Separation and Poincaré profiles

Corentin Le Coz

PARIS-SACLAY

Profils de séparation et de Poincaré

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Thèse de doctorat de l'Université Paris-Saclay

Ecole Doctorale de Mathématiques Hadamard (EDMH, ED 574)
Spécialité de doctorat : Mathématiques fondamentales
Unité de recherche : Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France.
Référent : Faculté des sciences d'Orsay

Thèse présentée et soutenue le 23 septembre 2020, à l'Institut de Mathématique d'Orsay, par

Corentin LE COZ

Composition du jury :

Pierre PANSU Professeur, Université Paris-Saclay	Président
Gábor PETE Chercheur sénior, Institut Alfréd Rényi	Rapporteur et examinateur
Laurent SALOFF-COSTE Professeur, Université Cornell	Rapporteur et examinateur
Goulnara ARZHANTSEVA Professeur, Université de Vienne	Examinatrice
Romain TESSERA Directeur de recherche, CNRS	Directeur
Jérémie BRIEUSSEL Maître de conférences, Université de Monpellier	Codirecteur

Thèse de doctorat

NNT: 2020UPASM014

Ce travail a été réalisé au Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay.
Il a été soutenu par l'Agence Nationale de la Recherche dans le cadre des projets
ANR-14-CE25-0004 GAMME et ANR-16-CE40-0022-01 AGIRA.

Remerciements

Je tiens à remercier mes directeurs de thèse pour leur accompagnement durant ces trois années de recherche. Depuis mon stage de Master, Romain Tessera a toujours su me communiquer son entousiasthme débordant. Sa vivacité d'esprit et son goût de transmettre m'ont permis de travailler d'une manière agréable. Je lui suis très reconnaissant d'avoir cru en moi, malgré les difficultés que j'ai pu manifester au départ, et de m'avoir incité à pouruivre ma voie dans la recherche en mathématiques.

J'ai eu la chance d'avoir Jérémie Brieussel comme codirecteur de thèse. Il a su me partager son goût du travail bien fait, en m'incitant à développer au maximum mes idées. Je veux le remercier d'avoir favorisé une atmosphère de confiance, qui m'a aidé à travailler sereinement.

Je remercie vivement les membres du jury de ma soutenance de thèse: Pierre Pansu, Goulnara Arzhantseva, et plus particulièrement Gábor Pete et Laurent Saloff-Coste qui ont accepté d'être rapporteurs et m'ont fourni des rapports très détaillés.

A Orsay, j'ai bénéficié d'un environnement d'exception. J'ai bien travaillé avec Gabriel et Suraj, puis avec Timothée, Pierre-Louis et Irving. A la piscine ou ailleurs, j'ai pu me changer les idées avec Guillaume et Camille. Il y a aussi eu les parties de tennis avec Salim, et tout simplement les déjeuners conviviaux entre doctorants.

Je remercie Mélanie pour la confiance qu'elle m'a accordée durant notre collaboration en Master MEEF, et de m'avoir donné goût à la vulgarisation des mathématiques.

J'ai aussi une pensée pour Mme Santoni, qui m'a donné goût aux mathématiques en classe préparatoire. J'ai compris avec elle que la beauté des mathématiques se révèle au sein même de leur rigueur. Cela a été pour moi un tournant majeur, dans ma vie étudiante ainsi que dans ma vie personnelle.

Je souhaite remercier mes beaux-parents de nous avoir accueilli pendant le confinement du printemps dernier. Leur aide précieuse m'a permis de rédiger ce manuscrit sereinement.

Je remercie enfin mes parents pour tout ce qu'ils m'ont apporté depuis mon enfance. Avec le recul des années et mon rôle de jeune Papa, je me rends compte de la générosité et de la patience dont j'ai bénéficié jusqu'à présent, et bénéficie encore aujourd'hui.

Cela fait presque trois ans que je partage ma vie avec Bertille. Trois années de complicité et de joie profonde. J'admire que tu me suives dans mes élans mathématiques, sans rechigner lorsque notre dîner se transforme en un exposé interminable. Enfin, il y a Clément et Marie qui me donnent du bonheur chaque jour.

"Il est plus beau d'éclairer que de briller seulement"

Saint Thomas d'Aquin

Résumé (en français)

Ce manuscrit de thèse récapitule mes travaux de recherche sur le profil de séparation et les profils de Poincaré. Le profil de séparation est apparu en 2012 dans un article fondateur de Benjamini, Schramm et Timár. La définition donnée tirait ses origines dans des travaux antérieurs, dans le domaine du calcul formel : principalement des études de Lipton et Trajan concernant les graphes planaires, et de Miller, Teng, Thurston et Vavasis concernant des graphes d'intersection. Le profil de séparation utile en théorie géométrique des groupes, mon domaine de recherche, à cause de sa propriété de monotonie par plongements grossiers. Il a été généralisé par Hume, Mackay et Tessera en 2019 en une gamme continue de profils, appelés profils de Poincaré.

La première partie de cette thèse constitue l'état de l'art concernant ces profils. J'y ai détaillé les résultats les plus marquants du point de vue de la théorie géométrique des groupes. J'ai souvent proposé des esquisses de preuves, me passant de certains détails techniques. Je renvoie aux articles originaux pour les démonstrations complètes. A l'opposé, j'ai plusieurs fois donné des preuves de propriétés bien connues des spécialistes, qui ne sont généralement pas démontrées dans les articles de recherche. J'espère que cela pourra rendre service aux débutants.

Les deux parties suivantes sont des adaptations de pré-publications, fruits de mes travaux des trois années passées. Le contenu de la deuxième partie a été écrit en collaboration avec Gournay. L'objectif y est de comparer le profil de séparation avec d'autres quantités mieux connues : le profil isopérimétrique, la compression des plongements dans les espaces de Hilbert, et la croissance du volume des boules.

La troisième partie correspond à une pré-publication qui porte sur la prescription de grands profils de Poincaré. Ceci est réalisé en utilisant une construction développée par Brieussel et Zheng en 2015. La compression des plongements dans les espacs L^p, et la théorie spectrale des graphes en degré non borné sont les principaux ingrédients de ce projet.

Abstract (english language)

The goal of this thesis report is to present my research concerning separation profile and Poincaré profiles. Separation profiles first appeared in 2012 in a seminal article written by Benjamini, Schramm and Timár. This definition was based on preceding research, in the field of computer science, mainly work of Lipton and Trajan concerning planar graphs, and of Miller, Teng, Thurston and Vavasis concerning overlap graphs. Separation profiles play now a role in geometric group theory, where my personal interests lies, because of their property of monotonicity under coarse embeddings. These were generalized by Hume, Mackay and Tessera in 2019 into a spectrum of profiles, called Poincaré profiles.

The first part of this thesis is a summary what we know on these profiles. I have given the most important results, in the point of view of geometric group theory. I often give sketches of proofs, and refer to references for complete proofs. On the other hand, we have several times detailed some basic facts that are likely to be easy exercises for specialists and are not usually proved in research articles. We hope that this will help novices.

The two next parts are adapted from two preprints, fruit of my research of the past three years. The content of the second part was written in collaboration with Gournay. Its purpose is to compare the separation profile with other well-studied quantities: isoperimetric profile, compression of embeddings in Hilbert spaces, volume growth.

The third part corresponds to a preprint concerning the prescription of high Poincaré profiles. This is done using a construction developed by Brieussel and Zheng in 2015. Compression of embeddings into L^p spaces and spectral graph theory in unbounded degrees context are the two main ingredients of this project.

Contents

Notations

- $\mathbf{N}$ is the ordered semigroup of natural integers, $\mathbf{Z}$ is the infinite cyclic group.
- $\mathbf{R}$ and $\mathbf{C}$ are respectively the ordered field of real numbers and the field of complex numbers.
- $\mathbf{R}_+$ is the ordered semi-group of non-negative real numbers.
- $\mathbf{H}$ is the separable infinite dimensional Hilbert space or, depending on the context, Hamilton's quaternion algebra.
- For any integer n, $\mathbf{Z}_n$ is the cyclic group of order n.
- $\mathbf{H}^d$ is the d-dimensional hyperbolic space, $\mathbf{R}^d$ is the d-dimensional Euclidean space.
- We adopt the convention that $\sup \emptyset = 0$, and $\inf \emptyset = \infty$.
- If $f, g \colon S \to [0, \infty]$ are two functions, where $S \in \{\mathbf{N}, \mathbf{R}_+\}$, we write $f \preceq_{A,B,\ldots} g$ if there exists a constant $C > 0$ depending only on $A, B, \ldots$ such that $f(x) \leq Cg(Cx)$ for all $x \geq C$. We write $f \simeq_{A,B,\ldots} g$ when we have $f \preceq_{A,B,\ldots} g$ and $g \preceq_{A,B,\ldots} f$, which defines an equivalence relation. Equivalence classes will be called *growth types*. We will drop the subscripts if the constants are understood.
- If γ and γ' are two growth types, we write $\gamma \leq \gamma'$ if there exists $(f, g) \in \gamma \times \gamma'$ such that $f \preceq g$.
- If $f, g \colon S \to \mathbf{R}_+$ are two functions, where $S \in \{\mathbf{N}, \mathbf{R}_+\}$, $f = \mathcal{O}(g)$ (respectively $f = o(g)$) means that there exists a bounded function (respectively a function of limit zero at infinity) h such that $f \leq hg$.
- When Γ is a graph, $V\Gamma$ and $E\Gamma$ are the sets of vertices and edges of Γ, respectively. The set $E\Gamma$ is a subset of $\mathcal{P}_2(V\Gamma)$, the collection of subsets of $V\Gamma$ containing 2 vertices. In particular, we only consider unoriented graphs, without self-loops.
- If u and v are two vertices of a graph, $u \sim v$ means that u and v are linked by an edge.
- A graphs of bounded degree is a graph such that there is a uniform bound on the degrees of its vertices. In statements concerning a graph of bounded degrees, such a bound on the degrees will be implicit in the notations $f \preceq g$ and $f \simeq g$.

- If x is a point of a metric space X, and r a non-negative real number, $B(x, r)$ is the closed ball of X centred at x of radius r. If A is a subset of X, then $[A]_r$ is the r-neighbourhood of A, *i.e.* the set $\{x \in X \mid \exists a \in A\ d(x, a) \leq r\}$.
- If (Z, ν) is a measured space of positive and finite total measure, and $f\colon Z \to \mathbf{R}$ is a measurable function such that $\int_Z f d\nu$ is well-defined, $⨍_Z f d\nu$ is the average integral $\frac{1}{\nu(Z)} \int_Z f d\nu$.

Introduction (en français)

Etant donnés deux espaces métriques, il est naturel de se demander si l'un peut se plonger dans l'autre, selon un plongement qui respecte leurs métriques respectives. Dans le cas des graphes, une première interprétation de ce problème est de savoir s'il existe un plongement au sens strict, *i.e.* une application injective qui préserve la relation d'adjacence. Il se trouve que dans le domaine de la théorie géométrique des groupes, il est bien plus naturel de considérer des notions de plongements plus flexibles, commes les plongements quasi-isométriques et plus généralement les plongements grossiers. En effet, l'interprétation *géométrique* des groupes de type fini est seulement valide à application bilipschitzienne près (à cause du choix de la partie génératrice). En outre, lorsqu'on étudie un groupe selon le prisme de ses actions géométriques, seule la géométrie à grande-échelle est retenue, ce qui donne lieu en général à une classe entière de quasi-isométrie.

Ici, nous nous intéresserons aux applications dites *régulières*, telles que définies par Benjamini, Schramm et Timár [19]: les applications lipschitziennes satisfaisant qu'il existe une borne uniforme sur le nombre de sommets qui ont même image. Cette notion de plongement est très grossière. En particulier, les plongements quasi-isométriques et grossiers sont des applications régulières, si le graphe initial est connexe.

En général, la question qui nous intéresse, à savoir de dire si un espace peut se plonger dans un autre, est une question difficile. Une réponse positive requiert souvent d'expliciter un plongement. Une réponse négative requiert de trouver une obstruction, qui peut être de nature géométrique, analytique, combinatoire, etc. Une manière d'encoder des obstructions est d'associer à tout espace un *invariant*, qui appartient à un ensemble ordonné (généralement un nombre ou une fonction), qui sera compatible avec la notion de plongement étudiée. On appelle cela un *invariant monotone*, qui est alors capable de fournir une obstruction à l'existence d'un plongement entre deux espaces. Dans le cas des applications régulières, très peu de tels invariants sont connus : la croissance du volume des boules, la dimension asymptotique, et plus récemment le profil de séparation et les profils de Poincaré. La croissance et la dimension asymptotique sont en général assez grossiers, d'où l'intérêt d'étudier ces profils.

Le **profil de séparation** a été défini par Benjamini, Schramm & Timár [19]. Comme l'a montré Hume [67], le profil de séparation d'un graphe (infini) G peut être défini, pour tout entier $n \geq 0$, par

$$\mathrm{sep}_G(n) = \sup \{|V\Gamma|\, h(\Gamma) \colon \Gamma \subset G, |V\Gamma| \leq n\},$$

où $h(\Gamma)$ est la constante de Cheeger du graphe Γ. Hume, Mackay et Tessera ont généralisé ce profil en définissant, pour tout $p \in [0, \infty]$ le **profil de Poincaré** L^p d'un graphe (infini) G par

$$\Pi_{G,p}(n) = \sup \{|V\Gamma| \, h_p(\Gamma) : \Gamma \subset G, |V\Gamma| \leq n\},$$

où $h_p(\Gamma)$ est la constante de Cheeger L^p du graphe Γ. Dans le cas des graphes de degré borné, le profil de Poincaré L^1 et le profil de séparation sont équivalents, à constante près.

Prescription de profils de séparation

Bien entendu, lorsqu'un tel invariant ne peut prendre qu'un nombre limité de valeurs, il est moins pertinent, étant donné que dans de nombreux cas il ne permettra pas de fournir une obstruction. Il est clair que, par définition, un profil de Poincaré est au moins constant et au plus linéaire. Il est alors naturel de se demander quelles sont les profils possibles dans cette plage de fonctions. Cette question est en quelque sorte orthogonale à la question initiale : elle interroge la finesse de l'invariant.

Il est déjà connu que le profil de séparation peut suivre une grande variété de comportements asymptotiques. En effet, les groupes hyperboliques peuvent avoit un profil constant (groupes libres [19, Théorème 2.1.]), un profil logarithmique (groupes Fuchsiens [19, Proposition 4.1.]), un profil polynomial (réseaux dans les espaces hyperboliques [19, Proposition 4.1.], [68, Théorème 12]). Parmi les groupes moyennables, on trouve des profils arbitrairement petits (le long d'une sous-suite) [69, Théorème 1.4], des profils polynomiaux (groupes nilpotents [68, Théorème 7]), ainsi que des profils entre $\frac{n}{(\log n)^2}$ et $\frac{n}{\log n}$ (groupes polycycliques [80]). Au sujet de la prescription (à constante près) du profil de séparation, on peut mentionner deux résultats majeurs :

- la prescription de "petits" profils par Hume and Mackay [69] (le long d'une sous-suite) à l'aide de groupes lacunairement hyperboliques [97] (qui donne des profils dominés par la fonction logarithme).
- la prescription de profils intermédiaires, qui a été en grande partie résolue par Hume, Mackay et Tessera [68], à l'aide de groupes agissant sur les immeubles de Bourdon-Pajot [24] (qui donnent des profils équivalents à n^α pour tout choix de α dans un sous-ensemble dense de l'intervalle $]0, 1[$).

La contribution majeure de l'auteur résoud partiellement cette question pour des "gros" profils de Poincaré : des profils situés entre $\frac{n}{\log \log n}$ et n (non atteint), voir le théorème A ci-dessous. Ces profils sont obtenus à l'aide de groupes moyennables construits par Brieussel et Zheng [29]. Ceci montre que les groupes moyennables peuvent avoir une grande variété de profils de Poincaré. En outre, ces groupes sont tous à croissance exponentielle et de dimension asymptotique égale à un, ce qui prouve que ces profils ne sont pas redondants vis-à-vis de ces invariants.

Théorème A (Théorème 1, Partie III.)**.** *Il existe deux constantes universelles* κ_1 *et* κ_2 *qui rendent la proposition suivante vraie. Soit* $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ *une fonction croissante, telle que* $\frac{x}{\rho(x)}$ *soit croissante, et satisfaisant* $\lim_\infty \rho = \infty$*. On suppose également que* ρ *est injective et qu'il existe* $\alpha > 0$ *tel que* $\frac{\rho^{-1}(x)}{\exp(x^\alpha)}$ *soit croissante. Alors, il existe un groupe moyennable de type fini* Δ *à croissance exponentielle et de dimension aymptotique égale à un tel que pour tout* $p \in [1, \infty)$*, on a*

$$\begin{aligned} &\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \textit{pour tout } n, \\ \textit{et} \quad &\Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\rho(\log n)} \quad \textit{pour une infinité d'entiers } n. \end{aligned}$$

Bornes sur les profils de Poincaré

Il est important de comparer les profils et de séparation, et plus généralement les profils de Poincaré, avec d'autres quantités mieux connues. En effet, cela donne la possibilité de s'appuyer sur des résultats antérieurs pour calculer des bornes sur les profils de séparation et de Poincaré. Nous donnons ici deux contributions majeures concernant ce problème. La première donne une borne supérieure sur les profils de Poincaré, et la seconde une borne inférieure sur le profil de séparation.

Compression des plongements lipschitziens Nous commençons par un théorème donnant une borne supérieure sur les profils de Poincaré à partir de la compression dans les espaces L^p. Ce théorème est à l'origine de la borne supérieure du théorème A. On définit la **compression** d'une application 1-lipschitzienne $f\colon G \to L^p$ comme

$$\rho_f(t) = \inf \left\{ \|f(g) - f(h)\|_p : d_G(g,h) \geq t \right\}.$$

Nous prouvons de théorème suivant :

Théorème B (Théorème 3.3.11, Partie III)**.** *Soit* G *un graphe de degré borné. Il existe deux constantes* $c_1, c_2 > 0$*, qui ne dépendent que du degré maximum des sommets de* G*, tel que pour toute application* 1*-lipschitzienne* $f\colon VG \to L^p$*, avec* $p \in [1, \infty)$*, on a*

$$\Pi_{G,p}(n) \leq c_1 \frac{n}{\rho_f(c_2 \log n)}, \quad \textit{pour tout } n \geq 0.$$

Cet énoncé est optimal pour les produits d'arbres (voir [19]), ainsi que dans le théorème A ci-dessus.

Profils isopérimétriques Dans la partie II, nous prouvons un théorème de comparaison entre le profil de séparation et les profil isopérimétrique (voir le théorème 8.2.1). Ce dernier est défini ainsi :

$$\Lambda(n) = \inf \left\{ \frac{|\partial F|}{|F|} : F \subset VG,\, |F| \leq n \right\},$$

où ∂F désigne le bord d'un sous-ensemble $F \subset VG$, que l'on peut définir par exemple comme l'ensemble des sommets de G à distance un de F. Nous nous contentons ici de donner des exemples d'applications.

Théorème C (Théorème 6.0.2, Partie II)**.** *Soit G un graphe degré borné tel que $\frac{K_1}{n^{1/d}} \leq \Lambda(n) \leq \frac{K_2}{n^{1/d}}$, pour certaines constantes positives K_1 et K_2. Alors, il existe une constante strictement positive K_3 telle que pour tout n, on a $\dfrac{\operatorname{sep}(n)}{n} \geq \dfrac{K_3}{n^{1/d}}$.*

L'hypothèse de ce théorème est satisfaite par les graphes de Cayley des groupes nilpotents de type fini. Dans ce cas on retrouve un résultat de Hume, Mackay & Tessera [68], qui est optimal. Cependant cet énoncé peut s'appliquer dans une plus grande généralité, comme les graphes pré-fractales des tapis de Sierpinski (voir Théorème 9.1.4, Partie II). Les méthodes sous-jacentes à ce théorème peuvent aussi s'appliquer aux graphes de profil isopérimétrique logarithmique, donnant lieu à l'énoncé ci-dessous.

Théorème D (Théorème 6.0.6, Partie II)**.** *Soit G un graphe de Cayley.*

Si, pour un certain $a > 0$, $\Lambda(N)$ *est ...*	*alors, pour une infinité d'entiers N,* $\dfrac{\operatorname{sep}(N)}{N}$ *est ...*
$\preccurlyeq \dfrac{1}{\log(N)^a}$	$\succcurlyeq \dfrac{\Lambda(N)}{\log(N)}$
$\preccurlyeq \dfrac{1}{\log^a\big(\log(N)\big)}$	$\succcurlyeq \dfrac{\Lambda(N)}{\log(N)^C}$ *(pour un certain C)*
$\preccurlyeq \dfrac{1}{(\log\ldots\log\log N)^a}$	$\succcurlyeq \dfrac{\Lambda(N)}{N^\epsilon}$ *, où ϵ peut être choisi arbitrairement petit.*

Ces estimées concernant le profil isoperimetrique sont connues pour les groupes polycycliques non virtuellement nilpotents (première ligne du tableau avec $a = 1$), les groupes à croissance sous-exponentielle (première ligne), les produits en couronne de la forme $F \wr N$, avec F un groupe fini et N un groupe nilpotent d'exposant de croissance d (première ligne avec $a = 1/d$), les produits en couronne itérés $F \wr (F \wr N)$, avec F et N comme précédemment (deuxième ligne avec $a = 1/d$), les groupes résolubles en général (troisième ligne).

Interactions avec l'algèbre

Un des principaux objectifs de la théorie géométrique des groupes est de donner des liens entre les propriétés algébriques et géométriques des groupes. Dans cette approche, le résultat le plus important connu et le fait qu'un groupe de type fini possède un profil de séparation borné si et seulement si il est virtuellement libre, voir Benjamini, Schramm & Timár [19] et Hume & Mackay [69].

Hume, Mackay & Tessera ont montré que tout groupe nilpotent possède un profil de Poincaré équivalent à $n^{\frac{d-1}{d}}$, où d est l'exposant de croissance du groupe [68]. Notre contribution majeure est un énoncé réciproque, dans le cas des groupes résolubles :

Théorème E (Théorème 6.0.7, Partie II)**.** *Soit G un groupe résoluble de type fini. S'il existe $\epsilon \in (0,1)$ et $c > 0$ tels que pour tout entier n suffisamment grand, on a*

$$\text{sep}_G(n) \leq cn^{1-\epsilon},$$

alors G est virtuellement nilpotent.

En combinant cet énoncé avec le calcul du profil des réseaux cocompacts des espaces hyperboliques [19], ainsi que le résultat de plongement de Bonk & Schamm [22], on obtient le corollaire suivant :

Corollaire F. *Soit G un groupe résoluble de type fini. S'il existe une application régulière de G vers un groupe hyperbolique de type fini, alors G est virtuellement nilpotent.*

Ce corollaire avait déjà été obtenu par Hume & Sisto [70, Corollary 1.3] dans le cas des plongements grossiers, avec une démonstration complètement différente.

Profils de séparation locaux

Les méthodes de la partie II peuvent également être mises à profit dans le contexte des composantes infinies de percolation de $\mathbf{Z}^d$, et plus généralement pour une classe importante de graphes à croissance polynômiale, appeleś graphes polynomiaux. Grossièrement, on appelle un graphe (d_1, d_2)-*polynomial* si, d'une part, sa croissance volumique est au plus en n^{d_2}, et d'autre part sa dimension isopérimétrique au moins égale à d_1. Etant donné qu'une composante de percolation contient presque sûrement les sous-ensembles réalisant le supremum du profil de séparation, il est davantage pertinent dans ce contexte d'introduire une version *locale* du profil de séparation, défini ainsi, pour tout sommet v d'un graphe G :

$$\text{sep}_G^v(n) := \sup_{F < B_G(v,r), |B_G(v,r)| \leq n} |F| \cdot h(F).$$

Dans la partie II, on montre que $\text{sep}_G^v(n)$ est borné inférieurement par une fonction du type n^α, pour tout sommet dans le cas des graphes polynômiaux, et pour les sommets exponentiellement proches de l'origine dans le cas de la percolation de $\mathbf{Z}^d$, comme l'indiquent les énoncés ci-dessous.

Théorème G (Théorème 6.0.3, Partie II)**.** *Soit G un graphe (d_1, d_2)-polynomial. Alors pour tout $\eta \in (0,1)$ il existe $c > 0$ tel que pour tout sommet v et tout entier n :*

$$\text{sep}^v(n) \geq cn^{(1-\eta)\frac{d_1^2(d_1-1)}{d_2^3}}.$$

Théorème H (Théorème 6.0.4, Partie II)**.** *Soit $\mathcal{C}_\infty$ une composante de percolation en phase supercritique de $\mathbf{Z}^d$. Alors, pour tout $\eta \in (0,1)$ il existe presque sûrement $c > 0$ tel que pour tout entier n suffisamment grand, si $\|x\|_\infty \leq \exp\left(n^{(1-\eta)\frac{d}{d-1}}\right)$, alors on a :*

$$\mathrm{sep}^x_{\mathcal{C}_\infty}(n) \geq cn^{\frac{d-1}{d}}.$$

L'inclusion dans $\mathbf{Z}^d$ prouve que cette borne inférieure est optimale.

Introduction (English language)

Given two metric spaces, it is a very natural question to wonder whether one can be embedded in the other, in a way that respects the distances. For graphs, a first obvious interpretation of this question is to ask for the existence of a strict graph embedding, *i.e.* an injective map on the vertices that preserves the edges. However, in the context of geometric group theory, it is more natural to consider more flexible notions of embeddings like quasi-isometric and coarse embeddings, since the geometric interpretation of finitely generated groups is only valid up to bilipschitz maps. Here, we will be interested in regular maps as defined by Benjamini, Schramm and Tímár in [19]: maps that are Lipschitz at large scale and such that the preimages of singletons have a uniformly bounded cardinality. This is a loose notion of embedding: in particular, quasi-isometric and coarse embeddings are regular maps (if the initial graph is connected).

It is usually a difficult question to decide whether one space can be embedded in another. To answer positively, one usually has to exhibit an embedding. To answer negatively, one needs to find an obstruction to the existence of such an embedding. An important idea of modern geometry is to associate to every space a data, belonging to a set endowed with a partial order (usually a number or a function), that will be compatible with the notion of embeddings we have chosen. This is called a *monotone invariant*, and it is then able to give obstructions to their existence. In the case of regular maps, few such invariants are known: volume growth, asymptotic dimension, and more recently, separation and Poincaré profiles. Volume growth and asymptotic dimension are very coarse, hence these profiles have great interest.

The **separation profile** was introduced by Benjamini, Schramm & Timár [19]. As remarked by Hume [67], the separation profile of an (infinite) graph G at $n \geq 0$ can be defined by

$$\mathrm{sep}_G(n) = \sup\left\{|V\Gamma|\, h(\Gamma) \colon \Gamma \subset G, |V\Gamma| \leq n\right\},$$

where $h(\Gamma)$ denotes the Cheeger constant of the graph Γ. Hume, Mackay and Tessera generalized this profile by defining, for any $p \in [0, \infty]$ the **L^p-Poincaré profile** of an (infinite) graph G by:

$$\Pi_{G,p}(n) = \sup\left\{|V\Gamma|\, h_p\left(\Gamma\right) : \Gamma \subset G, |V\Gamma| \leq n\right\},$$

where $h_p\left(\Gamma\right)$ denotes the L^p-Cheeger constant of the graph Γ. For graphs of bounded degree, the L^1-Poincaré profile and the separation profile are equivalent up to constants.

Prescription of high separation profiles

Indeed, when such an invariant takes few values, it is less relevant since in many cases it won't be able to give an obstruction. It is clear from the definition that any Poincaré profiles is least constant and at most linear. It is then natural to ask what are the possible profiles within this range. This issue is, in some sense, orthogonal to the initial question of the existence of embeddings: it asks the finesse of the invariant.

We already know that the separation profile can have variety of behaviour. Indeed, hyperbolic groups can have a constant profile (trees [19, Theorem 2.1.]), a logarithmic profile (Fuchsian groups [19, Proposition 4.1.]), a power profile (lattices in hyperbolic spaces [19, Proposition 4.1.], [68, Theorem 12]). Among amenable groups, we know that one can find arbitrary small unbounded profiles [69, Theorem 1.4] (up to subsequence), power profiles (nilpotent groups [68, Theorem 7]), and profiles bounded by $\frac{n}{(\log n)^2}$ and $\frac{n}{\log n}$ (polycyclic groups [80]). Concerning the prescription (up to constants) of separation profiles, we can mention two main results:

- the prescription of low profiles by Hume and Mackay [69], with lacunary hyperbolic groups from [97] (profiles arbitrarly low, below log).
- the prescription of medium profiles, mainly solved by Hume, Mackay and Tessera [68], with groups acting on Bourdon-Pajot buildings [24] (profiles $\simeq n^\alpha$ for any α in a dense subset of $(0,1)$).

Our main contribution solves this question for high separation and Poincaré profiles: profiles from $\frac{n}{(\log\log n)^a}$ (for any positive a) to n (not attained), see Theorem A. These examples are amenable groups constructed by Brieussel and Zheng [29]. This shows that amenable groups can have a variety of behaviours with respect to Poincaré profiles. Moreover, all our examples have exponential growth and asymptotic dimension one, which shows that those profiles are not redundant with respect to these invariants.

Theorem A (Theorem 1, Part III.)**.** *There exists two universal constants κ_1 and κ_2 such that the following is true. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing and $\lim_\infty \rho = \infty$. We assume that ρ is injective and that there exists some $\alpha > 0$ such that $\frac{\rho^{-1}(x)}{\exp(x^\alpha)}$ is non-decreasing. Then, there exists a finitely generated elementary amenable group Δ of exponential growth and of asymptotic dimension one such that for any $p \in [1,\infty)$,*

$$\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \text{for any } n,$$
$$\text{and} \quad \Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\rho(\log n)} \quad \text{for infinitely many } n\text{'s}.$$

Bounds on separation profiles

It is an important question to compare separation and Poincaré profiles with other known quantities. It gives the possibility to rely on known results to compute separation and Poincaré profiles. We give here our two major contributions to this problem. The first gives an upper bound on Poincaré profiles, and the second a lower bound on the separation profile.

Compression of embeddings We can start by a theorem giving an upper bound from compression in L^p spaces, that it at the origin of the upper bound on Theorem A. We define the **compression** of a 1-Lipschitz map $f\colon G \to L^p$ as

$$\rho_f(t) = \inf\left\{\|f(g) - f(h)\|_p \mid d_G(g,h) \geq t\right\}.$$

We showed the following theorem.

Theorem B (Theorem 3.3.11, Part III). *Let G be a graph of bounded degree. Then there exists two constants $c_1, c_2 > 0$, depending only on the maximum degree in G, such that if $f\colon VG \to L^p$ is a 1-Lipschitz map for some $p \in [1, \infty)$, then*

$$\Pi_{G,p}(n) \leq c_1 \frac{n}{\rho_f(c_2 \log n)}, \quad \textit{for all } n \geq 0.$$

This is optimal at least for product of trees (see [19]), and in Theorem A above.

Isoperimetric profiles In Part II, we give comparison statements between the separation profile and the isoperimetric profile, defined by:

$$\Lambda(n) = \inf\left\{\frac{|\partial F|}{|F|} : F \subset VG,\ |F| \leq n\right\},$$

where ∂F is the boundary of a subset $F \subset VG$, defined among other possibilities by the set of vertices of G at distance 1 from F. We detail here some examples.

Theorem C (Theorem 6.0.2, Part II). *Let G be a graph of bounded degree such that $\frac{K_1}{n^{1/d}} \leq \Lambda(n) \leq \frac{K_2}{n^{1/d}}$ for some constants K_1 and K_2, then, $\exists K_3 > 0$ such that for all n, $\frac{\mathrm{sep}(n)}{n} \geq \frac{K_3}{n^{1/d}}$.*

This theorem can be used on Cayley graphs of nilpotent groups (for which a sharp upper bound was already given by Hume, Mackay & Tessera [68]), but the method applies also to other type of graphs, such as pre-fractal Sierpinski carpets (see Theorem 9.1.4, Part II). They can also be applied to graphs with logarithmic isoperimetric profile, we obtained the following statement.

Theorem D (Theorem 6.0.6, Part II). *Let G be a Cayley graph.*

If, for some $a > 0$, $\Lambda(N)$ is ...	*then, for infinitely many N's, $\frac{\mathrm{sep}(N)}{N}$ is ...*
$\preccurlyeq \frac{1}{\log(N)^a}$	$\succcurlyeq \frac{\Lambda(N)}{\log(N)}$
$\preccurlyeq \frac{1}{\log^a\left(\log(N)\right)}$	$\succcurlyeq \frac{\Lambda(N)}{\log(N)^C}$ *(for some C)*
$\preccurlyeq \frac{1}{(\log\ldots\log\log N)^a}$	$\succcurlyeq \frac{\Lambda(N)}{N^\epsilon}$ *, where ϵ can be arbitrarily small*

These estimates on the isoperimetric profile are known for polycyclic groups which are not nilpotent (first row of the table with $a = 1$), groups with intermediate growth (first row), wreath products $F \wr N$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d (first row with $a = 1/d$), iterated wreath products $F \wr (F \wr N)$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d (second row with $a = 1/d$), solvable groups in general (third row).

Interaction with algebra

One of the main pupose of geometric group theory is to draw links between algebraic and geometric properties of groups. In this direction, the more important result is the fact that a finitely generated group has a bounded separation profile if an only of it is virtually free, see Benjamini, Schramm & Timár [19] and Hume & Mackay [69].

Hume, Mackay and Tessera showed that every nilpotent group has a Poincaré profile equivalent to $n^{\frac{d-1}{d}}$, where d is the volume growth rate of the group [68]. Our main contribution in this area is a reciprocal statement, among solvable groups:

Theorem E (Theorem 6.0.7, Part II). *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0,1)$ and $c > 0$ such that for any large enough integer n we have*

$$\mathrm{sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

Combining with the computaton of profiles of cocompact lattices in hyperbolic spaces [19] and Bonk & Schamm's embedding result [22], it has the following corollary.

Corollary F. *Let G be a finitely generated solvable group. If there exists a regular map from G to a finitely generated hyperbolic group, then G is virtually nilpotent.*

This corollary was already obtained by Hume & Sisto [70, Corollary 1.3] in the case of coarse embeddings, with a completely different proof.

Local separation profiles

The methods of Part II also yield results on the infinite percolation components of $\mathbf{Z}^d$, and more generally on a large class of graphs of polynomial growth, called polynomial graphs. Roughly speaking, a (d_1, d_2)-*polynomial graph* is a graph of volume growth bounded by n^{d_2} and of isoperimetric dimension at least d_1. Since the percolation component always includes arbitrary large balls, it is more interesting to introduce a local variant of the separation profile in this context, namely the *local separation at v*, where v is a vertex of the graph:

$$\mathrm{sep}_G^v(n) := \sup_{F < B_G(v,r),\, |B_G(v,r)| \leq n} |F| \cdot h(F).$$

In that case, we show that $\mathrm{sep}_G^v(n)$ is bounded below by a function of the type $n^{-\alpha}$, for every vertices in the polynomial case, and for vertices that stay exponentially close to the origin in the $\mathbb{Z}^d$ percolation case:

Theorem G (Theorem 6.0.3, Part II)**.** *Let G be a (d_1, d_2)-polynomial graph. Then for any $\eta \in (0,1)$ there exists $c > 0$ such that for any vertex v and any integer n:*

$$\mathrm{sep}^v(n) \geq cn^{(1-\eta)\frac{d_1^2(d_1-1)}{d_2^3}}$$

Theorem H (Theorem 6.0.4, Part II)**.** *Let $\mathcal{C}_\infty$ be a supercritical phase percolation cluster of $\mathbb{Z}^d$. Then for any $\varepsilon \in (0,1)$ there exists almost surely $c > 0$ such that for n large enough, if $\|x\|_\infty \leq \exp\left(n^{(1-\varepsilon)\frac{d}{d-1}}\right)$, then we have:*

$$\mathrm{sep}^x_{\mathcal{C}_\infty}(n) \geq cn^{\frac{d-1}{d}}$$

The inclusion in $\mathbf{Z}^d$ shows that this lower bound is optimal.

Part I

Separation profile and Poincaré profiles: a survey

This survey summarizes what we know on separation profile and Poincaré profiles. In Chapter 1, we will consider separation theorems that appeared before the formal definition of the separation profile in 2012, and its generalization to Poincaré profiles in 2019. Chapter 2 concerns coarse embeddings. We detail in particular two other preceding monotone invariants: volume growth and asymptotic dimension. In Chapter 3, we give definitions, first properties, and give a *toolbox* for seeking upper or lower bounds on separation/Poincaré profiles.

In Chapters 4 and 5, we detail the properties of separation and Poincaré profiles in the contexts of hyperbolic and amenable groups, respectively. Some progress have been done concerning relations between hyperbolicity and Poincaré profile. For amenability, nothing is clear at the moment. In both cases, state of research cannot tell if hyperbolicity or amenability has a *deep* link with Poincaré profiles. However, this is a common way of classifying groups in geometric group theory, and known theorems often concern one or the other of these families.

Profil de séparation et profils de Poincaré

Ce survol synthétise l'état de l'art du profil de séparation et des profils de Poincaré. Dans le chapitre 1, nous considérons les théorèmes de séparation qui sont parus avant la définition du profil de séparation en 2012. Le chapitre 2 porte sur les plongements grossiers; on y détaille en particulier deux invariants monotones antérieurs aux profils étudiés ici: la croissance du volume des boules et la dimension asymptotique. Dans le chapitre 3, nous détaillons les définitions du profil de séparation et des profils de Poincaré. Nous donnons ensuite leurs premières propriétés ainsi qu'une *boîte à outils* permettant d'obtenir des bornes inférieures et supérieures sur ces profils.

Dans les chapitres 4 et 5, nous explorons les propriétés du profil de séparation et des profils de Poincaré, respectivement dans les contextes des groupes hyperboliques et moyennables. Des progrès ont étés réalisés récemment au sujet des liens entre l'hyperbolicité et les profils de Poincaré. Cependant, force est de constater que rien n'est clair en ce qui concerne la moyennabilité. Dans les deux cas, l'état actuel des connaissances ne permet pas de donner de lien *profonds* entre l'hyperbolicité ou la moyennabilité et les profils de Poincaré. Quoi qu'il en soit, c'est un moyen commode de classer les groupes en théorie géométrique des groupes et en outre les théorèmes connus s'appliquent souvent à l'une ou l'autre de ces familles.

Chapter 1

Prehistory of the separation profile

Separator theorems are a useful tool for the design of efficient combinatorial algorithms by the divide-and-conquer paradigm. The starting point seems to be on planar graphs and trees. For trees, this problem was solved a long time ago: for example, Jordan showed in [75] that any tree on n vertices can be cut into pieces of size at most $\frac{2}{3}n$ by removing a single vertex. We will detail the important results on planar graphs, and their generalizations, in a first section. Then, we will detail the work of Miller, Teng, Thurston & Vavasis on overlap graphs that generalizes earlier work on planar graphs, with a more geometric approac.

The notion of separation that is involved in the definition of the separation profile is the following.

Definition 1.0.1. [19] For a finite graph G, let us write $L(G)$ the size of any largest component of G. For any $\epsilon \in (0,1)$, we define the **ϵ-cut** of G as

$$\operatorname{cut}^{\epsilon} G = \min\left\{|S| \colon S \subset VG \text{ and } |L(G \setminus S)| \leq \epsilon\,|VG|\right\}.$$

(we omit the "ϵ" for $\epsilon = 1/2$.)

Note that this notion of separation is quantitative: we are interested in subsets that cut a graph in relatively *big* pieces. This should not be confused with the well-studied theory of *graph separators* (see [108]), where there is no requirement on the size of connected components.

1.1 The planar separator theorem

For planar graphs, the starting point of quantitative separation seems to be a theorem of Ungar:

Theorem 1.1.1 ([117])**.** *Let G be a planar graph on n vertices. Then for any $\epsilon \in (\frac{2}{n}, 1)$,*

$$\operatorname{cut}^{\epsilon}(G) \leq \frac{12}{\sqrt{\epsilon}}(\log \epsilon n)^{\frac{3}{2}}\sqrt{n}.$$

The celebrated theorem of Lipton and Trajan improved this result:

Theorem 1.1.2 (the planar separator theorem, [83]). *Let G be a finite planar graph. Then*

$$\mathrm{cut}^{\frac{2}{3}}(G) \leq 2\sqrt{2n}.$$

See [41, 53, 90] for other proofs or generalizations. An $\mathcal{O}(n)$ algorithm for finding the partition follows as a consequence of this separator theorem. These results had profound consequences in the solution of a number of diverse problems. The major one is a dramatic generalization of the nested dissection technique of Alan George for the solution of sparse positive-definite symmetric systems of linear equations. In [84], other applications were developed. We refer the reader to the enlightening introduction of [91] for a presentation of different kind of applications. Later, Alon, Seymour and Thomas improved the planar separator theorem:

Theorem 1.1.3 ([4]). *Let G be a graph with n vertices, drawn in the sphere $\mathbf{S}_2$. Then there is a simple closed curve F in $\mathbf{S}_2$, meeting the drawing of F only in vertices, such that*

$$n_2 + n_3/2 \leq \frac{2}{3}n, \quad n_1 + n_3/2 \leq \frac{2}{3}n,$$

and

$$n_3 \leq \frac{3}{2}\sqrt{2n},$$

where F passes through n_3 vertices and the two open discs bounded by F contain n_1 and n_2 vertices, respectively.

This improves the constant of the planar separator theorem from $2\sqrt{2}$ to $\frac{3}{2}\sqrt{2}$, and yields to a a shorter proof of the original theorem of Lipton and Trajan. In [3], they generalize Theorem 1.1.2 to nonplanar graphs with a fixed excluded minor. They prove the following statement:

Theorem 1.1.4 ([3]). *Let G be a graph with n vertices and m edges, with no minor isomorphic to K_h, the complete graph of h vertices. Then*

$$\mathrm{cut}^{\frac{2}{3}}(G) \leq h\sqrt{hn}.$$

An $\mathcal{O}((m+n)\sqrt{hn})$-time algorithm for finding a cutset follows as a consequence of this separator theorem. Planar graphs cannot contain a graph isomorphic to K_5 as a minor, hence this generalizes the planar separator theorem 1.1.2.

Intense study of the separation of graphs of positive genus was also made. Apparently, the first breakthrough was this theorem by Albertson & Hutchinson:

Theorem 1.1.5 ([1]). *Let G be a graph with n vertices, of positive genus g. Then there exists $S \subset VG$ of size at most $\sqrt{2n}$ such that $G \setminus S$ is a graph of genus at most $g-1$.*

This immediately implies a separation theorem on graphs of a given genus (using the planar separator theorem 1.1.2), that was improved later by Gilbert, Hutchinson & Tarjan to the following theorem: (see also [71])

Theorem 1.1.6. *[56, Theorem 4] Let G be a graph with n vertices, of positive genus g. Then*

$$\mathrm{cut}^{\frac{3}{2}}(G) \leq 6\sqrt{gn} + 2\sqrt{2n} + 1.$$

1.2 The geometric separator theorem

The aforementioned approaches were mostly combinatorics. Miller, Teng, Thurston & Vavasis developed a geometrical approach in [91, 92, 93], based on the notion of radon point of a family of points. We detail here their definitions and statements, and give some consequences.

Definition 1.2.1. [91, Definition 2.1] Let $P = \{p_1, \ldots, p_n\}$ be some points in $\mathbf{R}^d$. A **k-ply neighbourhood system** for P is a set $\{B_1, \ldots, B_n\}$ of closed balls such that each B_i is centred at p_i and no point $p \in \mathbf{R}^d$ is strictly interior to more than k balls from B.

For each positive real α, if B is a ball of radius r in $\mathbf{R}^d$, then αB denotes the ball with the same center as B but of radius αr.

Definition 1.2.2. [91, Definition 2.3] Let $\alpha \geq 1$ and let $\{B_1, \ldots, B_n\}$ be a k-ply neighbourhood system for $P = \{p_1, \ldots, p_n\}$. The corresponding **(α, k)-overlap graph** is the undirected graph with vertices $V = \{1, \ldots, n\}$ and edges $E = \{\{i, j\} \mid B_i \cap (\alpha B_j) \neq \emptyset \text{ and } (\alpha B_i) \cap B_j \neq \emptyset\}$.

This is the geometric separator theorem:

Theorem 1.2.3 (the geometric separator theorem)**.** *[91, Theorem 2.4] Let G be an (α, k)-overlap graph for some fixed d, on n vertices. Then there exists $q(\alpha, k, d)$ such that*

$$\operatorname{cut}^{\frac{d-1}{d-2}}(G) = \mathcal{O}\left(\alpha \cdot k^{\frac{1}{d}} \cdot n^{\frac{d-1}{d}} + q(\alpha, k, d)\right).$$

In the case where $\alpha = 1$ and $k = 1$, and no two balls in the neighbourhood system have a common point in their interior, we have the family of graphs known as sphere-packings.

From Andreev and Thurston [7, 6, 116], each triangulated planar graph is isomorphic to a 2-dimensional sphere-packing graph. Therefore the geometric separator theorem 1.2.3 generalizes the planar separator theorem 1.1.2.

Definition 1.2.4. Let $G = (V, E)$ be a graph, and let π be an injective map from V to $\mathbf{R}^d$. We say π is an **embedding of density** α if the following inequality holds for all vertices v in G. Let u be the closest vertex to v. Let w be the farthest vertex from v that is connected to v by an edge. Then

$$\frac{\|\pi(w) - \pi(v)\|}{\|\pi(u) - \pi(v)\|} \leq \alpha.$$

We say G is an **α-density graph** in $\mathbf{R}^d$ if there exists an embedding of G in $\mathbf{R}^d$ with density α.

Recall moreover that the **aspect ratio** of a simplex can be defined, among other possibilities, by the ratio of the radius of the smallest containing sphere and the biggest inscribed sphere.

This theorem follows from the geometric separation theorem 1.2.3. Given the 1-skeleton G of a simplicial complex K, we shall say that a vertex of G is an **exterior vertex** if it lies in the boundary of K.

Theorem 1.2.5. *[91, Theorem 3.3] Let G be a graph on n vertices.*

- *If G is the* 1*-skeleton of a simplicial complex K in $\mathbf{R}^d$ with bounded aspect ratio, letting $\tilde{n}$ be the number of exterior vertices of G, then*

$$\operatorname{cut}^{\frac{d-1}{d-2}}(G) = \mathcal{O}\left(n^{\frac{d-1}{d}} + \tilde{n}\right).$$

- *If G has bounded density α in $\mathbf{R}^d$, then*

$$\operatorname{cut}^{\frac{d-1}{d-2}}(G) = \mathcal{O}\left(\alpha n^{\frac{d-1}{d}}\right).$$

Let us give the idea of how this theorem can be deduced from the geometric separation theorem 1.2.3. First, given an embedding of a graph $\pi\colon G \to \mathbf{R}^d$ of dentity α in $\mathbf{R}^d$, Miller, Teng, Thurston and Vavarsis showed that $\pi(F)$ is a spanning subgraph of an $(2\alpha, 1)$-overlap graph. Second, the bounded aspect ratio hypothesis on the simplicial complex K gives an upper bound on the degree of the vertices. Also, it implies that the ratio of the longest edge over the shortest edge of any simplex in K is also bounded. Up to removing the vertices that are on the external boundary of K we obtain a control on the density of the graph G. We conclude using the first point.

Chapter 2

Geometric obstructions to a coarse embedding

There are few invariants which can provide a general geometric obstruction to a coarse embedding, of which the most commonly studied are volume growth and asymptotic dimension. We start by giving the definition of coarse embeddings, then we will define these two invariants and explicit their monotonicity properties.

2.1 Coarse embeddings

The concept of coarse embedding was introduced by Gromov [62, p.218], in his investigation of the Novikov conjecture (1965).

2.1.1 Definition

Definition 2.1.1. [62, 106] Let X and Y be two metric spaces. A map $f\colon X \to Y$ is called a **coarse embedding** if there exists unbounded non-decreasing functions $\rho_1, \rho_2\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $x, x' \in X$,

$$\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')).$$

The function ρ_1 and ρ_2 are called the **distortion** functions.

For example, any subspace (endowed with the induced metric) coarsely embeds in the ambient space. In this case, the distortion function can be taken equal to the identity. Another example, different from the first, is the fact that any subgroup of a finitely generated group coarsely embeds in the ambient group, each one endowed with his own word metric. This relies on the fact that two proper left-invariant metrics on a group are always coarsely equivalent. Thus, given a word metric on a group, its restriction to a subgroup is coarsely equivalent to any word-metric on the subgroup itself (see [42, Corollary 1.2]). Here, the distortion functions need *not* be linear in general, we refer the reader to Remark 3.1.32.

We can notice that quotient groups do not coarsely embed in the ambient group. For example, for any finitely generating group, there is a free group that factors to it. Indeed, any non virtually free finitely generating group cannot coarsely embed in a free group (see Theorem 4.1.1, Corollary 3.1.35 and Proposition 3.1.31).

2.1.2 Coarse embeddings into a Hilbert space

Coarse embeddings into Hilbert spaces have been widely studied. Admitting a coarse embedding to an Hilbert space is an important property, even more since Yu proved that any group admitting one satisfies the Novikov and coarse Baum-Connes conjectures; two important open questions in topology [119].

Negative type kernels

A well-known characterization involves negative type kernels:

Definition 2.1.2. A negative type kernel on a metric space X is a symmetric function $k\colon X \times X \to \mathbf{R}$ such that for all m-tuples $x_1, \ldots, x_m$ of points of X and all real scalars $a_1, \ldots, a_m$, such that $\sum a_i = 0$, one has

$$\sum a_i a_j k(x_i, x_j) \leq 0.$$

The basic example of a negative type kernel in a Hilbert space is $k(x, y) = \|x - y\|^2$. In some sense every negative type kernel is of this form (see [106, Theorem 11.15]).

Theorem 2.1.3. *[106, Theorem 11.16] Let X be a metric space. X can be coarsely embedded into a Hilbert space if and only if there exists a negative type kernel on X such that for any $R > 0$ the set $\{(x, y)\colon |k(x, y)| \leq R\}$ contains a set of the form $\{(x, y)\colon d(x, y) \leq r\}$ for some $r > 0$.*

Poincaré inequalities

Closer to our topic, the notion of expanders has been pointed out by Gromov as an obstruction for a metric space to coarsely embed into a Hilbert space. Recall [72] (see also [87]) that a sequence of **expanders** is a sequence of finite connected graphs (X_n) with bounded degree, satisfying the following Poincaré inequality for all $f \in \ell^2(X_n)$

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} |f(x) - f(y)|^2 \leq \frac{C}{|X_n|} \sum_{x \sim y} |f(x) - f(y)|^2, \tag{2.1}$$

for some constant $C > 0$, and whose cardinality $|X_n|$ goes to infinity when $n \to \infty$. An equivalent formulation in ℓ^p [89] can be used to prove that expanders do not coarsely embed into L^p for any $1 \leq p < \infty$. Inequality (2.1) should be compared with the definition of L^p-Cheeger constants, see Definition 3.1.7.

Having a coarsely embedded expander is an obstruction for a metric space to coarsely embed into a Hilbert space. The reciprocal is false, but it becomes true if we consider generalized expanders, see definition and theorem below.

Definition 2.1.4. • Let X be a metric space. Let K and r be positive numbers. We set $\Delta_r(X) = \{(x,y) \in X^2, d(x,y) \geq r\}$. We say that X is a **generalized** (K,r)**-expander** if there exists a symmetric probability measure μ supported on $\Delta_r(X)$ with the following property. For every map $F : X \to \mathbf{H}$ satisfying $|F(x) - F(y)| \leq d(x,y)$ for all $(x,y) \in \Delta_1(X)$, we have

$$\mathrm{Var}_\mu(F) := \sum_{x,y} |F(x) - F(y)|^2 \mu(x,y) \leq K^2.$$

• A sequence of finite metric spaces (X_n) is called a **sequence of generalized** K**-expander** if for every $n \in \mathbf{N}$, X_n is a (K, r_n)-expanders, where $r_n \to \infty$.

An example of a generalized expander that is not an expander is given in [8].

Theorem 2.1.5. *[113] A metric space does not coarsely embed into a Hilbert space if and only if it has a coarsely-embedded sequence of generalized expanders.*

2.1.3 Quasi-isometric embeddings

Restricting distortion functions to affine functions gives the notion of quasi-isometric embedding. The concept of quasi-isometry is especially important in geometric group theory, following the work of Gromov.

Definition 2.1.6. Let X and Y be two metric spaces. Let $\lambda > 0$ and $c \geq 0$. A map $f : X \to Y$ is called a (λ, c)**-quasi-isometric embedding** if for every $x, x' \in X$,

$$\lambda^{-1}(d_X(x,x') - c) \leq d_Y(f(x), f(x')) \leq \lambda d_X(x,x') + c.$$

If for some $\lambda > 0$ and $c \geq 0$, there exists a (λ, c)-quasi-isometric embedding from X to Y, we say that X **quasi-isometrically embeds** in Y.

We can mention Bonk and Schramm's embedding theorem.

Theorem 2.1.7. *[22] Let X be a Gromov hyperbolic geodesic metric space with bounded growth at some scale. Then there exists an integer d such that X quasi-isometrically embeds in the d-dimensional hyperbolic space $\mathbf{H}^d$.*

In their terminology, a metric space X has bounded growth at some scale, if there are constants r, R with $R > r > 0$, and $N \in \mathbf{N}$ such that every open ball of radius R in X can be covered by N open balls of radius r. We can mention that the conclusion of the original theorem is actually much stronger, we refer the reader to their paper for details.

2.2 Monotone invariants

2.2.1 Volume growth

Volume growth comes in the context of measured metric spaces. Graphs are naturally endowed with such a structure, by choosing the shortest path metric and the counting measure.

Definition 2.2.1. Let X be a metric space. Let us write $\mathcal{B}(X)$ the family of borel sets of X. A map $\nu\colon \mathcal{B}(X) \to [0,\infty]$ is called a **pseudo-measure** if for any borel sets $A, A' \in \mathcal{B}(X)$ satisfying $A \subset A'$, we have $\nu(A) \le \nu(A')$.

Definition 2.2.2. Let (X, d, ν) be a metric space endowed with a pseudo-measure. The growth type of the function $r \mapsto \nu(B(x,r))$ does not depend on $x \in X$. We define the **volume growth** of X as the growth type of any of these functions, denoted vol_X or γ_X. When we want to emphasize on the measure, we will denote it $\mathrm{vol}_{X,\nu}$.

Remark 2.2.3. The independence on the base point is nothing but the fact that we have, from the triangle inequality, the inclusion $B(y,r) \subset B(x, r + d(x,y))$ for any $x, y \in X$.

Defined as above, the volume growth is indeed only interesting when the measure of the balls do not depend too much on the centre. The good definition for this is the following:

Definition 2.2.4. Let (X, d) be a metric space. We will say that a pseudo-measure ν on X is **uniform at large scale** if there are increasing functions $f, g\colon \mathbf{R}_+ \to \mathbf{R}_+$ and $r_0 > 0$ such that

$$0 < f(r) \le \nu((B(x,r)) \le g(r),$$

for any $x \in X$ and $r \ge r_0$. Up to rescaling the metric, we will always assume that we have $r_0 = 1/2$.

The volume growth with respect to a uniform at large scale pseudo-measure is always the same as a "metric volume growth", where balls of radius one have volume one, see definition and lemma below.

Definition 2.2.5. Let (X, d) be a metric space. We can define a pseudo-measure ν_d on X, called the **metric pseudo-measure**, defined by:

$$\nu_d(A) = \min\left\{k \mid A \text{ can be covered with } k \text{ balls of radius } 1\right\},$$

for any $A \subset X$. We define the **metric volume growth** of (X, d) as the volume growth given by the pseudo-measure ν_d.

Lemma 2.2.6. *Let (X, d, ν) be a metric space endowed with a pseudo-measure ν that is uniform at large scale. Then*

$$\mathrm{vol}_{X,\nu} = \mathrm{vol}_{X,\nu_d}.$$

Proof. Let f and g be as in Definition 2.2.4. Let $B \subset X$ be a ball of radius r.

If $B_1, \ldots, B_k$ is a family of k balls of radius 1 covering B, then $\nu(B) \le \sum_{i=1}^{k} \nu(B_i) \le k g(1)$. This implies $\nu(B) \le g(1)\nu_d(B)$.

We shall say that a set $S \subset X$ is **strictly 1-separated** if for any $s, s' \in S$ satisfying $s \neq s'$, we have $d(s, s') > 1$. Let $S \subset B$ be a strictly 1-separated subset. Then the family of balls $B(s, 1/2)$ for $s \in S$ are pairwise disjoints. Thus, S satisfies

$$|S|\, f(1/2) \le \nu(B(x, r + 1/2)). \tag{2.2}$$

In particular, this implies that B admits a subset S_m that is maximal strictly 1-separated, *i.e.* satisfying that S_m is strictly 1-separated and is not contained in any other strictly 1-separated subset (one cannot add a new point). Then, it is clear that B can be covered by the balls of radius one centred at the points of S_m. This implies, from (2.2),

$$\nu_d(B) \leq |S_m| \leq f(1/2)^{-1}\nu(B(x, r + 1/2)),$$

which proves the announced result. □

The next theorem establishes the monotonicity of the volume growth with respect to coarse embeddings. The hypotheses are indeed satisfied by connected graphs of bounded degree. We start with a definition.

Definition 2.2.7. We shall say that a metric space X is **coarsely geodesic** if there exist two positive constants r_0 and K such that any pair of points $x, y \in X$ lies in a connected union of at most $Kd(x, y)$ balls of radius r_0.

Theorem 2.2.8. *Let X and Y be two metric spaces endowed with uniform at large scale pseudo-measures. We assume that X is coarsely geodesic and that there exists a coarse embedding $f\colon X \to Y$. Then*

$$\mathrm{vol}_X \leq \mathrm{vol}_Y\,.$$

Proof. From Lemma 2.2.6, we can assume without loss of generality that X and Y are endowed with their metric pseudo-measure. Let ρ_1 and ρ_2 be as in Definition 2.1.1. Since X is coarsely geodesic, we can assume without loss of generality that we have $\rho_2(r) \leq K\rho_2(r_0)r$, for any $r \geq r_0$. Let B be a ball in X of radius $r \geq r_0$. The set $f(B)$ is included in a ball of radius $\rho_2(r)$. Let then $B_1, \ldots, B_k$ be a family of balls of radius 1 covering $f(B)$, with $k \leq \mathrm{vol}_Y(\rho_2(r))$. Then the family $f^{-1}(B_1), \ldots, f^{-1}(B_k)$ covers B. Since ρ_1 is unbounded, there exists $s > 0$ such that $\rho_1(s) \geq 1$. Each $f^{-1}(B_i)$ is contained in a ball of radius s, and therefore can be covered with $\mathrm{vol}_X(s)$ balls of radius 1. This proves

$$\begin{aligned}\mathrm{vol}_X(r) &\leq \mathrm{vol}_X(s) \times \mathrm{vol}_Y(\rho_2(r)) \\ &\leq \mathrm{vol}_X(s) \times \mathrm{vol}_Y(K\rho_2(r_0)r).\end{aligned}$$

□

The interest of Theorem 2.2.8 is limited in geometric group theory, since many groups have exponential growth; for example, non compact hyperbolic groups, non nilpotent solvable groups.

2.2.2 Asymptotic dimension

The asymptotic dimension theory was founded by Gromov [62]. See [13] for a survey on this topic. Let X be a metric space and $r > 0$. We shall say that a family of subsets $\mathcal{U} \subset \mathcal{P}(X)$ is r**-disjoint** if for every distinct $U, V \in \mathcal{U}$, if $x \in U$ and $v \in V$, then $d(u, v) \geq r$. We shall say that $\mathcal{U}$ is **uniformly bounded** if there exists D such that each element of $\mathcal{U}$ has diameter at most D.

Definition 2.2.9. Let X be a metric space. We say that the **asymptotic dimension** of X does not exceed n and write it $\operatorname{asdim} X \leq n$ if for every $r > 0$, there exists r-disjoint families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of uniformly bounded subsets of X such that $\cup_{0 \leq i \leq n} \mathcal{U}_i$ covers X.

We say that X has asymptotic dimension n and write it $\operatorname{asdim} X = n$ if we have $\operatorname{asdim} X \leq n$, and not $\operatorname{asdim} X \leq n-1$.

For example, the asymptotic dimension of the d-dimensional Euclidean space is d, as for the hyperbolic space $\mathbf{H}_d$ [62]. More generally,

Theorem 2.2.10. *Every finitely generated hyperbolic group has finite asymptotic dimension.*

This result was announced in Gromov's book [62] and an explicit proof of more general results appear in [106] and [107].

The asymptotic dimension defines a monotone coarse invariant:

Theorem 2.2.11. *Let X, Y be two metric spaces such that there exists a coarse embedding $f \colon X \to Y$. Then* $\operatorname{asdim} X \leq \operatorname{asdim} Y$.

Proof. Let us write ρ_1 and ρ_2 as in Definition 2.1.1. Let $n \leq \operatorname{asdim} Y$. Let $r > 0$. Let $\mathcal{U}_0, \ldots, \mathcal{U}_n$ be $\rho_2(r)$-disjoint families of uniformly bounded subsets of Y that cover it. Let D be a bound on the diameter of each element of each $\mathcal{U}_i$. Then it is easy to check that each $f^{-1}(\mathcal{U}_i)$ is r-disjoint, is formed of $\rho_1^{-1}(D)$ bounded subsets of X, and that $f^{-1}(\mathcal{U}_1), \ldots, f^{-1}(\mathcal{U}_n)$ cover X. □

We can mention that the asymptotic dimension takes only a *countable* number of values; this is not a very subtle invariant. To conclude, we can give two important constructive theorems:

Theorem 2.2.12. *[106] Let X be a metric space with finite asymptotic dimension. Then X admits a coarse embedding into a Hilbert space.*

We shall say that a metric space X has **bounded geometry** if for every $\epsilon > 0$ and every $r > 0$, there is a c such that for every $x \in X$, the ball $B(x, r)$ contains no more than c disjoint balls of radius ϵ.

Theorem 2.2.13. *[43] Let X be a metric space with bounded geometry, whose asymptotic dimension does not exceed n. Then X admits a coarse embedding into the product of $n+1$ locally finite trees.*

Chapter 3

Separation profile and L^p-Poincaré profiles

3.1 Definitions

In this section, we give the basic definitions of separation profile and Poincaré profiles. We give comparison theorems, following [68, Sections 6 and 7].

A graph will always be considered as a set of vertices endowed with the shortest path metric. In particular, we ignore the "points" of the edges.

3.1.1 Separation profile

We start with the definition of the separation profile. We give two equivalent definitions. The first is the original from Benjamini, Schramm & Timár [19], it uses the notion of "cut" of Definition 1.0.1, that we recall here. The second is an equivalent definition from Hume [67], that uses the notion of Cheeger constant.

Definition 3.1.1. [19]For a finite graph Γ, let us write $L(\Gamma)$ the size of any largest component of Γ. For any $\epsilon \in (0,1)$, we define the **ϵ-cut** of Γ as

$$\operatorname{cut}^{\epsilon} \Gamma = \min \left\{ |S| \colon S \subset V\Gamma \text{ and } |L(\Gamma \setminus S)| \leq \epsilon \, |V\Gamma| \right\}.$$

(we omit the "ϵ" for $\epsilon = 1/2$.)

For any (infinite) graph G, the **separation profile** of G is defined as

$$\operatorname{sep}_G(n) = \sup \left\{ \operatorname{cut}^{1/2} \Gamma \colon \Gamma \subset G \text{ and } |V\Gamma| \leq n \right\}.$$

As noticed in [19], the ratio ϵ of the cut is not crucial when we are interested in asymptotic behaviours. See Lemma A.1.7 for an explicit estimate of the constants involved.

To introduce the second definition, we first define Cheeger constants of finite graphs.

Definition 3.1.2. [67] For a finite graph Γ, we define the **(combinatorial) Cheeger constant** of Γ as:

$$h(\Gamma) = \inf \frac{|\partial A|}{|A|},$$

where the infimum is taken on the subsets A of $V\Gamma$ of size at most $\frac{|V\Gamma|}{2}$, and ∂A is the boundary of A, that we define as the set of vertices of Γ that are at distance 1 from A.

We define the **Cheeger-separation profile** of an (infinite) graph G as:

$$\mathrm{sep}^h_G(n) = \sup \left\{ |V\Gamma| \, h(\Gamma) \mid \Gamma \subset G \text{ and } |V\Gamma| \leq n \right\}.$$

These two separation profiles are closely equivalent:

Theorem 3.1.3. *[67] Let G be an infinite graph. Then, for any $n \geq 2$,*

$$\frac{1}{4} \mathrm{sep}_G(n) \leq \mathrm{sep}^h_G(n) \leq 4 \, \mathrm{sep}_G(n)$$

Proof. Given a finite graph Γ, we can remove a subset of size $\mathrm{cut}\,\Gamma$ and obtain connected components of size at most $|\Gamma|/2$. Then, making well-chosen unions of these connected components, it is easy to obtain $\mathrm{cut}\,\Gamma \geq \frac{1}{4} h(\Gamma)\,|\Gamma|$, and the right-hand side follows. (see [67, Proposition 2.2])

Using an iteration process, Hume shows in [67, Proposition 2.4], the following lemma: for any graph Γ with at least 2 vertices, there exists a subgraph $\Gamma' \subset \Gamma$ satisfying $|\Gamma'| \, h(\Gamma') \geq \frac{1}{4} \mathrm{cut}\,\Gamma$. It is straightforward to see that the left-hand side follows from this lemma. □

One may notice that we use here the vertex-boundaries, which is quite unusual (edge boundaries are more common), but give statements that do not depend on the maximal degree of the involved graph. In geometric group theory, we are used to working with graphs for which the degrees are bounded by absolute constants; in that case edge and vertex boundaries only differ by a constant factor. This is what happens in Part II, where we will use the edge-boundary instead of the vertex boundary. Nevertheless, in Part III, we will consider families of *model graphs* for which the degree tends to infinity, we will then have to use this precise definition of boundary.

3.1.2 Poincaré profiles

The separation profile can be naturally generalized to some other metric spaces than graphs: for example, in the case of locally compact groups having cocompact lattices, it can be defined as the separation profile of any of them; the invariance under quasi-isometry (see Corollary 3.1.35) implies that this is well-defined. However, a more satisfactory generalization was found by Hume, Mackay and Tessera in [68], called L^p-Neumann-Poincaré profiles, or for simplicity L^p-Poincaré profiles. They managed to define this profile in a very large framework including graphs, locally compact groups, Riemannian manifolds. This is an example of a fruitful interaction between coutinuous and discrete settings (see [76] for more on this subjet).

The general framework of Poincaré profiles is the following.

Definition 3.1.4. A **standard metric measure space** is a metric measure space (X, d, μ) with the following properties:

(i) (X, d) is a complete and separable metric space.

(ii) μ is a non-trivial, locally finite, Borel measure.

(iii) X has **bounded packing on large scales**: there exists $r_0 \geq 0$ such that for all $r \geq r_0$, there exists $K_r > 0$ such that

$$\forall x \in X, \ \mu(B(x, 2r)) \leq K_r \mu(B(x, r)).$$

We then say that X has **bounded packing on scales** $\geq r_0$.

(iv) X is k**-geodesic** for some $k > 0$: for every pair of points $x, y \in X$ there is a sequence $x = x_0, \ldots, x_n = y$ such that $d(x_{i-1}, x_i) \leq k$ for all i and $d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$.

Up to rescaling the metric we will always assume that X is 1-geodesic and has bounded packing on scales ≥ 1.

Definition 3.1.5. We will say that a subset of a standard metric measure space is **1-thick** if it is a union of closed balls of radius 1. Axioms (i) and (iii) imply in particular that a non-empty 1-thick subset has a positive measure. Such a subset $Z \subset X$ will be equipped with the **induced measure** and the **induced 1-distance**:

$$d_Z(z, z') = \inf \left\{ \sum_{i=1}^{n} d_X(z_{i-1}, z_i) \right\},$$

where the infimum is taken over all sequences $z = z_0, \ldots, z_n = z'$, such that each z_i is an element of Z, and $d_X(z_i, z_{i+1}) \leq 1$ for every i (this distance takes values in $[0, \infty]$).

Remark 3.1.6. In the case of bounded degree graphs, we will always choose d to be the shortest path metric and μ to be the (vertex) counting measure. Then, 1-thick subspaces are 1-thick subgraphs equipped with the counting measure and their own shortest path metric.

In a locally compact group G with compact generating set K, we equip G with a Haar measure (which is unique up to scaling) and the word metric $d = d_K$.

Definition 3.1.7. Let (X, d) be a metric space and let $a > 0$. Given a function $f : X \to \mathbf{R}$, we define its **upper gradient at scale** a as

$$|\nabla_a f|(x) = \sup_{y, y' \in B(x,a)} |f(y) - f(y')|.$$

Let (Z, d, ν) be a finite measure 1-thick subspace of a standard metric measure space, and fix a scale $a > 0$. We define the L^p**-Cheeger constant at scale** a of Z as

$$h_{a,p}(Z) = \inf_f \frac{\|\nabla_a f\|_p}{\|f\|_p},$$

where the infimum is taken over all $f \in L^p(Z, \nu)$ such that $f_Z := \fint_Z f d\nu = 0$ and $f \not\equiv 0$.

We finally give the definition of Poincaré profiles.

Definition 3.1.8. Let (X, d, μ) be a standard metric measure space, and fix some number $a \geq 2$. We define the **L^p-Poincaré profile at scale** a of X as

$$\Pi_{X,a,p}(v) = \sup \{\mu(A)h_{a,p}(A)\},$$

where the supremum is taken over all 1-thick subsets $A \subset X$ satisfying $\mu(A) \leq v$.

The choice of a is not important, as it is shown by the next proposition.

Proposition 3.1.9. *[68, Corollary 4.5] Let (X, d, ν) be a standard metric measure space. Then for all $a, a' \geq 2$ and all $p \in [1, \infty)$ we have*

$$\Pi_{X,a,p} \simeq_{a,a'} \Pi_{X,a',p}.$$

Then, we can denote by $\Pi_{X,p}$ the growth type of any $\Pi_{X,a,p}$ for some $a \geq 2$. This is what we refer to as "the" L^p-Poincaré profile of X. The most important Poincaré profiles are the L^1, L^2 and L^∞-Poincaré profiles. Each of them has a particular interpretation, that we consider now.

L^1-Poincaré profiles

L^1-Cheeger constant can be reinterpreted as the *minimum isoperimetric ratio.*

Definition 3.1.10. Given a metric measure space Z and a subset $A \subset Z$, we define the **boundary at scale** $a \geq 1$ of A as

$$\partial_a^Z A = [A]_a \cap [Z \setminus A]_a.$$

Definition 3.1.11. Let (Z, d, ν) be a metric measure space of finite total measure, and let $a \geq 2$. We define the **(geometric) Cheeger constant at scale** a of Z as

$$h_a(Z) = \inf \left\{ \frac{\nu(\partial_a^Z \Omega)}{\nu(\Omega)} \right\},$$

where the infimum is taken over measurable subsets Ω of Z of measure at most $\frac{\nu(Z)}{2}$.

Let (X, d, μ) be a standard metric measure space. We define the **(geometric) separation profile at scale** a of X as

$$\mathrm{sep}_{X,a}(v) = \sup \{\mu(Z)h_a(Z)\},$$

where the supremum is taken over all 1-thick subsets $Z \subset X$ (equipped with the induced measure and 1-metric), with $\mu(Z) \leq v$.

This geometric separation profile is equivalent to the L^1-Poincaré profile:

Theorem 3.1.12. *[68, Propositions 6.5 and 6.10] Let (Z, d, ν) be a metric measure space of finite total measure and let $a \geq 2$. Then*

$$h_{a,1}(Z) \leq h_a(Z) \leq 2h_{a,1}(Z)$$

Let (X, d, μ) be a standard metric measure space and let $a \geq 2$. Then

$$\frac{1}{2}\operatorname{sep}_{X,a} \leq \Pi_{X,a,1} \leq \operatorname{sep}_{X,a}.$$

The key tool for proving this theorem is the following (classical) co-area formula.

Proposition 3.1.13. *[68, Proposition 6.6] Let (X, d, μ) be a standard metric measure space, and let $a \geq 2$. The following co-area formula holds for every non-negative measurable function $f \colon X \to \mathbf{R}$:*

$$\int_X |\nabla_a f|(x)d\mu(x) = \int_{\mathbf{R}_+} \mu\left(\partial_a\{f > t\}\right) dt.$$

Theorem 3.1.12 then follows from Proposition 3.1.13 considering, on one hand, characteristic functions of measurable subsets, on the other hand, level sets of measurable functions.

It follows from Theorem 3.1.12 that, for bounded degree graphs, the L^1-Poincaré profile is equivalent to the separation profile (see Theorem 3.1.26).

L^2-Poincaré profiles

The L^2-Cheeger constant can be reinterpreted as the *spectral gap* of a Laplacian operator. If Γ is a finite graph, we can define the **Laplacian** Δ_Γ as the operator of $\ell^2(V\Gamma)$ satisfying:

$$\Delta_\Gamma f(i) = \sum_{j \sim i} f(i) - f(j),$$

for every $f \in \ell^2(V\Gamma)$ and $i \in V\Gamma$. The Laplacian Δ_Γ is a non-negative self-adjoint operator. Then, there is an orthonormal base of $\ell^2(V\Gamma)$ formed of eigenvectors of Δ_Γ. Moreover, the eigenvalues are non-negative, 0 included. Then, we can write λ the second smallest eigenvalue, which is positive if and only if Γ is connected. It is easy to see that we have, for every $f \in \ell^2(V\Gamma)$,

$$\frac{\langle \Delta_\Gamma f, f\rangle}{\|f\|_2^2} = \frac{\sum_{i\sim j}(f(i) - f(j))^2}{\sum_i f(i)^2},$$

where the sum of the numerator is taken on the set of edges of Γ; we recall that we have defined edges as unordered pairs of vertices $\{i, j\}$. Then, this theorem follows:

Theorem 3.1.14 (the Rayleigh principle)**.** *Let Γ be a finite graph, and let λ be the second smallest eigenvalue of Δ_Γ, the Laplacian of Γ. Then,*

$$\lambda = \inf_{f \in \ell^2(V\Gamma), \sum f = 0} \frac{\sum_{i\sim j}(f(i) - f(j))^2}{\sum_i f(i)^2}.$$

See details in [33], for example. The equality of Theorem 3.1.14 can be compared with Definition 3.1.7, with $p = 2$:

Proposition 3.1.15. *Let* Γ *be a finite graph,* D *be a bound on the degrees of the vertices of* Γ*, and* λ *be the second smallest eigenvalue of* Δ_Γ*, the Laplacian on* Γ*. Then,*

$$\frac{1}{4}h^2_{\Gamma,a=1,p=2} \leq \lambda \leq \frac{D}{2}h^2_{\Gamma,a=1,p=2}$$

The spectral gap can be used to bound mixing times of random walks on Γ. A related *spectral profile* was considered by Goel, Montenegro & Tetali [58].

L^∞-Poincaré profiles

There is a simple characterization of the L^∞-Poincaré profile in term of volume growth, see definitions and theorem below.

Definition 3.1.16. Let (X, d, ν) be a standard metric measure space. We define the **growth function** of X by

$$\gamma_X(r) = \sup_{x\in X} \nu(B(x, r)),$$

and we define its **(generalized) inverse** by

$$\kappa_X(v) = \inf\{r \mid \gamma_X(r) > v\}.$$

Theorem 3.1.17. *[68, Proposition 6.1] Let* (X, d, μ) *be a standard metric measure space with unbounded growth function, and let* $a \geq 3$*. Then*

$$\Pi_{X,a,\infty}(v) \simeq_a \sup\left\{\frac{w}{\kappa_X(w)} \;:\; \gamma_X(1) \leq w \leq v\right\}.$$

The right-hand sides of this theorem come from the following key lemma, where we see that the L^∞-Cheeger constant roughly detects the diameter of subspaces.

Lemma 3.1.18. *Let* Z *be a* 1*-thick subspace of* X *with diameter* $m \geq 3$ *and let* $a \geq 3$*. Then* $h^\infty_a(Z) \leq \frac{12a}{m}$*, and if every* $y, z \in Z$ *can be joined by a* 1*-path of length* $\leq 2m$ *then* $h^\infty_a(Z) \geq \frac{1}{2m}$*.*

The idea is to take a function of the form $x \mapsto d(x, z)$, for a well chosen z on one hand, and to bound the gradient along a 1-geodesic of maximal amplitude on the other hand. See [68] for details.

This theorem can be specified in the following context.

Definition 3.1.19. We shall say that a metric measure space (X, d, ν) is **uniformly non-sublinear** if there exists $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that $\frac{f(r)}{r}$ is non-decreasing and $K > 0$ such that

$$\nu(B(x, r)) \simeq_K f(r), \quad \text{for every } x \in X.$$

This can be compared with the definition of uniform measures for metric spaces (Definition 2.2.4). This is a very common property. In particular every metric measure space mentioned in this thesis report will be uniformly non-sublinear. Theorem 3.1.17 can be specified in this context, see corollary below.

Corollary 3.1.20. *Let (X, d, μ) be a standard metric measure space with unbounded growth function. If X is uniformly nonsublinear with constant K, then*

$$\Pi_{X,a,\infty}(n) \simeq_{a,K} \frac{n}{\kappa_X(n)}.$$

Proof. Let $n \geq \gamma_X(1)$ and $m \leq n$. Let r be such that $\nu(B(x, \frac{r}{2})) \leq n \leq \nu(B(x, r))$, and s be such that $s \leq r$ and $\nu(B(x, s)) \geq m$. Then $\frac{m}{s} \leq \frac{\nu(B(x,s))}{s} \leq K^2 \frac{\nu(B(x,r))}{r} \leq 2K^4 \frac{\nu(B(x,\frac{r}{2}))}{r} \leq 2K^4 \frac{n}{r}$. □

3.1.3 Comparing profiles

Separation profile and Poincaré profiles are strongly related, we summarize this in this section.

L^p and L^q-Poincaré profiles for $p \neq q$

Cheeger type inequalities hold in our context, they give some sharp inequalities between L^p-Cheeger constants for different p's.

Proposition 3.1.21. *[68, Proposition 7.2] Let (Z, d, ν) be a metric measure space of finite total measure. We assume that there exists a measurable subset Ω of Z such that $0 < \nu(\Omega) \leq \frac{1}{2}\nu(Z)$. Then for all $1 \leq p \leq q < \infty$ and all $a \geq 2$,*

$$h_{a,q}(Z) \succeq_{p,q} h_{a,p}(Z).$$

Let (X, d, μ) be a standard metric measure spaces (where μ is possibly infinite). Then, for all $1 \leq p \leq q < \infty$,

$$\Pi_{X,q} \succeq_{p,q} \Pi_{X,p}.$$

It is sharp in the case of nilpotent groups, for instance (see Theorem 5.1.2). This proposition cannot be extended to the case $q = \infty$ since there are bounded degree graphs containing expanders. Indeed, such a graph X satisfies that for every $p \in [1, \infty)$ $\Pi_{X,p}(v)/v \not\to 0$ as $v \to \infty$, while the volume growth of X is at most exponential and then $\Pi_{X,\infty}(v) \preceq v/\log(v)$. See details in §4.4. In the opposite direction we have the following.

Proposition 3.1.22. *[68, Proposition 6] Let (Z, d, ν) be a metric measure space of finite total measure. We assume that there exists a measurable subset Ω of Z such that $0 < \nu(\Omega) \leq \frac{1}{2}\nu(Z)$. Let $a \geq 2$ and $p \in [1, \infty)$. Then,*

$$h_{a,p}(Z) \leq 2^{\frac{p+1}{p}} h_{a,1}(Z)^{1/p}. \tag{3.1}$$

Let (X, d, μ) be a standard metric measure spaces (where μ is possibly infinite). Then, for all $1 \leq p < \infty$,

$$\Pi_{X,p} \preceq n^{\frac{p-1}{p}} \Pi_{X,1}^{1/p}.$$

Proof. Let Ω be such a subset of X, and let $\alpha = \frac{\nu(\Omega)}{\nu(Z)}$. Let f be the characteristic function of Ω. Using 3.1.13 for f, we get $\|\nabla_a f\|_p^p = \nu(\partial_a \Omega)$ on one hand, and

$$\|f - f_Z\|_p^p = \nu(\Omega) \cdot [\alpha(1-\alpha)^p + (1-\alpha)\alpha^p] \leq 2^{-p}\nu(\Omega),$$

on the other hand. Using the equivalence given by Theorem 3.1.12, the announced result follows. □

Asymptotically, this is sharp for the 3-regular tree, as we will see in Section 4.1.2. In that case, the supremum in the definition of the p-Poincaré profile is attained, up to constants, by balls. This shows that the inequality (3.1) is sharp.

Separation and L^p-Poincaré profile

In the context of graphs, we can actually take scale $a = 1$ in the definition of the L^p-Poincaré profile (Definition 3.1.8); unless we specify the scale, we will always assume that Poincaré profiles on graphs are computed at scale 1. In particular, for any graph G and any $p \in [1, \infty]$, we denote $\Pi_{G,p} := \Pi_{G,1,p}$. Here, we reformulate results of previous parts in the context of graphs.

Definition 3.1.23. For any finite graph Γ, we define the **majored combinatorial Cheeger constant** of Γ as

$$\tilde{h}(\Gamma) = \inf \frac{|\partial A|}{|A|},$$

where the infimum is taken on the subsets A of $V\Gamma$ of size at most $\frac{|V\Gamma|}{2}$, and ∂A is the boundary of A, that we define here as the set of vertices that are either in $\Gamma \setminus A$ and at distance 1 from A, or in A and at distance 1 from $\Gamma \setminus A$.

There is a slight difference with the combinatorial Cheeger constant h that we introduced in Definition 3.1.2. Here, the boundary of a subset A contains more vertices, since we consider the vertices at distance 1 from A (as in Definition 3.1.2), but also the vertices of A at distance 1 from $V\Gamma \setminus A$. In the context of bounded degree graphs, these two quantities are the same, up to constants:

Proposition 3.1.24. *Let Γ be a finite graph, and let D be a bound on the degrees of the vertices of Γ. Then,*

$$h(\Gamma) \leq \tilde{h}(\Gamma) \leq (D+1)h(\Gamma)$$

We introduce Definition 3.1.23 here in the only purpose of showing the precision of the next proposition. This particular case of Theorem 3.1.12 shows a very simple comparison between this majored Cheeger constant and the L^1-Cheeger constant at scale 1 (Definition 3.1.7).

Proposition 3.1.25. *([68, Proposition 6.10]) Let Γ be a finite graph. Then*

$$h_{p=1}(\Gamma) \leq \tilde{h}(\Gamma) \leq 2h_{p=1}(\Gamma)$$

Propositions 3.1.24, 3.1.25 and Theorem 3.1.3 imply:

Theorem 3.1.26. *Let G be an (infinite) graph, and D be a bound on the degrees of the vertices of G. Then for $n \geq 2$,*

$$\frac{1}{8}\operatorname{sep}_G(n) \leq \Pi_{G,1}(n) \leq 4(D+1)\operatorname{sep}_G(n).$$

Combining Theorem 3.1.26 and Proposition 3.1.21, and working all the constants, we get the following theorem.

Theorem 3.1.27. *Let G be an infinite graph. Then for any $p \in [1, \infty)$*

$$\Pi_{G,p} \geq \min\left(\frac{1}{96}, \frac{4^{-p}}{24}\right)\operatorname{sep}_G.$$

3.1.4 Regular and coarse regular maps

In geometric group theory, one of the main interests of separation and Poincaré profiles is their monotonicity under regular, and coarse regular maps. We introduce these notions and state the theorems of monotonicity. We start by considering coarse regular maps, in the context of metric measure spaces.

Definition 3.1.28. A map $F\colon (X, d_X, \mu) \to (Y, d_Y, \nu)$ between standard metric measure spaces is said to be **coarse regular** if it satisfies the following properties:

(i) F is coarse Lipschitz: there exists an increasing function $\rho : [0, \infty) \to [0, \infty)$ such that for all $x, x' \in X$,
$$d_Y(F(x), F(x')) \leq \rho(d_X(x, x'));$$

(ii) F is coarsely measure preserving: there exists δ_0 such that for all $\delta \geq \delta_0$ and for all 1-thick subspaces $A \subset X$,
$$\mu([A]_\delta) \simeq_\delta \nu([F(A)]_\delta) \simeq_\delta \mu([F^{-1}(F(A))]_\delta).$$

When we want to emphasize on the parameters, we call a map satisfying these two conditions a **(δ_0, ρ_+)-coarse regular map**.

For bounded degree graphs, we shall use the simpler definition of *regular maps*. Indeed, from [68, Lemma 5.4] a map between bounded degree graphs is coarse regular if and only if it is regular in the following sense:

Definition 3.1.29. Let X and Y be two graphs. We shall say that a map $f\colon X \to Y$ is a **regular map** if there exists $\kappa > 0$ such that

(a) $d(f(x), f(x')) \leq \kappa d(x, x')$ for any $x, x' \in X$,

(b) $|f^{-1}(\{y\})| \leq \kappa$ for any $y \in Y$.

When we want to emphasize on the constant, we call a map satisfying these two conditions a **κ-regular map**.

Regular maps appeared at least in [17], where they are called κ-quasimonomorphisms.

Proposition 3.1.30. *[68, Lemma 5.4] Let X and Y be two graphs of bounded degree. A map from X to Y is a regular map if and only if it is coarse regular.*

Coarse embeddings (Definition 2.1.1), and quasi-isometric embeddings (Definition 2.1.6), are examples of regular maps:

Proposition 3.1.31. *Let X and Y be two graphs. We assume that X is connected and has bounded degree. Any coarse embedding from X to Y is a regular map. In particular, any quasi-isometric embedding is a regular map.*

Proof. Let $f\colon X \to Y$ be a coarse embedding. Following Definition 2.1.1, there exists two unbounded non-decreasing functions $\rho_1, \rho_2\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $x, x' \in X$,

$$\rho_1(d_X(x,x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d(x,x')).$$

Since X is connected, it is 1-geodesic, and then we can assume without loss of generality that $\rho_2(r) \leq \rho(1)r$, which implies immediately (a).

For any y in Y, the set $f^{-1}(\{y\})$ is contained in a ball of radius s, for any s satisfying $\rho_1(s) > 0$. Then its cardinality can be bounded uniformly in y, because X has bounded degree, which gives (b). □

Remark 3.1.32. The following examples illustrate the fact that these three notions of embeddings are pairwise distinct.

- As we saw in §2.1.1, if G is a finitely generated group, then any subgroup $H \subset G$ coarsely embeds in G. If G is the Baumslag-Solitar defined by the presentation $\langle a, b \mid b^{-1}ab = a^2 \rangle$, and H is the subgroup generated by a, then the inclusion map $H \hookrightarrow G$ is a coarse embedding. Using the notation of Definition 2.1.1, we have for this map $\rho_1 \simeq \log$ (see [106, §11.1]). In particular, it is not a quasi-isometric embedding.

- If G is an infinite graph, and σ is a finite group acting on G, then the orbit map $G \to \sigma\backslash G$ is regular, but is not a coarse embedding in general. Indeed, the group σ can map vertices arbitrarily far away. For example, with $G = \mathbf{Z}$ and $\sigma = \langle n \mapsto -n \rangle$, we obtain the map $\mathbf{Z} \to \mathbf{N}$, $n \mapsto |n|$, which is not a coarse embedding.

Volume growth and asymptotic dimension are monotone invariants for regular maps:

Theorem 3.1.33. *Let X and Y be two graphs of bounded degree such that there exists a regular map $f\colon X \to Y$. Then,*

$$\mathrm{vol}_X \leq \mathrm{vol}_Y \quad \textit{and} \quad \operatorname{asdim} X \leq \operatorname{asdim} Y.$$

Proof. Let κ be as in Definition 3.1.29. Let $x \in X$ and $r > 0$. Then,

$$\frac{1}{\kappa} |B(x,r)| \leq |f(B(x,r)| \leq |B(x,\kappa r)| ,$$

which proves that $\mathrm{vol}_X \leq \mathrm{vol}_Y$.

Let n be such that $\operatorname{asdim} Y \leq n$. Let $r > 0$. Let $D > 0$ and $\mathcal{U}_0, \ldots, \mathcal{U}_n$ be D-bounded $r\kappa^{-1}$-disjoint families of subsets of Y, such that $\cup_{0 \leq i \leq n} \mathcal{U}_i$ covers Y. For every i, each $U \in \mathcal{U}_i$ has diameter at most D. Since Y has bounded degree, it implies that there exists C, that only depends on D and on the maximum degree of Y such that U has at most C elements.

Then, $f^{-1}(\mathcal{U}_0), \ldots, f^{-1}(\mathcal{U}_n)$ are r-disjoint families of subsets of X, and $\cup_{0 \leq i \leq n} f^{-1}(\mathcal{U}_i)$ covers X. For every i, each $U \in f^{-1}(\mathcal{U}_i)$ has at most κC element. A priori, we do not have a control on the diameter of such a subset U. Nevertheless, we can make a partition of U in r-disjoint subsets $U_1, \ldots, U_k$ such that, for every $j \in [1, k]$, U_j is r-connected[1]. Then, each U_j contains at most κC elements and is r-connected, which implies that its diameter is at most $\kappa r C$. Doing such partition of each element of $\mathcal{U}_i$, we obtain a family $\mathcal{V}_i$. Doing this for every i, we obtain r-disjoint families $\mathcal{V}_0, \ldots, \mathcal{V}_n$ of $\kappa r C$-bounded subsets of X such that $\cup_{0 \leq i \leq n} \mathcal{V}_i$ covers X. This proves that we have $\operatorname{asdim} X \leq n$, and, finally, that the asymptotic dimension of X cannot exceed that of Y. □

Theorem 3.1.34. *[68, Proposition 5.5] Let X and Y be two standard metric measure spaces such that there exists a coarse regular map $X \to Y$. Then for all $p \in [1, \infty]$,*

$$\Pi_{X,p} \preceq_p \Pi_{Y,p}.$$

The idea of its proof it to turn this problem into discrete problem, using a *discretization* process. The consequence for the separation profile is written below. Proposition 3.2.1 will be a quantitative version of this corollary.

Corollary 3.1.35. *Let X and Y be two graphs with bounded degree such that there exists a regular map $X \to Y$, then*

$$\mathrm{sep}_X \preceq \mathrm{sep}_Y .$$

Theorem 3.1.34 implies that for each p the L^p-Poincaré profile is a well-defined coarse invariant of a finitely generated group G. An important consequence of Theorem 3.1.34 is the following.

Proposition 3.1.36. *[68, Proposition 5.6] Let G and H be compactly generated locally compact groups, and let $f \colon H \to G$ be a proper continuous morphism (i.e. $\ker f$ is compact and $f(H)$ is a closed subgroup). We assume that both G and H are equipped with left-invariant Haar measures and word metrics with respect to some compact symmetric generating sets. Let $p \in [1, \infty)$. Then we have*

$$\Pi_{H,p} \preceq_p \Pi_{G,p}.$$

If moreover $f(H)$ is co-compact, then

$$\Pi_{H,p} \simeq_p \Pi_{G,p}.$$

[1] meaning that $\cup_{x \in U_j} B(x, r)$ is connected

The idea of the proof is that the morphism f itself is coarse regular (this is a consequence of the fact that it is a coarse embedding). In the case where $f(H)$ is co-compact, f has a coarse regular inverse.

We can define an equivalence relation associated with (coarse) regular maps:

Definition 3.1.37. We define two standard metric measure spaces X and Y to be **coarsely regularly equivalent** if there exist coarse regular maps $X \to Y$ and $Y \to X$.

We define two graphs X and Y to be **regularly equivalent** if there exist regular maps $VX \to VY$ and $VY \to VX$.

It is easy to see that both are equivalence relations. For example, $\mathbf{Z}$ is regularly equivalent to $\mathbf{N}$ (see Remark 3.1.32). We can also mention lamplighter groups (see Definition 5.3.5), for which we have the following proposition:

Proposition 3.1.38. *For every finite group F, there exists a regular map $F \wr \mathbf{Z} \to \mathbf{Z}_2 \wr \mathbf{Z}$.*

It follows that lamplighters over finite groups are contained in a single regular equivalence class, while they are contained in countably many quasi-isometry classes [51].

Proof of Proposition 3.1.38. We consider the Cayley graph of $F \wr \mathbf{Z}$ obtained with the generating set $\{(0, \pm 1)\} \cup \{(x\delta_0, 0), x \in F \setminus \{1\}\}$, and the same for $\mathbf{Z}_2 \wr \mathbf{Z}$ replacing F by $\mathbf{Z}_2$.

We can identify F with a subset of $\{0,1\}^n$ for some n, for example choosing a ordering on F and using binary code, with the prescription that the neutral element of F is identified with $(0, \ldots, 0)$. Then, we can map any $(f, i) \in F \wr \mathbf{Z}$ to an element $(g, j) \in \mathbf{Z}_2 \wr \mathbf{Z}$ in the following way: g is obtained concatenating the representations of each element of $(f_x)_{x \in \mathbf{Z}}$ (with f_0 at $[0, n-1]$), and j is equal to the product ni. Then this map is clearly injective, and it is an easy exercise to see that it is $3n$-Lipschitz. □

3.2 Lower bounds theorems

In the two next sections, we propose a *toolbox* for finding lower or upper bounds on separation or Poincaré profiles. We start with the lower bounds. We present here different possibilities that have been discovered to give a lower bound on the separation profile and/or on Poincaré profiles.

3.2.1 Subgraphs

Distorted subgraphs

In the litterature, there are some results on isometric embedding of finite graphs in infinite graphs. For example, certain expanders can be realized as subgraphs of Cayley graphs of finitely generated groups [99] (see 4.4). It is immediate from the definition that lower bounds can be obtained by estimating the cut of subgraphs. Using the behaviour of the separation profile, there is a little more flexibility, as shows the proposition below.

Proposition 3.2.1. *[19] Let X and Y be graphs with Y of bounded degree. Let d be a bound on the degrees of the vertices of Y. Let $f\colon X \to Y$ be a κ-regular map. Then for any n,*

$$\mathrm{sep}_X(n) \leq 8\kappa^2 d^{2\kappa+4}\,\mathrm{sep}_Y(nd^{2\kappa+2})$$

Proof. (adapted from [19]) Let $A \subset VX$ be a set of n vertices that induces a connected graph. Define $A' \subset VY$ as the κ-neighbourhood of $f(A)$. By κ-regularity, A' is connected, and it has size at most $|A|d^{\kappa+2}$. Let S' be a minimal subset of A' that separates it into pieces $C'_1, \ldots, C'_m$, each of size at most $A'/(2\kappa d^{\kappa+2})$. Then, (see Lemma A.1.7)

$$\begin{aligned}|S'| &= \mathrm{cut}^{(2\kappa d^{\kappa+2})^{-1}}(A')\\ &\leq 8\kappa d^{\kappa+2}\,\mathrm{sep}_Y(|A'|)\\ &\leq 8\kappa d^{\kappa+2}\,\mathrm{sep}_Y(|A|\,d^{\kappa+2})\end{aligned}$$

Let S_0 be the κ-neighbourhood of S' in Y. Then $S := f^{-1}(S_0)$ has size at most $\kappa d^{\kappa+2}\,|S'|$.

Let us prove that S is a 1/2-cut set of A. Let $C_i = f^{-1}(C'_i)$. We have $|C_i \setminus S| \leq \frac{1}{2}\,|A|$. We claim that S is a separating set in A between $C_1 \setminus S, C_2 \setminus S, \ldots, C_m \setminus S$. Suppose not: then there is a path P in $A \setminus S$ between some $C_i \setminus S$ and $C_j \setminus S$, with $i \neq j$. Then $f(P)$ is disjoint from $f(S) = S_0$, thus the κ-neighbourhood P' of $f(P)$ in A' does not intersect S'. Since P' is connected, this shows that some vertex of $f(C_i \setminus S) \subset C'_i$ and some vertex of $f(C_j \setminus S_0) \subset C'_j$ is connected by a path inside $A' \setminus S'$. This contradicts the fact that C'_i and C'_j are different components of $A' \setminus S'$. □

This proposition can apply even when X is a finite graph that maps regularly in an infinite graph Y. This process is used in Appendix A.2.

Product subgraphs

We can also mention an estimate of the cut of Cartesian products of two graphs. When such graphs regularly embed in a given graph, this gives a lower bound on its separation profile.

Theorem 3.2.2. *[19, Theorem 3.1] There exists a universal positive constant c such that for any finite graphs G and H, we have*

$$\mathrm{cut}(G \times H) \simeq_c \min(|H|\,\mathrm{cut}(G), |G|\,\mathrm{cut}(H)).$$

This is tight for computing the separation profile of $\mathbf{Z}^d$, for example. See Section 16.1 for a study of the Cheeger constants of graphs of the form G^k (with k "big").

3.2.2 Poincaré inequalities on subsets

Poincaré inequalities can be used to get lower bounds on Poincaré profiles. We call Poincaré inequality any inequality that gives a lower bound on the ratio between the gradient of a function and its norm. For example, a generalization of [77, Theorem 2.2] attributed to Diaconis and Saloff-Coste [39] gives:

Theorem 3.2.3. *[68, Corollary 8.5] Let G be a locally compact compactly generated group with polynomial growth. Then there exists a constant C such that, for any $p \geq 1$ and $a \geq 1$, for any metric ball $B = B(x_0, R)$ of radius R and any function $f \in L^p(G)$ we have*

$$\int_B |f(x) - f_B|^p d\mu(x) \leq CR^p \int_{3B} |\nabla_a f|(x)^p d\mu(x),$$

where μ denotes the Haar measure of G, and the distance is a word metric associated with a compact generating set.

See [68, Theorem 8.3] for a more general statement.

3.2.3 Poincaré inequalities on the boundary

In the case of hyperbolic graphs, lower bounds can be obtained using Poincaré inequalities not on the space itself but on its visual boundary. Example of such inequalities were found by Bourdon and Pajot [24], who showed that a family of Fuchsian buildings earlier studied by Bourdon [23] have boundaries that admit 1-Poincaré inequalities. We start by giving some definitions.

Definition 3.2.4. A metric space (Z, ρ) is called **Ahlfors Q-regular**, if there is a measure μ on Z so that for every ball $B(z, r)$ in Z with $r \leq \operatorname{diam}(Z)$, we have $\mu(B(z, r)) \simeq r^Q$. In particular, we may take μ to be the Hausdorff Q-measure on Z.

Definition 3.2.5. For $p \geq 1$, we say a metric space (Z, ρ) which is Ahlfors Q-regular with respect to a measure μ admits a **p-Poincaré inequality** (with constant $L \geq 1$) if for every Lipschitz function $f\colon Z \to \mathbf{R}$ and every ball $B(z, r) \subset Z$,

$$\fint_{B(z,r)} \left| f - f_{B(z,r)} \right| d\mu \leq Lr \left(\fint_{B(z,Lr)} (\operatorname{Lip}_x f)^p \, d\mu(x) \right)^{1/p},$$

where for $U \subset Z$, $f_U = \frac{1}{\mu(U)} \int_U f \, d\mu$, and

$$\operatorname{Lip}_x f = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}.$$

The next definition concerns Gromov hyperbolic spaces, for which the definition is given at the begging of Chapter 4.

Definition 3.2.6. Let (X, d, μ) be a δ-hyperbolic geodesic metric measure space. We shall say that it is **visual** with respect to x_0 if there exists $C \geq 0$ so that every $x \in X$ belongs to a C-quasi-geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x_0$.

We then can define the **boundary at infinity** $\partial_\infty X$ as the set of C-quasi-geodesic rays, identifying rays at bounded distance.

Let ρ be a metric on $\partial_\infty X$. We shall say that ρ is a **visual metric** on $\partial_\infty X$ based at $x_0 \in X$ with parameter $\epsilon > 0$ if $\rho(\cdot, \cdot) \simeq \exp(-\epsilon(\cdot \mid \cdot)_{x_0})$, where $(\cdot \mid \cdot)_{x_0}$ denotes the extension to $\partial_\infty X$ of the Gromov product in X with respect to x_0: $(x \mid y)_{x_0} = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y))$, for every $x, y \in X$.

For more background and discussions, see [22, 25]. The following theorems give upper bounds on Poincaré profiles of Gromov hyperbolic spaces, using Poincaré inequalities on the boundary at infinity.

Theorem 3.2.7. *[68, Theorem 11.1] Suppose that X is a visual Gromov hyperbolic graph with a visual metric ρ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a p-Poincaré inequality. Then for all $q \geq p$, $\Pi_{X,q}(r) \succeq r^{1-1/Q}$.*

Theorem 3.2.8. *[68, Theorem 11.3] Suppose that X is a visual Gromov hyperbolic graph with a visual metric ρ on $\partial_\infty X$ that is Ahlfors Q-regular and admits a Q-Poincaré inequality. Then $\Pi_{X,Q}(r) \succeq r^{1-1/Q}\log(r)^{1/Q}$.*

For example, these theorems apply to rank-one symmetric spaces (like hyperbolic spaces), see Theorem 4.3.5. Theorem 3.2.7 also gives the lower bound of Theorem 4.3.3, concerning some groups acting on Bourdon-Pajot buildings (see [68] and the references therein for details). In all these cases, the lower bounds are sharp.

3.2.4 Isoperimetric profile

The author and Gournay showed in [80] a comparison theorem between the isoperimetric and separation profiles. The content of this article is given in Part II, and is one of the most important results of this thesis.

Definition 3.2.9. Let G be a graph. We define the **isoperimetric profile** of G as:

$$\Lambda_G(n) = \inf_{\{A\subset VG\colon |A|\leq n\}} \frac{|\partial A|}{|A|},$$

for every natural integer n, where ∂A is the set of edges connecting A and its complementary in VG.
We shall say that the graph G is **amenable** if $\lim_\infty \Lambda_G = 0$.

Definition 3.2.10. Let $f\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a continuous non-increasing function such that $\lim_\infty f = 0$. For any $\delta \in (0,1)$, we define the **δ-geometric decay function** of f as:

$$p_f^\delta(x) := f^{-1}\left(\delta f(x)\right) = \min\left\{x' \mid f\left(x'\right) \leq \delta f(x)\right\}.$$

The theorem is the following:

Theorem 3.2.11. *Let G be an infinite connected amenable graph of bounded degree. Then for any $n \geq 1$ there exists an integer $N \in \left[n, p_{\Lambda_G}^{1/4}(n)\right]$ such that*

$$\frac{\mathrm{sep}_G(N)}{N} \geq \frac{1}{8}\frac{\Lambda_G(n)}{\log\left(\frac{p_{\Lambda_G}^{1/4}(n)}{n}\right)+1}.$$

This is optimal in the case of profiles of the form $\Lambda \simeq n^{-\alpha}$ (*e.g.* nilpotent groups). See Part II for a more general statement and more applications.

3.2.5 From growth - the gap theorem

We give here a theorem of Hume and Mackay showing a comparison between separation profile and volume growth, which applies in particular to non-elementary hyperbolic groups (see 4.2).

Theorem 3.2.12. *[69] A finitely presented group G which is not virtually free satisfies*

$$\mathrm{sep}_G(n) \succeq \kappa_G(n),$$

where κ_G is the ***inverse growth function*** *of G: $\kappa_G(n) = \max\{r \in \mathbb{N} : |B_r| \leq n\}$.*

Let us sketch the proof. Finitely presented groups are *accessible*, they can be written as a graph of groups where each edge-group is finite and each vertex-group has at most one end [44]. If we assume that the initial group is not virtually free, at least one of the vertex-groups is one-ended. It turns out that this vertex-group has to be finitely presented as well. Then we can use a lemma, stating that in a one-ended finitely presented group it is always possible to connect annuli of bounded radius. This implies that cutting a ball of radius r requires (at least) a set of size proportional to r. Theorem 3.2.12 can then be deduced using additional arguments, see [69] for details.

We can end with a gap theorem, that is a straightforward consequence of Theorem 3.2.12.

Theorem 3.2.13 (the gap theorem [69]). *If G is a finitely generated and finitely presented group, either*

- $\mathrm{sep}_G(n) \simeq 1$ *and G is virtually free, or*
- $\mathrm{sep}_G(n) \succeq \log n$ *and G is not virtually free.*

This is optimal in the case of surface groups, which have a logarithmic separation profile (see 4.2). Note that this theorem is not true if we remove the assumption that G is finitely presented (see Theorem 5.4.2).

3.3 Upper bounds theorems

Upperbounds are often trickier to obtain. Nevertheless, several theorems give upper bounds on the separation profile and/or the Poincaré profile.

3.3.1 Large-scale dimensions

Assouad-Nagata dimension

A useful tool, developed for the separation profile by Hume in [67], is the notion of Assouad-Nagata dimension, which is a quantitative refinement of the asymptotic dimension. The definition of asymptotic dimension given here is clearly equivalent to the definition given in §2.2.2.

Definition 3.3.1. Let X be a metric space. We say X has **asymptotic dimension** at most m if there exists a function $h : \mathbf{R}_+ \to \mathbf{R}_+$ such that for all $r > 0$ we can partition X into $m+1$ subsets $X_0, \dots, X_m$ and each X_i into sets $X_{i,j}$ with $\operatorname{diam}(X_{i,j}) \leq h(r)$ and such that $d(X_{i,j}, X_{i,j'}) > r$ whenever $j \neq j'$.

We say X has **Assouad-Nagata dimension** at most m if the above holds with $h(r) \leq Cr$ for some constant $C > 0$.

We define the **growth function** of a graph X to be the function $\gamma_X : \mathbf{N} \to \mathbf{N} \cup \{\infty\}$, where $\gamma_X(n)$ is the maximal cardinality of a closed ball of radius n in X. When graphs are regular enough, this is the same as Definition 2.2.2, when, as usual, we endow the graph X with the shortest path metric and the counting measure.

The following theorem holds:

Theorem 3.3.2. *[67, Theorem 3.2] Let X be a graph with asymptotic dimension at most $m-1$, let h be a non-decreasing function provided by the above definition and let $\gamma(n)$ be the growth function of X. There exists a constant $k = k(m)$ such that*

$$\operatorname{sep}_X(n) \leq \frac{kn}{f_h\left(\frac{n}{2m}\right)},$$

where we define $f_h(n) = \max\{k \,:\, \gamma(h(k)) \leq n\}$.

Let us give the main idea of the proof. First, we decompose $X = \sqcup_{i,j} X_{i,j}$ as before. Then, given a subgraph Γ of X, we cut it by removing vertices of the form $\{v \in V\Gamma \mid d_X(v, X_{i,j}) = r\}$, for some well-chosen i, j and r.

Theorem 3.3.3. *[67, Theorem 1.5] Let X be a graph of bounded degree and finite Assouad-Nagata dimension. Then $\operatorname{sep}_X(n) \preceq n/\log(n)$. If X is vertex transitive and has growth at most Cn^d, then $\operatorname{sep}_X(n) \preceq n^{(d-1)/d}$.*

This is known to be sharp, since the separation profile of $\mathbf{Z}^d$ is $n^{(d-1)/d}$ (see §5.1) and a direct product of two non-abelian free groups has separation profile $n/\log(n)$ (see §4.6).

Measurable dimension

In the context of measured metric spaces, which is the natural framework of Poincaré profiles, the above definition was modified by Hume, Mackay and Tessera [68, §9] to the following:

Definition 3.3.4. Let (X, d, μ) be a metric measure space. We say X has **measurable dimension at most** n ($\operatorname{mdim}(X) \leq n$) if, for all $r \geq 0$ we can write $X = X_0 \cup \cdots \cup X_n$ and decompose each $X_i = \bigcup X_{ij}$ so that each X_{ij} is 1-thick, $\sup(\mu(X_{ij})) < \infty$ and $d(X_{ij}, X_{ij'}) \geq r$ whenever $j \neq j'$.

If $\operatorname{mdim}(X) \leq n$ we define the function $\tilde{h}_n(r)$ to be the infimal value of $\sup(\mu(X_{ij})) + 1$ taken over all decompositions of X satisfying the above hypotheses.

A simple comparison can be made with asymptotic dimension when the metric measure space has **bounded geometry**: for all $r \geq 0$ there exists some C_r such that $\mu(B(x,r)) \leq C_r$ for all $x \in X$.

Lemma 3.3.5. *Let (X, d, μ) be a standard metric measure space with bounded geometry. Then,*

$$\operatorname{asdim} X \geq \operatorname{mdim}(X).$$

Lemma 3.3.6. *Let (X, d, μ) and (Y, d', μ') be standard metric measure spaces. If there exists a coarse regular map $F\colon X \to Y$, then $\operatorname{mdim}(X) \leq \operatorname{mdim}(Y)$. Moreover, for all suitable n we have $\tilde{h}_n^X \preceq_n \tilde{h}_n^Y$.*

Theorem 3.3.7. *Let (X, d, μ) be a metric measure space with $\mu(X) = \infty$ and measurable dimension at most n. Let $p \in [1, \infty)$. Then, for all $\delta > 0$*

$$\Pi_{X,p}(r) \preceq_n \sup\left\{\tilde{h}_n(t+\delta)/t \,:\, \tilde{h}_n(t) \leq r/(4n+4)\right\}.$$

As for the Assouad-Nagata dimension, this is optimal in the case of polynomial growth groups, and products of trees.

Equivariant conformal dimension

Inspired by [93], Benjamini, Schramm and Timár proved sharp upper bounds on the separation profiles of hyperbolic spaces (see the proof of Theorem 4.3.1). This was generalized by Hume, Mackay and Tessera to an upper bound for hyperbolic groups, relying on the notion of equivariant conformal dimension. This notion is related to the usual conformal dimension for metric spaces. On this topic, see [100, 88].

Definition 3.3.8. The **equivariant conformal dimension** of a hyperbolic group G is defined as the infimum of the Hausdorff dimension of $(\partial_\infty X, \rho)$ where $\partial_\infty X$ is the boundary at infinity of a space X on which G acts by isometries, cocompactly, and properly discontinuously, and ρ is a visual metric on $\partial_\infty X$. We say that the equivariant conformal dimension is **attained** if the infimum is realised.

See Definition 3.2.6 for the definition of a visual metric.

Theorem 3.3.9. *[68, Theorem 11] Let G be a hyperbolic group and let Q be its equivariant conformal dimension. Then, for any $\epsilon > 0$,*

$$\Pi_{G,p}(r) \preceq \begin{cases} r^{\frac{Q-1}{Q}+\epsilon} & \text{if } \; p \leq Q \\ r^{\frac{p-1}{p}} & \text{if } \; p > Q. \end{cases}$$

If the equivariant conformal dimension is attained, we have:

$$\Pi_{G,p}(r) \preceq \begin{cases} r^{\frac{Q-1}{Q}} & \text{if } \; 1 \leq p < Q \\ r^{\frac{Q-1}{Q}} \log^{\frac{1}{Q}}(r) & \text{if } \; p = Q \\ r^{\frac{p-1}{p}} & \text{if } \; p > Q. \end{cases}$$

See [68, Theorem 12.3] for details and a more general statement.

3.3.2 Compression in L^p spaces

The results of this subsection are detailed in Chapter 17, Part III. The author showed an upper bound theorem using the notion of compression in L^p-spaces. Let us give the definition:

Definition 3.3.10. Let $p \in [1, \infty)$, let $f\colon G \to L^p$ be a 1-Lipschitz map. We define the **compression** of f as:

$$\rho_f(t) = \inf \left\{ \|f(g) - f(h)\|_p \mid d_G(g, h) \geq t \right\}.$$

The theorem follows.

Theorem 3.3.11. *Let G be a graph of bounded degree. Then there exists two constants $c_1, c_2 > 0$, depending only on the maximum degree in G, such that if $f\colon VG \to L^p$ is a 1-Lipschitz map for some $p \in [1, \infty)$, then*

$$\Pi_{G,p}(n) \leq c_1 \frac{n}{\rho_f(c_2 \log n)}, \quad \textit{for all } n \geq 0.$$

This is optimal at least for products of trees (see 4.6), more generally for products of hyperbolic groups. It is also optimal for some groups constructed by Brieussel and Zheng in [29], along a subsequence (see Theorem 5.4.1). A more general statement and details are given in Part III. The following corollary is a straightforward consequence of Theorems 3.1.27 and 3.3.11.

Corollary 3.3.12. *Let G and $f\colon VG \to L^p$ be as in Theorem 3.3.11, then*

$$\operatorname{sep}_G(n) \leq c_1 \frac{n}{\rho_f(c_2 \log n)}, \quad \textit{for every } n \geq 0$$

where c_1 and c_2 are constants depending only on the maximal degree in G and on p.

3.3.3 Volume growth

The result of this subsection is detailed in Chapter 10, Part II. The author and Gournay showed in [80] the following upperbound using volume growth:

Theorem 3.3.13. *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq e^{f(n)}$ for a function f with $\lim_\infty \frac{f(n)}{n} = 0$. Assume the degree of the vertices is bounded by d. Then there is a constant K (depending on f) such that for any integer $N > K$,*

$$\frac{\operatorname{sep}(N)}{N} \leq 4d \frac{f\left(\frac{f^{-1}\left(\ln \frac{N}{2}\right) - 1}{2}\right)}{f^{-1}\left(\ln \frac{N}{2}\right) - 1}$$

The main idea of the proof is to cut any subgraph by removing a well-chosen sphere. This is close to be optimal for nilpotent groups (up to a factor log), and gives an interesting bound for intermediate growth groups (see details in Part II).

Chapter 4

Poincaré profiles and hyperbolicity

In this chapter, we detail what is known on Poincaré profiles of hyperbolic groups, with a focus on the separation profile. We will see three families of separation profiles for hyperbolic groups: constant, logarithmic and power profiles. Then, we will mention other related facts: acylindrically hyperbolic groups having a linear profile (along a subsequence), a criterion of hyperbolicity using the separation profile, and the profiles of products of hyperbolic groups.

Hyperbolic groups were introduced by Gromov [61]. They are among the central objects in geometric group theory. In any large isosceles triangle of the Euclidean plane, the midpoint of each side is far away from the other two sides. This cannot happen in a hyperbolic space. That observation led to the following definition of hyperbolicity:

Definition 4.0.1. A geodesic metric space (X, d) is **Gromov hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$: for every geodesic triangle $T = (\gamma_1, \gamma_2, \gamma_3)$, we have $\gamma_1 \subseteq [\gamma_2 \cup \gamma_3]_\delta$.

4.1 Free groups

In this part, we focus on profiles of (virtually) free groups.

4.1.1 Bounded separation profiles

We will say that a graph has **bounded separation** if its separation profile is a bounded function. Hume and Mackay showed the following theorem.

Theorem 4.1.1. *[69] A vertex transitive, bounded degree, connected graph has bounded separation if and only if it is quasi-isometric to a tree.*

In particular, a finitely generated group has bounded separation if and only if it is virtually free.

The fact that graphs that are quasi-isometric to trees have bounded separation is very simple. It relies on the following (easy) lemma:

Lemma 4.1.2. *Let T be a finite tree. Then* $\mathrm{cut}(T) \leq 1$.

It comes (at least) from [75]. For completeness, we give the proof.

Proof. Without loss of generality, we can assume that T is non-empty and connected. We then define a sequence $(u_n)_{n\geq 0}$ in the following way:

- $u_0 := \{\text{any vertex of } T \text{ chosen at random}\}$,
- for any $n \geq 0$, if a connected component of $\Gamma \setminus \{u_n\}$ contains more than $|\Gamma|/2$ vertices, then it is unique and we define u_{n+1} to be the unique neighbour of u_n that belongs to it. Otherwise, we set $u_{n+1} := u_n$.

We claim that this sequence, for any choice of u_0, is eventually constant. Then we obtain a cutset of Γ of size 1. □

The process of the proof of Lemma 4.1.2 was exploited by Shchur to show distortion results, for embeddings of balls in hyperbolic spaces into trees, see Theorem 4.1.8. The key notion around Theorem 4.1.1 is *treewidth*:

Definition 4.1.3. Let G be a finite graph, T be a tree, and consider a family $\mathcal{V} = (V_t)_{t\in T}$ such that $V_t \subset V(G)$ for every t. We say that (T, V) is a tree-decomposition of G if the following hold:

(i) $\cup_{t\in T} V_t = V(G)$,

(ii) for every $e \in E(G)$ there is a t such that both endpoints of e are in V_t,

(iii) for every $x \in V(G)$, the set $\{t \in V(T) \mid x \in V_t\}$ induces a connected subgraph of T.

The **width** of the tree decomposition is $\max_{t\in T} |V_t| - 1$. The **treewidth** of G is the minimum of the widths among all tree decompositions of G.

See [40, Theorem 12.4.4] for more details on treewidth. Benjamini, Schramm and Timár showed:

Theorem 4.1.4. *[19, Theorem 2.1] If an (infinite) bounded degree graph has finite separation, then it has a bounded treewidth. In that case, it admits a regular map to the* 3*-regular tree.*

Theorem 4.1.1 follows by combining this theorem with results of Kuske and Lohrey [78] on graphs with bounded treewidth.

The vertex-transitive assumption is necessary: as observed in [19, §2], the Sierpiński triangle graph has bounded separation but is not quasi-isometric to a tree (and it has bounded treewidth).

4.1.2 Poincaré profile of trees

Concerning Poincaré profiles, Hume, Mackay & Tessera proved the following theorem.

Theorem 4.1.5. *[68, Theorem 9, Proposition 6.1] Let T be the infinite 3-regular tree. Then*

- *For all $p \in [1, \infty)$, $\Pi_{T,p}(v) \simeq_p v^{\frac{p-1}{p}}$.*
- *For $p = \infty$, $\Pi_{T,p}(v) \simeq v/\log(v)$*

The proof shows that the balls (asymptotically) realize the supremum in the definition of Poincaré profiles. A corollary is that balls in T realize the equality, up to constants, of the Cheeger-type inequality of Proposition 3.1.22.

4.1.3 Shchur's embeddding theorem

Shchur proved that quasi-isometrically embedding hyperbolic balls into trees requires linear distortion (see Definition 2.1.6), with coefficients depending on the separation properties of these balls. We introduce some coarse notions of volume and separation (minimal volume of subsets dividing a metric space X into two pieces). These quantities are, in most cases, equivalent to the one we already defined.

Definition 4.1.6. Let $a > 0$. We define the **a-volume** of a metric space X as the quantity:

$$\mathrm{vol}_a(X) = \inf \{k \mid X \text{ can be covered with } k \text{ balls of radius } a\}.$$

We moreover define the **volume fonction** of X as:

$$\mathrm{vol}_{X,a}(c) = \sup \{\mathrm{vol}_a(B) \mid B \text{ is a closed ball in } X \text{ of radius } c.\}$$

Definition 4.1.7. Let $a > 0$. We define the **a-metric separation** of X, denoted $\tilde{\mathrm{sep}}_a(X)$, as the maximal number S such that the following property holds: for any partition of X with two subsets U_1 and U_2 of a-volume at most $\frac{1}{3}\mathrm{vol}_a(X)$, there exists a family of S pairwise disjoint balls of radius a intersecting both U_1 and U_2.

Theorem 4.1.8. *[112, Theorem 6] Let X be a bounded metric space, let T be a tree of degree at most d, and let $a > 0$. We assume that there exists a constant D such that any subset of X of a-volume at least $\frac{1}{3}\mathrm{vol}_a(X)$ has a diameter at least $\mathrm{diam}(X)/D$. Then, if $f : X \to T$ is a (λ, c)-quasi-isometric embedding, with $c < \mathrm{diam}(X)/D$, we have*

$$\lambda 2a + c \geq \log_d \frac{\tilde{\mathrm{sep}}_a(X)}{\mathrm{vol}_{X,a}(c)},$$

This theorem applies for balls in hyperbolic spaces, see corollary below. For every $n \geq 3$ and $r > 0$, we denote by $\mathbf{H}^n_r$ any closed ball of $\mathbf{H}^n$ of radius r. For every $d \geq 2$, we denote by T_d the infinite regular tree of degree d.

Corollary 4.1.9. *[112, Corollary 1] Let $n \geq 3$ and $d \geq 2$. There exists $\alpha > 0$, depending only on n and d, such that for any $r > 0$, every (λ, c)-quasi-isometric embedding $\mathbf{H}^n_r \to T_d$ satisfies $\lambda + c \geq \alpha r$.*

4.2 Surface groups and logarithmic profiles

In this section, we focus on hyperbolic groups with logarithmic profiles, like surface groups. It is remarkable that for hyperbolic groups, having an unbounded profile implies having a profile at least logaritmic. This is the following theorem, shown by Benjamini, Schramm & Timár [19].

Theorem 4.2.1. *[19, Theorem 4.2] A finitely generated hyperbolic group G which is not virtually free satisfies*

$$\mathrm{sep}_G(n) \succeq \log(n).$$

This theorem is also a consequence of Theorem 3.2.12 (the gap theorem [69]), since every finitely generated hyperbolic group can be finitely presented (see [34, Théorème 2.3]). Benjamini Schramm and Timár showed that Theorem 4.2.1 is sharp for the space $\mathbf{H}^2$:

Theorem 4.2.2. *[19]We have $sep_{\mathbf{H}^2} \simeq \log n$. Then, if G is a cocompact Fuchsian group or a surface group, then $sep_G \simeq \log n$.*

4.3 Hyperbolic spaces and power profiles

We give here the separation profile of hyperbolic spaces of dimension at least three. This follows from the method in [92, 93], which can also be used to give a proof for the separation of $\mathbf{R}^d$.

Theorem 4.3.1. *[19] For any $d \geq 3$, $sep_{\mathbf{H}^d}(n) \simeq n^{(d-2)/(d-1)}$.*

Let us sketch the proof of this theorem. The lower bound is immediate, it comes from the fact that $\mathbf{R}^{d-1}$ is a subspace of $\mathbf{H}^d$. For the upper bound, the idea of the proof of Benjamini, Schramm and Timár is to consider the dual graph G of a tiling of the hyperbolic space $\mathbf{H}^d$. Then, given a finite subgraph H of G, we consider the corresponding set of tiles, and define o as the center of mass of this finite union of tiles. Any hyperplane of $\mathbf{H}^d$ passing through o will separate the set of tiles into two pieces of equal volume. Then, the corresponding vertices will give a cut set of H. A counting argument shows that one can find a cutset with the required bound, and that is it maximized when H is a ball.

This theorem was generalized by Hume, Mackay and Tessera (see Theorem 4.3.5 below). In their proof, the lower bound is obtained from known Poincaré inequalities. For the upper bound, given a subspace Z of $\mathbf{H}^d$ and taking a well chosen basepoint x_0, one can find two subsets H^+ and H^- of the boundary at infinity $\partial_\infty \mathbf{H}^d$, that are at controlled distance from each other, and such that there is a lower bound on the proportion of points of Z lying in geodesic rays of H^+ and of H^-. We can define an appropriate function on $\partial_\infty \mathbf{H}^d$, taking the value 0 on H^- and 1 on H^+. Computing the Rayleigh quotient of the associated function on Z gives the required bound.

The following other example of hyperbolic groups with polynomial separation profile comes from a family of Fuchsian buildings $I_{m,n}$, studied by Bourdon and Pajot [23, 24].

Definition 4.3.2. Let $m \geq 5, n \geq 3$. We define the group

$$G_{m,n} = \langle s_1, \ldots, s_m \mid s_i^n, [s_i, s_{i+1}] \, \forall i \rangle, \quad \text{where indices are modulo } m.$$

For every m, n, $G_{m,n}$ acts cellularly and geometrically on $I_{m,n}$. Hume, Mackay and Tessera showed:

Theorem 4.3.3. *[68, Theorem 13.2] Let $m \geq 5, n \geq 3$. Then*

$$\mathrm{sep}_{G_{m,n}}(n) \simeq n^{1-1/Q_{m,n}},$$

with $Q_{m,n} = 1 + \frac{\log(n-1)}{\operatorname{arccosh}((m-2)/m)} \in (1, \infty)$.

Corollary 4.3.4. *There exists a dense subset A of $(0,1)$ such that for all $\alpha \in A$ there is a hyperbolic group G_α with $\mathrm{sep}_{G_\alpha}(n) \simeq n^\alpha$.*

The number $Q_{m,n}$ can be interpreted as the conformal dimension of the group $G_{m,n}$ (see Theorem 3.3.9).

Poincaré profiles

The difference of behaviour between $\mathbf{H}^2$ and $\mathbf{H}^d$ for $d \geq 3$ may seem to be pure coincidence. Actually, it is part of a more common phenomenon that can be understood with the Poincaré profiles (recall Theorem 3.3.9). We can mention the following theorem for rank-one symmetric spaces, that generalizes Theorem 4.3.1. Here, $\mathbf{O}$ denotes the Octonion algebra.

Theorem 4.3.5. *[68, Theorem 13.3] Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}\}$ be a real division algebra, and let $X = \mathbf{H}^m_{\mathbf{K}}$ be a rank-one symmetric space for $m \geq 2$ (and $m = 2$ when $\mathbf{K} = \mathbf{O}$). Let $Q = (m+1)\dim_{\mathbf{R}} \mathbf{K} - 2$ be the equivariant conformal dimension of $\mathbf{H}^m_{\mathbf{K}}$, then*

$$\Pi^p_{\mathbf{H}^m_{\mathbf{K}}}(v) \simeq \begin{cases} v^{\frac{Q-1}{Q}} & \text{if } p < Q \\ v^{\frac{Q-1}{Q}} \log(v)^{\frac{1}{Q}} & \text{if } p = Q \\ v^{\frac{p-1}{p}} & \text{if } p > Q \end{cases}$$

4.4 Groups containing expanders

We recall that for any finite graph X, we denote by $h(X)$ his Cheeger constant (see Definition 3.1.2).

Definition 4.4.1. Let d be a positive integer, and $\epsilon > 0$. A sequence of finite graphs $(X_n)_{n\geq 0}$ is called a (d, ϵ)**-expander** if the following conditions hold:

(i) for every n, the degrees of the vertices of X_n are bounded by d,

(ii) $\lim_{n\to\infty} |X_n| = \infty$,

(iii) for every n, $h(X_n) \geq \epsilon$.

Let $(X_n)_{n\geq 0}$ be a sequence of finite graphs. We say that a graph G **contains** $(X_n)_{n\geq 0}$ **as a subgraph** if each X_n is isomorphic to a subgraph of G.

Definition 3.1.2 implies that the separation profile detects the presence of expanders:

Theorem 4.4.2. *[67, Theorem 1.3] Let G be a graph. Then $sep_G(n)/n \not\to 0$ if and only if G contains an expander as a subgraph.*

As remarked by Hume, there exists a glass ceiling for hyperbolic graph, that implies in particular that they can never contain an expander as a subgraph.

Theorem 4.4.3. *[67, Theorem 2.9] Let X be a hyperbolic graph with bounded degree. There exists some k such that* $\mathrm{sep}_X(n) \preceq n^{\frac{k-1}{k}}$.

This follows from Theorem 4.3.1 and the fact that every hyperbolic group quasi-isometrically embeds in a hyperbolic space [22] (see Theorem 2.1.7).

However, using small cancellation labellings developed by Osajda [99], certain expanders of unbounded girth can be realized as subgraphs of Cayley graphs of *acylindrically* hyperbolic finitely generated groups. Hume proved that there are *many* such groups [67]:

Theorem 4.4.4. *[67, Theorem 1.1, 1.2] There exists a family of finitely generated groups $\{G_r : r \in \mathbf{R}\}$ such that*

- *for each $r \in \mathbf{R}$, G_r contains an expander as a subgraph,*
- *given $r \neq s$ there is no regular map $G_r \to G_s$.*

These groups are infinitely presented graphical $C'(1/6)$ small cancellation groups, and are therefore acylindrically hyperbolic [64, 37].

4.5 A criterion of hyperbolicity

We can end our chapter on hyperbolic groups with the following criterion of hyperbolicity, due to Hume and Mackay. It uses the notion of Dehn functions. These functions measure the complexity of the word problem in a given finitely presented group. We recall the definition:

Definition 4.5.1. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group. We denote F_S the free group generated by S, and $\pi\colon F_S \to \Gamma$ the projection map. The **Dehn function** $\delta_\Gamma(n)$ of Γ with respect to (S, R) is defined by

$$\delta_\Gamma(n) := \max_{\pi(w)=1, |w|\leq n} \min \left\{ k : w = \prod_{i=1}^{k} g_i r_i^{\pm 1} g_i^{-1}, r_i \in R, g_i \text{ word in } S \right\}.$$

Here, w is a word in the alphabet S and $|w|$ denotes length of the word w.

Theorem 4.5.2. *[69, Theorem 1.6]Let G be a finitely presented group with (exactly) quadratic Dehn function. Then there is an infinite subset $I \subseteq \mathbf{N}$ such that $\mathrm{sep}_G(n) \succeq n^{1/2}$ for all $n \in I$.*

Thus, if a finitely presented group G has Dehn function $\preceq n^2$, and separation function $o(n^{1/2})$, it must be hyperbolic.

We recall that a finitely presented group is hyperbolic if and only if its Dehn function is equivalent to n [61], and that if a finitely presented group satisfies a subquadratic Dehn function, then it is hyperbolic [61, 98, 27].

The class of groups with at-most-quadratic Dehn function is rich, including: CAT(0) groups, automatic and more generally combable groups [47], and free-by-cyclic groups [28].

4.6 Product of trees and of hyperbolic groups

Benjamini, Schramm and Timár studied product of trees [19].

Theorem 4.6.1. *[19] We denote by T be the infinite binary tree. Let $G = T \times \cdots \times T$ be a product of at least two copies of T. Then,*

$$sep_G(n) \simeq \frac{n}{\log n}.$$

Bounded degree trees always embed in the infinite regular tree T, this is the following lemma.

Lemma 4.6.2. *[19, Lemma 2.4]For any bounded degree tree A, there is a regular map $A \to T$.*

Then, this theorem follows:

Theorem 4.6.3. *[19] Let $T_1, \dots, T_n$ be bounded degree trees. Then,*

$$sep_{T_1 \times \cdots \times T_k}(n) \preceq \frac{n}{\log n}.$$

If moreover T regularly embeds in two of the T_i's, then there is equivalence.

From [22, 96], hyperbolic groups quasi-isometrically embed in $(\mathbf{R}^n, \ell^2)$ for some n. Using Theorem 3.3.11, this gives the following generalization of Theorem 4.6.3:

Theorem 4.6.4. *Let $G_1, \dots, G_n$ be finitely generated hyperbolic groups. Then, for any $p \in [1, \infty)$,*

$$\Pi_{G_1 \times \cdots \times G_k, p}(n) \preceq \frac{n}{\log n}.$$

Chapter 5

Poincaré profiles of amenable groups

In this chapter, we focus on results concerning separation profile, or Poincaré profiles, of amenable groups. In this family of groups, one can find a variety of behaviours. We will divide them into three categories: polynomial, subexponential, and exponential growth amenable groups. Then, we will mention two families of amenable groups of prescribed separation profile. The first gives close to linear profiles and the second close to constant.

Only two relationships between amenability and Poincaré profile are known at this stage. One the one hand, the Poincaré profile recognizes polynomial growth. Second, amenability is incompatible with a linear behaviour along a subsequence. In this chapter we shall see that a wide class of behaviors can be obtained within the class of amenable groups.

5.1 Nilpotent groups

We recall the definition of a nilpotent group.

Definition 5.1.1. Given two subgroups K, L of a group H, let $[K, L]$ be the subgroup generated by the commutators $[k, l] = klk^{-1}l^{-1}$, with $(k, l) \in K \times L$. The lower central series of a group G is the non-increasing sequence

$$G = G_1 \supset \cdots \supset G_i \ldots ,$$

defined inductively by $G_{i+1} = [G_i, G]$. A group is **nilpotent** if and only if its lower central series terminates, *i.e.*, there exists an integer c called the **class** of H such that $H_c \neq \{\mathrm{id}\}$ and $H_{c+1} = \{\mathrm{id}\}$.

The profiles of nilpotent groups was completely determined by Hume, Mackay and Tessera:

Theorem 5.1.2. *[68, Theorem 7] Let G be a finitely generated group such that $\gamma_G(r) \simeq r^d$. Then for all $p \in [1, \infty]$, $\Pi^p_G(r) \simeq_p r^{\frac{d-1}{d}}$.*

The first result in this context was the calculation of the separation of $\mathbf{Z}^d$ by Benjamini, Schramm and Timár [19, Corollary 3.3]. It can also be obtained from the geometric separator theorem (see Theorem 1.2.3), from [92].

For nilpotent groups, several proofs are now available: the lower bound can be deduced from Theorem 3.2.11 (comparison with isoperimetry), Theorem 3.2.3 (Poincaré inequalities on balls), Theorem 3.2.2 (Cartesian product, for $\mathbf{Z}^d$). Upper bounds can be deduced from Theorem 3.3.7 (measurable dimension), Theorem 3.3.11 (compression in L^p spaces, for $\mathbf{Z}^d$), Theorem 3.3.13 (volume growth, only for $p = 1$, and up to a log factor).

5.2 Intermediate growth groups

For intermediate growth groups, we present known upper and lower bounds. Both give a bound of the form $\frac{n}{(\log n)^\beta}$, but the lower bound is specific to Grigorchuk's groups and gives information only along an infinite sequence of integers. First, the upper bound. The following theorem is corollary of Theorem 3.3.11:

Theorem 5.2.1. *[80] Let G be a finitely generated group so that $|B_n(x)| \leq K_1 e^{K_2 n^\alpha}$ for some constants $K_1, K_2 > 0$ and $\alpha \in [0, 1)$. Then,*

$$\operatorname{sep}(n) \preceq \frac{n}{(\log n)^{1-\alpha}}.$$

This follows from embedding results for intermediate growth groups (see either [59, Theorem 1.3(b)] or [114, Proposition 14]). An upper bound on the separation profile can also be obtained using volume growth (Theorem 3.3.13). It applies to more general graphs, but it is less tight: it gives an exponent $\frac{1-\alpha}{\alpha}$ instead of $1 - \alpha$.

From Theorem 3.2.11 and [50, Example 2.4], we have the following lower bound for Grigorchuk's groups. These groups are known to have have sub-exponential volume growth [60].

Theorem 5.2.2. *Let G_ω be a Grigorchuk group, with ω not eventually constant. Then, there is a constant c (independent on ω) such that*

$$\operatorname{sep}_{G_\omega}(n) \geq c \frac{n}{(\log n)^2},$$

for infinitely many n's.

The reader who is not familiar with these groups can find the definitions in [50] or [60].

5.3 Exponential growth amenable groups

For exponential growth amenable groups, the two main tools are the following. First, Theorem 3.2.11 (comparison with isoperimetry) enables to get lower bounds on the separation profile on infinite sequences of integers. Second, Theorem 3.3.11 gives upper bounds on Poincaré profiles using compression in L^p spaces. Before coming into details, we can give the two following statements, consequences of Theorem 3.3.11, and of Theorems 3.2.11 and 3.1.27, respectively. These two statements will be useful in the two next sections, concerning solvable groups and wreath products.

Theorem 5.3.1. *Let G be a Cayley graph.*

	If, for some $a > 0$, $\Lambda(n)$ is ...	*then, for infinitely many n's, $\frac{\text{sep}(n)}{n}$ is ...*
*	$\preceq \frac{1}{\log(n)^a}$	$\succeq \frac{\Lambda(N)}{\log(N)}$
**	$\preceq \frac{1}{\log^a(\log n)}$	$\succeq \frac{\Lambda(N)}{\log(N)^C}$ *(for some C)*
***	$\preceq \frac{1}{(\log \ldots \log \log n)^a}$	$\succeq \frac{\Lambda(N)}{N^\epsilon}$, *for any $\epsilon > 0$.*

Theorem 5.3.2. *Let G be a graph of bounded degree such that there exists a 1-Lipschitz map $f : G \to L^p$, with $p \in [1, \infty)$, satisfying for some $\alpha \in (0, 1]$*

$$\|f(x) - f(y)\|_p \geq d(x, y)^\alpha, \quad \text{for any } x, y \in VG.$$

Then,

$$\text{sep}_G(n) \preceq 4^p \frac{n}{(\log n)^\alpha}.$$

(such an α is called a **compression exponent** of G.)

5.3.1 Solvable and nilpotent groups

We start with exponential growth solvable groups. We recall some definitions.

Definition 5.3.3. A group G is **solvable** (or soluble) if it admits a series of subgroups

$$G = G_1 \supset G_2 \supset \cdots \supset G_k = \{\text{id}\},$$

such that G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is an abelian group for $1 \leq i < k$.

A group G is **polycylic** if moreover the above series can be chosen in such a way that G_i/G_{i+1} is *cyclic* for $1 \leq i < k$.

Any finitely generated nilpotent group is polycyclic and any polycyclic group is solvable and finitely generated.

The following theorem gives a dichotomy between nilpotent and non-nilpotent solvable groups:

Theorem 5.3.4. *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0, 1)$ and $c > 0$ such that for any large enough integer n we have*

$$\text{sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

This theorem was obtained as a corollary of Theorem 5.3.1, using a universal upper bound on the isoperimetric profile of solvable groups (row *** of the tabular of Theorem 5.3.1, see [110] and [111]). It is very natural to wonder whether for every G amenable, $\mathrm{sep}_G(n) \preceq n^{1-\epsilon}$ implies that G is virtually nilpotent, which is not known at the moment.

5.3.2 Estimates for some amenable groups

Estimates are known on the isoperimetric profile and on the compression exponent for many amenable groups. We summarize them in the two next tabulars, and give the associated estimates for the separation profile. An interesting property is that the bounds are often of the form $\frac{n}{(\log n)^\alpha}$, which makes a real difference with the case of nilpotent or hyperbolic groups, where profiles are always dominated by some sublinear power function. We recall the definition of wreath products of groups, which is a common process (along with semi-direct products) for building amenable groups with interesting properties.

Definition 5.3.5. For any pair of groups L and S, we denote by $L \wr S$ the **wreath product** of L with S, *i.e.* the group $\oplus_S L \rtimes S$ where $\oplus_S L$ is the set of functions $f\colon S \to L$ with finite support (we define $\mathrm{support}(f) = \{s \in S \mid f(s) \neq 1_S\}$), and S acts by translation on the indices.

The group $L \wr S$ has a traditional interpretation, as a generalization of the lamplighter group $\mathbf{Z}_2 \wr \mathbf{Z}$. To avoid confusions, we adopt this notation: L is for "lamp", and S for "street".

First, we summarize lower bounds that can be obtained with Theorem 5.3.1.

Theorem 5.3.6. *Let G be a finitely generated group. We denote by F any finite group and by N a group of volume growth $\preceq r^d$.*

If G is ...	*then, for infinitely many n's,* sep(n) *is ...*
polycyclic and not nilpotent [(a)], *or* $BS(1,q)$ [(b)] *for* $q \geq 2$.	$\succeq \dfrac{n}{(\log n)^2}$
the wreath product $F \wr N$, [(c)]	$\succeq \dfrac{n}{\log(n)^{1+1/d}}$
an iterated wreath right-product of k copies of N: $G = (\dots(N \wr N)\cdots \wr N) \wr N$, [(d)]	$\succeq \dfrac{n}{\log(n)^{1+1/kd-\epsilon}}$, *for any* $\epsilon > 0$.
the iterated wreath product $F \wr (F \wr N)$, [(e)]	$\succeq \dfrac{n}{\log(n)^C}$ *(for some C)*
an iterated wreath left-product of k copies of N: $G = N \wr (\dots(N \wr (N \wr N))\dots)$, [(f)]	$\succeq n^{1-\epsilon}$, *for any* $\epsilon > 0$.

*(a) and (b) from Pittet & Saloff-Coste [102], and row * of Theorem 5.3.1 with* $a = 1$*. We recall that* $BS(p,q)$ *is the Baumslag-Solitar group defined by the presentation* $\langle a, b \mid ab^p a^{-1} = a^q \rangle$*. This bound is also valid for solvable groups with finite Prüfer rank and geometrically elementary solvable groups, see [115].*

*(c) and (d) from Pittet & Saloff-Coste [102, §4 and §7] or Erschler [48, Theorem 1] and row * of Theorem 5.3.1 with* $a = 1/d$ *and* $a = 1/kd - \epsilon$*, respectively.*

*(e) from Erschler [48, Theorem 1] and row ** of Theorem 5.3.1 with* $a = 1/d$*.*

*(f) from Erschler [48, Theorem 1] and row *** of Theorem 5.3.1.*

Second, we summarize upper bounds given by Theorem 3.3.11. For every finitely generated group G, we define $\alpha(G)$ as the supremum of the exponents of 1-Lipschitz maps into L^p spaces for some $p \in [1, \infty)$ (see definition in Theorem 5.3.1). When we consider maps to an L^p space only for a specific p, we write $\alpha_p(G)$.

Theorem 5.3.7. *Let* G *be a finitely generated group. We denote by* F *any finite group and by* N *a nilpotent group of volume growth* $\simeq r^d$*.*

If G is ...	*G embeds in* L^p *with* $\alpha = \ldots$	*then,* sep(n) *is ...*
polycyclic [(a)], $\mathbf{Z} \wr F$ [(b)], $BS(p,q)$ *with* $p, q \geq 1$ [(c)], *a 3-manifolds group* [(d)]	1	$\preceq \dfrac{n}{\log n}$
a wreath product $\mathbf{Z} \wr \mathbf{Z}$ [(e)], $F \wr N$ [(f)] *or* $\mathbf{Z} \wr N$ [(g)].	$1 - \epsilon$	$\preceq \dfrac{n}{(\log n)^{1-\epsilon}}$, *for every* $\epsilon > 0$.
$H \wr \mathbf{Z}^2$ *with lamp group* H *having* $\alpha_2(H) \geq \frac{1}{2}$ [(h)], *Thompson's group* F [(i)]	$\dfrac{1}{2}$	$\preceq \dfrac{n}{(\log n)^{1/2}}$
the free solvable groups $S_{r,d}$ *of length* d *when* $d > 1$ [(j)].	$\dfrac{1}{d-1} - \epsilon$	$\preceq \dfrac{n}{(\log n)^{\frac{1}{d-1} - \epsilon}}$, *for every* $\epsilon > 0$.
an iterated wreath right-product of copies of $\mathbf{Z}$*:* $(\ldots((\mathbf{Z} \wr \mathbf{Z}) \wr \mathbf{Z}) \ldots) \wr \mathbf{Z}$ *(with* k *"*$\mathbf{Z}$*")* [(k)]	$\dfrac{1}{2 - 2^{1-k}}$	$\preceq \dfrac{n}{(\log n)^{\frac{1}{2-2^{1-k}}}}$
a group with return probability after n *steps of a SRW* $\leq K_2 e^{-K_1 n^\gamma}$ [(l)]	$\dfrac{1-\gamma}{1+\gamma}$	$\preceq \dfrac{n}{(\log n)^{\frac{1-\gamma}{1+\gamma}}}$

(a),(b) from Tessera [114, Theorems 9 and 10]
(c) from Jolissaint & Pillon [73, Corollary 2], Cornulier & Valette [35]
(d) from Hume [66, Theorem 5.4]. (3-manifold are not amenable in general)
(e) Naor & Peres [94, Lemma 7.8] and [95, Theorem 6.1]; for every $p \in (1, 2)$*, the bound is* $\max\{\frac{p}{2p-1}, \frac{2}{3}\}$*, take* $p \to 1$*.*
(f),(g) from Naor & Peres [95, Theorem 3.1]; for every $p \in [1, 2]$*, the bound is* $\max\{\frac{1}{p}, \frac{1}{2}\}$*, take* $p = 1$*.*

(h) from Naor & Peres [94, Theorem 3.3]
(i) from Arzhantseva, Guba & Sapir [9, Theorem 1.3]
(j) from Sale [109, Corollary 4.2]; the bound on is $\frac{1}{p(d-1)}$ for $p \in [1,2]$, take $p = 1$.
(k) from Naor & Peres [94, Corollary 1.3]
(l) from Gournay [59, Theorem 1.1]

There are many other groups for which one can compute the compression (the above list does not exhaust the results in the references). For example, $\alpha(G \times H) = \min\big(\alpha(G), \alpha(H)\big)$. There are also further results: on HNN-extensions see Jollissaint & Pillon [73], on relatively hyperbolic groups see Hume [66]), on wreath products see Li [81], Brieussel & Zheng [29].

5.3.3 When p is large

We can mention a general lower bound (interesting for large p's) for elementary amenable groups with exponential growth. This follows from the fact that for any such group G, there always exists a regular map from the infinite 3-regular tree to G, see the proof below.

Proposition 5.3.8. *Let G be a finitely generated elementary amenable group with exponential growth. Then for all $p \in [1, \infty)$, $\Pi_{p,G}(r) \succeq_p r^{\frac{p-1}{p}}$.*

Proof. It follows from Theorem 3.2' of [32] that G contains a free subsemigroup on two generators H. We identify H with its Cayley graph (endowing it with the aforementioned generators). This is an infinite binary tree, that can be obtained from the 3-regular tree by choosing a vertex and identifying two branches touching it. From the discussion of Remark 3.1.32, this implies that the 3-regular tree regularly embeds in H. Still from Remark 3.1.32, H coarsely embeds in G. Then, Proposition 5.3.8 follows combining Theorem 4.1.5 with Theorem 3.1.34. □

From results of Benjamini and Schramm of embeddings in non amenable groups [18], Proposition 5.3.8 also applies for infinite non-amenable groups.

5.4 Prescription of the Poincaré profiles

We present here two results of prescription of the separation profile (or Poincaré profiles) using amenable groups. It is interesting to remark that the two constructions developed here are able to exhibit an amenable group with arbitrary *high* profile on one hand, or arbitrary *low* profile on the other hand. In both cases, these estimates on the separation profile are only valid on a subsequence. We can remark that the *average* profiles (*e.g.* power functions) can be largely prescribed with the Bourdon-Pajot hyperbolic groups (Corollary 4.3.4).

5.4.1 Diagonal lamplighter products

Brieussel and Zheng presented in [29] a method of construction of groups, that is based on wreath and diagonal products. In Part III, we study precisely their Poincaré profiles. This family of group is able to have almost any Poincaré profile between $\frac{n}{\log\log n}$ and n (not attained). This is a manifestation of the fact that Poincaré profiles are interesting tools to make asymptotic distinctions between amenable groups. This is summarized in the following statement.

Theorem 5.4.1. *There exist two universal constants κ_1 and κ_2 such that the following is true. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing and $\lim_\infty \rho = \infty$. We assume that ρ is injective and that there exists some $\alpha > 0$ such that $\frac{\rho^{-1}(x)}{\exp(x^\alpha)}$ is non-decreasing. Then, there exists a finitely generated elementary amenable group Δ of exponential growth and of asymptotic dimension one such that for any $p \in [1,\infty)$,*

$$\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \text{for any } n,$$
$$\text{and} \quad \Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\rho(\log n)} \quad \text{for infinitely many } n\text{'s}.$$

This theorem applies for example with $\rho = \log$, or “smaller” functions. The upper bound is obtained using Theorem 3.3.11, while the lower bounds are obtained for explicit subgraphs. See Part III for details.

As we mentioned, these groups are built using the construction of Brieussel and Zheng in [29]. As it is shown in this paper, the group Δ of Theorem 5.4.1 also have prescribed speed and entropy of the random walk equivalent to $\frac{n}{\rho(\sqrt{n})}$, ℓ^p-isoperimetric profile equivalent to $\rho(\log(n))^{-p}$, a return probability defined implicitly with ρ, and an L^p-equivariant compression gap of the form $\left(\frac{\rho}{\log^{1+\epsilon}(\rho)}, \rho\right)$. See [29, Theorem 1.1] for details.

5.4.2 Lacunary poorly connected amenable groups

The sort of hierarchy drawn by the dichotomy between exponential and polynomial growth solvable groups (Theorem 5.3.4) do not tell anything for more exotic amenable groups. Indeed, concerning elementary amenable groups, Hume & Mackay showed that the separation profile can be arbitrary small, on a subsequence.

Theorem 5.4.2. *[69, Theorem 1.4] Let $\rho : \mathbb{N} \to \mathbb{N}$ be an unbounded non-decreasing function. There is a finitely generated amenable group G such that*

$$1 \not\simeq \operatorname{sep}_G(n) \quad \text{and} \quad \operatorname{sep}_G(n) \not\gtrsim \rho(n).$$

The groups that were used by Hume & Mackay are the elementary amenable lacunary hyperbolic groups constructed in [97]. Their key property is that they are not virtually free, but are limits of virtually free groups (see details in [69]).

In §3.2.5, we mentioned the gap theorem, in the context of *finitely presented* groups. Theorem 5.4.2 shows that this theorem cannot be extended to finitely generated groups in general.

Part II

Separation profiles, isoperimetry, growth and compression

We give lower and upper bounds for the separation profile (introduced by Benjamini, Schramm & Timár) for various graphs using isoperimetric profile, volume growth and Hilbertian compression. For graphs which have polynomial isoperimetry and growth, we show that the separation profile is bounded by n^a and n^b for some $a, b \in (0, 1)$. For many amenable groups, we show a lower bound in $n/\log(n)^a$ and, for any group which has a non-trivial compression exponent in an L^p-space, an upper bound in $n/\log(n)^b$. We show that solvable groups of exponential growth cannot have a separation profile bounded above by a sublinear power function. We also introduce a notion of local separation, with applications for percolation clusters of $\mathbf{Z}^d$ and graphs which have polynomial isoperimetry and growth.

Profil de séparation, isopérimétrie, croissance et compression

Pour différents types de graphes, nous établissons des bornes inférieures et supérieures sur leur profil de séparation (introduit par Benjamini, Schramm & Timár), en utilisant le profil isopérimétrique, la fonction de croissance et la compression dans les espaces de Hilbert. Dans le cas des graphes de dimension isopérimétrique supérieure à 1 et de croissance polynomiale, nous montrons que le profil de séparation est borné par deux fonctions puissance, avec des exposants compris strictement entre 0 et 1. Pour de nombreux groupes moyennables, nous montrons une borne inférieure de la forme $n/\log(n)^a$ et, pour tout groupe ayant un exposant de compression positif dans un espace L^p, une borne supérieure de la forme $n/\log(n)^b$. Nous prouvons que le profil de séparation d'un groupe résoluble à croissance exponentielle n'est jamais dominé par une fonction puissance sous-linéaire. Nous introduisons également une notion de séparation locale, avec des applications aux composantes de percolation de $\mathbf{Z}^d$ et aux graphes de dimension isopérimétrique supérieure à 1 et à croissance polynomiale.

Chapter 6

Introduction

The separation profile was first defined by Benjamini, Schramm & Timár in [19]. The separation function at n is the supremum over all subgraphs of size n, of the number of vertices needed to be removed from the subgraph, in order to cut into connected pieces of size at most $n/2$. The introduction of this function was motivated by the study of regular maps between metric spaces, because the separation profile is monotonous under regular maps (up to constant factors). A map between two graphs of bounded degree is said to be regular if it is Lipschitz, and if the cardinality of the preimages of singletons is uniformly bounded (see Definition 11.0.2); for example, quasi-isometric and coarse embeddings between connected graphs are regular maps.

Hume's work about linear separation profiles (see [67]) led to an equivalent definition of the separation profile using the Cheeger constants h. The definition of the separation profile that we use is the following: (see section 7 for the details)

Definition 6.0.1 (Separation profile)**.**

$$\mathrm{sep}(n) = \sup_{F \subset V\Gamma, |F| \leq n} |F| \cdot h(F) \quad \text{where } h(F) \text{ denotes the Cheeger constant of } F.$$

This was studied by Hume in [67] with the aim of finding large classes of expanders. His work was continued by Hume, Mackay & Tessera [68] who introduced L^p-variants of these profiles and recently Hume & Mackay [69] studied the case of groups with low separation profiles ($\prec \log(n)$). On the opposite side, the separation profiles of expanders is linear (along a subsequence).

The subject matter of the current paper is to estimate the separation profile for various types of graphs, using other known information such as growth, isoperimetry and compression. Our focus is mostly on Cayley graphs, because this is where most information is available on other properties of the graph (such as growth, isoperimetry and compression). However our methods do not really require the high level of regularity that Cayley graphs possess.

A first group of results regards graphs G which are "polynomial" in some sense. Here Λ_G denotes the isoperimetric profile of G. For example, if for some $K > 0$, $\Lambda_G(n) \geq K n^{-1/d}$, one says the graph has d-dimensional isoperimetry. The growth will be measured by $b_n = \sup_x |B_n(x)|$ where $B_n(x)$ is the ball of radius n centred at x.

Theorem 6.0.2.
• *Let G be a graph such that $\frac{K_1}{n^{1/d}} \leq \Lambda_G(n) \leq \frac{K_2}{n^{1/d}}$ for some constants K_1 and K_2, then, $\exists K_3 > 0$ such that for all n, $\dfrac{\mathrm{sep}(n)}{n} \geq \dfrac{K_3}{n^{1/d}}$.*
• *Let G be a graph such that $b_n \leq K_1 n^d$ for some constant K_1, then, $\exists K_2 > 0$ such that for all n, $\dfrac{\mathrm{sep}(n)}{n} \leq \dfrac{K_2 \log n}{n^{1/d}}$.*
• *Let G be a graph with $(1+\epsilon)$-dimensional isoperimetry and $b_n \leq K_1 n^d$ then, for any $\delta > d$ there are constants K_1 and K_2 such that, for all n, $K_1 n^{\epsilon(1+\epsilon)/\delta^2} \leq \mathrm{sep}(n) \leq K_2 n^{(d-1)/d} \log n$.*

See Proposition 9.1.1, Proposition 9.1.3 and Corollary 10.1.4 for details.

This theorem can be used on Cayley graphs of nilpotent groups (for which a sharper upper bound was already given by Hume, Mackay & Tessera [68]), but our method applies also to other type of graphs, such as pre-fractal Sierpinski carpets (on this subject, see Gibson & Pivarski [55] for isoperimetry and Gladkova & Shum [57] for a study of the relationship between conformal dimension and separation profile in graphs of fractals).

Our methods also yield results on the infinite percolation components of $\mathbf{Z}^d$, and more generally on a large class of graphs of polynomial growth, called polynomial graphs. Roughly speaking, a (d_1, d_2)-*polynomial graph* is a graph of growth bounded by n^{d_2} and of isoperimetric dimension at least d_1, see Definition 11.2.1 for details. Since the percolation component always includes arbitrary large balls, it is more interesting to introduce a local variant of the separation profile in this context, namely the *local separation at* v, where v is a vertex of the graph (see Section 11):

$$\mathrm{sep}^v_G(n) := \sup_{F < B_G(v,r); |B_G(v,r)| \leq n} |F| \cdot h(F)$$

In that case, we show that $\frac{\mathrm{sep}^v(n)}{n}$ is bounded below by a function of the type $\frac{1}{n^\alpha}$, for every vertices in the polynomial case:

Theorem 6.0.3. *Let G be a (d_1, d_2)-polynomial graph. Then for any $\eta \in (0,1)$ there exists $c > 0$ such that for any vertex v and any integer n:*

$$\mathrm{sep}^v(n) \geq c n^{(1-\eta)\frac{d_1^2(d_1-1)}{d_2^3}}$$

(see Corollaries 11.2.3 and 11.4.3 for details), and for vertices that stay exponentially close to the origin in the $\mathbf{Z}^d$ percolation case:

Theorem 6.0.4. *Let $\mathcal{C}_\infty$ be a supercritical phase percolation cluster of $\mathbf{Z}^d$. Then for any $\varepsilon \in (0,1)$ There exists almost surely $c > 0$ such that for n large enough, if $\|x\|_\infty \leq \exp\left(n^{(1-\varepsilon)\frac{d}{d-1}}\right)$, then we have:*

$$\mathrm{sep}^x_{\mathcal{C}_\infty}(n) \geq c n^{\frac{d-1}{d}}$$

(see Corollary 11.2.5). In this case, the inclusion in $\mathbf{Z}^d$ shows that this lower bound is optimal.

The same methods as those of Theorem 6.0.2 can also be applied to groups of intermediate growth:

Theorem 6.0.5. *Let G be a Cayley graph of a groups of intermediate growth with $K_1 e^{n^a} \leq b_n \leq K_2 e^{n^b}$ where $K_1, K_2 > 0$ and $a, b \in]0,1[$. Then*

- $\exists K_3 > 0$ *such that there are infinitely many n with* $\dfrac{\text{sep}(n)}{n} \geq \dfrac{K_3}{(\log n)^{1+\frac{1}{a}}}$.
- $\exists K_4 > 0$ *such that for any n,* $\dfrac{\text{sep}(n)}{n} \leq \dfrac{K_4}{(\log n)^{\frac{1}{b}-1}}$.

See Proposition 9.2.1 and Corollary 10.1.3 for details. The upper bound is obtained here using the growth assumption. Relations between growth and separation is studied in subsection 10.1.

Inside the realms of graphs having a logarithmic isoperimetric profile, the lower bounds obtained are listed in the upcoming theorem. (The list is not exhaustive, one could also get a lower bound for any group whose isoperimetric profile is known, *e.g.* $\mathbf{Z} \wr \mathbf{Z}$.) Note that it is reasonable to compare $\frac{\text{sep}(n)}{n}$ with $\Lambda_G(n)$ for two reasons. First, for nilpotent groups those two functions coincide up to some multiplicative constants (see Hume, Mackay & Tessera [68]). Second, the underlying mechanism which allow us to provide such bounds relies on the known estimates of $\Lambda_G(n)$.

Theorem 6.0.6. *Let G be a Cayley graph.*

If, for some $a > 0$, $\Lambda_G(N)$ is ...	*then, for infinitely many N's, $\dfrac{\text{sep}(N)}{N}$ is ...*
$\preccurlyeq \dfrac{1}{\log(N)^a}$	$\succcurlyeq \dfrac{\Lambda_G(N)}{\log(N)}$
$\preccurlyeq \dfrac{1}{\log^a\big(\log(N)\big)}$	$\succcurlyeq \dfrac{\Lambda_G(N)}{\log(N)^C}$ *(for some C)*
$\preccurlyeq \dfrac{1}{(\log\ldots\log\log N)^a}$	$\succcurlyeq \dfrac{\Lambda_G(N)}{N^\epsilon}$ *, where ϵ can be arbitrarily small*

These estimates on the isoperimetric profile are known for polycyclic groups which are not nilpotent (first row of the table with $a = 1$), wreath products $F \wr N$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d (first row with $a = 1/d$), iterated wreath products $F \wr (F \wr N)$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d (second row with $a = 1/d$), solvable groups in general (third row, see [110] and [111]). See Propositions 9.2.1 and 9.3.1 for details.

Using the last row of the table, we prove the following theorem:

Theorem 6.0.7. *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0,1)$ and $c > 0$ such that for any large enough integer n we have*

$$\mathrm{sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

See Theorem 9.4.1 for details. It is a very natural question to wonder whether $\mathrm{sep}(n) \preceq n^{1-\epsilon}$ for G amenable implies that G is virtually nilpotent, see Question 12.0.7. However, our proofs techniques break down completely in the general case, see §9.5. Here is a nice application of Theorem 9.4.1 and Theorem 6.0.5: (see Definition 11.0.2 for the definition of a regular map)

Corollary 6.0.8. *Let G be a finitely generated group which is either solvable or of subexponential growth. If there exists a regular map from G to $\mathbf{H}^d$ (the $d-$dimensional hyperbolic space), then G is virtually nilpotent.*

The case of G solvable was already addressed by Hume & Sisto [70, Corollary 1.3] in the case of coarse embeddings, with a completely different proof.

Remark 6.0.9. In the context the current paper, limited to graphs, it might not be clear what a regular map to $\mathbf{H}^d$ is. Either replace $\mathbf{H}^d$ by any uniform lattice or see Hume, Mackay & Tessera [68] for the generalisation of the separation profile to Riemannian manifolds with bounded geometry.

It might be disappointing to see that our lower bound is sometimes much smaller that the original isoperimetric profile (*e.g.* a power of $\log N$ is much larger than $\log\log N$). However, our upper bounds show that such a dramatic loss cannot be avoided (see Corollary 10.2.2 and Remark 9.3.2).

In fact, for polycyclic groups, free metabelian groups, lamplighters on $\mathbf{Z}$ with finite lamps, lamplighters on $\mathbf{Z}^2$ with finite lamps and some iterated wreath products such as $F \wr (F \wr \mathbf{Z})$ our methods show that $\frac{\mathrm{sep}(n)}{n}$ is infinitely often $\geq K_1(\log n)^{-c_1}$ while it is (for all n) $\leq K_2(\log n)^{-c_2}$.

More precisely, $c_2 < 1$ can be arbitrarily close to 1 (but there is no control on the constant K_2 as far as we know) Furthermore, for polycyclic groups and lamplighters on $\mathbf{Z}$ with finite lamps $c_1 = 2$. For lamplighters on $\mathbf{Z}^2$ with finite lamps $c_1 = \frac{3}{2}$. For free metabelian groups $c_1 > 1 + \frac{1}{r}$ is arbitrarily close to $1 + \frac{1}{r}$, where r is the rank of the group.

The case of $F \wr (F \wr \mathbf{Z})$ is of particular interest, since it shows that the appearance of the logarithmic factor in the lower bound is necessary (see Remark 10.2.4). This also shows that there are amenable groups for which $\frac{\mathrm{sep}(n)}{n}$ decays much faster than $\Lambda_G(n)$.

Though our lower bounds apply to a vast array of groups, we show, using constructions of Erschler [49] and Brieussel & Zheng [29], that there are groups for which those methods fail to give a significant bound (see §9.5).

Our upper bounds on the separation profile come either from the growth of balls (for graphs which do not have exponential growth, see Theorems 10.1.3 ans 10.1.4) or from the compression exponents:

Theorem 6.0.10. *Let G be a bounded degree graph which has compression exponent (in some L^p-space) equal to $\alpha > 0$. Then, for any $c < \frac{\alpha}{2-\alpha}$, there is a constant K so that, for any N,* $\dfrac{\text{sep}(N)}{N} \leq \dfrac{K}{(\log N)^c}$.

See Corollary 10.2.2 for details.

In addition to the afore-mentioned examples, the previous theorem applies to products of hyperbolic groups (these have $\alpha = 1$ [114, Corollary 2]). This shows that the separation profile of a product of hyperbolic groups satisfies $\frac{\text{sep}(n)}{n} \leq \frac{K_1}{(\log n)^{1-\epsilon}}$ (for any $\epsilon \in (0,1)$). This is quite sharp, since such a group always have $\frac{\text{sep}(n)}{n} \simeq \frac{1}{\log n}$, as soon as at least two of these hyperbolic groups are non-elementary. Indeed, in this case it contains up to quasi-isometry the product $T \times T$, where T is the infinite binary tree (*e.g.* from the Tits alternative for hyperbolic groups). This implies that its separation profile satisfies $\frac{\text{sep}(n)}{n} \succeq \frac{1}{\log n}$ [19, Lemma 1.3, Theorem 3.5]. On the other hand, any product of hyperbolic groups coarsely embeds in a product of the form $T \times \cdots \times T$ [30], which implies that its separation profile satisfies $\frac{\text{sep}(n)}{n} \preceq \frac{1}{\log n}$.

Organisation of the paper: §7 contains the basic definitions. In §8.1, we make the first step towards a lower bound by looking at sets which have a small boundary to content ratio. These optimal set turn out to have a high separation. This estimate is then used in §8.2 to get lower bounds on the separation profile from the isoperimetric profile. §9 is devoted to concrete estimates in various Cayley graphs and self-similar graphs (§9.1.1). The proof of Theorem 6.0.7 (as well as some further lower bounds) is done in §8.3. Groups where the methods do not yield a lower bound are constructed in §9.5. Upper bounds on the separation profile are done in §10: via growth in §10.1 and via compression in §10.2. Local separation profiles are studied in §11, with applications to infinite percolation component in $\mathbf{Z}^d$ (§11.2) and to polynomial graphs (§11.2 and 11.4). Questions are presented in §12.

Acknowledgments: The authors would like to thank Jérémie Brieussel for having initiated and followed this project all along; Romain Tessera for various interesting questions, remarks and corrections; Itai Benjamini who proposed us to study local separation of polynomial graphs; and Todor Tsankov for noticing a mistake in a previous version of Corollary 11.2.5.

Chapter 7

Definitions

The set of vertices of a graph G will be denoted VG, while the set of edges will be written EG. The set of edges consists of ordered pairs of vertices: $EG \subset VG \times VG$. Since the separation profile is monotone under quasi-isometric embeddings only for graphs of bounded degree, we will always work with this hypothesis.

Definition 7.0.1. Let G be a graph. For any subset $A \subset VG$, its **boundary** is the set $\partial A = \{(a,b) \in EG, a \in A \Leftrightarrow b \in A^c\}$.

Definition 7.0.2. The **isoperimetric profile** of a graph G is the function $\Lambda_G : \mathbf{N} \to \mathbf{R}_{\geq 0}$ defined by

$$\Lambda_G(n) = \inf_{A \subset VG, |A| \leq n} \frac{|\partial A|}{|A|}$$

Note that, for an infinite connected graph, the isoperimetric profile never takes the value 0.

Definition 7.0.3. For any finite graph G, the **Cheeger constant** $h(G)$ of G is

$$h(G) = \min_{A \subset G, |A| \leq \frac{|G|}{2}} \frac{|\partial A|}{|A|}$$

Let Γ be an infinite graph. For any finite subset $F \subset V\Gamma$, let $\widetilde{F}$ be the subgraph of Γ induced on F:

- $V\widetilde{F} = F$
- $E\widetilde{F} = \{(a,b) \in V\Gamma \mid a, b \in F\}$

By a small abuse of notation, the Cheeger constant of $F \subset V\Gamma$ is $h(F) = h(\widetilde{F})$.

Definition 7.0.4. Let Γ be an infinite graph of bounded degree. The **separation profile** of Γ is the function $\mathrm{sep} : \mathbf{N} \to \mathbf{R}_{\geq 0}$ defined by

$$\mathrm{sep}(n) = \sup_{F \subset V\Gamma, |F| \leq n} |F| \cdot h(F).$$

As remarked by Hume, Mackay & Tessera in [68], it comes naturally from [67, Proposition 2.2. and Proposition 2.4.] that this definition is equivalent to the original one from Benjamini, Schramm & Timár [19].

One may notice that we use here the edge-boundaries, unlike Hume who uses vertex-boundaries. However, under the assumption that the graph has a bounded degree those two differ only by a constant factor.

Remark 7.0.5. In any graph with at least one edge there is a trivial lower bound on $\operatorname{sep}(n)$: $\operatorname{sep}(n) \geq 1$ or $\frac{\operatorname{sep}(n)}{n} \geq \frac{1}{n}$. Recall that Benjamini, Schramm & Timàr [19, Theorem 2.1] showed that a graph with bounded separation admits a regular map into a tree.

Lastly, the following convention will be used to compare functions: $f(n) \preceq g(n)$ if there is a constant $K > 0$ so that $f(n) \leq Kg(n)$. $f(n) \simeq g(n)$ if $f(n) \preceq g(n)$ and $g(n) \preceq f(n)$ (most of the time not with the same constant).

Chapter 8

Lower bound on the separation from isoperimetry

8.1 Optimal sets and their Cheeger constant

This section is devoted to the following question: given an infinite graph that has a rather large isoperimetric profile, does the same holds for the Cheeger constants of its finite subgraphs ?

There are two simple negative answers. First, in any infinite connected graph, there is an infinite subset L so that the graph induced to L is an infinite half-line. However, a finite subgraph of the half-line has the weakest Cheeger constant. Clearly, one needs to restrict a bit more the sets considered. It turns out the right thing to do is to restrict only to sets F which are "optimal" for the isoperimetric problem.

Second, consider the infinite regular tree; a graph with strong isoperimetric profile (*i.e.* the isoperimetric profile is bounded from below by a constant). Any graph induced by a finite subset has (again) the weakest possible Cheeger constant.

The aim of this section is to show that in graphs without strong isoperimetric profiles, the Cheeger constant on the optimal induced subsets is still fairly strong.

Definition 8.1.1 (Optimal sets and integers)**.** Let Γ be an infinite graph

- A subset F of $V\Gamma$ is called **optimal** if $\dfrac{|\partial F|}{|F|} = \Lambda_\Gamma(|F|)$, *i.e.*:

$$\forall G \subset V\Gamma, \qquad |G| \leq |F| \Rightarrow \frac{|\partial G|}{|G|} \geq \frac{|\partial F|}{|F|}.$$

- An integer n will be called **optimal** if there exists an optimal subset of cardinality n.

Lemma 8.1.2. *Assume F is optimal. Then:*

$$2h(F) \geq \Lambda_\Gamma\left(\frac{|F|}{2}\right) - \Lambda_\Gamma\left(|F|\right)$$

Proof. The Cheeger constant for the [finite] graph induced on F is given by looking at subsets F_1 of VF of size at most equal to $\frac{|F|}{2}$, and trying to minimise $\frac{|\partial_F F_1|}{|F_1|}$, where ∂_F denotes the boundary in F.

Let F_1 be a subset of VF of size at most equal to $\frac{|F|}{2}$, let $F_2 = F \setminus F_1$. For any (disjoint) subsets A and B of $V\Gamma$, we denote by $E(A, B)$ the set of edges of Γ that have one endpoint in A, and the other in B. We have:

$$\begin{aligned}\partial F_1 &= E\left(F_1, V\Gamma \setminus F_1\right) \\ &= E\left(F_1, V\Gamma \setminus F\right) \sqcup E\left(F_1, F_2\right) \\ &= E\left(F_1, V\Gamma \setminus F\right) \sqcup \partial_F F_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial F_2 &= E\left(F_2, V\Gamma \setminus F\right) \sqcup \partial_F F_2 \\ &= E\left(F_2, V\Gamma \setminus F\right) \sqcup \partial_F F_1.\end{aligned}$$

Then we have

$$2|\partial_F F_1| = |\partial F_1| + |\partial F_2| - |\partial F|.$$

Moreover, we have $\Lambda_\Gamma(|F_1|) \geq \Lambda_\Gamma(|F|/2)$, and as F is optimal, $\frac{|\partial F_2|}{|F_2|} \geq \frac{|\partial F|}{|F|}$ and $\Lambda_\Gamma(|F|) = \frac{|\partial F|}{|F|}$.

Using these facts, we can deduce that

$$\begin{aligned}2|\partial_F F_1| &\geq |F_1|\Lambda_\Gamma(|F_1|) + |F_2|.\frac{|\partial F|}{|F|} - |F|.\frac{|\partial F|}{|F|} \\ &= |F_1|\Lambda_\Gamma(|F_1|) - |F_1|.\frac{|\partial F|}{|F|} \\ &\geq |F_1|\left(\Lambda_\Gamma(|F|/2) - \Lambda_\Gamma(|F|)\right).\end{aligned}$$

Since this is true for any $F_1 \subset VF$ of size at most $|F|/2$, this concludes the proof. □

One already sees that these methods give basically nothing in graphs with a strong isoperimetric profile, since Λ_Γ is nearly constant. This is however to be expected since there can be no general reasonable bound in this family of graphs (the typical example would be infinite k-regular trees).

Corollary 8.1.3. *If $n > 0$ is optimal, then:*

$$2\frac{\text{sep}(n)}{n} \geq \Lambda_\Gamma(n/2) - \Lambda_\Gamma(n)$$

Proof. Assume F is optimal of cardinality n. Then $2\frac{\text{sep}(n)}{n} \geq 2h(F) \geq \Lambda_\Gamma(n/2) - \Lambda_\Gamma(n)$ □

8.2 A lower bound on the separation profile from isoperimetry

The following theorem is a consequence of the previous lemma and it applies to a large class of graphs, providing that the isoperimetric profile tends to 0 without reaching it (equivalently, the graph is amenable and has no finite connected components). It can be simplified in the case of graphs with "many symmetries", see Theorem 8.2.5.

Theorem 8.2.1. *Let G be an infinite connected amenable graph of bounded degree. Assume there is an increasing function $p : \mathbf{N} \to \mathbf{N}$ so that for any n there is a $k \in (n, p(n)]$ such that k is optimal. Choose $\epsilon \in (0,1)$. Let $n \geq 1$ be an integer. Let $m \geq 1$ be such that $\Lambda_G(m) \leq (1-\epsilon)\Lambda_G(n)$.*

Then there exists an $N \in [n, p(m)]$ such that

$$\boxed{\frac{\mathrm{sep}(N)}{N} \geq \epsilon \frac{\Lambda_G(n)}{4\log(\frac{p(m)}{n}) + 4}}.$$

For example, it applies to graphs where the isoperimetric profile is bounded above and below: $\frac{C_1}{n^a} \leq \Lambda_G(n) \leq \frac{C_2}{n^a}$. Given an optimal integer n_o the next optimal integer n_p happens at the latest when $\frac{C_1}{n_o^a} \geq \frac{C_2}{n_p^a}$, or in other words $n_p \leq (\frac{C_2}{C_1})^{1/a} n_o$ (and the function p is linear). We study this specific case in Theorem 9.1.1

Proof. We use $\Lambda = \Lambda_G$ throughout this proof. Let $n_0 = \min\{k \geq n \mid k \text{ is optimal}\}$. We define recursively $i \in \mathbf{N}$, $n_{i+1} = \max\{k \in (n_i, 2n_i] \mid k \text{ is optimal }\}$, if this set is non-empty, and, otherwise, $n_{i+1} = \min\{k \mid k \in [2n_i, p(n)] \text{ and } k \text{ optimal}\}$.

Notice that $\forall i, n_{i+2} \geq 2n_i$. Let $i_{\max}$ be the first index i for which $n_i \geq m$. Then

$$n_{i_{\max}-1} \leq m \leq n_{i_{\max}} \leq p(n_{i_{\max}-1}) \leq p(m).$$

Using Lemma 8.1.2,

$$\forall i \in [0, i_{\max}], \qquad 2\frac{\mathrm{sep}(n_i)}{n_i} \geq \Lambda(n_i/2) - \Lambda(n_i)$$

Summing up all these inequalities, one gets

$$2\sum_{i=0}^{i_{\max}} \frac{\mathrm{sep}(n_i)}{n_i} \geq \Lambda(n_0/2) - \Lambda(n_{i_{\max}}) + \sum_{i=0}^{i_{\max}-1} \Lambda(n_{i+1}/2) - \Lambda(n_i).$$

Either $\frac{1}{2}n_{i+1} \leq n_i$ (so that $\Lambda(n_{i+1}/2) \geq \Lambda(n_i)$ since Λ is decreasing) or $\Lambda(n_{i+1}/2) \geq \Lambda(n_i)$ because n_{i+1} is the next optimal integer after n_i. Either way, the sum on the right-hand side is positive, and consequently

$$\begin{aligned} 2\sum_{i=0}^{i_{\max}} \frac{\mathrm{sep}(n_i)}{n_i} &\geq \Lambda(n_0/2) - \Lambda(n_{i_{\max}}) \\ &\geq \Lambda(n) - \Lambda(m) \\ &\geq \epsilon\Lambda(n) \end{aligned}$$

To justify the second inequality, note that $\Lambda(m) \geq \Lambda(n_{i_{\max}})$ by monotonicity of Λ (and $m \leq n_{i_{\max}}$) To see that $\Lambda(n_0/2) \geq \Lambda(n)$ consider two cases:

- if $n_0 \leq 2n$, then (since Λ is non-increasing) $\Lambda(n_0/2) \geq \Lambda(n)$

- otherwise $n_0 \geq 2n$, therefore $\lfloor n_0/2 \rfloor$ is not optimal, so $\Lambda(n_0/2) = \Lambda(n)$.

From there, we can deduce that

$$\exists j \in [0, i_{\max}], \qquad \frac{\operatorname{sep}(n_j)}{n_j} \geq \frac{\epsilon}{2} \frac{\Lambda(n)}{i_{\max} + 1}$$

But recall that $\forall i, n_{i+2} \geq 2n_i$. Consequently, $n_{i_{\max}} \geq 2^{\lfloor \frac{i_{\max}}{2} \rfloor} n_0$. Furthermore, $n_{i_{\max}} \leq p(m)$. This implies: $2^{\lfloor \frac{i_{\max}}{2} \rfloor} n_0 \leq p(m)$. Since $n_0 = n$, $2^{\lfloor \frac{i_{\max}}{2} \rfloor} \leq \frac{p(m)}{n}$ Thus,

$$i_{\max} + 1 \leq 2 \lfloor \frac{i_{\max}}{2} \rfloor + 2 \leq 2 \log_2 \left(\frac{p(m)}{n} \right) + 2$$

The claim follows if we choose $N := n_j$. □

Definition 8.2.2. Let us say a graph G has **partial self-isomorphisms**, if, for every finite set $F \subset VG$, there exists another finite F' such that $F \cap F' = \varnothing$ and the graph induced on $F \cup \partial F$ is isomorphic (as a finite graph) to $F' \cup \partial F'$

Note that having partial self-isomorphisms implies the graph is infinite. This property is satisfied by fairly natural classes of graphs such as Cayley graphs, graphs with vertex-transitive (or edge-transitive) automorphisms and self-similar graphs.

Lemma 8.2.3. *Let G be a graph which has partial self-isomorphisms. Assume n is an optimal integer. The set $\{k \in (n, 2n] \mid k \text{ is optimal}\}$ is not empty.*

Proof. If there is an optimal subset F whose size is $(n, 2n]$, there is nothing to prove. Otherwise, let us construct such an optimal set of size $2n$.

Let F be an optimal set of size n. Using the property of partial self-isomorphisms, there is a set F' with $|F| = |F'|$, $|\partial F| = |\partial F'|$, and $F \cap F' = \varnothing$. Hence $|F \cup F'| = 2n$ and $\frac{|\partial(F \cup F')|}{|F \cup F'|} \leq \frac{|\partial F|}{|F|}$. Since we assumed there are no optimal sets whose size is in $(n, 2n]$, then for any $G \subset \Gamma$ such that $|G| \leq 2n$, we have $\frac{|\partial G|}{|G|} \geq \frac{|\partial F|}{|F|} \geq \frac{|\partial(F \cup F')|}{|F \cup F'|}$. Therefore $F \cup F'$ is optimal. □

Since there is always an optimal integer ($n = 1$!), any graph with partial self-isomorphisms always has infinitely many optimal n.

Remark 8.2.4. Even without the assumption that the graph G has partial self-isomorphisms, it is still possible to get some information on optimal integers, using the bounds on the isoperimetric profile. This is what is done in Theorem 8.3.2.

This strategy is very well adapted to polynomial graphs, so we applied it also in the proof of Theorem 11.2.2, taking a function $p(n)$ satisfying the more restrictive condition $\Lambda_G(p(n)) \leq \Lambda_G(n)/2$. It simplifies the proof without any loss.

Theorem 8.2.5. *Assume Γ is a connected amenable graph of bounded degree with partial self-isomorphisms. Let $n \geq 1$ and $\epsilon \in (0,1)$. Let $m \geq 1$ be such that $\Lambda_\Gamma(m) \leq (1-\epsilon)\Lambda_\Gamma(n)$.*

Then there exists an $N \in [n, 2m]$ such that

$$\boxed{\frac{\text{sep}(N)}{N} \geq \epsilon \frac{\Lambda_\Gamma(n)}{4\log(m/n)+8}.}$$

Proof. This result comes naturally from Theorem 8.2.1 and Lemma 8.2.3. □

8.3 A qualitative approach

In this subsection, we give an application of Theorem 8.2.1 using a single upper bound on the isoperimetric profile. In applications, this gives an improvement on the lower bounds obtained on the separation profile, but this comes to the cost of a weaker control on the frequency of the integers for which the bound holds. Inspired by the formulation of Theorem 8.2.1, we quantify the decreasing of real functions in the following way:

Definition 8.3.1. Let $f\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a continuous non-increasing function such that f tends to 0 at infinity. For any $\delta \in (0,1)$, we define the $\delta-$**geometric decay function** of f as:

$$p_f^\delta(x) := f^{-1}\left(\delta f(x)\right) = \min\left\{x' \mid f\left(x'\right) \leq \delta f(x)\right\}$$

We define the $\delta-$geometric decay function of a function from $\mathbf{N}^*$ to $\mathbf{R}_{>0}$ as the $\delta-$geometric decay function of either a natural extension, either a piecewise affine extension.

We can state the following theorem:

Theorem 8.3.2. *Let G be an infinite connected amenable graph of bounded degree. Let g be a continuous non-increasing positive function such that*

- $\lim\limits_{n\to\infty} g(n) = 0$
- *for any large enough n, $\Lambda_G(n) \leq g(n)$*

Then, for infinitely many integers n there exists $N \in \left[n, p_g^{1/8}(n)\right]$ such that

$$\frac{\text{sep}_G(N)}{N} \geq \frac{1}{8} \frac{\Lambda_G(n)}{\log\left(\frac{p_g^{1/8}(n)}{n}\right)+1}.$$

Proof of Theorem 8.3.2. This follows from the two lemmas below. □

Lemma 8.3.3. *Let G be an infinite connected amenable graph of bounded degree. Then for any $n \geq 1$ there exists an integer $N \in \left[n, p_{\Lambda_G}^{1/4}(n)\right]$ such that*

$$\frac{\text{sep}_G(N)}{N} \geq \frac{1}{8} \frac{\Lambda_G(n)}{\log\left(\frac{p_{\Lambda_G}^{1/4}(n)}{n}\right)+1}.$$

Proof. This is straightforward using Theorem 8.2.1, taking $\epsilon = \frac{1}{2}$, $m = \left\lfloor p_{\Lambda_G}^{1/2}(n) \right\rfloor \in \left[n, p_{\Lambda_G}^{1/4}(n)\right]$ and $p(m) = \left\lfloor p_{\Lambda_G}^{1/4}(n) \right\rfloor$. Note that in the degenerated cases where $m = p(m)$ the integer $p(m)$ is optimal since $\frac{\Lambda_G(p(m))}{\Lambda_G(p(m)-1)} \leq 1/2$. □

Lemma 8.3.4. *Let $f, g \colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ two continuous non-increasing functions such that* $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$.

We assume that for any $x > 0$ we have $f(x) \leq g(x)$. Then there exists infinitely many positive integers $(n_i)_{i\geq 0}$ such that for any i:

$$p_f^{1/4}(n_i) \leq p_g^{1/8}(n_i)$$

Proof. Suppose for a contradiction that there exists an integer N such that we have

$$p_f^{1/4}(n) > p_g^{1/8}(n) \quad \text{for any } n \geq N. \tag{8.1}$$

We claim that we have, for any $k \geq 0$,

$$p_f^{1/4^{\circ k}}(N) \geq p_g^{1/8^{\circ k}}(N). \tag{8.2}$$

We prove this by induction. If $k = 0$, then this is obvious. Now, if (8.2) is satisfied for some $k \geq 0$, we have:

$$\begin{aligned} p_f^{1/4^{\circ(k+1)}}(N) &= p_f^{1/4}\left(p_f^{1/4^{\circ k}}(N)\right) \\ &\geq p_f^{1/4}\left(p_g^{1/8^{\circ k}}(N)\right) \quad \text{by (8.2)} \\ &\geq p_g^{1/8}\left(p_g^{1/8^{\circ k}}(N)\right) \quad \text{by (8.1)} \\ &= p_g^{1/8^{\circ(k+1)}}(N). \end{aligned}$$

This proves (8.2) for every $k \geq 0$.

Let m be such that $2^m > \frac{g(N)}{f(N)}$. We compose m times the $1/4-$geometric decay function of f:

$$\begin{aligned} f\left(p_f^{1/4^{\circ m}}(N)\right) &= \frac{1}{4^m} f(N) \\ &> \frac{1}{8^m} g(N) \quad \text{by definition of } m, \\ &= g\left(p_g^{1/8^{\circ m}}(N)\right) \\ &\geq g\left(p_f^{1/4^{\circ m}}(N)\right) \quad \text{by (8.2) and } g \text{ non-increasing,} \end{aligned}$$

which contradicts the initial assumption that $f \leq g$.

□

Chapter 9

Applications

In this section, we give applications of Theorems 8.2.1, 8.2.5, and 8.3.2. We use isoperimetric profiles that have already been computed in the literature. In the course of this investigation, there are three factors that come into play:

- The geometry / the symmetries of the graph: the function $p(n)$ of Theorem 8.2.1.
- The decay of the isoperimetric profile.
- Inaccurate knowledge of the isoperimetric profile: when we only have loose bounds on the isoperimetric profile.

Our goal on this section is not to give an exhaustive overview of possible applications, but only to apply Theorem 8.2.1 in situations that seemed interesting to us. In §9.4, we give an application of Theorem 8.3.2 to solvable groups. In §9.5 we look at limit cases, where those theorems give no information.

Bendikov, Pittet & Sauer [14, Table 1 on p.52] contains many reference for the isoperimetric profile of groups. As noted in Erschler [48, §1] the isoperimetric profile Λ_G is connected to the Følner function F by the relation:

$$\Lambda_G(N) \simeq \frac{1}{F^{-1}(N)}$$

9.1 Isoperimetric profile decaying as a power of N

Recall that virtually nilpotent groups (equivalently groups for which the cardinality of a ball of radius r is bounded by polynomials in r) are the only groups where $\Lambda_G(n)$ is of the form $\frac{1}{n^{1/d}}$ (where d is the degree of the polynomial); see Pittet & Saloff-Coste [103, Theorem 7.1.5] or [102, Theorem 3.4].

A polynomial upper bound on the isoperimetric profile is given by Benjamini & Papasoglu [16, Theorem 2.1] for doubling planar graphs.

Proposition 9.1.1. *Let G be a graph of bounded degree such that if n is large enough, the following inequality holds:*

$$\frac{C_1}{n^\beta} \leq \Lambda_G(n) < \frac{C_2}{n^\beta} \tag{§}$$

for some constants $C_1, C_2, \beta > 0$.

Then if n is large enough:

$$\frac{\mathrm{sep}(n)}{n} \geq A \cdot \Lambda_G(n)$$

for some constant A.

One may notice that this proposition applies for very general graphs (although the hypotheses on G imply that it is amenable and has no finite connected component).

Proof. We use $\Lambda = \Lambda_G$ throughout this proof. Let n_0 be an integer such that (§) holds for any $n \geq n_0$. For any such n,

$$\Lambda\left(\left(\frac{C_2}{C_1}\right)^{1/\beta} \cdot n\right) \leq \frac{C_1}{n^\beta} \leq \Lambda(n)$$

Therefore with the notations of Theorem 8.2.1, we can take $p(n) = \left(\frac{C_2}{C_1}\right)^{1/\beta} \cdot n$.

Let C be the smallest integer larger than $\left(\frac{C_2}{\frac{1}{2} \cdot C_1}\right)^{1/\beta}$. Let n be an integer such that $n \geq n_0$ and let m be the smallest integer such that $m \geq Cn$. Since $m^\beta \geq 2\frac{C_2}{C_1}n^\beta$, it follows that $\Lambda(m) \leq \frac{C_2}{m^\beta} \leq \frac{\frac{1}{2} \cdot C_1}{n^\beta} \leq \frac{1}{2}\Lambda(n)$.

Then we can apply Theorem 8.2.1: there is a $N \in [n, 2m]$ such that

$$\frac{\mathrm{sep}(N)}{N} \geq \frac{\epsilon}{4} \frac{\Lambda(n)}{\log\left(\frac{p(m)}{n}\right) + 1} = \frac{\epsilon}{4} \frac{\Lambda(n)}{\log\left(\left(\frac{C_2}{C_1}\right)^{1/\beta} \cdot \frac{m}{n}\right) + 1}$$

Since $m/n - 1 \leq C$, and $\log(m/n) \leq \log(m/n - 1) + 1 \leq \log(C) + 1$, finally we get:

$$\frac{\mathrm{sep}(N)}{N} \geq K\Lambda(n)$$

with $K = \frac{\epsilon}{4\log\left(\left(\frac{C_2}{C_1}\right)^{1/\beta}\right) + \log(C) + 1}$.

The proof shows the lower bound for some (mysterious) infinite set of integers. To show it for any large enough integer: suppose that $k \geq 4C^2$, then (since k can be the upper bound of an interval with some integer N), we have:

$$\frac{\mathrm{sep}(k)}{k} \geq \frac{K}{2C+1} \frac{1}{k^\beta}$$

Indeed, assume $k \geq n_0$ and $k \geq 4C^2$. We know that there exists an integer $N \in \left[\lfloor \frac{k}{2C} \rfloor, k\right]$ such that $\frac{\mathrm{sep}(N)}{N} \geq K\Lambda(k)$. Then we have:

$$\frac{\mathrm{sep}(k)}{k} \geq \frac{\mathrm{sep}(N)}{(2C+1)N} \geq \frac{K}{2C+1} \cdot \frac{1}{N^\beta} \geq \frac{K}{2C+1} \cdot \frac{1}{k^\beta} \geq \frac{K}{(2C+1) \cdot C_2} \cdot \Lambda(k)$$

This concludes the proof. □

Note that in the case of virtually nilpotent groups, and more generally for vertex-transitive graphs with polynomial growth, Hume, Mackay & Tessera [68, Theorem 7] shows that the inequality of Proposition 9.1.1 is sharp.

Remark 9.1.2. If we assume that the graph G has partial self-isomorphisms then we can take $p(n) = 2n$ according to Lemma 8.2.3. Therefore we may improve the constant K_1 of Proposition 9.1.1.

Proposition 9.1.3. *If one assumes that for a graph of bounded degree there exist some positive constants C_1, C_2, α, and β, with $1 > \alpha > \beta$, such that we have*

$$\frac{C_1}{n^\alpha} \leq \Lambda_G(n) \leq \frac{C_2}{n^\beta},$$

for any positive integer n, then there exists $A > 0$ such that for any $n > 0$ we have:

$$\operatorname{sep}(n) \geq A \cdot \frac{n^\gamma}{\log(n)}$$

with:

- $\gamma = \frac{\beta(1-\alpha)}{\alpha}$ *if G has partial self-isomorphisms*
- $\gamma = \frac{\beta^2(1-\alpha)}{\alpha^2}$ *otherwise.*

Proof. Without assuming partial self-isomorphisms, we can apply Theorem 8.2.1, with $p(n) \simeq n^{\frac{\alpha}{\beta}}$ and $m \simeq n^{\frac{\alpha}{\beta}}$. Then for any integer n we have an integer $N \in \left[n, Cn^{\left(\frac{\alpha^2}{\beta^2}\right)}\right]$ such that $\operatorname{sep}(N) \succeq \dfrac{N \cdot \Lambda_G(n)}{\log(n)}$. Now, let k be an positive integer. Let $n = \left(\frac{k}{C}\right)^{\frac{\beta^2}{\alpha^2}}$. Then there exists some $N \in [n, k]$ such that

$$\operatorname{sep}(N) \succeq \frac{N \cdot \Lambda_G(n)}{\log(n)} \succeq \frac{n}{n^\alpha \log(n)} = \frac{n^{1-\alpha}}{\log(n)} \simeq \frac{k^\gamma}{\log(k)}$$

with $\gamma = \frac{\beta^2(1-\alpha)}{\alpha^2}$. Since $\operatorname{sep}(k) \geq \operatorname{sep}(N)$, we get the announced lower bound.

If the graph has partial self-isomorphisms, then Lemma 8.2.3 shows that $p(n) = 2n$ is a valid choice and the rest of the proof is similar. □

9.1.1 Application to pre-fractal Sierpinski carpets

Gibson & Pivarski showed in [55] some results on isoperimetry in pre-fractal graphical Sierpinski carpets.

Pre-fractal Sierpinski carpets are built using an iterating process. We consider a squared fundamental domain F_1 which is a union of little squares, obtained by removing subsquares in an admissible way which is satisfied by standard Sierpinski carpets (see Definition 2.1 and §2.2.2. of [55]). We can consider F_1 as a pattern. We make copies of F_1 in such a way that we reproduce this pattern at a larger scale. We get a bigger square that we can call F_2.

That is the first step of this process, and the pre-fractal Sierpinski carpet is the limit object that we get iterating the process indefinitely.

The carpet is then a subset of $\mathbf{R}^2$, which is a union of little squares. The associated graph is obtained putting a vertex in the centre of each of these squares, and linking vertices with an edge if and only if their squares share a common face in the carpet.

We use the notations of [55]: F_1 is the fundamental domain of the pre-fractal, m_F is the number of sub-squares in F_1 and R is the number of columns of F_1 with one or more squares removed.

Theorem 9.1.4 (Gibson & Pivarski [55])**.** *Let X be an admissible two-dimensional pre-fractal graphical Sierpinski carpet. Then*

$$\Lambda_X(n) \asymp n^{\frac{\log(R)}{m_F}-1}.$$

Proof. The lower bound comes from [55, Theorem 4.4], together with [55, Corollary 3.2], and [55, Corollary 3.8] to convert the result of the Theorem for the so-called graphical isoperimetry. The upper bound comes from the construction of explicit subsets [55, Lemma 4.1], together with [55, Lemma 3.3] (to get *graphical* subsets) and [55, Corollary 3.2]. □

The construction of pre-fractal Sierpinski carpets can be generalised in higher dimensions. The following theorem holds for standard Sierpinski carpets of any dimension.

Theorem 9.1.5 (see Gibson & Pivarski [55])**.** *Let X be the d-dimensional pre-fractal graphical standard Sierpinski carpet. Then*

$$\Lambda_X(n) \asymp n^{-\frac{\log\left(3^d-1\right)-\log\left(3^{d-1}-1\right)}{\log\left(3^d-1\right)}}.$$

Proof. The proof is very similar to the latest proof, using n-dimensional results: [55, Corollary 4.6] and [55, Corollary 4.2]. □

In this context, Proposition 9.1.1 applies, so we can deduce the following corollary:

Corollary 9.1.6. *Under the assumptions of Theorem 9.1.4 or of Theorem 9.1.5 , there exists $n_0, K_1 > 0$ such that*

$$\forall n \geq n_0 \ \frac{\operatorname{sep}_X(n)}{n} \geq K_1 \cdot \Lambda_X(n).$$

As pointed by the anonymous referee, this estimate is very close to a lower bound of Gladkova & Shum [57] (for the $S(d, 3, \{1\}, 1)$ graphical fractal in their terminology).

9.2 Isoperimetric profile with logarithmic decay

Before moving on to the next class of examples, let us recall that for polycyclic groups of exponential growth (as well as solvable groups with finite Prüfer rank and geometrically elementary solvable groups) the isoperimetric profile is known to be of the form $\frac{C_1}{\log(n)} \leq \Lambda_G(n) \leq \frac{C_2}{\log(n)}$;

see Pittet & Saloff-Coste [103, Theorem 7.2.1], [102, Theorem 3.4] for polycyclic groups, and Bendikov, Pittet & Sauer [14, Table 1], Pittet & Saloff-Coste [104] and Tessera [115] for more general statements.

Also a group of intermediate growth (*i.e.* a group where the cardinality of balls are such that $e^{n^a} \preccurlyeq |B_n| \preccurlyeq e^{n^b}$) are known to have a bound $\dfrac{C_1}{(\log n)^{\frac{1}{a}}} \preccurlyeq \Lambda_G(n) \preccurlyeq \dfrac{C_2}{(\log n)^{\frac{1}{b}-1}}$. The lower bound comes from Coulhon & Saloff-Coste [36, Theorem 1], the upper bound can be deduced from Lemma 10.1.1, and using the monotonicity of the isoperimetric profile.

Lastly, wreath products $F \wr N$ where F is finite and N has polynomial growth of degree d have an isoperimetric profile of the form $\dfrac{C_1}{(\log n)^{\frac{1}{d}}} \preccurlyeq \Lambda_G(n) \preccurlyeq \dfrac{C_2}{(\log n)^{\frac{1}{d}}}$; see Pittet & Saloff-Coste [102, §4 and §7] or Erschler [48, Theorem 1].

Proposition 9.2.1. *Let $C_1, C_2, \alpha, \beta > 0$. Let G be an infinite connected amenable graph of bounded degree with partial self-isomorphisms such that $\frac{C_1}{\log^{\alpha}(n)} \leq \Lambda_G(n) \leq \frac{C_2}{\log^{\beta}(n)}$ for any large enough n.*

Then there exists a constant K_1 such that for infinitely many N's, the following inequality holds:

$$\boxed{\frac{\mathrm{sep}(N)}{N} \geq K_1 \frac{\Lambda_G(N)}{\log N}}.$$

Remark 9.2.2. For this class of examples, we chose to apply Theorem 8.3.2. We could also have used Theorem 8.2.5. This would give a less interesting bound (of the form $\frac{\Lambda_G(N)}{(\log N)^{\alpha/\beta}}$), but, on the other hand, it gives an estimation of the frequency of the integers N satisfying the inequality.

We will use the following fact.

Fact 9.2.3. Let $g(n) = \frac{c}{(\log n)^{\beta}}$. Then we have $p_g^{1/8}(n) = n^{\left(8^{1/\beta}\right)}$, for any $n \geq 1$.

Proof. Let m, n be such that $m = n^{\left(8^{1/\beta}\right)}$. Then, $g(m) = \frac{1}{8}\frac{c}{(\log n)^{\beta}} = \frac{1}{8} g(n)$, which is the required equality. ☐

Proof of Proposition 9.2.1. Combining Theorem 8.3.2 with Fact 9.2.3, we obtain that there exists infinitely many integers $(n_i)_{i \geq 0}$ such that for any i there exists $N_i \in \left[n_i, p_g^{1/8}(n_i)\right]$ such that, if n_i is large enough,

$$\begin{aligned}
\frac{\mathrm{sep}_G(N_i)}{N_i} &\geq \Lambda_G(n_i) \frac{1/8}{\log\left(\frac{p_g^{1/8}(n_i)}{n_i}\right) + 1} \\
&\geq \Lambda_G(N_i) \frac{1/8}{\log\left(p_g^{1/8}(n_i)\right) + 1} \\
&= \Lambda_G(N_i) \frac{1/8}{8^{1/\beta}\log(n_i) + 1} \\
&\geq \Lambda_G(N_i) \frac{1/16}{8^{1/\beta}\log(N_i)}
\end{aligned}$$

☐

We deduce the three following corollaries.

Corollary 9.2.4. *Assume G is a polycyclic group of exponential growth (or, more generally, a solvable group with finite Prüfer rank or a geometrically elementary solvable group), then there exists a constant K so that for infinitely many N's we have:*

$$\frac{\operatorname{sep}(N)}{N} \geq K\frac{\Lambda_G(N)}{\log(N)} \simeq \frac{1}{\log(N)^2}$$

Corollary 9.2.5. *Assume G is a wreath product $F \wr N$ where F is finite and N has polynomial growth of degree d. Then there exists a constant K so that for infinitely many N's we have:*

$$\frac{\operatorname{sep}(N)}{N} \geq K\frac{\Lambda_G(N)}{\log(N)} \simeq \frac{1}{\log(N)^{\frac{d+1}{d}}}$$

Corollary 9.2.6. *Let G be a group of intermediate growth with $e^{n^a} \preccurlyeq |B_n| \preccurlyeq e^{n^b}$. Then there is a constant K so that for infinitely many N's we have:*

$$\frac{\operatorname{sep}(N)}{N} \geq K\frac{\Lambda_G(N)}{\log N} \succcurlyeq \frac{1}{(\log N)^{1+1/a}}$$

9.3 Isoperimetric profile with iterated logarithmic decay

There are explicit groups where the isoperimetric profile decays with a power of iterated logarithms. Example of such groups are iterated wreath products $F \wr (F \wr N)$ where F is finite and N is a nilpotent group whose growth is polynomial of degree d. For such groups, one has $\Lambda_G(n) \simeq \dfrac{1}{(\log\log n)^{1/d}}$. Iterating further the wreath products (with finite groups) gives a profile with more iterated logarithms; see Erschler [48, Theorem 1] or Gromov [63, §8.1].

Let $\log^{(k)}(x) := \underbrace{\log\log\cdots\log}_{k \text{ times}}(x)$ and $\exp^{(j)}(x) := \underbrace{\exp\exp\cdots\exp}_{j \text{ times}}(x)$.

Proposition 9.3.1. *Let G be an infinite connected amenable graph with partial self-isomorphisms such that there exists an integer k and $C, \beta > 0$ such that $\Lambda_G(n) \leq \dfrac{C}{\left(\log^{(k)} n\right)^{\beta}}$, for any large enough n. Then there exists some positive constants K and C such that the following inequality holds for infinitely many N's:*

$$\boxed{\frac{\operatorname{sep}(N)}{N} \geq K\frac{\Lambda_G(N)}{\exp^{(k-1)}\left(C\log^{(k)} N\right)}}$$

As in §9.2, we chose to apply Theorem 8.3.2. We could also have used Theorem 8.2.5, see Remark 9.2.2.

Remark 9.3.2. When $k = 2$, the denominator of the right-hand-side is a power of $\log(n)$. In that case, the lower bound is significantly weaker than Λ_G, because a power of $\log(N)$ is significantly bigger than $\log\log(N)$. This tendency continues for isoperimetric profiles which are even closer to being constant. For example, if $\Lambda_G(n) \simeq 1/\log\log\log(n)$, then the bound on $\mathrm{sep}(N)/N$ is of the form $\Lambda_G(N)/(\log N)^{\log^\eta(\log N)}$, with $\eta = 1 - C$.

However, this phenomenon is not only an artefact of the proof. Indeed, the upper bounds obtained in Corollary 10.2.2 together with Theorem 1.1 of [81] shows that the separation profile of $\mathbf{Z}_2(\mathbf{Z}_2 \wr \mathbf{Z})$ cannot dominate $\frac{n}{(\log n)^c}$ for some $c > 0$, while its isoperimetric profile is equivalent to $\frac{1}{\log\log n}$. Beyond amenable groups, the gap may be even larger: virtually free groups have a bounded separation profile (so $\mathrm{sep}(N)/N \to 0$), while their isoperimetric profile is equivalent to a positive constant.

Remark 9.3.3. Note that for any integer k and $C > 0$, we have for any $\epsilon \in (0, 1)$, we have

$$\exp^{(k-1)}\left(C \log^{(k)} N\right) \preceq N^\epsilon.$$

Indeed, let $\epsilon > 0$. For N large enough the following inequalities hold:

$$\log^{(k)}(N) \leq \frac{\log^{(k-1)}(N)}{2C} \leq \frac{1}{C}\log^{(k-1)}(N^\epsilon)$$

Therefore we have:

$$C\log^{(k)}(N) \leq \log^{(k-1)}(N^\epsilon)$$

i.e.:

$$\exp^{(k-1)}\left(C\log^{(k)}(N)\right) \leq N^\epsilon$$

To prove Proposition 9.3.1, we will use the following fact.

Fact 9.3.4. Let $g(n) = \frac{c}{\left(\log^{(d-1)}(n)\right)^\beta}$. Then for any large enough n we have

$$p_g^{1/8}(n) \leq \exp^{(k)}\left(8^{1/\beta}\log^{(k)}(n)\right).$$

Proof. Let m, n be such that $m \geq \exp^{(k)}\left(8^{1/\beta}\log^{(k)}(n)\right)$. Let us write $m' = \log^{(k)}(m)$ and $n' = \log^{(k)}(n)$. We have $m' \geq 8^{1/\beta}n'$. Hence, we have $g(m) \leq \frac{1}{8}g(n)$, which is the required inequality. □

Proof of Proposition 9.3.1. Combining Theorem 8.3.2 with Fact 9.3.4, we get that there exists infinitely many integers $(n_i)_{i\geq 0}$ such that for any i there exists $N_i \in \left[n_i, p_g^{1/8}(n_i)\right]$ such that, if n_i is large enough,

$$\begin{aligned}\frac{\mathrm{sep}_G(N_i)}{N_i} &\geq \Lambda_G(n_i)\frac{1/8}{\log\left(\frac{p_g^{1/8}(n_i)}{n_i}\right)+1}\\ &\geq \Lambda_G(N_i)\frac{1/8}{\log\left(p_g^{1/8}(n_i)\right)+1}\end{aligned}$$

$$\geq \Lambda_G(N_i)\frac{1/16}{\exp^{(k-1)}\left(8^{1/\beta}\log^{(k)}(n_i)\right)}$$

$$\geq \Lambda_G(N_i)\frac{1/16}{\exp^{(k-1)}\left(8^{1/\beta}\log^{(k)}(N_i)\right)}. \qquad \square$$

Corollary 9.3.5. *Assume $G = F \wr (F \wr N)$ where F is a finite group and N is a nilpotent group whose growth is polynomial of degree d. Then there are constants $K, C > 1$ so that for infinitely many integers N's, we have:*

$$\frac{\mathrm{sep}(N)}{N} \geq K\frac{\Lambda_G(N)}{\log(N)^C} \simeq \frac{1}{\log(\log(N))^{1/d}\cdot\log(N)^C}.$$

It is hard to tell if this lower bound is sharp. Note however, that some power of log need to be present as we exhibit an upper bound which also decays as a power of log, see Remark 10.2.4.

9.4 Solvable groups

In this subsection, we give an application of Proposition 9.3.1 to solvable groups. We prove the following theorem:

Theorem 9.4.1. *Let G be a finitely generated solvable group. If there exists $\epsilon \in (0,1)$ and $c > 0$ such that for any large enough integer n we have*

$$\mathrm{sep}_G(n) \leq cn^{1-\epsilon},$$

then G is virtually nilpotent.

It is known that any nilpotent group of rank d has a separation profile equivalent to $n^{\frac{d-1}{d}}$ (see [68, Theorem 7]). We show here that, among solvable groups, the separation profile is able to reveal nilpotence. This is quite sharp since the separation profile of the classical lamplighter group $\mathbf{Z}_2 \wr \mathbf{Z}$ (as well as any polycyclic group) is bounded above by $\frac{n}{\log(n)}$ (since it has finite Assouad dimension, see Hume [67, Theorem 1.5] ; our Corollary 10.2.2 gives a slightly weaker bound for these groups). Note that this result is definitively not true in general as non-amenable groups give counterexamples: for any $d \geq 3$, the d-dimensional hyperbolic space has a separation profile equivalent to $n^{\frac{d-2}{d-1}}$ (see [19, Proposition 4.1.]) or, more spectacularly, free groups has a bounded separation profile. This theorem partially answers a question posed to us independently by David Hume and Jérémie Brieussel:

Question 9.4.2. Is there an exponential growth solvable group Γ such that $\mathrm{sep}_\Gamma(n) \not\preceq \frac{n}{\log(n)}$?

See Question 12.0.4 in §12 for further discussions of this topic.

Proof of Theorem 9.4.1. Recall that for any group Δ we define the the derived series of Δ as the sequence of groups $\left(\Delta^{(i)}\right)_{i\geq 0}$ defined inductively by $\Delta^{(0)} = \Delta$, $\Delta^{(i)} = \left[\Delta^{(i-1)}, \Delta^{(i-1)}\right]$. A group is solvable if and only if $\Delta^{(i)} = \{e\}$ for some i and the smallest such i is called the derived length of Δ.

Let r be the size of a finite generating set of G and d be the derived length of G. If G is an abelian group, then the conclusion of Theorem 9.4.1 is valid, then we can assume that d is at least equal to 2. Let $\mathbf{F}_r$ be "the" free group on r generators, labelled by a generating set of G of size r, and let $\mathbf{S}_{d,r} := \mathbf{F}_r/\mathbf{F}_r^{(d)}$ be the free solvable on r generators of derived length d. G is a quotient of $\mathbf{S}_{d,r}$, considering the well-defined surjective group homomorphism

$$\pi_G \colon \mathbf{S}_{d,r} \to G.$$

From Tessera [115, Proposition 5.5], we have

$$\Lambda_G \leq \Lambda_{\mathbf{S}_{d,r}}.$$

Additionally, L. Saloff-Coste and T. Zheng explicited in the introduction of [111] the isoperimetric profile of the free solvable groups, namely:

$$\Lambda_{\mathbf{S}_{d,r}}(n) \simeq \left(\frac{\log^{(d)}(n)}{\log^{(d-1)}(n)} \right)^{1/r}.$$

Combining those two inequalities (and the fact that $\log^{(d)}(n) \leq \log^{(d-1)}(n)^{1/2}$ for $n \geq 4^{d-1}$), we get that there exists some constants c, d, r such that for any large enough n, we have

$$\Lambda_G(n) \leq \frac{c}{\left(\log^{(d-1)}(n)\right)^{1/2r}}. \tag{9.1}$$

Let us assume that G has an exponential volume growth. From Theorem 1 of [36], there exists a positive constant c_l such that the following inequality is true for any n:

$$\Lambda_G(n) \geq \frac{c_l}{\log(n)}. \tag{9.2}$$

Then combining Proposition 9.3.1 and (9.1), we get that there exists $K, C > 0$ such that for infinitely many integers N, we have:

$$\frac{\text{sep}(N)}{N} \geq K \frac{\Lambda_G(N)}{\exp^{(k-1)}\left(C \log^{(k)}(N)\right)}.$$

Let $\epsilon > 0$. From Remark 9.3.3, we then have, for any large enough such N,

$$\begin{aligned} \frac{\text{sep}_G(N)}{N} &\geq \Lambda_G(N) N^{-\epsilon/2} \\ &\geq \frac{c_l}{\log N} N^{-\epsilon/2} \quad \text{from (9.2)} \\ &\geq N^{-\epsilon}. \end{aligned}$$

We have shown that if the group G has exponential volume growth, then its separation profile dominates along a subsequence every sublinear power function. By contraposition, if the assumptions of Theorem 9.4.1 are satisfied, that means that the group G does not have exponential growth, meaning that it is virtually nilpotent according to the usual dichotomy for solvable groups (see for example [38]). □

9.5 Limitations of Theorem 8.2.5

Now we will be interested in graphs where the conclusion of Theorem 8.2.5 becomes trivial.

A first source of loss of sharpness is the uncertainty on the isoperimetric profile. This happens for example when the profile is bounded by different numbers of iterated logarithms. However, we will consider this limitation due to lack of information as superficial. The approach of §8.3 supports this point of view.

A second source, which we believe to be much deeper, is the decay of the isoperimetric profile. As we noticed before, Theorem 8.2.5 gives nothing in the case of graphs with almost constant isoperimetric profiles, since the integer m doesn't exist for n large enough. We will see in this subsection an example of isoperimetric profile for which the conclusion of the Theorem 8.2.5 is trivial.

Erschler [49] and Brieussel & Zheng [29] give an explicit construction of groups with up-to-constant prescribed[1] isoperimetric profile. We can deduce that the examples below has instances in Cayley graphs, which leaves little hope to generalise Theorem 9.4.1 to amenable groups. First, note that we have no particular control on N. Therefore we can consider that the conclusion of Theorem 8.2.5 is trivial if for any $\epsilon > 0$, $\frac{n\Lambda(n)}{\log(m_n)}$ is bounded, with $m_n = \min\{m : \Lambda(mn) \leq (1-\epsilon)\cdot\Lambda(n)\}$. Indeed, we would only be able to conclude that $\mathrm{sep}(n)$ - which is non-decreasing - is greater than a bounded function.

This condition is equivalent to the following:

$$\forall\epsilon > 0\ \exists\beta > 0\ \forall n \gg 1\ \Lambda\left(n\exp\left(\beta n\Lambda(n)\right)\right) \geq (1-\epsilon)\cdot\Lambda(n) \tag{*}$$

A first example Let us define a sequence $(a_k)_{k\geq 0}$ inductively: $a_0 = 1$ and $\forall k \geq 0\ a_{k+1} = a_k\exp(a_k)$.

Now we can define a unique piecewise affine function $f : \mathbf{R}_{\geq 1} \to \mathbf{R}$ such that $f(a_k) = \frac{1}{k+1}$ for any k.

For any integer k we denote by I_k the interval $[a_k, a_{k+1}]$. Let k be an integer. Let x be an element of I_k.

Then we have:

$$\begin{aligned} f\Big(x\exp\left(f(x)x\right)\Big) &\geq f\big(x\exp(x)\big) \\ &\geq f\left(a_{k+2}\right) \\ &\geq \frac{k}{k+2}f(a_k) \\ &\geq \frac{k}{k+2}f(x) \end{aligned}$$

Now let x is at most a_k, then x is in an interval $I_{k'}$ with k' beng at most equal to k. Then we can deduce that for any $x \geq a_k$ we have $f\left(x\exp\left(f\left(x\right)x\right)\right) \geq \frac{k}{k+2}f(x)$.

[1] In those examples, isoperimetric profiles aren't prescribed exactly, but up to constants. However, using Theorem 8.3.2 we can have the same bounds (up to constants) on infinitely many N's as using Theorem 8.2.1 if the isoperimetric profile was exactly prescribed.

From this fact we can deduce that (*) holds with $\beta = 1$ and $n \geq a_k$, where k is the smallest integer such that $\frac{k}{k+2} \geq 1 - \epsilon$

A second example A subtler counter-example is given by a variation of the previous example, with an exponential decay. We will see that if $\epsilon = 1/2$, there exists a β such that (*) holds, while for any $\epsilon < \frac{1}{2}$, (*) doesn't hold for any β.

Let us consider the sequence $(a_k)_{k \geq 0}$ defined previously, and let us define f recursively in the following way:

- $f(a_0) = 1$, $f(a_1) = 1/2$
- f is affine between a_0 and a_1
- $f(x \exp(x)) = \frac{1}{2} f(x)$ for any $x \geq 1$.

By construction, (*) holds for $\epsilon = \frac{1}{2}$.

Let us show that for any $\epsilon < 1/2$ we have $\forall \beta \in (0,1)$ $f(n \exp(\beta f(n) n)) \leq (1 - \epsilon) f(n)$, assuming n is large enough.

Note that for any $\alpha > 0$ we have $f(\beta f(n) n) \leq (1 + \alpha) f(n)$ for n large enough (this comes from the mean value inequality and the fact that $\lim_{n \to \infty} \frac{a_k}{2^k} = 0$). From this fact we can deduce:

$$\begin{aligned} f(n \exp(\beta f(n) n)) &\leq f(\beta f(n) n \exp(\beta f(n) n)) \\ &= \frac{1}{2} f(\beta f(n) n) \\ &\leq \frac{1 + \alpha}{2} f(n) \end{aligned}$$

Chapter 10

Upper bounds on the separation profile

10.1 From growth

The aim of this subsection is to obtain upper bound from on the separation profile using the growth of balls in the graphs. Let d denote the combinatorial distance in the graph, then $B_n(x) = \{v \in VG \mid d(x,v) \leq n\}$ is the ball of radius n with centre x.

In order to effectively apply this method, the upper bound on the size $B_n(x)$ should be independent of the choice of the ball's centre x.

Lemma 10.1.1. *Assume G is a graph such that $\sup_{x \in VG} |B_n(x)| \leq e^{f(n)}$ and $\frac{f(n)}{n} \to 0$. For any subgraph G' let $\beta_n(x)$ be the cardinality of a ball [inside the subgraph] of radius n centred at x. Let n_0 be such that $\sup\limits_{n \geq n_0} \dfrac{f(n)}{n} \leq 1$. Then for any $n \geq n_0$ and $x \in VG$ there is a $\ell \in [n, 2n]$ such that*

$$\frac{\beta_{\ell+1}(x) - \beta_\ell(x)}{\beta_\ell(x)} \leq \frac{2f(n)}{n}$$

Proof. Let us alleviate notation by using $\beta_j := \beta_j(x)$ and $\sigma_n = \beta_n - \beta_{n-1}$. Let $C_{n,k} = \min\limits_{i \in [n,n+k[} \dfrac{\beta_{i+1}}{\beta_i}$. Then $\beta_{n+k} \geq (C_{n,k})^k \beta_n$. However, since the growth of the extra k steps is bounded by $e^{f(k)}$, $\beta_{n+k} \leq \beta_{n-1} + \sigma_n e^{f(k)} = \beta_n + \sigma_n (e^{f(k)} - 1)$. Thus

$$(C_{n,k})^k \leq 1 + \frac{\sigma_n}{\beta_n}(e^{f(k)} - 1) \leq e^{f(k)}.$$

This implies that $C_{n,k} - 1 \leq e^{f(k)/k} - 1$. If $\frac{f(k)}{k} \leq 1$ then $C_{n,k} - 1 \leq e^{f(k)/k} - 1 \leq (e-1)\frac{f(k)}{k} \leq 2\frac{f(k)}{k}$. Taking $k = n$ yields the conclusion. □

For the next proposition f^{-1} denotes the generalised inverse (but one can also simply assume f is a continuous increasing function defined on the reals).

Proposition 10.1.2. *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq e^{f(n)}$ for a function f with $\lim\limits_{n \to \infty} \dfrac{f(n)}{n} = 0$. Assume the degree of the vertices is bounded by d. Then there is a constant K*

(depending on f) such that for any integer $N > K$,

$$\frac{\operatorname{sep}(N)}{N} \leq 4d \frac{f(\frac{f^{-1}(\ln \frac{N}{2})-1}{2})}{f^{-1}(\ln \frac{N}{2}) - 1}$$

Proof. For any subset $F \subset VG$ of cardinality N, consider the balls $B'_n(x)$ of radius n in the subgraph induced by F. (It does not matter where the centre x of the ball is, if one could choose, it would probably be best to choose a point that realises the diameter of F). Note that $|B'_n(x)| \leq |B_n(x)| \leq e^{f(n)}$. Let n_0 be the largest integer such that $e^{f(n_0)} \leq N/2$. Applying Lemma 10.1.1 with $n = \lfloor n_0/2 \rfloor$ implies that for some $k \in [\frac{n_0-1}{2}, n_0]$,

$$\frac{|\partial B'_k|}{|B'_k|} \leq d \frac{|B'_{k+1}| - |B'_k|}{|B'_k|} \leq d \frac{2f(n)}{n}$$

Consequently, the Cheeger constant of F is at most $2d\frac{f(n)}{n}$. This shows that $\frac{\operatorname{sep}(N)}{N} \leq 2d\frac{f(n)}{n}$, as long as $\frac{f(n)}{n} \leq 1$. The constant K is $e^{f(n_1)}$, where n_1 is the smallest integer so that $\frac{f(n)}{n} \leq 1$ for any $n \geq n_1$. □

Corollary 10.1.3. *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq K_1 e^{K_2 n^\alpha}$ for some constants $K_1, K_2 > 0$ and $\alpha \in [0, 1)$. Then, there are constants $L_1, L_2 > 0$ so that, for any $N > L_1$ large enough,*

$$\frac{\operatorname{sep}(N)}{N} \leq \frac{L_2}{(\ln N)^{\frac{1}{\alpha}-1}}$$

Proof. Use Proposition 10.1.2 with $f(x) = K_2 x^\alpha + \log K_1$. □

Corollary 10.1.4. *Let G be a graph so that $\sup_{x \in VG} |B_n(x)| \leq K n^d$ for some constants $K, d > 0$. Then, there are constants $L_1, L_2 > 0$ so that, for any $N > L_1$ large enough,*

$$\frac{\operatorname{sep}(N)}{N} \leq \frac{L_2 \log(N)}{N^{1/d}}$$

Proof. Use Proposition 10.1.2 with $f(x) \simeq d \log x + \log K$. □

10.2 From compression

Another upper bound on the separation profile can be obtained if the groups has a good embedding in L^p-spaces. Recall that for a finite graph X and a 1-Lipschitz embedding $F : X \hookrightarrow Y$ in a L^p-space, the distortion of F is

$$\operatorname{dist}(F) = \max_{x,y \in X, x \neq y} \frac{d(x,y)}{\|F(x) - F(y)\|_Y}$$

where d is the combinatorial distance on X. The L^p-distortion of X is $c_p(X) := \inf\{\operatorname{dist}(F) | F : X \hookrightarrow L^p\}$.

P.-N. Jolissaint and Valette [74, Theorem 1.1] (combined with [74, Proposition 3.3] as well as an estimate on the p-spectral gap from Amghibech [5]) give the following lower bound on the L^p distortion for a finite graph X with n vertices and Cheeger constant h:

$$c_p(X) \geq K \log(n) h \tag{††}$$

where the constant K depends only on p and the maximal degree.

Distortion can also be studied for infinite graphs, but we will here rely on the notion of compression. If G is a [connected] infinite graph, then a compression function for a 1-Lipschitz embedding $\Phi : G \hookrightarrow L^p$ is a function ρ so that

$$\forall x, y \in G, \qquad \rho(d(x,y)) \leq \|\Phi(x) - \Phi(y)\|_{L^p} \tag{‡‡}$$

The compression exponent of Φ is $\alpha(\Phi) = \liminf_{x\to\infty} \dfrac{\log \rho(x)}{\log x}$ and the compression exponent of the graph is $\alpha(G) = \sup_\Phi \alpha(\Phi)$.

Proposition 10.2.1. *Let G be a connected graph of bounded degree which admits an embedding in a L^p-space (for some p) with compression function $\rho(x) = k_1 + k_2 x^a$. Then there is a constant K so that*

$$\frac{\operatorname{sep}(n)}{n} \leq \frac{K}{(\log n)^{a/(2-a)}}$$

Proof. Assume X is a finite subgraph of G of cardinality n with Cheeger constant h, maximal degree k and diameter δ.

First note that the diameter of X is bounded by the other quantities. Indeed, pick $x, y \in X$, that realise the diameter and let $\delta' = \lfloor \frac{\delta-1}{2} \rfloor$. Then the balls of radius δ' around x and y are disjoint and at least one of them does not cover more than half the vertices. The ratio $\frac{h}{k}$ then dictates a minimal growth: $\frac{n}{2} \geq (1 + \frac{h}{k})^{\delta'}$, that is $\delta' \leq \frac{\log \frac{n}{2}}{\log(1+\frac{h}{k})}$. For $\delta \geq 6$, this gives $\frac{\delta}{3} \leq \frac{\log n}{\log(1+\frac{h}{k})}$.

Since G admits a 1-Lipschitz embedding Φ, this embedding restricts to X and (‡‡) can be rewritten as

$$\forall x, y \in X, \qquad \frac{d(x,y)}{\rho(d(x,y))} \geq \frac{d(x,y)}{\|\Phi(x) - \Phi(y)\|_{L^p}}.$$

The left-hand-side gets only bigger if one looks at $d(x,y) = \delta$. By taking the maximum on the right hand side, this leads to

$$\frac{\delta}{\rho(\delta)} \geq c_p(X).$$

Using the bound (††) of P.-N. Jolissaint and Valette [74], this gives

$$\frac{\delta}{\rho(\delta)} \geq K h \log(n).$$

For n large enough, δ is also large ($\delta \geq \frac{\log n}{\log k}$) so that $\frac{\delta}{\rho(\delta)} \leq k_3 \delta^{1-a}$ for some constant k_3. Next using the bound on δ above:

$$K h \log(n) \leq k_3 \delta^{1-a} \leq 3^{1-a} k_3 \frac{(\log n)^{1-a}}{(\log(1+\frac{h}{k}))^{1-a}}.$$

This can be rewritten as

$$Kh(\log(1+\frac{h}{k}))^{1-a} \leq 3^{1-a}k_3(\log n)^{-a}$$

Since $h \leq k$, $\log(1+\frac{h}{k}) \geq \frac{h}{k}\log 2$. With new constants, the inequality reads:

$$h^{2-a} \leq K'(\log n)^{-a}$$

This means that any [connected] subset X of cardinality n inside G has a Cheeger constant of at most $K''(\log n)^{-a/(2-a)}$. From the definition of $\mathrm{sep}(n)$ it follows that $\mathrm{sep}(n) \preceq n(\log n)^{-a/(2-a)}$.

□

Corollary 10.2.2. *Assume G is a graph with bounded degree and compression exponent α (in some L_p-space). Then for any $c < \frac{\alpha}{2-\alpha}$ there is a constant K so that*

$$\frac{\mathrm{sep}(n)}{n} \leq \frac{K}{(\log n)^c}$$

Remark 10.2.3. Using Theorem 3.3.11 instead of Proposition 10.2.1, one can actually take any $c < \alpha$. So the second column of the tabular below can actually be removed.

Here is a [non-exhaustive] list of Cayley graphs for which the compression exponent is known (references below). This table does not always use the case $p = 2$; in fact taking $p \to 1$ or $p \to \infty$ often gives better bounds, see Naor & Peres [94, Lemma 2.1].

$\alpha(G)$	$\frac{\alpha}{2-\alpha}$	Groups
1	1	polycyclic groups[(a)], the lamplighter group over $\mathbf{Z}$ with finite lamps[(a)], hyperbolic groups[(b)], Baumslag-Solitar groups[(c)], 3-manifolds groups[(d)]
$\to 1$	1	lamplighter group over $\mathbf{Z}$ with lamps in $\mathbf{Z}$[(e)], lamplighter over H of polynomial growth with finite lamps or lamps in $\mathbf{Z}$[(f)]
$\frac{1}{2}$	$\frac{1}{3}$	lamplighter over $\mathbf{Z}^2$ with lamp group H having $\alpha_2(H) \geq \frac{1}{2}$[(g)], Thompson's group F[(h)]
$\frac{1}{2-2^{1-k}}$	$\frac{1}{3-2^{2-k}}$	iterated wreath products of $\mathbf{Z}$: $(\ldots((\mathbf{Z}\wr\mathbf{Z})\wr\mathbf{Z})\ldots)\wr\mathbf{Z}$ (with k "$\mathbf{Z}$")[(i)]
$\geq \frac{1-\gamma}{1+\gamma}$	$\geq \frac{1-\gamma}{1+3\gamma}$	groups with return probability after n steps of a SRW $\leq K_2 e^{-K_1 n^\gamma}$[(j)]
$\geq 1-\nu$	$\geq \frac{1-\nu}{1+\nu}$	groups of intermediate growth with $b_n \leq e^{n^\nu}$[(k)]
$\to\geq \frac{1}{d-1}$	$\geq \frac{1}{2d-3}$	free solvable groups $\mathbf{S}_{d,r}$ of derived length d when $d > 1$[(l)]

Table's references:

(a) Tessera [114, Theorems 9 and 10]

(b) Bonk & Schramm [22] and Buyalo & Schroeder [31]

(c) Jolissaint & Pillon [73, Corollary 2]

(d) Hume [66, Theorem 5.4]

(e) Naor & Peres [94, Lemma 7.8] and [95, Theorem 6.1]; the bound is $\max\{\frac{p}{2p-1}, \frac{2}{3}\}$, take $p \to 1$

(f) Naor & Peres [95, Theorem 3.1]; the bound is $\max\{\frac{1}{p}, \frac{1}{2}\}$, take $p \to 1$

(g) Naor & Peres [94, Theorem 3.3]

(h) Arzhantseva, Guba & Sapir [9, Theorem 1.3]
(i) Naor & Peres [94, Corollary 1.3]
(j) see [59, Theorem 1.1]
(k) see either [59, Theorem 1.3(b)] or Tessera [114, Proposition 14]. In that case, the bound obtained on $\text{sep}(n)$ by Proposition 10.1.2 is better.
(l) see Sale [109, Corollary 4.2]; the bound on is $\frac{1}{p(d-1)}$ for $p \in [1,2]$, so take $p \to 1$.

There are many other groups for which one can compute the compression (the above list does not exhaust the results in the references). For example, $\alpha(G \times H) = \min\big(\alpha(G), \alpha(H)\big)$. There are also further results: on HNN-extensions see Jollissaint & Pillon [73], on relatively hyperbolic groups see Hume [66], on wreath products see Li [81].

Note that Proposition 10.2.1 is fairly sharp (in this generality). Indeed, if one looks at the product of two trees, then the compression exponent is 1. This means Proposition 10.2.1 ensures, for every $c < 1$, the existence of K_c such that $\frac{\text{sep}(n)}{n} \leq \frac{K_c}{(\log n)^c}$. On the other hand, it was shown by Benjamini, Schramm & Timár [19, Theorem 3.5] that the separation profile of such a space is $\frac{\text{sep}(n)}{n} \simeq \frac{1}{\log n}$.

Remark 10.2.4. The above corollary shows that there are amenable groups G for which $\frac{\text{sep}(n)}{n}$ decreases much more quickly than $\Lambda_G(n)$. For example, $G = F \wr (F \wr \mathbf{Z})$ has $\Lambda_G(n) \simeq \frac{1}{\log \log n}$. On the other hand this group has an isometric embedding in a Cayley graph of $\mathbf{Z} \wr (\mathbf{Z} \wr \mathbf{Z})$. In particular, its compression exponent is at least $\frac{4}{7}$. This implies that $\frac{\text{sep}(n)}{n} \preccurlyeq \frac{1}{(\log n)^c}$ for any $c < \frac{2}{5}$.

Remark 10.2.5. It is possible to show that if there is an embedding with $\rho(x) \geq K_1 (\log^{(k)} n)^\alpha$ (where $\log^{(k)}$ denotes k iterated logarithms) then the conclusion of Proposition 10.2.1 is that

$$\frac{\text{sep}(n)}{n} \leq \frac{K'}{(\log^{(k+1)} n)^{\alpha/2}}.$$

Compression function of this sort follow from the methods of [59, ¶ before Remark 3.4]. It can be shown that any [amenable] group where $P^n(e) \leq K_1 \exp(n/\log^{(k)} n)$ has an embedding in some Hilbert space with $\rho(x) \geq (\log^{(k)} n)^{1/2}$. In fact (thanks to Kesten's criterion for amenability), one can get for any amenable group an upper bound on the separation profile.

Chapter 11

Local separation profiles

In this section, we will study a local variant of the separation profile. We found it relevant in two contexts. First, in $\mathbf{Z}^d$ percolation clusters, where considering classical separation profile is trivial: since in almost every percolation configuration one can find arbitrary large balls, the profile is almost surely equal to the separation profile of $\mathbf{Z}^d$. Second, it can tackle the issue of the density of high separation subgraphs in non-vertex transitive graphs.

We will first define it, give a local version of Theorem 8.2.1 (Theorem 11.1.4), and some applications to percolation clusters in $\mathbf{Z}^d$, and to graphs of polynomial growth and of isoperimetric dimension larger than one, that we call *polynomial graphs*. Finally, we will give a theorem with a more abstract approach, that also applies to polynomial graphs, see Theorem 11.4.1.

Definition 11.0.1. Let (G, v) be a rooted graph. Let $\rho : \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function. We define the (ρ, v)-local separation profile as:

$$\mathrm{sep}_G^{\rho;v}(n) := \sup_{|F| \leq n \text{ and } F \subset B_G(v, \rho(n))} |F| \cdot h(F)$$

In comparison to the classical separation profile, which is defined as $\mathrm{sep}_G(n) = \sup_{|F|\leq n} |F| \cdot h(F)$, there is an extra condition restricting the subgraphs to lie in a given sequence of balls. One can think of it as searching for graph with big separation, but not too far from x; the "not too far"-part is quantified by the function ρ.

As for the classical separation profile, this local variant gives obstructions for the existence or regular maps (see Lemma 1.3 of [19]). We remind the reader the definition of a regular map:

Definition 11.0.2. Let X and Y be two graphs of uniformly bounded degrees. A map $f\colon X \to Y$ is said to be **regular** if there exists a constant $\kappa > 0$ such that the following two conditions are satisfied:

- $\forall x_1, x_2 \in X \; d(f(x_1), f(x_2)) \leq \kappa d(x_1, x_2)$,
- $\forall y \in Y \; \left|f^{-1}(\{y\})\right| \leq \kappa$.

The local separation profiles satisfies the following monotonicity:

Proposition 11.0.3. *Let (X, x_0) and (Y, y_0) be two rooted graphs of uniformly bounded degrees. Let $\rho : \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function. Let $f \colon X \to Y$ be a regular map such that $f(x_0) = y_0$. Then there exists a constant $K > 0$ such that for any n:*

$$\operatorname{sep}_X^{\rho, x_0}(n) \leq K \operatorname{sep}_Y^{\rho(K\cdot), y_0}(n)$$

Proof. The same proof as the proof of Lemma 1.3 of [19] works. □

Remark 11.0.4. Recall that the constant K appears both in factor of the separation profile, and in the argument of ρ (we define $\rho(K\cdot)$ by $n \mapsto \rho(Kn)$).

Note that if $\rho \succeq n$, the (v, ρ)-local separation profile coincides with the usual separation profile for vertex-transitive graphs.

The smallest (interesting) local profile is what we obtain choosing ρ to be the generalised inverse of the volume growth: $\rho(n) = \gamma_v^{-1}(n) = \sup\{x \geq 0 \mid \gamma_v(x) \leq n\}$, with $\gamma_v(n)$ denoting the size of the ball of radius n and centred at v. In that case we restrict the graphs investigated to lie in a ball of cardinality (almost) n.

In the situations we investigate, we get upper bounds in the case $\rho = \gamma_v^{-1}$. Then in what follows, we will restrict ourselves to this case. In this situation, the condition "$|F| \leq n$" is redundant and we will drop the ρ from our notation:

$$\operatorname{sep}_G^v(n) := \operatorname{sep}_G^{\gamma_v^{-1}, v}(n) = \sup_{F < B_G(v,r); |B_G(v,r)| \leq n} |F| \cdot h(F)$$

Therefore, we do not study this notion in its full generality, but we believe nevertheless that it can be relevant in some probabilistic contexts.

Local separation profile will be studied in two cases: first, $\mathbf{Z}^d$ percolation clusters, then, graphs of isoperimetric dimension greater that one and of polynomial growth.

11.1 A local version of Theorem 8.2.1

11.1.1 Statement of the Theorem

Before stating the theorem, we will introduce some notations for local isoperimetry.

Definition 11.1.1. We say that $F \subset G$ is (n, v)-optimal if:

- $F \subset B(v, n)$
- $\forall A \subset F \ \frac{|\partial A|}{|A|} \geq \frac{|\partial F|}{|F|}$

As before, we will say that an integer r is (n, v)-optimal if there exists an (n, v)-optimal set of cardinality r.

To adapt the previous result to our context, we need to introduce a local version of the isoperimetric profile:

Definition 11.1.2. Let G be a graph. Let $v \in G$ and n be a positive integer. We define for any $r > 0$:

$$\Lambda_n^v(r) = \inf_{A \subset B_G(v,n), |A| \leq r} \frac{|\partial A|}{|A|}$$

This is a mixed profile between the classical and the isoperimetric profile inside the balls introduced by Tessera in [114] .

We can now state the local versions of Lemma 8.1.2 and Theorem 8.2.1. The proofs of the corresponding statements still work in this local context, we will not write them again:

Lemma 11.1.3. *Let F be a (n, v)-optimal subset of a graph G. Then:*

$$2h(F) \geq \Lambda_n^v\left(\frac{|F|}{2}\right) - \Lambda_n^v(|F|)$$

Theorem 11.1.4. *Let G be a connected infinite graph of bounded degree. Let $v \in G$, n be a positive integer, and k be an integer. Assume there is a non-decreasing function $p : [0, k] \to [0, |B(x, n)|]$ so that for any $r \in [0, k]$ there is an $r_{op} \in (r, p(r)]$ such that r_{op} is optimal. Choose $\epsilon \in (0, 1)$. Let $r_1, r_2 \in [0, k]$ be such that $\Lambda_n^v(r_2) \leq (1 - \epsilon)\Lambda_n^v(r_1)$.*

Then there exists an $r' \in [r_1, p(r_2)]$ such that

$$\boxed{\frac{\text{sep}^v(r')}{r'} \geq \epsilon \frac{\Lambda_n^v(r_1)}{4\log(\frac{p(r_2)}{r_1}) + 4}.}$$

11.2 Application to polynomial graphs and $\mathbf{Z}^d$ percolation clusters.

We will apply Theorem 11.2.2 in graphs of polynomial growth and of dimension greater than one. We will call such a graph a *polynomial graph*. We will show that around any point the separation is bounded below by some power of n. We start with the definition of a polynomial graph:

Definition 11.2.1. Let G be a graph. Let d_1 and d_2 be two positive reals. We say that G is (d_1, d_2)**-polynomial** if there exist $b, g > 0$ such that:

- For any vertex v and any integer n $|B(v, n)| \leq bn^{d_2}$
- For any $V \subset VG$, $|\partial V| \geq g|V|^{\frac{d_1 - 1}{d_1}}$

The upcoming theorem will apply both to polynomial graphs and to percolation clusters of $\mathbf{Z}^d$. Therefore the assumptions of this theorem are less restrictive, and polynomial graphs will be a particular case where they are satisfied. In particular, we do not require every subset of vertices to satisfy the isoperimetric inequality, but only some large enough subset.

Theorem 11.2.2. *Let G be a connected infinite graph of bounded degree. We assume that there exist $d_1, d_2 > 1$ such that G locally has a growth rate at most d_2 and a large scale isoperimetric dimension at least d_1. Namely, we assume that there exist some functions $f, g, b > 0$ such that for any vertex v and any integer n:*

(i) $|B(v,n)| \leq b(v) \cdot n^{d_2}$,

(ii) For any $A \subset B(v,n)$ such that $|A| \geq f(v,n)$, $|\partial A| \geq g(v) \cdot |A|^{1-1/d_1}$.

We make the additional assumption that for any vertex v there exists an integer n_ω such that for any integer $n \geq n_\omega$ we have $f(v,n) \leq |B(v,n)|$.

Then for any $\eta \in (0,1)$, there exist $c(v), K(v), \beta > 0$ (with $\beta > d_1$), such that for any vertex v and any large enough integer n, when $f(v,n) \leq cg(v)^{\frac{\beta-d_1}{d_1-1}} n^{\frac{d_1^2(1-\eta)^2}{d_2^2}}$, then we have:

$$\mathrm{sep}_G^v(n) \geq K g(v)^\beta n^{\alpha(1-\eta)}, \text{ with } \alpha = \frac{d_1^2(d_1-1)}{d_2^3}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true with $\eta = 0$. In this case, the constant K depends on the logarithm of g.

Before proving this Theorem in 11.3, we will state the corollaries we obtain in the two particular cases that interest us. First, to polynomial graphs:

Corollary 11.2.3. *Let G be a (d_1, d_2)-polynomial graph. Then for any $\eta \in (0,1)$ there exists $c > 0$ such that for any vertex v and any integer n:*

$$\mathrm{sep}^v(n) \geq cn^{(1-\eta)\frac{d_1^2(d_1-1)}{d_2^3}}.$$

Moreover, if $d_1 = d_2$ the conclusion is also true with $\eta = 0$.

Remark 11.2.4. If d_1 equals d_2, we get the expected exponent $\frac{d_1-1}{d_1}$; this is optimal in the case of vertex-transitive graphs, see Benjamini, Schramm & Timár [19].

As a second application, we study local separation in $\mathbf{Z}^d$ percolation clusters. We obtain the following corollary:

Corollary 11.2.5. *Let $p > p_c\left(\mathbf{Z}^d\right)$. Let ω be a percolation configuration of $\mathbf{Z}^d$ of parameter p. Let $\mathcal{C}_\infty$ be an (almost surely unique) infinite connected component of ω. Let $\varepsilon \in (0,1)$. Then there exist $c(d,p) > 0$ and, for almost every ω, an integer l_ω such that for any $n \geq l_\omega$ and for any $x \in \mathcal{C}_\infty$ such that $\|x\|_\infty \leq \exp\left(n^{(1-\varepsilon)\frac{d}{d-1}}\right)$, we have:*

$$\mathrm{sep}_{\mathcal{C}_\infty}^x(n) \geq cn^{\frac{d-1}{d}}$$

This theorem will be deduced from a result on isoperimetry by G. Pete [101]. The forebears of this result can be found in the work of Barlow [12] and Benjamini & Mossel [15] (see [101] for more details on the history).

Theorem (Pete [101], Corollary 1.3.). *For all $p > p_c(Z^d)$ there exist $c_3(d,p) > 0$, $\alpha(d,p) > 0$ and (for almost all percolation configurations ω) an integer n_ω such that for all $n > n_\omega$, all connected subsets $S \subset \mathcal{C}_\infty \cap [-n,n]^d$ with size $|S| \geq c_3(\log n)^{\frac{d-1}{d}}$, we have $|\partial_{C_\infty} S| \geq \alpha |S|^{1-1/d}$.*

Proof of Corollary 11.2.5. From this theorem, one can deduce that we can apply Theorem 11.2.2 to almost every percolation configuration with $d_1 = d_2 = d$, $f(v,n) = c_3(\log(\|v\|_\infty + n))^{\frac{d-1}{d}}$, and $g(v) = \alpha$. □

11.3 Proof of Theorem 11.2.2

To show Theorem 11.2.2, we start with two lemmas. First, we can deduce from isoperimetry a lower bound on the growth of the graph:

Lemma 11.3.1. *Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 11.2.2. Let $v \in G$. Then there exists $b'(v) > 0$ such that for any large enough n, we have:*

$$|B(v,n)| \geq b'(v) \cdot g(v)^{d_1} \cdot n^{d_1}$$

Proof. We can substitute $n \mapsto |B(v,n)|$ with an piecewise affine function $B(t)$ that takes the same values on integer points. Then, for every $n > n_\omega$, we get:

$$\begin{aligned}
B(n)^{1/d_1} - B(n_\omega)^{1/d_1} &= \frac{1}{d_1}\int_{n_\omega}^{n} \frac{B'(t)}{B(t)^{1-1/d_1}} dt \\
&\geq \frac{1}{d_1}\sum_{r=n_\omega}^{n-1} \frac{B(r+1)-B(r)}{B(r+1)^{1-1/d_1}} \\
&= \frac{1}{d_1}\sum_{r=n_\omega}^{n-1} \frac{B(r+1)-B(r)}{B(r)^{1-1/d_1}} \left(\frac{B(r)}{B(r+1)}\right)^{1-1/d_1} \\
&\geq \frac{1}{d_1}\sum_{r=n_\omega}^{n-1} \frac{B(r+1)-B(r)}{B(r)^{1-1/d_1}} \frac{1}{D^{1-1/d_1}} \\
&\geq \frac{1}{d_1}\sum_{r=n_\omega}^{n-1} \frac{|\partial B(v,r)|}{B(r)^{1-1/d_1}} \frac{1}{D^{2-1/d_1}} \\
&\geq \frac{g(v)}{d_1 D^{2-1/d_1}} \cdot (n - n_\omega),
\end{aligned}$$

where D is a bound on the degrees of the vertices of G. This implies, for any large enough n,

$$B(n)^{1/d_1} \geq \frac{g(v)}{2d_1 D^{2-1/d_1}} n.$$

□

Second, we can deduce an upper bound on the isoperimetric ratio of balls using growth:

Lemma 11.3.2. *Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 11.2.2. Let $v \in G$ and $\eta \in (0,1)$.*

Then there exists $a > 0$ such that for any large enough integer n, there exists an integer r between $n^{1-\eta}$ and $2n$ such that:

$$\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{a}{|B(v,r)|^{1/d_1}}$$

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$.

To show this lemma, we will use the following facts, that we will prove later:

Fact 11.3.3. Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 11.2.2. Let $v \in G$ and $\eta \in (0,1)$.

Then there exists $A > 0$ such that for any non-negative integer n there exists $m \in [n^{1-\eta}, n]$ such that $|B(v,2m)| \leq A|B(v,m)|$.

Moreover, if $d_1 = d_2$ the conclusion is also true for $\eta = 0$: there exists $A > 0$ such that for any non-negative integer n we have $|B(v,2n)| \leq A|B(v,n)|$.

Fact 11.3.4. Let G be a connected infinite graph of bounded degree satisfying the assumptions of Theorem 11.2.2.

Let $A > 0$, v be a vertex of G and m be an integer such that $|B(v,2m)| \leq A|B(v,m)|$. Then there exists an integer r between m and $2m$ such that:

$$\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{\log(A)}{r}$$

Before proving those facts, we give a proof of Lemma 11.3.2:

Proof of Lemma 11.3.2. According to the Facts 11.3.3 and 11.3.4, there exists $A > 0$ such that for any non-negative integer n there exists $r \in [n^{1-\eta}, 2n]$ such that

$$\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{\log(A)}{r}$$

We assume that $n^{1-\eta}$ is large enough so that it satisfies the assumptions of Lemma 11.3.1. From this lemma, we have $r \leq \frac{|B(v,r)|^{1/d_1}}{b'(v)^{1/d_1} g(v)}$. Therefore $\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{a}{|B(v,r)|^{1/d_1}}$ with $a = g(v)\log(A) b'(v)^{1/d_1}$. □

We will now prove the facts.

Proof of Fact 11.3.3. Let A be such that $\frac{\eta}{2}\log(A) \geq d_2 + \log(b+1)$, and let n be a positive integer. Then:

- if $n \leq \exp\left(\frac{2}{\eta}\right)$, then up to taking a larger A, we can show that the conclusion of the fact holds, since is G is of bounded degree.

- otherwise, we assume by contradiction that for any integer m in the interval $[n^{1-\eta}, n]$, we have $|B(x,2m)| > A \times |B(x,m)|$. Then we have:

$$\begin{aligned}
|B(x,n)| &\geq A^{\log(n^\eta)-1}|B(x,n^{1-\eta})| \\
&\geq A^{\log(n^\eta)-1} \\
&\geq A^{\log(n^\eta)/2} \qquad \text{as } n \geq \exp\left(\frac{2}{\eta}\right) \\
&\geq \exp\left(\frac{\eta}{2}\log(n)\log(A)\right) \\
&\geq \exp(d_2\log(n) + \log(b+1)) \\
&= (b+1)n^{d_2}
\end{aligned}$$

(our logarithms and exponentials are in base 2)

This contradicts the assumption on the growth of the graph.

If $d_1 = d_2$, the assumption on the growth of G and the conclusion of Lemma 11.3.1 give the announced result with $A = \frac{b}{b'}2^{d_2}$. □

Proof of Fact 11.3.4. We assume by contradiction that for any r between n and $2n$ we have $\frac{|\partial B(v,r)|}{|B(v,r)|} > \frac{\log(A)}{r}$. That implies in particular the following inequality: $\frac{|B(v,r+1)|-|B(v,r)|}{|B(v,r)|} > \frac{\log(A)}{r}$. Summing-up those inequalities, we have:

$$\sum_{r=m}^{2m} \frac{|B(v,r+1)| - |B(v,r)|}{|B(v,r)|} > \log(A)\sum_{r=m}^{2m}\frac{1}{r}$$

Then we consider an piecewise affine function $B(t)$ that coincides with $|B(v,t)|$ on integer points. We get:

$$\log\left(\frac{B(2m)}{B(m)}\right) = \int_m^{2m}\frac{B'(t)}{B(t)}dt > \log(A)\int_m^{2m}\frac{1}{t}dt = \log(A)$$

Therefore $B(2m) > AB(m)$, which is a contradiction. □

We are now able to prove Theorem 11.2.2:

Proof of Theorem 11.2.2. Let v be a vertex of G and $n \geq n_\omega$ be an integer large enough so that we can apply Lemmas 11.3.1 and 11.3.2. We will require n to be (a priori) even larger in the following, satisfying some conditions that will appear later. Let η be a real of the interval $(0,1)$, that may be equal to zero if $d_1 = d_2$.

According to the isoperimetric assumption (ii) of Theorem 11.2.2, we have:

$$\forall r \in \left[g(v)^{d_1}f(v,n)^{d_1}, |B(v,n)|\right] \;\; \Lambda_n^v(r) \geq g(v)r^{-1/d_1} \qquad \text{(is1)}$$

Indeed, let r be such an integer and let F a subset of $B(v,n)$ of cardinality at most r. Two cases can occur:

- If $|F| \le f(v,n)$, then since G is infinite and connected, $|\partial F| \ge 1$. From the lower bound on r we can deduce that $\dfrac{|\partial F|}{|F|} \ge \dfrac{1}{|F|} \ge \dfrac{1}{f(v,n)} \ge g(v) r^{-1/d_1}$
- Otherwise, we have by assumption $\dfrac{|\partial F|}{|F|} \ge g(v)|F|^{-1/d_1} \ge g(v) r^{-1/d_1}$

Let r be an integer in $\left[\max\left(8^{d_2} b(v), 4^{d_2} b(v) n_\omega^{\frac{d_2}{1-\eta}}\right), |B(v,n)|\right]$. Let r' be the biggest integer such that $|B(v,2r')| \le r$. According to Lemma 11.3.2, there exists an integer r'' between $r'^{1-\eta}$ and $2r'$ such that $\dfrac{|\partial B(v,r'')|}{|B(v,r'')|} \le \dfrac{a}{|B(v,r'')|^{1/d_1}}$. Since $B(v, 2r'+2) \ge r$, we get from the growth assumption on G that $r' \ge \dfrac{1}{2}\left(\dfrac{r}{b(v)}\right)^{1/d_2} - 2 \ge \dfrac{1}{4}\left(\dfrac{r}{b(v)}\right)^{1/d_2}$. Then we have $r'' \ge \dfrac{r^{(1-\eta)/d_2}}{4^{(1-\eta)} b(v)^{(1-\eta)/d_2}} \ge n_\omega$. Therefore we have: $|B(v,r'')| \ge b'(v) \cdot g(v)^{d_1} \cdot r''^{d_1} \ge \dfrac{b'(v) g(v)^{d_1}}{4^{(1-\eta)d_1} b(v)^{(1-\eta)\frac{d_1}{d_2}}} r^{(1-\eta)\frac{d_1}{d_2}}$. We can deduce the following inequality, setting $g'(v) = \dfrac{a.4^{(1-\eta)} b(v)^{\frac{(1-\eta)}{d_2}}}{b'(v)^{1/d_1}}$:

$$\forall r \in \left[4^{d_2} b(v) n_\omega^{\frac{d_2}{1-\eta}}, |B(v,n)|\right] \ \Lambda_n^v(r) \le \frac{g'(v)}{g(v)} r^{-\frac{(1-\eta)}{d_2}}. \tag{is2}$$

Let us set $s = \left(\dfrac{2g'(v)}{g(v)^2}\right)^{\frac{d_2}{1-\eta}}$. From the inequalities (is1) and (is2), we can deduce that whenever r_1 and r_2, respectively in the validity domains of (is1) and (is2), satisfy $r_2 \ge s \cdot r_1^{\frac{d_2}{d_1(1-\eta)}}$, we have $\Lambda(r_2) \le \dfrac{1}{2}\Lambda(r_1)$. From this inequality we can deduce that $p\colon r \mapsto s \cdot r^{\frac{d_2}{d_1(1-\eta)}}$ is a suitable function to apply Theorem 11.1.4.

Let r_1 be the biggest integer such that $p(p(r_1)) \le |B(v,n)|$. Then we have $p(p(r_1+1)) \ge |B(v,n)|$. Since n is at least equal to n_ω, we can use Lemma 11.3.1, which gives $|B(v,n)| \ge b' \cdot g(v)^{d_1} \cdot n^{d_1}$. This yields:

$$\left(\frac{2g'(v)}{g(v)^2}\right)^{\left(\frac{d_2}{1-\eta} + \frac{d_2^2}{d_1(1-\eta)^2}\right)} \cdot (r_1+1)^{\frac{d_2^2}{d_1^2(1-\eta)^2}} \ge b' \cdot g(v)^{d_1} \cdot n^{d_1}$$

Therefore,

$$\begin{aligned} r_1 &\ge \frac{b'^{\frac{d_1^2(1-\eta)^2}{d_2^2}}}{(2g'(v))^{\frac{d_1^2(1-\eta)}{d_2} + \frac{1}{d_1}}} \cdot g(v)^{\left(\frac{d_1^3(1-\eta)^2}{d_2^2} + 2\frac{d_1^2(1-\eta)}{d_2} + 2d_1(1-\eta)\right)} n^{\frac{d_1^3(1-\eta)^2}{d_2^2}} - 1 \\ &\ge \frac{b'^{\frac{d_1^2(1-\eta)^2}{d_2^2}}}{(3g'(v))^{\frac{d_1^2(1-\eta)}{d_2} + \frac{1}{d_1}}} \cdot g(v)^{\left(\frac{d_1^3(1-\eta)^2}{d_2^2} + 2\frac{d_1^2(1-\eta)}{d_2} + 2d_1(1-\eta)\right)} n^{\frac{d_1^3(1-\eta)^2}{d_2^2}}, \text{ if } n \text{ is large enough.} \end{aligned}$$

Then if n is large enough r_1 is in the validity domain of (is2). Moreover, if we set

$$c = \frac{b'^{\frac{d_1(1-\eta)^2}{d_2^2}}}{(3g'(v))^{\frac{d_1(1-\eta)}{d_2}+\frac{1}{d_1^2}}}$$

$$\beta = \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2} + 2\frac{d_1(d_1-1)(1-\eta)}{d_2} + 2(d_1-1)(1-\eta) + 1$$

we get that if $f(v,n) \leq cg(v)^{\frac{\beta-d_1}{d_1-1}} n^{\frac{d_1^2(1-\eta)^2}{d_2^2}}$, then r_1 is in the validity domain of (is1). We find out the condition on $f(v,n)$ that is made in the statement of Theorem 11.2.2. Under this assumption, we can apply Theorem 11.1.4 with $r_2 = p(r_1)$. This gives:

$$\begin{aligned} \mathrm{sep}^v(|B(v,n)|) &\geq \mathrm{sep}^v(p(r_2)) \\ &\geq r_1 \frac{\Lambda_n^v(r_1)}{8\log(\frac{p(r_2)}{r_1}) + 8} \end{aligned}$$

First, from (is1) and the lower bound on r_1, we have:

$$r_1\Lambda_n^v(r_1) \geq g(v) r_1^{\frac{d_1-1}{d_1}} \geq \frac{b'^{\frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}}{(3g'(v))^{\frac{d_1(d_1-1)(1-\eta)}{d_2}+\frac{d_1-1}{d_1^2}}} \cdot g(v)^\beta n^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2}}$$

Second, we have $p(r_2) \leq |B(v,n)| \leq b(v)\cdot n^{d_2}$, from assumption (i) of Theorem 11.2.2. Then, we have from the lower bound on r_1:

$$\begin{aligned} 8\log(\frac{p(r_2)}{r_1}) + 8 &\leq 8\log\left(b(v)n^{d_2}\right) - 8\log\left(\frac{b'^{\frac{d_1^2(1-\eta)^2}{d_2^2}}}{(3g'(v))^{\frac{d_1^2(1-\eta)}{d_2}+\frac{1}{d_1}}} \cdot g(v)^{\left(\frac{d_1^3(1-\eta)^2}{d_2^2}+2\frac{d_1^2(1-\eta)}{d_2}+2d_1(1-\eta)\right)} n^{\frac{d_1^3(1-\eta)^2}{d_2^2}}\right) + 8 \\ &= 8\frac{d_2^3 - d_1^3(1-\eta)^2}{d_2^2}\log(n) + 8\log\left(\frac{b(v)(3g'(v))^{\frac{d_1^2(1-\eta)}{d_2}+\frac{1}{d_1}}}{b'^{\frac{d_1^2(1-\eta)^2}{d_2^2}} g(v)^{\left(\frac{d_1^3(1-\eta)^2}{d_2^2}+2\frac{d_1^2(1-\eta)}{d_2}+2d_1(1-\eta)\right)}}\right) + 8 \end{aligned}$$

Finally:

- if $d_1 \neq d_2$, we have for n large enough:

$$\begin{aligned} \mathrm{sep}^v(|B(v,n)|) &\geq \frac{b'^{\frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}}{9\frac{d_2^3-d_1^3(1-\eta)^2}{d_2^2}(3g'(v))^{\frac{d_1(d_1-1)(1-\eta)}{d_2}+\frac{d_1-1}{d_1^2}}} g(v)^\beta \frac{n^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2}}}{\log(n)} \\ &\geq \frac{b'^{\frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}}{9\frac{d_2^3-d_1^3(1-\eta)^2}{d_2^2}(3g'(v))^{\frac{d_1(d_1-1)(1-\eta)}{d_2}+\frac{d_1-1}{d_1^2}} b^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}}} g(v)^\beta \frac{|B(v,n)|^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}}}{\log(|B(v,n)|)} \end{aligned}$$

  Therefore we have:

$$\mathrm{sep}^v(N) \geq Kg(v)^\beta \frac{N^\alpha}{\log(N)}$$

if N is large enough, with: (D denotes a bound on the degrees of the vertices of G)

$$K = \frac{b'^{\frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}}{9^{\frac{d_2^3-d_1^3(1-\eta)^2}{d_2^2}}(3g'(v))^{\frac{d_1(d_1-1)(1-\eta)}{d_2}+\frac{d_1-1}{d_1^2}}(Db+b)^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}}}$$

$$\alpha = \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}$$

$$\beta = \frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2} + 2\frac{d_1(d_1-1)(1-\eta)}{d_2} + 2(d_1-1)(1-\eta)+1$$

Up to taking a larger η, we can substitute $(1-\eta)^2$ with $(1-\eta)$ and remove the $\log(n)$, we are done.

- if $d_1 = d_2$, we have for n large enough: (D denotes a bound on the degrees of the vertices of G)

$$\begin{aligned} \mathrm{sep}^v(|B(v,n)|) &\geq (Db+b)^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}} K \cdot g(v)^\beta n^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2}} \\ &\geq (D+1)^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^3}} K \cdot g(v)^\beta |B(v,n)|^{\frac{d_1^2(d_1-1)(1-\eta)^2}{d_2^2}} \end{aligned}$$

Therefore we have for any integer N:

$$\mathrm{sep}^v(N) \geq K g(v)^\beta N^\alpha$$

if N is large enough, with:

$$K = \frac{b'^{\frac{d_1(d_1-1)(1-\eta)^2}{d_2^2}}}{(3g'(v))^{\frac{d_1(d_1-1)(1-\eta)}{d_2}+\frac{d_1-1}{d_1^2}}(Db+b)^{\frac{(d_1^3-d_1^2)(1-\eta)^2}{d_2^3}}} \left(8\log\left(\frac{b(v)(3g'(v))^{\frac{d_1^2(1-\eta)}{d_2}+\frac{1}{d_1}}}{b'^{\frac{d_1^2(1-\eta)^2}{d_2^2}} g(v)^{\left(\frac{d_1^3(1-\eta)^2}{d_2^2}+2\frac{d_1^2(1-\eta)}{d_2}+2d_1(1-\eta)\right)}}\right)+8\right)^{-1}$$

$$\alpha = \frac{(d_1^3-d_1^2)(1-\eta)^2}{d_2^3}$$

$$\beta = \frac{-d_1^2(1-\eta)^2-d_1d_2(1-\eta)+d_1^3(1-\eta)^2+d_1^2d_2(1-\eta)+d_1d_2^2}{d_2^2}$$

(note that in this case K depends on g) □

11.4 Another approach for polynomial graphs.

In this subsection, we study local separation in graphs of polynomial growth and of isoperimetric dimension greater than 1. Using a more abstract and simple approach, we show again that around any point the separation is bounded below by a power of n, that improves Theorem 11.2.2 in some cases. We will prove a statement in a slightly more general context than polynomial graphs, with a local flavour, which is very natural regarding to the proof. We will then formulate the theorem in the setting of polynomial graphs (Corollary 11.4.3). Here is our theorem:

Theorem 11.4.1. *Let G be an infinite graph of bounded degree such that there exists $d_2 \geq d_1 > 1$ and two positive functions $b(v)$ and $g(v,n)$ such that for any vertex v and any positive integer n:*

- $\gamma_v(n) := |B(v,n)| \leq b(v)n^{d_2}$,
- *for any* $V \subset B(v,n)$, $|\partial V| \geq g(v, \gamma_v(n))|V|^{\frac{d_1-1}{d_1}}$.

We assume moreover that $d_1^2 > d_2 - d_1$. *Then for any* $\eta > 0$ *there exists* $s > 0$ *depending only on* d_1, d_2, b *and* η *such that for any positive integer* n *and any vertex* v*:*

$$\mathrm{sep}_G^v(n) \geq s \cdot g(v,n)^\beta \cdot n^{(1-\eta)\alpha} \quad \text{with } \alpha = \frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}$$
$$\text{and } \beta = \frac{d_1^2+d_1-1}{d_1}$$

Moreover, if $d_1 = d_2$ *the conclusion is also true for* $\eta = 0$.

Remark 11.4.2. The conclusion of the theorem implies in particular that the classical (or *global*) separation profile is bounded below: For any $\eta > 0$ there exists $s(v,\eta) > 0$ such that for any positive integer n:

$$\mathrm{sep}_G(n) \geq s \cdot g(v,n)^\beta \cdot n^{(1-\eta)\alpha}$$

This corollary follows, using the terminology introduced in Definition 11.2.1:

Corollary 11.4.3. *Let* G *be a* (d_1,d_2)*-polynomial graph such that* $d_2 - d_1 < d_1^2$. *Then for any* $\eta \in (0,1)$ *there exists* $c > 0$ *such that for any vertex* v *and any integer* n*:*

$$\mathrm{sep}_G^v(n) \geq cn^{(1-\eta)\alpha}, \quad \text{with } \alpha = \frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}.$$

Moreover, if $d_1 = d_2$ *the conclusion is also true for* $\eta = 0$.

Remark 11.4.4. As in Corollary 11.2.3, in the case where d_1 equals d_2 we get the expected exponent $\frac{d_1-1}{d_1}$, optimal in the case of vertex-transitive graphs. If d_1 is smaller than d_2 one can notice that Corollaries 11.2.3 and 11.4.3 do not give the same exponents (the best can be given by one or the other, depending on the values of d_1 and d_2), which is an interesting demonstration of the fact that, despite the use of the same ingredients, the two approaches are essentially different. Dropping the $(1-\eta)$ factor, the exponents of Corollaries 11.2.3 and 11.4.3 are respectively $\frac{d_1^2(d_1-1)}{d_2^3}$ and $\frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}$. For example, with $d_1 = 2$ and $d_2 = 3$, Corollary 11.4.3 gives $\frac{1}{4}$, which is bigger than $\frac{4}{27}$, given by Corollary 11.2.3. With $d_1 = 2$ and $d_2 = \frac{11}{2}$, Corollary 11.2.3 gives $\frac{32}{1331} \approx 0.024$, which is bigger than $\frac{1}{44} \approx 0.023$ given by Corollary 11.4.3. Moreover, Corollary 11.4.3 gives nothing when $d_2 \geq d_1^2 + d_1$, while Corollary 11.2.3 always gives a consistent result.

Let us explain the strategy of the proof of Theorem 11.4.1. We call *isoperimetric ratio* of a set the ratio between the size of its boundary and its size, $\frac{|\partial \cdot|}{|\cdot|}$. Our a goal is to find, for any n, a subset X of $B(v,n)$ for which we can bound below its cardinality and its Cheeger constant in order to get a bound on $|X|h(X)$. Adapting slightly the proof of Lemma 8.1.2, we see that to bound its Cheeger constant, it suffices for X to verify two conditions: first that it has a lower (or equal) isoperimetric ratio than its subsets, and second that the isoperimetric ratio of its small (less than a half) subsets is bigger, by a controlled factor greater than 1. To get those properties, we proceed recursively: starting from a ball $B(v,n)$, we take smaller and smaller subsets that violates the second condition, and when there is no such small subset, we finally take a subset of the resulting set that minimises the isoperimetric ratio. Our hypothesis on the growth of the graph gives an upper bound on the isoperimetric ratio the size of the boundary of $B(v,n)$, and the hypothesis on the isoperimetric dimension ensures a lower bound on the cardinality of the final set and on its isoperimetric ratio, leading to a bound on its Cheeger constant.

In the proof of Theorem 11.4.1, we will use the following lemma. As mentionned above, this is a local version of Lemma 8.1.2.

Lemma 11.4.5. *Let X be a finite subset of an infinite graph, satisfying the following properties:*

- $\forall Y \subset X \ \frac{|\partial Y|}{|Y|} \geq \frac{|\partial X|}{|X|}$,
- $\forall Y \subset X \ \left(|Y| \leq \frac{|X|}{2} \Rightarrow \frac{|\partial Y|}{|Y|} \geq (1+\epsilon) \frac{|\partial X|}{|X|} \right)$.

Then,

$$2h(X) \geq \epsilon \frac{|\partial X|}{|X|}.$$

Proof. The proof is very similar to the proof of Lemma 8.1.2. Let F_1 be a subset of X such that $|F_1| \leq \frac{|X|}{2}$. We denote $F_2 = X \setminus F_1$. Then we have:

$$\begin{aligned} 2|\partial_X F_1| &= |\partial F_1| + |\partial F_2| - |\partial X| \\ &\geq (1+\epsilon) \frac{|\partial X|}{|X|} |F_1| + \frac{|\partial X|}{|X|} |F_2| - |\partial X| \\ &= \epsilon \frac{|\partial X|}{|X|} |F_1| + \frac{|\partial X|}{|X|} (|F_1| + |F_2|) - |\partial X| \\ &= \epsilon \frac{|\partial X|}{|X|} |F_1| \end{aligned}$$

Then we have $2\frac{|\partial_X F_1|}{|F_1|} \geq \epsilon \frac{|\partial X|}{|X|}$. Since this is true for any subset F_1 of X containing at most half of its points, we have shown the announced inequality. □

Proof of Theorem 11.4.1. Let v be a vertex of G. We start by giving a doubling property of the graph G. Let η be a real of the interval $(0,1)$, that may be equal to zero if $d_1 = d_2$. Let n be an integer at least equal to 2. Let m, A, and r be given by Facts 11.3.3 and 11.3.4. Then we have:

- $n^{1-\eta} \leq r \leq 2n$
- $\frac{|\partial B(v,r)|}{|B(v,r)|} \leq \frac{\log(A)}{r}$

Let us write $F_1 = B(v, r)$ the ball of G centred at v of radius r. Let $g = g\left(v, |B(v, 2n)|\right)$ and ϵ be a positive real small enough so that $2^{\frac{1}{d_1+1}} \le \frac{2^{1/d_1}}{1+\epsilon}$.

We construct a finite decreasing sequence $(F_i)_i$ by induction, in the following way: let i be a positive integer. If F_i is defined, then:

- If there exists a subset of A of F_i such that $|A| \le \frac{|F_i|}{2}$ and $\frac{|\partial A|}{|A|} \le (1+\epsilon)\frac{|\partial F_i|}{|F_i|}$, then we take F_{i+1} being such a set.
- Otherwise, we stop the sequence.

Let k denote the number of terms of this sequence. From the isoperimetric dimension hypothesis we have: $|F_k|^{-1/d_1} \le \frac{1}{g}\frac{|\partial F_k|}{|F_k|} \le \frac{(1+\epsilon)^k}{g}\frac{|\partial F_1|}{|F_1|} \le \frac{(1+\epsilon)^k}{g}\frac{\log(A)}{r}$, therefore we can deduce that $|F_k| \ge \frac{g^{d_1} r^{d_1}}{\log(A)^{d_1}(1+\epsilon)^{kd_1}}$.

By construction, we have $|F_k| \le 2^{-k}|F_1|$. Hence, we can deduce that

$$2^{k/d_1}|F_1|^{-1/d_1} \le |F_k|^{-1/d_1} \le \frac{(1+\epsilon)^k}{g}\frac{\log(A)}{r},$$

which means that $2^{\frac{k}{d_1+1}} \le \left(\frac{2^{1/d_1}}{1+\epsilon}\right)^k \le \frac{\log(A)}{g}\frac{|F_1|^{1/d_1}}{r} \le \frac{\log(A)b^{1/d_1}}{g}\frac{r^{d_2/d_1}}{r} = \frac{\log(A)b^{1/d_1}}{g} r^{\frac{d_2-d_1}{d_1}}$.

Then, since $(1+\epsilon)^{kd_1} \le 2^{\left(\frac{1}{d_1}-\frac{1}{d_1+1}\right)kd_1} = 2^{\frac{k}{d_1+1}}$, we can deduce that, with $c = \log(A)^{-(d_1+1)}b^{-1/d_1}$,

$$|F_k| \ge c \cdot g^{d_1+1} \cdot \frac{r^{d_1}}{r^{\frac{d_2-d_1}{d_1}}} = c \cdot g^{d_1+1} \cdot r^{\frac{d_1^2-(d_2-d_1)}{d_1}}$$

We can take a final set X minimising $\frac{|\partial \cdot|}{|\cdot|}$ among subsets of F_k.

Therefore, X satisfies the following properties:

- $\forall Y \subset X \ \frac{|\partial Y|}{|Y|} \ge \frac{|\partial X|}{|X|}$
- $\forall Y \subset X \ \left(|Y| \le \frac{|X|}{2} \Rightarrow \frac{|\partial Y|}{|Y|} \ge (1+\epsilon)\frac{|\partial X|}{|X|}\right)$

Then, we can apply Lemma 11.4.5 to X. We get

$$2h(X) \ge \epsilon \frac{|\partial X|}{|X|}.$$

By construction of F_k, we have $|X| \ge |F_k|/2$. We have:

$$\begin{aligned}
|X|h(X) &\ge \frac{\epsilon}{2} \cdot |\partial X| \\
&\ge \frac{\epsilon}{2} g \cdot |X|^{\frac{d_1-1}{d_1}} \\
&\ge \frac{\epsilon}{2} g 2^{\frac{1-d_1}{d_1}} \cdot |F_k|^{\frac{d_1-1}{d_1}} \\
&\ge \frac{\epsilon}{2} g 2^{\frac{1-d_1}{d_1}} c^{\frac{d_1-1}{d_1}} g^{\frac{(d_1-1)(d_1+1)}{d_1}} \cdot r^{\frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2}} \\
&\ge c' \cdot g^{\frac{d_1^2+d_1-1}{d_1}} \cdot n^{d_2(1-\eta)\alpha}
\end{aligned}$$

With $c' = \frac{\epsilon}{2} \cdot 2^{\frac{1-d_1}{d_1}} \cdot c^{\frac{d_1-1}{d_1}}$ and $\alpha = \frac{(d_1-1)(d_1^2-(d_2-d_1))}{d_1^2 d_2}$.

We have shown that there exists a positive constant c' such that for any integer $n \geq 2$ and any vertex v, we have:

$$\begin{aligned}
\mathrm{sep}_G^v(|B(v,2n)|) &\geq c' g\left(v, |B(v,2n)|\right)^{\frac{d_1^2+d_1-1}{d_1}} \cdot n^{d_2(1-\eta)\alpha} \\
&\geq \frac{c'}{b^{(1-\eta)\alpha} 2^{d_2(1-\eta)\alpha}} g\left(v, |B(v,2n)|\right)^{\frac{d_1^2+d_1-1}{d_1}} \cdot |B(v,2n)|^{(1-\eta)\alpha}
\end{aligned}$$

The announced result follows.

□

Chapter 12

Questions

Although we showed that there are plenty of optimal integers, it turns out it's incredibly hard to describe optimal sets. In the case of $\mathbf{Z}^d$ this can probably be achieved with the Loomis-Whitney inequality (see [85]).

Question 12.0.1. Give an explicit description of the optimal sets in the discrete Heisenberg groups (or in any amenable group which is not virtually Abelian).

For the "continuous" version of the Heisenberg group, this is an old open question. But perhaps the discrete case is easier.

More generally, one could ask whether it is possible to find the optimal sets in semi-direct products of "well-known cases": assuming the optimal sets of the [finitely generated] groups G_1 and G_2 are known [for some generating sets S_1 and S_2], can the optimal sets of $G_1 \rtimes G_2$ be of the form $F_1 \times F_2$ (where F_i is an optimal set for the Cayley graph of G_i w.r.t. S_i)?

Another interesting question on optimal sets would be the following:

Question 12.0.2. If G is a graph whose isoperimetric profile is known up to a multiplicative constant, what can we say about the density of sets whose separation is good?

Let us shortly describe two interpretations of this question. First, Proposition 9.1.3 only uses the fact that $p(n) \leq Kn^c$ for some $K > 0$ and $c > 1$. This gives a fairly low density of optimal integers, leaving open the possibility for much higher densities. For example, if $K = 1$ and $c = 2$, then the sequence of optimal integers could be as sparse as $2, 4, 16, 256, \dots$

Second (in the spirit of local separation), one could also fix some n, r and K and look at the density of vertices x for which a ball of radius r contains a set of size n which is up to a multiplicative factor of K as hard to cut as the best set for that given n.

Here are many inequalities between the separation and isoperimetric profile which seem natural (they might be easy, or hard, to prove or disprove):

Question 12.0.3. 1. If G is the Cayley graph of a group, more generally a vertex-transitive graphs, $\dfrac{\mathrm{sep}(N)}{N} \overset{?}{\preccurlyeq} \Lambda_G(N)$.

2. If G is the Cayley graph of an amenable group, $\dfrac{\mathrm{sep}_G(N)}{N} \overset{?}{\succcurlyeq} \Lambda_G(N/2) - \Lambda_G(N)$

3. If G is the Cayley graph of a polycyclic group, $\frac{\mathrm{sep}_G(N)}{N} \overset{?}{\simeq} \frac{1}{\log n}$ (For such groups $\Lambda_G(N) \simeq \frac{1}{\log n}$.)

4. If G is the Cayley graph of a group, is $\frac{\mathrm{sep}(N)}{N} \overset{?}{\succcurlyeq} N\big(\Lambda_G(N-1) - \Lambda_G(N)\big)$

The following associated question was also posed to us in connection with Question 9.4.2:

Question 12.0.4. Does the classical lamplighter group $\mathbf{Z}_2 \wr \mathbf{Z}$ coarsely embeds in any exponential growth solvable group ?

A positive answer to this question would give a (negative) answer to Question 9.4.2. In fact, regular maps from the lamplighter to solvable groups (of exponential growth) would be enough (and should be easier to produce). Note that one cannot replace the lamplighter with a polycyclic group (of exponential growth) in Question 12.0.4. Indeed, the asymptotic dimension increases under a regular map (see Benjamini, Schramm and Timàr [19, §6]) and the classical lamplighter has asymptotic dimension 1 while polycyclic groups have dimension ≥ 2 (they are finitely presented; see Gentimis [54] for both results). Consequently, there are no regular maps from any polycyclic group to the classical lamplighter group (which is a solvable group).

It is very natural to ask Question 12.0.4 more generally for exponential growth amenable groups. However, in [69], Hume and Mackay gave examples of elementary hyperbolic groups with an arbitrary low profile, along a subsequence. Then, the lamplighter group cannot coarsely embed in those groups.

The following interesting question was suggested by the referee in light of Corollary 6.0.8:

Question 12.0.5. Are there amenable groups of exponential growth which have a regular embedding in an hyperbolic space of dimension $d > 1$.

Question 12.0.6. Does there exists a vertex-transitive graph G such that for some/any vertex v we have $\mathrm{sep}_G^{\gamma_v^{-1},v} \prec \mathrm{sep}_G$? With G amenable ?

This question in linked with the issue of controlling the diameter of high separation graphs. Indeed, we expect those graphs to have a small diameter but finding such a counter-example would be very interesting.

The following question is very natural, after Theorem 6.0.7 which answers positively when G is solvable and Theorem 6.0.5 which does it for groups of subexponential growth.

Question 12.0.7. For G amenable, does $\mathrm{sep}(n) \preceq n^{1-\epsilon}$ implies that G is virtually nilpotent?

Part III

Poincaré profiles of lamplighter diagonal products

We exhibit finitely generated groups with prescribed Poincaré profiles. It can be prescribed for functions between $n/\log n$ and linear, and is sharp for functions at least $n/\log\log n$. These profiles were introduced by Hume, Mackay and Tessera in 2019 as a generalization of the separation profile, defined by Benjamini, Schramm and Timár in 2012. The family of groups used is based on a construction of Brieussel and Zheng. As applications, we show that there exists bounded degrees graphs of asymptotic dimension one that do not coarsely embed in any finite product of bounded degrees trees, exhibit hyperfinite sequences of graphs of arbitrary large distortion in L^p-spaces, and prove the existence of a continuous family of pairwise uncomparable amenable groups.

Profils de Poincaré de produits diagonaux d'allumeurs de réverbères

Nous exhibons des groupes de type fini dont les profils de Poincaré sont prescrits. Ces derniers peuvent être prescrits pour une grande classe de fonctions situées entre $n/\log n$ et n. Les bornes obtenues sont optimales pour des fonctions minorées par $n/\log\log n$. Les profils de Poincaré ont étés introduits par Hume, Mackay et Tessera en 2019 et généralisent le profil de séparation défini par Benjamini, Schramm et Timár en 2012. Notre résultat est obtenu en utilisant une famille de groupes construits récemment par Brieussel et Zheng. En guise d'applications, nous obtenons qu'il existe des graphes de degré borné et de dimension asymptotique égale à un qui ne se plongent grossièrement dans aucun produit fini d'arbres de degrés ornés, exhibons des suites hyperfinies de graphes dont la p-distortion est arbitrairement grande, et prouvons l'existence d'une famille continue de groupes moyennables qui sont grossièrement incomparables deux-à-deux.

Chapter 13

Introduction

The **separation profile** was introduced by Benjamini, Schramm & Timár [19]. As remarked by Hume [67], the separation profile of an (infinite) graph G at $n \geq 0$ can be defined by

$$\mathrm{sep}_G(n) = \sup \{|V\Gamma|\, h(\Gamma) \colon \Gamma \subset G, |V\Gamma| \leq n\},$$

where $h(\Gamma)$ denotes the Cheeger constant of the graph Γ. Hume, Mackay and Tessera generalized this profile by defining, for any $p \in [0, \infty]$ the L^p**-Poincaré profile** of an (infinite) graph G by:

$$\Pi_{G,p}(n) = \sup \{|V\Gamma|\, h_p(\Gamma) : \Gamma \subset G, |V\Gamma| \leq n\},$$

where $h_p(\Gamma)$ denotes the L^p-Cheeger constant of the graph Γ (see Chapter 17 for details). For graphs of bounded degree, the L^1-Poincaré profile and the separation profile are equivalent up to constants.

A map between graphs of bounded degree is called **regular** if it is Lipschitz and if the preimage of singletons have a uniformly bounded cardinality. For example, coarse embeddings and quasi-isometric embeddings are regular maps. Separation and Poincaré profiles have the property to be *monotone* under regular maps, see Theorem 14.2.2. In this generality, the only other invariants known to have this property are volume growth and asymptotic dimension.

Separation and Poincaré profiles have interesting relations with other known properties or invariants: hyperbolicity [19, 68, 69], volume growth [68, 80], finite Assouad-Nagata dimension [67], isoperimetric profile [80]. Nevertheless, these profiles are able to give new information: here, we compute a variety of Poincaré profiles for groups all having exponential growth and asymptotic dimension one. On the other hand, the separation profile doesn't always detect the amenability of groups: for example polycyclic groups and product of free groups both have a separation profile $\simeq \frac{n}{\log n}$, and hyperbolic spaces $\mathbf{H}^d$ have the same separation profile as $\mathbf{Z}^{d-1}$, when d is at least three. In the latter example, it is worth noticing that Poincaré profiles can make a distinction between $\mathbf{H}^d$ and $\mathbf{Z}^{d-1}$.

It is clear from the definition that any Poincaré profile is at least constant and at most linear. It is then natural to ask what are the possible profiles within this range. Here, we obtain any Poincaré profile between $\frac{n}{\log \log n}$ and n, see Theorem 1 (the lower bounds on Poincaré profiles are only valid along a subsequence). To our knownledge, these are the first examples of amenable groups with profiles strictly between $\frac{n}{\log n}$ and n; it is worth noticing that our lower bounds are only valid along a subsequence. Our examples come from Brieussel and Zheng [29] and are amenable groups with exponential growth and asymptotic dimension one. This shows that amenable groups can have a variety of behaviours with respect to Poincaré profiles, even within families of groups that are indistinguishable by these classical invariants. As a corollary, we obtain a continuum of amenable groups with pairwise distinct regular classes, see Theorem 5.

Our main result is the following.

Theorem 1. *There exist two universal constants κ_1 and κ_2 such that the following is true. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing and $\lim_\infty \rho = \infty$. We assume that ρ is injective and that there exists some $\alpha > 0$ such that $\frac{\rho^{-1}(x)}{\exp(x^\alpha)}$ is non-decreasing. Then, there exists a finitely generated elementary amenable group Δ of exponential growth and of asymptotic dimension one such that for any $p \in [1, \infty)$,*

$$\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \textit{for any } n,$$
$$\textit{and} \quad \Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\rho(\log n)} \quad \textit{for infinitely many } n\textit{'s.}$$

This theorem applies for example with $\rho = \log$. These groups are built using the construction of Brieussel and Zheng in [29]. As it is shown in this paper, the group Δ of Theorem 1 also have prescribed speed and entropy of random walk equivalent to $\frac{n}{\rho(\sqrt{n})}$, ℓ^p-isoperimetric profile equivalent to $\rho(\log(n))^{-p}$, a return probability defined implicitly with ρ, and an L^p-equivariant compression gap of the form $\left(\frac{\rho}{\log^{1+\epsilon}(\rho)}, \rho\right)$. See [29, Theorem 1.1] for details.

Unfortunately, we were not able to make our upper and lower bounds match each other in all cases, but only on *high* separation profiles. In general, we have the following statement.

Theorem 2. *There exist two universal constants κ_1 and κ_2 such that the following is true. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing and $\lim_\infty \rho = \infty$. Then, there exists a finitely generated elementary amenable group Δ of exponential growth and of asymptotic dimension one such that for any $p \in [1, \infty)$,*

$$\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \textit{for any } n,$$
$$\textit{and} \quad \Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\rho(\log n)^2} \quad \textit{for infinitely many } n\textit{'s.}$$

The lower bound of Theorem 2 can be improved for functions ρ that grow slower than $\sqrt{x}$. This is the following theorem:

Theorem 3. *Under the setting of Theorem 2, there exists a universal constant $\kappa_3 > 0$ such that if ρ is injective and there exists $a \in (0, 1/2)$ such that $\frac{\rho^{-1}(x)}{x^{1/a}}$ is non-decreasing, then, for any $p \in [1, \infty)$,*

$$\Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_3 \frac{n}{\rho(\log n)^{\frac{1}{1-a}}} \quad \textit{for infinitely many } n\textit{'s.}$$

See Theorem 18.0.5 for a more general statement.

The upper bounds are obtained using compression in L^p spaces. The compression of a 1-Lipschitz embedding $f\colon G \to L^p$ is defined by

$$\rho_f(t) = \inf\left\{ \|f(g) - f(h)\|_p \mid d_G(g,h) \geq t \right\}.$$

The upper bounds of Theorems 1 and 2 are obtained from the following more general statement:

Theorem 4. *Let G be a graph of bounded degree. Then there exists two constants $c_1, c_2 > 0$, depending only on the maximum degree in G, such that if $f\colon VG \to L^p$ is a 1-Lipschitz map, then*

$$\Pi_{G,p}(n) \leq c_1 \frac{n}{\rho_f(c_2 \log n)},$$

for all $p \in [1,\infty)$ and $n \geq 0$.

This theorem is of independent interest, since it holds in great generality. Moreover, this inequality is known to be sharp for finite products of bounded degree trees. Indeed, they can be embedded in L^p spaces with compression function $\rho \succeq t^{1-\epsilon}$ (see [114, Corollary 2]). Then, Theorem 4 gives that their Poincaré profiles satisfy $\Pi_p \preceq \frac{n}{(\log n)^{1-\epsilon}}$ (for $p = 1$, one can actually take $\rho = t$). This is quite optimal since on the other hand, we have $\Pi_p \succeq_p \frac{n}{\log n}$, as soon as at least two of the trees coarsely contain the infinite binary tree, see [19] and Theorem 14.3.3.

More generally, the same reasoning applies to finite products of finitely generated hyperbolic groups (Tits alternative).

Other cases are examined in the more precise statement Theorem 17.1.2.

13.1 About the proofs

Lower bounds The lower bounds of Theorems 1, 2 and 3 are obtained by exhibiting particular subgraphs of the groups Δ. These subgraphs are compared to Cartesian powers of finite graphs. Along the way, we make a general study of these graphs in section 16.1. In particular, we prove the following proposition, that might be of independent interest:

Proposition 13.1.1. *Let G be a connected regular graph. Let k be a positive integer and $G^k = \underbrace{G \times \cdots \times G}_{k \text{ terms}}$ the Cartesian product of k copies of G. Then*

$$\frac{a}{k} \leq h(G^k) \leq \frac{b}{\sqrt{k}},$$

with $a = \left(\frac{h(G)}{2\deg G}\right)^2$ and $b = (2\sqrt{2}+2)\sqrt{\deg(G)h(G)}$.

We recall that for any finite graph H, $h(H)$ denotes the Cheeger constant of H (see Definition 16.1.1). Since G^k can have an arbitrary large degree, it is important to remark that Cheeger constants are defined using extern-vertex boundary, see Proposition 16.1.7. The proof relies on classical spectral graph theory, and results of Bobkov, Houdré and Tetali [20] on vertex-isoperimetry and L^∞-spectral gap.

Upper bounds As mentioned before, the upper bounds are obtained mapping graphs in L^p spaces. The basic idea is to use such an embedding as a "test" function in the definition of the L^p-Cheeger constant (see Definition 14.1.1, Proposition 17.1.6, Theorem 17.1.2). In the particular case of the groups studied in this paper, the upper bounds of Theorems 1 and 2 follow from explicit embeddings given in [29].

13.2 Applications

We present here some applications of the preceding statements.

A continuum of distinct regular classes Given two graphs of bounded degree G and H, let us recall that a map from G to H is called **regular** if it is Lipschitz and if the preimage of singletons of H have a uniformly bounded cardinality (see Definition 14.2.1). The following theorem is a corollary of Theorem 18.0.5, which is the technical version of Theorem 1.

Theorem 5. *There exists an uncountable family of amenable groups of asymptotic dimension one $(G_r)_{r\in\mathbf{R}}$ such that for any $r \neq s$ there is no regular map from G_s to G_r.*

Let us recall that quasi-isometric and coarse embeddings are regular maps. As stated above, this result is new. See Hume [67, Theorem 1.2] for an analog statement, with $C'(1/6)$ small cancelation groups. Our proof will use the following fact:

Fact 13.2.1. Let g be a function satisfying the hypothesis of Theorem 1. Then, there exists a sequence of integers $(v_n)_{n\geq 0}$ such that the following is true: for any function f satisfying the assumptions of Theorem 1 and such that $f \geq g$, there exists a group Δ_f and a sequence of integers $(u_m)_{m\geq 0}$ such that:

- $\Pi_{\Delta_f,p}(n) \leq \kappa_1 \frac{n}{f(\log n)}$ for any n,
- $\Pi_{\Delta_f,p}(u_m) \geq 4^{-p}\kappa_2 \frac{u_m}{f(\log u_m)}$ for any m and $p \in [1,\infty)$,
- for any large enough integer n, there exists an integer m such that $u_m \in [v_n, v_{n+1}]$.

This fact relies on the proof of Theorem 18.0.5. We refer the reader to Remark 18.0.7 for details.

Proof of Theorem 5. We will use a well known process, that comes at least from Grigorchuk [60, Theorem B.1, statement 4]. Let $(v_n)_{n\geq 0}$ be a sequence satisfying the lower bounds on the Poincaré profiles of Theorem 1 for $\rho = \log$. Up to extracting a subsequence, we can assume that we have, for any n,

$$\log(v_{n+1}) \leq (\log v_n)^2 \,. \tag{13.1}$$

Let $f_0 = (\log n)^2$ and $f_1 = (\log n)^3$. For any sequence $(\omega_n)_{n\geq 0} \in \{0,1\}^{\mathbf{N}}$, we claim that there exists a function ρ_ω such that for any $n \geq 0$ and any $x \in [v_{2n}, v_{2n+1}]$, we have $\rho_\omega(x) = f_{\omega_n}(x)$, and satisfying the assumptions of Theorem 1. To construct such a function, one just need to say what needs to be done when ω_n changes of value:

- If $\omega_n = 0$ and $\omega_{n+1} = 1$, then one can set $\rho_\omega(x) = \min\left\{\frac{\log^4 x}{(\log v_{2n+1})^2}, \log^3 x\right\}$, for every $x \in [v_{2n+1}, v_{2n+2}]$.
- If $\omega_n = 1$ and $\omega_{n+1} = 0$, then one can set $\rho_\omega(x) = \max\left\{(\log v_{2n+1})^2 \log x, \log^2 x\right\}$, for every $x \in [v_{2n+1}, v_{2n+2}]$.

The assumption (13.1) on the sequence $(v_n)_{n\geq 0}$ ensures that this gives a well-defined function, satisfying the assumptions of Theorem 1, and such that $\rho_\omega \geq \rho = \log$. Then, for each sequence $(\omega_n)_{n\geq 0}$, we obtain a group Δ_ω from Theorem 1. Each Δ_ω is a finitely gererated amenable group of asymptotic dimension one.

If, for some sequences ω and ω', there exists a regular map from Δ_ω to $\Delta_{\omega'}$, then, from the monotonicity of Poincaré profiles (see 14.2), we have $\Pi_{\Delta_\omega,1} \preceq \Pi_{\Delta_{\omega'},1}$. From the conclusion of Theorem 1, and Fact 13.2.1, this implies that we have $\omega_n \leq \omega'_n$, for any large enough n.

Equivalently, for each subset $N \subset \mathbf{N}$, we can consider the associated sequence $(\omega_n)_{n\geq 0} \in \{0,1\}^{\mathbf{N}}$ and we get a group that we call Δ_N. From the preceeding, if there is a regular map from Δ_N to $\Delta_{N'}$, this implies that $N \setminus N'$ is finite, and each Δ_N is a finitely gererated amenable group of asymptotic dimension one.

Following Hume [67], there exists a family $\mathcal{N}$ of $2^{\aleph_0}$ subsets of $\mathbf{N}$ with $M \setminus N, N \setminus M$ infinite for all distinct $M, N \in \mathcal{N}$. Then, the family of groups $(\Delta_N)_{N\in\mathcal{N}}$ satisfies that there exists no regular map from Δ_N to Δ_M, for all distinct M and N. □

Embeddings in products of trees Dranishnikov showed in [43] that any bounded degree graph can be coarsely embedded in a finite product of trees. Until now, the issue of knowing whether these trees can be chosen of bounded degree or not remained open. Theorem 1 is able to give a negative answer, see the statement below.

Theorem 6. *There exist bounded degree graphs of asymptotic dimension one that do not coarsely embed in any finite product of bounded degree trees.*

Proof. We recall that the L^1-Poincaré profile is equivalent to the separation profile. A finite product of bounded degree trees has a separation profile bounded above by $\frac{n}{\log(n)}$ (see [19, Theorem 3.5]). Taking any function ρ that is dominated by the identity function on $\mathbf{R}_{\geq 1}$, for example $\log(x)$, the separation profile of the group given by Theorem 1 dominates $\frac{n}{\log(n)}$ along a subsequence. Since the separation profile is monotone under coarse embeddings ([19, Lemma 1.3.]), this group cannot be embedded with a coarse embedding in any finite product of bounded degree trees. □

Embeddings in L^p spaces Given a graph Γ, say on n vertices, one can study how it can be embedded in L^p spaces. For any injective map $F\colon V\Gamma \hookrightarrow L^p$, we define the **distortion** of

F as:

$$\operatorname{dist} F = \sup_{a \neq b} \frac{d(a,b)}{\delta(F(a),F(b))} \sup_{a' \neq b'} \frac{\delta(F(a'),F(b'))}{d(a',b')},$$

where d and δ denote the distance in Γ and in L^p, respectively. We then can define $c_p := \inf \{\operatorname{dist}(F) \mid F\colon V\Gamma \hookrightarrow L^p\}$.

Bourgain showed in [26] that c_p is bounded by $O(\log n)$. It was proved that this is optimal for families of expander graphs [89, 82]. This was improved by Rao [105] to $O(\sqrt{\log n})$ in the case of planar graphs. Since any family of planar graphs is hyperfinite [84], it is natural to ask if this bound is also valid for hyperfinite graphs. Recall that a sequence of bounded degree graphs (G_n) is called **hyperfinite** if for any $\epsilon > 0$ there exists $K > 0$ such that for each $n \geq 1$, there exists a set $Z_n \subset VG_n$, with $|Z_n| \leq \epsilon\, |VG_n|$, such that $G_n \setminus Z_n$ consists of components of size at most K. This notion of hyperfiniteness was introduced by Elek in [45]. This question was posed to us by Gábor Pete, also motivated by the fact that that planar graphs conjecturally embed in L_1 with $O(1)$ distortion [65]. Theorem 1 is able to give a negative answer (see below). To our knowledge, this statement is new.

Theorem 7. *For any $\epsilon \in (0,1)$, there exists a hyperfinite sequence of bounded degree graphs $(\Gamma_n)_{n\geq 0}$, such that for any $p \in [1,\infty)$ there is a positive constant K' depending only on p such that for any n,*

$$c_p(\Gamma_n) \geq K'(\log|\Gamma_n|)^{1-\epsilon}.$$

This follows from the lemma below.

Lemma 13.2.2. *For any non-decreasing function $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ such that $\frac{x}{\rho(x)}$ is non-decreasing and $\lim_\infty \rho = \infty$, there exists a hyperfinite sequence of bounded degree graphs $(\Gamma_n)_{n\geq 0}$, such that for any $p \in [1,\infty)$ there is a positive constant K' depending only on p such that for any n,*

$$c_p(\Gamma_n) \geq K' \frac{\log|\Gamma_n|}{\rho(\log|\Gamma_n|)}.$$

Proof. Let Δ be the group associated with $\min(x, \sqrt{\rho})$, given by Theorem 2. Then there exists a sequence $(\Gamma_n)_{n\geq 0}$ of subgraphs of Δ such that for any $n \geq 0$,

$$h_p(\Gamma_n) \geq \frac{4^{-p}\kappa_1}{\rho(\log|\Gamma_n|)}.$$

Using [74, Theorem 1.1] together with [74, Proposition 3.3], there exists a positive constant $K'(p)$ such that for any $n \geq 0$,

$$\begin{aligned} c_p(\Gamma_n) &\geq K'(p) \log|\Gamma_n|\, h_p(\Gamma_n) \\ &\geq K(p) \frac{\log|\Gamma_n|}{\rho(\log|\Gamma_n|)}, \quad \text{with } K(p) = 4^{-p}\kappa_1 K'(p). \end{aligned}$$

The sequence $(\Gamma_n)_{n\geq 0}$ is made of finite subgraphs of a Cayley graph of an amenable group. Then, from [46, Theorem 2], it is hyperfinite.[a] □

[a] the fact that Δ has asymptotic dimension one also implies that the sequence $(\Gamma_n)_{n\geq 0}$ is hyperfinite (again from [46, Theorem 2]).

Upper bounds on Poincaré profiles We say that a graph G has a compression exponent α in L^p if there exists a 1-Lipschitz map $F\colon G \to L^p$ such that $\rho_F(t) \preceq t^\alpha$. Theorem 4 implies:

Corollary 13.2.3. *Assume G is a graph with bounded degree and compression exponent α in some L_p-space. Then there is a constant $K(p)$ so that*

$$\Pi_{G,p}(n) \leq K\frac{n}{(\log n)^\alpha}.$$

Compression exponents have been widely studied, see for example [80] for a tabular summarizing known results.

Organization of the paper In Chapter 14, we give the definitions of Poincaré and separation profiles, and give comparison theorems, following [68]. In Chapter 15, we give the construction of the groups Δ, following [29]. In Chapter 16, we prove the lower bounds on the separation profile of the groups Δ, and make a general study of Cartesian powers of graphs (section 16.1). In Chapter 17, we prove upper bounds on the Poincaré profiles using compression in L^p spaces. Finally, in Chapter 18, we prove Theorem 18.0.5, that generalizes Theorems 1, 2 and 3, by comparing the two bounds obtained in Chapters 16 and 17 in the case of the groups Δ.

In Appendix A, we consider generalisations of the study of the separation of distorted graphs, with three methods: combinatorics, geometric, and analytic.

Acknowledgements The author would like to thank Romain Tessera who initiated this project and gave the idea of using Lipschitz embeddings to get upper bounds on Poincaré profiles, and Jérémie Brieussel who helped him understand more deeply the diagonal lamplighter groups. The author is also grateful to Tianyi Zheng for interesting discussions about these groups and to David Hume and Gabor Pete for discussions about applications of Theorem 1.

Chapter 14

Definitions

In this chapter, we give the basic definitions of Poincaré and separation profiles. We give comparison theorems, following [68, Sections 6 and 7].

The set of vertices of a graph Γ will be denoted $V\Gamma$, while the set of edges will be written $E\Gamma$. Each edge is considered as a subset of $V\Gamma$ of cardinality 2, which means that they are not oriented and that we do not allow self-loops.

A graph will always be considered as a set of vertices endowed with the shortest path metric. We ignore the "points" of the edges.

14.1 Poincaré profiles

Definition of L^p-Poincaré profiles

We start with the definition of L^p-Cheeger constants and Poincaré profiles.

Definition 14.1.1. Let Γ be a finite graph. We define for any $p \geq 1$ the **L^p-Cheeger constant** of Γ as:

$$h_p(\Gamma) = \inf\left\{ \frac{\|\nabla f\|_p}{\|f - f_\Gamma\|_p} : f \in \mathrm{Map}(V\Gamma \to \mathbf{R}), \|f - f_\Gamma\|_p \not\equiv 0 \right\},$$

with $|\nabla f|(g) = \sup_{h,h' \in B(g,1)} |f(h) - f(h')|$ and $f_\Gamma := |V\Gamma|^{-1} \sum_{g \in V\Gamma} f(g)$.
Let G be an (infinite) graph. Following [68], we define the **L^p-Poincaré profile** of G as

$$\Pi_{G,p}(n) = \sup\left\{ |V\Gamma|\, h_p(\Gamma) : \Gamma \subset G, |V\Gamma| \leq n \right\}.$$

Interpretation of the L^1-Poincaré profile

The L^1-Cheeger constant can be reinterpreted as the *minimum isoperimetric ratio*, this is the purpose of this paragraph.

Definition 14.1.2. For any finite graph Γ, we define the **majored combinatorial Cheeger constant** of Γ as

$$\tilde{h}(\Gamma) = \inf \frac{|\tilde{\partial} A|}{|A|},$$

where the infimum is taken on the subsets A of $V\Gamma$ of size at most $\frac{|V\Gamma|}{2}$, and $\tilde{\partial}A$ is the boundary of A defined by the set of vertices that are either in $V\Gamma \setminus A$ and at distance 1 from A, or in A and at distance 1 from $V\Gamma \setminus A$.

This majored combinatorial Cheeger constant is strongly related with the L^1-Cheeger constant, see proposition below.

Proposition 14.1.3. *([68, Proposition 6.10]) Let Γ be a finite graph. Then*

$$h_1(\Gamma) \leq \tilde{h}(\Gamma) \leq 2h_1(\Gamma)$$

Remark 14.1.4. Our gradient is calculated "at scale 1", while [68, Proposition 6.10] concerns gradient at scales $a \geq 2$. However, in the context of graphs, it is easy to check that it is allowed to take $a = 1$.

Comparison of L^1 and L^p-Poincaré profile

Hume, Mackay & Tessera showed a lower bound on the L^p-Cheeger constants depending on the L^1-Cheeger constant ([68, Proposition 7.2]). Working all the constants of their proof, we get the following statement.

Proposition 14.1.5. *(from [68, Proposition 7.2]) Let Γ be a finite graph with at least 3 vertices. Then for any $p \in [1, \infty)$, we have:*

$$h_p(\Gamma) \geq \min\left(\frac{1}{12}, \frac{4^{-p}}{2}\right) h_1(\Gamma).$$

Let G be an infinite graph. Then for any $p \in [1, \infty)$,

$$\Pi_{G,p} \geq \min\left(\frac{1}{12}, \frac{4^{-p}}{2}\right) \Pi_{1,G}.$$

We can mention that, on the other hand, we have the following comparison theorem:

Proposition 14.1.6. *[68, Proposition 6] If Γ is a finite graph and $p \in [1, \infty)$, then*

$$h^p(\Gamma)^p \leq 2^p h^1(\Gamma).$$

14.2 Regular maps

Poincaré profiles have the nice property to be monotone under coarse embeddings and more generally under regular maps, see definition and theorem below.

Definition 14.2.1. A map $F\colon VX \to VY$ between bounded degree graphs is said to be **regular** if there exists a constant κ such that

- $d(f(x), f(x')) \leq \kappa d(x, x')$, for every $x, x' \in X$,

- and $|f^{-1}(\{y\})| \leq \kappa$, for every $y \in Y$.

Any coarse embedding is a regular map. The absolute value $\mathbf{Z} \to \mathbf{N}$ is an example of a regular map that is not a coarse embedding.

Theorem 14.2.2. *Let X, Y be graphs with bounded degree. If there is a regular map $f\colon VX \to VY$, then for all $p \in [1, \infty]$, there exists K depending only on p such that*

$$\Pi_{X,p}(n) \leq K\Pi_{Y,p}(Kn), \quad \textit{for any large enough } n.$$

Thus, for each $p \in [1, \infty]$, the growth type of the L^p-Poincaré profiles of the Cayley graphs of a finitely generated group G do not depend on the chosen finite generating set.

14.3 Separation profile

Poincaré profiles came up as a generalization of the separation profile defined by Benjamini, Schramm & Timár [19]. We give here the definition of this profile, and his relation with Poincaré profiles.

Definition 14.3.1. For a finite graph Γ', let $L(\Gamma')$ be the size of any largest component of Γ'. We first define the **ϵ-cut** of a finite graph Γ as

$$\operatorname{cut}^{\epsilon} \Gamma := \min\left\{|S|\colon S \subset V\Gamma \text{ and } |L(\Gamma - S)| \leq \epsilon\, |V\Gamma|\right\}.$$

(we omit the "ϵ" for $\epsilon = 1/2$.)

For an infinite graph G, the **separation profile** is defined as

$$\operatorname{sep}_G(n) := \sup\left\{\operatorname{cut}^{1/2} \Gamma\colon \Gamma \subset G \text{ and } |\Gamma| \leq n\right\}.$$

It corresponds to the Poincaré profile with $p = 1$, from the proposition below.

Proposition 14.3.2. *(from [68, Proposition 6.5]) Let G be an (infinite) graph, and D be a bound on the degrees of the vertices of G. Then for $n \geq 2$,*

$$\frac{1}{8}\operatorname{sep}_G(n) \leq \Pi_{G,1}(n) \leq 4(D+1)\operatorname{sep}_G(n).$$

Proof. From [67, Proposition 2.2] and Lemma 14.1.3, for any graph Γ with at least 2 vertices, we have

$$\operatorname{cut} \Gamma \geq \frac{1}{4(D+1)} h_1(\Gamma)\, |\Gamma|,$$

and the right-hand side follows.

From [67, Proposition 2.4] and Lemma 14.1.3, for any graph Γ with at least 2 vertices, there exists $\Gamma' \subset \Gamma$ satisfying

$$|\Gamma'|\, h_1(\Gamma') \geq \frac{1}{8}\operatorname{cut} \Gamma,$$

and the left-hand side follows. □

Combining Propositions 14.1.5 and 14.3.2, we deduce:

Theorem 14.3.3. *Let G be an infinite graph. Then for any $p \in [1, \infty)$*

$$\Pi_{G,p} \geq \min\left(\frac{1}{96}, \frac{4^{-p}}{24}\right) \operatorname{sep}_G.$$

Chapter 15

Construction of lamplighter diagonal products

We write here the construction of lamplighter diagonal products, following [29]. We start with some definitions.

Definition 15.0.1. Let Γ be a group. We denote by 1_Γ the identity element of Γ. For any function $f\colon \mathbf{Z} \to \Gamma$, we define the support of f by $\mathrm{support}(f) = \{j \in \mathbf{Z} \mid f(j) \neq 1_\Gamma\}$. We denote by $\Gamma^{(\mathbf{Z})}$ the set of functions $\mathbf{Z} \to \Gamma$ with finite support.

There is a natural action of $\mathbf{Z}$ on $\Gamma^{(\mathbf{Z})}$, by translation on the indices: for any $i \in \mathbf{Z}$ and $f \in \Gamma^{(\mathbf{Z})}$, we define $i.f$ so that $(i.f)_x = f_{x-i}$ for any $x \in \mathbf{Z}$.

We define the **wreath product** of Γ on $\mathbf{Z}$, denoted by $\Gamma \wr \mathbf{Z}$, as the semi-direct product $\Gamma^{(\mathbf{Z})} \rtimes \mathbf{Z}$. An element of $\Gamma \wr \mathbf{Z}$ is represented by a pair (f, i); we refer to f as the lamp configuration and to i as the position of the cursor. The product rule is:

$$(f, i)(g, j) = (h, i + j), \quad \text{with } h_x = f_x g_{x-i} \text{ for every } x \in \mathbf{Z}.$$

This group is also called the **lamplighter group** of Γ over $\mathbf{Z}$.

Definition 15.0.2. Let Γ be a group, For any $g \in \Gamma_s$ and $i \in \mathbf{Z}$, we define the **g-dirac function at** i, denoted by $g\delta_i$, as:

$$\begin{aligned} g\delta_i\colon \mathbf{Z} &\to \Gamma \\ n &\mapsto \begin{cases} g & \text{if } n = i, \\ 1_\Gamma & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 15.0.3. Let G be a group. Let $(G_i)_{i\in I}$ be a family of groups and such that there exists, for any $i \in I$, a surjective homomorphism $\pi_i\colon G \twoheadrightarrow G_i$. We define the **diagonal product** of $(G_i)_{i\in I}$ with respect to $(\pi_i)_{i\in I}$ as the quotient group $G/\cap_{i\in I} \ker(\pi_i)$.

Let A and B be two (non trivial) finite groups. Let $(\Gamma_s)_{s\geq 0}$ be a sequence of groups such that, for any $s \geq 0$, Γ_s possesses two subgroups A_s and B_s respectively isomorphic to A and B, such that $A_s \cup B_s$ generates Γ_s.

For any $s \geq 0$, let $a_s \colon A \to A_s$ and $b_s \colon B \to B_s$ be two group isomorphisms, and k_s be a non-negative integer.

Let $\mathbf{G}$ be the free product of A, B and $\mathbf{Z}$, and let $\tau \in \mathbf{G}$ be a generator of the copy of $\mathbf{Z}$. Let us fix $s \geq 0$. We denote by Δ_s the wreath product $\Gamma_s \wr \mathbf{Z}$. There exists a unique surjective homomorphism $\pi_s \colon \mathbf{G} \to \Delta_s$ such that

- $\pi_s(a) = (a_s(a)\delta_{-k_s}, 0)$ for any $a \in A$[(a)].
- $\pi_s(b) = (b_s(b)\delta_{k_s}, 0)$ for any $b \in B$,
- and $\pi_s(\tau) = (1_{\Gamma_s}, 1)$.

The symmetric set $\pi_s(A) \cup \pi_s(B) \cup \pi_s(\tau^{\pm 1})$ generates the group Δ_s. We can detail how each element of this generating set acts by right-translation. Let $(f, i) \in \Delta_s$.

- If $a \in A$, then $(f, i).\pi_s(a) = (g, i)$, with g satisfying $g_{i-k_s} = f_{i-k_s} a_s(a)$ and $g_x = f_x$ if $x \neq i - k_s$. In words, we "write" a at $i - k_s$.
- If $b \in B$, then $(f, i).\pi_s(b) = (g, i)$, with g such that $g_{i+k_s} = f_{i+k_s} b_s(b)$ and $g_x = f_x$ if $x \neq i + k_s$. In words, we "write" b at $i + k_s$.
- $(f, i).\pi_s(\tau^{\pm 1}) = (f, i \pm 1)$.

Definition 15.0.4. We define the associated **lamplighter diagonal product** Δ as the diagonal product of the sequence $(\Delta_s)_{s\geq 0}$ with respect to $(\pi_s)_{s\geq 0}$, *i.e.* Δ is the quotient group

$$\Delta = \mathbf{G}/\cap_{s\geq 0} \ker(\pi_s).$$

Assumption 15.0.5. Let $(\Gamma_s, a_s, b_s, k_s)_{s\geq 0}$ and $(\pi_s)_{s\geq 0}$ be as above. We we always assume that the following conditions are satisfied:

- the sequence $(k_s)_{s\geq 0}$ satisfies $k_0 = 0$, and $k_{s+1} > 2k_s$ for every $s \geq 0$.
- for every $s \geq 0$, the group $A_s \times B_s$ is a quotient of Γ_s, *i.e.* $\Gamma_s / [A_s, B_s]^{\Gamma_s}$ is isomorphic to $A_s \times B_s$.

The first assumption is an independence property between the quotients $(\Delta_s)_{s\geq 0}$ of Δ. The second assumption is more sutle and restrictive. It ensures the existence of projection maps $\Gamma_s \to \Gamma_s / [A_s, B_s]^{\Gamma_s} \simeq A \times B$ that plays a role in proving local finitess properties, see Paragraph 2.2.2. of [29] for details.

From the definition of diagonal products, an element of Δ is totally determined by its projections on the quotients Δ_s. Moreover, given an element of Δ, the position of the cursor in each of these projections is constant. Therefore we will denote the elements of Δ by $\left((f_s)_{s\geq 0}, i\right)$, where $i \in \mathbf{Z}$ and $f_s \colon \mathbf{Z} \to \Gamma_s$ is a finite support map, for each $s \geq 0$.

Let π the canonical projection map from $\mathbf{G}$ to Δ. Due to its quotient structure, the group Δ has the following universal property:

(a) In [29], $\pi_s(a)$ is defined as $(a_s(a)\delta_0, 0)$ instead of $(a_s(a)\delta_{-k_s}, 0)$. However, up to a factor 2 on k_s we obtain the same group.

Proposition 15.0.6. *For any group homomorphism $f\colon G \to X$ such that $\cap_{s\geq 0} \ker \pi_s \subset \ker f$, there exists a unique group homomorphism $\tilde{f}\colon \Delta \to G$ such that $f = \tilde{f} \circ \pi$.*

Example 15.0.7. An example of a family of groups satisfying the conditions above is the Lafforgue super expanders [79]. For any prime number q, let $A = \mathbf{Z}_q^2$, $B = \mathbf{Z}_3$, $\Gamma_0 = A \times B$, and, for every $s \geq 1$, Γ_s be the diagonal product of $\mathrm{SL}_3(\mathbf{F}_q[X]/(X^s - 1))$ and $A \times B$, with respect to the following surjective homomorphisms:

$$\pi_1\colon A * B \twoheadrightarrow A \times B,$$

and

$$\pi_2\colon A * B \twoheadrightarrow \mathrm{SL}_3(\mathbf{F}_q[X]/(X^s - 1)),$$

where π_2 is defined with the following identifications:

$$\mathbf{Z}_q^2 \simeq \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle, \text{ and } \mathbf{Z}_3 \simeq \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

Then, $(\Gamma_s)_{s\geq 1}$ satisfies the above properties, with $A = \mathbf{Z}_q^2$ and $B = \mathbf{Z}_3$.

This example is important because the sequence $(\Gamma_s)_{s\geq 1}$ is an expander. This will be used in applications. For simplicity, we denote by $(\Gamma_s)_{s\geq 1}$ the sequence $(\mathrm{Cay}(\Gamma_s, A_s \cup B_s))_{s\geq 1}$, which is a sequence of regular graphs. We have the following theorem,

Theorem 15.0.8. *[79] There exist $D, \epsilon > 0$ such that for every $s \geq 1$,*

- $\tilde{h}(\Gamma_s) > \epsilon$,
- $\deg \Gamma_s \leq D$,
- $(|\Gamma_s|)_{s\geq 1}$ *is unbounded.*

Chapter 16

A lower bound on Poincaré profiles

The goal of this chapter is to give a lower bound on the Poincaré profiles of diagonal lamplighter products. We fix a diagonal product of lamplighter groups Δ, keeping the same notations as above. We show the following theorem:

Theorem 16.0.1. *Let Δ be the lamplighter diagonal product of $(\Gamma_s, a_s, b_s, k_s)_{s\geq 0}$. Then for any $s \geq 0$ and $r \leq k_s/2$,*

$$\Pi_{\Delta,p}((2k_s + 2r + 1) |\Gamma_s|^{2r+1}) \geq 4^{-p} \frac{h(\Gamma_s)^2}{1536(\deg \Gamma_s)^2} \frac{|\Gamma_s|^{2r+1}}{2r+1}.$$

This theorem is the technical core of the lower bounds obtained in Theorems 1, 2 and 3, that will be proved in Chapter 18. To show it, we will exhibit subgraphs, that we call **distorted lamp groups**, and study their separation. We will make a comparison with Cartesian powers of finite graphs, that will play the role of model graphs. The lower bound will finally be extended to Poincaré profiles using Theorem 14.3.3. We start with a general study of Cartesian powers of a given finite graph.

16.1 Cheeger constants of Cartesian powers of a given graph

Here, we will consider sequences of graphs of unbounded maximal degree. We will use another definition of Cheeger constants, that is more relevant in this context, see definition and proposition below.

Definition 16.1.1. For any finite graph Γ, we define the **combinatorial Cheeger constant** of Γ as

$$h(\Gamma) = \inf \frac{|\partial A|}{|A|},$$

where the infimum is taken on the non-empty subsets A of $V\Gamma$ of size at most $\frac{|V\Gamma|}{2}$, and ∂A is the boundary of A defined as the set of vertices of $V\Gamma \setminus A$ and at distance 1 from A.

Mind the difference with the *majored* combinatorial Cheeger constant $\tilde{h}(\Gamma)$ of Definition 14.1.2, where the boundary includes more vertices. This definition is motivated by the following proposition:

Proposition 16.1.2. *[67, Proposition 2.2] For any graph Γ with at least 2 vertices,*

$$\operatorname{cut}(\Gamma) \geq \frac{1}{4} h(\Gamma) \, |\Gamma| \, .$$

This statement should be compared with Proposition 14.3.2, where the maximal degree of the graph appears in the inequality. Proposition 16.1.2 is more relevant here, as we work in an unbounded degree context. We have the following comparison between these two combinatorial Cheeger constants:

Proposition 16.1.3. *Let Γ be a finite graph of maximal degree D. Then,*

$$h(\Gamma) \leq \tilde{h}(\Gamma) \leq (D+1) h(\Gamma)$$

We will also use the notion of spectral gap.

Definition 16.1.4. If Γ is a finite graph, we can define the **Laplacian** Δ_Γ as the operator of $\ell^2(V\Gamma)$ satisfying:

$$\Delta_\Gamma f(i) = \sum_{j \sim i} f(i) - f(j),$$

for every $f \in \ell^2(V\Gamma)$ and $i \in V\Gamma$. We denote by $\lambda_2(\Gamma)$ the second smallest eigenvalue of Δ_Γ, called the **spectral gap** of Γ.

Spectral gaps and Cheeger constants are related by the Cheeger inequalies.

Theorem 16.1.5 (the Cheeger inequalities)**.** *Let Γ be a finite regular graph of degree D. Then*

$$\frac{h(\Gamma)^2}{2D} \leq \lambda_2(\Gamma) \leq 2D h(\Gamma).$$

See [33, Lemma 2.1, Theorem 2.2], and [2, Lemma 2.4] for detail.

Definition 16.1.6. Let G and H be two graphs. We define the **Cartesian product** of G and H, denoted by $G \times H$, as the graph with vertex set $VG \times VH$ satisfiying that (g,h) and (g',h') are linked with an edge if and only if: $\{g,g'\} \in EG$ and $h = h'$, or $g = g'$ and $\{h,h'\} \in EH$.

The following proposition gives lower and upper bounds on Cheeger constants of Cartesian powers of a given graph.

Proposition 16.1.7. *Let G be a finite connected regular graph. Let k be a positive integer and $G^k = \underbrace{G \times \cdots \times G}_{k \text{ terms}}$ the Cartesian product of k copies of G. Then we have*

$$\frac{a}{k} \leq h(G^k) \leq \frac{b}{\sqrt{k}},$$

with $a = \left(\frac{h(G)}{2 \deg G}\right)^2$ and $b = (2\sqrt{2}+2)\sqrt{\deg(G) h(G)}$.

From Proposition 16.1.2, we obtain the following lower bound for the separation of Cartesian powers of a given graph:

Corollary 16.1.8. *Let G be a finite connected regular graph with at least 2 vertices. Let k be a positive integer. Then,*

$$\mathrm{cut}(G^k) \geq \frac{h(G)^2}{16(\deg G)^2}\frac{|G|^k}{k}.$$

Remark 16.1.9. The k in the denominator will have an impact in Chapter 18 where we compare the lower and upper bounds obtained on the Poincaré profiles of the groups Δ. Without this term, the upper and lower bounds of Theorem 2 would match each other. However, the upper bound in Proposition 16.1.7, and the equivalence between Cheeger constants and cuts from [67], show that such a loss is probably unavoidable.

Proof of Proposition 16.1.7. We will use the following equality, from the statement 3.4 of Fiedler [52]:

$$\lambda_2(G^k) = \lambda_2(G). \tag{16.1}$$

We start with the lower bound. The degree of the graph G^k is $k \deg G$. From the Cheeger inequalities (Theorem 16.1.5), we have

$$h(G^k) \geq \frac{\lambda_2(G^k)}{2k \deg G} \text{ and } \lambda_2(G) > \frac{h(G)^2}{2 \deg G}. \tag{16.2}$$

Combining (16.1) and (16.2), we get $h(G^k) \geq \frac{1}{k}\left(\frac{h(G)}{2 \deg G}\right)^2$.

Let us prove the upper bound. In [20], Bobkov, Houdré and Tetali introduced another spectral quantity called λ_∞ that is linked with the vertex isoperimetry. It is defined by

$$\lambda_\infty(\Gamma) = 2 \inf_{f\colon V\Gamma \to \mathbf{R}} \frac{\frac{1}{n}\sum_{i\in V\Gamma} \sup_{j\sim i}(f(i)-f(j))^2}{\frac{1}{n^2}\sum_{i,j\in V\Gamma}(f(i)-f(j))^2},$$

where n is the size of the finite graph Γ (see [20, section 2]). From [20, Theorem 1] and a basic convexity argument, we have

$$h(G^k) \leq (2+\sqrt{2})\sqrt{\lambda_\infty(G^k)}.$$

Moreover, we have $\lambda_\infty(G^k) = \frac{\lambda_\infty(G)}{k}$ ([20, Concluding Remarks]), $\lambda_\infty(G) \leq \lambda_2(G)$ by definition, and $\lambda_2(G) \leq 2\deg(G)h(G)$ from Theorem 16.1.5. Then, we derive

$$h(G^k) \leq (2\sqrt{2}+2)\frac{\sqrt{\deg(G)h(G)}}{\sqrt{k}}.$$

□

Example 16.1.10. We do not know whether the lower bound is sharp or not, but the upper bound is sharp in the case where G is the path $[-n, n]$. Indeed, Wang & Wang showed in [118] that, up to constants, the following sets realize the infimum in the definition of the Cheeger constant of $[-n, n]^k$:

$$A_k = \left\{ (x_1, \ldots, x_k) \in [-n, n]^k, \sum_{i=1}^{k} x_i < 0 \right\}$$

Indeed, A_k contains roughly half of the points of $[-n, n]^k$, and its (vertex)-boundary is:

$$\partial A_k = \left\{ (x_1, \ldots, x_k) \in [-n, n]^k, \sum_{i=1}^{k} x_i = 0 \right\}$$

If we consider that $(x_i)_{i \geq 1}$ is a sequence of independent uniformly distributed random variables in $[-n, n]$, their partial sum $y_k = \sum_{i=1}^{k} x_i$ can be reinterpreted as a random walk in $\mathbf{Z}$. It is a well known fact that the probability of having $y_k = 0$ is, up to constants, equivalent to $\frac{1}{\sqrt{k}}$. This gives then an isoperimetric ratio $\frac{|\partial A_k|}{|A_k|}$ of the form $\frac{1}{\sqrt{k}}$.

Edge-Cheeger constants We give here the analogous of Proposition 16.1.7 in the context of edge-Cheeger constants. This paragraph will not be used in the proofs of our theorems. We detail this here for completeness, because this context is more usual and has more connections with analysis.

Definition 16.1.11. We define the **edge-Cheeger constant** of a graph Γ as

$$h_e(\Gamma) := \inf \frac{|E(A, V\Gamma \setminus A)|}{|A|},$$

where the infimum is taken on non-empty subsets A of $V\Gamma$ of size at most $\frac{V\Gamma}{2}$, and $E(A, V\Gamma \setminus A)$ denotes the set of edges between A and its complementary in $V\Gamma$.

The analogous of Proposition 16.1.7 in this context is:

Proposition 16.1.12. *Let G be a connected regular graph. Let k be a positive integer. Then*

$$a' \leq h_e(G^k) \leq b'\sqrt{k},$$

with $a' = \frac{1}{4}\frac{h_e(G)^2}{\deg G}$ and $b' = 2\sqrt{2}\sqrt{h(G)\deg G}$.

Proof. The proof uses the same ingredients as the proof of Proposition 16.1.7:

- The Cheeger inequalities for edge-Cheeger constants (see [33, Lemma 2.1, Theorem 2.2]) give
$$\frac{h_e^2(G)}{2\deg G} \leq \lambda_2(G) \leq 2h_e(G),$$
and
$$\frac{h_e^2(G^k)}{2k\deg G} \leq \lambda_2(G^k) \leq 2h_e(G^k),$$
- and [52] gives $\lambda_2(G^k) = \lambda_2(G)$. □

The lower bound in Proposition 16.1.12 is sharp. We can take again the example where G is the path $[-n, n]$. From [21], the half space $G^{k-1} \times [-n, 0]$ realizes (up to constants) the infimum in the definition of the (edge-)Cheeger constant of $[-n, n]^k$. Since its edge-boundary consists in $(2n+1)^{k-1}$ edges, the resulting Cheeger constant is, up to constants, equivalent to $1/n$, which is independent of k.

This paragraph shows a difference of behaviour, depending on the notion of isoperimetry that we consider. See [11] for more details on isoperimetric problems in the grid.

16.2 Distorted lamp groups and their separation

We fix a lamplighter diagonal product Δ as in Definition 15.0.4. In this subsection, we exhibit subgraphs of Δ, and study their separation. To do so, we compare these subgraphs with Cartesian powers of the lamp groups, that will play the role of model graphs.

16.2.1 Distorted lamp groups

Definition 16.2.1. Let Γ_s be a group generated by two subgroups A_s and B_s. We define $\Gamma_s^{k_s,r}$ as the graph with vertex set $(\Gamma_s)^{[-r,r]} \times [-(r+k_s), r+k_s]$, and the following edges:

- $[(x_{-r}, \ldots, \underset{(j)}{x_j}, \ldots, x_r), j-k_s] \sim [(x_{-r}, \ldots, \underset{(j)}{x_j b}, \ldots, x_r), j-k_s]$ (called "B-edges"),
- $[(x_{-r}, \ldots, x_r), i] \sim [(x_{-r}, \ldots, x_r), i+1]$ (called "**Z**-edge"),
- $[(x_{-r}, \ldots, \underset{(j)}{x_j}, \ldots, x_r), j+k_s] \sim [(x_{-r}, \ldots, \underset{(j)}{x_j a}, \ldots, x_r), j+k_s]$ (called "A-edges"),

for any $i \in [-(r+k_s), r+k_s-1]$, $j \in [-r, r]$, $a \in A_s$ and $b \in B_s$. The notation "$g \sim h$" means that $\{g, h\}$ is an edge of the graph $\Gamma_s^{k_s,r}$.

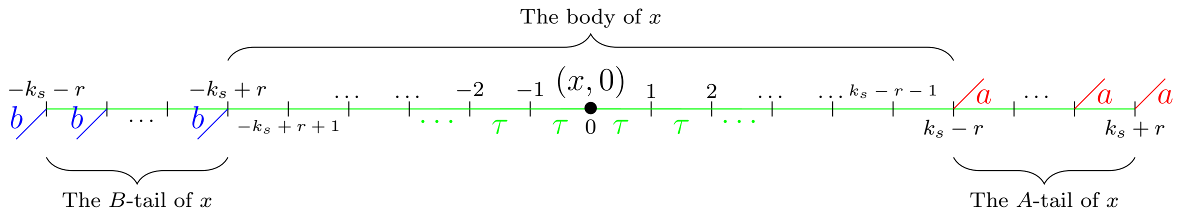
To figure out more clearly the shape of the graphs $\Gamma_s^{k_s,r}$, see Figure 16.1. Intuitively, we think of this graph as a distorted product of lamp groups: a product of copies of the group Γ_s where we have extended the edges by a factor $2k_s+1$. More precisely, a way of representing the graph $\Gamma_s^{k_s,r}$ is to partition it by subsets of the form $\{(x,i), i \in [-k_s-r, k_s+r]\}$. We call such a subset a *line*, see Figure 16.1. Then, we can distinguish three parts in such a *line*:

- For $i \in [\![-k_s-r, -k_s+r]\!]$, the B-*tail*, where vertices have **Z**-edges and B-edges.
- For $i \in [\![-k_s+r-1, k_s-r-1]\!]$, the *body*, where vertices only have **Z**-edges.
- For $i \in [\![k_s-r, k_s+r]\!]$, the A-*tail*, where vertices have **Z**-edges and A-edges.

Travelling through an A-edge or a B-edge changes one coordinate of x, and keeps the same value for i, and travelling through a **Z**-edge keeps the same value for x and adds or subtracts 1 from i (see §15 for details).

The case $r=0$ is particular, because $\Gamma_s^{k_s,0}$ is an homothetic copy of Γ_s. This is the following proposition.

Figure 16.1: the *line* in $\Gamma_s^{k_s,r}$ of x : $\{(x,i)\,,\, i \in [-k_s - r, k_s + r]\}$.



Proposition 16.2.2. *Let $\Gamma_s^{k_s,0}$ be as in Definition 16.2.1 with $r = 0$. We can define*

$$\begin{array}{rccc} \iota: & \Gamma_s & \longrightarrow & \Gamma_s^{k_s,0} \\ & x & \longmapsto & (x,0) \end{array}$$

Then, for any $x, y \in \Gamma_s$, we have

$$d(\iota(x), \iota(y)) = 2k_s d(x,y).$$

This observation will be exploited in Appendix A.2 to prove more general results concerning bilipschitz embeddings of graphs.

To show that this graph embeds in Δ, we start with a lemma. We remind the reader that a_s (respectively b_s) denotes a group isomorphism from A to A_s (respectively from B to B_s).

Lemma 16.2.3. *Let x be an element of Γ_s. Then there exists a couple $\left(x^{A_s}, x^{B_s}\right) \in A_s \times B_s$ such that for any decomposition of $x = \prod_{i=0}^{n} a_i b_i$, where $(a_i)_{i\in[0,n]}$ and $(b_i)_{i\in[0,n]}$ are some sequences of elements respectively of A_s and B_s, we have $\prod_{i=0}^{n} a_i = x^{A_s}$ and $\prod_{i=0}^{n} b_i = x^{B_s}$.*

Proof. According to the assumption that the groups $\Gamma_s / [A_s, B_s]^{\Gamma_s}$ and $A_s \times B_s$ are isomorphic, we have a well defined group homomorphism from $\Gamma_s / [A_s, B_s]^{\Gamma_s}$ to $A_s \times B_s$. Composing by the quotient map $\Gamma_s \twoheadrightarrow \Gamma_s / [A_s, B_s]^{\Gamma_s}$, we get a well defined group homomorphism from Γ_s to $A_s \times B_s$. The announced result follows. □

Proposition 16.2.4. *For any $r \leq k_s/2$, the graph $\Gamma_s^{k_s,r}$ is isomorphic to a subgraph of Δ.*

For simplicity, we will still denote by $\Gamma_s^{k_s,r}$ the corresponding subgraph of Δ.

Proof. We remind that the elements of Δ are denoted $\left((f_{s'})_{s'\geq 0}\,, i\right)$, where i is an integer, and for every s', $f_{s'}$ is a map of finite support from $\mathbf{Z}$ to $\Gamma_{s'}$.

For any $x \in \Gamma_s$ and $s' \geq 0$, we write $x^{A_{s'}} = a_{s'} \circ a_s^{-1}(x^{A_s})$ and $x^{B_{s'}} = b_{s'} \circ b_s^{-1}(x^{B_s})$. Let r be such that $r \leq k_s/2$. We define the following map:

$$\begin{aligned} \phi\colon\ (\Gamma_s)^{[-r,r]} \times [-(k_s + r), r + k_s] &\to \Delta \\ [(x_{-r}, \dots, x_r)\,, i] &\mapsto \left((f_{s'})_{s'\geq 0}\,, i\right), \\ &\quad \text{with } f_{s'} = \sum_{j\in[-r,r]} x_j^{A_{s'}} \delta_{j+k_s-k_{s'}} + \sum_{j\in[-r,r]} x_j^{B_{s'}} \delta_{j-k_s+k_{s'}} \text{ if } s' \neq s, \\ &\quad \text{and } f_s = \sum_{j\in[-r,r]} x_j \delta_j. \end{aligned}$$

When we define $f_{s'}$ for $s' \neq s$, we think of the two sum as "writing" some elements of $A_{s'}$ and of $B_{s'}$. The sum is valid if they are written at different places, *i.e.* if the supports of the two sums are disjoint, which is not clear *a priori*. However, under the assumption that $r \leq k_s/2$:

- If $s' < s$: the elements of $B_{s'}$ are written in the interval $[-r-(k_s-k_{s'}), r-(k_s-k_{s'})]$, and the elements of $A_{s'}$ are written in the interval $[-r+(k_s-k_{s'}), r+(k_s-k_{s'})]$. Since $k_s > 2k_{s'}$ by hypothesis, which implies $k_s/2 < k_s - k_{s'}$, these two intervals are disjoint.
- If $s' > s$: the elements of $A_{s'}$ are written in the interval $[-r-(k_{s'}-k_s), r-(k_{s'}-k_s)]$, and the elements of $B_{s'}$ are written in the interval $[-r+(k_{s'}-k_s), r+(k_{s'}-k_s)]$. Since $k_{s'} > 2k_s$ by hypothesis, which implies $k_s < k_{s'} - k_s$, these two intervals are disjoint.

Thus ϕ is well defined and is moreover injective. Let (v_1, v_2) be an edge of $\Gamma_s^{k_s,r}$. Using the terminology of Definition 16.2.1, three cases can occur:

- if (v_1, v_2) is a $\mathbf{Z}$-edge, then $(\phi(v_1), \phi(v_2))$ is clearly an edge of Δ.
- if (v_1, v_2) is a A-edge, then v_1 and v_2 are respectively of the form:

$$[(x_{-r}, \dots, \underset{(j)}{x_j}, \dots, x_r), j+k_s], \text{ and } [(x_{-r}, \dots, \underset{(j)}{x_j a}, \dots, x_r), j+k_s].$$

  This implies, in Δ_s, we have $\pi_s(\phi(v_1)) = \pi_s(\phi(v_2)) \times (a_s(a)\delta_{-k_s}, 0)$. Additionally, for any $s' \neq s$, $(x_j a)^{A_{s'}} = (x_j{}^{A_{s'}}) \times a_{s'}(a)$ and then we have the same equality in $\Delta_{s'}$: $\pi_{s'}(\phi(v_1)) = \pi_{s'}(\phi(v_2)) \times (a_{s'}(a)\delta_{-k_{s'}}, 0)$. Then, $\phi(v_1) = \phi(v_2)a$, which means that $(\phi(v_1), \phi(v_2))$ is an edge of Δ.
- if (v_1, v_2) is a B-edge, the same reasoning as for A-edges is valid.

Therefore ϕ is a graph embedding from $\Gamma_s^{k_s,r}$ to Δ. □

16.2.2 Comparison with Cartesian powers

For any $r \geq 0$, we denote ${\Gamma_s}^{[-r,r]}$ the (cartesian) product of $2r+1$ copies of Γ_s, indexed by $[-r, r]$. The following proposition compares the separation of ${\Gamma_s}^{[-r,r]}$ with that of the graph $\Gamma_s^{k_s,r}$ introduced above.

Proposition 16.2.5. *For any $r \geq 0$,*

$$\operatorname{cut}(\Gamma_s^{k_s,r}) \geq \operatorname{cut}\left({\Gamma_s}^{[-r,r]}\right).$$

Proof. Let C^{k_s} be a cutset of $\Gamma_s^{k_s,r}$. Let

$$C = \left\{ x \in {\Gamma_s}^{[-r,r]} \mid \exists i \in [-(r+k_s), r+k_s] \ \ (x,i) \in C^{k_s} \right\}.$$

We have $|C| \leq \left|C^{k_s}\right|$. Let us show that C is a cutset of $\Gamma_s^{[-r,r]}$. Let A be a connected subset of ${\Gamma_s}^{[-r,r]} \setminus C$. Let $A^{k_s} = \{(x,i) \mid x \in A \text{ and } i \in [-(r+k_s), r+k_s]\}$. We have $\left|A^{k_s}\right| =$

$(2r+2k_s+1)\times|A|$. Moreover, A^{k_s} does not meet C^{k_s} and induces a connected graph: any path in $\Gamma_s^{r+1}\setminus C$ can be followed in $\Gamma_s^{k_s,r}\setminus C^{k_s}$ since we are allowed to move the integer i in the whole interval $[-(r+k_s), r+k_s]$. Since C^{k_s} is a cutset of $\Gamma_s^{k_s,r}$, $\left|A^{k_s}\right| \leq \frac{\left|\Gamma_s^{k_s,r}\right|}{2} = \frac{2r+2k_s+1}{2}\left|{\Gamma_s}^{[-r,r]}\right|$. Since $\left|A^{k_s}\right| = (2r+2k_s+1)\times|A|$, we can deduce that $A \leq \frac{\left|{\Gamma_s}^{[-r,r]}\right|}{2}$. This means that C is a cutset of Γ_s^{r+1}. Therefore, $\mathrm{cut}\left({\Gamma_s}^{[-r,r]}\right) \leq \mathrm{cut}\left(\Gamma_s^{k_s,r}\right)$. □

In Appendix A, we study more general statements in the same spirit: in section A.1, we show a generalization of this proof in the context of coarsenings of graphs, and, in sections A.2 and A.3, two alternative proofs in the case $r=0$.

We can prove Theorem 16.0.1.

Proof of Theorem 16.0.1. Let $s \geq 0$ and $r \leq k_s/2$. Then, from Proposition 16.2.4, the graph $\Gamma_s^{k_s,r}$ is isomorphic to a subgraph of Δ. We have

$$\begin{aligned}
\mathrm{cut}(\Gamma_s^{k_s,r}) &\geq \mathrm{cut}\left({\Gamma_s}^{[-r,r]}\right), \quad \text{from Proposition 16.2.5,}\\
&\geq \frac{h(\Gamma_s)^2}{16(\deg\Gamma_s)^2}\frac{|\Gamma_s|^{2r+1}}{2r+1}, \quad \text{from Corollary 16.1.8.}
\end{aligned}$$

The graph $\Gamma_s^{k_s,r}$ has $(2k_s+2r+1)\,|\Gamma_s|^{2r+1}$ vertices. Then, we have

$$\mathrm{sep}_\Delta((2k_s+2r+1)\,|\Gamma_s|^{2r+1}) \geq \frac{h(\Gamma_s)^2}{16(\deg\Gamma_s)^2}\frac{|\Gamma_s|^{2r+1}}{2r+1}.$$

Finally, from Theorem 14.3.3,

$$\Pi_{\Delta,p}((2k_s+2r+1)\,|\Gamma_s|^{2r+1}) \geq 4^{-p}\frac{h(\Gamma_s)^2}{1536(\deg\Gamma_s)^2}\frac{|\Gamma_s|^{2r+1}}{2r+1}$$

□

Chapter 17

An upper bound on the Poincaré profiles

17.1 Compression in L^p spaces and Poincaré profiles

We show here an upper bound on L^p-Poincaré profiles of graphs, using embeddings into L^p spaces. Before stating our theorem, we define the compression function of such an embedding:

Definition 17.1.1. Let $f\colon G \to L^p$ be a $1-$Lipschitz map from a graph into an L^p space. We define the **compression function** of f, denoted ρ_f, as:

$$\rho_f(t) = \inf \left\{ \|f(g) - f(h)\|_p \mid d_G(g,h) \geq t \right\}.$$

We state our upper bound theorem:

Theorem 17.1.2. *Let G be a graph of bounded degree. Then there exist two constants $c_1, c_2 > 0$, depending only on the maximum degree in G, such that if $f\colon VG \to L^p$ is a 1-Lipschitz map, then*

$$\Pi_{G,p}(N) \leq c_1 \frac{N}{\rho_f(c_2 \log N)}, \tag{17.1}$$

for all $p \in [1, \infty)$ and $N \geq 0$.

More precisely, if there exists a function σ such that for any vertex x of G, the sphere centred at x of radius n contains at most $\sigma(n)$ vertices, then for any N we have:

$$\Pi_{G,p}(N) \leq 2^{\frac{2p-1}{p}} \sigma(1)^{1/p} \left(\frac{N^{p+1}}{\sum_{n=0}^{K} \sigma(n) \rho_f(n)^p} \right)^{1/p}, \tag{17.2}$$

where K is the biggest integer such that $\sum_{n=0}^{K} \sigma(n) \leq N$ (depends on N).

Remark 17.1.3. As mentionned in the introduction (see Theorem 4), the inequality (17.1) is known to be sharp. In this more precise statement, we can comment on inequality (17.2) which improves (17.1) when G doesn't have exponential growth. Indeed, one may notice that the inequality (17.2) is asymptotically optimal for the inclusion map $\mathbf{Z}^d \hookrightarrow \left(\mathbf{R}^d, \ell^1\right)$. In this case the compression function is $\rho(t) \simeq t$ and we can take $\sigma(n) = cn^{d-1}$. From Theorem 17.1.2, we can deduce that $\Pi_{\mathbf{Z}^d,1}(N) \preceq n^{\frac{d-1}{d}}$, which is optimal, using Proposition 16.1.7, or [68, Theorem 7].

In the case of the Heisenberg group, the inequality (17.2) is not asymptotically optimal if $p \geq 2$. Indeed, Austin, Naor and Tessera showed in [10] that any 1-Lipschitz embedding of the Heisenberg group in a superreflexive Banach space has a compression function at most equivalent to $t \mapsto \frac{t}{\log^c t}$ for some positive constant c. The inequality (17.2) gives, in this optimal case (with $\sigma(n) = c'n^3$ and assuming that $c < 1/p$), $\Pi_{\mathbb{H}^4,p}(N) \preceq \log(N)^{\frac{1}{p}-c} N^{\frac{3}{4}}$, while we have $\Pi_{\mathbb{H}^4,p}(N) \asymp N^{\frac{3}{4}}$, again from [68, Theorem 7].

We will see some cases where (17.1) is optimal in Chapter 18.

For the proofs, we will use another notion of gradient; we define the associated Poincaré profile:

Definition 17.1.4. Let $p \in [1, \infty)$.

- Let Γ be a finite graph. We define the **modified L^p-cheeger constant** of Γ as:

$$\tilde{h}_p(\Gamma) = \inf \left\{ \frac{\left\| \tilde{\nabla} f \right\|_p}{\| f - f_\Gamma \|_p} : f \in \mathrm{Map}(V\Gamma \to \mathbf{R}), \|f\|_p \not\equiv f_\Gamma \right\},$$

  with $|\nabla f|(g) = \left(\sum_{h \sim g} |f(g) - f(h)|^p \right)^{1/p}$ and $f_\Gamma = |V\Gamma|^{-1} \sum_{g \in \Gamma} f(g)$.

- Let G be an (infinite) graph. Following [68], we define the **modified L^p-Poincaré profile** of G as

$$\tilde{\Pi}_{G,p}(n) = \sup \left\{ |V\Gamma| \, \tilde{h}_p(\Gamma) : \Gamma \subset G, |V\Gamma| \leq n \right\}.$$

Remark 17.1.5. This definitions are equivalent to our previous ones (see Definition 14.1.1) in the following sense:

- If Γ is a finite graph, and D is a bound on the degrees of the vertices of Γ, then for any $p \in [1, \infty)$,

$$D^{-1/p} \tilde{h}_p(\Gamma) \leq h_p(\Gamma) \leq 2^{\frac{p-1}{p}} \tilde{h}_p(\Gamma).$$

- If G is an infinite graph of bounded degree, and D is a bound on the degrees of the vertices of G, then, for any $p \in [1, \infty)$,

$$D^{-1/p} \tilde{\Pi}_{G,p} \leq \Pi_{G,p} \leq 2^{\frac{p-1}{p}} \tilde{\Pi}_{G,p}.$$

Then, the proof of Theorem 17.1.2 can be done without loss of generality on the *modified* Poincaré profiles.

We give a property on modified L^p-Cheeger constants.

Proposition 17.1.6. *If $p > 1$, we do not change the value of $h_p(\Gamma)$ considering functions taking their values in an L^p space instead of $\mathbf{R}$, i.e.:*

If we define

$$\tilde{h}_p(\Gamma, L^p) = \inf \left\{ \frac{\left\| \tilde{\nabla} f \right\|_p}{\|f - f_\Gamma\|_p} : f \in \mathrm{Map}(V\Gamma \to \mathbf{L^p}), \|f\|_p \not\equiv f_\Gamma \right\},$$

with

- $\left| \tilde{\nabla} f \right| (g) = \left(\sum_{h \sim g} \|f(g) - f(h)\|_p^p \right)^{1/p}$,
- $f_\Gamma = |V\Gamma|^{-1} \sum_{g \in \Gamma} f(g)$,
- *and* $\|f - f_\Gamma\|_p = \left(\sum_{g \in VG} \|f(g) - f_\Gamma\|_p^p \right)^{1/p}$,

then, we have

$$\tilde{h}_p(\Gamma, L^p) = \tilde{h}_p(\Gamma).$$

Proof. The inequality $\tilde{h}_p(\Gamma, L^p) \leq \tilde{h}_p(\Gamma)$ is obvious. We prove the other inequality. Let us write $L^p = L^p(X, \mu)$, with (X, μ) a measured space. We denote by $\mathcal{L}^p$ the set of functions from X to $\mathbf{R}$ such that their p power is integrable (without quotienting by the almost everywhere equality equivalence relation). Let $f \colon V\Gamma \to \mathcal{L}^p$ be a non zero map. Without loss of generality, we can assume that $f_\Gamma = 0$. For every $x \in X$, we set

$$\begin{array}{rccc} f_x : & V\Gamma & \longrightarrow & \mathbf{R} \\ & g & \longmapsto & f(g)(x) \end{array}.$$

Since $f_\Gamma = 0$, we have $(f_x)_\Gamma = 0$ for every $x \in X$. Let $c \geq 0$ be such that for every $x \in X$ we have $\left\| \tilde{\nabla} f_x \right\|_p \geq c \|f_x\|_p$. Then we have for every vertex g of Γ:

$$\begin{aligned} \left(\tilde{\nabla} f(g) \right)^p &= \sum_{h \sim g} \|f(g) - f(h)\|_p^p \\ &= \sum_{h \sim g} \int_X |f_x(g) - f_x(h)|^p \, \mathrm{d}\mu(x) \\ &= \int_X \sum_{h \sim g} |f_x(g) - f_x(h)|^p \, \mathrm{d}\mu(x) \\ &= \int_X \left(\tilde{\nabla} f_x(g) \right)^p \mathrm{d}\mu(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \tilde{\nabla} f \right\|_p^p &= \sum_{g \in V\Gamma} \int_X \left(\tilde{\nabla} f_x(g) \right)^p \mathrm{d}\mu(x) \\ &= \int_X \sum_{g \in V\Gamma} \left(\tilde{\nabla} f_x(g) \right)^p \mathrm{d}\mu(x) \end{aligned}$$

$$
\begin{aligned}
&= \int_X \left\| \tilde{\nabla} f_x(g) \right\|_p^p \mathrm{d}\mu(x) \\
&\geq c^p \int_X \|f_x\|_p^p \mathrm{d}\mu(x) \\
&= c^p \int_X \sum_{g \in V\Gamma} |f_x(g)|^p \mathrm{d}\mu(x) \\
&= c^p \sum_{g \in V\Gamma} \|f(g)\|_p^p \\
&= c^p \|f\|_p^p .
\end{aligned}
$$

Then we deduce that $\left\|\tilde{\nabla} f\right\|_p \geq c \|f\|_p$.

Let now $c \geq 0$ satisfying $\left\|\tilde{\nabla} f\right\|_p < c \|f\|_p$. Then, from above, there exists $x \in X$ such that $\left\|\tilde{\nabla} f_x\right\|_p < c \|f_x\|_p$. This implies in particular $\|f_x\|_p \neq 0$. Then we have $\tilde{h}_p(\Gamma) \leq \frac{\|\tilde{\nabla} f_x\|_p}{\|f_x\|_p} < c$. Taking the infimum in c, we obtain $\tilde{h}_p(\Gamma) \leq \frac{\|\tilde{\nabla} f\|_p}{\|f\|_p}$. Taking the infimum in f, we obtain $\tilde{h}_p(\Gamma) \leq \tilde{h}_p(\Gamma, L^p)$. □

Before proving Theorem 17.1.2, we prove two lemmas.

Lemma 17.1.7. *Let Γ be a finite graph, let $p \in [1, \infty)$. We define the p-**variance** of a function $f \colon \Gamma \to L^p$ as:*

$$
\mathrm{Var}_p(f) = \left(\frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{h \in V\Gamma} \|f(g) - f(h)\|_p^p \right)^{1/p} .
$$

Then we have:

$$
\frac{1}{|V\Gamma|^{1/p}} \|f - f_\Gamma\|_p \leq \mathrm{Var}_p(f) \leq \frac{2}{|V\Gamma|^{1/p}} \|f - f_\Gamma\|_p .
$$

Proof.

$$
\begin{aligned}
\frac{1}{|V\Gamma|} \|f - f_\Gamma\|_p^p &= \frac{1}{|V\Gamma|} \sum_{g \in V\Gamma} \|f(g) - f_\Gamma\|_p^p \\
&= \frac{1}{|V\Gamma|^{p+1}} \sum_{g \in V\Gamma} \left\| \sum_{h \in \Gamma} f(g) - f(h) \right\|_p^p \\
&\leq \frac{1}{|V\Gamma|^{p+1}} \sum_{g \in V\Gamma} \left(\sum_{h \in \Gamma} \|f(g) - f(h)\|_p \right)^p \\
&\leq \frac{|V\Gamma|^{p-1}}{|V\Gamma|^{p+1}} \sum_{g \in V\Gamma} \sum_{h \in \Gamma} \|f(g) - f(h)\|_p^p \quad \text{since } \left(\sum_{i=1}^n x_i \right)^p \leq n^{p-1} \left(\sum_{i=1}^n x_i^p \right) \\
&= \frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{h \in V\Gamma} \|f(g) - f(h)\|_p^p \\
&= (\mathrm{Var}_p(f))^p
\end{aligned}
$$

$$\leq \frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{h \in V\Gamma} \Big(\|f(g) - f_\Gamma\|_p + \|f(h) - f_\Gamma\|_p \Big)^p \quad \text{(triangle inequality)}$$
$$\leq \frac{2^{p-1}}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{h \in V\Gamma} \|f(g) - f_\Gamma\|_p^p + \|f(h) - f_\Gamma\|_p^p$$
$$= \frac{2^p}{|V\Gamma|} \sum_{k \in V\Gamma} \|f(k) - f_\Gamma\|_p^p$$
$$= \frac{2^p}{|V\Gamma|} \|f - f_\Gamma\|_p^p \qquad \square$$

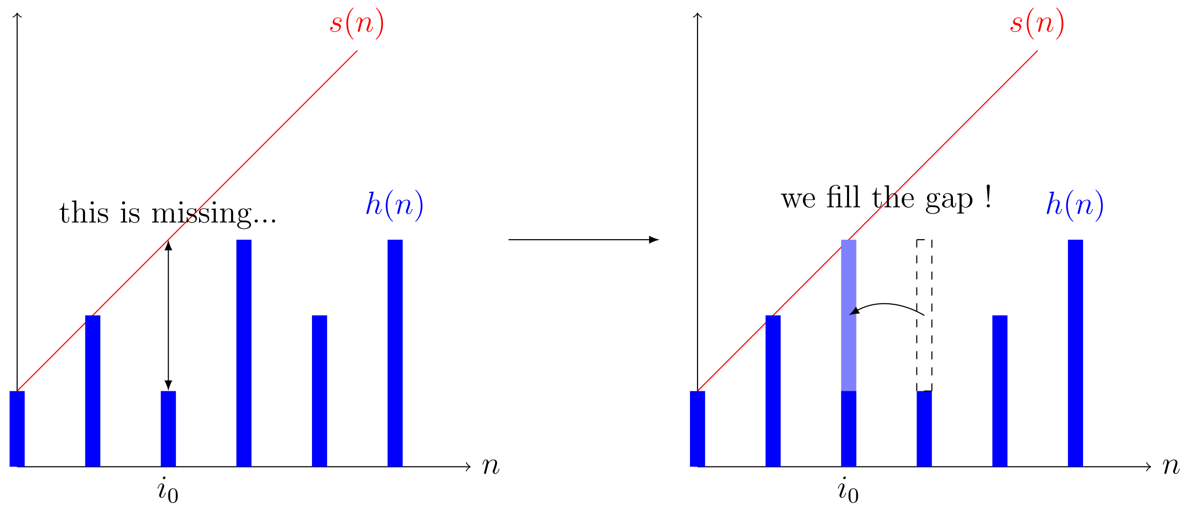
Therefore we could have written a variance time $|V\Gamma|^{1/p}$ instead of a norm in the definition of the Cheeger constant of Γ. This would give an equivalent notion, since we are only interested in asymptotic behaviours. The second lemma is the following.

Lemma 17.1.8. *Let $h, s \colon \mathbb{N} \to \mathbb{N}$ be such that for any $n \geq 0$, $h(n) \leq s(n)$. We assume that the sum $N := \sum_{n=0}^{k} h(n)$ is finite. Then for any non-decreasing function $\rho \colon \mathbb{N} \to \mathbf{R}$, we have:*

$$\sum_{n=0}^{+\infty} h(n)\rho(n) \geq \sum_{n=0}^{k} s(n)\rho(n), \quad \textit{for any } k \textit{ such that} \sum_{n=0}^{k} s(n) \leq N \ .$$

Proof. The proof is very elementary. The function $h(n)$ being at most equal to $s(n)$, we will modify inductively it by a series of elementary actions such that we conserve the sum of $h(n)$ equal to N, and such that there is an integer k such that $h(n)$ is equal to $s(n)$ in the interval $[0, k]$. At each step, this integer k will increase by 1, until we have $d(n) = 0$ for every $n \geq k + 1$. The algorithm is the following: (see Figure 17.1 for an illustration)

Figure 17.1: Illustration of Lemma 17.1.8



while *True* **do**

  if $\forall i \geq 0\ h(i) = s(i)$ **then**

    return h

  else

    let i_0 be the smallest integer such that $h(i_0) < s(i_0)$.

  end

  if $\forall i > i_0\ h(i) = 0$ **then**

    return h

  else

    if $\sum_{i=i_0}^{+\infty} h(i) < s(i_0)$ **then**

      $h(i_0) \longleftarrow \sum_{i=i_0}^{+\infty} h(i)$

      for any $i > i_0, h(i) \longleftarrow 0$

      return h

    else

      let j_0 be the smallest integer such that $\sum_{i=i_0}^{j_0} h(i) \geq s(i_0)$

      $\delta \longleftarrow \sum_{i=i_0}^{j_0} h(i) - s(i_0)$

      $h(i_0) \longleftarrow s(i_0)$,

      for any $i_0 < i < j_0$, $h(i) \longleftarrow 0$,

      $h(j_0) \longleftarrow \delta$,

    end

  end

end

Since ρ is non-decreasing, at each step of the process the quantity $\sum_{n=0}^{+\infty} h(n)\rho(n)$ won't increase.

At the end on the process, the function h satisfies the following properties:

- there exists an integer i_0 such that $h(i) = s(i)$ for any $i < i_0$, and $h(i) = 0$ for any $i > i_0$

- $\sum_{n=0}^{+\infty} h(n) = N$

This proves that the inequality

$$\sum_{n=0}^{+\infty} h(n)\rho(n) \geq \sum_{n=0}^{k} s(n)\rho(n)$$

is true for any k such that $\sum_{n=0}^{k} s(n) \leq N$, which is what we wanted to prove. □

We can start the proof of Theorem 17.1.2.

Proof of Theorem 17.1.2. Without loss of generality, we can use the *modified* Poincaré profile definition (Definition 17.1.4), see Remark 17.1.5 for details. We start with the second inequality. By definition, $\sigma(1)$ is a bound on the degrees on the vertices of G. Let n be a positive integer and Γ be a connected subgraph of G with at most n vertices. Then the restriction $f_{|V\Gamma}\colon \Gamma \to L^p$ is also 1-Lipschitz for the induced metric on Γ. For simplicity, we will still denote $f_{|V\Gamma}$ by f. Then we have:

$$\left\|\tilde{\nabla} f\right\|_p \leq \sigma(1)^{1/p} \left|V\Gamma\right|^{1/p} \tag{17.3}$$

We will now give an upper bound on the norm of $f_{|V\Gamma}$. We have the following inequalities:

$$\begin{aligned}
\mathrm{Var}_p(f_{|\Gamma})^p &= \frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{g' \in V\Gamma} \|f(g) - f(g')\|_p^p \\
&\geq \frac{1}{|V\Gamma|^2} \sum_{g,g' \in V\Gamma} (\rho_f(d(g,g')))^p \\
&\geq \frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{n \geq 0} \#\{g' \in V\Gamma \mid d(g,g') = n\} \rho_f(n)^p
\end{aligned}$$

We fix $g \in V\Gamma$. Using Lemma 17.1.8, with $h(n) = \#\{g' \in V\Gamma \mid d_G(g',g) = n\}$, $s(n) = \sigma(n)$ and $\rho = \rho_f^p$, we have $\sum_{n=0}^{+\infty} h(n) = |V\Gamma|$ and we can set K the biggest integer such that $\sum_{n=0}^{K} \sigma(n) \leq |V\Gamma|$. We obtain, for every $g \in V\Gamma$,

$$\sum_{n \geq 0} \#\{g' \in V\Gamma \mid d(g,g') = n\} \rho_f(n)^p \geq \sum_{n=0}^{K} \sigma(n) \rho_f(n)^p$$

We get

$$\begin{aligned}
\mathrm{Var}_p(f)^p &\geq \frac{1}{|V\Gamma|^2} \sum_{g \in V\Gamma} \sum_{n=0}^{K} \sigma(n)\rho_f(n)^p \\
&= \frac{1}{|V\Gamma|} \sum_{n=0}^{K} \sigma(n)\rho_f(n)^p. \qquad (17.4)
\end{aligned}$$

Combining (17.3), Lemma 17.1.7, and (17.4), we get:

$$\frac{\left\|\tilde{\nabla} f\right\|_p}{\|f - f_\Gamma\|_p} \leq 2 \frac{\left\|\tilde{\nabla} f\right\|_p}{|V\Gamma|^{1/p}\, \mathrm{Var}_p(f)}$$

$$\leq 2\frac{\sigma(1)^{1/p}\,|V\Gamma|^{1/p}}{\left(\sum_{n=0}^{K}\sigma(n)\rho_f(n)^p\right)^{1/p}}.$$

This implies

$$\begin{aligned}|V\Gamma|\,h_p(\Gamma) &\leq 2^{\frac{p-1}{p}}\,|V\Gamma|\,\tilde{h_p}(\Gamma), \quad \text{from Remark 17.1.5}\\ &\leq 2^{\frac{2p-1}{p}}\sigma(1)^{1/p}\left(\frac{|V\Gamma|^{p+1}}{\sum_{n=0}^{K}\sigma(n)\rho_f(n)^p}\right)^{1/p}.\end{aligned}$$

Since this is true for every subgraph $\Gamma \subset G$, we obtain, for every $N \geq 0$,

$$\Pi_{G,p}(N) \leq 2^{\frac{2p-1}{p}}\sigma(1)^{1/p}\left(\frac{N^{p+1}}{\sum_{n=0}^{K}\sigma(n)\rho_f(n)^p}\right)^{1/p}, \tag{17.5}$$

where K the biggest integer such that $\sum_{n=0}^{K}\sigma(n) \leq N$, which is the inequality (17.1).

Let us prove the second inequality (17.1). Let D be a bound on the degrees of the vertices of G. Inequality (17.1) is obtained by applying inequality (17.5) with $\sigma(n) = D^n$, which is possible by definition of D. Then we have $K \geq \frac{\log((D-1)N+1)}{\log D} - 2 \geq \frac{\log N}{\log D} - 2$, and $D^K \geq ND^{-2}$. We can deduce, keeping only the last term of the sum in (17.5),

$$\begin{aligned}\Pi_{G,p}(N) &\leq 2^{\frac{2p-1}{p}}D^{1/p}\left(\frac{N^{p+1}}{\sum_{n=0}^{K}D^n\rho_f(n)^p}\right)^{1/p}\\ &\leq 2^{\frac{2p-1}{p}}D^{1/p}\left(\frac{N^{p+1}}{D^K\rho_f(K)^p}\right)^{1/p}\\ &= 2^{\frac{2p-1}{p}}D^{1/p}\frac{N^{\frac{p+1}{p}}}{D^{K/p}\rho_f(K)}\\ &\leq 2^{\frac{2p-1}{p}}D^{3/p}\frac{N}{\rho_f\left(\frac{\log N}{2\log D}\right)}, \quad \text{if } N \geq D^4,\end{aligned}$$

When $N < D^4$, we have $\rho_f\left(\frac{\log N}{2\log D}\right) \leq \frac{\log N}{2\log D} + 1 \leq 3$ and $\Pi_{G,p}(N) \leq 6N \leq 6D^4$, from [68, Proposition 7.1].

Then, we deduce the inequality (17.1). One may notice that, in this situation, conserving only the last term of the sum can't lead to a dramatic loss, since $\sum_{n=0}^{K}D^n \asymp D^K$, and ρ_f is non-decreasing. This ends the proof of Theorem 17.1.2. □

17.2 Application to lamplighter diagonal products

In this section, we exhibit embeddings of lamplighter diagonal products and deduce an upper bound on their Poincaré profile, using Theorem 17.1.2. In [29], Brieussel and Zheng exhibit "global" embeddings into L^p spaces, meaning that they almost realize the compression upper bound at every scale. To do so, they use a process designed by Tessera in [114]: they sum up infinitely many cocycles, such that at each cocycle realizes the compression upper bound at a particular scale. Finally, the embedding obtained covers every scale. Unfortunately, this process costs a logarithmic factor in the compression function obtained. In our context, it happens that the conclusion of Theorem 17.1.2 only considers one particular value of the embedding f. Therefore we can take each one of these cocycles individually, and we will avoid this logarithmic factor. We will show the following theorem:

Theorem 17.2.1. *Let Δ be the lamplighter diagonal product of $(\Gamma_s, a_s, b_s, k_s)_{s\geq 0}$. For any $s \geq 0$, we set $l_s = \operatorname{diam}(\Gamma_s)$. We assume that there exists $m_0 \geq 2$ such that for any $s \geq 0$, we have $k_{s+1} \geq m_0 k_s$ and $l_{s+1} \geq m_0 l_s$.*

Let ϱ_Δ be defined as follows:

$$\begin{aligned} \varrho_\Delta \colon \mathbf{R}_{\geq 1} &\to \mathbf{R}_{\geq 1} \\ x &\mapsto \begin{cases} x/l_s & \text{if } x \in [k_s l_s, k_{s+1} l_s) \\ k_{s+1} & \text{if } x \in [k_{s+1} l_s, k_{s+1} l_{s+1}) \end{cases} \end{aligned}$$

Then there exists some positive constants c_1, c_2 depending only on m_0 and on the degree of Δ such that for any $p \in [1, \infty)$ and any positive integer N we have:

$$\Pi_{\Delta,p}(N) \leq c_1 \frac{N}{\varrho_\Delta(c_2 \log N)}.$$

We will simply adapt to our context the content of Section 6.2.3 of [29] "Basic test functions and 1-cocycles on Δ". We start with some definitions:

Definition 17.2.2. Let Δ be a lamplighter diagonal product.

- We define the $\mathbf{Z}$ projection as:

$$\begin{aligned} p_{\mathbf{Z}} \colon \Delta &\to \mathbf{Z} \\ \left((f_s)_{s\geq 0}, i\right) &\mapsto i \end{aligned}$$

  For any subset $S \subset \Delta$, we define $\operatorname{range}(S) = \operatorname{diam}\{p_{\mathbf{Z}}(z),\, z \in S\}$. For any $z \in \Delta$, we define its **range** as

$$\operatorname{range}(z) = \min\left\{\operatorname{range}(\gamma_{1,z}) \mid \gamma_{1,z} \text{ is a path from 1 to } z\right\}.$$

  Roughly speaking, it is the minimal diameter of the intervals of $\mathbf{Z}$ visited by the cursor when following a path linking 1 and z.

- We define for any $r \geq 2$ a subset U_r of Δ as

$$U_r = \{z \in \Delta \mid \operatorname{range}(z) \leq r\}.$$

- For any $g \in \Delta$, and $\varphi \colon \Delta \to X$, $\tau_g\varphi$ denotes the g-right translate of φ:

$$\tau_g\varphi(h) = \varphi\left(hg^{-1}\right), \quad \text{for any } h \in \Delta.$$

- We finally define

$$\varphi_r\left(\left(f_s\right), i\right) = \max\left\{0, 1 - \frac{|i|}{r}\right\} \mathbf{1}_{U_r}\left(\left(f_s\right), i\right),$$

and, for every $j \geq 1$,

$$\begin{aligned} \Phi_j \colon \Delta &\to \ell^2\left(\Delta\right) \\ Z &\mapsto \frac{\varphi_{2^j} - \tau_z\varphi_{2^j}}{\left\|\nabla\varphi_{2^j}\right\|_2}, \end{aligned}$$

As shown by the following lemma, the family of 1-cocycles $\left(\Phi_j\right)_{j\geq 1}$ captures the size of range(z).

Lemma 17.2.3. *Let $j \geq 1$. For any $z \in \Delta$ satisfying* range$(z) > 2^{j+1}$, *we have*

$$\left\|\Phi_j(z)\right\|_2 \geq \frac{2^j}{3}.$$

Proof. Let $j \geq 1$ and $z \in \Delta$ be such that range$(z) > 2^{j+1}$.

By definition, of φ_r, any element w of support(φ_{2^j}) satisfies range$(w) \leq 2^j$. Let now w be an element of support$(\tau_z\varphi_{2^j})$. It satisfies range$(wz^{-1}) \leq 2^j$. Then, there is a path $\gamma_{w,z}$ from w to z such that range$(\gamma_{w,z}) \leq 2^j$. Hence, if $\gamma_{1,w}$ is a path from 1 to w, then $\gamma_{1,z} = \gamma_{1,w} \cup \gamma_{w,z}$ is a path from 1 to z. By assumption, we can deduce that we have range$(\gamma_{1,z}) > 2^{j+1}$. This implies range$(\gamma_{1,w}) > 2^j$, and since this is true for any path from 1 to w, we obtain range$(w) > 2^j$. Then,

$$\text{support}(\varphi_{2^j}) \cap \text{support}(\tau_z\varphi_{2^j}) = \emptyset.$$

Therefore,

$$\left\|\varphi_{2^j} - \tau_z\varphi_{2^j}\right\|_2^2 = 2\left\|\varphi_{2^j}\right\|_2^2.$$

Let us write $r = 2^j$. We set $U_r^0 = \{g \in U_r \mid p_{\mathbf{Z}}(g) = 0\}$. Then, any element of U_r can be written $g\tau^i$, with $g \in U_r^0$ and $i \in [-r, r]$. Then,

$$\begin{aligned} \left\|\varphi_r\right\|_2^2 &= \sum_{g\in U_r^0} \sum_{i\in[-r,r]} \left(1 - \frac{|i|}{r}\right)^2 \\ &\geq \left|U_r^0\right| \frac{r}{6}. \end{aligned}$$

Let $g \in \Delta$. For any $a \in A$, and $b \in B$, we have range$(g) =$ range$(ga) =$ range(gb), which implies $\varphi_r(g) = \varphi_r(ga) = \varphi_r(gb)$. Then,

$$\begin{aligned} \left\|\nabla\varphi_r\right\|_2^2 &= \sum_{g\in\Delta} \left|\varphi_r(g) - \varphi_r(g\tau)\right|^2 \\ &= \sum_{g\in U_r} \left|\varphi_r(g) - \varphi_r(g\tau)\right|^2 \\ &\leq \frac{|U_r|}{r^2} \\ &\leq 3\frac{\left|U_r^0\right|}{r}. \end{aligned}$$

Therefore we have, for any $z \in \Delta$ satisfying $\mathrm{range}(z) > 2^{j+1}$,

$$\|\Phi_j(z)\|_2^2 = \frac{\|\varphi_{2^j} - \tau_z \varphi_{2^j}\|_2^2}{\|\nabla \varphi_{2^j}\|_2^2} \geq \frac{r^2}{9} = \frac{2^{2j}}{9}. \qquad \square$$

Proof of Theorem 17.2.1. For any $j \geq 0$, Φ_j satisfies the following identity:

$$\Phi_j(gh) = \Phi_j(g) + \tau_g \Phi_j(h), \tag{17.6}$$

for any $g, h \in \Delta$ (this is a cocycle identity). Moreover $\|\Phi_j(z)\|_2 = 0$ if z is a generator in $A \cup B$ and $\|\Phi_j(z)\|_2 \leq 1$ if z is a generator in $\mathbf{Z}$. Therefore Φ_j is $1-$Lipschitz.

As noticed in the proof of Lemma 6.9 of [29], we have, for any $z \in \Delta$,

$$\mathrm{range}(z) \in [k_s, k_{s+1}) \implies |z|_\Delta \leq \frac{9000(\mathrm{range}(z) + 1)l_s}{1 - 1/m_0}. \tag{17.7}$$

Let $s \geq 1$. Let $r \in [k_s, k_{s+1})$, and let j such that $2^{j+1} < r \leq 2^{j+2}$. We set $t = \frac{9000(1+2/m_0)}{1-1/m_0} r l_s$. We will show that we have

$$\rho_{\Phi_j}(t) \geq \frac{r}{12}. \tag{17.8}$$

Let then $z \in \Delta$ be such that $|z|_\Delta \geq t$. This implies in particular $|z|_\Delta \geq \frac{9000(r+1)l_s}{1-1/m_0}$. If $\mathrm{range}(z) < r$, then $|z|_\Delta > \frac{9000(\mathrm{range}(z)+1)l_s}{1-1/m_0}$. This implies, from (17.7), that we have $\mathrm{range}(z) \geq k_{s+1}$, which is a contradiction. Then, we have $\mathrm{range}(z) \geq r > 2^{j+1}$. From Lemma 17.2.3, we deduce $\|\Phi_j(z)\|_2 \geq \frac{2^j}{3} \geq \frac{r}{12}$. This implies, from the cocycle identity (17.6), that for any $z_1, z_2 \in \Delta$ such that $|z_1 z_2^{-1}| > t$, we have $\|\Phi_j(z_1) - \Phi_j(z_2)\|_2 = \|\Phi_j(z_1 z_2^{-1})\|_2 \geq \frac{r}{12}$, which proves (17.8).

Since ℓ^2 embeds isometrically in L^p for all $p \geq 1$ (see Lemma 2.3 of [94]), we obtain that for every $p \in [1, \infty)$, $s \geq 1$ and $r \in [k_s, k_{s+1})$, there exists a 1-Lipschitz map $\Phi_r^p \colon \Delta \to L^p$ such that, if we write ρ_r^p the compression function of Φ_r^p,

$$\rho_r^p(Crl_s) \geq \frac{r}{12}, \quad \text{with } C = \frac{9000(1 + 2/m_0)}{1 - 1/m_0}. \tag{17.9}$$

From Theorem 17.1.2, there exists two constants c_1 and c_2 depending only on the degree of Δ such that for each $p \in [1, \infty)$, $s \geq 1$ and $r \in [k_s, k_{s+1})$, we have for every $n \geq 0$,

$$\Pi_{\Delta,p}(n) \leq \frac{c_1}{\rho_r^p(c_2 \log n)}. \tag{17.10}$$

Let $n \geq 0$. There exists $s \geq 0$ such that $c_2 \log n \in [Ck_s l_s, Ck_{s+1} l_{s+1}]$. Without loss of generality, we can assume that $s \geq 1$. Two cases can occur:

1. If $c_2 \log n \in [Ck_s l_s, Ck_{s+1} l_s]$, then, if we set $r = \frac{c_2 \log n}{Cl_s}$, and $x = \frac{c_2 \log n}{C}$, we have

$$\begin{aligned}
\Pi_{\Delta,p}(n) &\leq \frac{c_1 n}{\rho_r^p(c_2 \log n)} \quad \text{from (17.10)} \\
&= \frac{c_1 n}{\rho_r^p(Crl_s)} \\
&\leq \frac{12 c_1 n}{r} \quad \text{from (17.9)} \\
&= \frac{12 c_1 n}{\varrho_\Delta(\frac{c_2}{C} \log n)}.
\end{aligned}$$

2. If $c_2 \log n \in [Ck_{s+1}l_s, Ck_{s+1}l_{s+1}]$, then $c_2 \log N \geq C\frac{k_{s+1}}{2}l_s \geq Ck_sl_s$. Then, we have

$$\begin{aligned}
\Pi_{\Delta,p}(n) &\leq \frac{c_1 n}{\rho^p_{\frac{1}{2}(k_{s+1})}(c_2 \log n)} \quad \text{from (17.10)} \\
&\leq \frac{c_1 n}{\rho^p_{\frac{1}{2}(k_{s+1})}(C\frac{k_{s+1}}{2}l_s)} \\
&\leq \frac{24 c_1 n}{k_{s+1}} \quad \text{from (17.9)} \\
&= \frac{24 c_1 n}{\varrho_\Delta(k_{s+1}l_{s+1})} \\
&\leq \frac{24 c_1 n}{\varrho_\Delta(\frac{c_2}{C}\log n)}.
\end{aligned}$$

This ends the proof of Theorem 17.2.1. □

Chapter 18

Comparison of the bounds

We compare the bounds obtained in Chapter 16 and 17 to prove Theorems 1, 2 and 3. We start with some definitions.

Definition 18.0.1. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be an non-decreasing function. For any $\alpha \in [0,1]$ and $\beta > 0$, we say that ρ satisfies the condition $(S_{\alpha,\beta})$ if it is injective and moreover there exists $C > 0$ such that

$$\rho^{-1}\left(\frac{x^{1/\beta}}{C}\right) \leq \frac{\rho^{-1}(x)}{x^{1-\alpha}}, \quad \text{for any large enough } x. \tag{$S_{\alpha,\beta}$}$$

Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be an non-decreasing function. We say that ρ is **strongly sublinear** if it is injective and moreover there exists $C > 0$ such that

$$\rho^{-1}\left(\frac{x}{C}\right) \leq \frac{\rho^{-1}(x)}{x}, \quad \text{for any large enough } x. \tag{SSL}$$

Remark 18.0.2. We can make two simple remarks. First, it is obvious that condition (SSL) is the same as $(S_{\alpha,\beta})$ with $\alpha = 0$ and $\beta = 1$. It has its own name because it will play a particular role in the proofs.

Second, it is clear that every function satisfies the condition $(S_{\alpha,\beta})$ with $\alpha = 1$ and $\beta = 1$, with $C = 1$.

Let us detail these two conditions.

Condition $(S_{\alpha,\beta})$ For every $a \in (0,1)$, $x \mapsto x^a$ satisfies condition $(S_{\alpha,\beta})$ with $\alpha = 0$, $\beta = \frac{1}{1-a}$ and $C = 1$. We have the following proposition:

Proposition 18.0.3. *Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be an increasing function such that there exists some $a \in (0,1)$ such that $\frac{\rho^{-1}}{x^{1/a}}$ is non-decreasing. Then ρ satisfies $(S_{\alpha,\beta})$ with $\alpha = 0$ and $\beta = \frac{1}{1-a}$, with $C = 1$.*

Proof. For any $x \geq 1$, we have $x \geq x^{1/\beta}$, which implies $\frac{\rho^{-1}(x)}{\rho^{-1}(x^{1/\beta})} \geq \frac{x^{1/a}}{x^{1/\beta a}} = x$. $\square$

Condition (SSL) The intuition behind condition (SSL) is the following: a change of scale for ρ^{-1} is able to compensate the division by the identity function. We think of ρ^{-1} as "big", and therefore think of ρ as "small". For example:

- if ρ is of the form $x \mapsto x^\alpha$, with $\alpha \in (0,1)$, condition (SSL) is *not* satisfied, since ρ^{-1} is a power function.

- if ρ is of the form $x \mapsto (\log x)^\alpha$, with $\alpha > 0$, condition (SSL) is satisfied, since ρ^{-1} is a power function composed with the exponential.

The following proposition gives more examples of functions satisfying (SSL). Roughly speaking, it states that any function ρ lower than $\log(n)^{1/\alpha}$ satisfies (SSL).

Proposition 18.0.4. *Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be an increasing function such that there exists some $\alpha > 0$ such that $\frac{\rho^{-1}}{\exp(x^\alpha)}$ is non-decreasing. Then ρ satisfies* (SSL) *for any $C > 1$.*

Proof. Let $C > 1$. Then, for any $x \geq 1$, we have $x \geq x/C$, which implies $\frac{\rho^{-1}(x)}{\rho^{-1}(x/C)} \geq \frac{\exp(x^\alpha)}{\exp(x^\alpha/C^\alpha)} = \exp\left(\frac{x^\alpha}{C^\alpha}(C^\alpha - 1)\right)$. We conclude by noticing that this last term is more than x, if x is large enough. □

We can state our main theorem.

Theorem 18.0.5. *There exist a universal constant κ_1 such that the following is true. Let $\rho\colon \mathbf{R}_{\geq 1} \to \mathbf{R}_{\geq 1}$ be a non-decreasing function such that $\frac{x}{\rho(x)}$ is non-decreasing and* $\lim_\infty \rho = \infty$. *We assume that ρ satisfies $(S_{\alpha,\beta})$ for $\alpha \in [0,1]$ and $\beta > 0$.*

Then, there exists a positive constant κ_2, that only depends on β, and a finitely generated elementary amenable group Δ of exponential growth and of asymptotic dimension one such that for any $p \in [1,\infty)$,

$$\Pi_{\Delta,p}(n) \leq \kappa_1 \frac{n}{\rho(\log n)} \quad \text{for any } n,$$
$$\text{and} \quad \Pi_{\Delta,p}(n) \geq 4^{-p}\kappa_2 \frac{n}{\big(\rho(\log n)\big)^{\beta(1+\alpha)}} \quad \text{for infinitely many } n\text{'s.}$$

Moreover, when $\beta \leq 2$, κ_2 can be chosen independent of β.

Remark 18.0.6.

- Theorem 1 is a particular case of Theorem 18.0.5, with $\alpha = 0$ and $\beta = 1$. Indeed, with these values for α and β, condition $(S_{\alpha,\beta})$ is the same as condition (SSL), and this condition is implied by the assumptions made on ρ, from Proposition 18.0.4. This gives an exponent $\beta(1+\alpha) = 1$ on the lower bound.

- Theorem 2 is a particular case of Theorem 18.0.5, with $\alpha = 1$ and $\beta = 1$. Indeed any function satisfies condition $(S_{\alpha,\beta})$ with these values for α and β. This gives an exponent $\beta(1+\alpha) = 2$ on the lower bound.

- Theorem 3 is a particular case of Theorem 18.0.5 with $\alpha = 0$, $\beta = \frac{1}{1-a}$. Indeeed, with these values for α and β, condition $(S_{\alpha,\beta})$ is implied by the assumptions made on ρ, from Proposition 18.0.3. In the statement Theorem 3, we make the assumption that $a \in (0, 1/2)$ because if $a \geq \frac{1}{2}$, if $\alpha = 0$ and $\beta = \frac{1}{1-a}$, then we have $\beta(1+\alpha) \geq 2$. In that case, Theorem 18.0.5 do not improve the lower bound of Theorem 2. When $a \in (0, 1/2)$, then $\beta = \frac{1}{1-a} \leq 2$ and κ_2 can be chosen universal. This gives an exponent $\beta(1+\alpha) = \frac{1}{1-a}$ on the lower bound.

We can prove Theorem 18.0.5.

Proof of Theorem 18.0.5. We set $(\Gamma'_{m_s})_{s\geq 0}$ to be the aforementioned sequence of Lafforgue super expanders (see Example 15.0.7), say with $q = 2$, indexed such that, for every $s \geq 0$, $\left|\Gamma'_{m_s}\right| = m_s$.

Let ρ be a function satisfying the assumptions of Theorem 18.0.5. We can model the process of [29, Proposition B.2.] and get two increasing sequences of integers k_s and n_s such that

(i) The sequence $(n_s)_{s\geq 0}$ is a subsequence of $(m_s)_{s\geq 0}$. Then, we can set $l_s = \operatorname{diam} \Gamma'_{n_s}$.

(ii) We have $k_0 = 0$, $k_1 \geq 3$, $k_{s+1} \geq 3k_s$ and $l_{s+1} \geq 3l_s$ for every $s \geq 0$.

(iii) There is a universal constant c such that if we define $\tilde{\rho}$ by:

$$\tilde{\rho}(x) = \begin{cases} x/l_s & \text{if } x \in [k_s l_s, k_{s+1} l_s) \\ k_{s+1} & \text{if } x \in [k_{s+1} l_s, k_{s+1} l_{s+1}), \end{cases}$$

then we have

$$c^{-1}\rho(x) \leq \tilde{\rho}(x) \leq c\rho(x), \quad \text{for any } x \geq 1.$$

Moreover, since the function $x \mapsto \frac{x}{\rho(x)}$ is non-decreasing, we have, for any $a, x \geq 1$,

$$\rho(ax) \leq a\rho(x). \tag{18.1}$$

For any s, we set $\Gamma_s := \Gamma'_{n_s}$. Let now Δ be the lamplighter diagonal product associated with $(\Gamma_s, a_s, b_s, k_s)_{s\geq 0}$, using the notations of Definition 15.0.4. To get the upper bound of Theorem 18.0.5, we can apply Theorem 17.2.1 to Δ. Then, by construction, $\varrho_\Delta = \tilde{\rho}$, and therefore $c^{-1}\rho \leq \varrho_\Delta \leq c\rho$. Then, there are universal constants c_1 and c_2 such that, for any $n \geq 0$,

$$\begin{aligned} \Pi_{\Delta,p}(n) &\leq c_1 \frac{n}{\varrho_\Delta(c_2 \log n)} \\ &\leq c_1 c_2^{-1} \frac{n}{\varrho_\Delta(\log n)} \quad \text{from (18.1)}, \end{aligned} \tag{18.2}$$

which gives the upper bound of Theorem 18.0.5.

The lower bound requires more calculation. We will use the following facts:

(iv) There is a constant c_3 such that $\operatorname{diam}\Gamma_s \leq c_3 \log|\Gamma_s|$, for every $s \geq 0$ (see [29, Example 2.3.]).

(v) From (iii), we have $c^{-1}k_s \leq \rho(k_s l_s) \leq ck_s$, for any s. In particular, since ρ is non-decreasing, this implies $l_s \geq \frac{\rho^{-1}(c^{-1}k_s)}{k_s}$.

(vi) The sequence $(\Gamma_s)_{\geq 0}$ is an expander: from Theorem 15.0.8 and Proposition 16.1.3, there is $D, \epsilon > 0$ such that we have $\deg\Gamma_s \leq D$ and $h(\Gamma_s) \geq \epsilon$, for every $s \geq 0$.

We fix $p \in [1,\infty)$. We assume that ρ satisfies $(S_{\alpha,\beta})$ with $\alpha \in [0,1]$, and $\beta > 0$. Let $s \geq 1$. We apply Theorem 16.0.1 with $r = \lfloor k_s^\alpha/2 \rfloor$. We get

$$\Pi_{\Delta,p}(N_s) \geq 4^{-p}\frac{h(\Gamma_s)^2}{1536(\deg\Gamma_s)^2}\frac{N_s}{(2k_s + 2\lfloor k_s^\alpha/2\rfloor + 1)(2\lfloor k_s^\alpha/2\rfloor + 1)}, \tag{18.3}$$

with $N_s = |\Gamma_s|^{2\lfloor k_s^\alpha/2\rfloor + 1} \times (2k_s + 2\lfloor k_s^\alpha/2\rfloor + 1) \geq |\Gamma_s|^{k^\alpha/2}$. Then,

$$\begin{aligned}
\log N_s &\geq \frac{k_s^\alpha}{2}\log|\Gamma_s| \\
&\geq (2c_3)^{-1}k_s^\alpha l_s \quad \text{from (iv)} \\
&\geq (2c_3)^{-1}\frac{\rho^{-1}(c^{-1}k_s)}{k_s^{1-\alpha}} \quad \text{from (v)} \\
&= (2c^{1-\alpha}c_3)^{-1}\frac{\rho^{-1}(c^{-1}k_s)}{(c^{-1}k_s)^{1-\alpha}} \\
&\geq (2c^{1-\alpha}c_3)^{-1}\rho^{-1}\left(\frac{c^{-1/\beta}k_s^{1/\beta}}{C}\right) \quad \text{from } (S_{\alpha,\beta}), \text{ if } s \text{ is large enough.}
\end{aligned}$$

Then, since ρ is non-decreasing, we obtain $k_s \leq C^\beta c\Big(\rho(2c^{1-\alpha}c_3\log N_s)\Big)^\beta$. Moreover, we have $(2k_s + 2\lfloor k_s^\alpha/2\rfloor + 1)(2\lfloor k_s^\alpha/2\rfloor + 1) \leq 8k_s^{1+\alpha}$. Therefore, combining with (vi) and (18.3), we obtain, for every large enough s:

$$\begin{aligned}
\Pi_{\Delta,p}(N_s) &\geq 4^{-p}\frac{\epsilon^2}{12288D^2C^{\beta(1+\alpha)}c^{1+\alpha}}\frac{N_s}{\Big(\rho(2c^{1-\alpha}c_3\log N_s)\Big)^{\beta(1+\alpha)}} \\
&\geq 4^{-p}\kappa_2(\alpha,\beta)\frac{N_s}{\Big(\rho(\log N_s)\Big)^{\beta(1+\alpha)}},
\end{aligned}$$

with

$$\kappa_2(\alpha,\beta) = \frac{\epsilon^2}{12288D^2C^{\beta(1+\alpha)}c^{(1+\alpha)(1+\beta(1-\alpha))}(2c_3)^{\beta(1+\alpha)}} \quad \text{(here, we use (18.1)).}$$

Since $\alpha \in (0,1)$, we can deduce

$$\kappa_2(\alpha,\beta) \geq \frac{\epsilon^2}{12288D^2C^{2\beta}c^{2(1+\beta)}(2c_3)^{2\beta}},$$

which proves that κ_2 can be chosen independent of α. If moreover $\beta \leq 2$,

$$\kappa_2(\alpha, \beta) \geq \frac{\epsilon^2}{49152 D^2 C^4 c^6 c_3^4},$$

which proves that, in that case, κ_3 can be chosen independent of β. This ends the proof of Theorem 18.0.5. □

Remark 18.0.7. Fact 13.2.1 (from the proof of Theorem 5) uses an important feature of this proof: we have explicit values for the integers N_s where the lower bounds on Poincaré profiles are known to be valid. More precisely, Theorem 5 relies on Theorem 18.0.5 with $\alpha = 0$ and $\beta = 1$. In that case, we have $N_s = |\Gamma_s| \times (2k_s + 1)$. The contruction of [29, Proposition B.2.] shows that, in the case of functions satisfying condition (SSL), we can take $k_s = 3^s$. Then, it is clear from the condition ((iii)) that the sequence N_s will be sparser when ρ grows slower. This is roughly what is stating Fact 13.2.1.

Remark 18.0.8. The lower bounds are obtained by exhibiting families of subgraphs of the group Δ. These subgraphs are isomorphic to graphs of the family $\Gamma_s^{k_s,r}$, which consist of Cartesian products of $2r + 1$ copies of the lamp groups Γ_s, "distorted" by a scale factor k_s, see Definition 16.2.1. From Proposition 16.2.4, these graphs are isomorphic to subgraphs of Δ when r is at most $k_s/2$. The choice of r is made so that we obtain the highest lower bound. In the proof of Theorem 18.0.5, we take r to be equal to $\lfloor k_s^\alpha/2 \rfloor$, where α is such that ρ satisfies condition $(S_{\alpha,\beta})$. Then, for such a ρ, we obtain the lower bound of Theorem 2 considering $1 + 2\lfloor k_s^\alpha/2 \rfloor$ copies of the lamp groups. To apply Theorem 18.0.5 to a given function ρ, one needs to find a couple (α, β) that minimizes the exponent of the lower bound $\beta(1+\alpha)$. Let us detail this fact in our applications.

In Theorem 2, we consider general functions ρ. This case corresponds to Theorem 18.0.5 with $\alpha = 1$ and $\beta = 1$, see Remark 18.0.6. Then $r \simeq k_s/2$. That means that the lower bound is obtained considering the maximal number of copies of the lamp groups. This gives a lower bound of the form $\frac{n}{(\rho \log n)^2}$, that doesn't match with (18.2).

In Theorem 1, we consider functions ρ growing slower than log, namely condition (SSL). This case corresponds to Theorem 18.0.5 with $\alpha = 0$ and $\beta = 1$, see Remark 18.0.6. Then $r = 0$ and $2r + 1 = 1$. That means that the lower bound is obtained considering single copies of the lamp groups, namely the graphs $\Gamma_s^{k_s,0}$, which are homothetic copies of Γ_s, see Proposition 16.2.2. This gives a lower bound of the form $\frac{n}{\rho(\log n)}$, which is optimal, from (18.2).

Nevertheless, when ρ grows faster than $\log(x)$ we loose this matching. Indeed, if we consider $a \in (0, 1)$, then x^a satisfies condition $(S_{\alpha,\beta})$ with $\alpha = 0$ and $\beta = \frac{1}{1-a}$. The lower obtained with Theorem 18.0.5 is of the form $\frac{n}{(\rho \log n)^{1/(1-a)}}$. As above, since $\alpha = 0$, it is obtained considering single copy of the lamp groups. We see that this lower bound gets worse when α increases, and that the exponent $\frac{1}{1-a}$ goes beyond 2 when a is more than 1/2. Hence, despite Theorem 3 also applies for $a > 1/2$, it is better to use the general Theorem 2.

The case of power functions is very instructive. Let $a \in (0, 1)$ and $\rho \colon x \mapsto x^a$, and let Δ be the associated group (as in the proof of Theorem 18.0.5). Then, as explained before, we can take for any $\alpha \in [0,1]$ a family of subgraphs of the form $\Gamma_s^{k_s,r}$, with $r \simeq k_s^\alpha$. Then, after a short calculation, we obtain a lower bound on the form $\frac{n}{(\log n)^\gamma}$, with $\gamma = \frac{1+\alpha}{1-a(1-\alpha)}$.

- If $a > 1/2$, γ is minimized with $\alpha = 0$. In this case $\gamma = \frac{1}{1-a}$. We recover Theorem 3.
- If $a < 1/2$, γ is minimized with $\alpha = 1$. In this case $\gamma = 2$. We recover Theorem 2.
- If $a = 1/2$, $\gamma = 2$ for any $\alpha \in [0,1]$. In this case, any subgraph of the form $\Gamma_s^{k_s,r}$, with $r \geq k_s/2$ gives a lower bound of the form $\frac{n}{(\rho \log n)^2}$.

Appendix A

Separation of distorted graphs.

In this appendix, we address the following question:

If a graph is distorted, how much can his separation decrease?

Indeed, the same question could be asked for Cheeger constants. The equivalence of Proposition 14.3.2 shows that these questions are closely related.

The toy example we have in mind is the following: let Λ be a finite graph. Let κ be an integer. Let Γ be the graph obtained adding κ vertices along each edge of Λ. How can be compared the separation properties of Γ with those of Λ?

We give three methods of answering this question. The first is called *combinatorial*. It is based on the notion of coarsening of graphs, and is very close to the proof of Proposition 16.2.5. The second is called *geometric* because it is based on a metric assumption. The third is called *analytic* because it concerns L^p-Cheeger constants of metric measure spaces, where graphs are considered as simplicial complexes. These three methods apply in the aforementioned toy example, see Corollaries A.1.5, A.2.3 and A.3.4. They can also provide alternative proofs of Proposition 16.2.5, see Corollaries A.1.6, A.2.4 and A.3.5.

A.1 Combinatorial method: coarsenings

In this section, we study the separation of coarsenings of graphs. See [86] for a more precise study of this notion, in the context of spectral graph theory.

For any graph Γ and any subset $A \subset V\Gamma$, we will still denote by A the graph of vertex set A obtained by taking every edge of Γ of the form $\{a, a'\}$, with $a, a' \in A$.

For any graph Γ and any subset $C \subset V\Gamma$, we denote $\Gamma \setminus C$ the graph obtained removing C, and the edges having an endpoint in C, to the graph Γ.

Definition A.1.1. Let Γ be a finite graph, let $s \in (0, 1)$. We will say that a subset $C \subset V\Gamma$ is an s**-cut set** if every connected component of $\Gamma \setminus C$ contain at most $s\,|V\Gamma|$ vertices.

We recall moreover that the s**-cut** of a finite graph Γ is the minimum size of an s-cut set of Γ, and that the s**-separation profile** of an infinite graphs maps, maps every positive

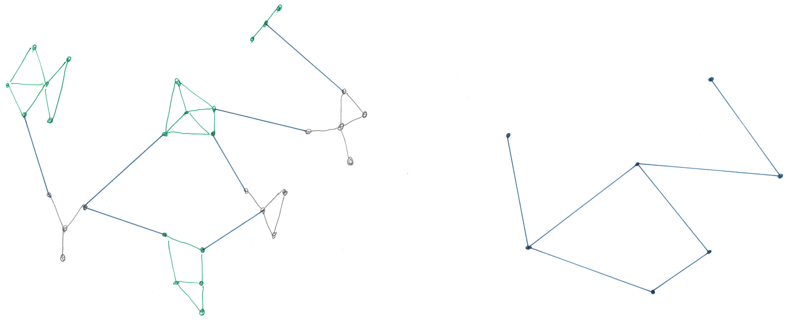


Figure A.1: An example of a regular coarsening Γ (left) and $\Gamma_{\mathcal{A}}$ (right)

integer n to the supremum of the s-cuts of the subgraphs of G having at most n vertices (see Definition 14.3.1 for details).

Definition A.1.2. Let Γ be a finite graph. A partition $(A_i)_{i\in I}$ of $V\Gamma$ is said to be **connected** if the graph A_i is connected, for every $i \in I$.

Given a connected partition $\mathcal{A} = (A_i)_{i\in I}$ of $V\Gamma$, we define the **coarsened graph**, denoted by $\Gamma_{\mathcal{A}}$, as the graph of vertex set $\{A_i, i \in I\}$, such that two distinct vertices A_i and A_j are linked by an edge if and only if there exists $(x, y) \in A_i \times A_j$ such that $\{x, y\}$ is an edge of Γ.

For any subset $A \subset V\Gamma$, we define its **boundary**, denoted by ∂A, as the set of $x \in A$ such that there exists $y \in V\Gamma \setminus A$ satisfying $y \sim x$.

Given a connected partition $\mathcal{A} = (A_i)_{i\in I}$ of $V\Gamma$, the cardinality of ∂A_i will be called the **anchoring** of the set A_i, denoted by $\operatorname{anch}(A_i)$.

See Figure A.1 for an example of a regular coarsening.

Theorem A.1.3. *Let Γ be a finite graph and $\Gamma_{\mathcal{A}}$ be coarsening associated with a partition $\mathcal{A} = (A_i)_{i\in I}$. Then*

$$\operatorname{sep}_\Gamma(|V\Gamma|) \geq \frac{\min(|A_i|)}{8\max(|A_i|)}\operatorname{cut}^{1/2}(\Gamma_{\mathcal{A}})$$

On the other hand, if for any $i \in I$ we have $|A_i| \leq \frac{|V\Gamma|}{2}$, then

$$\operatorname{cut}^{1/2}(\Gamma) \leq 8\frac{\max(|A_i|)}{\min(|A_i|)}\max(\operatorname{anch} A_i)\operatorname{sep}_{\Gamma_{\mathcal{A}}}(|V\Gamma_{\mathcal{A}}|).$$

Remark A.1.4. If an A_i contains more than $\frac{|V\Gamma|}{2}$ vertices, then Γ can be cut extracting A_i (removing at most $\operatorname{anch}(A_i)$ vertices), and cutting it (removing at most $\operatorname{cut}^{1/2}(A_i)$ vertices). This proves that, in this case, we have:

$$\operatorname{cut}^{1/2}(\Gamma) \leq \operatorname{anch}(A_i) + \max\left(\operatorname{cut}^{1/2}(A_i)\right)$$

Theorem A.1.3 has the two following corollaries. The first graph concerns the toy example of the introduction of Appendix A, the second is a variant of Proposition 16.2.5.

Corollary A.1.5. *Let Λ be a finite graph with no isolated vertex. Let $\kappa \geq 2$ be an integer. Let Γ be the graph obtained adding κ vertices along each edge of Λ. Let D be a bound on the degrees of the vertices of Λ. Then, Γ has a subgraph Γ' such that*

$$\operatorname{cut}\Gamma' \geq \frac{1}{24D}\operatorname{cut}\Lambda,$$

and

$$\operatorname{cut}\Gamma \leq 24D^2 \operatorname{sep}_\Lambda(|V\Lambda|). \tag{A.1}$$

Proof. Λ can be recovered from Γ by doing a coarsening, making a partition $(A_i)_{i\in I}$ of Γ using balls of radius $\kappa/2$ centred at the vertices of Λ (when κ is odd, the middle can be associated with any of the ends of his edge). Then, we have for every $i \in I$, $\frac{\kappa}{2}+1 \leq |A_i| \leq D\frac{\kappa}{2}+1$ when κ is even, and $\frac{\kappa-1}{2}+1 \leq |A_i| \leq D\frac{\kappa+1}{2}+1$ when κ is odd. Both imply $\frac{\max|A_i|}{\min|A_i|} \leq 3D$. Moreover, the anchoring of the A_i's is bounded by D. This implies inequality (A.1) and

$$\operatorname{sep}_\Gamma(|V\Gamma|) \geq \frac{1}{24D}\operatorname{cut}^{1/2}(\Lambda),$$

which implies that Γ has a subgraph Γ' such that

$$\operatorname{cut}\Gamma' \geq \frac{1}{24D}\operatorname{cut}\Lambda. \qquad \square$$

Corollary A.1.6. *Let $\Gamma_s^{k_s,0}$ be as in Definition 16.2.1, with $r = 0$. Then, $\Gamma_s^{k_s,0}$ has a subgraph Γ such that*

$$\operatorname{cut}(\Gamma) \geq \frac{1}{8}\operatorname{cut}(\Gamma_s).$$

Proof. This straightforward, considering the partition in *lines* explained in §16.2.1. $\square$

This statement should be compared with *Proposition* 16.2.5, which states, for $r = 0$, $\operatorname{cut}(\Gamma_s^{k_s,0}) \geq \operatorname{cut}(\Gamma_s)$.

To prove Theorem A.1.3, we will use the following lemma:

Lemma A.1.7. *Let G be a finite graph, let $s \leq 1/2$. Then*

$$\operatorname{cut}^s(G) \leq \frac{4}{s}\operatorname{sep}_G(|VG|).$$

Proof. We will show at first that for any positive integer k we have

$$\operatorname{cut}^{\frac{1}{2^k}}(G) \leq 2^{k+1}\operatorname{sep}_G(|VG|). \tag{A.2}$$

This is obtained by induction on k. If $k = 1$, this is immediate. Let k be a positive integer. By assumption, there exists a $\frac{1}{2^k}$-cut set of G of size at most $2^{k+1}\operatorname{sep}_G(|VG|)$. Let us call C such a set. In particular, C is non-empty. Then, taking unions of connected components of $VG \setminus C$, on can find a partition of $G \setminus C$ into l subgraphs $G_1, \ldots, G_l$ such that G_i contains at most $\frac{1}{2^k}|VG|$ vertices. Up to making unions of subgraphs of G_i's of size less than $\frac{1}{2^{k+1}}|VG|$, and to change the numbering, we can assume without loss of generality that for every $i \leq l-1$, G_i contains at least $\frac{1}{2^{k+1}} |VG|$ vertices. Then, we have

$$|VG| > |VG| - |C| \geq \sum_{i=1}^{l-1} |G_i| \geq \frac{l-1}{2^{k+1}} |VG| ,$$

which implies $l \leq 2^{k+1}$. Then, each G_i can be 1/2-cut removing a set C_i containing at most $\operatorname{sep}_G(|VG|)$ vertices. Then, the set $C' = C \cup C_1 \cup \ldots C_l$ is a $\frac{1}{2^{k+1}}$-cut set of G. We have

$$\begin{aligned}
|C'| &\leq |C| + \sum_{i=1}^{l} |C_i| \\
&\leq 2^{k+1} \operatorname{sep}_G(|VG|) + l \operatorname{sep}_G(|VG|) \\
&\leq \left(2^{k+1} + 2^{k+1}\right) \operatorname{sep}_G(|VG|) \\
&= 2^{k+2} \operatorname{sep}_G(|VG|),
\end{aligned}$$

which ends the proof of (A.2).

Let now $s \leq 1/2$. Let k be the smallest integer such that $\frac{1}{2^k} \leq s$. Then we have $2^{k+1} \leq 4/s$. Therefore,

$$\begin{aligned}
\operatorname{cut}^s(G) &\leq \operatorname{cut}^{\frac{1}{2^k}}(G) \\
&\leq 2^{k+1} \operatorname{sep}_G(|VG|) \quad \text{from (A.2)} \\
&\leq \frac{4}{s} \operatorname{sep}_G(|VG|)
\end{aligned}$$

□

Proof of Theorem A.1.3. For every vertex x of Γ, we denote by $\bar{x}$ the unique A_i that contains x. Then, $\bar{x}$ is a vertex of $\Gamma_{\mathcal{A}}$.

We start with the first inequality. Let $s \in (0,1)$. Let C be a s-cut set of Γ. Let C' be the set of vertices $\bar{c} \in V\Gamma_{\mathcal{A}}$ such that there exists some $x \in \bar{c}$ such that $x \in C$. We have $|C'| \leq |C|$.

Let $F' \subset V\Gamma_{\mathcal{A}} \setminus C'$ be such that the graph F' is connected. Then we can denote by F the set of vertices $x \in V\Gamma$ such that $\bar{x} \in F'$. F does not meet C, and moreover $\tilde{F}$ is connected: any path in F' can be followed identically, adding some steps to cross the A_i's, which are connected by assumption.

Since C is a s-cut set of Γ, we have:

$$|F| \leq s\,|V\Gamma| .$$

We have moreover $|V\Gamma| \leq \max(|A_i|) \times |V\Gamma_{\mathcal{A}}|$ and $|F'| \times \min(|A_i|) \leq |F|$. Therefore we can deduce

$$|F'| \leq \frac{\max(|A_i|)}{\min(|A_i|)} s \times |V\Gamma_{\mathcal{A}}| ,$$

which means that C' is a $\left(\frac{\max(|A_i|)}{\min(|A_i|)} s\right)$-cut set of $\Gamma_{\mathcal{A}}$. Then, we have shown that for any $s \in (0,1)$, we have

$$\operatorname{cut}^{\frac{\max(|A_i|)}{\min(|A_i|)} s}(\Gamma_{\mathcal{A}}) \leq \operatorname{cut}^s(\Gamma).$$

In particular, for $s = \frac{1}{2}\frac{\min(|A_i|)}{\max(|A_i|)}$, this gives

$$\begin{aligned}\mathrm{cut}^{1/2}(\Gamma_{\mathcal{A}}) &\leq \mathrm{cut}^s(\Gamma)\\ &\leq \frac{4}{s}\,\mathrm{sep}_\Gamma(|V\Gamma|) \quad \text{from Lemma A.1.7.}\\ &= 8\frac{\max(|A_i|)}{\min(|A_i|)}\,\mathrm{sep}_\Gamma(|V\Gamma|).\end{aligned}$$

We prove now the second inequality. Then we assume that for any i, A_i contains at most $\frac{|V\Gamma|}{2}$ vertices. Let $s \in (0,1)$. Let C' be a s-cut set of $\Gamma_{\mathcal{A}}$ of size $\mathrm{cut}^s(\Gamma_{\mathcal{A}})$. Let C be the set of vertices x such that $\bar{x} \in C'$ and $x \in \partial\bar{x}$. Then C contains at most $|C'|\max(\mathrm{anch}(A_i))$ vertices, and any connected subgraph of $\Gamma \setminus C$ is an union of at most $s\,|V\Gamma_{\mathcal{A}}|$ graphs A_i. Each of these contains at most $\max|A_i|$ vertices, and $\Gamma_{\mathcal{A}}$ contains at most $\frac{|V\Gamma|}{\min|A_i|}$ vertices. Then, each connected subgraph of $\Gamma \setminus C$ contains at most $s\frac{\max|A_i|}{\min|A_i|}\,|V\Gamma|$ vertices. Finally,

$$\mathrm{cut}^{s\times\frac{\max|A_i|}{\min|A_i|}}(\Gamma) \leq \max(\mathrm{anch}\,A_i) \times \mathrm{cut}^s(\Gamma_{\mathcal{A}}).$$

In particular, for $s = \frac{1}{2}\frac{\min(|A_i|)}{\max(|A_i|)}$,

$$\begin{aligned}\mathrm{cut}^{1/2}(\Gamma) &\leq \max(\mathrm{anch}\,A_i) \times \mathrm{cut}^s(\Gamma_{\mathcal{A}})\\ &\leq \frac{4}{s}\max(\mathrm{anch}\,A_i)\,\mathrm{sep}_{\Gamma_{\mathcal{A}}}(|V\Gamma_{\mathcal{A}}|\\ &= 8\frac{\max(|A_i|)}{\min(|A_i|)}\max(\mathrm{anch}\,A_i)\,\mathrm{sep}_{\Gamma_{\mathcal{A}}}(|V\Gamma_{\mathcal{A}}|\,.\end{aligned}$$

□

A.2 Geometric method: bi-Lipschitz embeddings

In this section, we adress the question in the case where the so-called *distorsion* satisfies some metric assumptions. More precisely, we assume that the initial graph embeds with a Lipschitz map, with some additional assumptions.

Theorem A.2.1. *Let Γ and X be two graphs, with Γ finite containing at least 4 vertices. Let $D \geq 2$ be a bound on the degrees of the vertices of Γ. Let $\kappa > 0$, $\alpha \in (0,1]$ and $c > 0$ be such that there exists a map $f\colon V\Gamma \to VX$ such that*

(i) $d(f(x), f(y)) \leq \kappa$, for every edge $\{x,y\}$ of Γ.

(ii) for any subset $F \subset V\Gamma$ satisfying $|F| \geq \frac{V\Gamma}{2}$, we have

$$\frac{1}{|EF|}\sum_{\{x,y\}\in EF} d(f(x), f(y)) \geq \alpha\kappa,$$

where EF is the set of edges of Γ of the form $\{x,y\}$ with $x, y \in F$.

(iii) for any ball B of X of radius κ, we have $|f^{-1}(B)| \leq c$.

Then

$$\mathrm{sep}_X\left(\kappa\frac{D}{2}|V\Gamma|\right)\geq\frac{\alpha}{4c^3D}\,\mathrm{cut}^{1/2}(\Gamma)\,.$$

Remark A.2.2. The assumptions of the theorem above are satisfied when Γ embeds in X with a bilipschitz map of constants $\alpha\kappa$ and κ, taking c to be the maximal size of a ball of radius $\frac{1}{\alpha}$ in Γ. This is the setting we have in mind. The assumptions on f are a little more general, allowing some local perturbations, such that f is still bilipschitz on average (assumption (ii)), and satisfies a loose notion of injectivity (assumption (iii)).

Corollary A.2.3. *Let Λ be a finite graph, and D be a bound on the degrees of the vertices of Λ. Let $\kappa\geq 2$ be an integer. Let Γ be the graph obtained adding κ vertices along each edge of Λ. Then, Γ has a subgraph Γ' such that*

$$\mathrm{cut}(\Gamma')\geq(4D)^{-1}\,\mathrm{cut}^{1/2}(\Lambda),$$

Proof. The canonical map $V\Lambda\hookrightarrow V\Gamma$ is clearly $\kappa+1$-bilipschitz, then we can apply Theorem A.2.1 with $\alpha=1$ and $c=0$. □

Corollary A.2.4. *Let $\Gamma_s^{k_s,0}$ be as in Definition 16.2.1, with $r=0$. Then, $\Gamma_s^{k_s,0}$ has a subgraph Γ' such that*

$$\mathrm{cut}(\Gamma')\geq\frac{1}{4}(|A|+|B|)^{-1}\,\mathrm{cut}(\Gamma_s).$$

Proof. The canonical map $V\Gamma_s\hookrightarrow\Gamma_s^{k_s,0}, x\mapsto(x,0)$ is $2k_s$-bilipschitz, then we can apply Theorem A.2.1 with $\alpha=1$ and $c=0$. Moreover, the degree of Γ_s is equal to $|A|+|B|$. □

Proof of Theorem A.2.1. Given a graph Λ, we will identify every subset of $V\Lambda$ with a subgraph of Λ, kepping every edge of Λ of the form $\{x,y\}$, with $x,y\in V\Lambda$.

We will define a subgraph Γ' of X, that will be considered as an avatar of Γ. For any edge $\{x,y\}$ of Γ, the vertices $f(x)$ and $f(y)$ are at distance at most κ, then we can choose a sequence of less than $\kappa-1$ vertices that link them along a geodesic. We will denote the set of these vertices by "geod$(f(x),f(y))$". We then define Γ' as the graph

$$\Gamma'=f(V\Gamma)\cup\bigcup_{\{x,y\}\in E\Gamma}\mathrm{geod}(f(x),f(y)).$$

We can define a projection map

$$\begin{array}{rccl}\pi_\Gamma: & V\Gamma' & \longrightarrow & \mathcal{P}(V\Gamma)\\ & x & \longmapsto & \{y\in V\Gamma\mid d(x,f(y))=d(x,f(V\Gamma)\}\,.\end{array}$$

For every $x\in V\Gamma'$, we have

$$\pi_\Gamma(x)\subset\{y\in V\Gamma\mid d(f(x),y)\leq\kappa\}\,. \tag{A.3}$$

The graph Γ has at most $\frac{1}{2}D\,|V\Gamma|$ edges. Therefore,

$$|V\Gamma'|\leq|V\Gamma|+(\kappa-1)\frac{D}{2}|V\Gamma|\leq\kappa\frac{D}{2}|V\Gamma|\,. \tag{A.4}$$

Let $s = \frac{\alpha}{Dc^2} \in (0,1)$. Let C' be a s-cut set of Γ'. We set $C = \{x \in V\Gamma \mid d(f(x), C') \leq \kappa\}$. We have

$$f^{-1}(C') \subset C \quad \text{and} \quad \pi_\Gamma(C') \subset C, \tag{A.5}$$

where the second inclusion comes from (A.3). Moreover, by assumption (iii), to each vertex of C' corresponds at most c vertices in C. Therefore

$$|C| \leq c\,|C'|\,.$$

We will show that C is a 1/2-cut of the graph Γ. Let F be a connected subgraph of $\Gamma \setminus C$. We need to show that F contains at most half of the vertices of Γ. Let us assume by contradiction that we have $|F| > |V\Gamma|/2$. Let F' be the following subset of $V\Gamma'$:

$$F' = f(F) \cup \bigcup_{(x,y)\in EF} \text{geod}(f(x), f(y)).$$

Since F is connected, F' is connected as well. Let us see that F' do not intersect C'. First, from the left inclusion of (A.5), $f(F)$ do not intersect C'. Second, if $\{v_1, v_2\}$ is an edge of F, and v' is a vertex of $\text{geod}(f(v_1), f(v_2))$, then we have $d(v', f(v_1)) \leq \kappa$. Therefore, from the definition of C, and since v_1 is not in C, v' is not in C'.

Then, F' is a connected subgraph of Γ and do not intersect C'. From the fact that C' is an s-cut set of Γ', we can deduce

$$|F'| \leq s\,|V\Gamma'|\,. \tag{A.6}$$

To each edge of the graph F corresponds some vertices in F': the images by f of the source and the target of the edge, and the vertices that link these two points along the geodesic "geod" we have chosen. We can call this set of vertices a "path". From assumption (ii) this gives in total at least $|EF|\,\alpha\kappa$ vertices, counted with multiplicity.

A single vertex of F' can lie in several of these paths. Precisely, if a vertex x appears in k paths, then we can call $v_1, \ldots, v_l$ the endpoints of these paths. Then, we have $k \leq C_l^2 = \frac{l(l-1)}{2}$. Moreover, for any i, the distance from x to $f(v_i)$ is at most κ. Then, from assumption (iii) we have $l \leq c$. So $k \leq \frac{c^2}{2}$. Finally, we can deduce

$$|F'| \geq \frac{2\alpha\kappa}{c^2}\,|EF|\,. \tag{A.7}$$

Then, since F is connected, we have $|F| \leq |EF| + 1$ and then, combining with the previous inequalities:

$$\begin{aligned}
|F| &\leq \frac{c^2}{2\alpha\kappa}\,|F'| + 1 \quad \text{from (A.7)} \\
&\leq \frac{sc^2}{2\alpha\kappa}\,|V\Gamma'| + 1 \quad \text{from (A.6)} \\
&\leq \frac{sc^2 D}{4\alpha}\,|V\Gamma| + 1 \quad \text{from (A.4)} \\
&= \frac{1}{4}\,|V\Gamma| + 1.
\end{aligned}$$

If Γ has at least 4 vertices, we deduce $|F| \leq \frac{|V\Gamma|}{2}$, which is a contradiction. Then, the graph Γ has a $\frac{1}{2}$-cut set of size at most $c\,\mathrm{cut}^s(\Gamma')$. We deduce

$$\begin{aligned}
\mathrm{cut}^{1/2}(\Gamma) &\leq c\,\mathrm{cut}^s\left(\Gamma'\right) \\
&\leq c\frac{4}{s}\,\mathrm{sep}_{\Gamma'}(|V\Gamma'|) \quad \text{from Lemma A.1.7} \\
&\leq c\frac{4}{s}\,\mathrm{sep}_{\Gamma'}\left(\kappa\frac{D}{2}\,|V\Gamma|\right) \quad \text{from Lemma A.4} \\
&\leq c\frac{4}{s}\,\mathrm{sep}_X\left(\kappa\frac{D}{2}\,|V\Gamma|\right) \\
&= \frac{4c^3D}{\alpha}\,\mathrm{sep}_X\left(\kappa\frac{D}{2}\,|V\Gamma|\right).
\end{aligned}$$

□

A.3 Analytic method: L^p-Cheeger constants

In this section, we adress the question from an analytic point of view. We will consider that both initial and distorted graphs describe the same metric space, but at different scales.

Statement and consequences

We start with some definitions.

Definition A.3.1. Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph, and $b \geq 2$. Let Y be a subset of X.

- We say that Y is **b-separated** if for every pair y, y' of distinct points of Y, we have $d(y, y') \geq b$.
- We say that Y is **maximal b-separated** if moreover it is maximal with this property: any subset Z of X that is b-separated and contains Y, is equal to Y.

Definition A.3.2. Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph, and $b > 0$. Let S be a maximal b-separated subset of $V\Gamma$. Then we can endow S with a *graph structure*, declaring that v and v' in S are neighbours if and only if $d_\Gamma(v, v') < 2b$.

Any graph obtained with this process will be called a **b-rescaling** of Γ.

Theorem A.3.3. *Let Γ be a finite graph of maximal degree D, let b be a positive integer and k be such that every ball of radius $8b$ in Γ have at most kb vertices. Let Λ be a b-rescaling of Γ. Then there exists a positive constant C that only depend on D and k such that for any $p \in [1, \infty)$,*

$$h_p(\Gamma) \geq \frac{C}{b} \cdot h_p(\Lambda),$$

Recall that $h_1^p(\Gamma)$ denotes the Cheeger constant of the graph Γ (see Definition 14.1.1). The theorem is only intersting when k is independent on b. This is the case in the following corollaries, which give examples of maps.

Corollary A.3.4. *Let Λ be a finite graph. Let κ be a positive integer. Let Γ be the graph obtained adding κ vertices along each edge of Λ. Then there exists a positive constant C depending only on the maximal degree of Λ such that for any $p \in [1, \infty)$,*

$$h_p(\Gamma) \geq \frac{C}{\kappa} \cdot h_p(\Lambda).$$

There is a κ^{-1} factor on the right-hand side, which differs from Corollaries A.1.5 and A.2.3. However, the equivalence between $\mathrm{cut}(\Gamma)$ and $|\Gamma|\, h(\Gamma)$ shown by Hume [67] (used in the proof of Theorem 14.3.2) shows that this result is not weaker.

Proof of Corollary A.3.4. Let us consider $V\Lambda$ as a subset of $V\Gamma$. For any distinct pair of vertices λ, λ' in $V\Lambda$, we have $d_\Gamma(\lambda, \lambda') \geq \kappa$. Then $V\Lambda$ is a κ-separated subset of $V\Gamma$. Moreover, any vertex of Γ in $\Gamma \setminus \Lambda$ is at distance less than κ from a vertex of Λ. Therefore $V\Lambda$ is maximal κ-separated in Γ. Is is clear that the corresponding b-rescaling is equal to the graph Λ. Finally, in Γ, the balls of radius 8κ contain at less than $D^9\kappa$ vertices, therefore the result follows from Theorem A.3.3. □

Corollary A.3.5. *Let $\Gamma_s^{k_s,0}$ be as in Definition 16.2.1, with $r = 0$. Let D be the degree of the graph Γ_s. Then, there exists a positive constant C' that only depend on D such that we have for any $p \in [1, \infty)$*

$$h_p(\Gamma_s^{k_s,0}) \geq \frac{C'}{k_s} h_p(\Gamma_s).$$

Proof. We recall that the vertex set of $\Gamma_s^{k_s,0}$ is $\Gamma_s \times [-k-s, k_s]$. The subset of elements of the form $(x, 0)$, with $x \in \Gamma_s$, is $2k_s$-separated. The $2k_s$-rescaling associated with this subset is isomorphic to Γ_s. Moreover, the balls of radius $16k_s$ in $\Gamma_s^{k_s,0}$ contain at most $2k_s D^9$ vertices. The inequality follows from Theorem A.3.3. □

Proof of Theorem A.3.3

We give the proof of Theorem A.3.3. For any r and y, we will denote by $B(y, r)$ the closed ball centred at y of radius r. When (Z, ν) is a positive finite measure space, we denote the averaged integral by $⨍_Z f d\nu := \frac{1}{\nu(Z)} \int_Z f d\nu$. After [68], we introduce a notion of metric measure spaces.

Definition A.3.6. A **standard metric measure space** is a metric measure space (X, d, μ) with the following properties:

(i) (X, d) is a complete and separable metric space.

(ii) μ is a non-trivial, locally finite, Borel measure.

(iii) X has **bounded packing on large scales**: there exists $r_0 \geq 0$ such that for all $r \geq r_0$, there exists $K_r > 0$ such that

$$\forall x \in X,\ \mu(B(x, 2r)) \leq K_r \mu(B(x, r)).$$

We then say that X has **bounded packing on scales** $\geq r_0$.

(iv) X is k-**geodesic** for some $k > 0$: for every pair of points $x, y \in X$ there is a sequence $x = x_0, \ldots, x_n = y$ such that $d(x_{i-1}, x_i) \leq k$ for all i and $d(x, y) = \sum_{i=1}^n d(x_{i-1}, x_i)$.

Up to rescaling the metric we will always assume that X is 1-geodesic and has bounded packing on scales ≥ 1.

Definition A.3.7. We will say that a subset of a standard metric measure space is **1-thick** if it is a union of closed balls of radius 1. Axioms (i) and (iii) imply in particular that a non-empty 1-thick subset has positive measure. Such a subset $Z \subset X$ will be equipped with the **induced measure** and the **induced and 1-distance**:

$$d(z, z') = \inf \left\{ \sum_{i=1}^n d(z_{i-1}, z_i) \right\},$$

where the infimum is taken over all sequences $z = z_0, \ldots, z_n = z'$, such that each z_i is an element of Z, and $d(z_i, z_{i+1}) \leq 1$ for every i. (this distance takes values in $[0, \infty]$.)

Remark A.3.8. In the case of a bounded degree graph, d is the shortest path metric and μ is the (vertex) counting measure. 1-thick subspaces are 1-thick subgraphs equipped with the vertex counting measure and their own shortest path metric.

The following definition is a generalization of Definition 14.1.1, for standard metric measure spaces, and different scales.

Definition A.3.9. Let (X, d, ν) be a measured metric space and let $a > 0$. Given a measurable function $f : X \to \mathbf{R}$, we define its **upper gradient at scale** a to be

$$|\nabla_a f|(x) = \sup_{y, y' \in B(x,a)} |f(y) - f(y')|.$$

Let (Z, d, ν) be a metric measure space with finite measure and fix a scale $a > 0$. We define the L^p-**Poincaré constant at scale** a of Z to be

$$h_{a,p}(Z) = \inf_f \frac{\|\nabla_a f\|_p}{\|f\|_p},$$

where the infimum is taken over all $f \in L^p(Z, \nu)$ such that $f_Z := \frac{1}{\nu(Z)} \int_Z f d\nu = 0$ and $f \not\equiv 0$. We adopt the convention that $h_{a,p}(Z) = 0$ whenever $\nu(Z) = 0$.

This generalizes Definition 14.1.1 in the following sense: if we endow a graph with shortest path distance and the (vertex) counting measure, we get the same definition. We now introduce a notion of discretization for metric measure spaces.

Definition A.3.10. Let (Z, d, ν) be a metric measured space and $b > 0$. A partition $\mathcal{A} = (A_y)_{y \in Y}$ of Z is called a **partition of scale** b if for any $A \in \mathcal{A}$, there exists $z \in Z$ such that

$$B(z, b) \subset A \subset B(z, 2b).$$

Any point z satisfying these inclusions is called a b-**centre** of A. We will always assume that such a partition $\mathcal{A}$ is indexed by a set of b-centres. This implies in particular that Y, which is a priori an abstract set, is a subset of Z.

Definition A.3.11. Let (Z, d, ν) be a metric measured space and $b > 0$. Let $\mathcal{A} = (A_y)_{y \in Y}$ be a measurable partition of scale b, such that for any $y \in Y$, y is a b-centre of A_y.

Then we can endow Y with the subset distance, and the unique measure ν_Y satisfying $\nu_Y(\{y\}) = \nu(A_y)$.

Let $\pi : Z \to Y$ be defined by "$\pi(z)$ is the only $y \in Y$ such that $z \in A_y$". Note that π is surjective, and a right-inverse of the inclusion $j : Y \to Z$. Moreover, $\pi^{-1}(\{y\}) = A_y$ for every $y \in Y$.

Any space $(Y, d_{|Y}, \nu_Y)$ obtained with this process will be called a **discretization of** Z **parameter** b.

Remark A.3.12.

1. Given a maximal b-separated subset Y of Z (see Definition A.3.1), there always exists a partition of scale b indexed by Y. Then we can consider Y as a metric measure space, up to choosing an appropriate partition. Indeed, since $\cup_{y \in Y} B(y, 2b)$ covers Z, one can find a measurable partition of scale b such that each element is b-centred at a point of Y.

2. As we mentioned above, any graph can also be considered as a metric measure space, where the distance takes only integer values. The notion of *b-rescaling* (Definition A.3.2) should not be confused with the *discretization of parameter* b presented here. Indeed, given a positive integer b and a maximal b-separated subset of a given graph, one can construct a b-rescaling (see details below in the proof of Theorem A.3.3), or, choosing an appropriate partition of scale b, a *discretization of parameter* b. These two metric measure spaces are different, but look alike when the initial graph has enough regularity; one may notice that the distances differ by a factor between b and $2b$.

Proposition A.3.13. *(see [68, Lemma 5.8]) Let (Z, d, ν) be a metric measure space of finite total measure. Assume there is no $z \in Z$ with $\nu(\{z\}) > \frac{2}{3}\nu(Z)$. Let Y be a discretization of Z of parameter $b \geq 1$. Then for all $p \in [1, \infty)$ and all $a \geq 2b$,*

$$h_{a,p}(Y) \leq 12 h_{2a,p}(Z), \quad \text{and} \quad h_{a,p}(Z) \leq h_{3a,p}(Y).$$

We will use the following lemma:

Lemma A.3.14. *(see [68, Proposition 7.1]) Let Z be as in Proposition A.3.13. Then for all $p \in [1, \infty)$ and all $a \geq 1$, we have $h_{a,p}(Z) \leq 6$.*

Proof of Lemma A.3.14. From our assumptions (Definition A.3.6), ν is measure isomorphic to a real interval and an at-most-countable collection of atoms. Then there exists a subset $Y \subset Z$ satisfying $\frac{1}{3}\nu(Z) \leq \nu(Y) \leq \frac{2}{3}\nu(Z)$. Let f be the characteristic function of Y.

Then $\|f - f_Z\|_p^p \geq \frac{\nu(Z)}{3 \cdot 2^p}$ and $\|\nabla_a f\|_p^p \leq \nu(Z)$, thus $h_{a,p}(Z) \leq 2 \cdot 3^{\frac{1}{p}} \leq 6$. □

Proof of Proposition A.3.13. This is the same proof as in [68], where we detail the constants involved.

Let $\mathcal{A} = (A_y)_{y\in Y}$ be a partition of scale b associated with Y. Let $f \in L^\infty(Z)$ be such that $\fint_Z f d\nu = 0$. We define $\phi \in \ell^\infty(Y)$ by $\phi(y) = \fint_{A_y} f d\nu$. Clearly $\fint_Y \phi d\nu_Y = 0$ and $\|\phi\circ\pi\|_{Z,p} = \|\phi\|_{Y,p}$. Write $f(z) = \phi(\pi(z)) + \fint_{A_{\pi(z)}}(f(z)-f(w))d\nu(w)$. Then

$$\begin{aligned}
\|f\|_{Z,p} &\le \|\phi\circ\pi\|_{Z,p} + \left(\int_Z \left|\fint_{A_{\pi(z)}} (f(z)-f(w))\, d\nu(w)\right|^p d\nu(z)\right)^{1/p} \\
&\le \|\phi\|_{Y,p} + \left(\int_Z \fint_{A_{\pi(z)}} |f(z)-f(w)|^p \, d\nu(w) d\nu(z)\right)^{1/p} \\
&\le \|\phi\|_{Y,p} + \left(\int_Z |\nabla_{2a} f|(z)^p d\nu(z)\right)^{1/p} \\
&= \|\phi\|_{Y,p} + \|\nabla_{2a} f\|_p \ .
\end{aligned}$$

On the other hand, for any y, y' in Y, $\phi(y')$ is in the interval $\left[\inf_{A_{y'}} f, \sup_{A_{y'}} f\right]$, and each $A_{y'}$ satisfying $d(y,y') \le a$ is contained in the ball $B(y, a+2b)$. Then, we have

$$|\nabla_a \phi|(y) \le |\nabla_{a+2b} f|(z) \le |\nabla_{2a} f|(z), \quad \text{for any } y \in Y \text{ and } z \in A_y.$$

We now prove the first inequality of Proposition A.3.13. If $h_{2a,p}(Z) \le \frac{1}{2}$, then for any $\epsilon \in (0, 1/6)$ we can find f as above so that

$$\frac{2}{3} \ge \frac{1}{2} + \epsilon \ge h_{2a,p}(Z) + \epsilon \ge \frac{\|\nabla_{2a} f\|_p}{\|f\|_p} \ge \frac{\|\nabla_{2a} f\|_p}{\|\phi\|_p + \|\nabla_{2a} f\|_p}.$$

Thus $\|\nabla_{2a} f\|_p \le 2\|\phi\|_p$ and

$$h^p_{2a}(Z) + \epsilon \ge \frac{\|\nabla_a \phi\|_p}{3\|\phi\|_p} \ge \frac{1}{3} h^p_a(Y).$$

Since ϵ was arbitrary, $h_{a,p}(Y) \le 3h_{2a,p}(Z)$. Moreover, from Lemma A.3.14, $h_{a,p}(Y) \le 6$, so if $h_{2a,p}(Z) \ge \frac{1}{2}$, then $h_{a,p}(Y) \le 12 h_{2a,p}(Z)$.

The other direction is easier: given $\psi \in \ell^\infty(Y)$ such that $\fint_Y \psi d\nu_Y = 0$, we define

$$g := \sum_{y\in Y} \psi(y) 1_{A_y},$$

where 1_{A_y} denotes the characteristic function of A_y. We clearly have $\fint g d\nu = 0$ and $\|g\|_p = \|\psi\|_p$. Hence we are left with comparing the gradients.

$$\begin{aligned}
\|\nabla_a g\|_p^p &= \sum_Y \nu(A_y) \fint_{A_y} \sup_{z',z''\in B(z,a)} |g(z') - g(z'')|^p d\nu(z) \\
&\le \sum_Y \nu(A_y) \sup_{z',z''\in B(y,a+2b)} |g(z') - g(z'')|^p \\
&\le \sum_Y \nu_Y(y) \sup_{y',y''\in B(y,a+4b)\cap Y} |\psi(y') - \psi(y'')|^p \\
&= \|\nabla_{3a} \psi\|_p^p.
\end{aligned}$$

□

We will need the following proposition to compare Poincaré constants at different scales.

Proposition A.3.15. *(see [68, Proposition 4.3]) Let (Z, d, ν) be a 1-geodesic metric measure space. Then for any $a \geq 3$ and all $p \in [1, \infty)$ we have*

$$\frac{\nu_{\min}(1/2)}{\nu_{\max}(2a)} \cdot h_{a,p}(Z) \leq h_{\frac{3}{2},p}(Z) \leq h_{a,p}(Z),$$

where $\nu_{\min}(1/2)$ denotes the minimal measure of a ball of Z of radius $1/2$, and $\nu_{\max}(2a)$ denotes the maximal measure of a ball of Z of radius $2a$.

Proof. This is the same proof as in [68], where we detail the constants involved.

The right-hand side inequality is obvious. Let us prove the left-hand side. Let f be a measurable function $Z \to \mathbf{R}$. Let $z \in Z$, and let x, y be two distinct points of $B(z, a)$. Then there exists $x = x_0, \ldots, x_n = y$ within $B(z, a)$ such that $d(x_{i+1}, x_i) \leq 1$ for all i, and $d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$. Up to removing vertices, we can make the assumption that this sequence is minimal in the following sense:

$$\forall i, j \in [\![0, n]\!] \ (|j - i| > 1 \implies d(x_i, x_j) > 1) .$$

Note that removing vertices may make the equality $d(x, y) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$ fail, but we keep the property that every x_i is at distance at most $a/2$ from x or y. We claim that the following inequality is true:

$$\int_{z' \in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right| d\nu \geq \nu_{\min}(1/2) \cdot |f(x) - f(y)| \tag{A.8}$$

We consider two cases:

- if n is even, let us call $Z_{x,y}$ the set of $z' \in Z$ that are in the $\frac{3}{2}$-neighbourhood of both x_{2i-2} and x_{2i} for some integer i between 1 and $n/2$. Then, since $a \geq 3$, $Z_{x,y}$ is contained in the ball $B(z, 2a)$. It contains the closed balls $B(x_{2i-1}, \frac{1}{2})$, for any such i. From the minimality assumption that we have made on the path $(x_i)_{0 \leq i \leq n}$, these balls are pairwise disjoints. Then,

$$\begin{aligned}
\int_{z' \in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right| d\nu &\geq \int_{z' \in Z_{x,y}} \left|\nabla_{\frac{3}{2}} f\right| d\nu \\
&\geq \sum_{i=1}^{n/2} \int_{B(x_{2i-1}, \frac{1}{2})} \left|\nabla_{\frac{3}{2}} f\right| d\nu \\
&\geq \sum_{i=1}^{n/2} \int_{B(x_{2i-1}, \frac{1}{2})} |f(x_{2i}) - f(x_{2i-2})| d\nu \\
&\geq \nu_{\min}(1/2) \sum_{i=1}^{n/2} |f(x_{2i}) - f(x_{2i-2})| \\
&\geq \nu_{\min}(1/2) \cdot |f(x) - f(y)|
\end{aligned}$$

- if n is odd, let us call $Z'_{x,y}$ the set of $z' \in Z$ that are in the $\frac{3}{2}$-neighbourhood of both x_{2i-2} and x_{2i} for some integer i between 1 and $(n-1)/2$, or that are in the $\frac{3}{2}$-neighbourhood of both x_{n-1} and y. Then, since $a \geq 3$, $Z_{x,y}$ is contained in the balls $B(z, 2a)$. It contains the closed ball $B(x_{2i-1}, \frac{1}{2})$, for any i from 1 to $(n+1)/2$ (note that the last ball is centred at y). From the minimality assumption that we have made on the path $(x_i)_{0\leq i\leq n}$, these balls are pairwise disjoints. Then,

$$\begin{aligned}
\int_{z'\in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right| d\nu &\geq \int_{z'\in Z'_{x,y}} \left|\nabla_{\frac{3}{2}} f\right| d\nu \\
&\geq \sum_{i=1}^{(n+1)/2} \int_{B(x_{2i-1},\frac{1}{2})} \left|\nabla_{\frac{3}{2}} f\right| d\nu \\
&\geq \sum_{i=1}^{(n-1)/2} \int_{B(x_{2i-1},\frac{1}{2})} |f(x_{2i}) - f(x_{2i-2})| d\nu + \int_{B(y,\frac{1}{2})} |f(x_{n-1}) - f(y)| d\nu \\
&\geq \nu_{\min}(1/2) \left(\sum_{i=1}^{(n-1)/2} |f(x_{2i}) - f(x_{2i-2})| + |f(x_{n-1}) - f(x_n)| \right) \\
&\geq \nu_{\min}(1/2) \cdot |f(x) - f(y)|
\end{aligned}$$

Since the inequality (A.8) is true for any $x, y \in B(z, 2a)$, we deduce

$$\int_{z'\in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right| d\nu \geq \nu_{\min}(1/2) \cdot |\nabla_a f|(z).$$

Integrating over z, we get:

$$\int_{z\in Z} \left(\int_{z'\in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right|(z') d\nu(z') \right)^p d\nu(z) \geq \nu_{\min}(1/2)^p \cdot \|\nabla_a f\|_p^p.$$

Moreover for any z,

$$\left(\int_{z'\in B(z,2a)} \left|\nabla_{\frac{3}{2}} f\right|(z') d\nu(z') \right)^p \leq \nu(B(z,2a))^{p-1} \int_{z'\in B(z,2a)} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p d\nu(z').$$

Then,

$$\begin{aligned}
\nu_{\min}(1/2)^p \cdot \|\nabla_a f\|_p^p &\leq \int_{z\in Z} \nu(B(z,2a))^{p-1} \int_{z'\in B(z,2a)} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p d\nu(z') d\nu(z) \\
&\leq \nu_{\max}(2a)^{p-1} \int_{z\in Z} \int_{z'\in B(z,2a)} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p d\nu(z') d\nu(z) \\
&= \nu_{\max}(2a)^{p-1} \int_{z,z'\in Z} \mathbf{1}_{d(z,z')\leq 2a} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p d\nu(z') d\nu(z) \\
&= \nu_{\max}(2a)^{p-1} \int_{z'\in Z} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p \left(\int_{z\in Z} \mathbf{1}_{z\in B(z',2a)} d\nu(z) \right) d\nu(z') \\
&\leq \nu_{\max}(2a)^p \int_{z'\in Z} \left(\left|\nabla_{\frac{3}{2}} f\right|(z') \right)^p d\nu(z')
\end{aligned}$$

$$= \nu_{\max}(2a)^p \left\| \nabla_{\frac{3}{2}} f \right\|_p^p .$$

Finally,

$$\|\nabla_{2a} f\|_p \leq \frac{\nu_{\max}(2a)}{\nu_{\min}(1/2)} \left\| \nabla_{\frac{3}{2}} f \right\|_p . \qquad \square$$

We now can prove Theorem A.3.3.

Proof of Theorem A.3.3. We can assume without loss of generality that Γ is connected, because otherwise $h_{a,p}(\Gamma) = h_{a,p}(\Lambda) = 0$.

Let $(\tilde{\Gamma}, d, \nu)$ be the "measured" simplicial complex obtained identifying each edge of Γ to the unit interval equipped with the Lebesgue measure. We define $\iota \colon V\Gamma \to \tilde{\Gamma}$ the natural map that maps the vertices of Γ in the simplicial complex $\tilde{\Gamma}$. For simplicity, for a given a vertex v of $V\Gamma$, we will still denote v the corresponding vertex $\iota(v)$ in the simplicial complex $\tilde{\Gamma}$.

By definition, $V\Lambda$ is a maximal b-separated subset of $V\Gamma$. $\iota(V\Lambda)$ is the subset of $\tilde{\Gamma}$ corresponding to $V\Lambda$. We claim that $\iota(V\Lambda)$ is also maximal b-separated. First, $\iota(V\Lambda)$ is clearly b-separated. Second, if x be a point of $\tilde{\Gamma}$, there exists a vertex v at distance at most $1/2$. By maximality, there exists $w \in V\Lambda$ such that $d(w,v) < b$, and, since both terms are integers, we have $d(w,v) \leq b-1$. Then we have $d(x,w) \leq b - 1/2 < b$, which shows that $\iota(V\Lambda)$ is maximal. Let $\mathcal{A} = (A_v)_{v \in \iota(V\Lambda)}$ be a measurable partition of scale b satisfying that each A_v is b-centred at v. We can identify $V\Lambda$ and $\iota(V\Lambda)$, then we have two different metric measure structures on $V\Lambda$:

- The graph $\Lambda = (V\Lambda, E\Lambda)$, which is b-rescaling associated with $V\Lambda$ (Definition A.3.2), endowed with the shortest-path metric and the counting measure,
- The b-discretization[(a)] $\Lambda_b = (\iota(V\Lambda), d_{|\iota(V\Lambda)}, \nu_b)$ associated with $\mathcal{A}$, that we will call Λ_b (Definition A.3.11).

Roughly speaking, the inequality (A.9) below states that taking the appropriate scale, their L^p Cheeger constant do not differ too much. Let us write $\nu_{\min}(b)$ be the minimal measure of a ball in $\tilde{\Gamma}$ of radius b, and $\nu_{\max}(2b)$ be the maximal measure of a ball in $\tilde{\Gamma}$ of radius $2b$. We have

$$\left(\frac{\nu_{\max}(2b)}{\nu_{\min}(b)}\right)^{-1/p} \times h_{2b,p}(\Lambda_b) \leq h_{1,p}(\Lambda) \leq \left(\frac{\nu_{\max}(2b)}{\nu_{\min}(b)}\right)^{1/p} \times h_{2b,p}(\Lambda_b). \tag{A.9}$$

Let us prove this inequality. By definition (see Definitions A.3.2, A.3.11), for any v in $V\Lambda$,

$$B_{\tilde{\Gamma}}(v,b) \subset A_v \subset B_{\tilde{\Gamma}}(v,2b). \tag{$*$}$$

Therefore:

$$\nu_{\min}(b) \leq \nu_b(\{v\}) \leq \nu_{\max}(2b), \quad \text{for any } v \text{ in } V\Lambda. \tag{$\dagger$}$$

(a) We use the notation Λ_b because this space is *close* from being the same space as Λ, where the distances are multiplied by b.

We can now prove (A.9). Let $\tilde{f}$ be a function from $\iota(V\Lambda)$ to $\mathbf{R}$. Let us write f the corresponding function from $V\Lambda$ to $\mathbf{R}$ (it is roughly the same function). From the right-hand side of $(*)$, we have $|\nabla_1 f|^p(v) \leq |\nabla_{2b}\tilde{f}|^p(v)$. Then,

$$\begin{aligned}\left\|\nabla_{2b}\tilde{f}\right\|_p^p &= \sum_{v\in V\Lambda} |\nabla_{2b}\tilde{f}|^p(v)\nu_b(\{v\}) \\ &\geq \sum_{v\in V\Lambda} |\nabla_{2b}\tilde{f}|^p(v)\nu_{\min}(b) \\ &\geq \nu_{\min}(b)\sum_{v\in V\Lambda} |\nabla_1 f|^p(v) \\ &= \nu_{\min}(b)\left\|\nabla_1 f\right\|_p^p\end{aligned}$$

Moreover, from right-hand side of (†), we have $\|\tilde{f}\|_p \leq \|f\|_p \times \nu_{\max}(2b)^{1/p}$, and the right-hand side of (A.9) follows. The left-hand side of (A.9) comes very similarly, we let the proof to the reader (we will not use this inequality).

From Proposition A.3.13, we can deduce

$$h_{2b,p}(\Lambda_b) \leq 12 h_{4b,p}(\tilde{\Gamma}). \tag{A.10}$$

From Proposition A.3.15, we can deduce

$$h_{4b,p}(\tilde{\Gamma}) \leq \frac{\nu_{\max}(8b)}{\nu_{\min}(1/2)} h_{\frac{3}{2},p}(\tilde{\Gamma}). \tag{A.11}$$

We claim that we have:

$$h_{\frac{3}{2},p}(\tilde{\Gamma}) \leq D^{2/p} h_{1,p}(\Gamma). \tag{A.12}$$

Indeed, if $f\colon V\Gamma \to \mathbf{R}$, then we can find $\tilde{f}\colon \tilde{\Gamma} \to \mathbf{R}$ such that for any x, $\tilde{f}(x) = f(v)$, where v is a vertex of Γ at distance at most $1/2$ from x. Since the degree of every vertex in Γ is between 1 and D, every ball in $\tilde{\Gamma}$ of radius $1/2$, centred at vertices, have a measure between $1/2$ and $D/2$. The inequality (A.12) follows from:

- $\left\|\tilde{f}\right\|_p^p = \sum_{v\in V\Gamma} |f(v)|^p \,\nu(B(v,1/2)) \geq \frac{1}{2}\left\|f\right\|_p^p$.
- For any z in $\tilde{\Gamma}$ that is not at the middle of an edge, let us write v its closest vertex. Then $\left|\nabla_{\frac{3}{2}}\tilde{f}(z)\right| \leq |\nabla_2 f(v)| \leq \sum_{w\sim v} |\nabla_1 f(w)|$, where the last sum is taken on the set of neighbours of v. Then,

$$\begin{aligned}\left\|\nabla_{\frac{3}{2}}\tilde{f}\right\|_p^p &= \int_{z\in\tilde{\Gamma}} \left|\nabla_{\frac{3}{2}}\tilde{f}(z)\right|^p d\nu(z) \leq \sum_{v\in V\Gamma}\int_{z\in B(v,1/2)} \left|\nabla_{\frac{3}{2}}\tilde{f}(z)\right|^p d\nu(z) \\ &\leq \sum_{v\in V\Gamma}\left(\sum_{w\sim v}|\nabla_1 f(w)|\right)^p \nu(B(v,1/2)) \\ &\leq \sum_{v\in V\Gamma} D^{p-1}\left(\sum_{w\sim v}|\nabla_1 f(w)|^p\right) D/2 \\ &= \frac{D^p}{2}\left\|\nabla_1 f\right\|_p^p.\end{aligned}$$

Theorem A.3.3 then follows from the chain of inequalities from (A.9) to (A.12):

$$\begin{aligned} h_{1,p}(\Lambda) &\leq \left(\frac{\nu_{\max}(2b)}{\nu_{\min}(b)}\right)^{1/p} \times h_{2b,p}(\Lambda_b) \\ &\leq k^{1/p} \times 12h_{4b,p}(\tilde{\Gamma}) \\ &\leq 12k^{1/p} \times \frac{\nu_{\max}(8b)}{\nu_{\min}(1/2)} h_{\frac{3}{2},p}(\tilde{\Gamma}) \\ &\leq 12k^{1/p}\frac{1}{2}kbD^{2/p}h_{1,p}(\Gamma) \\ &\leq \left(6k^{\frac{p+1}{p}}D^{2/p}\right)bh_{1,p}(\Gamma). \end{aligned}$$

□

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École doctorale de mathématiques Hadamard (EDMH)

Titre : Profils de séparation et de Poincaré

Mots clés : Graphes, Isoperimetrie, Plongements, Géométrie grossière

Résumé : Ce manuscrit de thèse récapitule mes travaux de recherche sur les profils de séparation et de Poincaré. Le profil de séparation est apparu en 2012 dans un l'article fondateur de Benjamini, Schramm et Timár. La définition donnée tirait ses origines dans des travaux antérieurs, dans le domaine du calcul formel : principalement des études de Lipton et Trajan concernant les graphes planaires, et de Miller, Teng, Thurston et Vavasis concernant des graphes d'intersection. Le profil de séparation est maintenant utilisé en théorie géométrique des groupes, mon domaine de recherche, à cause de sa propriété de monotonie par plongements grossiers. Il a été généralisé par Hume, Mackay et Tessera en 2019 en une gamme continue de profils, appelés profils de Poincaré.

Title: Separation and Poincaré profiles

Keywords: Graphs, Isoperimetry, Embeddings, Coarse geometry

Abstract: The goal of this thesis report is to present my research concerning separation and Poincaré profiles. Separation profile first appeared in 2012 in a seminal article written by Benjamini, Schramm and Timár. This definition was based on preceding research, in the field of computer science, mainly work of Lipton and Trajan concerning planar graphs, and of Miller, Teng, Thurston and Vavasis concerning overlap graphs. The separation profile plays now a role in geometric group theory, where my personal interests lies, because of its property of monotonicity under coarse embeddings. It was generalized by Hume, Mackay and Tessera in 2019 to a spectrum of profiles, called the Poincaré profiles.

Université Paris-Saclay, CNRS,
Laboratoire de mathématiques d'Orsay,
91405, Orsay, France.