

Estimation par Intervalles des Systèmes Cyber-physiques

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Interval Estimation for Cyber-Physical Systems

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Interval Estimation for Cyber-Physical Systems

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By

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Résumé en français

Motivations

Les systèmes cyber-physiques (CPS) et l'internet des objets marquent le début de la quatrième révolution industrielle, après la première révolution "mécanisation" résultant de l'invention de la machine à vapeur, la deuxième "production de masse" à l'aide de l'électricité et la troisième "numérisation" par utilisation de l'électronique et des technologies de l'information (Jazdi, 2014; Khaitan and McCalley, 2014). Le domaine des systèmes cyber-physiques devrait jouer un rôle majeur dans la conception et le développement des systèmes du futur. Les systèmes cyber-physiques reposent sur une interaction étroite entre des éléments cyber, c'est-à-dire des logiciels et des réseaux de communication, et des éléments physiques.

Aujourd'hui, ces systèmes se retrouvent dans nombreuses technologies innovantes telles que les systèmes énergétiques, les systèmes de soins et de santé, la robotique industrielle, les bâtiments intelligents et le transport. Les réseaux électriques modernes constituent un bon exemple de systèmes cyber-physiques. Dans de tels systèmes, les parcs éoliens et les fermes solaires constituent les composants physiques, alors que les calculateurs utilisés pour la décision constituent les composants cyber.

Du point de vue de l'Automatique, la classe de systèmes cyber-physiques décrite dans l'exemple ci-dessus est connue aussi sous le nom de systèmes contrôlés en réseau. L'intérêt croissant pour ces systèmes est motivé par le fait qu'ils disposent d'une grande flexibilité, de coûts réduits, d'un entretien et d'une installation plus facile, ainsi que d'un poids et d'un volume plus faibles par rapport aux systèmes de contrôle classiques (Postoyan and Nešić, 2012). Toutefois, ces flexibilités ne conduisent pas directement à une efficacité améliorée et ne peuvent pas être exploitées simplement. En général, l'insertion d'un réseau de communication dans la boucle de retour induit des contraintes et de nouvelles sources de variabilité. Par exemple, on observe des délais de transmission irréguliers, des retards dans la transmission, et la nécessité de planifier l'accès au réseau. Ces problématiques sont susceptibles d'affecter sérieusement le comportement et les performances du système contrôlé en réseau.

Dans cette thèse, nous nous intéressons à l'estimation d'état des systèmes dynamiques continus (composants physiques) en utilisant des systèmes informatiques reliés à travers un réseau de communication (composants cyber) dans le contexte "erreur bornée". Dans ce contexte, on suppose que les données transmises et le système sont soumis à des perturbations et bruits inconnus bornés avec bornes connues a priori. Afin d'atteindre cet objectif, nous proposons d'utiliser des approches d'observation par intervalles qui sont bien adaptées à l'absence d'information sur les distributions de probabilité des perturbations et des bruits. Les observateurs par intervalles sont des estimateurs d'état garantis, dans le sens où l'existence d'une solution peut être vérifiée et qu'aucune solution ne peut être perdue. Fondamentalement, les observateurs par intervalles calculent des tubes de trajectoires qui contiennent de manière garantie l'ensemble des trajectoires admissibles de l'état du système, tout en tenant compte de toutes les incertitudes et perturbations agissant sur le système et les mesures.

Dans la première partie de cette thèse, nous abordons les objectifs suivants dans le contexte "erreur bornée":

- La synthèse d'observateurs d'état qui prennent en compte le ryhtme irrégulier de transmission de données (mesures). Ceci va permettre d'améliorer la robustesse des systèmes cyberphysiques par rapport aux erreurs induites par le réseau de communication.

- La synthèse simultanée de l'observateur et du mécanisme d'échantillonnage de donnée qui ne transmet les données qu'en cas de besoin. Ce mécanisme va permettre également de réduire la charge sur le réseau de communication utilisé dans les systèmes cyberphysiques.

Nombreuses applications des systèmes cyberphysiques occupent des fonctions essentielles dans les infrastructures critiques, telles que la distribution d'énergie électrique, la distribution de gaz naturel, le traitement de l'eau, les systèmes de transport, les dispositifs de soins de santé et les systèmes d'armes. La perturbation de ces systèmes de contrôle pourrait avoir un impact significatif sur la santé et la sécurité publiques qui pourraient entraîner, par conséquence, des pertes économiques importantes. Les systèmes de contrôle ont été des cas confirmés de cyber-attaques et ils sont plus vulnérables aujourd'hui qu'avant aux attaques informatiques pour de nombreuses raisons (Cárdenas et al., 2008), (Humayed et al., 2017), par exemple:

Hypothèse d'isolement - Dans les systèmes de contrôle industriel (ICS) et les réseaux électriques, à titre d'exemple, l'accent a été mis sur la conception de systèmes fiables et sûrs, alors que la sécurité n'a pas été d'une grande importance. En effet, ces systèmes étaient censés être isolés du monde extérieur et les opérations de surveillance et de contrôle étaient effectuées localement, et donc considérées comme sécurisées (Ericsson, 2010).

Connectivité accrue - Les CPS sont plus connectés que jamais. Ces systèmes ont été améliorés en ajoutant des services qui reposent sur des réseaux ouverts et des technologies sans fil. En générale, les ICS et les réseaux intelligents sont connectés à des centres de contrôle connectés à Internet. La dépendance croissante des ICS aux protocoles à normes ouvertes, tels que TCP/IP, les rend vulnérables (Humayed et al., 2017). En effet, ces protocoles ont toujours des problèmes de sécurité car ils n'étaient pas censés être sécurisés par conception (Harris and Hunt, 1999).

En raison de ces faits, la sécurité des CPS contre les cyber-attaques devient désormais une étape primordiale et importante dans la conception des CPS. La sécurité de ces systèmes peut être vue de deux côtés: perspective informatique et perspective automatique. Dans cette thèse, nous nous intéressons au point de vue automatique. Ainsi, dans la suite de cette thèse, en utilisant les observateurs par intervalles conçus, nous visons à :

- Développer un estimateur sécurisé pour les CPS vis-à-vis les cyber-attaques.

Chapitre 1: Positivité et stabilité

Dans cette thèse nous développons des observateurs par intervalles pour des systèmes contrôlés via un réseau de communication. Ces observateurs sont des estimateurs ensemblistes qui estiment un ensemble admissible de l'état du système en calculant une borne supérieure et une borne inferieure de l'ensemble qui est sous forme d'un intervalle (boîte). Ce type d'observateur doit satisfaire deux propriétés; (i) la positivité des erreurs d'estimation et (ii) la stabilité de ces erreurs. La présence du réseau de communication dans la boucle d'estimation du système continu peut faire apparaitre des comportements discrets. C'est ainsi que nous fournissons dans ce chapitre les outils théoriques pour l'analyse de la positivité et la stabilité des systèmes qui présentent à la fois un comportement continu et un comportement discret, et donc un comportement "hybride".

Chapitre 2: Observateur par intervalles pour les systèmes linéaires avec mesures apériodiques

Aujourd'hui, l'étude des systèmes à données échantillonnées apériodiquement constitue un sujet de recherche très populaire en contrôle. Ceci est une conséquence de l'énorme développement des systèmes de contrôle embarqués et en réseau, où les données des capteurs et de contrôle sont transmises sur des canaux de communication numériques. Pour réduire la charge de communication sur ces canaux à bande passante limitée, il est tentant d'échanger des données de manière apériodique. En outre, la présence de gigue d'échantillonnage, de perte de paquets et de fluctuations dans l'accessibilité du réseau souligne encore plus l'intérêt pour l'échantillonnage variant dans le temps et apériodique (Postoyan and Nešić, 2012), (Hespanha et al., 2007). Dans le contexte de l'estimation de l'état, les estimateurs pour les systèmes à temps continu avec des sorties échantillonnées apériodiquement ont été étudiés sous plusieurs cadres ; par exemple, (i) sous le cadre du système hybride (Goebel et al., 2012) où la dynamique de l'erreur d'observation est représentée comme un système impulsif (Ferrante et al., 2016), ou (ii) sous le cadre du système à retard comme un système échantillonné (Efimov et al., 2016) où la sortie du système est maintenue constante entre les mesures.

Ce chapitre traite de la modélisation et la conception d'un observateur d'état par intervalles pour des systèmes linéaires invariants dans le temps en présence de mesures disponibles de manière sporadique. De plus, nous supposons que ces systèmes sont corrompus par des perturbations et du bruit inconnus mais bornés.

Fondamentalement, la conception des observateurs par intervalles doit garantir la positivité de l'erreur d'estimation ainsi que sa stabilité. En outre, l'avantage de ces observateurs par rapport à d'autres estimateurs garantis, tels que les approches de correction des prédictions ensemblistes (Alamo et al., 2005), (Meslem et al., 2010) par exemple, est que le gain de l'observateur peut être calculé hors ligne et la convergence de l'erreur d'estimation est prouvée a priori.

Pour pouvoir reconstruire un encadrement d'état garanti de l'état réel pour des systèmes linéaires à temps continu en présence de mesures sporadiques en temps discret, nous proposons un observateur impulsif par intervalles. Le comportement impulsif est le résultat de la nature discrète des mesures. Entre deux instants de mesure, l'observateur se comporte comme un prédicteur en temps continu basé uniquement sur le modèle d'évolution. Ensuite, à l'instant de mesure, une correction impulsive ajuste l'état estimé de l'encadrement. Ici, nous considérons

que le temps entre deux mesures consécutives est inconnu mais appartient à un intervalle de temps, contrairement à Mazenc and Dinh (2014) où il est choisi constant. La limite inférieure de l'intervalle de temps est choisie de manière à éviter le phénomène de Zénon (nombre infini d'échantillons en temps fini). Les limites inférieure et supérieure du temps entre deux mesures consécutives sont choisies a priori par l'utilisateur, puis la procédure de synthèse qu'on propose calcule le gain d'observation pour assurer à la fois la positivité et la stabilité de l'erreur d'estimation. L'analyse de stabilité de l'estimation des bornes s'inspire des travaux de Ferrante et al. (2016) alors que la positivité de l'erreur d'estimation est assurée en se basant sur la représentation interne positive des systèmes dynamiques comme dans Meslem and Ramdani (2011) et dans Cacace et al. (2015). Dans la littérature, l'étude de l'observateur par intervalles continu avec des mesures discrètes a été abordée dans Goffaux et al. (2009), dans Mazenc and Dinh (2014), et dans Efimov et al. (2016). Dans Goffaux et al. (2009), un observateur continu et discret à intervalle a été appliqué à un bio-processus spécifique à base de micro algues. Dans Mazenc and Dinh (2014), une analyse d'un observateur par intervalles pour des systèmes à temps continu avec des mesures discrètes a été proposée, où la période de mesure est supposée constante. À la connaissance des auteurs, il n'existe aucun travail traitant de la synthèse du gain d'observation dans le contexte de l'observation par intervalles de systèmes continus avec sortie discrète sporadique.

La nouveauté de ce chapitre réside dans une nouvelle méthodologie pour la synthèse du gain de l'observateur en présence de mesures apériodiques, qui garantit à la fois la positivité et la stabilité de l'erreur d'estimation par intervalles. De plus, le système étudié contient à la fois des perturbations du système et du bruit de mesure, alors que dans Ferrante et al. (2016), seul le bruit de sortie a été pris en compte. Le problème de synthèse est formulé comme un ensemble d'inégalités matricielles bilinéaires (ou problème BMI). La stabilité exponentielle et la stabilité entrée-état des erreurs d'estimation sont également prouvées.

Chapitre 3: Observateurs par intervalles à échantillonnage événementiel : approche basée stabilité gain L_1 fini

Des efforts énormes ont été déployés pour améliorer les performances des systèmes de contrôle en réseau tout en garantissant une utilisation minimale du réseau (partagé). Pour atteindre cet objectif, des approches alternatives à la méthode traditionnelle d'échantillonnage périodique des données ont été proposées. On peut

citer; le schéma d'échantillonnage apériodique des données où l'échantillonnage est autorisé dans un intervalle de temps prédéterminé (Hetel et al., 2017), et la stratégie d'échantillonnage événementiel (Tabuada, 2007) (Heemels et al., 2012) (Girard, 2015) où l'instant de l'échantillonnage dépend du comportement du système.

Dans ce travail, nous proposons la modélisation et la synthèse des observateurs par intervalles basées sur un mécanisme d'échantillonnage événementiel permettant de n'utiliser la mesure qu'en cas de besoin. L'estimation basée sur l'échantillonnage événementiel peut relaxer l'hypothèse de régularité de la disponibilité des mesures couramment utilisées par les méthodes traditionnelles d'estimation d'état. Grâce à ce mécanisme, la quantité de données transmises sur le réseau peut être considérablement réduite.

Dans le contexte d'échantillonnage événementiel, des approches d'estimation asymptotique ont été proposées pour les systèmes à temps discret (Muehlebach and Trimpe, 2017; Huang et al., 2019) et les systèmes à temps continu (Huang et al., 2017; Etienne and Di Gennaro, 2016), pour n'en citer que quelques-uns. L'estimation ensembliste a également été étudiée dans (Shi et al., 2014; Silvestre et al., 2018). Shi et al. (2014) ont proposé une approche d'estimation basée sur les événements, en utilisant le filtre de Kalman ensembliste pour les systèmes à sorties multiples, où les intervalles des mesures, l'état initial et les incertitudes sont supposés être gaussiens. Silvestre et al. (2018) ont proposé des stratégies basées sur les événements en utilisant des observateurs ensemblistes (Set valued observer). Les deux travaux cités ci-dessus sont consacrés aux systèmes à temps discret. Cependant, le problème de l'estimation ensembliste pour les systèmes à temps continu dans ce cadre n'a pas été entièrement étudié dans la littérature. Certains travaux préliminaires ont considéré l'échantillonnage événementiel (Rabehi et al., 2017) ou par auto-déclenchement (Meslem and Prieur, 2014) des mesures en utilisant les approches de prédiction-correction. Il est à noter que, dans ces deux travaux, seules les propriétés structurelles intrinsèques des systèmes sont exploitées afin de concevoir des estimateurs d'état ensemblistes convergents. Dans ce travail, nous proposons une nouvelle structure d'observateurs par intervalles, où la correction est effectuée à des instants à temps discret. De plus, ici, les ensembles possibles de l'état initial du système et des perturbations sont pris comme des vecteurs d'intervalles (boîtes) au lieu d'ellipsoïdes (Shi et al., 2014) ou de polytopes (Silvestre et al., 2018).

Le principal avantage de l'approche d'observation proposée réside dans la phase de correction qui repose sur un gain d'observation pré-calculé. Ce gain garantit la stabilité de l'erreur d'estimation ainsi que certaines spécifications de performance. Ces dernières sont obtenues grâce à l'analyse de la stabilité entrées-sorties, en particulier le concept de stabilité à gain $\mathcal{L}_p$ fini (Khalil, 2002).

Le concept de gain $\mathcal{L}_p$ fini est une approche intéressante pour analyser la stabilité ainsi que les performances des systèmes dynamiques. Ce concept a déjà été appliqué pour évaluer la performance des observateurs par intervalles (Chebotarev et al., 2015); (Briat and Khammash, 2016). Par exemple, des observateurs par intervalles ont été proposés pour les systèmes linéaires à paramètre variants (LPV) avec une analyse des performances $\mathcal{L}_1/\mathcal{L}_2$ par Chebotarev et al. (2015). Des observateurs par intervalles fournissant des encadrements étroits de l'état ont été conçus en utilisant l'approche du gain crête à crête $\mathcal{L}_\infty - \mathcal{L}_\infty$ par Briat and Khammash (2016) pour les systèmes à temps discret et à temps continu.

Dans ce chapitre, nous combinons l'approche du gain $\mathcal{L}_1$ pour les systèmes positifs (Briat, 2013 ; Ebihara et al., 2011) et l'approche du gain $\mathcal{L}_p$ pour les systèmes hybrides (Nešić et al., 2013) pour étudier la stabilité et la performance des observateurs proposés. Ces observateurs par intervalles sont à échantillonnage événementiel et la dynamique des erreurs d'estimation est modélisée comme un systèmes impulsif. La positivité de l'erreur d'estimation est garantie par l'utilisation de la positivité interne du système. De plus, les mesures ne sont pas disponibles en permanence et un mécanisme d'échantillonnage événementiel est conçu pour demander une mesure uniquement lorsque cela est nécessaire pour améliorer les estimations, c'est-à-dire les mesures sont demandées chaque fois qu'une condition, impliquant la largeur du domaine réalisable des incertitudes du système et la largeur des intervalles estimées, est satisfaite. Ainsi, la nouveauté de ce chapitre est la synthèse simultanée du mécanisme d'échantillonnage événementiel et du gain d'observateur qui assurent la stabilité et la positivité de la largeur des intervalles estimées.

Chapitre 4: Observateur sécurisé par intervalles pour les systèmes LTI avec mesures discrètes sous attaques

En raison du couplage cyber-physique et des conséquences perturbatrices des défaillances, la sécurité est ici l'une des principales préoccupations (Lun et al., 2019). Le problème de la sécurité n'est pas nouveau dans le domaine des systèmes

de contrôle, notamment dans le domaine de la détection et isolation des défauts (Massoumnia et al., 1989). Des travaux récents sur la cyber-sécurité des systèmes de contrôle se sont concentrés, en partie, sur l'effet de certains types de cyber-attaques sur la stabilité et/ou l'estimation du système, telles que les fausses attaques par injection de données (Liu et al., 2011), (Degue et al., 2018), les attaques par déni de service (DoS) (Amin et al., 2009) et les attaques d'intégrité (Mo et al., 2014), ou à une catégorie plus générale de cyber-attaques (Fawzi et al., 2014), (Chong et al., 2015) et (Shoukry et al., 2018), ce qui est le cas de notre approche.

Ce chapitre propose une méthodologie de synthèse d'observateurs d'état par intervalles pour des systèmes linéaires invariants dans le temps en présence de mesures discrètes périodiques affectées par un bruit inconnu mais borné avec des bornes connues et soumis à des cyber-attaques (probablement non bornés).

Pour reconstruire une estimation par intervalles de l'état réel des systèmes linéaires en temps continu en présence de mesures en temps discret, nous proposons d'utiliser une structure d'observateur impulsif par intervalles. A cette fin, nous exploiterons, dans ce chapitre, l'observateur impulsif par intervalles développé dans le chapitre 2 dans le cas où les mesures discrètes sont périodiques (l'hypothèse sur la périodicité sur l'échantillonnage des mesures est retenue pour simplifier la présentation, mais l'approche reste valide pour le cas d'échantillonnage apériodique).

Dans ce chapitre, nous considérons les systèmes LTI continus avec s sorties, chacunne d'elles est mesurée par un capteur qui est potentiellement sous cyber-attaque. Dans ce cadre, nous supposons que seul un sous-ensemble $\mathcal{S}$ de s_a capteurs peut être attaqué au même instant, avec $s > 2s_a$. Cette condition est issue de la M-observabilité (Chong et al., 2015) et de l'observabilité *s-sparse* (Shoukry et Tabuada, 2016). Sur la base de cette hypothèse, aux instants de correction, nous fournissons autant d'intervalles estimés que de capteurs, puis nous sélectionnons l'estimation sans attaque. Cette sélection se fait par une stratégie basée sur l'analyse par intervalles et la positivité de l'erreur d'estimation qui rejete les estimés corrompues par les attaques. La stratégie proposée est un algorithme en ligne tandis que la procédure de synthèse qui calcule le gain d'observation pour assurer à la fois la positivité et la stabilité de l'erreur d'estimation se fait hors ligne. L'analyse de stabilité et la positivité de l'erreur d'estimation est assurée sans tenir compte de l'influence de l'attaque. Ensuite, l'effet des attaques est traité par une stratégie en ligne basée sur l'analyse par intervalles.

La contribution de ce chapitre a deux volets: premièrement, une nouvelle méthodologie de synthèse basée sur des inégalités matricielles linéaires (LMI) du gain de l'observateur en présence de mesures en temps discret est proposée, qui garantit à la fois la positivité et la stabilité de l'erreur d'estimation par intervalles. Cette méthodologie est différente de celle proposée au chapitre 2 basée sur des inégalités matricielles bilinéaires (BMI). Deuxièmement, une nouvelle stratégie résistante aux attaques des capteurs qui sélectionne en ligne, aux moments des mesures, la bonne estimation parmi un ensemble d'estimations.

Conclusions

Dans cette thèse, nous nous sommes concentrés sur la synthèse des observateurs d'état par intervalles pour les systèmes en temps continu qui communiquent via un réseau. Ces systèmes sont soumis à des perturbations, et leurs mesures sont également soumises à des bruits et/ou des cyber-attaques. Ce problème est considéré sous deux angles. Premièrement, lorsque le réseau présente certaines contraintes de communication. Par conséquent, les données transmises sur ce réseau ne sont pas forcement périodiques. Deuxièmement, lorsque le réseau de communication est vulnérable aux cyber-attaques.

D'adord, dans le chapitre 2, nous avons introduit une nouvelle approche pour la synthèse des observateurs impulsifs par intervalles pour les systèmes linéaires à mesures discrètes apériodiques. En exploitant la représentation de la positivité interne du système, une nouvelle méthode a été proposée pour la synthèse du gain d'observation à l'aide de la résolution des inégalités matricielles bilinéaires.
Ensuite, dans le chapitre 3, nous avons proposé une méthode de synthèse simultané pour le mécanisme d'échantillonnage de mesures et le gain d'observateur par intervalles pour les systèmes linéaires à temps continu. Le mécanisme d'échantillonnage de mesures proposé est basé sur la propriété de positivité des erreurs d'observation par intervalles. Ces observateurs garantissent également un gain L_1 fini entre la largeur des bornes de perturbation et la largeur des intervalles d'état estimés.
Finalement, dans le chapitre 4, nous avons exploité l'observateur impulsif par intervalles développé dans le chapitre 2 afin de concevoir un estimateur sécurisé vis-à-vis des cyber-attaques. En utilisant la positivité des erreurs d'estimation par intervalles, une nouvelle stratégie de sélection d'un sous-ensemble des capteurs non attaqués a été proposée sous l'hypothèse de synchronisation des mesures. La synthèse des gains d'observations est formulée sous forme d'inégalitées matricielles linéaires.

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Abbreviations

CPS(s)	**C**yber-**P**hysical **S**ystem(s).
LTI	**L**inear **T**ime-**I**nvariant.
NCS(s)	**N**etworked **C**ontrol **S**ystem(s).
UBBE	**U**known-**B**ut-**B**ounded **E**rrors.
LP	**L**inear **P**rogramming.
BMI	**B**ilinear **M**atrix **I**nequality.
LMI	**L**inear **M**atrix **I**nequality.
IPR	**I**nternally **P**ositive **R**ealization.
GES	**G**lobal **E**xponential **S**tability.
ISS	**I**nput-to-**S**tate **S**tability.
DTS	**D**iscrete-**T**ime **S**ystem.
CS	**C**onvex **S**olver.
SDP	**S**emi-**D**efinite **P**rogramming.
ETM	**E**vent-**T**riggered **M**echanism.
MIET	**M**inimum **I**nter-**E**vent **T**ime.
FDI	**F**ault **D**etection and **I**dentification.
CQLF	**C**ommon **Q**uadratic **L**yapunov **F**unction.
LCLF	**L**inear **C**opositive **L**yapunov **F**unction.

Notations

$\mathbb{N}$	The set of nonnegative integers including zero.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{\geq 0}$	The set of positive real numbers $(\mathbb{R}_{\geq 0} = \{\tau \in \mathbb{R} : \tau \geq 0\})$.
$\mathbb{R}^n$	The n-dimensional Euclidean space.
$\lvert x\rvert_p$	Denotes the p-norm of the vector $x \in \mathbb{R}^n$, and simply by $\lvert x\rvert$ for the Euclidean norm .
$\lvert x\rvert_{\mathcal{A}}$	Denotes the distance of $x \in \mathbb{R}^n$ to the closed set $\mathcal{A} \subset \mathbb{R}^n$ and is defined by $\lvert x\rvert_{\mathcal{A}} := \inf_{y\in\mathcal{A}} \lvert x - y\rvert$.
$\lvert\lvert A\rvert\rvert$	For matrices, it stands for the induced matrix norm for $A \in \mathbb{R}^{n\times n}$.
$\mathbb{B}$	denotes the closed unit ball.
$\overline{1,k}$	Denotes the sequence of integers $1, \cdots, k$.
I_n	Denotes the identity matrix in $\mathbb{R}^{n\times n}$.
$\mathbb{1}_{p,m}$,	Denotes the $p \times m$ matrix whose elements are all one.
$\otimes$	Refers to the Kronecker product.
$M > (\geq) 0$	The matrix with all entries are positive (nonnegative) in $\mathbb{R}$.
$M \succ 0$	Stands for the positive-definite matrix $M \in \mathbb{R}^{n\times n}$ that satisfies $\upsilon^T M \upsilon > 0$ for all non-zero real vectors $\upsilon \in \mathbb{R}^n$.
$A^+ = \max\{A, 0\}$	The matrix with all positive entries of the matrix A.
$A^- = A^+ - A$	The matrix with all negative entries in absolute value of the matrix A.
$\lvert A\rvert = A^+ + A^-$	The element-wise absolute value of the matrix A.
$card\{\mathcal{I}\}$	The cardinality of the set of integers $\mathcal{I}$.
$\mathbf{int}(x_1, x_2)$	The interval set of admissible values bounded by the vectors x_1 and x_2 in $\mathbb{R}^n$.

Introduction

Motivations

Cyber-Physical Systems (CPSs) and the Internet of Things mark the dawn of the fourth Industrial Revolution following the first one "Mechanization" as a result of the invention of the steam engine, the second "Mass production" with the help of electricity, and the third "Digitization" by the use of Electronics and Information Technology (Jazdi, 2014) (Khaitan and McCalley, 2014). The concept of CPSs is expected to play a major role in the design and development of future systems. CPSs are systems that link the physical world through sensors or actuators with the virtual world of information processing. They are composed from diverse constituent parts that collaborate together to create some global behaviour. These components will include software systems, communications network technology, and sensors/actuators that interact with the real world.

Nowadays, the applications of CPSs are found in several innovated technologies such that energy systems, health care systems, industrial robotics, smart building, and transportation. An example of CPSs is seen in modern power grid. In such a system, wind farm and solar farm constitute the physical components. In the other hand, computers and communication networks represent the cyber components. The computations are carried out with the objective of taking a suitable decision based on which the physical components are further controlled and monitored, and the communication channels are involved to transmit data that are used to control and monitor the physical components.

From control theory perspective, the class of CPS described in the example above is also known as Networked control systems (NCSs). The particularity of NCSs is that they focus on the effect of the communication on the global behavior

(performance and stability) of the system. The growing interest for these systems is motivated by the fact that they show greater flexibility, lower costs, easier maintenance and installation, as well as lower weight and volume in comparison to the classical control systems (Postoyan and Nešić, 2012). However, the new flexibility does not directly lead to improved efficiency and cannot simply be exploited. The insertion of a network in the feedback loop induces *communication constraints* e.g., irregular transmission intervals, delays, and scheduling, which may seriously affect the whole system characteristics.

In this thesis, we are interested in the state estimation for continuous dynamical systems (physical components) using computers over network (cyber components) in the context of unknown-but-bounded errors (UBBE) where the transmitted data and the system are affected by unknown disturbances with a priori known bounds. In order to achieve this objective, we propose to use *inteval observer approaches* which address well the lack of knowledge on disturbances and noises. Interval observers are guaranteed state estimators in the sense that the existence of a solution can be verified and no solution can be lost. Basically, interval observers compute trajectory tubes that are proven to contain the plant state trajectory while taking into account all uncertainties and disturbances acting on the plant and the measurements.
In the first part of this thesis, we address the following objectives in the context of UBBE:

- The design of state observers that take into account the irregular transmission rate of data. This allows also to enhance the robustness of CPSs against the non-regularity sampling of the available data.

- The co-design of observers and event-triggered mechanisms that transmit data (measurements) only when needed. This allows reducing the load on the communication channel used in CPSs.

Several CPS applications perform vital functions in critical infrastructures, such as electric power distribution, natural gas distribution, water treatment, transportation systems, health-care devices, and weapon systems. The disruption of these control systems could have a significant impact on public health and safety,

and may lead to large economic losses. Control systems have been confirmed cases of *cyber-attacks* and they are more vulnerable now than before to computer vulnerabilities for many reasons (Cárdenas et al., 2008), (Humayed et al., 2017), for instance:

Isolation assumption – In industrial control systems (ICS) and power grids, as examples, the focus has been on designing reliable and safe systems, whereas the security has not been of a great importance. This is because the systems were supposed to be isolated from the outside world and the monitoring and control operations were performed locally, and therefore, considered secure (Ericsson, 2010).

Increased connectivity – CPSs are more connected than ever before. Manufacturers have improved CPS by adding services that rely on open networks and wireless technologies. For example, ICS and smart grids are connected to control centers which are connected to the Internet or some business-related networks. The increasing reliance of ICS on open standards protocols, such as TCP/IP, makes them vulnerable (Humayed et al., 2017). This is because these protocols still has security issues as it was not intended to be secure by design (Harris and Hunt, 1999).

Due to these facts, the security of CPSs against cyber-attacks now becomes a fundamental and important step in designing CPSs. The security of CPSs can be seen from two sides: Computer science perspective and control perspective. In this thesis we are interested in the control theory one. Thus in the second part of this thesis, using the designed interval observers, we aim to:

- Develop a secure estimator for CPS against cyber-attacks.

Main contributions

Apart from Chapter 1 where we give some control theory background on positivity and stability for interval observers, the contributions of this thesis are organized in three chapters. In Chapter 2, we propose a solution on the design of interval observer for LTI continuous-time systems with discrete aperiodic measurements, in which we take into account system and measurement disturbances. In Chapter 3, we propose a co-design of interval observers and event-triggered measurements mechanisms for the LTI continuous-time systems. In Chapter 4, we propose an

algorithm for secure state estimation for multiple-output LTI continuous-time systems subject to cyber-attacks using interval observers.
The main results of this thesis are summarized below.

Chapter 2 – addresses the modeling and the design of interval state observers for linear time-invariant systems in presence of sporadically available measurements corrupted by unknown-but-bounded errors and noise. The proposed interval observer is modelled as an impulsive system where an impulsive correction is made whenever a measurement is available. The nonnegativity of the observation error between two successive measurements is preserved by applying the internal positivity based on Müller's existence theorem, while at measurement times a linear programming constraint is added. A new methodology for designing the discrete-time observer gain is proposed that guarantees both nonnegativity and stability of the estimation error. The synthesis is performed by solving a set of Bilinear Matrix Inequalities (BMIs).

Chapter 3 – introduces a new approach based on an event-triggered mechanism to design finite-gain $\mathcal{L}_1$ interval observers for linear continuous-time systems in the presence of unknown-but-bounded uncertainties with *a priori* known bounds on state disturbances and measurement noises. In this setting, measurements are event-based sampled in order to reduce online communication between the sensors and the estimation algorithm. The proposed event-triggered mechanism relies on a dynamic condition that depends on the width of the feasible domain of the system's uncertainties and the width of the estimated state enclosures. Moreover, Further conditions are provided to guarantee the existence of a positive lower bound on the inter-event times, which avoids the Zeno phenomenon. Although the sensors data are provided in an irregular sampling way, the $\mathcal{L}_1$-stability performance of the estimation error is satisfied.

Chapter 4 – addresses the design of a secure interval state estimator for linear continuous-time multi-output systems in the bounded error context with discrete-time measurements subject to cyber-attacks. The attacker capabilities are assumed limited in the sense that only a subset of all the sensors can be attacked although this subset is unknown. For a given upper bound on the number of

attacked sensors, we propose a new selection strategy, which is able to achieve resiliency to cyber-attacks, using the width of estimated intervals. The interval observer is modelled as an impulsive system, where impulsive corrections are made periodically using measurement.
To illustrate the effectiveness of the proposed approaches, the presented theoretical results in Chapters 2, 3 and 4 are supported by numerical simulations.

Publications

The work presented in this thesis led to several papers either submitted or accepted.
Journal papers

- **D. Rabehi**, N. Meslem and N. Ramdani. Finite-gain $\mathcal{L}_1$ Event-triggered Interval Observers design for Continuous-time Linear Systems. (*submitted to International Journal of Robust and Nonlinear Control– 2^{nd} round of review* (Rabehi et al., 2019a))

- **D. Rabehi**, N. Meslem, A. El Amraoui and N. Ramdani. Interval Impulsive Observer for Linear Systems with Aperiodic Discrete Measurements. (*submitted to IEEE Transaction on Automatic Control– 3^{rd} round of review* (Rabehi et al., 2018))

International conference

- **D. Rabehi**, N. Meslem, N. Ramdani. Secure interval observer for linear continuous-time systems with discrete measurements subject to attacks. The 4th International Conference on Control and Fault-Tolerant Systems (SysTol'19), September $18 - 20^{th}$, 2019. (*published* (Rabehi et al., 2019c))

The following papers either published or submitted are also developed throughout the thesis working period, but they are not included.

- **D. Rabehi**, N. Meslem, A. El Amraoui and N. Ramdani. Event-Based Prediction-Correction State Estimator. In Proceedings of the 20th IFAC World Congress (IFAC'17), 4027–4032, 2019.(*published* (Rabehi et al., 2017))

- **D. Rabehi**, N. Meslem, N. Ramdani. An LMI approach to design interval observers for discrete-time linear switched systems. (*submitted to Nonlinear Analysis: Hybrid Systems* (Rabehi et al., 2019b))

In this thesis, we have included only the works based on interval impulsive observation approaches. Another work has been pursued during the thesis time as a continuation of the master's research project and published in the following paper.

- H. Ethabet, **D. Rabehi**, D. Efimov, T. Raïssi. Interval estimation for continuous-time switched linear systems. Automatica, 90 : 230–238. (Ethabet et al., 2018)

Chapter 1

Preliminaries

1.1 Interval observers

Interval observers are guaranteed state estimators in the sense that the existence of a solution can be verified and no solution can be lost. Basically, interval observers compute trajectory tubes that are proven to contain the plant state trajectory while taking into account all uncertainties and disturbances acting on the plant and the measurements. The design of interval observers must ensure by construction the positivity of the estimation error and its stability as well. These observers have been introduced in Gouzé et al. (2000) for continuous-time systems and extended to several classes of systems under the bounded-error framework (see the survey (Efimov and Raïssi, 2016)).
In the following section, we present the basics of positive systems and the design of interval observers.

1.1.1 Positivity

In general, the motivation behind studying positive systems lies in the behavioral nature of those systems. For instance, if the fluid levels in networks of reservoirs or the molecular concentrations in chemical reactors are chosen as state variables, then we will deal with positive state behaviour. On the other side, this thesis focuses on the positive dynamics of the estimation error regardless to the system's nature.
In this section, the definitions of monotonicity, internal positivity and external

positivity of systems will be given. Some of these definitions rely on the positive notion of functions, vectors and matrices.
We will say that a matrix A is greater than or equal to a matrix B (having the same number of rows and columns) and denote this by $A \geq B$, if and only if all the elements $A_{i,j}$ of A are greater than or equal to the corresponding elements $B_{i,j}$ of B. A matrix A is nonnegative, denoted by $A \geq 0$, if $A_{i,j} \geq 0$ for all (i, j).

1.1.1.1 Monotonicity

Monotonicity is an important property in analyzing positivity of dynamical systems. Monotone systems are differential equations whose solutions preserve some order relations with respect to initial conditions. In the next definition, the main property of monotone systems is introduced.
Consider an autonomous system described by

$$\dot{x} = f(x) \tag{1.1}$$

where $x \in \mathbb{R}^n$ and f is continuously differentiable on an open subset $D \subset \mathbb{R}^n$.

Definition 1.1 (Smith (2008))**.** Let $\phi(x(t_0), t)$ denote the solution of (1.1) that starts at point $x(t_0)$. The system (1.1) is said to be

- monotone if only if

$$\forall x_1, x_2 \in D : x_1(t_0) \leq x_2(t_0) \implies \phi(x_1(t_0), t) \leq \phi(x_2(t_0), t), \ \forall t \geq t_0,$$

- strictly monotone if only if

$$\forall x_1, x_2 \in D : x_1(t_0) < x_2(t_0) \implies \phi(x_1(t_0), t) < \phi(x_2(t_0), t), \ \forall t \geq t_0.$$

The theory of interval observers for continuous-time systems is based on a subclass of monotone systems, called cooperative systems.
Let us re-write the system (1.1) in the following form

$$\dot{x}_i = f_i(x_1, \cdots, x_n), \quad i \in \{1, \ldots, n\} \tag{1.2}$$

Definition 1.2 (Cooperativity)**.** The system (1.2) is said to be cooperative if the partial derivatives $\frac{\partial f_i(x)}{\partial x_j}$ have constant signs and satisfy

$$\forall x \quad \frac{\partial f_i(x)}{\partial x_j} \geq 0, \quad \forall i \neq j, \quad i, j \in \{1, \cdots, n\}.$$

1.1.1.2 External positivity

Consider a continuous-time linear system modeled by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.3}$$

Definition 1.3. The linear system (1.3) is said to be externally positive if and only if its forced output (i.e., the output corresponding to a zero initial state) is nonnegative for every nonnegative input function.

Theorem 1.4 (Farina and Rinaldi (2000))**.** *A linear system is called externally positive if and only if its impulse response is nonnegative.*

Proof. Consider a continuous-time linear system with zero initial state. The output is the convolution integral of the input and the impulse response, namely,

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

with

$$h(t) = Ce^{At}B, t \geq 0$$

Therefore, if the impulse response $h(t)$ is nonnegative, the output $y(t)$ is nonnegative for every nonnegative input $u(t)$, so that the system is externally positive. □

The step response of an externally positive system (starting from a zero state) is non-decreasing since it is the integral of the impulse response that is nonnegative. Therefore, when a constant input is applied to the system, its output tends toward an equilibrium without overshooting it. This is true also when there are oscillations in the impulse response of the system.

1.1.1.3 Internal positivity

Here, we mean by *internal* the internal state variables of the system. This term is used only to distinguish it from the external positivity. The internal positivity is also named, in the literature, as the positivity.

Definition 1.5. The continuous-time linear system (1.3) is said to be positive if and only if for every nonnegative initial state and for every nonnegative input its state and output are nonnegative.

This definition says that all trajectories starting from any point in the positive orthant $\mathbb{R}^n_{\geq 0}$ (boundary included) of the state space $\mathbb{R}^n$ obtained by applying a nonnegative input to system (1.3) remain in the positive orthant and yield a nonnegative output. We show, in the next theorem, how it is formally possible to determine if a given linear system is positive.

Theorem 1.6 (Condition of positivity (Mitkowski, 2008))**.** *The continuous-time linear system* (1.3) *is positive if and only if : (i) the matrix A is a Metzler matrix, that is, its non-diagonal elements are nonnegative $a_{i,j} \geq 0, \forall (i,j), i \neq j$, (ii) the matrices B and C are nonnegative, and (iii) has nonnegative input signal $u(t)$.*

Proof. The state solution is given as follows

$$x(t) = e^{At}x(0) + \int_0^t h(t-\tau)u(\tau)d\tau$$

where $h(t)$ is the impulsive response. It is intuitive that a sufficient condition on the positivity of the state $x(t)$ is that all of the initial condition $x(0)$, the input signal $u(t)$, the impulsive response, and the matrix e^{At} are positive. For positive matrices B and C, the impulsive response $h(t)$ is positive if the matrix e^{At} is.
By letting the matrix A be Metzler, there exists a real positive constant $\lambda \geq 0$ such that $A + \lambda I_n \geq 0$.
We have :

$$\begin{aligned} e^{At} &= e^{(A+\lambda I_n - \lambda I_n)t} \\ &= e^{(A+\lambda I_n)t} e^{-\lambda I_n t} \end{aligned} \tag{1.4}$$

From equation (1.4), we can deduce that $e^{At} \geq 0$, for all $t \geq 0$, since $e^{(A+\lambda I_n)t} \geq 0$ and $e^{-\lambda I_n t} \geq 0$, for all $t \geq 0$. □

We have presented the positivity condition of continuous-time linear systems which are governed by differential equations. These condition are not correct for discrete-time linear systems which have difference equation forms. Next we will show the equivalent positivity condition for discrete-time linear systems.
Consider the discrete-time linear system modeled by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.5}$$

Definition 1.7. The discrete-time linear system (1.5) is positive if and only if: (i) the state matrix A is nonnegative, that is, all elements are nonnegative $a_{i,j} \geq 0, \forall(i,j)$, (ii) the matrices B and C are nonnegative, and (iii) has nonnegative input signal $u(t)$.

At this stage, it is straightforward to know whether the system is positive or not. In the following sections, we will show how to use positivity property to design interval observers for LTI systems.

1.1.2 Interval observers for cooperative LTI systems

Historically, the concept of interval observers has been introduced by Rapaport and Gouzé (1999) and Gouzé et al. (2000) to deal with the guaranteed estimation problem of the non-measurable state variables of a class of uncertain biological systems, in the sense that all admissible values of the state are estimated in a set-membership manner. To illustrate the basics of interval observers, we recall some existing results in the literature.
Consider the linear system with output injection given as follows

$$\begin{aligned} \dot{x} &= Ax + \phi(y) \\ y &= Cx \end{aligned} \tag{1.6}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ are the state variable and system output, respectively. with the function ϕ is defined as $\phi : \mathbb{R}^p \to \mathbb{R}^n$.

The study of interval observers is based on the following assumptions:

Assumption 1.1. *The pair (A, C) is detectable.*

Assumption 1.2. *There exists a matrix $L \in \mathbb{R}^{n\times p}$ such that the matrix $(A - LC)$ is Metzler.*

Assumption 1.3. *There exist two functions $\underline{\phi} : \mathbb{R}^p \to \mathbb{R}^n$ and $\overline{\phi} : \mathbb{R}^p \to \mathbb{R}^n$ such that*

$$\underline{\phi}(y) \leq \phi(y) \leq \overline{\phi}(y), \quad \forall y \in \mathbb{R}^p. \tag{1.7}$$

The structure of interval observer The estimation of the upper bound $\overline{x} \in \mathbb{R}^n$ and the lower bound $\underline{x} \in \mathbb{R}^n$ of the state x of system (1.6) is realized by using an interval observer formed by a couple of dynamical systems, similar to the structure of Luenberger observer, as the following

$$\begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + \underline{\phi}(y(t)) + L(y(t) - C\underline{x}(t)), \\ \dot{\overline{x}}(t) = A\overline{x}(t) + \overline{\phi}(y(t)) + L(y(t) - C\overline{x}(t)), \\ \underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0). \end{cases} \tag{1.8}$$

Enclosure[1] **(existence condition)**
Defining the upper estimation error and the lower estimation error by $\overline{e}(t) = \overline{x}(t) - x(t)$ and $\underline{e}(t) = x(t) - \underline{x}(t)$, respectively. Thus, the bounds $\overline{x}(t)$ and $\underline{x}(t)$ frame in a guaranteed way the actual state trajectory $x(t)$ of system (1.6), i.e., $x(t) \in [\underline{x}(t),\ \overline{x}(t)]$, if for all $t \geq t_0 : \overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$.
To show the positivity properties of the estimation errors, let us consider the following dynamics

$$\begin{cases} \dot{\underline{e}}(t) = (A - LC)\underline{e}(t) + \phi(y(t)) - \underline{\phi}(y(t)), \\ \dot{\overline{e}}(t) = (A - LC)\overline{e}(t) + \overline{\phi}(y(t)) - \phi(y(t)), \end{cases}$$

By applying Theorem 1.6 on these equations, under Assumptions 1.2-1.3 and the initial conditions $\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0)$, the properties $\overline{e}(t) \geq 0$ and $\underline{e}(t) \geq 0$ are ensured. Hence, these conditions guarantee that the estimates $\underline{x}(t)$ and $\overline{x}(t)$ enclose the actuel state $x(t)$ of the system.

[1] It is also called Framer in the literature

Remark 1.8. To guarantee that system (1.8) is an interval observer we need to additionally satisfy the stability condition of the estimation error along with its positivity condition.

In the literature, several stability conditions are provided e.g., Hurwitz condition and the Lyapunov stability (Efimov and Raïssi, 2016). Moreover, other methods provide some performance level along with stability e.g., $\mathcal{L}_2$ in Chebotarev et al. (2015) and $\mathcal{L}_1/\mathcal{L}_\infty$ in Briat and Khammash (2016). The stability conditions that fit within the context of the study, in this thesis, will be provided later.
In this approach, we highlight the fact that it is necessary to determine the gain L such that the matrix $A - LC$ is simultaneously stable and Metzler. In general, this condition is not always satisfied, which limits the application of this design approach and might introduce some conservatism. Let us pick as an example the following case:

$$A = \begin{bmatrix} 1 & -0.5 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

we thus have

$$A - LC = \begin{bmatrix} 1 + l_1 & -0.5 \\ l_2 & -2 \end{bmatrix}.$$

It is clear that there does not exist a gain L that guarantees the Metzler property of the matrix $A - LC$. In the following subsection we will show how this problem has been managed for a class of systems in the literature.

1.1.3 Interval observers for non-cooperative LTI systems

The existence of a gain L that guarantees both stability and Metzler property of the matrix $A - LC$ is a strong condition in the design of interval observers. However, some works in the literature have relaxed this condition by using:

(a) Time-invariant change of coordinates (Raïssi et al., 2012),

(b) Time-varying change of coordinates (Mazenc and Bernard, 2011),

(c) Decomposition of the system into two coupled positive systems (Meslem and Ramdani, 2011), (Cacace et al., 2015).

Now, we will explicitly show the development of the first approach (a). Next, we will only discuss the two remainder approaches (b) and (c). Our designed interval observers will be mainly based on the last approach (c).
For the approach (a), we consider the following system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + w(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.9}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the state variable and the output of the system, respectively. $w(t) \in \mathbb{R}^n$ represents the system disturbance such that there exist two known vectors $\underline{w}(t)$ and $\overline{w}(t)$ that satisfy $w(t) \in [\underline{w}(t),\ \overline{w}(t)]$. The state and output matrices are defined by $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, respectively.

1.1.3.1 Time-invariant change of coordinates

This approach is based on a non singular change of coordinates $T \in \mathbb{R}^{n\times n}$ such that $z(t) = Tx(t)$. Thus, the system (1.9) can be written as follows

$$\begin{aligned} \dot{z}(t) &= TAT^{-1}z(t) + W(t) \\ y(t) &= CT^{-1}z(t) \end{aligned} \tag{1.10}$$

where $W(t) = Tw(t)$. So, the procedure consists in estimating an upper bound $\overline{z}(t)$ and a lower bound $\underline{z}(t)$ of the state $z(t)$ in the new basis. To do so, the following structure for interval observers is proposed:

$$\begin{cases} \dot{\underline{z}}(t) = TAT^{-1}\underline{z}(t) + TL[y(t) - CT^{-1}\underline{z}(t)] + \underline{W}(t), \\ \dot{\overline{z}}(t) = TAT^{-1}\overline{z}(t) + TL[y(t) - CT^{-1}\overline{z}(t)] + \overline{W}(t), \\ \underline{z}(t_0) \leq z(t_0) \leq \overline{z}(t_0), \end{cases}$$

where $\overline{W}(t) = T^+\overline{w}(t) - T^-\underline{w}(t)$ and $\underline{W}(t) = T^+\underline{w}(t) - T^-\overline{w}(t)$.
The estimation error dynamics are thus given as follows

$$\begin{cases} \dot{\underline{e}}_z(t) = T(A - LC)T^{-1}\underline{e}_z(t) + W(t) - \underline{W}(t), \\ \dot{\overline{e}}_z(t) = T(A - LC)T^{-1}\overline{e}_z(t) + \overline{W}(t) - W(t), \end{cases}$$

By definition, we have

$$\begin{aligned} W(t) = Tw(t) &= T^{+}w(t) - T^{-}w(t) \\ &\leq T^{+}\overline{w}(t) - T^{-}\underline{w}(t) = \overline{W}(t). \end{aligned}$$

By the same argument, the inclusion $\underline{W}(t) \leq W(t) \leq \overline{W}(t)$ is always satisfied (for more details about this property the reader can see (Efimov et al., 2012, Lemma 2)).

Under the condition that the matrix $T(A - LC)T^{-1}$ is stable and Metzler, and the initial condition satisfies $\underline{z}(t_0) \leq z(t_0) \leq \overline{z}(t_0)$, and according to Theorem 1.6, the estimation errors verify $\underline{e}_z(t) \geq 0$ and $\overline{e}_z(t) \geq 0$ for all $t \geq t_0$. Therefore, the state $z(t)$ belongs to the interval $[\underline{z}(t),\ \overline{z}(t)]$ for all $t \geq t_0$. Finally, the application of the property in (Efimov et al., 2012, Lemma 2)) allows to deduce the interval that encloses the state in the origin basis $x(t)$ as follows

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

where $\underline{x}(t) = M^{+}\underline{z}(t) - M^{-}\overline{z}(t)$ and $\overline{x}(t) = M^{+}\overline{z}(t) - M^{-}\underline{z}(t)$ with $M = T^{-1}$.

Contrary to the case of cooperative systems, the procedure relies on the appropriate choice of the change of coordinates T and the observer gain L such that the estimation errors are stable and cooperative.

Solution by pole assignment and diagonalization

In this paragraph, we present a simple method using pole placement technique which allows to determine the matrices L and T.

Assuming that the pair (A, C) is observable. There exists a gain L such that the matrix $A - LC$ is diagonalizable and Hurwitz with real poles, thus there exists a change of coordinates matrix T that diagonalizes the matrix $A - LC$ in the new basis. By definition, a diagonal matrix satisfies the Metzler property which allows to claim that $T(A - LC)T^{-1}$ is Hurwitz and Metzler.

The assumption of the observability of the pair (A, C) could be relaxed by considering only its detectability if the matrix $A - LC$ conserves the property of having real poles. Even though this approach is very simple, it is still conservative because the case of conjugate complex poles is not considered. It has been proved that no time-invariant changes of coordinates can transform the system (1.9) into

a Jordan one when the matrix $A - LC$ has complex eigenvalues (Mazenc and Bernard, 2010). Nevertheless, in the following approach, an alternative formulation proposed by Raïssi et al. (2012) that makes possible a time-invariant change of coordinates even for the case of complex eigenvalues.

Solution by solving a Sylvester equation

The work presented in Raïssi et al. (2012) proposes a less conservative solution comparing to the pole assignment and diagonalization presented early. Here, we will recall the primary steps of this approach.

The matrix T being non singular, the equation $R = T(A - LC)T^{-1}$ is equivalent to the form

$$TA - RT = QC, \quad Q = TL. \tag{1.11}$$

Choosing the matrix R Hurwitz and Metzler, the equation (1.11) is a Sylvester equation where the unknown variable is the matrix T. If the matrices A and R do not share any eigenvalues, then the solution to (1.11) is unique for any Q. The solution of Sylvester equation (1.11) can be obtained by using, e.g., the work of Golub et al. (1979).

The following lemma provides a simple procedure to compute the matrices L and T

Lemma 1.9. *(Raïssi et al., 2012, Lemma 1) Consider that the matrix $A - LC$ and the Metzler matrix R have the same eigenvalues for some gain L. If there exist two row vectors v_1 and v_2 such that the pairs $(A - LC,\ v_1)$ and $(R,\ v_2)$ are observable, then:*

$$T = O_2^{-1} O_1 \quad \text{and} \quad \mathrm{Q} = \mathrm{TL}$$

satisfy the Sylvester equation (1.11) with

$$O_1 = \begin{bmatrix} v_1 \\ \vdots \\ v_1(A - LC)^{n-1} \end{bmatrix}, \quad O_2 = \begin{bmatrix} v_2 \\ \vdots \\ v_2 R^{n-1} \end{bmatrix}. \tag{1.12}$$

Proof. Given that the pairs $(A - LC,\ v_1)$ and $(R,\ v_2)$ are observable, thus, the matrices O_1 and O_2 are non singular. In addition, the transformations O_1 and O_2 transform the matrices $A - LC$ and R into their observable canonical forms.

Therefore, one gets

$$O_1(A - LC)O_1^{-1} = O_2 R O_2^{-1}$$

By replacing the matrix R by its expression $T(A-LC)T^{-1}$, and by identification, one can obtain the equation $T = O_2^{-1}O_1$. □

1.1.3.2 Time-varying change of coordinates

This approach is similar to the previous one in using the change of coordinates. The difference is that when the observer closed loop dynamics $A-LC$ has complex eigenvalues. Mazenc and Bernard (2010) proposed to use time-varying change of coordinates which is based on rotation matrix blocks. This change of coordinates is able to compensate the negativity caused by the complex conjugate eigenvalues.

We have presented a review on the use of similarity transformations for continuous-time systems. For the case of discrete-time systems the reader can see (Efimov et al., 2013) and (Mazenc et al., 2014).

1.1.3.3 Decomposition of the system into coupled positive systems

This approach is totally different from the approaches (a) and (b). It does not use any change of coordinates even if the dynamics in the origin basis is not cooperative. The idea of this approach is the decompose the dynamics into two coupled positive and negative systems. Thus, by mild condition one can get a positive dynamics for observation error. For continuous-time nonlinear systems, a structure of interval observer is proposed by Meslem and Ramdani (2011), which is mainly based on the Müller's existence theorem (Müller, 1927) for the decomposition of the non-cooperative system. Similar idea has been proposed by Coogan and Arcak (2015) to compute a reachable set for discrete-time systems using mixed-monotonicity (Smith, 2006). Basically, the dynamics which are not monotone may nonetheless be decomposable into increasing and decreasing components. Such systems are called mixed monotone. Let us illustrate these ideas on linear systems.

For the continuous-time system $\dot{x} = Ax$ such that the matrix A is not Metzler (non-cooperative system), it is possible to decompose it into a Metzler matrix A^M and a nonnegative matrix A^N such that $A = A^M - A^N$. The matrices A^M and A^N

are any realization of A satisfying the previous conditions. However, the Müller's existence theorem based method proposed to decompose A as $A^M = d_A + (A - d_A)^+$ and $A^N = A^M - A$ with d_A is a diagonal matrix contains only the diagonal elements of A.
For the discrete-time system $x^+ = Ax$ such that the matrix A is not nonnegative, it is possible to decompose it into two nonnegative matrices A_1 and A_2 such that $A = A_1 - A_2$. The matrices A_1 and A_2 are any realization satisfying the previous conditions on A. In a particular case, the matrix A can be decomposed into its positive elements A^+ and negative elements $-A^-$ such that $A = A^+ - A^-$. This method is also known, for the case of linear systems, as Internally positive representation (Cacace et al., 2015).
In the design methodologies of interval observers that we propose in the future chapters, we will use the decomposition approach in the context of hybrid systems.

In the following section, we will present some theoretical background on stability and on the hybrid system framework that will be used throughout this thesis.

1.2 Hybrid systems stability

In this section, we will recall the classical stability properties for continuous-time and discrete-time systems, then, we will give the stability conditions for hybrid systems.

1.2.1 Classical stability

Intuitively, stability is the property of a system to return to its equilibrium position when it is removed promptly.
Consider the general case of a nonlinear time-invariant system

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.13}$$

with the assumption that the origin ($x^* = 0 \in \Omega \subset \mathbb{R}^n,\ u^* = 0 \in U \subset \mathbb{R}^m$) is an equilibrium point.

1.2.1.1 Input-to-State Stability

Before introducing some stability conditions, let us present some useful definitions about characterization comparison functions.

Definition 1.10 (Class $\mathcal{K}$ function)**.** A scalar continuous function $\alpha(x)$, defined for $x \in [0, a)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Definition 1.11 (Class $\mathcal{K}_\infty$ function)**.** A scalar continuous function $\alpha(x)$ is said to belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and defined for all $x \geq 0$ and $\lim_{x\to\infty} \alpha(x) \to \infty$.

Definition 1.12 (Class $\mathcal{KL}$ function)**.** A function $\beta(s, t)$, defined for $s \in [0, a)$ and $t \in [0, \infty)$, is said to belong to class $\mathcal{KL}$ if, (i) for each fixed $t \geq 0$, the mapping $\beta(s, t)$ belongs to class $\mathcal{K}$ with respect to s, (ii) for each fixed s, the mapping $\beta(s, t)$ is decreasing with respect to t and $\lim_{t\to\infty} \beta(s, t) \to 0$.

The objective of the characterization by comparison function is to express the fact that states remain bounded for bounded controls, with an ultimate bound which is a function of the input's magnitude, and in particular that states decay when inputs do.

Theorem 1.13 (Input-to-state stability (ISS), (Sontag, 2008))**.** *The system* (1.13) *is said to be ISS if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *such that, for all* $x_0 \in \mathbb{R}^n$ *and all* $u \in \mathbb{R}^m$*,*

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \gamma(|u|), \forall t \leq 0.$$

This concept of stability is closely linked to the Lyapunov stability theory. This theory is based on the fact that the systems whose state trajectories are attracted to a stable asymptotically equilibrium point lose gradually and monotonically the energy. Lyapunov generalized the notion of energy by using any positive definite function, with some assumption to be given later, which is independent of the state trajectories of the system.

1.2.1.2 Lyapunov stability

Theorem 1.14 (Continuous case)**.** *Consider system* (1.13) *without input* ($u(t) = 0$)*. If there exist a function* $V : \mathbb{R}^n \to \mathbb{R}$ *that admits continuous partial derivatives*

and two functions α_1 *and* α_2 *of class* $\mathcal{K}$ *such that*

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||), \forall x \in \Omega \subset \mathbb{R}^n,$$

then, the origin of the system is

- ***Stable*** *if*

$$\frac{dV(x)}{dt} \leq 0, \forall x \in \Omega, x \neq 0,$$

- ***Asymptotically stable*** *if there exist a function* ϕ *of class* $\mathcal{K}$ *such that*

$$\frac{dV(x)}{dt} \leq -\phi(||x||), \forall x \in \Omega, x \neq 0,$$

- ***Exponentially stable*** *if there exist positive constants* $\overline{\alpha}_1, \overline{\alpha}_2, \gamma, p$ *such that*

$$\alpha_1(||x||) = \overline{\alpha}_1||x||^p, \alpha_2(||x||) = \overline{\alpha}_2||x||^p, \phi(||x||) = \gamma\,||x||\,, \forall x \in \Omega, x \neq 0.$$

Remark 1.15. All the properties of the previous theorem are valid in the vicinity of the equilibrium point (local properties). It becomes global ($\Omega = \mathbb{R}$) if the functions are chosen of class $\mathcal{K}_\infty$.

We consider now the case of discrete-time autonomous nonlinear systems

$$x(k+1) = f(x(k)) \tag{1.14}$$

Theorem 1.16 (Discrete case)**.** *Consider system* (1.14)*. If there exist a function* $V : \mathbb{R}^n \to \mathbb{R}$ *and two functions* α_1 *and* α_2 *of class* $\mathcal{K}$ *such that*

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||), \forall x \in \Omega \subset \mathbb{R}^n,$$

then, the origin of the system is

- ***Stable*** *if*

$$\Delta V(x(k)) \leq 0, \forall x \in \Omega, x \neq 0,$$

where

$$\begin{aligned} \Delta V(x(k)) &= V(x(k+1)) - V(x(k)) \\ &= V(f(x(k))) - V(x(k)) \end{aligned} \tag{1.15}$$

- ***Asymptotically stable*** *if there exist a function* ϕ *of class* $\mathcal{K}$

$$\Delta V(x) \leq -\phi(||x||), \forall x \in \Omega, x \neq 0,$$

- ***Exponentially stable*** *if there exist positive constants* $\overline{\alpha}_1, \overline{\alpha}_2, \gamma, p$ *such that*

$$\alpha_1(||x||) = \overline{\alpha}_1 ||x||^p, \alpha_2(||x||) = \overline{\alpha}_2 ||x||^p, \phi(||x||) = \gamma \, ||x|| \, , \forall x \in \Omega, x \neq 0.$$

1.2.2 Hybrid systems framework

A dynamical system is usually classified as either a continuous-time dynamical system or a discrete-time dynamical system. For example, classical mechanical systems evolving in time according to physical laws can be viewed naturally as continuous-time dynamical system. Financial accounts can be viewed naturally as discrete-time dynamical system. Numerous dynamical systems escape such a clear-cut classification. In fact, there are dynamical systems that exhibit characteristics of both continuous-time and discrete-time systems. An example of such systems are the so-called Cyber-Physical System (CPS). Examples are provided by power electronic circuits that combine analog and digital components and by mechanical systems controlled by digital computers. Such systems are called, in control theory, hybrid dynamical systems.

Hybrid systems are modeled first by Hybrid Automata (Alur et al., 1992) in the computer science domain in the context of formal verification which helps proving the correctness of systems. Recently, a very elegant and concise framework that models hybrid systems was proposed in Goebel et al. (2012). This framework is general enough to study the class of systems considered in this work. We adopt this formalism in the sequel of this thesis.

In this section, we will present how complex systems with both continuous-time and discrete-time dynamics can be modeled by using the hybrid system formalism of Goebel et al. (2012) which allows us to use the well-defined notion of solutions and the tools provided within. Basic concepts and analysis of this class of dynamical systems are given below.
We consider the hybrid systems of the following form

$$\dot{x} = \mathcal{F}(x) \quad x \in \mathcal{C}, \quad x^+ = \mathcal{G}(x) \quad x \in \mathcal{D}, \tag{1.16}$$

where $x \in \mathbb{R}^n$ is the state, $\mathcal{C}, \mathcal{D} \in \mathbb{R}^n$ and $\mathcal{F}, \mathcal{G}$ are single-valued functions. This model shows that the state x of the hybrid system evolves according to the differential equation $\dot{x} = \mathcal{F}(x)$ as long as $x \in \mathcal{C}$, and it experiences an instantaneous change according to the difference equation $x^+ = \mathcal{G}(x)$ when $x \in \mathcal{D}$. When $x \in \mathcal{C} \cap \mathcal{D}$, the system behaves according to the differential equation $\dot{x} = \mathcal{F}(x)$ only if this evolution keeps $x \in \mathcal{C}$, otherwise the system experiences a discrete transition. To simplify the notation, the continuous behaviour and the discrete behaviour will be referred as *flow* and *jump*, respectively. Consequently, the elements of hybrid system (1.16) will now be named as follows: $\mathcal{F}$, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{D}$ are the *flow map*, the *flow set*, the *jump map* and the *jump set*, respectively. The solutions to system (1.16) are defined on so-called hybrid time domains.

Remark 1.17. The framework proposed in Goebel et al. (2012) is more general than what we have presented in (1.16) and considers *hybrid inclusions* of the following form

$$\dot{x} \in \mathcal{F}(x) \quad x \in \mathcal{C}, \quad x^+ \in \mathcal{G}(x) \quad x \in \mathcal{D},$$

But, in this work, only the simplified form (1.16) is needed.

Definition 1.18 (Goebel et al. (2012))**.** *(Hybrid time domains)* – A subset $\mathcal{E} \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$\mathcal{E} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in \mathcal{E}$, $\quad \mathcal{E} \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid domain.

Definition 1.19. *(Hybrid arc)* – A function $\phi : \mathcal{E} \to \mathbb{R}^n$ is a hybrid arc if $\mathcal{E}$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I^j = \{t : (t, j) \in \mathcal{E}\}$.

In the sequel, the hybrid time domain $\mathcal{E}$ of the hybrid arc ϕ will be noted by domϕ. A hybrid arc ϕ is a solution to the hybrid system $(\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ if:

(i) $\phi(0, 0) \in \mathcal{C} \cup \mathcal{D}$;

(ii) for all $j \in \mathbb{N}$, $\phi(t, j) \in \mathcal{C}$ and $\dot{\phi}(t, j) = \mathcal{F}(\phi(t, j))$ for almost all $t \in I^j$;

(iii) for all $(t, j) \in$ domϕ, such that $(t, j+1) \in$ domϕ, $\phi(t, j) \in \mathcal{D}$ and $\phi(t, j+1) = \mathcal{G}(\phi(t, j))$.

A solution ϕ to system (1.16) is

- *maximal* if it cannot be extended,
- *complete* if its domain, domϕ, is *unbounded*,
- *Zeno* if it is complete and $\sup_t$ dom$\phi < \infty$.

1.2.3 Lyapunov conditions for hybrid systems

As for continuous-time and discrete-time systems, Lyapunov functions are also useful in analyzing stability for hybrid systems. For now, it is not necessarily to require that a Lyapunov function be defined on all of $\mathbb{R}^n$ nor that it is continuously differentiable on all of $\mathbb{R}^n$ due to the constraints that are given in a hybrid system that limit where jumping and flowing is possible. The definition below gives the conditions required for a function V to be considered as a Lyapunov function candidate for hybrid systems.

Definition 1.20 (Lyapunov function candidate)**.** A function $V :$ dom$V \to \mathbb{R}$ is said to be a Lyapunov function candidate for the hybrid system $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ if the following conditions hold:

1. $\overline{\mathcal{C}} \cup \mathcal{D} \cup \mathcal{G}(\mathcal{D}) \subset$ dom$V$;
2. V is continuously differentiable on an open set containing $\overline{\mathcal{C}}$;

where $\overline{\mathcal{C}}$ denotes the closure[2] of C.

Definition 1.21 (Positive definite functions)**.** A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite, also written $\rho \in \mathcal{PD}$, if $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$.

The following theorem provides conditions on a Lyapunov function candidate that guarantee *uniform global pre-asymptotic stability.*

Remark 1.22. Pre-asymptotic stability is more general than asymptotic stability. In pre-asymptotic stability, the completeness of the maximal solution is not required. That is, it is not required that the solution tends to an equilibrium point as time goes to infinity, as in asymptotic stability, but to a closed set.

Theorem 1.23 (Sufficient Lyapunov conditions (Goebel et al., 2012))**.** *Let $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, and a continuous $\rho \in \mathcal{PD}$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}(\mathcal{D})\,, \tag{1.17a}$$

$$\langle \nabla V(x), \mathcal{F}(x) \rangle \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \mathcal{C}\,, \tag{1.17b}$$

$$V(\mathcal{G}(x)) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in \mathcal{D}\,. \tag{1.17c}$$

then $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$.

There is no loss of generality in using the same function $\rho(\cdot)$ in both inequality (1.17b) and (1.17c) in the Lyapunov conditions of Theorem 1.23. If the two functions are different, they can each be replaced by the point-wise minimum of the two functions, which will be another function belonging to the class-$\mathcal{PD}$.

In the context of cyber-physical system, where the continuous-time part is governed by the discrete-time one e.g., sampled data control, event-triggered estimation, the Lyapunov condition (1.17b) of Theorem 1.23, that imposes the convergence of the continuous-time part, is not always satisfied. In the following section, we will recall all possible relaxed conditions of Lyapunov conditions (1.17) as given in Goebel et al. (2012).

[2] A closure of a set of points consists of all points in this set together with all its limit points

1.2.3.1 Relaxed Lyapunov conditions

In this section, we will give several sufficient conditions for uniform global pre-asymptotic stability, in which the strict decrease assumptions of Theorem 1.23 are weakened. The assumptions of strict decrease of the Lyapunov function during both flows and jumps can be weakened in several ways. For example, this can be the case if the Lyapunov function is non-increasing during flows, strictly decreasing during jumps, and the jumps occur frequently enough, as stated in the proposition below.

Proposition 1.24 (Sufficient Lyapunov conditions: persistent jumping (Goebel et al., 2012))**.** *Let $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. Suppose that V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, and a continuous $\rho \in \mathcal{PD}$ such that (1.17a), (1.17c) hold and*

$$\langle \nabla V(x), \mathcal{F}(x) \rangle \leq 0 \quad \forall x \in \mathcal{C}. \tag{1.18}$$

If, for each $r > 0$, there exists $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for every solution ϕ to $\mathcal{H}$, $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, $(t, j) \in dom\phi$, $t + j \geq T$ imply $j \geq \gamma_r(T) - N_r$, then $\mathcal{A}$ is uniformly globally pre-asymptotically stable.

Proposition 1.25 (Sufficient Lyapunov conditions: increase balanced by decrease (Goebel et al., 2012))**.** *Let $\mathcal{H} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. Suppose that V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, such that (1.17a) holds, and*

$$\begin{aligned} \langle \nabla V(x), \mathcal{F}(x) \rangle &\leq \lambda_c V(x) \quad \forall x \in \mathcal{C}, \\ V(\mathcal{G}(x)) &\leq e^{\lambda_d} V(x) \quad \forall x \in \mathcal{D}. \end{aligned} \tag{1.19}$$

If there exist $\gamma > 0$ and $M > 0$ such that for each solution ϕ to $\mathcal{H}$, $(t, j) \in dom\phi$ implies $\lambda_c t + \lambda_d j \leq M - \gamma(t + j)$ then $\mathcal{A}$ is uniformly globally pre-asymptotically stable.

The result in Proposition 1.25 allows for the Lyapunov function to increase. The increases can be persistent but are compensated by strong decrease.

This relaxed result will be used in Chapter 2 in which an impulsive interval observer estimates the continuous-time state of the system based on aperiodic discrete measurements.

Chapter 2

Interval observer for linear systems with aperiodic measurements

2.1 Motivations and related works

Nowadays, the study of aperiodic sampled-data systems constitutes a very popular research topic in control. This is a consequence of the huge development of embedded and Networked Control Systems (NCS), where sensor and control data are transmitted over digital communication channels. To reduce the communication load on these limited bandwidth channels, it is tempting to exchange data in an aperiodic manner. Besides, the presence of sampling jitters, packet dropouts and fluctuations in network accessibility further emphasize the interest in time-varying and aperiodic sampling (Postoyan and Nešić, 2012), (Hespanha et al., 2007). In the context of state estimation, estimators for continuous-time systems with aperiodically sampled outputs have been studied under several frameworks; for instance, under *the hybrid system framework* (Goebel et al., 2012) where the whole system is represented as an impulsive system (Ferrante et al., 2016), or under *the time-delay system framework* as a sampled-data system (Efimov et al., 2016) where the system's output is held in between measurements.

This chapter addresses the modeling and the design of an interval state observer for a linear time-invariant plant in presence of sporadically available measurements

corrupted by unknown-but-bounded errors and noise.

Basically, the design of interval observers must ensure the nonnegativity and of the estimation error and its stability as well. Besides, the advantage of these observers compared to other guaranteed estimators, such as prediction-correction approaches (Alamo et al., 2005), (Meslem et al., 2010) for instance, is that the observer gain can be computed offline and the convergence of the estimation error can be proved a priori.

To be able to reconstruct a guaranteed state enclosure of the actual state for continuous-time linear systems in presence of sporadic discrete-time measurement, we propose an *interval impulsive observer*. The impulsive behavior is the result of the discrete nature of the measurements. In between two measurement time instants, the observer behaves as a continuous-time predictor based only on the evolution model. Then, at the measurement time instant, an impulsive correction adjusts the estimated state enclosure. Here, we consider that the inter-measurement time is unknown but belongs to an interval of times, contrariwise to Mazenc and Dinh (2014) where it is chosen constant. The lower bound of the time interval is chosen in a way to avoid Zeno phenomenon (infinite number of samples in finite time). Both lower and upper bounds of the inter-measurement time are chosen a priori by the user, then our synthesis procedure tunes the observation gain to ensure both positivity and stability of the estimation error. The stability analysis of the bounds estimate is inspired by the work of Ferrante et al. (2016) while the positivity of the estimation error is ensured based on the internal positivity for dynamical systems as in Meslem and Ramdani (2011) and in Cacace et al. (2015).

In the literature, the study of continuous interval observer with discrete measurements has been addressed in Goffaux et al. (2009), in Mazenc and Dinh (2014), and in Efimov et al. (2016). In Goffaux et al. (2009), a continuous-discrete interval observer has been applied to a specific microalgae-based bioprocess. In Mazenc and Dinh (2014), an analysis of an interval observer for continuous-time systems with discrete measurements has been proposed, where the measurement period is assumed constant. To the best of the authors' knowledge, there is no work dealing with the observation gain design in the context of continuous-discrete interval observation with sporadic discrete output.

The novelty of this chapter resides in a new methodology for the design of the

observer gain in presence of aperiodic measurements, that guarantees both positivity and stability of the interval estimation error. Moreover, the studied system includes both system perturbation and measurement noise while in Ferrante et al. (2016) only output noise has been considered. The synthesis problem is formulated as a set of BMI and Linear Programming (LP) constraints. An exponential stability and an input-to-state stability of the estimation error bounds are proved. The chapter is organized as follows. The stability verification of the interval observer for linear systems with aperiodic discrete measurements is performed in Section 2.2. The proposed method for observer gain synthesis is detailed in Section 2.3. Simulation results of numerical examples are presented in Section 2.4.

2.2 Interval impulsive observer analysis

Consider linear time-invariant systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + d(t), \\ y(t_k) = Cx(t_k) + v(t_k), \end{cases} \quad \forall k \in \mathbb{N} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ is the state variables, the input, the discrete-time output of the system, respectively. $d \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ represent the system disturbance and the output model uncertainty, respectively.

2.2.1 Assumptions

It is assumed that the discrete-time measurements are not necessarily periodic and there exist a minimal and a maximal time between two consecutive measurement instants as follows.

Assumption 2.1. *Let $\tau_{max} \geq \tau_{min}$ be two given real scalars satisfy: $t_{k+1} - t_k \in [\tau_{min}, \tau_{max}]$ $\forall k \in \mathbb{N}$.*

Based on aperiodic discrete-time measurements, the goal here is to estimate an upper and a lower bound of the system state while ensuring the convergence of the estimation error. To do so, let first introduce an assumption on the boundedness of both measurement noise and system disturbances.

Assumption 2.2.

(a) Let a constant vector $\overline{v} \in \mathbb{R}^p_{\geq 0}$ *be given such that*

$$|v(t)| \leq \overline{v} \ \forall t \in \mathbb{R}_{\geq 0},$$

where $\overline{v}_i = \sup_{t\geq 0} |v_i(t)| \quad \forall i \in \{1, \ldots, p\}$.

(b) Let two constant vectors $\underline{d}, \overline{d} \in \mathbb{R}^n$ *be given such that*

$$\underline{d} \leq d(t) \leq \overline{d}$$

is satisfied $\forall t \in \mathbb{R}_{\geq 0}$.

2.2.2 Observer structure

Before presenting the structure of the developed observer, we recall the following lemma which will help in the analysis of the observer dynamics.

Lemma 2.1 (Comparison Lemma (Khalil, 2002, Lemma 3.4))**.** *Consider the scalar differential equation*

$$\dot{x}_u(t) = f(t, x_u), \qquad x_u(t_0) = x_{u0}$$

where $f(t, x_u)$ *is continuous in* t *and locally Lipschitz in* x_u*, for all* $t \geq 0$ *and all* $x_u \in U \subset \mathbb{R}$*. Let* $[t_0, T_t)$ [1] *be the maximal interval of existence of the solution* $x_u(t)$*, and suppose* $x_u(t) \in U$ *for all* $t \in [t_0, T_t)$*. Let* $x_v(t)$ *be a continuous function whose upper right-hand derivative* $\mathbf{D}_{t^+} x_v(t)$ *satisfies the differential inequality*

$$\mathbf{D}_{t^+} x_v(t) \leq f(t, x_v(t)), \quad x_v(t_0) \leq x_{u0}$$

with $x_v(t) \in U \ \forall t \in [t_0, T_t)$*. Then,* $x_v(t) \leq x_u(t) \ \forall t \in [t_0, T_t)$.

Remark 2.2. A generalized version of the comparison theorem for $n-$dimensional system is introduced in Scott and Barton (2013) based on the Müller's existence theorem (Müller, 1927).

[1] The time interval $[t_0, T_t)$ could be $[t_j, t_{j+1}] \quad \forall j \in J$; the continuous part of the hybrid time domain where $f(t, x_u)$ is continuous in t

The interval observer that we propose for system (2.1) has two steps;
First step: the interval observer in-between two successive measure instants behaves like an open-loop estimator as follows

$$\begin{cases} \dot{\underline{x}}(t) = A^M \underline{x}(t) - A^N \overline{x}(t) + Bu(t) + \underline{d}, \\ \dot{\overline{x}}(t) = A^M \overline{x}(t) - A^N \underline{x}(t) + Bu(t) + \overline{d} \end{cases} \quad \forall t \in [t_k, t_{k+1}], k \in \mathbb{N} \tag{2.2}$$

where $A^M = d_A + (A - d_A)^+$ and $A^N = A^M - A$ with d_A is a diagonal matrix contains only the diagonal elements of A. In addition, the interval observer initial state at $k = 0$, i.e. at t_0, satisfies the inclusion

$$\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0) \tag{2.3}$$

The estimation error dynamics over the inter-measurement time for both bounds $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ can be obtained from equations (2.1) and (2.2) by

$$\begin{bmatrix} \dot{\underline{e}}(t) \\ \dot{\overline{e}}(t) \end{bmatrix} = \overline{A} \begin{bmatrix} \underline{e}(t) \\ \overline{e}(t) \end{bmatrix} + \begin{bmatrix} d(t) - \underline{d} \\ \overline{d} - d(t) \end{bmatrix}, \forall t \in [t_k, t_{k+1}] \; k \in \mathbb{N} \tag{2.4}$$

with $\overline{A} = \begin{bmatrix} A^M & A^N \\ A^N & A^M \end{bmatrix}$.

Let us denote by $\Xi(t) = \begin{bmatrix} d(t) - \underline{d} \\ \overline{d} - d(t) \end{bmatrix}$ and $\overline{\Xi} = \begin{bmatrix} \overline{d} - \underline{d} \\ \overline{d} - \underline{d} \end{bmatrix}$ the second term of (2.4) (due to the system disturbance) and its upper constant bound, respectively.
Note that, based on the construction of the matrices A^M and A^N as Metzler and nonnegative matrices, respectively, the matrix $\overline{A}$ is Metzler. In addition, the vector $\Xi(t)$ is nonnegative. Then, the solution to (2.4) is nonnegative which means that the lower and the upper bounds are nonnegative in the time interval $[t_k, t_{k+1}]$ provided that their initial conditions are nonnegative, that is, the inclusion $\underline{x}(t_k) \leq x(t_k) \leq \overline{x}(t_k)$ is satisfied.

Second step: using the output model in (2.1), the system state at the measurement time instants can be presented as

$$x(t_k^+) = x(t_k) + L[Cx(t_k) + v(t_k) - y(t_k)] \quad k \in \mathbb{N}$$

or equivalently

$$x(t_k^+) = (I + LC)x(t_k) + Lv(t_k) - Ly(t_k) \quad k \in \mathbb{N} \tag{2.5}$$

Equation (2.5) helps establishing the discrete-time dynamics of the estimation error which is used only for the synthesis phase. When the measurement is available, an impulsive correction of the estimated state enclosures will be done using the following correction equations

$$k \in \mathbb{N}, \quad \begin{cases} \underline{x}(t_k^+) = (I_n + LC)^+ \underline{x}(t_k) - (I_n + LC)^- \overline{x}(t_k) \\ \qquad\qquad - |L|\overline{v} - Ly(t_k) \\ \overline{x}(t_k^+) = (I_n + LC)^+ \overline{x}(t_k) - (I_n + LC)^- \underline{x}(t_k) \\ \qquad\qquad + |L|\overline{v} - Ly(t_k) \end{cases} \tag{2.6}$$

where $L \in \mathbb{R}^{n \times p}$ is an observer gain to be designed.
From (2.6) and (2.5), the estimation error dynamics at measurement instants can be described by the following dynamical system

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(L) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \Upsilon(t_k) \tag{2.7}$$

where

$$\Gamma(L) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix}; \Upsilon(t_k) = \begin{bmatrix} |L|\overline{v} + Lv(t_k) \\ |L|\overline{v} - Lv(t_k) \end{bmatrix}. \tag{2.8}$$

The matrix $\Gamma(L)$ is an Internally Positive Realization of the state reset matrix $(I_n + LC)$. The positivity property of the reset matrix allows to preserve the order relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ after experiencing the reset (for more details about IPR for linear systems, see (Cacace et al., 2015)).

Let us now consider the augmented error vector of the interval estimation as $\xi = [\underline{e}^\top, \overline{e}^\top]^\top$. Re-writing the system (2.4) and upper bounding its right-hand term using the upper bound of $\Xi(t)$ one gets $\forall t \in [t_k, t_{k+1}]$

$$\dot{\xi}(t) = \overline{A}\xi(t) + \Xi(t) \tag{2.9a}$$

$$\leq \overline{A}\xi(t) + \overline{\Xi} \tag{2.9b}$$

Now, using the results of the general comparison theorem (see Lemma 2.1 for scalar differential equation), the solution $\xi(t)$ of the system (2.9a) is upper bounded by the solution of the system

$$\dot{\xi}_u(t) = \overline{A}\xi_u(t) + \overline{\Xi} \tag{2.10}$$

provided that $\xi(t_0) \leq \xi_u(t_0)$, where $\xi_u(t) \in \mathbb{R}^{2n}$ is an auxiliary variable that represents an upper bound of $\xi(t)$. To characterize the steady state of the system (2.10), we assume that the matrix $\overline{A}$ is invertible. Thus, the steady state of the system (2.10) is as follows

$$\xi_{ueq} = -\overline{A}^{-1}\overline{\Xi}.$$

By introducing the shifted error $\xi_0 = \xi_u - \xi_{ueq}$, the ξ_0-system dynamics is given as follows

$$\dot{\xi}_0(t) = \overline{A}(\xi_0(t) + \xi_{ueq}) + \overline{\Xi} = \overline{A}\xi_0(t) \tag{2.11}$$

Remark 2.3. To simplify the stability analysis of the estimation error defined in (2.4) and (2.7), we will use the dynamics of the ξ_0-system. We justify our choice by the following reasons:

- The solution of the ξ_u-system in (2.10) is an upper bound of the solution of the $\xi-$system in (2.9a). Shifting the ξ_u-system by the value of its steady state ξ_{ueq} using the ξ_0-system, the convergence of ξ_0 to 0 means that ξ_u converges to ξ_{ueq}.

- The reset equation (2.7) is independent of the model disturbance $d(t)$. Consequently, the same reset equation (2.7) can be used for the reset of the ξ_u-system.

- In the case of system (2.1) without state disturbance ($d(t) = 0$), the trajectories $\xi(t)$, $\xi_u(t)$ and $\xi_0(t)$ are the same and have the same stability properties. Otherwise, when the state is under disturbance, then the trajectory $\xi(t)$ is upper bounded by the trajectory $\xi_u(t)$.

The stability analysis is studied under the hybrid system framework (Goebel et al., 2012).

From equations (2.4) and (2.7) and by using the shifted upper bound of the estimation error (2.11), after adding the time variable τ, the hybrid system modeling the dynamics of the upper bound of the estimation error is given by

$$\begin{cases} f(z_0) = \begin{bmatrix} \overline{A}\xi_0 \\ -1 \end{bmatrix} & \forall z_0 \in \mathcal{C} \\ g(z_0) = \begin{bmatrix} \Gamma(L)\xi_0 + \Upsilon_2(t_k) \\ \mu \end{bmatrix} & \forall z_0 \in \mathcal{D} \end{cases} \tag{2.12}$$

with

$$\Upsilon_2(t_k) = \Upsilon(t_k) + (\Gamma(L) - I)\xi_{ueq},$$

where $z_0 = [\xi_0^\top, \tau]^\top$ is the state variable of the hybrid system, $\mu \in [\tau_{min}, \tau_{max}]$ is the set of admissible values of the timer based on Assumption 2.1.

The flow and jump sets are defined as

$$\begin{aligned} \mathcal{C} &= \left\{(\xi_0, \tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \tau \in [0, \tau_{max}]\right\} \\ \mathcal{D} &= \left\{(\xi_0, \tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \tau = 0\right\}. \end{aligned} \tag{2.13}$$

It is worth noting that these sets do not force the system to jump until the timer reaches zero, then after the jump, the timer τ is reset to a value $\mu \in [\tau_{min}, \tau_{max}]$. Let us define the closed set $\mathcal{A}$ that contains all admissible values for the timer when the ξ_0-system state is at the origin

$$\mathcal{A} = \left\{z_0 = (\xi_0, \tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \xi_0 = 0, \tau \in [0, \tau_{max}]\right\}. \tag{2.14}$$

Remark 2.4. As discussed in Remark 2.3, the stability of the ξ_0-system is sufficient for the stability of the $\xi-$system. Thus, if the ξ_0-system is stable relatively to $\mathcal{A}$,

this implies that the $\xi-$system is practically stable relatively to $\mathcal{A}$

Remark 2.5. The hybrid system (2.12) can be considered for the perfect case (without noise and perturbation) by omitting the term $\Upsilon_2(t_k)$. That is, if the system has $d(t) = 0,\ \forall t \geq 0$ and $v(t_k) = 0,\ \forall k \in \mathbb{N}$, then we can pick $\underline{d} = \overline{d} = 0$ and $\overline{v} = 0$ which imply that $\overline{\Xi} = \mathbf{0}$ and $\Upsilon(t_k) = \mathbf{0}\ \forall k \in \mathbb{N}$. Thus we have $\Upsilon_2(t_k) = \mathbf{0}$.

We characterize the solutions' domain of (2.12) when $\Upsilon_2(t_k) = 0$. Indeed, the variable τ, acting as a timer, guarantees that for every initial condition $\phi(0,0) \in \mathcal{C} \cup \mathcal{D}$ the domain of every maximal solution ϕ to (2.12) when $\Upsilon_2(t_k) = 0$ can be written as follows:

$$\text{dom}\phi = \bigcup_{j\in\mathbb{N}}([t_j, t_{j+1}], j)$$

with $\tau_{min} \leq t_{j+1} - t_j \leq \tau_{max},\quad \forall j \in \mathbb{N} \setminus \{0\}$. Furthermore, assuming $t_0 = 0$, the structure of the above hybrid time domain implies that for each $(t, j) \in \text{dom}\phi$ we have

$$t \leq \tau_{max}(j + 1).$$

The latter relation will play a key role in establishing GES of the set $\mathcal{A}$ for hybrid system (2.12) when $\Upsilon_2(t_k) = \mathbf{0}$.
In the sequel, we consider the following definitions of the global exponential stability (GES) and the input-to-state stability (ISS) of closed sets for a general hybrid system.

Definition 2.6. (Teel et al., 2013) Let $\mathcal{A} \subset \mathbb{R}^{n_\phi}$ be closed. The set $\mathcal{A}$ is said to be GES for the hybrid system (2.12) when $\Upsilon_2(t_k) = \mathbf{0}$ if there exist $\lambda, \kappa \in \mathbb{R}_>$ such that every solution ϕ to (2.12) when $\Upsilon_2(t_k) = \mathbf{0}$ satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \kappa e^{-\lambda(t+j)}|\phi(0, 0)|_{\mathcal{A}} \quad \forall (t, j) \in \text{dom}\phi. \tag{2.15}$$

Definition 2.7. (Cai and Teel, 2009) Let $\mathcal{A} \subset \mathbb{R}^{n_\phi}$ be closed. The set $\mathcal{A}$ is said to be ISS for (2.12) with respect to Υ_2 if there exist $\beta \in \mathcal{KL}$ and $\alpha \in \mathcal{K}$ such that every solution ϕ to (2.12) satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \max\left\{\beta(|\phi(0, 0)|_{\mathcal{A}}, t + j), \alpha(||\Upsilon_2||_{(t,j)})\right\}\ \forall (t, j) \in \text{dom}\phi. \tag{2.16}$$

Before setting our first result in the next theorem, let introduce the following lemma.

Lemma 2.8 ((Ferrante et al., 2016)). *Let ρ be a strictly negative real number. Choose $\gamma \in \left(0, \frac{|\rho|}{1+\tau_{max}}\right]$ and $\omega \in \left[\frac{\tau_{max}|\rho|}{1+\tau_{max}}, \infty\right)$. Let ϕ be any solution to the hybrid system (2.12). Then for every $(t, j) \in \mathrm{dom}\phi$, one has $\rho j \leq \omega - \gamma(t + j)$.*

The idea behind using Lemma 2.8 is to allow for the Lyapunov function to increase locally. This increase is compensated by instantaneous decrease at jumps which renders the overall hybrid dynamics stable. This stability property is the main result of (Goebel et al., 2012, Proposition 3.29) which we have recalled in Proposition 1.25.

Theorem 2.9. *Let Assumption 2.1 and 2.2 hold. For a given gain matrix $L \in \mathbb{R}^{n\times p}$, if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ such that*

$$\Gamma(L)^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Gamma(L) - P \prec 0 \quad \forall \mu \in [\tau_{min}, \tau_{max}] \tag{2.17}$$

is satisfied, then the hybrid system (2.12)-(2.13) is Input-to-State-Stable (ISS) with respect to the set $\mathcal{A}$ defined in (2.14). Thus, the system (2.2), (2.6) is an interval observer for the system (2.1) with ISS estimation error relatively to $\mathcal{A}$ provided that $\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0)$. Moreover, if $v(t_k) = 0\ \forall k \in \mathbb{N}$ and $d(t) = 0\ \forall t \geq 0$ in (2.1), then the interval observer (2.2), (2.6) for the system (2.1) has a globally exponentially stable (GES) estimation error relatively to $\mathcal{A}$.

The stability proof of Theorem 2.9 follows the main lines of the proof of (Ferrante et al., 2016, Theorem 1) with appropriate modifications due to the fact that we study the stability of an over-approximation bound of the interval estimation error. These modifications have a double role. First, they have to guarantee the order preserving property of the estimation error. Then, they manage the effect of the state disturbance.

Proof of Theorem 2.9. The proof is split into two parts; studying the non-negativity of the estimation error, then the stability of the upper bound of the estimation error.

The cooperativity of the estimate error: Based on Assumption 2.2(b) and due to the Metzler property of the matrix $\overline{A}$ in (2.4), the continuous dynamics (2.4) is nonnegative. On the other hand, the observer updates the estimates at measurement times by means of (2.6), where its estimation error is represented by the discrete dynamics (2.7). The matrix $\Gamma(L)$ is structurally nonnegative while Assumption 2.2(a) ensures the nonnegativity of the vector $\Upsilon(t_k)$. Consequently, the estimation error preserves the order relation $\underline{x}(t_k) \leq x(t_k) \leq \overline{x}(t_k)$ after experiencing the reset. Finally, with the initial condition given as $\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0)$, the errors $\underline{e}(t)$ and $\overline{e}(t)$ of (2.12) are nonnegative for all $t \geq t_0$.

The stability of the estimation error: Instead of studying the stability of $z(t)$, we use its upper bound trajectory $z_u(t)$ induced by the system disturbance $d(t)$ as mentioned in Remark 2.5. The stability of the trajectory $z_u(t)$ can be analysed by studying the shifted solution $z_0(t)$. Let us consider the Lyapunov function candidate for the hybrid system (2.12)-(2.13) defined $\forall z_0 \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0}$

$$V(z_0) = \xi_0^\top e^{\overline{A}^\top \tau} P e^{\overline{A}\tau} \xi_0. \tag{2.18}$$

Note that there exist two positive scalars l_1, l_2 such that

$$l_1 |z_0|_{\mathcal{A}}^2 \leq V(z_0) \leq l_2 |z_0|_{\mathcal{A}}^2 \quad \forall z_0 \in \mathcal{C} \cup \mathcal{D}. \tag{2.19}$$

Due to the positive definiteness of P and the non-singularity of the matrix $e^{\overline{A}\tau}$ for every τ, by continuity arguments, one can set

$$l_1 = \min_{\tau \in [0, \tau_{max}]} \lambda_{min}\big(e^{\overline{A}^\top \tau} P e^{\overline{A}\tau}\big), \quad l_2 = \max_{\tau \in [0, \tau_{max}]} \lambda_{max}\big(e^{\overline{A}^\top \tau} P e^{\overline{A}\tau}\big) \tag{2.20}$$

where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the smallest and the largest eigenvalue of their matrix argument, respectively. From (2.18) we can find

$$\nabla V(z_0) = \Big(2e^{\overline{A}^\top \tau} P e^{\overline{A}\tau} \xi_0,\ \xi_0^\top \overline{A}^\top e^{\overline{A}^\top \tau} P e^{\overline{A}\tau} \xi_0 + \xi_0^\top e^{\overline{A}^\top \tau} P \overline{A} e^{\overline{A}\tau} \xi_0\Big) \tag{2.21}$$

For simplicity, throughout this proof, we write Γ and Υ_2 instead of $\Gamma(L)$ and $\Upsilon_2(t_k)$, respectively.

From system (2.12)-(2.13), and under the fact that $e^{\overline{A}\mu}$ and $\overline{A}$ commute, the variation of the Lyapunov function over the inter-sampling time is then $\langle \nabla V(z_0), f(z_0) \rangle = 0 \ \forall z_0 \in \mathcal{C}$.

Thus, the stability of the upper-bound of the error dynamics can be verified by considering the evolution of the Lyapunov function for every $z_0 \in \mathcal{D}$. In addition, whenever $z_0 \in \mathcal{D}$, we have $\tau = 0$, which implies

$$\begin{aligned} V(g(z_0)) - V(z_0) &= (\Gamma\xi_0 + \Upsilon_2)^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu}(\Gamma\xi_0 + \Upsilon_2) - \xi_0^\top P \xi_0 \\ &= \xi_0^\top (\Gamma e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Gamma - P)\xi_0 \\ &+ 2\Upsilon_2^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Gamma \xi_0 + \Upsilon_2^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Upsilon_2 \end{aligned} \tag{2.22}$$

GES stability

First, we discuss the case of system without noise and perturbation, i.e., when $v(t_k) = \mathbf{0}$ and $d(t) = 0$, which implies $\Upsilon_2 = \mathbf{0}$.

Based on inequality (2.17) there exists a constant $\eta \in \mathbb{R}_>$ small enough such that (2.22) becomes

$$V(g(z_0)) - V(z_0) \leq -\eta \xi_0^\top \xi_0 \quad \forall z_0 \in \mathcal{D} \tag{2.23}$$

Without loss of generality, let us assume that l_2 in (2.20) and η in (2.23) satisfy $1 - \frac{\eta}{l_2} > 0$. Define $\theta_1 = \ln\left(1 - \frac{\eta}{l_2}\right)$ and observe that $\theta_1 < 0$. Hence

$$V(g(z_0)) \leq e^{\theta_1} V(z_0) \ \ \forall z_0 \in \mathcal{D}. \tag{2.24}$$

Consider that ϕ is a maximal solution to (2.12)-(2.13).

As proven in (Goebel et al., 2012, Proposition 3.29), and similarly to (Ferrante et al., 2016, Theorem 1), using Lemma 2.8 one can find that the solution ϕ satisfies inequality (2.15) in Definition 2.6. Thus, the set $\mathcal{A}$ defined in (2.14) is GES for the hybrid system (2.12)-(2.13). Based on Remark 2.4, one can conclude that the interval observer (2.2), (2.6) with the system (2.1) have a GES estimation error relatively to $\mathcal{A}$ defined in (2.14).

ISS stability

Returning to the general case where $\Upsilon_2 \neq 0$. We apply Young's inequality $2a^\top b \leq \frac{\eta}{2} a^\top a + \frac{2}{\eta} b^\top b$ where $a = \xi_0$ and $b^\top = \Upsilon_2^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Gamma$ on the right-hand side of

equation (2.22). Then, we get

$$\begin{aligned} V(g(z_0)) - V(z_0) \leq & -\frac{\eta}{2}\xi_0^\top \xi_0 \\ & + \Upsilon_2^\top \Big[e^{\overline{A}^\top \mu} \Big(\frac{2}{\eta} P e^{\overline{A}\mu} \Gamma\Gamma^\top e^{\overline{A}^\top \mu} + I \Big) P e^{\overline{A}\mu} \Big] \Upsilon_2 \\ \leq & -\frac{\eta}{2l_2} V(z_0) + \Upsilon_2^\top \Upsilon_2 \Big|\Big| \frac{2}{\eta} P e^{\overline{A}\mu} \Gamma\Gamma^\top e^{\overline{A}^\top \mu} + I \Big|\Big| \\ & \times ||P|| \max_{\mu \in [\tau_{min}, \tau_{max}]} \big(||e^{\overline{A}\mu}||^2 \big) \end{aligned} \tag{2.25}$$

From (2.17), we have

$$\Big|\Big| \frac{2}{\eta} P e^{\overline{A}\mu} \Gamma\Gamma^\top e^{\overline{A}^\top \mu} + I \Big|\Big| \cdot ||P|| \leq \big(\frac{2}{\eta} ||P|| + 1 \big) ||P||. \tag{2.26}$$

In system (2.12), we have $\Upsilon_2 = \Upsilon + (\Gamma - I)\xi_{ueq}$. Using the triangle inequality, one obtain

$$\begin{aligned} \Upsilon_2^\top \Upsilon_2 &\leq 2\Upsilon^\top \Upsilon + 2\xi_{ueq}^\top (\Gamma^\top - I)(\Gamma - I)\xi_{ueq} \\ &\leq 2\Upsilon^\top \Upsilon + 2||\xi_{ueq}||^2 ||(\Gamma - I)||^2 \end{aligned} \tag{2.27}$$

The vector function Υ defined in (2.7) satisfies $\Upsilon^\top \Upsilon = 2\Big|\Big||L|\overline{v}\Big|\Big|^2 + 2||Lv||^2 \leq 4\Big|\Big||L|\overline{v}\Big|\Big|^2$, and based on Assumption 2.2.(a) we thus have

$$\Upsilon^\top \Upsilon \leq 4\Big|\Big||L|\Big|\Big|^2 \sup_t ||v(t)||^2 \tag{2.28}$$

Consequently, by replacing inequalities (2.26)-(2.28) in (2.25), the Lyapunov function at jump is obtained as

$$V(g(z_0)) \leq e^{\theta_2} V(z_0) + \zeta_1 \sup_t ||v||^2 + \zeta_2 \quad \forall z_0 \in \mathcal{D},\ v \in \mathbb{R}^p \tag{2.29}$$

where $\theta_2 = \ln\big(1 - \frac{\eta}{2l_2}\big)$ is a negative scalar while ζ_1 and ζ_2 are positive scalars defined as follows

$$\zeta_1 = 8\Big|\Big||L|\Big|\Big|^2 \big(\frac{2}{\eta} ||P|| + 1 \big) ||P|| \max_{\mu \in [\tau_{min}, \tau_{max}]} \big(||e^{\overline{A}\mu}||^2 \big)$$

$$\zeta_2 = 2\big|\big|\Gamma - L\big|\big|^2\big(\frac{2}{\eta}||P|| + 1\big)||P||\max_{\mu\in[\tau_{min},\tau_{max}]}\big(||e^{\overline{A}\mu}||^2\big)||\xi_{ueq}||^2.$$

Now, similarly to (Goebel et al., 2012, Proposition 3.29), and using Lemma 2.8 the solution ϕ verifies the inequality (2.16). Thus, the hybrid system (2.12)-(2.13) is ISS with respect to $v(t)$ relatively to the set $\mathcal{A}$. Finally, the interval observer (2.2), (2.3), (2.6) for the system (2.1) has ISS estimation error w.r.t $v(t)$ relatively to the set $\mathcal{A}$, and this concludes the proof. □

Remark 2.10. Note that assuming (2.17) to hold implies that the eigenvalues of $e^{\overline{A}\mu}\Gamma(L)$ are strictly contained in the unit circle for every μ belonging to $[\tau_{min}, \tau_{max}]$. Contrariwise to (Ferrante et al., 2016, Remark 3), it is not straightforward to connect this condition with a detectability property mainly because of the decomposition of $I + LC$ into matrix $\Gamma(L)$. However, one may conjecture that at least the following two necessary conditions are required for the feasibility of inequality (2.17):

(i) The detectability of the equivalent discrete-time system (DTS) of (2.1) given as: $\chi_{k+1} = e^{A\tau}\chi_k$, $y_k = C\chi_k$ $\forall\tau \in [\tau_{min}, \tau_{max}]$, where χ is the state and y_χ is the output. Similar to the criteria given in (Raff and Allgower, 2007, Remark 7), (Ferrante et al., 2016), (Mazenc and Dinh, 2014), the criterion should be the detectability of the pair $(e^{A\tau}, C)$ for every $\tau \in [\tau_{min}, \tau_{max}]$.

(ii) The second condition is due to our use of the IPR. In general, the stability of the above equivalent DTS does not imply the stability of its IPR. However, we can use the result of Cacace et al. (2015) on the relation between the eigenvalues of a system and the stability of its IPR. In Cacace et al. (2015), the authors have showed that the IPR of a system is stable if its eigenvalues remain in a restricted region of the complex plane defined by $\mathcal{P} = \{z \in \mathbb{C} : \mathfrak{Re}(z) + |\mathfrak{Im}(z)| < 1\}$. This second condition should also be satisfied by the equivalent DTS which restricts further the feasibility domain of the observer gain.

The combination of these two conditions deserves further analysis that goes beyond the scope of the chapter. This issue will be investigated in future work.

So far, a verification method has been given. The synthesis of the observation gain L cannot be achieved using convex solvers (CS) due to the decomposition of $(I_n + LC)$. However, using the positive realization of this matrix, the synthesis is still possible using CS. In the following section, we propose a synthesis methodology.

2.3 Synthesis methodology

In this section, we propose a new design methodology as second contribution of this chapter. We will show how to design the observer gain based on positive system theory.

2.3.1 Positive realization approach

Let us now re-consider the generic reset equation of the system state at measurement instant in (2.5). By introducing $G = [I + LC]$, the resetting matrix can be written as

$$\begin{aligned} x(t_k^+) &= Gx(t_k) + L[v(t_k) - y(t_k)] \\ &= (G^+ - G^-)x(t_k) + L[v(t_k) - y(t_k)] \quad k \in \mathbb{N} \end{aligned} \tag{2.30}$$

where G^+ and $-G^-$ are the positive and the negative part of the matrix G, respectively.

Let us note that for any two positive matrices $G_1,\ G_2 \in \mathbb{R}_{\geq 0}^{n\times n}$ that satisfy $G = G_1 - G_2$ there exists $\Delta \in \mathbb{R}_{\geq 0}^{n\times n}$ such that

$$G = (G^+ + \Delta) - (G^- + \Delta) \tag{2.31}$$

That is, matrices G_1 and G_2 are any positive realization of the matrices G^+ and G^-, respectively. Under the positive realization of the reset matrix G, the reset equation of the estimation error (2.7) can be generalized by the following difference equation

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(G_1, G_2) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \Upsilon(t_k) \tag{2.32}$$

where

$$\Gamma(G_1, G_2) = \begin{bmatrix} G_1 & G_2 \\ G_2 & G_1 \end{bmatrix} \tag{2.33}$$

Therefore, the idea for the synthesis is to calculate numerically the two positive matrices G_1 and G_2 that satisfy the stability conditions. Then, one can compute directly the matrices G^+ and G^- from the relation $G = G_1 - G_2$.
Using (2.33) instead of $\Gamma(L)$ in inequality (2.17), and introducing

$$\Phi(P, G_1, G_2) = \Gamma(G_1, G_2)^\top e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} \Gamma(G_1, G_2) - P, \tag{2.34}$$

the gain synthesis can now be performed by finding the solution set $\{P, G_1, G_2, L\}$ to the following feasibility problem

$$\Phi(P, G_1, G_2) \prec 0, \quad \forall \mu \in [\tau_{min}, \tau_{max}]\,, \tag{2.35a}$$

$$I_n + LC = G_1 - G_2\,, \tag{2.35b}$$

$$G_1 \geq 0\,, \tag{2.35c}$$

$$G_2 \geq 0\,, \tag{2.35d}$$

$$P \succ 0 \tag{2.35e}$$

From equation (2.31) and based on the definition of the positive matrices G^+ and G^- and their positive realization G_1 and G_2, respectively, the reset equation (2.32) can be seen as a positive discrete time system whose state matrix is perturbed by a nonnegative matrix as follows

$$\Gamma(G_1, G_2) = \Gamma(L) + \mathbb{1}_{2\times 2} \otimes \Delta. \tag{2.36}$$

Remark 2.11. Since the matrix Δ is nonnegative which implies that $\mathbb{1}_{2\times 2} \otimes \Delta$ is also nonnegative, it is always possible to enhance the interval observer dynamics at jumps in (2.32) by reducing the matrix $\Gamma(G_1, G_2)$ in (2.36) to its optimal realization $\Gamma(L)$. As a result, the observer gain matrix L in equations (2.6) of the interval observer can be synthesized using the intermediate auxiliary matrices G_1 and G_2, then the implementation is finally done using matrix $\Gamma(L)$ instead of $\Gamma(G_1, G_2)$.

2.3.2 Design procedure

The semi-definite programming (SDP) (2.35) is subjected to a Nonlinear Matrix inequality, which is hard to solve. The constraint $\Phi \prec 0$ can be relaxed to a Bilinear Matrix Inequality (BMI) using the following simplifications.
The constraints to be simplified are as follows

$$\begin{aligned} \Phi(P, G_1, G_2) &\prec 0 \\ -P &\prec 0 \end{aligned} \tag{2.37}$$

these two constraints can be combined in using projection lemma.

Lemma 2.12. *(Projection Lemma (Pipeleers et al., 2009)) Given $Z = Z^\top \in \mathbb{R}^{m\times m}$ and two matrices U and V of column dimension m; there exists an unstructured matrix F that satisfies*

$$U^\top F V + V^\top F^\top U + Z \prec 0, \tag{2.38}$$

if and only if the following projection inequalities with respect to F are satisfied

$$\begin{aligned} N_U^\top Z N_U &\prec 0, \\ N_V^\top Z N_V &\prec 0, \end{aligned} \tag{2.39}$$

where N_U and N_V are arbitrary matrices whose columns form a basis of the nullspaces of U and V, respectively.

Setting

$$Z = \begin{bmatrix} e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} & 0 \\ 0 & -P \end{bmatrix} \tag{2.40}$$

Thus, the equivalence between (2.39) and (2.37) is obtained with $N_U^\top = [\Gamma(G_1, G_2)^\top \;\; I_{2n}]$ and $N_V^\top = [0 \;\; I_{2n}]$. Picking $U = [-I_{2n} \;\; \Gamma(G_1, G_2)]$ and $V = [I_{2n} \;\; 0]$ based on their nullspaces basis N_U and N_V, respectively, the resulting inequality from (2.37) of the form (2.38) is given as follows

$$\begin{bmatrix} e^{\overline{A}^\top \mu} P e^{\overline{A}\mu} - F - F^\top & F\Gamma(G_1, G_2) \\ \star & -P \end{bmatrix} \prec 0, \quad \forall \mu \in [\tau_{min}, \tau_{max}] \tag{2.41}$$

Using Schur complement for the inequality (2.41) we have

$$\begin{bmatrix} -F-F^\top & F\Gamma(G_1,G_2) & e^{\overline{A}^\top \mu} \\ \star & -P & \mathbf{0} \\ \star & \star & -P^{-1} \end{bmatrix} \prec 0, \quad \forall \mu \in [\tau_{min}, \tau_{max}] \tag{2.42}$$

Pre-and-post multiplying (2.42) by the matrix $diag\{I_{2n}, I_{2n}, P\}$ on gets

$$\begin{bmatrix} -F-F^\top & F\Gamma(G_1,G_2) & e^{\overline{A}^\top \mu}P \\ \star & -P & \mathbf{0} \\ \star & \star & -P \end{bmatrix} \prec 0, \quad \forall \mu \in [\tau_{min}, \tau_{max}] \tag{2.43}$$

In the inequality (2.43) the time variable μ is in a compact interval defined as $\mu \in [\tau_{min}, \tau_{max}]$. So the term $e^{\overline{A}\mu}$ can be represented by its polytopic over-approximation. This over-approximation consists in determining a finite number of constant matrices $\{M_1, M_2, \ldots, M_v\} \in \mathbb{R}^{2n\times 2n}$ such that $e^{\overline{A}[\tau_{min},\tau_{max}]} \in \text{conv}\{M_1, M_2, \ldots, M_v\}$. Several methods exist in the literature that can be used to calculate the closed convex hull $\text{conv}\{M_i\}_{i\in\{1,\ldots,v\}}$ (Heemels et al., 2010). Here, the method developed in Hetel et al. (2007) based on Taylor's series is adopted.
Finally, to solve the matrix inequalities (2.37), it is sufficient to solve the set of BMIs given as follows

$$\begin{bmatrix} -F-F^\top & F\Gamma(G_1,G_2) & M_i^\top P \\ \star & -P & \mathbf{0} \\ \star & \star & -P \end{bmatrix} \prec 0 \quad \forall i \in \{1,\ldots,v\} \tag{2.44}$$

It is worth pointing that the matrices $\{P, G_1, G_2, L\}$ are the decision variables for this set of BMIs (2.44) with the constraints (2.35b)-(2.35d).
To sum up, the second contribution of this chapter about the design of the observer gain, is stated in the following proposition.

Proposition 2.13. *Let Assumption 2.1 and 2.2 hold. If there exist two nonnegative matrices G_1, $G_2 \in \mathbb{R}^{n\times n}$, and a matrix $L \in \mathbb{R}^{n\times p}$ such that the set of BMIs (2.44) and the constraints (2.35b)-(2.35d) are feasible, then the interval observer of the form (2.2), (2.3) and (2.6) for the system (2.1) has; (i) a GES estimation*

error relatively to the set $\mathcal{A}$ in the ideal case (system without noise and perturbation), (ii) an ISS estimation error relatively to the set $\mathcal{A}$ in the case of noisy and perturbed system.

Proof of Proposition 2.13. The proof is given in the previous lines by applying both positive realization method and projection lemma. □

2.4 Illustrative examples

In order to illustrate the performance of the proposed observer, we consider the following examples

2.4.1 Spring-mass-damper system

Consider the LTI system of the Spring-mass-damper system. Let p, κ_v, and f be the position, the velocity, and the force applied to the object, respectively. We have

$$\begin{aligned} \dot{p}(t) &= \kappa_v(t) + d_1(t) \\ m_s\dot{\kappa_v}(t) &= -k_s p(t) - c_s\kappa_v(t) + f(t) + d_2(t) \end{aligned}$$

where m_s, k_s and c_s stand for the mass of the object, the stiffness constant of the spring, and the damping ratio, respectively. $d(t) = [d_1(t)\ d_2(t)]^\top$ is the state disturbance. Defining $x_1 = p$, $x_2 = \kappa_v$, and $u = f$, with the practical parameters of the system which are given as $m_s = 1kg$, $k_s = 1N/m$ and $c_s = 1Ns/m$, one can introduce its LTI model as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + d(t)$$

where $d(t)$ is unknown-but-bounded as $-\overline{d} \leq d(t) \leq \overline{d}$ such that $\overline{d} = [1\ 1]^\top$, and the output is given by $y(t_k) = 2x_1(t_k) + v(t_k)$ where $v(t_k) = 0.4\cos(2t_k)$. Note that this system is not cooperative. So the solution that we have proposed is to use its IPR. Then, the dynamics of the interval estimation error between two consecutive

measurements in (2.4) is defined by the following matrices

$$A^M = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad A^N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that although the plant model is stable, the observer dynamics in between measurements is unstable. To illustrate the performance of the observer, we choose the time interval $[\tau_{min},\ \tau_{max}] = [0.04,\ 0.3]$, then we solve the feasibility problem given in Proposition 2.13. The set of BMIs (2.44) is solved using the YALMIP toolbox (Lofberg, 2004) based on the Penlab (v1.04) solver. The observation gain matrix is obtained as $L^\top = [-0.5004\ -0.0494]$. The implementation is made with matrix

$$(I_2 + LC) = G_1 - G_2 = \begin{bmatrix} -0.0008 & 0 \\ -0.0987 & 1 \end{bmatrix}.$$

The measurement times are generated randomly in the time interval $[\tau_{min},\ \tau_{max}]$. The system input is taken as $u(t) = 10[2\sin(10t) + \sin(16t) + \sin(24t)]$. The simulation results are given in Figure 2.1 and 2.2. In Figure 2.1, it is noticeable that the jump part of the interval impulsive observer prevents the estimate bounds from diverging, even though the open-loop dynamics of the interval estimator error is unstable.

The case of system without perturbation and noise In order to show the ideal case when the dynamics of the system and the available measurement are considered perfect (e.i., $d(t) = [0\ 0]^\top$ and $v(t_k) = 0$), we choose for the observer the bounds of perturbation and noise as $\overline{d} = [0\ 0]^\top$ and $\overline{v} = 0$, respectively. In Figure 2.2, the estimates converge exponentially to the actual state of the system, and the Lyapunov function of the state estimation error shows a faster convergence at jumps while stays almost constant when flowing.

TABLE 2.1: The relation between τ_{max} and the damping ratio.

c_s	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3
τ_{max}	1.1	1	0.9	0.8	0.5	0.3	0.05	–

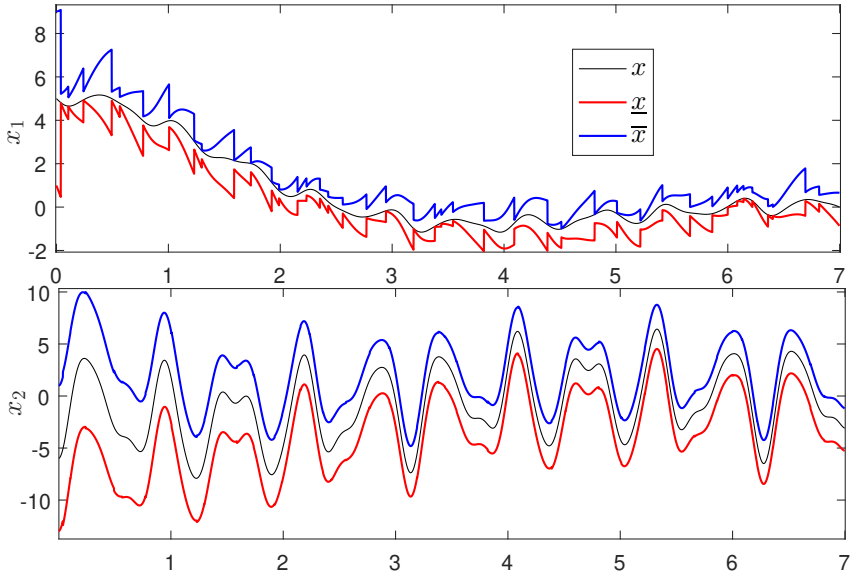


FIGURE 2.1: Simulation results for the spring-mass-damper system: the estimate bounds for the position (top), and velocity (bottom).

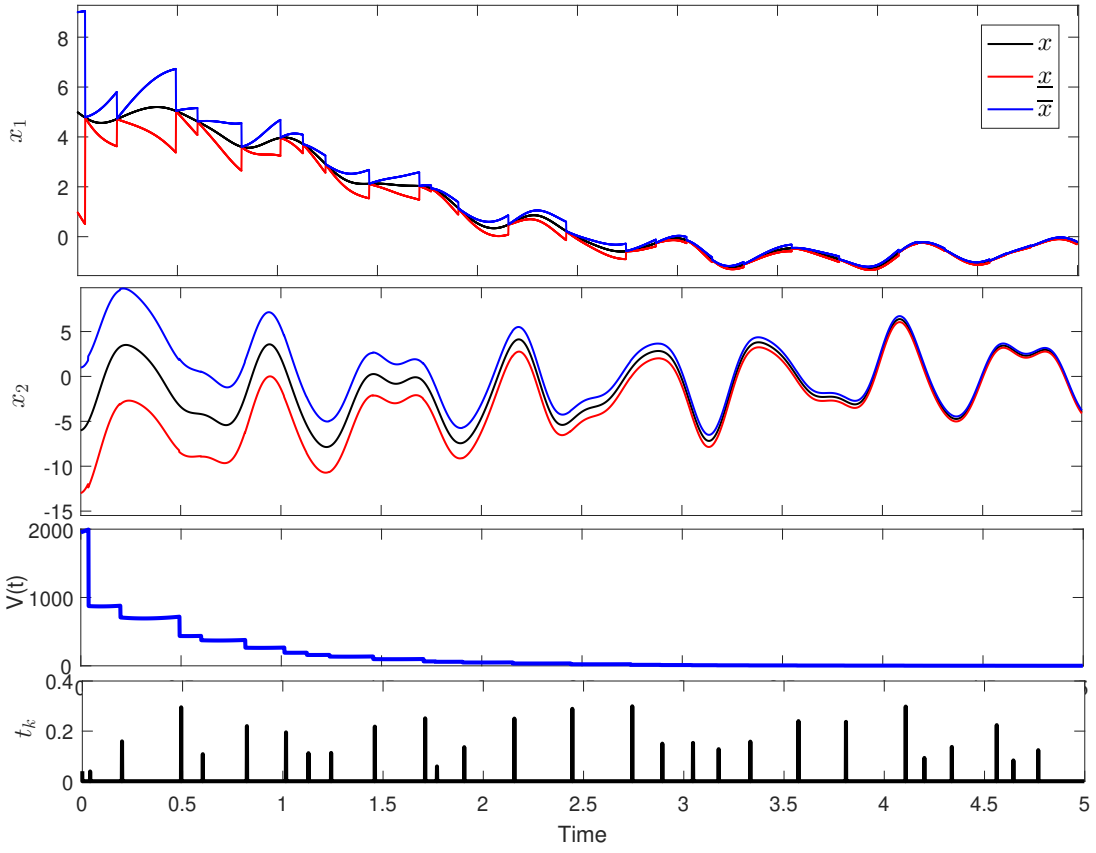


FIGURE 2.2: System without perturbations and noises : from top to bottom; state x_1, x_2, Lyapunov function of the error, and the inter-measurement times.

In order to show the maximum allowable inter-measurement time on the problem feasibility, we vary the damping ratio c_s which in turn varies the eigenvalues of the flow part. The maximum inter-measurement time τ_{max} that renders the design problem feasible, for different values of c_s, is given in Table 2.1. One can notice that the feasibility depends on the maximum inter-measurement time τ_{max}. The correction jumps are made to prevents the prediction from diverging, so the larger the divergence rate of the prediction error, the smaller feasible τ_{max}. For damping ratio $c_s \leq 0.3$, it was not possible to find a gain L that renders the eigenvalues of the equivalent discrete system (see Remark 2.10) in the restricted region of the complex plane given in Cacace et al. (2015).

2.4.2 Academic example: an unstable system

Consider the LTI system governed by equation (2.1) with

$$A = \begin{bmatrix} 0.5 & 0.6 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \end{bmatrix},$$

the input signal is the same as Example 1 and the measurement noise is $v(t_k) = 0.4\cos(2t_k) \leq \overline{v} = 0.4$ for each $k \in \mathbb{N}$. The system dynamics is clearly non-cooperative and unstable. The matrices A^M and A^N in (2.2) are obtained as follows

$$A^M = \begin{bmatrix} 0.5 & 0.6 \\ 0 & -2 \end{bmatrix}, \quad A^N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

One can see that the prediction part of the impulsive observer, which is based on the matrices A^M and A^N, has not the same stability properties of the original system. The observer gain matrix L is calculated by solving the set of BMIs (2.44). A feasible solution is obtained as $G_1 - G_2 = I + LC = \begin{bmatrix} -0.0016 & 0 \\ 0.0468 & 1.0000 \end{bmatrix}$ with $L^T = [-0.6678\ 0.0312]$.

The simulation result is presented in Figure 2.3 and Figure 2.4. The upper and lower estimate of the actual state are presented in Figure 2.3. Figure 2.4-(top) shows the Lyapunov function of the estimation error. Figure 2.4-(bottom) presents the measurement times, where the signal magnitude represents the time elapsed

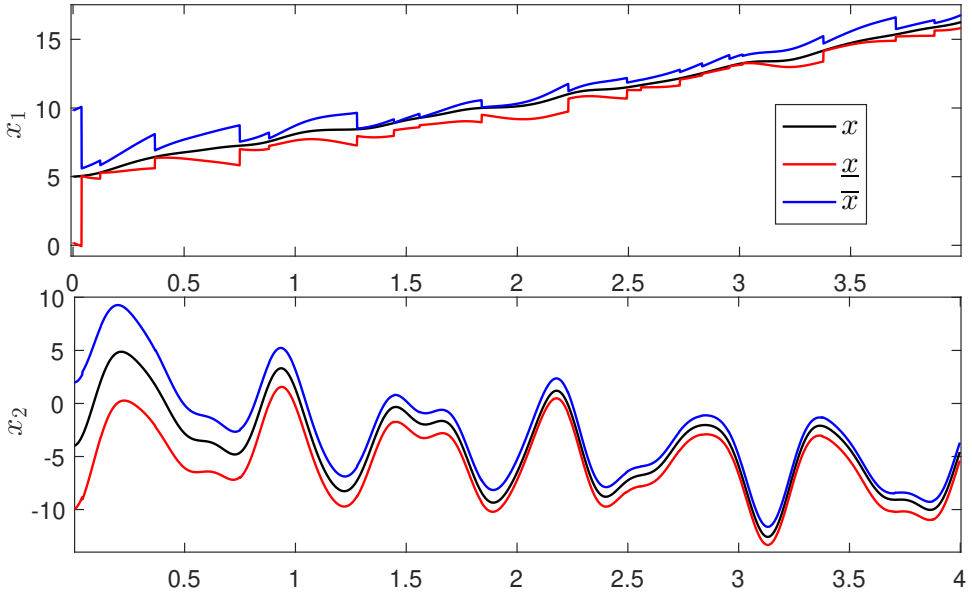


FIGURE 2.3: Simulation results: the estimate of the state; upper (blue) and lower (red) bounds.

after the last measurement. These inter-measurements intervals are chosen randomly in the interval $[\tau_{min},\ \tau_{max}] = [0.04,\ 0.4]$. The Lyapunov function shows a behavior similar to what described in Theorem 2.9; its value decreases faster at measurement times (at *jump*) compared to the prediction period (during *flow*). This simulation result shows that although the system is unstable, the proposed observer can estimate its state.

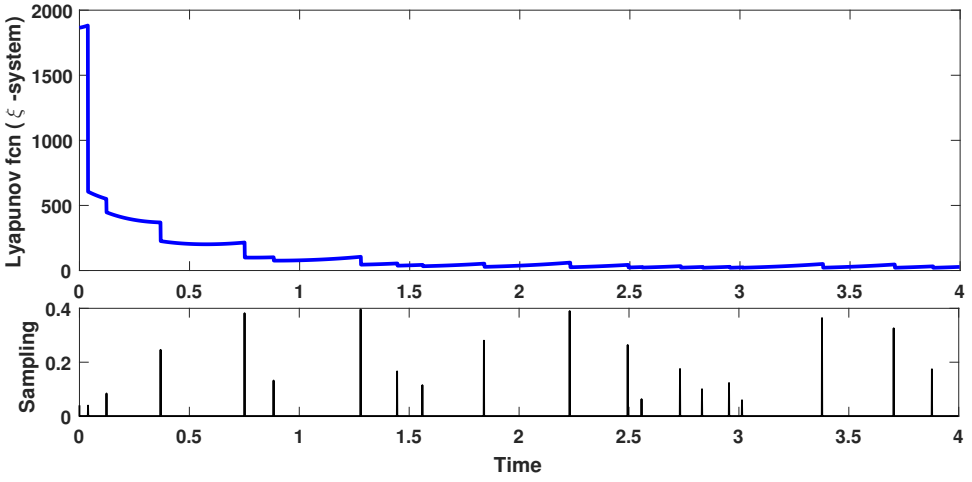


FIGURE 2.4: Simulation results: the Lyapunov function of the estimation error (top), the inter-measurement times (bottom).

2.5 Conclusion

In this chapter, the problem of interval observers synthesis for linear continuous-time systems with aperiodic discrete measurement has been investigated. The systems and the measurements have been assumed perturbed by unknown-but-bounded disturbances and noises, respectively. To deal with this problem, a new interval impulsive observer has been introduced. Exploring the internal positivity of the system, a new method has been proposed to design the observer gain. The synthesis of the observation gain is performed using BMIs. Simulation examples show the efficiency of the proposed interval impulsive observer for a class of linear perturbed systems.

Chapter 3

Finite-gain $\mathcal{L}_1$ Event-triggered Interval Observers for LTI Continuous-time Systems

3.1 Motivations and related works

Immense effort has been made to enhance the performance of networked control systems (NCS) while ensuring the minimal use of the (shared) network. To reach that objective, alternative approaches to the traditional periodic data-sampling method were proposed, such as, the aperiodic data-sampling scheme where sampling is allowed within a predetermined time interval (Hetel et al., 2017), and the event/self-triggered sampling strategy (Tabuada, 2007; Heemels et al., 2012; Girard, 2015) where the sampling time instant depends on the behaviour of the system.

In this work, we propose the design of interval observers based on an event-triggering mechanism allowing to use measurement only when necessary. The event-based estimation can relax the regularity assumption on the availability of the measurements commonly used by the traditional state estimation methods. Thanks to this mechanism, the amount of data transmitted over the network can be reduced considerably.

In the event-based context, point-valued estimation approaches have been proposed for discrete-time systems (Muehlebach and Trimpe, 2017; Huang et al.,

2019) and continuous-time systems (Etienne and Gennaro, 2016; Huang et al., 2017), to cite a few. The set-valued (set-membership) estimation has also been investigated (Shi et al., 2014; Silvestre et al., 2018). Shi et al. (2014) have proposed an event-based estimation approach based on the set-valued Kalman filter for systems with multiple outputs, where the set-valued measurements, the initial state and the uncertainties are assumed Gaussian. Silvestre et al. (2018) have proposed event and self-triggered strategies based on Set-Valued Observers. Both aforementioned works are dedicated to discrete-time systems. However, the set-membership state estimation problem for continuous-time systems has not been fully investigated in the literature. Some preliminary works have considered the self-triggered (Meslem and Prieur, 2014) and the event-based (Rabehi et al., 2017) sampling of measurements using the prediction-correction approaches. Note that, in both cited works, only the intrinsic structure properties of the systems are exploited in order to design converging set-membership state estimators. In this work, we propose a new structure of interval observers, where the correction is carried out at discrete-time instants. Furthermore, here, the feasible sets of the system initial state and the disturbances are taken as interval vectors (boxes) instead of ellipsoids (Shi et al., 2014) and polytopes (Silvestre et al., 2018).

The main advantage of the proposed observation approach is in the correction stage which is based on a pre-calculated observation gain that ensures the stability of the estimation error along with some performance specifications. The latter is achieved using Input-Output Stability (IOS) analysis, in particular the $\mathcal{L}_p$ stability concept (Khalil, 2002).

The $\mathcal{L}_p$-gain concept is an interesting approach for analyzing the stability along with the performance of dynamical systems. This concept has been already applied to evaluate the performance of interval observers (Chebotarev et al., 2015; Briat and Khammash, 2016). For instance, interval observers have been proposed for LPV systems with $\mathcal{L}_1/\mathcal{L}_2$ performance analysis by Chebotarev et al. (2015). Interval observers providing tight state enclosures have been designed using peak-to-peak approach $\mathcal{L}_\infty - \mathcal{L}_\infty$ in (Briat and Khammash, 2016) for discrete-time and continuous-time systems.

In this chapter, we combine the $\mathcal{L}_1$-gain approach for positive systems (Briat, 2013; Ebihara et al., 2011) and the $\mathcal{L}_p$-gain approach for hybrid systems (Nešić et al., 2013) to study the stability and the performance of the proposed event-triggered

interval observers viewed as impulsive systems. The positivity of the estimation error is guaranteed by using the internal positivity of the system. Furthermore, the measurements are not continuously available and a triggering mechanism is designed to request measurement only when needed to enhance the accuracy of the estimates, that is, the measurements are requested whenever a performance criterion, involving the width of the feasible domain of the system's uncertainties and the width of the estimated state enclosures, is violated. Thus, the novelty of this chapter is the co-design of the event-triggered mechanism for measurement sampling and the observer gains that ensure the convergence of the width of the estimated state enclosures.

The remainder of this chapter is organized as follows. Some preliminaries are given in Section 3.2. The structure of the interval observers for continuous-time linear systems and the design of the event-triggered mechanism with a guaranteed finite-gain $\mathcal{L}_1$ performance are proposed in Section 3.3. The co-design of the event-triggered mechanism and the interval observer gain for continuous-time linear systems is given in Section 3.4 which differs from Section 3.3 where we assume that the interval observer gains are given a priori. An illustrative numerical example shows the efficiency of the proposed approach in Section 3.5.

3.2 Preliminaries

3.2.1 Definitions

Definition 3.1. (Simple version of the S-procedure) (Boyd et al., 1994, Section 2.6.3) Let F_0, F_1 be functions (quadratic or linear) of the variable $z \in \mathbb{R}^n$. Consider the following condition on F_0, F_1 :

$$F_0(z) \geq 0 \quad \text{for all } z \quad \text{such that} \quad F_1(z) \geq 0$$

The above condition holds if there exists $\zeta \geq 0$ such that

$$\forall\ z, \quad F_0(z) - \zeta F_1(z) \geq 0.$$

Definition 3.2. (Moisan et al., 2009) Consider the system $\dot{x}(t) = f(x(t))$. If there exist two functions $\underline{f}$, $\overline{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then the following coupled dynamical systems

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \overline{x}(t)) \\ \dot{\overline{x}}(t) = \overline{f}(\underline{x}(t), \overline{x}(t)) \end{cases} \tag{3.1}$$

is a framer of this system, that is, the state trajectories satisfy $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$, provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$.

Remark 3.3. In the case of linear cooperative systems, a framer can be designed using copies of the system (i.e., $\underline{f}(\underline{x}, \overline{x}) = A\underline{x}$ and $\overline{f}(\underline{x}, \overline{x}) = A\overline{x}$ for the lower and the upper bounding system, respectively). In the case of linear non-cooperative systems, the two functions $\underline{f}, \overline{f}$ in (3.1) could be calculated using methods introduced in (Ramdani et al., 2009; Cacace et al., 2015) based on the Müller's existence theorem (Müller, 1927).

It is important to point out the fact that a framer is conceived to give an upper and a lower bounds for the unknown state. Stability can be considered as an additional feature for a framer to guarantee the convergence of the bounds to the system state and thus obtaining an interval observer.
In this chapter, we combine the notion of framer for continuous-time and discrete-time systems to obtain a framer for the so-called *impulsive (or hybrid)* system. Consider the following hybrid system:

$$\begin{cases} \dot{x} = f(x, d), & \forall (x, d) \in \mathcal{C} \\ x_{t_k^+} = g(x, d), & \forall (x, d) \in \mathcal{D} \\ y = h(x, d), \end{cases} \tag{3.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the state vector, the exogenous input and the output vector, respectively, and $d \in \mathbb{R}^{n_d}$ represents the exogenous input assumed to be unknown-but-bounded. $x_{t_k^+} = \lim_{t \to t_k^+} x(t)$ refers to the state variable after jumps. The following definition generalizes the definition of framer of discrete-time and continuous-time systems.

Definition 3.4. If there exist functions $\underline{f}, \overline{f}, \underline{g}, \overline{g} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_y} \to \mathbb{R}^n$ and $\underline{d}, \overline{d} \in \mathbb{R}^{n_d}$ such that the inclusions

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t), \qquad \forall t \geq 0\,, \tag{3.3a}$$

$$\underline{f}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y) \leq f(x, d) \leq \overline{f}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y), \qquad \forall (x, d) \in \mathcal{C}\,, \tag{3.3b}$$

$$\underline{g}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y) \leq g(x, d) \leq \overline{g}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y), \qquad \forall (x, d) \in \mathcal{D}\,. \tag{3.3c}$$

are satisfied, then the solution of the system

$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y), & \forall (x, d) \in \mathcal{C} \\ \dot{\overline{x}} = \overline{f}(\underline{x}, \overline{x}, \underline{d}, \overline{d}), & \\ \underline{x}_{t_k^+} = \underline{g}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y), & \forall (x, d) \in \mathcal{D} \\ \overline{x}_{t_k^+} = \overline{g}(\underline{x}, \overline{x}, \underline{d}, \overline{d}, y), & \\ \underline{x}(0) \leq x(0) \leq \overline{x}(0) & \end{cases} \tag{3.4}$$

is a *framer* for the system (3.2), which means $\underline{x}(t) \leq x(t) \leq \overline{x}(t) \quad \forall t \geq 0$.

Remark 3.5. In the linear case of a non-cooperative system (3.2) where

$$\begin{cases} f(x, d_f) = Ax + d_f, \\ g(x, d_g) = Jx + d_g, \end{cases}$$

based on the assumption that there exist $\underline{d}_f$, $\overline{d}_f$, $\underline{d}_g$, $\overline{d}_g$ satisfying $\underline{d}_f \leq d_f \leq \overline{d}_f$ and $\underline{d}_g \leq d_g \leq \overline{d}_g$, and using the method based on Müller's theorem, the bounding system (3.4) could be defined as follows

$$\forall x \in \mathcal{C} : \begin{cases} \dot{\underline{x}} = A_1 \underline{x} - A_2 \overline{x} + \underline{d}_f, \\ \dot{\overline{x}} = A_1 \overline{x} - A_2 \underline{x} + \overline{d}_f, \end{cases}$$

$$\forall x \in \mathcal{D} : \begin{cases} \underline{x}_{t_k^+} = J_1 \underline{x} - J_2 \overline{x} + \underline{d}_g, \\ \overline{x}_{t_k^+} = J_1 \overline{x} - J_2 \underline{x} + \overline{d}_g, \end{cases}$$

where A_1 is any Metzler matrix and the matrices A_2, J_1 and J_2 are any nonnegative matrices satisfying $A = A_1 - A_2$ and $J = J_1 - J_2$. These conditions guarantee the

nonnegativity of the estimation errors $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x$.

Definition 3.6. The impulsive system (3.4) is an interval observer for the system (3.2) if it is a framer for system (3.2) and the dynamics of the estimation errors $\underline{e} = x - \underline{x}$, $\overline{e} = \overline{x} - x$ are stable.

The stability of impulsive systems can be studied using, for example, Lyapunov techniques, or dissipativity approaches (Haddad et al., 2014). In this work, we adopt the $\mathcal{L}_p$ stability approach developed for hybrid systems (Nešić et al., 2013).

3.2.2 $\mathcal{L}_p$ stability: Input-Output sense

An Input-Output model relates the output of the system directly to its input, with no knowledge of the internal structure that can be represented by the state-space model. The $\mathcal{L}_p$ stability is a concept of stability in the input-output sense (for details see (Khalil, 2002, Chapter 5)). To exploit this stability analysis for hybrid systems, we denote by $||z||_p$, the $\mathcal{L}_p$-norm for the hybrid signal (arc) z as is defined by Nešić et al. (2013).

Definition 3.7 ($\mathcal{L}_p$-norm)**.** For a hybrid signal z, with domain $\text{dom}(z) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, and a scalar $T \in \mathbb{R} \geq 0$, the T-truncated $\mathcal{L}_p$-norm of z is given by

$$||z_{[T]}||_p := \left(\sum_{i=1}^{j(T)} |z(t_i, i-1)|^p + \sum_{i=0}^{j(T)} \int_{t_i}^{\sigma_i} |z(s,i)|^p ds \right)^{\frac{1}{p}} \tag{3.5}$$

where $t_0 = 0$, $j(T) = \max\{k : (t,k) \in \text{dom}(z),\ t + k \leq T\}$, and, $\forall i \in \{0, \ldots, j\}$, $\sigma_i = \min(t_{i+1}, T - i)$. Based on (3.5), the $\mathcal{L}_p$-norm of z is defined as

$$||z||_p = \lim_{T \to T^*} ||z_{[T]}||_p, \tag{3.6}$$

where $T^* = \sup\{t + j : (t,j) \in \text{dom}(z)\}$. Moreover, we have that $z \in \mathcal{L}_p$ whenever the above limit exists and is finite.

Remark 3.8. In (3.5), the value of the hybrid arc just before the jump $z(t_i, i-1)$ is considered. An alternative definition can be given in terms of the values of the hybrid arc after the jump, that is, $z(t_i, i)$.

The $\mathcal{L}_p$-norm for hybrid signals in Definition 3.7 is a generalized definition of the $\mathcal{L}_p$-norm for continuous-time signals (Khalil, 2002) and the l_p-norm for discrete-time signals (Vidyasagar, 2002). This norm can be simplified for particular cases; if the signal $z(t, j)$ is a continuous-time signal then the first part of the right-hand side of expression (3.5) equals zero, if $z(t, j)$ is discrete-time signal then the second part of the right hand side of (3.5) equals zero.

Remark 3.9. For multidimensional signals $z \in \mathbb{R}^{n_z}$, we write $z \in \mathcal{L}_p^{n_z}$, where $\mathcal{L}_1$-space is defined as the set of absolute-value integrable signals, the $\mathcal{L}_2$-space is defined as the set of square integrable signals, and the $\mathcal{L}_\infty$-space is defined as the set of signals which are bounded in amplitude. Note that the $\mathcal{L}_p^{n_z}$-norm of vector signals, $||z||_p$ as defined in (3.6) differs from the $p-$norm of vectors which is a norm at time t defined as $|z(t)|_p = \left(\sum_{i=1}^{n_z} |z_i(t)|^p\right)^{\frac{1}{p}}$. For instance the $1-$norm of the vector $z(t)$ is denoted by $|z(t)|_1$ which will be often used in the sequel.

The general definition of the input-output stability of hybrid systems that combines the $\mathcal{L}_p$ stability (Khalil, 2002) and the l_p stability (Vidyasagar, 2002), respectively for continuous-time and discrete-time systems is given in the following.

Definition 3.10. Given $p \in [1, +\infty)$. System (3.2) is finite-gain $\mathcal{L}_p$ stable from d to y with gain (upper bounded by) $\gamma_p \geq 0$, if there exists a scalar $\beta \geq 0$ such that any solution to (3.2) satisfies

$$||y||_p \leq \beta|x(0,0)|_p + \gamma_p||d||_p \tag{3.7}$$

for all $d \in \mathcal{L}_p^{n_d}$.

The $\mathcal{L}_p$ stability characterizes the input-to-output stability of dynamical systems. This characterization can be obtained using Lyapunov methods. The Lyapunov function in this context is called *storage function* which is defined next for hybrid systems.

Definition 3.11. (Nešić et al., 2013) Given $p \in [1, +\infty)$, a positive semi-definite continuously differentiable function $V : \mathbb{R}^{n_V} \to \mathbb{R}_{\geq 0}$ is a finite-gain $\mathcal{L}_p$ storage function for the system (3.2) if there exist positive constants c_2, γ_{yf} and γ_{yg}, and

nonnegative constants γ_{dg}, γ_{df}, such that

$$0 \leq V(x) \leq c_2|x|^p, \qquad \forall (x,d) \in \mathcal{C} \cup \mathcal{D}, \tag{3.8a}$$

$$\langle \nabla V(x), f(x,d) \rangle \leq -\gamma_{yf}|h(x,d)|^p + \gamma_{df}|d|^p, \qquad \forall (x,d) \in \mathcal{C}, \tag{3.8b}$$

$$V(g(x,d)) - V(x) \leq -\gamma_{yg}|h(x,d)|^p + \gamma_{dg}|d|^p, \qquad \forall (x,d) \in \mathcal{D}. \tag{3.8c}$$

Based on Definition 3.11, the $\mathcal{L}_p$ stability of the hybrid system (3.2) is set in the following proposition.

Proposition 3.12. *(Nešić et al., 2013) Consider system (3.2), and suppose that there exists a function V that satisfies (3.8). Then the system is finite gain $\mathcal{L}_p$ stable, and the gain of the operator $d \to y$ is upper bounded by $\gamma_p = \sqrt[p]{\gamma_d/\gamma_y}$, where $\gamma_d = \max\{\gamma_{df}, \gamma_{dg}\}$ and $\gamma_y = \min\{\gamma_{yf}, \gamma_{yg}\}$.*

In Definition 3.10 and 3.11, we have presented the existing results about the $\mathcal{L}_p$ stability of hybrid systems (Nešić et al., 2013). Next, we will use the finite-gain $\mathcal{L}_1$ stability for hybrid systems, which is a special case of $\mathcal{L}_p$ stability when $p = 1$. In the following section, we will exploit these results to analyze and design event-triggered interval observers with $\mathcal{L}_1$-gain performance for continuous-time linear systems.

3.3 Event-triggered interval observer for linear systems

Consider the linear time invariant system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t_k) = Cx(t_k) + Fd(t_k), \qquad k \in \mathbb{N} \end{cases} \tag{3.9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ is the state variables, the input, the output of the system, respectively, and $d \in \mathbb{R}^{n_d}$ represents the exogenous input assumed to be unknown-but-bounded with *a priori* know bounds. The boundedness assumption on the state and output disturbances is a standard assumption in interval estimation context.

Assumption 3.1. *Let two vectors* $\overline{d}(t),\ \underline{d}(t) \in \mathbb{R}^{n_d}$ *be given such that*

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t) \tag{3.10}$$

is verified $\forall t \geq 0$.

Remark 3.13. The exogenous input $d(t)$ in the system (3.9) gathers the measurements noises and the system disturbances with adapted matrices E and F.

In this section, the goal is to estimate an upper $\overline{x}$, and a lower bound $\underline{x}$ for the actual state of the system (3.9). More precisely, the aim is to compute a guaranteed enclosure of the set of admissible values for the actual state vector of the disturbed system. The measurements are supposed neither continuous nor periodic but are taken according to desired performance specifications on the estimation that will be introduced later. The advantage of this technique is two-fold: first, in networked systems, it may reduce the communication rate between the computers and the sensors, second, it can provide an estimate of the system state with only few sensor data.

The proposed interval observer includes two dynamic behaviors; the first part concerns the estimation without feedback information from the system, i.e., without measurement, and the second part improves the accuracy of the estimated state enclosure when a measurement is available.

The first part of the interval observer is proposed as follows:

$$\begin{cases} \dot{\underline{x}}(t) = A^M \underline{x}(t) - A^N \overline{x}(t) + Bu(t) \\ \qquad\quad + E^+ \underline{d}(t) - E^- \overline{d}(t) \\ \dot{\overline{x}}(t) = A^M \overline{x}(t) - A^N \underline{x}(t) + Bu(t) \\ \qquad\quad + E^+ \overline{d}(t) - E^- \underline{d}(t) \end{cases} \quad \forall t \in [t_k,\ t_{k+1}],\ \forall k \in \mathbb{N} \tag{3.11}$$

where $A^M = d_A + (A - d_A)^+$ and $A^N = A^M - A$ with d_A is a diagonal matrix contains only the diagonal elements of A, with the initial values

$$x(0) \in [\underline{x}(0), \overline{x}(0)] \tag{3.12}$$

The correction part at the time $t = t_k$ is introduced by the following discrete-time system:

$$\begin{cases} \underline{x}(t_k^+) = (I_n + LC)^+ \underline{x}(t_k) - (I_n + LC)^- \overline{x}(t_k) \\ \qquad\qquad + (LF)^+ \underline{d}(t_k) - (LF)^- \overline{d}(t_k) - Ly(t_k) \\ \overline{x}(t_k^+) = (I_n + LC)^+ \overline{x}(t_k) - (I_n + LC)^- \underline{x}(t_k) \\ \qquad\qquad + (LF)^+ \overline{d}(t_k) - (LF)^- \underline{d}(t_k) - Ly(t_k) \end{cases} \quad \forall k \in \mathbb{N} \tag{3.13}$$

Using the output model in (3.9), the system state can be re-written as

$$\begin{aligned} x(t_k^+) &= x(t_k) + L[Cx(t_k) + Fd(t_k) - y(t_k)] \\ &= (I + LC)x(t_k) + LFd(t_k) - Ly(t_k) \\ &= \left[(I + LC)^+ - (I + LC)^-\right] x(t_k) + \left[(LF)^+ - (LF)^-\right] d(t_k) - Ly(t_k) \end{aligned} \tag{3.14}$$

It worth noting that in (3.14), we have $x(t_k^+) = x(t_k)$ because $Cx(t_k) + Fd(t_k) - y(t_k) = 0$. Equation (3.14) is only a generic equation to find the estimation error at measurement time instants.

Similarly, without loss of generality, we can split the state matrix A of the system (3.9) into a Metzler A^M and a nonnegative A^N part, and split the perturbation matrix E into positive E^+ and negative $-E^-$ parts as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\ &= (A^M - A^N)x(t) + Bu(t) + (E^+ - E^-)d(t) \end{aligned}$$

The dynamics of the estimation error bounds $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ can be described as follows:

From (3.9) and (3.11), one has

$$\begin{bmatrix} \dot{\underline{e}}(t) \\ \dot{\overline{e}}(t) \end{bmatrix} = \mathcal{M}(A) \begin{bmatrix} \underline{e}(t) \\ \overline{e}(t) \end{bmatrix} + \tilde{E}\psi(t) \tag{3.15}$$

where

$$\mathcal{M}(A) = \begin{bmatrix} A^M & A^N \\ A^N & A^M \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} E^+ & E^- \\ E^- & E^+ \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} d(t) - \underline{d}(t) \\ \overline{d}(t) - d(t) \end{bmatrix} \tag{3.16}$$

And from (3.13) and (3.14) one obtains

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(L) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \tilde{F}(L)\psi(t_k) \tag{3.17}$$

where

$$\Gamma(L) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix}, \quad \tilde{F}(L) = \begin{bmatrix} (LF)^+ & (LF)^- \\ (LF)^- & (LF)^+ \end{bmatrix}.$$

Remark 3.14. The choice of matrices A^M and A^N in (3.11) is based on the Müller's existence theorem. In general, one can pick any Metzler matrix A^M and nonnegative matrix A^N that satisfy the constraint $A = A^M - A^N$. This choice guarantees the Metzler property of the matrix $\mathcal{M}(A)$ in (3.15).

Remark 3.15. The matrices $\tilde{E}$, $\tilde{F}(L)$ and $\Gamma(L)$ are structurally nonnegative. The matrix $\mathcal{M}(A)$ is Metzler. Thus, if Assumption 3.1 is satisfied then the dynamics (3.15) and (3.17) are nonnegative.

3.3.1 Event-triggered interval Observer formulation

In this subsection, we study the interval observer under the event-triggered mechanism (ETM) that we propose next.
Consider the augmented vector of the estimation errors given by

$$\xi = [\underline{e}^\top, \overline{e}^\top]^\top. \tag{3.18}$$

The dynamics (3.15)-(3.17) can be viewed as a hybrid system defined by

$$\begin{cases} \dot{\xi}(t) = \mathcal{M}(A)\xi(t) + \tilde{E}\psi(t) & \forall \xi \in \mathcal{C} \\ \xi(t_k^+) = \Gamma(L)\xi(t_k) + \tilde{F}(L)\psi(t_k) & \forall \xi \in \mathcal{D} \end{cases} \tag{3.19}$$

Before providing the sets $\mathcal{C}$ and $\mathcal{D}$ that represent the triggering mechanism, we define the width of the estimated state enclosure and the width of the feasible domain of uncertainties, respectively, as follows:

$$\begin{aligned} \omega(t) &= \overline{x}(t) - \underline{x}(t) = \overline{e}(t) + \underline{e}(t) \\ \delta(t) &= \overline{d}(t) - \underline{d}(t) \end{aligned} \tag{3.20}$$

Let us now define the flow and jump sets for the system (3.19) as

$$\begin{aligned} \mathcal{C} &= \{(\omega, \delta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \ : \ |\omega|_1 \leq \ \beta|\delta|_1\} \\ \mathcal{D} &= \{(\omega, \delta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \ : \ |\omega|_1 \geq \ \beta|\delta|_1\} \end{aligned} \tag{3.21}$$

where β is a positive scalar to be tuned.

Remark 3.16. Note that the variable $\delta(t)$ is assumed to be known a priori and the estimate width $\omega(t)$ for all $t \in [t_k, t_{k+1}]$ can be estimated from the last measurement $y(t_k)$. Consequently, we can consider that the proposed triggering mechanism (3.21) as an *implicit self-triggering mechanism*. It is implicit because the triggering time t_{k+1} is not given explicitly.

In order to analyze the stability of the estimation errors, we use the $\mathcal{L}_1$ stability (a particular case of the $\mathcal{L}_p$ stability introduced in Definition 3.10 with $p = 1$). To this aim, we consider the $\mathcal{L}_1$-gain of the operator $\delta \to \omega$, in other words, the $\mathcal{L}_1$-gain from the width of the known interval of the exogenous input $d(t)$ to the width of the estimated enclosure of $x(t)$. Note that the $\mathcal{L}_1$-gain of the operator $\delta \to \omega$ is the same $\mathcal{L}_1$-gain of the operator $\psi \to \xi$. This property is true if and only if the variables δ in (3.20), ω in (3.20), ψ in (3.16) and ξ in (3.18) are nonnegative, which is structurally satisfied in the interval estimation context. From (3.20) one can simply get

$$\omega(t) = \begin{bmatrix} I_n & I_n \end{bmatrix} \xi(t); \quad \delta(t) = \begin{bmatrix} I_{n_d} & I_{n_d} \end{bmatrix} \psi(t). \tag{3.22}$$

Finding an expression of an upper bound $\gamma_{\mathcal{L}_1}$ of the $\mathcal{L}_1$-gain of the operator $\delta \to \omega$ allows to compute this upper bound in a way to minimize the effect of the uncertainty width δ onto the state estimate width ω. Hence, we can obtain an interval estimate as tight as possible.

To simplify the computation procedure in the sequel, we use the $\mathcal{L}_1$-gain of the operator $\psi \to \xi$ instead of using the $\mathcal{L}_1$-gain of the operator $\delta \to \omega$. An equivalence of these two operators is based on the following equalities $|\delta(t)|_1 = |\psi(t)|_1$ and $|\omega(t)|_1 = |\xi(t)|_1$, which is detailed in Appendix A.1.
The triggering condition in (3.21), defined as $|\omega(t)|_1 \geq \beta|\delta(t)|_1$, is equivalent to

$$\begin{bmatrix} \xi(t) \\ \psi(t) \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta\mathbb{1}_{2n_d} \end{bmatrix} \geq 0.$$

Thus, the flow and jump sets (3.21) can be written as follows

$$\begin{aligned} \mathcal{C} &= \left\{ (\xi, \psi) \ : \ \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \right\} \\ \mathcal{D} &= \left\{ (\xi, \psi) \ : \ \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta\mathbb{1}_{2n_d} \end{bmatrix} \geq 0 \right\} \end{aligned} \tag{3.23}$$

Remark 3.17. In the evaluation of the triggering conditions we use (3.21) instead of (3.23) because we cannot evaluate the estimation error $\xi(t)$ which depends on the actual state $x(t)$. But we have always the width of estimation error $\omega(t)$ which depends only on the upper and the lower estimates $\overline{x}(t)$ and $\underline{x}(t)$, respectively. The same argument for the input side (i.e., $d(t)$).

Let us now state the next contribution of this chapter. The following theorem provides a design methodology of the ETM (3.23).

Theorem 3.18. *Let Assumption 3.1 hold. For a given matrix $L \in \mathbb{R}^{n\times n_y}$, if there exist a nonnegative vector $\lambda \in \mathbb{R}^{2n}_{\geq 0}$, and nonnegative scalars ζ_c, $\zeta_\mathcal{D}$, $\gamma_{\delta f}$, $\gamma_{\delta g}$, $\gamma_{\omega f}$, $\gamma_{\omega g}$ and β, satisfying the following inequalities*

$$\mathcal{M}^\top(A)\lambda + (\gamma_{\omega f} - \zeta_c)\mathbb{1}_{2n} \leq 0 \tag{3.24a}$$

$$\tilde{E}^\top\lambda - (\gamma_{\delta f} - \zeta_c\beta)\mathbb{1}_{2n_d} \leq 0 \tag{3.24b}$$

$$\Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_\mathcal{D})\mathbb{1}_{2n} \leq 0 \tag{3.24c}$$

$$\tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_\mathcal{D}\beta)\mathbb{1}_{2n_d} \leq 0 \tag{3.24d}$$

then, the system (3.11)-(3.13) *with the event-triggering mechanism* (3.21) *is a finite* $\mathcal{L}_1$*-gain interval observer for the system* (3.9)*. Furthermore, the* $\mathcal{L}_1$*-gain from* δ *to* ω *is upper bounded by* $\gamma_{\mathcal{L}_1} = \gamma_\delta/\gamma_\omega$ *where* $\gamma_\delta = \max\{\gamma_{\delta f}, \gamma_{\delta g}\}$ *and* $\gamma_\omega = \min\{\gamma_{\omega f}, \gamma_{\omega g}\}$.

Remark 3.19. It is worth noting that the design procedure in Theorem 3.18 is based on the S-procedure given in Definition 3.1, which includes the triggering conditions (3.23) as inequalities into the stability conditions in Definition 3.11. This allows to design the triggering parameter independently of the variables $\xi(t)$ and $\psi(t)$. The technical details are given in the following proof.

Proof. This proof is split into two main parts; the first part shows the nonnegativity of the observation error, and the second one is dedicated to the $\mathcal{L}_1$ stability and the performance of the interval observation error.

Nonnegativity of the observation error This is based on Definition 3.4. For given initial conditions $\underline{x}(0)$ and $\overline{x}(0)$ that satisfy $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, the initial values of the estimation error $\underline{e}(0) = x(0) - \underline{x}(0)$ and $\overline{e}(0) = \overline{x}(0) - x(0)$ are non-negative. The matrix $\mathcal{M}(A)$ and $\tilde{E}$ are Metzler and non-negative matrices, respectively. In addition, based on Assumption 3.1, the vector $\psi(t)$ is non-negative. Thus, the continuous dynamics (3.15) of the impulsive observer is non-negative between two successive measurement time instants (i.e. $\forall t \in [t_k^+, t_{k+1}]$). To ensure the non-negativity of the estimation error for all $t \in \mathbb{R}_{\geq 0}$ we add a condition that guarantees that the inclusion $\underline{x}(t_k^+) \leq x(t_k) \leq \overline{x}(t_k^+)$ is satisfied provided that $\underline{x}(t_k) \leq x(t_k) \leq \overline{x}(t_k)$ is true. That is, at measurement time instants the observer updates the values of the interval bounds by means of a correction jump represented by the discrete dynamics (3.17), which has to stay non-negative after the jump. The matrices $\tilde{F}(L)$ and $\Gamma(L)$ are non-negative and based on Assumption 3.1, the vector $\tilde{F}(L)\psi(t_k)$ is non-negative. This allows to preserve the ordering relation for the estimation error after experiencing the reset. Consequently, the errors $\underline{e}(t)$, $\overline{e}(t)$ of the system (3.19) are non-negative $\forall t \geq 0$ provided that $\underline{e}(0) \geq 0$, $\overline{e}(0) \geq 0$.

$\mathcal{L}_1$ stability of the observation error This is based on the non-negativity nature of the used variables, which allows us to rely on some properties of positive systems. In the sequel, we will use linear co-positive Lyapunov functions of the form $V(\xi) = \xi^\top \lambda$ where $\lambda \in \mathbb{R}^{2n}_{\geq 0}$ which can reduce the complexity of the design problem. First of all, we pick $p = 1$ as a special case of the results obtained by Nešić et al. (2013) and recalled in Definition 3.11. Based on the first part of the proof, we have $\xi(t, j) \geq 0$, $\forall (t, j) \in \text{dom}\xi$, thus, the function $V(\xi)$ is non-negative. By choosing $c_2 = \max\{\lambda_i\}$, the inclusion (3.8a) is satisfied.

Now, we first analyze the behavior of the interval observer between two successive measurements (the continuous dynamics of (3.19)). For simplicity of presentation, we drop the time index for all variables (e.g. $|\omega(t)|_1 = |\omega|_1$). The variation of the proposed Lyapunov function is given by

$$\langle \nabla V(\xi), \mathcal{M}(A)\xi + \tilde{E}\psi \rangle = \xi^\top \mathcal{M}^\top(A)\lambda + \psi^\top \tilde{E}^\top \lambda \tag{3.25}$$

Using the fact that $|\omega|_1 = |\xi|_1 = \xi^\top \mathbb{1}_{2n}$ and $|\delta|_1 = |\psi|_1 = \psi^\top \mathbb{1}_{2n_d}$ and by designing an upper bound of the $\mathcal{L}_1$-gain of the operator $\delta \to \omega$ as defined by (3.8b), one can write

$$\langle \nabla V(\xi), \mathcal{M}(A)\xi + \tilde{E}\psi \rangle \leq -\gamma_{\omega f}|\xi|_1 + \gamma_{\delta f}|\psi|_1$$

Now using (3.25), we obtain

$$\xi^\top \mathcal{M}^\top(A)\lambda + \psi^\top \tilde{E}^\top \lambda \leq -\gamma_{\omega f}\xi^\top \mathbb{1}_{2n} + \gamma_{\delta f}\psi^\top \mathbb{1}_{2n_d} \tag{3.26}$$

and (3.26) can be represented under a vector form as follows

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A)\lambda + \gamma_{\omega f}\mathbb{1}_{2n} \\ \tilde{E}^\top \lambda - \gamma_{\delta f}\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \tag{3.27}$$

The inequality (3.27) should be satisfied when the observer is flowing (i.e., $\forall (\xi, \psi) \in \mathcal{C}$ in (3.23)). Using the S-procedure (see Definition 3.1), with a similar reasoning

as in (Nešić et al., 2013, Corollary 1), this is equivalent to

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A)\lambda + \gamma_{\omega f}\mathbb{1}_{2n} \\ \tilde{E}^\top\lambda - \gamma_{\delta f}\mathbb{1}_{2n_d} \end{bmatrix} - \zeta_{\mathcal{C}} \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathbb{1}_{2n} \\ -\beta\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \tag{3.28}$$

Similarly, the stability condition for the discrete dynamics (3.8c), using the proposed Lyapunov function, is given as

$$[\Gamma(L)\xi + \tilde{F}(L)\psi]^\top\lambda - \xi^\top\lambda \leq -\gamma_{\omega g}\xi^\top\mathbb{1}_{2n} + \gamma_{\delta g}\psi^\top\mathbb{1}_{2n_d}, \quad \forall(\xi,\psi) \in \mathcal{D} \tag{3.29}$$

which is equivalent to

$$\begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \Gamma^\top(L)(A)\lambda - \lambda + \gamma_{\omega g}\mathbb{1}_{2n} \\ \tilde{F}^\top(L)\lambda - \gamma_{\delta g}\mathbb{1}_{2n_d} \end{bmatrix} - \zeta_{\mathcal{D}} \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} -\mathbb{1}_{2n} \\ \beta\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \tag{3.30}$$

Finally, inequalities (3.28) and (3.30) can be written in the following form

$$\begin{cases} \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \mathcal{M}^\top(A)\lambda + (\gamma_{\omega f} - \zeta_{\mathcal{C}})\mathbb{1}_{2n} \\ \tilde{E}^\top\lambda - (\gamma_{\delta f} - \zeta_{\mathcal{C}}\beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \\ \begin{bmatrix} \xi \\ \psi \end{bmatrix}^\top \begin{bmatrix} \Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_{\mathcal{D}})\mathbb{1}_{2n} \\ \tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0 \end{cases} \tag{3.31}$$

Based on the fact that $\begin{bmatrix} \xi \\ \psi \end{bmatrix} \geq 0$ then inequalities (3.31) hold if inequalities (3.24) are satisfied. And this concludes the proof. □

Discussion on Theorem 3.18

The conditions in (3.24) are sufficient conditions to satisfy inequalities (3.31). For a given observer gain matrix L, if the inequalities (3.24) are satisfied, then there exist a co-positive storage (Lyapunov) function of the form $V(\xi) = \xi^\top\lambda$, and an upper bound γ of the $\mathcal{L}_1$-gain between the known width of perturbation and the width of interval state estimate. Besides, solving (3.24) tunes the positive scalar β characterizing the event-triggered mechanism.

Inequalities (3.24) ensure also that the stability margin of the observer's impulsive part can compensate, in $\mathcal{L}_1$-gain sense, any instability related to the observer's

continuous part, which is based on the internal positive representation $\mathcal{M}(A)$ of the matrix A. This can be explained as follows: By definition, a Metzler matrix M is stable if there exists a nonnegative vector ν satisfying $M\nu \leq 0$ (Efimov and Raïssi, 2016). Based on inequality (3.24a), the Hurwitz stability of the Metzler matrix $\mathcal{M}(A)$ depends of the parameters $\gamma_{\omega f}$ and ζ_c. Thus, even if the matrix $\mathcal{M}(A)$ is unstable, the inequality (3.24a) can be satisfied for some parameters $\gamma_{\omega f}$ and ζ_c. On the other hand, the matrix $\Gamma^\top(L)$ in inequality (3.24c) is always stable because we have $\Gamma^\top(L)\lambda - \lambda \leq -(\gamma_{\omega g}+\zeta_{\mathcal{D}})\mathbb{1}_{2n} \leq 0$. Inequalities (3.24b) and (3.24d) make an indirect relation between inequalities (3.24a) and (3.24c) to establish stability in the $\mathcal{L}_1$-gain sense. More precisely, inequalities (3.24a)-(3.24b) represent a sufficient condition of the $\mathcal{L}_1$-gain stability (for continuous-time systems) of the continuous part when $(\xi, \psi) \in \mathcal{C}$, and inequalities (3.24c)-(3.24d) gives a sufficient condition of the l_1-gain stability (for discrete-time systems) of the impulsive part when $(\xi, \psi) \in \mathcal{D}$. Consequently, inequalities (3.24) combine these definitions by taking the maximum gain for all (ξ, ψ).

In Theorem 3.18, the stability and the non-negativity of the interval observer errors are proved.
In the following, we propose another event-triggered mechanism that can reduce the amount of transmitted measurements

3.3.2 Dynamic event-triggered interval observer

The flow and jump dynamics of the hybrid system (3.19) are linear. Using the fact that linear systems are globally Lipschitz (Khalil, 2002) we can bound the flow of (3.19) as follows[1]

$$|\dot{\xi}(t)|_1 \leq \mathcal{K}_\xi(|\xi(t)|_1 + |\psi(t)|_1) \tag{3.32}$$

where $\mathcal{K}_\xi = \max\{|\mathcal{M}(A)|_\infty, |\tilde{E}|_\infty\}$, and by the definition of the 1-norm[2] of ξ and ψ we can write

$$|\dot{\omega}(t)|_1 \leq \mathcal{K}_\xi(|\omega(t)|_1 + |\delta(t)|_1) \tag{3.33}$$

[1]The Lipschitz property is correct with any norm (Khalil, 2002).
[2]This norm is for a vector at time t. It defers from the signal $\mathcal{L}_1$-norm.

The new dynamic event-triggering mechanism that we propose is as follows

$$\begin{aligned}\mathcal{C}_\eta &= \left\{(\omega,\delta,\eta)\in\mathbb{R}^n\times\mathbb{R}^{n_d}\times\mathbb{R}\ :\ |\omega|_1\leq\beta|\delta|_1+\frac{\eta}{\theta}\right\}\\ \mathcal{D}_\eta &= \left\{(\omega,\delta,\eta)\in\mathbb{R}^n\times\mathbb{R}^{n_d}\times\mathbb{R}\ :\ |\omega|_1\geq\beta|\delta|_1+\frac{\eta}{\theta}\right\}\end{aligned} \tag{3.34}$$

where η is the state of an auxiliary scalar dynamical system given as follows

$$\begin{aligned}\dot{\eta}(t) &= -\alpha\eta(t)+\beta|\delta(t)|_1-|\omega(t)|_1\\ \eta(0) &\geq |\omega(0)|_1-\beta|\delta(0)|_1\end{aligned} \tag{3.35}$$

where the initial condition of the auxiliary system is chosen in a way to initialize the observer in the flow set $\mathcal{C}_\eta$.

Remark 3.20. The solution of the auxiliary dynamics (3.35) with the event-triggering condition (3.34) is nonnegative. Its nonnegativity is proved similarly to (Girard, 2015, Lemma 1). The proof is as follows: When the hybrid system is flowing (i.e. $(\omega,\delta,\eta)\in\mathcal{C}_\eta$) the auxiliary variable dynamics (3.35) satisfies the inequality $\dot{\eta}\geq-\alpha\eta-\frac{1}{\theta}\eta$. And using the comparison theorem, the solution of the obtained dynamics is lower bounded by the solution of $\dot{\underline{\eta}}=-\alpha\underline{\eta}-\frac{1}{\theta}\underline{\eta}$ which has a nonnegative dynamics provided that $(\omega,\delta,\eta)\in\mathcal{C}_\eta$ and the initial condition $\eta(0)$ satisfies (3.35). The initial condition of $\eta(t)$ is the main difference between the estimation problem in this study and the control one in Girard (2015) (e.g., in general the initialization of estimate which width is $\omega(0)$ cannot always satisfy $(\omega,\delta,\eta)\in\mathcal{C}_\eta$ if $\eta(0)=0$).

In the hybrid framework, in general, we have the choice to flow or jump when the variables are in $\mathcal{C}_\eta\cap\mathcal{D}_\eta$, thus in our study we force the ETM to jump when $(\omega,\delta,\eta)\in\mathcal{C}_\eta\cap\mathcal{D}_\eta$. In the following, we will show how the use of this dynamic event-triggering mechanism can ensure stability of the estimation error.

Assumption 3.2. *Let $\mathcal{K}_\delta<\infty$ be a bounded positive scalar. The perturbation width $\delta(t)$ defined in (3.20) satisfies*

$$|\dot{\delta}(t)|_1\leq\mathcal{K}_\delta|\delta(t)|_1. \tag{3.36}$$

Corollary 3.21. *Let Assumption 3.1 and 3.2 hold. For a given matrix $L \in \mathbb{R}^{n\times n_y}$ and a given positive scalar θ, if there exist a nonnegative vector $\lambda \in \mathbb{R}^{2n}_{\geq 0}$, and nonnegative scalars ζ_c, $\zeta_{\mathcal{D}}$, $\gamma_{\delta f}$, $\gamma_{\delta g}$, $\gamma_{\omega f}$, $\gamma_{\omega g}$, α and β, satisfying the following inequalities*

$$\left.\begin{aligned} \mathcal{M}^\top(A)\lambda + (-1+\gamma_{\omega f}-\zeta_c)\mathbb{1}_{2n} &\leq 0 \\ \tilde{E}^\top\lambda + (\beta-\gamma_{\delta f}+\zeta_c\beta)\mathbb{1}_{2n_d} &\leq 0 \\ -\alpha+\zeta_c\tfrac{1}{\theta} &\leq 0 \end{aligned}\right\} \tag{3.37a}$$

$$\left.\begin{aligned} \Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g}+\zeta_{\mathcal{D}})\mathbb{1}_{2n} &\leq 0 \\ \tilde{F}^\top(L)\lambda - (\gamma_{\delta g}+\zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} &\leq 0 \end{aligned}\right\} \tag{3.37b}$$

$$\gamma_{\delta g} - \beta\gamma_{\omega g} \leq 0 \tag{3.37c}$$

then, the hybrid system (3.19), (3.34)–(3.35) is finite $\mathcal{L}_1$-gain stable. Thus, the system (3.11)-(3.13) with the triggering mechanism (3.34)-(3.35) is a finite $\mathcal{L}_1$-gain interval observer for the system (3.9)Furthermore, the $\mathcal{L}_1$-gain from δ to ω is upper bounded by $\gamma_{\mathcal{L}_1} = \gamma_\delta/\gamma_\omega$ where $\gamma_\delta = \max\{\gamma_{\delta f}, \gamma_{\delta g}\}$, $\gamma_\omega = \min\{\gamma_{\omega f}, \gamma_{\omega g}\}$.

Proof. The nonnegativity property of $\eta(t)$ is provided in Remark 3.20. The estimation error $\xi(t)$ is nonnegative as shown in Theorem 3.18. Based on the nonnegativity property of η and ξ, we can analyse the stability of the hybrid system (3.19) under the jump mechanism (3.34)–(3.35) by using a new Lyapunov function of the form $W(\xi,\eta) = V(\xi) + \sigma\eta = \xi^\top\lambda + \sigma\eta$, with $\eta \in \mathbb{R}_{\geq 0}$, where σ is a positive constant. Without loss of generality, we pick $\sigma = 1$.

The function $V(\xi)$ is no longer a Lyapunov function for the event-triggered mechanism (3.34) because the decrease of the auxiliary variable η allows the function $V(\xi)$ to increase while $W(\xi,\eta)$ decreases.

Similarly to the proof of Theorem 3.18, the stability condition on the continuous dynamics is given as follows

$$\begin{aligned}\langle \nabla W(\xi), \begin{bmatrix}\dot{\xi}\\ \dot{\eta}\end{bmatrix}\rangle =& \xi^\top \mathcal{M}^\top(A)\lambda + \psi^\top \tilde{E}^\top \lambda + (-\alpha\eta + \beta\psi^\top \mathbb{1}_{2n_d} - \xi^\top \mathbb{1}_{2n}) \\ &\leq -\gamma_{\omega f}\xi^\top \mathbb{1}_{2n} + \gamma_{\delta f}\psi^\top \mathbb{1}_{2n_d}\end{aligned} \quad \forall(\xi,\psi,\eta)\in\mathcal{C}_\eta \tag{3.38}$$

which is equivalent to

$$\begin{bmatrix}\xi\\ \psi\\ \eta\end{bmatrix}^\top \begin{bmatrix}\mathcal{M}^\top(A)\lambda - \mathbb{1}_{2n} + \gamma_{\omega f}\mathbb{1}_{2n}\\ \tilde{E}^\top\lambda + \beta\mathbb{1}_{2n_d} - \gamma_{\delta f}\mathbb{1}_{2n_d}\\ -\alpha\end{bmatrix} - \zeta_c \begin{bmatrix}\xi\\ \psi\\ \eta\end{bmatrix}^\top \begin{bmatrix}\mathbb{1}_{2n}\\ -\beta\mathbb{1}_{2n_d}\\ -\frac{1}{\theta}\end{bmatrix} \leq 0 \tag{3.39}$$

Finally, if the inequality (3.37a) is satisfied then the inequality (3.39) holds.
It is worth noting that jumps do not impact the auxiliary variable η (i.e., $\eta(t_k,k) = \eta(t_k,k-1)$). Similarly, the stability condition on the discrete dynamics can thus be written as

$$W(\xi^+,\eta^+) - W(\xi,\eta) = V(\xi^+) - V(\xi) \leq -\gamma_{\omega g}|\omega|_1 + \gamma_{\delta g}|\delta|_1, \quad \forall(\omega,\delta,\eta)\in\mathcal{D}_\eta \tag{3.40}$$

which is satisfied if the inequality (3.37b) holds. □

So far, the existence of positive inter-event times has not been discussed. In the following section, we will provided some conditions that ensure the existence of MIET.

3.3.3 Further conditions to guarantee the existence of minimum inter-event times (MIET)

Corollary 3.22. *The conditions in Corollary 3.21 with the storage function $W(\xi,\eta) = \xi^\top\lambda + \eta$ that satisfies $\lambda = \varrho\mathbb{1}_{2n}$ where ϱ is a positive scalar, guarantee the existence of minimum inter-event times.*

Proof. In order to investigate the existence of MIET, we study the variation of the following ratio:

$$\kappa(t,j) = \frac{|\omega(t,j)|_1}{\beta|\delta(t,j)|_1 + \frac{\eta(t,j)}{\theta}} \tag{3.41}$$

It is the ratio between the $1-$norm of the estimation error width and the threshold $\beta|\delta(t,j)|_1 + \frac{\eta(t,j)}{\theta}$. Based on the ETM (3.34), this ratio is larger than or equal to 1 when $(\omega,\delta,\eta) \in \mathcal{D}_\eta$, and it is lower than 1 when $(\omega,\delta,\eta) \in \mathcal{C}_\eta$. To simplify the analysis, we consider that at times of jump, this ratio satisfies $\kappa(t_k, k-1) = 1$ which fits the triggering condition in (3.34). After the jump, it will be reset to $\kappa(t_k,k) \in [0,\ 1)$ based on the fact that the width of estimate is contracted using measurement, as it will be shown in the sequel.

This behavior is shown in two steps as following: In the first step, we show that the width of the estimation error is contracting at jump times. Then, in the second step, by analyzing the dynamics of the ratio $\kappa(t,j)$ we will show how this contraction can guarantee the existence of a lower bound of the inter-event times.

Step 1: Contraction of the estimated state enclosure after jump.

As mentioned above the stability condition at jump instants satisfies (3.40). Replacing the event-triggered condition $|\omega|_1 \geq \beta|\delta|_1 + \frac{\eta}{\theta}$ in the right hand side of the inequality (3.40), one gets

$$\xi^{+\top}\lambda - \xi^{\top}\lambda \leq -(\gamma_{\omega g} - \frac{\gamma_{\delta g}}{\beta})|\omega|_1 - \frac{\gamma_{\delta g}}{\beta\theta}\eta \tag{3.42}$$

Thus, satisfying the inequality (3.37c) implies that $\xi^{+\top}\lambda - \xi^{\top}\lambda < 0$.

Based on the definition of the vector $\lambda = \varrho\mathbb{1}_{2n}$ where $\varrho > 0$ and the nonnegativity of the estimation error ξ, one can obtain that $(|\xi^+|_1 - |\xi|_1)\varrho < 0$. Consequently, one can deduce that the norm-1 of ξ decreases at measurement times which implies the contraction of $|\omega|_1$.

Step 2: The existence of a positive time interval that lets the state estimate width to increase before entering into the jump set $\mathcal{D}_\eta$.

Now, we return to study the dynamics of the ratio $\kappa(t,j)$ in between two successive measurement times.

$$\begin{aligned}\frac{d}{dt}\kappa(t) = \frac{d}{dt}\frac{|\omega(t)|_1}{\beta|\delta(t)|_1+\frac{\eta(t)}{\theta}} &= \frac{d}{dt}\frac{|\xi(t)|_1}{\beta|\delta(t)|_1+\frac{\eta(t)}{\theta}}\\ &= \frac{\dot{\xi}(t)^\top \mathbb{1}_{2n}(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta}) - |\xi|_1(\beta\dot{\delta}(t)^\top\mathbb{1}_{n_d}+\frac{\dot{\eta}(t)}{\theta})}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\end{aligned}$$

Using the property (3.32), the dynamics (3.35), and Assumption 3.2, one gets

$$\begin{aligned}\frac{d}{dt}\kappa(t) \leq& \frac{\mathcal{K}_\xi(|\omega(t)|_1+|\delta(t)|_1)}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2} - \frac{|\omega(t)|_1(-\alpha\eta(t)+\beta|\delta(t)|_1-|\omega(t)|_1)}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\\ \leq& \mathcal{K}_\xi\frac{|\omega(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} + \frac{\mathcal{K}_\xi}{\beta} + \frac{|\omega(t)|_1\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2} + \frac{|\omega(t)|_1\alpha\eta(t)}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\\ &- \frac{|\omega(t)|_1(\beta|\delta(t)|_1)}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\\ \leq& \frac{\mathcal{K}_\xi}{\beta} + \left[\mathcal{K}_\xi + \frac{\alpha\eta(t)}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} + \frac{\beta\mathcal{K}_\delta|\delta(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} - \frac{(\beta|\delta(t)|_1)}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})}\right]\\ &\times\frac{|\omega(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\\ \leq& \frac{\mathcal{K}_\xi}{\beta} + \left[\mathcal{K}_\xi+\alpha+\mathcal{K}_\delta-\frac{1}{\theta}\right]\frac{|\omega(t)|_1}{(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})} + \frac{|\omega(t)|_1^2}{\theta(\beta|\delta(t)|_1+\frac{\eta(t)}{\theta})^2}\\ \leq& \frac{\mathcal{K}_\xi}{\beta} + \left[\mathcal{K}_\xi+\alpha+\mathcal{K}_\delta\right]\kappa(t) + \frac{1}{\theta}\kappa^2(t)\end{aligned}$$

Based on the fact that $0 \leq \kappa(t) < 1$, one can simplify the above inequality as following

$$\frac{d}{dt}\kappa(t) \leq \frac{\mathcal{K}_\xi}{\beta} + \left[\mathcal{K}_\xi+\alpha+\mathcal{K}_\delta+\frac{1}{\theta}\right]\kappa(t) \tag{3.43}$$

Thus, an upper bound trajectory for the ratio $\kappa(t), \forall t \in [t_k, t_{k+1}]$ is given by

$$\dot{\overline{\kappa}} = a + b\overline{\kappa} \tag{3.44}$$

where $a = \frac{\mathcal{K}_\xi}{\beta}$ and $b = \mathcal{K}_\xi+\alpha+\mathcal{K}_\delta+\frac{1}{\theta}$ with $\overline{\kappa}(t_k,k) = \kappa(t_k) < 1$ and $\overline{\kappa}(t_{k+1},k) = \kappa(t_{k+1},k) = 1$. By using the fact that $\kappa(t,j)$ is a monotone increasing solution function as shown by lemma A.1 in Appendix A.2, one can deduce that the ratio $\kappa(t,j)$ solution to (3.44) guarantees the existence of τ_{min} such that $0 < \tau_{min} \leq$

$t_{k+1} - t_k$ for all $k \in \mathbb{N}$ when initial and final conditions are given as $\overline{\kappa}(t_k, k) = \kappa(t_k) \in [0, 1)$, $\overline{\kappa}(t_{k+1}, k) = 1$. Consequently, as the solution $\overline{\kappa}(t, j)$ is an over-approximation of the ratio $\kappa(t, j)$, the time τ_{min} is the MIET for the ETM (3.34). □

The previous result shows the event-triggered mechanism design for a pre-calculated observer gain. In the following section we will show how to co-design the ETM and the observer gain.

3.4 Co-design of event-triggered mechanisms and interval Observer gains

In this section, we will co-design the ETM and the observation gain using an over-approximate of the reset matrix $\Gamma(L)$ of the estimation error dynamics (3.19). This reset matrix is not easy to synthesized due to the non-smooth operator in (3.17) that uses the gain L. To tackle this problem we propose to use a nonnegative realization of the matrix $[I_n + LC]$ to over-approximate the reset matrix.

Proposition 3.23. *Consider the following nonnegative discrete-time system*

$$\chi_D(k+1) = A_D \chi_D(k) \tag{3.45}$$

where $\chi_D \in \mathbb{R}^n$ is the state variables, with $A_D \in \mathbb{R}^{n\times n}_{\geq 0}$. Let assume that the matrix A_D is Schur stable[3]*. If there exist two nonnegative matrices A_d, $E_d \in \mathbb{R}^{n\times n}_{\geq 0}$ such that $A_D = A_d + E_d$, then $\chi_D(k)$ solution to (3.45) and $\chi_d(k)$ solution to the system $\chi_d(k+1) = A_d\chi_d(k)$ satisfy $\chi_D(k) \geq \chi_d(k)$ $\forall k \in \mathbb{N}$ provided that $\chi_D(0) \geq \chi_d(0) \geq 0$.*

Proof. Starting from the solution sequence $\chi_D(k)$, we have

$$\chi_D(k+1) = (A_d + E_d)\chi_D(k) = A_d\chi_D(k) + E_d\chi_D(k), \tag{3.46}$$

[3]All its eigenvalues are contained in the open unit disk in the complex plane.

Based on the nonnegativity of the matrices E_d and A_d and the initial conditions $\chi_d(0) \geq \chi(0) \geq 0$, one have that the system $\chi_D(k+1) = A_D\chi_D(k)$ has a nonnegative dynamics and the term $E_d\chi_D(k)$ is nonnegative. Thus, from (3.46) one gets

$$\chi_D(k+1) \geq A_d\chi_D(k) \quad \geq \quad A_d\chi_d(k) = \chi_d(k+1),$$

this concludes the proof. □

Proposition 3.24. *Let $H \in \mathbb{R}^{n\times n}$, for any two nonnegative matrices $H_p, H_n \in \mathbb{R}^{n\times n}_{\geq 0}$ satisfy $H = H_p - H_n$, there exits a nonnegative matrix $\Delta \in \mathbb{R}^{n\times n}_{\geq 0}$ such that $H_p = (H^+ + \Delta)$ and $H_n = (H^- + \Delta)$.*

Proof. For any element H_{ij} of the matrix H, if $H_{ij} \leq 0$ then $H^-_{ij} = |H_{ij}|$ and $H^+_{ij} = 0$, and we have $H_{ij} = H_{pij} - H_{nij} = (\Delta_{ij}) - (H^-_{ij} + \Delta_{ij})$. Consequently, the condition $H_{pij} \geq 0$ implies $\Delta_{ij} \geq 0$. Similarly, in the case where $H_{ij} \geq 0$, we have $H^-_{ij} = 0$ and $H^+_{ij} = H_{ij}$, and we have $H_{ij} = H_{pij} - H_{nij} = (H^+_{ij} + \Delta_{ij}) - (\Delta_{ij})$. Thus, the condition $H_{nij} \geq 0$ implies that $\Delta_{ij} \geq 0$. So, one can conclude that the matrix Δ is always nonnegative. □

In the following theorem, we will provide a co-design methodology of the observer gain and some parameters of the event-triggered mechanism. Compared to Corollary 3.21 where the interval observer gain is given a priori, here we co-design it along with the ETM.

Theorem 3.25. *Let Assumption 3.1 hold, if there exist a matrix $L \in \mathbb{R}^{n\times n_y}$, and nonnegative matrices $G_p, G_n \in \mathbb{R}^{n\times n}_{\geq 0}$ and $R_p, R_n \in \mathbb{R}^{n\times n_d}_{\geq 0}$, a nonnegative vector $\lambda \in \mathbb{R}^{2n}_{\geq 0}$, and nonnegative scalars ζ_c, $\zeta_\mathcal{D}$, $\gamma_{\delta f}$, $\gamma_{\delta g}$, $\gamma_{\omega f}$, $\gamma_{\omega g}$ and β, satisfying inequalities (3.37a) and (3.37c) and the following inequality*

$$\left.\begin{aligned} \Gamma^\top(G_p, G_n)\lambda - \lambda + (\gamma_{\omega g} + \zeta_\mathcal{D})\mathbb{1}_{2n} &\leq 0 \\ \tilde{F}^\top(R_p, R_n)\lambda - (\gamma_{\delta g} + \zeta_\mathcal{D}\beta)\mathbb{1}_{2n_d} &\leq 0 \end{aligned}\right\}, \tag{3.47a}$$

$$G_p - G_n = I_n + LC\,, \tag{3.47b}$$

$$R_p - R_n = LF\,, \tag{3.47c}$$

where

$$\Gamma(G_p, G_n) = \begin{bmatrix} G_p & G_n \\ G_n & G_p \end{bmatrix}, \quad \tilde{F}(R_p, R_n) = \begin{bmatrix} R_p & R_n \\ R_n & R_p \end{bmatrix}$$

Then, the system (3.11)-(3.13) *with the event-triggered mechanism* (3.34) *is a finite* $\mathcal{L}_1$*-gain interval observer for the system* (3.9)*, where the ETM* (3.34), (3.35) *guarantees the existence of a positive lower bound on the inter-measurement times. Furthermore, the* $\mathcal{L}_1$*-gain from* δ *to* ω *is upper bounded by* $\gamma_{\mathcal{L}_1}$ *given in Theorem 3.18.*

Proof. The difference between Corollary 3.21 and Theorem 3.25 is the design of the observer gain L. Thus we need to prove that the constraints (3.47) imply (3.37b). Pick $G = [I_n + LC]$ and $R = LF$, based on equations 3.47b and (3.47c), an upper bound of the estimation error of the correction part (3.13) can be written as follows:

$$\begin{cases} \underline{x}(t_k^+) = G_p\underline{x}(t_k) - G_n\overline{x}(t_k) + R_p\underline{d}(t_k) - R_n\overline{d}(t_k) - Ly(t_k) \\ \overline{x}(t_k^+) = G_p\overline{x}(t_k) - G_n\underline{x}(t_k) + R_p\overline{d}(t_k) - R_n\underline{d}(t_k) - Ly(t_k) \end{cases} \quad \forall k \in \mathbb{N} \qquad (3.48)$$

Satisfying inequality (3.47a), the correction part (3.48) with the obtained parameters G_p, G_n, R_p and R_n can ensure the stability and the positivity of the interval observer error as given in Corollary 3.21.

Using the result of Proposition 3.24, the existence of nonnegative matrices G_p, G_n, R_p and R_n satisfying (3.47b) implies the existence of nonnegative matrices Δ_G and Δ_R such that $G_p = (I_n + LC)^+ + \Delta_G$, $G_n = (I_n + LC)^- + \Delta_G$, $R_n = (LF)^- + \Delta_R$ and $R_n = (LF)^- + \Delta_R$. Consequently, the estimation error at jump (3.17) can be seen as

$$\Gamma(G_p, G_n) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix} + \begin{bmatrix} \Delta_G & \Delta_G \\ \Delta_G & \Delta_G \end{bmatrix} = \Gamma(L) + \begin{bmatrix} \Delta_G & \Delta_G \\ \Delta_G & \Delta_G \end{bmatrix}$$

$$\tilde{F}(R_p, R_n) = \tilde{F}(L) + \begin{bmatrix} \Delta_R & \Delta_R \\ \Delta_R & \Delta_R \end{bmatrix}$$

Now, based on the property of uncertain discrete time system given in Proposition 3.23, the estimation error corresponding to the correction part (3.13) is upper bounded by the one corresponding to the correction part (3.48). Finally, the constraints (3.47) imply the inequalities (3.37b) which allows to implement the correction part given by (3.13). This ends the proof. □

Remark 3.26. It is worth noting that the matrices G_p, G_n, R_p and R_n are only intermediate variables allowing the synthesis of the interval observer gains L.

Remark 3.27. To minimize the impact of the uncertainty width δ onto the estimates width ω, the synthesis is performed, in practice, by minimizing the upper bound $\gamma_{\mathcal{L}_1}$ on the $\mathcal{L}_1-$gain. From Theorem 3.18, this can be achieved by the following optimization problem

$$\begin{aligned} \min \quad & \gamma_{\delta f} + \gamma_{\delta g} - \gamma_{\omega f} - \gamma_{\omega g} \\ \text{s.t.} \quad & \text{constraints (3.37a), (3.37c), (3.47).} \end{aligned} \tag{3.49}$$

3.5 Illustrative example

In order to illustrate the performance of the proposed observer, we consider the following example.

3.5.1 Double spring-mass-damper system

Consider a mechanical system consisting of two masses m_1 and m_2 that are sliding over an horizontal surface. Suppose that the masses are attached to one another, and to two immovable walls, by means of three horizontal springs of stiffness constants "k_1, k_2 and k_3" and dampers of damping ratio "c_1, c_2 and c_3".
Let $x^\top = [x_{m_1}\ \dot{x}_{m_1}\ x_{m_2}\ \dot{x}_{m_2}]$ be the state variables representing the position and the velocity of each mass and $u^\top = [f_{m_1}\ f_{m_2}]$ be the force applied to the object, respectively. We introduce an LTI model (3.9) for the double spring-mass-damper

system as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_2}{m_2} & -\frac{(k_2+k_3)}{m_2} & -\frac{(c_2+c_3)}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 & -0.2 \\ -0.7 & 0.6 \\ 0.2 & -0.2 \\ -0.5 & 0.6 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & -0.8 \\ -0.4 & 0.5 \end{bmatrix}$$

with $d(t) = [d_1(t)\ d_2(t)]^\top$ is the disturbance which is assumed unknown-but-bounded $-\overline{d} \leq d(t) \leq \overline{d}$ with $\overline{d} = [0.5\ 0.5]^\top$. For simulation, we pick the values of disturbances as $d_1(t) = 0.5\cos(10t)$ and $d_2(t) = 0.5\sin(6t)$.

The practical parameters of the system are given as $m_1 = 0.6kg, m_2 = 1kg, k_1 = k_2 = k_3 = 1N/m$ and $c_1 = 2Ns/m, c_2 = 1.4Ns/m, c_3 = 1.2Ns/m$.

The dynamics of the interval estimation error in between two consecutive measurements defined in (3.15) has the following matrices

$$A^M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_2}{m_2} & 0 & -\frac{(c_2+c_3)}{m_2} \end{bmatrix}, \quad A^N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{(k_1+k2)}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(k_2+k_3)}{m_2} & 0 \end{bmatrix}.$$

Note that although the plant model is stable, the state estimation given by the predictor in between measurements is unstable.

The synthesis problem of the observer gain in Theorem 3.25 is solved using the YALMIP toolbox Lofberg (2004) based on the FMINCON solver. For $\theta = 2$, the solution of the co-design of the event-triggered mechanism and the observer gain is as follows: the obtained coefficients of the ETM (3.35) are $\alpha = 1.3081$ and $\beta = 3.9244$, and the computed observation gain matrix is

$$L = \begin{bmatrix} -0.4535 & -0.0558 \\ 0 & 0 \\ -0.0528 & -0.5166 \\ 0 & 0 \end{bmatrix}$$

The upper bound of the $\mathcal{L}_1$-gain from δ to ω is found as $\gamma_{\mathcal{L}_1} = 190$.

For generating the pseudo-actual data, the system inputs are taken as $f_{m_1}(t) = 14[1+2\sin(10t)+\cos(40t)]$, $f_{m_2}(t) = 10[2\sin(15t)+\sin(30t)]$, and the initial values of the system state taken as $x(0) = [10 \;\; 4 \;\; 15 \;\; 4]^\top$.

The observer uses the following lower and upper bounds: $\underline{x}(0) = [6 \;\; -1 \;\; 11 \;\; -1]^\top$ and $\overline{x}(0) = [14 \;\; 9 \;\; 19 \;\; 9]^\top$, respectively. The simulation results are given in Figure 3.1 and 3.2. In Figure 3.1, it is noticeable that the correction part of

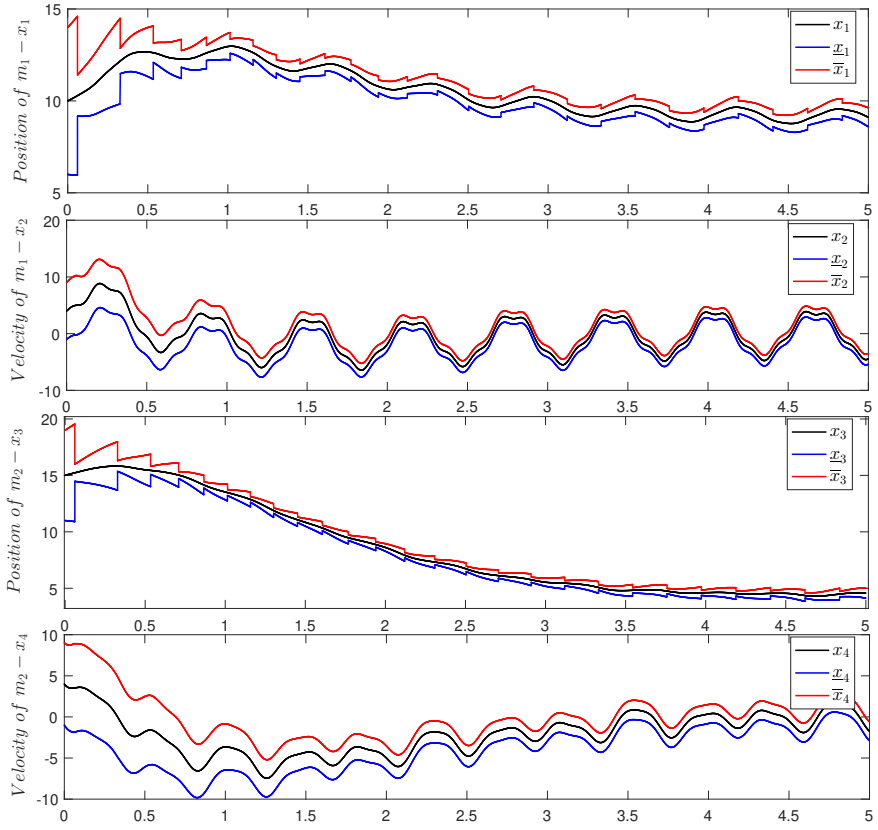


FIGURE 3.1: Simulation results for the double spring-mass-damper system: the upper and the lower estimate bounds for the masses position (x_1, x_3), and masses velocity (x_2, x_4).

the interval observer contracts the estimate bounds, even though the open-loop dynamics of the interval estimator error is unstable. This behaviour corresponds to the condition given in Corollary 3.21.

In Figure 3.2, we can see that the observer triggers the measurements whenever the norm of the width violates the dynamic threshold as described by (3.34).

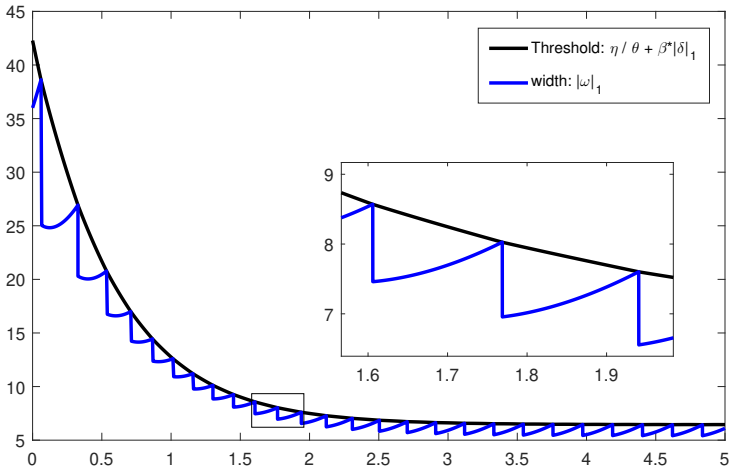


FIGURE 3.2: The evolution of the triggering mechanism : (blue) the width of the state estimate, (black) the threshold for the width.

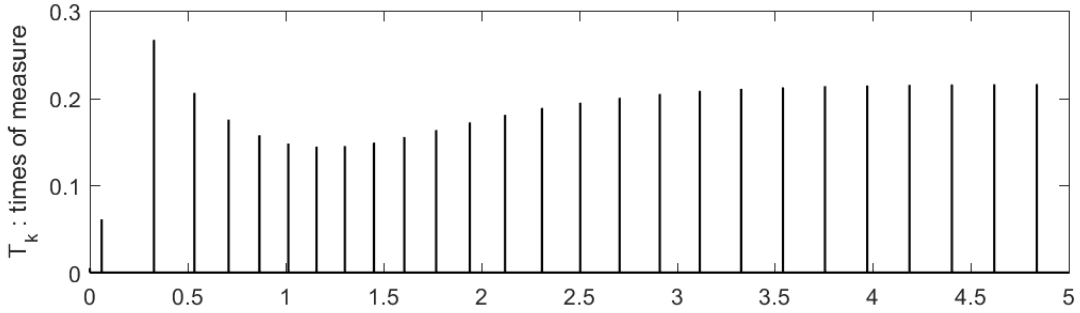


FIGURE 3.3: The Inter-event times of the triggering mechanism.

3.6 Conclusion

In this chapter, co-design of event-triggered mechanisms and interval observers for linear continuous-time systems have been proposed. These interval observers have impulsive structures in which the estimation is performed in two steps. First, in between two consecutive events, open-loop interval estimates are calculated based only on the system's model. Then, at event instants, the interval estimates are corrected in impulsive way. The proposed event-triggered mechanism is based on the positivity property of the interval observation errors. Moreover, they guarantee a finite $\mathcal{L}_1$-gain between the width of the perturbation bounds and the width of the estimated state intervals. Further conditions have also been given on the existence of positive inter-event times for a particular Storage function. The generalization of these conditions will be investigated in future works.

Chapter 4

Secure interval observer for LTI systems with discrete measurements under attacks

4.1 Motivations and related works

Cyber-physical systems (CPS) are smart integrations of computation and networking resources, and physical processes (Lee and Seshia, 2016). Due to the cyber-physical coupling and to the disrupting consequences of failures, security here is one of the primary concerns (Lun et al., 2019). The problem of security is not new to the control systems field, particularly in the area of fault detection and isolation (FDI) (Massoumnia et al., 1989). Recent works on the cyber security of control systems have been focused, in part, on the effect of specific types of attacks on stability and/or estimation, such that false data injection attacks (Liu et al., 2011), (Degue et al., 2018), denial-of-service attacks (Amin et al., 2009) and integrity attacks (Mo et al., 2014), or to more general class of adversarial attacks (Fawzi et al., 2014), (Chong et al., 2015) and (Shoukry et al., 2018), which is the case of our work.

This chapter addresses the design of an interval state observer for a linear time-invariant plant in presence of periodic discrete measurements affected by unknown-but-bounded noise with known bounds and subject to cyber-attacks (probably unbounded).

To be able to reconstruct a guaranteed state enclosure of the actual state for continuous-time linear systems in presence of discrete-time measurement, we propose an *interval impulsive observer*. To this aim, we will exploit, in this chapter, the impulsive interval observer developed in Chapter 2 in the case where the discrete measurements are periodic.

In this chapter, we consider continuous LTI systems with $\mathbf{s}$ outputs, each of them is measured by a sensor which is potentially under cyber-attack. In this framework, we assume that only a subset S of $\mathbf{s_a}$ sensors can be attacked in the same instant, with $\mathbf{s} > 2\mathbf{s_a}$. This condition is issued from the *M-observability* (Chong et al., 2015) and the *s-sparse observability* (Shoukry and Tabuada, 2016). Based on this assumption, at correction times, we provide as many interval estimate as sensors, then we select the attack-free estimate by a proposed attack-resilient strategy using interval analysis and the positivity of the interval estimation error. The proposed strategy is an *online* algorithm while the synthesis procedure that tunes the observation gain to ensure both positivity and stability of the estimation error is *offline*. The stability analysis of estimation error is inspired by the work of Ferrante et al. (2016) while the positivity of the estimation error is ensured based on the internal positivity for dynamical systems as in Meslem and Ramdani (2011), Cacace et al. (2015) with taking into account the attack influence. Then the effect of the attacks is treated by an online set-membership strategy.

The *novelty* of this chapter is twofold: First, a new LMI-design methodology of the observer gain in presence of discrete-time measurements is proposed, that guarantees both positivity and stability of the interval estimation error. This methodology is different from the one proposed in chapter 2, which is based on BMIs. Second, a new sensor attack-resilient strategy that selects online, at measurements times, the correct estimate among a set of estimates.

The rest of this chapter is organized as follows. The structure of the proposed interval impulsive observer is introduced in Section 4.2. The observer synthesis method is presented in Section 4.3. The attack-resilient strategy is detailed in Section 4.4. Numerical illustrative examples are presented in Section 4.5.

4.2 Interval impulsive observer analysis

4.2.1 Problem statement

Consider the multi-output linear time invariant system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y_\sigma(t_k) = C_\sigma x(t_k) + v_\sigma(t_k) + a_\sigma(t_k), \quad \forall k \in \mathbb{N}, \sigma \in \mathcal{I} \end{cases} \tag{4.1}$$

where $\mathcal{I} = \{1, \ldots, \mathbf{s}\}$ such that $\mathbf{s}$ is the number of sensors. $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y_\sigma \in \mathbb{R}$ is the state variables, the input, the discrete output of the system, respectively. $v_\sigma \in \mathbb{R}$ and $a_\sigma \in \mathbb{R}$ represent the output sensor noise and sensor attack, respectively. In this chapter, we assume that the system dynamics is perfectly known, that is, without perturbation, and we focus on the output imperfections (noise and cyber-attack). The goal is to provide a secure estimate of the system state from noisy discrete measurements and under sensors attack. To reach this objective, we propose the following strategy

4.2.2 Secure observation strategy

In order to obtain a secure interval estimate for the multi-output LTI system (4.1), we propose a two-stage policy:

First – We design an interval impulsive observer for each output y_σ separately with $\sigma \in \mathcal{I}$ in the absence of attacks. To simplify notation we drop the subscript σ in this section and the next one. So the system (4.1) without attack will be in the following form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t_k) = Cx(t_k) + v(t_k), \end{cases} \quad \forall k \in \mathbb{N}, \tag{4.2}$$

In this step, we design as many observers as outputs. In this step, we assume that the system is detectable from each output.

Second – After designing observers for every output without attack, we propose a strategy based on interval analysis to recover the state estimate against the sensors attacks presented in (4.1). This result is based on the following assumption.

Assumption 4.1. *The number of attacked sensors denoted by* $\mathbf{s_a}$ *is strictly lower than the half of the total number of sensors* $\mathbf{s}$ *without knowing which sensors are attacked (i.e.,* $\mathbf{s_a} < \mathbf{s/2}$*).*

This assumption is the main condition for the *M-observability* for continuous-time systems (Chong et al., 2015) and the *s-sparse observability* for discrete-time systems (Shoukry and Tabuada, 2016).

4.2.3 Observer analysis

The observer is constructed to estimate the continuous state of the system from discrete measurements. To this aim, it is assumed that there exists a regular period of time between two consecutive measurement instants as follows.

Assumption 4.2. *Let* τ_m *be real positive scalars satisfy:*

$$t_{k+1} - t_k = \tau_m \ \forall k \in \mathbb{N}.$$

The goal of interval observers is to estimate an upper and a lower bound of the system state from noisy measurements. In this chapter, we adopt the structure of the interval impulsive observer developed in chapter 2, with $\tau = \tau_m$, and attribute different observer gain matrices for the lower and the upper estimates. For ease of reading we recall the interval impulsive observer in the new condition.
Let introduce an assumption on the boundedness of the measurement noise.

Assumption 4.3. *Let* $\overline{v} \in \mathbb{R}$ *be a given positive constant such that*

$$|v(t)| \leq \overline{v} \ \forall t \in \mathbb{R}_{\geq 0}.$$

The interval observer that we propose for system (4.2) works with two steps.
First step: the interval observer in-between two successive measurement instants behaves like an open-loop estimator as follows

$$\begin{cases} \dot{\underline{x}}(t) = A^M \underline{x}(t) - A^N \overline{x}(t) + Bu(t), \\ \dot{\overline{x}}(t) = A^M \overline{x}(t) - A^N \underline{x}(t) + Bu(t) \end{cases} \quad \forall t \in [t_k, t_{k+1}], k \in \mathbb{N} \tag{4.3}$$

where $A^M = d_A + (A - d_A)^+$ and $A^N = A^M - A$ with d_A is a diagonal matrix contains only the diagonal elements of A. In addition, the interval observer initial state at $k = 0$, i.e. at t_0, satisfies the inclusion

$$\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0). \tag{4.4}$$

The dynamics of the estimation error over the inter-measurement time for both bounds $\underline{e}(t) = x(t) - \underline{x}(t)$ and $\overline{e}(t) = \overline{x}(t) - x(t)$ can be obtained from equations (4.2) and (4.3) by

$$\begin{bmatrix} \dot{\underline{e}}(t) \\ \dot{\overline{e}}(t) \end{bmatrix} = \overline{A} \begin{bmatrix} \underline{e}(t) \\ \overline{e}(t) \end{bmatrix}, \forall t \in [t_k, t_{k+1}] \; k \in \mathbb{N} \tag{4.5}$$

with $\overline{A} = \begin{bmatrix} A^M & A^N \\ A^N & A^M \end{bmatrix}$.

Note that, based on the construction of the matrices A^M and A^N as Metzler and nonnegative matrices, respectively, the matrix $\overline{A}$ is Metzler. Then, the solution to (4.5) is nonnegative which means that the lower and the upper bounds do not cross each other in the time interval $[t_k, t_{k+1}]$ provided that their initial conditions satisfy the inclusion $\underline{x}(t_k) \leq x(t_k) \leq \overline{x}(t_k)$.

Second step: using the output model in (4.2), the system state at the measurement time instants can be presented as

$$x(t_k^+) = x(t_k) + L^\bullet[Cx(t_k) + v(t_k) - y(t_k)] \quad k \in \mathbb{N} \tag{4.6}$$

with $L^\bullet \in \{\underline{L}, \overline{L}\}$, where $\underline{L}, \overline{L} \in \mathbb{R}^{n \times 1}$ are observer gains to be designed for the lower, upper bound estimate, respectively.

Equation (4.6) helps establishing the discrete-time dynamics of the estimation error which is used only for synthesis phase. When the measurement is available, an impulsive correction of the estimated state enclosures will be done using the

following correction equations

$$k \in \mathbb{N}, \quad \begin{cases} \underline{x}(t_k^+) = (I_n + \underline{L}C)^+ \underline{x}(t_k) - (I_n + \underline{L}C)^- \overline{x}(t_k) \\ \qquad\qquad - |\underline{L}|\overline{v} - \underline{L}y(t_k) \\ \overline{x}(t_k^+) = (I_n + \overline{L}C)^+ \overline{x}(t_k) - (I_n + \overline{L}C)^- \underline{x}(t_k) \\ \qquad\qquad + |\overline{L}|\overline{v} - \overline{L}y(t_k) \end{cases} \tag{4.7}$$

From (4.7) and (4.6), the estimation error dynamics at measurement instants can be described by the following dynamical system

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(\underline{L}, \overline{L}) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \Upsilon(t_k) \tag{4.8}$$

where

$$\Gamma(\underline{L}, \overline{L}) = \begin{bmatrix} (I_n + \underline{L}C)^+ & (I_n + \underline{L}C)^- \\ (I_n + \overline{L}C)^- & (I_n + \overline{L}C)^+ \end{bmatrix}; \Upsilon(t_k) = \begin{bmatrix} |\underline{L}|\overline{v} + \underline{L}v(t_k) \\ |\overline{L}|\overline{v} - \overline{L}v(t_k) \end{bmatrix}. \tag{4.9}$$

The positivity property of the reset matrix allows to preserve the order relation $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ after experiencing the reset (for more details about IPR for linear systems, see (Cacace et al., 2015)).

Let us now define the augmented vector of the interval estimation error as $\xi = [\underline{e}^\top, \overline{e}^\top]^\top$.

From equations (4.5) and (4.8), and after adding the time variable τ, the hybrid system modeling the dynamics of the estimation error is given by

$$\mathcal{H} : \begin{cases} f(z) = \begin{bmatrix} \overline{A}\xi \\ -1 \end{bmatrix} & \forall z \in \mathcal{C} \\ g(z) = \begin{bmatrix} \Gamma(\underline{L}, \overline{L})\xi + \Upsilon(t_k) \\ \tau_m \end{bmatrix} & \forall z \in \mathcal{D} \end{cases} \tag{4.10}$$

where $z = [\xi^\top, \tau]^\top$ is the state variable of the hybrid system, and τ_m is the reset value of the timer based on Assumption 4.2.

The flow and jump sets are defined as

$$\begin{aligned} \mathcal{C} &= \left\{(\xi,\tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \tau \in [0,\tau_m]\right\} \\ \mathcal{D} &= \left\{(\xi,\tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \tau = 0\right\}. \end{aligned} \tag{4.11}$$

It is worth noting that these sets do not force the system to jump until the timer reaches the zero, then after the jump, the timer τ is reset to τ_m.
The stability analysis of the variable z is based on the notion of distance to a set. Thus, with mild conditions, the stability analysis is straightforward under the hybrid system framework Goebel et al. (2012).
Let us define the closed set $\mathcal{A}$ that contains all admissible values for the timer when the $\xi-$system state is at the origin

$$\mathcal{A} = \left\{z = (\xi,\tau) \in \mathbb{R}^{2n} \times \mathbb{R}_{\geq 0} \quad | \quad \xi = 0, \tau \in [0,\tau_m]\right\}. \tag{4.12}$$

Remark 4.1. The hybrid system (4.10) can be considered for the case of perfect measurement by omitting the term $\Upsilon(t_k)$.

We characterize the domain of solutions of (4.10) when $\Upsilon(t_k) = 0$. Indeed, the variable τ, acting as a timer, guarantees that for every initial condition $\phi(0,0) \in \mathcal{C} \cup \mathcal{D}$ the domain of every maximal solution ϕ to (4.10) when $\Upsilon(t_k) = 0$ can be written as follows:

$$\text{dom}\phi = \bigcup_{j\in\mathbb{N}} ([t_j, t_{j+1}], j)$$

with $t_{j+1} - t_j = \tau_m, \quad \forall j \in \mathbb{N} \setminus \{0\}$. Furthermore, assuming $t_0 = 0$, the structure of the above hybrid time domain implies that for each $(t,j) \in \text{dom}\phi$ we have

$$t \leq \tau_m(j+1).$$

The latter relation will play a key role in establishing GES of the set $\mathcal{A}$ for hybrid system (4.10) when $\Upsilon(t_k) = \mathbf{0}$.
The idea of the stability proof in the following theorem is from (Goebel et al., 2012, Proposition 3.29) which is given in Proposition 1.25. It allows for the Lyapunov function to increase locally, then, this increase is compensated by instantaneous decrease at jumps which renders the overall hybrid dynamics stable.

Theorem 4.2. *Let Assumption 4.2 and 4.3 hold. For given gain matrices* $\underline{L}, \overline{L} \in \mathbb{R}^{n\times 1}$*, if there exists a symmetric positive definite matrix* $P \in \mathbb{R}^{2n\times 2n}$ *such that*

$$\Gamma(\underline{L}, \overline{L})^\top e^{\overline{A}^\top \tau_m} P e^{\overline{A}\tau_m} \Gamma(\underline{L}, \overline{L}) - P \prec 0 \tag{4.13}$$

is satisfied, then the hybrid system (4.10)-(4.11) *is Input-to-State-Stable (ISS) with respect to the set* $\mathcal{A}$ *defined in* (4.12)*. Thus, the system* (4.3), (4.7) *is an interval observer for the system* (4.2) *with ISS estimation error relatively to* $\mathcal{A}$ *provided that* $\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0)$*. Moreover, if* $v(t_k) = 0\ \forall k \in \mathbb{N}$ *in* (4.2)*, then the interval observer* (4.3), (4.7) *for the system* (4.2) *has a globally exponentially stable (GES) estimation error relatively to* $\mathcal{A}$*.*

Proof of Theorem 4.2. The proof of this theorem is similar to the one of Theorem 2.9 in Chapter 2. The difference is that the time between two consecutive measurements τ_m is constant. In addition, we use here different observer gain matrices $\underline{L}$ and $\overline{L}$ for the lower and the upper estimates, respectively. □

Remark 4.3. A necessary condition on the existence of observers for the system (4.2) is the detectability of the pair $(e^{A\tau_m}, Ce^{A\tau_m})$. More details about this condition are given in Raff and Allgower (2007).

So far, a verification method has been given. The synthesis of the observation gains $\underline{L}, \overline{L}$ cannot be achieved using convex solvers (CS) due to the decomposition of $(I_n + L^\bullet C)$. However, using the positive realization of these matrices, the synthesis is still possible using CS. In the following section, we propose a synthesis methodology.

4.3 Synthesis method

In this section, we propose a new design methodology as second contribution of this chapter. We will show how to design the observer gain based on positive system theory.

4.3.1 Positive realization based approach

Let us now re-consider the generic reset equation of the system state at measurement instant in (4.6). By introducing $\underline{G} = [I + \underline{L}C]$ and $\overline{G} = [I + \overline{L}C]$, the resetting matrix can be written as

$$\begin{aligned} x(t_k^+) &= G^\bullet x(t_k) + L^\bullet[v(t_k) - y(t_k)] \\ &= (G^{\bullet+} - G^{\bullet-})x(t_k) + L^\bullet[v(t_k) - y(t_k)] \quad k \in \mathbb{N} \end{aligned} \tag{4.14}$$

where $G^{\bullet+}$ and $-G^{\bullet-}$ are the positive and the negative part of the matrix $G^\bullet \in \{\underline{G}, \overline{G}\}$, respectively.

Let us note that for any positive matrices $\underline{G}_p$, $\underline{G}_n$, $\overline{G}_p$, $\overline{G}_n \in \mathbb{R}^{n\times n}_{\geq 0}$ satisfying $\underline{G} = \underline{G}_p - \underline{G}_n$ and $\overline{G} = \overline{G}_p - \overline{G}_n$ there exist $\underline{\Delta}, \overline{\Delta} \in \mathbb{R}^{n\times n}_{\geq 0}$ such that

$$G^\bullet = (G^{\bullet+} + \Delta^\bullet) - (G^{\bullet-} + \Delta^\bullet) \tag{4.15}$$

that is, the matrices $G_p^\bullet$ and $G_n^\bullet$ are any positive realization of the matrices $G^{\bullet+}$ and $G^{\bullet-}$, respectively. Under the positive realization of the reset matrix G, the reset equation of the estimation error (4.8) can be generalized by the following difference equation

$$\begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(G_p^\bullet, G_n^\bullet) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \Upsilon(t_k) \tag{4.16}$$

where

$$\Gamma(G_p^\bullet, G_n^\bullet) = \begin{bmatrix} \underline{G}_p & \underline{G}_n \\ \overline{G}_n & \overline{G}_p \end{bmatrix} \tag{4.17}$$

Therefore, the idea for the synthesis is to calculate numerically the positive matrices $G_p^\bullet$ and $G_n^\bullet$ that satisfy the stability conditions. Then, one can compute directly the matrices $G^{\bullet+}$ and $G^{\bullet-}$ from the relation $G^\bullet = G_p^\bullet - G_n^\bullet$.

Using (4.17) instead of $\Gamma(\underline{L}, \overline{L})$ in inequality (4.13), and introducing

$$\Phi(P, G_p^\bullet, G_n^\bullet) = \Gamma(G_p^\bullet, G_n^\bullet)^\top e^{\overline{A}^\top \tau_m} P e^{\overline{A}\tau_m} \Gamma(G_p^\bullet, G_n^\bullet) - P, \tag{4.18}$$

the gain synthesis can now be performed by finding a solution $\{P, \underline{G}_p, \underline{G}_n, \overline{G}_p, \overline{G}_n, \underline{L}, \overline{L}, \}$ to the following feasibility problem

$$\Phi(P, G_p^\bullet, G_n^\bullet) \prec 0\,, \tag{4.19a}$$

$$I_n + \underline{L}C = \underline{G}_p - \underline{G}_n\,, \tag{4.19b}$$

$$I_n + \overline{L}C = \overline{G}_p - \overline{G}_n\,, \tag{4.19c}$$

$$\underline{G}_p \geq 0,\ \underline{G}_n \geq 0\,, \tag{4.19d}$$

$$\overline{G}_p \geq 0,\ \overline{G}_n \geq 0\,, \tag{4.19e}$$

$$P \succ 0 \tag{4.19f}$$

From equation (4.15) and based on the definition of the positive matrices $G^{\bullet+}$ and $G^{\bullet-}$ and their positive realization $G_p^\bullet$ and $G_n^\bullet$, respectively, the reset equation (4.16) can be seen as a positive discrete time system whose state matrix is perturbed by a nonnegative matrix as follows

$$\Gamma(G_p^\bullet, G_n^\bullet) = \Gamma(\underline{L}, \overline{L}) + \begin{bmatrix} \underline{\Delta} & \underline{\Delta} \\ \overline{\Delta} & \overline{\Delta} \end{bmatrix} \tag{4.20}$$

Remark 4.4. Since the matrices $\Delta^\bullet$ are nonnegative which implies that $\begin{bmatrix} \underline{\Delta} & \underline{\Delta} \\ \overline{\Delta} & \overline{\Delta} \end{bmatrix}$ is also nonnegative, it is always possible to enhance the interval observer dynamics at jumps in (4.16) by reducing the matrix $\Gamma(G_p^\bullet, G_n^\bullet)$ in (4.20) to its optimal realization $\Gamma(\underline{L}, \overline{L})$.

4.3.2 Design procedure

The semi-definite programming (SDP) (4.19) is subjected to a Nonlinear Matrix inequality, which is hard to solve. The constraint $\Phi \prec 0$ can be relaxed to a Linear Matrix Inequality (LMI) in the following Corollary. This relaxed constraints rely also on M-matrices which have by construction nonnegative inverse matrices (Berman and Plemmons, 1994, Chapter 6).

Corollary 4.5. *Let Assumption 4.2 and 4.3 hold. If there exist nonnegative matrices* $\underline{U}_p, \underline{U}_n, \overline{U}_p, \overline{U}_n \in \mathbb{R}^{n\times n}$, *M-matrices* $F_1, F_2 \in \mathbb{R}^{2n\times 2n}$ *and two matrices*

$\underline{X}, \overline{X} \in \mathbb{R}^{n\times 1}$ *such that the constraints*

$$\begin{bmatrix} e^{\overline{A}^\top \tau_m} P e^{\overline{A}\tau_m} - F - F^\top & U \\ \star & -P \end{bmatrix} \prec 0\,, \tag{4.21a}$$

$$F_1 + \underline{X}C = \underline{U}_p - \underline{U}_n\,, \tag{4.21b}$$

$$F_2 + \overline{X}C = \overline{U}_p - \overline{U}_n\,, \tag{4.21c}$$

with $U = \begin{bmatrix} \underline{U}_p & \underline{U}_n \\ \overline{U}_n & \overline{U}_p \end{bmatrix}$, *and* $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$ *are feasible, then the interval observer of the form* (4.3), (4.4) *and* (4.7) *with gains* $\underline{L} = F_1^{-1}\underline{X}, \overline{L} = F_2^{-1}\overline{X}$ *for the system (respectively, noise-free system)* (4.2) *has: (i) a GES estimation error with respect to the set* $\mathcal{A}$ *in the ideal case (noise-free system), (ii) an ISS estimation error with respect to the set* $\mathcal{A}$ *in the presence of measurement noise.*

Proof of Corollary 4.5. The semi-definite constraints to be simplified are the following

$$\begin{aligned} \Phi(P, G_p^\bullet, G_n^\bullet) &\prec 0 \\ -P &\prec 0 \end{aligned} \tag{4.22}$$

These constraints can be combined in one inequality using the projection lemma Pipeleers et al. (2009). The resulting inequality from (4.22) is given as follows

$$\begin{bmatrix} e^{\overline{A}^\top \tau_m} P e^{\overline{A}\tau_m} - F - F^\top & F\Gamma(G_p^\bullet, G_n^\bullet) \\ \star & -P \end{bmatrix} \prec 0 \tag{4.23}$$

It is worth noting that the matrix F is an unstructured matrix issued from the projection lemma procedure, but in our corollary we make it M-matrix which has positive inverse matrix. Pick $F\Gamma(G_p^\bullet, G_n^\bullet) = U$ this implies that the matrix $\Gamma(G_p^\bullet, G_n^\bullet) = F^{-1}U$ is nonnegative because F is M-matrix. Pre-multiplying (4.21b) and (4.21c) by F_1^{-1} and F_2^{-1}, respectively, one gets (4.19b) and (4.19c). This completes the proof □

4.4 Online observer selection strategy

So far, we have designed an interval observer for each sensor under the assumption of attack-free sensors. Now, we return to the initial problem where an unknown subset of sensors in (4.1) are under attack. This subset is defined as $S \subset \mathcal{I}$ with $\text{card}(S) = \mathbf{s_a}$. The complement set of S relatively to $\mathcal{I}$ is $S^c = \mathcal{I} \setminus S$ such that $a_\sigma(t_k) = \mathbf{0}$ if $\sigma \in S^c$. Based on this assumption, there exist at least $\mathbf{s} - \mathbf{s_a}$ attack-free sensors that can provide the true estimate. The idea of the proposed selection strategy is from $\mathbf{s}$ sensors select a combination of $\mathbf{s} - \mathbf{s_a}$ sensors and check there intersection using interval analysis. Thus, the number of combination of sensors sets in which only one set contains attack-free sensors is $N_b = \binom{\mathbf{s}}{\mathbf{s} - \mathbf{s_a}} = \frac{\mathbf{s}}{(\mathbf{s}-\mathbf{s_a})!\mathbf{s_a}!}$.

Definition 4.6. A sensor attack a_i is called distinguishable if the attacked estimates $(\underline{x}_i^+(a_i), \overline{x}_i^+(a_i))$ and the free-attack estimates $(\underline{x}_\sigma^+(a_{\sigma,0}), \overline{x}_\sigma^+(a_{\sigma,0}))$ satisfy

$$\mathbf{int}(\underline{x}_i^+(a_i), \overline{x}_i^+(a_i)) \cap \mathbf{int}(\underline{x}_\sigma^+(a_{\sigma,0}), \overline{x}_\sigma^+(a_{\sigma,0})) = \emptyset \quad \forall \sigma \in \mathcal{I} = \{1, \ldots, \mathbf{s}\}$$

where $a_i \neq 0$ and $a_{\sigma,0} = 0$.

Remark 4.7. By nonnegativity argument of interval estimation errors, the estimate enclosures from attack-free sensors always intersect each others. Based on Assumption 4.1, if there exist $\mathbf{s_a}$ attacked sensors with distinguishable attacks s.t. $\frac{\mathbf{s}}{2} - 1 \leq \mathbf{s_a} < \frac{\mathbf{s}}{2}$, then there exists only one set of $\mathbf{s} - \mathbf{s_a}$ free sensors for which the intersection of the interval estimates in not empty. This makes the main idea of the proposed attack-resilient strategy in Algorithm 1. In general situations where the $\mathbf{s_a}$ sensors are not fully attacked, then there exist at least one set of $\mathbf{s} - \mathbf{s_a}$ free sensors which provides interval estimates with non-empty intersection.

Discussion on Algorithm 1 In this algorithm, it is assumed that all attacks are distinguishable. The algorithm receives corrections $(\underline{x}_\sigma^+, \overline{x}_\sigma^+)$ from $\mathbf{s}$ sensors. The combination of estimated intervals to be tested is calculated offline based on the knowledge of the number of attacked sensors $\mathbf{s_a}$. We define Σ as the family of sets $S^c \subset \mathcal{I}$ such that $\text{card}(S^c) = \mathbf{s} - \mathbf{s_a}$.

For instance, if we have a system with $\mathbf{s} = \mathbf{5}$ sensors among which $\mathbf{s_a} = \mathbf{2}$ sensors are under cyber-attack, then we have $\mathcal{I} = \{1, 2, 3, 4, 5\}$ and $N_b = 10$ combination

Algorithm 1: Selection strategy for attack-free estimate

Input : $(\underline{x}_\sigma^+, \overline{x}_\sigma^+) = Jump(\underline{x}_\sigma, \overline{x}_\sigma) \quad \sigma \in \mathcal{I},\ card\{I\} = \mathbf{s}$,
Number of sensor combination: $N_b = \frac{\mathbf{s}}{(\mathbf{s}-\mathbf{s_a})!\mathbf{s_a}!}$,
Set Σ contains all possible sets.

Output: Selection of the attack-resilient correction: $(\underline{x}_{\sigma^*}^+, \overline{x}_{\sigma^*}^+)$

1 **for** $i = 1$ **to** N_b **do**
2 $\quad S^c \in \Sigma$;
3 $\quad \overline{\pi}_{S^c} := \{\overline{x}_\sigma^+ \ \mid \ \sigma \in S^c\}$;
4 $\quad \underline{\pi}_{S^c} := \{\underline{x}_\sigma^+ \ \mid \ \sigma \in S^c\}$;
5 $\quad W_{S^c} := \max\left\{\mathbf{0}, \min(\overline{\pi}_{\mathbf{S^c}}) - \max(\underline{\pi}_{\mathbf{S^c}})\right\}$;
6 **end**
7 Define the sets of intersected estimates

$$\Sigma_\cap := \{S^c \in \Sigma \ : \ ||W_{S^c}||_0 = n\}$$

8 Select the set $\Sigma_\cap^*$ which provides an intersection of minimum size
9 Find the best estimate from attack-free sensors

$$\sigma^* := \underset{\sigma \in S^c, S^c \in \Sigma_\cap^*}{\operatorname{argmin}} \ ||W_{S^c} - (\overline{x}_\sigma^+ - \underline{x}_\sigma^+)||_2$$

of sets

$$\begin{aligned} S^c \in \Sigma = &\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\\ &\{2,4,5\},\{2,3,5\},\{3,4,5\}\}. \end{aligned}$$

There exist thus at least one set with attack-free sensors.

- In line 1 - line 6, we compute the intersection of interval estimates $(\underline{x}_\sigma^+, \overline{x}_\sigma^+)\ \forall \sigma \in S^c$. This procedure is repeated for each combination set $S^c \subset \mathcal{I}$ with $\text{card}(S^c) = \mathbf{s} - \mathbf{s_a}$ for a total of N_b possible combinations which is calculated *a priori*.

- In line 7, we select only sets whose estimates intersect by checking the $zero-$norm of the vector W_{S^c}. If the intervals $[\underline{x}_\sigma^+, \overline{x}_\sigma^+],\ \sigma \in S^c$ have empty intersection, then $\min(\overline{\pi}_{S^c}) \not> \max(\underline{\pi}_{S^c})$, thus W_{S^c} has at least one zero element. If any vector W_{S^c} has a zero element, then its $zero-$norm is less than n. Thus, its corresponding set S^c is excluded from $\Sigma_\cap$.

- In line 8, we select the set S^c which provides the intersection of estimate of minimum size. This step is only executed in the case when the actual

number of attacked sensors is less than $\mathbf{s_a}$ which is only an upper bound on the number of attacked sensors. By recalling the above example, if the number of actual attacked sensors is exactly $\mathbf{s_a} = \mathbf{2}$, and assuming that the set of attacked sensors is $S = \{1, 2\}$, then the sets with attack-free sensors are only $S^c = \{3, 4, 5\}$. Hence, we have $\Sigma_{\cap}^* = \Sigma_{\cap} = S^c$. Contrariwise, if the set of attacked sensors is $S = \{1\}$, then the sets with attack-free sensors are $S^c \in \{\{2, 3, 4\}, \{2, 4, 5\}, \{2, 3, 5\}, \{3, 4, 5\}\}$. In this case, we need to find the best set. To this end, we select the one which provides an interval estimate with the smallest size.

- In line 9, we select the tightest estimate in the selected set.

4.5 Illustrative examples

In order to illustrate the performance of the proposed observer against cyber-attacks, we consider the following examples

4.5.1 Example 1: Unmanned Ground Vehicle (UGV) system

This example is borrowed from (Xie and Yang, 2018) and (Shoukry and Tabuada, 2016)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix}}_{\text{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\text{B}} F$$

where $x_1(t)$ and $x_2(t)$ are the UGV position and the linear velocity, respectively. m and b are the mechanical mass and the translational friction coefficient, respectively. The input to the UGV is the force F. The UGV is equipped with 3 GPS sensors, which measure its position in discrete times. The considered outputs are as follows

$$y_\sigma(t_k) = C_\sigma x(t_k) + v_\sigma(t_k) + a_\sigma(t_k), \quad \sigma \in \mathcal{I} = \{1, 2, 3\}$$

with $C_1 = C_2 = C_3 = [1 \;\; 0]$, where $a_\sigma(t_k)$ are attack signals. $v_\sigma(t_k)$ are measurement noises. In our experiments, the parameters are specified as $m = 0.8$ and

$b = 1$, the measurement period $t_{k+1} - t_k = \tau_m = 0.5$. We have the number of sensors $\mathbf{s} = card\{\mathcal{I}\} = 3$, thus the maximum attacked sensors is $\mathbf{s_a} = 1 < \frac{\mathbf{s}}{2}$.
The detectability of the pairs $(e^{A\tau_m}, Ce^{A\tau_m})\ \ \forall\sigma \in \mathcal{I}$ are satisfied. To synthesis our set of interval observers, we solve the design problem in Corollary 4.5 only once due to the fact that $C = C_1 = C_2 = C_3$. The constraints (4.21) are solved using the YALMIP toolbox (Lofberg, 2004) based on the SDPT3 solver. The obtained observer gains are as follows $\underline{L}_1 = \underline{L}_2 = \underline{L}_3 = [-1 \quad 0.0006]^\top$ and $\overline{L}_1 = \overline{L}_2 = \overline{L}_3 = [-1 \quad 0.0002]^\top$.
The number of combination is $N_b = 3$ with the sets of possible attacked sensors are $S \in \{\{1\}, \{2\}, \{3\}\}$, their complement are $S^c \in \{\{1,2\}, \{2,3\}, \{1,3\}\}$. In sets S^c there exists only one set with attack-free sensors. Our proposed selection strategy in Algorithm 1 selects the set S^c whose sensors provide non-empty intersected interval estimates, then in the selected set, the best estimate is selected based on the criterion of line 9 in Algorithm 1.
For simulation, the output noise is $v_\sigma(t_k) = \cos(2t_k) \leq \overline{v}_\sigma = 1 \quad \forall\sigma$, and $F = 10(\sin(10t) + \cos(40t))$. The attack is simulated as $[a_1(t_k)\ a_2(t_k)\ a_3(t_k)]^\top = [0\ 0\ 0]^\top \forall t_k < 1.5s$ and $[a_1(t_k)\ a_2(t_k)\ a_3(t_k)]^\top = [0\ 0\ -20]^\top \forall t_k \geq 1.5s$
The simulation results are given in Figure 4.1 and 4.2. In Figure 4.1, the attack-free estimate bounds are selected by Algorithm 1, which guarantees the nonnegativity of the estimation errors. It is noticeable that the jump part of the interval impulsive observer contracts significantly the estimation errors comparing to the open-loop estimation. In Figure 4.2, it is shown how the attacked estimate position behaves compared to the attack-resilient one.
The UGV system in Example 1 is a cooperative system. In order to show the efficiency of the proposed method, we apply it on a non-cooperative system in the following example.

4.5.2 Example 2: Academic system (Non-cooperative system)

Let us consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

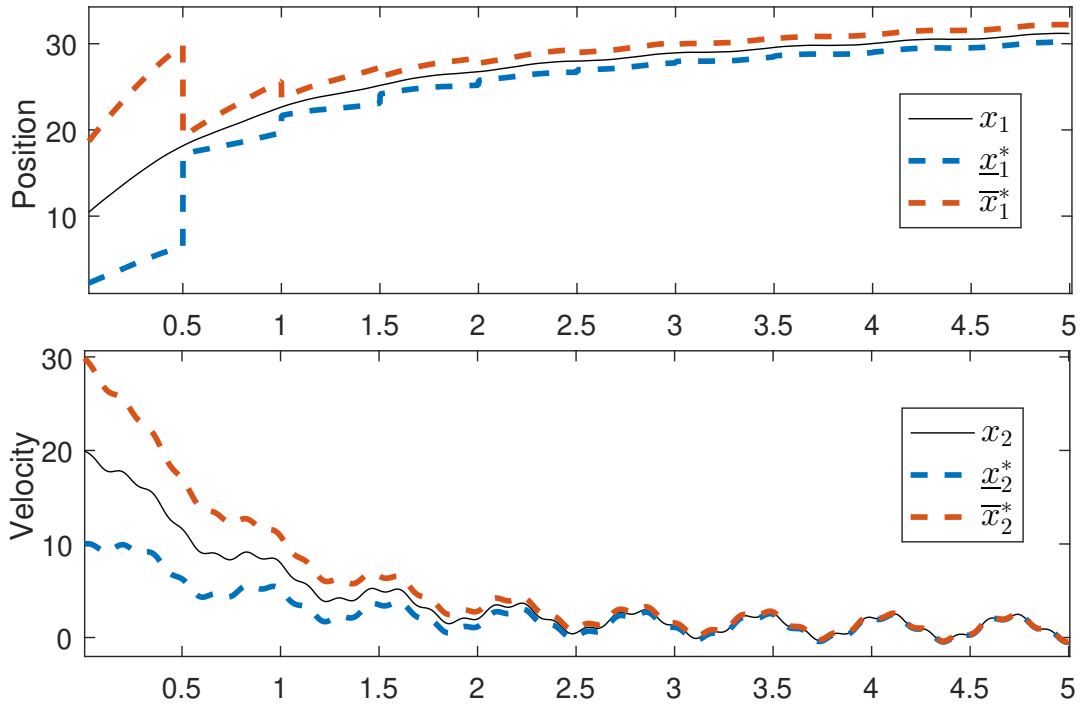


FIGURE 4.1: Simulation results for the UGV system: the attack-resilient estimate bounds for the position (top), and velocity (bottom).

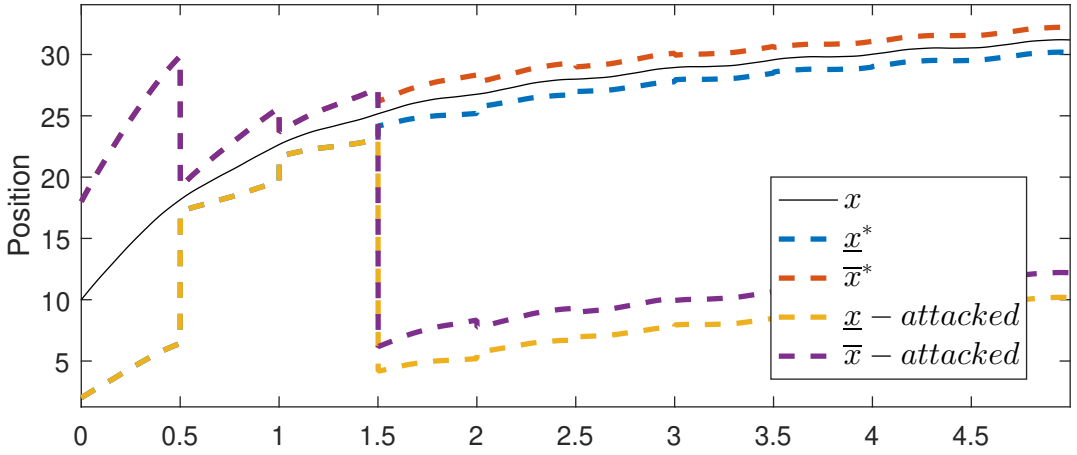


FIGURE 4.2: Position estimate bounds $(\cdot)^*$ selected by Algorithm 1 and the attacked position.

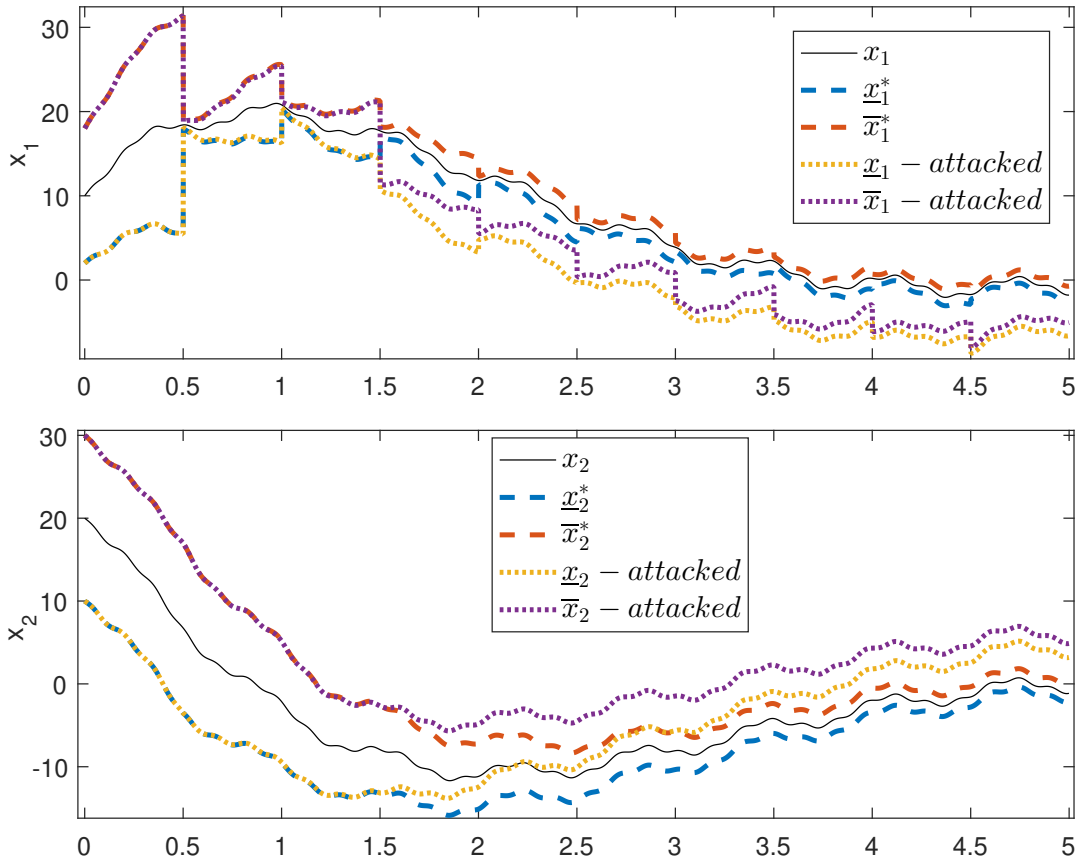


FIGURE 4.3: Simulation result for the academic system: the estimate bounds for the states x_1 (top) and x_2 (bottom), both attack-resilient estimate (dashed line) and attacked one (dotted line).

the outputs have the same form of (4.1) with $C_1 = [2 \quad 0]$, $C_2 = [1 \quad 0]$ and $C_3 = [3 \quad 0]$ which satisfy the detectability condition in Remark 4.3. We solve the design problem in Corollary 4.5 for each output matrix ($\forall \sigma \in \mathcal{I}$) separately by picking $C = C_\sigma$. The designed observation gains are obtained as

$$\underline{L}_1 = \begin{bmatrix} -0.5 \\ 0.0004 \end{bmatrix}, \quad \underline{L}_2 = \begin{bmatrix} -1 \\ 0.0007 \end{bmatrix}, \quad \underline{L}_3 = \begin{bmatrix} -0.3333 \\ 0.0001 \end{bmatrix}$$

$$\overline{L}_1 = \begin{bmatrix} -0.5 \\ 0.0002 \end{bmatrix}, \quad \overline{L}_2 = \begin{bmatrix} -1 \\ 0.0001 \end{bmatrix}, \quad \overline{L}_3 = \begin{bmatrix} -0.3333 \\ 0.0002 \end{bmatrix}.$$

For brevity of presentation, we use the same simulation conditions as of Example 1 with $u = F$. The simulation results are given in Figure 4.3. In Figure 4.3, it is clear that the observer whose sensor is under attack provides erroneous estimate

bounds. On the other hand, our proposed algorithm is able to provide a secure estimate bounds among the set of available ones under cyber-attacks.

4.6 Conclusion

In this chapter, the attack-resilient state estimation problem for linear systems with adversarial cyber-attacks has been studied. The interval impulsive observers for linear continuous-time systems with discrete measurement developed in Chapter 2 has been exploited to provide a secure estimator. Using the positivity of the interval estimation errors, a new strategy for sensor attack-resilient state estimation has been proposed. Although, the strategy is independent to the periodicity of measurements, to simplify the problem presentation, we have considered that measurements are periodic. The synthesis of the observation gains is performed using LMIs. The proposed approach have relaxed the continuity of measurement in Chong et al. (2015) while ensuring a continuous estimate. Simulation examples show the efficiency of the proposed secure estimation approach for a class of linear systems.

Conclusion and perspectives

Conclusions

In this thesis, we have focused on the design of interval state observers for continuous-time systems connected over network. The systems are subject to disturbances, and their measurements are also subject to noises and/or adversary cyber-attacks. This problem is considered from two angles. First, when the network presents some communication constraints; such as the data transmitted over this network need not be obtained periodically. Second, when the communication network is vulnerable to cyber-attacks.
Here, we outline the key contributions of the thesis as follows.

In Chapter 2, we have introduced a new approach for designing interval impulsive observers for linear systems with aperiodic discrete measurement. Exploiting the internal positivity representation of the system, a new method has been proposed for the design of the interval impulsive observer gain. The synthesis of the observation gain is performed using BMIs method.

In Chapter 3, we have proposed a co-design method for both the event-triggered mechanism and the interval observer for linear continuous-time systems. The proposed event-triggered mechanism is based on the positivity property of the interval observation errors. Moreover, Further conditions are provided to guarantee the existence of a positive lower bound on the inter-event times. The proposed observers also ensure a finite $\mathcal{L}_1$-gain between the width of perturbation bounds and the width of the estimated state intervals.

In Chapter 4, based on the interval impulsive observers for linear continuous-time systems with discrete measurement developed in Chapter 2, the case of periodic discrete measurement has been exploited to provide a secure estimator. Using the positivity of the interval estimation errors, a new strategy for sensor attack-resilient state estimation has been proposed. The synthesis of the observation gains is performed using LMIs method. The proposed approach has relaxed the continuity assumption on the measurement required in Chong et al. (2015), while ensuring a continuous estimate.

Perspectives

The work presented in this thesis opens some directions for future developments. We describe below those that we think are promising ones.

Optimal interval predictor In the framework of the proposed interval impulsive observer, the design of the open-loop predictor in between two consecutive measurements is based on the Müller's existence theorem which is not necessarily the optimal way to get the tightest state enclosure. Thus, it is of interest to develop an optimization algorithm that helps finding the optimal realization which generates the tightest state enclosure.
It is also of great interest, to extend this approaches to nonlinear system, e.g., systems that can be over-approximated by linear system as in Etienne et al. (2017).

Robust control The developed approaches have considered the connection over network only on the measurements while the control input is assumed continuously available. A future direction of this thesis is to consider the case where both control inputs and measurements are connected over network. Then, the design of an event-triggered output feedback stabilizing law can also be intended.

Secure estimation In the case of multiple output systems, the M-observability (Chong et al., 2015) of the system can be also considered as perspective in the interval observation approaches. This property can relax the assumption on the observability of the system from each output.

Appendix A

Tools for chapter 3

A.1 Relation between errors $\psi(t)$, $\xi(t)$ and widths $\delta(t)$, $\omega(t)$ norms

Suppose that Assumption 3.1 is satisfied, then we have $\psi(t) = \begin{bmatrix} d(t) - \underline{d}(t) \\ \overline{d}(t) - d(t) \end{bmatrix} \geq 0$. By definition $|\delta(t)|_1 = \sum_{i=1}^{n_d} |\delta_i(t)| = \mathbb{1}_{n_d}^\top \delta(t)$ and using (3.22) we obtain

$$\begin{aligned} |\delta(t)|_1 &= \left| [I_{n_d} \; I_{n_d}] \psi(t) \right|_1 \\ &= \left| [I_{n_d} \; I_{n_d}] \begin{bmatrix} d(t) - \underline{d}(t) \\ \overline{d}(t) - d(t) \end{bmatrix} \right|_1 \\ &= \left| (d(t) - \underline{d}(t)) + (\overline{d}(t) - d(t)) \right|_1 \\ &= \left| \overline{d}(t) - \underline{d}(t) \right|_1 \end{aligned}$$

and

$$\begin{aligned} |\psi(t)|_1 &= \left| \begin{bmatrix} d(t) - \underline{d}(t) \\ \overline{d}(t) - d(t) \end{bmatrix} \right|_1 \\ &= \left| d(t) - \underline{d}(t) \right|_1 + \left| \overline{d}(t) - d(t) \right|_1 \\ &= \mathbb{1}_{n_d}^\top [d(t) - \underline{d}(t)] + \mathbb{1}_{n_d}^\top [\overline{d}(t) - d(t)] \\ &= \mathbb{1}_{n_d}^\top \left[(d(t) - \underline{d}(t)) + (\overline{d}(t) - d(t)) \right] = \left| \overline{d}(t) - \underline{d}(t) \right|_1 \end{aligned}$$

Thus, it is explicit that $|\delta(t)|_1 = |\psi(t)|_1$. By the same steps, based on the inclusions $\underline{x} \leq x \leq \overline{x}$, we deduce that $|\omega(t)|_1 = |\xi(t)|_1$.

A.2 Integral of rational functions

Lemma A.1. *Given the differential equation* $\dot{\overline{\kappa}} = a + b\overline{\kappa}$ *with* $\overline{\kappa}(t_0) = \overline{\kappa}_0$, $\forall t_0 \geq 0$. *If* a *and* b *are positive, then the differential equation has a monotone increasing solution* $\overline{\kappa}(t, \overline{\kappa}_0)$, $\forall t \geq t_0$.

Proof. The system can be rewritten as

$$\frac{d\overline{\kappa}}{dt} = a + b\overline{\kappa}$$

By integration from $\overline{\kappa}_0$ to $\overline{\kappa}(t, \overline{\kappa}_0)$, one gets

$$t - t_0 = \Big[\ln(|a + b\overline{\kappa}|) \Big]_{\overline{\kappa}(t_0)}^{\overline{\kappa}(t)}$$

The function $f_1(\overline{\kappa}) = \ln(|a+b\overline{\kappa}|)$ is an increasing function, thus, for any $t_1, t_2 \in \mathbb{R}_{\geq 0}$ the relation $\overline{\kappa}(t_1) < \overline{\kappa}(t_2)$ implies $t_1 < t_2$. □

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Djahid RABEHI

Estimation par intervalles des systèmes cyber-physiques

Résumé :

Les systèmes cyber-physiques sont des intégrations intelligentes de calculateurs, de réseaux de communications, et de processus physiques. Dans cette thèse, nous travaillons dans le contexte erreur inconnue mais bornée de borne connue, et nous nous intéressons à l'estimation d'état des systèmes dynamiques sous contraintes de communication. Nous proposons des méthodes de synthèse d'observateurs par intervalles pour des systèmes linéaires à temps continu, et dont les mesures à temps discret sont transmises à travers un réseau de communication.

Les contributions de cette thèse sont les suivantes: (i) nous concevons un observateur impulsif par intervalles pour des systèmes linéaires à temps continu avec des mesures sporadiques; (ii) nous proposons un observateur impulsif par intervalles avec gain L_1 fini et échantillonnage contrôlé, puis, nous développons une méthode de synthèse pour concevoir simultanément le gain d'observation et la condition de contrôle de l'échantillonnage des mesures; (iii) en utilisant l'observateur impulsif par intervalles proposé dans (i), nous développons une stratégie d'estimation sécurisée pour des systèmes soumis à des cyber-attaques.

Mots clés : Estimation par intervalles ; Estimation à échantillonnage contrôlées ; Systèmes contrôlés en réseau ; Estimation sécurisé ; Systèmes dynamiques hybrides ; Systèmes linéaire à temps continu.

Interval estimation for cyber-physical systems

Abstract:

Cyber-Physical Systems are smart integrations of computation, networking, and physical processes. In this thesis, we deal with interval observers for cyber-physical systems in which the continuous-time physical systems are estimated and monitored using discrete-time data transmitted over network.

The contributions of the presented material are threefold: (i) we design an interval impulsive observer for continuous-time linear systems with sporadic discrete outputs; (ii) we propose a finite L_1-gain event-triggered interval observer for continuous-time linear systems, in which we develop a co-design procedure to simultaneously design the observer gain and the event-triggering condition; (iii) using the interval impulsive observer, we develop a secure estimation strategy for multi-output system under cyber-attacks

Keywords: Interval estimation; Event-triggered estimation; Networked controlled systems; Secure estimation; Hybrid dynamical systems; Linear continuous-time systems.

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