l)\in\Gamma}, z) - m\left(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il} + \bar{g}_{jl}(s)]\right)^+.$$

The following step is to prove that the sequence $(\bar{Y}^{ij,m})_{(i,j)\in\Gamma}$ convergences to Y^{ij} in $\mathcal{S}^2, \forall (i,j)\in\Gamma$. Here the difficulty derives from the continuity of this limit process. In another word, if we can prove the uniform convergence of $(\bar{Y}^{ij,m})_{(i,j)\in\Gamma}$ in $\mathcal{S}^2$, then Y^{ij} holds the continuity. However the penalized term $m\left(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il} + \bar{g}_{jl}(s)]\right)^+$ is a little bit troublesome when making calculus. Then we introduce the equivalent RBSDEs $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma}$ where the driver is the following:

$$f^{ij,m}(t, \vec{y}, z) := f^{ij}(t, (y^{kl})_{(k,l)\in\Gamma}, z) - m\left(y^{ij} - \sum_{l\in(\Gamma^2)^{-j}}[y^{il} + \bar{g}_{jl}(t)]\right)^+.$$

The equivalence (if one converges, the other one does so to the same limit) between $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma}$ and $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m}, \bar{K}^{ij,m})_{(i,j)\in\Gamma}$ is proved by the comparison theorem.

In the following we need to prove the convergence of $(Y^{ij,m})_{(i,j)\in\Gamma}$. For this purpose we rely on the link with switching problems and introduce the following definitions:

- $(\sigma_n)_{n\geq 0}$: increasing sequence of stopping times s.t. $\mathbb{P}[\sigma_n < T, \forall n \geq 0] = 0$;
- $\delta_n : \Gamma^1-$valued and $\mathcal{F}_{\sigma_n}-$measurable random variable;
- $u := (\sigma_n, \delta_n)_{n\geq 0}$ an admissible switching strategy;
- $A^u_t := \sum_{n\geq 1} g_{\delta_{n-1}\delta_n}(\sigma_n) 1_{(\sigma_n \leq t)}$ the cumulative switching cost and $A^u_T := \lim_{t\to T} A^u_t$;
- $a_t := \delta_0 1_{(\sigma_0)}(t) + \sum_{n\geq 1} \delta_{n-1} 1_{(\sigma_{n-1},\sigma_n]}(t)$;
- $\mathcal{A}^i_t := \{u = (\sigma_n, \delta_n)_{n\geq 0}$ admissible strategy such that $\sigma_0 = t, \delta_0 = i$ and $\mathbb{E}[(A^u_T)^2] < \infty\}$

Thus there exists $(U^{aj,m}, V^{aj,m}), \forall a \in \mathcal{A}^i_t, j \in \Gamma^2$, the unique solution of a non-standard type BSDE associated with $(\xi^{a_t j}, \underline{f}^{ij,m}, A^a)$:

$$\begin{cases} U^{aj,m} \text{ is rcll, } \mathbb{E}\left[\sup_{t\leq T} |U^{aj,m}_t|^2\right] < \infty \text{ and } V^{aj,m} \in \mathcal{H}^{2,d}; \\ U^{aj,m}_t = \xi^{a_T j} + \int_t^T \mathbb{1}_{(s\geq\sigma_0)} \underline{f}^{aj,m}\left(s, (Y^{kl,m}_s)_{(k,l)\in\Gamma}, V^{aj,m}_s\right) ds - \int_t^T V^{aj,m}_s dB_s + A^a_T - A^a_t, \end{cases}$$

where

$$\begin{aligned} \underline{f}^{aj,m}(s, (Y^{kl,m}_s)_{(k,l)\in\Gamma}, z) = \sum_{n\geq 1} \Bigg(\sum_{q\in\Gamma^1} \Big\{ f^{qj}(s, (Y^{kl,m}_s)_{(k,l)\in\Gamma}, z) \\ - m \sum_{l\in(\Gamma^2)^{-j}} (Y^{qj,m}_s - Y^{ql,m}_s - \overline{g}_{jl}(t))^+ \Big\} 1_{\{\delta_{n-1}=q\}} \Bigg) 1_{\{\sigma_{n-1}\leq s<\sigma_n\}}. \end{aligned} \tag{1.17}$$

Then we have:

$$Y^{ij,m}_t = \operatorname*{ess\,sup}_{a\in\mathcal{A}^i_t} \left\{ U^{aj,m}_t - A^a_t \right\}.$$

Afterwards by Itô's calculus, we obtain the following estimate:

Proposition 1.3.3. *For any $(i,j) \in \Gamma, t \leq T$,*

$$m^2 \mathbb{E}\left[\sum_{l\in\Gamma^2-\{j\}} \{(Y^{ij,m}_t - Y^{il,m}_t - \overline{g}_{jl}(t))^+\}^2 \right] \leq C. \tag{1.18}$$

This estimate implies that the discontinuity points of $Y^{ij} := \lim_{m\to\infty} Y^{ij,m}$ stem from the discontinuities of $\lim_{m\to\infty} K^{ij,m,+}$. But by the uniqueness of the solution $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma}$ of the system of RBSDEs associated with
$(\xi^{ij}, f^{ij,m}, (\underline{g}_{ik})_{k\in\Gamma^1})_{(i,j)\in\Gamma}$, the estimates on f^{ij} (Assumption 1.3.1, 1.a)) and the regularity of the switching costs, as in Proposition 1.3.3, we show that $K^{ij,m,+}$ has a density w.r.t. dt which uniformly bounded in $\mathcal{H}^{2,1}$. Therefore $\lim_{m\to\infty} K^{ij,m,+}$ is a continuous non-decreasing process. It implies that Y^{ij} is a continuous process and the convergence of $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma}$ and $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m}, \bar{K}^{ij,m})_{(i,j)\in\Gamma}$ to $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ in the appropriate spaces holds (especially the convergence of $(Y^{ij,m})_m$ to Y^{ij} in $\mathcal{S}^2$). Finally we obtain:

Theorem 1.3.4. *Under Assumption 1.3.1, the process $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is a solution of the system of reflected BSDEs* (1.15).

Existence and Uniqueness without monotonicity

The results of this subsection are based on the existence of the value function of a zero-sum stochastic differential game whose payoff is given by:

$$J_t^{ij}(\gamma(u,v)) = \mathbb{E}\left[\xi^{\pi_T} + \int_t^T f^{\pi}(s)ds - \sum_{n\geq 1}\left(\underline{g}_{\gamma^{(1)}_{n-1}\gamma^{(1)}_n}(\rho_n) - \overline{g}_{\gamma^{(2)}_{n-1}\gamma^{(2)}_n}(\rho_n)\right)\middle|\mathcal{F}_t\right], \tag{1.19}$$

where $\gamma(u,v)$ is the coupling of (u,v). The construction of the model is well detailed in [33] where the authors relate the solution Y^{ij} of the system of DRBSDEs (1.15) to this zero-sum stochastic switching game, when the generators f^{ij} do not depend neither on y nor on z. They show that the value of the game is nothing but Y^{ij}, solution of the DRBSDEs, when the controlled system under the switchings of the two players starts from (i,j).

For this purpose we introduce the following assumptions:

Assumption 1.3.5.

1. *The processes $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\overline{g}_{j,\ell})_{j,\ell\in\Gamma^2}$ verify the non free loop property, that is to say, if $(i_k, j_k)_{k=1,2,...,N}$ is a loop in Γ, i.e., $(i_N, j_N) = (i_1, j_1)$,* ***card*** $\{(i_k, j_k)_{k=1,2,...,N}\} = N-1$ *and for any $k = 1,2,...,N-1$, either $i_{k+1} = i_k$ (resp. $j_{k+1} = j_k$), we have:*

$$\mathbb{P}-a.s., \ \forall t \leq T, \quad \sum_{k=1}^{N-1} G_{i_k j_k}(t) \neq 0, \tag{1.20}$$

where $\forall k = 1,...N-1$, $G_{i_k j_k}(t) = -\underline{g}_{i_k i_{k+1}}(t)1_{i_k\neq i_{k+1}} + \overline{g}_{j_k j_k}(t)1_{j_k\neq j_{k+1}}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1 \times \Gamma^2$, of the players (or deci-

sion makers), is not free i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode. Note that (1.20) also implies: for any $(i_1, ..., i_N) \in (\Gamma^1)^N$ *such that* $i_N = i_1$ *and* $\textbf{card}\{i_1, i_2, ..., i_N\} = N - 1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t) = 0\right] = 0,\ \forall t \leq T,$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ *such that* $j_N = j_1$ *and* $\textbf{card}\{j_1, j_2, ..., j_N\} = N - 1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t) = 0\right] = 0,\ \forall t \leq T.$$

2. *For any* $(i, j) \in \Gamma$, *the function* f^{ij} *does not depend on* z.

Theorem 1.3.6. *Assume that Assumption 1.3.1-(1.)-(4.) and Assumption 1.3.5 are fulfilled. Then system of reflected BSDEs* (1.15) *has a solution* $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$, *i.e., for any* $(i, j) \in \Gamma$ *and* $t \leq T$,

$$\begin{cases} Y^{ij} \in \mathcal{S}^2, Z^{ij} \in \mathcal{H}^{2,d}, K^{ij,\pm} \in \mathcal{A}^2\ ; \\ Y^{ij}_t = \xi^{ij} + \int_t^T f^{ij}\left(s, \omega, (Y^{kl}_s)_{(k,l)\in\Gamma^1\times\Gamma^2}\right) ds - \int_t^T Z^{ij}_s dB_s + K^{ij,+}_T - K^{ij,+}_t - (K^{ij,-}_T - K^{ij,-}_t)\ ; \\ L^{ij}_t \leq Y^{ij}_t \leq U^{ij}_t\ ; \\ \int_0^T \left(Y^{ij}_t - L^{ij}_t\right) dK^{ij,+}_t = 0 \text{ and } \int_0^T \left(U^{ij}_t - Y^{ij}_t\right) dK^{ij,-}_t = 0 \end{cases} \tag{1.21}$$

where $L^{ij}_t := \max_{k\in(\Gamma^1)^{-i}} \left\{Y^{kj}_t - \underline{g}_{ik}(t)\right\}$ *and* $U^{ij}_t := \min_{l\in(\Gamma^2)^{-j}} \left\{Y^{il}_t + \overline{g}_{jl}(t)\right\}$. *Moreover it is unique in the following sense: if* $(\overline{Y}^{ij}, \overline{Z}^{ij}, \overline{K}^{ij,+}, \overline{K}^{ij,-})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is another solution of* (1.21), *then for any* $(i, j) \in \Gamma$,

$$\overline{Y}^{ij} = Y^{ij},\ \overline{Z}^{ij} = Z^{ij},\ \overline{K}^{ij,+} - \overline{K}^{ij,-} = K^{ij,+} - K^{ij,-}.$$

The sketch of the proof is the following: We firstly define a mapping Φ from $\mathcal{H}^{2,\Lambda}$ to itself by $\Phi(\phi) := (\vec{Y^{\phi,ij}})_{(i,j)\in\Gamma}$. Then we consider two different solutions $(Y^{\phi,ij})_{(i,j)\in\Gamma}$ and $(Y^{\psi,ij})_{(i,j)\in\Gamma}$ of the systems of DRBSDEs (the existence of the solution is proved in the previous subsection). When calculating the difference between $(Y^{\phi,ij})_{(i,j)\in\Gamma}$ and $(Y^{\psi,ij})_{(i,j)\in\Gamma}$, we relate to the corresponding values of the games whose payoffs are $J^{\phi,ij}$ and $J^{\psi,ij}$ to get rid of the switching cost. Next using stantard Itô's calculus we prove that Φ is a contraction mapping from $\mathcal{H}^{2,\Lambda}$ into itself under an appropriate norm, then the solutin of the DRBSDEs is the unique fixed point of Φ.

Connection with system of PDEs with bilateral interconnected obstacles

We are now going to decline Assumptions 1.3.1 and 1.3.5 in the markovian framework of randomness. Let us introduce deterministic functions $f^{ij}(t,x,\vec{y})$, $h^{ij}(x)$, $\underline{g}_{ik}(t,x)$ and $\bar{g}_{jl}(t,x)$, $i,k \in \Gamma^1$, $j,l \in \Gamma^2$ and $t,x,\vec{y}$ in $[0,T]$, $\mathbb{R}^k$ and $\mathbb{R}^\Lambda$ respectively.

Assumption 1.3.7.

1. *For any $(i,j) \in \Gamma$,*

   i) *There exists non negative constants C and γ such that*

   $$|f^{ij}(t,x,\vec{y})| \leq C(1+|x|^\gamma + |\vec{y}\,|).$$

   ii) *f^{ij} is Lipschitz continuous w.r.t. $\vec{y}$ uniformly in (t,x), i.e. there exists a constant C such that for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^\Lambda$,*

   $$|f^{ij}(t,x,\vec{y}_1) - f^{ij}(t,x,\vec{y}_2)| \leq C|\vec{y}_1 - \vec{y}_2|.$$

2. *For any $(i,j) \in \Gamma$, the function h^{ij}, which stands for the terminal condition, is continuous w.r.t. x, belongs to class Π_g and satisfies the following consistency condition: $\forall (i,j) \in \Gamma$ and $x \in \mathbb{R}^k$,*

   $$\max_{k\in(\Gamma^1)^{-i}} (h^{kj}(x) - \underline{g}_{ik}(T,x)) \leq h^{ij}(x) \leq \min_{l\in(\Gamma^2)^{-j}} (h^{il}(x) + \bar{g}_{jl}(T,x)). \tag{1.22}$$

3. *For all $i_1, i_2 \in \Gamma^1$ (resp. $j_1, j_2 \in \Gamma^2$), the function $\underline{g}_{i_1 i_2}$(resp. $\bar{g}_{j_1 j_2}$)*

   i) *is non-negative, continuous and belong to Π_g ;*

   ii) *For any $k \in \Gamma^1$ (resp. $\ell \in \Gamma^2$) such that $|\{i_1, i_2, k\}| = 3$ (resp. $|\{j_1, j_2, \ell\}| = 3$) it holds: $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,*

   $$\underline{g}_{i_1 i_2}(t,x) < \underline{g}_{i_1 k}(t,x) + \underline{g}_{k i_2}(t,x) \ \Big(\text{resp. } \bar{g}_{j_1 j_2}(t,x) < \bar{g}_{j_1 \ell}(t,x) + \bar{g}_{\ell j_2}(t,x)\Big)\,; \tag{1.23}$$

   iii) *The functions $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\bar{g}_{jl})_{j,l\in\Gamma^2}$ verify the non free loop property, that is to say, if $(i_k, j_k)_{k=1,2,\dots,N}$ is a loop in Γ, i.e., $(i_N, j_N) = (i_1, j_1)$, **card** $\{(i_k, j_k)_{k=1,2,\dots,N}\} = N-1$ and for any $k = 1,2,\dots,N-1$, either $i_{k+1} = i_k$ or $j_{k+1} = j_k$, we have:*

   $$\forall t \leq T, \quad \sum_{k=1}^{N-1} G_{i_k j_k}(t,x) \neq 0, \tag{1.24}$$

   where $\forall k = 1,\dots N-1$, $\quad G_{i_k j_k}(t,x) = -\underline{g}_{i_k i_{k+1}}(t,x) 1_{(i_k \neq i_{k+1})} + \bar{g}_{j_k j_k}(t,x) 1_{(j_k \neq j_{k+1})}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1 \times \Gamma^2$ is not *free, i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode.*

Note that (1.24) also implies: for any $(i_1, ..., i_N) \in (\Gamma^1)^N$ *such that* $i_N = i_1$ *and* $\textbf{card}\{i_1, i_2, ..., i_N\} = N-1$,

$$\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k,$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ *such that* $j_N = j_1$ *and* $\textbf{card}\{j_1, j_2, ..., j_N\} = N-1$,

$$\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

4. *For any* $i,k \in \Gamma^1$ *(resp.* $j,l \in \Gamma^2$*),* $\underline{g}_{ik}$*(resp.* $\overline{g}_{jl}$*) is* $\mathcal{C}^{1,2}$ *and* $D_x\underline{g}_{ik}, D^2_{xx}\underline{g}_{ik}$ *(resp.* $D_x\overline{g}_{jl}, D^2_{xx}\overline{g}_{jl}$*) belong to* Π_g*. Thus by Itô's formula we have:*

$$\begin{cases} \underline{g}_{ik}(s, X_s^{t,x}) = \underline{g}_{ik}(t,x) + \displaystyle\int_t^s \mathcal{L}^X(\underline{g}_{ik})(r, X_r^{t,x})dr \\ \qquad + \int_t^s D_x\underline{g}_{ik}(r, X_r^{t,x})\sigma(r, X_r^{t,x})dB_r, \ s \in [t,T]; \\ \underline{g}_{ik}(s, X_s^{t,x}) = \underline{g}_{ik}(s,x), \ s \leq t. \end{cases}$$

$$\left(\textit{resp.} \begin{cases} \overline{g}_{jl}(s, X_s^{t,x}) = \overline{g}_{jl}(t,x) + \displaystyle\int_t^s \mathcal{L}^X(\overline{g}_{jl})(r, X_r^{t,x})dr \\ \qquad + \int_t^s D_x\overline{g}_{jl}(r, X_r^{t,x})\sigma(r, X_r^{t,x})dB_r, \ s \in [t,T]; \\ \overline{g}_{jl}(s, X_s^{t,x}) = \overline{g}_{jl}(s,x), \ s \leq t. \end{cases} \right)$$

In this subsection, we study the existence of the unique solution of the following system of PDEs in the viscosity sense: for any $(i,j) \in \Gamma, t \in [0,T]$,

$$\begin{cases} \min \Big\{ v^{ij}(t,x) - \max_{k \in (\Gamma^1)^{-i}}[v^{kj}(t,x) - \underline{g}_{ik}(t,x)]; \max \Big[v^{ij}(t,x) - \min_{l \in (\Gamma^2)^{-j}}[v^{il}(t,x) + \overline{g}_{jl}(t,x)]; \\ -\partial_t v^{ij}(t,x) - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma}) \Big] \Big\} = 0; \\ \\ v^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{1.25}$$

The infinitesimal generator $\mathcal{L}^X$ is given by: for any $(t,x) \in [0,T] \times \mathbb{R}^k, \phi \in \mathcal{C}^{1,2}$ ($(.)^\top$ is the transpose),

$$\mathcal{L}^X\phi(t,x) := \frac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^\top(t,x))_{i,j}\partial^2_{x_i x_j}\phi(t,x) + \sum_{i=1}^k b_i(t,x)\partial_{x_i}\phi(t,x). \tag{1.26}$$

Before proving the existence of the unique solution of (1.25), we firstly show the connection between the unique solution $(Y^{ij;t,x})_{(i,j)\in\Gamma}$ of DRBSDEs and a deterministic function $(v^{ij})_{(i,j)\in\Gamma}$, the so-called Feynman-Kac formula:

Proposition 1.3.8. *For any $(i,j) \in \Gamma, (t,x) \in [0,T] \times \mathbb{R}^k$, there exists deterministic continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ such that*

$$\mathbb{P}-a.s. \quad \forall s \in [t,T],\ Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x}) \tag{1.27}$$

where $Y_s^{ij;t,x}$ is the unique solution of the following system of DRBSDEs with doubly interconnected obstacles:

$$\begin{cases} Y^{ij;t,x} \in \mathcal{S}^2, Z^{ij;t,x} \in \mathcal{H}^{2,1}, K^{ij,\pm;t,x} \in \mathcal{A}^2\ ; \\ Y_s^{ij;t,x} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}\left(r, X_r^{t,x}, (Y_r^{kl;t,x})_{(k,l)\in\Gamma}\right) dr - \int_s^T Z_r^{ij;t,x} dB_r + K_T^{ij,+;t,x} - K_s^{ij,+;t,x} \\ \qquad -(K_T^{ij,-;t,x} - K_s^{ij,-;t,x}); \\ L_s^{ij;t,x} \leq Y_s^{ij;t,x} \leq U_s^{ij;t,x}; \\ \int_0^T (Y_s^{ij;t,x} - L_s^{ij;t,x}) dK_s^{ij,+;t,x} = 0 \text{ and } \int_0^T (Y_s^{ij;t,x} - U_s^{ij;t,x}) dK_s^{ij,-;t,x} = 0 \end{cases} \tag{1.28}$$

where $L_s^{ij;t,x} := \max_{k\in(\Gamma^1)^{-i}} \left[Y_s^{kj;t,x} - \underline{g}_{ik}(s, X_s^{t,x})\right]$ and $U_s^{ij;t,x} := \min_{l\in(\Gamma^2)^{-j}} \left[Y_s^{il;t,x} + \overline{g}_{jl}(s, X_s^{t,x})\right]$, $s \in [0,T]$.

The proof is composed of two steps. Firstly we assume that the coefficients $f^{ij}(t,x,0,0)$ and $h^{ij}(x)$ are bounded. By constructing a Picard iterations process $(Y^{ij,n,t,x})_{(i,j)\in\Gamma} = \Phi((Y^{ij,n-1,t,x})_{(i,j)\in\Gamma})$, we prove the uniform convergence of $(Y^{ij,n,t,x})_{n\geq 0}$ in $\mathcal{S}^2$ and $\forall s \in [t,T], Y_s^{ij,n,t,x} = v^{ij,n}(s, X_s^{t,x})$ with $v^{ij,n}$ a continuous function. Then we show the uniform convergence of $v^{ij,n}$ to v^{ij} which implies that v^{ij} is continuous. In the second place, we relax the boundedness condition of $f^{ij}(t,x,0,0)$ and $h^{ij}(x)$, i.e. they are of polynomial growth. By applying Itô's formula with $\tilde{Y}^{ij} := Y_s^{ij;t,x}(1+|X_s^{t,x}|^2)^{-\gamma}$, we fall in the previous framework. Therefore we deduce that Y^{ij} has the previous representation (1.27) with v^{ij} continuous and of polynomial growth.

Then we prove the existence of the unique viscosity solution of (1.25).

Theorem 1.3.9. *Assume that Assumptions 1.3.5-(2.) and 1.3.7 are fulfilled. Then the Λ-tuple of continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ of* (1.27) *is a viscosity solution of* (1.25). *Moreover it is unique in the class of continuous functions which belong to Π_g.*

The proof is divided into two steps. Firstly thanks to Proposition (1.3.8) we show that $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (1.25) . Secondly we prove the uniqueness of this solution by using the uniqueness of the solution $(Y^{ij;t,x})_{(i,j)\in\Gamma}$ of the system (1.28).

1.4 Zero-sum Switching Game, Systems of Reflected Backward SDEs and Parabolic PDEs with bilateral interconnected obstacles

Chapter 3 is a preprint joint work with Hamadène (ref.[36]). In this work we study a specific zero-sum switching game and its verification theorems expressed in terms of either a system of RBSDEs with bilateral interconnected obstacles or a system of parabolic PDEs with bilateral interconnected obstacles as well. The framework is markovian. We show that each one of the systems has a unique solution. Then we show that the game has a value.

1.4.1 Framework setting

First let us additionally denote by:

- $\mathcal{A}_{loc}$: the set of $\mathcal{P}$-measurable continuous non-decreasing processes $K = (K_t)_{t\le T}$ with $K_0 = 0$ such that $\mathbb{P} - a.s.$, $K_T(\omega) < \infty$;
- $\mathcal{H}^{2,d}_{loc}(d \geq 1)$: the set of $\mathcal{P}$-measurable $\mathbb{R}^d$-valued processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{P} - a.s.$, $\int_0^T |\phi_t|^2 dt < \infty$;

Next we define $\Gamma := \{1, 2, ..., p\}$ and for any $i \in \Gamma$, let us set $\Gamma^{-i} := \Gamma - \{i\}$. For $\vec{y} := (y^i)_{i\in\Gamma} \in \mathbb{R}^p$ and $\hat{y} \in \mathbb{R}$, we denote by $(\vec{y}^{-i}, \hat{y})$ the element of $\mathbb{R}^p$ obtained in replacing the i-th component of $\vec{y}$ with $\hat{y}$.

We now introduce the following deterministic functions: for any $i \in \Gamma$,

- f^i: $(t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^{k+p+d} \mapsto f^i(t, x, \vec{y}, z) \in \mathbb{R}$,
- $\underline{g}_{i,i+1}$: $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto \underline{g}_{i,i+1}(t, x) \in \mathbb{R}$,
- $\overline{g}_{i,i+1}$: $(t, x) \in [0, T] \times \mathbb{R}^k \mapsto \overline{g}_{i,i+1}(t, x) \in \mathbb{R}$,
- $h^i : x \in \mathbb{R}^k \mapsto h^i(x) \in \mathbb{R}$.

Next let us consider the following assumptions which, sometimes, we use only partly.

Assumption 1.4.1.
(H1) *For any $i \in \Gamma$, f^i does not depend on $(\vec{y}, z)$, is continuous in (t, x) and belongs to class Π_g ;*

(H2) *For any $i \in \Gamma$, the function h^i, which stands for the terminal payoff, is continuous w.r.t. x, belongs to class Π_g and satisfies the following consistency condition: $\forall i \in \Gamma, \forall x \in \mathbb{R}^k$,*

$$h^{i+1}(x) - \underline{g}_{i,i+1}(T, x) \le h^i(x) \le h^{i+1}(x) + \overline{g}_{i,i+1}(T, x). \tag{1.29}$$

(H3) *a) For all $i \in \Gamma$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, the functions $\underline{g}_{i,i+1}$ and $\overline{g}_{i,i+1}$ are continuous, non-negative, belong to Π_g and verify:*

$$\underline{g}_{i,i+1}(t,x) + \overline{g}_{i,i+1}(t,x) > 0.$$

b) They satisfy the non-free loop property, i.e., for any $j \in \Gamma$ and $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\varphi_{j,j+1}(t,x) + ... + \varphi_{p-1,p}(t,x) + \varphi_{p,1}(t,x) + ... + \varphi_{j-1,j}(t,x) \neq 0, \tag{1.30}$$

where $\varphi_{\ell,\ell+1}(t,x)$ is either $-\underline{g}_{\ell,\ell+1}(t,x)$ or $\overline{g}_{\ell,\ell+1}(t,x)$. Let us notice that (3.14) *also implies:*

$$\overline{g}_{j,j+1}(t,x) + ... + \overline{g}_{p-1,p}(t,x) + \overline{g}_{p,1}(t,x) + ... + \overline{g}_{j-1,j}(t,x) > 0, \tag{1.31}$$

and

$$\underline{g}_{j,j+1}(t,x) + ... + \underline{g}_{p-1,p}(t,x) + \underline{g}_{p,1}(t,x) + ... + \underline{g}_{j-1,j}(t,x) > 0. \tag{1.32}$$

(H5) *For any $i \in \Gamma$,*

a) f^i is Lipschitz in $(\vec{y}, z)$ uniformly in (t,x) i.e. for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^p$, $z_1, z_2 \in \mathbb{R}^d$ and $t \in [0,T]$,

$$|f^i(t,x,\vec{y}_1,z_1) - f^i(t,x,\vec{y}_2,z_2)| \leq C(|\vec{y}_1 - \vec{y}_2| + |z_1 - z_2|);$$

b) $\forall j \in \Gamma^{-i}$, the mapping $y^j \mapsto f^i(t,x,\vec{y},z)$ is non-decreasing when the other components $(y^k)_{k \in \Gamma^{-j}}$, t,x,z are fixed.

c) f^i is continuous in (t,x) uniformly in $(\vec{y}, z)$ and $f^i(t,x,0,0)$ belongs to Π_g.

1.4.2 Motivation

This paper is related to zero-sum switching games, systems of reflected backward differential equations (RBSDEs) with bilateral interconnected obstacles and systems of variational inequalities of min-max type with interconnected obstacles, namely the Hamilton-Jacobi-Bellman (HJB for short) system associated with the game.

First let us describe the zero-sum switching game which we will consider in this paper. Let Γ be the set $\{1, ..., p\}$. Assume we have a system which has p working modes indexed by Γ. This system can be switched from one working mode to another one, e.g. due to economic, financial, ecological reasons, etc, by two players or decision makers C_1 and C_2. The main feature of the switching actions is that when the system is in mode $i \in \Gamma$, and one of the players decides to switch it, then it is switched to mode $i+1$ (hereafter $i+1$ is 1 if $i = p$). It means that the decision makers do not have their proper modes to which they can switch the system when they decide to switch (see e.g. [33] for more details on this model). Therefore a switching strategy for the players are sequences of stopping times $u = (\sigma_n)_{n\geq 0}$ for C_1 and $v = (\tau_n)_{n\geq 0}$ for C_2 such that $\sigma_n \leq \sigma_{n+1}$ and $\tau_n \leq \tau_{n+1}$ for any $n \geq 0$. On the other hand, the switching actions are not free and generate expenditures for the players. Loosely speaking at time $t \leq T$, they amount to A_t^u (resp. B_t^v) given by:

$$A_t^u = \sum_{\sigma_n \leq t} \underline{g}_{\zeta_n,\zeta_n+1}(\sigma_n) \text{ (resp. } B_t^v = \sum_{\tau_n \leq t} \bar{g}_{\theta_n,\theta_n+1}(\tau_n)).$$

The process $\underline{g}_{i,i+1}(s)$ (resp. $\bar{g}_{i,i+1}(s)$) is the switching cost payed by C_1 (resp. C_2) is she makes the decision to switch the system from mode i to mode $i+1$ at time s while ζ_n (resp. θ_n) is the mode in which the system is at time σ_n (resp. τ_n). Next when the system is run under the control u (resp. v) for C_1 (resp. C_2), there is a payoff $J(u,v)$ which is a profit (resp. cost) for C_1 (resp. C_2) given by:

$$J(u,v) = \mathbb{E}[\int_0^T f^{\delta_s}(s)ds - A_T^u + B_T^v + \zeta^{\delta_T}].$$

where $\delta := (\delta_s)_{s\leq T}$ is the process valued in Γ which indicates the working modes of the system along with time. If at time s the system is in mode i_0, then $\delta_s = i_0$. It is bind to the controls u and v implemented by both players. On the other hand, for $i \in \Gamma$, the process f^i is the utility of the system in mode i and finally ζ^{δ_T} is the terminal payoff or bequest.

The problem we are interested in is to know whether or not the game has a value, i.e., roughly speaking, if the following equality holds:

$$\inf_v \sup_u J(u,v) = \sup_u \inf_v J(u,v).$$

In case of equality we say that the game has a value. Finally we say that the game has a saddle-point (u^*, v^*) if, for any u and v, controls of C_1 and C_2 respectively, we have:

$$J(u,v^*) \leq J(u^*,v^*) \leq J(u^*,v).$$

Note that in such a case, the game has a value.
From the probabilistic point of view, this zero-sum switching game problem turns into looking for a solution of its associated system of reflected BSDEs with interconnected bilateral obstacles. A solution for such a system are adapted processes $(Y^i, Z^i, K^{i,\pm})_{i\in\Gamma}$ such that for any $i \in \Gamma$ and $s \leq T$,

$$\begin{cases} Y^i \text{ and } K^{i,\pm} \text{ continuous; } K^{i,\pm} \text{ increasing; } (Z^i(\omega)_t)_{t\leq T} \text{ is } dt- \text{ square integrable;} \\ Y_s^i = \xi^i + \int_s^T f^i(r)dr - \int_s^T Z_r^i dB_r + K_T^{i,+} - K_s^{i,+} - (K_T^{i,-} - K_s^{i,-}); \\ L^i(\vec{Y})_s \leq Y_s^i \leq U^i(\vec{Y})_s; \\ \int_0^T (Y_s^i - L^i(\vec{Y})_s)dK_s^{i,+} = 0 \text{ and } \int_0^T (Y_s^i - U^i(\vec{Y})_s)dK_s^{i,-} = 0, \end{cases} \tag{1.33}$$

where: a) $B := (B_t)_{t\leq T}$ is a Brownian motion; b) $\vec{Y} := (Y^i)_{i\in\Gamma}$; c) $L^i(\vec{Y})_s = Y_s^{i+1} - \underline{g}_{i,i+1}(s)$ and $U^i(\vec{Y})_s = Y_s^{i+1} + \bar{g}_{i,i+1}(s)$.

Actually the solution of the previous system provides the value of the zero-sum switching game which is equal to Y_0^i if the starting mode of the system is i. Roughly speaking, system (1.33) is the verification theorem for the zero-sum switching game problem. Usually it is shown that the value functions of the game is the unique solution of (1.33).

In the Markovian framework, i.e., when randomness stems from a diffusion process $X^{t,x}$ ($(t,x) \in [0,T] \times \mathbb{R}^k$) which satifies:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s,\ s \in [t,,T] \text{ and } X_s^{t,x} = x, \ \text{ for } s \leq t, \tag{1.34}$$

and the data of the game are deterministic functions of $(s, X_s^{t,x})$, the Hamilon-Jacobi-Bellman system associated with this switching game is the following system of partial differential equations (PDEs in short) with a bilateral interconnected obstacles: $\forall i \in \Gamma$, $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} \min\{v^i(t,x) - L^i(\vec{v})(t,x); \max\left[v^i(t,x) - U^i(\vec{v})(t,x);\ -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x)\right]\} = 0; \\ \\ v^i(T,x) = h^i(x). \end{cases} \tag{1.35}$$

where: a) $\vec{v} = (v^i)_{i\in\Gamma}$; b) $L^i(\vec{v})(t,x) := v^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x), U^i(\vec{v})(t,x) := v^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$; c) $\mathcal{L}^X$ is the infinitesimal generator of X.

This work is originated by an article by N.Yamada [68] where the author deals with the system of PDEs (1.35) in the case when the switching costs are constant and for bounded domains $\bar{\Omega}$. By penalization method, the author proved existence and uniqueness of the solution of (1.35) in a weak sense (actually in a Sobolev space). Then he gives an interpretation of the solution of this system as a value function of the zero-sum switching game described previously. A saddle-point of the game is also given. However neither this interpretation nor the existence of the saddle-point are clear because the question of admissiblity of the controls which are supposed to realize the saddle-point property is not addressed. In zero-sum switching games this issue of admissibility of those controls, defined implicitely through $(Y^i)_{i\in\Gamma}$, is crucial (see e.g. [33]). Note also that there is another paper by N.Yamada [67] where the solution of system (1.35) is considered in viscosity sense. Once more by penalization, he shows existence and uniqueness of the solution on bounded domains $\bar{\Omega}$.

Therefore the main objective of this paper is to show that:

i) the system of reflected BSDEs with interconnected obstacles (1.33) has a unique solution in the Markovian framework.
ii) the zero-sum switching game described above has a value in different settings.

iii) The system of PDEs (3.3) has a unique solution.

1.4.3 Main results

Actually under Assumptions 1.4.1, (H2), (H3) and (H5) we show that the following system has a unique solution in the class Π_g:

$$\begin{cases} \min\{v^i(t,x) - L^i(\vec{v})(t,x); \max\left[v^i(t,x) - U^i(\vec{v})(t,x);\right. \\ \qquad \left. -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x,(v^l(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x v^i(t,x)\right]\} = 0; \\ v^i(T,x) = h^i(x), \end{cases} \tag{1.36}$$

where for any $i \in \Gamma$, $L^i(\vec{v})(t,x) := v^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$ and $U^i(\vec{v})(t,x) := v^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$. This system generalizes system (1.35).

The proof is based on Perron's method and the construction of this solution (more or less the same as in [19]) proceeds as follows: a) we first introduce the processes $(Y^{i,m}, Z^{i,m}, K^{\pm,i,m})_{i\in\Gamma}$, $m \geq 1$, solution of the system of reflected BSDEs with interconnected lower barriers associated with
$\{f^i(r, X_r^{t,x}, \vec{y}, z^i) - m(y^i - y^{i+1} - \bar{g}_{i,i+1}(r, X_r^{t,x}))^+, h^i(X_T^{t,x}), \underline{g}_{i,i+1}(r, X_r^{t,x})\}_{i\in\Gamma}$ (see (3.57)). It is a decreasing penalization scheme. As the framework is Markovian then there exist deterministic functions continuous and of polynomial growth $(v^{i,m})_{i\in\Gamma}$ such that the following Feynman-Kac representation holds: For any $i \in \Gamma$, $m \geq 1$ and $s \in [t,T]$,

$$Y_s^{i,m} = v^{i,m}(s, X_s^{t,x}).$$

As for any $i \in \Gamma$, $m \geq 1$, $Y^{i,m} \geq Y^{i,m+1}$ then we have also $v^{i,m} \geq v^{i,m+1}$. Now if we define $v^i = \lim_m v^{i,m}$, then $(v^i)_{i\in\Gamma}$ is a subsolution of (3.3) and for any fixed m_0, $(v^{i,m_0})_{i\in\Gamma}$ is a supersolution of (3.3). Next it is enough to use Perron's method to show that (3.3) has a unique solution since comparison principle holds. Finally, by uniqueness this solution does not depend on m_0 and is $(v^i)_{i\in\Gamma}$. Additionally for any $i \in \Gamma$, v^i is of polynomial growth and continuous.

Next for $(t,x) \in [0,T] \times \mathbb{R}^k$, $i \in \Gamma$ and $s \in [t,T]$, let us set:

$$Y_s^{i,t,x} = v^i(s, X_s^{t,x}).$$

With the help of the previous result, mainly continuity of $(v^i)_{i\in\Gamma}$, we show the following theorem:

Theorem 1.4.2. *Assume that assumptions 1.4.1-(H2), (H3) and (H5) are fulfilled and that for any $i \in \Gamma$, f^i does not depend on z. Then for any $(t,x) \in [0,T] \times \mathbb{R}^k$, there exists adapted processes $K^{i,\pm,t,x}$*

and $Z^{i,t,x}$ valued respectively in $\mathbb{R}^+$ and $\mathbb{R}^d$ such that, in combination with $Y^{i,t,x}$, verify: For any $i \in \Gamma$,

i) $K^{i,\pm,t,x}$ are continuous non decreasing and $\mathbb{P}$-a.s. $\int_0^T |Z_s^{i,t,x}|^2 ds < \infty$;

ii) $\forall s \in [t,T]$,

$$\begin{cases} Y_s^i = h^i(X_T^{t,x}) + \int_s^T f^i(r, X_r^{t,x}, (Y_r^l)_{l\in\Gamma})dr - \int_s^T Z_r^{i,t,x} dB_r \\ \qquad\qquad +K_T^{i,+,t,x} - K_s^{i,+,t,x} - (K_T^{i,-,t,x} - K_s^{i,-,t,x}); \\ L_s^i((Y^l)_{l\in\Gamma}) \le Y_s^i \le U_s^i((Y^l)_{l\in\Gamma}); \\ \int_t^T (Y_s^i - L_s^i((Y^l)_{l\in\Gamma}))dK_s^{i,+} = 0 \text{ and } \int_0^T (Y_s^i - U_s^i((Y^l)_{l\in\Gamma}))dK_s^{i,-} = 0, \end{cases} \tag{1.37}$$

where for $s \in t \le T$, $L_s^i((Y^l)_{l\in\Gamma}) := Y_s^{i+1} - \underline{g}_{i,i+1}(s, X_s^{t,x})$ and $U_s^i((Y^l)_{l\in\Gamma}) := Y_s^{i+1} + \bar{g}_{i,i+1}(s, X_s^{t,x})$.

The existence of the solution of (1.37) is proved by the penalization method combined with the continuous solution of (1.36). Indeed, we introduce the following processes $(\bar{Y}^{i,m})_{m\ge 0} : \forall i \in \Gamma$,

$$\begin{cases} \overline{Y}^{i,m} \in \mathcal{S}^2, \overline{Z}^{i,m} \in \mathcal{H}^2, \overline{K}^{i,m,+} \in \mathcal{A}^2; \\ \overline{Y}_s^{i,m} = h^i(X_T^{t,x}) + \int_s^T \overline{f}^{i,m}(r, X_r^{t,x}, (\overline{Y}^{l,m})_{l\in\Gamma})dr - \int_s^T \overline{Z}_r^{i,m} dB_r + \overline{K}_T^{i,m,+} - \overline{K}_s^{i,m,+}, s \le T; \\ \overline{Y}_s^{i,m} \ge L^i((\bar{Y}_s^{l,m})_{l\in\Gamma}), s \le T; \\ \int_0^T (\overline{Y}_s^{i,m} - L^i((\bar{Y}_s^{l,m})_{l\in\Gamma}))d\overline{K}_s^{i,m,+} = 0, \end{cases} \tag{1.38}$$

where $\overline{f}^{i,m}(s, X_s^{t,x}, \vec{y}) = f^i(s, X_s^{t,x}, \vec{y}) - m(y^i - [y^{i+1} + \overline{g}_{i,i+1}(s, X_s^{t,x})])^+$.

Then we have: For any $i \in \Gamma$ and $m \ge 0$, the processes $\bar{Y}^{i,m,t,x}$ have the following representation (see e.g. A4 in [18] for more details): For any $s \in [t,T]$,

$$\begin{aligned} \bar{Y}_s^{i,m,t,x} = \operatorname*{ess\,sup}_{\sigma\ge s} \operatorname*{ess\,inf}_{\tau\ge s} \mathbb{E}[h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X_r^{t,x}, (\bar{Y}_r^{l,m,t,x})_{l\in\Gamma})dr \\ + L_\sigma^i((\bar{Y}^{l,m,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + \{U_\tau^i((\bar{Y}^{l,m})_{l\in\Gamma}) \vee \bar{Y}_\tau^{i,m,t,x}\}1_{(\tau\le\sigma,\tau<T)}|\mathcal{F}_s]. \end{aligned} \tag{1.39}$$

Moreover there exist deterministic continuous functions $(\bar{v}^{i,m})_{i\in\Gamma}$ such that for any $m \ge 0$, $i \in \Gamma$ and $s \in [t,T]$ we have:

$$\bar{Y}_s^{i,m,t,x} = v^{i,m}(s, X_s^{t,x}).$$

But for any $i \in \Gamma$, $v^{i,m} \searrow v^i$ which, by Dini's Theorem, implies that this convergence holds uniformly on compact subsets of $[0,T] \times \mathbb{R}^k$. Take now the limit w.r.t. m in (1.39) and use the facts that $X^{t,x}$ has moments of any order and that $v^{i,m}$ and v^i have uniform polynomial growth to obtain that:

$$
\begin{aligned}
Y_s^{i,t,x} = \operatorname*{ess\,sup}_{\sigma \geq s} \operatorname*{ess\,inf}_{\tau \geq s} \mathbb{E}[h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X_r^{t,x}, (Y_r^{l,t,x})_{l\in\Gamma})dr \\
+ L_\sigma^i((Y^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + U_\tau^i((Y^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s].
\end{aligned}
\tag{1.40}
$$

But we have also the following inequalities: For any $s \in [t, T]$ and $i \in \Gamma$,

$$
U_s^i((Y^l)_{l\in\Gamma}) \geq Y_s^i \geq L_s^i((Y^l)_{l\in\Gamma}).
$$

On the other hand by Assumption (H3)-a),

$$
U_s^i((Y^{l,t,x})_{l\in\Gamma}) - L_s^i((Y^{l,t,x})_{l\in\Gamma}) = \bar{g}_{i,i+1}(s, X_s^{t,x}) + \underline{g}_{i,i+1}(s, X_s^{t,x}) > 0,
$$

which means that the obstacles $U^i((Y^{l,t,x})_{l\in\Gamma})$ and $L^i((Y^{l,t,x})_{l\in\Gamma})$, for any $i \in \Gamma$, are completely separated. Therefore by Theorem 3.7 in [27], there exist progressively measurable processes $\underline{Y}^{i,t,x}$, $K^{i,\pm,t,x}$ and $Z^{i,t,x}$ valued respectively in $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^d$ such that:

i) $\underline{Y}^{i,t,x} \in \mathcal{S}^2([t, T])$, $K^{i,\pm,t,x}$ are continuous non decreasing and $K_t^{i,\pm,t,x} = 0$; $\mathbb{P}$-a.s. $\int_t^T |Z_s^{i,t,x}|^2 ds < \infty$;

ii) The processes $(\underline{Y}^{i,t,x}, K^{i,\pm,t,x}, Z^{i,t,x})$ verify: $\forall s \in [t, T]$,

$$
\begin{cases}
\underline{Y}_s^{i,t,x} = h^i(X_T^{t,x}) + \int_s^T f^i(r, X_r^{t,x}, (Y_r^{l,t,x})_{l\in\Gamma})dr - \int_s^T Z_r^{i,t,x} dB_r \\
\qquad\qquad + K_T^{i,+,t,x} - K_s^{i,+,t,x} - (K_T^{i,-,t,x} - K_s^{i,-,t,x}); \\
L_s^i((Y^{l,t,x})_{l\in\Gamma}) \leq \underline{Y}_s^{i,t,x} \leq U_s^i((Y^{l,t,x})_{l\in\Gamma}); \\
\int_t^T (\underline{Y}_s^{i,t,x} - L_s^i((Y^{l,t,x})_{l\in\Gamma}))dK_s^{i,+,t,x} = 0 \text{ and } \int_0^T (\underline{Y}_s^{i,t,x} - U_s^i((Y^{l,t,x})_{l\in\Gamma}))dK_s^{i,-,t,x} = 0.
\end{cases}
\tag{1.41}
$$

Moreover $\underline{Y}^{i,t,x}$ has the following representation: $\forall s \in [t, T]$,

$$
\begin{aligned}
\underline{Y}_s^{i,t,x} = \operatorname*{ess\,sup}_{\sigma \geq s} \operatorname*{ess\,inf}_{\tau \geq s} \mathbb{E}[h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X_r^{t,x}, (Y_r^l)_{l\in\Gamma})dr \\
+ L_\sigma^i((Y^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + U_\tau^i((Y^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s].
\end{aligned}
\tag{1.42}
$$

Thus for any $s \in [t, T]$, $\underline{Y}^{i,t,x} = Y^{i,t,x}$ and then by (1.41), $(Y^{i,t,x}, K^{i,\pm,t,x}, Z^{i,t,x})$ verifies (1.37) for fixed i. Finally as i is arbitrary then $(Y^{i,t,x}, K^{i,\pm,t,x}, Z^{i,t,x})_{i\in\Gamma}$ is a solution for the system of reflected BSDEs with double obstacles (1.37). The proof of existence is then stated.

In the case when f^i, $i \in \Gamma$, do not depend on $\vec{y}$, the link with a specific zero-sum switching game is the following:

Theorem 1.4.3. *Assume Assumption 1.4.1-(H1)-(H3). Then for any $i = 1, ..., m$, then as previously one can show that for any (t, x) and $s \in [t, T]$,*

$$Y_s^{i,t,x} = \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^{(1)}} \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^{(1)}} J_i^{t,x}(\theta(u,v))_s = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^{(1)}} \operatorname*{ess\,inf}_{v\in\mathcal{B}_i^{(1)}} J_i^{t,x}(\theta(u,v))_s,$$

where

$$J_i^{t,x}(\theta(u,v))_s := \mathbb{E}\{h^{\theta(u,v)_T}(X_T^{t,x}) + \int_t^T f^{\theta(u,v)_r}(r, X_r^{t,x})dr - C_\infty^{\theta(u,v)}|\mathcal{F}_s\},$$

and $\mathcal{A}_t^{(1)}$ (resp. $\mathcal{B}_t^{(1)}$) is the set of admissible integrable controls which start from i at t, and finally $\theta(u, v)$ is the coupling of (u, v). □

This theorem tells us also that the solution of (1.37) is unique when f^i, $i \in \Gamma$, do not depend on $\vec{y}$. Next to show that the solution is unique in the general framework of Theorem 1.4.2 it is enough to consider the mapping Φ from $\mathcal{H}^{2,p}$ into itself by $\Phi(\vec{\phi}) := (Y^{\phi,i})_{i\in\Gamma}$. Then we consider two different solutions $(Y^{\phi,i})_{i\in\Gamma}$ and $(Y^{\psi,i})_{i\in\Gamma}$ of the systems of DRBSDEs (the existence of the solution is already proved). When calculating the difference between $(Y^{\phi,i})_{i\in\Gamma}$ and $(Y^{\psi,i})_{i\in\Gamma}$, we relate to the relations of Theorem 1.4.3 to get rid of the switching cost. Next using standard Itô's calculus we prove that Φ is a contraction mapping from $\mathcal{H}^{2,p}$ into itself, then the solution of the DRBSDEs (1.37) is unique. □

1.5 Mean-field Doubly Reflected backward stochastic differential equations

Chapter 4 is a preprint joint work with Chen and Hamadène (ref.[12]). In this work we investigate into a Doubly reflected BSDE of Mean-field type (MF-DRBSDE in short). In two different frameworks, we show the existence and uniqueness of the MF-DRBSDE where the two barriers are interconnected to the solution .

1.5.1 Overview of Mean-field theories

Mean-field games and MF-BSDEs

Motivated by classical mean-field approaches in Statistical Mechanics and Physics, in particular the study of systems composed of a very large number of particles, Lasry and Lions [45] introduced the so-called mean-field model. Later the mean-field game has attracted a significant attentions in the last decades, in particular motivated by the linear McKean-Vlasov PDE, Buckdahn et al. [7, 6, 5] introduced a new class of BSDEs of Mean-field type with the driver $f := f(\omega', \omega, t, y', z', y, z) : \Omega^{1+1} \times [0, T] \times \mathbb{R}^{1+1} \times \mathbb{R}^{d+d} \to \mathbb{R}$ satisfying the following conditions:

Assumption 1.5.1.

1. *For any* $t \in [0,T], y_1, y_2, y'_1, y'_2 \in \mathbb{R}, z_1, z_2, z'_1, z'_2 \in \mathbb{R}^d$, *there exists a constant* C *such that*

$$|f(t,y'_1,z'_1,y_1,z_1) - f(t,y'_2,z'_2,y_2,z_2)| \leq C(|y'_1 - y'_2| + |z'_1 - z'_2| + |y_1 - y_2| + |z_1 - z_2|)$$

2. $f(.,0,0,0,0) \in \mathcal{H}^{2,1}_{\mathcal{F}\otimes\mathcal{F}_t}(0,T;\mathbb{R})$

Theorem 1.5.2. *[Buckdahn et al. [6], [7]] Under Assumptions 1.5.1 the following MF-BSDE has a unique adapted solution: for any* $t \in [0,T]$,

$$\begin{cases} \xi \in L^2(\Omega,\mathcal{F}_t,P), Y \in \mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R}), Z \in \mathcal{H}^{2,d}_{\mathbb{F}}(0,T;\mathbb{R}^d); \\ Y_t = \xi + \int_t^T \mathbb{E}'[f(s,Y'_s,Z'_s,Y_s,Z_s)]ds - \int_t^T Z_s dB_s \end{cases} \tag{1.43}$$

where $\mathbb{E}'$ *is an operator defined by* $\mathbb{E}'(\gamma(.,\omega)) := \int_\Omega \gamma(\omega',\omega)P(d\omega'), \forall \gamma \in L^1(\Omega^{1+1},\mathcal{F}\otimes\mathcal{F},P\otimes P)$.

They also provided the corresponding comparison result and the converse comparison result for MF-BSDE, as well as the research to McKean-Vlasov PDE and related Dynamic Programming Principle (DPP in short).

In this context, Li [50] introduced a class of MF-Reflected BSDEs which makes the connection between the results of classical RBSDEs (e.g. [23]) and those of MF-FBSDEs (e.g. [7, 6]). Later after that, Djehiche, Elie and Hamadène [17] deepened the MF-RFBSDEs results by adding the dependence on the distribution of the Y-component of the solution in the barrier. The motivation comes from insurance problems. So let us consider the following system of reflected MF-BSDEs:

Definition 1.5.3. *[Djehiche et al. [17]]*
The triple of progressively measurable processes $(Y_t, Z_t, K_t)_{t\in[0,T]}$ *is called a solution of the MF-reflected BSDE associated with* (f,ξ,h) *if:*
(1) When $p > 1$,

$$\begin{cases} Y \in \mathcal{S}^p, Z \in \mathcal{H}^{p,d} \text{ and } K \in \mathcal{S}^p_i; \\ Y_t = \xi + \int_t^T f(s,Y_s,\mathbb{E}[Y_s])ds + K_T - K_t - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T; \\ Y_t \geq h(t,Y_t,\mathbb{E}[Y_t]), \quad \forall t \in [0,T]; \\ \int_0^T (Y_t - h(t,Y_t,\mathbb{E}[Y_t]))dK_t = 0. \end{cases} \tag{1.44}$$

(2) When $p = 1$,

$$\begin{cases} Y \in \mathcal{D}, Z \in \cup_{q\in(0,1)}\mathcal{M}^q \text{ and } K \in \mathcal{S}_i^1; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T - K_t - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ Y_t \ge h(t, Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\ \int_0^T (Y_t - h(t, Y_t, \mathbb{E}[Y_t]))dK_t = 0, \end{cases} \tag{1.45}$$

where $\mathcal{D}$ is the class of adapted continuous of class [D] processes and $\mathcal{M}^q$ is the set of $\mathcal{P}$-measurable processes $(z_t)_{t\le T}$ such that $\mathbb{E}[(\int_0^T |z_s|^2 ds)^{q/2}] < \infty$.

Next let us recall the following necessary conditions:

Assumption 1.5.4. *[Djehiche et al. [17]]*
The coefficients f, h and ξ satisfy:

1. *$f := f(t, y, z) : [0, T] \times \mathbb{R}^{1+1} \to \mathbb{R}$ such that $(f(t, 0, 0))_{t\in[0,T]}$ is $\mathcal{P}$-measurable and belongs to $\mathcal{H}^{p,1}$, in addition f is Lipschitz continuous w.r.t (y, y') uniformly in (t, ω), i.e. there exists a positive constant C such that*

$$\forall t \in [0, T], y_1, y_1', y_2, y_2' \in \mathbb{R}, |f(t, y_1, y_1') - f(t, y_2, y_2')| \le C(|y_1 - y_2| + |y_1' - y_2'|);$$

2. *The mapping $h := h(y, y') : \mathbb{R}^{1+1} \to \mathbb{R}$ is Lipschitz continuous w.r.t. (y, y'), i.e. there exists two positive constants γ_1 and γ_2 such that*

$$\forall x, y, x', y' \in \mathbb{R}, |h(x, x') - h(y, y')| \le \gamma_1 |x - y| + \gamma_2 |x' - y'|;$$

3. *ξ is $\mathcal{F}_T-$measurable and $\mathbb{R}-$valued random variable such that $\mathbb{E}[\xi^p] < \infty$ and $\xi \ge h(\xi, \mathbb{E}[\xi])$.*

Theorem 1.5.5. *[Djehiche et al. [17]] Suppose that Assumptions 1.5.4 is fulfilled,*

1. *for $p > 1$, the mean-field reflected BSDE (1.44) has a unique solution if the following condition holds true:*

$$(\gamma_1 + \gamma_2)^{\frac{p-1}{p}} \left[\left(\frac{p}{p-1}\right)^p \gamma_1 + \gamma_2 \right]^{\frac{1}{p}} < 1;$$

2. *for $p = 1$, the mean-field reflected BSDEs (1.45) has a unique solution if $\gamma_1 + \gamma_2 < 1$.*

Based on the results of [17], we investigate into the mean-field doubly reflected BSDEs.

1.5.2 Motivation and problem setting

Motivation

In this work we are concerned with the problem of existence and uniqueness of a solution of the doubly reflected BSDE associated with the quadruple (f, ξ, h, g):

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0,T]; \\ \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = 0, \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{1.46}$$

Those BSDEs are of mean-field type because the generator f and the barriers depend on the law of Y_t through its expectation.

There have been several papers on mean-field BSDEs including ([7, 6, 4, 17, 50]). Those equations are connected with several motivations of which the representation of a utility of an agent inside an economy ([7, 6, 50]), the assesment of the risk of a financial position ([4]), the representation of set of portfolios in life-insurance ([17]), etc.

As previously mentioned, in [17], the authors consider the case of one reflecting barrier of (1.46). They prove existence and uniqueness of a solution via the fixed point method and the penalization one as well. Those methods do not allow for the same framework. For example, the fixed point method does not allow generators which depend on z while the penalization does at the price of some additional regularity properties which are not required by the use of the first method.

Notations

Let T be a fixed positive constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with $B = (B_t)_{t\in[0,T]}$ a d-dimensional Brownian motion whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \le t\})_{0\le t\le T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions. On the other hand, let $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of the $\mathbb{F}$-progressively measurable sets.

For $p \ge 1$ and $0 \le s_0 < t_0 \le T$, we define the following spaces:

- $L^p := \{\xi : \mathcal{F}_T -$ measurable radom variable s.t. $\mathbb{E}[|\xi|^p] < \infty\}$;
- $\mathcal{H}_{loc}^m := \{(z_t)_{t\in[0,T]} : \mathcal{P} -$ measurable process and $\mathbb{R}^m -$ valued s.t. $\mathbb{P} - a.s. \int_0^T |z_s(\omega)|^2 ds < \infty\}$;
- $\mathcal{S}^p := \{(y_t)_{t\in[0,T]} :$ continuous and $\mathcal{P}$-measurable process s.t. $\mathbb{E}[\sup_{t\in[0,T]} |y_t|^p] < \infty\}$;

- $\mathcal{A} := \{(k_t)_{t\in[0,T]} :$ continuous, $\mathcal{P}$-measurable and non-decreasing process s.t. $k_0 = 0\}$;
- $\mathcal{T}_t := \{\tau, \mathbb{F} -$ stopping time s.t. $\mathbb{P} - a.s.\tau \geq t\}$;
- $\mathcal{D} := \{(\phi)_{t\in[0,T]} : \mathbb{F} -$ adapted, $\mathbb{R} -$ valued continuous process s.t. $\|\phi\|_1 = \sup_{\tau\in\mathcal{T}_0} \mathbb{E}[|y_\tau|] < \infty\}$. Note that the normed space $(\mathcal{D}, \|.\|_1)$ is complete.

We introduce the following assumptions on (f, ξ, h, g):

Assumption 1.5.6.
(i) The coefficients f, h, g and ξ satisfy:

(a) the process $(f(t,0,0))_{t\leq T}$ is $\mathcal{P}$- measurable and such that $\int_0^T |f(t,0,0)|dt \in L^p(d\mathbb{P})$;

(b) f is Lipschitz w.r.t (y,y') uniformly in(t,ω), i.e., there exists a positive constant C_f such that $\mathbb{P}$- a.s. for all $t \in [0,T]$, y_1, y_1', y_2 and y_2' elements of $\mathbb{R}$,

$$|f(t,\omega,y_1,y_1') - f(t,\omega,y_2,y_2')| \leq C_f(|y_1 - y_1'| + |y_2 - y_2'|). \tag{1.47}$$

(ii) h and g are mappings from $\mathbb{R}^2$ into $\mathbb{R}$ which satisfy:

(a) h and g are Lipschitz w.r.t. (y,y') i.e., there exist pairs of positive constants (γ_1,γ_2), (β_1,β_2) such that for any x, x', y and $y' \in \mathbb{R}$,

$$\begin{aligned} |h(x,x') - h(y,y')| &\leq \gamma_1|x-y| + \gamma_2|x'-y'|, \\ |g(x,x') - g(y,y')| &\leq \beta_1|x-y| + \beta_2|x'-y'|. \end{aligned} \tag{1.48}$$

(b) $h(x,x') < g(x,x')$, for any $x, x' \in \mathbb{R}$;

(iii) ξ is an $\mathcal{F}_T$- measurable, $\mathbb{R}$-valued r.v., $\mathbb{E}[\xi^p] < \infty$ and satisfies $h(\xi,\mathbb{E}[\xi]) \leq \xi \leq g(\xi,\mathbb{E}[\xi])$.

Definition 1.5.7. *We say that the quaternary of $\mathcal{P}$-measurable processes $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ is a solution of the mean-field reflected BSDE associated with (f,ξ,h,g) if :*

Case: $p > 1$

$$\begin{cases} Y \in \mathcal{S}^p, \quad Z \in \mathcal{H}^d_{loc} \quad and \quad K^+, K^- \in \mathcal{A}; \\ Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T; \\ h(Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0,T]; \\ \displaystyle\int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = 0, \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{1.49}$$

Case: $p = 1$,

$$\begin{cases} Y \in \mathcal{D}, \quad Z \in \mathcal{H}^d_{loc} \quad and \quad K^+, K^- \in \mathcal{A}; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\ \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = 0, \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{1.50}$$

1.5.3 Main results of this paper

This work is devoted to studying the solvability of MF-DRBSDE for the case $p > 1$ (1.49) and for the case $p = 1$ (1.50). By means of the associated zero-sum stochastic switching games and the Snell envelope argument, we prove the existence of the local fixed point Y over $t \in [T-\delta, T]$ where δ is a parameter independent to the terminal condition ξ. Next by concatenating of all small intervals $[T - i\delta, T - (i-1)\delta], \forall i = 1, ..., \frac{T}{\delta}$, we then obtain the global fixed point Y on $[0, T]$. However some supplementary conditions on Lipschitz constants $\gamma_1, \gamma_2, \beta_1, \beta_2$ are required.

Theorem 1.5.8. *Assume that Assumption 1.5.6 holds for some $p > 1$. If γ_1 and γ_2 satisfy*

$$(\gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{\frac{p-1}{p}} \left[\left(\frac{p}{p-1} \right)^p (\gamma_1 + \beta_1) + (\gamma_2 + \beta_2) \right]^{\frac{1}{p}} < 1 \tag{1.51}$$

then the mean-field doubly reflected BSDE (4.2) *has a unique solution (Y, Z, K^+, K^-).*

Theorem 1.5.9. *Let f, h, g and ξ satisfy Assumption 1.5.6 for $p = 1$ and suppose that*

$$\gamma_1 + \gamma_2 + \beta_1 + \beta_2 < 1. \tag{1.52}$$

Then, there exists $\delta > 0$ only depending on $C_f, \gamma_1, \gamma_1, \beta_1$ and β_2 such that (1.50) *has a unique solution $(Y, Z, K^+, K^-) \in \mathcal{D} \times \mathcal{H}^d_{loc} \times \mathcal{A} \times \mathcal{A}$.* □

CHAPTER 2

PAPER 1: SYSTEM OF REFLECTED BSDES WITH INTERCONNECTED BILATERAL OBSTACLES: EXISTENCE, UNIQUENESS AND APPLICATIONS

This chapter is a published joint work with Hamadène (ref.[35]).

2.1 Introduction

This paper is related to the study of systems of reflected backward stochastic differential equations (BSDEs in short) with interconnected bilateral obstacles. A solution for such a system is a family of adapted processes $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ such that: For any $(i,j) \in \Gamma$ and $t \le T$,

$$\begin{cases} Y_t^{ij} = \xi^{ij} + \int_t^T f^{ij}\left(s, \omega, (Y_s^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij}\right) ds - \int_t^T Z_s^{ij} dB_s + \int_t^T (dK_s^{ij,+} - dK_s^{ij,-})\,; \\ L_t^{ij} \le Y_t^{ij} \le U_t^{ij}\,; \\ \int_0^T (Y_t^{ij} - L_t^{ij}) dK_t^{ij,+} = 0 \quad \text{and} \quad \int_0^T (U_t^{ij} - Y_t^{ij}) dK_t^{ij,-} = 0, \end{cases} \tag{2.1}$$

where:

a) $\Gamma := \Gamma_1 \times \Gamma_2 = \{1, ..., m_1\} \times \{1, ..., m_2\}$;
b) $L_t^{ij} := \max_{k\in\Gamma^1-\{i\}} \{Y_t^{kj} - \underline{g}_{ik}(t)\}$ and $U_t^{ij} := \min_{l\in\Gamma^2-\{j\}} \{Y_t^{il} + \overline{g}_{jl}(t)\}$;
c) f^{ij}, ξ^{ij}, $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are given data of the problem which are described precisely later;
d) $K^{ij,\pm}$ are non-decreasing processes such that $K_0^{ij,\pm} = 0$.

This system introduced first in [41] is related to the zero-sum stochastic switching game, as shown later in some papers including [19, 33]. On the other hand, note that the above BSDEs have two reflecting barriers which depend on the solution $(Y^{ij})_{(i,j)\in\Gamma}$.

A stochastic optimal switching control problem of a system (which can be a portfolio in market, a power plant, etc.) is a discrete stochastic optimal control where a strategy σ is pair of sequences $((\tau_n)_{n\geq 0}, (\zeta_n)_{n\geq 0})$ such that for any $n \geq 0$, τ_n is a stopping time such that $\tau_n \leq \tau_{n+1}$ and ζ_n are random variables valued in the set of modes under which the system is run. Roughly speaking at time τ_n the controller decides to switch the system from its current mode to the new one denoted by ζ_n. The switching actions are not free and generate expenditures. When a strategy σ is implemented, it induces a payoff which is equal to $J(\sigma)$ and then the problem is to find a strategy σ^* which realizes $\sup_\sigma J(\sigma)$. This problem is related to systems of reflected backward stochastic differential equations (RBSDEs in short) with interconnected one lower obstacles to which reduces (2.1) in the case when $\overline{g}_{jl} = +\infty$. There are several papers on this topic including [11, 10, 20, 34, 29, 37, 40, 64, 66, 69, 43] (see also the references therein) in connection with energy, finance, etc.

Next, one has a zero-sum switching game if there are two decision makers π_1 and π_2 which intervene on the system by both choosing its joint working mode $(i, j) \in \Gamma$ (π_1 and π_2 choose $i \in \Gamma^1$ and $j \in \Gamma^2$ respectively). The interests of the decision makers are antagonistic, that is to say, when π_1 (resp. π_2) implements the strategy σ_1 (resp. σ_2) there is in-between a payoff $J(\sigma_1, \sigma_2)$ which is a profit (resp. cost) for π_1 (resp. π_2). The zero-sum switching game (especially issues of existence of the value, a saddle point, etc.) is connected with the solutions of systems of reflected BSDEs of types (2.1) (see e.g. [19, 33]). This is the main motivation to study this system (2.1).

There are only very few papers which deal with the problem of existence of a solution for system (2.1). The question of uniqueness is even less studied. According to our best knowledge, system (2.1) is studied in two papers only which are [41] and [19]. In [41], the authors have shown existence of a solution for this system (2.1) when the switching costs $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are constant. The question of uniqueness is not addressed and remained open. On the other hand, in [19], Djehiche et al. have considered system (2.1) in the markovian framework of randomness. By using tools which combine results on partial differential equations (PDEs for short) with results on BSDEs, the authors have shown existence and uniqueness of the solution of system (2.1). The switching costs $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are not constant.

Therefore the main objective of this work is to complete the existing literature on the problem of existence and uniqueness of a solution for the system of RBSDEs with bilateral interconnected obstacles (2.1) and to provide an application in the field of PDEs. Actually the novelties of this paper are the following:

i) We show that system (2.1) has a solution in the case when the processes $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are of Itô type and under the monotonicity assumption of the functions f^{ij} (see [H5] below) ;

ii) We show that system (2.1) has a unique solution in the case when the processes $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are Itô processes and the functions f^{ij} do no depend on z. We do not require the monotonicity assumption on these latter functions ;

iii) When randomness is Markovian and comes from a diffusion process $X^{t,x}$, we show that the Feynman-Kac representation formula holds for $(Y^{ij})_{(i,j)\in\Gamma}$, the first component of the solution of system (2.1), i.e., there exist deterministic continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ such that for any $(i,j)\in\Gamma$, $s\in[t,T]$, $Y_s^{ij;t,x}=v^{ij}(s,X_s^{t,x})$. Moreover the functions $(v^{ij})_{(i,j)\in\Gamma}$ are the unique solution of the following system of PDEs with bilateral interconnected obstacles: $\forall (i,j)\in\Gamma$,

$$\begin{cases} \min\{v^{ij}(t,x)-\max\limits_{k\in\Gamma^1-\{i\}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)];\max[v^{ij}(t,x)-\min\limits_{l\in\Gamma^2-\{j\}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big]\}=0; \\ v^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{2.2}$$

The monotonicity assumption of the functions $(f^{ij})_{(i,j)\in\Gamma}$ is no longer required as in [18, 19, 42, 68], etc. This result on PDEs improves also substantially the existing literature in this domain (see the previous references). System (2.2) can be seen as the Hamilton-Jacobi-Bellman-Isaacs system associated with the zero-sum switching game when utilities are implicit or depend on the values.

The chapter is organized as follows: In Section 2.2, we give some statements and assumptions. In Section 2.3 we introduce and analyse, under the monotonicity assumption on the functions $(f^{ij})_{(i,j)\in\Gamma}$, the approximating schemes of (2.1) obtained by penalization. We show that the penalization terms are bounded in appropriate space. We then show that the penalization schemes converge and their limits provide solutions for (2.1). In Section 2.4, by the zero-sum stochastic representation, we show that, the system (2.1) has a unique solution when $(f^{ij})_{(i,j)\in\Gamma}$ does not depend on z. Finally in Section 2.5, we deal with application of the result of Section 2.4 in the field of PDEs. We first show that the processes $(Y^{ij})_{(i,j)\in\Gamma}$ enjoy the Feynman-Kac representation through deterministic continuous with polynomial growth functions $(v^{ij})_{(i,j)\in\Gamma}$. Moreover the functions $(v^{ij})_{(i,j)\in\Gamma}$ are the unique solution in viscosity of system of PDEs with obstacles (2.2) of min-max type. They are also the unique solution of the dual system to (2.2) which is of max-min type.

2.2 Statements, assumptions and preliminaries

Let $T>0$ be a fixed real constant. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space which carries a d-dimensional Brownian motion $B=(B_t)_{t\in[0,T]}$ whose natural filtration is $\mathcal{F}_t^0:=\sigma\{B_s, s\le t\}_{0\le t\le T}$. We denote by $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions, i.e., it is complete and right continuous. On the other hand, we define $\mathcal{P}$ as the σ-algebra on $[0,T]\times\Omega$ of the $\mathbb{F}$-progressively measurable sets. Next, we denote by:

- $\mathcal{S}^2$: the set of $\mathcal{P}$-measurable continuous processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\sup_{t\in[0,T]} |\phi_t|^2) < \infty$;
- $\mathcal{A}^2$: the subset of $\mathcal{S}^2$ of non-decreasing processes $K = (K_t)_{t\leq T}$ such that $K_0 = 0$;
- $\mathcal{H}^{2,k}(k \geq 1)$: the set of $\mathcal{P}$-measurable, $\mathbb{R}^k$-valued processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\int_0^T |\phi_t|_k^2 dt) < \infty$.

To proceed, let Γ^1, Γ^2 be the finite sets of the whole switching modes available for the controllers or players. As mentioned previously $\Gamma := \Gamma^1 \times \Gamma^2$ and denote by Λ its cardinal, i.e., $\Lambda := |\Gamma| = |\Gamma^1| \times |\Gamma^2|$. On the other hand for $(i,j) \in \Gamma^1 \times \Gamma^2$, we define $(\Gamma^1)^{-i} := \Gamma^1 - \{i\}$ and $(\Gamma^2)^{-j} := \Gamma^2 - \{j\}$.

Next let us denote by $\vec{y}$ the generic element $(y^{ij})_{(i,j)\in\Gamma}$ of $\mathbb{R}^\Lambda$ and let us introduce the following items: For any $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$,

i) f^{ij}: $(t,\omega,\vec{y},z) \in [0,T] \times \Omega \times \mathbb{R}^\Lambda \times \mathbb{R}^d \mapsto f^{ij}(t,\omega,\vec{y},z) \in \mathbb{R}$;

ii) $\underline{g}_{ik}$: $(t,\omega) \in [0,T] \times \Omega \mapsto \underline{g}_{ik}(t,\omega) \in \mathbb{R}^+$;

iii) $\overline{g}_{jl}$: $(t,\omega) \in [0,T] \times \Omega \mapsto \overline{g}_{jl}(t,\omega) \in \mathbb{R}^+$.

iv) ξ^{ij} is a r.v. valued in $\mathbb{R}$ and $\mathcal{F}_T$-measurable.

Finally let us introduce the following assumptions on $f^{ij}, \overline{g}_{ik}$ and $\underline{g}_{jl}$ for $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$:

[H1] For any $(i,j) \in \Gamma^1 \times \Gamma^2$,

a) There exists a positive constant C and a non negative $\mathcal{P}$-measurable process $(\eta_t)_{t\leq T}$ which satisfies $\mathbb{E}[\sup_{s\leq T} |\eta_s|^2] < \infty$ and such that: $\mathbb{P}$-a.s, $\forall(\vec{y},z) \in \mathbb{R}^{\Lambda+d}$, $t \in [0,T]$,

$$|f^{ij}(t,\vec{y},z)| \leq C(1 + \eta_t + |\vec{y}|),$$

where $|\vec{y}|$ refers to the standard Euclidean norm of $\vec{y}$ in $\mathbb{R}^\Lambda$ (the same for $|z|$ below). Note that this implies that $\mathbb{E}[\int_0^T |f^{ij}(t,0,0)|^2 dt] < \infty$;

b) f^{ij} is Lipschitz continuous with respect to (w.r.t for short) $(\overrightarrow{y}, z)$ uniformly in (t,ω), i.e. $\mathbb{P}$-a.s., for any $t \in [0,T]$, $(\overrightarrow{y_1}, z_1)$ and $(\overrightarrow{y_2}, z_2)$ elements of $\mathbb{R}^{\Lambda+d}$, we have

$$|f^{ij}(t, \overrightarrow{y_1}, z_1) - f^{ij}(t, \overrightarrow{y_2}, z_2)| \leq C(|\overrightarrow{y_1} - \overrightarrow{y_2}| + |z_1 - z_2|)$$

where C is a fixed constant.

[H2] For any $(i,j) \in \Gamma$,

a) $\mathbb{E}(|\xi^{ij}|^2) < \infty$;

b) ξ^{ij}, as the terminal condition at time T of system (2.1), satisfies the following consistency condition: $\mathbb{P}$-a.s.,

$$\max_{k\in(\Gamma^1)^{-i}} \left(\xi^{kj} - \underline{g}_{ik}(T)\right) \le \xi^{ij} \le \min_{l\in(\Gamma^2)^{-j}} \left(\xi^{jl} + \overline{g}_{jl}(T)\right).$$

[H3] a) For all $i_1, i_2 \in \Gamma^1$ (resp. $j_1, j_2 \in \Gamma^2$) and $t \in [0, T]$, the process $\underline{g}_{i_1 i_2}$(resp. $\overline{g}_{j_1 j_2}$),

(i) is non-negative and continuous;

(ii) For any $k \in \Gamma^1$ (resp. $\ell \in \Gamma^2$) such that $|\{i_1, i_2, k\}| = 3$ (resp. $|\{j_1, j_2, \ell\}| = 3$) it holds:

$$\mathbb{P} - a.s.,\ \forall t \le T,\ \underline{g}_{i_1 i_2}(t) < \underline{g}_{i_1 k}(t) + \underline{g}_{k i_2}(t) \ \left(\text{resp. } \overline{g}_{j_1 j_2}(t) < \overline{g}_{j_1 \ell}(t) + \overline{g}_{\ell j_2}(t)\right); \tag{2.3}$$

iii) By convention we set $\forall (i,j) \in \Gamma, \underline{g}_{ii} = 0$ and $\bar{g}_{jj} = 0$. Note that this convention implies the so-called non loop free property (see (2.39) and (2.40)).

[H4] For any $(i,j), (k,\ell) \in \Gamma$, $\underline{g}_{ik}$(resp. $\overline{g}_{j\ell}$) is an Itô process, i.e.,

$$\begin{cases} \underline{g}_{ik}(t) = \underline{g}_{ik}(0) + \int_0^t \underline{b}_{ik}(s)ds + \int_0^t \underline{\sigma}_{ik}(s)dB_s,\ t \le T, \\ \text{with } \underline{\sigma}_{ik} \in \mathcal{H}^{2,d} \text{ and } \underline{b}_{ik}, \mathcal{P}\text{-measurable and } \mathbb{E}[\sup_{s\le T} |\underline{b}_{ik}(s)|^2] < \infty. \end{cases}$$

$$\left(\text{resp.} \begin{cases} \overline{g}_{j\ell}(t) = \overline{g}_{j\ell}(0) + \int_0^t \overline{b}_{j\ell}(s)ds + \int_0^t \overline{\sigma}_{j\ell}(s)dB_s,\ t \le T, \\ \text{with } \overline{\sigma}_{j\ell} \in \mathcal{H}^{2,d} \text{ and } \overline{b}_{j\ell}, \mathcal{P}\text{-measurable and } \mathbb{E}[\sup_{s\le T} |\overline{b}_{j\ell}(s)|^2] < \infty. \end{cases}\right).$$

[H5] Monotonicity:

For any $(i,j) \in \Gamma$ and $(k,l) \in \Gamma^{-ij} := \Gamma - \{(i,j)\}$, the mapping $y^{kl} \mapsto f^{ij}(t, \overrightarrow{y}, z)$ is non-decreasing when the other components $(y^{pq})_{(p,q)\neq(k,l)}$ and z are fixed.

Definition 2.2.1. *A family $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is said to be a solution of the system of reflected BSDEs with doubly interconnected barriers associated with $((f^{ij})_{(i,j)\in\Gamma}, (\xi^{ij})_{(i,j)\in\Gamma}, (\underline{g}_{ik})_{i,k\in\Gamma^1}, (\overline{g}^{j,\ell})_{j,\ell\in\Gamma^2})$, if it satisfies the followings: $\forall (i,j) \in \Gamma$,*

$$\begin{cases} Y^{ij} \in \mathcal{S}^2, Z^{ij} \in \mathcal{H}^{2,d}, K^{ij,\pm} \in \mathcal{A}^2; \\ Y_t^{ij} = \xi^{ij} + \int_t^T f^{ij}(s, \omega, (Y_s^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij})ds - \\ \qquad \int_t^T Z_s^{ij} dB_s + K_T^{ij,+} - K_t^{ij,+} - (K_T^{ij,-} - K_t^{ij,-}),\ \forall t \le T; \\ L_t^{ij} \le Y_t^{ij} \le U_t^{ij},\ \forall t \in [0,T]; \\ \int_0^T (Y_t^{ij} - L_t^{ij})dK_t^{ij,+} = 0 \quad and \quad \int_0^T (U_t^{ij} - Y_t^{ij})dK_t^{ij,-} = 0, \end{cases} \tag{2.4}$$

where $L_t^{ij} := \max_{k\in(\Gamma^1)^{-i}}\{Y_t^{kj} - \underline{g}_{ik}(t)\}$ *and* $U_t^{ij} := \min_{l\in(\Gamma^2)^{-j}}\{Y_t^{il} + \overline{g}_{jl}(t)\}, \forall t \le T.$

2.3 Existence under the monotonicity condition [H5]

In this part we prove the existence of a solution for the system of reflected BSDEs (2.4) under Assumptions [H1]-[H5]. For this we first introduce penalization schemes which we analyse and show properties of the penalizing terms. Then by using the monotonicity assumption of the generator $f^{ij}(s,\vec{y},z)$, namely [H5], and comparison of the solutions we prove that the approximating schemes converge and their limits provide solutions of the system of reflected BSDEs with bilateral interconnected obstacles (2.4).

So let us consider the following sequence of BSDEs : $\forall m,n\in\mathbb{N}$, $(i,j)\in\Gamma$,

$$\begin{cases} Y^{ij,m,n}\in\mathcal{S}^2, Z^{ij,m,n}\in\mathcal{H}^{2,d};\\ Y_t^{ij,m,n} = \xi^{ij} + \int_t^T f^{ij,m,n}\left(s,(Y_s^{kl,m,n})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij,m,n}\right)ds - \int_t^T Z_s^{ij,m,n}dB_s,\ t\le T, \end{cases} \tag{2.5}$$

where

$$f^{ij,m,n}\left(t,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z\right) = f^{ij}(t,\vec{y},z) + n\left\{y_t^{ij} - \max_{k\in(\Gamma^1)^{-i}}\left[y_t^{kj} - \underline{g}_{ik}(t)\right]\right\}^- \\ -m\left\{y_t^{ij} - \min_{l\in(\Gamma^2)^{-j}}\left[y_t^{il} + \overline{g}_{jl}(t)\right]\right\}^+$$

($x^+ = x\vee 0$ and $x^- = (-x)\vee 0, \forall x\in\mathbb{R}$).

Since (2.5) is a standard BSDE without obstacles, thanks to the results by Pardoux-Peng [52], the solution exists and is unique. Moreover we have the following comparison result based on a paper by Hu-Peng [39] related to comparison of solutions of multi-dimensional BSDEs.

Proposition 2.3.1 ([18], pp.143). *For any* $(i,j)\in\Gamma$, f^{ij} *satisfies [H1] and [H5],* ξ^{ij} *satisfies [H2] and* $(\underline{g}_{ik})_{i,k\in\Gamma^1}, (\bar{g}_{jl})_{j,l\in\Gamma^2}$ *satisfy [H3]-a), then for* $m,n\ge 0$, *we have*

$$\mathbb{P}-a.s.\quad Y^{ij,m+1,n}\le Y^{ij,m,n}\le Y^{ij,m,n+1}. \tag{2.6}$$

Next we are interested in discussing the limit of $Y^{ij,m,n}$ in $\mathcal{S}^2$ when n goes to $+\infty$ for fixed m. Some similar results are already discussed in [37], [34], [18], [40], etc. Here we apply the same method as in Hamadène et al. [18] to prove the convergence of $Y^{ij,m,n}$ in $\mathcal{S}^2$ as $n\to\infty$ and then we have:

Lemma 2.3.2. *a) For any $(i,j) \in \Gamma^1 \times \Gamma^2$, the sequence $(Y^{ij,m,n}, Z^{ij,m,n})_{n\geq 0}$ converges, as n tends to infinity, to $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m})$ in $\mathcal{S}^2 \times \mathcal{H}^{2,d}$;*

b) For any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $m \geq 0$, let $\bar{K}^{ij,m,+}$ be the following limit in $\mathcal{S}^2$ (which exists, one can see [18] for more details):

$$\forall t \leq T,\ \overline{K}_t^{ij,m,+} := \lim_{n\to\infty} \int_0^t n \left\{ Y_s^{ij,m,n} - \max_{k\in(\Gamma^1)^{-i}} \left[Y_s^{kj,m,n} - \underline{g}_{ik}(s) \right] \right\}^- ds$$

Then the triples $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m}, \bar{K}^{ij,m,+})_{(i,j)\in\Gamma}$ is the unique solution of the following system of RBSDEs with lower interconnected obstacles: For any $(i,j) \in \Gamma$ and $t \leq T$,

$$\begin{cases} \overline{Y}^{ij,m} \in \mathcal{S}^2, \overline{Z}^{ij,m} \in \mathcal{H}^2, \overline{K}^{ij,m,+} \in \mathcal{A}^2; \\ \overline{Y}_t^{ij,m} = \xi^{ij} + \int_t^T \overline{f}^{ij,m}\left(s, (\overline{Y}_s^{kl,m})_{(k,l)\in\Gamma}, \overline{Z}_s^{ij,m}\right) ds - \int_t^T \overline{Z}_s^{ij,m} dB_s + \overline{K}_T^{ij,m,+} - \overline{K}_t^{ij,m,+}\ ; \\ \overline{Y}_t^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \left[\overline{Y}_t^{kj,m} - \underline{g}_{ik}(t) \right]\ ; \\ \int_0^T \left\{ \overline{Y}_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} [\overline{Y}_t^{kj,m} - \underline{g}_{ik}(t)] \right\} d\overline{K}_t^{ij,m,+} = 0 \end{cases} \tag{2.7}$$

where $\overline{f}^{ij,m}(s,(y^{kl})_{(k,l)\in\Gamma}, z) = f^{ij}\left(s,(y^{kl})_{(k,l)\in\Gamma}, z\right) - m\left(y^{ij} - \min_{l\in(\Gamma^2)^{-j}} \left[y^{il} + \overline{g}_{jl}(s) \right] \right)^+.$

c) For any $m \geq 0$ and $(i,j) \in \Gamma, \overline{Y}^{ij,m} \geq \overline{Y}^{ij,m+1}$.

Let us just point out that the function $(t,\omega,(y^{kl})_{(k,l)\in\Gamma}) \mapsto -m\left\{y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il} + \overline{g}_{jl}(t)]\right\}^+$ enjoys the same properties as f^{ij} w.r.t $\vec{y}$, hence $\overline{f}^{ij,m}$ keeps the same monotonicity properties as f^{ij} displayed in [H1] and [H5]. Therefore to prove that $(\overline{Y}^{ij,m}, \overline{Z}^{ij,m}, \overline{K}^{ij,m,+})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique solution of the RBSDEs (2.7) can be performed in the same way as in Hamadène and Zhang [37], we then omit the proof. □

Next, we introduce another equivalent approximating scheme defined as follows : for $m \geq 0$, let $(Y^{ij,m}, Z^{ij,m}, K^{ij,m,+})_{(i,j)\in\Gamma}$ be the unique solution of the following system of RBSDEs with lower interconnected obstacle: $\forall (i,j) \in \Gamma$,

$$\begin{cases} Y^{ij,m} \in \mathcal{S}^2, Z^{ij,m} \in \mathcal{H}^2, K^{ij,m,+} \in \mathcal{A}^2; \\ Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) ds - \int_t^T Z_s^{ij,m} dB_s + K_T^{ij,m,+} - K_t^{ij,m,+},\ t \leq T; \\ Y_t^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \left(Y_t^{kj,m} - \underline{g}_{ik}(t) \right),\ t \leq T; \\ \int_0^T \left[Y_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} (Y_t^{kj,m} - \underline{g}_{ik}(t)) \right] dK_t^{ij,m,+} = 0 \end{cases} \tag{2.8}$$

where $f^{ij,m}(t,\overrightarrow{y},z) := f^{ij}(t,\overrightarrow{y},z) - m\sum_{l\in(\Gamma^2)^{-j}}\left(y^{ij}-y^{il}-\overline{g}_{il}(t)\right)^+$.

To proceed we are going to analyse the properties of this scheme (2.8) and its relationship with system (2.7) as well.

First note that for any $(i,j)\in\Gamma$, the sequence $(f^{ij,m})_{m\geq 0}$ is non decreasing w.r.t. m, since for all $m\geq 0$,

$$f^{ij,m}(t,\vec{y},z) - f^{ij,m+1}(t,\vec{y},z) = \sum_{l\in(\Gamma^2)^{-j}}\left(y^{ij}-y^{il}-\overline{g}_{il}(.)\right)^+ \geq 0.$$

Therefore by applying comparison theorem of systems of reflected BSDEs (see [34]) we obtain

$$\forall m\geq 0, (i,j)\in\Gamma^1\times\Gamma^2,\quad Y^{ij,m}\geq Y^{ij,m+1} \tag{2.9}$$

i.e. $(Y^{ij,m})_{m\geq 0}$ is a non increasing sequence. Besides the following inequalities hold:

$$\overline{f}^{ij,|\Gamma^2|m} = f^{ij}(t,\overrightarrow{y},z) - |\Gamma^2|m\left\{y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il}+\overline{g}_{jl}(t)]\right\}^+ \leq f^{ij,m}\leq \overline{f}^{ij,m}$$

where $|\Gamma^2|$ is the cardinal of Γ^2. Therefore once more by the comparison result of solutions of systems we have

$$\forall m\geq 0, (i,j)\in\Gamma^1\times\Gamma^2,\ \overline{Y}^{ij,|\Gamma^2|m}\leq Y^{ij,m}\leq \overline{Y}^{ij,m}. \tag{2.10}$$

Consequently, as the sequences $(Y^{ij,m})_{m\geq 0}$ and $(\overline{Y}^{ij,m})_{m\geq 0}$ are decreasing then if one of them converges then is so the other one to the same limit.

Finally we have the following estimate of the penalization term in (2.8). This estimate plays a crucial role in the proof of existence of the solution of (2.4).

Proposition 2.3.3. *For any $(i,j)\in\Gamma$, $\forall t\leq T$,*

$$m^2\mathbb{E}\left[\sum_{l\in\Gamma^2-\{j\}}\{(Y_t^{ij,m}-Y_t^{il,m}-\overline{g}_{jl}(t))^+\}^2\right]\leq C \tag{2.11}$$

where the constant C is independent of m.

Proof. First let us show that there exists a constant C independent of m such that for any $(i,j)\in\Gamma$,

$$\mathbb{E}\left[\sup_{s\leq T}|Y_s^{ij,m}|^2\right]\leq C. \tag{2.12}$$

Actually taking into account of (2.10), it is enough to show that $\bar{Y}^{ij,m}$ satisfies the same estimate.

But from (2.6) we have

$$\mathbb{P}-\text{a.s.}\quad \tilde{Y}^{ij} \leq Y^{ij,m,0} \leq Y^{ij,m,n} \tag{2.13}$$

and the sequences $(Y^{ij,m,0})_{m\geq 0}$, $(i,j)\in\Gamma$, converge in $\mathcal{S}^2$ respectively to $\tilde{Y}^{ij}$ (one can see [18], Prop.3.3, pp.149, for more details) where $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij})_{(i,j)\in\Gamma}$ is the unique solution of the system of reflected BSDEs wih interconnected upper obstacles associated with $\Big((f^{ij})_{(i,j)\in\Gamma},(\xi^{ij})_{(i,j)\in\Gamma},(\bar{g}_{jl})_{j,l\in\Gamma^2}\Big)$. Now the claim follows since $\bar{Y}^{ij,m}\overset{\mathcal{S}^2}{=}\lim_n Y^{ij,m,n}$ and $\bar{Y}^{ij,m+1}\leq \bar{Y}^{ij,m}$.

Next in order to prove the boundedness of the penalized part of (2.8), we rely on the link between solutions of systems of reflected BSDEs with lower interconnected obstacles and optimal stochastic switching, which is well studied in the literature (see e.g. [11, 29, 34, 37, 40] etc). For this purpose, we set $u:=(\sigma_n,\delta_n)_{n\geq 0}$ an admissible strategy of switching, i.e., $(\sigma_n)_{n\geq 0}$ is an increasing sequence of stopping times such that $\mathbb{P}[\sigma_n<T,\forall n\geq 0]=0$, δ_n is Γ^1-valued and $\mathcal{F}_{\sigma_n}-$measurable random variable. Next when u is implemented, we set the cumulative switching cost $A_t^u:=\sum_{n\geq 1}\underline{g}_{\delta_{n-1}\delta_n}(\sigma_n)\mathbb{1}_{(\sigma_n\leq t)}$ for $t<T$ and $A_T^u:=\lim_{t\to T}A_t^u$. On the other hand, for $t\leq T$, we set $a_t:=\delta_0\mathbb{1}_{(\sigma_0)}(t)+\sum_{n\geq 1}\delta_{n-1}\mathbb{1}_{(\sigma_{n-1},\sigma_n]}(t)$ which stands for the indicator of the mode in which the system under switching is at time t. Note that a is in bijection with the strategy u. Finally denote by $\mathcal{A}_t^i$ ($t\in[0,T]$ and $i\in\Gamma$) the following set:

$$\mathcal{A}_t^i:=\{u=(\sigma_n,\delta_n)_{n\geq 0}\text{ admissible strategy such that }\sigma_0=t,\delta_0=i\text{ and }\mathbb{E}\left[(A_T^u)^2\right]<\infty\}.$$

Next for $j\in\Gamma^2$ and $a\in\mathcal{A}_t^i$, let $(U^{aj,m},V^{aj,m})$ be the unique solution of the following BSDE which is not of standard form since A^a is only rcll: $\forall t\leq T$,

$$\begin{cases} U^{aj,m}\text{ is rcll, }\mathbb{E}\left[\sup_{t\leq T}|U_t^{aj,m}|^2\right]<\infty\text{ and }V^{aj,m}\in\mathcal{H}^{2,d};\\ U_t^{aj,m}=\xi^{a_Tj}+\int_t^T\mathbb{1}_{(s\geq\sigma_0)}\underline{\mathrm{f}}^{aj,m}\left(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{aj,m}\right)ds-\int_t^T V_s^{aj,m}dB_s+A_T^a-A_t^a.\end{cases}\tag{2.14}$$

where for any $s\leq T$, $\underline{\mathrm{f}}^{aj,m}$ is defined by:

$$\begin{aligned}\underline{\mathrm{f}}^{aj,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z)=\sum_{n\geq 1}\Bigg(&\sum_{q\in\Gamma^1}\Big\{f^{qj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z)\\ &-m\sum_{l\in(\Gamma^2)^{-j}}(Y_s^{qj,m}-Y_s^{ql,m}-\bar{g}_{jl}(t))^+\Big\}1_{\{\delta_{n-1}=q\}}\Bigg)1_{\{\sigma_{n-1}\leq s<\sigma_n\}}\end{aligned}\tag{2.15}$$

i.e. $\underline{f}^{aj,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z) = f^{qj,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z)$ if at time s, $a(s) = q$. Let us notice that the arguments of $\underline{f}^{aj,m}$ are s,ω and z since $(Y_s^{kl,m})_{(k,l)\in\Gamma}$ is already fixed. Then the following representation holds true (see e.g.[34]): $\forall t \in [0,T]$,

$$Y_t^{ij,m} = \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i}\left(U_t^{aj,m} - A_t^a\right) \tag{2.16}$$

since, mainly, the switching costs verify the non free loop property (2.39).
Indeed let $(\underline{Y}^{ij,m},\underline{Z}^{ij,m},\underline{K}^{ij,m})_{(i,j)\in\Gamma}$ be the unique solution of the following system:

$$\begin{cases} \underline{Y}^{ij,m} \in \mathcal{S}^2, \underline{Z}^{ij,m} \in \mathcal{H}^2, \underline{K}^{ij,m,+} \in \mathcal{A}^2; \\ \underline{Y}_t^{ij,m} = \xi^{ij} + \displaystyle\int_t^T \left\{ f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},\underline{Z}_s^{ij,m}) - m\sum_{l\in(\Gamma^2)^{-j}}(\underline{Y}_s^{ij,m} - \underline{Y}_s^{il,m} - \overline{g}_{il}(s))^+ \right\} ds \\ \qquad - \displaystyle\int_t^T \underline{Z}_s^{ij,m} dB_s + \underline{K}_T^{ij,m,+} - \underline{K}_t^{ij,m,+},\ t\le T; \\ \underline{Y}_t^{ij,m} \ge \displaystyle\max_{k\in(\Gamma^1)^{-i}}\left(\underline{Y}_t^{kj,m} - \underline{g}_{ik}(t)\right),\ t\le T; \\ \displaystyle\int_0^T \left[\underline{Y}_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}}\left(\underline{Y}_t^{kj,m} - \underline{g}_{ik}(t)\right)\right] d\underline{K}_t^{ij,m,+} = 0. \end{cases} \tag{2.17}$$

Therefore (see e.g.[34]): $\forall t \in [0,T]$,

$$\underline{Y}_t^{ij,m} = \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i}(U_t^{aj,m} - A_t^a). \tag{2.18}$$

But $(Y^{ij,m},Z^{ij,m},K^{ij,m})_{(i,j)\in\Gamma}$ is also solution of (2.17), then by uniqueness of the solution of system (2.17) we have $Y^{ij,m} = \underline{Y}^{ij,m}$ which combined with (2.18) implies (2.16).
Next as a consequence of (2.16) we have: For any $t\in[0,T]$, $i\in\Gamma^1$ and $j,l\in\Gamma^2$,

$$\left(Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t)\right)^+ \le \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i}\left(U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t)\right)^+. \tag{2.19}$$

Now for $t\le T$, let us set $W_t^{a,jl,m} := U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t)$, $W_t^{a,jl,m,+} := (U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t))^+$ and let θ be a real constant which will be chosen appropriately later. Then applying Itô-Tanaka's formula with $e^{-\theta t}W_t^{a,jl,m,+}$ yields (note that $W_T^{a,jl,m,+} = 0$ by [H2]): $\forall t\le T$,

$$\begin{aligned} & e^{-\theta t}W_t^{a,jl,m,+} + \frac{1}{2}\int_t^T e^{-\theta s} dL_s^w \\ & = \theta\int_t^T e^{-\theta s}W_s^{a,jl,m,+} ds \\ & + \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s}\left\{\underline{f}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{aj,m}) - \underline{f}^{al}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{al,m}) + \overline{b}_{jl}(s)\right\} ds \end{aligned}$$

$$
\begin{aligned}
&- \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} \left(V_s^{aj,m} - V_s^{al,m} - \overline{\sigma}_{jl}(s) \right) dB_s \\
&- m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} \left\{ \sum_{k\in(\Gamma^2)^{-j}} W_s^{a,jk,m,+} - \sum_{k\in(\Gamma^2)^{-l}} W_s^{a,lk,m,+} \right\} ds \qquad (2.20)
\end{aligned}
$$

where L^w is the local time of $W^{a,jl,m,+}$ at 0 and $\underline{f}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z) := \underline{f}^{aj,0}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z)$ (see (2.15)). Next let us focus on the last term of the right side of (2.20): $\forall t \le T$

$$
\begin{aligned}
&- m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} \left\{ \sum_{k\in(\Gamma^2)^{-j}} W_s^{a,jk,m,+} - \sum_{k\in\Gamma^2-\{l\}} W_s^{a,lk,m,+} \right\} ds \\
&= m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} \left\{ W_s^{a,lj,m,+} - W_s^{a,jl,m,+} + \sum_{k\in\Gamma^2-\{j,l\}} (W_s^{a,lk,m,+} - W_s^{a,jk,m,+}) \right\} ds. \qquad (2.21)
\end{aligned}
$$

Note that $\mathbb{1}_{(W_s^{a,jl,m}>0)} W_s^{a,lj,m,+} = 0$ since $\{W_s^{a,jl,m} > 0\} \cap \{W_s^{a,lj,m} > 0\} = \emptyset$ as $\bar{g}_{jl} \ge 0$. Next by applying the inequality $a^+ - b^+ \le (a-b)^+$ we have: $\forall s \le T$

$$
\mathbb{1}_{(W_s^{a,jl,m}>0)} \sum_{k\in\Gamma^2-\{j,l\}} \left(W_s^{a,lk,m,+} - W_s^{a,jk,m,+} \right) \le \mathbb{1}_{(W_s^{a,jl,m}>0)} \sum_{k\in\Gamma^2-\{j,l\}} \left(U_s^{al,m} - \overline{g}_{lk}(s) - U_s^{aj,m} + \overline{g}_{jk}(s) \right)^+.
$$

Using the fact that $\overline{g}_{jl}(s) + \overline{g}_{lk}(s) > \overline{g}_{jk}(s)$, by Assumption [H3]-(a),(ii), we deduce that

$$
W_s^{a,jl,m} < U_s^{aj,m} - U_s^{al,m} + \overline{g}_{lk}(s) - \overline{g}_{jk}(s)
$$

and then

$$
\begin{aligned}
0 &\le \mathbb{1}_{(W_s^{a,jl,m}>0)} \sum_{k\in\Gamma^2-\{j,l\}} (U_s^{al,m} - \overline{g}_{lk}(s) - U_s^{aj,m} + \overline{g}_{jk}(s))^+ \\
&\le \sum_{k\in\Gamma^2-\{j,l\}} \mathbb{1}_{(U_s^{aj,m} - U_s^{al,m} + \overline{g}_{lk}(s) - \overline{g}_{jk}(s) > 0)} (U_s^{al,m} - \overline{g}_{lk}(s) - U_s^{aj,m} + \overline{g}_{jk}(s))^+ \\
&= 0.
\end{aligned}
$$

Now going back to (2.21) we obtain: $\forall t \le T$,

$$
\begin{aligned}
-m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} &\left\{ \sum_{k\in(\Gamma^2)^{-j}} W_s^{a,jk,m,+} - \sum_{k\in(\Gamma^2)^{-l}} W_s^{a,lk,m,+} \right\} ds \\
&\le -m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} W_s^{a,jl,m,+} ds \qquad (2.22)
\end{aligned}
$$

and consequently from (2.20) we have: $\forall t \leq T$,

$$\begin{aligned} &e^{-\theta t} W_t^{a,jl,m,+} + m \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} W_s^{a,jl,m,+} ds + \frac{1}{2} \int_t^T e^{-\theta s} dL_s^w \\ &\leq - \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} (V_s^{aj,m} - V_s^{al,m} - \overline{\sigma}_{jl}(s)) dB_s + \theta \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} W_s^{a,jl,m,+} ds \\ &+ \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)} e^{-\theta s} \left\{ \underline{\mathrm{f}}^{aj}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, V_s^{aj,m}) - \underline{\mathrm{f}}^{al}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, V_s^{al,m}) + \overline{b}_{jl}(s) \right\} ds. \end{aligned} \tag{2.23}$$

Next by taking $\theta = m$, recall that [H1] implies the boundedness of $(f^{ij}(t, \vec{y}, z))_{(i,j)\in\Gamma}$ by $|\vec{y}|$ and [H4] represents $(\bar{g}_{jl})_{jl\in\Gamma^2}$ as Itô process, hence by taking the conditional expectation we deduce: $\forall t \leq T$,

$$\begin{aligned} W_t^{a,jl,m,+} &\leq \mathbb{E}\left[\int_t^T e^{-m(s-t)} |\underline{\mathrm{f}}^{aj}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, V_s^{aj,m}) - \underline{\mathrm{f}}^{al}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, V_s^{al,m}) + \overline{b}_{jl}(s)| ds \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E}\left[C \left\{ 1 + \sup_{s\leq T} |\eta_s| + \sum_{(k,l)\in\Gamma} \sup_{s\leq T} |Y_s^{kl,m}| + \sup_{s\leq T} |\overline{b}_{jl}(s)| \right\} \int_t^T e^{-m(s-t)} ds \middle| \mathcal{F}_t \right] \\ &= \frac{1}{m}(1 - e^{-m(T-t)}) \mathbb{E}\left[C \left\{ 1 + \sup_{s\leq T} |\eta_s| + \sum_{(k,l)\in\Gamma} \sup_{s\leq T} |Y_s^{kl,m}| + \sup_{s\leq T} |\overline{b}_{jl}(s)| \right\} \middle| \mathcal{F}_t \right]. \end{aligned}$$

Now by (2.19), we get

$$\forall t \leq T,\ m(Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t))^+ \leq C\mathbb{E}\left[\left\{ 1 + \sup_{s\leq T} |\eta_s| + \sum_{(k,l)\in\Gamma} \sup_{s\leq T} |Y_s^{kl,m}| + \sup_{s\leq T} |\overline{b}_{jl}(s)| \right\} \middle| \mathcal{F}_t \right]$$

and then squaring, using conditional Jensen's inequality and finally taking expectation to obtain: $\forall t \leq T$,

$$m^2 \mathbb{E}\left[\left\{ (Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t))^+ \right\}^2 \right] \leq C\mathbb{E}\left[1 + \sup_{s\leq T} |\eta_s|^2 + \sum_{(k,l)\in\Gamma} \sup_{s\leq T} |Y_s^{kl,m}|^2 + \sup_{s\leq T} |\overline{b}_{jl}(s)|^2 \right]$$

which implies the desired result since the processes η and $\overline{b}_{jl}$ are uniformly square integrable and by estimate (2.12). □

Next we are going to show that $K^{ij,m,+}$ is absolutely continuous w.r.t time and its density $(\frac{dK_s^{ij,m,+}}{ds})_{s\leq T}$ belongs to $\mathcal{H}^{2,1}$ uniformly in m.

Proposition 2.3.4. *For any $m \geq 0$ and $(i,j) \in \Gamma$, there exists a $\mathcal{P}$-measurable process $(\alpha_t^{ij,m})_{t\leq T}$ such* *that for any $t \le T$,*

$$K_t^{ij,m,+} = \int_0^t \alpha_s^{ij,m} ds.$$

Moreover there exists a constant C independent of m such that $\mathbb{E}\left[\int_0^T |\alpha_s^{ij,m}|^2 ds\right] \le C.$

Proof. Let us consider the following system of BSDEs: for any $(i,j) \in \Gamma$,

$$\begin{cases} \tilde{Y}^{ij,m,n} \in \mathcal{S}^2, \tilde{Z}^{ij,m,n} \in \mathcal{H}^{2,d} \ ; \\ \tilde{Y}_s^{ij,m,n} = \xi^{ij} + \int_t^T \Big\{ f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) - m \sum_{l \ne j} (Y_s^{ij,m} - Y_s^{il,m} - \overline{g}_{jl}(s))^+ \\ \qquad + n \sum_{k\in(\Gamma^1)^{-i}} (\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s))^- \Big\} ds - \int_t^T \tilde{Z}_s^{ij,m,n} dB_s, \ t \le T. \end{cases} \tag{2.24}$$

For $(i,j) \in \Gamma$, $m \ge 0$ and $s \le T$ let us set:

$$\Phi^{ij,m}(s) = f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) - m \sum_{l \ne j} (Y_s^{ij,m} - Y_s^{il,m} - \overline{g}_{jl}(s))^+.$$

First note that by [H1], (2.11) and (2.12), there exists a constant C independent of m such that

$$\mathbb{E}\left[\int_0^T |\Phi^{ij,m}(s)|^2 ds\right] \le C. \tag{2.25}$$

On the other hand the sequences $\left(\tilde{Y}^{ij,m,n}, \tilde{Z}^{ij,m,n}, n\int_0^{\cdot} \sum_{k\in(\Gamma^1)^{-i}} (\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s))^- \} ds\right)_{n\ge 0}$, $(i,j) \in \Gamma$, converge when n goes to $+\infty$ in $\mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{S}^2$ to $(\tilde{Y}^{ij,m}, \tilde{Z}^{ij,m}, \tilde{K}^{ij,m,+})$, $(i,j) \in \Gamma$, respectively. Moreover $(\tilde{Y}^{ij,m}, \tilde{Z}^{ij,m}, \tilde{K}^{ij,m,+})_{(i,j)\in\Gamma}$ (see e.g. [18] for more details) is solution of the following system: $\forall t \le T$,

$$\begin{cases} \tilde{Y}_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) ds - \int_t^T \tilde{Z}_s^{ij,m} dB_s + \tilde{K}_T^{ij,m,+} - \tilde{K}_t^{ij,m,+} \ ; \\ \tilde{Y}_t^{ij,m} \ge \max_{k\in(\Gamma^1)^{-i}} \left(\tilde{Y}_t^{kj,m} - \underline{g}_{ik}(t)\right) \ ; \\ \int_0^T \left[\tilde{Y}_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} (\tilde{Y}_t^{kj,m} - \underline{g}_{ik}(t))\right] d\tilde{K}_t^{ij,m,+} = 0. \end{cases} \tag{2.26}$$

As the solution of this latter is unique and by (2.8), $(Y^{ij,m}, Z^{ij,m}, K^{ij,m,+})_{(i,j)\in\Gamma}$ is also a solution then, $\tilde{Y}^{ij,m} = Y^{ij,m}$, $\tilde{Z}^{ij,m} = Z^{ij,m}$ and $\tilde{K}^{ij,m,+} = K^{ij,m,+}$ for any $(i,j) \in \Gamma$.

Next for $s \le T$, $i,k \in \Gamma^1$ and $j \in \Gamma^2$, let us set

$$\rho_s^{ikj,m,n} := (\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s))^-.$$

Note that by Assumption [H2], $\rho_T^{ikj,m,n} = 0$. Now if $(X_s)_{s\le T}$ is a continuous semimartingale then by the use of Itô-Tanaka formula (see e.g. [61], pp.231) we have that: $\forall t \leq T$,

$$(X_t^-)^2 + \int_t^T 1_{\{X_s<0\}} d\langle X\rangle_s = (X_T^-)^2 + 2\int_t^T X_s^- dX_s.$$

Therefore for any $t \leq T$,

$$\begin{aligned}
&(\rho_t^{ikj,m,n})^2 + \int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \left(\tilde{Z}_s^{ij,m,n} - \tilde{Z}_s^{kj,m,n} + \underline{\sigma}_{ik}(s)\right)^2 ds \\
&= -2\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \left\{\Phi^{ij,m}(s) - \Phi^{kj,m}(s) - \underline{b}_{ik}(s)\right\} ds \\
&+ 2\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \left(\overline{Z}_s^{ij,m,n} - \overline{Z}_s^{kj,m,n} + \underline{\sigma}_{ik}(s)\right) dB_s \\
&- 2n\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \left\{\sum_{l\in(\Gamma^1)^{-i}} \rho_s^{ilj,m,n} - \sum_{l\in(\Gamma^1)^{-k}} \rho_s^{klj,m,n}\right\} ds.
\end{aligned} \tag{2.27}$$

We now focus on the last term of (2.27).

$$\begin{aligned}
&- 2n\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \left\{\sum_{l\in(\Gamma^1)^{-i}} \rho_s^{ilj,m,n} - \sum_{l\in(\Gamma^1)^{-k}} \rho_s^{klj,m,n}\right\} ds \\
&= -2n\int_t^T 1_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} (\rho_s^{ikj,m,n})^2 ds + 2n\int_t^T 1_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \underbrace{\rho_s^{ikj,m,n}\rho_s^{kij,m,n}}_{=0} ds \\
&\quad + 2n\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \sum_{l\in\Gamma^1-\{i,k\}} \left(-\rho_s^{ilj,m,n} + \rho_s^{klj,m,n}\right) ds
\end{aligned} \tag{2.28}$$

since by positiveness of $\underline{g}_{ki}$ and $\underline{g}_{ik}$, $\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s) < 0\} \cap \{\tilde{Y}_s^{kj,m,n} - \tilde{Y}_s^{ij,m,n} + \underline{g}_{ki}(s) < 0\} = \emptyset$. Next by applying the inequality $a^- - b^- \leq (a-b)^-$ we have

$$\begin{aligned}
&\rho_s^{ikj,m,n} \sum_{l\in\Gamma^1-\{i,k\}} \left(\rho_s^{klj,m,n} - \rho_s^{ilj,m,n}\right) \\
&= \rho_s^{ikj,m,n} \sum_{l\in\Gamma^1-\{i,k\}} \left\{\tilde{Y}_s^{kj,m,n} - \tilde{Y}_s^{lj,m,n} + \underline{g}_{kl}(s))^- - (\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{lj,m,n} + \underline{g}_{il}(s))^-\right\} \\
&\leq \rho_s^{ikj,m,n} \sum_{l\in\Gamma^1-\{i,k\}} \left(\tilde{Y}_s^{kj,m,n} - \tilde{Y}_s^{ij,m,n} + \underline{g}_{kl}(s) - \underline{g}_{il}(s)\right)^- \\
&= \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} \rho_s^{ikj,m,n} \sum_{l\in\Gamma^1-\{i,k\}} \left(\tilde{Y}_s^{kj,m,n} - \tilde{Y}_s^{ij,m,n} + \underline{g}_{kl}(s) - \underline{g}_{il}(s)\right)^- = 0
\end{aligned}$$

since by Assumption [H3]-(a),(ii), for any $l \in \Gamma^1 - \{i,k\}$, $\mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s)<0\}} (\tilde{Y}_s^{kj,m,n} - \tilde{Y}_s^{ij,m,n} +$

$\underline{g}_{kl}(s) - \underline{g}_{il}(s))^- = 0$. We then deduce from (2.27) that, after taking expectation,

$$\begin{aligned}
&2n\mathbb{E}\left[\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n} - \tilde{Y}_s^{kj,m,n} + \underline{g}_{ik}(s) < 0\}} (\rho_s^{ikj,m,n})^2 ds\right] = 2n\mathbb{E}\left[\int_t^T (\rho_s^{ikj,m,n})^2 ds\right] \\
&\leq 2\mathbb{E}\left[\int_t^T \rho_s^{ikj,m,n} |\Phi^{ij,m}(s) - \Phi^{kj,m}(s) - \underline{b}_{ik}(s)| ds\right] \\
&\leq n\mathbb{E}\left[\int_t^T (\rho_s^{ikj,m,n})^2 ds\right] + \frac{1}{n}\mathbb{E}\left[\int_t^T |\Phi^{ij,m}(s) - \Phi^{kj,m}(s) - \underline{b}_{ik}(s)|^2 ds\right]
\end{aligned} \tag{2.29}$$

which implies that

$$n^2\mathbb{E}\left[\int_t^T (\rho_s^{ikj,m,n})^2 ds\right] \leq C\mathbb{E}\left[\int_t^T \left\{|\Phi^{ij,m}(s)|^2 + |\Phi^{kj,m}(s)|^2 + |\underline{b}_{ik}(s)|^2\right\} ds\right]. \tag{2.30}$$

Then by (2.25) and Assumption [H4] on $\underline{b}_{ik}$ we obtain:

$$n^2\mathbb{E}\left[\int_0^T (\rho_s^{ikj,m,n})^2 ds\right] \leq \underline{C} \text{ and } n^2\mathbb{E}\left[\int_0^T (\sum_{k\neq i} \rho_s^{ikj,m,n})^2 ds\right] \leq \underline{C}$$

for some constant $\underline{C}$ independent of n, m. It implies that for any $(i,j) \in \Gamma$, the sequence $((\alpha_s^{ij,m,n} := n\sum_{k\in\Gamma^1-\{i\}} \rho_s^{ikj,m,n})_{s\leq T})_{n\geq 0}$ is bounded in $\mathcal{H}^{2,1}$. Thus one can extract a subsequence (still denoted by n) such that for any $(i,j) \in \Gamma$, $((\alpha_s^{ij,m,n})_{s\leq T})_{n\geq 0}$ converges weakly in $\mathcal{H}^{2,1}$ to some $\mathcal{P}$-measurable process $(\alpha_t^{ij,m})_{t\leq T}$ which moreover satisfy: For any $(i,j) \in \Gamma$ and $m \geq 0$,

$$\mathbb{E}\left[\int_0^T (\alpha_s^{ij,m})^2 ds\right] \leq \underline{C}. \tag{2.31}$$

Additionally for any $(i,j) \in \Gamma$ and any stopping time τ it holds:

$$K_\tau^{ij,m,+} = \int_0^\tau \alpha^{ij,m}(s) ds. \tag{2.32}$$

Actually this is due to the fact that the sequence $(\int_0^\tau \alpha_s^{ij,m,n} ds)_{n\geq 0}$ is also weakly convergent in $L^2_{\mathbb{R}}(\Omega, \mathcal{F}_T, d\mathbb{P})$ and since, as pointed out previously, $K^{ij,m,+} \stackrel{\mathcal{S}^2}{=} \lim_{n\to\infty} \int_0^\cdot \alpha_s^{ij,m,n} ds$.

Indeed let us show the weak convergence of $(\int_0^\tau \alpha_s^{ij,m,n} ds)_{n\geq 0}$. Let ζ be a random variable of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}_T, d\mathbb{P})$. By the representation property there exists a $\mathcal{P}$-measurable process $(\bar{\eta}_t)_{t\leq T}$ of $\mathcal{H}^{2,d}$ such that:

$$\forall t \leq T,\ \mathbb{E}[\zeta|\mathcal{F}_t] = \mathbb{E}[\zeta] + \int_0^t \bar{\eta}_s dB_s.$$

Next by Itô's formula we have

$$\mathbb{E}\left[\zeta\int_0^\tau \alpha_s^{ij,m,n} ds\right] = \mathbb{E}\left[\mathbb{E}[\zeta|\mathcal{F}_\tau]\int_0^\tau \alpha_s^{ij,m,n} ds\right] = \mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m,n} ds\right]$$

since by Burkholder et al.'s inequality ([62], pp.160) $(\int_0^t(\int_0^s \alpha_r^{ij,m,n}dr)\bar{\eta}_s dB_s)_{t\le T}$ is a martingale due to $\mathbb{E}[\{\int_0^T(\int_0^s \alpha_r^{ij,m,n}dr)^2|\bar{\eta}_s|^2ds\}^{\frac{1}{2}}] < \infty$. As the sequence $((\alpha_s^{ij,m,n})_{s\le T})_{n\ge 0}$ converges weakly in $\mathcal{H}^{2,1}$ to $\alpha^{ij,m}$ then

$$\mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m,n}ds\right] \longrightarrow_{n\to\infty} \mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m}ds\right] = \mathbb{E}\left[\zeta\int_0^\tau \alpha_s^{ij,m}ds\right]$$

which is the claim. □

Proposition 2.3.5. *There exist continuous adapted processes $(Y^{ij})_{(i,j)\in\Gamma}$ and $\mathcal{P}$-measurable processes $(Z^{ij})_{(i,j)\in\Gamma}$, such that for $(i,j)\in\Gamma^1\times\Gamma^2$:*

i) $(Y^{ij,m})_{m\ge 0}$ uniformly converges to Y^{ij} in $\mathcal{S}^2$.
ii) $(Z^{ij,m})_{m\ge 0}$ converges to Z^{ij} in $\mathcal{H}^{2,d}$.

Proof. First let us recall the process $(Y^{ij,m})_{(i,j)\in\Gamma}$ in (2.8). Next fix $(i,j)\in\Gamma$ and let Y^{ij} be the optional process such that

$$\mathbb{P}\text{-a.s}, \forall t\le T, Y_t^{ij} = \lim_{m\to\infty} Y_t^{ij,m}$$

which exists since the sequence $(Y^{ij,m})_{m\ge 0}$ is decreasing (see (2.9)). On the other hand for any $m\ge 0$ we have: $\forall t\le T$,

$$Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m})ds + \int_t^T \alpha^{ij,m}(s)ds - \int_t^T Z_s^{ij,m}dB_s.$$

Then using Itô formula with $(Y^{ij,m})^2$ and taking into account of (2.25)-(2.31), one deduces the existence of a constant C independent of m such that

$$\mathbb{E}\left[\int_0^T \left|Z_s^{ij,m}\right|^2 ds\right] \le C. \tag{2.33}$$

Next, let $\{m\}$ be a sequence such that:

i) $\left(f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}))_{s\le T}\right)_{m\ge 0}$ converges weakly in $\mathcal{H}^{2,1}$ to Φ^{ij} ;

ii) $\left(m\sum_{l\in\Gamma^2-\{j\}}\left(Y_s^{ij,m} - Y_s^{il,m} - \bar{g}_{jl}(s))^+\right)_{s\le T}\right)_{m\ge 0}$ converges weakly to θ^{ij} is $\mathcal{H}^{2,1}$;

iii) $\left(\alpha^{ij,m}\right)_{m\ge 0}$ converges weakly to α^{ij} is $\mathcal{H}^{2,1}$;

iv) $\left(Z^{ij,m}\right)_{m\ge 0}$ converges weakly to Z^{ij} is $\mathcal{H}^{2,d}$.

This sequence exists thanks to Assumption [H1] on f^{ij} and (2.12), (2.11), (2.31) and finally (2.33). Next let τ be a stopping time. Then as in the proof of Proposition 2.3.4, the following weak convergences in $L^2(d\mathbb{P})$, as $m\to\infty$, hold true:

$$\text{a) } \int_0^\tau f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m})ds \rightharpoonup \int_0^\tau \Phi^{ij}(s)ds,$$

b) $\int_0^\tau m \sum_{l\in\Gamma^2-\{j\}} (Y_s^{ij,m} - Y_s^{il,m} - \bar{g}_{jl}(s))^+ ds \rightharpoonup \int_0^\tau \theta^{ij}(s)ds,$

c) $\int_0^\tau \alpha^{ij,m}(s)ds \rightharpoonup \int_0^\tau \alpha^{ij}(s)ds,$

d) $\int_0^\tau Z_s^{ij,m} dB_s \rightharpoonup \int_0^\tau Z_s^{ij} dB_s.$

Therefore for any stopping time τ, we have:

$$Y_\tau^{ij} = Y_0^{ij} - \int_0^\tau \Phi^{ij}(s)ds + \int_0^\tau \theta^{ij}(s)ds - \int_0^\tau \alpha^{ij}(s)ds - \int_0^\tau Z_s^{ij} dB_s.$$

As Y^{ij} is an optional process and this equality holds for any stopping time then the processes of the left and right-hand side are indistinguishable which means that $\mathbb{P}-a.s., \forall t \leq T$,

$$Y_t^{ij} = Y_0^{ij} - \int_0^t \Phi^{ij}(s)ds + \int_0^t \theta^{ij}(s)ds - \int_0^t \alpha^{ij}(s)ds - \int_0^t Z_s^{ij} dB_s \tag{2.34}$$

and the process Y^{ij} is continuous. Thus by Dini's Theorem the convergence of the sequence of $(Y^{ij,m})_{m\geq 0}$ to Y^{ij} holds in $\mathcal{S}^2$ i.e. $\lim_{m\to\infty} \mathbb{E}\left[\sup_{t\leq T} |Y_t^{ij,m} - Y_t^{ij}|^2\right] = 0.$

Next once more by the use of Itô's formula with $(Y^{ij,m} - Y^{ij,n})^2$ and taking into account of (2.25)-(2.31) one deduces that $(Z^{ij,m})_{m\geq 0}$ is a Cauchy sequence in $\mathcal{H}^{2,d}$ and then $(Z^{ij,m})_{m\geq 0}$ converges strongly to Z^{ij} is $\mathcal{H}^{2,d}$. □

To proceed let us define for any $(i,j) \in \Gamma$, $t \leq T$,

$$K_t^{ij,-} = \int_0^t \theta_s^{ij} ds \text{ and } K_t^{ij,+} = \int_0^t \alpha_s^{ij} ds.$$

We then give the main result of this section:

Theorem 2.3.6. *The process $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is a solution of the system of reflected BSDEs* (2.4).

Proof. First note that by (2.34) and since $Y_T^{ij} = \xi^{ij}$ then for any $(i,j) \in \Gamma$,

$$Y_\tau^{ij} = \xi^{ij} + \int_\tau^T \Phi^{ij}(s)ds - \int_\tau^T \theta^{ij}(s)ds + \int_\tau^T \alpha^{ij}(s)ds - \int_\tau^T Z_s^{ij} dB_s$$

Now recall the definition of Φ^{ij} and since the convergences of $(Y^{ij,m})_{m\geq 0}$ and $(Z^{ij,m})_{m\geq 0}$ hold in strong sense then

$$\Phi^{ij}(s) = f^{ij}\left(s, (Y_s^{kl})_{(k,l)\in\Gamma}, Z_s^{ij}\right), \ ds \otimes d\mathbb{P}$$

which implies that for any $(i,j) \in \Gamma$, $\mathbb{P}$-a.s. for any $t \leq T$,

$$Y_t^{ij} = \xi^{ij} + \int_t^T f^{ij}\left(s, (Y_s^{kl})_{(k,l)\in\Gamma}, Z_s^{ij}\right) ds + (K_T^{ij,+} - K_t^{ij,+}) - (K_T^{ij,-} - K_t^{ij,-}) - \int_t^T Z_s^{ij} dB_s.$$

Next from (2.8) we have

$$Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) ds - \int_t^T Z_s^{ij,m} dB_s + K_T^{ij,m,+} - K_t^{ij,m,+}$$

which implies in taking expectation

$$\begin{aligned} &m\mathbb{E}\left[\int_0^T \sum_{\ell\in\Gamma-\{i\}} (Y_s^{ij,m} - Y_s^{i\ell,m} - \bar{g}_{j\ell}(s))^+\right] \\ &= \mathbb{E}\left[-Y_0^{ij,m} + \xi^{ij} + \int_0^T f^{ij}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) ds + K_T^{ij,m,+}\right]. \end{aligned} \tag{2.35}$$

Then by Assumption [H1], (2.12),(2.31) and (2.32), there exists a constant C such that

$$\mathbb{E}\left[\int_0^T \sum_{\ell\in\Gamma^2-\{j\}} (Y_s^{ij,m} - Y_s^{i\ell,m} - \bar{g}_{j\ell}(s))^+\right] \leq Cm^{-1} \tag{2.36}$$

which implies that, in taking the limit as $m \to \infty$, for any $(i,j) \in \Gamma$ and $s \leq T$, $Y_s^{ij} \leq Y_s^{i\ell} + \bar{g}_{j\ell}(s)$ for any $\ell \in \Gamma_2 - \{j\}$. Then

$$\mathbb{P} - a.s., \forall s \leq T,\ Y_s^{ij} \leq \min_{\ell\in\Gamma^2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s)).$$

Next

$$\mathbb{E}\left[\int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right) dK_s^{ij,-}\right] = -\mathbb{E}\left[\int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right)^- \alpha_s^{ij} ds\right] \tag{2.37}$$

$$= \lim_{m\to\infty} \mathbb{E}\left[\int_0^T \left(Y_s^{ij,m} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell,m} + \bar{g}_{j\ell}(s))\right)^- \alpha_s^{ij,m} ds\right] = 0$$

since $(\alpha^{ij,m})_m$ is weakly convergent to α^{ij} and $(Y^{ij,m} - \min_{\ell\in\Gamma_2-\{j\}}(Y^{i\ell,m} + \bar{g}_{j\ell}))_m$ converges strongly in $\mathcal{S}^2$ to $Y^{ij} - \min_{\ell\in\Gamma_2-\{j\}}(Y^{i\ell} + \bar{g}_{j\ell}))^-$. As $\int_0^T (Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}}(Y_s^{i\ell} + \bar{g}_{j\ell}(s)))dK_s^{ij,-} \leq 0$ then

$$\mathbb{P} - a.s., \int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right) dK_s^{ij,-} = 0.$$

In the same way one can show that

$$\mathbb{P}-a.s., \int_0^T \left(Y_s^{ij} - \max_{k \in \Gamma_1 - \{i\}} (Y_s^{kj} - \underline{g}_{kj}(s)) \right) dK_s^{ij,+} = 0.$$

Thus the processes $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is a solution of the system of reflected BSDEs (2.4).
$\square$

Remark 2.3.7.

(i) The constant C *such that for any* $(i,j) \in \Gamma$,

$$\mathbb{E}\left[\int_0^T (|\alpha_s^{ij}|^2 + |\theta_s^{ij}|^2) ds\right] \leq C$$

depends only on $(f^{ij})_{(i,j)\in\Gamma}$, $(\xi^{ij})_{(i,j)\in\Gamma}$, $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ *and* $(\bar{g}_{jl})_{j,l\in\Gamma^2}$.

(ii) In our construction of the solution of (2.4) *through the penalization scheme* (2.8), *we have penalized the upper barriers. Had we taken the dual scheme of* (2.8) *where, instead, the lower barriers are penalized, we would have obtained another solution* $(\check{Y}^{ij}, \check{Z}^{ij}, \check{K}^{ij,\pm})_{(i,j)\in\Gamma}$ *of system* (2.4). *Additionally we have* $\check{Y}^{ij} \leq Y^{ij}$ *for any* $(i,j) \in \Gamma$.

(iii) The solutions of systems (2.4) *which we have constructed are comparable. Actually let us consider* $(\underline{f}^{1,ij})_{(i,j)\in\Gamma}$, $(\underline{\xi}^{1,ij})_{(i,j)\in\Gamma}$, $(\underline{g}^1_{ik})_{i,k\in\Gamma^1}$ *and* $(\bar{g}^1_{jl})_{j,l\in\Gamma^2}$ *items which satisfy the same assumptions [H1]-[H5] receptively as* $(f^{ij})_{(i,j)\in\Gamma}$, $(\xi^{ij})_{(i,j)\in\Gamma}$, $(g_{ik})_{i,k\in\Gamma^1}$ *and* $(\bar{g}_{jl})_{j,l\in\Gamma^1}$. *Let us denote by* $(Y^{1,ij}, Z^{1,ij}, K^{1,ij,+}, K^{1,ij,-})_{(i,j)\in\Gamma}$ *the solution of system* (2.4) *associated with* $\{(\underline{f}^{1,ij})_{(i,j)\in\Gamma}, (\underline{\xi}^{1,ij})_{(i,j)\in\Gamma}, (\underline{g}^1_{ik})_{i,k\in\Gamma^1}, (\bar{g}^1_{jl})_{j,l\in\Gamma^2}\}$ *(which exists by Theorem 2.3.6). Assume that for any:*

a) $(i,j) \in \Gamma$, $f^{ij} \leq f^{1,ij}$ *and* $\xi^{ij} \leq \xi^{1,ij}$;
b) $i,k \in \Gamma^1$, $\underline{g}_{ik} \geq \underline{g}^1_{ik}$;
c) $j,l \in \Gamma^2$, $\bar{g}_{ik} \leq \bar{g}^1_{ik}$.

Then we have: For any $(i,j) \in \Gamma$,

$$\mathbb{P}-a.s., Y^{ij} \leq Y^{1,ij}.$$

This is actually a direct consequence of the constructions of Y^{ij} *and* $Y^{1,ij}$ *since for any* $(i,j) \in \Gamma$,

$$Y^{ij} = \lim_{m\to\infty} Y^{ij,m} \text{ and } Y^{1,ij} = \lim_{m\to\infty} Y^{1,ij,m}$$

where $(Y^{1,ij,m})_{(i,j)\in\Gamma}$ *are defined in the same way as* $(Y^{ij,m})_{(i,j)\in\Gamma}$ *in* (2.7) *but with the items* $\{(\underline{f}^{1,ij})_{(i,j)\in\Gamma}, (\underline{\xi}^{1,ij})_{(i,j)\in\Gamma},$
But by comparison ([34], pp.190 for more details) we have for any $(i,j) \in \Gamma$, $Y^{ij,m} \leq Y^{1,ij,m}$ *which implies the result in taking the limit as* $m \to \infty$. $\square$

2.4 Existence and uniqueness without monotonicity

In this section, we focus on the second main result of this paper. Actually we are going to show that system of reflected BSDEs with inter-connected obstacles (2.4) has a unique solution without assuming the monotonicity Assumption [H5] on the functions $(f^{ij})_{(i,j)\in\Gamma}$. Meanwhile in this section we shall need the following assumptions:

[H3] b) The processes $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\overline{g}_{j,\ell})_{j,\ell\in\Gamma^2}$ verify the non free loop property, that is to say, if $(i_k, j_k)_{k=1,2,...,N}$ is a loop in Γ, i.e., $(i_N, j_N) = (i_1, j_1)$, $\mathbf{card}\,\{(i_k, j_k)_{k=1,2,...,N}\} = N-1$ and for any $k = 1, 2, ..., N-1$, either $i_{k+1} = i_k$ (resp. $j_{k+1} = j_k$), we have:

$$\mathbb{P}-a.s.,\ \forall t \leq T, \quad \sum_{k=1}^{N-1} G_{i_k j_k}(t) \neq 0 \tag{2.38}$$

where $\forall k = 1, ...N-1$, $G_{i_k j_k}(t) = -\underline{g}_{i_k i_{k+1}}(t)\mathbb{1}_{i_k\neq i_{k+1}} + \overline{g}_{j_k j_k}(t)\mathbb{1}_{j_k\neq j_{k+1}}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1\times\Gamma^2$, of the players (or decision makers), is not free i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode. Note that

(2.38) also implies: For any $(i_1, ..., i_N) \in (\Gamma^1)^N$ such that $i_N = i_1$ and $\mathbf{card}\{i_1, i_2, ..., i_N\} = N-1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t) = 0\right] = 0,\ \forall t \leq T, \tag{2.39}$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ such that $j_N = j_1$ and $\mathbf{card}\{j_1, j_2, ..., j_N\} = N-1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t) = 0\right] = 0,\ \forall t \leq T. \tag{2.40}$$

[H6] For any $(i,j) \in \Gamma$, the function f^{ij} does not depend on z.

We highlight that in this section, the generator $(f^{ij})_{(i,j)\in\Gamma}$ is not monotonic any more, i.e. it does not verify [H5].

First let us temporarily assume that for any $(i,j) \in \Gamma$, the function f^{ij} does not depend on $(\vec{y}, z)$. Therefore by Theorem 2.3.6, there is a solution $(\underline{Y}^{ij}, \underline{Z}^{ij}, \underline{K}^{ij,\pm})_{(i,j)\in\Gamma}$ of the following system: $\forall (i,j) \in \Gamma$,

$$\begin{cases} \underline{Y}^{ij} \in \mathcal{S}^2, \underline{Z}^{ij} \in \mathcal{H}^{2,d}, \underline{K}^{ij,+} \in \mathcal{A}^2, \underline{K}^{ij,-} \in \mathcal{A}^2; \\ \underline{Y}^{ij}_t = \xi^{ij} + \int_t^T f^{ij}(s)ds - \int_t^T \underline{Z}^{ij}_s dB_s + \underline{K}^{ij,+}_T - \underline{K}^{ij,+}_t - (\underline{K}^{ij,-}_T - \underline{K}^{ij,-}_t), \forall t \le T; \\ \underline{L}^{ij}_t \le \underline{Y}^{ij}_t \le \underline{U}^{ij}_t, \forall t \in [0,T]; \\ \int_0^T \left(\underline{Y}^{ij}_t - \underline{L}^{ij}_t \right) d\underline{K}^{ij,+}_t = 0 \text{ and } \int_0^T \left(\underline{U}^{ij}_t - \underline{Y}^{ij}_t \right) d\underline{K}^{ij,-}_t = 0. \end{cases} \tag{2.41}$$

where $\underline{L}^{ij}_t := \max_{k \in (\Gamma^1)^{-i}} \left\{ \underline{Y}^{kj}_t - \underline{g}_{ik}(t) \right\}$ *and* $\underline{U}^{ij}_t := \min_{l \in (\Gamma^2)^{-j}} \left\{ \underline{Y}^{il}_t + \bar{g}_{jl}(t) \right\}$, $t \le T$.

As pointed out previously we are going to represent the process $\underline{Y}^{ij}$ as the value function of a zero-sum switching game which we describe briefly now.

Let us consider a system which has $\Lambda = |\Gamma^1 \times \Gamma^2|$ working modes indexed by $\Gamma^1 \times \Gamma^2$. It means that a working mode is a pair (i,j) such that $i \in \Gamma^1$ and $j \in \Gamma^2$. This system is controlled by two agents or players P1 and P2 by choosing their own appropriate working mode of the system and switch to another one when they make the decision to do so (e.g. according to profitability, etc.). The player P1 (resp. P2) chooses her modes in Γ^1 (resp. Γ^2). The features of the system is that when it works in mode (i,j) from time t to $t + dt$, it comes with a payoff which amounts to $f^{ij}(t)dt$ and which is a profit (resp. cost) for P1 (resp. P2). On the other hand when the player P1 (resp. P2) makes the decision at time t to switch from mode i (resp. j) to $k \in \Gamma^1 - \{i\}$ (resp. $l \in \Gamma^2 - \{j\}$), she pays an amount which equals to $\underline{g}_{ik}(t)$ (resp. $\bar{g}_{jl}(t)$). Therefore a switching control for P1 (resp. P2), denoted by u (resp. v) is a sequence of pairs $u := (\sigma_n, \delta_n)_{n\ge 0}$ (resp. $v := (\tau_n, \zeta_n)_{n \ge 0}$) such that: $\forall n \ge 0$,

i) σ_n is an $\mathbb{F}$-stopping time such that $\sigma_n \le \sigma_{n+1}$ and δ_n is a r.v. with values in Γ^1 and $\mathcal{F}_{\sigma_n}$-measurable (resp. τ_n is an $\mathbb{F}$-stopping time such that $\tau_n \le \tau_{n+1}$ and ζ_n is a r.v. with values in Γ^2 and $\mathcal{F}_{\tau_n}$-measurable) ;

ii) $\mathbb{P}[\sigma_n < T, \forall n \ge 0] = 0$ (resp. $\mathbb{P}[\tau_n < T, \forall n \ge 0] = 0$) ;

iii) Let us define the process A^u (resp. B^v) by

$$A^u_t := \sum_{n \ge 1} \underline{g}_{\delta_{n-1}\delta_n}(\sigma_n) \mathbb{1}_{(\sigma_n \le t)} \text{ for } t < T \text{ and } A^u_T := \lim_{t \to T} A^u_t$$

(resp. $B^v_t := \sum_{n \ge 1} \bar{g}_{\zeta_{n-1}\zeta_n}(\tau_n) \mathbb{1}_{(\zeta_n \le t)}$ for $t < T$ and $B^v_T := \lim_{t \to T} B^v_t$)
then $\mathbb{E}[|A^u_T|^2] < \infty$ (resp. $\mathbb{E}[|B^v_T|^2] < \infty$).

A control which satisfies the properties i)-iii) is called admissible.

Next let $\mathcal{A}^i_t$ (resp. $\mathcal{B}^j_t$) be the set of admissible controls $u := (\sigma_n, \delta_n)_{n \ge 0}$ (resp. $v := (\tau_n, \zeta_n)_{n \ge 0}$) for P1 (resp. P2) satisfying $\sigma_0 = t, \delta_0 = i$ (resp. $\tau_0 = t, \xi_0 = j$).

To proceed let $(u,v) \in \mathcal{A}^i_t \times \mathcal{B}^j_t$ be a pair of switching controls of the players. We define the coupling of (u,v) by $\gamma(u,v) = (\rho_n, \gamma_n)_{n\geq 0}$ as the modes under which the system is run along with time after t when P1 (resp. P2) implements u (resp. v). In our definition we give the priority of switching to player P1 in the case when both players make the decision to switch at the same time.

Precisely let:

i) $r_0 = s_0 = 1, r_1 = s_1 = 1$ and for $n \geq 2$,

$$r_n = r_{n-1} + \mathbb{1}_{(\sigma_{r_{n-1}} \leq \tau_{s_{n-1}})}, \quad s_n = s_{n-1} + \mathbb{1}_{(\tau_{s_{n-1}} < \sigma_{r_{n-1}})};$$

ii)

$$\forall n \geq 0, \rho_n = \sigma_{r_n} \wedge \tau_{s_n};$$

iii) $(\gamma_n := (\gamma_n^{(1)}, \gamma_n^{(2)}))_{n\geq 0}$ is a sequence of $\Gamma-$valued random variables defined as follows: $\gamma_0 = (\delta_0, \zeta_0)$ and for all $n \geq 1$,

$$\gamma_n = \begin{cases} (\delta_{r_n}, \gamma_{n-1}^{(2)}) & \text{if } \sigma_{r_n} \leq \tau_{s_n} \text{and } \sigma_{r_n} < T; \\ (\gamma_{n-1}^{(1)}, \zeta_{s_n}) & \text{if } \tau_{s_n} < \sigma_{r_n}; \\ \gamma_{n-1} & \text{if } \tau_{s_n} = \sigma_{r_n} = T. \end{cases}$$

We associate with $\gamma(u,v)_t$ the following process $(\pi_s)_{s\in[t,T]}$ which indicates in which pair of modes the system is along with time: $\forall s \in [t,T]$,

$$\pi_s = \gamma_0 \mathbb{1}_{[\rho_0,\rho_1]}(s) + \sum_{n\geq 1} \gamma_n \mathbb{1}_{(\rho_n,\rho_{n+1}]}(s)$$

where $(\rho_n, \rho_{n+1}] = \emptyset$ on $\{\rho_n = \rho_{n+1}\}$.

Finally when the player P1 (resp. P2) implements the control $u \in \mathcal{A}_t^i$ (resp. $v \in \mathcal{B}_t^j$), the payoff in-between, which is a reward for P1 and a cost for P2, is given by:

$$J_t^{ij}(\gamma(u,v)) = \mathbb{E}\left[\xi^{\pi_T} + \int_t^T f^{\pi}(s)ds - \sum_{n\geq 1}\left(\underline{g}_{\gamma_{n-1}^{(1)}\gamma_n^{(1)}}(\rho_n) - \overline{g}_{\gamma_{n-1}^{(2)}\gamma_n^{(2)}}(\rho_n)\right)\middle| \mathcal{F}_t\right] \tag{2.42}$$

where $\xi^{\pi_T} = \xi^{ij}$ if at time T, $\pi_T = (i,j)$ and $f^{\pi}(s) = f^{ij}$ if at time s, $\pi(s) = (i,j)$, for any $s \leq T$.

The following result is stated in [33]:

Theorem 2.4.1. *([33], Theorem 3.1) For any $t \in [0,T]$ and $(i,j) \in \Gamma$,*

$$\underline{Y}_t^{ij} = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} J_t^{ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} J_t^{ij}(\gamma(u,v)).$$

As a by-product of this result we have the following one related to uniqueness of the solution of system (2.41) which stems from the above characterization of the component Y^{ij} as the value function of the zero-sum switching game.

Corollary 2.4.2. *Let* $(\underline{Y}_1^{ij}, \underline{Z}_1^{ij}, \underline{K}_1^{ij,\pm})_{(i,j)\in\Gamma}$ *be another solution of system* (2.41)*, then for any* $(i,j) \in \Gamma$*,*

$$\underline{Y}^{ij} = \underline{Y}_1^{ij},\ \underline{Z}^{ij} = \underline{Z}_1^{ij} \text{ and } \underline{K}_1^{ij,+} - \underline{K}_1^{ij,-} = \underline{K}^{ij,+} - \underline{K}^{ij,-}.$$

Finally thanks to Theorems 2.3.6 and 2.4.1, we will prove the existence and uniqueness of the solution for the system of reflected BSDEs with bilateral interconnected obstacles (2.4) without assuming Assumption [H5] on monotonicity and we instead assume [H6].

Theorem 2.4.3. *Assume that [H1]-[H4] and [H6] are fulfilled. Then system of reflected BSDEs* (2.4) *has a solution* $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$*, i.e., for any* $(i,j) \in \Gamma$ *and* $t \leq T$*,*

$$\begin{cases} Y^{ij} \in \mathcal{S}^2, Z^{ij} \in \mathcal{H}^{2,d}, K^{ij,\pm} \in \mathcal{A}^2 \ ; \\ Y_t^{ij} = \xi^{ij} + \displaystyle\int_t^T f^{ij}\left(s, \omega, (Y_s^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}\right) ds - \int_t^T Z_s^{ij} dB_s + K_T^{ij,+} - K_t^{ij,+} - (K_T^{ij,-} - K_t^{ij,-})\ ; \\ L_t^{ij} \leq Y_t^{ij} \leq U_t^{ij}\ ; \\ \displaystyle\int_0^T \left(Y_t^{ij} - L_t^{ij}\right) dK_t^{ij,+} = 0 \text{ and } \int_0^T \left(U_t^{ij} - Y_t^{ij}\right) dK_t^{ij,-} = 0, \end{cases} \tag{2.43}$$

where $L_t^{ij} := \max\limits_{k\in(\Gamma^1)^{-i}} \left\{Y_t^{kj} - \underline{g}_{ik}(t)\right\}$ *and* $U_t^{ij} := \min\limits_{l\in(\Gamma^2)^{-j}} \left\{Y_t^{il} + \overline{g}_{jl}(t)\right\}$*. Moreover it is unique in the following sense: If* $(\overline{Y}^{ij}, \overline{Z}^{ij}, \overline{K}^{ij,+}, \overline{K}^{ij,-})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is another solution of* (2.43)*, then for any* $(i,j) \in \Gamma$*,*

$$\overline{Y}^{ij} = Y^{ij},\ \overline{Z}^{ij} = Z^{ij},\ \overline{K}^{ij,+} - \overline{K}^{ij,-} = K^{ij,+} - K^{ij,-}.$$

Proof. First let us define the following operator:

$$\begin{aligned} \Phi: \quad & \mathcal{H}^{2,\Lambda} \to \mathcal{H}^{2,\Lambda} \\ \vec{\phi} := (\phi^{ij})_{(i,j)\in\Gamma} & \mapsto \Phi(\vec{\phi}) := \left(Y^{\phi,ij}\right)_{(i,j)\in\Gamma} \end{aligned} \tag{2.44}$$

where $(Y^{\phi,ij}, Z^{\phi,ij}, K^{\phi,ij,\pm})_{(i,j)\in\Gamma}$ is the solution of the following system (this solution exists and is unique by Theorem 2.3.6 and Corollary 2.4.2): $\forall (i,j) \in \Gamma$,

$$\begin{cases} Y^{\phi,ij} \in \mathcal{S}^2, Z^{\phi,ij} \in \mathcal{H}^{2,d}, K^{\phi,ij,\pm} \in \mathcal{A}^2; \\ Y_t^{\phi,ij} = \xi^{ij} + \displaystyle\int_t^T f^{ij}(s, \vec{\phi}(s)) ds - \int_t^T Z_s^{\phi,ij} dB_s + K_T^{\phi,ij,+} - K_t^{\phi,ij,+} - (K_T^{\phi,ij,-} - K_t^{\phi,ij,-}),\ \forall t \leq T; \\ L_t^{\phi,ij} \leq Y_t^{\phi,ij} \leq U_t^{\phi,ij},\ \forall t \in [0,T]; \\ \displaystyle\int_0^T \left(Y_t^{\phi,ij} - L_t^{\phi,ij}\right) dK_t^{\phi,ij,+} = 0 \quad \text{and} \quad \int_0^T \left(U_t^{\phi,ij} - Y_t^{\phi,ij}\right) dK_t^{\phi,ij,-} = 0 \end{cases} \tag{2.45}$$

where $L^{\phi,ij}$ and $U^{\phi,ij}$ are defined as previously but with the processes $(Y^{\phi,ij})_{(i,j)\in\Gamma}$. Let $\vec{\psi} := (\psi^{ij})_{(i,j)\in\Gamma}$ be another element of $\mathcal{H}^{2,\Lambda}$ and let $(Y^{\psi,ij}, Z^{\psi,ij}, K^{\psi,ij,\pm})_{(i,j)\in\Gamma}$ be defined as in (2.45) but where $\vec{\phi}$ is replaced with $\vec{\psi}$.

Next let us introduce the following norm on $\mathcal{H}^{2,\Lambda}$, denoted by $\| \,.\, \|_{2,\alpha}$, and defined by

$$\| y \|_{2,\alpha} := \left[\mathbb{E}\left(\int_0^T e^{\alpha t}|y_t|^2 dt\right)\right]^{\frac{1}{2}}.$$

The space $(\mathcal{H}^{2,\Lambda}, \| \,.\, \|_{2,\alpha})$ is of Banach type. If the map Φ is a contraction on $(\mathcal{H}^{2,\Lambda}, \| \,.\, \|_{2,\alpha})$, then it has a fixed point which is the unique solution of (2.43). So let us show that Φ is a contraction. By Theorem 2.4.1, the following representation holds true: $\forall (i,j) \in \Gamma$ and $t \leq T$,

$$Y_t^{\phi,ij} = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} J_t^{\phi,ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} J_t^{\phi,ij}(\gamma(u,v))$$

where

$$J_t^{\phi,ij}(\gamma(u,v)) = \mathbb{E}\left[\xi^{\pi_T} + \int_t^T f^{\pi}(s,\vec{\phi}(s))ds - \sum_{n\geq 1}\left(\underline{g}_{\gamma_{n-1}^{(1)}\gamma_n^{(1)}}(\rho_n) - \overline{g}_{\gamma_{n-1}^{(2)}\gamma_n^{(2)}}(\rho_n)\right)\middle|\, \mathcal{F}_t\right]. \quad (2.46)$$

Next let $\vec{\psi} := (\psi^{ij})_{(i,j)\in\Gamma}$ be another element of $\mathcal{H}^{2,\Lambda}$. Once again, for any $(i,j) \in \Gamma$ and $t \leq T$,

$$Y_t^{\psi,ij} = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} J_t^{\psi,ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} J_t^{\psi,ij}(\gamma(u,v))$$

where $J_t^{\psi,ij}$ is defined similarly as $J_t^{\phi,ij}$ but with $\vec{\psi}$ instead of $\vec{\phi}$. Therefore

$$\forall t \leq T,\ |Y_t^{\psi,ij} - Y_t^{\phi,ij}| \leq \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} \operatorname*{ess\,sup}_{v\in\mathcal{B}_t^j} \left| J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right|. \quad (2.47)$$

First, by the martingale representation theorem, there exists a predictable process $\Delta Z^{\psi,\phi,\pi} \in \mathcal{H}^{2,d}$ (π depends on (i,j)) which is adapted with respect to $(\mathcal{F}_t)_{t\leq T}$ such that: $\forall t \leq T$,

$$\begin{aligned} J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) &= \mathbb{E}\left[\int_t^T (f^{\pi}(s,\vec{\psi}(s)) - f^{\pi}(s,\vec{\phi}(s)))ds \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_0^T (f^{\pi}(s,\vec{\psi}(s)) - f^{\pi}(s,\vec{\phi}(s)))ds \,\middle|\, \mathcal{F}_t\right] - \int_0^t (f^{\pi}(s,\vec{\psi}(s)) - f^{\pi}(s,\vec{\phi}(s)))ds \\ &= C^{\psi,\phi,\pi} + \int_0^t \Delta Z_s^{\psi,\phi,\pi} dB_s - \int_0^t (f^{\pi}(s,\vec{\psi}(s)) - f^{\pi}(s,\vec{\phi}(s)))ds. \end{aligned} \quad (2.48)$$

where $C^{\psi,\phi,\pi} := \mathbb{E}\left[\int_0^T (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds\right]$. Thus $\forall t \leq T$,

$$d(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))) = -(f^\pi(t,\vec{\psi}(t)) - f^\pi(t,\vec{\phi}(t)))dt + \Delta Z_t^{\psi,\phi,\pi} dB_t. \tag{2.49}$$

Next by applying Itô's formula, one has: $\forall t \leq T$,

$$\begin{aligned} d\left[e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2\right] &= \alpha e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2 dt \\ &+ 2\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)\left\{-\left(f^\pi(t,\vec{\psi}(t)) - f^\pi(t,\vec{\phi}(t))\right)dt + \Delta Z_t^{\psi,\phi,\pi} dB_t\right\}. \end{aligned} \tag{2.50}$$

Now let $t \in [0,T]$ fixed. By integrating in (2.50) from t to T we obtain:

$$\begin{aligned} e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2 + \int_t^T e^{\alpha s}|\Delta Z_s^{\psi,\phi,\pi}|^2 ds &= -\alpha \int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)^2 ds \\ &+ 2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)\left(f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s))\right)ds \\ &- 2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi} dB_s. \end{aligned} \tag{2.51}$$

Now let us apply the inequality $2ab \leq \alpha a^2 + \alpha^{-1}b^2, \forall \alpha > 0, a, b \in \mathbb{R}$, then (2.51) yields

$$\begin{aligned} &e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2 + \int_t^T e^{\alpha s}|\Delta Z_s^{\psi,\phi,\pi}|^2 ds \\ &\leq \frac{1}{\alpha}\int_t^T e^{\alpha s}\left(f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s))\right)^2 ds - 2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi} dB_s. \end{aligned}$$

Then by Lipschitz condition of f we have

$$\begin{aligned} e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2 &\leq \frac{C^2(f)}{\alpha}\int_t^T e^{\alpha s}|\vec{\psi}(s) - \vec{\phi}(s)|^2 ds \\ &- 2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi} dB_s \end{aligned} \tag{2.52}$$

where $C(f) = \sum_{(i,j)\in\Gamma} C_{ij}$ with C_{ij} is the Lipschitz constant w.r.t. f^{ij}. Next $(\int_t^s e^{\alpha r}(J_r^{\psi,ij}(\gamma(u,v)) - J_r^{\phi,ij}(\gamma(u,v)))\Delta Z_r^{\psi,\phi,\pi} dB_r)_{s\in[t,T]}$ is a martingale. Then by taking the conditional expectation on both sides of (2.52) we obtain

$$\mathbb{E}\left[e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v))\right)^2 \middle| \mathcal{F}_t\right] \leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_s^T e^{\alpha r}|\vec{\psi}(r) - \vec{\phi}(r)|^2 dr \middle| \mathcal{F}_t\right]. \tag{2.53}$$

Take now the limit as $s \to t$ in (2.53) yields

$$e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))\right)^2 \le \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha r}|\vec{\psi}(r)-\vec{\phi}(r)|^2 dr \,\middle|\, \mathcal{F}_t\right], \ \forall t \le T. \quad (2.54)$$

Let us recall now (2.47), then (2.54) implies that: $\forall t \le T$,

$$e^{\alpha t}\left(Y_t^{\psi,ij} - Y_t^{\phi,ij}\right)^2 \le \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2 ds \,\middle|\, \mathcal{F}_t\right] \quad (2.55)$$

Next take the expectation in both hand-sides of (2.56) (and replace t with 0 in the right one) to obtain:

$$\mathbb{E}\left[e^{\alpha t}\left(Y_t^{\psi,ij} - Y_t^{\phi,ij}\right)^2\right] \le \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_0^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2 ds\right]. \quad (2.56)$$

Finally by integrating (2.55) from 0 to T and summing over $(i,j) \in \Gamma$ we get

$$\int_0^T \sum_{(i,j)\in\Gamma} e^{\alpha t}\left(Y_t^{\psi,ij} - Y_t^{\phi,ij}\right)^2 dt \le \frac{C^2(f)T\Lambda}{\alpha}\mathbb{E}\left[\int_0^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2 ds\right] \quad (2.57)$$

Now if we take $\alpha > C^2(f)T\Lambda$ then $\dfrac{C^2(f)T\Lambda}{\alpha} < 1$. This implies that Φ is a contraction from $\mathcal{H}^{2,\Lambda}$ into itself, and then it has a fixed point which is the unique solution of (2.43). The proof is complete.

□

As a by-product of the above result we also have:

Corollary 2.4.4. *The Λ-tuple of processes $(Y^{ij})_{(i,j)\in\Gamma}$ is the unique fixed point of the mapping Φ on $\mathcal{H}^{2,\Lambda}$.*

Remark 2.4.5. *Assume that for any $(i,j) \in \Gamma$, the function f^{ij} does not depend on z and verify the monotonicity Assumption [H5], then the solution constructed in Section 3, Theorem 2.3.6, is unique.*

2.5 Connection with systems of PDEs with bilateral interconnected obstacles

It is well-known that BSDEs, through the Feynman-Kac representation of solutions in the Markovian framework of randomness, provide solutions for partial differential equations. Similarly, in this section we are going to show that, in this very Markovian framework, the component $(Y^{ij})_{ij\in\Gamma}$ of the solution of system (2.43), has a Feynman-Kac representation which, besides, provides a unique solution in viscosity sense of the following system of PDEs with bilateral interconnected obstacles: For any $(i,j) \in \Gamma$,

$$\begin{cases} \min\Big\{v^{ij}(t,x) - \max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x) - \underline{g}_{ik}(t,x)]; \max\Big[v^{ij}(t,x) - \min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x) + \overline{g}_{jl}(t,x)]; \\ -\partial_t v^{ij}(t,x) - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big]\Big\} = 0; \\ \\ v^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{2.58}$$

So first let us fix the framework:

i) A function $\varrho : (t,x) \in [0,T] \mapsto \varrho(t,x) \in \mathbb{R}^m$ $(m \geq 1)$ has of polynomial growth if there exist two non-negative real constants C and γ such that $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$|\varrho(t,x)| \leq C(1 + |x|^\gamma).$$

Hereafter this class of functions is denoted by Π_g.

ii) Let $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$(or $\mathcal{C}^{1,2}$ for short) denote the set of real-valued functions defined on $[0,T] \times \mathbb{R}^k$ which are respectively once and twice differentiable w.r.t. t and x, with continuous derivatives.

iii) Let $b(t,x)$ and $\sigma(t,x)$ be two functions from $[0,T] \times \mathbb{R}^k$ into $\mathbb{R}^k$ jointly continuous and Lipschitz w.r.t x, i.e., for any $(t,x,x') \in [0,T] \times \mathbb{R}^{k+k}$, there exists a non-negative constant C such that

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'|. \tag{2.59}$$

Therefore b and σ are of linear growth w.r.t x, i.e.,

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|). \tag{2.60}$$

Under (2.59)-(2.60), for any $(t,x) \in [0,T] \times \mathbb{R}^k$, there exists a unique process $X^{t,x}$ solution of the following standard SDE:

$$\begin{aligned} &dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \ s \in [t,T]; \\ &X_s^{t,x} = x, \ \forall s \leq t. \end{aligned} \tag{2.61}$$

Besides, $X^{t,x}$ satisfies the following estimates: $\forall \gamma \geq 1$,

$$\mathbb{E}\left[\sup_{s\leq T} |X_s^{t,x}|^\gamma\right] \leq C(1 + |x|^\gamma) \tag{2.62}$$

and its infinitesimal generator $\mathcal{L}^X$ is given by: for any $(t,x) \in [0,T] \times \mathbb{R}^k, \phi \in \mathcal{C}^{1,2}$ $((.)^\top$ is the transpose),

$$\mathcal{L}^X\phi(t,x) := \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^\top(t,x))_{i,j}\partial^2_{x_ix_j}\phi(t,x) + \sum_{i=1}^{k} b_i(t,x)\partial_{x_i}\phi(t,x). \tag{2.63}$$

We are now going to decline the assumptions [H1]-[H4] of Section 2.2 in this markovian framework of randomness. So let us introduce deterministic functions $f^{ij}(t,x,\vec{y})$, $h^{ij}(x)$, $\underline{g}_{ik}(t,x)$ and $\bar{g}_{jl}(t,x)$, $i,k\in\Gamma^1$, $j,l\in\Gamma^2$ and $t,x,\vec{y}$ in $[0,T]$, $\mathbb{R}^k$ and $\mathbb{R}^\Lambda$ respectively.

[H1b]: For any $(i,j)\in\Gamma$,

i) There exist non negative constants C and γ such that

$$|f^{ij}(t,x,\vec{y})| \le C(1+|x|^\gamma+|\vec{y}\,|).$$

ii) f^{ij} is Lipschitz continuous w.r.t. $\vec{y}$ uniformly in (t,x), i.e. there exists a constant C such that for any $\vec{y}_1,\vec{y}_2\in\mathbb{R}^\Lambda$,

$$|f^{ij}(t,x,\vec{y}_1) - f^{ij}(t,x,\vec{y}_2)| \le C|\vec{y}_1-\vec{y}_2|.$$

[H2b]: For any $(i,j)\in\Gamma$, the function h^{ij}, which stands for the terminal condition, is continuous w.r.t. x, belongs to class Π_g and satisfies the following consistency condition: $\forall (i,j)\in\Gamma$ and $x\in\mathbb{R}^k$,

$$\max_{k\in(\Gamma^1)^{-i}}(h^{kj}(x)-\underline{g}_{ik}(T,x)) \le h^{ij}(x) \le \min_{l\in(\Gamma^2)^{-j}}(h^{il}(x)+\overline{g}_{jl}(T,x)). \tag{2.64}$$

[H3b]: For all $i_1,i_2\in\Gamma^1$ (resp. $j_1,j_2\in\Gamma^2$), the function $\underline{g}_{i_1i_2}$(resp. $\overline{g}_{j_1j_2}$)

iii) is non-negative, continuous and belong to Π_g ;

iv) For any $k\in\Gamma^1$ (resp. $\ell\in\Gamma^2$) such that $|\{i_1,i_2,k\}|=3$ (resp. $|\{j_1,j_2,\ell\}|=3$) it holds: $\forall (t,x)\in[0,T]\times\mathbb{R}^k$,

$$\underline{g}_{i_1i_2}(t,x) < \underline{g}_{i_1k}(t,x)+\underline{g}_{ki_2}(t,x) \ \Big(\text{resp. } \overline{g}_{j_1j_2}(t,x) < \overline{g}_{j_1\ell}(t,x)+\overline{g}_{\ell j_2}(t,x)\Big)\,; \tag{2.65}$$

v) The functions $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\overline{g}_{jl})_{j,l\in\Gamma^2}$ verify the non free loop property, that is to say, if $(i_k,j_k)_{k=1,2,\ldots,N}$ is a loop in Γ, i.e., $(i_N,j_N)=(i_1,j_1)$, **card** $\{(i_k,j_k)_{k=1,2,\ldots,N}\}=N-1$ and for any $k=1,2,\ldots,N-1$, either $i_{k+1}=i_k$ or $j_{k+1}=j_k$, we have:

$$\forall t\le T,\quad \sum_{k=1}^{N-1} G_{i_kj_k}(t,x)\ne 0 \tag{2.66}$$

where $\forall k = 1, ...N-1, \quad G_{i_k j_k}(t,x) = -\underline{g}_{i_k i_{k+1}}(t,x)\mathbb{1}_{(i_k \neq i_{k+1})} + \overline{g}_{j_k j_k}(t,x)\mathbb{1}_{(j_k \neq j_{k+1})}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1 \times \Gamma^2$ is not free, i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode.

Note that (2.66) also implies: For any $(i_1, ..., i_N) \in (\Gamma^1)^N$ such that $i_N = i_1$ and $\mathbf{card}\{i_1, i_2, ..., i_N\} = N-1$,

$$\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ such that $j_N = j_1$ and $\mathbf{card}\{j_1, j_2, ..., j_N\} = N-1$,

$$\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

[H4b] For any $i, k \in \Gamma^1$ (resp. $j, l \in \Gamma^2$), $\underline{g}_{ik}$(resp. $\overline{g}_{jl}$) is $\mathcal{C}^{1,2}$ and $D_x \underline{g}_{ik}, D^2_{xx}\underline{g}_{ik}$ (resp. $D_x \overline{g}_{jl}, D^2_{xx}\overline{g}_{jl}$) belong to Π_g. Thus by Itô's formula we have:

$$\begin{cases} \underline{g}_{ik}(s, X^{t,x}_s) = \underline{g}_{ik}(t,x) + \int_t^s \mathcal{L}^X(\underline{g}_{ik})(r, X^{t,x}_r)dr + \int_t^s D_x \underline{g}_{ik}(r, X^{t,x}_r)\sigma(r, X^{t,x}_r)dB_r, \ s \in [t,T]; \\ \underline{g}_{ik}(s, X^{t,x}_s) = \underline{g}_{ik}(s,x), \ s \leq t. \end{cases}$$

$$\left(\text{resp.} \begin{cases} \overline{g}_{jl}(s, X^{t,x}_s) = \overline{g}_{jl}(t,x) + \int_t^s \mathcal{L}^X(\overline{g}_{jl})(r, X^{t,x}_r)dr \int_t^s D_x \overline{g}_{jl}(r, X^{t,x}_r)\sigma(r, X^{t,x}_r)dB_r, \ s \in [t,T]; \\ \overline{g}_{jl}(s, X^{t,x}_s) = \overline{g}_{jl}(s,x), \ s \leq t. \end{cases} \right).$$

Remark 2.5.1. *Since $D_x \underline{g}_{ik}, D^2_{xx}\underline{g}_{ik}$ (resp. $D_x \overline{g}_{jl}, D^2_{xx}\overline{g}_{jl}$) belong to Π_g, taking into account of assumptions* (2.60) *on linear growth of b and σ and finally estimate* (2.62)*, one gets that* $\sup_{s \leq T} |D_x \underline{g}_{ik}(s, X^{t,x}_s)|$ *(resp.* $\sup_{s \leq T} |D_x \overline{g}_{jl}(s, X^{t,x}_s)|$*) belongs to $L^2(d\mathbb{P})$.* □

To begin with we first give the following result which stems from Theorem 2.4.3 under assumptions [H1b]-[H4b].

Proposition 2.5.2. *Assume that Assumptions [H1b]-[H4b] are fulfilled. Then for any $(t,x) \in [0,T] \times \mathbb{R}^k$, there exist processes $(Y^{ij;t,x}, Z^{ij;t,x}, K^{ij,+;t,x}, K^{ij,-;t,x})_{(i,j)\in\Gamma}$ unique solution of system of reflected BSDEs with bilateral interconnected obstacles associated with $(f^{ij}, h^{ij}, \underline{g}_{ik}, \overline{g}_{jl})$, i.e., for any $(i,j) \in \Gamma$*

and $s \in [0,T]$,

$$\begin{cases} Y^{ij;t,x} \in \mathcal{S}^2, Z^{ij;t,x} \in \mathcal{H}^{2,1}, K^{ij,\pm;t,x} \in \mathcal{A}^2 \ ; \\ Y_s^{ij;t,x} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}\left(r, X_r^{t,x}, (Y_r^{kl;t,x})_{(k,l)\in\Gamma}\right) dr - \int_s^T Z_r^{ij;t,x} dB_r + K_T^{ij,+;t,x} - K_s^{ij,+;t,x} \\ \qquad -(K_T^{ij,-;t,x} - K_s^{ij,-;t,x}); \\ L_s^{ij;t,x} \le Y_s^{ij;t,x} \le U_s^{ij;t,x}; \\ \int_0^T (Y_s^{ij;t,x} - L_s^{ij;t,x}) dK_s^{ij,+;t,x} = 0 \text{ and } \int_0^T (Y_s^{ij;t,x} - U_s^{ij;t,x}) dK_s^{ij,-;t,x} = 0 \end{cases} \tag{2.67}$$

where $L_s^{ij;t,x} := \max_{k\in(\Gamma^1)^{-i}} \left[Y_s^{kj;t,x} - \underline{g}_{ik}(s, X_s^{t,x})\right]$ *and* $U_s^{ij;t,x} := \min_{l\in(\Gamma^2)^{-j}} \left[Y_s^{il;t,x} + \overline{g}_{jl}(s, X_s^{t,x})\right]$, $s \in [0,T]$.

We are now going to focus on the properties of $(Y^{ij;t,x})_{(i,j)\in\Gamma}$. For simplicity reasons the quadruple of processes $(Y^{ij;t,x}, Z^{ij;t,x}, K^{ij,+;t,x}, K^{ij,-;t,x})$ will be sometimes simply denoted by $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})$.

Theorem 2.5.3. *Assume that Assumptions [H1b]-[H4b] are fulfilled. Then there exist deterministic continuous functions* $(v^{ij})_{(i,j)\in\Gamma}$ *of polynomial growth, defined on* $[0,T]\times\mathbb{R}^k$ *such that for any* $(i,j) \in \Gamma$, $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$\mathbb{P} - a.s., \forall s \in [t,T], \quad Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x}). \tag{2.68}$$

Proof. The proof is given in several steps.

A) We first assume that $\forall (i,j) \in \Gamma$, $(t,x) \in [0,T]\times\mathbb{R}^k$, $f^{ij}(t,x,0,0)$ and $h^{ij}(x)$ are bounded.

We will prove that for any $(i,j) \in \Gamma$, for a fixed δ_1 there exists a bounded continuous deterministic function v^{ij} defined on $[T-\delta_1, T]\times\mathbb{R}^k$ such that for any $(t,x) \in [T-\delta_1,T]\times\mathbb{R}^k$ we have:

$$\mathbb{P}- \text{ a.s. for any } s \in [t,T], Y_s^{ij} = v^{ij}(s, X_s^{t,x}).$$

Let us recall the system (2.67) and let $(\bar{Y}, \bar{Z})$ be the unique solution in $\mathcal{S}^2 \times \mathcal{H}^{2,d}$ of the following BSDE (it depends on t,x which we omit as there is no confusion):

$$\bar{Y}_s = \bar{h}(X_T^{t,x}) + \int_s^T \Psi(\bar{Y}_r) dr - \int_s^T \bar{Z}_r dB_r, \ s \le T,$$

where $\bar{h}(x) = \sum_{(i,j)\in\Gamma} |h^{ij}(x)|$ and $\Psi(y) := \Lambda^2 C^\sharp (1+|y|)$ where $C^\sharp = \max\{C(f), \underline{C}\}$ with $\underline{C}$ is a uniform constant of boundedness of $|f^{ij}(t,x,0)|$. It is well-known that there exists a bounded deterministic continuous function $\bar{v}$ such that $\mathbb{P}$-a.s., $\forall s \in [t,T]$, $\bar{Y}_s = \bar{v}(s, X_s^{t,x})$ (see e.g. [24]). Finally note that $\bar{Y} \ge 0$ and then $\bar{v} \ge 0$.

Now for any $(i,j) \in \Gamma$, we set $(\hat{Y}^{ij}, \hat{Z}^{ij}, \hat{K}^{ij,+}, \hat{K}^{ij,-}) := (\bar{Y}, \bar{Z}, 0, 0)$. Therefore $(\hat{Y}^{ij}, \hat{Z}^{ij}, \hat{K}^{ij,+}, \hat{K}^{ij,-})$ is the unique solution of doubly reflected BSDEs associated with $(\bar{h}^{ij}, \hat{\Psi}, (\underline{g}_{ik})_{k\in(\Gamma^1)^{-i}},$

$(\bar{g}_{jl})_{l\in(\Gamma^2)^{-j}})$ where $\hat{\Psi}(y) := \Lambda^2 C^\sharp(1+(y)^+)$. This actually holds in taking into account of: i) the backward equation satisfied by $(\bar{Y},\bar{Z})$; ii) the fact that $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative ; iii) the fact that $\bar{Y}\geq 0$ and then $|\bar{Y}| = \bar{Y}^+$. Lastly let us notice that by Theorem 2.4.3, the solution of this system exists and is unique and then it is equal to $(\bar{Y},\bar{Z},0,0)_{(i,j)\in\Gamma}$. Hence we also have $\mathbb{P}$-a.s., for any $s\in[t,T]$, $\hat{Y}^{ij}_s = \bar{v}(s,X^{t,x}_s)$.

In the same way, setting $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij,+},\tilde{K}^{ij,-}) = (-\bar{Y},-\bar{Z},0,0)$ for any $(i,j)\in\Gamma$, we obtain that the family $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij,+},\tilde{K}^{ij,-})_{(i,j)\in\Gamma}$ is the unique solution of reflected BSDEs associated with $(-\bar{h}^{ij},\hat{\Psi}_2,(\underline{g}_{ik})_{k\in(\Gamma^1)^{-i}},(\bar{g}_{jl})_{l\in(\Gamma^2)^{-j}})$ where $\hat{\Psi}_2(y) = -C^\sharp\Lambda^2(1+(y)^-)$. Next let us consider the following Picard iterations: for any $(i,j)\in\Gamma$, $Y^{ij,0;t,x}=0$ and for all $n\geq 1$, $(Y^{ij,n;t,x})_{(i,j)\in\Gamma} = \Phi((Y^{ij,n-1;t,x})_{(i,j)\in\Gamma})$, where Φ is defined in (2.44). In other words the family $(Y^{ij,n;t,x},Z^{ij,n;t,x},K^{ij,n,+;t,x},$ $K^{ij,n,-;t,x})_{(i,j)\in\Gamma}$ (which sometimes is simply denoted by $(Y^{ij,n},Z^{ij,n},K^{ij,n,+},K^{ij,n,-})_{(i,j)\in\Gamma}$ as no confusion is possible) is the unique solution of the following system of BSDEs: $\forall(i,j)\in\Gamma$ and $s\in[0,T]$,

$$\begin{cases} Y^{ij,n;t,x}_s = h^{ij}(X^{t,x}_T) + \displaystyle\int_s^T f^{ij}(r,X^{t,x}_r,(Y^{kl,n-1;t,x}_r)_{(k,l)\in\Gamma})dr - \int_s^T Z^{ij,n;t,x}_r dB_r \\ \qquad\qquad +K^{ij,n,+;t,x}_T - K^{ij,n,+;t,x}_s - (K^{ij,n,-;t,x}_T - K^{ij,n,-;t,x}_s); \\ \displaystyle\max_{k\in(\Gamma^1)^{-i}}\left[Y^{kj,n;t,x}_s - \underline{g}_{ik}(s,X^{t,x}_s)\right] \leq Y^{ij,n;t,x}_s \leq \min_{l\in(\Gamma^2)^{-j}}\left[Y^{il,n;t,x}_s + \bar{g}_{jl}(s,X^{t,x}_s)\right]; \\ \displaystyle\int_0^T \left\{Y^{ij,n;t,x}_s - \max_{k\in(\Gamma^1)^{-i}}\left[Y^{kj,n;t,x}_s - \underline{g}_{ik}(s,X^{t,x}_s)\right]\right\} dK^{ij,n,+;t,x}_s = 0; \\ \displaystyle\int_0^T \left\{Y^{ij,n;t,x}_s - \min_{l\in(\Gamma^2)^{-j}}\left[Y^{il,n;t,x}_s + \bar{g}_{jl}(s,X^{t,x}_s)\right]\right\} dK^{ij,n,-;t,x}_s = 0. \end{cases} \tag{2.69}$$

Then we have the following inequalities: for any $n\geq 0$, $(i,j)\in\Gamma$,

$$-\bar{Y}\leq Y^{ij,n}\leq \bar{Y} \tag{2.70}$$

Indeed when $n=0$, (2.70) holds true since for any $(i,j)\in\Gamma$, $(t,x)\in[0,T]\times\mathbb{R}^k$, $-\bar{Y}\leq 0\leq\bar{Y}$. Next we assume that (2.70) holds for some $n-1$, i.e. for any $(i,j)\in\Gamma$, $\tilde{Y}^{ij} = -\bar{Y}\leq Y^{ij,n-1}\leq \hat{Y}^{ij} = \bar{Y}$. Then by [H1b]-ii), the boundedness of $f^{ij}(t,x,\vec{0})$ and the induction hypothesis we have:

$$f^{ij}(s,X^{t,x}_s,(Y^{kl,n-1}_s)_{(k,l)\in\Gamma})) \leq C^\sharp(1+\sum_{(k,l)\in\Gamma}|Y^{kl,n-1}_s|) \leq \Psi(\hat{Y}^{ij}_s).$$

As $h^{ij}(x)\leq \bar{h}(x)$, then by the comparison result (Remark 2.3.7, iii)) between the solutions of equations $(\hat{Y}^{ij})_{ij}$ and (2.69), one deduces that for any $(i,j)\in\Gamma$, $Y^{ij,n}\leq\hat{Y}^{ij}$. Similarly by the induction steps, one deduces that for any $(i,j)\in\Gamma$, $Y^{ij,n}\geq\tilde{Y}^{ij} = -\bar{Y}$. The proof of the claim (2.70) is complete.

Next once more by induction, using the result by Djehiche et al. [19] there exist deterministic continuous functions $(v^{ij,n})_{(i,j)\in\Gamma}$, $n \geq 0$, such that $\forall (i,j) \in \Gamma, (t,x) \in [0,T] \times \mathbb{R}^k$ we have

$$\mathbb{P} - a.s.,\ \forall s \in [t,T], Y_s^{ij,n} = v^{ij,n}(s, X_s^{t,x}). \tag{2.71}$$

Therefore from (2.70), we deduce that for any $(t,x) \in [0,T] \times \mathbb{R}^k$, $-\bar{v}(t,x) \leq v^{ij,n}(t,x) \leq \bar{v}(t,x)$, for any $(i,j) \in \Gamma$. As a by-product the sequence $(v^{ij,n}(t,x))_{n\geq 0}$ is uniformly bounded since $\bar{v}$ is so. Afterwards we just need to prove that $((v^{ij,n})_{(i,j)\in\Gamma})_{n\geq 0}$ is a Cauchy sequence for the uniform convergence norm.

Actually as shown in the proof of Theorem 2.4.3, the sequence $((Y^{ij,n})_{(i,j)\in\Gamma})_{n\geq 0}$ converges in $\mathcal{H}^{2,d}_{[0,T]}$ to $(Y^{ij})_{(i,j)\in\Gamma}$ since $(Y^{ij})_{(i,j)\in\Gamma}$ is the fixed point in $\mathcal{H}^{2,\Lambda}$. On the other hand, for any $t \in [0,T]$ and $x \in \mathbb{R}^k$, by (2.55) we have:

$$e^{\alpha t}|v^{ij,n}(t,x) - v^{ij,q}(t,x)|^2 \leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T \sum_{(i,j)\in\Gamma} |Y_s^{ij,n-1;t,x} - Y_s^{ij,q-1;t,x}|^2 ds\right]. \tag{2.72}$$

But, as mentioned previously, the last term converges to 0 as n, q go to infinite. It follows that for any $(i,j) \in \Gamma$, the sequence $(v^{ij,n})_{n\geq 0}$ is of Cauchy type point-wise on $[0,T] \times \mathbb{R}^k$. Therefore there exists a function v^{ij} defined on $[0,T] \times \mathbb{R}^k$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$, $v^{ij}(t,x) = \lim_{n\to\infty} v^{ij,n}(t,x)$. Moreover, $-\bar{v}(t,x) \leq v^{ij}(t,x) \leq \bar{v}(t,x)$ which implies that the function v^{ij} is bounded. Finally we have

$$\forall\ (i,j) \in \Gamma,\ Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x}),\ ds \otimes d\mathbb{P} \text{ on } [t,T] \times \mathbb{R}^k.$$

Next by the inequality (2.55) and taking expectation to obtain: For any $(t,x) \in [0,T] \times \mathbb{R}^k$, $(i,j) \in \Gamma$ and $n, q \geq 1$,

$$|v^{ij,n}(t,x) - v^{ij,q}(t,x)|^2 \leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha(s-t)} \sum_{(i,j)\in\Gamma} |v^{ij,n-1}(s, X_s^{t,x}) - v^{ij,q-1}(s, X_s^{t,x})|^2 ds\right]. \tag{2.73}$$

Recall (2.73), for any $(i,j) \in \Gamma$ and $t \in [T-\delta_1, T]$ we have

$$\begin{aligned}\sum_{(i,j)\in\Gamma} \|v^{ij,n} - v^{ij,q}\|^2_{\infty,\delta_1} &\leq \frac{C^2(f)\Lambda}{\alpha} e^{\alpha(T-\delta_1)} \int_{T-\delta_1}^T e^{\alpha s} ds \sum_{(i,j)\in\Gamma} \|v^{ij,n-1}(t,x) - v^{ij,q-1}(t,x)\|^2_{\infty,\delta_1} \\ &= \frac{C^2(f)\Lambda(e^{\alpha\delta_1} - 1)}{\alpha^2} \sum_{(i,j)\in\Gamma} \|v^{ij,n-1} - v^{ij,q-1}\|^2_{\infty,\delta_1}\end{aligned} \tag{2.74}$$

Choose now δ_1 such that $\dfrac{C^2(f)\Lambda(e^{\alpha\delta_1}-1)}{\alpha^2}=\dfrac{3}{4}$, then, as a result, the sequence of continuous functions $(v^{ij,n})_{(i,j)\in\Gamma}$ is uniformly convergent on $[T-\delta_1,T]\times\mathbb{R}^k$ which implies that $(v^{ij})_{(i,j)\in\Gamma}$ is continuous on $[T-\delta_1,T]\times\mathbb{R}^k$.

Next by (2.55) and since $Y_t^{ij,n}$ is deterministic then for any $t\in[0,T-\delta_1]$, $x\in\mathbb{R}^k$, we have:

$$\begin{aligned}|v^{ij,n}(t,x)-v^{ij,q}(t,x)|^2 &= \mathbb{E}[|Y_t^{ij,n}-Y_t^{ij,q}|^2] \qquad (2.75)\\ &\le \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha(s-t)}\sum_{(i,j)\in\Gamma}|v^{ij,n-1}(s,X_s^{t,x})-v^{ij,q-1}(s,X_s^{t,x})|^2ds\right]\\ &\le \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^{T-\delta_1} e^{\alpha(s-t)}\sum_{(i,j)\in\Gamma}|v^{ij,n-1}(s,X_s^{t,x})-v^{ij,q-1}(s,X_s^{t,x})|^2ds\right]\\ &\quad+\frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}-v^{ij,q-1}\|^2_{\infty,\delta_1} \qquad (2.76)\end{aligned}$$

The last inequality is valid thanks to (2.74). Now let $(t,x)\in[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$. Taking the supremum on (t,x) in (2.73) and summing over $(i,j)\in\Gamma$, yields:

$$\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|^2_{\infty,2\delta_1}\le\frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|^2_{\infty,\delta_1}+\frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}-v^{ij,q-1}\|^2_{\infty,2\delta_1}.$$

But we know that $\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|_{\infty,\delta_1}\to 0$ as $n,q\to\infty$, therefore we have also:

$$\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|_{\infty,2\delta_1}\to 0,\ \text{as } n,q\to\infty.$$

It follows that for any $(i,j)\in\Gamma$, the sequence $(v^{ij,n})_n$ converges uniformly to v^{ij} in $[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$. Consequently v^{ij} is continuous in $[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$ and then also on $[T-2\delta_1,T]\times\mathbb{R}^k$ since we have already shown that it continuous on $[T-\delta_1,T]\times\mathbb{R}^k$. Repeating now this procedure as many times as necessary on $[T-3\delta_1,T-2\delta_1]\times\mathbb{R}^k$, $[T-4\delta_1,T-3\delta_1]\times\mathbb{R}^k$ and so on, we obtain that for any $(i,j)\in\Gamma$, v^{ij} is continuous on $[0,T]\times\mathbb{R}^k$ and then the processes $(Y_s^{ij;t,x})_{s\in[0,T]}$ and $(v^{ij}(s,X_s^{t,x}))_{s\in[0,T]}$ are indistinguishable, i.e.,

$$\forall\ (i,j)\in\Gamma,\ \mathbb{P}-a.s.,\forall s\in[0,T],Y_s^{ij;t,x}=v^{ij}(s,X_s^{t,x}).$$

B) The general case: The functions $f^{ij}(t,x,0)$ and $h^{ij}(x)$, $(i,j)\in\Gamma$, are of polynomial growth.

Let γ be a positive constant such that for any $(i,j)\in\Gamma$,

$$|f^{ij}(t,x,0)|+|h^{ij}(x)|+|\underline{g}_{ij}(t,x)|+|\overline{g}_{ij}(t,x)|\le C(1+|x|^\gamma).$$

Let $\rho(x) := (1+|x|^2)^{-\gamma}$, $x \in \mathbb{R}^k$, and for any $(i,j) \in \Gamma, s \in [t,T]$, set

$$\tilde{Y}_s^{ij} := Y_s^{ij}\rho(X_s^{t,x}). \tag{2.77}$$

Then by Itô's formula we have: $\forall s \in [t,T]$,

$$\begin{aligned} d\tilde{Y}_s^{ij} &= Y_s^{ij} d\rho(X_s^{t,x}) + \rho(X_s^{t,x})dY_s^{ij} + d\langle Y^{ij}, \rho(X^{t,x})\rangle_s \\ &= \Big[Y_s^{ij}\mathcal{L}\rho(X_s^{t,x}) - \rho(X_s^{t,x})f^{ij}(s, X_s^{t,x}, (Y_s^{kl})_{(k,l)\in\Gamma}) + D_x\rho(X_s^{t,x})\sigma(s, X_s^{t,x})Z_s^{ij}\Big]\, ds \\ &\quad + \Big[Y_s^{ij}D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x}) + \rho(X_s^{t,x})Z_s^{ij}\Big]\, dB_s - \rho(X_s^{t,x})dK_s^{ij,+} + \rho(X_s^{t,x})dK_s^{ij,-}. \end{aligned} \tag{2.78}$$

Next for $(i,j) \in \Gamma$ and $s \in [t,T]$, let us set:

a) $\tilde{Z}_s^{ij} := Y_s^{ij}D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x}) + \rho(X_s^{t,x})Z_s^{ij}$;

b) $d\tilde{K}_s^{ij,+} := \rho(X_s^{t,x})dK_s^{ij,+}$ and $d\tilde{K}_s^{ij,-} := \rho(X_s^{t,x})dK_s^{ij,-}$;

c) $\tilde{f}^{ij}(s, X_s^{t,x}, \overrightarrow{y}) := \rho(X_s^{t,x})f^{ij}(s, X_s^{t,x}, (\rho^{-1}(X_s^{t,x})y^{kl})_{(k,l)\in\Gamma}) - \rho^{-1}(X_s^{t,x})y^{ij}\mathcal{L}\rho(X_s^{t,x})$
$\quad - D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x})\rho^{-1}(X_s^{t,x})[\tilde{Z}_s^{ij} - y^{ij}\rho^{-1}(X_s^{t,x})D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x})]$;

d) $\underline{\tilde{g}}_{ij}(s,X_s^{t,x}) := \rho(X_s^{t,x})\underline{g}_{ij}(s,X_s^{t,x})$ and $\tilde{g}_{ij}(s,X_s^{t,x}) := \rho(X_s^{t,x})\overline{g}_{ij}(s,X_s^{t,x})$;

e) $\tilde{h}^{ij}(X_T^{t,x}) := \rho(X_T^{t,x})h^{ij}(X_T^{t,x})$.

Then the family $(\tilde{Y}^{ij}, \tilde{Z}^{ij}, \tilde{K}^{ij,+}, \tilde{K}^{ij,-})_{(i,j)\in\Gamma}$ is the unique solution of the system of reflected BSDEs associated with $((\tilde{f}^{ij})_{ij}, (\tilde{h}^{ij})_{ij}, (\underline{\tilde{g}}_{ik})_{i,k\in\Gamma^1}, (\tilde{g}_{jl})_{j,l\in\Gamma^2})$.

But for any $(i,j) \in \Gamma$, $\tilde{h}^{ij}, \tilde{f}^{ij}(t,x,0), \underline{\tilde{g}}_{ik}, \tilde{g}_{jl}$ are bounded. Then thanks to the previous step, for any $(i,j) \in \Gamma$, one can find continuous bounded functions $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ defined on $[0,T] \times \mathbb{R}^k$ such that $\tilde{Y}_s^{ij,t,x} = \tilde{v}^{ij}(s, X_s^{t,x})$, $\forall s \in [t,T]$. Therefore in setting, for $(i,j) \in \Gamma$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, $v^{ij}(t,x) = \rho^{-1}(x)\tilde{v}^{ij}(t,x)$ makes that $(v^{ij}(t,x))_{(i,j)\in\Gamma}$ is continuous on $[0,T]\times\mathbb{R}^k$, is of polynomial growth and verifies for any $(i,j) \in \Gamma$, $Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x})$, $\forall s \in [t,T]$. The proof is now complete. □

We are now ready to give the main result of this section.

Theorem 2.5.4. *Assume that Assumptions [H1b]-[H4b] and [H6] are fulfilled. Then the Λ-tuple of continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution (see Appendix for the definition) of the following* *system of variational inequalities with bilateral interconnected obstacles: For any* $(i,j) \in \Gamma$,

$$\begin{cases} \min\{v^{ij}(t,x) - \max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x) - \underline{g}_{ik}(t,x)]; \max\Big\{v^{ij}(t,x) - \min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x) + \overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x) - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big\}\} = 0; \\ \\ v^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{2.79}$$

Moreover it is unique in the class of continuous functions which belong to Π_g.

Proof. We first prove that $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution, then we prove the uniqueness.

Step 1: $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (2.79).
For convenience we recall the unique solution $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ of (2.67): For any $(i,j) \in \Gamma$ and $s \le T$,

$$\begin{cases} Y^{ij}_s = h^{ij}(X^{t,x}_T) + \int_s^T f^{ij}\left(r, X^{t,x}_r, (Y^{kl}_r)_{(k,l)\in\Gamma}\right) dr - \int_s^T Z^{ij}_r dB_r + \int_s^T d(K^{ij,+}_s - dK^{ij,-}_s); \\ \\ L^{ij}_s \le Y^{ij}_s \le U^{ij}_s; \\ \\ \int_0^T \left(Y^{ij}_s - L^{ij}_s\right) dK^{ij,+}_s = 0 \text{ and } \int_0^T \left(Y^{ij}_s - U^{ij}_s\right) dK^{ij,-}_s = 0. \end{cases} \tag{2.80}$$

By (2.68), the system (2.80) can be decoupled as follows: for any $(i,j) \in \Gamma$ and $s \in [t,T]$,

$$\begin{cases} Y^{ij}_s = h^{ij}(X^{t,x}_T) + \int_s^T f^{ij}(r, X^{t,x}_r, (v^{kl}(r, X^{t,x}_r))_{(k,l)\in\Gamma}) dr - \int_s^T Z^{ij}_r dB_r + \int_s^T d(K^{ij,+}_s - dK^{ij,-}_s); \\ \\ \max\limits_{k\in(\Gamma^1)^{-i}} \left[v^{kj}(s, X^{t,x}_s) - \underline{g}_{ik}(s, X^{t,x}_s)\right] \le Y^{ij}_s \le \min\limits_{l\in(\Gamma^2)^{-j}} \left[v^{il}(s, X^{t,x}_s) + \overline{g}_{jl}(s, X^{t,x}_s)\right]; \\ \\ \int_t^T \left\{Y^{ij}_s - \max\limits_{k\in(\Gamma^1)^{-i}}[v^{kj}(s, X^{t,x}_s) - \underline{g}_{ik}(s, X^{t,x}_s)]\right\} dK^{ij,+}_s = 0 \\ \\ \int_t^T \left\{Y^{ij}_s - \min\limits_{l\in(\Gamma^2)^{-j}}[v^{il}(s, X^{t,x}_s) + \overline{g}_{jl}(s, X^{t,x}_s)]\right\} dK^{ij,-}_s = 0. \end{cases} \tag{2.81}$$

Applying Theorem 6.2 in [27] (see also Theorem A.3 in [19]), for any arbitrary (i,j) in Γ, v^{ij} is a viscosity solution of

$$\begin{cases} \min\{v^{ij}(t,x) - \max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x) - \underline{g}_{ik}(t,x)]; \max\Big\{v^{ij}(t,x) - \min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x) + \overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x) - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big\}\} = 0; \\ v^{ij}(T,x) = h^{ij}(x). \end{cases}$$

As (i,j) is arbitrary then $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (2.79).

<u>*Step 2*</u>: Uniqueness

Firstly let us suppose the existence of another solution $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ of system (2.79) which is continuous and of polynomial growth. Next let $(\tilde{y}^{ij})_{(i,j)\in\Gamma}$ be the process of $\mathcal{H}^{2,\Lambda}$ such that for any $(i,j)\in\Gamma$ and $s\le T$,

$$\tilde{y}^{ij}_s = \tilde{v}^{ij}(s, X^{t,x}_s) \tag{2.82}$$

We can now define another process $(\tilde{Y}^{ij})_{(i,j)\in\Gamma}$ via the mapping Φ of (2.44) as follows:

$$(\tilde{Y}^{ij})_{(i,j)\in\Gamma} := \Phi\left((\tilde{y}^{ij})_{(i,j)\in\Gamma}\right) \tag{2.83}$$

By the definition of Φ, $(\tilde{Y}^{ij})_{(i,j)\in\Gamma}$ is the first component of the unique solution of following doubly RBSDEs: For any $(i,j)\in\Gamma$ and $s\le T$,

$$\begin{cases} \tilde{Y}^{ij}_s = h^{ij}(X^{t,x}_T) + \int_s^T f^{ij}(r, X^{t,x}_r, (\tilde{v}^{kl}_r(r, X^{t,x}_r))_{(k,l)\in\Gamma})dr - \int_s^T \tilde{Z}^{ij}_r dB_r + \int_s^T d(\tilde{K}^{ij,+}_s - \tilde{K}^{ij,-}_s); \\ \max\limits_{k\in(\Gamma^1)^{-i}}\left[\tilde{Y}^{kj}_s - \underline{g}_{ik}(s, X^{t,x}_s)\right] \le \tilde{Y}^{ij}_s \le \min\limits_{l\in(\Gamma^2)^{-j}}\left[\tilde{Y}^{il}_s + \overline{g}_{jl}(s, X^{t,x}_s)\right]; \\ \int_0^T \left\{\tilde{Y}^{ij}_s - \max\limits_{k\in(\Gamma^1)^{-i}}\left[\tilde{Y}^{kj}_s - \underline{g}_{ik}(s, X^{t,x}_s)\right]\right\} d\tilde{K}^{ij,+}_s = 0; \\ \int_0^T \left\{\tilde{Y}^{ij}_s - \min\limits_{l\in(\Gamma^2)^{-j}}\left[\tilde{Y}^{il}_s + \overline{g}_{jl}(s, X^{t,x}_s)\right]\right\} d\tilde{K}^{ij,-}_s = 0. \end{cases}$$

As a result, by Theorem 2.5.3, there exist deterministic functions of polynomial growth, denoted $(u^{ij})_{(i,j)\in\Gamma}$, such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(i,j)\in\Gamma$ and $s\in[t,T]$,

$$\tilde{Y}^{ij}_s = u^{ij}(s, X^{t,x}_s).$$

Moreover by the result of Step 1, $(u^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of the following system of variational inequalities with bilateral interconnected obstacles: $\forall (i,j)\in\Gamma$,

$$\begin{cases} \min\{u^{ij}(t,x) - \max\limits_{k\in(\Gamma^1)^{-i}}[u^{kj}_t - \underline{g}_{ik}(t,x); \max[u^{i}(t,x) - \min\limits_{l\in(\Gamma^2)^{-j}}[u^{il}_t + \overline{g}_{jl}(t,x); \\ \qquad -\partial_t u^{ij}(t,x) - \mathcal{L}u^{ij}(t,x) - f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma})]\} = 0; \\ u^{ij}(T,x) = h^{ij}(x) \end{cases} \tag{2.84}$$

since the generators $f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma})$, $(i,j)\in\Gamma$, do not depend on the solution $(u^{ij})_{(i,j)\in\Gamma}$. But the solution of system (2.84) is unique in the class of continuous functions of Π_g (see Theorem 3.2 in [18] for more details) and $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ is a solution in this class. Therefore, for any $(i,j)\in\Gamma$, $u^{ij}=\tilde{v}^{ij}$ and then

$$\mathbb{P}-a.s., \forall s\in[t,T], \tilde{y}^{ij}_s=\tilde{Y}^{ij}_s, \forall (i,j)\in\Gamma. \tag{2.85}$$

Next by (2.83) we obtain on $[t,T]$,

$$(\tilde{y}^{ij})_{(i,j)\in\Gamma} := \Phi\left((\tilde{y}^{ij})_{(i,j)\in\Gamma}\right)$$

However, by Corollary 2.4.4, $(Y^{ij})_{(i,j)\in\Gamma}$ is the only fixed point of Φ in $(\mathcal{H}^{2,\Lambda}_{[t,T]},\|.\|_2)$. Therefore for any $(i,j)\in\Gamma,(t,x)\in[0,T]\times\mathbb{R}^k$, $\mathbb{P}-a.s.$, $\forall\ s\in[t,T]$,

$$\tilde{v}^{ij}(s,X^{t,x}_s)=\tilde{y}^{ij}_s=Y^{ij}_s=v^{ij}(s,X^{t,x}_s). \tag{2.86}$$

Take now $s=t$, leads to $\tilde{v}^{ij}(t,x)=v^{ij}(t,x)$ for any $(i,j)\in\Gamma$ which means that the solution is unique. □

Remark 2.5.5. *The functions $(v^{ij})_{(i,j)\in\Gamma}$ are also the unique solution in the class of continuous functions which belong to Π_g, of the following system which is of max-min type and dual to* (2.79)*: $\forall(i,j)\in\Gamma$,*

$$\begin{cases} \max\{v^{ij}(t,x)-\max\limits_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)];\min[v^{ij}(t,x)-\min\limits_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{kl\in\Gamma})]\}=0; \\ v^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{2.87}$$

This can be shown in considering $(-Y^{ij},-Z^{ij},K^{ij,\pm})_{(i,j)\in\Gamma}$ which is the solution of the system of reflected BSDEs with inter-connected bilateral obstacles associated with $((-f^{ij}(t,x,-\vec{y}))_{(i,j)\in\Gamma},(-h^{ij}(x))_{(i,j)\in\Gamma}$, $(\overline{g}_{jl}(t,x))_{j,l\in\Gamma^2},(\underline{g}_{ik}(t,x))_{i,k\in\Gamma^1})$ and then use the result of the previous Theorem 2.5.4 with $(-v^{ij})_{(i,j)\in\Gamma}$ which implies that $(v^{ij})_{(i,j)\in\Gamma}$ is also he unique solution of (2.87).

2.6 Appendix

The definition of the viscosity solution of system (2.79) is the following:

Definition 2.6.1. *Let $\vec{v}:=(v^{ij})_{(i,j)\in\Gamma}$ be a Λ-tuple of continuous functions on $[0,T]\times\mathbb{R}^k$.*

A) We say that $\vec{v}$ is a viscosity supersolution (resp. subsolution) of (2.79) *if for any fixed (i_0,j_0) in Γ,*

$v^{i_0j_0}$ is a viscosity supersolution (resp. subsolution) of the following PDE with bilateral obstacles:

$$\begin{cases} \min\{v^{i_0j_0}(t,x) - \max_{k\in(\Gamma^1)^{-i_0}}[v^{kj_0}(t,x) - \underline{g}_{i_0k}(t,x)]; \max\Big\{v^{i_0j_0}(t,x) - \min_{l\in(\Gamma^2)^{-j_0}}[v^{i_0l}(t,x) + \overline{g}_{j_0l}(t,x)]; \\ \qquad -\partial_t v^{i_0j_0}(t,x) - \mathcal{L}^X(v^{i_0j_0})(t,x) - f^{i_0j_0}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big\}\} = 0; \\ \\ v^{i_0j_0}(T,x) = h^{i_0j_0}(x), \end{cases} \tag{2.88}$$

that is to say:

i) $v^{i_0j_0}(T,x) \geq h^{i_0j_0}(x)$ (resp. $v^{i_0j_0}(T,x) \leq h^{i_0j_0}(x)$) ;
ii) if $(t,x) \in [0,T) \times \mathbb{R}^k$ and $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ such that (t,x) is a local minimum (resp. maximum) point of $v^{i_0j_0} - \phi$ then

$$\begin{cases} \min\{v^{i_0j_0}(t,x) - \max_{k\in(\Gamma^1)^{-i_0}}[v^{kj_0}(t,x) - \underline{g}_{i_0k}(t,x)]; \max\Big\{v^{i_0j_0}(t,x) - \min_{l\in(\Gamma^2)^{-j_0}}[v^{i_0l}(t,x) + \overline{g}_{j_0l}(t,x)]; \\ \qquad -\partial_t \phi(t,x) - \mathcal{L}^X(\phi)(t,x) - f^{i_0j_0}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\Big\}\} \geq 0 \ (\textit{resp.} \ \leq 0). \end{cases} \tag{2.89}$$

B) We say that $\vec{v} := (v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (2.79) *if it is both a supersolution and subsolution of* (2.79).

CHAPTER 3

PAPER 2: ZERO-SUM SWITCHING GAME, SYSTEMS OF REFLECTED BACKWARD SDES AND PARABOLIC PDES WITH BILATERAL INTERCONNECTED OBSTACLES

This chapter is a preprint joint work with Hamadène (ref.[36]).

3.1 Introduction

This paper is related to zero-sum switching games, systems of reflected backward differential equations (RBSDEs) with bilateral interconnected obstacles and systems of variational inequalities of min-max type with interconnected obstacles, namely the Hamilton-Jacobi-Bellman (HJB for short) system associated with the game.

First let us describe the zero-sum switching game which we will consider in this paper. Let Γ be the set $\{1, ..., p\}$. Assume we have a system which has p working modes indexed by Γ. This system can be switched from one working mode to another one, e.g. due to economic, financial, ecological reasons, etc, by two players or decision makers C_1 and C_2. The main feature of the switching actions is that when the system is in mode $i \in \Gamma$, and one of the players decides to switch it, then it is switched to mode $i+1$ (hereafter $i+1$ is 1 if $i = p$). It means that the decision makers do not have their proper modes to which they can switch the system when they decide to switch (see e.g. [33] for more details on this model). Therefore a switching strategy for the players are sequences of stopping times $u = (\sigma_n)_{n\geq 0}$ for C_1 and $v = (\tau_n)_{n\geq 0}$ for C_2 such that $\sigma_n \leq \sigma_{n+1}$ and $\tau_n \leq \tau_{n+1}$ for any $n \geq 0$. On the other hand, the switching actions are not free and generate expenditures for the players. Loosely speaking at time $t \leq T$, they amount to A^u_t

(resp. B_t^v) given by:

$$A_t^u = \sum_{\sigma_n \le t} \underline{g}_{\zeta_n, \zeta_n+1}(\sigma_n) \text{ (resp. } B_t^v = \sum_{\tau_n \le t} \bar{g}_{\theta_n, \theta_n+1}(\tau_n)).$$

The process $\underline{g}_{i,i+1}(s)$ (resp. $\bar{g}_{i,i+1}(s)$) is the switching cost payed by C_1 (resp. C_2) is she makes the decision to switch the system from mode i to mode $i+1$ at time s while ζ_n (resp. θ_n) is the mode in which the system is at time σ_n (resp. τ_n). Next when the system is run under the control u (resp. v) for C_1 (resp. C_2), there is a payoff $J(u,v)$ which is a profit (resp. cost) for C_1 (resp. C_2) given by:

$$J(u,v) = \mathbb{E}[\int_0^T f^{\delta_s}(s)ds - A_T^u + B_T^v + \zeta^{\delta_T}].$$

where $\delta := (\delta_s)_{s \le T}$ is the process valued in Γ which indicates the working modes of the system along with time. If at time s the system is in mode i_0, then $\delta_s = i_0$. It is bind to the controls u and v implemented by both players. On the other hand, for $i \in \Gamma$, the process f^i is the utility of the system in mode i and finally ζ^{δ_T} is the terminal payoff or bequest.

The problem we are interested in is to know whether or not the game has a value, i.e., roughly speaking, if the follwoing equality holds:

$$\inf_v \sup_u J(u,v) = \sup_u \inf_v J(u,v)$$

In case of equality we say that the game has a value. Finally we say that the game has a saddle-point (u^*, v^*) if, for any u and v, controls of C_1 and C_2 respectively, we have:

$$J(u, v^*) \le J(u^*, v^*) \le J(u^*, v).$$

Note that in such a case, the game has a value.

From the probabilistic point of view, this zero-sum switching game problem turns into looking for a solution of its associated system of reflected BSDEs with interconnected bilateral obstacles (see e.g. [33] for the case of proper modes of players). A solution for such a system are adapted processes $(Y^i, Z^i, K^{i,\pm})_{i\in\Gamma}$ such that for any $i \in \Gamma$, and $s \le T$,

$$\begin{cases} Y^i \text{ and } K^{i,\pm} \text{ continuous; } K^{i,\pm} \text{ increasing; } (Z^i(\omega)_t)_{t\le T} \text{ is } dt- \text{ square integrable;} \\ Y_s^i = \zeta^i + \int_s^T f^i(r)dr - \int_s^T Z_r^i dB_r + K_T^{i,+} - K_s^{i,+} - (K_T^{i,-} - K_s^{i,-}); \\ L^i(\vec{Y})_s \le Y_s^i \le U^i(\vec{Y})_s; \\ \int_0^T (Y_s^i - L^i(\vec{Y})_s) dK_s^{i,+} = 0 \text{ and } \int_0^T (Y_s^i - U^i(\vec{Y})_s) dK_s^{i,-} = 0 \end{cases} \tag{3.1}$$

where: a) $B := (B_t)_{t\le T}$ is a Brownian motion; b) $\vec{Y} := (Y^i)_{i\in\Gamma}$; c) $L^i(\vec{Y})_s = Y_s^{i+1} - \underline{g}_{i,i+1}(s)$ and $U^i(\vec{Y})_s = Y_s^{i+1} + \bar{g}_{i,i+1}(s)$.

Actually the solution of the previous system provides the value of the zero-sum switching game which is equal to Y_0^i if the starting mode of the system is i. Roughly speaking, system (3.1) is the verification theorem for the zero-sum switching game problem.

In the Markovian framework, i.e., when randomness stems from a diffusion process $X^{t,x}$ ($(t,x) \in [0,T] \times \mathbb{R}^k$) which satifies:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \ s \in [t,,T] \text{ and } X_s^{t,x} = x \text{ for } s \leq t \tag{3.2}$$

and the data of the game are deterministic functions of $(s, X_s^{t,x})$, the Hamilon-Jacobi-Bellman system associated with this switching game is the following system of partial differential equations (PDEs in short) with a bilateral interconnected obstacles: $\forall i \in \Gamma$, $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} \min\{v^i(t,x) - L^i(\vec{v})(t,x); \max\left[v^i(t,x) - U^i(\vec{v})(t,x); \ -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x)\right]\} = 0; \\ \\ v^i(T,x) = h^i(x). \end{cases} \tag{3.3}$$

where: a) $\vec{v} = (v^i)_{i\in\Gamma}$; b) $L^i(\vec{v})(t,x) := v^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$, $U^i(\vec{v})(t,x) := v^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$; c) $\mathcal{L}^X$, the infinitesimal generator of X, is given by:

$$\mathcal{L}^X\phi(t,x) := \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)D^2_{xx}\phi(t,x)] + b(t,x)^\top D_x\phi(t,x).$$

Usually it is shown that the value functions of the game is a unique solution of (3.3).

This work is originated by an article by N.Yamada [68] where the author deals with the system of PDEs (3.3) in the case when the switching costs are constant and for bounded domains $\bar{\Omega}$. By penalization method, the author proved existence and uniqueness of the solution of (1.35) in a weak sense (actually in a Sobolev space). Then he gives an interpretation of the solution of this system as a value function of the zero-sum switching game described previously. A saddle-point of the game is also given. However neither this interpretation nor the existence of the saddle-point are clear because the question of admissiblity of the controls which are supposed to realize the saddle-point property is not addressed. In zero-sum switching games this issue of admissibility of those controls, defined implicitely through $(Y^i)_{i\in\Gamma}$, is crucial (see e.g. [33]). Note also that there is another paper by N.Yamada [67] where the solution of system (3.3) is considered in viscosity sense. Once more by penalization, he shows existence and uniqueness of the solution on bounded domains $\bar{\Omega}$.

Therefore the main objectif of this work is to show that:

i) the system of reflected BSDEs with interconnected obstacles (3.1) has a unique solution in the Markovian framework.

ii) the zero-sum switching game described above has a value.
iii) The system of PDEs (3.3) has a unique solution.

Actually in this paper we show that system of PDEs (3.3) has a unique continuous with polynomial growth solution $(v^i)_{i\in\Gamma}$ in viscosity sense on $[0,T]\times\mathbb{R}^k$. Mainly this solution is constructed by using Perron's method in combination with systems of reflected BSDEs with one lower interconnected obstacle and the Feynman-Kac representation of their solutions in the Markovian framework. Then we show that the following system of RBSDEs with interconnected bilateral obstacles has a unique solution: For any $i\in\Gamma$,

$$\begin{cases} Y^i \text{ and } K^{i,\pm} \text{ are continuous; } K^{i,\pm} \text{ are increasing; } (Z^i(\omega)_t)_{t\le T} \text{ is } dt- \text{ square integrable;}\\ Y^i_s = h^i(X^{t,x}_T) + \int_s^T f^i(r,X^{t,x}_r)dr - \int_s^T Z^i_r dB_r + K^{i,+}_T - K^{i,+}_s - (K^{i,-}_T - K^{i,-}_s),\ s\le T;\\ L^i(\vec{Y})_s \le Y^i_s \le U^i(\vec{Y})_s;\\ \int_0^T (Y^i_s - L^i(\vec{Y})_s)dK^{i,+}_s = 0 \text{ and } \int_0^T (Y^i_s - U^i(\vec{Y})_s)dK^{i,-}_s = 0 \end{cases} \tag{3.4}$$

where X is the Markov process solution of (3.2), $L^i(\vec{Y})_s = Y^{i+1}_s - \underline{g}_{i,i+1}(s,X^{t,x}_s)$ and $U^i(\vec{Y})_s = Y^{i+1}_s + \bar{g}_{i,i+1}(s,X^{t,x}_s)$.

Finally we consider the zero-sum switching game and we show that when the processes Z^i, $i\in\Gamma$, of (3.4) are:

a) $dt\otimes d\mathbb{P}$-square integrable then Y^i_0 is the value of the game under square integrable controls, i.e., $\mathbb{E}[(A^u_T)^2+(B^v_T)^2]<\infty$.

b) only ω by ω, dt-square integrable then Y^i_0 is the value of the game under integrable controls, i.e., $\mathbb{E}[A^u_T+B^v_T]<\infty$.

The paper is organized as follows:

In Section 2, we introduce the zero-sum switching game and especially the notion of coupling which is already used in several papers including [33, 63]. In Section 3, we show that the solution of (3.4) is the value of the zero-sum switching game over square integrable controls when Z^i, $i\in\Gamma$, are $dt\otimes d\mathbb{P}$-square integrable. Without additional assumptions on the data of the problem, this property is rather tough to check in practice because it depends on the room between the barriers $L^i(\vec{Y})$ and $U^i(\vec{Y})$ which depend on the solution $\vec{Y}$. For example, it is not clear how to assume an hypothesis like Mokobodski's one (see e.g. [14, 30]) since the barriers depend on the solution and this latter is not explicit. However by localiztion, we can show that in some cases, e.g. when the switching costs are constant, Y^i_0 is actually the value function over square integrable controls even when we do not know that Z^i, $i\in\Gamma$, are $dt\otimes d\mathbb{P}$-square integrable. In the case when for any $i\in\Gamma$ and $\mathbb{P}$-a.s. $(Z^i_s(\omega))_{s\le T}$ is dt-square integrable only, which is the minimum condition to define the stochastic integral, Y^i_0 is the value function of the zero-sum switching game over integrable controls. To show this property we proceed by localization. Section 4 is devoted to existence and uniqueness of the solution of system of PDEs (3.3) in a more general form. The result is given in Theorem 3.4.3, but the main steps of its proof are postponed to Appendix. This proof is based on Perron's method and the construction of this solution (more or less the same as in [19]) proceeds as follows: a) we first introduce the processes $(Y^{i,m}, Z^{i,m}, K^{\pm,i,m})_{i\in\Gamma}$, $m \geq 1$, solution of the system of reflected BSDEs with interconnected lower barriers associated with $\{f^i(r, X_r^{t,x}, \vec{y}, z^i) - m(y^i - y^{i+1} - \bar{g}_{i,i+1}(r, X_r^{t,x}))^+, h^i(X_T^{t,x}), \underline{g}_{i,i+1}(r, X_r^{t,x})\}_{i\in\Gamma}$ (see (3.57)). It is a decreasing penalization scheme. As the framework is Markovian then there exist deterministic functions continuous and of polynomial growth $(v^{i,m})_{i\in\Gamma}$ such that the following Feynman-Kac representation holds: For any $i \in \Gamma$, $m \geq 1$ and $s \in [t, T]$,

$$Y_s^{i,m} = v^{i,m}(s, X_s^{t,x}).$$

As for any $i \in \Gamma$, $m \geq 1$, $Y^{i,m} \geq Y^{i,m+1}$ then we have also $v^{i,m} \geq v^{i,m+1}$. Now if we define $v^i = \lim_m v^{i,m}$, then $(v^i)_{i\in\Gamma}$ is a subsolution of (3.3) and for any fixed m_0, $(v^{i,m_0})_{i\in\Gamma}$ is a supersolution of (3.3). Next it is enough to use Perron's method to show that (3.3) has a unique solution since comparison principle holds. Finally, by uniqueness this solution does not depend on m_0 and is $(v^i)_{i\in\Gamma}$. Additionally for any $i \in \Gamma$, v^i is of polynomial growth and continuous. In Section 5, we show existence and uniqueness of the solution of system of RBSDEs (3.1) and give some extensions. This proof is based on results on zero-sum Dynkin games and standard two barriers reflected BSDEs. The component Y^i, $i \in \Gamma$, is just the limit of the processes $(Y^{i,m})_m$. We make use of the fact that, by Dini's Theorem, $(v^{i,m})_m$ converges to v^i uniformly on compact sets since v^i is continuous and then the sequence $(Y^{i,m})_m$ converges uniforly in $L^2(d\mathbb{P})$ to Y^i, $i \in \Gamma$. As mentionned previously, this latter property stems from the PDE part. Note also that the following representation holds:

$$\forall s \in [t, T], Y_s^i = v^i(s, X_s^{t,x}).$$

Here we should point out that since the switching of the system is made from i to $i+1$ and the players do not have their proper sets of switching modes, then the method used e.g. in [33] cannot be applied in our framework. As a consequence of this fact, the question of a solution of (3.1) outside the Markovian framework still open. At the end of the paper there is the Appendix. □

3.2 Preliminaries. Setting of the stochastic switching game

Let T be a fixed positive constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space, $B = (B_t)_{t\in[0,T]}$ a d-dimensional Brownian motion whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$

and we denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$. Then it satisfies the usual conditions. On the other hand, let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of the $\mathbb{F}$-progressively measurable sets.

Next, we denote by:

- $\mathcal{S}^2$: the set of $\mathcal{P}$-measurable continuous processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\sup_{t\in[0,T]}|\phi_t|^2) < \infty$;
- $\mathcal{A}^2$: the subset of $\mathcal{S}^2$ with all non-decreasing processes $K = (K_t)_{t\leq T}$ with $K_0 = 0$;
- $\mathcal{A}_{loc}$: the set of $\mathcal{P}$-measurable continuous non-decreasing processes $K = (K_t)_{t\leq T}$ with $K_0 = 0$ such that $\mathbb{P}-a.s.\ K_T(\omega) < \infty$;
- $\mathcal{H}^{2,d}_{loc}(d \geq 1)$: the set of $\mathcal{P}$-measurable $\mathbb{R}^d$-valued processes $\psi = (\psi_t)_{t\in[0,T]}$ such that $\mathbb{P}-a.s.$, $\int_0^T |\psi_t|^2 dt < \infty$.
- $\mathcal{H}^{2,d}$: the subset of $\mathcal{H}^{2,d}_{loc}(d \geq 1)$ of processes $\psi = (\psi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\int_0^T |\psi_t|^2 dt) < \infty$.
- For $s \leq T$, $\mathcal{T}_s$ is the set of stopping times ν such that $\mathbb{P}$-a.s., $s \leq \nu \leq T$.

Now for any $(t,x) \in [0,T]\times\mathbb{R}^k$, let us consider the process $(X_s^{t,x})_{s\in[t,T]}$ solution of the following standard SDEs:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s,\ s\in[t,T]; \\ \\ X_s^{t,x} = x,\ s\leq t \end{cases} \tag{3.5}$$

where, throughout this paper, b and σ satisfy the following conditions:

(H0) The functions b and σ are Lipschitz continuous w.r.t. x uniformly in t, i.e. for any $(t,x,x') \in [0,T]\times\mathbb{R}^{k+k}$, there exists a non-negative constant C such that

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x-x'|. \tag{3.6}$$

Moreover we assume that they are jointly continuous in (t,x). The continuity of b and σ imply their linear growth w.r.t. x, i.e. there exists a constant C such that for any $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|). \quad \square \tag{3.7}$$

Therefore under assumption (H0), the SDE (3.5) has a unique solution $X^{t,x}$ which satisfies the following estimates: $\forall\gamma\geq 1$,

$$\mathbb{E}[\sup_{s\leq T}|X_s^{t,x}|^\gamma] \leq C(1+|x|^\gamma). \quad \square \tag{3.8}$$

Next a function $\Phi : (t,x) \in [0,T] \times \mathbb{R}^k \mapsto \Phi(t,x) \in \mathbb{R}$ is called of polynomial growth if there exist two non-negative real constants C and γ such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k, |\Phi(t,x)| \leq C(1+|x|^\gamma).$$

Hereafter this class of functions is denoted by Π_g.

3.2.1 Description of the zero-sum stochastic switching game

Let $\Gamma := \{1,2,...,p\}$ and for $i \in \Gamma$, let us set $\Gamma^{-i} := \Gamma - \{i\}$. For $\vec{y} := (y^i)_{i\in\Gamma} \in \mathbb{R}^p$ and $\hat{y} \in \mathbb{R}$, we denote by $[\vec{y}_{-i}, \hat{y}]$ or $[(y^k)_{k\in\Gamma^{-i}}, \hat{y}]$, the element of $\mathbb{R}^p$ obtained in replacing the i-th component of $\vec{y}$ with $\hat{y}$.

We now introduce the following deterministic functions: for any $i \in \Gamma$,

- f^i: $(t,x,\vec{y},z) \in [0,T] \times \mathbb{R}^{k+p+d} \mapsto f^i(t,x,\vec{y},z) \in \mathbb{R}$
- $\underline{g}_{i,i+1}$: $(t,x) \in [0,T] \times \mathbb{R}^k \mapsto \underline{g}_{i,i+1}(t,x) \in \mathbb{R}$
- $\overline{g}_{i,i+1}$: $(t,x) \in [0,T] \times \mathbb{R}^k \mapsto \overline{g}_{i,i+1}(t,x) \in \mathbb{R}$
- $h^i : x \in \mathbb{R}^k \mapsto h^i(x) \in \mathbb{R}$

Next let us consider a system with p working modes indexed by the set Γ. On the other hand, there are two agents or controllers C_1 and C_2, whose interests are antagonistic and who act on this system, along with time, by switching its working mode from the current one, say i_0, to the next one i_0+1 if $i_0 \leq p-1$ and 1 if $i_0 = p$, whatever which agent decides to switch first. Therefore a switching control for C_1 (resp. C_2) is $u := (\sigma_n)_{n\geq 0}$ (resp. $v := (\tau_n)_{n\geq 0}$) an increasing sequence of stopping times which correspond to the successive times where C_1 (resp. C_2) decides to switch the system. The control u (resp. v) is called *admissible* if

$$\mathbb{P}[\sigma_n < T, \forall n \geq 0] = 0 \text{ (resp. } \mathbb{P}[\tau_n < T, \forall n \geq 0] = 0). \tag{3.9}$$

The set of admissible controls of C_1 (resp. C_2) is denoted $\mathbf{A}$ (resp. $\mathbf{B}$).

Now let $u := (\sigma_n)_{n\geq 0}$ (resp. $v := (\tau_n)_{n\geq 0}$) be an admissible control of C_1 (resp. C_2). Let $(r_n)_{n\geq 0}$ and $(s_n)_{n\geq 0}$ be the sequences defined by: $r_0 = s_0 = 0$, $r_1 = s_1 = 1$ and for $n \geq 2$,

$$r_n = r_{n-1} + \mathbf{1}_{\{\sigma_{r_{n-1}} \leq \tau_{s_{n-1}}\}} \text{ and } s_n = s_{n-1} + \mathbf{1}_{\{\tau_{s_{n-1}} < \sigma_{r_{n-1}}\}}.$$

For $n \geq 0$, let us set $\rho_n = \sigma_{r_n} \wedge \tau_{s_n}$. It is a stopping time and it stands for the time when the n-th switching of the system, by one of the players, occurs. On the other hand, the piecewise process $(\theta(u,v)_s)_{s\le T}$ which indicates in which mode the system is at time s is given by: $\forall s \le T$,

$$\theta(u,v)_s = \theta_0 \mathbf{1}_{[\rho_0,\rho_1]}(s) + \sum_{n\ge 1} \theta_n \mathbf{1}_{(\rho_n,\rho_{n+1}]}(s)$$

where:

i) $(\rho_n,\rho_{n+1}] = \varnothing$ on $\{\rho_n = \rho_{n+1}\}$;

ii) $\theta_0 = i$ if at $t = 0$, the system is in mode i ;

iii) For $n \ge 1$, $\theta_n = \theta_{n-1} + 1$ if $\theta_{n-1} \le p-1$ and $\theta_n = 1$ if $\theta_{n-1} = p$.

The sequence $\Theta(u,v) := (\rho_n,\theta_n)_{n\ge 0}$, called the *coupling* of (u,v), indicates the successive times and modes of switching of the system operated by the players.

When the players implement the pair of admissible controls (u,v), this incurs switching costs which amount to A_T^u and B_T^v, for C_1 and C_2 respectively, and given by:

$$\forall s < T, A_s^u = \sum_{n\ge 1} \underline{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \mathbf{1}_{\{\rho_n = \sigma_{r_n} \le s\}} \text{ and } A_T^u = \lim_{s\to T} A_s^u;$$

$$\forall s < T, B_s^v = \sum_{n\ge 1} \overline{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \mathbf{1}_{\{\rho_n = \tau_{s_n} \le s\}} \text{ and } B_T^v = \lim_{s\to T} B_s^v.$$

The admissible control u (resp. v) of C_1 (resp. C_2) is called *square integrable* if

$$\mathbb{E}[(A_T^u)^2] < \infty \text{ (resp. } \mathbb{E}[(B_T^v)^2] < \infty).$$

The set of square integrable admissible controls of C_1 (resp. C_2) is denoted by $\mathcal{A}$ (resp. $\mathcal{B}$).

The admissible control u (resp. v) of C_1 (resp. C_2) is called *integrable* if

$$\mathbb{E}[A_T^u] < \infty \text{ (resp. } \mathbb{E}[B_T^v] < \infty).$$

The set of integrable admissible controls of C_1 (resp. C_2) is denoted by $\mathcal{A}^{(1)}$ (resp. $\mathcal{B}^{(1)}$).

The coupling $\theta(u,v)$, of a pair (u,v) of admissible controls, is called *square integrable* (resp. *integrable*) if

$$C_\infty^{\theta(u,v)} := \lim_{n\to\infty} C_N^{u,v} \in L^2(d\mathbb{P}) \text{ (resp. } \in L^1(d\mathbb{P}))$$

where for any $N \ge 1$,

$$C_N^{\theta(u,v)} := \sum_{n=1,N} \underline{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \mathbf{1}_{\{\rho_n = \sigma_{r_n} < T\}} - \sum_{n=1,N} \overline{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \mathbf{1}_{\{\rho_n = \tau_{s_n} < T\}}.$$

Note that $C_\infty^{\theta(u,v)}$, defined as the pointwise limit of $C_N^{\theta(u,v)}$, exists since the controls u and v are admissible. On the other hand, the quantity $C_N^{\theta(u,v)}$ is nothing but the switching costs associated with the N first switching actions of both players.

Next when C_1 (resp. C_2) implements $u \in \mathbf{A}$ (resp. $v \in \mathbf{B}$), there is a payoff which a is reward for C_1 and a cost for C_2 which is given by (we suppose that $\theta_0 = i$):

$$J_i(\theta(u,v)) = \mathbb{E}\left[h^{\theta(u,v)_T}(X_T^{0,x}) + \int_0^T f^{\theta(u,v)_r}(r, X_r^{0,x})dr - C_\infty^{\theta(u,v)}\right]. \tag{3.10}$$

It means that between C_1 and C_2 there is a game of zero-sum type. The main objective of this section is to deal with the issue of existence of a value for this zero-sum switching game, i.e., whether or not it holds

$$\inf_{v\in\mathcal{B}}\sup_{u\in\mathcal{A}} J_i(\theta(u,v)) = \sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i(\theta(u,v)) \tag{3.11}$$

or

$$\inf_{v\in\mathcal{B}^{(1)}}\sup_{u\in\mathcal{A}^{(1)}} J_i(\theta(u,v)) = \sup_{u\in\mathcal{A}^{(1)}}\inf_{v\in\mathcal{B}^{(1)}} J_i(\theta(u,v)). \tag{3.12}$$

Remark 3.2.1. *In our framework when the players decide to switch at the same time, we give priority to the maximizer C_1. This appears through the definition of r_n for $n \geq 2$. On the other hand, for the well-posedness of $J_i(\theta(u,v))$, it is enough that the coupling $\theta(u,v)$ is integrable.*

To proceed we are going to define the notion of admissible square integrable and integrable strategies.

Definition 3.2.2 (Non-anticipative switching strategies)**.** *Let $s \in [0,T]$ and ν a stopping time such that $\mathbb{P}$-a.s. $\nu \geq s$. Two controls $u^1 = (\sigma_n^1)_{n\geq 0}$ and $u^2 = (\sigma_n^2)_{n\geq 0}$ in $\mathbf{A}$ are said to be equivalent, denoting this by $u^1 \equiv u^2$, on $[s,\nu]$ if we have $\mathbb{P}$-a.s.,*

$$\mathbf{1}_{[\sigma_0^1,\sigma_1^1]}(r) + \sum_{n\geq 1}\mathbf{1}_{(\sigma_n^1,\sigma_{n+1}^1]}(r) = \mathbf{1}_{[\sigma_0^2,\sigma_1^2]}(r) + \sum_{n\geq 1}\mathbf{1}_{(\sigma_n^2,\sigma_{n+1}^2]}(r), \quad s \leq r \leq \nu.$$

A non-anticipative strategy for C_1 is a mapping $\overline{\alpha}\colon \mathbf{B} \to \mathbf{A}$ such that for any $s \in [0,T]$, $\nu \in \mathcal{T}_s$, and $v^1, v^2 \in \mathbf{B}$ such that $v^1 \equiv v^2$ on $[s,\nu]$, we have $\overline{\alpha}(v^1) \equiv \overline{\alpha}(v^2)$ on $[s,\nu]$.

The non-anticipative strategy $\overline{\alpha}$ for C_1 is called $square-integrable$ (resp. integrable) if for any $v \in \mathcal{B}$ we have $\overline{\alpha}(v) \in \mathcal{A}$ (resp. for any $v \in \mathcal{B}^{(1)}$ we have $\overline{\alpha}(v) \in \mathcal{A}^{(1)}$).

In a similar manner we define non-anticipative strategies, square integrable and integrable strategies for C_2 denote by $\overline{\beta}$.

We denote by $\mathbb{A}$ and $\mathbb{B}$ (resp. $\mathbb{A}^{(1)}$ and $\mathbb{B}^{(1)}$) the set of non-anticipative square integrable (resp. integrable) strategies for C_1 and C_2 respectively. □

3.3 Existence of a value of the zero-sum switching game. Link with systems of reflected BSDEs

We are now going to deal with the issue of existence of a value for the zero-sum switching game described previously. For that let us introduce the following assumptions on the functions f^i, h^i, $\underline{g}_{i,i+1}$ and $\overline{g}_{i,i+1}$. Some assumptions will be only applied in the next sections.

Assumptions (H):

(H1) For any $i \in \Gamma$, f^i does not depend on $(\vec{y}, z)$, is continuous in (t, x) and belongs to class Π_g ;

(H2) For any $i \in \Gamma$, the function h^i, which stands for the terminal payoff, is continuous w.r.t. x, belongs to class Π_g and satisfies the following consistency condition: $\forall i \in \Gamma, \forall x \in \mathbb{R}^k$,

$$h^{i+1}(x) - \underline{g}_{i,i+1}(T, x) \leq h^i(x) \leq h^{i+1}(x) + \overline{g}_{i,i+1}(T, x). \tag{3.13}$$

(H3) a) For all $i \in \Gamma$ and $(t, x) \in [0, T] \times \mathbb{R}^k$, the functions $\underline{g}_{i,i+1}$ and $\overline{g}_{i,i+1}$ are continuous, non-negative, belong to Π_g and verify:

$$\underline{g}_{i,i+1}(t, x) + \overline{g}_{i,i+1}(t, x) > 0.$$

b) They satisfy the non-free loop property, i.e., for any $j \in \Gamma$ and $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\varphi_{j,j+1}(t, x) + ... + \varphi_{p-1,p}(t, x) + \varphi_{p,1}(t, x) + ... + \varphi_{j-1,j}(t, x) \neq 0 \tag{3.14}$$

where $\varphi_{\ell,\ell+1}(t, x)$ is either $-\underline{g}_{\ell,\ell+1}(t, x)$ or $\overline{g}_{\ell,\ell+1}(t, x)$. Let us notice that (3.14) also implies:

$$\overline{g}_{j,j+1}(t, x) + ... + \overline{g}_{p-1,p}(t, x) + \overline{g}_{p,1}(t, x) + ... + \overline{g}_{j-1,j}(t, x) > 0 \tag{3.15}$$

and

$$\underline{g}_{j,j+1}(t, x) + ... + \underline{g}_{p-1,p}(t, x) + \underline{g}_{p,1}(t, x) + ... + \underline{g}_{j-1,j}(t, x) > 0. \tag{3.16}$$

(H4) For any $i = 1, ..., m$, the processes $(\bar{g}_{i,i+1}(s, X_s^{0,x}))_{s \leq T}$ and $(\underline{g}_{i,i+1}(s, X_s^{0,x}))_{s \leq T}$ are non decreasing. $\square$

(H5) For any $i \in \Gamma$,

a) f^i is Lipschitz continuous in $(\vec{y}, z)$ uniformly in (t, x), i.e. for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^p$, $z_1, z_2 \in \mathbb{R}^d$, $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$|f^i(t, x, \vec{y}_1, z_1) - f^i(t, x, \vec{y}_2, z_2)| \leq C(|\vec{y}_1 - \vec{y}_2| + |z_1 - z_2|);$$

b) $\forall j \in \Gamma^{-i}$, the mapping $\bar{y} \mapsto f^i(t,x,[(y^k)_{k\in\Gamma^{-j}},\bar{y}],z)$ is non-decreasing when the other components $t,x,(y^k)_{k\in\Gamma^{-j}}$ and z are fixed.

c) f^i is continuous in (t,x) uniformly in $(\vec{y},z)$ and $f^i(t,x,0,0)$ belongs to Π_g.

In order to deal with the zero-sum switching game we rely on solutions of systems of reflected BSDEs with oblique reflection or inter-connected bilateral obstacles of type below. The following result whose proof is given in Section 5 will allow us to show that the zero-sum switching game has a value.

Theorem 3.3.1. *Assume that assumptions (H1), (H2) and (H3) are fulfilled. Then there exist processes $(Y^i,Z^i,K^{i,\pm})_{i\in\Gamma}$ such that: For any $i\in\Gamma$ and $(t,x)\in[0,T]\times\mathbb{R}^k$, $\forall s\le T$,*

$$\begin{cases} Y^i\in\mathcal{S}^2; K^{i,\pm}\in\mathcal{A}_{loc} \text{ and } Z^i\in\mathcal{H}^{2,d}_{loc};\\ Y^i_s=h^i(X^{t,x}_T)+\int_s^T f^i(r,X^{t,x}_r)dr-\int_s^T Z_r dB_r+K^{i,+}_T-K^{i,+}_s-(K^{i,-}_T-K^{i,-}_s);\\ L^i(\vec{Y})_s\le Y^i_s\le U^i(\vec{Y})_s;\\ \int_0^T(Y^i_s-L^i(\vec{Y})_s)dK^{i,+}_s=0 \text{ and } \int_0^T(Y^i_s-U^i(\vec{Y})_s)dK^{i,-}_s=0; \end{cases} \tag{3.17}$$

where for any $s\le T$, $L^i(\vec{Y})_s:=Y^{i+1}_s-\underline{g}_{i,i+1}(s,X^{t,x}_s)$ and $U^i(\vec{Y})_s:=Y^{i+1}_s+\bar{g}_{i,i+1}(s,X^{t,x}_s)$.

Note that obviously the solution $(Y^i,Z^i,K^{i,\pm})_{i\in\Gamma}$ of (3.17) depends also on (t,x) which we omit as there is no possible confusion.

To proceed let $(Y^i,Z^i,K^{i,\pm})_{i\in\Gamma}$ be the solution of (3.17) when $t=0$. We then have (see e.g. [32], for more details):

Proposition 3.3.2. *For all $i\in\Gamma$ and $s\le T$,*

(a)

$$Y^i_0=\operatorname*{ess\,inf}_{\tau\in\mathcal{T}_0}\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_0}\mathcal{J}^i_0(\sigma,\tau)=\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_0}\operatorname*{ess\,inf}_{\tau\in\mathcal{T}_0}\mathcal{J}^i_0(\sigma,\tau), \tag{3.18}$$

where,

$$\mathcal{J}^i_s(\sigma,\tau)=\mathbb{E}\Big[\int_s^{\sigma\wedge\tau}f^i(r,X^{0,x}_r)dr+\mathbf{1}_{\{\tau<\sigma\}}U^i_\tau(Y)+\mathbf{1}_{\{\sigma\le\tau,\,\sigma<T\}}L^i_\sigma(Y)+h^i(X^{0,x}_T)\mathbf{1}_{\{\sigma=\tau=T\}}\;\big|\;\mathcal{F}_s\Big]. \tag{3.19}$$

(b) *We have $Y^i_s=\mathcal{J}^i_s(\sigma^i_s,\tau^i_s)$ where $\sigma^i_s\in\mathcal{T}_s$ and $\tau^i_s\in\mathcal{T}_s$ are stopping times defined by,*

$$\begin{cases}\sigma^i_s=\inf\{s\le t\le T\colon Y^i_t=L^i_t(\vec{Y})\}\wedge T,\\ \tau^i_s=\inf\{s\le t\le T\colon Y^i_t=U^i_t(\vec{Y})\}\wedge T,\end{cases} \tag{3.20}$$

and we use the convention that $\inf \emptyset = +\infty$*. Moreover,* (σ_s^i, τ_s^i) *is a saddle-point for the zero-sum Dynkin game,*

$$\mathcal{J}_s^i(\sigma, \tau_s^i) \le \mathcal{J}_s^i(\sigma_s^i, \tau_s^i) \le \mathcal{J}_s^i(\sigma_s^i, \tau) \quad \forall \sigma,\ \tau \in \mathcal{T}_s. \quad \square \tag{3.21}$$

Remark 3.3.3. *For any* $s < T$ *and* $i \in \Gamma$, $\mathbb{P}[\sigma_s^i = \tau_s^i < T] = 0$ *due to assumption [H3]-a) on* $\underline{g}_{i,i+1}$ *and* $\overline{g}_{i,i+1}$.

3.3.1 Value of the zero-sum switching game on square integrable admissible controls

We are now going to focus on the link between Y^i, $i \in \Gamma$, with the value function of the zero-sum switching game over square integrable controls, namely the relation (3.11). For that we are going to make another supplementary assumption on the solution $(Y^i, Z^i, K^{i,\pm})_{i\in\Gamma}$ of system (3.17) which is related to integrability of Z^i, $i \in \Gamma$. Later on we will show that we have also the relation (3.11) without this latter assumption, but at the price of some additional regularity properties of the switching costs $\underline{g}_{i,i+1}$ and $\bar{g}_{i,i+1}$ (see (H4)).

To proceed, consider the following sequence $(\rho_n, \theta_n)_{n\ge 0}$ defined as following: $\rho_0 = 0$, $\theta_0 = i$ and for $n \ge 1$,

$$\rho_n = \sigma_{\rho_{n-1}}^{\theta_{n-1}} \wedge \tau_{\rho_{n-1}}^{\theta_{n-1}} \text{ and } \theta_n = \begin{cases} 1 + \theta_{n-1} \text{ if } \theta_{n-1} \le p-1, \\ 1 \text{ if } \theta_{n-1} = p; \end{cases}$$

where $\sigma_{\rho_{n-1}}^{\theta_{n-1}}$ and $\tau_{\rho_{n-1}}^{\theta_{n-1}}$ are defined using (3.20). Next let $u^{(1)} := (u_s^{(1)})_{s\le T}$ (resp. $u^{(2)} := (u_s^{(2)})_{s\le T}$) be the piecewise process defined by: $u_s^{(1)} = 0$ for $s < \rho_1$ and for $n \ge 1, s \in [\rho_n, \rho_{n+1})$,

$$u_s^{(1)} = \begin{cases} 1 + u_{\rho_n-}^{(1)} \text{if } Y_{\rho_n}^{\theta_{n-1}} = Y_{\rho_n}^{\theta_n} - \underline{g}_{\theta_{n-1},\theta_n}(\rho_n, X_{\rho_n}^{0,x}), \\ u_{\rho_n-}^{(1)} \text{ if } Y_{\rho_n}^{\theta_{n-1}} > Y_{\rho_n}^{\theta_n} - \underline{g}_{\theta_{n-1},\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \end{cases}$$

where $u_{\rho_n-}^{(1)}$ is the left limit of $u^{(1)}$ at ρ_n (resp. $u_s^{(2)} = 0$ for $s < \rho_1$ and for $n \ge 1, s \in [\rho_n, \rho_{n+1})$,

$$u_s^{(2)} = \begin{cases} 1 + u_{\rho_n-}^{(2)} \text{if } Y_{\rho_n}^{\theta_{n-1}} = Y_{\rho_n}^{\theta_n} + \overline{g}_{\theta_{n-1},\theta_n}(\rho_n, X_{\rho_n}^{0,x}), \\ u_{\rho_n-}^{(2)} \text{ if } Y_{\rho_n}^{\theta_{n-1}} < Y_{\rho_n}^{\theta_n} + \overline{g}_{\theta_{n-1},\theta_n}(\rho_n, X_{\rho_n}^{0,x}) \end{cases}$$

where $u_{\rho_n-}^{(2)}$ is the left limit of $u^{(2)}$ at ρ_n). Next let u^* and v^* be the following sequences of stopping times: $\sigma_0^* = \tau_0^* = 0$ and for $n \ge 1$,

$$\sigma_n^* = \inf\{s \ge \sigma_{n-1}^*, u_s^{(1)} > u_{s-}^{(1)}\} \wedge T \text{ and } \tau_n^* = \inf\{s \ge \tau_{n-1}^*, u_s^{(2)} > u_{s-}^{(2)}\} \wedge T.$$

We then have:

Proposition 3.3.4. *Assume that (H1), (H2), (H3) and $(Z^i)_{i\in\Gamma} \in \mathcal{H}^{2,d}$. Then the following properties of $u^* = (\sigma_n^*)_{n\geq 0}$ and $v^* = (\tau_n^*)_{n\geq 0}$ hold true:*

i) u^* and v^* are admissible ;

ii) the coupling $\theta(u^*, v^*)$ is square integrable ;

iii)

$$Y_0^i = J_i(\theta(u^*, v^*)).$$

Proof. i) Let us show that u^* is admissible. Assume that $\mathbb{P}[\sigma_n^* < T, \forall n \geq 0] > 0$. As the σ_n^*'s are defined through the $\rho_n's$, then there exists a loop $\{j, j+1, ..., p-1, p, 1, ..., j-1, j\}$ such that

$$\begin{aligned}\mathbb{P}[\omega, \exists \text{ a subsequence } (n_\ell)_{\ell\geq 0} \text{ such that } Y^j_{\rho_{n_\ell}} = Y^{j+1}_{\rho_{n_\ell}} + \varphi_{j,j+1}(\rho_{n_\ell}, X^{0,x}_{\rho_{n_\ell}}), \ldots,\\ Y^{j-1}_{\rho_{n_\ell+p-1}} = Y^j_{\rho_{n_\ell+p-1}} + \varphi_{j-1,j}(\rho_{n_\ell+p-1}, X^{0,x}_{\rho_{n_\ell+p-1}}), \forall \ell \geq 0] > 0\end{aligned}$$

where $\varphi_{i,i+1}$ is the same as in (3.14) and equal to either $-\underline{g}_{i,i+1}$ or $\bar{g}_{i,i+1}$ depending on whether C_1 or C_2 makes the decision to switch from the current state j_0 to the next one. Next let us set $\gamma = \lim_{\ell\to\infty} \rho_{n_\ell}$. Take the limit w.r.t ℓ in the previous equalities to deduce that:

$$\mathbb{P}[\varphi_{j,j+1}(\gamma, X^{0,x}_\gamma) + ... + \varphi_{p-1,p}(\gamma, X^{0,x}_\gamma) + \varphi_{p,1}(\gamma, X^{0,x}_\gamma) + ... + \varphi_{j-1,j}(\gamma, X^{0,x}_\gamma) = 0] > 0$$

which is contradictory with the non free loop property (3.14). By the same reasoning we obtain the admissibility of v^*.

ii) Let us recall the definition of the square integrability for $\theta(u^*, v^*)$. As u^* and v^* are proved admissible in i), then the coupling $\theta(u^*, v^*)$ exists. Next we will prove that $\lim_{N\to\infty} C_N^{u^*,v^*} \in \mathbb{L}^2(d\mathbb{P})$.

For this recall that i is fixed, $\rho_0 = 0$ and $\theta_0 = i$. Next let us consider the equation satisfied by Y^i on $[0, \rho_1]$. We then have:

$$\begin{aligned}
Y_0^i &= h^i(X_T^{0,x})1_{(\rho_1=T)} + Y^i_{\rho_1}1_{(\rho_1<T)} + \int_0^{\rho_1} f^i\left(r, X_r^{0,x}\right) dr - \int_0^{\rho_1} Z_r^i dB_r + \int_0^{\rho_1} dK_r^{i,+} - \int_0^{\rho_1} dK_r^{i,-}\\
&= h^i(X_T^{0,x})1_{(\rho_1=T)} + \left(Y^{i+1}_{\sigma_0^i} - \underline{g}_{i,i+1}(\sigma_0^i, X^{0,x}_{\sigma_0^i})\right) 1_{(\sigma_0^i\leq\tau_0^i)}1_{(\sigma_0^i<T)} + \left(Y^{i+1}_{\tau_0^i} + \bar{g}_{i,i+1}(\tau_0^i, X^{0,x}_{\tau_0^i})\right) 1_{(\tau_0^i<\sigma_0^i)}\\
&\qquad + \int_0^{\rho_1} f^i\left(r, X_r^{0,x}\right) dr - \int_0^{\rho_1} Z_r^i dB_r\\
&= h^{\theta_0}(X_T^{0,x})1_{(\rho_1=T)} + Y^{\theta_1}_{\rho_1}1_{(\rho_1<T)} - \left[\underline{g}_{\theta_0\theta_1,}(\rho_1, X^{0,x}_{\rho_1})1_{(\rho_1=\sigma_0^{\theta_0})} - \bar{g}_{\theta_0\theta_1}(\rho_1, X^{0,x}_{\rho_1})1_{(\rho_1=\tau_0^{\theta_0})}\right] 1_{(\rho_1<T)}\\
&\qquad + \int_0^{\rho_1} f^{\theta_0}\left(r, X_r^{0,x}\right) dr - \int_0^{\rho_1} Z_r^{\theta_0} dB_r \qquad (3.22)
\end{aligned}$$

Next we deal with $Y^{\theta_1}_{\rho_1}$ by considering the doubly RBSDEs (3.17) in the interval $[\rho_1, \rho_2]$, i.e.

$$Y^{\theta_1}_{\rho_1} = Y^{i+1}_{\rho_1}$$

$$= h^{\theta_1}(X_T^{0,x})1_{(\rho_2=T)} + Y_{\rho_2}^{\theta_2}1_{(\rho_2<T)} - \left[\underline{g}_{\theta_1\theta_2}(\rho_2, X_{\rho_2}^{0,x})1_{(\rho_2=\sigma_{\rho_1}^{\theta_1})} - \bar{g}_{\theta_1\theta_2}(\rho_2, X_{\rho_2}^{0,x})1_{(\rho_2=\tau_{\rho_1}^{\theta_1})}\right]1_{(\rho_2<T)}$$
$$+ \int_{\rho_1}^{\rho_2} f^{\theta_1}\left(r, X_r^{0,x}\right) dr - \int_{\rho_1}^{\rho_2} Z_r^{\theta_1} dB_r \tag{3.23}$$

By replacing $Y_{\rho_1}^{\theta_1}$ in (3.22) with (3.23), then (3.22) yields

$$Y_0^i = \sum_{n=1}^{2} h^{\theta_{n-1}}(X_T^{0,x})1_{(\rho_n=T)}1_{(\rho_{n-1}<T)} + Y_{\rho_2}^{\theta_2}1_{(\rho_2<T)} + \int_0^{\rho_2} f^{\theta(u^*,v^*)_r}\left(r, X_r^{0,x}\right) dr - \int_{\rho_1}^{0} Z_r^{\theta(u^*,v^*)_r} dB_r$$
$$- \sum_{n=1}^{2}\left[\underline{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x})1_{(\rho_n=\sigma_{\rho_{n-1}}^{\theta_{n-1}},\rho_n<T)} - \bar{g}_{\theta_{n-1}\theta_n}(\rho_n, X_{\rho_n}^{0,x})1_{(\rho_n=\tau_{\rho_{n-1}}^{\theta_{n-1}},\rho_n<T)}\right] \tag{3.24}$$

Following (3.24) we replace iteratively $Y_{\rho_n}^{\theta_n}$ for $n = 1, 2, ..., N$ we deduce that

$$Y_0^i = \sum_{n=1}^{N} h^{\theta_{n-1}}(X_T^{0,x})1_{(\rho_n=T)}1_{(\rho_{n-1}<T)} + Y_{\rho_N}^{\theta_N}1_{(\rho_N<T)} - C_N^{\theta(u^*,v^*)} + \int_0^{\rho_N} f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})dr$$
$$- \int_0^{\rho_N} Z_r^{\theta(u^*,v^*)_r} dB_r \tag{3.25}$$

From (3.25) we obtain: $\forall N \geq 1$,

$$|C_N^{\theta(u^*,v^*)}| \leq \sum_{n=1}^{N} |h^{\theta_{n-1}}(X_T^{0,x})|1_{(\rho_n=T)}1_{(\rho_{n-1}<T)} + |Y_{\rho_N}^{\theta_N}1_{(\rho_N<T)}| + |Y_0^i| + |\int_0^{\rho_N} f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})dr|$$
$$+ |\int_0^{\rho_N} Z_r^{\theta(u^*,v^*)_r} dB_r|$$
$$\leq \max_{i\in\Gamma}\left|h^i(X_T^{0,x})\right| + 2\max_{i\in\Gamma}\sup_{s\in[0,T]}|Y_s^i| + \int_0^T |f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})|dr + \sup_{s\in[0,T]}|\int_0^s Z_r^{\theta(u^*,v^*)_r} dB_r|$$

Finally by taking the supremum over N we obtain:

$$\sup_{N\geq 1}\left|C_N^{\theta(u^*,v^*)}\right| \leq \max_{ß\in\Gamma}\left|h^i(X_T^{0,x})\right| + 2\max_{ß\in\Gamma}\sup_{s\in[0,T]}|Y_s^i|$$
$$+ \int_0^T |f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})|dr + \sup_{s\in[0,T]}|\underbrace{\int_0^s Z_r^{\theta(u^*,v^*)_r} dB_r}_{M_s^{\theta(u^*,v^*)_s}}|. \tag{3.26}$$

As $(Z^i)_{i\in\Gamma}$ are $dt \otimes d\mathbb{P}$-square integrable, then

$$\mathbb{E}[\sup_{s\leq T}|M_s^{\theta(u^*,v^*)_s}|^2] \leq C\mathbb{E}[\sum_{i=1,m}\int_0^T |Z_s^i|^2 ds] < \infty.$$

It implies that the right-hand side of (3.26) belongs to $\mathbb{L}^2(d\mathbb{P})$ and then $\lim_{N\to\infty} C_N^{\theta(u^*,v^*)}$ is square integrable and then $\theta(u^*, v^*)$ is square integrable.

Finally for iii), by directly taking the expectation on both sides of (3.25) we obtain

$$Y_0^i = \mathbb{E}\left[\sum_{n=1}^{N} h^{\theta_{n-1}}(X_T^{0,x})1_{(\rho_n=T)}1_{(\rho_{n-1}<T)} + Y_{\rho_N}^{\theta_N}1_{(\rho_N<T)} - C_N^{\theta(u^*,v^*)} + \int_0^{\rho_N} f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})dr\right] \tag{3.27}$$

Now it is enough to take the limit w.r.t. N in (3.27) and to use the Lebesgue dominated convergence theorem since $\lim_{N\to\infty}\rho_N = T$ and considering (3.26), to deduce that

$$Y_0^i = \mathbb{E}\left[h^{\theta(u^*,v^*)_T}(X_T^{0,x}) + \int_0^T f^{\theta(u^*,v^*)_r}(r, X_r^{0,x})dr - C_\infty^{\theta(u^*,v^*)}\right] = J_i(\theta(u^*, v^*))$$

as $\lim_{N\to\infty} C_N^{\theta(u^*,v^*)} = C_\infty^{\theta(u^*,v^*)}$. □

Let i be the starting mode of the system which is fixed. Let $\sigma = (\sigma_n)_{n\geq 0}$ be an admissible control of C_1 (which then belongs to **A**) and $v^*(\sigma) =: (\bar{\tau}_n)_{n\geq 0}$ be the optimal response strategy of C_2 which we are going to define below. Indeed let $(\rho_n, \theta_n)_{n\geq 0}$ be the sequence defined as follows: $\rho_0 = 0$, $\theta_0 = i$ and for $n \geq 1$

$$\rho_0 = 0,\ \theta_0 = i,\ \text{ and for } n \geq 1,$$

$$\rho_n = \sigma_{\check{r}_n} \wedge \tilde{\tau}_n,\ \theta_n = \begin{cases} 1 + \theta_{n-1} & \text{if } \theta_{n-1} \leq p-1 \\ 1 & \text{if } \theta_{n-1} = p \end{cases} \tag{3.28}$$

where

$$\tilde{\tau}_n := \tau_{\rho_{n-1}}^{\theta_{n-1}} := \inf\left\{s \geq \rho_{n-1}, Y_s^{\theta_{n-1}} = Y_s^{\theta_n} + \bar{g}_{\theta_{n-1}\theta_n}(s)\right\} \wedge T \ \text{(according to (3.20))}$$

and $\check{r}_n$ is defined by $\check{r}_0 = 0, \check{r}_1 = 1$, for $n \geq 2$,

$$\check{r}_n = \check{r}_{n-1} + 1_{\{\sigma_{\check{r}_{n-1}} \leq \tilde{\tau}_{n-1}\}}.$$

Next let $\check{v}$ be the piecewise process defined by: $\check{v}_s = 0$ for $s < \rho_1$ and for $n \geq 1, s \in [\rho_n, \rho_{n+1})$,

$$\check{v}_s = \begin{cases} 1 + \check{v}_{\rho_n-}\text{if } \rho_n = \tilde{\tau}_n < \sigma_{\check{r}_n} \\ \\ \check{v}_{\rho_n-} \text{ if } \rho_n = \sigma_{\check{r}_n} \leq \tilde{\tau}_n \end{cases}$$

where $\check{v}_{\rho_n -} = \lim_{s \nearrow \rho_n} \check{v}_s$. Now the stopping times $\bar{\tau}_n$, $n \geq 0$, are defined as follows:

$$\bar{\tau}_0 = 0 \text{ and for } n \geq 1, \bar{\tau}_n = \inf\{s \geq \bar{\tau}_{n-1},\ \check{v}_s > \check{v}_{s-}\} \wedge T \tag{3.29}$$

where $\check{v}_{s-} = \lim_{r \nearrow s} \check{v}_r$.

Next we are going to define the notion of optimal responce $u^*(v) = (\bar{\sigma}_n)_{n\geq 0}$ of C_1 to an admissible control $v = (\tau_n)_{n\geq 0}$ of the second player C_2. Indeed let $(\rho_n, \theta_n)_{n\geq 0}$ be the sequence defined as follows: $\rho_0 = 0$, $\theta_0 = i$ and for $n \geq 1$

$$\rho_0 = 0,\ \theta_0 = i,\ \text{ and for } n \geq 1,$$

$$\rho_n = \tilde{\sigma}_n \wedge \tau_{\check{s}_n},\ \theta_n = \begin{cases} 1 + \theta_{n-1} & \text{if } \theta_{n-1} \leq p - 1 \\ 1 & \text{if } \theta_{n-1} = p \end{cases} \tag{3.30}$$

where

$$\tilde{\sigma}_n := \sigma^{\theta_{n-1}}_{\rho_{n-1}} := \inf\left\{s \geq \rho_{n-1}, Y^{\theta_{n-1}}_s = Y^{\theta_n}_s - \underline{g}_{\theta_{n-1}\theta_n}(s)\right\} \wedge T \text{ (according to (3.20))}$$

and $\check{s}_n$ is defined by $\check{s}_0 = 0, \check{s}_1 = 1$, for $n \geq 2$,

$$\check{s}_n = \check{s}_{n-1} + 1_{\{\tilde{\sigma}_{n-1} > \tau_{\check{s}_{n-1}}\}}.$$

Next let $\check{u}$ be the piecewise process defined by: $\check{u}_s = 0$ for $s < \rho_1$ and for $n \geq 1, s \in [\rho_n, \rho_{n+1})$,

$$\check{u}_s = \begin{cases} 1 + \check{u}_{\rho_n -} \text{if } \rho_n = \tilde{\sigma}_n \leq \tau_{\check{s}_n} \\ \\ \check{u}_{\rho_n -} \text{ if } \rho_n = \tau_{\check{s}_n} < \tilde{\sigma}_n \end{cases}$$

where $\check{u}_{\rho_n -} = \lim_{s \nearrow \rho_n} \check{u}_s$. Now the stopping times $\bar{\sigma}_n$, $n \geq 0$, are defined as follows:

$$\bar{\sigma}_0 = 0 \text{ and for } n \geq 1, \bar{\sigma}_n = \inf\{s \geq \bar{\sigma}_{n-1},\ \check{u}_s > \check{s}_{s-}\} \wedge T \tag{3.31}$$

where $\check{u}_{s-} = \lim_{r \nearrow s} \check{u}_r$. We then have:

Proposition 3.3.5. *Assume (H1), (H2), (H3) and $(Z^i)_{i\in\Gamma} \in \mathcal{H}^{2,d}$. Then for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$, we have:*
i) $u^*(v) \in \mathcal{A}, v^*(u) \in \mathcal{B}$;
ii)

$$J_i\left(\theta(u, v^*(u))\right) \leq Y^i_0 \leq J_i\left(\theta(u^*(v), v)\right). \tag{3.32}$$

Proof. i) In order to show $u^*(v) \in \mathcal{A}$, when $v = (\tau_n)_{n\geq 0} \in \mathcal{B}$, we need to prove that $u^*(v) = (\bar{\sigma}_n)_{n\geq 0}$ is admissible and $\mathbb{E}\left[(A_T^{u^*(v)})^2\right] < \infty$.

Indeed if $u^*(v) = (\bar{\sigma}_n)_{n\geq 0}$ is not admissible then there would exist a loop $\{j, j+1, ..., p-1, p, 1, ..., j-1, j\}$ which is visited infinitley many times, i.e.,

$$\mathbb{P}[\omega, \exists \text{ a subsequence } (n_\ell)_{\ell\geq 0} \text{ such that } Y^j_{\bar{\sigma}_{n_\ell}} = Y^{j+1}_{\bar{\sigma}_{n_\ell}} - \underline{g}_{j,j+1}(\bar{\sigma}_{n_\ell}, X^{0,x}_{\bar{\sigma}_{n_\ell}}), \ldots,$$
$$Y^{j-1}_{\bar{\sigma}_{n_\ell+p-1}} = Y^{j}_{\bar{\sigma}_{n_\ell+p-1}} - \underline{g}_{j-1,j}(\bar{\sigma}_{n_\ell+p-1}, X^{0,x}_{\bar{\sigma}_{n_\ell+p-1}}), \forall \ell \geq 0] > 0.$$

Next let us set $\eta = \lim_{\ell\to\infty} \bar{\sigma}_{n_\ell}$. Take the limit in the previous equalities yield:

$$\mathbb{P}[\underline{g}_{j,j+1}(\eta, X^{0,x}_\eta) + ... + \underline{g}_{p-1,p}(\eta, X^{0,x}_\eta) + \underline{g}_{p,1}(\eta, X^{0,x}_\eta) + ... + \underline{g}_{j-1,j}(\eta, X^{0,x}_\eta) = 0] > 0.$$

But this is contradictory with the non free loop property (3.16).

Next let us show that $\mathbb{E}\left[(A_T^{u^*(v)})^2\right] < \infty$. Proceeding similarly as in the proof of Proposition 3.3.4, in the interval $[0, \rho_1]$ we have

$$Y^i_0 = h^i(X^{0,x}_T)1_{(\rho_1=T)} + Y_{\rho_1}1_{(\rho_1<T)} + \int_0^{\rho_1} f^i(r, X^{0,x}_r)dr - \int_0^{\rho_1} Z^i_r dB_r + \int_0^{\rho_1} dK^{i,+}_r - \int_0^{\rho_1} dK^{i,-}_r \tag{3.33}$$

Note that the minimizer C_2's control $v = (\tau_n)_{n\geq 0}$ is not necessarily optimal, then $\int_0^{\rho_1} dK^{i,-}_r \geq 0$ and we know that for any $s \in [0, T]$, $Y^i_s \leq Y^{i+1}_s + \bar{g}_{i,i+1}(s, X^{0,x}_s)$. On the other hand, since $\rho_1 = \bar{\sigma}_1 \wedge \tau_{\check{s}_1}$ then $\int_0^{\rho_1} dK^{i,+}_r = 0$. It follows that:

$$\begin{aligned}
Y^i_0 &\leq h^i(X^{0,x}_T)1_{(\rho_1=T)} + Y^i_{\rho_1}1_{(\rho_1<T)} + \int_0^{\rho_1} f^i(r, X^{0,x}_r)dr - \int_0^{\rho_1} Z^i_r dB_r \\
&\leq h^i(X^{0,x}_T)1_{(\rho_1=T)} + 1_{(\rho_1<T)}(Y^i_{\bar{\sigma}_1}1_{\{\rho_1=\bar{\sigma}_1\}} + Y^i_{\tau_{\check{s}_1}}1_{\{\rho_1=\tau_{\check{s}_1}\}}) + \int_0^{\rho_1} f^i(r, X^{0,x}_r)dr - \int_0^{\rho_1} Z^i_r dB_r \\
&\leq h^{\theta_0}(X^{0,x}_T)1_{(\rho_1=T)} + Y^{\theta_1}_{\rho_1}1_{(\rho_1<T)} - \left[\underline{g}_{\theta_0\theta_1}(\rho_1, X^{0,x}_{\rho_1})1_{(\rho_1=\bar{\sigma}_1<T)} - \bar{g}_{\theta_0\theta_1}(\rho_1, X^{0,x}_{\rho_1})1_{(\rho_1=\tau_{\check{s}_1}<T)}\right] \\
&\quad + \int_0^{\rho_1} f^{\theta_0}(r, X^{0,x}_r)dr - \int_0^{\rho_1} Z^{\theta_0}_r dB_r
\end{aligned} \tag{3.34}$$

Proceeding then iteratively for $n = 1, 2, ..., N$ to obtain

$$\begin{aligned}
Y^i_0 \leq& \sum_{n=1}^{N} h^{\theta_{n-1}}(X^{0,x}_T)1_{(\rho_{n-1}<T,\rho_n=T)} + Y^{\theta_N}_{\rho_N}1_{(\rho_N<T)} + \int_0^{\rho_N} f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr - \int_0^{\rho_N} Z^{\theta(u^*(v),v)_r}_r dB_r \\
&- \sum_{n=1}^{N}\left[\underline{g}_{\theta_{n-1}\theta_n}(\rho_n, X^{0,x}_{\rho_n})1_{(\rho_n=\bar{\sigma}_n<T)} - \bar{g}_{\theta_{n-1}\theta_n}(\rho_n, X^{0,x}_{\rho_n})1_{(\rho_n=\tau_{\check{s}_n}<T)}\right].
\end{aligned} \tag{3.35}$$

Then we have

$$A^{u^*(v)}_{\rho_N} \leq \sum_{n=1}^{N} h^{\theta_{n-1}}(X^{0,x}_T)1_{(\rho_{n-1}<T,\rho_n=T)} + Y^{\theta_N}_{\rho_N}1_{(\rho_N<T)} + \int_0^{\rho_N} f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr \\ - \int_0^{\rho_N} Z^{\theta(u^*(v),v)_r}_r dB_r - Y^i_0 + B^v_{\rho_N}. \tag{3.36}$$

Next as $v \in \mathcal{B}$ and since $(Z^i)_{i\in\Gamma} \in \mathcal{H}^{2,d}$, taking the squares of each hand-side of the previous inequality to deduce that:

$$\mathbb{E}[(A^{u^*(v)}_{\rho_N})^2] \leq C$$

for some real constant C. Finally to conclude it is enough to use Fatou's Lemma since $\rho_N \to T$ as $N \to \infty$.

In the same way we show that $v^*(u)$ belongs to $\mathcal{B}$ when u belongs to $\mathcal{A}$.

iii) Let $v \in \mathcal{B}$. Going back to (3.47), take expectation to obtain:

$$Y^i_0 = \mathbb{E}[Y^i_0] \leq \mathbb{E}[\sum_{n=1}^{N} h^{\theta_{n-1}}(X^{0,x}_T)1_{(\rho_{n-1}<T,\rho_n=T)} + Y^{\theta_n}_{\rho_n}1_{(\rho_n<T)} + \int_0^{\rho_N} f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr - C^{\theta(u^*(v),v)}_N]$$

As $v \in \mathcal{B}$ and $u^*(v) \in \mathcal{A}$, then for any $N \geq 1$, $|C^{\theta(u^*(v),v)}_N| \leq A^{u^*(v)}_T + B^v_T \in L^2(d\mathbb{P})$. Take now the limit w.r.t N in the right-hand side of the previous inequality and using dominated convergence theorem to deduce that:

$$Y^i_0 \leq \mathbb{E}[h^{\theta_T(u^*(v),v)}(X^{0,x}_T) + \int_0^T f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr - C^{\theta(u^*(v),v)}_\infty] = J_i(\theta(u^*(v),v)),\ \forall v \in \mathcal{B}.$$

The other inequality is shown in a similar fashion. □

As a by-product we obtain the following result:

Theorem 3.3.6. *Assume (H1), (H2), (H3) and $(Z^i)_{i\in\Gamma} \in \mathcal{H}^{2,d}$. Then for any $i = 1, ..., m$,*

$$Y^i_0 = \sup_{u\in\mathcal{A}} \inf_{v\in\mathcal{B}} J_i(\theta(u,v)) = \inf_{v\in\mathcal{B}} \sup_{u\in\mathcal{A}} J_i(\theta(u,v)).$$

Proof. By (3.32), we know that for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$,

$$J_i\left(\theta(u, v^*(u))\right) \leq Y^i_0 \leq J_i\left(\theta(u^*(v), v)\right).$$

Therefore

$$\sup_{u\in\mathcal{A}} J_i\left(\theta(u, v^*(u))\right) \leq Y^i_0 \leq \inf_{v\in\mathcal{B}} J_i\left(\theta(u^*(v), v)\right).$$

As when $u \in \mathcal{A}$ (resp. $v \in \mathcal{B}$), $v^*(u) \in \mathcal{B}$ (resp. $u^*(v) \in \mathcal{A}$) then

$$\inf_{v\in\mathcal{B}} \sup_{u\in\mathcal{A}} J_i\left(\theta(u,v)\right) \le \sup_{u\in\mathcal{A}} J_i\left(\theta(u,v^*(u))\right) \le Y_0^i \le \inf_{v\in\mathcal{B}} J_i\left(\theta(u^*(v),v)\right) \le \sup_{u\in\mathcal{A}} \inf_{v\in\mathcal{B}} J_i\left(\theta(u,v)\right)$$

which implies the desired result since the right-hand side is smaller that the left-hand one. □

Remark 3.3.7. *Note that we have also the following equalities: For any $i \in \Gamma$,*

$$\begin{aligned} Y_0^i &= \sup_{u\in\mathcal{A}} J_i\left(\theta(u,v^*(u))\right) = \inf_{v\in\mathcal{B}} J_i\left(\theta(u^*(v),v)\right) \\ &= \inf_{v\in\mathbb{B}} \sup_{u\in\mathcal{A}} J_i\left(\theta(u,v(u))\right) = \sup_{u\in\mathbb{A}} \inf_{v\in\mathcal{B}} J_i\left(\theta(u(v),v)\right). \end{aligned}$$

Actually let us show the fourth equality. Let $\tilde{u}(.) \in \mathbb{A}$. Then

$$\inf_{v\in\mathcal{B}} J_i\left(\theta(\tilde{u}(v),v)\right) \le \inf_{v\in\mathcal{B}} \sup_{u\in\mathcal{A}} J_i\left(\theta(u,v)\right) = Y_0^i = \inf_{v\in\mathcal{B}} J_i\left(\theta(u^*(v),v)\right)$$

which implies the fourth equality since $u^*(.) \in \mathbb{A}$. The third one is proved similarly. □

As mentioned before, the bottleneck for proving the existence of a value for the zero-sum switching game over square integrable controls is the square integrability of $(Z^i)_{i\in\Gamma}$. The point now is whether or not it is possible to characterize Y^i as the value of the zero-sum switching game without assuming the square integrability of $(Z^i)_{i\in\Gamma}$. At least at the cost of adding some supplementary conditions on the data of the game. The answer is affirmative if we require assumption (H4) on the switching costs. Finally note that this assumption (H4) is satisfied if $\bar{g}_{i,i+1}$ and $\underline{g}_{i,i+1}$, $i = 1,\dots,p$, do not depend on x and are non decreasing w.r.t t (e.g. they are constant).

We then have:

Theorem 3.3.8. *Assume (H1), (H2) and (H3). Then for any $i \in \Gamma$,*

$$Y_0^i = \sup_{u\in\mathcal{A}} \inf_{v\in\mathcal{B}} J_i(\theta(u,v)) = \inf_{v\in\mathcal{B}} \sup_{u\in\mathcal{A}} J_i(\theta(u,v)).$$

Proof. First recall the processes $(Y^i, Z^i, K^{i,\pm})_{i\in\Gamma}$ that satisfy: For any $i \in \Gamma$ and $s \le T$,

$$\begin{cases} Y^i \in \mathcal{S}^2; K^{i,\pm} \in \mathcal{A}_{loc} \text{ and } Z^i \in \mathcal{H}^{2,d}_{loc}; \\ Y_s^i = h^i(X_T^{0,x}) + \int_s^T f^i(r, X_r^{0,x})dr - \int_s^T Z_r dB_r + K_T^{i,+} - K_s^{i,+} - (K_T^{i,-} - K_s^{i,-}); \\ L^i(\vec{Y})_s \le Y_s^i \le U^i(\vec{Y})_s; \\ \int_0^T (Y_s^i - L^i(\vec{Y})_s)dK_s^{i,+} = 0 \text{ and } \int_0^T (Y_s^i - U^i(\vec{Y})_s)dK_s^{i,-} = 0 \end{cases} \tag{3.37}$$

where for $s \leq T$, $L^i(\vec{Y})_s := Y_s^{i+1} - \underline{g}_{i,i+1}(s, X_s^{0,x})$ and $U^i(\vec{Y})_s := Y_s^{i+1} + \bar{g}_{i,i+1}(s, X_s^{0,x})$.

Next for any $k \geq 0$, let us define the following stopping time:

$$\gamma_k := \inf\{s \geq 0, \int_0^s \{ \sum_{i=1,m} |Z_r^i|^2\} dr \geq k\} \wedge T. \tag{3.38}$$

First note that the sequence $(\gamma_k)_{k\geq 1}$ is increasing, of stationnary type and converges to T. Next we have $\int_0^{\gamma_k} |Z_r^i|^2 dr \leq k$, which means that the processes $(Z_s^i \mathbf{1}_{\{s \leq \gamma_k\}})_{s \leq T}$ belong to $\mathcal{H}^{2,d}$. Let us now define $(\bar{Y}^i, \bar{Z}^i, \bar{K}^{i,\pm})_{i\in\Gamma}$ as follows: For all $i \in \Gamma$ and $s \leq T$,

$$\bar{Y}_s^i := Y_{s\wedge\gamma_k}^i, \bar{Z}_s^i = Z_s^i \mathbf{1}_{\{s\leq\gamma_k\}}, \bar{K}_s^{i,+} := K_{s\wedge\gamma_k}^{i,+} \text{ and } \bar{K}_s^{i,-} := K_{s\wedge\gamma_k}^{i,-}. \tag{3.39}$$

Thus the family $\left(\bar{Y}^i, \bar{Z}^i, \bar{K}^{i,+}, \bar{K}^{i,-}\right)_{i\in\Gamma}$ is the solution of the following system: $\forall i \in \Gamma$,

$$\begin{cases} \text{i) } \bar{Y}^i \in \mathcal{S}^2, \bar{Z}^i \in \mathcal{H}^{2,d}, \bar{K}^{i,\pm} \in \mathcal{A}_{loc}; \\ \text{ii) } \bar{Y}_s^i = Y_{\gamma_k}^i + \int_s^T 1_{(r\leq\gamma_k)} f^i(r, X_r^{0,x}) dr - \int_s^T \bar{Z}_r^i dB_r + \bar{K}_T^{i,+} - \bar{K}_s^{i,+} - (\bar{K}_T^{i,-} - \bar{K}_s^{i,-}), \forall s \leq T; \\ \text{iii) } \bar{Y}_s^{i+1} - \underline{g}_{i,i+1}(s, X_s^{0,x}) \leq \bar{Y}_s^i \leq \bar{Y}_s^{i+1} + \bar{g}_{i,i+1}(s, X_s^{0,x}), \forall s \leq T; \\ \text{iv) } \int_0^T \left(\bar{Y}_s^i - L^i(\vec{\bar{Y}})_s\right) d\bar{K}_s^{i,+} = 0 \text{ and } \int_0^T \left(\bar{Y}_s^i - U^i(\vec{\bar{Y}})_s\right) d\bar{K}_s^{i,-} = 0 \end{cases} \tag{3.40}$$

where $U^i(\vec{\bar{Y}})$ and $L^i(\vec{\bar{Y}})$ are defined as in (3.37). Let us amphazise that here we need the assumption [H4] to show the inequalities in point iii) which actually hold true. Indeed for $s \leq \gamma_k$, the inequalities hold true by the definition of the processes $\left(\bar{Y}^i, \bar{Z}^i, \bar{K}^{i,+}, \bar{K}^{i,-}\right)_{i\in\Gamma}$ and (3.37). If $s > \gamma_k$, by (H4) we have,

$$\begin{aligned} \bar{Y}_s^{i+1} - \underline{g}_{i,i+1}(s, X_s^{0,x}) &= Y_{\gamma_k}^{i+1} - \underline{g}_{i,i+1}(s, X_s^{0,x}) \leq Y_{\gamma_k}^{i+1} - \underline{g}_{i,i+1}(\gamma_k, X_{\gamma_k}^{0,x}) \\ &\leq Y_{\gamma_k}^i = \bar{Y}_s^i \leq Y_{\gamma_k}^{i+1} + \bar{g}_{i,i+1}(s, X_s^{0,x}) = \bar{Y}_s^{i+1} + \bar{g}_{i,i+1}(s, X_s^{0,x}). \end{aligned}$$

On the other hand, by definition of $\bar{K}^{\pm,i}$ and $\bar{Y}^i$, $i \in \Gamma$, we have

$$\int_0^T \left(\bar{Y}_s^i - L^i(\vec{\bar{Y}})_s\right) d\bar{K}_s^{i,+} = \int_0^{\gamma_k} (Y_s^i - L^i(\vec{Y})_s) dK_s^{i,+} = 0.$$

Similarly we have also $\int_0^T (\bar{Y}_s^i - U^i(\vec{\bar{Y}})_s) d\bar{K}_s^{i,-} = 0$. Therefore the processes $(\bar{Y}^i, \bar{Z}^i, \bar{K}^{i,\pm})_{i\in\Gamma}$ verify (3.40).

Now using the result of Theorem 3.3.6, we obtain: For any $i \in \Gamma$,

$$Y_0^i = \bar{Y}_0^i = \sup_{u\in\mathcal{A}} \inf_{v\in\mathcal{B}} J_i^k(\theta(u,v)) = \inf_{v\in\mathcal{B}} \sup_{u\in\mathcal{A}} J_i^k(\theta(u,v)).$$

with
$$J_i^k(\theta(u,v)) = \mathbb{E}\left[Y_{\gamma_k}^{\theta(u,v)_T} + \int_0^T 1_{(r\le\gamma_k)} f^{\theta(u,v)_r}(r, X_r^{0,x})dr - C_\infty^{\theta(u,v)}\right]$$
where $\theta(u,v)$ is the coupling of the pair (u,v) of controls and
$$C_\infty^{\theta(u,v)} := \lim_{n\to\infty} C_N^{u,v}.$$

Next let
$$\breve{Y}_0^i = \sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i(\theta(u,v)) \text{ and } \tilde{Y}_0^i = \inf_{v\in\mathcal{B}}\sup_{u\in\mathcal{A}} J_i(\theta(u,v)).$$

Therefore
$$\begin{aligned}
|\breve{Y}_0^i - Y_0^i| =& |\sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i(\theta(u,v)) - \sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i^k(\theta(u,v))| \\
\le& \sup_{(u,v)\in\mathcal{A}\times\mathcal{B}} \mathbb{E}[|Y_{\gamma_k}^{\theta(u,v)_T} - h^{\theta(u,v)_T}(X_T^{0,x})| \\
&\qquad + \int_0^T |1_{(r\le\gamma_k)} f^{\theta(u,v)_r}(r, X_r^{0,x})dr - f^{\theta(u,v)_r}(r, X_r^{0,x})|dr] \\
\le& \mathbb{E}[\sum_{i=1,m} |Y_{\gamma_k}^i - h^i(X_T^{0,x})| + \int_{\gamma_k}^T \sum_{i=1,m} |f^i(r, X_r^{0,x})|dr].
\end{aligned}$$

But the right-hand side converges to 0 as $k\to\infty$. Therefore
$$\breve{Y}_0^i = Y_0^i = \sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i(\theta(u,v)).$$

In the same way we obtain also that
$$\tilde{Y}_0^i = Y_0^i = \inf_{v\in\mathcal{B}}\sup_{u\in\mathcal{A}} J_i(\theta(u,v)).$$

It follows that
$$Y_0^i = \sup_{u\in\mathcal{A}}\inf_{v\in\mathcal{B}} J_i(\theta(u,v)) = \inf_{v\in\mathcal{B}}\sup_{u\in\mathcal{A}} J_i(\theta(u,v)).$$

Thus the zero-sum switching game has a value on square integrable controls which is equal to Y_0^i. □

3.3.2 Value of the zerosum switching game on integrable admissible controls

In this part, we are not going to assume the square integrability of $(Z^i)_{i\in\Gamma}$ neither (H4) and show that the relation (3.12) holds true and this common value is equal to Y_0^i, where $(Y^i, Z^i, K^{i,\pm})_{i\in\Gamma}$ is the solution of system (3.17). Actually we have the following result:

Theorem 3.3.9. *Assume (H1), (H2) and (H3). Then for any $i \in \Gamma$,*

$$Y_0^i = \inf_{v \in \mathcal{B}^{(1)}} \sup_{u \in \mathcal{A}^{(1)}} J_i(\theta(u,v)) = \sup_{u \in \mathcal{A}^{(1)}} \inf_{v \in \mathcal{B}^{(1)}} J_i(\theta(u,v)). \tag{3.41}$$

Proof. Let $u = (\sigma_n)_{n\geq 0}$ and $v = (\tau_n)_{n\geq 0}$ be two admissible controls which belong to $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ respectively. Next recall the optimal responses $u^*(v) = (\bar{\sigma}_n)_{n\geq 0}$ and $v^*(u) = (\bar{\tau}_n)_{n\geq 0}$ defined in (3.31) and (3.29) respectively. First note that, as shown in Proposition 3.3.5, the controls $u^*(v)$ and $v^*(u)$ are admissible. Let us now show $u^*(v)$ belongs to $\mathcal{A}^{(1)}$. A similar procedure will show that $v^*(u)$ belongs to $\mathcal{B}^{(1)}$.

Indeed for $k \geq 1$, recall the stopping time γ_k defined in (3.38) and the sequences $(\rho_n)_{n\geq 0}$ and $(\theta_n)_{n\geq 0}$ defined in (3.3.5). Next for $k \geq 1$, let us define: $\forall\ n \geq 0$,

$$\rho_n^k = \rho_n 1_{\{\rho_n < \gamma_k\}} + T 1_{\{\rho_n \geq \gamma_k\}} \text{ and } \theta_n^k = \theta_n 1_{\{\rho_n < \gamma_k\}} + \theta_{n_k} 1_{\{\rho_n \geq \gamma_k\}}$$

where $n_k = \inf\{n \geq 0, \rho_n \geq \gamma_k\} - 1$. Note that ρ_n^k is a stopping time and $\{\rho_n^k < T\} = \{\rho_n < \gamma_k\}$. The sequences $(\rho_n^k)_{n\geq 0}$ and $(\theta_n^k)_{n\geq 0}$ constitute the fact that we freeze the actions of the controllers when γ_k is reached. Next going back to the system of equations (3.17) satisfied by the family $\left(Y^i, Z^i, K^{i,+}, K^{i,-}\right)_{i\in\Gamma}$ and as in (3.45) we have:

$$\begin{aligned} Y_0^i &= h^i(X_T^{0,x}) 1_{(\rho_1^k = T)} + Y_{\rho_1^k}^i 1_{(\rho_1^k < T)} + \int_0^{\rho_1^k} f^i(r, X_r^{0,x}) dr - \int_0^{\rho_1^k} Z_r^i dB_r + \underbrace{\int_0^{\rho_1^k} dK_r^{i,+}}_{=0} - \int_0^{\rho_1^k} dK_r^{i,-} \\ &\leq h^i(X_T^{0,x}) 1_{(\rho_1^k = T)} + Y_{\rho_1^k} 1_{(\rho_1^k < T)} + \int_0^{\rho_1^k} f^i(r, X_r^{0,x}) dr - \int_0^{\rho_1^k} Z_r^i dB_r \end{aligned} \tag{3.42}$$

But $\{\rho_1^k < T\} = \{\rho_1 < \gamma_k\}$. Therefore

$$Y_{\rho_1^k}^i 1_{(\rho_1^k < T)} = Y_{\rho_1}^i 1_{(\rho_1 < \gamma_k)} = (Y_{\bar{\sigma}_1}^i 1_{\{\rho_1 = \bar{\sigma}_1\}} + Y_{\tau_{\bar{s}_1}}^i 1_{\{\rho_1 = \tau_{\bar{s}_1}\}}) 1_{(\rho_1 < \gamma_k)}$$

and then

$$Y_0^i \leq h^i(X_T^{0,x}) 1_{(\rho_1^k = T)} + (Y_{\bar{\sigma}_1}^i 1_{\{\rho_1 = \bar{\sigma}_1\}} + Y_{\tau_{\bar{s}_1}}^i 1_{\{\rho_1 = \tau_{\bar{s}_1}\}}) 1_{(\rho_1 < \gamma_k)} + \int_0^{\rho_1^k} f^i(r, X_r^{0,x}) dr - \int_0^{\rho_1^k} Z_r^i dB_r \tag{3.43}$$

But for any $s \in [0,T]$, $Y_s^i \leq Y_s^{i+1} + \bar{g}_{i,i+1}(s, X_s^{0,x})$ and

$$Y_{\bar{\sigma}_1}^i 1_{\{\rho_1 = \bar{\sigma}_1\}} 1_{(\rho_1 < \gamma_k)} = (Y_{\bar{\sigma}_1}^{i+1} - \underline{g}_{i,i+1}(\bar{\sigma}_1, X_{\bar{\sigma}_1}^{0,x})) 1_{\{\rho_1 = \bar{\sigma}_1\}} 1_{(\rho_1 < \gamma_k)}.$$

Plug now this in (3.43) to obtain:

$$Y_0^i \le h^i(X_T^{0,x})1_{(\rho_1^k=T)} + (Y_{\bar{\sigma}_1}^{i+1} - \underline{g}_{i,i+1}(\bar{\sigma}_1, X_{\bar{\sigma}_1}^{0,x}))1_{\{\rho_1=\bar{\sigma}_1\}}1_{(\rho_1<\gamma_k)} \\ + (Y_{\tau_{\check{s}_1}}^{i+1} + \bar{g}_{i,i+1}(\tau_{\check{s}_1}, X_{\tau_{\check{s}_1}}^{0,x}))1_{\{\rho_1=\tau_{\check{s}_1}\}}1_{(\rho_1<\gamma_k)} + \int_0^{\rho_1^k} f^{\theta_0}(r, X_r^{0,x})dr - \int_0^{\rho_1^k} Z_r^{\theta_0} dB_r. \tag{3.44}$$

As

$$(Y_{\bar{\sigma}_1}^{i+1}1_{\{\rho_1=\bar{\sigma}_1\}} + Y_{\tau_{\check{s}_1}}^{i+1}1_{\{\rho_1=\tau_{\check{s}_1}\}})1_{(\rho_1<\gamma_k)} = Y_{\rho_1^k}^{\theta_1^k}1_{(\rho_1^k<\gamma_k)}$$

and

$$(-\underline{g}_{i,i+1}(\bar{\sigma}_1, X_{\bar{\sigma}_1}^{0,x})1_{\{\rho_1=\bar{\sigma}_1\}} + \bar{g}_{i,i+1}(\tau_{\check{s}_1}, X_{\tau_{\check{s}_1}}^{0,x})1_{\{\rho_1=\tau_{\check{s}_1}\}})1_{(\rho_1<\gamma_k)} \\ = (-\underline{g}_{\theta_0,\theta_1^k}(\bar{\sigma}_1, X_{\bar{\sigma}_1}^{0,x})1_{\{\rho_1^k=\bar{\sigma}_1\}} + \bar{g}_{\theta_0,\theta_1^k}(\tau_{\check{s}_1}, X_{\tau_{\check{s}_1}}^{0,x})1_{\{\rho_1^k=\tau_{\check{s}_1}\}})1_{(\rho_1^k<\gamma_k)}$$

then from (3.45), we obtain:

$$Y_0^i \le h^{\theta_0}(X_T^{0,x})1_{(\rho_1^k=T)} + Y_{\rho_1^k}^{\theta_1^k}1_{(\rho_1^k<\gamma_k)} + (-\underline{g}_{\theta_0,\theta_1^k}(\bar{\sigma}_1, X_{\bar{\sigma}_1}^{0,x})1_{\{\rho_1^k=\bar{\sigma}_1\}} + \bar{g}_{\theta_0,\theta_1^k}(\tau_{\check{s}_1}, X_{\tau_{\check{s}_1}}^{0,x})1_{\{\rho_1^k=\tau_{\check{s}_1}\}})1_{(\rho_1^k<\gamma_k)} \\ + \int_0^{\rho_1^k} f^{\theta_0}(r, X_r^{0,x})dr - \int_0^{\rho_1^k} Z_r^{\theta_0} dB_r. \tag{3.45}$$

But we can do the same with $Y_{\rho_1^k}^{\theta_1^k}1_{(\rho_1^k<\gamma_k)}$ to obtain:

$$Y_{\rho_1^k}^{\theta_1^k}1_{(\rho_1^k<\gamma_k)} \le h^{\theta_2^k}(X_T^{0,x})1_{(\rho_1^k<\gamma_k,\rho_2^k=T)} + Y_{\rho_2^k}^{\theta_2^k}1_{(\rho_2^k<\gamma_k)} + \\ (-\underline{g}_{\theta_1^k,\theta_2^k}(\bar{\sigma}_2, X_{\bar{\sigma}_2}^{0,x})1_{\{\rho_2^k=\bar{\sigma}_2\}} + \bar{g}_{\theta_1^k,\theta_2^k}(\tau_{\check{s}_2}, X_{\tau_{\check{s}_2}}^{0,x})1_{\{\rho_2^k=\tau_{\check{s}_2}\}})1_{(\rho_2^k<\gamma_k)} \\ + \int_{\rho_1^k}^{\rho_2^k} f^{\theta_2^k}(r, X_r^{0,x})dr - \int_{\rho_1^k}^{\rho_2^k} Z_r^{\theta_2^k} dB_r. \tag{3.46}$$

Plug now (3.46) in (3.45) and repeat this procedure N times to obtain:

$$Y_0^i \le \sum_{n=1}^{N} h^{\theta_{n-1}^k}(X_T^{0,x})1_{(\rho_{n-1}^k<T,\rho_n^k=T)} + Y_{\rho_N^k}^{\theta_N^k}1_{(\rho_N^k<\gamma_k)} + \int_0^{\rho_N^k} f^{\theta(u^*(v),v)_r}(r, X_r^{0,x})dr - \int_0^{\rho_N^k} Z_r^{\theta(u^*(v),v)_r} dB_r \\ \underbrace{- \sum_{n=1}^{N} \left[\underline{g}_{\theta_{n-1}^k\theta_n^k}(\rho_n^k, X_{\rho_n^k}^{0,x})1_{(\rho_n^k=\bar{\sigma}_n<\gamma_k)} - \bar{g}_{\theta_{n-1}^k\theta_n^k}(\rho_n^k, X_{\rho_n^k}^{0,x})1_{(\rho_n^k=\tau_{\check{s}_n}<\gamma_k)} \right]}_{A_{\rho_N^k}^{u^*(v)} - \tilde{B}_{\rho_N^k}^v} \tag{3.47}$$

where $0 \le \tilde{B}_{\rho_N^k}^v \le B_{\rho_N^k}^v$, since C_1 has priority when the two players decide to switch at the same time. Then take expectation in both hand-sides to obtain:

$$\mathbb{E}[A^{u^*(v)}_{\rho^k_N}] \leq -Y^i_0 + \mathbb{E}[\sum_{n=1}^{N} h^{\theta^k_{n-1}}(X^{0,x}_T)1_{(\rho^k_{n-1}<T,\rho^k_n=T)} + Y^{\theta^k_N}_{\rho^k_N}1_{(\rho^k_N<\gamma_k)} + \int_0^{\rho^k_N} f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr + B^v_{\rho^k_N}]. \tag{3.48}$$

As $v \in \mathcal{B}^{(1)}$, then $\mathbb{E}[B^v_{\rho^k_N}] \leq \mathbb{E}[B^v_T]$ and the right hand side of (3.47) is bounded. Then there exists a constant C such that

$$\mathbb{E}[A^{u^*(v)}_{\rho^k_N}] \leq C + \mathbb{E}[B^v_T].$$

Finally by using twice Fatou's Lemma (w.r.t k then N) we deduce that $\mathbb{E}[A^{u^*(v)}_T] < \infty$ which is the claim.

iii) Let $v \in \mathcal{B}^{(1)}$. Going back to (3.47), take expectation to obtain:

$$\begin{aligned} Y^i_0 \leq \mathbb{E}\{ &\sum_{n=1}^{N} h^{\theta^k_{n-1}}(X^{0,x}_T)1_{(\rho^k_{n-1}<T,\rho^k_n=T)} + Y^{\theta^k_N}_{\rho^k_N}1_{(\rho^k_N<\gamma_k)} + \int_0^{\rho^k_N} f^{\theta(u^*(v),v)_r}(r, X^{0,x}_r)dr \\ &- \sum_{n=1}^{N}\left[\underline{g}_{\theta^k_{n-1}\theta^k_n}(\rho^k_n, X^{0,x}_{\rho^k_n})1_{(\rho^k_n=\bar{\sigma}_n<\gamma_k)} - \bar{g}_{\theta^k_{n-1}\theta^k_n}(\rho^k_n, X^{0,x}_{\rho^k_n})1_{(\rho^k_n=\tau_{\bar{s}_n}<\gamma_k)}\right]\}. \end{aligned} \tag{3.49}$$

By taking the limit w.r.t k then N we obtain that

$$Y^i_0 \leq J_i(u^*(v), v), \forall v \in \mathcal{B}^{(1)}.$$

In the same way as previously, for any $u \in \mathcal{A}^{(1)}$, $v^*(u)$ belongs to $\mathcal{B}^{(1)}$ and

$$Y^i_0 \geq J_i(u, v^*(u)).$$

It follows that for any $u \in \mathcal{A}^{(1)}$ and $v \in \mathcal{B}^{(1)}$,

$$J_i(u, v^*(u)) \leq Y^i_0 \leq J_i(u^*(v), v).$$

Therefore

$$\sup_{u\in\mathcal{A}^{(1)}} J_i(u, v^*(u)) \leq Y^i_0 \leq \inf_{v\in\mathcal{B}^{(1)}} J_i(u^*(v), v).$$

As $u^*(v)$ (resp. $v^*(u)$) belongs to $\mathcal{A}^{(1)}$ (resp. $\mathcal{B}^{(1)}$) when $v \in \mathcal{B}^{(1)}$ (resp. $u \in \mathcal{A}^{(1)}$), then

$$\underbrace{\inf_{v\in\mathcal{B}^{(1)}} \sup_{u\in\mathcal{A}^{(1)}} J_i(u,v)}_{V^+} \leq \sup_{u\in\mathcal{A}^{(1)}} J_i(u, v^*(u)) \leq Y^i_0 \leq \inf_{v\in\mathcal{B}^{(1)}} J_i(u^*(v), v) \leq \underbrace{\sup_{u\in\mathcal{A}^{(1)}} \inf_{v\in\mathcal{B}^{(1)}} J_i(u,v)}_{V^-}$$

and the claim is proved since $V^+ \geq V^-$. $\square$

Remark 3.3.10. *a) As in Remark 3.3.7 we have also the following equalities: For any $i \in \Gamma$,*

$$Y_0^i = \inf_{v \in \mathbb{B}^{(1)}} \sup_{u \in \mathcal{A}^{(1)}} J_i\left(\theta(u, v(u))\right) = \sup_{u \in \mathbb{A}^{(1)}} \inf_{v \in \mathcal{B}^{(1)}} J_i\left(\theta(u(v), v)\right).$$

b) Let $(Y^{i,t,x}, Z^{i,t,x}, K^{i,\pm,t,x})_{i\in\Gamma}$ be the measurable processes such that: For any $i \in \Gamma$,
i) $Y^{i,t,x} \in \mathcal{S}^2$, $K^{i,\pm,t,x}$ are continuous non decreasing and $\mathbb{P}$-a.s. $\int_t^T |Z_s^{i,t,x}|^2 ds < \infty$;
ii) $\forall s \in [t,T]$,

$$\begin{cases} Y_s^i = h^i(X_T^{t,x}) + \int_s^T f^i(r, X_r^{t,x})dr - \int_s^T Z_r^{i,t,x} dB_r + K_T^{i,+,t,x} - K_s^{i,+,t,x} - (K_T^{i,-,t,x} - K_s^{i,-,t,x}); \\ L_s^i((Y^l)_{l\in\Gamma}) \leq Y_s^i \leq U_s^i((Y^l)_{l\in\Gamma}); \\ \int_t^T (Y_s^i - L_s^i((Y^l)_{l\in\Gamma}))dK_s^{i,+} = 0 \text{ and } \int_0^T (Y_s^i - U_s^i((Y^l)_{l\in\Gamma}))dK_s^{i,-} = 0. \end{cases} \tag{3.50}$$

These processes exist by Theorem 3.3.1. Then as previously one can show that for any (t,x) and $s \in [t,T]$,

$$Y_s^{i,t,x} = \operatorname*{ess\,inf}_{v \in \mathcal{B}_t^{(1)}} \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^{(1)}} J_i^{t,x}(\theta(u,v))_s = \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^{(1)}} \operatorname*{ess\,inf}_{v \in \mathcal{B}_i^{(1)}} J_i^{t,x}(\theta(u,v))_s$$

where

$$J_i^{t,x}(\theta(u,v))_s := \mathbb{E}\{h^{\theta(u,v)_T}(X_T^{t,x}) + \int_t^T f^{\theta(u,v)_r}(r, X_r^{t,x})dr - C_\infty^{\theta(u,v)} | \mathcal{F}_s\}$$

and $\mathcal{A}_t^{(1)}$ (resp. $\mathcal{B}_t^{(1)}$) is the set of admissible integrable controls which start from i at t. $\square$

3.4 System of PDEs of min-max type with interconnected obstacles

We are going now to deal with the problem of existence and uniqueness of a solution in viscosity sense for the following system of PDEs of min-max type with interconnected obstacles:

$$\begin{cases} \min\{v^i(t,x) - L^i(\vec{v})(t,x); \max\left[v^i(t,x) - U^i(\vec{v})(t,x);\right. \\ \qquad \left. -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x,(v^l(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x v^i(t,x)\right]\} = 0; \\ v^i(T,x) = h^i(x) \end{cases} \tag{3.51}$$

where for any $i \in \Gamma$, $L^i(\vec{v})(t,x) := v^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$ and $U^i(\vec{v})(t,x) := v^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$. Note that f^i is more general w.r.t. the HJB system of (3.3) since it depends also on $\vec{y}$ and z^i.

The result is given in Theorem 3.4.3 but its proof, based on Perron's method, is postponed to

Appendix. Nonetheless in this section we will introduce some notions which we need also in Section 5 when we deal with system of RBSDEs (3.1) or more generally (3.5.2).

For any locally bounded deterministic function $u : [0,T] \times \mathbb{R}^k \to \mathbb{R}$, we denote by u_* (resp. u^*) the lower semi-continuous (lsc) (resp. upper semi-continuous (usc)) envelope of u as follows: $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$u_*(t,x) = \liminf_{(t',x') \mapsto (t,x), t'<T} u(t',x') \text{ and } u^*(t,x) = \limsup_{(t',x') \mapsto (t,x), t'<T} u(t',x').$$

Next for an lsc (resp. usc) function u we denote by $\bar{J}^- u(t,x)$ (resp. $\bar{J}^+ u(t,x)$), the parabolic limiting subjet (resp. superjet) of u at (t,x) (see e.g. [13] for the definition and more details).

Definition 3.4.1. *: Viscosity solution to* (3.51)
Let $\vec{v} := (v^i)_{i\in\Gamma}$ be a p-tuple of $\mathbb{R}$-valued, locally bounded functions defined on $[0,T] \times \mathbb{R}^k$.

A) We say that $\vec{v}$ is a viscosity supersolution (resp. subsolution) of (3.51) *if for any $i \in \Gamma$:*
(i) $v^i_*(T,x) \geq h^i(x)$ (resp. $v^{i*}(T,x) \leq h^i(x)$), for any $x \in \mathbb{R}^k$;
(ii) For any $(t,x) \in [0,T) \times \mathbb{R}^k$ and for any $(p,q,M) \in \bar{J}^- v^i_*(t,x)$ (resp. $\bar{J}^+ v^{i*}(t,x)$), we have:

$$\begin{aligned} &\min\{v^i_*(t,x) - L^i(\vec{v}_*)(t,x), \\ &\quad \max\{-p - b(t,x).q - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)M] - f^i(t,x,\vec{v}_*(t,x),\sigma^\top(t,x)q); \\ &\quad v^i_*(t,x) - U^i(\vec{v}_*)(t,x)\}\} \ \geq 0 \end{aligned} \tag{3.52}$$

where $\vec{v}_* = (v^i_*)_{i\in\Gamma}$ (resp.

$$\begin{aligned} &\min\{v^{i*}(t,x) - L^i(\vec{v}^*)(t,x), \\ &\quad \max\{-p - b(t,x).q - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)M] - f^i(t,x,\vec{v}^*(t,x),\sigma^\top(t,x)q); \\ &\quad v^{i*}(t,x) - U^i(\vec{v}^*)(t,x)\}\} \ \leq 0 \end{aligned} \tag{3.53}$$

where $\vec{v}^* = (v^{i*})_{i\in\Gamma}$).

B) A locally bounded function $\vec{v} = (v^i)_{i\in\Gamma}$ is called a viscosity solution of (3.51) *if $(v^i_*)_{i\in\Gamma}$ and $(v^{i*})_{i\in\Gamma}$ are viscosity supersolution and viscosity subsolution of* (3.51) *respectively.* □

Next (t,x) be fixed and let us consider the following sequence of BSDEs: $\forall m,n \in \mathbb{N}, \forall i \in \Gamma$,

$$\begin{cases} Y^{i,m,n} \in \mathcal{S}^2, Z^{i,m,n} \in \mathcal{H}^{2,d}; \\ Y^{i,m,n}_s = h^i(X^{t,x}_T) + \int_s^T f^{i,m,n}(r, X^{t,x}_r, (Y^{l,m,n}_r)_{l\in\Gamma}, Z^{i,m,n}_r)dr - \int_s^T Z^{i,m,n}_r dB_r, \ s \leq T; \\ Y^{i,m,n}_T = h^i(X^{t,x}_T) \end{cases} \tag{3.54}$$

where

$$f^{i,m,n}(s,X_s^{t,x},\vec{y},z)=f^i(s,X_s^{t,x},\vec{y},z)+n\left\{y^i-[y^{i+1}-\underline{g}_{i,i+1}(s,X_s^{t,x})]\right\}^- - m\left\{y^i-[y^{i+1}+\overline{g}_{i,i+1}(s,X_s^{t,x})]\right\}^+.$$

As (3.54) is a classical BSDE without obstacle, thanks to the results by Pardoux-Peng [24], the solution exists and is unique. In addition there exist deterministic functions $(v^{i,m,n})_{i\in\Gamma}$ (see Theorem 4.1. in [24]) such that:

$$\forall s\in[t,T],\ Y_s^{i,m,n}=v^{i,m,n}(s,X_s^{t,x}). \tag{3.55}$$

On the other hand, we have the following properties which we collect in the following proposition.

Proposition 3.4.2 (see [34],[18]). *Assume that (H2), (H3) and (H5) are fulfilled. Then we have:*

a) $\mathbb{P}-a.s.,\forall s\leq T, Y_s^{i,m+1,n}\leq Y_s^{i,m,n}\leq Y_s^{i,m,n+1},\ \forall i\in\Gamma, n,m\geq 0$, which also implies the same property for $(v^{i,m,n})_{i\in\Gamma}$, i.e. for any $(t,x)\in[0,T]\times\mathbb{R}^k, i\in\Gamma$,

$$v^{i,m+1,n}(t,x)\leq v^{i,m,n}(t,x)\leq v^{i,m,n+1}(t,x). \tag{3.56}$$

b) The sequence $((Y^{i,m,n})_{i\in\Gamma})_{n\geq 0})$ (resp. $((Y^{i,m,n})_{i\in\Gamma})_{m\geq 0}$) converges in $(\mathcal{S}^2)^p$ to $(\bar{Y}^{i,m})_{i\in\Gamma}$ (resp. $(\underline{Y}^{i,n})_{i\in\Gamma}$) which verifies the following system of reflected RBSDEs:

$$\begin{cases}\overline{Y}^{i,m}\in\mathcal{S}^2,\overline{Z}^{i,m}\in\mathcal{H}^2,\overline{K}^{i,m,+}\in\mathcal{A}^2;\\ \overline{Y}_s^{i,m}=h^i(X_T^{t,x})+\int_s^T\overline{f}^{i,m}(r,X_r^{t,x},(\overline{Y}_r^{l,m})_{l\in\Gamma},\overline{Z}_r^{i,m})dr-\int_s^T\overline{Z}_r^{i,m}dB_r+\overline{K}_T^{i,m,+}-\overline{K}_s^{i,m,+},s\leq T;\\ \overline{Y}_s^{i,m}\geq L^i((\bar{Y}_s^{l,m})_{l\in\Gamma}),s\leq T;\\ \int_0^T(\overline{Y}_s^{i,m}-L^i((\bar{Y}_s^{l,m})_{l\in\Gamma}))d\overline{K}_s^{i,m,+}=0\end{cases} \tag{3.57}$$

where

$$\overline{f}^{i,m}(s,X_s^{t,x},\vec{y},z^i)=f^i(s,X_s^{t,x},\vec{y},z^i)-m(y^i-[y^{i+1}+\overline{g}_{i,i+1}(s,X_s^{t,x})])^+.$$

(resp.

$$\begin{cases}\underline{Y}^{i,n}\in\mathcal{S}^2,\underline{Z}^{i,n}\in\mathcal{H}^2,\underline{K}^{i,n,-}\in\mathcal{A}^2;\\ \underline{Y}_s^{i,n}=h^i(X_T^{t,x})+\int_s^T\underline{f}^{i,n}(r,X_r^{t,x},(\underline{Y}_r^{l,n})_{l\in\Gamma},\underline{Z}_r^{i,n})dr-\int_s^T\underline{Z}_r^{i,n}dB_r+\underline{K}_T^{i,n,-}-\underline{K}_s^{i,n,-},s\leq T;\\ \underline{Y}_s^{i,n}\leq U^i((\underline{Y}_s^{l,m})_{l\in\Gamma}),s\leq T;\\ \int_0^T(\underline{Y}_s^{i,m}-U^i((\underline{Y}_s^{l,m})_{l\in\Gamma}))d\underline{K}_s^{i,n,-}=0\end{cases} \tag{3.58}$$

where

$$\underline{f}^{i,n}(s,X_s^{t,x},\vec{y},z^i)=f^i(s,X_s^{t,x},\vec{y},z^i)+n(y^i-[y^{i+1}-\underline{g}_{i,i+1}(s,X_s^{t,x})])^+.$$

c) There exist deterministic continuous functions $(\overline{v}^{i,m})_{i\in\Gamma}$ (resp. $(\underline{v}^{i,n})_{i\in\Gamma}$) such that for any $(t,x) \in [0,T]\times\mathbb{R}^k, s\in[t,T]$,

$$\overline{Y}_s^{i,m} = \overline{v}^{i,m}(s, X_s^{t,x}) \tag{3.59}$$

(resp.

$$\underline{Y}_s^{i,n} = \underline{v}^{i,n}(s, X_s^{t,x})) \tag{3.60}$$

In addition the sequence $((\overline{v}^{i,m})_{m\geq 0})_{i\in\Gamma}(resp.((\underline{v}^{i,n})_{n\geq 0})_{i\in\Gamma})$ are decreasing w.r.t. m (resp. increasing w.r.t. n).

d) $(\overline{v}^{i,m})_{i\in\Gamma}$ (resp. $(\underline{v}^{i,n})$) belong to class Π_g and is the unique viscosity solution of following system of variational inequalities with a reflected obstacle:

$$\begin{cases} \min\{\overline{v}^{i,m}(t,x) - L^i((\overline{v}^{l,m})_{l\in\Gamma})(t,x); \\ -\partial_x \overline{v}^{i,m}(t,x) - \mathcal{L}^X(\overline{v}^{i,m})(t,x) - f^{i,m}(t,x,(\overline{v}^{l,m}(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x\overline{v}^{i,m}(t,x))\} = 0 \\ \overline{v}^{i,m}(T,x) = h^i(x). \end{cases} \tag{3.61}$$

(resp.

$$\begin{cases} \max\{\underline{v}^{i,n}(t,x) - U^i((\underline{v}^{l,m})_{l\in\Gamma})(t,x); \\ -\partial_x \underline{v}^{i,n}(t,x) - \mathcal{L}^X(\underline{v}^{i,n})(t,x) - f^{i,n}(t,x,(\underline{v}^{l,n}(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x\underline{v}^{i,n}(t,x))\} = 0 \\ \underline{v}^{i,n}(T,x) = h^i(x)). \end{cases} \tag{3.62}$$

Proof. This proof can be found in [34] and [18] so we omit it. □

Next for any $i\in\Gamma$ and $(t,x)\in[0,T]\times\mathbb{R}^k$, we denote by

$$\overline{v}^i(t,x) := \lim_{m\to\infty} \overline{v}^{i,m}(t,x) \text{ and } \underline{v}^i(t,x) := \lim_{n\to\infty} \underline{v}^{i,n}.$$

Then from (3.56) we deduce that for any $(t,x)\in[0,T]\times\mathbb{R}^k$

$$\underline{v}^i(t,x) \leq \overline{v}^i(t,x).$$

Note that since for any $i\in\Gamma$,

$$\underline{v}^{i,0} \leq \underline{v}^i \leq \bar{v}^i \leq \bar{v}^{i,0}$$

then $\underline{v}^i$ and $\bar{v}^i$ belong to Π_g. Additionnaly we have:

Theorem 3.4.3. *Assume (H2),(H3) and (H5). Then the p-tuple of functions $(\overline{v}^i)_{i\in\Gamma}$ are continuous, of polynomial growth and unique viscosity solution, in the class Π_g, of the following systems: $\forall i \in \Gamma$ and $(t,x) \in [0,T]\times \mathbb{R}^k$,*

$$\begin{cases} \min\{v^i(t,x) - L^i(\vec{v})(t,x); \max\left[v^i(t,x) - U^i(\vec{v})(t,x);\right. \\ \qquad \left. -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x,(v^l(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x v^i(t,x)\right]\} = 0; \\ v^i(T,x) = h^i(x). \end{cases} \tag{3.63}$$

Proof. It is rather long and then postponed to Appendix. □

As a consequence we have the following result for the increasing scheme:

Corollary 3.4.4. *The p-tuple of functions $(\underline{v}^i)_{i\in\Gamma}$ is also continuous and the unique viscosity solution, in the class Π_g, of the following system of max-min type: $\forall i \in \Gamma$ and $(t,x) \in [0,T]\times\mathbb{R}^k$,*

$$\begin{cases} \max\{v^i(t,x) - U^i(\vec{v})(t,x); \min\left[v^i(t,x) - L^i(\vec{v})(t,x);\right. \\ \qquad \left. -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) - f^i(t,x,(v^l(t,x))_{l\in\Gamma}, \sigma(t,x)^\top D_x v^i(t,x)\right]\} = 0; \\ v^i(T,x) = h^i(x). \end{cases} \tag{3.64}$$

To obtain the proof of this result it is enough to consider $(-\underline{v}^i)_{i\in\Gamma}$ which becomes a decreasing scheme associated with $((-f^i(t,x,-\vec{y},-z))_{i\in\Gamma}, (-h^i)_{i\in\Gamma}, (\bar{g}_i)_{i\in\Gamma}, (\underline{g}_i)_{i\in\Gamma})$, to use the previous theorem and finally a result by G.Barles ([1], pp.18). □

3.5 Systems of Reflected BSDEs with bilateral interconnected barriers

First recall the system of RBSDEs $(\bar{Y}^{i,m,t,x}, \bar{Z}^{i,m,t,x}, \bar{K}^{i,m,+,t,x})$ in Proposition 3.4.2-b)-c) and the representation (3.59). As the sequence $((\bar{v}^{i,m})_{\geq 0})_{i\in\Gamma}$ converges pointwise decreasingly to the continuous functions $(v^i)_{i\in\Gamma}$. Then, by Dini's theorem, this convergence is uniform on compact sets of $[0,T]\times\mathbb{R}^k$. Next, the uniform polynomial growths of $(v^i)_{i\in\Gamma}$ and $((\bar{v}^{i,m})_{\geq 0})_{i\in\Gamma}$ combined with estimate (3.8) of $X^{t,x}$ imply that for any $i \in \Gamma$,

$$\mathbb{E}(\sup_{s\in[t,T]} |\bar{Y}_s^{i,m,t,x} - Y_s^{i,t,x}|^2) \to_{m\to\infty} 0 \tag{3.65}$$

where we set: For any $s \leq T$ and $i \in \Gamma$,

$$Y_s^{i,t,x} = v^i(s\vee t, X_{s\vee t}^{t,x}). \tag{3.66}$$

Proposition 3.5.1. *For any* $(t,x) \in [0,T] \times \mathbb{R}^k, s \in [t,T], i \in \Gamma$,

$$Y_s^i \le U^i((Y_s^l)_{l\in\Gamma}) := Y_s^{i+1} + \overline{g}_{i,i+1}(s, X_s^{t,x}). \tag{3.67}$$

Proof. According to (3.66), it is enough to show the following inequality: for any $i \in \Gamma, (t,x) \in [0,T] \times \mathbb{R}^k$,

$$v^i(t,x) \le v^{i+1}(t,x) + \bar{g}_{i,i+1}(t,x). \tag{3.68}$$

Indeed, we assume by contradiction that there exists some $(t_0,x_0) \in [0,T) \times \mathbb{R}^k$ and a strictly positive $\epsilon > 0$ such that

$$v^i(t_0,x_0) - v^{i+1}(t_0,x_0) - \bar{g}_{i,i+1}(t_0,x_0) \ge \epsilon > 0. \tag{3.69}$$

By the uniform convergence of $(\bar{v}^{i,m})_{i\in\Gamma}$ to the functions $(v^i)_{i\in\Gamma}$ on compact subsets, we can find some $\rho > 0$ and a ball defined by

$$\mathcal{B}((t_0,x_0),\rho) := \{(t,x) \in [0,T] \times \mathbb{R}^k, \text{ s.t. } |t-t_0| \le \rho \text{ and } |x-x_0| \le \rho\}$$

and some m_0 large enough such that for any $m \ge m_0$,

$$\bar{v}^{i,m}(t,x) - \bar{v}^{i+1,m}(t,x) - \bar{g}_{i,i+1}(t,x) \ge \frac{\epsilon}{8} > 0, \ \forall (t,x) \in \mathcal{B}((t_0,x_0),\rho). \tag{3.70}$$

Next let us introduce the following stopping time

$$\tau_{t_0,x_0} ;= \inf\{s \ge t_0, X_s^{t,x} \notin \mathcal{B}((t_0,x_0),\rho)\} \wedge (t_0+\rho)$$

Notice that for any $s \in [t_0, \tau_{t_0,x_0}]$,

$$\begin{aligned}
\bar{Y}_s^{i,m,t_0,x_0} &= \bar{v}^{i,m}(s, X_s^{t_0,x_0}) \\
&> \bar{v}^{i+1,m}(s, X_s^{t_0,x_0}) + \bar{g}_{i,i+1}(s, X_s^{t_0,x_0}) \\
&> \bar{v}^{i+1,m}(s, X_s^{t_0,x_0}) - \underline{g}_{i,i+1}(s, X_s^{t_0,x_0}) \\
&= \bar{Y}_s^{i+1,m,t_0,x_0} - \underline{g}_{i,i+1}(s, X_s^{t_0,x_0})
\end{aligned}$$

As a result for $s \in [t_0, \tau_{t_0,x_0}]$, $d\bar{K}_s^{i,m,+,t_0,x_0} = 0$ and then from (3.57) we deduce that: $\forall\, s \in [t_0, \tau_{t_0,x_0}]$,

$$\begin{aligned}
\overline{Y}_s^{i,m,t_0,x_0} =& \overline{Y}_{\tau_{t_0,x_0}}^{i,m,t_0,x_0} + \int_s^{\tau_{t_0,x_0}} \{f^{i,m}(r, X_r^{t_0,x_0}, (\overline{Y}_r^{l,m,t_0,x_0})_{l\in\Gamma}, \overline{Z}_r^{i,m,t_0,x_0}) \\
&- m(\overline{Y}_r^{i,m,t_0,x_0} - [\overline{Y}_r^{i+1,m,t_0,x_0} + \overline{g}_{i,i+1}(r, X_r^{t_0,x_0})])^+\}dr - \int_s^{\tau_{t_0,x_0}} \overline{Z}_r^{i,m,t_0,x_0} dB_r.
\end{aligned} \tag{3.71}$$

Next as in [23], since $\overline{g}_{i,i+1}$, $\bar{v}^{i,m}$ and $\bar{v}^{i+1,m}$ are of polynomial growth (uniformly for these latter) and by using (3.8) we deduce that:

$$
\begin{aligned}
&m^2\mathbb{E}[\{\int_{t_0}^{\tau_{t_0,x_0}}(\bar{Y}_s^{i,m,t_0,x_0}-\bar{Y}_s^{i+1,m,t_0,x_0}-\bar{g}_{i,i+1}(s,X_s^{t_0,x_0}))^+ds\}^2]\\
&\qquad\leq C\mathbb{E}[\sup_{s\in[t_0,\tau_{t_0,x_0}],i\in\Gamma}|\bar{Y}_s^{i,m,t_0,x_0}|^2]+C\mathbb{E}[\{\int_{t_0}^{\tau_{t_0,x_0}}f^i(s,X_s^{t_0,x_0},0,0)ds\}^2].
\end{aligned}
\tag{3.72}
$$

for some cosntant C which is independant of m. Therefore using (3.70) we have

$$
m^2\frac{\epsilon^2}{64}\mathbb{P}[t_0<\tau_{t_0,x_0}]\leq C\mathbb{E}[\sup_{s\in[t_0,\tau_{t_0,x_0}],i\in\Gamma}|\bar{Y}_s^{i,m,t_0,x_0}|^2]+C\mathbb{E}[\{\int_{t_0}^{\tau_{t_0,x_0}}f^i(s,X_s^{t,x},0,0)ds\}^2]. \tag{3.73}
$$

which implies, in sending m to $+\infty$, $\mathbb{P}[t_0<\tau_{t_0,x_0}]=0$, i.e. $\mathbb{P}[t_0=\tau_{t_0,x_0}]=1$. But this is contradictory since $\rho>0$ and then (t_0,x_0) satisfying (3.69) does not exists. The proof of the claim is complete. □

We now give the main result of this section.

Theorem 3.5.2. *Assume that the assumptions (H2),(H3) and (H5) are fulfilled and for any $i\in\Gamma$, f^i does not depend on z^i. Then for any $(t,x)\in[0,T]\times\mathbb{R}^k$, there exist adapted processes $K^{i,\pm,t,x}$ and $Z^{i,t,x}$ valued respectively in $\mathbb{R}^+$ and $\mathbb{R}^d$ such that, in combination with $Y^{i,t,x}$, verify: For any $i\in\Gamma$,*

i) $K^{i,\pm,t,x}$ are continuous, non decreasing and $K_t^{i,\pm,t,x}=0$; $\mathbb{P}$-a.s. $K_T^{i,\pm,t,x}<\infty$ and $\int_t^T|Z_s^{i,t,x}|^2ds<\infty$;

ii) $\forall s\in[t,T]$,

$$
\begin{cases}
Y_s^{i,t,x}=h^i(X_T^{t,x})+\int_s^T f^i(r,X_r^{t,x},(Y_r^{l,t,x})_{l\in\Gamma})dr-\int_s^T Z_r^{i,t,x}dB_r\\
\qquad\qquad +K_T^{i,+,t,x}-K_s^{i,+,t,x}-(K_T^{i,-,t,x}-K_s^{i,-,t,x});\\
L_s^i((Y^{l,t,x})_{l\in\Gamma})\leq Y_s^{i,t,x}\leq U_s^i((Y^{l,t,x})_{l\in\Gamma});\\
\int_t^T(Y_s^{i,t,x}-L_s^i((Y^{l,t,x})_{l\in\Gamma}))dK_s^{i,+,t,x}=0 \text{ and } \int_0^T(Y_s^{i,t,x}-U_s^i((Y^{l,t,x})_{l\in\Gamma}))dK_s^{i,-,i,t,x}=0
\end{cases}
\tag{3.74}
$$

where for $s\in t\leq T$, $L_s^i((Y^{l,t,x})_{l\in\Gamma}):=Y_s^{i+1,t,x}-\underline{g}_{i,i+1}(s,X_s^{t,x})$ and $U_s^i((Y^{l,t,x})_{l\in\Gamma}):=Y_s^{i+1,t,x}+\bar{g}_{i,i+1}(s,X_s^{t,x})$.

Moreover if there exists another quadruple $(\bar{Y}^{i,t,x},\bar{Z}^{i,t,x},\bar{K}^{i,\pm,t,x})$ which satisfies (i)-(ii), then for any $s\in[t,T]$ and $i\in\Gamma$, $\bar{Y}_s^{i,t,x}=Y_s^{i,t,x}$, $\bar{K}_s^{i,\pm,t,x}=K_s^{i,\pm,t,x}$ and $\bar{Z}^{i,t,x}=Z_s^{i,t,x}$, $ds\otimes d\mathbb{P}$ on $[t,T]\times\Omega$.

Proof. Existence

For any $i \in \Gamma$ and $m \geq 0$, the processes $\bar{Y}^{i,m,t,x}$ have the following representation (see e.g. A4 in [18] for more details): For any $s \in [t, T]$,

$$\begin{aligned} \bar{Y}^{i,m,t,x}_s = \operatorname*{ess\,sup}_{\sigma \geq s} \operatorname*{ess\,inf}_{\tau \geq s} \mathbb{E}[h^i(X^{t,x}_T) 1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X^{t,x}_r, (\bar{Y}^{l,m,t,x}_r)_{l\in\Gamma}) dr \\ + L^i_\sigma((\bar{Y}^{l,m,t,x})_{l\in\Gamma}) 1_{(\sigma<\tau)} + \{U^i_\tau((\bar{Y}^{l,m})_{l\in\Gamma}) \vee \bar{Y}^{i,m,t,x}_\tau\} 1_{(\tau\leq\sigma,\tau<T)} | \mathcal{F}_s]. \end{aligned} \tag{3.75}$$

Now the convergence of $(\bar{Y}^{i,m,t,x})_m$ to $Y^{i,,t,x}$ in $\mathcal{S}^2([t, T])$ (by (3.65)) and the inequalities (3.67) imply that, in taking the limits in both hand-sides of (3.75): $\forall s \in [t, T]$,

$$\begin{aligned} Y^{i,t,x}_s = \operatorname*{ess\,sup}_{\sigma \geq s} \operatorname*{ess\,inf}_{\tau \geq s} \mathbb{E}[h^i(X^{t,x}_T) 1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X^{t,x}_r, (Y^{l,t,x}_r)_{l\in\Gamma}) dr \\ + L^i_\sigma((Y^{l,t,x})_{l\in\Gamma}) 1_{(\sigma<\tau)} + U^i_\tau((Y^{l,t,x})_{l\in\Gamma}) 1_{(\tau\leq\sigma,\tau<T)} | \mathcal{F}_s]. \end{aligned} \tag{3.76}$$

Next the third inequality in (3.57) and (3.67) imply that: For any $s \in [t, T]$ and $i \in \Gamma$,

$$U^i_s((Y^l)_{l\in\Gamma}) \geq Y^i_s \geq L^i_s((Y^l)_{l\in\Gamma}).$$

On the other hand by Assumption (H3)-a),

$$U^i_s((Y^{l,t,x})_{l\in\Gamma}) - L^i_s((Y^{l,t,x})_{l\in\Gamma}) = \bar{g}_{i,i+1}(s, X^{t,x}_s) + \underline{g}_{i,i+1}(s, X^{t,x}_s) > 0$$

which means that the obstacles $U^i((Y^{l,t,x})_{l\in\Gamma})$ and $L^i((Y^{l,t,x})_{l\in\Gamma})$, for any $i \in \Gamma$, are completely separated. Therefore by Theorem 3.7 in [27], there exist progressively measurable processes $\underline{Y}^{i,t,x}$, $K^{i,\pm,t,x}$ and $Z^{i,t,x}$ valued respectively in $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^d$ such that:
i) $\underline{Y}^{i,t,x} \in \mathcal{S}^2([t, T])$, $K^{i,\pm,t,x}$ are continuous non decreasing and $K^{i,\pm,t,x}_t = 0$; $\mathbb{P}$-a.s. $\int_t^T |Z^{i,t,x}_s|^2 ds < \infty$;
ii) The processes $(\underline{Y}^{i,t,x}, K^{i,\pm,t,x}, Z^{i,t,x})$ verify: $\forall s \in [t, T]$,

$$\begin{cases} \underline{Y}^{i,t,x}_s = h^i(X^{t,x}_T) + \int_s^T f^i(r, X^{t,x}_r, (Y^{l,t,x}_r)_{l\in\Gamma}) dr - \int_s^T Z^{i,t,x}_r dB_r \\ \qquad + K^{i,+,t,x}_T - K^{i,+,t,x}_s - (K^{i,-,t,x}_T - K^{i,-,t,x}_s); \\ L^i_s((Y^{l,t,x})_{l\in\Gamma}) \leq \underline{Y}^{i,t,x}_s \leq U^i_s((Y^{l,t,x})_{l\in\Gamma}); \\ \int_t^T (Y^{i,t,x}_s - L^i_s((Y^{l,t,x})_{l\in\Gamma})) dK^{i,+,t,x}_s = 0 \text{ and } \int_0^T (Y^{i,t,x}_s - U^i_s((Y^{l,t,x})_{l\in\Gamma})) dK^{i,-,t,x}_s = 0. \end{cases} \tag{3.77}$$

Moreover $\underline{Y}^{i,t,x}$ has the following representation: $\forall s \in [t,T]$,

$$\underline{Y}_s^{i,t,x} = \operatorname*{ess\,sup}_{\sigma\geq s} \operatorname*{ess\,inf}_{\tau\geq s} \mathbb{E}[h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X_r^{t,x}, (Y_r^l)_{l\in\Gamma})dr + L_\sigma^i((Y^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + U_\tau^i((Y^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s]. \tag{3.78}$$

Thus for any $s \in [t,T]$, $\underline{Y}^{i,t,x} = Y^{i,t,x}$ and by (3.77), $(Y^{i,tx}, K^{i,\pm,t,x}, Z^{i,t,x})$ verify (3.74). Finally as i is arbitrary then $(Y^{i,t,x}, K^{i,\pm,t,x}, Z^{i,t,x})_{i\in\Gamma}$ is a solution for the system of reflected BSDEs with double obstacles (3.74). The proof of existence is then stated. It remains to show uniqueness.

Uniqueness: In this part we apply the fixed point argument over the value of the stochastic game representation (Theorem 3.3.9), and the proof is similar to [35]. In the following proof, the defined processes $(Y^{\phi,i}, Z^{\phi,i}, K^{\phi,i,\pm})_{i\in\Gamma}$ and $(Y^{\psi,i}, Z^{\psi,i}, K^{\psi,i,\pm})_{i\in\Gamma}$ depend on (t,x), but for simplicity of notations we omit it as there is no confusion.

Firstly let us define the following operator:

$$\begin{aligned} &\Phi : \mathcal{H}^{2,p} \to \mathcal{H}^{2,p} \\ &\vec{\phi} := (\phi^i)_{i\in\Gamma} \mapsto \Phi(\vec{\phi}) := (Y^{\phi,i})_{i\in\Gamma} \end{aligned}$$

where $(Y^{\phi,i}, Z^{\phi,i}, K^{\phi,i,\pm})_{i\in\Gamma}$ is the unique solution of

$$\begin{cases} Y^{\phi,i} \in \mathcal{S}^2([t,T]), \mathbb{P}-a.s. \int_t^T |Z_s^{\phi,i}|^2 ds < \infty \text{ and } K_T^{\phi,i,+} + K_T^{\phi,i,-} < \infty \ (K_t^{\phi,i,+} + K_t^{\phi,i,-} = 0); \\ Y_s^{\phi,i} = h^i(X_T^{t,x}) + \int_s^T f^i(r, X_r^{t,x}, \vec{\phi}(r))dr - \int_s^T Z_r^{\phi,i} dB_r + K_T^{\phi,i,+} - K_s^{\phi,i,+} - (K_T^{\phi,i,-} - K_s^{\phi,i,-}), t \leq s \leq T; \\ L_s^i((Y^{\phi,l})_{l\in\Gamma}) \leq Y_s^{\phi,i} \leq U_s^i((Y^{\phi,l})_{l\in\Gamma}), s \in [t,T]; \\ \int_t^T \left(Y_s^{\phi,i} - L_s^i((Y^{\phi,l})_{l\in\Gamma})\right) dK_s^{\phi,i,+} = 0 \text{ and } \int_t^T \left(Y_s^{\phi,i} - U_s^i((Y^{\phi,l})_{l\in\Gamma})\right) dK_s^{\phi,i,-} = 0. \end{cases} \tag{3.79}$$

In the similar way we define another element of $\mathcal{H}^{2,p}$ by $\vec{\psi} := (\psi^i)_{i\in\Gamma}$ and let $(Y_s^{\psi,i}, Z_s^{\psi,i}, K_s^{\psi,i,\pm})_{s\in[t,T]}$ be a solution of (3.79) where its driver is replaced with $f^i(t,x,\vec{\psi}(t)), \forall i \in \Gamma$.

Next we set the following norm, denoted by $\|.\|_{2,\beta}$ on $\mathcal{H}^{2,p}$:

$$\|y\|_{2,\beta} := (\mathbb{E}[\int_t^T e^{\beta s}|y_s|^2 ds])^{1/2}.$$

The following calculus is dedicated to prove that Φ is a contraction on $(\mathcal{H}^{2,p}, \|.\|_{2,\beta})$ where the appropriate value of β is determined in the following.

Let us recall Theorem 3.3.9 and Remark 3.3.10, for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and $t \leq s \leq T$, the following representation holds true:

$$Y_s^{\phi,i} = \operatorname*{ess\,inf}_{v\in\mathcal{B}_s^{(1)}} \operatorname*{ess\,sup}_{u\in\mathcal{A}_s^{(1)}} J_i^{\phi}(\Theta(u,v))_s = \operatorname*{ess\,sup}_{u\in\mathcal{A}_s^{(1)}} \operatorname*{ess\,inf}_{v\in\mathcal{B}_s^{(1)}} J_i^{\phi}(\Theta(u,v))_s. \tag{3.80}$$

where $J_i^{\phi}(\Theta(u,v))_s = \mathbb{E}\left[h^{\theta(u,v)_T}(X_T^{t,x}) + \int_s^T f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - C_\infty^{\theta(u,v)} \middle| \mathcal{F}_s \right].$

In the same way $Y^{\psi,i}$ has also the stochastic game representation by replacing ϕ to ψ.

Now we study the difference of $|Y^{\phi,i} - Y^{\psi,i}|$. Indeed, $\forall i \in \Gamma, t \in [0,T], t \le s \le T$,

$$|Y_s^{\phi,i} - Y_s^{\psi,i}| \le \operatorname*{ess\,sup}_{u\in\mathcal{A}_s^{(1)}} \operatorname*{ess\,sup}_{v\in\mathcal{B}_s^{(1)}} |J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s| \tag{3.81}$$

Thanks to the martingale representation theorem, there exists an $(\mathcal{F}_s)_{s\le T}-$adapted process $\Delta^{\phi,\psi,\theta(u,v)} \in \mathcal{H}^{2,d}$ such that

$$\begin{aligned}
&J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s = \mathbb{E}\left[\int_s^T f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) dr \middle| \mathcal{F}_s \right] \\
&= \mathbb{E}\left[\int_0^T f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) dr \middle| \mathcal{F}_s \right] \\
&- \int_0^s f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) dr \\
&= \mathbb{E}\left[\int_0^T f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) dr \right] + \int_0^s \Delta_r^{\phi,\psi,\theta(u,v)} dB_r \\
&- \int_0^s f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) dr
\end{aligned}$$

Therefore we obtain the following differential form for the difference of the two value functions:

$$d(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s) = \left[f^{\theta(u,v)_s}(s, X_s^{t,x}, \vec{\phi}(s)) - f^{\theta(u,v)_s}(s, X_s^{t,x}, \vec{\psi}(s)) \right] ds + \Delta_s^{\phi,\psi,\theta(u,v)} dB_s$$

Next for any $s \in [t,T]$, we apply Itô's formula on $e^{\beta s}\left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s\right)^2$ yielding

$$\begin{aligned}
&d\left[e^{\beta s}\left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s\right)^2 \right] = \beta e^{\beta s}\left[J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s \right]^2 \\
&+ 2e^{\beta s}\left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s\right)\left[-\left(f^{\theta(u,v)_s}(s, X_s^{t,x}, \vec{\phi}(s)) - f^{\theta(u,v)_s}(s, X_s^{t,x}, \vec{\psi}(s)) \right) \Delta_s^{\phi,\psi,\theta(u,v)} dB_s \right] \\
&+ e^{\beta s}\left(\Delta_s^{\phi,\psi,\theta(u,v)}\right)^2 ds
\end{aligned} \tag{3.82}$$

By integrating (3.82) over $[s,T]$ we obtain

$$e^{\beta s}\left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s\right)^2 + \int_s^T e^{\beta r}\left(\Delta_r^{\phi,\psi,\theta(u,v)}\right)^2 dr$$

$$\begin{aligned}
&= -\beta \int_s^T e^{\beta r} \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right)^2 dr \\
&\quad + 2 \int_s^T e^{\beta r} \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right) \left(f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) \right) dr \\
&\quad - 2 \int_s^T \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right) \Delta_r^{\phi,\psi,\theta(u,v)} dB_r \qquad (3.83)
\end{aligned}$$

By applying the inequality $2ab \leq \beta a^2 + \dfrac{b^2}{\beta}$, (3.83) yields

$$\begin{aligned}
&e^{\beta s} \left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s \right)^2 + \int_s^T e^{\beta r} \left(\Delta_r^{\phi,\psi,\theta(u,v)} \right)^2 dr \\
&\leq \frac{1}{\beta} \int_s^T e^{\beta s} \left(f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\phi}(r)) - f^{\theta(u,v)_r}(r, X_r^{t,x}, \vec{\psi}(r)) \right)^2 dr \\
&\quad - 2 \int_s^T \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right) \Delta_r^{\phi,\psi,\theta(u,v)} dB_r
\end{aligned}$$

By the Lipschitz condition on the driver $f^{\theta(u,v)}$, and using the fact that $\int_s^T e^{\beta r} \left(\Delta_r^{\phi,\psi,\theta(u,v)} \right)^2 dr \geq 0$, we then obtain

$$\begin{aligned}
&e^{\beta s} \left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s \right)^2 \\
&\leq \frac{C^2}{\beta} \int_s^T |\vec{\phi}(r) - \vec{\psi}(r)|^2 dr - 2 \int_s^T \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right) \Delta_r^{\phi,\psi,\theta(u,v)} dB_r \qquad (3.84)
\end{aligned}$$

where $C = \sum_{i \in \Gamma} C_i$ with C_i the Lipschitz constant w.r.t. $f^i, \forall i \in \Gamma$. On the other hand since $(2 \int_s^u \left(J_i^{\phi}(\Theta(u,v))_r - J_i^{\psi}(\Theta(u,v))_r \right) \Delta_r^{\phi,\psi,\theta(u,v)} dB_r)_{u \in [s,T]}$ is a martingale, then taking the conditional expectation w.r.t. $\mathcal{F}_s$ on both sides of (3.84) we have

$$e^{\beta s} \left(J_i^{\phi}(\Theta(u,v))_s - J_i^{\psi}(\Theta(u,v))_s \right)^2 \leq \frac{C^2}{\beta} \mathbb{E} \left[\int_s^T |\vec{\phi}(r) - \vec{\psi}(r)|^2 dr | \mathcal{F}_s \right] \qquad (3.85)$$

Let us recall (3.81), then by taking the expectation on both sides of (3.85) we obtain: $\forall\, s \in [t,T]$,

$$\mathbb{E} \left[e^{\beta s} \left(Y_s^{\phi,i} - Y_s^{\psi,i} \right)^2 \right] \leq \frac{C^2}{\beta} \mathbb{E} \left[\int_t^T |\vec{\phi}(r) - \vec{\psi}(r)|^2 dr \right] \qquad (3.86)$$

The last step is integrating (3.86) over $s \in [t,T]$ and then summing over all $i \in \Gamma$ to obtain:

$$\mathbb{E} \left[\int_t^T \sum_{i \in \Gamma} e^{\beta s} \left(Y_s^{\phi,i} - Y_s^{\psi,i} \right)^2 ds \right] \leq \frac{C^2 TP}{\beta} \mathbb{E} \left[\int_t^T |\vec{\phi}(r) - \vec{\psi}(r)|^2 dr \right] \qquad (3.87)$$

Obviously it is enough to take $\beta > C^2TP$ (for example we can let $\beta := 4C^2TP$) then the operator

Φ is a contraction on $\mathcal{H}^{2,p}$ to itself. As a consequence, there exists a fixed point which is nothing but the unique solution of (3.74).

Next we suppose that there exists another solution $(\hat{Y}^i, \hat{Z}^i, \hat{K}^{i,\pm})_{i\in\Gamma}$ of (3.74), i.e.

$$\begin{cases} \hat{Y}^i_s = h^i(X^{t,x}_T) + \int_s^T f^i(r, X^{t,x}_r, (\hat{Y}^l_r)_{l\in\Gamma})dr - \int_s^T \hat{Z}^{i,t,x}_r dB_r \\ \qquad\qquad + \hat{K}^{i,+,t,x}_T - \hat{K}^{i,+,t,x}_s - (\hat{K}^{i,-,t,x}_T - \hat{K}^{i,-,t,x}_s), s \in [t,T]; \\ L^i_s((\hat{Y}^l)_{l\in\Gamma}) \leq Y^i_s \leq U^i_s((\hat{Y}^l)_{l\in\Gamma}), s\in[t,T]; \\ \int_t^T (\hat{Y}^i_s - L^i_s((\hat{Y}^l)_{l\in\Gamma}))d\hat{K}^{i,+}_s = 0 \text{ and } \int_t^T (\hat{Y}^i_s - U^i_s((\hat{Y}^l)_{l\in\Gamma}))d\hat{K}^{i,-}_s = 0 \end{cases} \tag{3.88}$$

Thanks to the fixed point result (3.87) we have immediately $Y^i = \hat{Y}^i, \forall i \in \Gamma$. By applying the equivalence of Y^i and $\hat{Y}^i$, we also have $Z^i = \hat{Z}^i$ since from the representation of (3.74) and (3.88), their martingale parts should be equal, i.e. for any $i \in \Gamma, s \in [t,T]$, $\int_s^T Z^i_s dB_s = \int_s^T \hat{Z}^i_s dB_s$. Moreover by (3.74) and (3.88) we have $\forall s \in [t,T], i\in\Gamma, K^{i,+}_s - K^{i,-}_s = \hat{K}^{i,+}_s - \hat{K}^{i,-}_s$. It remains us now to prove the equivalence of the barriers processes.

For any $s \in [t,T], i \in \Gamma$ we have

$$\int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)(dK^{i,+}_r - dK^{i,-}_r) = \int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)(d\hat{K}^{i,+}_r - d\hat{K}^{i,-}_r) \tag{3.89}$$

On the other hand by the minimality conditions we have

$$\begin{aligned} \forall s \in [t,T], i\in\Gamma, \int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)(dK^{i,+}_r - dK^{i,+}_r) &= -\int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)dK^{i,-}_r \\ &= -\int_t^s \left(U^i_r((Y_l)_{l\in\Gamma}) - L^i_r((Y_l)_{l\in\Gamma})\right)dK^{i,-}_r \end{aligned} \tag{3.90}$$

This last equality is due to the fact that $\forall r \in [t,s], dK^{i,-}_r \neq 0$ only if Y^i touches the upper obstacle.

In the same way we have also the following condition for $\hat{K}^{i,-}$: $\forall i \in \Gamma, s \in [t,T]$,

$$\begin{aligned} &\int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)(d\hat{K}^{i,+}_r - d\hat{K}^{i,+}_r) = -\int_t^s \left(Y^i_r - L^i_r((Y_l)_{l\in\Gamma})\right)d\hat{K}^{i,-}_r \\ &= -\int_t^s \left(U^i_r((Y_l)_{l\in\Gamma}) - L^i_r((Y_l)_{l\in\Gamma})\right)d\hat{K}^{i,-}_r \end{aligned} \tag{3.91}$$

Combining (3.89)-(3.91) and (H3)-a)(the two obstacles are totally separated), we finally obtain

$$\forall i \in \Gamma,\ s \in [t,T],\ K^{i,-}_s = \hat{K}^{i,-}_t$$

since $K_t^{i,-} = \hat{K}_t^{i,-} = 0$. Finally the equality $K_s^{i,+} - K_s^{i,-} = \hat{K}_s^{i,+} - \hat{K}_s^{i,-}$, $s \in [t, T]$, implies $K^{i,+} = \hat{K}^{i,+}$. The proof of uniqueness is now finished. □

We now go back to systems (3.63) and (3.64) and the question is whether or not they have the same solution. We have the following result:

Proposition 3.5.3. *Assume that the assumptions (H2),(H3) and (H5) are fulfilled and for any $i \in \Gamma$, f^i does not depend on z^i. Then for any $i \in \Gamma$, $\bar{v}^i = \underline{v}^i$.*

Proof. : Actually $(-\underline{v}^i)_{i\in\Gamma}$ is the unique solution of the following system of PDEs with obstacles:

$$\begin{cases} \min\{v^i(t,x) - \check{L}^i(\vec{v})(t,x); \max\big[v^i(t,x) - \check{U}^i(\vec{v})(t,x); \\ \qquad -\partial_t v^i(t,x) - \mathcal{L}^X(v^i)(t,x) + f^i(t,x,(-v^l(t,x))_{l\in\Gamma}, -\sigma(t,x)^\top D_x v^i(t,x)\big]\} = 0; \\ v^i(T,x) = -h^i(x) \end{cases} \tag{3.92}$$

where $\check{L}^i(\vec{v})(t,x) = v^i(t,x) - \bar{g}_{i,i+1}(t,x)$ and $\check{U}^i(\vec{v})(t,x) = v^i(t,x) - \underline{g}_{i,i+1}(t,x)$. Therefore $-\underline{v}^i$, has accordingly, the representation (3.76), i. e. for any (t,x) and $i \in \Gamma$, setting $\underline{Y}_s^{i,t,x} = \underline{v}^i(s \vee t, X_{s\vee t}^{t,x})$ for $s \in [t,T]$, we have:

$$\begin{aligned} -\underline{Y}_s^{i,t,x} &= \operatorname*{ess\,sup}_{\sigma\geq s} \operatorname*{ess\,inf}_{\tau\geq s} \mathbb{E}[-h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} -f^i(r, X_r^{t,x}, (-\underline{Y}_r^{l,t,x})_{l\in\Gamma})dr \\ &\qquad + \check{L}_\sigma^i((-\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + \check{U}_\tau^i((-\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s] \\ &= \operatorname*{ess\,inf}_{\tau\geq s} \operatorname*{ess\,sup}_{\sigma\geq s} \mathbb{E}[-h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} -f^i(r, X_r^{t,x}, (-\underline{Y}_r^{l,t,x})_{l\in\Gamma})dr \\ &\qquad + \check{L}_\sigma^i((-\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + \check{U}_\tau^i((-\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s] \end{aligned} \tag{3.93}$$

since the barriers are completely separated (see e.g. [27]). Therefore

$$\begin{aligned} \underline{Y}_s^{i,t,x} &= \operatorname*{ess\,sup}_{\sigma\geq s} \operatorname*{ess\,inf}_{\tau\geq s} \mathbb{E}[h^i(X_T^{t,x})1_{(\sigma=\tau=T)} + \int_s^{\sigma\wedge\tau} f^i(r, X_r^{t,x}, (\underline{Y}_r^{l,t,x})_{l\in\Gamma})dr \\ &\qquad + L_\sigma^i((\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\sigma<\tau)} + U_\tau^i((\underline{Y}^{l,t,x})_{l\in\Gamma})1_{(\tau\leq\sigma,\tau<T)}|\mathcal{F}_s] \end{aligned} \tag{3.94}$$

Which means that $((\underline{Y}_s^{i,t,x})_{s\in[t,T]})_{i\in\Gamma}$ verifes (3.74). As the solution of this latter is unique then for any $i \in \Gamma$, $\underline{Y}^{i,t,x} = \underline{Y}^{i,t,x}$ which means that for $i \in \Gamma$, $\bar{v}^i = \underline{v}^i$. □

3.6 Appendix: Proof of Theorem 3.4.3

In this section, we prove that the system of (3.51) has a unique continuous solution in viscosity sense in the class Π_g. Indeed, we firstly provide a comparison result of subsolution and supersolution of (3.51) if they exist, then we show that $(\bar{v}^i)_{i\in\Gamma}$ is a solution by Perron's method. We recall once for all that the results in this section are constructed under (H2),(H3) and (H5).

3.6.1 A comparison result

Before investigating (3.51), we provide some a priori results and a comparison principle for sub. and supersolutions of system (3.51). To begin with let us show the following:

Lemma 3.6.1. *Let $\vec{u} := (u^i)_{i\in\Gamma}$ (resp.$\vec{\hat{u}} := (\hat{u}^i)_{i\in\Gamma}$) be an usc subsolution (resp. sci supersolution) of* (3.51)*. For any $(t,x) \in [0,T] \times \mathbb{R}^k$, let $\hat{\Gamma}(t,x)$ be the following set:*

$$\hat{\Gamma}(t,x) := \{i \in \Gamma, u^i(t,x) - \hat{u}^i(t,x) = \max_{l\in\Gamma}(u^l(t,x) - \hat{u}^l(t,x))\}.$$

Then there exists $i_0 \in \hat{\Gamma}(t,x)$ such that

$$u^{i_0}(t,x) > u^{i_0+1}(t,x) - \underline{g}_{i_0,i_0+1}(t,x) \text{ and } \hat{u}^{i_0}(t,x) < \hat{u}^{i_0+1}(t,x) + \overline{g}_{i_0,i_0+1}(t,x).$$

Proof. Let $(t,x) \in [0,T] \times \mathbb{R}^k$ be fixed. As Γ is a finite set then $\hat{\Gamma}$ is not empty. To proceed, we assume, by contradiction that for any $i \in \hat{\Gamma}(t,x)$, either

$$u^i(t,x) \le u^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x) \tag{3.95}$$

or

$$\hat{u}^i(t,x) \ge \hat{u}^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x) \tag{3.96}$$

holds.

Assume first that (3.95) holds true i.e. $u^i(t,x) \le u^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$. As $\vec{\hat{u}}$ is a supersolution of (3.51), we deduce that

$$\hat{u}^i(t,x) \ge \hat{u}^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x) \tag{3.97}$$

By taking into account of (3.95) we have

$$\hat{u}^{i+1}(t,x) - \hat{u}^i(t,x) \le \underline{g}_{i,i+1}(t,x) \le u^{i+1}(t,x) - u^i(t,x)$$

which implies

$$u^i(t,x) - \hat{u}^i(t,x) \le u^{i+1}(t,x) - \hat{u}^{i+1}(t,x).$$

However as $i \in \hat{\Gamma}(t,x)$, then the previous inequality is an equality and then

$$\hat{u}^{i+1}(t,x) - \hat{u}^i(t,x) = u^{i+1}(t,x) - u^i(t,x) = \underline{g}_{i,i+1}(t,x) \tag{3.98}$$

and

$$u^i(t,x) - \hat{u}^i(t,x) = u^{i+1}(t,x) - \hat{u}^{i+1}(t,x).$$

As a result we deduce that $(i+1) \in \hat{\Gamma}(t,x)$ and also the equality (3.98) holds.

Next if $u^i(t,x) \leq u^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$ does not hold, then $u^i(t,x) > u^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x)$. On the other hand, assume that (3.96) holds true, i.e., $\hat{u}^i(t,x) \geq \hat{u}^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$. Since u^i is a subsolution of (3.51), we have

$$u^i(t,x) \leq u^{i+1}(t,x) + \overline{g}_{i,i+1}(t,x)$$

which implies

$$\hat{u}^{i+1}(t,x) - \hat{u}^i(t,x) \leq -\overline{g}_{i,i+1}(t,x) \leq u^{i+1}(t,x) - u^i(t,x)$$

and then

$$u^i(t,x) - \hat{u}^i(t,x) \leq u^{i+1}(t,x) - \hat{u}^{i+1}(t,x).$$

However as $i \in \hat{\Gamma}(t,x)$, then the last inequality is an equality and $(i+1) \in \hat{\Gamma}(t,x)$. Moreover

$$u^{i+1}(t,x) - u^i(t,x) = -\overline{g}_{i,i+1}(t,x) = \hat{u}^{i+1}(t,x) - \hat{u}^i(t,x). \tag{3.99}$$

It means that (3.95) or (3.96) imply that $(i+1) \in \hat{\Gamma}(t,x)$ and one of the equalities (3.98), (3.99). Repeat now this reasonning as many times as necessary (actually p times) to find a loop such that $\sum_{i\in\Gamma} \varphi_{i,i+1}(t,x) = 0$ ($\varphi_{i,i+1}$ is defined in (3.14)) and which is contradictory to assumption (H3). $\square$

Next we give the comparison result.

Proposition 3.6.2. *Let $\vec{u} := (u^i)_{i\in\Gamma}$ be an usc subsolution (resp. $\vec{w} := (w^i)_{i\in\Gamma}$ be a lsc supersolution) of the system* (3.51) *and for any $i \in \Gamma$, both u^i and w^i belong to class Π_g i.e. there exists two constants γ and C such that*

$$\forall i \in \Gamma, (t,x) \in [0,T] \times \mathbb{R}^k,\ |u^i(t,x)| + |w^i(t,x)| \leq C(1 + |x|^\gamma)$$

Then it holds true that

$$u^i(t,x) \leq w^i(t,x),\ \forall i \in \Gamma, (t,x) \in [0,T] \times \mathbb{R}^k. \tag{3.100}$$

Proof. Let us show the result by contradiction, i.e. there exists $\epsilon_0 > 0$ and some $(t_0, x_0) \in [0, T) \times \mathbb{R}^k$ such that

$$\max_{i\in\Gamma}(u^i(t_0, x_0) - w^i(t_0, x_0)) \geq \epsilon_0 \tag{3.101}$$

Next without loss of generality we assume that there exists $R > 0$ such that for $t \in [0, T]$, $|x| \geq R$ we have for any $i \in \Gamma$,

$$(u^i - w^i)(t, x) < 0. \tag{3.102}$$

Actually if (3.102) does not hold, it is enough to consider the following functions $w^{i,\theta,\mu}$ defined by

$$w^{i,\theta,\mu} = w^i(t, x) + \theta e^{-\bar{\lambda}t}(1 + |x|^{2\gamma+2}),\ (t, x) \in [0, T] \times \mathbb{R}^k$$

which still a supersolution of (3.51) for any $\theta > 0$ and $\bar{\lambda} \geq \lambda_0$ (λ_0 is fixed). Then to show that $u^i - w^{i,\theta,\mu} \leq 0$ for any $i \in \Gamma$ and finally to take the limit as $\theta \to 0$ to obtain (3.100). But for any $i \in \Gamma$, $u^i - w^{i,\theta,\mu}$ is negative uniformly in t when $|x|$ is large enough since u^i belongs to Π_g with polynomial exponent γ.

To proceed, let (3.101)-(3.102) be fulfilled. Then

$$\begin{aligned}\max_{(t,x)\in[0,T]\times\mathbb{R}^k}\max_{i\in\Gamma}\{u^i(t, x) - w^i(t, x)\} &= \max_{(t,x)\in[0,T]\times B(0,R)}\max_{i\in\Gamma}\{u^i(t, x) - w^i(t, x)\}\\ &:= \max_{i\in\Gamma}(u^i - w^i)(t^*, x^*) \leq \epsilon_0 > 0\end{aligned}$$

where $B(0, R)$ is the ball centered in the origin with radius R. Note that $t^* < T$ since $u^i(T, x) \leq h^i(x) \leq w^i(T, x)$.

Next the proof will be divided into two steps:

Step 1: To begin with, we introduce the following auxiliary condition: There exists $\lambda > (p - 1)\max_{i\in\Gamma} C_{f^i}$ such that for any $i \in \Gamma, (t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{p+d}$, and $(v^1, v^2) \in \mathbb{R}^2$ such that $v^1 \geq v^2$ we have

$$f^i(t, x, [\vec{y}^{-i}, v^1], z) - f^i(t, x, [\vec{y}^{-i}, v^2], z) \leq -\lambda(v^1 - v^2) \tag{3.103}$$

and where C_{f^i} is the Lipschitz constant of f^i w.r.t. $\vec{y}$.

So let i_0 be an element of $\hat{\Gamma}(t^*, x^*)$ such that

$$u^{i_0}(t^*, x^*) > u^{i_0+1}(t^*, x^*) - \underline{g}_{i_0,i_0+1}(t^*, x^*) \tag{3.104}$$

and

$$w^{i_0}(t^*, x^*) < w^{i_0+1}(t^*, x^*) + \overline{g}_{i_0,i_0+1}(t^*, x^*) \tag{3.105}$$

which exists by Lemma 3.6.1. Next we define the following function: For any $n \geq 1$,

$$\Phi_n^{i_0}(t,x,y) := (u^{i_0}(t,x) - w^{i_0}(t,y)) - \phi_n(t,x,y), \quad (t,x,y) \in [0,T] \times \mathbb{R}^{k+k}$$

where

$$\phi_n(t,x,y) := n|x-y|^{2\gamma+2} + |x-x^*|^{2\gamma+2} + (t-t^*)^2.$$

The function $\Phi_n^{i_0}(t,x,y)$ is usc, then we can find a triple $(t_n, x_n, y_n) \in [0,T] \times \bar{B}(0,R)^2$ such that

$$\Phi_n^{i_0}(t_n, x_n, y_n) = \max_{(t,x,y) \in [0,T] \times \bar{B}(0,R)^2} \Phi_n^{i_0}(t,x,y)$$

($\bar{B}(0,R)$ is the closure of $B(0,R)$). Then we have

$$\Phi_n^{i_0}(t^*, x^*, x^*) \leq \Phi_n^{i_0}(t_n, x_n, y_n).$$

From which we deduce that

$$\begin{aligned}
\Phi_n^{i_0}(t^*, x^*, x^*) &= u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*) \\
&\leq \Phi_n^{i_0}(t_n, x_n, y_n) \\
&= u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n) - \phi_n(t_n, x_n, y_n) \\
&\leq u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n) \leq C_R
\end{aligned} \tag{3.106}$$

(C_R is a constant which may depend on R) since the sequences $(t_n)_n$, $(x_n)_n$ and $(y_n)_n$ are bounded and u^{i_0} and w^{i_0} are of polynomial growth. As a result $(x_n - y_n)_{n \geq 0}$ converges to 0. On the other hand, by boundedness of the sequences, we can find a subsequence, which we still denote by $(t_n, x_n, y_n)_n$, converging to a point denoted $(\hat{t}, \hat{x}, \hat{x})$. By (3.108) it satisfies:

$$\begin{aligned}
u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*) &\leq \liminf_n (u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n)) \\
&\leq \limsup_n (u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n)) \\
&\leq \limsup_n u^{i_0}(t_n, x_n) - \liminf_n w^{i_0}(t_n, y_n) \\
&\leq u^{i_0}(\hat{t}, \hat{x}) - w^{i_0}(\hat{t}, \hat{x})
\end{aligned} \tag{3.107}$$

since u^{i_0} (resp. w^{i_0}) is usc (resp. lsc). As the maximum of $u^{i_0} - w^{i_0}$ on $[0,T] \times \mathbb{R}^k$ is reached in (t^*, x^*), then $u^{i_0}(\hat{t}, \hat{x}) - w^{i_0}(\hat{t}, \hat{x}) = u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*)$ and consequently the sequence $(u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n))_n$ converges to $u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*)$. Next as we have

$$\begin{aligned}
\Phi_n^{i_0}(t^*, x^*, x^*) &= u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*) \\
&\leq \Phi_n^{i_0}(t_n, x_n, y_n)
\end{aligned}$$

$$= u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n) - \phi_n(t_n, x_n, y_n) \tag{3.108}$$

then $(\phi_n(t_n, x_n, y_n))_n$ converges to 0 as $n \to \infty$ and then $(t_n)_n$, $(x_n)_n$ and (y_n) converge respectively to t^*, x^* and x^*. Finally

$$\begin{aligned}\liminf_n u^{i_0}(t_n, x_n) &= u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*) + \liminf_n w^{i_0}(t_n, y_n)\\ &\geq u^{i_0}(t^*, x^*) \geq \limsup_n u^{i_0}(t_n, x_n)\end{aligned}$$

which implies that the sequence $(u^{i_0}(t_n, x_n))_n$ converges to $u^{i_0}(t^*, x^*)$ and then also the sequence $(w^{i_0}(t_n, y_n))_n$ converges to $w^{i_0}(t^*, x^*)$.

Next, we recall the definition of $i_0 \in \hat{\Gamma}(t^*, x^*)$. By (3.104)-(3.105), for n large enough we can find a subsequence $(t_n, x_n)_n$ such that

$$u^{i_0}(t_n, x_n) > u^{i_0+1}(t_n, x_n) - \underline{g}_{i_0 i_0+1}(t_n, x_n) \tag{3.109}$$

and

$$w^{i_0}(t_n, y_n) < w^{i_0+1}(t_n, y_n) + \overline{g}_{i_0 i_0+1}(t_n, y_n). \tag{3.110}$$

Next we apply Crandall-Ishii-Lions's Lemma (see e.g. [25], pp.216) and then there exist $(p_u^n, q_n^n, M_u^n) \in \bar{J}^+(u^{i_0})(t_n, x_n)$ *and* $(p_w^n, q_w^n, M_w^n) \in \bar{J}^-(w^{i_0})(t_n, y_n)$ such that

$$\begin{cases} p_u^n - p_w^n = \partial_t \phi_n(t_n, x_n, y_n) = 2(t_n - t^*), \\ q_u^n = \partial_x \phi_n(t_n, x_n, y_n), \\ q_w^n = -\partial_y \phi_n(t_n, x_n, y_n) \text{ and} \\ \begin{pmatrix} M_u^n & 0 \\ 0 & -M_w^n \end{pmatrix} \leq A_n + \dfrac{1}{2n} A_n^2 \end{cases} \tag{3.111}$$

where $A_n = D^2_{xy}\phi_n(t_n, x_n, y_n)$. Next by taking into account that $(u^i)_{i\in\Gamma}$ and $(w^i)_{i\in\Gamma}$ are respectively subsolution and supersolution of (3.51) and the inequalities (3.109)-(3.110), we obtain

$$-p_u^n - b(t_n, x_n)^\top q_u^n - \frac{1}{2} Tr[(\sigma\sigma^\top(t_n, x_n))(t_n, x_n) M_u^n] - f^{i_0}(t_n, x_n, (u^l(t_n, x_n))_{l\in\Gamma}, \sigma(t_n, x_n)^\top q_u^n) \leq 0 \tag{3.112}$$

and

$$-p_w^n - b(t_n, y_n)^\top q_w^n - \frac{1}{2} Tr[(\sigma\sigma^\top(t_n, y_n))(t_n, y_n) M_w^n] - f^{i_0}(t_n, y_n, (w^l(t_n, y_n))_{l\in\Gamma}, \sigma(t_n, y_n)^\top q_w^n) \geq 0. \tag{3.113}$$

By taking the difference of (3.112) and (3.113), one deduces that

$$-(p_u^n - p_w^n) - (b(t_n, x_n)^\top q_u^n - b(t_n, y_n)^\top q_w^n) - \frac{1}{2}Tr[\{\sigma\sigma^\top(t_n, x_n)M_u^n - \sigma\sigma^\top(t_n, y_n)M_w^n\}]$$
$$-\{f^{i_0}(t_n, x_n, (u^l(t_n, x_n))_{l\in\Gamma}, \sigma(t_n, x_n)^\top q_u^n) - f^{i_0}(t_n, y_n, (w^l(t_n, y_n))_{l\in\Gamma}, \sigma(t_n, y_n)^\top q_w^n)\} \leq 0.$$

Combining with (3.111), there exists some appropriate ρ_n with $\limsup_n \rho_n \leq 0$ such that the last inequality yields the following one:

$$-\{f^{i_0}(t_n, x_n, (u^l(t_n, x_n))_{l\in\Gamma}, \sigma(t_n, x_n)^\top q_u^n) - f^{i_0}(t_n, x_n, (w^l(t_n, y_n))_{l\in\Gamma}, \sigma(t_n, x_n)^\top q_u^n)\} \leq \rho_n$$

Next by linearising f^{i_0} we obtain

$$\lambda(u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n)) - \sum_{k\in\Gamma^{-i_0}} \Theta_n^k(u^k(t_n, x_n) - w^k(t_n, y_n)) \leq \rho_n \qquad (3.114)$$

where Θ_n^k is the increment rate of f^{i_0} w.r.t. y^k, which is uniformly bounded w.r.t. n and is non negative by the monotonicity assumption of f^i. Therefore (3.114) becomes

$$\begin{aligned}\lambda(u^{i_0}(t_n, x_n) - w^{i_0}(t_n, y_n)) &\leq \sum_{k\in\Gamma^{-i_0}} \Theta_n^k(u^k(t_n, x_n) - w^k(t_n, y_n)) + \rho_n \\ &\leq C_{f^{i_0}} \sum_{k\in\Gamma^{-i_0}} (u^k(t_n, x_n) - w^k(t_n, y_n))^+ + \rho_n.\end{aligned}$$

Then by taking $n \to \infty$ the inequality yields

$$\begin{aligned}\lambda(u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*)) &\leq \limsup_n C_{f^{i_0}}[\sum_{k\in\Gamma^{-i_0}} (u^k(t_n, x_n) - w^k(t_n, y_n))^+] \\ &\leq C_{f^{i_0}}[\sum_{k\in\Gamma^{-i_0}} (\limsup_n (u^k(t_n, x_n) - w^k(t_n, y_n)))^+] \\ &\leq C_{f^{i_0}}[\sum_{k\in\Gamma^{-i_0}} (u^k(t^*, x^*) - w^k(t^*, x^*)]\end{aligned}$$

Next as $i_0 \in \hat{\Gamma}(t^*, x^*)$, we deduce that

$$\lambda(u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*)) \leq C_{f^{i_0}}(p-1)(u^{i_0}(t^*, x^*) - w^{i_0}(t^*, x^*)$$

which is contradictory with the definiton of λ given in (3.103). As a consequence for any $i \in \Gamma, u^i \leq w^i$.

Step 2: the general case

For any arbitrary $\lambda \in \mathbb{R}$, let us define

$$\hat{u}^i(t,x) := e^{\lambda t} u^i(t,x) \text{ and } \hat{w}^i(t,x) := e^{\lambda t} w^i(t,x).$$

Note that $(\hat{u}^i)_{i\in\Gamma}$ and $(\hat{w}^i)_{i\in\Gamma}$ is respectively the subsolution and the supersolution of the following system of PDEs: for any $i \in \Gamma$ and $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\min\{v^i(t,x) - v^{i+1}(t,x) + e^{\lambda t}\underline{g}_{i,i+1}(t,x); \max[v^i(t,x) - v^{i+1}(t,x) - e^{\lambda t}\overline{g}_{i,i+1}(t,x);$$

$$-\partial_t v^i(t,x) - \mathcal{L}^X v^i(t,x) + \lambda v^i(t,x) - e^{\lambda t} f^i(t,x,(e^{-\lambda t}v^l(t,x))_{l\in\Gamma}, e^{-\lambda t}\sigma^\top(t,x)D_x v^i(t,x))]\} = 0$$

and $v^i(T,x) = e^{\lambda T}h_i(x)$. For λ large enough, the condition (3.103) holds, then we go back to the result in Step 1 and we obtain, for any $i \in \Gamma$, $\hat{u}^i \le \hat{w}^i$, which also yields $u^i \le w^i$. The proof of comparison is now complete. □

3.6.2 Existence and uniqueness of viscosity solution of (3.51)

Let us recall $(\bar{v}^i)_{i\in\Gamma}$ and $(\bar{v}^{i,m})_{i\in\Gamma}$ the functions defined in Proposition 3.4.2. We firstly prove that $(\bar{v}^i)_{i\in\Gamma}$ is a subsolution of (3.51), then we show that for a fixed m_0, $(\bar{v}^{i,m_0})_{i\in\Gamma}$ is a supersolution of (3.51), finally by Perron's method we show that $(\bar{v}^i)_{i\in\Gamma}$ is the unique solution of (3.51).

Proposition 3.6.3. *The family $(\bar{v}^i)_{i\in\Gamma}$ is a viscosity subsolution of* (3.51).

Proof. We first recall that $\forall i \in \Gamma, \bar{v}^i := \lim_{m\to\infty} \bar{v}^{i,m}$, is usc function since the sequence $(\bar{v}^{i,m})_{m\ge 0}$ is decreasing and $(\bar{v}^{i,m})_{i\in\Gamma}$ is continuous. Then thanks to the definition we have $\bar{v}^* = \bar{v}^i$, hence when $t = T$ we have $\bar{v}^i(T,x) = \lim_{m\to\infty} \bar{v}^{i,m}(T,x) = h^i(x)$.
Next let us recall Definition 3.4.1, for any $(t,x) \in [0,T) \times \mathbb{R}^k$, $i \in \Gamma$, $(\underline{p}, \underline{q}, \underline{M}) \in \bar{J}^+\bar{v}^i(t,x)$, we shall prove either

$$\bar{v}^i(t,x) - L^i(\vec{\bar{v}})(t,x) \le 0 \tag{3.115}$$

or

$$\begin{aligned} &\max[\bar{v}^i(t,x) - U^i(\vec{\bar{v}})(t,x);\\ &\quad -\underline{p} - b^\top(t,x)\underline{q} - \frac{1}{2}Tr(\sigma\sigma^\top)(t,x)\underline{M}) - f^i(t,x,(\bar{v}^l(t,x))_{l\in\Gamma}, \sigma^\top(t,x).\underline{q})] \le 0. \end{aligned} \tag{3.116}$$

To proceed, we first assume that there exists $\epsilon_0 > 0$ such that

$$\bar{v}^i(t,x) \ge \bar{v}^{i+1}(t,x) - \underline{g}_{i,i+1}(t,x) + \epsilon_0$$

then we need to prove (3.116).
As for any $i \in \Gamma$, $(\bar{v}^{i,m})_{m\ge 0}$ decreasingly converges to $\bar{v}^i$, then there exists m_0 such that for any

$m \geq m_0$ we have

$$\bar{v}^{i,m}(t,x) \geq \bar{v}^{i+1,m}(t,x) - \underline{g}_{i,i+1}(t,x) + \frac{\epsilon_0}{2}$$

By the continuity of $(\bar{v}^{i,m})_{i\in\Gamma}$ and $\underline{g}_{i,i+1}$, we can find a neighbourhood O_m of (t,x) such that

$$\bar{v}^{i,m}(t',x') \geq \bar{v}^{i+1,m}(t',x') - \underline{g}_{i,i+1}(t',x') + \frac{\epsilon_0}{4}, \ \forall (t',x') \in O_m. \tag{3.117}$$

Next by Lemma 6.1 in [13] there exists a subsequence $(t_k, x_k)_{k\geq 0}$ such that

$$(t_k, x_k) \to_{k\to\infty} (t,x) \text{ and } \lim_{k\to\infty} \bar{v}^{i,k}(t_k,x_k) = \bar{v}^i(t,x).$$

In addition we can also find a sequence which we still denote by $(p_k, q_k, M_k) \in \bar{J}^+\bar{v}^{i,k}(t_k,x_k)$ such that

$$\lim_{k\to\infty}(p_k,q_k,M_k) = (\underline{p},\underline{q},\underline{M})$$

As the sequence (t_k,x_k) can be chosen in the neighbourhood O_k, by applying the fact that $(\bar{v}^{i,k})_{i\in\Gamma}$ is the unique viscosity solution of the following system: For any $i \in \Gamma$,

$$\begin{aligned}
&\min\{\bar{v}^{i,m}(t,x) - L^i((\bar{v}^{l,m})_{l\in\Gamma})(t,x);\\
&- \partial_t \bar{v}^{i,m}(t,x) - b^\top(t,x) D_x \bar{v}^{i,m}(t,x) - f^{i,m}(t,x,(\bar{v}^{l,m}(t,x))_{l\in\Gamma}, \sigma^\top(t,x) D_x \bar{v}^{i,m}(t,x))\} = 0\\
&\bar{v}^{i,m}(T,x) = h_i(x).
\end{aligned} \tag{3.118}$$

we obtain

$$-p_k - b^\top(t_k,x_k).q_k - \frac{1}{2}Tr(\sigma\sigma^\top(t_k,x_k)M_k) - f^{i,k}(t,x,(\bar{v}^{l,k}(t_k,x_k))_{l\in\Gamma}, \sigma^\top(t_k,x_k)q_k) \leq 0 \tag{3.119}$$

where $f^{i,k}(t,x,(v^l(t,x))_{l\in\Gamma},z) := f^i(t,x,(v^l(t,x))_{l\in\Gamma},z) - k(v^i(t,x) - U^i(\vec{v})(t,x))^+$.
Moreover as the sequence $(t_k,x_k,p_k,q_k,M_k)_k$ is bounded and $(\bar{v}^{i,m})_{i\in\Gamma}$ is uniformly of polynomial growth, then we deduce from (3.119) that

$$\epsilon_k := (\bar{v}^{i,k}(t_k,x_k) - \bar{v}^{i+1,k}(t_k,x_k) - \bar{g}_{i,i+1}(t_k,x_k))^+ \to_{k\to\infty} 0$$

However for any fixed (t,x) and k_0, $(\bar{v}^{i,k}(t,x))_{k\geq k_0}$ is decreasing, then for $k \geq k_0$,

$$\begin{aligned}
\bar{v}^{i,k}(t_k,x_k) &\leq \bar{v}^{i+1,k}(t_k,x_k) + \bar{g}_{i,i+1}(t_k,x_k) + \epsilon_k\\
&\leq \bar{v}^{i+1,k_0}(t_k,x_k) + \bar{g}_{i,i+1}(t_k,x_k) + \epsilon_k
\end{aligned}$$

As $\bar{v}^{i,k_0}$ is continuous, by taking $k \to \infty$ we obtain that

$$\lim_{k\to\infty} \bar{v}^{i,k}(t_k, x_k) = \bar{v}^i(t,x) \le \bar{v}^{i+1,k_0}(t,x) + \bar{g}_{i,i+1}(t,x).$$

We then take $k_0 \to \infty$ yielding

$$\bar{v}^i(t,x) \le \bar{v}^{i+1}(t,x) + \bar{g}_{i,i+1}(t,x). \tag{3.120}$$

In the second place we consider a subsequence (k_l) of (k) such that for any $a \in \Gamma$, $(\bar{v}^{a,k_l}(t_{k_l}, x_{k_l}))_l$ converges, then by taking $l \to \infty$ in (3.119) we obtain

$$\lim_{l\to\infty}\{-p_{k_l} - b(t_{k_l}, x_{k_l})q_{k_l} - \frac{1}{2}Tr(\sigma\sigma^\top(t_{k_l}, x_{k_l})M_{k_l}) - f^i(t_{k_l}, x_{k_l}, (\bar{v}^{a,k_l}(t_{k_l}, x_{k_l}))_{a\in\Gamma}, \sigma^\top(t_{k_l}, x_{k_l}).q_{k_l})\} \le 0.$$

Then we deduce that

$$\begin{aligned}
&-\underline{p} - b^\top(t,x)\underline{q} - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)\underline{M}) \\
&\le \lim_{l\to\infty} f^i(t_{k_l}, x_{k_l}, (\bar{v}^{a,k_l}(t_{k_l}, x_{k_l}))_{a\in\Gamma}, \sigma^\top(t_{k_l}, x_{k_l})q_{k_l}) \\
&= f^i(t, x, \lim_{l\to\infty}(\bar{v}^{a,k_l}(t_{k_l}, x_{k_l}))_{a\in\Gamma}, \sigma^\top(t,x)\underline{q}) \\
&\le f^i(t, x, (\bar{v}^a(t,x))_{a\in\Gamma}, \sigma^\top(t,x)\underline{q}).
\end{aligned} \tag{3.121}$$

The last inequality holds true by the monotonicity assumption (H5) of f^i and the fact that for any $a \in \Gamma$, $\bar{v}^a$ verifies

$$\bar{v}^a(t,x) = \bar{v}^{*,a}(t,x) = \limsup_{(t',x')\to(t,x), m\to\infty} \bar{v}^{a,m}(t',x'), \ (t,x) \in [0,T] \times \mathbb{R}^k$$

Thus for any $a \in \Gamma^{-i}$ we have

$$\bar{v}^a(t,x) \ge \lim_{l\in\infty} \bar{v}^{a,k_l}(t_{k_l}, x_{k_l})$$

and

$$\bar{v}^i(t,x) = \lim_{l\to\infty} \bar{v}^{i,k_l}(t_{k_l}, x_{k_l}).$$

Thus (3.121) becomes

$$-\underline{p} - b^\top(t,x)\underline{q} - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)\underline{M}) \le f^i(t, x, (\bar{v}^a(t,x))_{a\in\Gamma}, \sigma^\top(t,x).\underline{q}). \tag{3.122}$$

Hence under (3.120) and (3.122), (3.116) is satisfied, then $(\bar{v}^i)_{i\in\Gamma}$ is a viscosity subsolution of (3.51). □

Proposition 3.6.4. *Let us fix $m_0 \in \mathbb{N}$. Then the family $(\bar{v}^{i,m_0})_{i\in\Gamma}$ is a viscosity supersolution of* (3.51).

Proof. We first recall that the triple $(\bar{Y}^{i,m_0}, \bar{Z}^{i,m_0}, \bar{K}^{i,m_0,+})_{i\in\Gamma}$ is the unique solution of the system of RBSDEs associated with $(f^{i,m_0}, h^i, \underline{g}_{i,i+1})_{i\in\Gamma}$ where

$$f^{i,m_0}(s, X_s^{t,x}, \vec{y}, z) := f^i(s, X_s^{t,x}, \vec{y}, z) - m_0(y^i - y^{i+1} - \bar{g}_{i,i+1}(s, X_s^{t,x}))^+.$$

In addition there exist unique deterministic continuous functions with polynomial growth $(\bar{v}^{i,m_0})_{i\in\Gamma}$ such that for any $i \in \Gamma, s \in [t,T]$,

$$\bar{Y}_s^{i,m_0} = \bar{v}^{i,m_0}(s, X_s^{t,x}) \;\; ((t,x) \in [0,T] \times \mathbb{R}^k \text{ is fixed}).$$

Now let us define the following processes: $\forall i \in \Gamma,\ s \in [t,T]$,

$$\begin{aligned}
\tilde{U}_s^{i,m_0} &:= Y_s^{i,m} \vee (Y_s^{i+1,m_0} + \overline{g}_{i,i+1}(s, X_s^{t,x})) \\
\bar{K}_s^{i,m_0,-} &:= m_0 \int_0^s (Y_s^{i,m_0} - Y_s^{i+1,m_0} - \bar{g}_{i,i+1}(s, X_s^{t,x}))^+ ds.
\end{aligned}$$

Then $(\bar{Y}^{i,m_0}, \bar{Z}^{i,m_0}, \bar{K}^{i,m_0,+}, \bar{K}^{i,m_0,-})_{i\in\Gamma}$ solves the following doubly reflected BSDEs: for any $i \in \Gamma$, $s \in [t,T]$,

$$\begin{cases}
\bar{Y}_s^{i,m_0} = h^i(X_T^{t,x}) + \int_s^T f^i(r, X_r^{t,x}, (\bar{Y}_r^{l,m_0})_{l\in\Gamma}, \bar{Z}_r^{i,m_0})dr - \int_s^T \bar{Z}_r^{i,m_0} dB_r \\
\qquad\qquad + \bar{K}_T^{i,m_0,+} - \bar{K}_s^{i,m_0,+} - (\bar{K}_T^{i,m_0,-} - \bar{K}_s^{i,m_0,-}); \\
\\
L_s^{i,m_0} \le \bar{Y}_s^{i,m_0} \le \tilde{U}_s^{i,m_0} \\
\\
\int_t^T (\bar{Y}_s^{i,m_0} - L_s^{i,m_0}) d\bar{K}_s^{i,m_0,+} = 0 \text{ and } \int_t^T (\bar{Y}_s^{i,m_0} - \tilde{U}_s^{i,m_0}) d\bar{K}_s^{i,m_0,-} = 0.
\end{cases}$$

Accordingly by the results of [14] and [32], $\bar{Y}^{i,m_0}$ is also associated with a zero-sum Dynkin game as follow: For any $s \in [t,T]$,

$$\begin{aligned}
\bar{Y}_s^{i,m_0} = \operatorname*{ess\,sup}_{\sigma \ge s} \operatorname*{ess\,inf}_{\tau \ge s} \mathbb{E}[\textstyle\int_s^{\sigma\wedge\tau} & f^i(r, X_r^{t,x}, (\bar{Y}_r^{l,m_0})_{l\in\Gamma}, \bar{Z}_r^{i,m_0})dr \\
& + L_\sigma^{i,m_0} 1_{(\sigma<\tau)} + \tilde{U}_\tau^{i,m_0} 1_{(\tau\le\sigma<T)} + h^i(X_T^{t,x}) 1_{(\tau=\sigma=T)} | \mathcal{F}_s]
\end{aligned}$$

Next following Theorem 3.7 and Theorem 6.2 in [27], $\bar{v}^{i,m_0}$ is the unique solution in viscosity sense of the following PDE with obstacle:

$$\begin{cases} \min\{w(t,x) - L^i((\bar{v}^{l,m_0})_{l\in\Gamma})(t,x); \max[w(t,x) - \tilde{U}((\bar{v}^{l,m_0})_{l\in\Gamma})(t,x); \\ \quad -\partial_t w(t,x) - b^\top(t,x)D_x w(t,x) - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)D^2_{xx}w(t,x)] \\ \quad -f^i(t,x,(\bar{v}^{l,m_0})_{l\in\Gamma},\sigma^\top(t,x)D_x w(t,x))]\} = 0; \\ \\ w(T,x) = h^i(X^{t,x}_T) \end{cases}$$

where $\tilde{U}((\bar{v}^{l,m_0})_{l\in\Gamma})(t,x) := \bar{v}^{i,m_0}(t,x) \vee (\bar{v}^{i+1,m_0} + \bar{g}_{i,i+1})(t,x)$.
In other words, for any $(t,x) \in [0,T) \times \mathbb{R}^k$ and for any $(p,q,M) \in \bar{J}^-(\bar{v}^{i,m_0})(t,x)$, it still holds that

$$\bar{v}^{i,m_0}(t,x) \geq L^i((v^{l,m_0})_{l\in\Gamma})(t,x) \tag{3.123}$$

and

$$\begin{aligned} &\max[\bar{v}^{i,m_0}(t,x) - \tilde{U}^i((\bar{v}^{l,m_0})_{l\in\Gamma})(t,x); \\ &\quad - p - b^\top(t,x).q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)M) - f^i(t,x,(\bar{v}^{l,m_0})_{l\in\Gamma},\sigma^\top(t,x)q)] \geq 0. \end{aligned} \tag{3.124}$$

Next apply the inequality $a - a \vee b \leq a - b$, then (3.124) yields

$$\begin{aligned} &\max[\bar{v}^{i,m_0}(t,x) - (\bar{v}^{i+1,m_0} + \bar{g}_{i,i+1})(t,x); \\ &\quad - p - b^\top(t,x).q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)M) - f^i(t,x,(\bar{v}^{l,m_0})_{l\in\Gamma},\sigma^\top(t,x)q)] \geq 0 \end{aligned}$$

Hence, with (3.123), this implies that $(\bar{v}^{i,m_0})_{i\in\Gamma}$ is a viscosity supersolution of (3.51). □

We are now ready to use Perron's method to provide a solution for (3.51). So let us consider the following functions denoted by $({}^{m_0}v^i)_{i\in\Gamma}$ and defined as: Let

$$\mathcal{U}^{m_0} := \{\vec{u} := (u^i)_{i\in\Gamma}, \vec{u} \text{ is a subsolution of (3.51) and for any } i \in \Gamma, \bar{v}^i \leq u^i \leq \bar{v}^{i,m_0}\}$$

Note that $\mathcal{U}^{m_0}$ is not empty since $(\bar{v}^i)_{i\in\Gamma} \in \mathcal{U}^{m_0}$. Next for $i \in \Gamma$, $(t,x) \in [0,T] \times \mathbb{R}^k$ we set

$${}^{m_0}v^i(t,x) := \sup\{u^i(t,x),\ (u^i)_{i\in\Gamma} \in \mathcal{U}^{m_0}\}.$$

We then have:

Theorem 3.6.5. *Assume (H2),(H3) and (H5) hold true, the functions $({}^{m_0}v^i)_{i\in\Gamma}$ is the unique viscosity solution of* (3.51). *Moreover the solution does not depend on m_0. Finally for any $i \in \Gamma$, ${}^{m_0}v^i = \bar{v}^i$.*

Proof. It is obvious that for any $i \in \Gamma$, the function ${}^{m_0}v^i$ belongs to class Π_g since $(\bar{v}^i)_{i\in\Gamma}$ and $(\bar{v}^{i,m_0})_{i\in\Gamma}$ are functions of Π_g.
To proceed, we divide the main proof into three steps. On the other hand, to simplify the notation, we replace $({}^{m_0}v^i)_{i\in\Gamma}$ with $(v^i)_{i\in\Gamma}$ as there is no possible confusion.

Step 1: $(v^i)_{i\in\Gamma}$ is a viscosity subsolution of (3.51).
For any $i \in \Gamma, v^i \in \mathcal{U}^{m_0}$ and then it satisfies:

$$\bar{v}^i \leq v^i \leq \bar{v}^{i,m_0}.$$

The inequalities still valid for the upper semicontinuous envelops, i.e.,

$$\bar{v}^i \leq v^{i,*} \leq \bar{v}^{i,m_0}$$

since $\bar{v}^i$ is usc and $\bar{v}^{i,m_0}$ is continuous. Therefore we have

$$\bar{v}^i(T,x) = v^{i,*}(T,x) = \bar{v}^{i,m_0}(T,x) = h^i(x).$$

It means that $(v^{i,*})_{i\in\Gamma}$ verify the subsolution property of system (3.63) at time T.

Next let $(\tilde{v}^k)_{k\in\Gamma}$ be an arbitrary element of $\mathcal{U}^{m_0}$ and let $i \in \Gamma$ be fixed. Since $(\tilde{v}^k)_{k\in\Gamma}$ is a subsolution of (3.51), then for any $(t,x) \in [0,T) \times \mathbb{R}^k$ and $(p,q,M) \in \bar{J}^+\tilde{v}^{i,*}(t,x)$ we have

$$\begin{aligned}&\min\{\tilde{v}^{i,*}(t,x) - L^i((\tilde{v}^{l,*})_{l\in\Gamma})(t,x); \max[\tilde{v}^{i,*}(t,x) - U^i((\tilde{v}^{l,*})_{l\in\Gamma})(t,x);\\ &-p - b^\top(t,x)q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)M) - f^i(t,x,(\tilde{v}^{l,*}(t,x))_{l\in\Gamma}, \sigma^\top(t,x)q)]\} \leq 0.\end{aligned} \tag{3.125}$$

But for any $k \in \Gamma$, $\tilde{v}^k \leq v^k$, then $\tilde{v}^{k,*} \leq v^{k,*}$. On the other hand, we notice that the operators $(w^l)_{l\in\Gamma} \mapsto \tilde{v}^{i,*} - L^i((w^l)_{l\in\Gamma})$ and $(w^l)_{l\in\Gamma} \mapsto \tilde{v}^{i,*} - U^i((w^l)_{l\in\Gamma})$ are decreasing, then by the monotonicity of f^i ((H5)) and (3.125) we have

$$\begin{aligned}&\min\{(\tilde{v}^{i,*} - L^i((v^{l,*})_{\text{ł}\in\Gamma}))(t,x); \max[(\tilde{v}^{i,*} - U^i((v^{l,*})_{\text{ł}\in\Gamma}))(t,x);\\ &-p - b^\top(t,x)q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)M) - f^i(t,x,[(v^{l,*}(t,x))_{l\in\Gamma^{-i}}, \tilde{v}^{i,*}], \sigma^\top(t,x)q)]\} \leq 0.\end{aligned} \tag{3.126}$$

It means that $\tilde{v}^i$ is a subsolution of the following PDE:

$$\begin{cases}\min\{(w - L^i((v^{l,*})_{\text{ł}\in\Gamma}))(t,x); \max[(w - U^i((v^{l,*})_{\text{ł}\in\Gamma}))(t,x);\\ \quad -p - b^\top(t,x)q - \dfrac{1}{2}Tr(\sigma\sigma^\top(t,x)M) - f^i(t,x,[(v^{l,*}(t,x))_{l\in\Gamma^{-i}}, w], \sigma^\top(t,x)q)]\} = 0\\ w(T,x) = h^i(x)\end{cases} \tag{3.127}$$

In addition, the following function is lsc:

$$
\begin{aligned}
&(t,x,w,p,q,M) \in [0,T] \times \mathbb{R}^{k+1+1+k} \times \mathbb{S}^k \\
&\qquad \mapsto \min\{w - L^i((v^{l,*})_{l\in\Gamma})(t,x); \max[w - U^i((v^{l,*})_{l\in\Gamma})(t,x); \\
&\qquad\qquad - p - b^\top(t,x)q - f^i(t,x,[(v^{l,*}(t,x))_{l\in\Gamma^{-i}}, w], \sigma^\top(t,x).q)]\}.
\end{aligned}
$$

As v^i is the supremum of $\bar{v}^i$, thanks to Lemma 4.2 in [13], v^i is a viscosity subsolution of (3.127). But i is arbitrary, then $(v^i)_{i\in\Gamma}$ is a viscosity subsolution of system (3.51).

Step 2: *$(v^i)_{i\in\Gamma}$ is a viscosity supersolution of* (3.51).

We first focus on the terminal condition. For any $i \in \Gamma$, $v^i_*(T,x) = h^i(x)$ from the inequality $\underline{v}^i = \underline{v}^i_* \le \bar{v}^i_* \le v^i_* \le \bar{v}^{i,m_0}_* = \bar{v}^{i,m_0}$ since $\underline{v}^i$ is lsc and $\bar{v}^{i,m_0}$ is continuous.

Next by contradiction we assume that $(v^i)_{i\in\Gamma}$ is not a supersolution of (3.51), i.e. there exists at least one $i \in \Gamma$ and for some $(t_0,x_0) \in [0,T) \times \mathbb{R}^k$ and $(p,q,M) \in J^-(v^i_*)(t,x)$ we have

$$
\begin{aligned}
&\min\{v^i_*(t_0,x_0) - L^i((v^l_*)_{l\in\Gamma})(t_0,x_0); \max[v^i_*(t_0,x_0) - U^i((v^l_*)_{l\in\Gamma})(t_0,x_0); \\
&- p - b^\top(t_0,x_0)q - \frac{1}{2}Tr(\sigma\sigma^\top(t_0,x_0)M) - f^i(t_0,x_0,(v^l_*(t_0,x_0))_{l\in\Gamma}, \sigma^\top(t_0,x_0)q)]\} < 0.
\end{aligned} \tag{3.128}
$$

Next for any positive constants δ, γ and r let us define:

$$
\begin{aligned}
&u_{\delta,\gamma}(t,x) := v^i_*(t_0,x_0) + \delta + < q, x - x_0 > + p(t - t_0) + \frac{1}{2} < (M - 2\gamma)(x - x_0), (x - x_0) > \\
&\text{and } B_r := \{(t,x) \in [0,T] \times \mathbb{R}^k \text{ such that } |t - t_0| + |x - x_0| < r\}.
\end{aligned} \tag{3.129}
$$

By choosing δ and γ small enough, we deduce from (3.128) that

$$
\begin{aligned}
&\min\{v^i_*(t_0,x_0) - L^i((v^l_*)_{l\in\Gamma})(t_0,x_0) + \delta; \max[v^i_*(t_0,x_0) - U^i((v^l_*)_{l\in\Gamma})(t_0,x_0) + \delta; \\
&- p - b^\top(t_0,x_0)q - \frac{1}{2}Tr(\sigma\sigma^\top(t_0,x_0)(M - 2\gamma)) \\
&- f^i(t_0,x_0,[(v^l_*(t_0,x_0))_{l\in\Gamma^{-i}}, v^i_*(t_0,x_0) + \delta], \sigma^\top(t_0,x_0)q)]\} < 0
\end{aligned} \tag{3.130}
$$

Next let us define the following function:

$$
\begin{aligned}
\Theta(t,x) := &\min\{u_{\delta,\gamma}(t,x) - L^i((v^l_*)_{l\in\Gamma})(t,x); \max[u_{\delta,\gamma}(t,x) - U^i((v^l_*)_{l\in\Gamma})(t,x); \\
&- p - b^\top(t,x).q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x))(M - 2\gamma)
\end{aligned}
$$

$$- f^i(t,x,[(v^l_*(t,x))_{l\in\Gamma^{-i}}, u_{\delta,\gamma}(t,x)], \sigma^\top(t,x)q)]\}$$

According to (3.130) we have $\Theta(t_0,x_0) < 0$. On the other hand, Θ is usc since the functions v^i_*, $i \in \Gamma$, are lsc, $u_{\delta,\gamma}$ is continuous and f^i is continuous and verifies the monotonicity property. Therefore for any $\epsilon > 0$, there is some $\eta > 0$ such that for any $(t,x) \in B_\eta$ we have

$$\Theta(t,x) \le \Theta(t_0,x_0) + \epsilon$$

Next as $\Theta(t_0,x_0) < 0$, we can choose ϵ small enough to obtain $\Theta(t,x) \le 0$ for any $(t,x) \in B_\eta$. Thus for any $(t,x) \in B_\eta$, $u_{\delta,\gamma}$ is nothing but a viscosity subsolution of the following PDE(on B_η):

$$\begin{aligned}\min\{&w(t,x) - L^i((v^l_*)_{l\in\Gamma})(t,x); \max[w(t,x) - U^i((v^l_*)_{l\in\Gamma})(t,x);\\ &- \partial_t w(t,x) - b^\top(t,x)D_x w(t,x) - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)D^2_{xx}w(t,x))\\ &- f^i(t,x,[(v^l_*(t,x))_{l\in\Gamma^{-i}}, w(t,x)], \sigma^\top(t,x)D_x w(t,x))]\} = 0.\end{aligned} \tag{3.131}$$

As for any $i \in \Gamma, v^i_* \le v^{i,*}$, then $u_{\delta,\gamma}$ is also a viscosity subsolution of (3.131) by replacing $(v^i_*)_{i\in\Gamma}$ with $(v^{i,*})_{i\in\Gamma}$, i.e.

$$\begin{aligned}\min\{&w(t,x) - L^i((v^{l,*})_{l\in\Gamma})(t,x); \max[w(t,x) - U^i((v^{l,*})_{l\in\Gamma})(t,x);\\ &- \partial_t w(t,x) - b^\top(t,x)D_x w(t,x) - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)D^2_{xx}w(t,x))\\ &- f^i(t,x,[(v^{l,*}(t,x))_{l\in\Gamma^{-i}}, w(t,x)], \sigma^\top(t,x)D_x w(t,x))]\} = 0.\end{aligned}$$

On the other hand since $(p,q,M) \in J^-(v^i_*(t_0,x_0))$, by the definition of the subjet ([13]) we have: $\forall i \in \Gamma$,

$$\begin{aligned}v^i(t,x) &\ge v^i_*(t,x)\\ &\ge v^i_*(t_0,x_0) + p(t-t_0) + < q, x-x_0 > + \frac{1}{2} < M(x-x_0),(x-x_0) >\\ &\quad + o(|t-t_0|) + o(|x-x_0|^2).\end{aligned}$$

Next let us set $\delta = \frac{r^2}{8}\gamma$ and let us go back to the definition of $u_{\delta,\gamma}$ yielding

$$\begin{aligned}v^i(t,x) > u_{\delta,\gamma}(t,x) = v^i_*(t_0,x_0) &+ \frac{r^2}{8}\gamma + < q, x-x_0 > + p(t-t_0) + \frac{1}{2} < M(x-x_0),(x-x_0) >\\ &- < \gamma(x-x_0),(x-x_0) >\end{aligned}$$

when $\frac{r}{\sqrt{2}} < |x-x_0| \le r$ and r small enough.

Next let us take $r \leq \eta$ and let us define the function $\tilde{u}^i$ by:

$$\tilde{u}^i(t,x) = \begin{cases} \max(v^i(t,x), u_{\delta,\gamma}(t,x)), & \text{if } (t,x) \in B_r; \\ v^i(t,x) \text{ otherwise.} \end{cases}$$

Then according to (3.131) and Lemma 1.2 in [13], $\tilde{u}^i$ is also a subsolution of the following PDE:

$$\begin{cases} \min\{w(t,x) - L^i((v^{l,*})_{l\in\Gamma})(t,x); \max[w(t,x) - U^i((v^{l,*})_{l\in\Gamma})(t,x); \\ \\ \quad -p - b^\top(t,x)q - \frac{1}{2}Tr(\sigma\sigma^\top(t,x)M) \\ \\ \quad -f^i(t,x,[(v^{l,*}(t,x))_{l\in\Gamma^{-i}}, w(t,x)], \sigma^\top(t,x)q)]\} = 0 \\ \\ w(T,x) = h^i(x). \end{cases}$$

Once more by the monotonicity of f^i and the fact that $\tilde{u}^i \geq v^i$, $[(v^l)_{l\in\Gamma^{-i}}, \tilde{u}^i]$ is also a subsolution of (3.51) which belongs to Π_g. Then by comparison we obtain that $[(v^l)_{l\in\Gamma^{-i}}, \tilde{u}^i]$ belongs to $\mathcal{U}^{m_0}$. Next by the definition of v^i_*, we can find a sequence $(t_n, x_n, v^i(t_n, x_n))_{n\geq 1}$ which converges to $(t_0, x_0, v^i_*(t_0, x_0))$, then we have

$$\begin{aligned} \lim_{n\to\infty}(\tilde{u}^i - v^i)(t_n, x_n) &= \lim_{n\to\infty}(u_{\delta,\gamma} - v^i_*)(t_n, x_n) \\ &= v^i_*(t_0, x_0) + \delta - v^i_*(t_0, x_0) > 0 \end{aligned}$$

This result implies that we can find some points (t_n, x_n) such that $\tilde{u}^i(t_n, x_n) > v^i(t_n, x_n)$, which is contradictory against the fact that $[(v^l)_{l\in\Gamma^{-i}}, \tilde{u}^i]$ belongs to $\mathcal{U}^{m_0}$ and $(v^i)_{i\in\Gamma}$ is the supremum element in the latter set. Hence $(v^i)_{i\in\Gamma}$ is a supersolution of (3.51).

Step 3: Continuity and uniqueness of $(v^i)_{i\in\Gamma}$.

Following the definition of usc envelop $(v^{i,*})_{i\in\Gamma}$ (resp. lsc envelop $(v^i_*)_{i\in\Gamma}$) and Remark 4.2.2 in [58], $(v^{i,*})_{i\in\Gamma}$ (resp.$(v^i_*)_{i\in\Gamma}$) is a usc subsolution (resp. lsc supersolution) of (3.51), then by Proposition 3.6.2 we obtain $\forall i \in \Gamma$,

$$v^{i,*} \leq v^i_*$$

Meanwhile it holds true that $v^i_* \leq v^i \leq v^{i,*}$ then $v^i_* = v^{i,*}$, which implies the continuity of v^i.

Next we assume that there exists another solution $(\hat{v}^i)_{i\in\Gamma}$ of (3.51) which belongs to class Π_g. As $(v^i)_{i\in\Gamma}$ and $(\hat{v}^i)_{i\in\Gamma}$ are both subsolutions and supersolutions, by the comparison result we obtain both $v^i \leq \hat{v}^i$ and $v^i \geq \hat{v}^i$ with al $i \in \Gamma$, as a result the solution is unique. The uniqueness of solution leads us directly to the fact that the solution $(v^i)_{i\in\Gamma}$ does not depend on m_0. Finally for any $i \in \Gamma$ and m_0 we have

$$\bar{v}^i \leq v^i \leq v^{i,m_0}.$$

Just send m_0 to $+\infty$ to obtain that for any $i \in \Gamma$, $\bar{v}^i = v^i$. $\square$

CHAPTER 4

PAPER 3: MEAN-FIELD DOUBLY REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

This chapter is a preprint joint work with Chen and Hamadène (ref.[12]).

4.1 Introduction

In this paper we are concerned with the problem of existence and uniqueness of a solution of the doubly reflected BSDE of the following type:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0,T]; \\ \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0 \ (K^\pm \text{ are increasing processes }). \end{cases} \tag{4.1}$$

It is said associated with the quadruple (f, ξ, h, g). Those BSDEs are of mean-field type because the generator f and the barriers h and g depend on the law of Y_t through its expectation. For simplicity reasons we stick to this framework, however it can be generalized (see Remark 4.3.5).

Since the introduction by Lasry and Lions [45] of the general mathematical modeling approach for high-dimensional systems of evolution equations corresponding to a large number of "agents" (the mean-field model), the interest to the mean-field models grows steadily in connection with several applications. Later standard mean-field BSDEs have been introduced in [6]. Since then, there have been several papers on mean-field BSDEs including ([7, 8, 4, 17, 50, 9, 59, 51], etc) in relation with several fields and motivations in mathematics and economics, such stochastic control, games, mathematical finance, utility of an agent inside an economy, PDEs, actuaries, etc.

Mean-field one barrier reflected BSDEs have been considered first in the paper [50]. This latter generalizes the work in [7] to the reflected framework. Later Briand et al. [4] have considered another type of one barrier mean-field reflected BSDEs. Actually in [4], the reflection of the component Y of the solution holds in expectation. They show existence and uniqueness of the solution when the increasing process, which makes the constraint on Y satisfied, is deterministic. Otherwise the solution is not necessarily unique. The main motivation is the assessment of the risk of a position in a financial market.

In [17], Djehiche et al. consider the above problem (4.1) when there is only one reflecting barrier (e.g. take $g \equiv +\infty$). The authors show existence and uniqueness of the solution in several contexts of integrability of the data (f, ξ, h). The methods are the usual ones: Fixed point and penalization. Those methods do not allow for the same framework. For example, the fixed point method does not allow generators which depend on z while the penalization does at the price of some additional regularity properties which are not required by the use of the first method. The main motivation for considering such a problem comes from the assessment of the prospective reserve of a representative contract in life-insurance.

In this paper we consider the extension of the framework of [17] to the case of two reflecting barriers. We show existence and uniqueness of a solution of (4.1), by the fixed point method. We deal with the case when the data of the problem are only integrable or p-integrable with $p > 1$. Those cases are treated separalety because one cannot deduce one of them from the other one.

The paper is organized as follows: In Section 2, we fix the notations and the frameworks. In Section 3, we deal with the case when $p > 1$ and finally with the case $p = 1$.

4.2 Notations and formulation of the problems

4.2.1 Notations

Let T be a fixed positive constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with $B = (B_t)_{t\in[0,T]}$ a d-dimensional Brownian motion whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions. On the other hand, let $\mathcal{P}$ be the σ-algebra on $[0, T] \times \Omega$ of the $\mathbb{F}$-progressively measurable sets.

For $p \geq 1$ and $0 \leq s_0 < t_0 \leq T$, we define the following spaces:

- $L^p := \{\xi : \mathcal{F}_T -$ measurable radom variable s.t. $\mathbb{E}[|\xi|^p] < \infty\}$;
- $\mathcal{H}^m_{loc} := \{(z_t)_{t\in[0,T]} : \mathcal{P} -$ measurable process and $\mathbb{R}^m -$ valued s.t. $\mathbb{P} - a.s.\ \int_0^T |z_s(\omega)|^2 ds < \infty\}$; $\bar{z} \in \mathcal{H}^m_{loc}([s_0, t_0])$ if $\bar{z}_r = z_r 1_{[s_0,t_0]}(r)$, $dr \otimes d\mathbb{P}$-a.e. with $z \in \mathcal{H}^m_{loc}$.

- $\mathcal{S}^p := \{(y_t)_{t\in[0,T]}$: continuous and $\mathcal{P}$-measurable process s.t. $\mathbb{E}[\sup_{t\in[0,T]}|y_t|^p] < \infty\}$; $\mathcal{S}^p([s_0,t_0])$ is the space $\mathcal{S}^p$ reduced to the interval $[s_0,t_0]$. If $y \in \mathcal{S}^p([s_0,t_0])$, we denote by $\|y\|_{\mathcal{S}^p_c([s_0,t_0])} := \{\mathbb{E}[\sup_{s_0\le u\le t_0}|y_u|^p]\}^{1/p}$.

- $\mathcal{A} := \{(k_t)_{t\in[0,T]}$: continuous, $\mathcal{P}$-measurable and non-decreasing process s.t. $k_0 = 0\}$; $\mathcal{A}([s_0,t_0])$ is the space $\mathcal{A}$ reduced to the interval $[s_0,t_0]$ (with $k_{s_0} = 0$);

- $\mathcal{T}_t := \{\tau, \mathbb{F}-\text{stopping time s.t. } \mathbb{P}-a.s. \tau \ge t\}$;

- $\mathcal{D} := \{(\phi)_{t\in[0,T]} : \mathbb{F}-\text{adapted},\ \mathbb{R}-\text{valued continuous process s.t. } \|\phi\|_1 = \sup_{\tau\in\mathcal{T}_0}\mathbb{E}[|y_\tau|] < \infty\}$. Note that the normed space $(\mathcal{D}, \|.\|_1)$ is complete (e.g. [15], pp.90). We denote by $(\mathcal{D}([s_0,t_0]), \|.\|_1)$, the restriction of $\mathcal{D}$ to the time interval $[s_0,t_0]$. It is a complete metric space when endowed with the norm $\|.\|_1$ on $[s_0,t_0]$, i.e.,

$$\|X\|_{1,[s,t]} := \sup_{\tau\in\mathcal{T}_0,\ s_0\le\tau\le t_0}\mathbb{E}[|X_\tau|] < \infty.$$

4.2.2 The class of doubly reflected BSDEs

In this paper we aim at finding $\mathcal{P}$-measurable processes (Y, Z, K^+, K^-) solution of the doubly reflected BSDE of mean-field type associated with the generator $f(t,\omega,y,y')$, the terminal condition ξ, the lower barrier $h(y,y')$, and the upper barrier $g(y,y')$, in the cases $p > 1$ and $p = 1$ respectively. The two cases should be considered separately since one cannot deduce one case from another one. So to begin with let us make precise the definitions:

Definition 4.2.1. *We say that the quaternary of $\mathcal{P}$-measurable processes $(Y_t, Z_t, K_t^+, K_t^-)_{t\le T}$ is a solution of the mean-field reflected BSDE associated with (f,ξ,h,g) if :*

Case: $p > 1$

$$\begin{cases} Y \in \mathcal{S}^p_c, \quad Z \in \mathcal{H}^d_{loc} \quad \textit{and} \quad K^+, K^- \in \mathcal{A}; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0,T]; \\ \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{4.2}$$

Case: $p = 1$,

$$\begin{cases} Y \in \mathcal{D}, \quad Z \in \mathcal{H}^d_{loc} \quad and \quad K^+, K^- \in \mathcal{A}; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\ \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{4.3}$$

4.2.3 Assumptions on (f, ξ, h, g)

We now make precise the requirements on the items (f, ξ, h, g) which define the two reflecting barriers backward stochastic differential equation of mean-field type.

Assumption (A1):

(i) The coefficients f, h, g and ξ satisfy:

(a) the process $(f(t, 0, 0))_{t \le T}$ is $\mathcal{P}$- measurable and such that $\int_0^T |f(t, 0, 0)|dt \in L^p(d\mathbb{P})$;

(b) f is Lipschitz w.r.t (y, y') uniformly in(t, ω), i.e., there exists a positive constant C_f such that $\mathbb{P}$- a.s. for all $t \in [0, T]$, y_1, y_1', y_2 and y_2' elements of $\mathbb{R}$,

$$|f(t, \omega, y_1, y_1') - f(t, \omega, y_2, y_2')| \le C_f(|y_1 - y_1'| + |y_2 - y_2'|). \tag{4.4}$$

(ii) h and g are mappings from $\mathbb{R}^2$ into $\mathbb{R}$ which satisfy:

(a) h and g are Lipschitz w.r.t. (y, y') i.e., there exist pairs of positive constants (γ_1, γ_2), (β_1, β_2) such that for any x, x', y and $y' \in \mathbb{R}$,

$$\begin{aligned} |h(x, x') - h(y, y')| &\le \gamma_1|x - y| + \gamma_2|x' - y'|, \\ |g(x, x') - g(y, y')| &\le \beta_1|x - y| + \beta_2|x' - y'|. \end{aligned} \tag{4.5}$$

(b) $h(x, x') < g(x, x')$, for any $x, x' \in \mathbb{R}$;

(iii) ξ is an $\mathcal{F}_T$- measurable, $\mathbb{R}$-valued r.v., $\mathbb{E}[\xi^p] < \infty$ and satisfies $h(\xi, \mathbb{E}[\xi]) \le \xi \le g(\xi, \mathbb{E}[\xi])$.

4.3 Existence and Uniqueness of a Solution of the Doubly Reflected BSDE of Mean-Field type

Let $Y = (Y_t)_{t\le T}$ be an $\mathbb{R}$-valued, $\mathcal{P}$-measurable process and Φ the mapping that associates to Y the following process $(\Phi(Y)_t)_{t\le T}$: $\forall t \le T$,

$$\Phi(Y)_t := \operatorname*{ess\,sup}_{\tau\ge t} \operatorname*{ess\,inf}_{\sigma\ge t} \mathbb{E}\{\int_t^{\sigma\wedge\tau} f(s, Y_s, \mathbb{E}[Y_s])ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} \\ + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\le\sigma,\tau<T\}} + \xi\mathbb{1}_{\{\tau=\sigma=T\}}|\mathcal{F}_t\}.$$

For the well-posedness of $\Phi(Y)$ one can see e.g. [49], Theorem 7.

The following result is related to some properties of $\Phi(Y)$.

Lemma 4.3.1. *Assume that assumptions (A1) are satisfied for $p = 1$ and $Y \in \mathcal{D}$. Then the process $\Phi(Y)$ belongs to $\mathcal{D}$. Moreover there exist processes $(\underline{Z}_t)_{t\le T}$ and $(\underline{A}_t^{\pm})_{t\le T}$ such that:*

$$\begin{cases} \underline{Z} \in \mathcal{H}^m_{loc};\ \underline{A}^{\pm} \in \mathcal{A}; \\ \Phi(Y)_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + \underline{A}_T^+ - \underline{A}_t^+ - \underline{A}_T^- + \underline{A}_t^- - \int_t^T \underline{Z}_s dB_s,\ t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le \Phi(Y)_t \le g(Y_t, \mathbb{E}[Y_t]),\ t \le T; \\ \int_0^T (\Phi(Y)_t - h(Y_t, \mathbb{E}[Y_t])d\underline{A}_t^+ = \int_0^T (\Phi(Y)_t - g(Y_t, \mathbb{E}[Y_t])d\underline{A}_t^- = 0. \end{cases} \tag{4.6}$$

Proof. First note that since $Y \in \mathcal{D}$ and g, h are Lipschitz then the processes $(h(Y_t, \mathbb{E}[Y_t]))_{t\le T}$ and $(g(Y_t, \mathbb{E}[Y_t]))_{t\le T}$ belong also to $\mathcal{D}$. Next as $h < g$ then, using a result by [38], Theorem 4.1 or [65], Theorem 3.1, there exist $\mathcal{P}$-measurable processes $(\underline{Y}_t)_{t\le T}$, $(\underline{Z}_t)_{t\le T}$ and $(\underline{A}_t^{\pm})_{t\le T}$ such that:

$$\begin{cases} \underline{Y} \in \mathcal{D};\ \underline{Z} \in \mathcal{H}^m_{loc}; \underline{A}^{\pm} \in \mathcal{A}; \\ \underline{Y}_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + \underline{A}_T^+ - \underline{A}_t^+ - \underline{A}_T^- + \underline{A}_t^- - \int_t^T \underline{Z}_s dB_s,\ t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le \underline{Y}_t \le g(Y_t, \mathbb{E}[Y_t]),\ t \le T; \\ \int_0^T (\underline{Y}_t - h(Y_t, \mathbb{E}[Y_t])d\underline{A}_t^+ = \int_0^T (\underline{Y}_t - g(Y_t, \mathbb{E}[Y_t])d\underline{A}_t^- = 0. \end{cases}$$

Let us point out that in [65], Theorem 3.1, the result is obtained in the discontinuous framework, namely the obstacles are right continuous with left limits processes. However since in our situation the processes $(h(Y_t, \mathbb{E}[Y_t]))_{t\le T}$ and $(g(Y_t, \mathbb{E}[Y_t]))_{t\le T}$ are continuous then $\underline{Y}$ and $\underline{A}^{\pm}$ are continuous, and the frameworks of [38] and [65] are the same (one can see e.g. [46], pp.60). Finally the process $\underline{Y}$ has the following representation as the value of a zero-sum Dynkin game: $\forall t \leq T$,

$$\begin{aligned}\underline{Y}_t := \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{\sigma \geq t} \mathbb{E}\{\int_t^{\sigma\wedge\tau} f(s, Y_s, \mathbb{E}[Y_s])ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} \\ + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\leq\sigma,\tau<T\}} + \xi\mathbb{1}_{\{\tau=\sigma=T\}}|\mathcal{F}_t\}.\end{aligned} \tag{4.7}$$

Therefore $\underline{Y} = \Phi(Y)$ and the claim is proved. □

4.3.1 The case $p > 1$

We will first show that Φ is well defined from $\mathcal{S}^p$ to $\mathcal{S}^p$.

Lemma 4.3.2. *Let f, h, g and ξ satisfy Assumption (A1) for some $p > 1$. If $Y \in \mathcal{S}^p$ then $\Phi(Y) \in \mathcal{S}^p$.*

Proof. Let $Y \in \mathcal{S}^p$. For σ and τ two stopping times, let us define:

$$\mathcal{L}(\tau,\sigma) = \int_0^{\tau\wedge\sigma} f(r, Y_r, \mathbb{E}[Y_r])dr + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\leq\sigma,\tau<T\}} + \xi\mathbb{1}_{\{\tau=\sigma=T\}}.$$

Then for any $t \leq T$,

$$\Phi(Y)_t + \int_0^t f(s, Y_s, \mathbb{E}[Y_s])ds = \operatorname*{ess\,sup}_{\tau\geq t} \operatorname*{ess\,inf}_{\sigma\geq t} \mathbb{E}[\mathcal{L}(\tau,\sigma)|\mathcal{F}_t] = \operatorname*{ess\,inf}_{\sigma\geq t} \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}[\mathcal{L}(\tau,\sigma)|\mathcal{F}_t]. \tag{4.8}$$

As pointed out previously when Y belongs to $\mathcal{S}^p$ with $p > 1$, then it belongs to $\mathcal{D}$. Therefore, under assumptions (A1), the process $\Phi(Y)$ is continuous. On the other hand, the second equality in (4.8) is valid since by (A1)-(ii), (a)-(b), $h < g$ and the processes $(h(Y_s, \mathbb{E}[Y_s]))_{s\leq T}$ and $(g(Y_s, \mathbb{E}[Y_s]))_{s\leq T}$ belongs to $\mathcal{S}^p$ since Y belongs to $\mathcal{S}^p$ (see e.g. [22] for more details).

Next let us define the martingale $M := (M_t)_{0\leq t\leq T}$ as follows:

$$\begin{aligned}M_t := \mathbb{E}\Bigg\{\int_0^T [|f(s,0,0)| + C_f(|Y_s| + \mathbb{E}|Y_s|)]\, ds + |g(0,0)| + \beta_1 \sup_{s\leq T}|Y_s| + \beta_2 \sup_{s\leq T}\mathbb{E}|Y_s| \\ +|h(0,0)| + \gamma_1 \sup_{s\leq T}|Y_s| + \gamma_2 \sup_{s\leq T}\mathbb{E}|Y_s| + |\xi| \ \ |\mathcal{F}_t\Bigg\}.\end{aligned} \tag{4.9}$$

As Y belongs to $\mathcal{S}^p$ and by (A1)-(1)(a), the term inside the conditional expectation belongs to $L^p(d\mathbb{P})$. As the filtration $\mathbb{F}$ is Brownian then M is continuous and by Doob's inequality with $p > 1$ one deduces that M belongs also to $\mathcal{S}^p$. Next as f, g and h are Lipschitz, then by a linearization proccedure of those functions one deduces that:

$$|\mathbb{E}[\mathcal{L}(\tau,\sigma)|\mathcal{F}_t]| \leq M_t$$

for any $t \leq T$ and any stopping times $\sigma, \tau \in \mathcal{T}_t$. Then we obtain

$$\forall t \leq T, \ \left| \Phi(Y)_t + \int_0^t f(s, Y_s, \mathbb{E}[Y_s]) ds \right| \leq M_t.$$

Therefore,

$$\mathbb{E}\{\sup_{t \leq T} |\Phi(Y)_t|^p\} \leq C_p \left\{ \mathbb{E}\left(\int_0^T |f(s, Y_s, \mathbb{E}[Y_s])| ds \right)^p + \mathbb{E}[\sup_{t \leq T} |M_t|^p] \right\}$$

where C_p is a positive constant that only depends on p and T. It holds that $\Phi(Y) \in \mathcal{S}_c^p$ since $Y \in \mathcal{S}_c^p$ and f is Lipschitz. $\square$

Next we have the following result.

Proposition 4.3.3. *Let Assumption (A1) holds for some $p > 1$. If $\gamma_1, \gamma_2, \beta_1$ and β_2 satify*

$$(\gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{\frac{p-1}{p}} \left[\left(\frac{p}{p-1} \right)^p (\gamma_1 + \beta_1) + (\gamma_2 + \beta_2) \right]^{\frac{1}{p}} < 1 \tag{4.10}$$

then there exists $\delta > 0$ depending only on $p, C_f, \gamma_1, \gamma_2, \beta_1$ and β_2 such that Φ is a contraction on the time interval $[T - \delta, T]$.

Proof. Let $Y, Y' \in \mathcal{S}_c^p$. Then, for any $t \leq T$, we have,

$$\begin{aligned}
&|\Phi(Y)_t - \Phi(Y')_t| \\
&= |\operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{\sigma \geq t} \{\mathbb{E}\left[\int_t^{\sigma \wedge \tau} f(s, Y_s, \mathbb{E}[Y_s]) ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma}) \mathbb{1}_{\{\sigma < \tau\}} \right. \\
&\qquad \left. + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) \mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} + \xi \mathbb{1}_{\{\tau = \sigma = T\}} | \mathcal{F}_t \right] \} - \operatorname*{ess\,sup}_{\tau \leq t} \operatorname*{ess\,inf}_{\sigma \leq t} \{\mathbb{E}\left[\int_t^{\sigma \wedge \tau} f(s, Y'_s, \mathbb{E}[Y'_s]) ds \right. \\
&\qquad \left. + g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma}) \mathbb{1}_{\{\sigma < \tau\}} + h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau}) \mathbb{1}_{\{\tau \leq \sigma < T\}} + \xi \mathbb{1}_{\{\tau = \sigma = T\}} | \mathcal{F}_t \right] \}| \\
&\leq \operatorname*{ess\,sup}_{\tau \geq t, \sigma \geq t} \mathbb{E} \left\{ \int_t^{\sigma \wedge \tau} |f(s, Y_s, \mathbb{E}[Y_s]) - f(s, Y'_s, \mathbb{E}[Y'_s])| ds + |g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma}) \right. \\
&\qquad \left. - g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})| \mathbb{1}_{\{\sigma < \tau\}} + |h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) - h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})| \mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} | \mathcal{F}_t \right\} \\
&\leq \mathbb{E} \left\{ \int_t^T |f(s, Y_s, \mathbb{E}[Y_s]) - f(s, Y'_s, \mathbb{E}[Y'_s])| ds + (\beta_1 + \gamma_1) \sup_{t \leq s \leq T} |Y_s - Y'_s| | \mathcal{F}_t \right\} \\
&\qquad + (\beta_2 + \gamma_2) \sup_{t \leq s \leq T} \mathbb{E}[|Y_s - Y'_s|].
\end{aligned} \tag{4.11}$$

Fix now $\delta > 0$ and let $t \in [T-\delta, T]$. By the Lipschitz condition of f, (4.11) implies that

$$\begin{aligned}
&|\Phi(Y)_t - \Phi(Y')_t| \\
&\leq \mathbb{E}\left[\delta C_f\{\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s| + \sup_{T-\delta\leq s\leq T}\mathbb{E}[|Y_s - Y'_s|]\} + (\beta_1+\gamma_1)\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|\right. \\
&\left.+(\beta_2+\gamma_2)\sup_{T-\delta\leq s\leq T}\mathbb{E}[|Y_s - Y'_s|]\big|\mathcal{F}_t\right] \\
&= (\delta C_f + \gamma_1 + \beta_1)\mathbb{E}\left[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|\big|\mathcal{F}_t\right] + (\delta C_f + \gamma_2 + \beta_2)\sup_{T-\delta\leq s\leq T}\mathbb{E}\{|Y_s - Y'_s|\}.
\end{aligned} \tag{4.12}$$

As $p > 1$, thanks to the convexity inequality $(ax_1 + bx_2)^p \leq (a+b)^{p-1}(ax_1^p + bx_2^p)$ holding for any non-negative real constants a, b, x_1 and x_2, (4.12) yields

$$\begin{aligned}
|\Phi(Y)_t - \Phi(Y')_t|^p &\leq (2\delta C_f + \gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p-1}\left\{(\delta C_f + \gamma_1 + \beta_1)\right. \\
&\left.\left(\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s||\mathcal{F}_t]\right)^p + (\delta C_f + \gamma_2 + \beta_2)\left(\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|]\right)^p\right\}.
\end{aligned} \tag{4.13}$$

Next, by taking expectation of the supremum over $t \in [T-\delta, T]$ on the both hand-sides of (4.13), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T-\delta\leq s\leq T}|\Phi(Y)_s - \Phi(Y')_s|^p\right] \\
&\leq (2\delta C_f + \gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p-1}\left\{(\delta C_f + \gamma_1 + \beta_1)\,\mathbb{E}[\left(\sup_{T-\delta\leq t\leq T}\left\{\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s||\mathcal{F}_t]\right\}\right)^p]\right. \\
&\qquad\left.+(\delta C_f + \gamma_2 + \beta_2)\left\{\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|]\right\}^p\right\}.
\end{aligned} \tag{4.14}$$

By applying Doob's inequality we have:

$$\mathbb{E}[\left(\sup_{T-\delta\leq t\leq T}\left\{\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s||\mathcal{F}_t]\right\}\right)^p] \leq (\frac{p}{p-1})^p\mathbb{E}\left[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|^p\right] \tag{4.15}$$

and by Jensen's one we have also

$$\left\{\mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|]\right\}^p \leq \mathbb{E}[\sup_{T-\delta\leq s\leq T}|Y_s - Y'_s|^p]. \tag{4.16}$$

Plug now (4.15) and (4.16) in (4.14) to obtain:

$$\|\Phi(Y)-\Phi(Y')\|_{\mathcal{S}^p([T-\delta,T])} \leq \Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(\delta)\|Y-Y'\|_{\mathcal{S}^p([T-\delta,T])}$$

where

$$\Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(\delta) = (2\delta C_f+\gamma_1+\gamma_2+\beta_1+\beta_2)^{\frac{p-1}{p}}\left[\left(\frac{p}{p-1}\right)^p(\delta C_f+\gamma_1+\beta_1)\right.$$
$$\left.+(\delta C_f+\gamma_2+\beta_2)\right]^{\frac{1}{p}}.$$

Note that (4.10) is just $\Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(0)<1$. As $\lim_{\delta\to 0}\Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(\delta) = \Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(0) < 1$. Then there exists δ small enough which depends only on $C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2$ and not on ξ nor T such that $\Lambda(C_f,p,\gamma_1,\gamma_2,\beta_1,\beta_2)(\delta)<1$. It implies that Φ is a contraction on $\mathcal{S}_c^p([T-\delta,T])$. Then there exists a process which belongs to $\mathcal{S}_c^p([T-\delta,T])$ such that

$$Y_t=\Phi(Y)_t,\ \forall t\in[T-\delta,T].$$

□

We now show that the mean-field reflected BSDE (4.2) has a unique solution.

Theorem 4.3.4. *Assume that Assumption (A1) holds for some $p>1$. If γ_1 and γ_2 satisfy*

$$(\gamma_1+\gamma_2+\beta_1+\beta_2)^{\frac{p-1}{p}}\left[\left(\frac{p}{p-1}\right)^p(\gamma_1+\beta_1)+(\gamma_2+\beta_2)\right]^{\frac{1}{p}}<1 \tag{4.17}$$

then the mean-field doubly reflected BSDE (4.2) *has a unique solution* (Y,Z,K^+,K^-).

Proof. Let δ be as in Proposition 4.3.3. Then there exists a process $Y\in\mathcal{S}^p([T-\delta,T])$, which is the fixed point of Φ in this latter space and verifies: For any $t\in[T-\delta,T]$,

$$Y_t=\operatorname*{ess\,sup}_{\tau\geq t}\operatorname*{ess\,inf}_{\sigma\geq t}\mathbb{E}\left\{\int_t^{\sigma\wedge\tau}f(s,Y_s,\mathbb{E}[Y_s])ds+g(Y_\sigma,\mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}}\right.$$
$$\left.+h(Y_\tau,\mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\leq\sigma,\tau<T\}}+\xi\mathbb{1}_{\{\tau=\sigma=T\}}|\mathcal{F}_t\right\}.$$

Next since $\xi\in L^p(d\mathbb{P})$, $\mathbb{E}[(\int_0^T|f(s,\omega,Y_s,\mathbb{E}[Y_s])|ds)^p]<\infty$, the processes $(h(Y_t,\mathbb{E}[Y_t]))_{T-\delta\leq t\leq T}$ and $(g(Y_t,\mathbb{E}[Y_t]))_{T-\delta\leq t\leq T}$ belong to $\mathcal{S}^p([T-\delta,T])$ since Y is so, and finally since $h<g$, then there exist processes $\bar{Y}\in\mathcal{S}^p([T-\delta,T])$, $\bar{Z}\in\mathcal{H}^d_{loc}([T-\delta,T])$ and $\bar{K}^\pm\in\mathcal{A}([T-\delta,T])$ (see e.g.

[22] for more details) such that for any $t \in [T-\delta, T]$, it holds:

$$\begin{cases} \bar{Y}_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + \bar{K}_T^+ - \bar{K}_t^+ - \bar{K}_T^- + \bar{K}_t^- - \int_t^T \bar{Z}_s dB_s; \\ h(Y_t, \mathbb{E}[Y_t]) \le \bar{Y}_t \le g(Y_t, \mathbb{E}[Y_t]); \\ \int_{T-\delta}^T (\bar{Y}_s - h(Y_s, \mathbb{E}[Y_s]))d\bar{K}_s^+ = 0, \int_{T-\delta}^T (\bar{Y}_s - g(Y_s, \mathbb{E}[Y_s]))d\bar{K}_s^- = 0. \end{cases} \tag{4.18}$$

Therefore the process $\bar{Y}$ has the following representation: $\forall t \in [T-\delta, T]$,

$$\begin{aligned} \bar{Y}_t = \operatorname*{ess\,sup}_{\tau \ge t} \operatorname*{ess\,inf}_{\sigma \ge t} \mathbb{E} \Big\{ \int_t^{\sigma\wedge\tau} & f(s, Y_s, \mathbb{E}[Y_s])ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} \\ & + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\le\sigma, \tau<T\}} + \xi \mathbb{1}_{\{\tau=\sigma=T\}} | \mathcal{F}_t \Big\} . \end{aligned} \tag{4.19}$$

It follows that for any $t \in [T-\delta, T]$, $Y_t = \bar{Y}_t$. Thus $(Y, \bar{Z}, \bar{K}^\pm)$ verifies (4.2) and (4.18) on $[T-\delta, T]$, i.e., for $t \in [T-\delta, T]$

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + \bar{K}_T^+ - \bar{K}_t^+ - \bar{K}_T^- + \bar{K}_t^- - \int_t^T \bar{Z}_s dB_s; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]); \\ \int_{T-\delta}^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))d\bar{K}_s^+ = 0, \int_{T-\delta}^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))d\bar{K}_s^- = 0. \end{cases} \tag{4.20}$$

But δ of Proposition 4.3.3 does not depend on the terminal condition ξ nor on T, therefore there exists another process Y^1 which is a fixed point of Φ in $\mathcal{S}^p([T-2\delta, T-\delta])$ with terminal condition $Y_{T-\delta}$, i.e., for any $t \in [T-2\delta, T-\delta]$,

$$\begin{aligned} Y_t^1 = \operatorname*{ess\,sup}_{\tau \in [t, T-\delta]} \operatorname*{ess\,inf}_{\sigma \in \in [t, T-\delta]} \mathbb{E} \Big\{ \int_t^{\sigma\wedge\tau} & f(s, Y_s^1, \mathbb{E}[Y_s^1])ds + g(Y_\sigma^1, \mathbb{E}[Y_t^1]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} \\ & + h(Y_\tau^1, \mathbb{E}[Y_t^1]_{t=\tau})\mathbb{1}_{\{\tau\le\sigma, \tau<T-\delta\}} + Y_{T-\delta}\mathbb{1}_{\{\tau=\sigma=T-\delta\}} | \mathcal{F}_t \Big\} . \end{aligned} \tag{4.21}$$

Then as previously, there exist processes $(\bar{Z}^1, \bar{K}^{1,\pm})$ ($\bar{K}^{1,\pm} \in \mathcal{A}([T-2\delta, T-\delta])$) suh that $(Y^1, \bar{Z}^1, \bar{K}^{1,\pm})$ verify: For any $t \in [T-2\delta, T-\delta]$,

$$\begin{cases} Y_t^1 = Y_{T-\delta} + \int_t^{T-\delta} f(s, Y_s^1, \mathbb{E}[Y_s^1])ds + \bar{K}_{T-\delta}^{1,+} - \bar{K}_t^{1,+} - \bar{K}_{T-\delta}^{1,-} + \bar{K}_t^{1,-} - \int_t^{T-\delta} \bar{Z}_s^1 dB_s; \\ h(Y_t^1, \mathbb{E}[Y_t^1]) \le Y_t^1 \le g(Y_t^1, \mathbb{E}[Y_t^1]); \\ \int_{T-2\delta}^{T-\delta} (Y_s^1 - h(Y_s^1, \mathbb{E}[Y_s^1]))d\bar{K}_s^{1,+} = 0, \int_{T-2\delta}^{T-\delta} (Y_s^1 - g(Y_s^1, \mathbb{E}[Y_s^1]))d\bar{K}_s^{1,-} = 0. \end{cases} \tag{4.22}$$

Concatenating now the solutions $(Y, \bar{Z}, \bar{K}^\pm)$ and $(Y^1, \bar{Z}^1, \bar{K}^{1,\pm})$ we obtain a solution of (4.2) on

$[T-2\delta, T]$. Actually for $t \in [T-2\delta, T]$, let us set:

$$\begin{aligned}
\tilde{Y}_t &= Y_t 1_{[T-\delta,T]}(t) + Y_t^1 1_{[T-2\delta,T-\delta)}(t),\\
\tilde{Z}_t &= \bar{Z}_t 1_{[T-\delta,T]}(t) + \bar{Z}_t^1 1_{[T-2\delta,T-\delta)}(t),\\
\int_{T-2\delta}^{t} d\tilde{K}_t^{1,\pm} &= \int_{T-2\delta}^{t} \{1_{[T-\delta,T]}(s) d\bar{K}_s^{0,\pm} + 1_{[T-2\delta,T-\delta]}(s) d\bar{K}_s^{1,\pm}\}.
\end{aligned}$$

Then $\tilde{Y} \in \mathcal{S}^p([T-2\delta, T]$, $\tilde{Z} \in \mathcal{H}^d_{loc}([T-2\delta, T])$ and $\tilde{K}^{\pm} \in \mathcal{A}([T-2\delta, T])$ and they verify: For any $t \in [T-2\delta, T]$,

$$\begin{cases}
\tilde{Y}_t = \xi + \int_t^T f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s]) ds + \tilde{K}_T^+ - \tilde{K}_t^+ - \tilde{K}_T^- + \tilde{K}_t^- - \int_t^T \tilde{Z}_s dB_s;\\
h(\tilde{Y}_t, \mathbb{E}[\tilde{Y}_t]) \le \tilde{Y}_t \le g(\tilde{Y}_t, \mathbb{E}[\tilde{Y}_t]);\\
and \int_{T-2\delta}^T (\tilde{Y}_s - h(\tilde{Y}_s, \mathbb{E}[\tilde{Y}_s])) d\tilde{K}_s^+ = 0, \int_{T-2\delta}^T (\tilde{Y}_s - g(\tilde{Y}_s, \mathbb{E}[\tilde{Y}_s])) d\tilde{K}_s^- = 0.
\end{cases} \tag{4.23}$$

But we can do the same on $[T-3\delta, T-2\delta]$, $[T-4\delta, T-3\delta]$, etc. and at the end, by concatenation of those solutions, we obtain a solution $(Y, Z, K^{\pm})$ which satisfy (4.2).

Let us now focus on uniqueness. Assume there is another solution $(\underline{Y}, \underline{Z}, \underline{K}^{\pm})$ of (4.2). It means that $\underline{Y}$ is a fixed point of Φ on $\mathcal{S}^p([T-\delta, T])$, therefore for any $t \in [T-\delta, T]$, $Y_t = \underline{Y}_t$. Next writing equation (4.2) for Y and $\underline{Y}$ on $[T-2\delta, T-\delta]$, using the link with zeros-sum Dynkin games (see Lemma 4.3.1) and finally the uniqueness of the fixed point of Φ on $\mathcal{S}^p([T-2\delta, T-\delta])$ to obtain that for any $t \in [T-2\delta, T-\delta]$, $Y_t = \underline{Y}_t$. By continuig this procedure on $[T-3\delta, T-2\delta]$, $[T-4\delta, T-3\delta]$, etc. we obtain that $Y = \underline{Y}$. The equality between the stochastic integrales imply that $Z = \underline{Z}$. Finally as $h < g$ and since $Y = \underline{Y}$, then $K^+ = \underline{K}^+$ and $K^- = \underline{K}^-$ (see e.g. [22]) for more details. Thus the solution is unique. The proof is complete. $\square$

Remark 4.3.5. *i) We have the same result if we replace the function h (resp. g) with $h(t, \omega, y, y')$ (resp. $g(t, \omega, y, y')$) with $(h(t, \omega, 0, 0))_{t \le T}$ (resp. $(g(t, \omega, 0, 0))_{t \le T}$) is a process of $\mathcal{S}^p$.*
ii) There is no specific difficulty to consider the following more general framework of equations (4.2) *and* (4.3).

$$\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{P}_{Y_s}) ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T;\\
h(Y_t, \mathbb{P}_{Y_s}) \le Y_t \le g(Y_t, \mathbb{P}_{Y_s}), \quad \forall t \in [0, T];\\
and \int_0^T (Y_s - h(Y_s, \mathbb{P}_{Y_s})) dK_s^+ = 0, \int_0^T (Y_s - g(Y_s, \mathbb{P}_{Y_s})) dK_s^- = 0
\end{cases}$$

where the Lipschitz property of f, h and g w.r.t. $\mathbb{P}_{Y_t}$ should be read as: for $\Psi \in \{f, g, h\}$ for any ν, ν'

probabilities

$$|\Psi(\nu) - \Psi(\nu')| \leq C d_p(\nu, \nu')$$

where $d_p(.,.)$ is the p-Wasserstein distance on the subset $\mathcal{P}_p(\mathbb{R})$ of probability measures with finite p-th moment, formulated in terms of a coupling between two random variables X and Y defined on the same probability space:

$$d_p(\mu, \nu) := \inf \left\{ (\mathbb{E}\left[|X - Y|^p\right])^{1/p}, \ law(X) = \mu, \ law(Y) = \nu \right\}. \quad \square$$

4.3.2 The case p=1

We proceed as we did in the case when $p > 1$. We have the following result.

Proposition 4.3.6. *Let Assumptions (A1) hold for some $p = 1$. If $\gamma_1, \gamma_2, \beta_1$ and β_2 satify*

$$\gamma_1 + \gamma_2 + \beta_1 + \beta_2 < 1 \tag{4.24}$$

then there exists $\delta > 0$ depending only on $C_f, \gamma_1, \gamma_2, \beta_1, \beta_2$ such that Φ is a contraction on the space $\mathcal{D}([T - \delta, T])$.

Proof. Let δ be a positive constant and θ a stopping time which belongs to $[T - \delta, T]$. Therefore

$$\begin{aligned}
|\Phi(Y)_\theta - \Phi(Y')_\theta| &= |\operatorname*{ess\,sup}_{\tau \geq \theta} \operatorname*{ess\,inf}_{\sigma \geq \theta} \{\mathbb{E}\left[\int_\theta^{\sigma \wedge \tau} f(s, Y_s, \mathbb{E}[Y_s])ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma < \tau\}}\right. \\
&\quad + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} + \xi \mathbb{1}_{\{\tau = \sigma = T\}} |\mathcal{F}_t]\} - \operatorname*{ess\,sup}_{\tau \geq \theta} \operatorname*{ess\,inf}_{\sigma \geq \theta} \{\mathbb{E}\left[\int_t^{\sigma \wedge \tau} f(s, Y'_s, \mathbb{E}[Y'_s])ds\right. \\
&\quad + g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})\mathbb{1}_{\{\sigma < \tau\}} + h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})\mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} + \xi \mathbb{1}_{\{\tau = \sigma = T\}} |\mathcal{F}_\theta]\}| \\
&\leq \operatorname*{ess\,sup}_{\tau \geq \theta} \operatorname*{ess\,sup}_{\sigma \geq \theta} \mathbb{E}\left\{\int_\theta^{\sigma \wedge \tau} |f(s, Y_s, \mathbb{E}[Y_s]) - f(s, Y'_s, \mathbb{E}[Y'_s])|ds + |g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\right. \\
&\quad - g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})|\mathbb{1}_{\{\sigma < \tau\}} + |h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) - h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})|\mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} |\mathcal{F}_\theta\} \\
&\leq \mathbb{E}\left\{\int_{T-\delta}^T |f(s, Y_s, \mathbb{E}[Y_s]) - f(s, Y'_s, \mathbb{E}[Y'_s])|ds |\mathcal{F}_\theta\right\} \\
&\quad + \operatorname*{ess\,sup}_{\sigma \geq \theta} \mathbb{E}\{|g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma}) - g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})| |\mathcal{F}_\theta\} \\
&\quad + \operatorname*{ess\,sup}_{\tau \geq \theta} \mathbb{E}\{|h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) - h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})| |\mathcal{F}_\theta\}.
\end{aligned}$$

Take now expectation in both hand-sides to obtain:

$$\begin{aligned}\mathbb{E}[|\Phi(Y)_\theta - \Phi(Y')_\theta|] \leq & 2\delta C_f \sup_{\tau\in[T-\delta,T]} \mathbb{E}[|Y_\tau - Y'_\tau] + \sup_{\sigma\geq\theta} \mathbb{E}\{|g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma}) - g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})|\} \\ & + \sup_{\tau\geq\theta} \mathbb{E}\{|h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) - h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})|\} \\ & \leq 2\delta C_f \sup_{\tau\in[T-\delta,T]} \mathbb{E}[|Y_\tau - Y'_\tau] + \sup_{\sigma\geq\theta} \mathbb{E}\{|g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma}) - g(Y'_\sigma, \mathbb{E}[Y'_t]_{t=\sigma})|\} \\ & + \sup_{\tau\geq\theta} \mathbb{E}\{|h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau}) - h(Y'_\tau, \mathbb{E}[Y'_t]_{t=\tau})|\}.\end{aligned}$$

Then for any θ a stopping time valued in $[T-\delta, T]$, we have:

$$\mathbb{E}[|\Phi(Y)_\theta - \Phi(Y')_\theta|] \leq \underbrace{(2\delta C_f + \beta_1 + \beta_2 + \gamma_1 + \gamma_2)}_{\Sigma(\delta)} \sup_{\tau\in[T-\delta,T]} \mathbb{E}[|Y_\tau - Y'_\tau|].$$

Next since $\beta_1 + \beta_2 + \gamma_1 + \gamma_2 < 1$, then for δ small enough we have $\Sigma(\delta) < 1$ (δ does not depend neither on ξ nor on T) and Φ is a contraction on the space $\mathcal{D}([T-\delta, T])$. Therefore it has a fixed point Y, which then verifies:

$$\begin{aligned} & Y \in \mathcal{D}([T-\delta, T]) \text{ and } \forall t \in [T-\delta, T], \\ & Y_t = \operatorname*{ess\,sup}_{\tau\geq t} \operatorname*{ess\,inf}_{\sigma\geq t} \{\mathbb{E}\{\int_t^{\sigma\wedge\tau} f(s, Y_s, \mathbb{E}[Y_s])ds + g(Y_\sigma, \mathbb{E}[Y_t]_{t=\sigma})\mathbb{1}_{\{\sigma<\tau\}} \\ & \quad + h(Y_\tau, \mathbb{E}[Y_t]_{t=\tau})\mathbb{1}_{\{\tau\leq\sigma, \tau<T\}} + \xi\mathbb{1}_{\{\tau=\sigma=T\}}|\mathcal{F}_t\}\}. \end{aligned} \tag{4.25}$$

□

As a by-product we have the following result which stems from the link between the value of a zero-sum Dynkin game and doubly reflected BSDE given in (4.6).

Corollary 4.3.7. *Let Assumption (A1) hold for some $p = 1$. If $\gamma_1, \gamma_2, \beta_1$ and β_2 satisfy* (4.24) *then there exists $\delta > 0$, depending only on $C_f, \gamma_1, \gamma_2, \beta_1, \beta_2$, and $\mathcal{P}$-measurable processes Z^0, $K^{0,\pm}$ such that:*

$$\begin{cases} \mathbb{P} - a.s., \ \int_{T-\delta}^T |Z_s^0|^2 ds < \infty; \ K^{0,\pm} \in \mathcal{A} \text{ and } K^{0,\pm}_{T-\delta} = 0; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^{0,+} - K_t^{0,+} - K_T^{0,-} + K_t^{0,-} - \int_t^T Z_s^0 dB_s, \ T-\delta \leq t \leq T; \\ h(Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(Y_t, \mathbb{E}[Y_t]), \ T-\delta \leq t \leq T; \\ \int_{T-\delta}^T (Y_t - h(Y_t, \mathbb{E}[Y_t]))dK_t^{0,+} = \int_{T-\delta}^T (Y_t - g(Y_t, \mathbb{E}[Y_t]))dK_t^{0,-} = 0. \end{cases} \tag{4.26}$$

We now give the main result of this subsection.

Theorem 4.3.8. *Let f, h, g and ξ satisfying Assumption (A1) for $p = 1$. Suppose that*

$$\gamma_1 + \gamma_2 + \beta_1 + \beta_2 < 1. \tag{4.27}$$

Then, there exist $\mathcal{P}$-mesurable processes $(Y, Z, K^{\pm})$ unique solution of the mean-field reflected BSDE (4.3)*, i.e.,*

$$\begin{cases} Y \in \mathcal{D}, \quad Z \in \mathcal{H}^d_{loc} \quad \text{and} \quad K^+, K^- \in \mathcal{A}; \\ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s])ds + K_T^+ - K_t^+ - K_T^- + K_t^- - \int_t^T Z_s dB_s, \quad 0 \le t \le T; \\ h(Y_t, \mathbb{E}[Y_t]) \le Y_t \le g(Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\ \text{and} \int_0^T (Y_s - h(Y_s, \mathbb{E}[Y_s]))dK_s^+ = 0, \int_0^T (Y_s - g(Y_s, \mathbb{E}[Y_s]))dK_s^- = 0. \end{cases} \tag{4.28}$$

Proof. Let δ be as in Proposition 4.3.6 and Y the fixed point of Φ on $\mathcal{D}([T-\delta, T])$ which exists since (4.24) is satisfied. Next let Y^1 be the fixed point of Φ on $\mathcal{D}([T-2\delta, T-\delta])$ with terminal condition $Y_{T-\delta}$, i.e., for any $t \in [T-2\delta, T-\delta]$,

$$\begin{aligned} Y_t^1 = \operatorname*{ess\,sup}_{\tau \in [t, T-\delta]} \operatorname*{ess\,inf}_{\sigma \in \in [t, T-\delta]} \mathbb{E} \Big\{ \int_t^{\sigma \wedge \tau} f(s, Y_s^1, \mathbb{E}[Y_s^1])ds + g(Y_\sigma^1, \mathbb{E}[Y_t^1]_{t=\sigma}) \mathbb{1}_{\{\sigma < \tau\}} \\ + h(Y_\tau^1, \mathbb{E}[Y_t^1]_{t=\tau}) \mathbb{1}_{\{\tau \le \sigma, \tau < T-\delta\}} + Y_{T-\delta} \mathbb{1}_{\{\tau = \sigma = T-\delta\}} | \mathcal{F}_t \Big\}. \end{aligned} \tag{4.29}$$

The process Y^1 exists since condition (4.24) is satisfied and δ does not depend neither on T nor on the terminal condition. Once more the link between reflected backward equations and zero-sum Dynkin games (see Lemma 4.3.1) implies the existence of $\mathcal{P}$-measurable processes Z^1, $K^{1,\pm}$ such that:

$$\begin{cases} \mathbb{P} - a.s., \int_{T-2\delta}^{T-\delta} |Z_s^1|^2 ds < \infty; \ K^{1,\pm} \in \mathcal{A} \text{ and } K^{1,\pm}_{T-2\delta} = 0; \\ Y_t^1 = Y_{T-\delta} + \int_t^{T-\delta} f(s, Y_s^1, \mathbb{E}[Y_s^1])ds + K^{1,+}_{T-\delta} - K_t^{1,+} - K^{1,-}_{T-\delta} + K_t^{1,-} - \int_t^{T-\delta} Z_s^1 dB_s, \ t \in [T-2\delta, T-\delta]; \\ h(Y_t^1, \mathbb{E}[Y_t^1]) \le Y_t^1 \le g(Y_t^1, \mathbb{E}[Y_t^1]), \ t \in [T-2\delta, T-\delta]; \\ \int_{T-2\delta}^{T-\delta} (Y_t^1 - h(Y_t^1, \mathbb{E}[Y_t^1]))dK_t^{1,+} = \int_{T-2\delta}^{T-\delta} (Y_t^1 - g(Y_t^1, \mathbb{E}[Y_t^1]))dK_t^{1,-} = 0. \end{cases} \tag{4.30}$$

Concatenating now the solutions $(Y, Z^0, K^{0,\pm})$ of (4.26) and $(Y^1, Z^1, K^{1,\pm})$ we obtain a solution of (4.3) on $[T-2\delta, T]$. Actually for $t \in [T-2\delta, T]$, let us set:

$$\begin{aligned}
\tilde{Y}_t &= Y_t 1_{[T-\delta,T]}(t) + Y_t^1 1_{[T-2\delta,T-\delta)}(t), \\
\tilde{Z}_t &= Z_t^0 1_{[T-\delta,T]}(t) + Z_t^1 1_{[T-2\delta,T-\delta)}(t), \\
\int_{T-2\delta}^t d\tilde{K}_t^{1,\pm} &= \int_{T-2\delta}^t \{1_{[T-\delta,T]}(s) dK_s^{0,\pm} + 1_{[T-2\delta,T-\delta]}(s) dK_s^{1,\pm}\}.
\end{aligned}$$

Then $\tilde{Y} \in \mathcal{D}([T-2\delta, T]$, $\tilde{Z} \in \mathcal{H}^d_{loc}([T-2\delta, T])$ and $\tilde{K}^\pm \in \mathcal{A}([T-2\delta, T])$ and they verify: For any $t \in [T-2\delta, T]$,

$$\begin{cases} \tilde{Y}_t = \xi + \int_t^T f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s])ds + \tilde{K}_T^+ - \tilde{K}_t^+ - \tilde{K}_T^- + \tilde{K}_t^- - \int_t^T \tilde{Z}_s dB_s; \\ h(\tilde{Y}_t, \mathbb{E}[\tilde{Y}_t]) \leq \tilde{Y}_t \leq g(\tilde{Y}_t, \mathbb{E}[\tilde{Y}_t]); \\ \textit{and} \int_{T-2\delta}^T (\tilde{Y}_s - h(\tilde{Y}_s, \mathbb{E}[\tilde{Y}_s])) d\tilde{K}_s^+ = 0, \int_{T-2\delta}^T (\tilde{Y}_s - g(\tilde{Y}_s, \mathbb{E}[\tilde{Y}_s])) d\tilde{K}_s^- = 0. \end{cases} \tag{4.31}$$

But we can do the same on $[T-3\delta, T-2\delta]$, $[T-4\delta, T-3\delta]$, etc. and at the end, by concatenation of those solutions, we obtain a solution $(Y, Z, K^\pm)$ which satisfy (4.2).

Let us now focus on uniqueness. Assume there is another solution $(\underline{Y}, \underline{Z}, \underline{K}^\pm)$ of (4.2). It means that $\underline{Y}$ is a fixed point of Φ on $\mathcal{D}([T-\delta, T])$, therefore for any $t \in [T-\delta, T]$, $Y_t = \underline{Y}_t$. Next writing equation (4.2) for Y and $\underline{Y}$ on $[T-2\delta, T-\delta]$, using the link with zeros-sum Dynkin games (Lemma 4.3.1) and finally the uniqueness of the fixed point of Φ on $\mathcal{D}([T-2\delta, T-\delta])$ to obtain that for any $t \in [T-2\delta, T-\delta]$, $Y_t = \underline{Y}_t$. By continuing this procedure on $[T-3\delta, T-2\delta]$, $[T-4\delta, T-3\delta]$, etc. we obtain that $Y = \underline{Y}$. The equality between the stochastic integrals imply that $Z = \underline{Z}$. Finally as $h < g$ and since $Y = \underline{Y}$, then $K^+ = \underline{K}^+$ and $K^- = \underline{K}^-$ (see e.g. [22]) for more details. Thus the solution is unique. The proof is complete. □

Finally let us notice that the same Remark 4.3.5 is valid for this case $p = 1$.

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

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Titre : Équations différentielles stochastiques rétrogrades et applications : switching optimal, jeux stochastiques, EDP et mean-field

Mots clés : EDSRs, jeux stochastiques de somme nulle, EDSR réfléchie à deux obstacles interconnectés, EDPs, Méthode de pénalisation, méthode de point-fixe, switching optimal stochastique.

Résumé : Cette thèse est relative aux Equations Différentielles Stochastique Rétrogrades (EDSRs) réfléchies avec deux obstacles et leurs applications aux jeux de switching de somme nulle, aux systèmes d'équations aux dérivées partielles, aux problèmes de mean-field. Il y a deux parties dans cette thèse. La première partie porte sur le switching optimal stochastique et est composée de deux travaux. Dans le premier travail, nous montrons l'existence de la solution d'un système d'EDSR réfléchies à obstacles bilatéraux interconnectés dans le cadre probabiliste général. Ce problème est lié à un jeu de switching de somme nulle. Ensuite nous abordons la question de l'unicité de la solution. Et enfin nous appliquons les résultats obtenus pour montrer que le système d'EDP associé à une unique solution au sens viscosité, sans la condition de monotonie habituelle. Dans le second travail, nous considérons aussi un système d'EDSRs réfléchies à obstacles bilatéraux interconnectés dans le cadre markovien. La différence avec le premier travail réside dans le fait que le switching ne s'opère pas de la même manière. Cette fois-ci quand le switching est opéré, le système est mis dans l'état suivant importe peu lequel des joueurs décide de switcher. Cette différence est fondamentale et complique singulièrement le problème de l'existence de la solution du système. Néanmoins, dans le cadre markovien nous montrons cette existence et donnons un résultat d'unicité en utilisant principalement la méthode de Perron. Ensuite, le lien avec un jeu de switching spécifique est établi dans deux cadres.
Dans la seconde partie nous étudions les EDSR réfléchies unidimensionnelles à deux obstacles de type mean-field. Par la méthode du point fixe, nous montrons l'existence et l'unicité de la solution dans deux cadres, en fonction de l'intégrabilité des données.

Title : Backward Stochastic Differential Equations and applications: optimal switching, stochastic games, partial differential equations and mean-field

Keywords : BSDEs, zero-sum stochastic games, DRBSDEs with doubly interconnected barriers, PDEs, penalization method, fixed point method, stochastic optimal switching.

Abstract : This thesis is related to Doubly Reflected Backward Stochastic Differential Equations (DRBSDEs) with two obstacles and their applications in zero-sum stochastic switching games, systems of partial differential equations, mean-field problems. There are two parts in this thesis. The first part deals with optimal stochastic switching and is composed of two works. In the first work we prove the existence of the solution of a system of DRBSDEs with bilateral interconnected obstacles in a probabilistic framework. This problem is related to a zero-sum switching game. Then we tackle the problem of the uniqueness of the solution. Finally, we apply the obtained results and prove that, without the usual monotonicity condition, the associated PDE system has a unique solution in viscosity sense. In the second work, we also consider a system of DRBSDEs with bilateral interconnected obstacles in the markovian framework. The difference between this work and the first one lies in the fact that switching does not work in the same way. In this second framework, when switching is operated, the system is put in the following state regardless of which player decides to switch. This difference is fundamental and largely complicates the problem of the existence of the solution of the system. Nevertheless, in the Markovian framework we show this existence and give a uniqueness result by the Perron's method. Later on, two particular switching games are analyzed. In the second part we study a onedimensional Reflected BSDE with two obstacles of mean-field type. By the fixed point method, we show the existence and uniqueness of the solution in connection with the integrality of the data.