Qualitative study of three nonlinear parabolic problems

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Étude qualitative de trois problèmes paraboliques non-linéaires

Résumé

Cette thèse est consacrée à l'étude de trois problèmes paraboliques non linéaires :

Premièrement, nous considérons un modèle de systèmes micro-électro-mécaniques (MEMS) avec permittivité diélectrique variable. Le modèle est basé sur une équation parabolique avec non-linéarité singulière, qui décrit la déformation dynamique d'une plaque élastique sous les effets d'un potentiel électrostatique. Nous étudions le phénomène de touchdown, ou quenching. Avec le but de contrôler l'ensemble de touchdown, nous donnons des résultats concernant la localisation du touchdown, en termes du profil de permittivité.

Dans la deuxième partie de la thèse, nous étudions une équation de Hamilton-Jacobi avec diffusion dans un domaine borné avec conditions de Dirichlet nulles au bord. On analyse l'explosion du gradient (GBU) qui peut avoir lieu sur le bord du domaine. Dans un article précédent, il a été démontré, pour des domaines très particuliers (domaines localement plats et disques), qu'il est possible de construire des solutions pour lesquelles l'ensemble de GBU est réduit à un seul point. Nous démontrons qu'il est possible de construire ce type de solutions pour une large classe de domaines, où la courbure n'est pas forcement constante près du point de GBU.

Dans la dernière partie de la thèse, nous étudions le problème d'évolution associé à la j-ème valeur propre de la matrice Hessienne. On démontre tout d'abord l'existence d'une (unique) solution de viscosité, qui peut être approximée par la fonction valeur d'un jeu à deux joueurs et somme nulle, quand la longueur du pas du jeu tend vers 0. On démontre ensuite la convergence exponentielle des solutions du problème d'évolution vers l'unique solution stationnaire. Finalement, pour des cas particuliers (avec données au bord affines), on démontre que la solution coïncide avec la solution stationnaire en temps fini.

Mots clés :

- Équations aux dérivées partielles paraboliques
- Touchdown pour un modèle de MEMS
- Équation de Hamilton-Jacobi diffusive
- Évolution pour les valeurs propres de la matrice Hessienne

Qualitative study of three nonlinear parabolic problems

Abstract

This thesis is concerned with the study of three nonlinear parabolic problems :

We start with a mathematical model for a micro-electro-mechanical system (MEMS) with variable dielectric permittivity. The model is based on a parabolic equation with singular nonlinearity which describes the dynamic deflection of an elastic plate under the effect of an electrostatic potential. We study the touchdown, or quenching, phenomenon. With the aim of controlling the touchdown set, we give results concerning the touchdown localization in terms of the permittivity profile.

In the second part of the thesis, we study a diffusive Hamilton-Jacobi equation in a bounded domain with zero Dirichlet boundary conditions. We analyze the gradient blow-up (GBU) that solutions can exhibit on the boundary of the domain. In a previous work, it was shown that single-point GBU solutions can be constructed in very particular domains, namely, locally flat domains and disks. We prove the existence of this kind of solutions for a large family of domains, for which the curvature of the domain may be nonconstant near the GBU point.

In the last part of the thesis, we study the evolution problem associated to the j-th eigenvalue of the Hessian matrix. First, we show the existence of a (unique) viscosity solution, which can be approximated by the value function of a two-player zero-sum game as the step length of the game goes to zero. Then, we show that solutions to this evolution problem converge exponentially fast to the unique stationary solution as t goes to ∞. Finally, we show that in some special cases (for affine boundary data) the solution coincides with the stationary solution in finite time.

Keywords :

- Nonlinear parabolic partial differential equations
- Touchdown for a MEMS model
- Diffusive Hamilton-Jacobi equation
- Evolution for the eigenvalues of the hessian matrix

Liste des publications issues de la thèse

[1] C. Esteve, Single-point Gradient Blow-up on the Boundary for Diffusive Hamilton-Jacobi Equation in domains with non-constant curvature. Soumis.
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Preprint arXiv :1901.01052

[3] C. Esteve, Ph. Souplet, No touchdown at points of small permittivity and non-trivial touchdown sets for the MEMS problem. *Advances in Differential Equations,* Vol 24, Number 7-8(2019), 465-500.

[4] C. Esteve, Ph. Souplet, Quantitative touchdown localization for the MEMS problem with variable dielectric permittivity. *Nonlinearity* 31 4883 (2018).

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Chapitre 1

Introduction générale

Cette introduction est consacrée à la présentation des différents objets d'étude de cette thèse, résultats précédents, objectifs qui motivent notre recherche, résultats obtenus et questions qui restent ouvertes.

La thèse porte sur l'étude de trois problèmes différents, qui peuvent être traités de façon indépendante. Par conséquent, l'introduction est divisée en trois sections.

- Étude qualitative et quantitative d'un modèle de systèmes micro-électromécaniques avec permittivité diélectrique variable. (voir Chapitre 2 et Chapitre 3)
- Explosion du gradient pour une équation de Hamilton-Jacobi diffusive. (voir Chapitre 4)
- Une approche basée sur les Tug-of-war games pour le problème d'évolution associé aux valeurs propres de la matrice hessienne. (voir Chapitre 5)

1 Étude d'un modèle de systèmes micro-électromécaniques (MEMS) avec permittivité diélectrique variable

1.1 Intérêt physique du modèle

Les systèmes micro-électromécaniques (MEMS en anglais) englobent une grande variété de dispositifs à échelle microscopique qui font usage de forces électrostatiques dans leur fonctionnement. Aujourd'hui, la variété de ce type de systèmes et de leurs domaines d'application croît de jour en jour, et a entraîné une révolution dans nombreuses branches de la technologie, ainsi que dans notre vie quotidienne. On peut trouver des MEMS comme composants de capteurs modernes, par exemple, ceux utilisés dans les airbags des voitures, mais l'usage des MEMS intervient aussi dans un grand nombre de domaines, aussi divers que l'industrie biomédicale, l'exploration de l'espace ou les télécommunications. Un aperçu du développement des MEMS est donné dans le livre [45].

On peut voir, dans la figure 1.1, une représentation schématique d'un de ces dispositifs. Ici, la partie supérieure est une plaque rigide fixe, alors que la partie inférieure est une plaque élastique, pouvant être déformée, qui est fixée uniquement au bord. Ces deux plaques sont connectées à un circuit électrique, la plaque élastique étant fabriquée à partir d'un matériau diélectrique. Quand un voltage V est appliqué, la différence

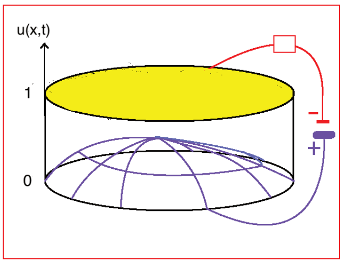

FIGURE 1.1 – Dispositif de MEMS

de potentiel entre les deux plaques produit un champ électrique qui déforme la plaque élastique dans la direction de la plaque fixe.

Dans ce type de systèmes on a deux forces qui entrent en compétition. D'une part, on a la force électrostatique attractive entre les deux plaques, et d'autre part, on a la force de rappel vers la position de repos due à la tension de la plaque élastique.

Il est connu (voir [14, 20, 21, 26, 44]) qu'il existe un voltage critique V^*, appelé *pull-in voltage*, tel que si le voltage appliqué est inférieur à V^*, le système se stabilisera en un état stationnaire. Par contre, si le voltage est augmenté au-delà de V^*, cet état d'équilibre sera perdu et la plaque élastique touchera la plaque fixe en un temps fini. Ce phénomène est connu dans la littérature comme *touchdown* ou *pull-in instability*.

Dans le but de contrôler la zone de contact entre les deux plaques, en pratique, les ingénieurs recouvrent la plaque élastique d'un matériau à permittivité diélectrique variable, conduisant à un champ électrique inhomogène pour une différence de potentiel donnée. Notre but est donc de donner des informations sur la zone de contact, dans le cas où il y a touchdown, en termes du profil de permittivité diélectrique choisi.

Nous allons tout d'abord présenter l'équation aux dérivées partielles qui gouverne la dynamique de la plaque élastique de notre système et qui est l'objet d'étude de cette première partie de la thèse. Pour plus de détails sur la dérivation du modèle mathématique, voir par exemple [14, 26, 35, 45, 46]. On commence par déterminer l'équation pour le potentiel électrostatique. Ensuite, on intégrera la force produite par ce potentiel dans l'équation qui gouverne la dynamique de la plaque élastique. Comme le potentiel électrostatique se produit dans l'espace entre les deux plaques et que la plaque supérieure est déformable, le problème est posé sur un domaine à frontière libre. Après, on fera une hypothèse simplificatrice sur le modèle, qui ramène le problème de frontière libre à une seule EDP. Cette simplification est connue sous le nom de *small aspect ratio*.

On suppose que la plaque fixe est placée à hauteur 1 et que $\Omega \subset \mathbb{R}^2$ représente la forme de cette plaque dans la direction horizontale (voir Figure 1.1). La plaque élastique a la même forme Ω dans la direction horizontale et son bord est fixé à hauteur 0. À chaque point $(x, y) \in \Omega$ et chaque instant $t \geq 0$, la hauteur de la plaque élastique est déterminée par $u(x, y, t) \in]-\infty, 1]$. Alors, pour chaque $t \geq 0$, le domaine entre les deux plaques est défini par

$$\Omega(u(t)) := \{(x, y, z) \in \Omega \times]-\infty, 1] \ ; \ u(t, x, y) < z < 1\}.$$

Dans ce domaine, le potentiel électrostatique $\psi = \psi(x, y, z, t)$ satisfait l'équation de Laplace

$$\varepsilon^2 \psi_{xx} + \varepsilon^2 \psi_{yy} + \psi_{zz} = 0, \qquad t > 0, \ (x, y, z) \in \Omega(u(t)), \tag{1.1.1}$$

où $\varepsilon > 0$ est le ratio entre la hauteur et la longueur du dispositif (*aspect ratio* en anglais). Si l'on considère que la plaque fixe est à potentiel 0 et que la plaque élastique est à potentiel 1, les conditions au bord pour ψ sont données par

$$\psi(t,x,y,z) = \frac{1-z}{1-u(t,x,y)}, \quad t>0,\ (x,y,z) \in \partial\Omega(u(t)). \tag{1.1.2}$$

Maintenant, on intègre la force produite par ce potentiel dans l'équation d'ondes pour une plaque élastique amortie. On obtient que la dynamique de la plaque élastique est gouvernée par l'équation

$$\gamma^2 u_{tt} + u_t - \Delta u + \beta\Delta^2 u = \lambda f(x,y)(\varepsilon^2\psi_x^2 + \varepsilon^2\psi_y^2 + \psi_z^2), \quad t>0,\ (x,y) \in \Omega, \tag{1.1.3}$$

avec données initiales et au bord nulles (on considère qu'à l'instant initial la plaque est au repos). Voir [31, 35] pour plus de détails sur la dérivation de cette équation.

Le paramètre γ dans (1.1.3) représente l'importance relative des effets d'inertie par rapport à l'amortissement. Le terme $\beta\Delta^2 u$ fait référence à la force de flexion due à la rigidité de la plaque et le terme Δu est dû à la tension de la plaque. Le paramètre $\lambda > 0$ est proportionnel au carré du voltage appliqué, qui dans notre modèle est constant. Finalement, la fonction $f(x,y) = \dfrac{\varepsilon_0}{\varepsilon_1(x,y)}$ caractérise la permittivité diélectrique du matériau, qui n'est pas homogène et c'est, en effet, le degré de liberté qui nous permet de contrôler la zone de touchdown. Ici, la constante ε_0 représente la permittivité diélectrique du vide, et $\varepsilon_1(x,y)$, la permittivité diélectrique du matériau. Alors, pour éviter que le touchdown se produise sur une certaine région de Ω, on recouvrira cette zone avec un matériau avec permittivité diélectrique ε_1 grande, donnant lieu à un champ électrique petit sur cette zone. Dans la suite, et sans risque de confusion, nous utiliserons toujours la fonction $f(x,y)$ pour faire référence à la permittivité diélectrique.

Comme annoncé, on va considérer la simplification connue comme *small aspect ratio*, qui nous permet de réduire le problème de frontière libre à l'étude d'une seule EDP. Cette simplification consiste à supposer que la distance entre les deux plaques est négligeable par rapport à la longueur du dispositif. C'est-à-dire, on va supposer $\varepsilon = 0$. On voit que la solution explicite de (1.1.1),(1.1.2) est

$$\psi(t,x,y,z) = \frac{1-z}{1-u(t,x,y)},$$

que l'on peut substituer directement dans l'équation (1.1.3) pour obtenir

$$\gamma^2 u_t^2 + u_t - \Delta u + \beta\Delta^2 u = \frac{\lambda f(x,y)}{(1-u)^2}.$$

Maintenant on va supposer que le système est très amorti et on va négliger les effets d'inertie, c.-à-d. $\gamma = 0$. On peut faire cette simplification en supposant que l'épaisseur de la plaque élastique est nulle. On va supposer aussi que la plaque n'a pas de rigidité, c.-à-d. $\beta = 0$.

L'équation obtenue pour modéliser la déviation de la membrane est

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{f(x,y)}{(1-u)^2}, & (x,y) \in \Omega, \quad t>0, \\ u(t,x,y) = 0, & (x,y) \in \partial\Omega, \quad t>0, \\ u(t,x,y) = 0, & (x,y) \in \Omega. \end{cases} \tag{1.1.4}$$

On voit que le phénomène de touchdown, $u = 1$ se traduit dans le modèle mathématique par l'apparition d'une singularité dans la partie droite de l'équation (1.1.4).

1.2 Intérêt mathématique et contexte

D'un point de vue mathématique, le problème (1.1.4) peut être vu comme un exemple d'équation parabolique semi-linéaire avec non-linéarité singulière, et a été l'objet de nombreuses études pendant les vingt dernières années. On considère le problème plus général

$$\begin{cases} u_t - \Delta u = f(x)(1-u)^{-p}, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(0,x) = 0, & x \in \Omega, \end{cases} \qquad (Q_f)$$

où Ω est un domaine borné de $\mathbb{R}^n$, $n \geq 1$, avec bord régulier, $p > 0$ et $f \in E$, où

$$E = \big\{ f : \overline{\Omega} \to [0,\infty[;\ f \text{ est une fontion Hölderienne} \big\}.$$

Par la théorie standard des équations paraboliques, voir par exemple [19], l'équation (Q_f) admet une unique solution classique pour t petit. Par le principe du maximum, comme f est positive, on a $u \geq 0$ pour tout $x \in \Omega$ et $t > 0$. On voit aussi que si u prend la valeur 1, le second membre de l'EDP devient singulier. La solution classique peut donc être prolongée tant que $u < 1$ sur Ω. On appelle $T = T_f \in]0,\infty]$ le temps maximal d'existence de la solution classique u de (Q_f). Alors, on déduit que si $T < \infty$, on a

$$\lim_{t \uparrow T} \|u\|_\infty = 1.$$

Ce phénomène est connu dans la littérature comme *quenching* ou *touchdown*, et dans le modèle physique présenté précédemment, correspond au contact entre les deux plaques.

L'apparition ou non de touchdown est déterminée par l'existence d'une solution stationnaire. Soit $f(x) = \lambda g(x)$, avec $g \in E$ et $\lambda > 0$, et considérons le problème elliptique associé à (Q_f)

$$\begin{cases} -\Delta z = \lambda g(x)(1-z)^{-p}, & x \in \Omega, \\ 0 < z < 1, & x \in \Omega, \\ z = 0, & x \in \partial\Omega. \end{cases} \qquad (S_\lambda)$$

On définit le *pull-in voltage* $\lambda^* = \lambda^*(\Omega, g)$ comme

$$\lambda^* = \sup\{\lambda > 0 \ : \ (S_\lambda) \text{ possède au moins une solution}\}.$$

On note que λ^* est proportionnel au carré du voltage critique V^* dont on a parlé dans la section 1.1.

Dans le cas où (S_λ) admet une solution, cette solution est une sur-solution du problème (Q_f). De plus, la solution de (Q_f) converge vers la seule solution minimale de (S_λ). On peut en déduire que si la solution minimale de (S_λ) est uniformément inférieure à 1, alors la solution du problème d'évolution (Q_f) est globale en temps, c.-à-d. $T = \infty$.

Il est connu que, pour tout g et Ω, on a $0 < \lambda^* < \infty$. Des estimations supérieures et inférieures pour λ^* sont données en [14]. On peut résumer la relation entre le pull-in voltage et l'apparition ou non de touchdown dans la propriété suivante :

- Si $\lambda < \lambda^*$, la solution minimale de (S_λ) est strictement plus petite que 1 et alors la solution de (Q_f) est globale en temps.

- Si $\lambda > \lambda^*$, la solution de (Q_f) n'est pas globale en temps, c'est-à-dire, il y a *touchdown* en temps fini.

Le cas $\lambda = \lambda^*$ est traité dans [14] et le comportement est étroitement lié à la dimension de Ω. Pour ce cas critique, il existe une unique solution stationnaire z^*, qui peut être obtenue comme la limite de la branche de solutions minimales z_λ de (S_λ) quand $\lambda \uparrow \lambda^*$.

Ici, nous avons deux scénarios possibles. D'une part, on peut avoir une solution minimale de (S_{λ^*}) régulière (strictement inférieure à 1), ce qui implique que la solution de (Q_f) avec $f(x) = \lambda^* g(x)$ est globale en temps. C'est le cas quand, par exemple, $n \leq 7$. D'autre part, si z^* n'est pas régulière, c.-à-d. si la branche de solutions minimales z_λ de (S_λ) satisfait

$$\lim_{\lambda \uparrow \lambda^*} \|z_\lambda\|_\infty = 1,$$

alors la solution de (Q_f) avec $f(x) = \lambda^* g(x)$ est globale en temps et converge vers un état stationnaire singulier, donnant lieu au *touchdown* en temps infini (voir [14] et les références citées).

Dans ce travail, on s'intéresse uniquement au cas où il y a touchdown en temps fini, c.-à-d. on considérera toujours $\lambda > \lambda^*$, ou autrement dit, $f \in E$ telle que $T < \infty$. Notre but sera d'étudier la localisation en espace des points de touchdown. On définit un point de touchdown comme $x_0 \in \overline{\Omega}$ tel qu'il existe une suite $(x_n, t_n) \in \Omega \times]0, T[$ satisfaisant

$$t_n \uparrow T, \quad x_n \to x_0, \quad \text{et} \quad u(x_n, t_n) \to 1.$$

Ainsi, on peut définir l'ensemble de touchdown $\mathcal{T} = \mathcal{T}_f \subset \overline{\Omega}$ comme l'ensemble des points satisfaisant cette propriété. On voit que $\mathcal{T}$ est fermé par définition. On remarque aussi que, d'après la définition des points de touchdown, la condition nulle au bord n'exclut pas la possibilité d'avoir des points de touchdown sur le bord.

Si l'on pense au modèle de MEMS, le champ électrique sera plus petit aux points où le profil f est petit, ce qui intuitivement nous amène à penser que le touchdown ne se produira pas sur ces points. En conséquence, la première question que l'on peut se poser est

"peut-on avoir touchdown en des points où f s'annule ?"

Une réponse négative à cette question a été donnée en 2015 par J.S. Guo et Ph. Souplet dans [28] pour les points intérieurs à Ω. Ce résultat n'est pas trivial car, pour le problème analogue de *blow-up* $u_t - \Delta u = f(x)u^p$ avec $f(x) = |x|^\sigma$, des exemples de solutions explosant à l'origine ont été donnés dans [17, 27] pour $\sigma > 0$, $p > 1$ et des données initiales bien choisies.

On rappelle que dans le modèle physique, la permittivité est caractérisée par la fonction $f(x) = \frac{\varepsilon_0}{\varepsilon_1(x)}$. Ainsi, dans la vie réelle il n'est pas possible de construire un MEMS pour lequel f s'annule. Il est donc nécessaire d'améliorer ce résultat en donnant des conditions suffisantes qui nous permettent d'éliminer le touchdown aux points où la permittivité f est strictement positive.

1.3 Résultats I : non-touchdown aux points de permittivité petite

Les démonstrations des résultats de cette section, ainsi que d'autres détails et références, sont inclus dans le Chapitre 2.

Le premier but de la thèse est d'améliorer le résultat de Guo et Souplet [28] en donnant des conditions de petitesse suffisantes pour éliminer le touchdown aux points où f est suffisamment petite. C'est le contenu du théorème suivant, qui assure l'existence d'un seuil strictement positif pour chaque point intérieur du domaine nous permettant d'éliminer le touchdown aux points où f est inférieure à ce seuil. Ici, nous définissons la fonction distance au bord comme

$$\delta(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \overline{\Omega}.$$

Théorème 1.1. *(Non-touchdown aux points intérieurs de permittivité petite) Soit $p > 0$, $\Omega \subset \mathbb{R}^n$ un domaine borné avec bord régulier et $f \in E$. On suppose*

$$\begin{cases} T_f \leq M, \quad \|f\|_\infty \leq M, \quad f \geq r\chi_B, \\ \text{où } M, r > 0 \text{ et } B \subset \Omega \text{ est une boule de rayon } r. \end{cases} \tag{1.3.1}$$

Il existe $\gamma_0 > 0$ dépendant uniquement de p, Ω, M, r tel que, pour tout $x_0 \in \Omega$, si

$$f(x_0) < \gamma_0\, \delta^{p+1}(x_0), \tag{1.3.2}$$

alors $x_0 \notin \mathcal{T}_f$.

Mais ce résultat a un clair inconvénient. Le seuil de la formule (1.3.2) dépend de la fonction distance au bord, et la condition de petitesse nous permettant d'éliminer le touchdown devient plus restrictive pour les points près du bord. De plus, ce résultat ne permet en aucun cas d'éliminer le touchdown aux points du bord.

En effet, la question sur la possibilité d'avoir des points de touchdown sur le bord reste toujours ouverte. Un résultat partiel sur ce problème a été donné dans [24], où il est prouvé, en utilisant un argument de *moving planes*, que pour des domaines convexes, si f est décroissante près du bord dans la direction normale, alors l'ensemble de touchdown est compact dans Ω. Dans [28] Guo et Souplet ont apporté deux nouvelles conditions sous lesquelles le touchdown ne se produit pas sur le bord. Une condition est de supposer que f décroît près du bord plus rapidement que $\delta^{p+1}(x)$, et l'autre est le cas $0 < p < 1$.

Nous donnons une nouvelle contribution partielle à ce problème. Plus précisément, le résultat suivant prouve que le touchdown peut être localisé dans n'importe quel sous-ensemble compact de Ω à condition que f soit suffisamment petite en dehors de ce sous-ensemble. Ce résultat est donc d'un caractère moins local que celui du Théorème 1.1, mais il nous permet d'éliminer le touchdown sur le bord sans faire aucune hypothèse de monotonie sur f ni supposer que f s'annule au bord.

Théorème 1.2. *(Non-touchdown sur le bord pour des permittivités petites sur un voisinage du bord) Soit $p > 0$, $\Omega \subset \mathbb{R}^n$ un domaine borné avec bord régulier et $f \in E$. Supposons* (1.3.1). *Il existe $\gamma_0 > 0$ dépendant uniquement de p, Ω, M, r tel que pour tout $\omega \subset\subset \Omega$, si*

$$\sup_{x \in \overline{\Omega} \setminus \omega} f(x) < \gamma_0 \operatorname{dist}^{p+1}(\omega, \partial\Omega), \tag{1.3.3}$$

alors $\mathcal{T}_f \subset \omega$.

Ces deux théorèmes sont conséquence de la propriété suivante, qui implique que le touchdown à l'intérieur du domaine est de *type I*. Ceci signifie que, sur les points intérieurs à Ω, la solution croît au plus comme la solution de l'EDO associée, c.-à-d. l'équation (Q_f) sans le terme Δu.

Proposition 1.3. *(Estimation de type I) Sous l'hypothèse* (1.3.1), *la solution u du problème (Q_f) satisfait*

$$u(t, x) \leq 1 - \gamma\delta(x)(T - t)^{\frac{1}{p+1}}, \qquad \text{pour tout } t \in [0, T) \text{ et } x \in \Omega, \tag{1.3.4}$$

où γ ne dépend que de p, Ω, M, r.

On remarque que la fonction $\delta(x)$ dans l'estimation (1.3.4) est la raison pour laquelle la distance au bord apparaît dans les seuils des Théorèmes 1.1 et 1.2.

Motivés par les applications pratiques au design de MEMS, une fois obtenues les conditions de petitesse des Théorèmes 1.1 et 1.2 nous permettant l'élimination du touchdown sur certaines régions du domaine, nous nous intéressons à étudier la taille des seuils intervenant dans ces conditions. On fera cette étude pour le cas unidimensionnel en espace, où il est possible de faire les calculs analytiques plus précisément, donnant des estimations assez satisfaisantes pour les seuils. En effet, pour le cas physique $p = 2$ et sous certaines hypothèses sur f, nos méthodes donnent des seuils qui ne sont pas petits mais de l'ordre de 0.3 fois le maximum de f.

1.4 Résultats II : une approche quantitative aux seuils de petitesse en dimension 1

Les démonstrations des résultats de cette section, ainsi que d'autres détails et références, sont inclus dans le Chapitre 3.

Afin de donner de bonnes estimations du seuil, on considérera deux situations typiques, qui correspondent à des profils f avec une et deux bosses respectivement, les résultats pouvant être étendus à des profils avec un nombre fini de bosses. On verra que le touchdown peut être éliminé sur un intervalle situé entre une bosse et le bord, ou entre deux bosses.

L'idée, au niveau pratique, est de considérer une plaque à permittivité constante et de recouvrir les zones où l'on veut éliminer le touchdown d'un matériau qui fait que le profil f soit suffisamment petit. Notre but est donc d'estimer le ratio entre les permittivités des deux zones. Nos méthodes pour l'estimation de ce ratio sont basées sur l'étude minutieuse de la preuve de la Proposition 1.3, pour laquelle on a fait un raffinement assez important des techniques. Ensuite, nous réduisons le calcul à un problème d'optimisation de dimension finie avec trois ou quatre paramètres, où les paramètres sont liés aux choix de certaines fonctions auxiliaires intervenant dans la preuve de l'estimation de type I.

Avant d'énoncer les résultats précis, on va les illustrer avec deux exemples concrets, pour le cas $p = 2$, qui montrent deux situations typiques d'application. Ces deux exemples correspondent à un profil avec une et deux bosses respectivement. La solution du problème d'optimisation qui nous donne le ratio a été approximée numériquement en utilisant une méthode d'approximation assez simple (voir une description dans le Chapitre 3), qui de plus, assure que l'estimation donnée est inférieure à la vraie valeur du ratio. Sur les dessins, la zone où le touchdown peut apparaitre est marquée en trais épais.

Ici, le profil f satisfait

$$\|f\|_\infty \leq 2.25, \qquad f(x) \geq 2 \ \text{ dans } \omega, \qquad f(x) \leq 0.42 \ \text{ dans } \Omega \setminus \tilde{\omega},$$

avec respectivement $\Omega = (-6, 6)$, $\omega = (-2, 0)$, $\tilde{\omega} = (-2.1, 0.1)$ et $\Omega = (-10, 10)$, $\omega = (-1, 1) \cup (4, 6)$, $\tilde{\omega} = (-1.1, 1.1) \cup (3.9, 6.1)$. L'ensemble de touchdown est alors contenu dans $\tilde{\omega}$.

On énonce ci-dessous deux de nos résultats principaux concernant l'estimation quantitative des seuils de petitesse pour la localisation du touchdown. Du premier théorème, il est possible de déduire la localisation du touchdown pour les profils à une bosse, alors que pour les profils à deux bosses ou plus, il faut combiner les deux théorèmes. On

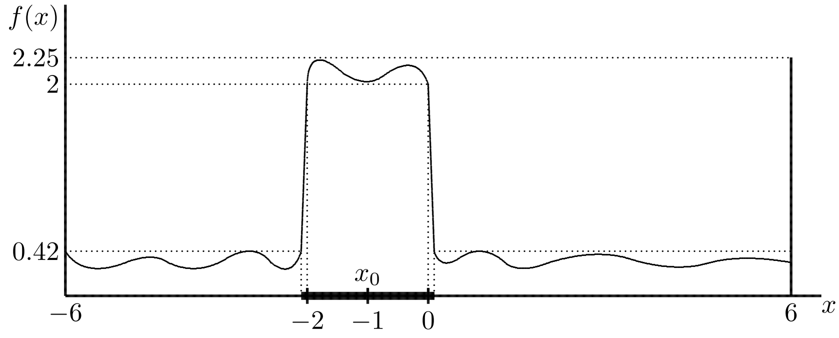

FIGURE 1.2 – Exemple de localisation du touchdown pour un profil à une bosse avec $p = 2$.

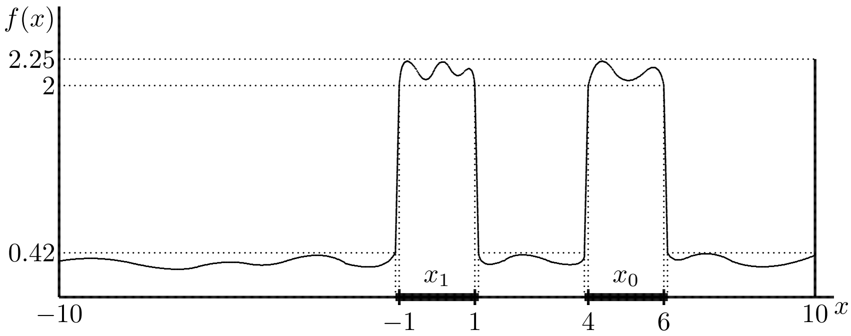

FIGURE 1.3 – Exemple de localisation du touchdown pour un profil à deux bosses avec $p = 2$.

rappelle la fonction d'erreur de Gauss qui est donnée par l'expression

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

On définit aussi $\overline{\cot}\, s = \cot s$ si $0 < s \leq \pi/2$, et $\overline{\cot}\, s = 0$ si $s > \pi/2$.

Théorème 1.4. *On considère le problème (Q_f) avec $p > 0$ et $\Omega = \,]-R, R[$. Soit $x_0 \in \Omega$ et supposons*

$$\mu > \mu_1(p) := \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{2}, \qquad d_0 := R - |x_0| - 1 > \frac{p+1}{\sqrt{p\mu}} \overline{\cot}\big[\sqrt{p\mu}\big]. \tag{1.4.1}$$

Pour tout $d \in \,]0, d_0[$, il existe $\rho = \rho(p, \mu, \|f\|_\infty, d_0, d) \in]0, 1[$ tel que, si f satisfait

$$f \geq \mu \quad \text{dans } \big]x_0 - 1, x_0 + 1\big[\quad \text{et} \quad f < \rho\mu \quad \text{dans } D := [x_0 + 1 + d, R], \tag{1.4.2}$$

alors $T_f < \infty$ et il n'y a pas de points de touchdown dans D.

De plus, ρ peut être choisi comme solution du problème d'optimisation suivant :

$$\rho = \frac{1}{2} \operatorname*{Sup}_{(\tau,\beta,K)\in\mathcal{A}} \left(\frac{\beta - d}{\beta}\right)^{p+1} \frac{S(t_0(\tau), \beta)}{K + \tau^{-p}} \min\Big\{H\big(t_0(\tau), \beta\big), G\big(t_0(\tau), \beta, K\big)\Big\} \tag{1.4.3}$$

avec

$$\mathcal{A} = \Big\{ (\tau, \beta, K) \in]0,1[\times]d, d_0[\times]0, \infty[;\ \tau \geq \tfrac{\mu}{2\mu-\mu_1},\ K \geq \tfrac{p}{\mu\beta^2} - \tfrac{1}{p+1},\ \delta(\beta, K) \leq 1 \Big\}.$$

Ici, on choisit

$$t_0(\tau) = \tfrac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}, \quad L = 1 + (p+1)K, \quad \Gamma = \sqrt{\tfrac{(p+1)L}{pK\mu\beta^2}}, \quad A = \arctan\Gamma, \quad \alpha = 1 + \tfrac{p}{L}$$

et les fonctions S, H, G, δ *sont définies par*

$$\begin{aligned}
&S(t,\beta) = e^{-\frac{\pi^2 t}{4(d_0+1)^2}} \left[1 - e^{-\frac{d_0(d_0-\beta)}{t}}\right], \qquad \delta(\beta, K) = A(1+K)\sqrt{\tfrac{p+1}{pLK\mu}},\\
&H(t,\beta) = \inf_{0<x<1} \frac{\operatorname{erf}\left(\frac{1}{\sqrt{t}}\left(1+\frac{\beta}{2}x\right)\right) - \operatorname{erf}\left(\frac{\beta}{2\sqrt{t}}x\right)}{(1-x)^{p+1}}, \qquad (1.4.4)\\
&G(t,\beta,K) = (\Gamma^2+1)^{-\alpha/2} \inf_{0<x<1} \frac{\operatorname{erf}\left(\frac{2-(1-x)\delta}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{(1-x)\delta}{2\sqrt{t}}\right)}{\cos^\alpha(Ax)}.
\end{aligned}$$

On voit qu'avec le Théorème 1.4, il est possible d'éliminer le touchdown sur un intervalle localisé entre une bosse et le bord. Le Théorème suivant nous permet d'éliminer le touchdown sur un intervalle localisé entre deux bosses. La combinaison de ces deux théorèmes nous permet donc de localiser le touchdown près des bosses pour un profil avec un nombre fini de bosses.

Théorème 1.5. *On considère le problème* (Q_f) *avec* $p > 0$ *et* $\Omega = \,]-R, R[$. *Soient* $x_0, x_1 \in \Omega$ *tels que* $|x_0| \geq |x_1|$ *et supposons* (1.4.1). *Pour tout* $d \in \,]0, d_0[$, *il existe* $\rho = \rho(p, \mu, \|f\|_\infty, d_0, d) \in]0,1[$ *tel que, si* $x_0 - x_1 > 2(1+d)$ *et* f *satisfait*

$$f \geq \mu \quad \textit{dans } \big]x_1 - 1, x_1 + 1\big[\cup \big]x_0 - 1, x_0 + 1\big[\qquad (1.4.5)$$

et

$$f < \rho\mu \quad \textit{dans } D := \big[x_1 + 1 + d, x_0 - 1 - d\big], \qquad (1.4.6)$$

alors $T_f < \infty$ *et il n'y a pas de points de touchdown dans* D. *De plus,* ρ *peut être choisi comme la solution du problème d'optimisation* (1.4.3).

Une des hypothèses que l'on fait dans ces deux théorèmes est $\mu > \mu_1 := \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{2}$. Cette hypothèse assure l'apparition de touchdown en temps fini. Par contre, cette hypothèse peut être améliorée, nous permettant de traiter les cas où $\mu > \mu_0 := \mu_1/2$, mais au détriment d'un problème d'optimisation plus complexe, qui a quatre paramètres au lieu de trois. C'est le contenu du Théorème 2.3 au Chapitre 3. L'intérêt d'améliorer cette hypothèse sur μ est de pouvoir appliquer nos résultats aux cas où le voltage est petit, c.-à-d. près du pull-in voltage (voir Discussion 1.3(e) dans le Chapitre 3).

Bien sûr, pour appliquer les Théorèmes 1.4 et 1.5, on n'a pas besoin de calculer la solution exacte ρ du problème d'optimisation. On peut utiliser n'importe quel $\overline{\rho} \in \,]0, \rho[$ au lieu de ρ dans les hypothèses (1.4.2),(1.4.6). Il suffit donc d'évaluer la fonction du second membre en (1.4.3) pour un triplet admissible $(\tau, \beta, K) \in \mathcal{A}$. Un choix plus ou moins optimal de ce triplet peut se faire par exploration de l'espace des paramètres admissibles $\mathcal{A}$. Dans le Chapitre 3, on propose une méthode numérique pour cette exploration. On peut trouver, dans l'appendice A, les scripts utilisés sur Matlab pour calculer les exemples du tableau 1.1 et du Chapitre 3.

Dans le tableau 1.1, pour le cas physique $p = 2$ et pour des valeurs raisonnables des paramètres $\mu, \|f\|_\infty, d, d_0$, on donne des estimations du ratio ρ, solution du problème d'optimisation (1.4.3) (voir colonne $\overline{\rho}_1$). On voit qu'il est possible d'obtenir des estimations jusqu'à l'ordre

$$\rho \sim 0.17,$$

ce qui est assez satisfaisant d'un point de vue pratique, où l'on voudrait éviter le touchdown sur certaines zones spécifiques du domaine. Dans la colonne $\overline{\rho}_2$ on voit des estimations améliorées du ratio, qui peuvent arriver jusqu'à l'ordre

$$\rho \sim 0.3.$$

Ces estimations approximent la solution d'un problème d'optimisation plus complexe, qui est présenté dans les Théorèmes 2.1-2.2 au Chapitre 3, et qui permet aussi de calculer le ratio ρ. Ce problème d'optimisation provient d'un raffinement supplémentaire de la méthode de preuve de la Proposition 1.3, qui permet d'obtenir une meilleure constante γ dans l'estimation de type I (1.3.4). On a décidé de ne pas présenter l'énoncé de ces théorèmes dans l'introduction à cause de leur longueur et de leur complexité au niveau de la notation. Néanmoins, ce problème d'optimisation est du même type que (1.4.3) mais avec quatre paramètres au lieu de trois, et la méthode pour estimer la solution est similaire.

μ	$\|f\|_\infty$	d	d_0	$\overline{\rho}_1$	$\overline{\rho}_2$
1	1.1	0.1	5	**0.1050**	**0.2249**
2	2.25	0.1	4	**0.1182**	**0.2111**
3	3.5	0.01	5	**0.1554**	**0.2698**
6	6.2	0.01	10	**0.1682**	**0.2856**
10	10	0.005	10	**0.1732**	**0.2921**

TABLE 1.1 – Estimations inférieures du ratio ρ pour $p = 2$. On a utilisé les Théorèmes 1.4-1.5 (colonne $\overline{\rho}_1$) et les Théorèmes 2.1-2.2 du Chapitre 3 (colonne $\overline{\rho}_2$).

L'évaluation de $\overline{\rho}_1$ et $\overline{\rho}_2$ a été faite avec l'aide du logiciel Matlab (voir l'appendice A pour les scripts utilisés). Dans le Chapitre 3, on décrit en détail les procédures numériques utilisées. En particulier, on utilise une discrétisation monotone pour évaluer les infima des fonctions G et H dans (1.4.4). Ceci garantit que l'estimation trouvée n'est pas supérieure à ρ, et les seules sources d'erreur possibles en excès sont celles liées aux problèmes d'arrondi de la machine ou celles produites par l'approximation numérique de la fonction erf.

D'après les résultats précédents, il est naturel de se demander si les conditions de petitesse obtenues pour éviter le touchdown sont vraiment nécessaires ou si au contraire, le touchdown aurait toujours lieu près des points où f est plus grande. Autrement dit, on se demande si, par exemple dans le cas des théorèmes 1.4 et 1.5, le résultat resterait vrai pour tout $\rho < 1$. Dans la section suivante, on montre que ce n'est pas le cas. En effet, on donne des exemples où le touchdown se produit loin des points de maximum de f, voir en l'unique point de minimum. On décrit aussi des comportements intéressants pour certaines familles de profils. Par exemple, on présente des situations où l'ensemble de touchdown est réduit à un seul point, ou contient au moins deux composantes connexes concentrées arbitrairement près de deux points donnés, ou contient deux sphères de dimension $n - 1$.

1.5 Résultats III : Ensembles de touchdown non-triviaux et profils en forme de "M"

Les démonstrations des résultats de cette section, ainsi que d'autres détails et références, sont inclus dans le Chapitre 2.

Ici, on considère à nouveau le cas $n-$dimensionnel en espace. On commence en donnant une définition de ce qu'on appelle profil en forme de "M". L'étude de ce type de profil a une double motivation : d'une part, on peut construire des profils pour lesquels le touchdown se produit loin des points de maximum de f, ce qui justifie la nécessité des hypothèses de petitesse pour la localisation du touchdown dans les théorèmes précédents. D'autre part, on est capable de confirmer rigoureusement, par des arguments analytiques, certains comportements intéressants concernant la localisation de l'ensemble de touchdown, qui ont été suggérés numériquement dans le livre [14, Section 7.4] pour ce type de profils.

Pour un domaine $\Omega = B_R \subset \mathbb{R}^n$ $(n \geq 1)$, où B_R est la boule de rayon R centrée à l'origine, on appelle *profil en forme de "M"* une fonction f satisfaisant

$$\begin{aligned} &f \text{ est radiale, croissante en } |x| \text{ sur } [0, L] \text{ et} \\ &\text{décroissante en } |x| \text{ sur } [L, R], \text{ pour un certain } L \in]0, R[. \end{aligned} \tag{1.5.1}$$

Pour ce type de fonctions, si $L = 0$ ou si $f(0) = f(L)$, c'est-à-dire, si f est radiale décroissante, il est bien connu que le touchdown ne peut avoir lieu qu'à l'origine, ce qui est connu sous le nom de *single point touchdown*, voir [23, 24]. Comme conséquence des Théorèmes 1.1 et 1.2, on peut avoir la situation opposée. En effet, pour des profils en forme de "M", si $f(0)$ est suffisamment petit, l'origine ne sera pas un point de touchdown, et par symétrie radiale, on peut affirmer que l'ensemble de touchdown contiendra une sphère de dimension $n - 1$. De plus, cette sphère peut être arbitrairement proche d'une sphère centrée à l'origine de rayon donné.

Corollaire 1.6. *Soit $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$.*

(i) (Touchdown contenant une sphère) *Soit $f \in E$ un profil en forme de "M", c.-à-d. f satisfait* (1.5.1)*, et supposons* (1.3.1)*. Si $f(0)$ est suffisamment petit (dépendant uniquement de p, n, R, M, r), alors 0 n'est pas un point de touchdown. En particulière, $\mathcal{T}_f$ contient une sphère de dimension $n - 1$.*

(ii) (Touchdown concentré près d'une sphère donnée) *Soit $r > 0$ et $0 < \varepsilon < \min(r, R - r)$. Il existe un profil f en forme de "M" avec deux bosses, tel que $T_f < \infty$ et*

$$\mathcal{T}_f \subset \{r - \varepsilon < |x| < r + \varepsilon\}.$$

Plus précisément, il existe $\eta, A > 0$, dépendant uniquement de p, R, r, ε, tels que la propriété ci-dessus est vraie pour tout profil radial $f \in E$ satisfaisant

$$\begin{cases} f(x) \geq A, & \text{pour } |x| \in [r - \varepsilon/2, r + \varepsilon/2], \\ f(x) \leq \eta, & \text{pour } |x| \in [0, r - \varepsilon] \cup [r + \varepsilon, R]. \end{cases}$$

Dans la Figure 1.4, on peut voir une illustration du Corollaire 1.6. Maintenant, on va répondre la question suivante, qui se pose naturellement :

L'hypothèse $f(0) < f(L)$, est-elle suffisante
pour éliminer le touchdown à l'origine ?

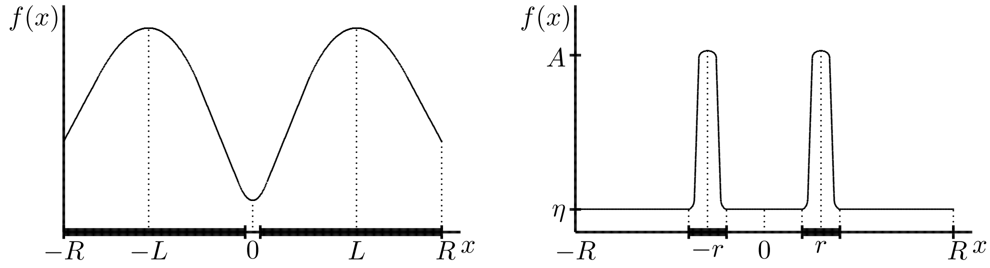

FIGURE 1.4 – Illustration du Corollaire 1.6(i) et (ii) – dans cette figure, et celles d'après, le touchdown doit être contenu dans les régions représentées en trait épais.

Le résultat suivant, qui montre la stabilité du *single point touchdown* à l'origine sous de petites perturbations de f, donne une réponse négative à cette question. Il est alors nécessaire de considérer $f(0)$ plus petit qu'un certain seuil, si l'on veut éliminer la possibilité de touchdown à l'origine. De plus, ceci montre que les profils radiaux décroissants ne sont pas les seuls qui présentent *single point touchdown* à l'origine, ce qui confirme les prédictions numériques dans [14, Remark 7.4.2].

Dans la suite, on notera $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ et

$$\mu_0(p,n) := \frac{p^p}{(p+1)^{p+1}}\lambda_1, \tag{1.5.2}$$

où λ_1 est la première valeur propre de $-\Delta$ dans $H_0^1(B_1)$ et B_1 est la boule unité de $\mathbb{R}^n$.

Théorème 1.7 (Stabilité du *single point touchdown* sous perturbations)**.**
Soit $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$, $1 \le q \le \infty$ avec $q > \frac{n}{2}$, $M > 0$, $\rho \in\,]0, R[$. Soit $f \in E \cap C^1(\overline{B}_\rho)$ une fonction radiale décroissante, avec $f(r) > \mu_0(p,n)\rho^{-2}$ on $\overline{B}_\rho$. Il existe $\varepsilon > 0$ tel que, si $g \in E \cap C^1(\overline{B}_\rho)$ est radiale et satisfait

$$\begin{aligned}
&\|g\|_\infty \le M,\\
&-M \le g'(r) \le \varepsilon r, \quad \textit{pour tout } r \in [0,\rho],\\
&\|g-f\|_q \le \varepsilon,
\end{aligned}$$

alors $T_g < \infty$ et $\mathcal{T}_g = \{0\}$.

Une autre conséquence surprenante de ce résultat est la possibilité de construire des profils *strictement convexes* pour lesquels, le touchdown se produit uniquement sur le seul *point de minimum* de f. Considérons, par exemple, une fonction $f_\lambda(x)$ définie sur $\Omega = B(0,R)$ par

$$f_\lambda(x) := \mu + \lambda\frac{|x|^2}{R^2}, \qquad \text{avec } \mu > \mu_0(p,n)\rho^{-2} \text{ et } \lambda \ge 0.$$

Comme que f_0 est constante, on a $\mathcal{T}_{f_0} = \{(0,0)\}$. On voit que, pour $\lambda > 0$ suffisamment petit, f_λ satisfait les hypothèses du Théorème 1.7. Alors l'origine, c'est-à-dire, le seul point de minimum de f, reste le seul point de touchdown. Cet exemple répond négativement à la question ouverte de [14, Section 7.5] sur la possibilité que l'ensemble de touchdown soit une sphère de dimension $n-1$ pour tout profil radial croissant.

Dans [14], pour des profils en forme de "M", des situations similaires à celles décrites dans le Corollaire 1.6 et le Théorème 1.7 ont été observées numériquement, avec soit

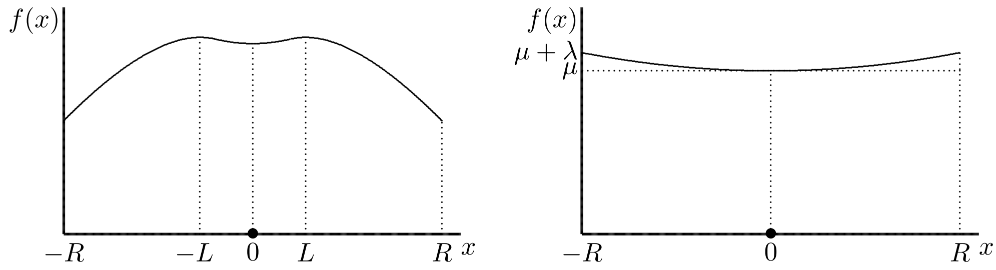

FIGURE 1.5 – Illustration du Théorème 1.7 en dimension 1 pour un profil en forme de 'M" et un profil strictement convexe.

un ou deux points de touchdown. Dans le cas du Corollaire 1.6, en revanche, on n'est pas en mesure jusqu'à présent de déterminer si l'ensemble de touchdown est formé par deux points ou plus. Dans [14, Section 7.4], des simulations numériques ont suggéré l'existence de situations intermédiaires entre celle de la Figure 1.4 et celle de la Figure 1.5, pour lesquelles l'ensemble de touchdown pourrait être formé par un segment contenant l'origine. Cette situation semble actuellement difficile à confirmer analytiquement.

Notre dernier exemple montre que l'ensemble de touchdown peut avoir deux composantes connexes concentrées arbitrairement près de deux points donnés. Pour le cas radial dans une boule, il est donc possible de construire des profils pour lesquels l'ensemble de touchdown contient deux sphères de rayon arbitrairement proches de deux rayons donnés.

Théorème 1.8. *Soit $p > 0$. Soit $\Omega \subset \mathbb{R}^n$ un domaine borné avec bord régulier.*

(i) (Ensemble de touchdown concentré près de deux points donnés)
Pour tout $x_1, x_2 \in \Omega$ et tout $\rho > 0$, il existe des profils $f \in E$ tels que

$$\mathcal{T}_f \subset B(x_1,\rho) \cup B(x_2,\rho), \qquad \mathcal{T}_f \cap B(x_1,\rho) \neq \emptyset, \qquad \mathcal{T}_f \cap B(x_2,\rho) \neq \emptyset.$$

(ii) (Ensemble de touchdown concentré près de deux sphères données)
Soit $\Omega = B_R \subset \mathbb{R}^n$, $0 < r_1 < r_2 < R$, $\rho > 0$ et soit

$$A_i = \{x \in \mathbb{R}^n;\ |x| \in (r_i - \rho, r_i + \rho)\}.$$

Il existe des profils radiaux $f \in E$ tels que

$$\mathcal{T}_f \subset A_1 \cup A_2, \qquad \mathcal{T}_f \cap A_1 \neq \emptyset, \qquad \mathcal{T}_f \cap A_2 \neq \emptyset.$$

La construction des profils pour le Théorème 1.8(i) est basée sur un argument limite appliqué à un profil à deux bosses, où l'on fait varier de façon continue la hauteur de chaque bosse. Pour le Théorème 1.8(ii), nous utilisons la même stratégie appliquée à des profils radiaux. Il est remarquable que, pour l'équation de la chaleur non-linéaire en dimension 1 avec des coefficients constants, il est prouvé dans [41] que pour n'importe quel ensemble fini de points, il existe une donnée initiale pour laquelle l'ensemble d'explosion coïncide avec cet ensemble de points. Une telle propriété reste un problème ouvert pour les problèmes à coefficients variables (MEMS ou chaleur non-linéaire).

La plupart des résultats de cette section dépendent de façon cruciale du résultat suivant, qui prouve la stabilité du temps et de l'ensemble de touchdown sous de petites perturbations du profil.

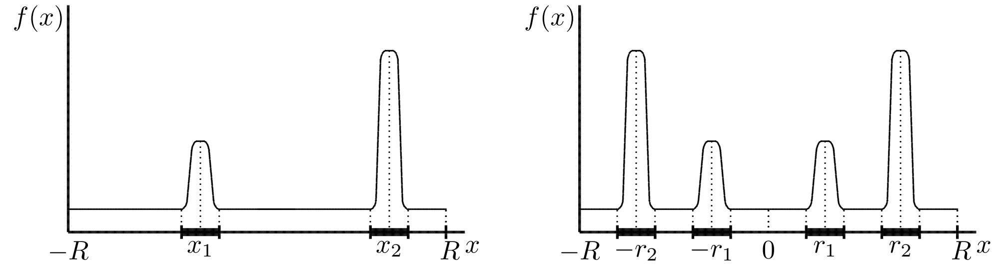

FIGURE 1.6 – Illustration du Théorème 1.8(i) et (ii) pour $n = 1$.

Théorème 1.9. ***(Continuité du temps de touchdown et semi-continuité inférieure de l'ensemble de touchdown)*** *Soit $p > 0$ et $\Omega \subset \mathbb{R}^n$ un domaine borné avec bord régulier. Soit $1 \leq q \leq \infty$ avec $q > \frac{n}{2}$, $B \subset \Omega$ une boule de rayon $r > 0$, $M \geq \mu > \mu_0(p,n)r^{-2}$ et soit*

$$\tilde{E} = \big\{f \in E;\ M \geq f \geq \mu\chi_B\big\}. \tag{1.5.3}$$

Pour tout $f \in \tilde{E}$ avec $\mathcal{T}_f \subset\subset \Omega$ et tout $\sigma > 0$, il existe $\varepsilon > 0$ tel que,

$$\textit{si } g \in \tilde{E} \textit{ et } \|g - f\|_q \leq \varepsilon, \quad \textit{alors } |T_g - T_f| \leq \sigma \textit{ et } \mathcal{T}_g \subset \mathcal{T}_f + B(0,\sigma).$$

On remarque que la continuité de l'ensemble de touchdown n'est pas vraie en général. Considérons par exemple un profil à deux bosses comme celui de la Figure 1.6, pour lequel on a touchdown près des deux points x_1 et x_2. Ensuite, on baisse de façon continue la hauteur d'une des deux bosses. Comme conséquence du Théorème 1.1 et d'un argument limite, on voit qu'au dessous d'une certaine valeur, les points de touchdown près de cette bosse vont disparaître tout d'un coup. Ceci prouve la non-continuité de l'ensemble de touchdown par rapport au profil f.

2 Explosion du gradient pour une équation de Hamilton-Jacobi diffusive

La deuxième partie de cette thèse se situe encore dans le cadre des équations paraboliques semi-linéaires. Cette fois-ci, nous allons considérer une non-linéarité qui dépend du gradient. Plus précisément, on s'intéresse à l'équation suivante :

$$u_t - \Delta u = |\nabla u|^p, \qquad \text{avec } p > 2. \tag{DHJ}$$

Cette équation représente un modèle typique dans la théorie des équations paraboliques non-linéaires, étant le cas le plus simple d'une non-linéarité du type gradient et peut être vue comme l'analogue de l'équation

$$u_t - \Delta u = u^p, \qquad \text{avec } p > 1,$$

qui est largement étudiée dans la littérature. Par ailleurs, l'équation (DHJ) intervient dans certains modèles physiques, par exemple de dépôt balistique, où u décrit la croissance d'une surface, voir [29, 32, 33]. L'équation (DHJ) est aussi une équation de type Hamilton-Jacobi qui intervient en théorie du contrôle stochastique, voir [37].

2.1 Théorie du contrôle stochastique et équation de Hamilton-Jacobi diffusive

Pour deux fonctions $f : \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$ et $\alpha : [0, \infty[\longrightarrow A$, où $A \subset \mathbb{R}^m$, un point de départ $x \in \mathbb{R}^n$ et un intervalle de temps $[t, T]$ donnés, on considère le système dynamique contrôlé qui est gouverné par l'équation différentielle ordinaire suivante :

$$\begin{cases} \dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \alpha(s)), & s > t, \\ \mathbf{x}(t) = x, \end{cases} \tag{ODE}$$

où $\dot{\mathbf{x}}(s) = \frac{d}{ds}\mathbf{x}(s)$. L'inconnue est la courbe $\mathbf{x}(\cdot) : [t, T] \to \mathbb{R}^n$, que l'on peut interpréter comme l'évolution dynamique d'un système qui a comme point de départ x à l'instant $t > 0$, par exemple, le mouvement d'une particule dans l'espace. Cette dynamique est décrite par la fonction f, qui représente la vitesse de la particule à chaque instant. La fonction $\alpha(\cdot)$ est appelée contrôle, et on peut choisir n'importe quelle fonction $\alpha(\cdot)$ dans l'ensemble des contrôles admissibles

$$\mathcal{A} := \{\alpha(\cdot) : [0, \infty[\to A \mid \alpha(\cdot) \text{ est mesurable}\}.$$

On voit que la courbe $\mathbf{x}(\cdot)$ solution de (ODE) dépend du point de départ (x, t) et du choix de $\alpha(\cdot)$.

Notre but est de déterminer quel est le contrôle "optimal" pour notre système. À cet effet, il convient de définir un critère d'optimalité ou récompense. Pour chaque $x \in \mathbb{R}^n$ et $t \in]0, T]$, on définit la *fonctionnelle de récompense totale*

$$P_{x,t}[\alpha(\cdot)] = \int_t^T r(\mathbf{x}(s), \alpha(s))ds + g(\mathbf{x}(T)),$$

où $r : \mathbb{R}^n \times A \to \mathbb{R}$ et $g : \mathbb{R}^n \to \mathbb{R}$ sont des fonctions données qu'on appelle respectivement *récompense courante* et *récompense terminale.* Ici, $\mathbf{x}(\cdot)$ est la solution de (ODE) avec point de départ $\mathbf{x}(t) = x$ et contrôle $\alpha(\cdot)$. Le problème revient donc à trouver un contrôle $\alpha^*(\cdot) \in \mathcal{A}$ qui maximise la récompense totale, c'est-à-dire,

$$P[\alpha^*(\cdot)] \geq P[\alpha(\cdot)], \qquad \text{pour tous les contrôles } \alpha(\cdot) \in \mathcal{A}. \tag{P}$$

Maintenant, on définit la *fonction valeur* pour ce problème de contrôle comme

$$v(x, t) := \sup_{\alpha(\cdot) \in \mathcal{A}} P_{x,t}[\alpha(\cdot)], \qquad \text{pour tout } x \in \mathbb{R}^n,\ 0 \leq t \leq T. \tag{2.1.1}$$

Autrement dit, v est la fonction qui prend la plus grande récompense possible pour chaque $(x, t) \in \mathbb{R}^n \times [0, T]$.

Supposons que $v(x, t)$, définie en (2.1.1), est une fonction $C^{1,1}(\mathbb{R}^n \times [0, T[)$. Alors, v satisfait l'équation aux dérivées partielles suivante :

$$v_t(x, t) + \sup_{a \in A}\{f(x, a) \cdot \nabla v(x, t) + r(x, a)\} = 0, \quad x \in \mathbb{R}^n,\ 0 \leq t < T, \tag{HJB}$$

avec donnée finale $v(x, T) = g(x)$, pour tout $x \in \mathbb{R}^n$. On peut trouver une preuve de ce résultat dans [15, Section 10.3].

L'équation (HJB) est appelée équation de Hamilton-Jacobi-Bellman associée a ce problème de contrôle et on peut la réécrire comme

$$v_t + H(x, \nabla v) = 0, \quad x \in \mathbb{R}^n,\ 0 \leq t < T, \tag{HJB}$$

où le Hamiltonien $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ est défini par

$$H(x,p) := \sup_{a\in A} H(x,p,a) = \sup_{a\in A}\{f(x,a)\cdot p + r(x,a)\}, \qquad \text{pour } x,p \in \mathbb{R}^n. \tag{2.1.2}$$

Évidemment, nous n'avons justifié ni l'existence ni la régularité d'une telle fonction, qui sont soumises à des hypothèses sur l'ensemble A et les fonctions f, r et g. On ne rentrera pas dans ce genre de questions, car on va considérer un cas particulier de cette équation, pour laquelle notre intérêt est d'étudier des propriétés qualitatives de la solution.

On va traiter le cas d'un système dynamique, où à chaque instant, on peut choisir le vecteur vitesse du système. On va considérer une récompense courante négative, proportionnelle à une puissance de la norme du vecteur choisi (on peut l'appeler aussi coût courant). C'est-à-dire, on considère l'ensemble de contrôle $A = \mathbb{R}^n$ et les fonctions $f(x,a) = a$ et $r(x,a) = -\frac{K}{\beta}|a|^\beta$, avec $\beta > 1$ et $K > 0$. Comme récompense finale, on considérera une fonction $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$.

Alors, on a le système contrôlé

$$\begin{cases} \dot{\mathbf{x}}(s) = \alpha(s), \quad t < s < T, \\ \mathbf{x}(t) = x, \end{cases} \tag{ODE}$$

où le contrôle est une fonction $\alpha(\cdot) : [0,\infty[\to \mathbb{R}^n$ mesurable. Pour chaque $x \in \mathbb{R}^n$ et $0 \le t \le T$, la fonctionnelle récompense à maximiser est définie par

$$P_{x,t}[\alpha(\cdot)] = -\int_t^T \frac{K}{\beta}|\alpha(s)|^\beta ds + g(\mathbf{x}(T)).$$

Pour trouver l'équation (HJB) associée à ce problème, il faut déterminer le Hamiltonien défini en (2.1.2), c.-à-d. pour chaque $x,p \in \mathbb{R}^n$, il faut déterminer $a^* \in \mathbb{R}^n$ qui maximise la fonction

$$H(x,p,a) = p\cdot a - \frac{K}{\beta}|a|^\beta.$$

On peut calculer

$$\nabla_a H = p - K|a|^{\beta-2}a.$$

Comme la fonction H est concave par rapport à a (car $\beta > 1$), on déduit que le maximum global de H est atteint sur le seul a qui annule $\nabla_a H$. Avec un calcul simple, on obtient

$$a^* = K^{-\frac{1}{\beta-1}}|p|^{-\frac{\beta-2}{\beta-1}}p \qquad \text{et} \qquad |a^*|^\beta = K^{-\frac{\beta}{\beta-1}}|p|^{\frac{\beta}{\beta-1}},$$

ce qui donne comme Hamiltonien la fonction

$$H(x,p) = \frac{\beta-1}{\beta}K^{-\frac{1}{\beta-1}}|p|^{\frac{\beta}{\beta-1}}.$$

Pour $q > 1$, on peut choisir $\beta = \frac{q}{q-1}$ et $K = q^{q(q-1)}$. Le Hamiltonien est donc défini par

$$H(x,p) = |p|^q.$$

Alors, d'après (HJB), la fonction valeur définie en (2.1.1) satisfait l'équation de Hamilton-Jacobi

$$v_t + |\nabla v|^q = 0, \qquad 0 \le t < T,$$

avec condition finale $v(x,T) = g(x)$. On peut définir la fonction $u(x,t) = v(x,T-t)$, qui satisfait l'équation

$$u_t = |\nabla u|^q, \qquad 0 < t \le T \tag{HJ}$$

avec donnée initiale $u(x,0) = g(x)$.

Le terme de diffusion Δu dans l'équation (DHJ) provient de l'ajout d'un bruit blanc à l'équation différentielle déterministe (ODE). On considère maintenant, pour chaque $x \in \mathbb{R}^n$ et $0 \le t \le T$, l'équation différentielle stochastique contrôlée suivante :

$$\begin{cases} \dot{\mathbf{X}}(s) = f(\mathbf{X}(s), \alpha(s)) + \sigma\xi(s), \quad 0 \le t \le T, \\ \mathbf{X}(t) = x, \end{cases} \tag{SODE}$$

où $\sigma > 0$ et $\xi(\cdot)$ est un terme de bruit blanc. On peut penser, au moins formellement, que $\xi(s) = \frac{dW(s)}{ds}$, où $W(\cdot)$ est un mouvement Brownien.

Maintenant, la solution $\mathbf{X}(\cdot)$ de (SODE) est un processus stochastique. Pour notre exemple avec $f(x,a) = a$, on peut interpréter que, à chaque instant, le vecteur vitesse de notre système est donné par un vecteur a que l'on peut choisir dans $\mathbb{R}^n$ plus une perturbation de ce vecteur produite par le bruit blanc qu'on a ajouté. On remarque que maintenant, pour (x,t) donnés, on ne peut pas déterminer la trajectoire du système $\mathbf{X}(\cdot)$. Ici, le contrôle optimal cherché $\alpha^*(\cdot)$ ne dépendra pas uniquement de s, mais aussi de la trajectoire que le système a suivi jusqu'à l'instant s, c.-à-d., de la courbe $\{\mathbf{X}(\tau),\ t \le \tau < s\}$.

La fonctionnelle récompense totale pour $x \in \mathbb{R}^n$, $0 \le t \le T$ et un contrôle $\alpha(\cdot)$, est définie par

$$P_{x,t}[\alpha(\cdot)] = \mathbb{E}\left\{\int_t^T r(\mathbf{X}(s), \alpha(s))ds + g(\mathbf{X}(T))\right\},$$

c'est-à-dire, l'espérance de la récompense totale. La fonction valeur pour ce problème est donc définie par

$$v(x,t) := \sup_{\alpha(\cdot)\in\mathcal{A}} P_{x,t}[\alpha(\cdot)], \qquad \text{pour tout } x \in \mathbb{R}^n,\ 0 \le t \le T.$$

L'équation de Hamilton-Jacobi-Bellman qui est satisfaite par la fonction valeur v, dans le cas où elle appartient à $C^{2,1}(\mathbb{R}^n \times [0,T[)$ est

$$v_t(x,t) + \frac{\sigma^2}{2}\Delta v(x,t) + \sup_{a\in A}\{f(x,a)\cdot\nabla v(x,t) + r(x,a)\} = 0, \quad x \in \mathbb{R}^n,\ t \ge 0, \tag{HJB}$$

avec condition finale $v(x,T) = g(x)$ (voir [18, Chapitre IV] pour une preuve de cette dérivation). Finalement, avec les choix de f et r qu'on a faits précédemment, on obtient que $u(x,t) = v(x,T-t)$ satisfait l'équation de Hamilton-Jacobi diffusive (DHJ), qui est l'objet d'étude de cette partie de la thèse, avec donnée initiale $u(x,0) = g(x)$.

2.2 Croissance de surfaces par déposition balistique

Dans [32], Kardar-Parisi-Zhang ont proposé l'équation

$$h_t = \nu\Delta h + \frac{\lambda}{2}|\nabla h|^2 + \eta(x,t) \tag{KPZ}$$

pour décrire l'évolution d'une surface qui croît par un processus de déposition balistique. La fonction $h(x,t)$ représente la hauteur de la surface à chaque instant. Le premier terme du second membre de l'équation décrit la relaxation de la surface due à la tension superficielle ν. Le deuxième terme est le terme non-linéaire de plus petit ordre qui décrit la déposition balistique. Il peut y apparaitre aussi des termes d'ordre supérieur qui, dans l'article de Kardar-Parisi-Zhang, sont négligés. Le bruit $\eta(x,t)$ a une distribution Gaussienne.

Pour justifier le terme non-linéaire de l'équation, on considère qu'à chaque fois qu'une particule est déposée, la surface croît dans la direction normale à son profil. Alors, l'accroissement de la surface dans la direction verticale est donné par

$$\delta h = \sqrt{(\lambda \delta t)^2 + (\lambda \delta t)^2 |\nabla h|^2},$$

d'où

$$\frac{\delta h}{\delta t} = \lambda \sqrt{1 + |\nabla h|^2}.$$

Ici, $\lambda > 0$ est proportionnel à la vitesse de déposition de particules. On déduit alors l'équation

$$h_t = \lambda \sqrt{1 + |\nabla h|^2} = \lambda + \frac{\lambda}{2} |\nabla h|^2 + \dots$$

Après avoir fait la transformation $\tilde{h}(x,t) = h(x,t) - \lambda t$ et inclus le terme de diffusion (on peut considérer que les particules se diffusent sur le domaine), on obtient l'équation (KPZ).

Dans [33], Krug et Spohn ont généralisé l'équation (KPZ) (déterministe) sous la forme

$$u_t = \nu \Delta u + \lambda |\nabla u|^p,$$

avec $p \geq 1$ pour étudier l'effet du terme non-linéaire.

2.3 Contexte mathématique et motivation

Notre intérêt pour l'équation (DHJ) est motivé par l'explosion en temps fini du gradient. Considérons le problème à valeur initiale et conditions au bord suivant :

$$\begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega, \quad t > 0, \\ \qquad\quad u = 0, & x \in \partial\Omega, \quad t > 0, \\ \ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{DHJ}$$

où Ω est un domaine borné dans $\mathbb{R}^2$ avec bord régulier, $p > 2$ et

$$u_0 \in X_+ := \{v \in C^1(\overline{\Omega});\ v \geq 0,\ v|_{\partial\Omega} = 0\}.$$

Par la théorie classique des équations paraboliques, voir par exemple [19], on sait que le problème (DHJ) admet une unique solution classique maximale $u \in C^{2,1}(\overline{\Omega} \times]0,T[) \cap C^{1,0}(\overline{\Omega} \times [0,T[)$, où $T = T(u_0)$ est le temps maximal d'existence de la solution classique. De plus, par le principe du maximum, on déduit que

$$0 \leq u(x,t) \leq \|u_0\|_\infty, \qquad 0 < t < T,\ x \in \Omega.$$

Comme (DHJ) est bien posé dans X_+, si $T < \infty$, on a

$$\lim_{t \to T} \|\nabla u\|_\infty = \infty.$$

Ce phénomène d'explosion du gradient, tandis que la solution reste bornée, est connu dans la littérature sous le nom de *gradient blow-up* (GBU). L'équation (DHJ) a attiré beaucoup d'attention pendant les vingt dernières années. Il est bien connu que, si $p \leq 2$ ou si $\Omega = \mathbb{R}^n$, toutes les solutions sont globales en temps, c.-à-d. $T = \infty$ (voir [2, 5, 22, 34, 53]). Par contre, pour le cas $p > 2$, si le bord de Ω n'est pas vide, il existe des solutions dont le gradient explose en temps fini (voir [1, 3, 4, 12, 16, 25, 36, 52, 54]). Dans ce cas, on ne peut pas continuer la solution indéfiniment en temps au sens classique.

Dans [54, Théorème 3.2], il est démontré que le GBU ne peut avoir lieu que sur le bord du domaine (voir aussi [1, 3]). De plus, l'estimation suivante est donnée :

$$|\nabla u| \leq C_1 \delta^{-\frac{1}{p-1}}(x,y) + C_2 \quad \text{sur } \Omega \times [0,T[, \tag{2.3.1}$$

où $C_1 = C_1(n,p) > 0$ et $C_2 = C_2(p, \Omega, \|u_0\|_{C^1}) > 0$. Ici, $\delta(x,y)$ est la fonction distance au bord de Ω.

L'ensemble de gradient blow-up est défini par

$$GBUS(u_0) := \{x_0 \in \partial\Omega; \quad \limsup_{t \to T,\ x \to x_0} |\nabla u(x,t)| = \infty\}.$$

Dans le Chapitre 4, on s'intéresse à la possibilité d'avoir des points de GBU isolés. Les seuls résultats précédemment connus sur cette question sont ceux de [36]. Dans cet article, pour des domaines très particuliers, à savoir, les disques et les domaines avec un morceau de bord plat, il est prouvé l'existence de solutions dont l'ensemble de GBU est réduit à un seul point. Une caractéristique qui semble essentielle pour les domaines considérés dans [36] est le fait que la courbure du bord soit constante près du point de GBU. Notre but sera d'obtenir des résultats du même type, mais qui s'appliquent à une large classe de domaines, pour lesquels la courbure du bord n'est pas forcement constante sur un voisinage du point de GBU.

2.4 Résultats : *Single-point Gradient Blow-up* sur le bord pour des domaines à courbure non-constante

Les démonstrations des résultats de cette section, ainsi que d'autres détails et références, sont inclus dans le Chapitre 4.

Notre stratégie pour prouver les résultats de cette section est basée sur l'utilisation d'un système de coordonnées adaptées au bord du domaine, combinée avec des fonctions auxiliaires appropriées et des propriétés de monotonie de la solution. La dérivation et l'analyse des équations paraboliques satisfaites par les fonctions auxiliaires nécessitent des calculs longs et techniques, qui font usage de ces coordonnées adaptées au bord.

Mais, avant de rentrer dans les détails techniques et d'introduire la notation dont on aura besoin par la suite, on va donner une illustration simple de nos résultats principaux avec les deux théorèmes suivants qui, pour deux classes de domaines typiques, assurent l'existence de données initiales pour lesquelles l'ensemble d'explosion du gradient ne contient qu'un seul point. Des résultats plus généraux sont donnés dans la section 2.5 ci-dessous. On commence avec le cas des ellipses.

Théorème 2.1. *Soit $p > 2$ et $\Omega \subset \mathbb{R}^2$ une ellipse. Alors, il existe des données initiales $u_0 \in X_+$ telles que $T(u_0) < \infty$ et $GBUS(u_0)$ ne contient qu'un seul point de courbure minimale.*

Pour notre deuxième classe de domaines, la caractéristique principale est que le point de GBU soit un point du bord dont le centre de courbure est situé en dehors du domaine et de plus, qu'il soit un point de minimum local pour la courbure. Plus précisément, on suppose :

$$\Omega \text{ est symétrique par rapport à l'axe } OY \text{ et convexe dans la direction } x, \tag{2.4.1}$$

$$\partial\Omega \text{ est tangent à la droite } \{y = 0\} \text{ à l'origine et } \Omega \subset \{y > 0\}, \tag{2.4.2}$$

$$\text{le rayon de courbure de } \partial\Omega \text{ est une fonction décroissante de } x, \text{ pour } x > 0 \text{ près de l'origine, et } \overline{\Omega} \subset \{y < R\}, \text{ où } R \text{ est le rayon de courbure de } \partial\Omega \text{ à l'origine,} \tag{2.4.3}$$

$$\text{pour tout } X_0 \in \partial\Omega \cap \{x > 0\} \text{ près de l'origine, le symétrique de } \Omega_{X_0} \text{ par rapport à } \Lambda_{X_0} \text{ est contenu dans } \Omega, \text{ où } \Lambda_{X_0} \text{ est la droite normale à } \partial\Omega \text{ en } X_0, \text{ et } \Omega_{X_0} \text{ est la partie de } \Omega \text{ située à droite de } \Lambda_{X_0}. \tag{2.4.4}$$

La Figure 1.7 représente un exemple de domaine de cette classe.

Théorème 2.2. *Soit $p > 2$ et supposons que $\Omega \subset \mathbb{R}^2$ est un domaine satisfaisant* (2.4.1)–(2.4.4). *Alors, il existe des données initiales $u_0 \in X_+$ telles que $T(u_0) < \infty$ et $GBUS(u_0)$ ne contient que l'origine.*

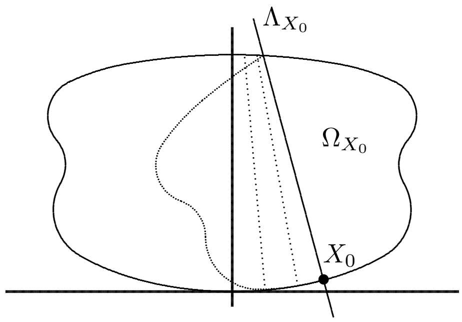

FIGURE 1.7 – Exemple de domaine satisfaisant les hypothèses (2.4.1)–(2.4.4).

On remarque que, pour les domaines localement plats près de l'origine qui sont traités dans [36], l'hypothèse (2.4.4) est une conséquence de (2.4.1). On voit que dans ce cas-là, pour tout X_0 près de l'origine, Λ_{X_0} est parallèle à l'axe OY. L'hypothèse (2.4.3) devient également triviale pour ce type de domaines.

2.5 Résultats plus généraux et coordonnées adaptées au bord.

On va maintenant introduire une classe de domaines, contenant ceux des Théorèmes 2.1 et 2.2, pour lesquels nous sommes capables de construire des solutions dont l'ensemble de GBU est réduit à un seul point. Un pas important dans notre stratégie est l'étude du signe de la dérivée dans la direction parallèle au bord. En conséquence, il est naturel d'utiliser un système de coordonnées adaptées au bord. Ce système de coordonnées est connu sous le nom de *"boundary-fitted" coordinate system* ou *flow coordinates.*

Pour commencer, on a besoin d'établir la notation suivante, qui sera utilisée dans la suite pour décrire les hypothèses de nos résultats.

Notation 2.3. - *Soit Ω un domaine borné de $\mathbb{R}^2$ avec bord régulier et $\nu = (\nu_x, \nu_y)$ le vecteur unitaire normal extérieur à $\partial\Omega$.*

- *Soit $\Gamma \subset \partial\Omega$ un morceau connexe du bord avec $(0,0) \in \Gamma$ et supposons que*

$$\Omega \textit{ et } \Gamma \textit{ sont symétriques par rapport à l'axe } OY. \tag{2.5.1}$$

- *Pour un certain $s_0 > 0$, soit l'application*

$$\gamma(s) = (\alpha(s), \beta(s)), \qquad s \in [-s_0, s_0],$$

une paramétrisation de Γ par longueur d'arc (c.-à-d. $\alpha'(s)^2 + \beta'(s)^2 = 1$), avec $\gamma(0) = (0,0)$.

- *On note, pour chaque $s \in [-s_0, s_0]$,*

$$T(s) = (\alpha'(s), \beta'(s)), \qquad N(s) = T^{\perp}(s) = (-\beta'(s), \alpha'(s)).$$

On voit que $T(s)$ est un vecteur unitaire tangent à $\partial\Omega$ au point $\gamma(s)$. Alors, sans aucune perte de généralité (en remplaçant au besoin s par $-s$), on peut supposer que

$$N(s) \textit{ est le vecteur unitaire normal intérieur à } \partial\Omega \textit{ au point } \gamma(s) \tag{2.5.2}$$

et que

$$\gamma(0) = (0,0), \qquad T(0) = (1,0), \qquad N(0) = (0,1).$$

- *La fonction courbure est donnée par*

$$K(s) := det(\gamma', \gamma'') = \alpha'\beta'' - \beta'\alpha'', \quad \textit{pour } s \in [-s_0, s_0].$$

Par la régularité de $\partial\Omega$, cette fonction est bornée et régulière.

- *On introduit l'application $M := \gamma + rN$, c.-à-d.*

$$\begin{array}{llcl} M: & [0,\infty[\times[-s_0, s_0] & \longrightarrow & \mathbb{R}^2 \\ & (r,s) & \longmapsto & M(r,s) = \gamma(s) + rN(s). \end{array} \tag{2.5.3}$$

Pour un domaine Ω et un morceau de bord Γ comme ceux décrits ci-dessus, notre but sera de montrer l'existence de données initiales pour lesquelles l'ensemble de GBU est réduit à l'origine. Afin d'appliquer nos méthodes, on a besoin de supposer que Ω est localement convexe près de l'origine et que l'origine est un point de minimum local pour la fonction courbure du bord, c.-à-d.

$$K(0) \geq 0 \quad \text{et} \quad K'(s) \geq 0 \qquad \text{pour tout } s \in [0, s_0], \tag{2.5.4}$$

et

$$\alpha'(s), \beta'(s) > 0, \qquad \text{pour tout } s \in]0, s_0[. \tag{2.5.5}$$

La raison de faire ces deux hypothèses vient de la nécessité d'avoir un sous-domaine où la paramétrisation (2.5.3) soit bien définie et de contrôler les termes associés à la courbure non-constante, qui apparaissent quand on dérive les fonctions auxiliaires. Il faut souligner que l'hypothèse (2.5.5) ne permet pas de traiter les domaines localement plats. Cependant, les domaines de ce type ont déjà été traités dans [36].

Sous les hypothèses de la Notation 2.3, on définit

$$R(s) = 1/K(s) \in]0, \infty], \qquad s \in [0, s_0], \tag{2.5.6}$$

qui est la fonction rayon de courbure de $\partial\Omega$ au point $\gamma(s)$. On définit aussi les régions naturelles suivantes :

$$Q_\Gamma = \{(r,s) \in \mathbb{R}^2;\ 0 \le r < R(s),\ 0 \le s \le s_0\} \quad \text{et} \quad D_\Gamma = M(Q_\Gamma). \tag{2.5.7}$$

On voit que D_Γ est la région entourée par les quatre courbes : Γ, l'axe OY, la droite normale à $\partial\Omega$ en $\gamma(s_0)$, et par dessus, la développée de Γ, c.-à-d. le lieu des centres de courbure, qui est donnée par

$$C(s) = \gamma(s) + R(s)N(s), \qquad \text{pour chaque } s \in [0, s_0]. \tag{2.5.8}$$

Dans la Figure 1.8, on peut trouver une illustration de la Notation 2.3 et de $D_\Gamma, C(s)$ définis en (2.5.7), (2.5.8) respectivement. Dans le Chapitre 4, il est démontré que la région D_Γ est bien paramétrée par l'application (2.5.3), et par conséquent, on peut définir u_s, la dérivée dans la direction parallèle au bord.

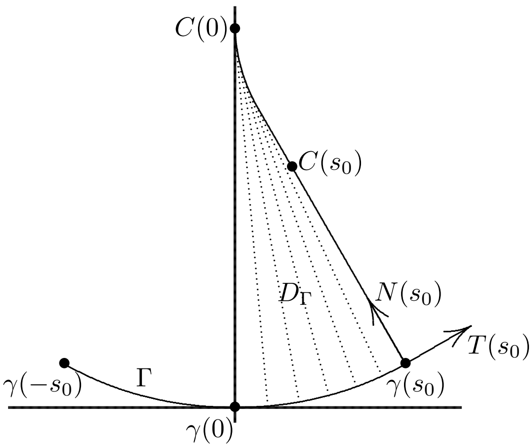

FIGURE 1.8 – Exemple de $\Gamma, \gamma(s), T(s), N(s)$ suivant la Notation 2.3 et $D_\Gamma, C(s)$ définis par (2.5.7), (2.5.8).

Le résultat suivant assure que, sous certaines hypothèses de monotonie de la solution près de l'ensemble d'explosion du gradient, on n'aura qu'un seul point de GBU.

Théorème 2.4. *Soit $p > 2$ et considérons $\Omega, \Gamma, \gamma, M$ comme dans la Notation 2.3. Supposons* (2.5.4) *et* (2.5.5). *Si $u_0 \in X_+$ est une fonction symétrique par rapport à l'axe OY, telle que $T = T(u_0) < \infty$, qui satisfait*

$$GBUS(u_0) \subset \gamma\left(\left]-\tfrac{s_0}{2}, \tfrac{s_0}{2}\right[\right) \tag{2.5.9}$$

et que pour un certain $t_0 \in (0,T)$ et $r_0 \in (0, R(s_0))$, on a

$$u_x, u_s < 0 \quad \textit{dans } \omega_0 \times\,]t_0, T[, \ \textit{avec } \omega_0 := \Omega \cap M(]0, r_0[\times]0, s_0[), \tag{2.5.10}$$

alors, $GBUS(u_0)$ contient uniquement l'origine.

L'hypothèse (2.5.9) n'est pas difficile à assurer. Il suffit, par exemple, de choisir une fonction u_0 à support concentré près de l'origine (voir [36] et la Proposition 4.2 dans le Chapitre 4). Par contre, la condition $u_s < 0$ de (2.5.10) est, en général, plus difficile à assurer et a besoin d'hypothèses de nature plus globale. Dans le reste de cette section, on décrira les hypothèses suffisantes sous lesquelles on est capable de construire des données initiales satisfaisant (2.5.10).

On commence par établir la notation suivante, qui est motivée par des arguments de type *moving planes* qu'on utilisera dans la démonstration.

Notation 2.5. *Pour chaque* $s \in [0, s_0]$

- Λ_s *est la droite* $\gamma(s) + \mathbb{R}N(s)$
- $\mathcal{T}_s(\cdot)$ *est la symétrie par rapport à* Λ_s
- H_s *est le demi-espace situé à droite de* Λ_s*, c.-à-d. :*

$$H_s = \{P \in \mathbb{R}^2;\ T(s) \cdot (P - \gamma(s)) > 0\}.$$

- $\Omega_s = \Omega \cap H_s$.

En utilisant les Notations 2.3 et 2.5, on fait les hypothèses suivantes :

$$\overline{\omega}_0 \subset D_\Gamma, \quad \text{où } \omega_0 := \Omega \cap D_\Gamma \cap \{y < y_0\} \text{ pour un certain } y_0 \in]0, \infty], \tag{2.5.11}$$

$$\nu_x \geq 0 \quad \text{sur } \partial\Omega \cap \{x > 0\}, \tag{2.5.12}$$

$$\nu_y \geq 0 \text{ sur } \partial\Omega \cap \partial\omega_0 \cap \{r > 0\}, \tag{2.5.13}$$

$$\mathcal{T}_{s_0}(\Omega_{s_0}) \subset \Omega, \tag{2.5.14}$$

$$\begin{aligned} &\mathcal{T}_+(\Omega^+) \subset \Omega, \quad \text{où } \Omega^+ := \Omega \cap \{y > y_0\} \text{ et } \mathcal{T}_+(\cdot) \text{ est} \\ &\text{la symétrie par rapport à la droite } y = y_0. \end{aligned} \tag{2.5.15}$$

Les ellipses et les exemples représentés dans les Figures 1.7 et 1.9 sont des domaines qui satisfont ces hypothèses. Le résultat principal de cette section est le suivant :

Théorème 2.6. *Soit* $p > 2$ *et considérons* $\Omega, \gamma, s_0, \mathcal{T}_s, \Omega_s$ *comme dans les Notations 2.3 et 2.5. Soit* D_Γ *défini par* (2.5.7) *et supposons* (2.5.4), (2.5.5), (2.5.11)–(2.5.15). *Alors, on a :*

i. Il existe des données initiales $u_0 \in X_+$ *telles que* $T(u_0) < \infty$ *et*

$$u_0 \text{ est symétrique par rapport à la droite } x = 0, \tag{2.5.16}$$

$$u_{0,x} \leq 0 \text{ dans } \Omega \cap \{x > 0\} \quad \text{et} \quad u_{0,s} \leq 0 \text{ dans } \omega_0, \tag{2.5.17}$$

$$u_0(P) \leq u_0(\mathcal{T}_{s_0}(P)) \quad \text{pour tout } P \in \Omega_{s_0}, \tag{2.5.18}$$

$$u_0(P) \leq u_0(\mathcal{T}_+(P)) \quad \text{pour tout } P \in \Omega \cap \{y > y_0\}. \tag{2.5.19}$$

$$GBUS(u_0) \subset \gamma\left(\left]-\tfrac{s_0}{2}, \tfrac{s_0}{2}\right[\right). \tag{2.5.20}$$

ii. Pour une telle donnée initiale u_0*,* $GBUS(u_0)$ *contient uniquement l'origine.*

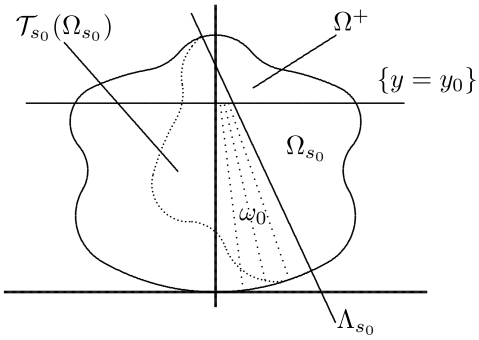

FIGURE 1.9 – Exemple de domaine satisfaisant les hypothèses (2.5.11)–(2.5.15).

Commentaires :

i. Si le domaine est suffisamment étroit dans la direction y, le centre de courbure ne rentrera pas dans le domaine pour les points du bord près de l'origine. Dans ce cas, on a le droit de prendre $y_0 = \infty$ dans (2.5.11). Par conséquent, les hypothèses (2.5.15) et (2.5.19) ne seront pas nécessaires. C'est le cas, par exemple, du domaine représenté dans la Figure 1.7.

   Par contre, si le domaine n'est pas suffisamment étroit, il faudra le tronquer avec la droite $y = y_0$, afin de pouvoir définir les coordonnées adaptées au bord. Cependant, il est alors nécessaire de payer le prix des hypothèses de symétrie (2.5.15) et (2.5.19).

ii. Considérons un domaine Ω comme dans le Théorème 2.6 et soit $B_\rho^+ := B_\rho(0,0) \cap \{x > 0\}$, avec $\rho > 0$, telle que

$$\Omega \cap B_\rho^+ \subset \omega_0, \quad \partial\Omega \cap B_\rho^+ \subset \gamma(]0, s_0/2[).$$

   On peut déduire du Théorème 2.6 et la Proposition 4.2 au Chapitre 4 que $T(u_0) < \infty$ et $GBUS(u_0) = \{(0,0)\}$ pour tout $u_0 \in X_+$ satisfaisant (2.5.16), (2.5.17) et

$$\begin{aligned} &\mathrm{supp}(u_0) \subset \overline{\Omega} \cap \overline{B}_{\rho/2}, \\ &\|u_0\|_\infty \leq C_2, \\ &\inf_{\tilde{B}_\varepsilon} u_0 \geq C_1 \varepsilon^k \quad \text{où } \tilde{B}_\varepsilon = B_{\varepsilon/2}(0,\varepsilon), \text{ avec } \varepsilon \in (0, \rho/2), \end{aligned}$$

   où $C_1(p) > 0$ et $C_2(p, \Omega, \rho) > 0$. De plus, des données initiales satisfaisant ces hypothèses peuvent se construire très facilement (voir la preuve du Théorème 2.6(i) et la Remarque 2.7 au Chapitre 4 pour plus de détails).

iii. Une hypothèse qui est nécessaire dans toutes nos preuves est (2.5.4), c.-à-d., supposer que le point de GBU est un minimum local de la fonction courbure. On ne sait pas si cette restriction est nécessaire ou s'il s'agit juste d'une hypothèse technique. Cette condition provient du fait que la dérivée par rapport à s ne commute pas avec l'opérateur Laplacien et par conséquent, on a besoin de contrôler des termes liés à la courbure qui apparaissent quand on dérive les fonctions auxiliaires. Nous n'avons pas trouvé d'autres systèmes de coordonnées adaptées au bord, ou d'autres fonctions auxiliaires qui commutent mieux avec le Laplacien.

iv. Dans l'article [49], pour $2 < p \leq 3$ et des domaines localement plats, le profil exact de la solution près d'un point de GBU isolé est décrit. Il est donné par :

$$u_y(x,y,T) \sim d_p \left[y + C|x|^{2(p-1)/(p-2)} \right]^{-1/(1-p)}, \qquad \text{quand } (x,y) \to (0,0).$$

Ce qui est intéressant est que le profil est différent dans les directions parallèle et normale au bord. Même si la preuve donnée dans [49] est très compliquée et nécessite beaucoup de calculs, avec la machinerie des coordonnées adaptées au bord qu'on a développée, il semble possible d'attaquer ce problème pour des domaines plus généraux. Ceci fera l'objet de recherches futures. Il serait intéressant de voir si le profil présente le même comportement anisotrope que celui décrit dans [49] ou si des effets de la courbure peuvent être décelés dans le profil.

3 Le problème d'évolution associé aux valeurs propres de la matrice hessienne

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Les démonstrations des résultats de cette section, ainsi que d'autres détails et références, sont inclus dans le Chapitre 5.

Dans cette partie de la thèse, nous étudions le problème à valeur initiale et conditions au bord suivant :

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{dans } \Omega \times]0,\infty[, \\ u(x,t) = g(x,t), & \text{sur } \partial\Omega \times]0,\infty[, \\ u(x,0) = u_0(x), & \text{dans } \Omega, \end{cases} \tag{P_j}$$

où Ω est un domaine borné de $\mathbb{R}^N$, avec $N \geq 1$, et j est un entier naturel entre 1 et N. Pour la matrice hessienne $D^2u = (\partial^2_{x_i,x_j}u)_{ij}$, on note

$$\lambda_1(D^2u) \leq \ldots \leq \lambda_N(D^2u)$$

ses valeurs propres ordonnées. On considérera que u_0 et g satisfont la condition de compatibilité $g(x,0) = u_0(x)$ pour tout $x \in \partial\Omega$.

Le problème (P_j) est la version d'évolution du problème de Dirichlet

$$\begin{cases} \lambda_j(D^2z(x)) = 0, & \text{dans } \Omega, \\ z(x) = g(x), & \text{sur } \partial\Omega, \end{cases} \tag{S_j}$$

qui a été largement étudié dans [8, 6, 7, 9, 10, 30, 42, 43]. Il faut souligner que $\lambda_j(D^2\cdot)$ est un opérateur elliptique, complètement non-linéaire et dégénéré. Des opérateurs similaires sont, par exemple,

$$\mathcal{P}_k^-(D^2u) = \sum_{i=1}^{k} \lambda_i(D^2u) \qquad \text{et} \qquad \mathcal{P}_k^+(D^2u) = \sum_{i=1}^{k} \lambda_{N+1-i}(D^2u),$$

qui dans la littérature sont connus sous le nom de *Laplaciens tronqués*. Pour $k = N$, ces deux opérateurs coïncident avec le Laplacien ordinaire. Les laplaciens tronqués ont été introduits, dans les années 80, par Sha [50, 51] et Wu [55] afin d'étudier les variétés avec bord k-convexe, c'est-à-dire, celles dont la somme des k premières courbures principales à chaque point du bord est positive. De même, les problèmes (P_j) et (S_j) ont aussi un intérêt géométrique important.

3.1 Interprétation géométrique de l'équation (S_j)

Moralement, une solution du problème (S_j) est une fonction qui, à chaque point, a la j-ème courbure principale nulle et coïncide avec g sur le bord de Ω, avec g une fonction donnée. En particulier, pour $j = 1$ et $j = N$, le problème (S_j) est l'équation pour l'enveloppe convexe et l'enveloppe concave respectivement, c.-à-d., la plus plus grande fonction convexe (plus petite fonction concave) qui reste en dessous (au-dessus) de la fonction g sur le bord, voir [42, 43].

Nous donnons plus loin l'interprétation géométrique de l'équation (S_j) présentée par Blanc et Rossi dans [9]. Mais d'abord, on va considérer les deux cas triviaux suivants :

Si l'on pense au problème (S_j) sur un domaine $\Omega =]a, b[$ dans $\mathbb{R}$, on ne peut que considérer $j = 1 = N$. Le graphe de la solution est donc le segment joignant les points $(a, g(a))$ et $(b, g(b))$. Celle-ci est, évidemment, la plus grande fonction convexe qui passe par-dessous ces deux points et aussi la plus petite fonction concave qui passe par-dessus. On peut généraliser ce raisonnement au cas N-dimensionnel si l'on considère une fonction g qui est la restriction à $\partial\Omega$ d'une fonction affine. Dans ce cas, les enveloppes convexe et concave, qui correspondent à $j = 1$ et $j = N$ respectivement, sont la même fonction affine.

Maintenant, on va traiter les cas extrêmes $j = 1$ et $j = N$, qui sont plus simples à comprendre et nous seront utiles plus tard pour comprendre les cas plus complexes $1 < j < N$.

Le cas $j = 1$: On appelle H_1 l'ensemble des fonctions v telles que

$$v \leq g \qquad \text{sur } \partial\Omega,$$

et qui satisfont la propriété suivante : pour tout segment $D =]x_1, x_2[\subset \Omega$, on a

$$v \leq z, \qquad \text{dans } D,$$

où z est la fonction linéaire dans D qui prend les valeurs $v|_{\partial D}$ sur le bord du segment. Autrement dit, z est l'enveloppe concave de $v|_{\partial D}$ dans D. Dans ce cas, le graphe de z est le segment joignant les points $(x_1, v(x_1))$ et $(x_2, v(x_2))$. Alors, on a

$$v(tx_1 + (1-t)x_2) \leq tv(x_1) + (1-t)v(x_2), \qquad \text{pour tout } t \in]0, 1[.$$

On déduit que H_1 est l'ensemble des fonctions convexes sur Ω qui sont plus petites que g sur $\partial\Omega$. La fonction définie par

$$u(x) = \sup_{v \in H_1} v(x),$$

est l'enveloppe convexe de g dans Ω et est aussi la solution de (S_1) au sens de viscosité (voir la définition 3.1).

Le cas $j = N$: Ce cas est très similaire au cas précédent. Si l'on considère $v = -u$, où u est l'enveloppe convexe de $-g$ dans Ω, alors v est l'enveloppe concave de g dans Ω.

Le cas $1 < j < N$: On commence par rappeler la caractérisation de la j-ème valeur propre d'une matrice A symétrique de taille $N \times N$

$$\lambda_j(A) = \inf_{\dim(S)=j} \sup_{v \in S, |v|=1} \{\langle Av, v\rangle\}. \tag{3.1.1}$$

Maintenant, on va caractériser l'ensemble de fonctions H_j, qui est l'analogue de H_1 défini ci-dessus. Soit H_j l'ensemble des fonctions v telles que

$$v \leq g, \qquad \text{sur } \partial\Omega,$$

et satisfont la propriété suivante : pour tout S espace affine de dimension j et tout sous-domaine j-dimensionnel $D \subset S \cap \Omega$, on a

$$v \leq z, \qquad \text{dans } D,$$

où z est l'enveloppe concave de $v|_{\partial\Omega}$ dans D.

D'après (3.1.1), on voit que H_j est l'ensemble de sous-solutions des (S_j). La fonction définie par

$$u(x) = \sup_{v \in H_j} v(x),$$

est la plus grande solution de viscosité de $\lambda_j(D^2u) = 0$ en Ω qui satisfait $u \leq g$ sur $\partial\Omega$. Dans [9], il est prouvé que la condition géométrique suivante sur Ω est nécessaire et suffisante pour l'existence d'une solution continue de (S_j) pour toute g continue :

Pour tout $y \in \partial\Omega$ on suppose que, pour tout $r > 0$ il existe un $\delta > 0$ tel que, pour tout $x \in B_\delta(y) \cap \Omega$ et tout sous-espace $S \subset \mathbb{R}^N$ de dimension j, il existe $v \in S$ unitaire tel que

$$\{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \tag{G_j}$$

On dit que le domaine Ω satisfait la condition (G) s'il satisfait au même temps (G_j) et (G_{N-j+1}).

L'équation (S_j) a une solution continue pour tout g continue si et seulement si Ω satisfait la condition (G). Cette condition (G), pour un domaine borné avec bord régulier (au moins C^2), est équivalente à

$$\kappa_j(x) > 0 \quad \text{et} \quad \kappa_{N-j+1}(x) > 0, \qquad \text{pour tout } x \in \partial\Omega, \tag{3.1.2}$$

où $\kappa_1(x) \leq \kappa_2(x) \leq \ldots \leq \kappa_{n-1}(x)$ sont les courbures principales de $\partial\Omega$ en x.

On remarque que u est une solution continue de (S_j) pour g si et seulement si $-u$ est une solution continue de (S_{N-j+1}) pour $-g$. Cela explique pourquoi il faut supposer (G_j) et (G_{N-j+1}) dans la condition (G). Pour les cas extrêmes $j = 1$ et $j = N$, la condition (G) est équivalente à la convexité stricte de Ω.

3.2 Solutions de viscosité

Pour les problèmes (P_j) et (S_j), on ne peut pas, en général, espérer avoir une solution suffisamment régulière pour laquelle on puisse calculer la matrice hessienne au sens classique. Ainsi, il est nécessaire d'introduire un type de solution plus faible qui nous permette de considérer des solutions qui ne sont pas deux fois dérivables.

Énonçons la définition de solution de viscosité qu'on utilisera pour le problème (P_j). Ce type de solutions est typiquement utilisé pour traiter des problèmes complètement non-linéaires et/ou dégénérés (voir Crandall-Ishii-P.L. Lions [13] pour plus de détails sur la théorie des solutions de viscosité).

Définition 3.1. *Une fonction $u : \Omega_T := \Omega \times\ ]0, \infty[\to \mathbb{R}$ vérifie l'équation*

$$u_t - \lambda_j(D^2u) = 0$$

au sens de viscosité si les enveloppes semi-continues supérieure et inférieure de u, définies respectivement par

$$u_*(x,t) = \sup_{r>0} \ \inf\{u(y,s); \quad y \in B_r(x),\ |s-t| < r\},$$
$$u^*(x,t) = \inf_{r>0} \ \sup\{u(y,s); \quad y \in B_r(x),\ |s-t| < r\},$$

satisfont :

i. Pour toute $\phi \in C^2(\Omega_T)$ *telle que* $u_* - \phi$ *a un minimum strict au point* $(x,t) \in \Omega_T$ *avec* $u_*(x,t) = \phi(x,t)$*, on a*

$$\phi_t(x,t) - \lambda_j(D^2\phi(x,t)) \geq 0.$$

ii. Pour toute $\psi \in C^2(\Omega_T)$ *telle que* $u^* - \psi$ *a un maximum strict au point* $(x,t) \in \Omega_T$ *avec* $u^*(x,t) = \psi(x,t)$*, on a*

$$\psi_t(x,t) - \lambda_j(D^2\psi(x,t)) \leq 0.$$

Pour nos résultats, on obtiendra des solutions qui sont continues sur $\overline{\Omega_T}$ de sorte que l'on n'utilisera pas la notation u^* et u_* dans la suite, car les deux fonctions sont égales. Cependant, on ne rentrera pas dans les questions concernant la régularité des solutions. Pour l'équation (S_j), Oberman et Silvestre ont prouvé dans [42] que la régularité $C^{1+\alpha}$ des données au bord g implique la même régularité uniquement à l'intérieur. Ils ont de plus obtenu un contre-exemple prouvant que ce résultat de régularité intérieure est optimal et ne peut pas s'étendre jusqu'au bord. On espère un résultat du même type pour (P_j), mais on ne l'a pas encore prouvé.

3.3 Approche basée sur un jeu de type *Tug-of-war* et existence

On présente ici l'approche qu'on a choisie pour démontrer l'existence d'une solution du problème (P_j). Dans [30], l'existence d'une solution de viscosité de (S_j) est démontrée en utilisant la méthode de Perron. Dans [9], une preuve différente est donnée, où les auteurs introduisent un jeu de type "Tug-of-war" dont la fonction valeur approche la solution de l'EDP quand la longueur du pas du jeu tend vers zero. Voir [11, 38, 39, 40, 47, 48] pour plus de références concernant les jeux de type Tug-of-war et EDPs complètement non-linéaires.

Pour notre problème parabolique (P_j), il semble naturel d'utiliser la méthode de Perron pour essayer de démontrer l'existence d'une solution de viscosité. Néanmoins, on a décidé d'introduire une version parabolique du jeu utilisé en [9] pour prouver l'existence. Cette approche se révèle en effet très utile pour ensuite étudier le comportement asymptotique de la solution, car l'interprétation à l'aide du jeu donne beaucoup d'intuition.

Règles du jeu

Il s'agit d'un jeu de somme nulle à deux joueurs. On fixe $\varepsilon > 0$ et un entier j entre 1 et N. La position initiale du jeu est déterminée par un jeton placé sur un point de départ $x_0 \in \Omega$ à un temps initial donné $t_0 > 0$. Le Joueur I, qui veut minimiser le prix final, choisit un sous-espace S de dimension j dans $\mathbb{R}^N$. Ensuite, le Joueur II, qui veut maximiser le prix final, choisit un vecteur unitaire $v \in S$. Après un tirage au sort, la nouvelle position du jeton sera $(x_0 + \varepsilon v, t_0 - \varepsilon^2/2)$ ou $(x_0 - \varepsilon v, t_0 - \varepsilon^2/2)$ avec même probabilité. À la fin du premier tour, le jeu continue depuis la nouvelle position (x_1, t_1) avec les mêmes règles. Le jeu finit quand le jeton sort du domaine $\Omega \times]0, t_0]$.

Une fonction prix h est définie en dehors du domaine $\Omega \times]0, \infty[$. Compte tenu du problème (P_j), il convient que $h(x,t) = g(x,t)$ pour tout $(x,t) \in \partial\Omega \times]0, \infty[$, et que $h(x,0) = u_0(x)$ pour tout $x \in \Omega$. Autrement dit, on considère une extension continue des données initiale et au bord. On appelle (x_τ, t_τ) la position finale du jeu, c'est-à-dire, le plus petit $\tau \in \mathbb{N}$ tel que $x_\tau \notin \Omega$ ou $t_\tau \leq 0$. Finalement, le prix final que le Joueur I doit payer au Joueur II est donné par $h(x_\tau, t_\tau)$.

Stratégies

À chaque tour k, si $x_k \in \Omega$ et $t_k > 0$, le Joueur I a la possibilité de choisir le sous-espace S de dimension j dans lequel se situera x_{k+1}, sachant la position actuelle du jeton et toutes les positions précédentes. Par conséquent, le choix du Joueur I au k-ème tour dépend de $(x_0, \dots, x_k, t_0)$. On voit que, comme le pas dans la variable t est fixe, pour t_0 donné, toutes les positions t_k sont déterminées par k. Il est nécessaire donc que $t_0 > k\varepsilon^2/2$. Sinon, le jeu serait fini avant le tour k.

Pour le Joueur I, une stratégie S_I est une collection de fonctions mesurables $S_I = \{S_k\}_{k=0}^{\infty}$ avec

$$\begin{array}{rccc} S_k : & \Omega^{k+1} \times]k\varepsilon^2/2, \infty[& \longrightarrow & Gr(j, \mathbb{R}^N) \\ & (x_0, \dots, x_k, t_0) & \longmapsto & S. \end{array}$$

De même, pour le Joueur II, une stratégie S_{II} est une collection de fonctions mesurables $S_{II} = \{S_k\}_{k=0}^{\infty}$ avec

$$\begin{array}{rccc} S_k : & \Omega^{k+1} \times Gr(j, \mathbb{R}^N) \times]k\varepsilon^2/2, \infty[& \longrightarrow & S \\ & (x_0, \dots, x_k, S, t_0) & \longmapsto & v, \end{array}$$

où v est un vecteur unitaire de S.

Fonction valeur

Une fois que les deux joueurs ont choisi leurs stratégies, on peut calculer l'espérance du prix final, qu'on désigne par

$$\mathbb{E}_{S_I, S_{II}}^{x_0, t_0}[h(x_\tau, t_\tau)].$$

La fonction valeur du jeu pour chaque joueur est la meilleure valeur espérée du prix final en utilisant une de ses stratégies respectives et en supposant que l'autre joueur utilise sa meilleure stratégie. Comme le Joueur I veut minimiser le prix final et le Joueur II veut le maximiser, on peut écrire la fonction valeur du jeu pour chaque joueur comme

$$u_I^\varepsilon(x_0, t_0) = \inf_{S_I} \sup_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0, t_0}[h(x_\tau, t_\tau)], \quad \text{et} \quad u_{II}^\varepsilon(x_0, t_0) = \sup_{S_{II}} \inf_{S_I} \mathbb{E}_{S_I, S_{II}}^{x_0, t_0}[h(x_\tau, t_\tau)].$$

On remarque que les espérances ci-dessus sont bien définies car le jeu finit en moins de $\lceil 2t_0/\varepsilon^2 \rceil$ tours avec probabilité 1.

Dans le Chapitre 5, on démontrera que pour ce jeu on a $u_I^\varepsilon(x_0, t_0) = u_{II}^\varepsilon(x_0, t_0)$. Alors, on définit la fonction valeur du jeu comme

$$u^\varepsilon(x, t) = u_I^\varepsilon(x, t) = u_{II}^\varepsilon(x, t), \tag{3.3.1}$$

pour $(x, t) \in \Omega \times]0, \infty[$.

Dynamic Programming Principle et Théorème d'existence

On verra dans le Chapitre 5 que la fonction valeur vérifie une équation qui est connue dans la littérature sous le nom de *Dynamic Programming Principle* (voir Théorème 3.2).

De plus, sous une hypothèse géométrique similaire à (G) qu'on précisera ci-dessous, la fonction $u^\varepsilon(x,t)$ converge uniformément dans $\overline{\Omega}\times[0,T]$, pour tout $T>0$, vers une fonction continue $u(x,t)$, qui est la seule solution de viscosité du problème (P_j).

L'hypothèse géométrique pour la convergence du jeu est la suivante : Pour tout $y\in\partial\Omega$, on suppose qu'il existe $r>0$ tel que pour tout $\delta>0$ il existe $T\subset\mathbb{R}^n$ un sous-espace de dimension j, $w\in\mathbb{R}^n$ un vecteur unitaire, $\lambda>0$ et $\theta>0$ tels que

$$\{x\in\Omega\cap B_r(y)\cap T_\lambda : \langle w, x-y\rangle<\theta\}\subset B_\delta(y) \tag{F_j}$$

où

$$T_\lambda=\{x\in\mathbb{R}^n : \operatorname{dist}(x-y,T)<\lambda\}.$$

De façon analogue à la condition (G), pour la convergence du jeu avec j, on supposera la condition (F), c.-à-d., on supposera (F_j) et (F_{N-j+1}). On remarque que tout domaine strictement convexe satisfait (F_j) pour tout $j\in\{1,\dots,N\}$, mais des domaines plus généraux peuvent aussi vérifier la condition (F) avec $1<j<N$, voir [9]. On donne maintenant le théorème d'existence de solution pour le problème (P_j).

Théorème 3.2. *Il existe une fonction u^ε définie en* (3.3.1)*, qui est la fonction valeur du jeu décrit ci-dessus. Cette fonction peut être caractérisée comme la seule solution du Dynamic Programming Principle suivant :*

$$\begin{cases} u^\varepsilon(x,t)=\inf\limits_{dim(S)=j}\ \sup\limits_{v\in S,|v|=1}\left\{\dfrac{1}{2}u^\varepsilon(x+\varepsilon v,t-\dfrac{\varepsilon^2}{2})+\dfrac{1}{2}u^\varepsilon(x-\varepsilon v,t-\dfrac{\varepsilon^2}{2})\right\}, \\ \hfill si\ x\in\Omega,\ t>0, \\ u^\varepsilon(x,t)=h(x,t), \hfill si\ x\notin\Omega,\ ou\ t\le 0. \end{cases}$$

De plus, si Ω satisfait l'hypothèse (F), alors il existe une fonction $u\in C(\overline{\Omega}\times[0,\infty[)$ telle que

$$u^\varepsilon\to u \qquad uniformement\ dans\quad \overline{\Omega}\times[0,T],$$

quand $\varepsilon\to 0$ pour tout $T>0$. La limite u est la seule solution de viscosité de

$$\begin{cases} u_t(x,t)-\lambda_j(D^2u(x,t))=0, & dans\ \Omega\times]0,\infty[,\\ u(x,t)=g(x,t), & sur\ \partial\Omega\times]0,\infty[,\\ u(x,0)=u_0(0), & dans\ \Omega. \end{cases}$$

3.4 Comportement asymptotique

Maintenant, on s'intéresse au comportement asymptotique de la solution de (P_j) quand $t\to\infty$. Notre but est de démontrer la convergence de la solution du problème d'évolution vers celle du problème stationnaire. Pour cela, on se limite à étudier le cas des données au bord qui ne dépendent pas du temps, c.-à-d., on considère maintenant le problème

$$\begin{cases} u_t(x,t)-\lambda_j(D^2u(x,t))=0, & \text{dans } \Omega\times]0,\infty[,\\ u(x,t)=g(x), & \text{sur } \partial\Omega\times]0,\infty[,\\ u(x,0)=u_0(x), & \text{dans } \Omega, \end{cases} \tag{3.4.1}$$

où u_0 est une fonction continue définie dans $\overline{\Omega}$ et $g=u_0|_{\partial\Omega}$.

Á l'aide d'arguments de comparaison on peut démontrer que $u(x,t)$ converge exponentiellement vers la solution du problème stationnaire avec même donnée au bord. En particulier, pour les cas $j=1$ et $j=N$, ce résultat affirme que l'on peut approcher l'enveloppe convexe ou concave de g sur Ω par la solution du problème d'évolution (P_j).

Théorème 3.3. *Soit $\Omega \subset \mathbb{R}^N$ un domaine borné, soit u_0 une fonction continue définie sur $\overline{\Omega}$ et $g = u_0|_{\partial\Omega}$. Alors, il existe deux constantes positives, $\mu > 0$ (qui ne dépendent que de Ω) et C (qui dépend de la donnée initiale u_0) telles que l'unique solution de viscosité u de* (3.4.1) *satisfait*

$$\|u(\cdot,t) - z(\cdot)\|_\infty \leq Ce^{-\mu t}, \tag{3.4.2}$$

où z est l'unique solution de viscosité de (S_j).

Dans la preuve de ce théorème, afin de construire des sur- et sous-solutions pour le problème (P_j), on utilise le problème de valeurs propres associé aux opérateurs $-\lambda_1(D^2\cdot)$ et $-\lambda_n(D^2\cdot)$. Ce problème de valeurs propres a été étudié précédemment dans [6, 7].

Dans le Chapitre 5, on donne aussi une preuve alternative du Théorème 3.3 qui n'utilise que des arguments probabilistes liés au jeu décrit dans la section 3.3. Cette preuve est basée sur le fait que si la position initiale du jeu se situe à un temps initial t_0 suffisamment grand, la probabilité de ne pas sortir de Ω en moins de $\lceil 2t_0/\varepsilon^2 \rceil$ pas devient très petite. En effet, cette probabilité décroît exponentiellement avec t_0.

Comportement asymptotique plus précis pour des cas spéciaux

Ici, on décrit un comportement de la solution, pour les cas où g est la restriction à $\partial\Omega$ d'une fonction affine, qui possède des propriétés spécifiques par rapport au comportement décrit dans le Théorème 3.3 pour les données au bord plus générales. Plus précisément, on arrive à montrer que, pour ces cas spéciaux, l'estimation (3.4.2) peut être largement améliorée. Commençons par présenter la situation pour le cas le plus simple en considérant $j = 1$ et $g \equiv 0$. On a donc le problème

$$\begin{cases} u_t(x,t) - \lambda_1(D^2u(x,t)) = 0, & \text{dans } \Omega\times]0,\infty[, \\ u(x,t) = 0, & \text{sur } \partial\Omega\times]0,\infty[, \\ u(x,0) = u_0, & \text{dans } \Omega. \end{cases}$$

Dans ce cas, on a $z \equiv 0$ et, par le Théorème 3.3, on déduit qu'il existe deux constantes $\mu, C > 0$ telles que

$$-Ce^{-\mu t} \leq u(x,t) \leq Ce^{\mu t}.$$

Dans ce scénario, on peut améliorer l'estimation supérieure. On démontre qu'il existe un temps fini $T > 0$, qui dépend uniquement de Ω, tel que $u(x,t) \leq 0$ pour tout $t > T$. Autrement dit, u est inférieure à l'enveloppe convexe de g dans Ω, pour tout temps au-delà de T. Ce résultat est une conséquence du fait que, pour tout Ω borné, le problème de valeur propre

$$-\lambda_1(D^2\varphi(x)) = \mu\varphi(x), \qquad \text{dans } \Omega, \tag{3.4.3}$$

admet une solution positive pour tout $\mu > 0$. En fait, on peut décrire le même comportement asymptotique si g est la restriction à $\partial\Omega$ d'une fonction affine. Il suffit d'appliquer le même argument à $\tilde{u} = u - g$.

Si l'on considère $j = n$, et g une fonction affine, on obtient la situation analogue. C'est-à-dire, il existe un temps $T > 0$ au-delà duquel la solution de (P_j) est supérieure à l'enveloppe concave de g dans Ω. Par contre, la situation est différente pour les cas $1 < j < n$. Dans ce cas, l'équation (3.4.3) admet une solution positive et une autre négative pour tout $\mu > 0$, et comme conséquence, on peut démontrer que, si g est une fonction affine, il existe $T > 0$ tel que $u(x,t) = z(x)$ pour tout $t > T$, où z est la solution de (S_j). On rassemble ces trois scénarios dans le théorème suivant.

Théorème 3.4. *Soit $\Omega \subset \mathbb{R}^n$ un domaine borné. Soit g la restriction à $\partial\Omega$ d'une fonction affine et u_0 une fonction continue dans Ω. Si $u(x,t)$ est la solution de viscosité de (3.4.1) et $z(x)$ est la fonction affine qui coïncide avec g sur le bord (autrement dit, la solution de viscosité de (S_j)), alors il existe un $T > 0$, qui dépend uniquement de Ω, tel que*

i. *Si $j = 1$, alors $u(x,t) \leq z(x)$, pour tout $t > T$.*

ii. *Si $j = N$, alors $u(x,t) \geq z(x)$, pour tout $t > T$.*

iii. *Si $1 < j < N$, alors $u(x,t) = z(x)$, pour tout $t > T$.*

Interprétation du résultat

Pour simplifier l'exposé, on va considérer uniquement le cas $g \equiv 0$. Si l'on pense au jeu qu'on a décrit pour approcher les solutions de (P_j), on peut interpréter ce résultat de la façon suivante :

Pour le cas $j = 1$, le Joueur I a une stratégie qui lui permet de sortir le jeton du domaine Ω avec probabilité 1 en un nombre fini de pas. Alors, pour t_0 suffisamment grand, on peut assurer que le Joueur II ne va pas toucher $u_0(x)$ comme prix final.

En effet, considérons par exemple le cas où $\Omega = B_1(0)$ la boule unité dans $\mathbb{R}^N$. Si le jeton n'est pas à l'origine, c.-à-d. $x_k \in B_1(0) \setminus \{0\}$, le Joueur I peut choisir comme sous-espace $S = \text{span}\{v\}$, où v est un vecteur unitaire tangent en x_k à la boule de rayon $|x_k|$ et centre à l'origine. Ici, le Joueur II ne peut choisir comme vecteur que v ou $-v$, et en tout cas, après le tirage au sort, la nouvelle position du jeton sera éloignée de l'origine une distance proportionnelle à ε^2. Comme à chaque pas, on s'approche au bord de Ω une distance constante, il existe $T > 0$, qui ne dépend pas de x_0 ni de t_0, telle que $x_{k+1} \notin \Omega$, pour $k = \lceil 2T/\varepsilon^2 \rceil$. Alors, pour tout $t_0 > T$, on a $t_\tau > 0$, ce qui implique $h(x_\tau, t_\tau) = g(x_\tau)$.

Par conséquent, si $g \equiv 0$, le Joueur I a une stratégie avec laquelle le prix final sera 0 avec probabilité 1. On en déduit, que la fonction valeur du jeu sera plus petite que 0 pour tout $t_0 > T$.

Le même principe s'applique au cas $j = N$. Maintenant, c'est le Joueur II qui a une stratégie lui permettant de sortir le jeton de Ω en un nombre fini de pas. Alors, son prix gagné sera au pire 0. Dans ce cas, comme le Joueur I doit choisir un sous-espace de dimension n, le seul choix est de choisir $S = \mathbb{R}^N$, et alors, le Joueur II peut choisir un vecteur tangent a la boule de rayon $|x_k|$ et centre à l'origine, qui va ramener le jeton vers le bord en un nombre fini de pas.

Pour le cas $1 < j < N$, on a une situation similaire. Ici, les deux joueurs ont des stratégies permettant de sortir le jeton de Ω rapidement indépendamment de la stratégie choisie par l'autre joueur. En effet, si $x_k \in B_1(0) \setminus \{0\}$, le Joueur I peut choisir un sous-espace S de dimension j tangent en x_k à la boule de rayon $|x_k|$ et centre l'origine, mais aussi, quel que soit le sous-espace S choisi par le Joueur I, le Joueur II peut toujours choisir un vecteur unitaire tangent à cette boule. Ces deux stratégies vont finir le jeu en un nombre de pas, indépendant de t_0. Par conséquent, si t_0 est suffisamment grand, le pire prix pour tous les deux joueurs sera la donnée au bord, qui est nulle.

Bien que le Théorème 3.4 implique que pour certains cas, l'estimation (3.4.2) n'est pas optimale, dans le Chapitre 5, on décrit des exemples (avec données au bord qui ne sont pas affines) pour lesquels la solution u ne devient jamais ni plus petite ni plus grande que la solution du problème stationnaire en temps fini. Le résultat suivant prouve que, en général, on ne peut pas espérer une convergence meilleure que la convergence exponentielle du Théorème (3.4).

Théorème 3.5. *Soit Ω un domaine borné de $\mathbb{R}^N$, et soit $1 \leq j \leq N$. Pour tout $x_0 \in \Omega$, il existe g et u_0 deux fonctions continues dans $\partial\Omega$ et $\overline{\Omega}$ respectivement, satisfaisant $u_0|_{\partial\Omega} = g$, telles que la solution du problème (P_j) satisfait*

$$u(x_0, t) \geq z(x_0) + ke^{-\mu_1 t}, \qquad \textit{pour tout } t > 0,$$

où $\mu_1, k > 0$ sont deux constantes et z est la solution de (S_j).

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Chapter 2

No touchdown at points of small permittivity and nontrivial touchdown sets for the MEMS problem

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Abstract. We consider a well-known model for micro-electromechanical systems (MEMS) with variable dielectric permittivity, involving a parabolic equation with singular nonlinearity. We study the touchdown, or quenching, phenomenon. Recently, the question whether or not touchdown can occur at zero points of the permittivity profile f, which had long remained open, was answered negatively for the case of interior points.

The first aim of this article is to go further by considering the same question at points of positive but small permittivity. We show that, in any bounded domain, touchdown cannot occur at an interior point where the permittivity profile is suitably small. We also obtain a similar result in the boundary case, under a smallness assumption on f in a neighborhood of the boundary. This allows in particular to construct f producing touchdown sets concentrated near any given sphere.

Our next aim is to obtain more information on the structure and properties of the touchdown set. In particular, we show that the touchdown set need not in general be localized near the maximum points of the permittivity profile f. In the radial case in a ball, we actually show the existence of "M"-shaped profiles f for which the touchdown set is located *far away* from the maximum points of f and we even obtain strictly convex f for which touchdown occurs only at the unique *minimum* point of f. These results give analytical confirmation of some numerical simulations from the book *[P. Esposito, N. Ghoussoub, Y. Guo, Mathematical analysis of partial differential equations modeling electrostatic MEMS, Courant Lecture Notes in Mathematics, 2010]* and solve some of the open questions therein. They also show that some kind of smallness condition as above cannot be avoided in order to rule out touchdown at a point.

On the other hand, we construct profiles f producing more complex behaviors: in any bounded domain the touchdown set may be concentrated near two arbitrarily given points,

or two arbitrarily given $(n-1)$-dimensional spheres in a ball. These examples are obtained as a consequence of stability results for the touchdown time and touchdown set under small perturbations of the permittivity profile.

1 Introduction

1.1 Mathematical problem and physical background

We consider the problem

$$\begin{cases} u_t - \Delta u = f(x)(1-u)^{-p}, & t>0, \quad x \in \Omega, \\ u = 0, & t>0, \quad x \in \partial\Omega, \\ u(0,x) = 0, & x \in \Omega, \end{cases} \tag{1.1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, $p > 0$ and $f \in E$, where

$$E = \big\{ f : \overline{\Omega} \to [0,\infty);\ f \text{ is Hölder continuous} \big\}. \tag{1.1.2}$$

Problem (1.1.1) with $p = 2$ is a known model for micro-electromechanical devices (MEMS) and has received a lot attention in the past 15 years. An idealized version of such device consists of two conducting plates, connected to an electric circuit. The lower plate is rigid and fixed while the upper one is elastic and fixed only at the boundary. Initially the plates are parallel and at unit distance from each other. When a voltage (difference of potential between the two plates) is applied, the upper plate starts to bend down and, if the voltage is large enough, the upper plate eventually touches the lower one. This is called *touchdown* phenomenon. Such device can be used for instance as an actuator, a microvalve (the touching-down part closes the valve), or a fuse.

In the mathematical model, $u = u(t,x)$ measures the vertical deflection of the upper plate and the function $f(x)$ represents the dielectric permittivity of the material which, as a key feature, may be possibly inhomogeneous. (Actually f is also proportional to the – constant – applied voltage.)

It is well known that problem (1.1.1) admits a unique maximal classical solution u. We denote its maximal existence time by $T = T_f \in (0,\infty]$. Moreover, under some largeness assumption on f, it is known that the maximum of u reaches the value 1 at a finite time, so that u ceases to exist in the classical sense, i.e. $T < \infty$ (see, e.g., Lemma 2.2 below). This property, known as quenching, is the mathematical counterpart of the touchdown phenomenon.

A point $x = x_0 \in \overline{\Omega}$ is called a *touchdown* or *quenching point* if there exists a sequence $\{(t_n, x_n)\} \in (0,T) \times \Omega$ such that

$$x_n \to x_0, \quad t_n \uparrow T \quad \text{and} \quad u(x_n,t_n) \to 1 \text{ as } n \to \infty.$$

The set of all such points is closed. It is called the *touchdown* or *quenching set*, denoted by $\mathcal{T} = \mathcal{T}_f \subset \overline{\Omega}$.

MEMS problems, including system (1.1.1) and the related touchdown issues, have received considerable attention in the physical and engineering as well as in the mathematical communities. We refer to [6], [29] for more details on the physical background, and to, e.g., [16], [9] [28], [24], [10], [12], [13], [18], [22], [23], [25], [19], [35], [17], [21] for mathematical studies. See also [30], [26], [4], [15], [7], [8] for earlier mathematical work on the case of constant f.

As a question of particular interest, it has long remained open whether touchdown could occur at zero points of the permittivity profile. This has been answered negatively in [21] for the case of interior points. This is by no means obvious since, for the analogous blowup problem $u_t - \Delta u = f(x)u^p$ with $f(x) = |x|^\sigma$, examples of solutions blowing up at the origin have been constructed in [9], [20] for suitable $\sigma > 0$, $p > 1$ and suitable initial data $u_0 \geq 0$.

To go further, natural questions are then:

• can one rule out touchdown at points of positive but *small* permittivity?

• can one obtain more information on the structure and properties of the touchdown set?

These are the main motivations of the present article.

1.2 Results (I): no touchdown at points of small permittivity

Our first main result shows that touchdown cannot occur at an interior point of small permittivity $f(x_0)$, and we provide a suitable smallness condition in terms of f and x_0. In the sequel, we denote by

$$\delta(x) := \mathrm{dist}(x, \partial\Omega), \qquad x \in \overline{\Omega},$$

the function distance to the boundary.

Theorem 1.1 (No touchdown at interior points of small permittivity). *Let $p > 0$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and $f \in E$. Assume*

$$\begin{cases} T_f \leq M, \quad \|f\|_\infty \leq M, \quad f \geq r\chi_B, \\ \textit{where } M, r > 0 \textit{ and } B \subset \Omega \textit{ is a ball of radius } r. \end{cases} \tag{1.2.1}$$

There exists $\gamma_0 > 0$ depending only on p, Ω, M, r such that, for any $x_0 \in \Omega$, if

$$f(x_0) < \gamma_0\, \delta^{p+1}(x_0), \tag{1.2.2}$$

then $x_0 \notin \mathcal{T}_f$.

As a drawback of Theorem 1.1, boundary points are not covered, and the threshold value vanishes when x_0 approaches the boundary. Actually, it remains an open problem whether touchdown can occur on the boundary, including at boundary points of zero permittivity. Some partial results can be found in [22], [21], where $f(x)$ is assumed to either be monotonically decreasing or to vanish sufficiently fast, as x approaches the boundary. Our second main result gives another contribution to that question. It shows that touchdown can be localized in any compact subdomain of Ω under the assumption that f is small enough outside this subdomain. It is thus of a more global nature than the local criterion in Theorem 1.1 for interior points. As a consequence, it rules out touchdown on the boundary when f is small enough on a neighborhood of the boundary. We stress that for this result, unlike in [22], [21], we do not require any monotonicity or decay of f near $\partial\Omega$.

Theorem 1.2 (No touchdown for small permittivity near the boundary). *Let $p > 0$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and $f \in E$. Assume* (1.2.1). *There exists $\gamma_0 > 0$ depending only on p, Ω, M, r such that, for any smooth open subset $\omega \subset\subset \Omega$, if*

$$\sup_{x \in \overline{\Omega} \setminus \omega} f(x) < \gamma_0\, \mathrm{dist}^{p+1}(\omega, \partial\Omega), \tag{1.2.3}$$

then $\mathcal{T}_f \subset \omega$.

In view of Theorems 1.1 and 1.2, it is a natural question whether smallness conditions, such as (1.2.2) and (1.2.3), are actually necessary, or whether touchdown could be shown to occur only at or near the maximum points of the permittivity profile f. This, among other related issues, is the subject of our next subsection, where a number of results on the structure and properties of the touchdown set are obtained.

1.3 Results (II): Nontrivial touchdown sets and "M"-shaped profiles.

We will pay special attention to the following class of permittivity profiles. For $\Omega = B_R \subset \mathbb{R}^n$ $(n \geq 1)$, we call "M"-shaped permittivity profile a function f such that

$$\begin{array}{l} f \text{ is radially symmetric, nondecreasing in } |x| \text{ on } [0,L] \\ \text{and nonincreasing in } |x| \text{ on } [L,R], \text{ for some } L \in (0,R). \end{array} \tag{1.3.1}$$

In the book [6, Section 7.4], for particular "M"-shaped profiles, numerical simulations were carried out, which suggest some interesting phenomena regarding the location of touchdown points. In this paper we are able to confirm some of them by rigorous analytical arguments. In this connection, we shall construct "M"-shaped profiles, and variants thereof, giving rise to various types of touchdown sets: single-point, touchdown set concentrated near a sphere, near two points, near two spheres. We point out that such properties may be useful in the practical design of MEMS devices, at least on a qualitative level.

To begin with, as a consequence of Theorems 1.1 and 1.2, we have the following corollary.

Corollary 1.3. *Let $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$.*

(i) (Touchdown containing a sphere.) *Let $f \in E$ be an "M"-shaped profile, i.e.* (1.3.1) *holds, and assume* (1.2.1). *If $f(0)$ is small enough (depending only on p, n, R, M, r), then 0 is not a touchdown point. In particular, $\mathcal{T}_f$ contains an $(n-1)$-dimensional sphere.*

(ii) (Touchdown concentrated near a given sphere.) *Let $r > 0$ and $0 < \varepsilon < \min(r, R-r)$. There exist two-bump, "M"-shaped profiles f such that $T_f < \infty$ and*

$$\mathcal{T}_f \subset \{r - \varepsilon < |x| < r + \varepsilon\}.$$

More precisely, there exist $\eta, A > 0$, depending only on p, R, r, ε, such that this is true for any radially symmetric $f \in E$ satisfying

$$\begin{cases} f(x) \geq A, & \textit{for } |x| \in [r - \varepsilon/2, r + \varepsilon/2], \\ f(x) \leq \eta, & \textit{for } |x| \in [0, r - \varepsilon] \cup [r + \varepsilon, R]. \end{cases}$$

When Ω is a ball and f is constant or radial nonincreasing, it is well known that touchdown can occur only at the origin (see [4], [15], [22]). In particular, this is the case if we take $f(0) = f(L)$ instead of $f(0)$ small in Corollary 1.3(i). A natural question is then, whether the assumption "$f(0)$ small enough" in Corollary 1.3(i) could be replaced by $f(0) < f(L)$. The following theorem, which shows the stability of single point touchdown under suitable perturbation of f, answers this question negatively. In the sequel we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ and

$$\mu_0(p,n) := \frac{p^p}{(p+1)^{p+1}} \lambda_1, \tag{1.3.2}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(B_1)$ and B_1 is the unit ball in $\mathbb{R}^n$.

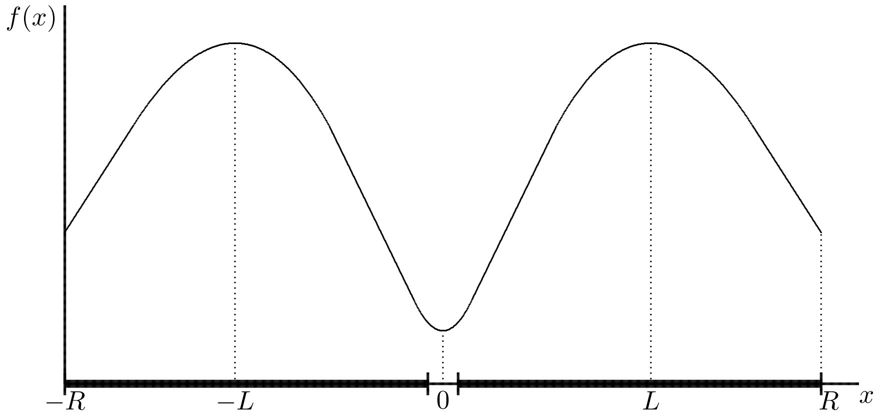

Figure 2.1 – Illustration of Corollary 1.3(i) – in this and the subsequent figures, the touchdown set must be a subset of the fat lines

Theorem 1.4 (Stability of single point touchdown under perturbation)**.** *Let* $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$, $1 \le q \le \infty$ *with* $q > \frac{n}{2}$, $M > 0$, $\rho \in (0, R)$. *Let* $f \in E \cap C^1(\overline{B}_\rho)$ *be radially symmetric nonincreasing, with* $f(r) > \mu_0(p,n)\rho^{-2}$ *on* $\overline{B}_\rho$. *There exists* $\varepsilon > 0$ *such that, if* $g \in E \cap C^1(\overline{B}_\rho)$ *is radially symmetric and satisfies*

$$\|g\|_\infty \le M, \tag{1.3.3}$$
$$-M \le g'(r) \le \varepsilon r, \quad \textit{for all } r \in [0, \rho], \tag{1.3.4}$$
$$\|g - f\|_q \le \varepsilon, \tag{1.3.5}$$

then $T_g < \infty$ *and* $\mathcal{T}_g = \{0\}$.

As a direct consequence of Theorem 1.4, there exist genuine "M"-shaped profiles g (i.e., such that $g(0) < g(L)$, with $g(0)$ close to $g(L)$), for which touchdown occurs at the single point $x = 0$ (see figure 2.3). This shows that some kind of smallness condition, such as (1.2.2) or (1.2.3), is required in order to rule out touchdown in a given region of the domain.

In turn, this provides examples of profiles f for which the touchdown set is located *far away* from the maximum points of f. It also shows that the radial nonincreasing monotonicity of f is sufficient but *not necessary* for single point touchdown at the origin. This confirms some of the numerical predictions from [6] (see [6, Remark 7.4.2]). Such a behavior must be interpreted as an effect of the diffusion (and of the boundary conditions), since in the absence of diffusion the explicit computation immediately shows that touchdown occurs only at the maximum points of f.

Another, rather surprising, consequence of Theorem 1.4, is the possibility of constructing *strictly convex* profiles producing single point touchdown at the unique *minimum* point of f. Indeed, let $f_\lambda(x)$ be the function defined in $\Omega = B(0, R)$ by

$$f_\lambda(x) := \mu + \lambda \frac{|x|^2}{R^2}, \qquad \text{with } \mu > \mu_0(p,n)\rho^{-2} \text{ and } \lambda \ge 0.$$

We see that f_0 is radially nonincreasing and, for $\lambda > 0$ small enough, f_λ satisfies the hypothesis of the Theorem. Therefore, the only touchdown point is the origin, i.e. the

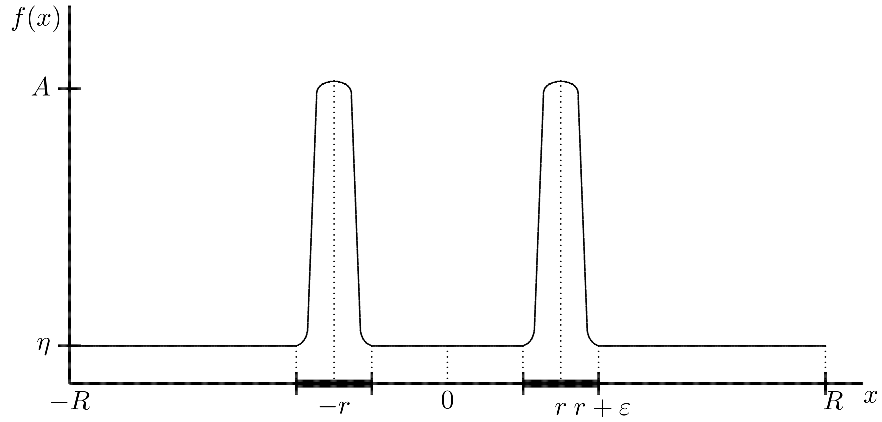

Figure 2.2 – Illustration of Corollary 1.3(ii)

unique minimum point of f_λ (see figure 2.4). This solves negatively the open question in [6, Section 7.5], on whether the touchdown set must consist of an $(n-1)$ dimensional sphere when $f(x) = f(|x|)$ is increasing in $|x|$. This example also shows that the monotonicity or decay hypotheses on f near the boundary are not necessary in general for the compactness of $\mathcal{T}_f$.

In Corollary 1.3(ii) we saw that the touchdown set can be concentrated near any $(n-1)$-dimensional sphere, where f achieves its maxima. As a consequence of the following result, which shows the stability of unfocused touchdown concentrated near the origin, we obtain profiles g whose touchdown set contains an $(n-1)$-dimensional sphere and is arbitrarily concentrated near the origin, *far away* from the maxima of g. Such g can take the form of an "M"-shaped profile with a narrow "well" near the origin (see fig. 2.5).

Theorem 1.5 (Stability of unfocused touchdown concentrated near the origin)**.** *Let $p > 0$, $\Omega = B_R \subset \mathbb{R}^n$, $1 \le q \le \infty$ with $q > \frac{n}{2}$, $0 < \eta < R$. Let $M, \rho > 0$, $B := B(x_0, \rho) \subset \Omega$ and $\mu > \mu_0(n,p)\rho^{-2}$. Let $f \in E$ be radially symmetric nonincreasing. There exists $\varepsilon > 0$ such that, if $g \in E$ is radially symmetric and satisfies*

$$\mu\chi_B \le g \le M,$$
$$g(0) < \varepsilon,$$
$$\|g - f\|_q \le \varepsilon,$$

then $T_g < \infty$ and $\mathcal{T}_g \subset B_\eta \backslash \{0\}$. In particular $\mathcal{T}_g$ contains at least an $(n-1)$-dimensional sphere.

In [12], for "M"-shaped profiles in dimension one, situations similar to Corollary 1.3 and Theorem 1.4 (cf. figures 2.1–2.4) were observed numerically, with respectively two and a single touchdown point. In the case of Corollary 1.3 we here do not know whether there are two points or more. On the other hand, for some other "M"-shaped profiles (roughly, intermediate between figure 2.1 and 2.3), touchdown on a whole interval containing 0 was observed numerically, which we are presently unable to confirm analytically. These seem to be difficult questions. In this connection, we stress that

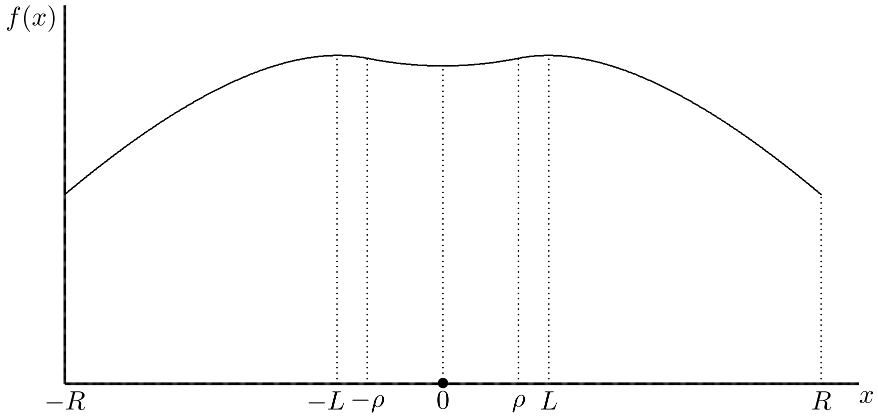

Figure 2.3 – An illustration of Theorem 1.4 in one space dimension for an "M"-shaped profile

results asserting the finiteness of the singular set for one-dimensional or radial problems (see [3]), based on reflection techniques, are essentially restricted to the case of constant or monotone coefficients. Also, it was shown in [33] that for the nonlinear heat equation in $\mathbb{R}^n$ with constant coefficients, the blowup set has Hausdorff dimension at most $n-1$, but the methods in [33] do not seem to apply to the present situation.

Our last example shows that more complicated behaviors can occur. To be specific, the touchdown set can be concentrated near two arbitrarily given points. In the case when Ω is a ball, we can construct radially symmetric profiles for which the touchdown set is concentrated near two arbitrarily given $(n-1)$-dimensional spheres.

Theorem 1.6. *Let $p > 0$. Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain.*

(i) (Touchdown set concentrated near two arbitrary points.)
For any $x_1, x_2 \in \Omega$ and any $\rho > 0$, there exist positive profiles $f \in E$ such that

$$\mathcal{T}_f \subset B(x_1,\rho) \cup B(x_2,\rho), \qquad \mathcal{T}_f \cap B(x_1,\rho) \neq \emptyset, \qquad \mathcal{T}_f \cap B(x_2,\rho) \neq \emptyset.$$

(ii) (Touchdown set concentrated near two arbitrary spheres.)
Let $\Omega = B_R \subset \mathbb{R}^n$, $0 < r_1 < r_2 < R$, $\rho > 0$ and set

$$A_i = \{x \in \mathbb{R}^n;\ |x| \in (r_i - \rho, r_i + \rho)\}.$$

There exist positive, radially symmetric profiles $f \in E$ such that

$$\mathcal{T}_f \subset A_1 \cup A_2, \qquad \mathcal{T}_f \cap A_1 \neq \emptyset, \qquad \mathcal{T}_f \cap A_2 \neq \emptyset.$$

Remark 1.7. *(i) In Theorem 1.6, the touchdown set in particular has at least two connected components if ρ is sufficiently small (at least four if $n = 1$). The profile in Theorem 1.6(i) is obtained, by a limiting argument, by constructing a two-bump profile, where each bump is contained in $B(x_1,\rho)$, $B(x_2,\rho)$ respectively, and smoothly varying the height in each bump. For (ii), we follow the same idea but considering radially symmetric profiles and replacing balls with annuli.*

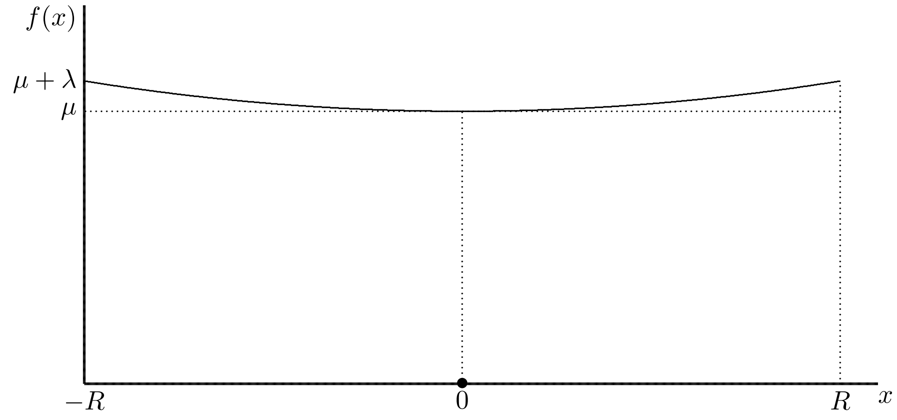

Figure 2.4 – An illustration of Theorem 1.4 for a strictly convex profile

(ii) In the case of the one-dimensional nonlinear heat equation with constant coefficients, for any prescribed finite set, it was shown in [27] that there exists an initial data for which the solution blows up exactly on this set. Such a construction does not seem easy to transpose to problem (1.1.1).

The proofs of the results in this subsection crucially depend on, rather delicate, stability properties of the touchdown set and time under small perturbations of the potential. We here state the following result, which may be of independent interest. Further results are given and proved in Section 5.

Theorem 1.8 (Continuity of the touchdown time and upper semi-continuity of the touchdown set)**.** *Let $p > 0$ and $\Omega \subset \mathbb{R}^n$ a smooth bounded domain. Let $1 \le q \le \infty$ with $q > \frac{n}{2}$, $B \subset \Omega$ a ball of radius $r > 0$, $M \ge \mu > \mu_0(p,n)r^{-2}$ and set*

$$\tilde{E} = \big\{f \in E;\ M \ge f \ge \mu\chi_B\big\}. \tag{1.3.6}$$

For all $f \in \tilde{E}$ with $\mathcal{T}_f \subset\subset \Omega$ and all $\sigma > 0$, there exists $\varepsilon > 0$ such that,

$$\text{if } g \in \tilde{E} \text{ and } \|g - f\|_q \le \varepsilon, \quad \text{then } |T_g - T_f| \le \sigma \text{ and } \mathcal{T}_g \subset \mathcal{T}_f + B(0,\sigma).$$

On the other hand, we can show that the *continuity* of the touchdown set with respect to f fails in general – see Proposition 5.6, which will be a consequence of the proof of Theorem 1.6. Actually, considering the profile constructed in that proof and depicted in fig. 2.6 for $n = 1$, it is shown that the touchdown points in the inner bumps immediately disappear as soon as the height of this plateau is decreased.

Remark 1.9. *For results on continuity of the existence time in the case of blow-up problems, see [2], [14], [31], [32] and the references therein. For results on the semi-continuity of the blow-up set, see [27], [1]. We note that the latter are restricted to one-dimensional problems, due to the lack of estimates near every possible point in the blow-up set. We are here able to avoid such restriction in the case of quenching problems, taking advantage of a time integrability property of the RHS of the PDE in* (1.1.1) *up to the quenching time (see the first step of the proof of Theorem 5.3). However we have to face some additional difficulties due to the lack of a type I estimate up to the boundary.*

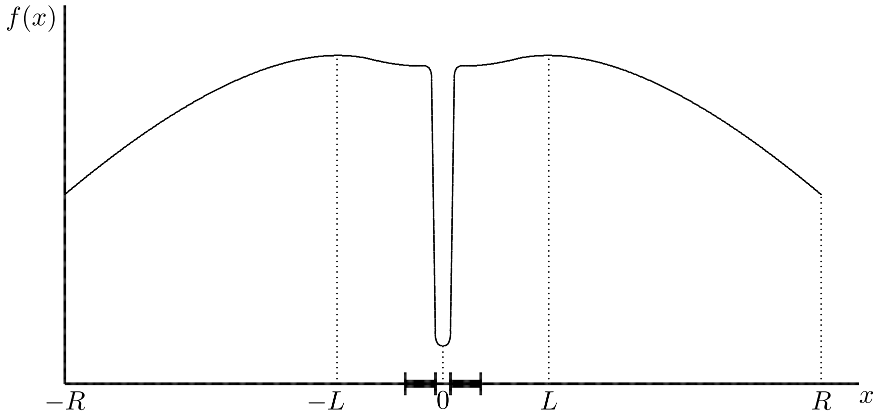

Figure 2.5 – Illustration of Theorem 1.5 with touchdown far away from the maxima of f.

The outline of the rest of the paper is as follows. In Section 2, we give some basic estimates for the touchdown time T, which will be useful in the sequel. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, based on refinements of the approach in [21]. Namely, in Section 3, we establish a type I estimate for the touchdown rate away from the boundary. In Section 4, we establish a no-touchdown criterion under an (optimal) smallness condition on f, assuming a local type I estimate. We then combine it with the estimate obtained in Section 3 to conclude the proof. In Section 5 we prove results on the continuity of the touchdown time and the semicontinuity of the touchdown set under small perturbations of the permittivity profile f. In Section 6, we then apply them, along with Theorem 1.1 and our type I estimate, to establish Theorems 1.4, 1.5 and 1.6.

2 Basic estimates for the touchdown time

The following simple estimates will be useful in the sequel.

Lemma 2.1 (Lower estimate for T)**.** *Let u be the solution of* (1.1.1)*. Then,* $T \geq T_* := \dfrac{1}{(p+1)\|f\|_\infty}$ *and, for any $\tau \in (0,1)$ we have $\|u(t_0)\|_\infty \leq 1-\tau$, where* $t_0 = t_0(\tau) = \dfrac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}$.

Proof. Let $y(t) \in C^1(0, T_*)$ be the solution of the problem

$$\begin{cases} y' = \dfrac{\|f\|_\infty}{(1-y)^p}, & \text{for } t > 0, \\ y(0) = 0. \end{cases}$$

We have

$$\int_0^t y'(1-y)^p dt = \|f\|_\infty t$$

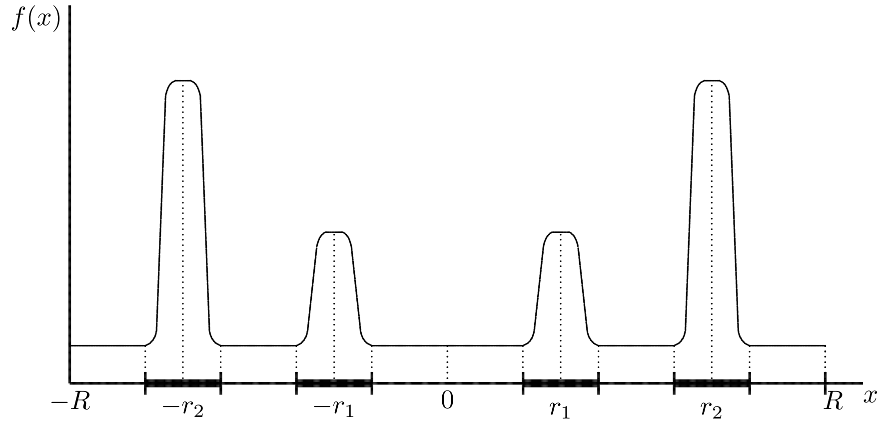

Figure 2.6 – Illustration of Theorem 1.6(ii) for $n = 1$.

that is,

$$\frac{1}{p+1} - \frac{(1-y(t))^{p+1}}{p+1} = \|f\|_\infty t, \qquad \text{for all } t \in [0, T_*).$$

Since, by the comparison principle, $T \geq T_*$ and $u(t,x) \leq y(t)$ for all $(t,x) \in [0,T) \times \Omega$, it follows that $\|u(t_0(\tau))\|_\infty \leq y(t_0(\tau)) = 1 - \tau$. □

Lemma 2.2 (Upper estimate for T). *Assume that $B(0,r) \subset \Omega$, $f \geq \mu\chi_{B(0,r)}$, with $\mu > \mu_0(p,n)r^{-2}$, where $\mu_0(p,n)$ is defined in* (1.3.2)*. Let u be the solution of problem* (1.1.1)*. Then $T < \infty$ and T satisfies the upper bound*

$$T \leq \frac{1}{(p+1)(\mu - \mu_0(p,n)r^{-2})}.$$

Proof. Let $\tilde{\varphi}$ denote the first eigenfunction of $-\Delta$ in $H_0^1(B(0,r))$, with $\|\tilde{\varphi}\|_{L^1} = 1$ and let λ_r be the corresponding eigenvalue, i.e $\lambda_r = \lambda_1 r^{-2}$. Set $y(t) = \int_{B(0,r)} u(t)\tilde{\varphi}$ and note that $y < 1$ on $[0,T)$. Multiplying (1.1.1) by $\tilde{\varphi}$, integrating by parts over $B(0,r)$ and using Jensen's inequality (in view of the convexity of the function $(1-s)^{-p}$), we obtain

$$y'(t) \geq \mu(1-y(t))^{-p} - \lambda_r y(t), \quad 0 < t < T.$$

An elementary computation shows that

$$\max_{0 \leq s < 1} s(1-s)^p = \max_{0 \leq X < 1} X^p - X^{p+1} = \frac{p^p}{(p+1)^{p+1}}.$$

It follows that

$$y'(t) \geq \Big(\mu - \frac{p^p}{(p+1)^{p+1}}\lambda_r\Big)(1-y(t))^{-p}, \quad 0 < t < T.$$

The conclusion follows by integration. □

3 Qualitative Type I estimate

Following the approach in [21], a key ingredient in the proofs of Theorems 1.1 and 1.2 is the following type I estimate for u away from the boundary.

Proposition 3.1 (type I estimate). *Under assumption* (1.2.1)*, the solution u of problem* (1.1.1) *satisfies*

$$u(t,x) \le 1 - \gamma\delta(x)(T-t)^{\frac{1}{p+1}}, \qquad \text{for all } t \in [0,T) \text{ and } x \in \Omega, \tag{3.0.1}$$

where γ depends only on p, Ω, M, r.

A similar estimate is given in [21, Theorem 1.2], except that the constant γ depends on u in an unspecified way. We stress that the precise dependence of γ is here a key feature, not only for the proofs of Theorems 1.1 and 1.2, but also in view of the stability results for the touchdown time and set in Section 5, which require uniform type I estimates with respect to the permittivity profile f. Proposition 3.1 will be proved by means of the maximum principle applied to an auxiliary function of the form

$$J(t,x) = u_t - \varepsilon a(x)h(u). \tag{3.0.2}$$

We here follow the approach of [21], which was a modification of a method from Friedman-McLeod ([11]; see also [15]). The main new ideas in [21] were to construct h as a suitable perturbation of the nonlinearity and $a(x)$ as an appropriate function vanishing on the boundary. In order to obtain the precise dependence of γ, special care is here necessary in the construction and in the estimates of the function a.

3.1 Basic computation for the function J.

The basic computation for the function J is contained in the following lemma. Although it is close to [21, Lemma 2.1], we give the proof for convenience and completeness.

Lemma 3.2. *Let $a \in C^2(\Omega)$ be a positive function. Let u be the solution of* (1.1.1) *and let J be given by* (3.0.2) *in $(0,T)\times\Omega$, where*

$$h(u) = (1-u)^{-p} + 1, \qquad 0 \le u < 1. \tag{3.1.1}$$

Then

$$J_t - \Delta J - pf(x)(1-u)^{-p-1}J = \varepsilon\Theta \qquad \text{in } (0,T)\times\Omega, \tag{3.1.2}$$

where

$$\Theta = pa(x)f(x)(1-u)^{-p-1} + ah''(u)|\nabla u|^2 + 2h'(u)\nabla a\cdot\nabla u + h(u)\Delta a. \tag{3.1.3}$$

Moreover, we have $h''(u) > 0$ for all $u \in [0,1)$ and

$$\Theta \ge \underbrace{pa(x)f(x)(1-u)^{-p-1}}_{\tau_1} + \underbrace{h(u)\Delta a(x)}_{\tau_2} - \underbrace{\frac{h'^2(u)|\nabla a(x)|^2}{a(x)h''(u)}}_{\tau_3}. \tag{3.1.4}$$

Proof. We compute

$$\begin{aligned}
J_t &= u_{tt} - \varepsilon a(x)h'(u)u_t,\\
\nabla J &= \nabla u_t - \varepsilon\big(a(x)h'(u)\nabla u + h(u)\nabla a(x)\big),\\
\Delta J &= \Delta u_t - \varepsilon\big(a(x)h'(u)\Delta u + a(x)h''(u)|\nabla u|^2\\
&\qquad + 2h'(u)\nabla a(x)\cdot\nabla u + h(u)\Delta a(x)\big).
\end{aligned}$$

Setting $g(u) = (1-u)^{-p}$ and omitting the variables x, u without risk of confusion, we get

$$\begin{aligned} J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon a h'(u_t - \Delta u) \\ &\quad + \varepsilon(a h'' |\nabla u|^2 + 2h' \nabla a \cdot \nabla u + h \Delta a) \\ &= f g' u_t - \varepsilon f a h' g + \varepsilon(a h'' |\nabla u|^2 + 2h' \nabla a \cdot \nabla u + h \Delta a). \end{aligned}$$

Using $u_t = J + \varepsilon a h$, we have

$$J_t - \Delta J - f g' J = \varepsilon \Theta,$$

where

$$\Theta = f a(g' h - h' g) + a h'' |\nabla u|^2 + 2h'(u) \nabla a \cdot \nabla u + h \Delta a.$$

On the other hand, we have

$$h'(u) = g'(u) = p(1-u)^{-p-1}, \tag{3.1.5}$$

hence

$$g' h - h' g = h'(h - g) = h' = p(1-u)^{-p-1},$$

which yields (3.1.3). Also, we have

$$h'' = p(p+1)(1-u)^{-p-2} > 0. \tag{3.1.6}$$

Finally, since $a > 0$, we may write

$$\Theta = p a(x) f(x)(1-u)^{-p-1} + h \Delta a + a h'' \left[|\nabla u|^2 + 2 \frac{h'(u) \nabla a \cdot \nabla u}{a h''} \right].$$

Since $|\nabla u|^2 + 2 \dfrac{h'(u) \nabla a \cdot \nabla u}{a h''} \geq -\dfrac{h'^2 |\nabla a|^2}{a^2 (h'')^2}$, inequality (3.1.4) follows. □

3.2 Construction of the function $a(x)$.

We shall apply Lemma 3.2. In order to guarantee $\Theta \geq 0$, the negative term τ_3 on the right-hand side of (3.1.4) must be absorbed by a positive contribution coming either from the term τ_1, provided $f(x) > 0$, or from the term τ_2, provided $\Delta a(x) > 0$. But $a(x)$ is positive and we require that it vanishes at the boundary, so we cannot have $\Delta a > 0$ everywhere. Therefore, we shall consider a function $a(x)$ which is positive in Ω and suitably convex everywhere, except in a ball B where f is bounded away from zero. A key point is here to obtain estimates of a in terms of the radius of B, but independent of its location.

The following lemma gives the construction of the appropriate function $a(x)$. In what follows we set

$$\Omega_r := \{x \in \Omega;\ \delta(x) > r\}, \qquad \omega_r := \{x \in \Omega;\ \delta(x) < r\}.$$

Lemma 3.3. *Let*

$$h(u) = (1-u)^{-p} + 1. \tag{3.2.1}$$

Let $r > 0$, $y \in \Omega_{2r}$ and set $B = B_r(y)$. Then there exists a function $a \in C^2(\overline{\Omega})$ with the following properties:

$$h h'' a \Delta a - h'^2 |\nabla a|^2 \geq 0, \qquad \text{for all } x \in \overline{\Omega \setminus B} \text{ and all } 0 \leq u < 1, \tag{3.2.2}$$

$$C_1 \delta^{p+1}(x) \leq a(x) \leq C_2 \delta^{p+1}(x), \qquad \text{for all } x \in \overline{\Omega}, \tag{3.2.3}$$

$$\|a\|_{C^2(\overline{\Omega})} \leq C_3, \tag{3.2.4}$$

for some constants $C_1, C_2, C_3 > 0$ depending only on p, Ω, r (and not on y).

Proof. **Step 1.** *Construction of $a(x)$ in $\Omega \setminus B$ and proof of* (3.2.2). We introduce a suitable harmonic function $\phi = \phi_y$, the unique smooth solution of the problem

$$\left.\begin{aligned} &\Delta\phi = 0, && x \in \Omega \setminus B,\\ &\phi = 0, && x \in \partial\Omega,\\ &\phi = 1, && x \in \partial B. \end{aligned}\right\} \tag{3.2.5}$$

By the strong maximum principle, we have $0 < \phi < 1$ in $\Omega \setminus B$. Now, we set

$$a(x) = \phi^{p+1}(x), \qquad x \in \overline{\Omega \setminus B} \tag{3.2.6}$$

and we compute

$$\nabla a = (p+1)\phi^p \nabla\phi, \qquad \Delta a = (p+1)p\phi^{p-1}|\nabla\phi|^2 + \underbrace{(p+1)\phi^p\Delta\phi}_{=0}$$

in $\Omega \setminus B$. It follows that

$$a\Delta a = (p+1)p\phi^{p+1}\phi^{p-1}|\nabla\phi|^2 = \frac{p}{p+1}|\nabla a|^2, \qquad \text{in } \Omega \setminus B.$$

Since, on the other hand, we have

$$hh'' = p(p+1)(1-u)^{-2p-2} + p(p+1)(1-u)^{-p-2} \geq \frac{p+1}{p}(h')^2, \qquad 0 < u < 1,$$

due to (3.2.1), property (3.2.2) follows.

Step 2. *Uniform estimates in $\Omega \setminus B$.* We shall prove that

$$a(x) \geq C_1\delta^{p+1}(x) \quad \text{for all } x \in \overline{\Omega} \setminus B, \tag{3.2.7}$$

and

$$\|a\|_{C^2(\overline{\Omega}\setminus B)} \leq C_3, \tag{3.2.8}$$

for some constants $C_1, C_3 > 0$ depending only on p, Ω, r.

For each $y \in \overline{\Omega}_{2r}$, the function $\phi_y(y+\cdot)$ is harmonic in $\{r < |z| < 2r\}$ with $\phi_y(y+\cdot) = 1$ on $\{|z| = r\}$. Consequently, by elliptic regularity, there exists a constant $C = C(n,r) > 0$ such that

$$\|\phi_y\|_{C^2(\{r \leq |x-y| \leq 3r/2\})} \leq C, \qquad \text{for all } y \in \overline{\Omega}_{2r}. \tag{3.2.9}$$

Since $\phi_y = 1$ on $\partial B_r(y)$, we deduce that there exists $\sigma = \sigma(n,r) \in (0, r/6)$ such that

$$\phi_y \geq 1/2 \quad \text{in } \{r \leq |x-y| \leq r+3\sigma\}, \quad \text{for all } y \in \overline{\Omega}_{2r}. \tag{3.2.10}$$

Next we claim that there exists $c > 0$ such that

$$-\frac{\partial\phi_y}{\partial\nu} \geq c \quad \text{on } \partial\Omega, \quad \text{for all } y \in \overline{\Omega}_{2r}. \tag{3.2.11}$$

Assume for contradiction that there exist sequences $y_i \in \overline{\Omega}_{2r}$ and $x_i \in \partial\Omega$ such that

$$\frac{\partial\phi_{y_i}}{\partial\nu}(x_i) \to 0. \tag{3.2.12}$$

We may assume $y_i \to y_0 \in \overline{\Omega}_{2r}$ and $x_i \to x_0 \in \partial\Omega$. Set

$$d = \delta(y_0) - r - 2\sigma \geq r - 2\sigma > 0.$$

For all large i, we have $\delta(y_i) > \delta(y_0) - \sigma$, hence $\delta(y_i) - r > d + \sigma$, so that ϕ_{y_i} is harmonic in $\omega_{d+\sigma} \subset \Omega \setminus B_r(y_i)$ with $\phi_{y_i} = 0$ on $\partial\Omega$. Applying elliptic regularity again, it follows that there exist $\alpha \in (0,1)$ and $C > 0$ such that

$$\|\phi_{y_i}\|_{C^{2+\alpha}(\overline{\omega}_d)} \le C, \quad \text{for all large } i. \tag{3.2.13}$$

Up to extracting a subsequence, it follows that

$$\phi_{y_i} \to \phi \quad \text{in } C^2(\overline{\omega}_d), \tag{3.2.14}$$

where $\phi \ge 0$ is harmonic in ω_d and satisfies $\phi = 0$ on $\partial\Omega$. Moreover, by (3.2.12) we have $\frac{\partial \phi}{\partial \nu}(x_0) = 0$. By Hopf's Lemma, we deduce that

$$\phi \equiv 0 \quad \text{in } \omega_d. \tag{3.2.15}$$

Now, for large i, we have $\delta(y_i) < \delta(y_0) + \sigma$, hence $\delta(y_i) - r - 3\sigma < d$, so that

$$\{r \le |x - y_i| \le r + 3\sigma\} \cap \overline{\omega}_d \ne \emptyset.$$

But (3.2.14) and (3.2.15) then yield a contradiction with (3.2.10). The claim (3.2.11) follows.

Now, arguing as for (3.2.13), we have

$$\|\phi_y\|_{C^2(\overline{\omega}_{r/2})} \le C(\Omega, r), \quad \text{for all } y \in \overline{\Omega}_{2r}. \tag{3.2.16}$$

Combining this with (3.2.11), we deduce that there exists $\eta \in (0, r/2)$ such that

$$\phi_y(x) \ge \frac{c}{2}\delta(x), \quad \text{for all } x \in \omega_\eta \text{ and all } y \in \overline{\Omega}_{2r}. \tag{3.2.17}$$

Since ϕ_y now satisfies

$$\left.\begin{array}{ll} \Delta\phi_y = 0, & x \in \Omega_\eta \setminus B_r(y) \\ \phi_y \ge \frac{c\eta}{2}, & x \in \partial\Omega_\eta, \\ \phi_y = 1, & x \in \partial B_r(y), \end{array}\right\}$$

we deduce from the maximum principle that $\phi \ge \frac{c\eta}{2}$ in $\Omega_\eta \setminus B_r(y)$. This along with (3.2.17) guarantees (3.2.7).

Finally, for $x \in \Omega_\eta \setminus B_{3r/2}(y)$, we observe that ϕ_y is harmonic in $B_\varepsilon(x)$ with $\varepsilon = \min(\eta, r/2)$ and $0 \le \phi_y \le 1$. It follows from elliptic regularity that that there exists a constant $C > 0$ such that

$$\|\phi_y\|_{C^2(B_{\varepsilon/2}(x))} \le C, \quad \text{for all } x \in \Omega_\eta \setminus B_{3r/2}(y) \text{ and all } y \in \overline{\Omega}_{2r}. \tag{3.2.18}$$

Property (3.2.8) is then a consequence of (3.2.9), (3.2.16) and (3.2.18).

Step 3. *Extension to B.* Since $a \in C^2(\overline{B}_{2r}(y) \setminus B_r(y))$ and a satisfies (3.2.8), by standard properties of extension operators, the function a can be extended in $\overline{B}_r(y)$ to a function $\tilde{a}$ such that

$$\|\tilde{a}\|_{C^2(\overline{B}_{2r}(y))} \le C_3. \tag{3.2.19}$$

On the other hand, since $a = 1$ on $\partial B_r(y)$, there exists $r_1 \in (0, r)$ depending only on C_3 such that

$$\tilde{a}(x) \ge 1/2 \quad \text{for } r_1 \le |x - y| \le r. \tag{3.2.20}$$

Fix a cutoff function $\psi \in C^2([0,\infty))$ such that $0 \le \psi \le 1$, $\psi(s) = 0$ for $s \in [0, r_1]$ and $\psi(s) = 1$ for $s \in [(r + r_1)/2, r]$, and define a in $B_r(y)$ by

$$a(x) := 1 + (\tilde{a}(x) - 1)\psi(|x - y|).$$

We thus obtain a function which satisfies $\|a\|_{C^2(\overline{B}_r(y))} \le C_4(p, \Omega, r)$ and $a(x) \ge 1/2$ in $\overline{B}_r(y)$, owing to (3.2.19) and (3.2.20). This, along with (3.2.7) and (3.2.8), guarantees (3.2.4) and the lower estimate in (3.2.3). Finally, the upper estimate in (3.2.3) follows from (3.2.4), (3.2.6), (3.2.16) and $\phi = 0$ on $\partial\Omega$. □

3.3 Proof of Proposition 3.1

We shall also use the following lower bound for u_t.

Lemma 3.4. *Under assumption* (1.2.1)*, for a given* $t_0 \in (0,T)$*, the solution* u *of problem* (1.1.1) *satisfies*

$$u_t(t,x) \geq c_0 e^{-c_1 t}\delta(x), \qquad \text{for all } t \in [t_0,T) \text{ and } x \in \Omega,$$

with $c_0 = c_0(\Omega, r, t_0) > 0$ *and* $c_1 = c_1(\Omega) > 0$.

Proof. Let $x_0 \in \Omega$ be such that $B = B(x_0, r)$. First, we observe that the function $v = u_t$ is a (classical) solution of the problem:

$$\begin{cases} v_t - \Delta v = pf(x)(1-u)^{-p-1}v, & \text{in } (0,T)\times\Omega, \\ v = 0, & \text{in } [0,T)\times\partial\Omega, \\ v(0,x) = f(x), & \text{in } \Omega. \end{cases} \tag{3.3.1}$$

By the maximum principle, we thus have

$$u_t \geq e^{t\Delta_\Omega} f \quad \text{in } [0,T)\times\Omega. \tag{3.3.2}$$

By (1.2.1), we deduce that

$$u_t \geq r\int_\Omega G_\Omega(t,x,y)\chi_B(y)\,dy.$$

Here, $e^{t\Delta_\Omega}$ and G_Ω are respectively the Dirichlet heat semigroup and heat kernel of Ω. It is known (see [5] and also [34]) that

$$G_\Omega(t,x,y) \geq ce^{-c_1 t}\delta(x)\delta(y), \quad t \geq t_0$$

with $c = c(t_0, \Omega) > 0$ and $c_1 = c_1(\Omega) > 0$. Consequently, since $\delta(x_0) \geq r$, we have

$$u_t \geq cre^{-c_1 t}\delta(x)\int_{B(x_0,r/2)}\delta(y)\,dy \geq \frac{cr^2}{2}|B(0,r/2)|e^{-c_1 t}\delta(x)$$

and the lemma follows. □

Proof of Proposition 3.1. It is done in three steps.

Step 1: Preparations. Let J and h be given by (3.0.2) and (3.2.1). Owing to assumption (1.2.1), upon replacing r by $r/2$, we may assume that there exists $y \in \Omega$ such that $\delta(y) \geq 2r$ and

$$f \geq r \quad \text{on } B := B_r(y). \tag{3.3.3}$$

Consider the function $a \in C^2(\overline{\Omega})$ given by Lemma 3.3. By (3.2.3), we have

$$\inf_{x\in B} a(x) \geq \sigma = \sigma(\Omega, p, r) := C_1 r^{p+1}. \tag{3.3.4}$$

Next, let $t_0 = \dfrac{1}{2(p+1)M}$, where M is given by (1.2.1). By Lemma 2.1 we have $0 < t_0 < T$ and

$$\|u(t,\cdot)\|_\infty \leq 1 - 2^{-1/(p+1)}, \quad 0 \leq t \leq t_0. \tag{3.3.5}$$

We split the cylinder $\Sigma := (t_0, T) \times \Omega$ into three subregions as follows:

$$\begin{aligned} \Sigma_1 &= (t_0,T) \times [\Omega \setminus B], \\ \Sigma_2^\eta &= \{(t,x) \in (t_0,T) \times B;\ u(t,x) \geq 1-\eta\}, \\ \Sigma_3^\eta &= \{(t,x) \in (t_0,T) \times B;\ u(t,x) < 1-\eta\}, \end{aligned} \tag{3.3.6}$$

where $\eta \in (0,1)$ will be specified later.

Step 2: Parabolic inequality for J in the regions Σ_1 and Σ_2^η. It follows from properties (3.1.4) in Lemma 3.2 and (3.2.2) in Lemma 3.3, along with $a > 0$, $f \geq 0$ in Ω, and $h'' > 0$, that

$$J_t - \Delta J - pf(x)(1-u)^{-p-1} J \geq 0 \quad \text{in } \Sigma_1. \tag{3.3.7}$$

Next, in view of (3.2.1) and property (3.2.4) in Lemma 3.3, we have

$$|h\Delta a| \leq C_4 (1-u)^{-p}, \quad |h' \nabla a| \leq C_4(1-u)^{-p-1} \quad \text{in } \Sigma,$$

for some $C_4 = C_4(\Omega, p, r) > 0$. Also, from (3.2.1) and (3.3.4) we get

$$ah'' \geq \sigma p(p+1)(1-u)^{-p-2} \quad \text{in } (0,T) \times B.$$

Consequently, recalling the definition (3.1.3) of Θ, it follows from (3.1.4), (3.3.3), (3.3.4) that

$$\begin{aligned} (1-u)^{p+1}\Theta &\geq pf(x)a(x) + h\Delta a (1-u)^{p+1} - \frac{(h'|\nabla a|)^2}{ah''}(1-u)^{p+1} \\ &\geq pr\sigma - C_5(1-u) \geq pr\sigma - C_5\eta \quad \text{in } \Sigma_2^\eta, \end{aligned}$$

for some $C_5 = C_5(\Omega, p, r) > 0$. Choosing $\eta = \eta(\Omega, p, r) \in (0,1)$ small enough, we then deduce from (3.1.2) that

$$J_t - \Delta J - pf(x)(1-u)^{-p-1} J \geq 0 \quad \text{in } \Sigma_2^\eta. \tag{3.3.8}$$

Step 3: Control of J on Σ_3^η and conclusion. Now that η has been fixed, using Lemma 3.4 and (1.2.1), (3.2.1), (3.2.3), (3.3.5), we may choose $\varepsilon = \varepsilon(\Omega,\ p, r, M) > 0$ small enough, such that

$$\begin{aligned} J &\geq \delta(x)\left[c_0 e^{-c_1 M} - 2C_2 \varepsilon \delta^p(x)(1-u)^{-p}\right] \\ &\geq \delta(x)\left[c_0 e^{-c_1 M} - 2C_2 \varepsilon \delta^p(x)\eta^{-p}\right] \geq 0 \quad \text{in } \Sigma_3^\eta \end{aligned} \tag{3.3.9}$$

and

$$\begin{aligned} J(t_0,x) &\geq \delta(x)\left[c_0 e^{-c_1 M} - 2C_2 \varepsilon \delta^p(x)(1-\|u(t_0,\cdot)\|_\infty)^{-p}\right] \\ &\geq \delta(x)\left[c_0 e^{-c_1 M} - 2^{1+\frac{p}{p+1}} C_2 \varepsilon \delta^p(x)\right] \geq 0 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{3.3.10}$$

where c_0, c_1 are the constants in Lemma 3.4 and C_2 is the constant in (3.2.3). Observe now that, as a consequence of (3.3.9) and $\Sigma = \Sigma_1 \cup \Sigma_2^\eta \cup \Sigma_3^\eta$, we have

$$\{(t,x) \in \Sigma;\ J(t,x) < 0\} \subset \Sigma_1 \cup \Sigma_2^\eta. \tag{3.3.11}$$

Also, since $a = 0$ on $\partial\Omega$, we have

$$J = 0 \quad \text{on } (t_0,T) \times \partial\Omega. \tag{3.3.12}$$

On the other hand, by standard parabolic regularity, we have

$$J \in C^{1,2}(\Sigma) \cap C([t_0,T) \times \overline{\Omega}).$$

It follows from (3.3.7), (3.3.8), (3.3.10)-(3.3.12), and the maximum principle (see, e.g., [32], Proposition 52.4 and Remark 52.11(a)) that

$$J \geq 0 \quad \text{in } \Sigma.$$

Then, for $t_0 < t < s < T$ and $x \in \Omega$, we have

$$u_t \geq \varepsilon a(x)h(u) \geq \varepsilon a(x)(1-u)^{-p}$$

and an integration in time gives

$$(1-u(t,x))^{p+1} \geq (p+1)\int_t^s u_t(1-u)^p \geq \varepsilon a(x)(s-t).$$

Letting $s \to T$, we get

$$(1-u(t,x))^{p+1} \geq (p+1)\varepsilon a(x)(T-t) \quad \text{in } \Sigma. \tag{3.3.13}$$

In view of (3.2.3), this implies (3.0.1) in $[t_0,T)\times\Omega$ with $\gamma = \gamma(\Omega,p,r,M) > 0$. Due to (3.3.5), (1.2.1), the estimate (3.0.1) is true in $[0,t_0)\times\Omega$ as well, for a possibly smaller constant $\gamma = \gamma(\Omega,p,r,M) > 0$. □

4 Proof of Theorems 1.1 and 1.2

4.1 No touchdown criterion under a local type I estimate.

The following lemma enables one to exclude touchdown at a given interior point or on a neighborhood of $\partial\Omega$, under a suitable type I estimate and a smallness assumption on f.

Lemma 4.1. *Let u be the solution of problem* (1.1.1). *Let either*

(i) $$D = B(x_0,b) \subset\subset \Omega \quad \textit{and} \quad \Gamma = \partial D,$$

or

(ii) $$D = \Omega\setminus\overline{\omega} \ \ \textit{for some smooth open subset } \omega \subset\subset \Omega \quad \textit{and} \quad \Gamma = \partial\omega,$$

or

(iii) $$\Omega = (-R,R), \quad D = (a,R) \ \ \textit{for some } a \in (-R,R) \quad \textit{and} \quad \Gamma = \{a\}.$$

Assume

$$u \leq 1 - k(T-t)^{\frac{1}{p+1}} \quad \textit{on } [0,T)\times\Gamma \tag{4.1.1}$$

for some $k > 0$. If

$$\|f\|_{L^\infty(D)} < \frac{k^{p+1}}{p+1}, \tag{4.1.2}$$

then $\mathcal{T}\cap D = \emptyset$. In addition, in case (ii) we have $\mathcal{T}\cap\partial\Omega = \emptyset$, and in case (iii), $R \notin \mathcal{T}$.

Remark 4.2. *Condition* (4.1.2) *is essentially optimal. Indeed, considering* (1.1.1) *with $f(x) \equiv 1$ and leaving the boundary conditions apart, we see that the ODE solution $y(t) = 1-[1-(p+1)t]^{1/(p+1)}$ satisfies* (4.1.1) *with $k = (p+1)^{\frac{1}{p+1}}$ and $T = 1/(p+1)$, so that one could not take a larger value of the constant in the RHS of* (4.1.2).

Proof. We use a simplification of a comparison argument from [21] (where the comparison was done with a selfsimilar supersolution, instead of a separated variable supersolution). We define the comparison function

$$w(t,x) := y(t)\psi(x) \quad \text{for } (t,x) \in [0,T) \times \overline{D},$$

where $y(t)$ is defined by

$$y(t) = 1 - k(T-t)^{\frac{1}{p+1}}.$$

Here, in case (i), ψ is given by

$$\psi(x) := 1 - \sigma\left(1 - \frac{|x-x_0|^2}{b^2}\right)$$

for $\sigma \in (0,1)$ to be chosen below and, in case (ii), $\psi \in C^2(D) \cap C(\overline{D})$ is the solution of the problem

$$\begin{cases} \Delta\psi = 0, & x \in D, \\ \psi = 1, & x \in \partial\omega, \\ \psi = 1-\sigma, & x \in \partial\Omega. \end{cases}$$

Observe that

$$1 - \sigma < \psi(x) < 1, \quad x \in D,$$

by the strong maximum principle. In case (iii), similarly to (ii), we set

$$\psi(x) = 1 - \sigma(x-a)/(R-a) \qquad \text{for } x \in [a,R].$$

In particular, owing to (4.1.1), we note that in all cases,

$$w \geq 0 \quad \text{in } [0,T) \times \overline{D}. \tag{4.1.3}$$

We compute, in $(0,T) \times D$:

$$\begin{aligned} & w_t - \Delta w - f(x)(1-u)^{-p} \\ &= y'(t)\psi(x) - y(t)\Delta\psi(x) - f(x)(1 - y(t)\psi(x))^{-p} \\ &\geq \tfrac{k}{p+1}(T-t)^{-\frac{p}{p+1}}\psi(x) - y(t)\Delta\psi(x) - f(x)(1-y(t))^{-p} \\ &= \big(\tfrac{k}{p+1}\psi(x) - f(x)k^{-p}\big)(T-t)^{-\frac{p}{p+1}} - y(t)\Delta\psi(x). \end{aligned}$$

Moreover, we have $\Delta\psi = \frac{2\sigma}{b^2}$ in $B(x_0,b)$ in case (i), and $\Delta\psi = 0$ in D in cases (ii) and (iii). In all cases, using assumption (4.1.2) and taking $\sigma > 0$ small enough, it follows that

$$w_t - \Delta w - f(x)(1-w)^{-p} \geq \big(\tfrac{k}{p+1}(1-\sigma) - f(x)k^{-p}\big)T^{-\frac{p}{p+1}} - \tfrac{2\sigma}{b^2} \geq 0 \tag{4.1.4}$$

in $[0,T) \times D$.

We next look at the comparison on the parabolic boundary of $[0,T) \times D$. On the one hand, by (4.1.3), we have

$$w(0,x) \geq 0 = u(0,x) \quad \text{in } \overline{D}. \tag{4.1.5}$$

On the other hand, using $\psi = 1$ on Γ and (4.1.1), we have

$$w(t,x) = 1 - k(T-t)^{\frac{1}{p+1}} \geq u(t,x) \quad \text{in } [0,T) \times \Gamma. \tag{4.1.6}$$

Moreover, in case (ii) (resp., (iii)), we have, by (4.1.3),

$$w(t,x) \geq 0 = u(t,x) \quad \text{in } [0,T) \times \partial\Omega \text{ (resp., } [0,T) \times \{-R\}). \tag{4.1.7}$$

By (4.1.4)–(4.1.6) and (4.1.7) (in cases (ii) and (iii)), along with the comparison principle and $y(t) \leq 1$ for all $t \in [0,T)$, we conclude that

$$u(t,x) \leq w(t,x) \leq \psi(x) \quad \text{ in } (0,T) \times D. \tag{4.1.8}$$

In all cases, since ψ is uniformly smaller than 1 in compact subsets of D, it follows from (4.1.8) that $\mathcal{T} \cap D = \emptyset$. We also see that in case (ii), ψ is uniformly smaller than 1 in a neighborhood of $\partial\Omega$, so we can rule out quenching at the boundary. For the case (iii), the conclusion follows similarly. □

4.2 Proof of Theorem 1.1.

We shall apply case (i) of Lemma 4.1. Let γ be given by estimate (3.0.1), and assume

$$f(x_0) < \frac{(\gamma\delta(x_0))^{p+1}}{p+1}.$$

Pick $k \in (0, \gamma\delta(x_0))$ such that

$$f(x_0) < \frac{k^{p+1}}{p+1} < \frac{(\gamma\delta(x_0))^{p+1}}{p+1}.$$

By estimate (3.0.1), together with the continuity of f, conditions (4.1.1) and (4.1.2) are satisfied in $D = B(x_0, b)$ for $b > 0$ sufficiently small. We can then conclude from Lemma 4.1 that $x_0 \notin \mathcal{T}$, which proves Theorem 1.1 with $\gamma_0 = \frac{\gamma^{p+1}}{p+1}$. □

4.3 Proof of Theorem 1.2.

Let γ be given by estimate (3.0.1), and assume (1.2.3) with $\gamma_0 := \frac{\gamma^{p+1}}{p+1}$. Applying case (ii) of Lemma 4.1 with $k = \gamma \operatorname{dist}(\omega, \partial\Omega)$, it follows that $\mathcal{T} \subset \overline{\omega}$. Finally, we note that for any $x \in \partial\omega$, our assumption implies $f(x) < \gamma_0 \delta^{p+1}(x)$, so that $x \notin \mathcal{T}$ by Theorem 1.1. Therefore $\mathcal{T} \subset \omega$ and the theorem is proved. □

4.4 Proof of Corollary 1.3.

Assertion (i) follows from Theorem 1.1. Assertion (ii) follows by applying Lemma 2.2, and then Theorems 1.1 and 1.2. □

5 Stability results for the touchdown time and touchdown set

One of the main ingredients in the proofs of Theorems 1.4–1.6 is the stability of the touchdown time and touchdown set under small perturbations of the potential f.

Recalling the definition in (1.1.2), we denote by $U : E \ni f \mapsto U_f$ the semiflow generated by problem (1.1.1). Namely, $u = U_f(t,\cdot)$ is the maximal classical solution of (1.1.1). We recall that its existence time and touchdown set are respectively denoted by $T_f \in (0,\infty]$ and $\mathcal{T}_f \subset \overline{\Omega}$. We start with a more or less standard continuous dependence property of the solution itself with respect to f.

Proposition 5.1 (Continuity of U from L^q to L^∞). *Let $1 \le q \le \infty$ with $q > \frac{n}{2}$. Let $f \in E$ and let $0 < t_0 < T_f$. For all $\sigma > 0$, there exists $\varepsilon > 0$ such that*

$$\text{if } g \in E \text{ and } \|g - f\|_q \le \varepsilon, \text{ then } T_g > t_0 \text{ and } \sup_{t \in [0,t_0]} \|U_g - U_f\|_\infty \le \sigma.$$

For the stability of the touchdown time and set, the local type I estimate (3.0.1) in Proposition 3.1 plays a crucial role. A uniform version is actually needed. To this end, for given $\gamma > 0$, we set

$$E_\gamma = \Big\{ g \in E;\ T_g < \infty \text{ and } U_g(t,x) \le 1 - \gamma\delta(x)(T_g - t)^{\frac{1}{p+1}} \\ \text{for all } (t,x) \in [0,T_g) \times \Omega \Big\}.$$

Proposition 5.2 (Continuity of the touchdown time). *Let $1 \le q \le \infty$ with $q > \frac{n}{2}$, $\gamma > 0$ and let $f \in E$ be such that $T_f < \infty$ and $\mathcal{T}_f \cap \Omega \ne \emptyset$. For all $\sigma > 0$, there exists $\varepsilon > 0$ such that*

$$\text{if } g \in E_\gamma \text{ and } \|g - f\|_q \le \varepsilon, \text{ then } |T_g - T_f| < \sigma.$$

Theorem 5.3 (Upper semi-continuity of the touchdown set). *Let $1 \le q \le \infty$ with $q > \frac{n}{2}$, $\gamma, M > 0$ and let $f \in E$ be such that $T_f < \infty$ and $\mathcal{T}_f \subset\subset \Omega$. For all $\sigma > 0$, there exist $\varepsilon, \kappa > 0$ such that, if*

$$g \in E_\gamma,\ \|g\|_\infty \le M \text{ and } \|g - f\|_q \le \varepsilon,$$

then

$$U_g(t,x) \le 1 - \kappa \qquad \text{in } [0,T_g) \times \big(\overline{\Omega} \setminus (\mathcal{T}_f + B(0,\sigma))\big),$$

hence in particular

$$\mathcal{T}_g \subset \mathcal{T}_f + B(0,\sigma).$$

Remark 5.4. *The assumption $g \in E_\gamma$, i.e. estimate* (3.0.1) *with a uniform constant, can be guaranteed by assuming $\mu\chi_B \le g \le M$, where $M, r > 0$, $B \subset \Omega$ is a ball of radius r and $\mu > \mu_0(p,n)r^{-2}$ (cf. Lemma 2.2). This is a consequence of Proposition 3.1.*

Remark 5.5. *(i) Theorem 5.3 in particular proves that $\mathcal{T}_g$ is also a compact subset of Ω, provided g is close enough to f in L^q norm. It is unknown whether the compactness assumption on $\mathcal{T}_f$ can be removed. This would be true if we knew the analogue of estimate* (3.0.1) *without the factor distance to the boundary.*

(ii) To ensure that the touchdown set $\mathcal{T}_f$ is compact, we can consider f small enough near the boundary (apply Theorem 1.2), or Ω convex and f non-increasing near the boundary in the outer direction (this is proved in [22] by a moving planes argument). Also, if we consider $0 < p < 1$, then the touchdown set is compact for any f (see [21]).

We note that Theorem 1.8 is a direct consequence of Proposition 5.2 and Theorem 5.3, together with Proposition 3.1.

The semi-continuity property of $\mathcal{T}_f$ in Theorem 1.8 can be expressed as

$$d\big(\mathcal{T}_g, \mathcal{T}_f\big) \to 0, \qquad \text{as } g \to f \text{ in } L^q,\ g \in \tilde{E},$$

where $\tilde{E}$ is defined in (1.3.6) and

$$d(A,B) = \sup_{x\in A} d(x,B) \tag{5.0.1}$$

denotes the usual Hausdorff semi-distance. Our next result shows that the *continuity* of the touchdown set with respect to f fails in general.

Proposition 5.6 (Non continuity of the touchdown set)**.** *Let $p > 0$ and $\Omega = B_R \subset \mathbb{R}^n$. Let $1 \le q < \infty$ with $q > \frac{n}{2}$. One can find $B \subset \Omega$ a ball of radius $r > 0$, $M \ge \mu > \mu_0(p,n)r^{-2}$, a function $f \in \tilde{E}$ with $\mathcal{T}_f \subset\subset \Omega$ and a sequence $g_i \in \tilde{E}$, such that*

$$g_i \to f \text{ in } L^q \quad \text{and} \quad \liminf_{i\to\infty} d\big(\mathcal{T}_f, \mathcal{T}_{g_i}\big) > 0.$$

Proposition 5.6 will be proved in the next section, along with Theorem 1.6.

Proof of Proposition 5.1. Set $h(z) = (1-z)^{-p}$. Note that, for any $0 < M < 1$, we have $0 < h'(z) < L(M) := p(1-M)^{-p-1}$ for $0 < z < M$. Now, fix

$$M := \max_{0\le t\le t_0} \|U_f(t)\|_\infty < 1$$

and define

$$\tau_g := \sup\left\{t \in [0,T_g);\ \|U_g(s)\|_\infty \le \frac{1+M}{2} \quad \text{for all } s \in [0,t]\right\}.$$

In this proof, $C_1, C_2, \ldots$ will denote positive constants independent of g.

By the variation-of-constants formula and the L^p-L^q-estimates for the linear heat semigroup, we get

$$\begin{aligned}
\|(U_f - U_g)(t)\|_\infty &\le \int_0^t \|e^{(t-s)\Delta}(f(x)h(U_f) - g(x)h(U_g))\|_\infty\, ds \\
&\le \int_0^t (4\pi(t-s))^{-\frac{n}{2q}} \|f(x)h(U_f) - g(x)h(U_g)\|_q\, ds.
\end{aligned}$$

Now, for all $0 < t \le \min\{t_0, \tau_g\}$, we have

$$\begin{aligned}
&\|f(\cdot)h(U_f(t,\cdot)) - g(\cdot)h(U_g(t,\cdot))\|_q \\
&\quad= \|f(\cdot)h(U_f(t,\cdot)) - f(\cdot)h(U_g(t,\cdot)) + f(\cdot)h(U_g(t,\cdot)) - g(\cdot)h(U_g(t,\cdot))\|_q \\
&\quad\le \|f\|_q \|h(U_f(t)) - h(U_g(t))\|_\infty + \|h(U_g(t))\|_\infty \|f-g\|_q \\
&\quad\le \|f\|_q L\left(\tfrac{1+M}{2}\right) \|(U_f - U_g)(t)\|_\infty + \left(\tfrac{1-M}{2}\right)^{-p} \|f-g\|_q.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(U_f - U_g)(t)\|_\infty \\
&\quad\le (4\pi)^{-\frac{n}{2q}} \|f\|_q L\left(\tfrac{1+M}{2}\right) \int_0^t (t-s)^{-\frac{n}{2q}} \|(U_f - U_g)(s)\|_\infty\, ds \\
&\qquad + (4\pi)^{-\frac{n}{2q}} \|f-g\|_q \left(\tfrac{1-M}{2}\right)^{-p} \int_0^t (t-s)^{-\frac{n}{2q}}\, ds \\
&\quad\le C_1 \int_0^t (t-s)^{-\frac{n}{2q}} \|(U_f - U_g)(s)\|_\infty\, ds + C_2 \|f-g\|_q,
\end{aligned}$$

for all $0 < t \le \min\{t_0, \tau_g\}$, Now, applying Gronwall's Lemma, we obtain

$$\|(U_f - U_g)(t)\|_\infty \le C_3 \|f-g\|_q, \qquad 0 < t \le \min\{t_0, \tau_g\}, \tag{5.0.2}$$

If we consider $\varepsilon > 0$ small enough, then, for all g such that $\|f-g\|_q < \varepsilon$, we have

$$\|(U_f - U_g)(t)\|_\infty \le \frac{1-M}{4}, \qquad \text{for } 0 < t \le t_1 := \min\{t_0, \tau_g\}.$$

Therefore, $\|U_g(t_1)\|_\infty < \dfrac{1+M}{2}$, hence $\tau_g > t_1$, i.e. $\tau_g > t_0$. We deduce that $T_g > t_0$ and the result then follows from (5.0.2). □

Proof of Proposition 5.2. The lower semicontinuity of the touchdown time is a consequence of Proposition 5.1. Therefore we may always assume $T_g > T_f$. By assumption, there exist sequences $x_i \in \Omega$ and $t_i < T_f$ such that $x_i \to x_0 \in \Omega$, $t_i \to T_f$ and $U_f(t_i, x_i) \to 1$. We may assume $\delta(x_i) \ge c > 0$.

Now fix $0 < \lambda < 1$, pick i such that $U_f(t_i, x_i) \ge \lambda$ and next take $0 < \alpha < 1$. As a consequence of Proposition 5.1, there exists $\varepsilon > 0$ such that if $\|g-f\|_q < \varepsilon$, then $U_g(t_i, x_i) \ge \alpha\lambda$. Since $g \in E_\gamma$, we have

$$1 - \alpha\lambda \ge 1 - U_g(t_i, x_i) \ge c\gamma (T_g - t_i)^{\frac{1}{p+1}} \ge c\gamma (T_g - T_f)^{\frac{1}{p+1}}.$$

Therefore,

$$\limsup_{\|g-f\|_q \to 0} T_g \le T_f + \left(\frac{1-\alpha\lambda}{c\gamma}\right)^{p+1}.$$

The result follows by letting $\alpha \to 1$ and then $\lambda \to 1$. □

Proof of Theorem 5.3. The proof is more delicate. It is based on parabolic regularity, comparison arguments and uniform Hölder estimates in time for u up to the touchdown time. The latter follow from a key integrability property in time for the RHS of the PDE in (1.1.1) (see (5.0.4)), which is a consequence of the type I estimate.

Step 1. *Uniform Hölder estimates in time.* For each $\eta > 0$, we recall the notation

$$\Omega_\eta := \{x \in \Omega; \delta(x) > \eta\}.$$

We claim that for all $\eta > 0$ and $\beta \in (0, 1/(p+1))$, there exists $C(\eta, \beta) > 0$ such that for all $g \in E_\gamma$ with $\|g\|_\infty \le M$, we have

$$U_g(T_g, x) \le U_g(t, x) + C(\eta, \beta)(T_g - t)^\beta, \quad 0 < t < T_g, \ x \in \Omega_\eta. \tag{5.0.3}$$

Set $a = \dfrac{p}{p+1} < 1$ and take $\eta > 0$. Let $g \in E_\gamma$ with $\|g\|_\infty \le M$. We have

$$\begin{aligned} |\partial_t U_g - \Delta U_g| &= g(x)(1 - U_g(t,x))^{-p} \\ &\le C(\eta)(T_g - t)^{-a} \qquad \text{in } (0, T_g) \times \Omega_{\eta/2}, \end{aligned} \tag{5.0.4}$$

with $C(\eta) > 0$ independent of g. Moreover, by Lemma 2.1, we have $T_g \ge \tau_0$ with $\tau_0 \in (0,1)$ depending on M but not on g. In addition, there exists $\overline{T}_\gamma > 0$ such that $T_g \le \overline{T}_\gamma$ for all $g \in E_\gamma$.

Now, for $\tau \in (0, \tau_0/2)$, we define $w(t,x) = \tau^a U_g(t,x)$, which satisfies

$$|w_t - \Delta w| \le C(\eta) \quad \text{and} \quad 0 \le w \le 1 \quad \text{in } (0, T_g - \tau) \times \Omega_{\eta/2},$$

along with $w(0, \cdot) \equiv 0$. By interior parabolic regularity (see e.g. [32, Theorem 48.1]), for all $r \in (1, \infty)$, noting that $\tau_0/2 \le T_g - \tau \le \overline{T}_\gamma$, we deduce

$$\|w_t\|_{L^r(\Sigma_{g,\tau})} + \|\nabla w\|_{L^r(\Sigma_{g,\tau})} \le C(\eta, r), \qquad \text{where } \Sigma_{g,\tau} := (0, T_g - \tau) \times \Omega_\eta,$$

with $C(\eta, r) > 0$ independent of $g \in E_\gamma$ and $\tau \in (0, \tau_0/2)$. Using Sobolev embedding, we obtain, for all $\alpha \in (0,1)$,

$$\|w\|_{C^\alpha(\overline{\Sigma}_{g,\tau})} \le C(\eta, r).$$

Consequently, for all $T_g - \frac{\tau_0}{2} < t < s < T_g$ and $x \in \Omega_\eta$, choosing $\tau = T_g - s \in (0, \tau_0/2)$, we obtain

$$U_g(s,x) - U_g(t,x) \le C(s-t)^\alpha (T_g - s)^{-a}. \tag{5.0.5}$$

Here and until the end of Step 1, $C > 0$ denotes a positive constant, depending on η, α, but independent of g.

Now, for fixed $T_g - \frac{\tau_0}{2} < t < T_g$, we consider the sequence

$$s_i = T_g - (T_g - t)2^{-i}, \qquad i = 0, 1, 2, \ldots,$$

which satisfies

$$s_{i+1} - s_i = (T_g - t)2^{-i-1} = T_g - s_{i+1}. \tag{5.0.6}$$

Fix $\alpha \in (a, 1)$. From (5.0.5) and (5.0.6), we have

$$\begin{aligned} U_g(s_{i+1}, x) - U_g(s_i, x) &\le C(s_{i+1} - s_i)^\alpha (T_g - s_{i+1})^{-a} \\ &= C(T_g - s_{i+1})^{\alpha - a} = C[(T_g - t)2^{-i-1}]^{\alpha - a} \end{aligned}$$

and iterating, we obtain

$$U_g(s_{i+1}, x) - U_g(t, x) \le C(T_g - t)^{\alpha - a} \sum_{j=0}^{i} 2^{-(j+1)(\alpha - a)}.$$

Claim (5.0.3) for $t \in \big(T_g - \frac{\tau_0}{2}, T_g\big)$ follows by letting $i \to \infty$. Finally, taking $C(\eta, \beta) \ge \big(\frac{\tau_0}{2}\big)^{-\beta}$, we see that (5.0.3) is satisfied for $t \in \big(0, T_g - \frac{\tau_0}{2}\big]$ as well.

Step 2. *No touchdown away from $\mathcal{T}_f$ and from $\partial\Omega$.* Let $\sigma, \eta > 0$. We claim that there exists $\kappa = \kappa(\sigma) > 0$ (independent of η) and $\varepsilon = \varepsilon(\sigma, \eta) > 0$, such that for all $g \in E_\gamma$ with $\|g\|_\infty \le M$, if $\|g - f\|_q \le \varepsilon$, then

$$U_g(T_g, x) \le 1 - 2\kappa \qquad \text{in } \overline{\Omega_\eta} \setminus A_\sigma, \tag{5.0.7}$$

where $A_\sigma := \mathcal{T}_f + B(0, \sigma)$.

Choose any $\beta \in (0, 1/(p+1))$. As a consequence the definition of $\mathcal{T}_f$, there exists $\kappa = \kappa(\sigma) \in (0, 1/5)$ such that

$$U_f(t, x) \le 1 - 5\kappa \qquad \text{in } [0, T_f) \times (\Omega \setminus A_\sigma). \tag{5.0.8}$$

Set $t_0 = \max\big(0, T_f - \big(\frac{\kappa}{C_\eta}\big)^{1/\beta}\big)$, where $C_\eta = C(\eta, \beta)$ is given by (5.0.3). By Proposition 5.1, there exists $\varepsilon = \varepsilon(\sigma, \eta)$ such that, for all $g \in E$, if $\|g - f\|_q \le \varepsilon$, then

$$U_g(t_0, x) \le U_f(t_0, x) + \kappa \le 1 - 4\kappa \qquad \text{in } [0, t_0] \times (\Omega \setminus A_\sigma).$$

Applying (5.0.3), it follows that

$$\begin{aligned} U_g(T_g, x) &\le U_g(t_0, x) + C_\eta (T_g - T_f)^\beta + C_\eta (T_f - t_0)^\beta \\ &\le 1 - 3\kappa + C_\eta (T_g - T_f)^\beta \qquad \text{in } \overline{\Omega_\eta} \setminus A_\sigma. \end{aligned}$$

The Claim then follows from Proposition 5.2.

Step 3. *No touchdown near $\partial\Omega$ and conclusion.* We shall use a supersolution argument to exclude touchdown on $\Omega\setminus\Omega_\eta$ for g close to f and some $\eta>0$ (independent of g).

Since $\mathcal{T}_f\subset\subset\Omega$, we may fix $\sigma_0>0$ sufficiently small, such that $A_{\sigma_0}\subset\subset\Omega$. Set $\kappa=\kappa(\sigma_0)$, given by Step 2. Let $W(x)=1-2\kappa+K\phi(x)$, where $K=M\kappa^{-p}$ and ϕ is the solution of $-\Delta\phi=1$ in Ω, with $\phi=0$ on $\partial\Omega$. We can choose $\eta>0$ small enough such that $K\phi(x)<\kappa$ in $\Omega\setminus\Omega_\eta$, so that

$$-\Delta W=K\geq\frac{M}{(1-W)^p}\quad\text{in }\Omega\setminus\Omega_\eta.$$

Taking $\eta>0$ smaller if necessary, we may also assume $\partial\Omega_\eta\cap A_{\sigma_0}=\emptyset$. For all $g\in E_\gamma$ with $\|g\|_\infty\leq M$ and $\|g-f\|_q\leq\varepsilon(\sigma_0,\eta)$, it then follows from (5.0.7) and $\partial_tU_g\geq 0$ that

$$U_g(t,x)\leq 1-2\kappa\leq W(x)\qquad\text{on }[0,T_g)\times\partial\Omega_\eta.$$

Since $W\geq 0$, it follows from the comparison principle, applied on $[0,T_g)\times(\Omega\setminus\Omega_\eta)$, that

$$U_g(t,x)\leq W(x)\leq 1-\kappa\qquad\text{in }[0,T_g)\times(\overline{\Omega}\setminus\Omega_\eta).\tag{5.0.9}$$

Finally combining (5.0.9) and (5.0.7) with the η just chosen and any $\sigma>0$, we conclude that, for all $g\in E_\gamma$ with $\|g\|_\infty\leq M$, if $\|g-f\|_q\leq\min(\varepsilon(\sigma,\eta),\varepsilon(\sigma_0,\eta))$, then

$$U_g(t,x)\leq W(x)\leq 1-\kappa\qquad\text{in }[0,T_g)\times(\overline{\Omega}\setminus A_\sigma).$$

The Theorem follows. □

6 Proof of Theorems 1.4, 1.5 and 1.6

Proof of Theorem 1.4. **Step 1.** *Estimates of U_g.* Let g satisfy the assumptions of the Theorem for some $\varepsilon>0$. Since U_g (and U_f) is radially symmetric, we shall indifferently write $U_g(t,x)$ or $U_g(t,r)$ with $r=|x|$. It is known from [4], [15], [22] that $\mathcal{T}_f=\{0\}$.

We first observe that, taking $\varepsilon>0$ small enough, assumptions (1.3.4) and (1.3.5) guarantee that

$$g(x)>\mu_0\rho^{-2}\quad\text{on }B_\rho.\tag{6.0.1}$$

Indeed for given $\delta>0$, we may choose $s_0\in(0,(R-\rho)/2]$ sufficiently small (depending only on f,δ), such that

$$\sup\big\{|f(r+s)-f(r)|;\,0\leq r\leq\rho,\,0\leq s\leq 2s_0\big\}\leq\delta/2.$$

By (1.3.4), for all $r\in[0,\rho]$ and $s\in[0,2s_0]$, we then have

$$\begin{aligned}g(r)&\geq g(r+s)-\varepsilon R\\&\geq f(r)-|f(r+s)-f(r)|-|g(r+s)-f(r+s)|-\varepsilon R\\&\geq f(r)-|g(r+s)-f(r+s)|-\varepsilon R-\delta/2.\end{aligned}$$

Averaging in $s\in[s_0,2s_0]$ and using Hölder's inequality and (1.3.5), we obtain, for all $r\in[0,\rho]$,

$$\begin{aligned}g(r)&\geq f(r)-s_0^{-1}\int_{s_0}^{2s_0}|g(r+s)-f(r+s)|\,ds-\varepsilon R-\delta/2\\&\geq f(r)-s_0^{-n}\int_{s_0}^{2s_0}|g(r+s)-f(r+s)|(r+s)^{n-1}\,ds-\varepsilon R-\delta/2\\&\geq f(r)-C(n,R,q)s_0^{-n}\|g-f\|_q-\varepsilon R-\delta/2\geq f(r)-\delta,\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. In view of our assumptions on f, property (6.0.1) follows by choosing δ sufficiently small.

Owing to (6.0.1), we have $T_g < \infty$ and, by Remark 5.4, $g \in E_\gamma$ for some $\gamma > 0$ independent of g. Also, in view of Theorem 5.3, we may assume $T_g > t_0 := T_f/2$ and

$$U_g(t,x) \le 1-\kappa \qquad \text{in } [0,T_g) \times \{\rho/4 \le |x| \le R\}, \tag{6.0.2}$$

for some $\kappa > 0$ independent of g. By parabolic estimates, it follows that

$$\|U_g\|_{C^{1+\nu/2,2+\nu}([0,T_g]\times\{\rho/2\le|x|\le R\})} \le C_1. \tag{6.0.3}$$

for some $C_1, \nu > 0$ independent of g.

Next we claim that, for any given $t_1 \in (0,T_f)$, we have

$$T_g > t_1 \text{ and } U_g(t_1,\cdot) \text{ converges to } U_f(t_1,\cdot) \text{ in } C^2(\overline{B}(0,\rho/2)) \text{ as } \varepsilon \to 0. \tag{6.0.4}$$

The fact that $T_g > t_1$ for $\varepsilon > 0$ small follows from Proposition 5.2. To prove the convergence, we first note that, by Proposition 5.1, we have

$$\sup_{t\in[0,t_1]} \|U_g(t,\cdot) - U_f(t,\cdot)\|_\infty \to 0, \qquad \text{as } \varepsilon \to 0. \tag{6.0.5}$$

In particular we can assume $\sup_{t\in[0,t_1]} \|U_g(t,\cdot)\|_\infty \le c < 1$. Moreover, our assumptions guarantee that $\|g\|_{C^1(\overline{B}(0,\rho))} \le C$, with $C > 0$ independent of g. It then follows from standard parabolic estimates that

$$\|U_g\|_{C^{\nu/2,\nu}([0,t_1]\times\overline{B}(0,3\rho/4))} \le C$$

for some $\nu > 0$, and next that

$$\|U_g\|_{C^{1+\nu/2,2+\nu}([0,t_1]\times\overline{B}(0,\rho/2))} \le C.$$

Using compact embeddings, we deduce that, for any sequence g_i satisfying the assumptions of the Theorem with $\varepsilon = \varepsilon_i \to 0$, there exists a subsequence of $U_{g_i}(t_1,\cdot)$ which converges in $C^2(\overline{B}(0,\rho/2))$ to some limit W. By (6.0.5) we must have $W = U_f(t_1,\cdot)$ and property (6.0.4) follows.

Step 2. *Monotonicity properties of* U_g. We first claim that

$$\partial_r U_f < -c_0 r \quad \text{in } [t_0,T_f) \times (0,\rho], \tag{6.0.6}$$

for some constant $c_0 > 0$.

To prove (6.0.6), we set $B_+(0,R) = B(0,R) \cap \{x_1 > 0\}$. It follows from our assumptions that $z := \partial_{x_1} U_f \le 0$ in $(0,T_f) \times B_+(0,R)$. Recalling that $f \in C^1(\overline{B}_\rho)$ and using parabolic regularity, we deduce that z is a (strong) subsolution of the heat equation in $Q := (0,T_f) \times B_+(0,\rho)$, namely:

$$z_t - \Delta z = \frac{pf(0)z}{(1-U_f)^{p+1}} \le 0 \quad \text{a.e. in } Q.$$

Since $z = 0$ on $\{x_1 = 0\}$ it follows from the strong maximum principle and the Hopf Lemma that

$$z(t,x_1,0,\cdots,0) \le -c_0 x_1 \quad \text{for all } (t,x_1) \in [t_0,T_f) \times [0,\rho].$$

Claim (6.0.6) follows by observing that $\partial_r U_f(t,x) = \partial_{x_1} U_f(t,|r|,0,\cdots,0)$.

We next choose $t_1 \in (t_0, T_f)$ such that

$$C_1|T_f - t_1|^{\nu/2} \le \frac{c_0\rho}{8}, \tag{6.0.7}$$

where the constants C_1, c_0 are given by (6.0.3), (6.0.6), respectively. We claim that if ε is sufficiently small, then

$$\partial_r U_g(t, \rho/2) \le -\frac{c_0\rho}{16}, \qquad \text{for all } t \in [t_1, T_g). \tag{6.0.8}$$

To prove (6.0.8), we observe that, by (6.0.6) and (6.0.4), if ε is sufficiently small, then we have

$$\partial_r U_g(t_1, \rho/2) \le \partial_r U_f(t_1, \rho/2) + \frac{c_0\rho}{4} \le -\frac{c_0\rho}{4}.$$

Applying (6.0.3), (6.0.7) and Proposition 5.2, we deduce that, if ε is sufficiently small then, for all $t \in [t_1, T_g)$,

$$\begin{aligned}
\partial_r U_g(t, \rho/2) &\le \partial_r U_g(t_1, \rho/2) + C_1|T_f - t_1|^{\nu/2} + C_1|T_g - T_f|^{\nu/2} \\
&\le -\frac{c_0\rho}{4} + \frac{c_0\rho}{8} + C_1|T_g - T_f|^{\nu/2} \\
&\le -\frac{c_0\rho}{4} + \frac{c_0\rho}{8} + \frac{c_0\rho}{16} = -\frac{c_0\rho}{16},
\end{aligned}$$

which proves (6.0.8).

Then we claim that there exists a constant $c_1 > 0$ such that, if ε is sufficiently small, then

$$\partial_r U_g(t_1, r) \le -c_1 r, \qquad \text{for all } r \in [0, \rho/2]. \tag{6.0.9}$$

To prove (6.0.9), we note that, by (6.0.6), there exists $\ell \in (0, \rho)$ such that $\partial_r^2 U_f(t_1, r) \le -c_0/2$ for all $r \in [0, \ell]$ and $\partial_r U_f(t_1, r) \le -c_0\ell$ for all $r \in [\ell, \rho/2]$. By property (6.0.4), we deduce that if ε is sufficiently small, then $\partial_r^2 U_g(t_1, r) \le -c_0/4$ for all $r \in [0, \ell]$, which, after integration, gives

$$\partial_r U_g(t_1, r) \le -\frac{c_0}{4} r, \qquad \text{for all } r \in [0, \ell].$$

Also by property (6.0.4), if ε is small enough, we have

$$\partial_r U_g(t_1, r) \le -\frac{c_0\ell}{2} \le -\frac{c_0\ell}{\rho} r, \qquad \text{for all } r \in [\ell, \rho/2].$$

Hence, (6.0.9) follows by taking $c_1 = \min\{\frac{c_0}{4}, \frac{c_0\ell}{\rho}\}$.

Step 3. *Auxiliary function and conclusion.* In what follows, omitting the subscript g without risk of confusion, we will use the notation $u = U_g$. Following the method in [11], we define the auxiliary function

$$J(t, r) := w(t, r) + \eta a(r) h(u), \quad (t, r) \in [t_1, T_g) \times [0, \rho/2],$$

where $w(t, r) = r^{n-1} u_r$, $a(r) = r^n$, $h(u) = (1 - u)^{-\gamma}$ and $\eta, \gamma > 0$ are constants to be chosen later.

We first look at the parabolic boundary of $[t_1, T_g) \times (0, \rho/2)$. By (6.0.9), for all $r \in [0, \rho/2)$, we have

$$\begin{aligned}
J(t_1, r) &= r^{n-1} u_r(t_1, r) + \eta \tfrac{r^n}{(1-u(t_1,r))^\gamma} \\
&\le r^n \big(-c_1 + \tfrac{\eta}{(1-\|u(t_1)\|_\infty)^\gamma}\big) \le 0,
\end{aligned} \tag{6.0.10}$$

provided $\eta \le c_1(1-\|u(t_1)\|_\infty)^\gamma$. We also have $J(t,0)=0$, for all $t\in[t_1,T_g)$, and by (6.0.8) and (6.0.2), we have, for all $t\in[t_1,T_g)$,

$$\begin{aligned} J\big(t,\rho/2\big) &= \left(\tfrac{\rho}{2}\right)^{n-1} u_r(t,\rho/2)+\eta\tfrac{(\rho/2)^n}{(1-u(t,\rho/2))^\gamma} \\ &\le \left(\tfrac{\rho}{2}\right)^{n}\left(-\tfrac{c_0}{8}+\tfrac{\eta}{\kappa^\gamma}\right)\le 0, \end{aligned} \tag{6.0.11}$$

provided $\eta \le \dfrac{c_0\kappa^\gamma}{8}$.

Next, we note that u and w respectively solve the equations

$$\begin{aligned} u_t-u_{rr}-\frac{n-1}{r}u_r &= g(r)H(u),\\ w_t-w_{rr}+\frac{n-1}{r}w_r &= g(r)H'(u)w+g'(r)r^{n-1}H(u), \end{aligned}$$

where $H(u)=(1-u)^{-p}$. Omitting the variables r and u from now on without risk of confusion, we compute

$$\begin{aligned} J_t &= w_t+\eta a h' u_t,\\ J_r &= w_r+\eta a' h+\eta a h' u_r,\\ J_{rr} &= w_{rr}+\eta a'' h+2\eta a' h' u_r+\eta a h'' u_r^2+\eta a h' u_{rr}. \end{aligned}$$

Using these identities and $w=J-\eta a h$, we obtain (a.e. in $[t_1,T_g)\times(0,\rho/2)$):

$$\begin{aligned} J_t-J_{rr}+\tfrac{n-1}{r}J_r &\\ &= w_t-w_{rr}+\tfrac{n-1}{r}w_r+\eta a h'\left(u_t-u_{rr}-\tfrac{n-1}{r}u_r\right)\\ &\qquad +\tfrac{n-1}{r}\eta a' h-\eta a'' h-2\eta a' h' u_r-\eta a h'' u_r^2+2\eta a h'\tfrac{n-1}{r}u_r\\ &= gH'w+g'r^{n-1}H+\eta a h' gH-2\eta h' w-\eta a h'' u_r^2\\ &\le [pg(1-u)^{-p-1}-2\gamma\eta(1-u)^{-1-\gamma}]w+\gamma\eta g r^n(1-u)^{-p-1-\gamma}\\ &\qquad +g'r^{n-1}(1-u)^{-p}\\ &= b(t,r)J+r^{n-1}(1-u)^{-p}[g'-(p-\gamma)\eta g r(1-u)^{-\gamma-1}\\ &\qquad +2\gamma\eta^2 r(1-u)^{p-2\gamma-1}]\\ &= b(t,r)J+r^{n-1}(1-u)^{-p}\big[g'-\eta r(1-u)^{-\gamma-1}\big((p-\gamma)g\\ &\qquad -2\gamma\eta(1-u)^{p-\gamma}\big)\big], \end{aligned}$$

where b is a bounded function on $(0,\rho/2)\times[t_1,T_g-\tau]$ for each $\tau>0$. For any $\gamma\in[0,p)$, using assumption (1.3.4), it follows that

$$J_t-J_{rr}+\frac{n-1}{r}J_r-b(t,r)J\le r^n(1-u)^{-p}\big[\varepsilon-\eta(1-u)^{-\gamma-1}\big((p-\gamma)g-2\gamma\eta\big)\big].$$

Recalling from Step 1 that g is uniformly positive in B_ρ, we can choose η small enough such that

$$(p-\gamma)g(r)\ge 3\gamma\eta \quad \text{on } B_{\rho/2}.$$

Taking $\varepsilon\le\gamma\eta^2$, we finally obtain

$$J_t-J_{rr}+\frac{n-1}{r}J_r-b(t,r)J\le r^n(1-u)^{-p}\big(\varepsilon-\gamma\eta^2\big)\le 0,$$

a.e. in $[t_1,T_g)\times(0,\rho/2)$.

In view of this inequality, together with (6.0.10) and (6.0.11), it follows from the maximum principle that $J \leq 0$, hence

$$u_r \leq -\eta r(1-u)^{-\gamma}, \qquad \text{for all } (t,r) \in [t_1, T_g) \times [0, \rho/2).$$

After integrating this inequality in space, we obtain

$$(1-u(t,r))^{\gamma+1} \geq \frac{\gamma+1}{2}\eta r^2, \qquad \text{for all } (t,r) \in [t_1, T_g) \times [0, \rho/2),$$

which guarantees $\mathcal{T}_g = \{0\}$. □

Remark 6.1. *(i) In view of the previous proof, we obtain the following estimate of the final touchdown profile of the solution near the origin:*

$$1 - u(T,r) \geq c r^{\frac{2}{p+1}+\varepsilon}, \qquad \text{as } r \to 0,$$

for any $\varepsilon > 0$ and $c = c(\varepsilon) > 0$. For more accurate results regarding the quenching profile in the case $f = Const.$, see [8].

(ii) We point out that under the stronger, global assumption that $g \in C^1(\overline{\Omega})$ and $-M \leq g'(r) \leq \varepsilon r$ on $[0,R]$, we can prove single point touchdown at the origin without using the semicontinuity property of the touchdown set. To this end, in the above proof, one considers J on the whole cylinder $[t_1, T_g) \times (0,R)$ and uses the Hopf lemma to ensure $J \leq 0$ at $r = R$. In this case, we only use Proposition 5.1, along with the hypothesis $\|g - f\|_q \leq \varepsilon$, to ensure that $J(t_1, \cdot) \leq 0$ at some time $t_1 < T_g$.

Proof of Theorem 1.5. It is a direct consequence of Theorem 5.3, Remark 5.4 and Theorem 1.1. □

Proof of Theorem 1.6 and Proposition 5.6. We just need to prove assertion (i) of Theorem 1.6. Assertion (ii) follows by the same arguments together with the radial symmetry of the domain and profile.

In order to construct this example, we set $B_i = B(x_i, 2r)$, with $r \in (0, \rho/2)$ chosen sufficiently small so that

$$\min\Big(\mathrm{dist}(B_1, B_2), \mathrm{dist}(B_1, \partial\Omega), \mathrm{dist}(B_2, \partial\Omega)\Big) > r.$$

Denote $\tilde{B}_i = B(x_i, r)$. Choose $\mu > \mu_0(p,n) r^{-2}$. By Lemma 2.2, for any $g \in E$ such that $g \geq \mu \chi_{\tilde{B}_i}$ with $i = 1$ or 2, we have

$$T_g \leq \frac{1}{(p+1)(\mu - \mu_0(p,n) r^{-2})} < \infty. \tag{6.0.12}$$

Set

$$\hat{E} = \big\{g \in E;\ \text{there exists } j \in \{1,2\} \text{ such that } 2\mu \geq g \geq \mu \chi_{\tilde{B}_j}\big\}.$$

By Theorems 1.1 and 1.2, we deduce the existence of $\eta = \eta(p, \Omega, \mu, r) \in (0, \mu/2)$ such that

$$\text{if } g \in \hat{E} \text{ and } g \leq 2\eta \text{ in } \overline{B}_i \text{ for some } i \in \{1,2\}, \text{ then } \mathcal{T}_g \cap B_i = \emptyset \tag{6.0.13}$$

and

$$\text{if } g \in \hat{E} \text{ and } g \leq 2\eta \text{ in } \overline{\Omega} \setminus [B_1 \cup B_2], \text{ then } \mathcal{T}_g \subset B_1 \cup B_2. \tag{6.0.14}$$

Now, for a fixed $q \geq 1$ and $\frac{n}{2} < q < \infty$, we consider a continuous map $h \mapsto f_h$ from $[\eta, 2\mu]$ to $(\hat{E}, \|\cdot\|_q)$ with the following properties. Each f_h satisfies

$$f_h(x) = \begin{cases} h & \text{for } x \in \tilde{B}_1, \\ 2\mu + \eta - h & \text{for } x \in \tilde{B}_2, \\ \eta & \text{for } x \in \overline{\Omega} \setminus [B_1 \cup B_2], \end{cases}$$

together with

$$\max_{\overline{B}_i} f_h = f_h(x_i), \qquad \text{for } i = 1, 2.$$

By (6.0.12), (6.0.14), we have

$$T_{f_h} < \infty \quad \text{and} \quad \mathcal{T}_{f_h} \subset B_1 \cup B_2, \qquad \text{for all } h \in [\eta, 2\mu]. \tag{6.0.15}$$

In addition, by (6.0.13), (6.0.14), we have

$$\mathcal{T}_{f_h} \subset B_2, \qquad \text{for all } h \in [\eta, 2\eta] \tag{6.0.16}$$

and

$$\mathcal{T}_{f_h} \subset B_1, \qquad \text{for all } h \in [2\mu - \eta, 2\mu]. \tag{6.0.17}$$

Now, we define

$$h^* := \inf\big\{h \in [\eta, 2\mu], \text{ such that } \mathcal{T}_{f_h} \cap B_1 \neq \emptyset\big\}. \tag{6.0.18}$$

By (6.0.16), (6.0.17), we know that $2\eta \leq h^* \leq 2\mu - \eta$. By the definition of h^* and (6.0.15), we have

$$\mathcal{T}_{f_h} \subset B_2, \qquad \text{for all } h \in [\eta, h^*),$$

and there exists a sequence $h_i \downarrow h^*$ such that

$$\mathcal{T}_{f_{h_i}} \cap B_1 \neq \emptyset.$$

Since $h \mapsto f_h$ is continuous in L^q with q as above, we may apply Theorem 1.8 and (6.0.15) to deduce that

$$\mathcal{T}_{f_{h^*}} \cap B_1 \neq \emptyset \quad \text{and} \quad \mathcal{T}_{f_{h^*}} \cap B_2 \neq \emptyset. \tag{6.0.19}$$

This completes the proof of Theorem 1.6. Finally, in view of (6.0.18) and the second part of (6.0.19), Proposition 5.6 follows by considering the sequence $g_i = f_{h^* - 1/i}$. $\square$

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Chapter 3

Quantitative touchdown localization for the MEMS problem with variable dielectric permittivity

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Abstract. We consider a well-known model for micro-electromechanical systems (MEMS) with variable dielectric permittivity, based on a parabolic equation with singular nonlinearity. We study the touchdown or quenching phenomenon. Recently, the question whether or not touchdown can occur at zero points of the permittivity profile f, which had long remained open, was answered negatively in [15] for the case of interior points, and we then showed in [2] that touchdown can actually be ruled out in subregions of Ω where f is positive but suitably small.

The goal of this paper is to further investigate the touchdown localization problem and to show that, in one space dimension, one can obtain quite quantitative conditions. Namely, for large classes of typical, one-bump and two-bump permittivity profiles, we find good lower estimates of the ratio ρ between $f(x)$ and its maximum, below which no touchdown occurs outside of the bumps. The ratio ρ is rigorously obtained as the solution of a suitable **finite-dimensional optimization problem** (with either three or four parameters), which is then numerically estimated. Rather surprisingly, it turns out that the values of the ratio ρ are not "small" but actually **up to the order** ~ 0.3, which could hence be quite appropriate for robust use in practical MEMS design.

The main tool for the reduction to the finite-dimensional optimization problem is a quantitative type I, temporal touchdown estimate. The latter is proved by maximum principle arguments, applied to a multi-parameter family of refined, nonlinear auxiliary functions with cut-off.

1 Introduction

1.1 Mathematical problem and physical background

We consider the problem

$$\begin{cases} u_t - u_{xx} = f(x)(1-u)^{-p}, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(0,x) = 0, & x \in \Omega, \end{cases} \tag{1.1.1}$$

where $\Omega = (-R, R) \subset \mathbb{R}$, $p > 0$ and

$$f \geq 0 \text{ is a Hölder continuous function in } \overline{\Omega}. \tag{1.1.2}$$

Problem (1.1.1) with $p = 2$ is a known model for micro-electromechanical devices (MEMS) and has received a lot attention in the past 15 years. An idealized version of such device consists of two conducting plates, connected to an electric circuit. The lower plate is rigid and fixed while the upper one is elastic and fixed only at the boundary. Initially the plates are parallel and at unit distance from each other. When a voltage (difference of potential between the two plates) is applied, the upper plate starts to bend down and, if the voltage is large enough, the upper plate eventually touches the lower one. This is called *touchdown* phenomenon. Such device can be used for instance as an actuator, a microvalve (the touching-down part closes the valve), or a fuse.

In the mathematical model, $u = u(t, x)$ measures the vertical deflection of the upper plate and the function $f(x)$ represents the dielectric permittivity of the material (and is also proportional to the – constant – applied voltage). As a key feature, the permittivity f may be inhomogeneous and this can be used to trigger the properties of the device. We refer to [1] and the references therein for the full details of the model derivation.

It is well known that problem (1.1.1) admits a unique maximal classical solution u. We denote its maximal existence time by $T = T_f \in (0, \infty]$. Moreover, under some largeness assumption on f, it is known that the maximum of u reaches the value 1 at a finite time, so that u ceases to exist in the classical sense, i.e. $T < \infty$. This property, known as quenching, is the mathematical counterpart of the touchdown phenomenon.

A point $x = x_0$ is called a *touchdown* or *quenching point* if there exists a sequence $\{(x_n, t_n)\} \in \Omega \times (0, T)$ such that

$$x_n \to x_0, \quad t_n \uparrow T \quad \text{and} \quad u(x_n, t_n) \to 1 \text{ as } n \to \infty.$$

The set of all such points is called the *touchdown* or *quenching set*, denoted by $\mathcal{T} = \mathcal{T}_f \subset \overline{\Omega}$.

In the past decades, MEMS problems, including system (1.1.1) and the related touchdown issues, have received considerable attention in the physical and engineering as well as in the mathematical communities. We refer to [1], [23] for more details on the physical background, and to, e.g., [10], [4] [22], [18], [5], [7], [8], [12], [16], [17], [19], [13], [11], [15] for mathematical studies. See also [24], [20], [9], [3] for earlier mathematical work on the case of constant f.

1.2 Motivation

The question whether or not touchdown can occur at zero points of the permittivity profile f, raised in [17], [8], [16], [1], was answered negatively in [15] for the case of interior points. This is by no means obvious since, for the analogous blowup problem $u_t - \Delta u = f(x)u^p$ with $f(x) = |x|^\sigma$, examples of solutions with single-point blowup at the origin have been constructed in [4], [14] for suitable $\sigma > 0$, $p > 1$ and suitable initial data $u_0 \geq 0$. We then showed in [2] that touchdown can actually be ruled out in subregions of Ω where f is positive but suitably small. The following theorem collects the two smallness criteria given in [2].

Theorem. *Let $p > 0$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and f a function satisfying* (1.1.2) *and*

$$\begin{cases} T_f \leq M, \quad \|f\|_\infty \leq M, \quad f \geq r\chi_B, \\ \textit{where } M, r > 0 \textit{ and } B \subset \Omega \textit{ is a ball of radius } r. \end{cases}$$

There exists $\gamma_0 > 0$ depending only on p, Ω, M, r such that:

(i) For any $x_0 \in \Omega$, if $f(x_0) < \gamma_0 \, dist^{p+1}(x_0, \partial\Omega)$, then x_0 is not a touchdown point.

(ii) For any $\omega \subset\subset \Omega$, if $\sup_{x \in \overline{\Omega} \setminus \omega} f(x) < \gamma_0 \, dist^{p+1}(\omega, \partial\Omega)$, then the touchdown set is contained in ω.

Motivated by practical considerations of MEMS design, our aim in this article is to further investigate the touchdown localization problem and to show that in one space dimension, where analytic computations can be made more precise, one can obtain quite quantitative conditions. Namely, we look for a lower estimate of the ratio ρ between f and its maximum, below which no touchdown occurs on a subregion of Ω. Rather surprisingly, it turns out that in the physical case $p = 2$, under suitable assumptions on f, our methods yield values of the ratio ρ which are not "small" but can actually be up to the order $\rho \sim 0.3$, which could hence be quite appropriate for robust practical use.

1.3 Reduction to a finite-dimensional optimization problem and quantitative results

In order to give good estimates of the ratio ρ, we shall consider two typical situations, which roughly correspond to a "one-bump" or a "two-bump" shape for the profile f. The touchdown is ruled out in a subinterval respectively located between a bump and an endpoint of Ω, or between two bumps.

The idea behind this is that the plate can be covered with two dielectric materials, one with a high permittivity and the other with a lower permittivity. We then seek for a ratio between the two permittivities, allowing to rule out touchdown in the low permittivity region.

We point out that, as a consequence of our method, the ratio ρ is rigorously obtained as the solution of a suitable *finite-dimensional optimization problem*, with either three or four parameters. Such kind of reduction in nonlinear parabolic problems is new, as far as we know. An advantage of our results is that they apply to classes of configurations, so that detailed numerics need not be carried out to localize the quenching set for particular cases.

In spite of the rather awkward shape of the optimization problem, it turns out to yield quite reasonable practical values of the threshold ratio ρ in concrete cases. Before presenting our rigorous statements, let us illustrate the results for the physical case

$p = 2$ by some concrete examples, that can be deduced from them by a relatively simple numerical procedure applied to the finite-dimensional optimization problem (see also Table 3.1 below for more applications). The following two figures represent some typical permittivity profiles $f(x)$ and the localization of the corresponding touchdown sets in the one-bump and two-bump cases respectively. The touchdown sets are localized in a neighborhood of the bumps, represented by the fat lines,

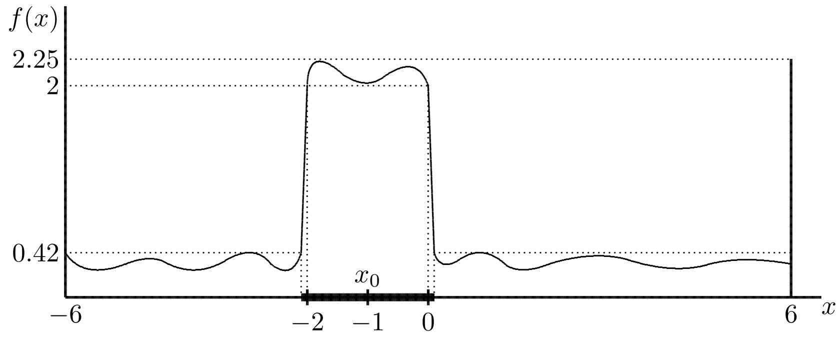

Figure 3.1 – An illustration of the localization of the touchdown set in the one-bump case for $p = 2$

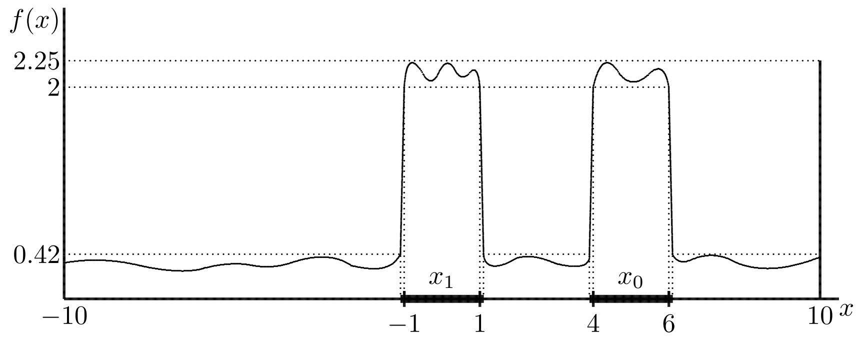

Figure 3.2 – An illustration of the localization of the touchdown set in the two-bump case for $p = 2$

Here, the profile f satisfies

$$\|f\|_\infty \le 2.25, \qquad f(x) \ge 2 \;\text{ in } \omega, \qquad f(x) \le 0.42 \;\text{ in } \Omega \setminus \tilde{\omega}, \tag{1.3.1}$$

with respectively $\Omega = (-6, 6)$, $\omega = (-2, 0)$, $\tilde{\omega} = (-2.1, 0.1)$ and $\Omega = (-10, 10)$, $\omega = (-1, 1) \cup (4, 6)$, $\tilde{\omega} = (-1.1, 1.1) \cup (3.9, 6.1)$. The touchdown set is then contained in $\tilde{\omega}$.

Let us now state our first two main theorems. The following result, although stated in a more general form, applies in particular to "one-bump" profiles. Here we recall that the error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and we also set $\overline{\cot}\, s = \cot s$ if $0 < s \le \pi/2$, and $\overline{\cot}\, s = 0$ if $s > \pi/2$.

Theorem 1.1. *Consider problem* (1.1.1) *with* $p > 0$ *and* $\Omega = (-R, R)$. *Let* $x_0 \in \Omega$ *and assume*

$$\mu > \mu_1(p) := \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{2}, \qquad d_0 := R - |x_0| - 1 > \frac{p+1}{\sqrt{p\mu}} \overline{\cot}\big[\sqrt{p\mu}\big]. \tag{1.3.2}$$

For each $d \in (0, d_0)$, *there exists* $\rho = \rho(p, \mu, \|f\|_\infty, d_0, d) \in (0, 1)$ *such that, if* f *satisfies*

$$f \geq \mu \quad \text{in } \big(x_0 - 1, x_0 + 1\big) \quad \text{and} \quad f < \rho\mu \quad \text{in } D := [x_0 + 1 + d, R], \tag{1.3.3}$$

then $T_f < \infty$ *and there are no touchdown points in* D.

In addition, ρ *can be chosen as the solution of the following optimization problem:*

$$\rho = \frac{1}{2} \operatorname*{Sup}_{(\tau,\beta,K)\in\mathcal{A}} \left(\frac{\beta - d}{\beta}\right)^{p+1} \frac{S(t_0(\tau), \beta)}{K + \tau^{-p}} \min\Big\{H\big(t_0(\tau), \beta\big), G\big(t_0(\tau), \beta, K\big)\Big\} \tag{1.3.4}$$

with

$$\mathcal{A} = \Big\{(\tau, \beta, K) \in (0, 1) \times (d, d_0) \times (0, \infty);\ \tau \geq \tfrac{\mu}{2\mu - \mu_1},\ K \geq \tfrac{p}{\mu\beta^2} - \tfrac{1}{p+1},\ \delta(\beta, K) \leq 1\Big\}.$$

Here we set

$$t_0(\tau) = \tfrac{1 - \tau^{p+1}}{(p+1)\|f\|_\infty}, \quad L = 1 + (p+1)K, \quad \Gamma = \sqrt{\tfrac{(p+1)L}{pK\mu\beta^2}}, \quad A = \arctan\Gamma, \quad \alpha = 1 + \tfrac{p}{L}$$

and the functions S, H, G, δ *are defined by*

$$\begin{aligned}
S(t, \beta) &= e^{-\frac{\pi^2 t}{4(d_0+1)^2}} \Big[1 - e^{-\frac{d_0(d_0 - \beta)}{t}}\Big], \qquad \delta(\beta, K) = A(1 + K)\sqrt{\tfrac{p+1}{pLK\mu}}, \\
H(t, \beta) &= \inf_{0<x<1} \frac{\operatorname{erf}\Big(\frac{1}{\sqrt{t}}\Big(1 + \frac{\beta}{2}x\Big)\Big) - \operatorname{erf}\Big(\frac{\beta}{2\sqrt{t}}x\Big)}{(1 - x)^{p+1}}, \\
G(t, \beta, K) &= (\Gamma^2 + 1)^{-\alpha/2} \inf_{0<x<1} \frac{\operatorname{erf}\Big(\frac{2 - (1-x)\delta}{2\sqrt{t}}\Big) + \operatorname{erf}\Big(\frac{(1-x)\delta}{2\sqrt{t}}\Big)}{\cos^\alpha(Ax)}.
\end{aligned} \tag{1.3.5}$$

As for the next theorem, it enables one to treat “two-bump” profiles, in combination with Theorem 1.1.

Theorem 1.2. *Consider problem* (1.1.1) *with* $p > 0$ *and* $\Omega = (-R, R)$. *Let* $x_0, x_1 \in \Omega$ *be such that* $|x_0| \geq |x_1|$ *and assume* (1.3.2). *For each* $d \in (0, d_0)$, *there exists* $\rho = \rho(p, \mu, \|f\|_\infty, d_0, d) \in (0, 1)$ *such that, if* $x_0 - x_1 > 2(1 + d)$ *and* f *satisfies*

$$f \geq \mu \quad \text{in } \big(x_1 - 1, x_1 + 1\big) \cup \big(x_0 - 1, x_0 + 1\big) \tag{1.3.6}$$

and

$$f < \rho\mu \quad \text{in } D := \big[x_1 + 1 + d, x_0 - 1 - d\big], \tag{1.3.7}$$

then $T_f < \infty$ *and there are no touchdown points in* D. *In addition,* ρ *can be chosen as the solution of the optimization problem* (1.3.4).

Of course, in order to apply Theorems 1.1-1.2, it is not necessary in practice to determine the value of ρ itself. Any number $\overline{\rho} < \rho$ can be used instead of ρ in assumptions (1.3.3) and (1.3.7). It therefore suffices to evaluate the function in the RHS of (1.3.4) for suitable choices of $(\tau, \beta, K) \in \mathcal{A}$.

In Table 3.1, for the physical case $p = 2$ and physically reasonable values of the parameters, we present some numerical lower estimates of the threshold ratio ρ (see the column $\overline{\rho}_1$). They show that Theorems 1.1-1.2 allow to reach ratios up to the order of

$$\rho \sim 0.17,$$

which seems quite satisfactory in view of robust practical conception of MEMS, in which one would like to prevent touchdown in specific parts of the device by proper design of the permittivity profile. As for the results in the column $\overline{\rho}_2$ of Table 3.1, they even give values up to

$$\rho \sim 0.3.$$

However, they are based on a more complicated optimization problem, whose lengthy statement is therefore postponed to Section 2 (see Theorem 2.1). Figures 3.1 and 3.2 above are based on the second line in Table 3.1 (using $\overline{\rho}_2$ as lower estimate for ρ). Note that in the example of Figure 3.2, we are applying the localization criteria from the two-bump and one-bump cases at the same time (between and at the exterior of the two bumps).

μ	$\|f\|_\infty$	d	d_0	$\overline{\rho}_1$	$\overline{\rho}_2$
1	1.1	0.1	5	**0.1050**	**0.2249**
2	2.25	0.1	4	**0.1182**	**0.2111**
3	3.5	0.01	5	**0.1554**	**0.2698**
6	6.2	0.01	10	**0.1682**	**0.2856**
10	10	0.005	10	**0.1732**	**0.2921**

Table 3.1 – Lower estimates of the threshold ratio ρ for $p = 2$, using Theorems 1.1-1.2 (column $\overline{\rho}_1$) and Theorems 2.1-2.2 (column $\overline{\rho}_2$).

The evaluation of $\overline{\rho}_1$ and $\overline{\rho}_2$ in Table 3.1 is done with the help of the computational tool *Matlab*, and this can be done with very good accuracy. See Section 6 for details on the numerical procedure. In particular we stress that we use a "monotone" discretization scheme to evaluate the infima in (1.3.5), which guarantees that the discrete infima are not larger than the exact ones. In this way, the only possible sources of errors in excess on $\overline{\rho}$ are the round-off machine errors and the numerical errors in the *Matlab* evaluations (for instance those of erf). In principle this can be guaranteed with any reasonably prescribed safety margin.

Discussion 1.3. *(a) It is a natural question whether the above touchdown localization behavior could be true whenever* $\rho < 1$ *in* (1.3.3), (1.3.7). *Actually, we show in [2] that this is* **not** *the case. Indeed, among other things, we construct examples showing that, for some class of symmetric "M"-shaped profiles in* $\Omega = (-R, R)$, *where* $f(0)$ *is less but close enough to the maximum of* f, *touchdown does occur at the origin, and only there. Moreover, interestingly, the touchdown set is then located* **far away** *from the points of maximum of* f. *This is illustrated in Figure 3.3. Although the function* f *is also a two-bump profile, it clearly presents a reverse situation to that described in Theorems 1.1-1.2 and illustrated in Figure 3.2. This shows that the threshold ratio* ρ *cannot exceed a certain value* ρ_0 *less than* 1 *and that there is an intermediate range* $(\rho_0, 1]$ *where results of the type of Theorems 1.1-1.2 cannot hold.*

(b) However, the values of ρ *found in Theorems 1.1, 1.2, 2.1–2.3 are probably not optimal and we have no indication what the value of the optimal threshold* ρ *should be. This seems to be a difficult problem.*

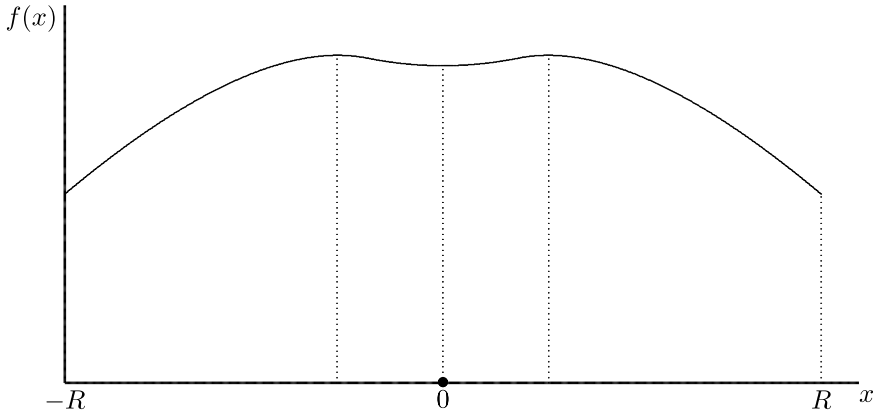

Figure 3.3 – Touchdown only at the origin for an "M"-shaped profile corresponding to the intermediate range of ρ (cf. Discussion 1.3(a)).

(c) For notational simplicity, we have chosen intervals of radius 1 in conditions (1.3.3), (1.3.6), *but by a straightforward scaling argument, one can see that this entails no loss of generality.*

(d) When Ω is a ball and f is monotonically decreasing with respect to $|x|$, the touchdown set is reduced to the origin (see [9], [16]). We stress that in Theorems 1.1-1.2 we do not make any kind of monotonicity or symmetry assumptions on f. On the other hand, for general nonmonotone profiles f, it remains an open – and probably difficult – problem to determine the finer structure of the touchdown set beyond the localization properties in Theorems 1.1-1.2. Numerical simulations in [1] suggest that it need not consist of isolated points but might contain intervals.

(e) Consider the reference case when $\Omega = (-1, 1)$ and $f = k = Const.$, and recall that there exists a number k^* such that $T < \infty$ if $k > k^*$ and $T = \infty$ if $k < k^*$. This is the square of the so-called (adimensionalized) pull-in voltage, for a unit profile (see e.g. [1]). We have

$$\frac{2p^p}{(p+1)^{p+1}} \leq k^* \leq \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{4}$$

(see [1, Proposition 2.2.1 and 2.2.2] and their proofs). In particular for $p = 2$, we have $0.30 \sim \frac{8}{27} \leq k^* \leq \frac{\pi^2}{27} \sim 0.37$, *whereas Theorems 1.1-1.2 require $\mu \geq \frac{2\pi^2}{27} \sim 0.73$. Although this condition is a bit more restrictive, we note that* 0.73 *still remains of the same order of magnitude as the reference squared pull-in voltage. On the other hand, Theorem 2.3 below applies whenever $\mu > \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{4}$ (~ 0.37 for $p = 2$).*

The article is structured as follows. In Section 2 we present further localization results, either giving more precise estimates of ρ (Theorems 2.1-2.2), or requiring weaker assumptions on μ, d_0 (Theorem 2.3). In Section 3, we state and prove a type I touchdown estimate (Proposition 3.1), which is a key ingredient in the proof of the localization results. Here is where our basic, multi-parameter auxiliary function is introduced, which will eventually lead to the optimization problem. The proofs of Theorems 1.1, 1.2 and 2.3 are given in Section 4, as a consequence of a no-touchdown criterion (Lemma 4.1), combined with the previous type I estimate. In Section 5, we prove Theorems 2.1-2.2, as a consequence of a more precise type I estimate, obtained by refining various ingredients from the proof of Theorems 1.1-1.2. In Section 6, we describe the numerical procedures that we use to handle the optimization problem. Finally, the article is completed by two appendices. The first one provides some useful quantitative comparison estimates for heat semigroups, and the second one is devoted to establishing the optimality of the cut-off functions $a(r)$ appearing in our proofs of localization.

2 Further quantitative results

2.1 Improved ratio

As announced in introduction, the following theorem allows one to obtain better estimates for the ratio ρ (cf. the last column of Table 3.1), at the expense of a more complicated optimization problem. Although the statement may seem somewhat lengthy, we stress that this result allows for quite good estimates of ρ (cf. Table 3.1). For simplicity, we first state the result in the case of one-bump profiles (cf. Fig. 3.1). In what follows, we set

$$\mu_0(p) := \frac{p^p}{(p+1)^{p+1}}\frac{\pi^2}{4}.$$

Theorem 2.1. *Consider problem* (1.1.1) *with* $p > 0$ *and* $\Omega = (-R, R)$. *Assume*

$$\mu > \max\left\{\mu_0(p), \frac{\arctan^2(\sqrt{p+1})}{p}\right\}, \quad 0 < d < \sqrt{\frac{p+1}{p\mu}} < d_0 := R - |x_0| - 1. \tag{2.1.1}$$

If f satisfies

$$f \ge \mu \ \text{ in } J := \big[x_0 - 1, x_0 + 1\big] \quad \text{and} \quad f < \rho\mu \ \text{ in } D := \overline{\Omega} \setminus \big(x_0 - 1 - d, x_0 + 1 + d\big), \tag{2.1.2}$$

then $T_f < \infty$ and there are no touchdown points in D. Here ρ is given by the solution of the following optimization problem:

$$\rho = \operatorname*{Sup}_{(\beta,K,\tau,\lambda)\in\mathcal{A}} \min\left\{\frac{1}{2}\left(\frac{\beta-d}{\beta}\right)^{p+1} S(t_0(\tau),\beta)G^*(\tau,t_0(\tau),\beta,K,\lambda),\ \frac{1}{p+1}\frac{\tau^{p+1}}{(T-t_0(\tau))\mu},\ \lambda\right\} \tag{2.1.3}$$

with

$$\mathcal{A} = \left\{(\beta,K,\tau,\lambda);\ \beta\in(d,d_0),\ K\in(0,p],\ \tau,\lambda\in(0,1),\ K\mu\beta^2 \le \frac{p(p+2)-K}{p},\ \delta_1+\delta_2\le 1\right\}, \tag{2.1.4}$$

$$G^*(\tau,t,\beta,K,\lambda) = \inf_{r\in(r_0,1+\beta)}\Big(1+p\mu S(t,\beta)\Lambda(t,r)\Big)\frac{\operatorname{erf}\left(\frac{r+1}{2\sqrt{t}}\right)+\operatorname{erf}\left(\frac{1-r}{2\sqrt{t}}\right)}{W_{\tau,K,\lambda}(r)a_{\beta,K}(r)}, \tag{2.1.5}$$

$$\Lambda(t,r) = \frac{1}{2}\int_0^t (1-Y(s))^{-\frac{p}{p+1}-1}\left[\operatorname{erf}\left(\frac{r+1}{2\sqrt{t-s}}\right)+\operatorname{erf}\left(\frac{1-r}{2\sqrt{t-s}}\right)\right]ds, \tag{2.1.6}$$

where the function S is defined in (1.3.5) *and we set* $Y(s) = S(s,0)\operatorname{erf}\left(\frac{1}{\sqrt{s}}\right)\frac{(p+1)\mu}{2}s$,

$$W_{\tau,K,\lambda}(r) = K\big(1-(1-\tau)\tilde{u}(r)\big) + \big(1-(1-\tau)\tilde{u}(r)\big)^{-p}, \tag{2.1.7}$$

$$\tilde{u}(r) = \frac{\lambda\mu}{\|f\|_\infty} + \Big(1-\frac{\lambda\mu}{\|f\|_\infty}\Big)\frac{1}{\cosh\big(\sqrt{c_p\|f\|_\infty}\,(r-1-d)_+\big)}, \tag{2.1.8}$$

$$a_{\beta,K}(r) = \begin{cases} D_1 \cos^{\alpha}\big(A_0\sqrt{K\mu}(r-r_0)\big), & r\in[r_0,r_1),\\ D_2 \cos^{p+1}\left(\sqrt{K\mu}(r-1)+A_3\right), & r\in[r_1,1],\\ \big(1-\beta^{-1}(r-1)\big)^{p+1}, & r\in(1,1+\beta], \end{cases} \tag{2.1.9}$$

$$t_0(\tau)=\frac{1-\tau^{p+1}}{(p+1)\|f\|_\infty},\qquad \overline{T}=\frac{1}{(p+1)(\mu-\mu_0(p))},\qquad c_p=\frac{(p+1)^{p+1}}{p^p},\qquad L=p(p+2)-K,$$

$$A_0=\sqrt{\tfrac{p(1+K)+KL}{p(1+K)^2}},\ \ A_1=\arctan\Big(\sqrt{\tfrac{p(1+K)}{L}}+K\Big),\ \ A_2=\arctan\Big(\sqrt{\tfrac{p}{L}}\Big),\ \ A_3=\arctan\Big(\tfrac{1}{\sqrt{K\mu\beta^2}}\Big),$$

$$\delta_1(K)=\frac{A_1}{A_0\sqrt{K\mu}},\quad \delta_2(\beta,K)=\frac{A_3-A_2}{\sqrt{K\mu}},\quad r_0=1-\delta_1-\delta_2,\quad r_1=1-\delta_2,\quad \alpha=\frac{p+1}{(1+K)A_0^2},$$

$$D_2=\left(1+\frac{1}{K\mu\beta^2}\right)^{\frac{p+1}{2}},\quad D_1=D_2\frac{D_{11}^{\alpha}}{D_{12}^{p+1}},\quad D_{11}=\sqrt{1+K+\tfrac{p(1+K)}{L}},\quad D_{12}=\sqrt{1+\tfrac{p}{L}}.$$

We will see in our numerical examples that formula (2.1.3) in practice simplifies to $\rho\approx\frac12 G^*$ once the parameters τ,β,K,λ have been selected (cf. Remark 6.2, and see Figure 3.6 for a plot of the RHS of (2.1.5) as a function of r).

On the other hand, whereas, in Theorems 1.1-1.2, excluding touchdown on a single interval required the smallness condition to be only assumed in that interval, this is no longer the case here, due to additional arguments in the proof (see Remark 5.7(i) for details). For simplicity we thus made the smallness assumption in the global form (2.1.2).

We now give the corresponding global result in the case of multi-bump permittivity profiles (note that Theorems 1.1-1.2 were applicable to multi-bump profiles as well, in view of their local character).

Theorem 2.2. *Consider problem* (1.1.1) *with* $p>0$ *and* $\Omega=(-R,R)$, *let* $-R<x_m<\cdots<x_0<R$ *for some* $m\geq 1$, *and assume*

$$d_0:=\min_{0\le i\le m} R-|x_i|-1>0,\qquad d_1:=\min_{0\le i\le m-1}\tfrac12|x_i-x_{i+1}|-1>0.$$

Then the result of Theorem 2.1 remains valid with

$$J:=\bigcup_{0\le i\le m}\big[x_i-1,x_i+1\big],\qquad D:=\overline{\Omega}\setminus\bigcup_{0\le i\le m}\big(x_i-1-d,x_i+1+d\big),$$

and d_0 *replaced with* $d_2:=\min(d_0,d_1)$ *in formulas* (2.1.1) *and* (2.1.4).

2.2 Weaker conditions on μ and d_0

We next state a variant of Theorems 1.1-1.2, which is valid under less restrictive conditions on μ and for any $d_0>0$. It actually allows one to handle values of μ which are close to the reference squared pull-in voltage (cf. Discussion 1.3(e)). For instance, in the case $p=2$, for which the reference squared pull-in voltage is known to be comprised between $8/27\sim 0.30$ and $\pi^2/27\sim 0.37$, the following theorem only requires $\mu>\pi^2/27\sim 0.37$, whereas Theorems 1.1-1.2 and 2.1-2.2 respectively required $\mu>2\pi^2/27\sim 0.73$ and $\mu>\pi^2/18\sim 0.55$ (along with additional restrictions on d_0). The corresponding optimization problem for ρ is similar to that in Theorems 1.1-1.2 (and simpler than in Theorems 2.1-2.2), but it now has four instead of three parameters.

Theorem 2.3. *Consider problem* (1.1.1) *with* $p > 0$ *and* $\Omega = (-R, R)$. *Let either* $x_0 \in \Omega$ *or* $x_0, x_1 \in \Omega$ *with* $|x_0| \geq |x_1|$, *and assume*

$$\mu > \mu_0(p) = \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{4}, \qquad d_0 := R - |x_0| - 1 > 0. \tag{2.2.1}$$

Then the conclusions of Theorems 1.1-1.2 remain valid, where ρ *can now be chosen as the solution of the following optimization problem.*

$$\rho = \operatorname*{Sup}_{(\tau,\beta,K,\eta)\in\mathcal{A}} \min\left\{\rho_1(\tau,\beta,K,\eta), \frac{1}{p+1}\frac{\tau^{p+1}}{(T-t_0(\tau))\mu}\right\} \tag{2.2.2}$$

where

$$\begin{aligned}\rho_1(\tau,\beta,K,\eta) = \frac{1}{2}\left(\frac{\beta-d}{\beta}\right)^{p+1} \min\Bigg\{&\frac{S(\overline{T},0)}{K+\eta^{-p}} G(\overline{T},\beta,K,\eta),\\ &\frac{S(t_0,\beta)}{K+\tau^{-p}} \min\Big[H\big(t_0,\beta\big), G\big(t_0,\beta,K,\eta\big)\Big]\Bigg\},\end{aligned}$$

$$\mathcal{A} = \left\{(\tau,\beta,K,\eta) \in (d,d_0)\times(0,\infty)\times(0,1)^2,\ K \geq \frac{p\eta}{\mu\beta^2} - \frac{1}{(p+1)\eta^p},\ \delta(\beta,K,\eta) \leq 1\right\}. \tag{2.2.3}$$

Here we set

$$t_0(\tau) = \tfrac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}, \qquad \overline{T} = \tfrac{1}{(p+1)(\mu-\mu_0(p))}, \qquad L = 1 + (p+1)K\eta^p,$$

$$\Gamma = \sqrt{\tfrac{(p+1)\eta L}{pK\mu\beta^2}}, \qquad A = \arctan\Gamma, \qquad \alpha = 1 + \tfrac{p}{L},$$

the function H *and* S *are defined by* (1.3.5) *and the functions* G, δ *are defined by*

$$\begin{aligned}&\delta(\beta,K,\eta) = A(1+K\eta^p)\sqrt{\tfrac{(p+1)\eta}{pLK\mu}},\\ &G(t,\beta,K,\eta) = (\Gamma^2+1)^{-\alpha/2} \inf_{0<x<1} \frac{\operatorname{erf}\left(\frac{2-(1-x)\delta}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{(1-x)\delta}{2\sqrt{t}}\right)}{\cos^\alpha(Ax)}.\end{aligned} \tag{2.2.4}$$

In Table 3.2 below, for the physical case $p = 2$, we present some examples of numerical lower estimates of the threshold ratio ρ, based on Theorem 2.3, for values of μ close to the reference squared pull-in voltage ($0.37 < \mu < 0.73$), for which Theorems 1.1-1.2 are not applicable.

μ	$\|f\|_\infty$	d	d_0	$\overline{\rho}$
0.7	0.8	0.01	8	**0.0815**
0.6	0.65	0.05	10	**0.0714**
0.5	0.6	0.001	6	**0.0137**
0.5	0.5	0.01	7	**0.0228**

Table 3.2 – Examples for Theorem 2.3 with $p = 2$.

3 Type I estimate and auxiliary optimization problem

Following the approach in [15] and [2], a key ingredient in the proofs of Theorems 1.1, 1.2 and 2.1-2.3 is the following type I estimate for u away from the boundary, which we here refine in a nontrivial way in order to allow good quantitative estimates.

3.1 The estimate and the auxiliary optimization problem for Theorems 1.1 and 1.2

Proposition 3.1. *Let $R > 1$, $x_0 \in \Omega = (-R, R)$ with $d_0 := R - 1 - |x_0| > 0$, and $\mu > \mu_0(p)$. Assume that f satisfies*

$$f \geq \mu \quad \text{in } (x_0 - 1, x_0 + 1) \tag{3.1.1}$$

and let u be the solution of problem (1.1.1)*. Let $d \in (0, d_0)$, $\tau \in (0, 1)$. Then the touchdown time T of u satisfies*

$$T > t_0 := \frac{1 - \tau^{p+1}}{(p+1)\|f\|_\infty} \tag{3.1.2}$$

and u satisfies the type I estimate

$$\big[1 - u\big(t, x_0 \pm (1+d)\big)\big]^{p+1} \geq (p+1)\overline{\varepsilon}\mu(T - t) \quad \text{for all } t \in [t_0, T), \tag{3.1.3}$$

where

$$\overline{\varepsilon} = \sup_{(\beta,K)\in\mathcal{A}_1} \hat{\varepsilon}(\beta, K), \quad \hat{\varepsilon}(\beta, K) = \frac{1}{2}\left(\frac{\beta - d}{\beta}\right)^{p+1} \frac{S(t_0, \beta)}{K + \tau^{-p}} \min\big\{H(t_0, \beta), G(t_0, \beta, K)\big\}, \tag{3.1.4}$$

$$\mathcal{A}_1 := \left\{(\beta, K) \in (d, d_0) \times (0, \infty),\ \ K \geq \frac{p}{\mu\beta^2} - \frac{1}{p+1},\ \ \delta(\beta, K) \leq 1\right\}, \tag{3.1.5}$$

and H, G, S, δ are defined in (1.3.5)*.*

Remark 3.2. *Proposition 3.1 (and the analogous Propositions 4.2 and 5.6 below), are of course useful only if $\mathcal{A}_1 \neq \emptyset$ (since otherwise $\overline{\varepsilon} = -\infty$ and ((3.1.3)) is void). The condition $\mathcal{A}_1 \neq \emptyset$ will be checked when we apply these propositions in the proofs of Theorems 1.1, 1.2 and 2.1-2.3.*

For the proof of Proposition 3.1 (and of Propositions 4.2 and 5.6), our strategy is to use a parametrized auxiliary function of the form

$$J(t, x) = u_t - \varepsilon\mu a_{\beta,K,\eta}(x - x_0)h(u) \quad \text{in } \Sigma := [t_0, T) \times I_\beta \tag{3.1.6}$$

$$h(u) = (1 - u)^{-p} + K(1 - u)^q, \tag{3.1.7}$$

(cf. Lemma 3.3), where I_β is a subinterval of Ω, namely $I_\beta := [x_0 - 1 - \beta, x_0 + 1 + \beta]$ with $\beta > d$, and $a_{\beta,K,\eta}$ is a suitable family of cut-off functions with $a(1 + \beta) = 0$ (see in particular Figure 3.5). We shall assume for simplicity that a is an even function, with

$$a > 0 \text{ and } a' \leq 0 \text{ on } [0, 1 + \beta). \tag{3.1.8}$$

A key feature in order to reach good values of the threshold ratio ρ is the possibility to optimize with respect to the various parameters which appear in the function J, namely:

$$\beta, t_0, q, K, \varepsilon,$$

as well as η, which defines the subregions of Σ in (3.3.10).

3.2 Basic computation for the function J

The basic computation for the function J is contained in the following lemma. This computation was already done in [15] and [2] for specific choices of the parameters q and K (and we recall that earlier versions of functions of type J, without cut-off and perturbation terms, go back to [26], [6], [9]). In this paper, varying these parameters will be useful in the proof of Propositions 3.1, 4.2 and 5.6.

Lemma 3.3. *Let ω be a subdomain of Ω and let $a \in C^2(\omega)$ be a positive function. Let u be the solution of (1.1.1), $t_0 \in [0, T)$, and let J be given by (3.1.6) in $(t_0, T) \times \omega$, where*

$$h(u) = (1-u)^{-p} + K(1-u)^q, \qquad 0 \le u < 1, \tag{3.2.1}$$

with $q \in [0,1]$ and

$$0 < K < \frac{p(p+1)}{q(1-q)}. \tag{3.2.2}$$

Then

$$J_t - J_{xx} - pf(x)(1-u)^{-p-1}J = \mu\varepsilon\Theta \qquad in\ (t_0,T)\times\omega, \tag{3.2.3}$$

where

$$\Theta = (p+q)Ka(x)f(x)(1-u)^{-p+q-1} + ah''(u)u_x^2 + 2h'(u)a_xu_x + h(u)a_{xx}. \tag{3.2.4}$$

Moreover, we have $h''(u) > 0$ for all $u \in [0,1)$ and

$$\Theta \ge \underbrace{(p+q)Ka(x)f(x)(1-u)^{-p+q-1}}_{\tau_1} + \underbrace{h(u)a_{xx}(x)}_{\tau_2} - \underbrace{\frac{h'^2(u)a_x^2(x)}{a(x)h''(u)}}_{\tau_3}. \tag{3.2.5}$$

Proof. We compute

$$\begin{aligned}
J_t &= u_{tt} - \varepsilon a(x)h'(u)u_t,\\
J_x &= u_{xt} - \varepsilon\mu\big(a(x)h'(u)u_x + h(u)a_x(x)\big),\\
J_{xx} &= u_{xxt} - \varepsilon\mu\big(a(x)h'(u)u_{xx} + a(x)h''(u)u_x^2 + 2h'(u)a_x(x)u_x + h(u)a_{xx}(x)\big).
\end{aligned}$$

Setting $g(u) = (1-u)^{-p}$ and omitting the variables x, u without risk of confusion, we get

$$\begin{aligned}
J_t - J_{xx} &= (u_t - u_{xx})_t - \varepsilon\mu ah'(u_t - u_{xx}) + \varepsilon\mu(ah''u_x^2 + 2h'a_xu_x + ha_{xx})\\
&= fg'u_t - \varepsilon\mu fah'g + \varepsilon\mu(ah''u_x^2 + 2h'a_xu_x + ha_{xx}).
\end{aligned}$$

Using $u_t = J + \varepsilon\mu ah$, we have

$$J_t - J_{xx} - fg'J = \varepsilon\mu\Theta,$$

where

$$\Theta = fa(g'h - h'g) + ah''u_x^2 + 2h'(u)a_xu_x + ha_{xx}.$$

On the other hand, we have

$$h'(u) = p(1-u)^{-p-1} - qK(1-u)^{q-1}, \tag{3.2.6}$$

hence

$$\begin{aligned}
g'h - h'g &= p(1-u)^{-p-1}[(1-u)^{-p} + K(1-u)^q]\\
&\quad -(1-u)^{-p}[p(1-u)^{-p-1} - qK(1-u)^{q-1}]\\
&= (p+q)K(1-u)^{-p+q-1},
\end{aligned}$$

which yields (3.2.4). Also, owing to (3.2.2), we have

$$\begin{aligned} h'' &= [p(p+1) - q(1-q)K(1-u)^{p+q}](1-u)^{-p-2} \\ &\geq [p(p+1) - q(1-q)K](1-u)^{-p-2} > 0. \end{aligned} \tag{3.2.7}$$

Finally, since $a > 0$ in ω, we may write

$$\Theta = (p+q)Kaf(1-u)^{-p+q-1} + ha_{xx} + ah'' \left[u_x^2 + 2\frac{h' a_x u_x}{ah''} \right].$$

Since $u_x^2 + 2\frac{h' a_x u_x}{ah''} \geq -\frac{h'^2 a_x^2}{a^2 (h'')^2}$, inequality (3.2.5) follows. □

Remark 3.4. *(a) We observe that no loss of information seems to occur from inequality* (3.2.5)*. Indeed, by* (3.2.4)*, this inequality becomes an equality at any point x such that $u_x + \frac{h' a_x}{h'' a} = 0$ i.e., $[\log(ah'(u)]_x = 0$. But since, in order to apply the maximum principle in the proofs of Proposition 3.1 (and Propositions 4.2 and 5.6 below), the function a will be required to vanish on $\partial\omega$, such points x must exist for each $t \in (0,T)$.*

(b) The restriction $q \leq 1$ is necessary to guarantee that the key term $h''(u)u_x^2$ in (3.2.4) *remains positive. Similarly, the positivity of the key term τ_1 in* (3.2.5) *imposes $p+q > 0$. Although the values $q \in (-p, 0)$ would be also admissible, we shall not consider them. Indeed, when looking for quantitative estimates in Section 6, they seem to lead to worse results due to smaller constant $p+q$ (and to more complicated expressions than $q = 0$ or $q = 1$).*

We also recall the following simple lemma, that will be used in the sequel.

Lemma 3.5. *Let u be the solution of* (1.1.1)*.*

(i) We have $T \geq T_* := \frac{1}{(p+1)\|f\|_\infty}$ and

$$\|u(t)\|_\infty \leq y(t) := 1 - \big(1 - (p+1)\|f\|_\infty t\big)^{\frac{1}{p+1}}, \qquad \text{for all } t \in [0, T_*]. \tag{3.2.8}$$

(ii) Assume that $I := (x_0-1, x_0+1) \subset \Omega$ and $f \geq \mu\chi_I$, with $\mu > \mu_0(p) := \frac{p^p}{(p+1)^{p+1}} \frac{\pi^2}{4}$. Then

$$T \leq \overline{T} := \frac{1}{(p+1)(\mu - \mu_0(p))} < \infty.$$

Proof. Since $y(t)$ is the solution of the ODE

$$y'(t) = \|f\|_\infty (1 - y(t))^{-p}, \quad t \in (0, T_*), \qquad \text{with } y(0) = 0, \tag{3.2.9}$$

and T_* is the maximal existence time for $y(t)$, assertion (i) follows immediately from the comparison principle.

Assertion (ii) follows from a simple eigenfunction argument, see e.g., Lemma 2.2 in [2]. □

3.3 Construction of the family of cut-off functions $a(x)$ and parametrized type I estimate

The function J needs to satisfy a basic parabolic inequality (cf. (3.3.13) below). In one space dimension, the study of this parabolic inequality can be made quite precise.

It actually leads to the following, natural and optimal, differential inequality for the function $a(r)$:

$$a''(r) \geq a(r)\, F\Big(r, \frac{a'(r)}{a(r)}\Big), \qquad 0 \leq r < 1+\beta, \tag{3.3.1}$$

where

$$F(r,\xi) = \begin{cases} \displaystyle\sup_{u\in(0,1)} \left[\frac{h'^2(u)}{hh''(u)}\right] \xi^2, & r > 1, \\ \displaystyle\sup_{u\in(1-\eta,1)} \left[\frac{h'^2(u)}{hh''(u)}\,\xi^2 - \frac{(p+q)K\mu}{(1-u)^{p+1-q}h(u)}\right], & r < 1. \end{cases} \tag{3.3.2}$$

This is the contents of the following lemma which gives a family of type I estimates, corresponding to each admissible value of the parameters. We note that only the choice $q = 0$ and $\eta = 1$ will be used in the proof of Proposition 3.1 and Theorems 1.1-1.2. Other values of the parameters q, η will be used in the proofs of the results of Section 2.

Lemma 3.6. *Let $R > 1$, $x_0 \in \Omega = (-R,R)$ with $d_0 := R-1-|x_0| > 0$, and $\mu > \mu_0(p)$. Assume that f satisfies* (3.1.1) *and let u be the solution of problem* (1.1.1). *Let*

$$q \in [0,1], \quad \tau \in (0,1), \quad \eta \in (0,1], \quad \beta \in (0,d_0), \quad K > 0 \tag{3.3.3}$$

and let $h(u)$ be defined by (3.1.7). *Set $I_0 = (x_0-1, x_0+1)$, $I_\beta = (x_0-1-\beta, x_0+1+\beta)$ and*

$$t_0 = t_0(\tau) = \frac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}.$$

Assume that there exists a solution $a \in W^{2,2}([0,1+\beta])$ of (3.3.1)-(3.3.2), *with* (3.1.8) *and the boundary conditions*

$$a'(0) = 0, \quad a(1+\beta) = 0. \tag{3.3.4}$$

Assume also that $\varepsilon > 0$ satisfies

$$\varepsilon \leq \varepsilon_1 := \inf_{x\in I_\beta} \frac{e^{t_0\Delta_\Omega}\chi_{I_0}(x)}{h(u(t_0,x))a(|x-x_0|)} \tag{3.3.5}$$

where $e^{t\Delta_\Omega}$ denotes the Dirichlet heat semigroup on Ω, and

$$\eta = 1 \quad \text{or} \quad \varepsilon \leq \varepsilon_2 := \frac{1}{W(q,\eta,K)} \inf_{(t,x)\in[t_0,\overline{T}]\times I_0} \frac{e^{t\Delta_\Omega}\chi_{I_0}(x)}{a(|x-x_0|)}, \tag{3.3.6}$$

with $\overline{T} = \frac{1}{(p+1)(\mu-\mu_0(p))}$ and $W(q,\omega,K) := \sup_{u\in(0,1-\omega)} h(u)$. Then

$$(1-u(t,x))^{p+1} \geq (p+1)\varepsilon\mu a(|x-x_0|)(T-t) \quad \text{in } [t_0,T)\times I_\beta. \tag{3.3.7}$$

By Lemma 3.5(i) we have $\|u(t_0)\|_\infty \leq 1-\tau$, so we can estimate

$$h(u(t_0,x)) \leq W(q,\tau,K). \tag{3.3.8}$$

We use this estimate in Theorems 1.1, 1.2, 2.3 for simplicity. However, in Theorems 2.1-2.2 we will use a better upper estimate of u by taking advantage of the smallness hypothesis in (2.1.2).

By the monotonicity properties of the function $h(u)$, we note that if $p \geq qK$, then

$$W(q,\omega,K) = h(1-\omega) = \omega^{-p} + K\omega^q, \quad \omega \in (0,1). \tag{3.3.9}$$

This will be the case in the proofs of Theorems 1.1, 1.2 and 2.3 since we are considering $q = 0$. For Theorems 2.1-2.2 we will also restrict ourselves to this case for simplicity and since we have observed numerically that the optimal choice of K is less than p.

Proof. Set $\Sigma := [t_0, T) \times I_\beta$ and let

$$J(t,x) = u_t - \varepsilon\mu b(x)h(u) \quad \text{in } \Sigma,$$

where

$$h(u) = (1-u)^{-p} + K(1-u)^q, \qquad b(x) = a(x - x_0).$$

We split the cylinder Σ into three subregions as follows:

$$\begin{array}{rcl} \Sigma_1 & = & [t_0, T) \times (I_\beta \setminus \bar{I}_0), \\ \Sigma_2^\eta & = & \{(t,x) \in [t_0,T) \times \bar{I}_0;\ u(t,x) \geq 1-\eta\}, \\ \Sigma_3^\eta & = & \{(t,x) \in [t_0,T) \times \bar{I}_0;\ u(t,x) < 1-\eta\}. \end{array} \tag{3.3.10}$$

By Lemma 3.3 (still valid for $a \in W^{2,2}([0, 1+\beta])$), we have

$$J_t - J_{xx} - pf(x)(1-u)^{-p-1}J \geq \varepsilon\tilde{\Theta} \quad \text{a.e. in } (t_0,T) \times I_\beta, \tag{3.3.11}$$

where

$$\tilde{\Theta} = (p+q)Kf(x)b(x)(1-u)^{-p+q-1} + b''(x)h - \frac{|b'(x)|^2 h'^2}{b(x)h''}. \tag{3.3.12}$$

Since our only lower assumption on f is $f \geq \mu\chi_{I_0}$ the property $\tilde{\Theta} \geq 0$ a.e. in Σ_1 (resp. Σ_2^η) amounts to requiring

$$\frac{(p+q)K\mu\chi_{(-1,1)}(r)}{(1-u)^{p+1-q}h(u)} + \frac{a''(r)}{a(r)} \geq \left[\frac{a'(r)}{a(r)}\right]^2 \frac{h'^2(u)}{hh''(u)}$$

for a.e. $(r,u) \in (1, 1+\beta) \times (0,1)$ (resp., $[0,1) \times [1-\eta, 1)$). Since we assumed that a solves (3.3.1)-(3.3.2), this precisely guarantees that

$$J_t - J_{xx} - pf(x)(1-u)^{-p-1}J \geq 0 \quad \text{a.e. in } \Sigma_1 \cup \Sigma_2^\eta. \tag{3.3.13}$$

On the other hand, we claim that

$$u_t \geq \mu e^{t\Delta_\Omega}\chi_{I_0} \qquad \text{in } [0,T) \times \Omega. \tag{3.3.14}$$

The claim follows from the comparison principle applied to the function $v = u_t$, which solves the problem

$$\begin{cases} v_t - v_{xx} = pf(x)(1-u)^{-p-1}v, & \text{in } (0,T) \times \Omega, \\ v = 0, & \text{in } [0,T) \times \partial\Omega, \\ v(0,x) = f(x), & \text{in } \Omega \end{cases} \tag{3.3.15}$$

(see Lemma 3.4 in [2]). In view of (3.3.5), (3.3.14), we have

$$J(t_0,x) \geq \mu e^{t_0\Delta_\Omega}\chi_{I_0}(x) - \varepsilon\mu h(u(t_0,x))a(x-x_0) \geq 0 \quad \text{in } I_\beta. \tag{3.3.16}$$

Next observe that if $\eta = 1$, then the subregion Σ_3^η is empty. If $\eta \in (0,1)$, using assumption (3.3.6) and $T \leq \overline{T}$ by Lemma 3.5(ii), we have

$$J(t,x) = u_t - \varepsilon\mu b(x)h(u) \geq \mu e^{t\Delta_\Omega}\chi_{I_0}(x) - \varepsilon\mu a(x-x_0)W(q,\eta,K) \geq 0 \quad \text{in } \Sigma_3^\eta. \tag{3.3.17}$$

Now, as a consequence of (3.3.17) and $\Sigma = \Sigma_1 \cup \Sigma_2^\eta \cup \Sigma_3^\eta$, we have

$$\{(t,x) \in \Sigma;\quad J(t,x) < 0\} \subset \Sigma_1 \cup \Sigma_2^\eta. \tag{3.3.18}$$

Also, since $b = 0$ on ∂I_β, we have

$$J \geq 0 \quad \text{on } [t_0, T) \times \partial I_\beta. \tag{3.3.19}$$

On the other hand, using standard parabolic regularity, we observe that $J \in C^1\big([t_0, T) \times \overline{I_\beta}\big)$, with $J_{xx} \in L^2(\Sigma)$. It then follows from (3.3.13), (3.3.16), (3.3.18), (3.3.19) and the maximum principle (see, e.g., [25, Proposition 52.8 and Remark 52.11(a)]) that

$$J \geq 0 \quad \text{in } \Sigma.$$

Integrating in time we obtain

$$(1 - u(t,x))^{p+1} \geq (p+1)\varepsilon\mu b(x)(T - t) \quad \text{in } \Sigma,$$

which concludes the proof of the lemma. □

Our next task is to identify appropriate solutions of the differential inequality (3.3.1)-(3.3.2). First of all, without loss of generality, we may assume the normalization condition

$$a(1) = 1. \tag{3.3.20}$$

Indeed, the inequality (3.3.1), the boundary conditions (3.3.4) and (owing to assumptions (3.3.5), (3.3.6)) the estimate (3.3.7) are not affected by multiplication of a by a positive constant.

Now, it can be shown (see Proposition 8.1) that, among all possible solutions of (3.3.1), (3.3.2), (3.3.4), the optimal choice is to actually look for a solution of the corresponding ODE:

$$a''(r) = a(r)\, F\Big(r, \frac{a'(r)}{a(r)}\Big), \quad r_0 \leq r < 1 + \beta, \tag{3.3.21}$$

for some $r_0 \in [0, 1)$, with boundary conditions

$$a'(r_0) = 0, \quad a(1 + \beta) = 0, \tag{3.3.22}$$

and to extend it to be constant on the remaining part of the interval:

$$a(r) = a(r_0), \quad 0 \leq r \leq r_0. \tag{3.3.23}$$

Indeed, Proposition 8.1 shows that, fixing the reference value in (3.3.20), the ratios $\varepsilon_1, \varepsilon_2$ in (3.3.5), (3.3.6) are largest when the function $a(r)$ is chosen in this way. In order not to interrupt the main line of argument, Proposition 8.1 and its proof are postponed to Appendix 2.

Next, it turns out that, for the special values $q = 0$ and $q = 1$, the solution of (3.3.20)-(3.3.23) can be explicitly computed whenever it exists. We start with the case $q = 0$ (keeping $\eta \in (0, 1]$ for future use in the proof of Theorem 2.3.) As for the more complicated case $q = 1$, it will be studied in Section 5.1 for the proof of Theorems 2.1-2.2.

Lemma 3.7. *Let $\beta > 0$, $q = 0$, $\eta \in (0, 1]$, $K > 0$ and assume*

$$K \geq \frac{p\eta}{\mu\beta^2} - \frac{1}{(p+1)\eta^p} \tag{3.3.24}$$

Let h, F be given by (3.1.7), (3.3.2) and set

$$m = \frac{p}{(p+1)(1 + K\eta^p)}, \quad M = \frac{pK\mu}{\eta(1 + K\eta^p)}, \quad \delta_0 = \frac{\arctan\Big(\frac{p+1}{\beta}\sqrt{\frac{1-m}{M}}\Big)}{\sqrt{M(1-m)}}.$$

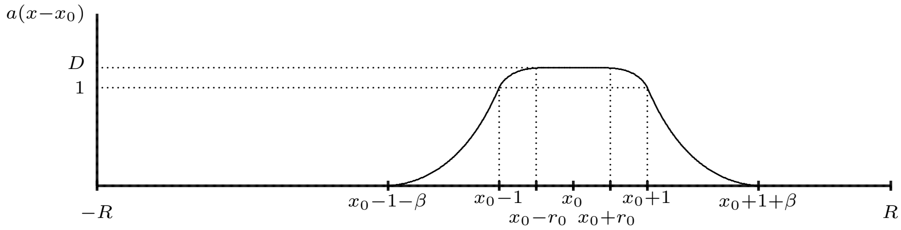

Figure 3.4 – The function $a_{\beta,K,\eta}(x-x_0)$.

There exist $r_0 \in [0,1)$ and a solution $a \in W^{2,2}([0,1+\beta])$ of (3.3.20)-(3.3.23), (3.1.8) if and only if $\delta_0 \le 1$. The couple (r_0, a) is then unique and it is given by

$$a_{\beta,K,\eta}(r) = \begin{cases} D, & r \in [0, r_0), \\ D\cos^{\frac{1}{1-m}}\left(\sqrt{M(1-m)}(r-r_0)\right), & r \in [r_0, 1], \\ \left(\dfrac{1+\beta-r}{\beta}\right)^{p+1}, & r \in (1, 1+\beta], \end{cases} \tag{3.3.25}$$

where $r_0 = 1 - \delta_0$ and $D = \left[1 + \frac{1-m}{M}\left(\frac{p+1}{\beta}\right)^2\right]^{\frac{1}{2(1-m)}}$.

Proof. Step 1: Determination of a in $[1, 1+\beta]$. Since

$$hh'' = p(p+1)(1-u)^{-2p-2}\left[1 + K(1-u)^p\right], \quad h'^2 = p^2(1-u)^{-2p-2},$$

we have

$$\sup_{u\in(0,1)} \left[\frac{h'^2(u)}{hh''(u)}\right] = \frac{p}{p+1}. \tag{3.3.26}$$

We are thus left with the ODE

$$aa'' = \frac{p}{p+1}a'^2,$$

with boundary conditions $a(1) = 1$, $a(1+\beta) = 0$. It is easy to solve this by setting $\phi = a^{1-\sigma}$, $\sigma = p/(p+1) \in (0,1)$, hence $\phi' = (1-\sigma)a^{-\sigma}a'$ and

$$\phi'' = (1-\sigma)a^{-\sigma-1}[aa'' - \sigma(a')^2] = 0.$$

This leads to $\phi(r) = (1+\beta-r)/\beta$, hence $a(r) = \phi^{p+1}(r)$ given by the last part of (3.3.25).

Step 2: Determination of a in $[0,1]$ and existence condition. For $r \in [0,1]$, we compute

$$\begin{aligned} F(r,\xi) &= \sup_{u\in(1-\eta,1)} \left[\frac{h'^2(u)}{hh''(u)}\,\xi^2 - \frac{pK\mu}{(1-u)^{p+1}h(u)}\right] \\ &= \sup_{y\in(0,\eta)} \left[\frac{p\xi^2}{(p+1)\left(1+Ky^p\right)} - \frac{pK\mu}{y\left(1+Ky^p\right)}\right] \\ &= \sup_{y\in(0,\eta)} Z(\xi,y), \quad \text{where } Z(\xi,y) = \left(\frac{\xi^2}{p+1} - \frac{K\mu}{y}\right)\frac{p}{1+Ky^p}. \end{aligned}$$

Computing

$$\frac{\partial Z}{\partial y} = \frac{K\mu}{y^2}\frac{p}{1+Ky^p} - \left(\frac{\xi^2}{p+1} - \frac{K\mu}{y}\right)\frac{p^2Ky^{p-1}}{(1+Ky^p)^2},$$

we see that

$$\frac{\partial Z}{\partial y} \geq 0 \Leftrightarrow \frac{\mu(1+Ky^p)}{y} \geq \Big(\frac{\xi^2}{p+1} - \frac{K\mu}{y}\Big) py^p \Leftrightarrow \xi^2 \leq \frac{(p+1)\mu}{py}\Big[\frac{1}{y^p} + K(p+1)\Big].$$

Consequently, for $r \in [0,1]$, we have

$$F(r,\xi) = \Big(\frac{\xi^2}{p+1} - \frac{K\mu}{\eta}\Big)\frac{p}{1+K\eta^p}, \qquad \text{for } \xi^2 \leq \frac{(p+1)\mu}{p\eta}\Big[\frac{1}{\eta^p} + K(p+1)\Big]. \tag{3.3.27}$$

Furthermore, owing to (3.3.2) and (3.3.26), any solution of (3.3.21) on some interval $[r_0, 1]$ with $a > 0$ and $a' \leq 0$ must satisfy

$$\Big(\frac{a'}{a}\Big)' = \frac{a''}{a} - \Big(\frac{a'}{a}\Big)^2 \leq 0$$

hence, by the C^1 continuity conditions

$$a(1_-) = 1, \qquad a'(1_-) = a'(1_+) = -\frac{p+1}{\beta}, \tag{3.3.28}$$

we must have

$$\Big|\frac{a'(r)}{a(r)}\Big| \leq \Big|\frac{a'(1)}{a(1)}\Big| = \frac{p+1}{\beta}, \quad r_0 \leq r \leq 1. \tag{3.3.29}$$

By (3.3.27), (3.3.29) and assumption (3.3.24), we are thus left with the ODE

$$aa'' = m{a'}^2 - Ma^2, \quad r \leq 1, \tag{3.3.30}$$

with

$$m = \frac{p}{(p+1)(1+K\eta^p)} \in (0,1), \quad M = \frac{pK\mu}{\eta(1+K\eta^p)}.$$

Setting $\phi = a^{1-m}$, (3.3.30) is equivalent to

$$\phi'' = (1-m)a^{-m-1}[aa'' - m(a')^2] = -(1-m)M\phi. \tag{3.3.31}$$

Since all solutions of (3.3.31) are given by cosine functions, and since we are looking for a solution ϕ such that $\phi'(1) < 0$, ϕ' must have a first zero on the left of $r = 1$. Since we also impose $\phi > 0$, $\phi' \leq 0$ on $[r_0, 1]$ and $\phi'(r_0) = 0$, this first zero must coincide with r_0 and the solution must be of the form

$$\phi(r) = D_0 \cos\big[\sqrt{(1-m)M}(r - r_0)\big], \tag{3.3.32}$$

for some $D_0 > 0$, and we must have

$$\sqrt{(1-m)M}(1 - r_0) < \pi/2. \tag{3.3.33}$$

This, along with (3.3.28), yields

$$1 = \phi(1) = D_0 \cos\big[\sqrt{(1-m)M}(1 - r_0)\big] \tag{3.3.34}$$

and

$$-(1-m)(p+1)\beta^{-1} = \phi'(1) = -D_0\sqrt{(1-m)M}\sin\big[\sqrt{(1-m)M}(1 - r_0)\big],$$

hence

$$\tan\big[\sqrt{(1-m)M}(1 - r_0)\big] = \frac{p+1}{\beta}\sqrt{\frac{1-m}{M}}. \tag{3.3.35}$$

It follows that

$$1 - r_0 = \delta_0 := \frac{1}{\sqrt{M(1-m)}} \arctan\left(\frac{p+1}{\beta}\sqrt{\frac{1-m}{M}}\right) \tag{3.3.36}$$

(which in particular guarantees (3.3.33)) and, by (3.3.34), (3.3.35), we have

$$D_0 = \frac{1}{\cos\big[\sqrt{(1-m)M}(1-r_0)\big]} = \sqrt{1+\tan^2\big[\sqrt{(1-m)M}(1-r_0)\big]},$$

hence

$$D_0 = \sqrt{1 + \frac{1-m}{M}\left(\frac{p+1}{\beta}\right)^2}. \tag{3.3.37}$$

From the above, we see that the existence of a solution of (3.3.20)-(3.3.23), (3.1.8) for some $r_0 \in [0,1)$ is equivalent to $\delta_0 \le 1$. Under this condition, we deduce from (3.3.32) and (3.3.37) that $a(r)$ is given by (3.3.25). Moreover, in view of (3.3.36), the couple (r_0, a) is then unique. The lemma is proved. □

3.4 Proof of Proposition 3.1

To complete the proof of Proposition 3.1, it essentially remains to express the infima in (3.3.5) and (3.3.6) in terms of the error function.

This relies on quantitative comparison estimates for the Dirichlet heat semigroups, given in Proposition 7.1, combined with the following elementary lemma.

Lemma 3.8. *Denote by $e^{t\Delta_{\mathbb{R}}}$ the heat semigroup on $\mathbb{R}$. Let the functions G, H, δ be defined by* (1.3.5). *Let $\beta, K > 0$ satisfy $K \ge \frac{p}{\mu\beta^2} - \frac{1}{p+1}$ and $\delta(\beta, K) \le 1$, and let $a(x)$ be defined by* (3.3.25) *with $\eta = 1$. We have*

$$\inf_{0<r<1} \frac{e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)}{a(r)} = \frac{1}{2}G(t,\beta,K) \tag{3.4.1}$$

and

$$\inf_{1<r<1+\beta} \frac{e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)}{a(r)} = \frac{1}{2}H(t,\beta), \quad t>0. \tag{3.4.2}$$

Moreover,

$$t \mapsto G(t,\beta,K) \ \text{ is nonincreasing for } t>0. \tag{3.4.3}$$

Proof. Denote by L_1 the LHS of (3.4.1). We note that $a(r)$ is constant in the interval $[0, r_0)$. Since $e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)$ is even and monotonically decreasing for $r>0$, we thus need only consider the interval $[r_0, 1)$. Using the definition of $a(r)$ in this interval and the heat kernel on the real line, we obtain

$$L_1 = \frac{1}{\sqrt{4\pi t}} \inf_{r_0<r<1} \frac{1}{a(r)} \int_{-1}^{1} e^{-\frac{(r-y)^2}{4t}}\,dy.$$

After the change of variable $Z = \frac{r-y}{2\sqrt{t}}$, and then $x = \frac{r-r_0}{1-r_0}$, we obtain

$$L_1 = \frac{1}{\sqrt{\pi}} \inf_{r_0<r<1} \frac{1}{a(r)} \int_{\frac{r-1}{2\sqrt{t}}}^{\frac{r+1}{2\sqrt{t}}} e^{-Z^2}\,dZ = \frac{1}{2}\inf_{0<x<1} \frac{\operatorname{erf}\left(\frac{1+r_0+(1-r_0)x}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{1-r_0-(1-r_0)x}{2\sqrt{t}}\right)}{a((1-r_0)x+r_0)}.$$

Comparing (3.3.25), where $\eta = 1$, with (1.3.5), we deduce

$$L_1 = \frac{1}{2} \inf_{0<x<1} \frac{\operatorname{erf}\left(\frac{2-(1-x)\delta_0}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{(1-x)\delta_0}{2\sqrt{t}}\right)}{D\cos^\alpha(Ax)} = \frac{1}{2} G(t,\beta,K).$$

Now denote by L_2 the LHS of (3.4.2). Using the definition of $a(r)$ in this interval and the heat kernel on the real line, we find

$$L_2 = \frac{1}{\sqrt{4\pi t}} \inf_{1<r<1+\beta} \left(\frac{\beta}{1+\beta-r}\right)^{p+1} \int_{-1}^{1} e^{-\frac{(r-y)^2}{4t}}\,dy.$$

After the changes of variables $X = \dfrac{r-1}{\beta}$, $Y = \dfrac{y-1}{\beta}$, and then $Z = \dfrac{\beta(Y-X)}{2\sqrt{t}}$, we obtain

$$\begin{aligned} L_2 &= \frac{1}{\sqrt{4\pi t}} \inf_{0<X<1} \frac{\beta \int_{-\frac{2}{\beta}}^{0} e^{-\frac{(X-Y)^2}{4t}\beta^2}\,dY}{(1-X)^{p+1}} = \frac{1}{\sqrt{\pi}} \inf_{0<X<1} \frac{\int_{X\frac{\beta}{2\sqrt{t}}}^{\left(\frac{2}{\beta}+X\right)\frac{\beta}{2\sqrt{t}}} e^{-Z^2}\,dZ}{(1-X)^{p+1}} \\ &= \frac{1}{2} \inf_{0<X<1} \frac{\operatorname{erf}\left((\frac{2}{\beta}+X)\frac{\beta}{2\sqrt{t}}\right) - \operatorname{erf}\left(X\frac{\beta}{2\sqrt{t}}\right)}{(1-X)^{p+1}} = \frac{1}{2} H(t,\beta). \end{aligned}$$

Let us finally verify (3.4.3). For any $t > 0$ and $x \in [-1,1]$, using the change of variables $x - y = z\sqrt{4t}$, we obtain

$$e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(x) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4t}}\,dy = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4t}}}^{\frac{x+1}{\sqrt{4t}}} e^{-z^2}\,dz.$$

Since $x-1 \le 0 \le x+1$, this quantity is nonincreasing as t increases and property (3.4.3) then immediately follows from (3.4.1). $\square$

Proof of Proposition 3.1. Let $d \in (0, d_0)$, $\tau \in (0,1)$ and let $(\beta, K) \in \mathcal{A}_1$, $\eta = 1$. Estimate (3.1.2) follows from Lemma 3.5(i). We may apply Lemma 3.6 for $q = 0, \eta = 1$ with the function $a(r)$ given by Lemma 3.7; see Figure 3.5.

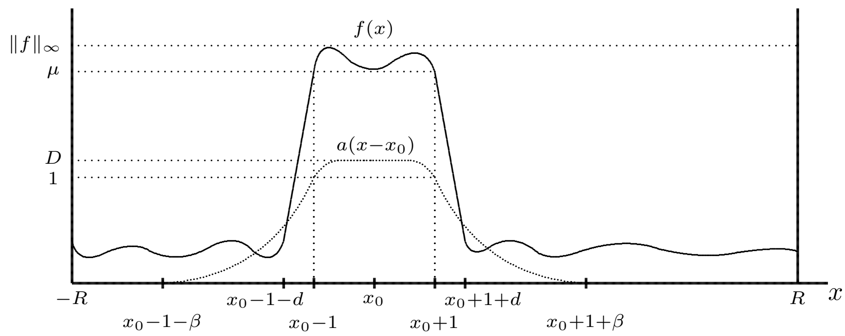

Figure 3.5 – Location of the support of the cut-off function a in the proof of Proposition 3.1 (d is fixed and β is one of the optimization parameters).

It follows from (3.3.7) and (3.3.25) for $\eta = 1$ that, for all $t \in [t_0, T)$,

$$\left[1 - u\big(t, x_0 \pm (1+d)\big)\right]^{p+1} \ge (p+1)\left(\frac{\beta-d}{\beta}\right)^{p+1} \varepsilon\,\mu(T-t), \qquad (3.4.4)$$

where $\varepsilon = \varepsilon_1$, given in (3.3.5).

On the other hand, assuming $x_0 \geq 0$ without loss of generality and recalling the definition of S in (1.3.5), it follows from the comparison properties for the Dirichlet heat semigroup in Proposition 7.1 that, for all $\lambda \in [1, R - x_0]$,

$$\begin{aligned}\left(e^{t\Delta_\Omega}\chi_{I_0}\right)(x) &\geq e^{-\frac{\pi^2}{4(R-x_0)^2}}\left[1 - e^{-(R-x_0-1)(R-x_0-\lambda)/t}\right]\left(e^{t\Delta_{\mathbb{R}}}\chi_{I_0}\right)(x)\\ &= S(t, \lambda - 1)\left(e^{t\Delta_{\mathbb{R}}}\chi_{I_0}\right)(x) \quad \text{for all } x \in [x_0 - \lambda, x_0 + \lambda].\end{aligned}$$

This along with estimate (3.3.8) guarantees that

$$\begin{aligned}\inf_{|x-x_0|\leq\lambda} \frac{[e^{t_0\Delta_\Omega}\chi_{I_0}](x)}{h(u(t_0,x))a(|x-x_0|)} &\geq \frac{S(t_0,\lambda-1)}{W(0,\tau,K)} \inf_{|x-x_0|\leq\lambda} \frac{[e^{t_0\Delta_{\mathbb{R}}}\chi_{I_0}](x)}{a(|x-x_0|)}\\ &= \frac{S(t_0,\lambda-1)}{W(0,\tau,K)} \inf_{|y|\leq\lambda} \frac{[e^{t_0\Delta_{\mathbb{R}}}\chi_{(-1,1)}](y)}{a(|y|)}.\end{aligned} \tag{3.4.5}$$

Applying (3.4.5) with $\lambda = 1 + \beta$, (3.3.9), (3.4.1) and (3.4.2), it follows that

$$\varepsilon_1 \geq \frac{1}{2}\frac{S(t_0,\beta)}{K + \tau^{-p}} \min\{G(t_0,\beta,K), H(t_0,\beta)\}. \tag{3.4.6}$$

Combining (3.4.4) and (3.4.6), the conclusion then follows by taking the supremum over $(\beta, K) \in \mathcal{A}_1$. □

Remark 3.9. *(a) The choice $\eta = 1$ allows one to get rid of condition $\varepsilon \leq \varepsilon_2$ in* (3.3.6)*, leading to an important simplification of the expression for $\hat{\varepsilon}$ in Proposition 3.1.*

(b) Concerning the definition of the subregions in (3.3.10)*, we observe that it would not be of any use to separate the cases $1 - u \geq \eta$ and $1 - u < \eta$ outside the interval $I_0 = (x_0 - 1, x_0 + 1)$. Indeed, this would not lead to a better function $a(x)$ outside I_0, since the supremum in* (3.3.26) *is achieved for $u \sim 1$.*

4 Proof of Theorems 1.1, 1.2 and 2.3

The proofs will use the following no-touchdown criterion, which enables one to exclude touchdown on a given subinterval D of Ω, under a type I estimate on ∂D and a suitable smallness assumption on f in D.

Lemma 4.1. *Let $\Omega = (-R, R)$, $\tau \in (0,1)$ and u be the solution of problem* (1.1.1)*. Let either*

(i) $\qquad D = (x_0 - b, x_0 + b) \subset\subset \Omega \quad$ *and* $\quad \Gamma = \partial D,$

or

(ii) $\qquad D = (a, R)$ *for some* $a \in (-R, R) \quad$ *and* $\quad \Gamma = \{a\}.$

Let $\tau \in (0,1)$, $t_0 = t_0(\tau) = \frac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}$ and assume that

$$(1-u)^{p+1} \geq k(T-t) \quad \text{on } [t_0, T) \times \Gamma \tag{4.0.1}$$

for some $k > 0$. If

$$\|f\|_{L^\infty(D)} < \frac{1}{p+1} \min\left\{k, \frac{\tau^{p+1}}{T - t_0}\right\}, \tag{4.0.2}$$

then $\mathcal{T} \cap D = \emptyset$. In addition, $R \notin \mathcal{T}$ in case (ii).

The proof of this lemma is given in [2] for the case when the type I estimate (4.0.1) is satisfied in the whole time interval $[0, T)$. Here, we make a slight modification of the proof in order to use Propositions 3.1, 4.2, 5.6, where the quantitative type I estimate only holds in the interval $[t_0, T)$.

Proof. We define the comparison function

$$w(t,x) := y(t)\psi(x) \quad \text{for } (t,x) \in [t_0,T) \times \overline{D},$$

where $y(t)$ is defined by

$$y(t) = 1 - A(T-t)^{\frac{1}{p+1}}, \qquad \text{with } A = \min\left\{k^{\frac{1}{p+1}}, \frac{\tau-\sigma}{1-\sigma}(T-t_0)^{-\frac{1}{p+1}}\right\}$$

for some $\sigma \in (0,\tau)$ to be chosen later, and $\psi(x)$ is given by $\psi(x) := 1-\sigma\left(1-\dfrac{(x-x_0)^2}{b^2}\right)$ in case (i), and by $\psi(x) := 1 - \frac{\sigma}{R-a}(x-a)$ in case (ii).

Considering σ small enough, we obtain, in $(t_0,T) \times D$

$$w_t - w_{xx} - f(x)(1-w)^{-p} \geq \left(\frac{A}{p+1}(1-\sigma) - f(x)A^{-p}\right) T^{-\frac{p}{p+1}} - \psi''(x) \geq 0, \quad (4.0.3)$$

noting that $\|f\|_{L^\infty(D)} < \frac{A^{p+1}}{p+1}$ by (4.0.2) for σ small enough.

We next look at the parabolic boundary of $[t_0,T) \times D$. On the one hand, by Lemma 3.5(i), we have

$$w(t_0,x) \geq (1 - A(T-t_0)^{\frac{1}{p+1}})(1-\sigma) \geq 1-\tau \geq u(t_0,x) \quad \text{in } \overline{D}. \quad (4.0.4)$$

On the other hand, using $\psi = 1$ on Γ and $A \leq k^{\frac{1}{p+1}}$, we apply (4.0.1) to obtain

$$w(t,x) = 1 - A(T-t)^{\frac{1}{p+1}} \geq 1 - k^{\frac{1}{p+1}}(T-t)^{\frac{1}{p+1}} \geq u(t,x) \quad \text{in } [t_0,T) \times \Gamma. \quad (4.0.5)$$

In case (ii), we also note that due to the boundary conditions on u, we have

$$w(t,x) \geq 0 = u(t,x) \quad \text{in } [t_0,T) \times \{R\}. \quad (4.0.6)$$

By (4.0.3), (4.0.4), (4.0.5) and (4.0.6) (in case (ii)), along with the comparison principle and $y(t) \leq 1$ for all $t \in [0,T)$, we conclude that

$$u(t,x) \leq w(t,x) \leq \psi(x) \quad \text{in } (0,T) \times D. \quad (4.0.7)$$

In both cases, since ψ is uniformly smaller than 1 in compact subsets of D, it follows from (4.0.7) that $\mathcal{T} \cap D = \emptyset$. We also see that in case (ii), ψ is uniformly smaller than 1 in a neighborhood of $\{R\}$, so we can rule out touchdown at this point. □

Proof of Theorems 1.1 and 1.2. Let $\rho, \mathcal{A}$ be given by (1.3.4). We first claim that $\mathcal{A}$ is nonempty, so that ρ is well defined and positive. We notice that, due to the assumption $\mu > \mu_1(p)$, there exists τ such that

$$\frac{\mu}{2\mu - \mu_1(p)} \leq \tau < 1. \quad (4.0.8)$$

Next we see that the condition $\delta(\beta, K) \leq 1$ is equivalent to

$$\phi_1(K) := \arctan\left[\sqrt{\frac{p+1}{p\mu\beta^2}(K^{-1}+p+1)}\right] \leq \sqrt{\frac{p\mu K(K+\theta)}{(1+K)^2}} =: \phi_2(K), \quad (4.0.9)$$

where $\theta = (p+1)^{-1} \in (0,1)$. We have

$$\frac{\partial}{\partial K}\left[\frac{K(K+\theta)}{(1+K)^2}\right] = \frac{(2K+\theta)(1+K) - 2K(K+\theta)}{(1+K)^3} > \frac{2(1-\theta)K}{(1+K)^3} > 0,$$

so that $\phi_2(K)$ is monotonically increasing on $(0,\infty)$. Moreover, $\phi_1(K)$ is monotonically decreasing on $(0,\infty)$. Therefore, there exists $K > 0$ satisfying (4.0.9) if and only if

$$\lim_{K\to\infty} \phi_2(K) > \lim_{K\to\infty} \phi_1(K) \iff \sqrt{p\mu} > \arctan\Big[\frac{p+1}{\beta\sqrt{p\mu}}\Big] \iff \beta > \frac{p+1}{\sqrt{p\mu}}\cot\big[\sqrt{p\mu}\big].$$

By assumption (1.3.2), we may thus find $\beta \in (d, d_0]$ satisfying this condition. Moreover, (4.0.9) is then true for any sufficiently large $K > 0$. It follows that $\mathcal{A}$ is nonempty.

Next set

$$\begin{cases} D = (x_0+1+d, R), \quad \Gamma = \{x_0+1+d\} & \text{if } f \text{ satisfies (1.3.3)}, \\ D = (x_1+1+d, x_0-1-d), \quad \Gamma = \partial D & \text{if } f \text{ satisfies (1.3.6)}. \end{cases}$$

By our assumption on f, we may select $(\tau, \beta, K) \in \mathcal{A}$ such that

$$\|f\|_{L^\infty(D)} \le \mu\hat{\varepsilon}(\beta, K), \tag{4.0.10}$$

with

$$\hat{\varepsilon}(\beta, K) = \frac{1}{2}\left(\frac{\beta-d}{\beta}\right)^{p+1} \frac{S(t_0,\beta)}{K+\tau^{-p}} \min\big\{H(t_0,\beta), G(t_0,\beta,K)\big\}$$

and

$$t_0 = t_0(\tau) = \frac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}.$$

Under assumption (1.3.3), it follows from Proposition 3.1 that

$$(1-u)^{p+1} \ge (p+1)\mu\hat{\varepsilon}(\beta, K)(T-t) \quad \text{on } [t_0, T) \times \Gamma. \tag{4.0.11}$$

Moreover, (4.0.11) remains true under assumption (1.3.6). Indeed, we may apply Proposition 3.1 with x_0 replaced by x_1, recalling $|x_1| \le |x_0|$ and noticing that $S(t,\beta)$ in (1.3.5), hence $\hat{\varepsilon}$ depends on $d_0 = R - 1 - |x_0|$ in a monotonically increasing way.

Now, in view of applying Lemma 4.1, we claim that

$$(p+1)\mu\hat{\varepsilon}(\beta, K) \le \frac{\tau^{p+1}}{T-t_0}. \tag{4.0.12}$$

Indeed, since $H(t_0,\beta) \le \operatorname{erf}\big(\frac{1}{\sqrt{t_0}}\big) \le 1$ by (1.3.5), $K > 0$ and $S(d_0+1, t_0, d_0-\beta) \le 1$, we have $\hat{\varepsilon} \le \frac{\tau^p}{2}$. Therefore, using (4.0.8), $\mu_1(p) = 2\mu_0(p)$ and $(p+1)(\mu - \mu_0(p))T \le 1$ by Lemma 3.5(ii), we obtain

$$(p+1)\mu\hat{\varepsilon}(\beta, K) \le 2(p+1)(\mu-\mu_0(p))\tau\hat{\varepsilon}(\beta, K) \le \frac{2\hat{\varepsilon}(\beta,K)\tau}{T} \le \frac{\tau^{p+1}}{T-t_0}.$$

By Lemma 4.1, it follows that that $\mathcal{T} \cap D = \emptyset$, and that $R \notin \mathcal{T}$ in case of (1.3.3). Finally, let us show that $\mathcal{T} \cap \Gamma = \emptyset$. By the continuity of f, assumptions (1.3.3) and (1.3.6) remain true for some $\tilde{d} < d$ close to d. Also, since the set $\mathcal{A}$ and the quantity $\big(\frac{\beta-d}{\beta}\big)^{p+1}$ increase as d decreases, it follows that the supremum in (1.3.4) is a nonincreasing function of $d \in (0, d_0)$, so we deduce that $\mathcal{T} \cap \Gamma = \emptyset$. This concludes the proof. $\square$

In view of the proof of Theorem 2.3, we first establish the following quantitative type I estimate, which is a version of Proposition 3.1, with the three parameters (β, K, η) instead of (β, K).

Proposition 4.2. *Under the assumptions of Proposition 3.1, u satisfies the type I estimate* (3.1.3) *where now*

$$\overline{\varepsilon} = \sup_{(\beta,K,\eta)\in\widehat{\mathcal{A}}} \hat{\varepsilon}(\beta, K, \eta), \tag{4.0.13}$$

with

$$\begin{aligned}\hat{\varepsilon}(\beta, K, \eta) = \tfrac{1}{2}\big(\tfrac{\beta-d}{\beta}\big)^{p+1} \min\Big\{ &\tfrac{S(\overline{T},0)}{K+\eta^{-p}} G(\overline{T}, \beta, K, \eta), \\ &\tfrac{S(t_0,\beta)}{K+\tau^{-p}} \min\big[H(t_0, \beta), G(t_0, \beta, K, \eta)\big]\Big\},\end{aligned} \tag{4.0.14}$$

$$\widehat{\mathcal{A}} = \mathcal{A}_2 := \Big\{(\beta, K, \eta) \in (d, d_0) \times (0, \infty) \times (0, 1],\ K \ge \tfrac{p\eta}{\mu\beta^2} - \tfrac{1}{(p+1)\eta^p},\ \delta(\beta, K, \eta) \le 1\Big\},$$

where $\overline{T} = \frac{1}{(p+1)(\mu-\mu_0(p))}$, *the functions S, H are defined in* (1.3.5) *and G, δ are defined in* (2.2.4)*,*

Proof. Let $d \in (0, d_0)$ and $\tau \in (0, 1)$. Let $(\beta, K, \eta) \in \mathcal{A}_2$ and let $a(r) = a_{\beta,K,\eta}(r)$ be given by (3.3.25). As in the proof of Proposition 3.1, we shall rely on Lemma 3.6 with $q = 0$, and we first express the infima in (3.3.5) and (3.3.6) in terms of the error function. Denote by $e^{t\Delta_{\mathbb{R}}}$ the heat semigroup on $\mathbb{R}$. By the proof of Lemma 3.8 we have

$$\inf_{0<r<1} \frac{e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)}{a(r)} = \frac{1}{2}G(t, \beta, K, \eta), \quad t > 0, \tag{4.0.15}$$

and

$$\inf_{1<r<1+\beta} \frac{e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)}{a(r)} = \frac{1}{2}H(t, \beta), \quad t > 0. \tag{4.0.16}$$

Moreover,

$$t \mapsto S(t, \beta),\ G(t, \beta, K, \eta) \ \text{ are nonincreasing for } t > 0. \tag{4.0.17}$$

The proof of Proposition 4.2 is then completely similar to that of Proposition 3.1, applying Lemma 3.6 with the function $a(r)$ given by Lemma 3.7. The only difference is that, since now η may be less than 1, we need to choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ in Lemma 3.6. To estimate ε_2, we use (3.3.6), (3.3.9), (3.4.1), (3.4.5) with $\lambda = 1$, and (4.0.15) to deduce

$$\begin{aligned}\varepsilon_2 &= \frac{1}{W(0, \eta, K)} \inf_{(t,x)\in[t_0,\overline{T}]\times I_0} \frac{e^{t\Delta_\Omega}\chi_{I_0}(x)}{a(|x - x_0|)} \\ &\ge \frac{1}{2}\frac{1}{\eta^{-p} + K} \inf_{t\in[t_0,\overline{T}]} S(t, 0)\, G(t, \beta, K, \eta) = \frac{1}{2}\frac{S(\overline{T}, 0)}{\eta^{-p} + K}\, G(\overline{T}, \beta, K, \eta).\end{aligned}$$

The conclusion follows by taking the supremum over $(\beta, K, \eta) \in \mathcal{A}_2$. □

Proof of Theorem 2.3. Let $\mathcal{A}$ be given by (2.2.3). Picking any $\tau \in (0, 1)$, $\beta \in (d, d_0)$, $K > 0$ and then $\eta \in (0, 1)$ small, we see that $\mathcal{A}$ is nonempty, so that ρ is well defined and positive.

Now arguing as in the second paragraph of the proof of Theorems 1.1-1.2, we may select $(\tau, \beta, K, \eta) \in \mathcal{A}$ satisfying (4.0.10) and (4.0.11), where $\hat{\varepsilon} = \hat{\varepsilon}(\beta, K, \eta)$ is now given by (4.0.14) and t_0 is given by (2.1). Applying Lemma 4.1 in the same way as before, we finally conclude that $\mathcal{T} \cap \overline{D} = \emptyset$. □

5 Proof of Theorems 2.1-2.2

It is based on refinements of various ingredients of the proof of Theorems 1.1 and 1.2. Namely, we introduce a more precise cut-off function $a(x)$, corresponding to the choice $q = 1$ in the function h (cf. (3.1.7) and Lemma 3.6). We also use an improved lower estimate on u_t (see Lemma 5.3), and an upper estimate of u for t small at points of small permittivity (see Lemma 5.5).

5.1 Determination of the family of functions $a(r)$ in the case $q = 1$

In this subsection, we compute the solution of the auxiliary problem (3.3.20)-(3.3.23) in the case $q = 1$ and $\eta = 1$, which will be used to prove Theorems 2.1-2.2. The following Lemma 5.1 is the analogue of Lemma 3.7 for $q = 0$. We see that the expression of a in (5.1.2) is significantly more complicated than for $q = 0$. As for the choices of $q \in [0, 1]$ other than $q = 0$ or $q = 1$, they seem quite difficult to investigate and have been left out of this study.

Lemma 5.1. *Let $\mu, \beta, K > 0$, $q = 1$, $\eta = 1$ and let h, F be given by (3.1.7), (3.3.2). Assume*

$$K\mu\beta^2 \leq \frac{p(p+2) - K}{p} \tag{5.1.1}$$

and let

$$A_0 = \sqrt{\frac{p(1+K) + (p(p+2) - K)K}{p(1+K)^2}}, \quad A_1 = \arctan\left(\sqrt{\frac{p(1+K)}{p(p+2) - K}} + K\right),$$

$$A_2 = \arctan\left(\sqrt{\frac{p}{p(p+2) - K}}\right), \quad A_3 = \arctan\left(\frac{1}{\sqrt{K\mu\beta^2}}\right),$$

$$\delta_1(K) = \frac{A_1}{A_0\sqrt{K\mu}}, \quad \delta_2(\beta, K) = \frac{A_3 - A_2}{\sqrt{K\mu}} \geq 0.$$

Assume in addition that $\delta_1 + \delta_2 \leq 1$. Then there exist $r_0 \in [0, 1)$ and a solution $a \in W^{2,2}([0, 1 + \beta])$ of (3.3.20)-(3.3.23), (3.1.8). The pair (r_0, a) is then unique and it is given by

$$a_{\beta,K}(r) = \begin{cases} D_1, & r \in [0, r_0), \\ D_1 \cos^\alpha\left(A_0\sqrt{K\mu}(r - r_0)\right), & r \in [r_0, r_1), \\ D_2 \cos^{p+1}\left(\sqrt{K\mu}(r-1) + A_3\right), & r \in [r_1, 1], \\ \left(\dfrac{1+\beta - r}{\beta}\right)^{p+1}, & r \in (1, 1+\beta], \end{cases} \tag{5.1.2}$$

where $r_0 = 1 - \delta_1 - \delta_2$, $r_1 = 1 - \delta_2$,

$$\alpha = \frac{p+1}{(1+K)A_0^2}, \quad D_2 = \left(1 + \frac{1}{K\mu\beta^2}\right)^{\frac{p+1}{2}}, \quad D_1 = D_2\frac{D_{11}^\alpha}{D_{12}^{p+1}},$$

$$D_{11} = \sqrt{1 + K + \frac{p(1+K)}{p(p+2) - K}}, \quad D_{12} = \sqrt{1 + \frac{p}{p(p+2) - K}}.$$

In view of the proof of Lemma 5.1, we first compute the function F in (3.3.2).

Lemma 5.2. *Let $\mu, \beta, K > 0$, $q = 1$, $\eta = 1$ and let h, F be given by* (3.1.7), (3.3.2). *Assume* (5.1.1) *and set*

$$m_1 = \frac{(p-K)^2}{p(p+1)(1+K)}, \quad M_1 = \frac{(p+1)K\mu}{1+K}, \quad m_2 = \frac{p}{p+1}, \quad M_2 = (p+1)K\mu.$$

Then we have

$$0 < M_1 < M_2, \quad 0 \le m_1 < m_2 < 1, \tag{5.1.3}$$

$$F(r,\xi) = \frac{p}{p+1}, \quad r > 1, \tag{5.1.4}$$

and

$$F(r,\xi) = \begin{cases} m_1\xi^2 - M_1 & \text{if } |\xi| < \xi_0 \\ m_2\xi^2 - M_2 & \text{if } |\xi| \ge \xi_0, \end{cases} \qquad 0 \le r < 1, \tag{5.1.5}$$

where

$$\xi_0 := \sqrt{\frac{M_2 - M_1}{m_2 - m_1}} = (p+1)\sqrt{\frac{pK\mu}{p(p+2) - K}}. \tag{5.1.6}$$

Proof. *Step 1: Computation of $F(r,\xi)$ for $r > 1$.* In this step, we keep $q \in (0,1]$, since the computation requires the same amount of effort. Since $K < p(p+2)$ by (5.1.1), we have in particular

$$K \le \frac{p(p+q+1)}{q}. \tag{5.1.7}$$

We claim that, under condition (5.1.7), we have

$$\sup_{u\in(0,1)} \left[\frac{h'^2(u)}{hh''(u)}\right] = \frac{h'^2(1)}{hh''(1)} = \frac{p}{p+1}, \tag{5.1.8}$$

which immediately yields (5.1.4).

To show (5.1.8), we compute

$$hh'' = (1-u)^{-2p-2}\big[1 + K(1-u)^{q+p}\big]\big[p(p+1) + Kq(q-1)(1-u)^{q+p}\big]$$

and

$$h'^2 = (1-u)^{-2p-2}\big[p - Kq(1-u)^{q+p}\big]^2,$$

hence

$$\frac{h'^2}{hh''} = \frac{\big[p - Kq(1-u)^{q+p}\big]^2}{\big[1 + K(1-u)^{q+p}\big]\big[p(p+1) + Kq(q-1)(1-u)^{q+p}\big]}. \tag{5.1.9}$$

Now setting $X = K(1-u)^{q+p}$, we see that (5.1.8) is equivalent to

$$Q(X) := p(1+X)[p(p+1) + q(q-1)X] - (p+1)(p-qX)^2 \ge 0 \quad \text{for all } X \in [0,K].$$

Since

$$\begin{aligned} Q(X) &= \big[pq(q-1) + p^2(p+1) + 2(p+1)pq\big]X - \big[pq(1-q) + (p+1)q^2\big]X^2 \\ &= p\big[q^2 + p^2 + 2pq + p + q\big]X - q(p+q)X^2 = (p+q)X\big[p(p+q+1) - qX\big] \end{aligned}$$

and $0 \le X \le K$, claim (5.1.8) follows from (5.1.7).

Step 2: Computation of $F(r,\xi)$ for $r<1$. In view of (5.1.1), properties (5.1.3) and (5.1.6) follow from

$$M_2 - M_1 = \frac{(p+1)K^2\mu}{1+K} > 0, \qquad m_2 - m_1 = \frac{(p(p+2)-K)K}{p(p+1)(1+K)}. \tag{5.1.10}$$

Next, taking $q=1$ and setting $y=(1-u)^{p+1}$, we have

$$F(r,\xi) = \sup_{y\in(0,1)} \hat{F}(\xi,y), \qquad \hat{F}(\xi,y) := \frac{(p-Ky)^2}{p(p+1)(1+Ky)}\xi^2 - \frac{(p+1)K\mu}{1+Ky}.$$

Setting $s=\frac{1}{1+Ky}$, i.e., $Ky = s^{-1}-1$, we may rewrite

$$\hat{F}(\xi,y) = \frac{s(s^{-1}-p-1)^2}{p(p+1)}\xi^2 - (p+1)K\mu s = C_1(\xi)s + C_2(\xi) \ + \ \xi^2\frac{s^{-1}}{p(p+1)}.$$

Since, for each fixed ξ, $\hat{F}$ is a convex function of s, it follows that $\sup_{y\in[0,1]}\hat{F}(\xi,y)$ is achieved for $y=0$ or 1. Consequently,

$$F(r,\xi) = \max\left\{m_1\xi^2 - M_1,\ m_2\xi^2 - M_2\right\}, \tag{5.1.11}$$

which immediately yields (5.1.5). □

Proof of Lemma 5.1. We split the proof in three steps.

Step 1: Preliminaries and monotonicity of $\xi(r)$. Assume that there exist $r_0\in[0,1)$ and a solution $a\in W^{2,2}([0,1+\beta])$ of (3.3.20)-(3.3.23), (3.1.8). We note in particular that $a\in C^1([r_0,1+\beta])$. We shall show that a is necessarily given by (5.1.2). It will then be a simple matter to show that this is indeed a solution.

First, in view of (5.1.4), it follows from the proof of Lemma 3.7 that a is necessarily given by (5.1.2) on $[1,1+\beta]$.

We next claim that

$$\xi(r) := \frac{a'(r)}{a(r)} \le 0 \text{ is a monotone decreasing function of } r\in[r_0,1]. \tag{5.1.12}$$

Indeed we can compute (a.e.)

$$\begin{aligned}\xi'(r) &= \frac{a''(r)}{a(r)} - \left(\frac{a'(r)}{a(r)}\right)^2 = F(r,\xi(r)) - \xi^2(r)\\ &= \max\left\{(m_1-1)\xi^2(r) - M_1,\ (m_2-1)\xi^2(r) - M_2\right\} < 0,\end{aligned}$$

by (5.1.11) and (5.1.3). On the other hand, in view of (5.1.1) and (5.1.6), we have $\xi(1) = -\frac{p+1}{\beta} \le -\xi_0$. By (5.1.12), it follows that there exists a unique $r_1\in(r_0,1]$ such that $\xi(r_1)=-\xi_0$, with

$$\begin{cases} \xi(r) > -\xi_0 & \text{for all } r\in[r_0,r_1),\\ \xi(r) < -\xi_0 & \text{for all } r\in(r_1,1],\end{cases} \tag{5.1.13}$$

and that $r_1<1$ unless we have equality in (5.1.1).

Step 2: Determination of r_1 and of a in $[r_1,1]$. If we have equality in (5.1.1), then $\xi(1)=-\xi_0$ and we go directly to Step 3. In the rest of this step, we thus assume that the inequality in (5.1.1) is strict, hence $r_1 < 1$. By (5.1.5), (5.1.13), Step 1 and the fact that $a \in C^1([r_0, 1+\beta])$, it follows that a solves the problem

$$\frac{a''(r)}{a(r)} = m_2 \left(\frac{a'(r)}{a(r)}\right)^2 - M_2, \quad \text{for } r_1 < r < 1, \qquad a(1-) = 1, \qquad a'(1-) = -\frac{p+1}{\beta}. \tag{5.1.14}$$

Similarly as for (3.3.30), setting $\phi_2(r) = a^{1-m_2}(r)$, the ODE in (5.1.14) is reduced to the equation

$$\phi_2'' = (1-m_2)a^{-m_2-1}[aa'' - m_2(a')^2] = -(1-m_2)M_2\phi_2. \tag{5.1.15}$$

The solution of (5.1.15) is of the form

$$\phi_2(r) = D_2^{1-m_2} \cos\left(\sqrt{M_2(1-m_2)}(r-1)+\theta\right) = D_2^{1-m_2} \cos\left(\sqrt{K\mu}(r-1)+\theta\right), \tag{5.1.16}$$

for some constant $\theta \in [0, 2\pi)$. Since $\phi_2 > 0$ and $\phi_2' < 0$ on $[r_1, 1]$, this imposes

$$\left[\sqrt{K\mu}(r_1-1)+\theta, \theta\right] \subset \left(0, \frac{\pi}{2}\right),$$

that is,

$$r_1 > 1 - \frac{\theta}{\sqrt{K\mu}}, \quad \theta < \frac{\pi}{2}.$$

In particular, we have

$$\xi(r) = \frac{1}{1-m_2}\frac{\phi_2'(r)}{\phi_2(r)} = -\sqrt{\frac{M_2}{1-m_2}} \tan\left(\sqrt{K\mu}(r-1)+\theta\right), \quad r_1 < r < 1. \tag{5.1.17}$$

By a simple computation, the boundary conditions in (5.1.14) give

$$\theta = \arctan\left(\tfrac{p+1}{\beta}\sqrt{\tfrac{1-m_2}{M_2}}\right) = A_3 \in \left(0, \tfrac{\pi}{2}\right),$$

$$D_2^{1-m_2} = \sqrt{1 + \tfrac{1-m_2}{M_2}\left(\tfrac{p+1}{\beta}\right)^2} = \sqrt{1 + \tfrac{1}{K\mu\beta^2}}.$$

On the other hand, by (5.1.13), we have $\xi(r_1) = -\xi_0$. By (5.1.17), $\xi(r_1) = -\xi_0$ and (5.1.10), we deduce

$$\tan\left(\sqrt{K\mu}(r_1-1)+\theta\right) = \sqrt{\frac{(M_2-M_1)(1-m_2)}{(m_2-m_1)M_2}} = \sqrt{\frac{p}{p(p+2)-K}}$$

hence,

$$r_1 = 1 - \frac{1}{\sqrt{K\mu}}\left[\arctan\left(\sqrt{\frac{1}{K\mu\beta^2}}\right) - \arctan\left(\sqrt{\frac{p}{p(p+2)-K}}\right)\right] = 1 - \delta_2.$$

Since we assume $\delta_1 + \delta_2 \le 1$, hence $0 \le \delta_2 < 1$, we note that we do have $0 < r_1 \le 1$.

Step 3: Determination of r_0 *and of* a *in* $[r_0, r_1]$ *and conclusion.* By (5.1.5), (5.1.13), Step 2 and the fact that $a \in C^1([r_0, 1+\beta])$, it follows that a solves the problem

$$\begin{cases} \dfrac{a''(r)}{a(r)} = m_1 \left(\dfrac{a'(r)}{a(r)}\right)^2 - M_1, \quad \text{for } r_0 < r < r_1, \\ a(r_{1-}) = a(r_{1+}), \qquad a'(r_{1-}) = a'(r_{1+}). \end{cases} \tag{5.1.18}$$

Similarly as before, setting $\phi_1(r) = a^{1-m_1}(r)$, we are left with

$$\phi_1'' = (1-m_1)a^{-m_1-1}[aa'' - m_1(a')^2] = -(1-m_1)M_1\phi_1,$$

whose solution is of the form

$$\phi_1(r) = D_1^{1-m_1}\cos\left(\sqrt{M_1(1-m_1)}(r-\theta_0)\right),$$

for some constant $\theta_0 \in \mathbb{R}$. Since $\phi_1 > 0$ and $\phi_1' < 0$ on $(r_0, r_1]$, this imposes (modulo 2π)

$$\left(\sqrt{M_1(1-m_1)}(r_0-\theta_0), \sqrt{M_1(1-m_1)}(r_1-\theta_0)\right] \subset \left(0, \frac{\pi}{2}\right).$$

Since $\theta_0 = r_0$ owing to $\phi_1'(r_0) = 0$, we thus have

$$\left(0, \sqrt{M_1(1-m_1)}(r_1-r_0)\right] \subset \left(0, \frac{\pi}{2}\right).$$

Now, we use the boundary condition $a'(r_{1-}) = a'(r_{1+})$ to obtain the value of r_0. Since

$$a'(r_{1-}) = \frac{1}{1-m_1}\left[\phi_1'\phi_1^{\frac{m_1}{1-m_1}}\right](r_1) = -D_1\sqrt{\frac{M_1}{1-m_1}}\left[\sin\cos^{\frac{m_1}{1-m_1}}\right]\left(\sqrt{M_1(1-m_1)}(r_1-r_0)\right)$$

and

$$\begin{aligned} a'(r_{1+}) &= -\xi_0 a(r_{1+}) = -\sqrt{\frac{M_2-M_1}{m_2-m_1}}a(r_1) \\ &= -\sqrt{\frac{M_2-M_1}{m_2-m_1}}D_1\cos^{\frac{1}{1-m_1}}\left(\sqrt{M_1(1-m_1)}(r_1-r_0)\right), \end{aligned}$$

r_0 is determined by

$$\tan\left(\sqrt{M_1(1-m_1)}(r_1-r_0)\right) = \sqrt{\frac{(M_2-M_1)(1-m_1)}{(m_2-m_1)M_1}} \tag{5.1.19}$$

hence,

$$r_0 = r_1 - \frac{1}{\sqrt{M_1(1-m_1)}}\arctan\left(\sqrt{\frac{(M_2-M_1)(1-m_1)}{(m_2-m_1)M_1}}\right) = r_1 - \delta_1,$$

where we used (5.1.10) in the last equality.

Finally, to obtain the value of D_1 we use $a(r_{1-}) = a(r_{1+})$. Using

$$\begin{aligned} a(r_{1-}) &= D_1\cos^{\frac{1}{1-m_1}}\left(\sqrt{M_1(1-m_1)}(r_1-r_0)\right) \\ &= D_1\cos^{\frac{1}{1-m_1}}\left(\arctan\sqrt{\frac{(M_2-M_1)(1-m_1)}{(m_2-m_1)M_1}}\right), \\ a(r_{1+}) &= D_2\cos^{p+1}\left(\sqrt{K\mu}(r_1-1)+\theta\right) \\ &= D_2\cos^{p+1}\left(\arctan\sqrt{\frac{(M_2-M_1)(1-m_2)}{(m_2-m_1)M_2}}\right), \end{aligned}$$

we obtain the expression for D_1 in the statement after a straightforward calculation.

We have thus proved that (r_0, a) is necessarily given by (5.1.2). Conversely, an immediate inspection of the above proof shows that (5.1.2) does define a solution $a \in W^{2,2}([0, 1+\beta])$ of (3.3.20)-(3.3.23), (3.1.8). $\square$

5.2 Improved lower bound for u_t

In this subsection, we improve the lower bound on u_t used in (3.3.14) for Propositions 3.1, 4.2, by exploiting the contribution coming from the nonlinear term in (3.3.15). This will be used in the proof of Theorems 2.1-2.2.

Lemma 5.3. *Let $x_0 \in \Omega = (-R, R)$ with $\tilde{R} := R - |x_0| > 1$. Set $I_\beta = (x_0 - 1 - \beta, x_0 + 1 + \beta) \subset \Omega$ for $\beta \geq 0$. Assume that f satisfies*

$$f \geq \mu > 0 \quad \text{in } I_0.$$

Then the solution u of problem (1.1.1) *satisfies*

$$u_t(t,x) \geq \big(1 + p\mu\Lambda_0(t,x)\big)\mu e^{t\Delta_{I_\beta}}\chi_{I_0} \qquad \text{in } (0,T)\times I_\beta, \tag{5.2.1}$$

where $\Lambda_0(t,x)$ is given by

$$\Lambda_0(t,x) = \int_0^t \big(1-(p+1)\mu K_s s\big)^{-\frac{p}{p+1}} e^{(t-s)\Delta_{I_\beta}} \left[\frac{\chi_{I_0}}{1-(p+1)\mu\varphi(s,\cdot)}\right] ds. \tag{5.2.2}$$

Here

$$K_s = \inf_{0<\tau<s}\left(e^{\tau\Delta_{I_\beta}}\chi_{I_0}\right)(x_0+1), \;\; \varphi(s,\cdot) = \int_0^s e^{\tau\Delta_{I_\beta}}\chi_{I_0}d\tau,$$

where $1-(p+1)\mu K_s s > 0$ for $s \in (0,T)$, and the denominator in (5.2.2) *is positive in $(0,T)\times I_\beta$.*

In particular, we can estimate

$$\Lambda_0(t,x) \geq \int_0^t \big(1-(p+1)\mu K_s s\big)^{-\frac{p}{p+1}-1} e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}ds \geq \varphi(t,x) \qquad \text{in } (0,T)\times I_\beta. \tag{5.2.3}$$

Proof. Step 1. We first claim that

$$(1-u)^{p+1} \leq 1-(p+1)\mu\int_0^t \big(e^{s\Delta_{I_\beta}}\chi_{I_0}\big)ds \quad \text{in } (0,T)\times I_\beta, \tag{5.2.4}$$

which in particular guarantees that the denominator in (5.2.2) is positive in $(0,T)\times I_\beta$ and $1-(p+1)\mu K_s s > 0$ for $s \in (0,T)$.

Indeed, let $w = \frac{1-(1-u)^{p+1}}{p+1} \geq 0$. We compute

$$w_t = (1-u)^p u_t, \qquad w_x = (1-u)^p u_x, \qquad w_{xx} = (1-u)^p u_{xx} - p(1-u)^{p-1}u_x^2.$$

Hence,

$$w_t - w_{xx} \geq f(x) \geq \mu\chi_{I_0}, \qquad \text{with} \quad w|_{\partial I_\beta} \geq 0, \quad w|_{t=0} = 0.$$

Therefore, $w \geq \mu\int_0^t \big(e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}\big)ds = \mu\int_0^t \big(e^{s\Delta_{I_\beta}}\chi_{I_0}\big)ds$, and (5.2.4) follows.

Step 2. Let $v := u_t$. We claim that

$$v(t) \;\geq\; \mu e^{t\Delta_{I_\beta}}\chi_{I_0} + p\mu\int_0^t e^{(t-s)\Delta_{I_\beta}}\left[\frac{\chi_{I_0}v(s,\cdot)}{1-(p+1)\mu\int_0^s(e^{\tau\Delta_{I_\beta}}\chi_{I_0})d\tau}\right]ds \quad \text{in } (0,T)\times I_\beta. \tag{5.2.5}$$

By (3.3.15) and (5.2.4), v satisfies

$$v_t - v_{xx} = p(1-u)^{-p-1} f(x) v \geq \frac{p\mu\chi_{I_0} v}{1-(p+1)\mu\int_0^t (e^{s\Delta_{I_\beta}}\chi_{I_0})ds} \quad \text{in } (0,T)\times I_\beta.$$

Therefore, (5.2.5) follows from the variation of constants formula.

Step 3. We next claim that

$$v(t) \geq \mu\gamma(t) e^{t\Delta_{I_\beta}}\chi_{I_0} \quad \text{in } (0,T)\times I_0, \tag{5.2.6}$$

with $\gamma(s) = \big(1-(p+1)\mu K_s s\big)^{-\frac{p}{p+1}}$. Since

$$e^{s\Delta_{I_\beta}}\chi_{I_0} \text{ is symmetric w.r.t. } x_0 \text{ and decreasing in } |x-x_0|, \tag{5.2.7}$$

we have $\int_0^s \big(e^{\tau\Delta_{I_\beta}}\chi_{I_0}\big)(x)d\tau \geq sK_s$ for all $0<s<T$ and all $x\in I_0$. Since the numerator of the bracket in (5.2.5) is supported in I_0, it follows from (5.2.5) that

$$v(t) \geq \mu e^{t\Delta_{I_\beta}}\chi_{I_0} + p\mu\int_0^t \theta(s)\big(e^{(t-s)\Delta_{I_\beta}}(\chi_{I_0}v(s,\cdot))\big)ds \quad \text{in } (0,T)\times I_\beta,$$

where $\theta(s) = (1-(p+1)\mu K_s s)^{-1}$. In particular, $\phi := \chi_{I_0} v$ satisfies

$$\phi(t) \geq \mu\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0} + p\mu\chi_{I_0}\int_0^t \theta(s)\big(e^{(t-s)\Delta_{I_\beta}}\phi(s)\big)ds \quad \text{in } (0,T)\times I_\beta. \tag{5.2.8}$$

We want to show that $\psi(t) = \mu\gamma(t)\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0}$ is a subsolution of (5.2.8). This is equivalent to

$$\begin{aligned}\mu\gamma(t)\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0} &\leq \mu\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0} \\ &+ p\mu^2\chi_{I_0}\int_0^t \theta(s)\gamma(s) e^{(t-s)\Delta_{I_\beta}}\Big(\chi_{I_0} e^{s\Delta_{I_\beta}}\chi_{I_0}\Big)ds \quad \text{in } (0,T)\times I_\beta.\end{aligned} \tag{5.2.9}$$

Next, for any $s>0$, using (5.2.7), it follows from Lemma 7.3 that, for all $t>s$,

$$\begin{aligned} e^{(t-s)\Delta_{I_\beta}}\big[\chi_{I_0} e^{s\Delta_{I_\beta}}\chi_{I_0}\big] &\geq \Big(e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}\Big)\Big(e^{(t-s)\Delta_{I_\beta}}\big[e^{s\Delta_{I_\beta}}\chi_{I_0}\big]\Big) \\ &= \Big(e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}\Big)\Big(e^{t\Delta_{I_\beta}}\chi_{I_0}\Big) \quad \text{in } I_\beta. \end{aligned}$$

Therefore, a sufficient condition for (5.2.9) is given by

$$\mu\gamma(t)\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0} \leq \left[1 + p\mu\int_0^t \theta(s)\gamma(s)\big(e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}\big)ds\right]\mu\chi_{I_0} e^{t\Delta_{I_\beta}}\chi_{I_0} \quad \text{in } (0,T)\times I_\beta,$$

which is equivalent to

$$\gamma(t) \leq 1 + p\mu\int_0^t \theta(s)\gamma(s)\big(e^{(t-s)\Delta_{I_\beta}}\chi_{I_0}\big)(x)ds \quad \text{in } (0,T)\times I_0.$$

For this, by (5.2.7), it is sufficient to have

$$\gamma(t) \leq 1 + p\mu K_t\int_0^t \theta(s)\gamma(s)ds \quad \text{for all } t\in(0,T). \tag{5.2.10}$$

Note that K_t is continuous and nonincreasing w.r.t. $t>0$. Now, for each $0<t\leq\tau<T$, we set

$$\gamma_\tau(t) = [1-(p+1)\mu K_\tau t]^{-\frac{p}{p+1}}, \qquad \theta_\tau(t) = [1-(p+1)\mu K_\tau t]^{-1},$$

which are well defined by Step 1 and the monotonicity of K_t. We have $\gamma_\tau'(t) = p\mu K_\tau \theta_\tau(t)\gamma_\tau(t)$, hence

$$\gamma_\tau(t) \le 1 + p\mu K_\tau \int_0^t \theta_\tau(s)\gamma_\tau(s)\,ds, \qquad 0 < t \le \tau < T.$$

Letting $\tau \to t$ and using the continuity and monotonicity of K_t, we obtain

$$\gamma(t) \le 1 + p\mu K_t \int_0^t \theta_t(s)\gamma_t(s)\,ds \le 1 + p\mu K_t \int_0^t \theta(s)\gamma(s)\,ds, \quad 0 < t < T.$$

Therefore, (5.2.10), hence (5.2.9), is satisfied. Property (5.2.6) then follows from the comparison principle (in variation of constants form).

Step 4: Since, for any $s > 0$, $\left(1 - (p+1)\mu \int_0^s (e^{\tau\Delta_{I_\beta}}\chi_{I_0})d\tau\right)^{-1} \chi_{I_0}$ is symmetric w.r.t. x_0 and decreasing in $|x - x_0|$, we deduce from Lemma 7.3 that, for all $t > s$,

$$\begin{aligned}
e^{(t-s)\Delta_{I_\beta}} & \left[\frac{\chi_{I_0} v(s,\cdot)}{1 - (p+1)\mu \int_0^s (e^{\tau\Delta_{I_\beta}}\chi_{I_0})d\tau}\right] \ge \\
& \ge \mu\gamma(s) e^{(t-s)\Delta_{I_\beta}} \left[\frac{\chi_{I_0} e^{s\Delta_{I_\beta}}\chi_{I_0}}{1 - (p+1)\mu \int_0^s (e^{\tau\Delta_{I_\beta}}\chi_{I_0})d\tau}\right] \\
& \ge \mu\gamma(s) \big(e^{t\Delta_{I_\beta}}\chi_{I_0}\big) e^{(t-s)\Delta_{I_\beta}} \left[\frac{\chi_{I_0}}{1 - (p+1)\mu \int_0^s (e^{\tau\Delta_{I_\beta}}\chi_{I_0})d\tau}\right].
\end{aligned}$$

Estimate (5.2.1) then follows from (5.2.5) and (5.2.6). Finally, the first part of estimate (5.2.3) follows from (5.2.7), and the second inequality from the fact that $0 < \gamma(s) \le 1$. □

Lemma 5.4. *Under the assumptions of Lemma 5.3, the solution u of problem* (1.1.1) *satisfies*

$$u_t(t,x) \ge \big(1 + p\mu\tilde\Lambda(t,x)\big)\mu S(t,\beta) e^{t\Delta_{\mathbb{R}}}\chi_{I_0} \qquad \text{in } (0,T)\times I_\beta, \tag{5.2.11}$$

where S is defined in (1.3.5) *and $\tilde\Lambda(t,x)$ is given by*

$$\tilde\Lambda(t,x) = S(t,\beta) \int_0^t \big(1 - Y(s)\big)^{-\frac{p}{p+1}} e^{(t-s)\Delta_{\mathbb{R}}} \left[\frac{\chi_{I_0}}{1 - (p+1)\mu S(t,0)\psi(s,\cdot)}\right] ds, \tag{5.2.12}$$

with $Y(s) = S(s,0)\,\mathrm{erf}\left(\frac{1}{\sqrt{s}}\right)\frac{(p+1)\mu}{2}\, s$ and $\psi(s,\cdot) = \displaystyle\int_0^s e^{\tau\Delta_{\mathbb{R}}}\chi_{I_0} d\tau$.

In particular, we can estimate

$$\tilde\Lambda(t,x) \ge S(t,\beta) \int_0^t \big(1 - Y(s)\big)^{-\frac{p}{p+1}-1} e^{(t-s)\Delta_{\mathbb{R}}}\chi_{I_0} ds \qquad \text{in } (0,T)\times I_\beta. \tag{5.2.13}$$

Proof. This follows by combining Lemma 5.3 and Proposition 7.1, using in particular

$$\begin{aligned}
\left(e^{s\Delta_{I_\beta}}\chi_{I_0}\right)(x_0+1) &\ge S(s,0)\left(e^{s\Delta_{\mathbb{R}}}\chi_{(-1,1)}\right)(1) \ge \frac{S(s,0)}{\sqrt{4\pi s}} \int_{-1}^1 e^{-\frac{(1-y)^2}{4s}} dy \\
&= \frac{S(s,0)}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{s}}} e^{-z^2} dz = \frac{S(s,0)}{2}\mathrm{erf}\left(\frac{1}{\sqrt{s}}\right).
\end{aligned}$$

□

5.3 Control of u at $t = t_0(\tau)$ for points of small permittivity

Our goal in this subsection is to take advantage of the smallness assumption in (2.1.2) to improve the upper estimate (3.2.8) of u that was used in the proof of Propositions 3.1 and 4.2. Estimate (3.2.8) followed from a mere comparison with the associated ODE problem $y'(t) = \|f\|_\infty(1-y(t))^{-p}$ and thus did not take advantage of the possible smaller values of f. The following lemma provides a more precise control of $h(u(t_0,\cdot))$, which will allow a better lower estimate of the ratio ε_1 in (3.3.5) (cf. the proof of Proposition 5.6).

Lemma 5.5. *Let $I \subset \Omega = (-R,R)$, $N \in (0, \|f\|_\infty]$ and assume that*

$$f \le N \quad \text{in } \overline{I}. \tag{5.3.1}$$

Then we have

$$u(t,x) \le y(t)\theta_N(x) \quad \text{in } [0,T_*) \times \Omega, \tag{5.3.2}$$

where $c_p = \frac{(p+1)^{p+1}}{p^p}$, $T_* := \frac{1}{(p+1)\|f\|_\infty} \le T$ and

$$\begin{aligned} y(t) &= 1 - \big(1-(p+1)\|f\|_\infty t\big)^{\frac{1}{p+1}}, \\ \theta_N(x) &= \tfrac{N}{\|f\|_\infty} + \Big(1 - \tfrac{N}{\|f\|_\infty}\Big)\frac{1}{\cosh\big[\sqrt{c_p\|f\|_\infty}\,\mathrm{dist}(x,\Omega\setminus I)\big]}. \end{aligned} \tag{5.3.3}$$

Proof. By Lemma 3.5(i), we have $T_* \ge T$, as well as $u(t,x) \le y(t)$ in $[0,T_*)$. It suffices to show that, if $z_0 \in \Omega$ and $\delta > 0$ are such that

$$f(x) \le N \quad \text{for all } x \in [z_0-\delta, z_0+\delta] \cap [-R,R], \tag{5.3.4}$$

then

$$u(t,z_0) \le y(t)\left[\frac{N}{\|f\|_\infty} + \left(1 - \frac{N}{\|f\|_\infty}\right)\frac{1}{\cosh\big[\sqrt{c_p\|f\|_\infty}\,\delta\big]}\right] \quad \text{for all } t \in [0,T_*). \tag{5.3.5}$$

To prove (5.3.5), we look for a supersolution of problem (1.1.1) in $\mathcal{Q} := [0,T_*) \times ((z_0-\delta, z_0+\delta)\cap\Omega)$. We define the comparison function

$$v(t,x) := y(t)\psi(x), \qquad \text{for } (t,x) \in \tilde{\mathcal{Q}} := [0,t_0(\tau)] \times [z_0-\delta, z_0+\delta],$$

where ψ is a function to be chosen, satisfying $0 \le \psi \le 1$, $\psi(z_0 \pm \delta) = 1$ and $\psi'' \ge 0$. By (3.2.9), (5.3.4) and using $0 \le \psi(x) \le 1$ and $0 \le y(t) < 1$, we have

$$\begin{aligned} Pv := v_t - v_{xx} - \frac{f(x)}{(1-v)^p} &= y'(t)\psi(x) - y(t)\psi''(x) - \frac{f(x)}{(1-y(t)\psi)^p} \\ &\ge \frac{\|f\|_\infty\psi(x) - N}{(1-y(t))^p} - y(t)\psi''(x). \end{aligned}$$

Since $\psi'' \ge 0$, a sufficient condition to guarantee $Pv \ge 0$ in $\tilde{\mathcal{Q}}$ is thus

$$\|f\|_\infty\psi(x) - \psi''(x)\max_{t\in[0,T)} y(t)(1-y(t))^p \ge N.$$

An elementary computation shows that

$$\max_{0\le s\le 1} s(1-s)^p = \max_{0\le X\le 1} X^p - X^{p+1} = \frac{p^p}{(p+1)^{p+1}},$$

so we are left with the following differential inequality for ψ:

$$-\frac{\psi''(x)}{c_p\|f\|_\infty}+\psi(x)\geq\frac{N}{\|f\|_\infty}.$$

Solving the corresponding ODE, symmetrically in $[z_0-\delta,z_0+\delta]$, we obtain the solution

$$\psi(x)=\frac{N}{\|f\|_\infty}+B\cosh\Big[\sqrt{c_p\|f\|_\infty}(x-z_0)\Big],$$

where B is a constant. From the boundary conditions $\psi(z_0\pm\delta)=1$, we finally get

$$\psi(x)=\frac{N}{\|f\|_\infty}+\left(1-\frac{N}{\|f\|_\infty}\right)\frac{\cosh\big[\sqrt{c_p\|f\|_\infty}(x-z_0)\big]}{\cosh\big[\sqrt{c_p\|f\|_\infty}\delta\big]},\qquad x\in[z_0-\delta,z_0+\delta],$$

and we see that the requirements $0\leq\psi\leq 1$ and $\psi''\geq 0$ are satisfied. Therefore, $Pv\geq 0$ in $\tilde{\mathcal{Q}}$.

Now, we look at the parabolic boundary of $\mathcal{Q}$. On the one hand we have $y(0)=0$, so $v(0,x)=u(0,x)=0$. On the other hand, for any point (t,x) on the lateral boundary of $\mathcal{Q}$, we have either $x=\pm R$ or $x=z_0\pm\delta$. In the first case, we have $v(t,x)=y(t)\geq 0=u(t,x)$. In the second case, since $y(t)$ is a supersolution of (1.1.1) on $[0,t_0(\tau)]\times\Omega$, we have $v(t,x)=y(t)\geq u(t,x)$.

We thus deduce from the comparison principle that $u\leq v$ in $\mathcal{Q}$, and the lemma follows. □

5.4 Proof of Theorems 2.1-2.2

We first establish the following quantitative type I estimate, which is the analogue of Propositions 3.1 and 4.2.

Proposition 5.6. *Consider problem* (1.1.1) *with* $\Omega=(-R,R)$ *and let* $\xi\in\Omega$ *satisfy* $|\xi|<R-1$. *Let* $0<d<d_0:=R-1-|\xi|$, $\mu>\mu_0(p)$, $e\in(d,\infty]$ *and* $\tau,\lambda\in(0,1)$. *Set* $I_\ell=(\xi-1-\ell,\xi+1+\ell)$ *and assume that*

$$f\geq\mu\quad\text{in }\overline{I}_0\qquad\text{and}\qquad f<\lambda\mu\quad\text{in }(\overline{\Omega}\cap\overline{I}_e)\setminus I_d.\tag{5.4.1}$$

Then the touchdown time T *verifies* $T>t_0:=\frac{1-\tau^{p+1}}{(p+1)\|f\|_\infty}$ *and* u *satisfies the type I estimate*

$$\big[1-u\big(t,\xi\pm(1+d)\big)\big]^{p+1}\geq(p+1)\overline{\varepsilon}\mu(T-t)\quad\text{for all }t\in[t_0,T),\tag{5.4.2}$$

with

$$\overline{\varepsilon}=\sup_{(\beta,K)\in\widehat{\mathcal{A}}}\hat{\varepsilon}(\beta,K),\qquad\hat{\varepsilon}(\beta,K)=\frac{1}{2}\left(\frac{\beta-d}{\beta}\right)^{p+1}S(t_0,\beta)G_1^*(\tau,t_0,\beta,K,\lambda),\tag{5.4.3}$$

$$\widehat{\mathcal{A}}=\mathcal{A}_3:=\Big\{(\beta,K)\in(d,\bar{d})\times(0,p],\quad K\mu\beta^2\leq\tfrac{p(p+2)-K}{p},\quad\delta_1(K)+\delta_2(\beta,K)\leq 1\Big\},\tag{5.4.4}$$

where $\bar{d}=\min\big[d_0,(d+e)/2\big]$, S *is defined in* (1.3.5) *and* G_1^*,δ_1,δ_2 *are defined in Theorem 2.1.*

Proof. Let $(\beta, K) \in \mathcal{A}_3$ and let $a(r) = a_{\beta,K}(r)$ be given by (5.1.2). By the proof of Lemma 3.6 with $q = 1$, $\eta = 1$, using the lower bound (5.2.11), (5.2.13) for u_t in Lemma 5.4, instead of (3.3.14), we have

$$(1-u(t,x))^{p+1} \geq (p+1)\varepsilon\mu a(|x-\xi|)(T-t) \quad \text{in } [t_0,T) \times I_\beta,$$

with

$$\varepsilon = S(t_0,\beta) \inf_{x \in I_\beta} \Big(1 + p\mu\tilde{\Lambda}(t_0,x)\Big) \frac{e^{t_0\Delta_{\mathbb{R}}}\chi_{I_0}(x)}{h(u(t_0,x))a(|x-\xi|)}. \tag{5.4.5}$$

In particular, by (5.1.2), we have

$$\big[1-u\big(t,\xi \pm (1+d)\big)\big]^{p+1} \geq (p+1)\left(\frac{\beta-d}{\beta}\right)^{p+1} \varepsilon\,\mu(T-t), \qquad \text{for all } t \in [t_0,T).$$

We now estimate ε from below. Recalling

$$\begin{aligned} e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r) &= \frac{1}{\sqrt{4\pi t}}\int_{-1}^{1} e^{-\frac{(r-y)^2}{4t}}\,dy = \frac{1}{\sqrt{\pi}}\int_{\frac{r-1}{2\sqrt{t}}}^{\frac{r+1}{2\sqrt{t}}} e^{-Z^2}\,dZ \\ &= \frac{1}{2}\Big[\mathrm{erf}\Big(\frac{r+1}{2\sqrt{t}}\Big) + \mathrm{erf}\Big(\frac{1-r}{2\sqrt{t}}\Big)\Big] \end{aligned} \tag{5.4.6}$$

and setting $r = |x-\xi|$, we have

$$\begin{aligned} \tilde{\Lambda}(t,x) &\geq \frac{1}{2}S(t,\beta)\int_0^t \big(1-Y(s)\big)^{-\frac{p}{p+1}-1}\left[\mathrm{erf}\left(\frac{r+1}{2\sqrt{t-s}}\right) + \mathrm{erf}\left(\frac{1-r}{2\sqrt{t-s}}\right)\right] ds \\ &= S(t,\beta)\Lambda(t,r), \end{aligned}$$

where Λ is defined in the statement of Theorem 2.1.

We next proceed to estimate the factor $h(u(t_0,x))$ in (5.4.5) from above. We recall from (3.3.9) that, since $K \leq p$, the function h is monotone increasing as a function of u. Hence, we shall use the upper estimate given in Lemma 5.5 in $I = \big(\xi+1+d, \min(\xi+1+e, R)\big)$ with $N = \lambda\mu$. For this, we note that, for all $x \in [\xi, \xi+1+\beta]$, we have $\mathrm{dist}(x, \Omega \setminus I) = (x-\xi-1-d)_+$ if $R \leq \xi+1+e$, and that this is still true if $R > \xi+1+e$ due to $\beta < \bar{d} \leq (d+e)/2$. It follows from (5.3.2) that, for all $x \in [\xi, \xi+1+\beta]$, we have

$$u(t_0,x) \leq y(t_0)\theta_{\lambda\mu}(x) = (1-\tau)\tilde{u}(|x-\xi|),$$

hence

$$h(u(t_0,x)) \leq W(|x-\xi|), \tag{5.4.7}$$

where $\tilde{u}$ and $W = W_{\tau,K,\lambda}$ are respectively given by (2.1.8) and (2.1.7). Moreover, (5.4.7) remains true for all $x \in I_\beta$ (applying Lemma 5.5 in $I = \big(\max(\xi-1-e,-R), \xi-1-d\big)$).

Since $e^{t\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)$ and $\Lambda(t,r)$ are even and nonincreasing with respect to $r > 0$ and $a(r)$ is even and constant on $[0, r_0]$, it follows from (5.4.7) that

$$\varepsilon \geq S(t_0,\beta) \inf_{r \in (r_0, 1+\beta)} \Big(1 + p\mu S(t_0,\beta)\Lambda(t_0,r)\Big) \frac{e^{t_0\Delta_{\mathbb{R}}}\chi_{(-1,1)}(r)}{W(r)a(r)}.$$

This combined with (5.4.6) yields

$$\begin{aligned} \varepsilon &\geq \frac{1}{2}S(t_0,\beta) \inf_{r \in (0,1+\beta)} \Big(1 + p\mu S(t_0,\beta)\Lambda(t_0,r)\Big) \frac{\mathrm{erf}\Big(\frac{r+1}{2\sqrt{t}}\Big) + \mathrm{erf}\Big(\frac{1-r}{2\sqrt{t}}\Big)}{W(r)a(r)} \\ &= \frac{1}{2}S(t_0,\beta)G_1^*(\tau, t_0, \beta, K, \lambda). \end{aligned}$$

The conclusion then follows by taking the supremum over $(\beta, K) \in \mathcal{A}_3$. $\square$

Proof of Theorems 2.1-2.2. So as to prove both results at the same time, we set $m = 0$, $d_2 = d_0$ and $d_1 = \infty$ in the case of Theorem 2.1. It suffices to show that there are no touchdown points in $\overline{\mathcal{D}}$, where either:

(i) $\mathcal{D} = (x_{i+1} + 1 + d, x_i - 1 - d)$ for some $i \in \{1, \dots, m-1\}$ (in Theorem 2.2), or

(ii) $\mathcal{D} = (x_0 + 1 + d, R)$ (the case $\mathcal{D} = (-R, x_m - 1 - d)$ is similar).

We set $\Gamma = \partial\mathcal{D}$ in case (i) and $\Gamma = \{x_0 + 1 + d\}$ in case (ii).

Let $\mathcal{A}$ be given by (2.1.4). We first claim that $\mathcal{A}$ is nonempty, so that ρ is well defined and positive. Note that if we take

$$\beta^2 = \frac{p(p+2) - K}{\mu K p},$$

then $\delta_2(\beta, K) = 0$. Now, we can pick $K = p$ and $\beta = \sqrt{\frac{p+1}{p\mu}} \in (d, d_2)$. For this choice of K, β, we have

$$\delta_1(K) = \frac{\arctan(\sqrt{p+1})}{\sqrt{p\mu}} < 1,$$

provided $\mu > \frac{\arctan^2(\sqrt{p+1})}{p}$. The claim follows.

Next, by our assumption on f, we may select $(\beta, K, \tau, \lambda) \in \mathcal{A}$ such that

$$\|f\|_{L^\infty(D)} < \mu \min\left\{\tilde{\varepsilon}(\beta, K, \tau, \lambda), \frac{1}{p+1}\frac{\tau^{p+1}}{(T - t_0(\tau))\mu}, \lambda\right\}$$

with

$$\tilde{\varepsilon}(\beta, K, \tau, \lambda) = \frac{1}{2}\left(\frac{\beta - d}{\beta}\right)^{p+1} S(t_0(\tau), \beta) G^*(\tau, t_0(\tau), \beta, K, \lambda).$$

Let $j \in \{i, i+1\}$ in case (i) and $j = 0$ in case (ii). We shall apply Proposition 5.6 with $\xi = x_j$ and $e = 2d_1 - d \geq 2d_2 - d > d$. By assumption (2.1.2), we have

$$f < \lambda\mu \quad \text{on } \big([x_j - 1 - e, x_j - 1 - d] \cup [x_j + 1 + d, x_j + 1 + e]\big) \cap \overline{\Omega} = (\overline{I}_e \cap \overline{\Omega}) \setminus I_d \subset D.$$

Since $\bar{d} = \min(R - 1 - |\xi|, d_1) \geq \min(d_0, d_1) = d_2 > \beta > d$, it follows from (5.4.2)-(5.4.4) that

$$(1-u)^{p+1} \geq (p+1)\mu\tilde{\varepsilon}(\beta, K, \tau, \lambda)(T - t) \quad \text{on } [t_0, T) \times \Gamma.$$

As $\|f\|_{L^\infty(D)} < \min\{\mu\tilde{\varepsilon}(\beta, K, \tau, \lambda), \frac{1}{p+1}\frac{\tau^{p+1}}{T - t_0(\tau)}\}$, we may apply Lemma 4.1, to deduce that $\mathcal{T} \cap \mathcal{D} = \emptyset$.

Finally, let us show that $\mathcal{T} \cap \Gamma = \emptyset$. By the continuity of f, assumption (1.3.3) remains true for some $\tilde{d} < d$ close to d. Moreover, since $\lambda\mu < \mu \leq \|f\|_\infty$ and since $h(u) = K(1-u) + (1-u)^{-p}$ is nondecreasing on $[0, 1)$ due to $K \leq p$, the function $W_{\tau,K,\lambda}(r)$ defined by (2.1.7), (2.1.8) is increasing with respect to $d > 0$. It follows that the supremum in (1.3.4) is a nonincreasing function of $d \in (0, d_2)$, and we deduce that $\mathcal{T} \cap \Gamma = \emptyset$. This concludes the proof. □

Remark 5.7. *(i) Due to the search for a more precise control of $h(u(t_0, \cdot))$ to increase ε in* (5.4.5)*, we have to make a smallness assumption on f on both sides of the bump in Proposition 5.6 (cf.* (5.4.1)*). For this reason, excluding touchdown on a single interval in Theorems 2.1-2.2 would require a smallness condition on some additional intervals,* *unlike in Theorems 1.1-1.2. For simplicity, we have refrained from giving such a formulation of Theorems 2.1-2.2 and have restricted ourselves to a more global statement.*

(ii) We could use the more precise formula (5.2.12) *instead of* (5.2.13). *However, numerical tests indicate that the difference is extremely small, while the computational time is considerably larger.*

6 Numerical procedures

6.1 Iterative procedure for the optimization problem

In this section, we describe the iterative procedure that we use to find an accurate lower estimate for the solution of the optimization problem (1.3.4) giving the threshold ρ in Theorems 1.1-1.2. The procedure consists of three steps:

Step 1: First exploration. We first use a simple discretized exploration of the optimization set $\mathcal{A}$ for (τ, β, K). For this, we iterate in β, τ and K as follows:

• We initialize β by setting

$$\beta_0 = \min\left(1 + d, \frac{d_0 + d}{2}\right).$$

Then, for a chosen value ε_β of the discretization parameter in β, we increment β in the interval (d, d_0), first increasingly (i.e., $\beta_{i+1} = \beta_i + \varepsilon_\beta$) and then decreasingly (i.e., $\beta_{j+1} = \beta_j - \varepsilon_\beta$). Note that additional stopping conditions will be given below.

• For each β, we initialize K by setting

$$K_0 = \max\left(\frac{p}{\mu\beta^2} - \frac{1}{p+1}, \varepsilon_K\right),$$

where ε_K is a chosen value of the discretization parameter in K, and then increment K increasingly, i.e. $K_{i+1} = K_i + \varepsilon_K$.

• For each couple (β, K), we compute $\delta(\beta, K)$ defined in (1.3.5). If it is less than or equal to 1, we then iterate in τ. This is done by picking n_τ equidistant points in the interval $\left(\frac{\mu}{2\mu-\mu_0}, 1\right)$, where n_τ is a chosen number of discretization points. (The corresponding ε_τ is thus $\left(1 - \frac{\mu}{2\mu-\mu_0}\right)/n_\tau$.)

• For each such (β, K, τ), we then compute an approximation of ρ, given by

$$\tilde{\rho} := \frac{1}{2}\left(\frac{\beta - d}{\beta}\right)^{p+1} \frac{S\big(t_0(\tau), \beta\big)}{K + \tau^{-p}} \min\Big\{\tilde{H}\big(t_0(\tau), \beta\big), \tilde{G}\big(t_0(\tau), \beta, K\big)\Big\}, \tag{6.1.1}$$

where $\tilde{H}(t, \beta)$ (resp., $\tilde{G}(t, \beta, K)$) is a suitable approximation of $H(t, \beta)$ (resp., $G(t, \beta, K)$), and $S(t, \beta)$ is given in (1.3.5). To define $\tilde{G}$, $\tilde{H}$, we set

$$\begin{aligned} N_H(x) &= \operatorname{erf}\left(\frac{1}{\sqrt{t}}\left(1 + \frac{\beta x}{2}\right)\right) - \operatorname{erf}\left(\frac{\beta x}{2\sqrt{t}}\right), \quad D_H(x) = (1-x)^{p+1}, \\ N_G(x) &= \operatorname{erf}\left(\frac{2 - (1-x)\delta}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{(1-x)\delta}{2\sqrt{t}}\right), \quad D_G(x) = (\Gamma^2 + 1)^{\frac{\alpha}{2}} \cos^\alpha(Ax) \end{aligned} \tag{6.1.2}$$

and recall that

$$H(t,\beta)=\inf_{0<x<1}\frac{N_H(x)}{D_H(x)},\qquad G(t,\beta,K)=\inf_{0<x<1}\frac{N_G(x)}{D_G(x)},$$

where $\delta=\delta(\beta,K),\Gamma,A,\alpha$ are defined in (1.3.5). We then set

$$\tilde H(t,\beta):=\min_{0\le i\le n_x}\frac{N_H(x_i)}{D_H(x_i)},\qquad \tilde G(t,\beta,K):=\min_{0\le i\le n_x}\frac{N_G(x_i)}{D_G(x_i)},\tag{6.1.3}$$

where $x_i=i/n_x$ for $i=0,1,\cdots,n_x$, and n_x is the chosen number of discretization points in the interval $[0,1]$. For each i, the corresponding quotients are computed using the error function provided by $Matlab$.

• We define the variables ρ_{opt} and β_1^*,K_1^*,τ_1^*, which respectively stand for the largest value of $\tilde\rho$ obtained so far, and for the corresponding values of the parameters β,K,τ. These variables are updated after each iteration.

• To avoid unnecessary computations, we also observe that we can (dynamically) further restrict the ranges of β,K,τ, as follows:

$$\beta\ge\frac{d}{1-(2\rho_{opt})^{1/(p+1)}},\qquad K\le\frac{1}{2\rho_{opt}}-1,\qquad \tau\ge(2\rho_{opt})^{1/p}.\tag{6.1.4}$$

Indeed, since

$$\left(\tfrac{\beta-d}{\beta}\right)^{p+1}\le 1,\qquad S(t_0(\tau),\beta)\le 1\tag{6.1.5}$$

and

$$H(t,\beta)\le\frac{N_H(0)}{D_H(0)}=\mathrm{erf}\Big(\frac{1}{\sqrt t}\Big)\le 1,\quad G(t,\beta,K)\le\frac{N_H(1)}{D_H(1)}=\mathrm{erf}\Big(\frac{1}{\sqrt t}\Big)\le 1,\tag{6.1.6}$$

we have

$$\tilde\rho\le\frac12\min\Big[\Big(1-\frac{d}{\beta}\Big)^{p+1},\frac{1}{K+1},\tau^p\Big],$$

so that any choice of (K,β,τ) violating at least one of the conditions in (6.1.4) will lead to values $\tilde\rho<\rho_{opt}$.

Step 2: Refined exploration. We make a finer second exploration near the parameters β_1^*,K_1^* and τ_1^* obtained in Step 1. This is done by repeating Step 1 on the new ranges

$$[\beta_1^*-\varepsilon_\beta,\beta_1^*+\varepsilon_\beta],\quad [K_1^*-\varepsilon_K,K_1^*+\varepsilon_K],\quad [\tau_1^*-\varepsilon_\tau,\tau_1^*+\varepsilon_\tau],\tag{6.1.7}$$

taking a chosen number of equidistant points in each interval. The values of β,K,τ corresponding to the largest $\tilde\rho$ obtained are denoted β^*,K^*,τ^*.

Step 3: Lower estimate of ρ. Finally, for the parameters (β^*,K^*,τ^*) selected in Step 2, we recompute a "safer" approximation of the supremum ρ by looking this time for a lower estimate. This is done by setting

$$\overline\rho:=\frac12\left(\frac{\beta^*-d}{\beta^*}\right)^{p+1}\frac{S\big(t_0(\tau^*),\beta\big)}{K^*+\tau^{*-p}}\min\Big\{\overline H\big(t_0(\tau^*),\beta^*\big),\overline G\big(t_0(\tau^*),\beta^*,K^*\big)\Big\},\tag{6.1.8}$$

where $\overline H(t,\beta)$ (resp., $\overline G(t,\beta,K)$) is now a suitably accurate lower bound of $H(t,\beta)$ (resp., $G(t,\beta,K)$).

To compute $\overline G$, $\overline H$, this time we choose another (larger) number n_x of equidistant discretization points of the interval $[0,1]$, we denote $x_i=i/n_x$ for $i=0,1,\cdots,n_x$, and then set

$$\overline H(t,\beta):=\min_{0\le i\le n_x-1}\frac{N_H(x_{i+1})}{D_H(x_i)},\qquad \overline G(t,\beta,K):=\min_{0\le i\le n_x-1}\frac{N_G(x_{i+1})}{D_G(x_i)},\tag{6.1.9}$$

where N_H, D_H, N_G, D_G are given by (6.1.2). For each i, the corresponding quotients are computed using the error function provided by *Matlab*. Observe that the functions $N_H(x), D_H(x)$, $N_G(x), D_G(x)$) are monotonically decreasing in $[0,1]$ (owing to $A \in (0,\pi/2)$ and $\delta \in (0,1]$ for $(\beta,\tau,K) \in \mathcal{A}$). Consequently,

$$\inf_{x_i \le x \le x_{i+1}} \frac{N_H(x)}{D_H(x)} \ge \frac{N_H(x_{i+1})}{D_H(x_i)}, \qquad \inf_{x_i \le x \le x_{i+1}} \frac{N_G(x)}{D_G(x)} \ge \frac{N_G(x_{i+1})}{D_G(x_i)},$$

hence

$$H(t,\beta) \ge \overline{H}(t,\beta), \qquad G(t,\beta,K) \ge \overline{G}(t,\beta,K). \tag{6.1.10}$$

Moreover, the discretization errors can be estimated by

$$H - \overline{H} \le \frac{N_H(x_{i_0}) - N_H(x_{i_0+1})}{D_H(x_{i_0})}, \qquad G - \overline{G} \le \frac{N_G(x_{i_1}) - N_G(x_{i_1+1})}{D_G(x_{i_1})},$$

where i_0, i_1 are the indices for which the respective minima in (6.1.9) are achieved, so that n_x can be adjusted to guarantee a satisfactory error level (say, 10^{-4}).

Remark 6.1. *(i) In Step 3, it is consistent to choose a larger n_x in order to have good lower estimates of H, G, while in Steps 1 and 2 we need to choose coarser partitions of the interval* $[0,1]$*, in order to keep the computational cost of the method within feasible limits.*

(ii) In the above procedure, the only possible sources of errors in excess on $\overline{\rho}$ are the round-off machine errors and the numerical errors in the Matlab evaluations (for instance those of erf*). In principle this can be guaranteed with any reasonably prescribed safety margin.*

In the following table, for $p = 2$ and each of the values of $\mu, \|f\|_\infty, d, d_0$ considered in Table 3.1, we give the numerical approximation of the optimal parameters τ^*, β^*, K^* found by the above procedure, as well as the lower bounds $\overline{H}(t_0(\tau^*), \beta^*)$, $\overline{G}(t_0(\tau^*), \beta^*, K^*)$ for H, G and the approximated semigroup comparison constant $S\big(t_0(\tau^*), \beta^*\big)$.

p	μ	$\|f\|_\infty$	d	d_0	τ^*	β^*	K^*	$\overline{H}$	$\overline{G}$	S	$\overline{\rho}$
2	1	1.1	0.1	5	0.7904	1.7400	1.4787	0.9220	0.9140	0.8452	**0.1050**
2	1.25	1.3	0.1	3	0.8094	1.5600	1.1117	0.8807	0.8754	0.7429	**0.1010**
2	2	2.25	0.1	4	0.8111	1.2200	0.7184	0.7629	0.7650	0.8966	**0.1182**
2	2	2.25	0.05	4	0.8201	1.1900	0.8228	0.7825	0.7869	0.9004	**0.1341**
2	3	3.5	0.01	5	0.8036	0.9900	0.7402	0.7757	0.7710	0.9510	**0.1554**
2	4	4.1	0.05	5	0.8001	0.9100	0.7407	0.8211	0.8182	0.9574	**0.1436**
2	4	4.1	0.01	5	0.8286	0.8700	0.6705	0.7582	0.7517	0.9623	**0.1643**
2	4	7	0.01	5	0.7905	0.7300	0.9385	0.7137	0.7132	0.9739	**0.1313**
2	6	6.2	0.01	10	0.8063	0.7300	0.7879	0.8252	0.8223	0.9917	**0.1682**
2	10	10	0.005	10	0.8037	0.5850	0.6331	0.7794	0.7832	0.9948	**0.1732**
1.5	10	10	0.005	10	0.7461	0.5850	0.6298	0.8390	0.8335	0.9932	**0.1857**
1	10	10	0.005	10	0.6611	0.5850	0.6000	0.8643	0.8643	0.9909	**0.1992**
0.5	10	10	0.005	10	0.5724	0.5850	0.4800	0.7972	0.7991	0.9877	**0.2157**

Table 3.3 – Numerical parameters corresponding to the examples for Theorems 1.1-1.2.

In practice we use $\varepsilon_\beta = \varepsilon_K = 0.1, n_\tau = 10$ for Step 1, whereas for Step 2 we take 10 equidistant points in the intervals (6.1.7). As for the approximations of H, G we take $n_x = 20$ in Steps 1 and 2. For the lower estimates in Step 3 we have chosen $n_x = 50000$ to compute $\overline{H}$ and $n_x = 2000$ to compute $\overline{G}$, which guarantees an error level no larger than 10^{-4}.

6.2 Numerical lower estimates for Theorem 2.3

The numerical procedure is similar to that in Section 6.1, this time for the optimization problem (2.2.2). The main difference is that we also need to iterate in the parameter $\eta \in (0,1)$ for each couple (β, K). As before, if $\delta(\beta, K, \eta) \leq 1$, we then iterate in τ, but now in the whole interval $(0,1)$.

In the following table, for $p = 2$ and each of the values of $\mu, \|f\|_\infty, d, d_0$ considered in Table 3.2, we give the numerical approximation of the optimal parameters $\tau^*, \eta^*, \beta^*, K^*$ found by the above procedure, as well as the lower bounds $\overline{H}(t_0(\tau^*), \beta^*)$, $\overline{G}(t_0(\tau^*), \beta^*, K^*, \eta^*)$, $\overline{G}(\bar{T}, \beta^*, K^*, \eta^*)$ for H, G and the approximated semigroup comparison constants $S\big(t_0(\tau^*), \beta^*\big)$, $S\big(\overline{T}), \beta^*\big)$. We also give the second term of the minimum in (2.2.2):

$$\rho_2(\tau^*) := \frac{1}{p+1} \frac{\tau^{p+1}}{(T - t_0(\tau^*))\mu}. \tag{6.2.1}$$

In practice, this term is observed to be larger than the first one. However, we have been unable to find a proof of this without assuming $\tau \geq \frac{\mu}{2\mu - \mu_1}$, which induces the extra hypothesis $\mu > \mu_1$ (cf. Theorems 1.1-1.2).

μ	$\|f\|_\infty$	d	d_0	τ^*	η^*	β^*	K^*	$\overline{G}(\overline{T})$	$\overline{H}(t_0)$	$\overline{G}(t_0)$	$S(\overline{T})$	$S(t_0)$	$\rho_2(\tau)$	$\overline{\rho}$
0.7	0.8	0.01	8	0.5800	0.8000	2.7100	0.8000	0.6352	0.7322	0.9465	0.6152	0.8492	0.1405	**0.0815**
0.6	0.65	0.05	10	0.5600	0.8400	3.0500	1.000	0.5770	0.7230	0.9311	0.6289	0.8712	0.0977	**0.0714**
0.5	0.6	0.001	6	0.3800	0.8800	3.8010	1.1000	0.4824	0.3440	0.9323	0.1357	0.6551	0.0187	**0.0137**
0.5	0.5	0.01	7	0.4400	0.8400	4.0100	0.9000	0.4907	0.4068	0.8983	0.2167	0.6865	0.0304	**0.0228**

Table 3.4 – Numerical parameters corresponding to the examples for Theorem 2.3.

6.3 Numerical lower estimates for Theorems 2.1-2.2

We use a similar numerical procedure to that presented in Section 6, this time for the optimization problem (2.1.3). As a difference, the range for K is now

$$\varepsilon_K \leq K \leq \min\Big(p, \frac{p(p+2)}{1 + p\mu\beta^2}\Big). \tag{6.3.1}$$

Also, we need to test the condition $\delta_1(K) + \delta_2(\beta, K) \leq 1$ instead of $\delta(\beta, K) \leq 1$. Moreover, the additional stopping conditions in (6.1.4) are not available. However, unlike in Section 6.1, this is not essential since the range for K is already bounded, owing to (6.3.1).

We also need to iterate the procedure with respect to the additional parameter λ. Namely, we carry out the exploration of the parameter λ in a certain subinterval of $(0,1)$ for each fixed admissible (β, K, τ). To restrict the range of λ, let us first rewrite (2.1.3) under the form $\rho = \min(\hat{\rho}, \lambda)$ and observe that G^* in (2.1.3) is a nonincreasing function of $\lambda \in (0,1)$ (due to (2.1.7), (2.1.8)). Since our numerical tests reveal that the method does not produce values of $\hat{\rho}$ larger than 0.3 for $p = 2$, we intitialize with $\lambda_0 = 0.3$. We then compute the corresponding numerical value of $\hat{\rho}$. As long as $\hat{\rho}_i < \lambda_i$, we iterate by setting $\lambda_{i+1} = \lambda_i - 0.01$. Once $\hat{\rho}_i \geq \lambda_i$, we stop and retain the largest between $\rho_i = \min(\hat{\rho}_i, \lambda_i)$ and $\rho_{i-1} = \min(\hat{\rho}_{i-1}, \lambda_{i-1})$. The values of λ that we used in pratice remain in the interval $[0.2, 0.3]$ (for $p = 2$) and the step $\Delta\lambda = 0.01$ turns out to be sufficient since the results are not very sensitive to the variations of λ.

Let us turn our attention to the main term in (2.1.3), that we rewrite as

$$G^* = \inf_{r\in(r_0,1+\beta)} \mathcal{G}(r), \qquad \text{with } \mathcal{G}(r) = \Big(1 + p\mu S(t,\beta)\Lambda(t,r)\Big)\,\frac{\operatorname{erf}\Big(\frac{r+1}{2\sqrt{t}}\Big) + \operatorname{erf}\Big(\frac{1-r}{2\sqrt{t}}\Big)}{W_{\tau,K,\lambda}(r)a_{\beta,K}(r)} \tag{6.3.2}$$

(for fixed values of the parameters $\tau, t, \beta, K, \lambda$), where

$$\Lambda(t,r) = \frac{1}{2}\int_0^t \big(1 - Y(s)\big)^{-\frac{p}{p+1}-1}\left[\operatorname{erf}\left(\frac{r+1}{2\sqrt{t-s}}\right) + \operatorname{erf}\left(\frac{1-r}{2\sqrt{t-s}}\right)\right] ds \tag{6.3.3}$$

and the functions S, W, a, Y are defined in Theorem 2.1.

As for the time integral in $\Lambda(t,r)$, in the exploration process we use a coarser partition of the interval $[0,t]$ and the Simpson method, in order to have good approximations while keeping a reasonable computational time. However, in the last step, once we have chosen the approximated optimal parameters $(\tau, \beta, K, \lambda)$, we use the following monotonicity properties of the integrand and the rectangle rule in order to give a safe lower estimate of the ratio ρ.

First, although $Y(s)$ is not monotone in general, we however note that, since $S(s,0)$ decreases with s, we can estimate

$$Y(s) \geq \tilde{Y}(t,s) = S(t,0)\operatorname{erf}\left(\frac{1}{\sqrt{s}}\right)\frac{(p+1)\mu}{2}s, \qquad 0 < s < t. \tag{6.3.4}$$

We will see numerically in Table 3.5 that $S(t_0,\beta)$, which satisfies $S(t_0,\beta) \leq S(t_0,0) \leq 1$, is of the order ~ 0.99 in our examples, so that the loss from from estimate (6.3.4) is quite small. Now, we observe that $\tilde{Y}(t,s)$ is monotonically increasing with respect to $s \in (0,t)$, due to

$$\frac{\partial}{\partial s}\left[\sqrt{\pi}\operatorname{erf}\left(\frac{1}{\sqrt{s}}\right)s\right] = 2\int_0^{\frac{1}{\sqrt{s}}} e^{-t^2}dt - \frac{e^{-\frac{1}{s}}}{\sqrt{s}} \geq 2\frac{e^{-\frac{1}{s}}}{\sqrt{s}} - \frac{e^{-\frac{1}{s}}}{\sqrt{s}} > 0.$$

For fixed (t,r), let $\{[t_j, t_{j+1}],\ j \in J\}$ be a partition of the interval $[0,t]$, and set $\tau_j = t_j$ if $r \leq 1$ and $\tau_j = t_{j+1}$ if $r > 1$. We can therefore estimate $\Lambda(t,r)$ in (6.3.3) by

$$\Lambda(t,r) \geq \frac{1}{2}\sum_{j\in J}(t_{j+1}-t_j)(1-\tilde{Y}(t,t_j))^{-\frac{p}{p+1}-1}\Big[\operatorname{erf}\Big(\frac{r+1}{2\sqrt{t-t_j}}\Big) + \operatorname{erf}\Big(\frac{1-r}{2\sqrt{t-\tau_j}}\Big)\Big]_+. \tag{6.3.5}$$

Since we are computing a lower estimate only in the last step, we can choose a much finer partition of the interval $[0,t]$, so as to keep enough accuracy in the rectangle rule. For the examples of Table 3.5 we have used a partition of $[0,t]$ in 100 equal subintervals, while for the exploration process, we have used only 10.

Finally, we observe in (6.3.3) that $\Lambda(t,r)$ is monotonically decreasing as a function of $r > 0$, as well as the function $W_{\tau,\beta,K}$ defined in (2.1.7) (owing to $K \leq p$). Therefore, like in the numerical procedure described in Section 6, both the numerator and denominator of $\mathcal{G}(r)$ in (6.3.2) are monotonically decreasing as functions of r. Hence, applying the same strategy as in Section 6 (cf. (6.1.9)-(6.1.10)) and then using (6.3.5) at the discretization points $r = r_i$, we can compute a safe lower estimate of the infimum G^* in (6.3.2) once the parameters β, K, τ are chosen. For the examples in Table 3.5 we used a partition of 5000 subintervals, while in the exploration process, we used only 20.

The following Table 3.3 is the analogue for Theorems 2.1-2.2 of Table 3.3, using the numerical procedure described above. We point out that, due to the finer mesh in the final step, the difference between the lower estimate $\overline{\rho}(\tau^*,\beta^*,K^*)$ and the explored value $\tilde{\rho}(\tau^*,\beta^*,K^*)$ is no larger than 10^{-3} for any of these examples.

p	μ	$\|f\|_\infty$	d	d_0	τ	β	K	$\overline{G^*}$	$S(t_0,\beta)$	$\rho_2(\tau)$	$\overline{\lambda}$	$\overline{\rho}$
2	1	1.1	0.1	5	0.8000	1.6600	0.6800	0.5474	0.9899	0.4521	0.23	**0.2249**
2	1.25	1.3	0.1	3	0.8000	1.5400	0.5600	0.5735	0.9809	0.5423	0.24	**0.2299**
2	2	2.25	0.1	4	0.8000	1.1400	0.6200	0.5601	0.9929	0.6482	0.22	**0.2111**
2	2	2.25	0.05	4	0.7800	1.2300	0.5000	0.5712	0.9923	0.6272	0.26	**0.2502**
2	3	3.5	0.01	5	0.8000	0.9300	0.6400	0.5591	0.9968	0.7106	0.28	**0.2698**
2	4	4.1	0.05	5	0.7800	0.9100	0.4200	0.5930	0.9971	0.8071	0.26	**0.2495**
2	4	4.1	0.01	5	0.7200	1.0100	0.3200	0.5726	0.9965	0.7631	0.28	**0.2769**
2	4	7	0.01	5	0.7400	0.7700	0.6600	0.4653	0.9981	0.5327	0.23	**0.2232**
2	6	6.2	0.01	10	0.7800	0.7300	0.4600	0.5957	0.9994	0.8529	0.29	**0.2856**
2	10	10	0.005	10	0.8000	0.5450	0.5200	0.6007	0.9997	0.9310	0.3	**0.2921**
1.5	10	10	0.005	10	0.7400	0.5450	0.3800	0.6349	0.9996	0.9074	0.32	**0.3101**
1	10	10	0.005	10	0.6800	0.5250	0.3000	0.6762	0.9995	0.8755	0.34	**0.3315**
0.5	10	10	0.005	10	0.6200	0.4650	0.2600	0.7503	0.9993	0.8231	0.37	**0.3689**

Table 3.5 – Numerical parameters corresponding to the examples for Theorems 2.1-2.2.

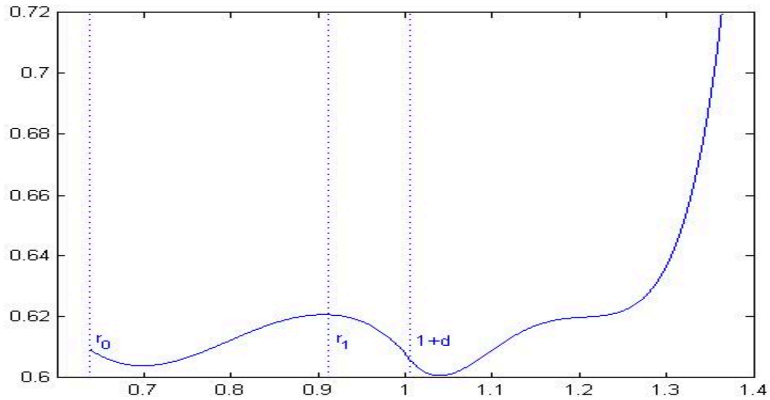

Figure 3.6 – The function $\mathcal{G}(r)$ in (6.3.2) for $p = 2$, $\mu = \|f\|_\infty = 10$, $d = 0.005$, $d_0 = 10$.

Remark 6.2. *(i) We observe from Tables 3.3 and 3.5 that, in our examples, the factors* $\left(\frac{\beta-d}{\beta}\right)^{p+1}$ *and* $S = S(t_0,\beta)$ *in formulae* (1.3.4) *and* (2.1.3) *are close to* 1 *and have only small influence on the value of* ρ. *We also see that the quantity* $\rho_2(\tau)$ *in* (6.2.1) *does not affect* ρ *(indeed, it corresponds to the second term in* (2.1.3)*, and turns out to be larger than the first term). It follows that* $\rho \approx \frac{1}{2}G^*$ *in* (2.1.3)*, once the optimal numerical parameters have been selected.*

(ii) Recall that G^* *is obtained as the infimum of the function* $\mathcal{G}(r)$ *in* (6.3.2). *A plot of* $\mathcal{G}(r)$ *is given in Figure 3.6 (for the last example with* $p = 2$ *in Table 3.5, and following the numerical method described above). We observe that* $\mathcal{G}(r)$ *appears to be neither monotone nor convex.*

(iii) The quantity G^* *could possibly be increased by taking into account the enhancing effect on* u_t *of the positive values of* f *outside the interval* $I_0 = (x_0 - 1, x_0 + 1)$*, so as to improve estimate* (5.2.11) *and the ratio in* (5.4.5). *This could be done at the expense of* **lower** *assumptions on* f *in the region where we want to rule out touchdown. We have not pursued this further, since this would deviate too much from the main line of the article.*

7 Appendix 1: Comparison estimates for the heat semigroup

In this appendix, we establish the quantitative comparison properties for the heat semigroup, that we have used in order to express the infima in (3.3.5) and (3.3.6) in terms of the error function. Here $e^{t\Delta_\Omega}$ and $e^{t\Delta_{\mathbb{R}}}$ respectively denote the Dirichlet heat semigroup on Ω and the heat semigroup on $\mathbb{R}$.

Proposition 7.1. *Let $\Omega = (-R, R)$ and assume that $I_0 := (x_0 - 1, x_0 + 1) \subset\subset \Omega$. Let $\phi \in L^\infty(\mathbb{R})$ be a nonnegative function, symmetric with respect to x_0, nonincreasing in $|x - x_0|$ and supported in I_0. Then, for all $t > 0$, we have*

$$\left(e^{t\Delta_\Omega}\phi\right)(x) \geq e^{-\lambda_\ell t}\Big[1 - e^{-d_1(R-x)/t}\Big]\left(e^{t\Delta_{\mathbb{R}}}\phi\right)(x), \quad x_0 \leq x \leq R, \tag{7.0.1}$$

and

$$\left(e^{t\Delta_\Omega}\phi\right)(x) \geq e^{-\lambda_\ell t}\Big[1 - e^{-d_2(R+x)/t}\Big]\left(e^{t\Delta_{\mathbb{R}}}\phi\right)(x), \quad -R \leq x \leq x_0, \tag{7.0.2}$$

where $\ell = R - |x_0|$, $d_1 = R - x_0 - 1$, $d_2 = R + x_0 - 1$ and $\lambda_\ell = \left(\dfrac{\pi}{2\ell}\right)^2$.

Remark 7.2. *(i) We note that similar qualitative results follow from known estimates of the Dirichlet heat kernel estimates from [27]:*

$$G(t, x, y) \geq c_1 \min\left[1, \frac{\delta(x)\delta(y)}{t}\right](4\pi t)^{-n/2} e^{-c_2|x-y|^2/t}, \quad 0 < t < T. \tag{7.0.3}$$

The lower bound (7.0.3) is valid in any sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^n$ and the corresponding upper bound is also true. Although estimate (7.0.3) is quite powerful (and requires sophisticated methods), it is not suitable for our needs, since the constants $c_1, c_2, T > 0$ depending on Ω in [27] are not quantitatively estimated.

(ii) Another key feature of estimates (7.0.1)-(7.0.2) in view of Theorems 1.1-1.2 is that they imply

$$\frac{\left(e^{t\Delta_\Omega}\chi_{I_0}\right)(x)}{\left(e^{t\Delta_{\mathbb{R}}}\chi_{I_0}\right)(x)} \to 1, \quad \text{as } t/\delta(x) \to 0, \tag{7.0.4}$$

with quantitative control of the convergence, whereas (7.0.4) does not follow from (7.0.3). Qualitative properties similar to (7.0.4), valid also in higher dimensions, were obtained in [21] by different methods.

For the proof of Proposition 7.1 we will use the following lemma.

Lemma 7.3. *Let $\Omega = (x_0 - \ell, x_0 + \ell)$ and $\phi_1, \phi_2 \in L^\infty(\mathbb{R})$ be nonnegative, symmetric with respect to x_0 and nonincreasing in $|x - x_0|$. Then we have*

$$e^{t\Delta_\Omega}(\phi_1\phi_2) \geq \left(e^{t\Delta_\Omega}\phi_1\right)\left(e^{t\Delta_{\mathbb{R}}}\phi_2\right) \geq \left(e^{t\Delta_\Omega}\phi_1\right)\left(e^{t\Delta_\Omega}\phi_2\right) \quad \text{in } (0,\infty)\times\Omega. \tag{7.0.5}$$

Proof. Assume $x_0 = 0$ without loss of generality and set

$$v(t,\cdot) = e^{t\Delta_\Omega}\phi_1, \quad w(t,\cdot) = e^{t\Delta_{\mathbb{R}}}\phi_2, \quad \underline{\varphi}(t,x) = v(t,x)\,w(t,x), \qquad t > 0, \quad x \in \Omega.$$

Note that for each $t > 0$, the functions $x \mapsto v(t,x)$ and $x \mapsto w(t,x)$ are even in x and nonincreasing for $x \in [0, \ell]$. Therefore,

$$\underline{\varphi}_t - \underline{\varphi}_{xx} = v(w_t - w_{xx}) + w(v_t - v_{xx}) - 2v_x w_x = -2v_x w_x \leq 0, \qquad t > 0, \quad x \in (-\ell, \ell).$$

Since $\underline{\varphi}(0,\cdot) = \phi_1\phi_2$ and $\underline{\varphi}(t,\pm\ell) = 0$, it follows from the maximum principle that $e^{t\Delta_\Omega}(\phi_1\phi_2) \geq \underline{\varphi}$ in $[0,\infty)\times(-\ell,\ell)$, i.e. the first inequality in (7.0.5). The second follows from the maximum principle. $\square$

Proof of Proposition 7.1. It suffices to prove (7.0.1) (changing x to $-x$). Let $v(t,x) = \left(e^{t\Delta_\Omega}\phi\right)(x)$ and let w be the solution of the problem

$$\begin{cases} w_t - w_{xx} = 0, & t > 0,\ x \in (-\infty, R), \\ w(t,R) = 0, & t > 0, \\ w(0,x) = \phi(x), & x \in (-\infty, R). \end{cases}$$

Set $\Omega_1 = (x_0 - \ell, x_0 + \ell) \subset \Omega$. By the maximum principle and (7.0.5), we have

$$v \geq e^{t\Delta_{\Omega_1}}\phi \geq \left(e^{t\Delta_{\Omega_1}}\chi_{\Omega_1}\right)\left(e^{t\Delta_{\mathbb{R}}}\phi\right). \tag{7.0.6}$$

On the other hand, setting $\varphi(x) = \cos\Bigl(\dfrac{\pi(x-x_0)}{2\ell}\Bigr)$, we have

$$e^{t\Delta_{\Omega_1}}\chi_{\Omega_1} \geq e^{t\Delta_{\Omega_1}}\varphi = e^{-\lambda_\ell t}\varphi. \tag{7.0.7}$$

In particular, it follows from (7.0.6), (7.0.7) and the maximum principle that

$$v(t,x_0) \geq e^{-\lambda_\ell t}\left(e^{t\Delta_{\mathbb{R}}}\phi\right)(x_0) \geq e^{-\lambda_\ell t} w(t,x_0).$$

For each $t > 0$, we thus have

$$v(s,x_0) \geq e^{-\lambda_\ell t} w(s,x_0), \quad 0 < s \leq t.$$

It then follows from the maximum principle, applied to $z(s,x) := v(s,x) - e^{-\lambda_\ell t}w(s,x)$ on $[0,t]\times[x_0,R]$ for each $t>0$, that

$$v(t,x) \geq e^{-\lambda_\ell t} w(t,x), \quad t > 0,\ x_0 \leq x \leq R. \tag{7.0.8}$$

Now, w admits the representation

$$w(t,x) = \int_{x_0-1}^{x_0+1} K(t,x,y)\phi(y)\,dy,$$

where

$$K(t,x,y) = (4\pi t)^{-1/2} e^{-(x-y)^2/4t}\Bigl[1 - e^{-(R-x)(R-y)/t}\Bigr]$$

is the Dirichlet heat kernel of the half-line $(-\infty,R)$. For all $t>0$ and $x\in[x_0,R)$, we have

$$\begin{aligned} w(t,x) &\geq \Bigl[1 - e^{-(R-x_0-1)(R-x)/t}\Bigr]\int_{x_0-1}^{x_0+1}(4\pi t)^{-1/2}e^{-(x-y)^2/4t}\phi(y)\,dy \\ &= \Bigl[1 - e^{-(R-x_0-1)(R-x)/t}\Bigr]\left(e^{t\Delta_{\mathbb{R}}}\phi\right)(x). \end{aligned}$$

This combined with (7.0.8) yields the desired estimate. $\square$

8 Appendix 2: Optimality of the cut-off functions $a(r)$

We here justify the claim, made in Section 3.3, about the optimality of the functions $a(r)$ involved in the main auxiliary functional J from (3.1.6), among all possible solutions of the differential inequality (3.3.1).

Proposition 8.1. *Let $q \in [0,1]$, $\eta \in (0,1]$, $R_1 > 1$, $K > 0$, with $K \leq (p+q+1)p/q$ if $q > 0$. Let $h = h(u)$ and $F = F(r,\xi)$ be defined by* (3.1.7) *and* (3.3.2). *Assume that there exists a solution $a \in W^{2,2}([0,R_1])$ of*

$$a''(r) \geq a(r)\, F\Big(r, \frac{a'(r)}{a(r)}\Big), \quad 0 \leq r < R_1, \tag{8.0.1}$$

$$a'(0) = 0, \quad a(1) = 1, \quad a(R_1) = 0, \tag{8.0.2}$$

$$a > 0 \ \text{ and } \ a' \leq 0 \quad \text{ in } [0,R_1). \tag{8.0.3}$$

(i) Then there exist $r_0 \in [0,1)$, $\beta > 0$ with $1+\beta \leq R_1$, and a solution $\bar a \in W^{2,2}([r_0, 1+\beta])$ of

$$\begin{cases} \bar a''(r) = \bar a(r)\, F\Big(r, \dfrac{\bar a'(r)}{\bar a(r)}\Big), & \text{for a.e. } r \in (r_0, 1+\beta), \\[2mm] \bar a'(r_0) = 0, \quad \bar a(1) = 1, \quad \bar a(1+\beta) = 0, \end{cases} \tag{8.0.4}$$

such that

$$0 < \bar a \leq a \ \text{ and } \ \bar a' \leq 0 \quad \text{ in } [r_0, 1+\beta).$$

(ii) Let $\bar a$ be extended by setting $\bar a(r) = \bar a(r_0)$ on $[0,r_0]$. Then $\bar a \in W^{2,2}([0,1+\beta])$ and $\bar a$ is a solution of (8.0.1)*-*(8.0.3) *with R_1 replaced by $1+\beta$. Moreover, for any open interval Ω containing $(-1-\beta, 1+\beta)$, any $\ell \in [1,1+\beta]$ and any $t > 0$, we have*

$$\inf_{x\in(-\ell,\ell)} \frac{e^{t\Delta_\Omega}\chi_{(-1,1)}(x)}{\bar a(|x|)} \geq \inf_{x\in(-\ell,\ell)} \frac{e^{t\Delta_\Omega}\chi_{(-1,1)}(x)}{a(|x|)}. \tag{8.0.5}$$

Proof. (i) **Step 1**. *Preliminaries.* Set

$$F_1(X) = \sup_{u\in(1-\eta,1)} f_1(u,X), \qquad f_1(u,X) = m_1(u)X^2 - M_1(u), \tag{8.0.6}$$

where

$$m_1(u) = \frac{h'^2(u)}{hh''(u)}, \quad M_1(u) = \frac{(p+q)K\mu}{(1-u)^{p+1-q}h(u)}. \tag{8.0.7}$$

We claim that

$$F_1 \text{ is locally Lipschitz continuous on } \mathbb{R}. \tag{8.0.8}$$

We have $m_1 \in C([0,1])$, owing to (5.1.9) and using $K \leq (p+q+1)p/q \leq p(p+1)/q(1-q)$ if $q \in (0,1)$. On the other hand, if $q \in [0,1)$, then $M_1 \in C([0,1))$ and $\lim_{u\to 1} M_1(u) = +\infty$, whereas $M_1 \in C([0,1])$ if $q = 1$. In both cases, for all $X \in \mathbb{R}$, there exists $u(X) \in [1-\eta,1]$ such that $F_1(X) = f_1(u(X),X)$ (with $u(X) \in [1-\eta,1)$ if $q \in [0,1)$). This, combined with (8.0.6) and (8.0.7), yields

$$\begin{aligned} F_1(X) - F_1(Y) &= f_1(u(X),X) - f_1(u(Y),Y) \leq f_1(u(X),X) - f_1(u(X),Y) \\ &= m_1(u(X))(X^2 - Y^2) \leq \|m_1\|_\infty(|X|+|Y|)|X-Y|. \end{aligned}$$

Exchanging the roles of X, Y, claim (8.0.8) follows.

Step 2. *Resolution of* (8.0.4) *for $r < 1$ and comparison.* Set $\hat F_1(\phi) := F_1(\phi) - \phi^2$. Let ψ be the maximal solution of the Cauchy problem

$$\psi' = \hat F_1(\psi), \quad r < 1, \qquad \text{with } \psi(1) = a'(1).$$

Denote by $r^* \in [0,1)$ the endpoint of its interval of existence and set

$$\bar a(r) = \exp\Big[\int_1^r \psi(\tau)\,d\tau\Big] > 0.$$

We claim that there exists $r_0 \in [0,1)$ such that $\bar a \in C^2([r_0,1])$,

$$\frac{\bar a''}{\bar a} = F_1(\psi), \quad r_0 < r < 1, \qquad \bar a'(r_0) = 0,\ \bar a(1) = 1 \tag{8.0.9}$$

and

$$\bar a \le a \text{ and } \bar a' \le 0 \text{ on } [r_0,1]. \tag{8.0.10}$$

Using $\psi = \frac{\bar a'}{\bar a}$, we have $\frac{\bar a''}{\bar a} = \psi' + \psi^2 = \hat F_1(\psi) + \psi^2 = F_1(\psi)$ for all $r \in (r^*,1)$. Setting $\phi = (\log a)' = \frac{a'}{a} \le 0$, by (8.0.1), we have $\phi' = \frac{a''}{a} - \phi^2 \ge \hat F_1(\phi)$ for all $r \in (0,1)$, and $\phi(1) = a'(1) < 0$. Using (8.0.8) and the fact that ϕ, ψ are locally bounded on $(r^*,1]$, for each $r_2 \in (r^*,1)$, it follows that there exists $L > 0$ such that

$$(\phi-\psi)'_+(r) \ge [\hat F_1(\phi(r)) - \hat F_1(\psi(r))]\chi_{\{\phi>\psi\}} \ge -L(\phi-\psi)_+ \quad \text{a.e. in } (r_2,1).$$

Since $(\phi-\psi)(1) = 0$, we conclude that $(\phi-\psi)_+ = 0$ on $(r^*,1]$, hence

$$\phi(r) \le \psi(r), \quad r^* < r < 1. \tag{8.0.11}$$

It follows that

$$a(r) = \exp\Big[\int_1^r \phi(\tau)\,d\tau\Big] \ge \exp\Big[\int_1^r \psi(\tau)\,d\tau\Big] = \bar a(r), \quad r^* < r < 1.$$

Now first consider the case when

$$r^* = 0 \text{ and } \psi \text{ is continuous on } [0,1]. \tag{8.0.12}$$

Then, since $\psi(0) \ge \phi(0) = 0$ and $\psi(1) < 0$, it follows that there exists a largest $r_0 \in [0,1)$ such that $\psi(r_0) = 0$.

Next consider the case when (8.0.12) is not true. Then we must have $\lim_{r\to r^*} |\psi(r)| = \infty$. On the other hand, since $m_1(u) \le 1$ by (3.3.26) and (5.1.8), we have $\psi' = \hat F_1(\psi) = F_1(\psi) - \psi^2 \le 0$ on $(r^*,1]$. It follows that $\lim_{r\to r^*} \psi(r) = +\infty$ and there again exists a largest $r_0 \in (r^*,1)$ such that $\psi(r_0) = 0$.

In both cases we have $\bar a \in C^2([r_0,1])$, (8.0.9) and (8.0.10).

Step 3. *Resolution of* (8.0.4) *for* $r > 1$ *and comparison.* By (3.3.26), (5.1.8), we have

$$F(r,X) = F_2(X) := \frac{p}{p+1}X^2, \qquad r > 1.$$

In particular, we have $a''(r) \ge 0$ due to (8.0.1). Since $a(R_1) = 0 < a(1)$, it follows that $a'(1) < 0$. Now set

$$\bar a(r) = \Big(\frac{1+\beta-r}{\beta}\Big)^{p+1}, \quad 1 \le r \le 1+\beta \qquad \text{where } \beta = -\frac{p+1}{\bar a'(1)} > 0.$$

An immediate computation shows that

$$\frac{\bar a''}{\bar a} = F_2\Big(\frac{\bar a'}{\bar a}\Big), \quad 1 < r < 1+\beta. \tag{8.0.13}$$

Moreover, $\bar{a}'(1) = -(p+1)/\beta = a'(1)$ and $\bar{a} \in W^{2,2}(1, 1+\beta)$. Now define $\hat{F}_2(X) = F_2(X) - X^2 = -\frac{X^2}{p+1}$, along with

$$\phi = \frac{a'}{a}, \quad 1 \le r < R_1 \quad \text{ and } \quad \psi(r) = \frac{\bar{a}'(r)}{\bar{a}(r)} = -\frac{p+1}{1+\beta-r}, \quad 1 \le r < 1+\beta.$$

Using (8.0.1) and (8.0.13), we easily obtain $\phi' \ge \hat{F}_2(\phi)$, $1 \le r < R_1$, and $\psi' = \hat{F}_2(\psi)$, $1 \le r < 1+\beta$, with $\psi(1) = \phi(1)$. Applying the argument leading to (8.0.11), this time for $r > 1$, we obtain $\phi(r) \ge \psi(r)$ for all $r \in (1, \min(R_1, 1+\beta))$, hence

$$a(r) = \exp\Big[\int_1^r \phi(\tau)\, d\tau\Big] \ge \exp\Big[\int_1^r \psi(\tau)\, d\tau\Big] = \bar{a}(r) > 0, \quad 1 < r < \min(R_1, 1+\beta). \tag{8.0.14}$$

Since $a(R_1) = 0$, it follows that $R_1 \ge 1+\beta$.

Finally, since $\bar{a} \in C^2([r_0, 1])$, $\bar{a} \in W^{2,2}([1, 1+\beta])$ and $\bar{a}'_-(1) = a'(1) = \bar{a}'_+(1)$, we have $\bar{a} \in W^{2,2}([r_0, 1+\beta])$. In view of (8.0.9), (8.0.10), (8.0.13), (8.0.14), and noting that $\bar{a}(1+\beta) = 0$, this completes the proof of assertion (i).

(ii) Since $\bar{a} \in W^{2,2}([r_0, 1+\beta])$ and $\bar{a}'_+(r_0) = 0$, we have $\bar{a} \in W^{2,2}([0, 1+\beta])$. By (3.3.2), for all $r \in (0, r_0)$, we have $F(r, 0) \le 0$, hence $\bar{a}''(r) = 0 \ge F(r, 0) = F(r, \frac{\bar{a}'}{\bar{a}})$. This along with Steps 2 and 3 guarantees that $\bar{a}$ is a solution of (8.0.1)-(8.0.3) with R_1 replaced by $1+\beta$.

Next, since $a(r) \ge a(r_0) \ge \bar{a}(r_0) = \bar{a}(r)$ on $[0, r_0]$ due to $a' \le 0$, we deduce from (8.0.10), (8.0.14) that $0 < \bar{a} \le a$ in $[0, 1+\beta)$ and property (8.0.5) follows. $\square$

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Chapter 4

Single-point gradient blow-up on the boundary for diffusive Hamilton-Jacobi equation in domains with non-constant curvature.

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Abstract. We consider the diffusive Hamilton-Jacobi equation $u_t - \Delta u = |\nabla u|^p$ in a bounded planar domain with zero Dirichlet boundary condition. It is known that, for $p > 2$, the solutions to this problem can exhibit gradient blow-up (GBU) at the boundary. In this paper we study the possibility of the GBU set being reduced to a single point. In a previous work *[Y.-X. Li, Ph. Souplet, 2009]*, it was shown that single point GBU solutions can be constructed in very particular domains, i.e. locally flat domains and disks. Here, we prove the existence of single point GBU solutions in a large class of domains, for which the curvature of the boundary may be nonconstant near the GBU point.

Our strategy is to use a boundary-fitted curvilinear coordinate system, combined with suitable auxiliary functions and appropriate monotonicity properties of the solution. The derivation and analysis of the parabolic equations satisfied by the auxiliary functions necessitate long and technical calculations involving boundary-fitted coordinates.

1 Introduction and first results

We consider the initial-boundary value problem for the diffusive Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, $p > 2$ and

$$u_0 \in X_+ := \{v \in C^1(\overline{\Omega});\ v \geq 0,\ v|_{\partial\Omega} = 0\}.$$

Equation (1.0.1) is a typical model-case in the theory of nonlinear parabolic equations, being the simplest example of a parabolic equation with a nonlinearity depending on the gradient of the solution. It has been extensively studied in the past twenty years and it is well known that if $p \leq 2$ or if $\Omega = \mathbb{R}^n$, then all solutions exist globally in the classical sense, see [3], [7], [8], [9], [14], [15], [21], [29], [31]. On the contrary, for the case of superquadratic growth of the nonlinearity, i.e. $p > 2$, with $\Omega \neq \mathbb{R}^n$, solutions exhibit singularities for large enough initial data. The nature of this singularity is of *gradient blow-up* type, and occurs on some subset of the boundary of the domain, see [1], [2], [4], [6], [10], [11], [16], [18], [22], [29], [30], [32], [33].

In addition, equation (1.0.1) arises in stochastic control theory [23], and is involved in certain physical models, for example of ballistic deposition processes, where the solution describes the growth of an interface, see [17], [19], [20].

It follows from classical theory, see for example [12, Theorem 10, p. 206], that problem (1.0.1) admits a unique maximal, nonnegative classical solution $u \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap C^{1,0}(\overline{\Omega} \times [0,T))$, where $T = T(u_0)$ is the maximal existence time. By the maximum principle, for problem (1.0.1) we have

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \qquad 0 < t < T.$$

Since (1.0.1) is well posed in X_+, it follows that, if $T < \infty$, then

$$\lim_{t \to T} \|\nabla u(t)\|_\infty = \infty.$$

This phenomenon of ∇u blowing up with u remaining uniformly bounded is known as *gradient blow-up*. The gradient blow-up set of u is defined by

$$GBUS(u_0) = \{x_0 \in \partial\Omega;\ \limsup_{t \to T,\, x \to x_0} |\nabla u(x,t)| = \infty\}.$$

We call gradient blow-up point (GBU point for short) any point in $GBUS(u_0)$. The space profile at $t = T$ is investigated in [10], [4], [33], [16], [24]. For results on the GBU rate, we refer to [10], [16], [34], [26]. Also, the existence and properties of a weak continuation of the solution after GBU are studied in [11], [6], [27], [25], [28].

From [33, Theorem 3.2], it follows that gradient blow-up can only occur at the boundary (see also [2], [4]). More precisely, the following estimate is given:

$$|\nabla u| \leq C_1 \delta^{-\frac{1}{p-1}}(x,y) + C_2 \quad \text{in } \Omega \times [0,T), \tag{1.0.2}$$

where $C_1 = C_1(n,p) > 0$ and $C_2 = C_2(p, \Omega, \|u_0\|_{C^1}) > 0$. Here, $\delta(x,y)$ is the distance function to the boundary.

In this paper we are interested in the possibility of having isolated gradient blow-up points at the boundary. Up to now, the only available results of this kind, ensuring single-point GBU for suitable initial data, are those from [22], and they are restricted to very particular domains, namely disks and locally flat domains with some symmetry assumptions (see also [5] for a related problem with nonlinear diffusion in locally flat domains).

As it turns out, a key feature in the proofs in [22], [5] is the fact that the curvature of the boundary is constant near the GBU point. In this paper we are able to show that this can be considerably relaxed and we cover large classes of domains.

In order to give a good illustration of our main results without entering into too much technicality, let us right away formulate a single point gradient blow-up result for two typical classes of domains. More general results will be given in Section 2. We first treat the case of ellipses.

Theorem 1.1. *Let $p > 2$ and $\Omega \subset \mathbb{R}^2$ be an ellipse. Then, there exist initial data $u_0 \in X_+$ such that $T(u_0) < \infty$ and $GBUS(u_0)$ contains only a boundary point of minimal curvature.*

For our second class of domains, the main feature is that the GBU point has its center of curvature lying outside Ω and is a local minimum of the curvature, along with suitable geometric conditions. Namely, we assume:

Ω is symmetric with respect to the line $x = 0$ and convex in the x-direction, (1.0.3)

$\partial\Omega$ is tangent to the line $y = 0$ at the origin and $\Omega \subset \{y > 0\}$, (1.0.4)

The radius of curvature $R(x)$ of $\partial\Omega$ is a nonincreasing function for $x > 0$ small and $\overline{\Omega} \subset \{y < R(0)\}$, (1.0.5)

For all $X_0 \in \partial\Omega \cap \{x > 0\}$ close to the origin, the symmetric of Ω_{X_0} with respect to Λ_{X_0} is contained in Ω, where Λ_{X_0} is the normal line to $\partial\Omega$ at X_0, and Ω_{X_0} is the part of Ω to the right of Λ_{X_0}. (1.0.6)

See Figure 4.1 for an example of a domain satisfying these hypotheses. We point out that the function $R(x)$ in (1.0.5) is valued in $(0, \infty]$.

Theorem 1.2. *Let $p > 2$ and suppose $\Omega \subset \mathbb{R}^2$ is a domain satisfying* (1.0.3)–(1.0.6). *Then, there exist initial data $u_0 \in X_+$ such that $T(u_0) < \infty$ and $GBUS(u_0)$ contains only the origin.*

Remark 1.3.

i. *Observe that in the case of the locally flat domains studied in [22], condition* (1.0.6) *is a consequence of* (1.0.3). *In this case, for any $X_0 \in \partial\Omega \cap \{x > 0\}$ near the origin, Λ_{X_0} will be parallel to the line $x = 0$. Also hypothesis* (1.0.5) *is trivially satisfied by locally flat domains.*

ii. *Although it is possible to construct initial data for which the GBU set is arbitrarily concentrated close to any given boundary point (see Proposition 4.2), it is presently a (probably difficult) open question whether single point GBU may occur on points other than local minima of the curvature.*

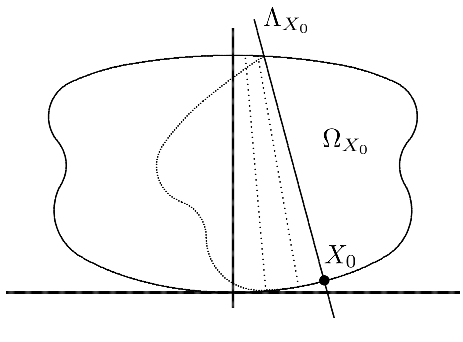

Figure 4.1 – Example of domain satisfying hypotheses (1.0.3)–(1.0.6).

In the next section we give single point GBU results more general than Theorems 1.1 and 1.2, at the expense of more technical statements (see Theorems 2.3 and 2.5). The technical complexity of the statements comes from the fact that, in order to describe the hypotheses involved, we need to introduce a coordinate system adapted to the boundary near the gradient blow-up point (and actually this coordinate system is crucially used in the proof of our results).

2 General results

We introduce a class of symmetric domains with respect to the line $x = 0$, containing those described in the previous theorems, and for which we can construct single-point GBU solutions. A first step of our strategy is to prove that the solution u is monotone in the parallel direction to the boundary in a neighborhood of the GBU point. It is therefore natural to introduce a curvilinear coordinate system adapted to the domain, allowing us to study the sign of the derivative of the solution in the parallel direction to the boundary. This coordinate system is sometimes called "boundary-fitted" coordinate system or "flow coordinates". We point out that the use of these coordinates brings some technical difficulties, and that long computations and quite delicate arguments are required in order to control the terms related to the non-constant curvature (under appropriate assumptions on the domain). However, our attempts to prove such results, on single-point GBU in domains with nonconstant curvature, by merely using cartesian coordinates or local charts have turned out to be unsuccessful.

Next, we set the notation used throughout the rest of the paper and introduce the curvilinear coordinate system mentioned above. See Figure 4.2 for an illustration of this notation.

Notation 2.1.

- *Ω is a smoothly bounded domain of $\mathbb{R}^2$ and $\nu = (\nu_x, \nu_y)$ denotes the unit normal outward vector to $\partial\Omega$.*

- *$\Gamma \subset \partial\Omega$ is a connected boundary piece, with $(0,0) \in \Gamma$, and we assume that*

$$\Omega \text{ and } \Gamma \text{ are symmetric with respect to the line } x = 0. \tag{2.0.1}$$

- *For given $s_0 > 0$, the map*

$$\gamma(s) = (\alpha(s), \beta(s)), \qquad s \in [-s_0, s_0],$$

is an arclength parametrization of Γ (i.e. $\alpha'(s)^2 + \beta'(s)^2 = 1$), with $\gamma(0) = (0,0)$.

- *We denote*

$$T(s) = (\alpha'(s), \beta'(s)), \qquad N(s) = T^{\perp}(s) = (-\beta'(s), \alpha'(s)), \qquad \text{for all } s \in [-s_0, s_0].$$

We see that $T(s)$ is a unit tangent vector to $\partial\Omega$ at the point $\gamma(s)$ and, without loss of generality (replacing s by $-s$ if necessary), we can assume that

$$N(s) \text{ is the inward normal vector to } \partial\Omega \text{ at the point } \gamma(s) \tag{2.0.2}$$

and that

$$\gamma(0) = (0,0), \qquad T(0) = (1,0), \qquad N(0) = (0,1).$$

- *We denote the curvature of the boundary by*

$$K(s) := det(\gamma', \gamma'') = \alpha'\beta'' - \beta'\alpha'', \quad \text{for all } s \in [-s_0, s_0].$$

By the regularity of $\partial\Omega$, this function is bounded and smooth.

- *We introduce the map $M := \gamma + rN$, i.e.*

$$\begin{array}{rccc} M: & [0,\infty) \times [-s_0, s_0] & \longrightarrow & \mathbb{R}^2 \\ & (r,s) & \longmapsto & M(r,s) = \gamma(s) + rN(s). \end{array} \tag{2.0.3}$$

For a given domain Ω and a boundary piece Γ as in Notation 2.1, our goal will be to prove the existence of initial data for which the GBU set is reduced to the origin. Using the coordinates given by the map M, we will use auxiliary functions to estimate the derivative of u with respect to s. Then, an integration over the coordinate curves parallel to the boundary will give an upper estimate on u which is sufficient to apply a nondegeneracy result (see Lemma 4.1 below) for each $s > 0$, proving that gradient blow-up can only take place at the origin.

In order to apply our methods, we need to make some extra geometric assumptions on the domain. Namely, we need to assume that Ω is locally convex near the origin and that the origin is a local minimum for the curvature of the boundary, i.e.

$$K(0) \geq 0 \quad \text{and} \quad K'(s) \geq 0 \qquad \text{for all } s \in [0, s_0], \tag{2.0.4}$$

along with

$$\alpha'(s), \beta'(s) > 0, \qquad \text{for all } s \in (0, s_0). \tag{2.0.5}$$

We note that (2.0.4) implies $K(s) \geq 0$ for $s \in (0, s_0]$. We point out that condition (2.0.5) excludes domains which are flat near the origin, but this case is comparatively easier and was treated in [22]. Hypotheses (2.0.4) and (2.0.5) are necessary for two reasons. On the one hand, they are needed to define a region where the parameterization M is well defined. On the other hand, when deriving the parabolic inequalities satisfied by the auxiliary functions, they are needed to control some terms coming from the non-constant curvature.

Under the above assumptions, let us denote

$$R(s) = 1/K(s) \in (0, \infty], \qquad s \in [0, s_0], \tag{2.0.6}$$

the radius of curvature of $\partial\Omega$ at $\gamma(s)$, and define the natural regions

$$Q_\Gamma = \big\{(r, s) \in \mathbb{R}^2;\ 0 \leq r < R(s),\ 0 \leq s \leq s_0\big\} \quad \text{and} \quad D_\Gamma = M(Q_\Gamma). \tag{2.0.7}$$

We observe that D_Γ is the region bordered by the four curves: Γ, the y-axis, the normal line at $\gamma(s_0)$ and, from above, the evolute of Γ, i.e. the locus of the curvature centers

$$C(s) = \gamma(s) + R(s)N(s). \tag{2.0.8}$$

The following proposition shows that the region D_Γ is well parametrized by M and, consequently, that one can define there the derivative u_s, in the parallel direction to the boundary. Although this fact is more or less standard, we give a proof in Section 3 for convenience.

Proposition 2.2. *Let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume* (2.0.4), (2.0.5).

(i) Then, the map M is a diffeomorphism from Q_Γ to D_Γ.

(ii) As a consequence, for any solution u of (1.0.1), *the derivative*

$$u_s := \frac{\partial}{\partial s}\big[u(M(r, s), t)\big]$$

is well defined in $(\overline{\Omega} \cap D_\Gamma) \times [0, T(u_0))$.

The following result ensures that single-point GBU occurs for symmetric solutions satisfying a monotonicity condition near the origin.

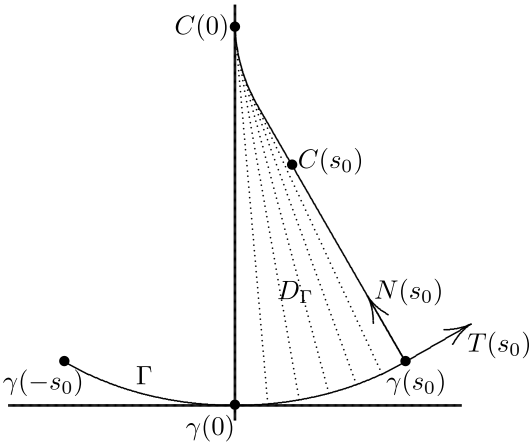

Figure 4.2 – Example of $\Gamma, \gamma(s), T(s), N(s)$ as in Notation 2.1 and $D_\Gamma, C(s)$ defined in (2.0.7), (2.0.8).

Theorem 2.3. *Let $p > 2$, let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume* (2.0.4), (2.0.5). *Let $u_0 \in X_+$ be a symmetric function with respect to the line $x = 0$, such that $T = T(u_0) < \infty$. Suppose that*

$$GBUS(u_0) \subset \gamma\big(-\tfrac{s_0}{2}, \tfrac{s_0}{2}\big) \tag{2.0.9}$$

and that, for some $t_0 \in (0,T)$, $r_0 \in (0, R(s_0))$, we have

$$u_x, u_s < 0 \quad \textit{in } \omega_0 \times (t_0,T), \;\textit{with } \omega_0 := \Omega \cap M\big((0,r_0) \times (0,s_0)\big). \tag{2.0.10}$$

Then, $GBUS(u_0)$ contains only the origin.

Hypothesis (2.0.9) is not difficult to guarantee. It is in fact satisfied whenever u_0 is sufficiently concentrated near the origin (cf. [22] and Proposition 4.2 below). On the contrary, the hypothesis $u_s < 0$ in (2.0.10) is in general more difficult to verify, and requires assumptions of more global nature.

The assumption $u_x < 0$ in (2.0.10) is required by the fact that the Laplace operator does not commute with the derivative in the $s-$direction. Therefore, we need to control a term involving u_r. This can be done by writing u_r as a linear combination of u_x and u_s, see formula (3.0.10). The term u_x is obviously more tractable since the $x-$derivative does commute with the Laplace operator. This requires the use of two auxiliary functions J and $\bar{J}$ in the proof of this Theorem (section 5), the first to control u_s and the second to control u_x. The derivation and analysis of the parabolic equations satisfied by J and $\bar{J}$ necessitate long and technical calculations involving boundary-fitted coordinates. At the beginning of section 5, we give an outline of the proof of Theorem 2.3 which summarizes the main points of the proof and makes clear our strategy.

We next introduce the geometric hypotheses on the domain Ω under which we are able to construct solutions satisfying condition (2.0.10). To this end we set the following further notation, which is motivated by moving plane arguments that we rely on.

Notation 2.4. *For each $s \in [0, s_0]$, we denote*

- *Λ_s the line $\gamma(s) + \mathbb{R}N(s)$*
- *$\mathcal{T}_s(\cdot)$ the symmetry with respect to Λ_s*

- *H_s the half-plane at the right of the line Λ_s, i.e.:*

$$H_s = \{P \in \mathbb{R}^2;\ T(s) \cdot (P - \gamma(s)) > 0\}.$$

- *$\Omega_s = \Omega \cap H_s$.*

Using Notations 2.1 and 2.4, the hypotheses that we shall assume are the following:

$$\overline{\omega}_0 \subset D_\Gamma, \qquad \text{where } \omega_0 := \Omega \cap D_\Gamma \cap \{y < y_0\}, \text{ for some } y_0 \in (0, \infty], \tag{2.0.11}$$

$$\nu_x \geq 0 \quad \text{on } \partial\Omega \cap \{x > 0\}, \tag{2.0.12}$$

$$\nu_y \geq 0 \text{ on } \partial\Omega \cap \partial\omega_0 \cap \{r > 0\}, \tag{2.0.13}$$

$$\mathcal{T}_{s_0}(\Omega_{s_0}) \subset \Omega, \tag{2.0.14}$$

$$\begin{array}{l} \mathcal{T}_+(\Omega^+) \subset \Omega, \quad \text{where } \Omega^+ := \Omega \cap \{y > y_0\}, \\ \text{and } \mathcal{T}_+(\cdot) \text{ is the symmetry with respect to the line } y = y_0. \end{array} \tag{2.0.15}$$

See Figure 4.1 in section 1 and Figures 4.3 and 4.4 in section 6 for examples of domains satisfying these hypotheses. In view of Proposition 2.2, assumption (2.0.11) ensures that u_s is well defined in $\overline{\omega}_0$. The necessity of constructing solutions satisfying the monotonicity condition (2.0.10) prevents us of using purely local arguments as in the proof of Theorem 2.3, where (2.0.10) is assumed. Hence, we need here to set the boundary-fitted coordinates in a region sufficiently big with respect to the entire domain Ω (see Remark 2.6 i.), for instance, when using moving planes arguments. Our result reads as follows.

Theorem 2.5. *Let $p > 2$ and let $\Omega, \gamma, s_0, \mathcal{T}_s, \Omega_s$ be as in Notations 2.1 and 2.4. Let D_Γ be defined by* (2.0.7) *and assume* (2.0.4), (2.0.5), (2.0.11)–(2.0.15).

i. There exist initial data $u_0 \in X_+$ such that $T(u_0) < \infty$ and

$$u_0 \textit{ is symmetric with respect to the line } x = 0, \tag{2.0.16}$$

$$u_{0,x} \leq 0 \textit{ in } \Omega \cap \{x > 0\} \quad \textit{and} \quad u_{0,s} \leq 0 \textit{ in } \omega_0, \tag{2.0.17}$$

$$u_0(P) \leq u_0(\mathcal{T}_{s_0}(P)) \quad \textit{for all } P \in \Omega_{s_0}, \tag{2.0.18}$$

$$u_0(P) \leq u_0(\mathcal{T}_+(P)) \quad \textit{for all } P \in \Omega \cap \{y > y_0\}. \tag{2.0.19}$$

$$GBUS(u_0) \subset \gamma\big(-\tfrac{s_0}{2}, \tfrac{s_0}{2}\big). \tag{2.0.20}$$

ii. For any such u_0, $GBUS(u_0)$ contains only the origin.

Remark 2.6. *i. If the domain Ω is sufficiently thin in the y-direction, then the center of curvature of the boundary lies outside Ω for all $s \in [0, s_0]$. In that case we can consider $y_0 = +\infty$ in* (2.0.11) *and conditions* (2.0.15) *and* (2.0.19) *disappear. When this is not the case, we can restrict ω_0 to $\{y < y_0\}$, for some $y_0 > 0$, in order to be able to define the boundary-fitted coordinates. However, we then have to pay the price of assuming the reflection assumption* (2.0.15)*, which allows us to prove $u_y \leq 0$ on $\Omega \cap \{y = y_0\}$ by a moving planes argument.*

ii. Hypothesis (2.0.12) *implies that the domain is convex in the x direction, and this, together with* (2.0.1)*, allows one to construct solutions such that $u_x \leq 0$ in $\Omega \cap \{x > 0\}$.*

iii. *On the other hand, hypotheses* (2.0.13) *and* (2.0.14) *are useful to construct solutions such that* $u_s < 0$ *in* ω_0. *In particular, hypothesis* (2.0.13) *implies that on the upper piece of* $\partial\omega_0$ *which coincides with* $\partial\Omega$, u_s *represents the derivative in a direction pointing outside* Ω, *and therefore* $u_s \leq 0$. *Then, we prove that* $u_s \leq 0$ *on* $\Lambda \cap \partial\omega_0$ *by a moving planes argument, which can be applied only under hypothesis* (2.0.14).

iv. *On* $\partial\omega_0 \cap \{y = y_0\}$, *we prove* $u_s \leq 0$ *by expressing it as a linear combination of* u_x *and* u_y, *that we can prove to be negative, see (i) and (ii) in this remark.*

Observe that in figure 4.1 the domain is sufficiently thin so that we can consider $y_0 = +\infty$. Ellipses with non-zero eccentricity, i.e. ellipses which are not disks, are also examples of domains where it is possible to apply this result. In that case, we choose y_0 such that the line $\{y = y_0\}$ coincides with the major axis of the ellipse. The case of a disk is excluded since, in order to satisfy hypothesis (2.0.15), we must consider an y_0 bigger or equal than the radius of curvature of the disk, but then, hypothesis (2.0.11) cannot hold. However, the case of the disk can be treated using polar coordinates (see [22]).

Remark 2.7. *Let* $p > 2$ *and* Ω *be as in Theorem 2.5, denote* $B_\rho^+ := B_\rho(0,0) \cap \{x > 0\}$ *and let* $\rho > 0$ *be such that*

$$\Omega \cap B_\rho^+ \subset \omega_0, \quad \partial\Omega \cap B_\rho^+ \subset \gamma(0, s_0/2).$$

It follows from Theorem 2.5 and Proposition 4.2 below that $T(u_0) < \infty$ *and* $GBUS(u_0) = \{(0,0)\}$ *whenever* $u_0 \in X_+$ *for instance satisfies* (2.0.16), (2.0.17) *and*

$$\begin{aligned}
&\operatorname{supp}(u_0) \subset \overline{\Omega} \cap \overline{B}_{\rho/2}, \\
&\|u_0\|_\infty \leq C_2, \\
&\inf_{\tilde{B}_\varepsilon} u_0 \geq C_1 \varepsilon^k \quad \textit{with } \tilde{B}_\varepsilon = B_{\varepsilon/2}(0, \varepsilon), \ \textit{for some } \varepsilon \in (0, \rho/2),
\end{aligned}$$

where $C_1(p) > 0$ *and* $C_2(p, \Omega, \rho) > 0$. *Moreover, initial data satisfying these assumptions can be easily constructed. See the proof of Theorem 2.5(i) for details.*

The outline of the rest of the paper is as follows. In section 3 we give some basic computations and notation on the "boundary-fitted" curvilinear coordinate system and we give the proof of Proposition 2.2. In section 4 we give some useful preliminary results, concerning nondegeneracy and localization of GBU as well as a Serrin type corner lemma. Theorems 2.3 and 2.5 are respectively proved in sections 5 and 6. Finally in section 7, we deduce Theorems 1.1 and 1.2 from Theorem 2.5.

3 Preliminary results I: basic computations in boundary-fitted curvilinear coordinates

In this section we give some basic computations in the coordinate system given by the map M in (2.0.3). Here Ω and Γ are as in Notation 2.1 and we assume conditions (2.0.4) and (2.0.5). By Proposition 2.2, that we will prove at the end of this section, M is a diffeomorphism from Q_Γ to D_Γ, where Q_Γ and D_Γ are defined in (2.0.7). To facilitate the change of coordinates throughout the paper, we adopt the following notation and conventions.

Notation 3.1. *For any function $\psi(x,y)$ defined on (a part of) D_Γ, we express ψ in terms of the variables (r,s) by setting*

$$\tilde{\psi} := \psi \circ M,$$

i.e. $\tilde{\psi}(r,s) = \psi(M(r,s))$ for $(r,s) \in Q_\Gamma$. The derivatives with respect to the variables (r,s) of a function $\psi = \psi(x,y) \in C^1(D_\Gamma)$ are then defined by

$$\psi_r := \tilde{\psi}_r, \qquad \psi_s := \tilde{\psi}_s. \tag{3.0.1}$$

Similarly, for any function $\varphi(r,s)$ defined on (a part of) Q_Γ, we denote

$$\hat{\varphi} = \varphi \circ M^{-1}.$$

In the rest of the paper, for any functions $\psi = \psi(x,y)$ and $\varphi = \varphi(r,s)$, when no risk of confusion arises, we will drop the tilde and the hat and will just write $\psi(r,s)$ in place of $\tilde{\psi}(r,s)$ and $\varphi(x,y)$ in place of $\hat{\varphi}(x,y)$.

Also, the gradient and the Laplacian operators will always be understood as

$$\nabla\psi = (\psi_x, \psi_y)$$

and

$$\Delta\psi = div(\nabla\psi) = \psi_{xx} + \psi_{yy},$$

either as functions of (x,y), or as functions of (r,s) (i.e., implicitly considering $(\nabla\psi)\circ M$ and $(\Delta\psi)\circ M$).

According to the chain rule, we have

$$\psi_r = \nabla\psi(M(r,s)) \cdot N(s) \quad \text{and} \quad \psi_s = \nabla\psi(M(r,s)) \cdot (\gamma'(s) + rN'(s)). \tag{3.0.2}$$

Using (3.0.7) and (3.0.8), we obtain

$$N'(s) = -K(s)(\alpha'(s), \beta'(s)) = -K(s)T(s),$$

and then, we can rewrite (3.0.2) as

$$\psi_r = \nabla\psi \cdot N(s) \quad \text{and} \quad \psi_s = (1 - rK(s))\nabla\psi \cdot T(s). \tag{3.0.3}$$

Note that

$$1 - rK(s) > 0 \quad \text{in } D_\Gamma \tag{3.0.4}$$

owing to (2.0.6), (2.0.7). Since the vectors $N(s)$ and $T(s)$ are orthonormal, we then have

$$\nabla\psi(r,s) \equiv (\nabla\psi)\circ M = \psi_r N(s) + \frac{\psi_s}{1 - rK(s)}T(s), \tag{3.0.5}$$

as well as

$$\nabla\psi \cdot \nabla\varphi = \psi_r\varphi_r + \frac{\psi_s\varphi_s}{(1 - rK)^2}. \tag{3.0.6}$$

We next recall two alternative expressions for the function curvature of the boundary $K(s)$. Since $\gamma(s) = (\alpha(s), \beta(s))$ is an arclength parametrization, we have

$$\alpha'(s)\alpha''(s) + \beta'(s)\beta''(s) = \frac{(\alpha'(s)^2 + \beta'(s)^2)'}{2} = 0,$$

and then we have $\alpha'(s)\alpha''(s) = -\beta'(s)\beta''(s)$. Using this identity, we can obtain

$$K(s) = \alpha'(s)\beta''(s) - \beta'(s)\alpha''(s) = \alpha'(s)\beta''(s) + \frac{\beta'(s)\beta''(s)}{\alpha'(s)} = \frac{\beta''(s)}{\alpha'(s)}, \tag{3.0.7}$$

and in a similar way, recalling (2.0.5), we obtain

$$K(s) = -\frac{\alpha''(s)}{\beta'(s)}, \qquad s \neq 0. \tag{3.0.8}$$

Now, we give some further identities relating the derivatives in boundary-fitted coordinates with the derivatives in cartesian coordinates. As we will see in our proofs, we have particular interest in expressing, when possible, ψ_r as a linear combination of ψ_x and ψ_s. In the following computations, and without risk of confusion, we omit the dependence on s of the functions K, α', β'. In view of (3.0.3), we have

$$\begin{aligned} \psi_r &= -\beta'\psi_x + \alpha'\psi_y, \\ \frac{\psi_s}{1-rK} &= \alpha'\psi_x + \beta'\psi_y. \end{aligned} \tag{3.0.9}$$

Then, recalling (2.0.5), we obtain the identity

$$\psi_r = -\frac{1}{\beta'}\psi_x + \frac{\alpha'}{\beta'}\frac{\psi_s}{1-rK}, \qquad s \neq 0. \tag{3.0.10}$$

We note that it is possible to write ψ_r as a linear combination of ψ_x and ψ_s only when $\beta'(s) \neq 0$ (i.e., $s \neq 0$). This makes sense since, if $\beta'(s) = 0$, then $\psi_x = \psi_s$ and ψ_r is the derivative in the y direction, which is then orthogonal to the x and s directions.

The next result is a very useful expression of the Laplacian in flow coordinates.

Proposition 3.2. *(i) Let $\psi = \psi(x,y) \in C^2(D_\Gamma)$. We have*

$$\Delta\psi \equiv (\Delta\psi)\circ M = \psi_{rr} - \frac{K}{1-rK}\psi_r + \frac{1}{(1-rK)^2}\psi_{ss} + \frac{rK'}{(1-rK)^3}\psi_s, \qquad (r,s) \in Q_\Gamma. \tag{3.0.11}$$

(ii) If $\varphi = \varphi(r,s) \in C^2(Q_\Gamma)$, then $\Delta\varphi \equiv [\Delta(\varphi \circ M^{-1})]\circ M$ is also given by (3.0.11) *with ψ replaced by φ.*

Proof. (i) For $\varphi = \varphi(r,s)$ recall the notation $\hat\varphi = \hat\varphi(x,y) := \varphi \circ M^{-1}$. For any $\psi = \psi(x,y) \in C^2(D_\Gamma)$, using (3.0.5), we obtain

$$\nabla\psi = \widehat{\psi_r}\,\hat N + \frac{\widehat{\psi_s}}{1-\hat r\hat K}\,\hat T \quad \text{ in } D_\Gamma.$$

It follows that

$$\begin{aligned} \Delta\psi = \operatorname{div}(\nabla\psi) &= \nabla(\widehat{\psi_r})\cdot\hat N + \widehat{\psi_r}\operatorname{div}\hat N + \frac{1}{1-\hat r\hat K}\nabla(\widehat{\psi_s})\cdot\hat T \\ &\quad + \widehat{\psi_s}\nabla\left(\frac{1}{1-\hat r\hat K}\right)\cdot\hat T + \frac{1}{1-\hat r\hat K}\widehat{\psi_s}\operatorname{div}\hat T. \end{aligned} \tag{3.0.12}$$

By (3.0.5), we have

$$[(\nabla\varphi)\circ M]\cdot N = \varphi_r \quad \text{and} \quad (1-rK)[(\nabla\varphi)\circ M]\cdot T = \varphi_s,$$

hence

$$\nabla\varphi \cdot \hat{N} = \widehat{\varphi_r} \equiv (\varphi \circ M)_r \circ M^{-1} \quad \text{and} \quad (1 - \hat{r}\hat{K})(\nabla\varphi \cdot \hat{T}) = \widehat{\varphi_s} \equiv (\varphi \circ M)_s \circ M^{-1}.$$

Using this with $\varphi = \widehat{\psi_r}$, we can thus identify

$$\nabla(\widehat{\psi_r}) \cdot \hat{N} \equiv \nabla((\psi \circ M)_r \circ M^{-1}) \cdot \hat{N} = (\psi \circ M)_{rr} \circ M^{-1} \equiv \psi_{rr} \circ M^{-1}, \tag{3.0.13}$$

$$(1 - \hat{r}\hat{K})\nabla(\widehat{\psi_s}) \cdot \hat{T} \equiv (1 - \hat{r}\hat{K})\nabla((\psi \circ M)_s \circ M^{-1}) \cdot \hat{T} = (\psi \circ M)_{ss} \circ M^{-1} \equiv \psi_{ss} \circ M^{-1} \tag{3.0.14}$$

and

$$\nabla\left(\frac{1}{1 - \hat{r}\hat{K}}\right) \cdot \hat{T} = \frac{1}{1 - \hat{r}\hat{K}}\left[\frac{1}{1 - rK}\right]_s \circ M^{-1} = \frac{rK'}{(1 - rK)^3} \circ M^{-1}. \tag{3.0.15}$$

On the other hand, since $N(s) = -\beta'(s)(1,0) + \alpha'(s)(0,1)$, we have

$$\operatorname{div}(\hat{N}) = -\nabla\widehat{\beta'} \cdot (1,0) + \nabla\widehat{\alpha'} \cdot (0,1).$$

Applying (3.0.5) with $\psi = \widehat{\beta'}$ and $\psi = \widehat{\alpha'}$, we obtain

$$\begin{aligned} [\operatorname{div}(\hat{N})] \circ M &= -\frac{\beta''}{1 - rK}T(s) \cdot (1,0) + \frac{\alpha''}{1 - rK}T(s) \cdot (0,1) \\ &= -\frac{\beta''\alpha'}{1 - rK} + \frac{\alpha''\beta'}{1 - rK} = -\frac{K}{1 - rK}. \end{aligned} \tag{3.0.16}$$

Similarly, since $T(s) = \alpha'(s)(1,0) + \beta'(s)(0,1)$, hence

$$\operatorname{div}(\hat{T}) = \nabla\widehat{\alpha'} \cdot (1,0) + \nabla\widehat{\beta'} \cdot (0,1),$$

we have

$$\begin{aligned} [\operatorname{div}(\hat{T})] \circ M &= \frac{\alpha''}{1 - rK}T(s) \cdot (1,0) + \frac{\beta''}{1 - rK}T(s) \cdot (0,1) \\ &= \frac{\alpha''\alpha'}{1 - rK} + \frac{\beta''\beta'}{1 - rK} = \frac{(\alpha'^2 + \beta'^2)'}{2(1 - rK)} = 0. \end{aligned} \tag{3.0.17}$$

Finally, plugging (3.0.13)–(3.0.17) in (3.0.12), we obtain (3.0.11).

(ii) It suffices to apply assertion (i) to $\psi := \varphi \circ M^{-1}$, using (3.0.1) and the fact that $\tilde{\psi} \equiv \psi \circ M = \varphi$. □

We end this section with the proof of Proposition 2.2.

Proof of Proposition 2.2. It suffices to show assertion (i). We first establish the injectivity of M on Q_Γ. Let $C(s) = \gamma(s) + R(s)N(s)$ be the center of curvature. We note that D_Γ can be written as the union of half-open segments:

$$D_\Gamma = \bigcup_{s \in [0,s_0]} \Sigma(s), \qquad \text{where } \Sigma(s) = [\gamma(s), C(s)).$$

To show the injectivity, it suffices to verify that for any $0 \le s_1 < s_2 \le s_0$, the segments $\Sigma(s_1)$ and $\Sigma(s_2)$ do not intersect. This amounts to showing that $\Sigma(s_2)$ lies entirely in the open half-plane to the right of the line Λ_{s_1}, defined as in Notation 2.4, which is the line containing the segment $\Sigma(s_1)$. This half-plane is defined by the inequality

$$T(s_1)\cdot(x-\gamma(s_1))>0, \quad \text{with } x\in\mathbb{R}^2.$$

Considering the extremes of the segment $\Sigma(s_2)$, this is thus equivalent to

$$T(s_1)\cdot(\gamma(s_2)-\gamma(s_1))>0 \quad \text{and} \quad T(s_1)\cdot(C(s_2)-\gamma(s_1))\geq 0. \tag{3.0.18}$$

To show (3.0.18), using $\gamma'(s)=T(s)$ and (2.0.5), we first compute

$$\frac{d}{ds}\Big(T(s_1)\cdot(\gamma(s)-\gamma(s_1))\Big)=T(s_1)\cdot T(s)>0, \qquad s_1<s\leq s_0,$$

hence the first inequality in (3.0.18) follows. On the other hand, using $N'(s)=-K(s)T(s)$, (2.0.4) and (2.0.5), we get

$$\frac{d}{ds}\Big(T(s_1)\cdot N(s)\Big)=-K(s)T(s_1)\cdot T(s)\leq 0,$$

Since $T(s_1)\cdot N(s_1)=0$, we deduce that

$$T(s_1)\cdot N(s)\leq 0, \qquad s_1<s\leq s_0. \tag{3.0.19}$$

Also, using $\gamma'(s)=T(s)$ and $N'(s)=-K(s)T(s)$, we have

$$C'(s)=(1-K(s)R(s))T(s)+R'(s)N(s)=R'(s)N(s).$$

Since $R'(s)\leq 0$ due to (2.0.4), it follows from (3.0.19) that

$$\frac{d}{ds}\Big(T(s_1)\cdot(C(s)-C(s_1))\Big)=R'(s)T(s_1)\cdot N(s)\geq 0, \qquad s_1<s\leq s_0,$$

hence, it follows from $\gamma(s_1)=C(s_1)-R(s_1)N(s_1)$ that

$$T(s_1)\cdot(C(s_2)-\gamma(s_1))=T(s_1)\cdot(C(s_2)-C(s_1))\geq 0,$$

which guarantees the second inequality in (3.0.18). This completes the proof of the injectivity.

To prove that M is a diffeomorphism from Q_Γ to $D_\Gamma=M(Q_\Gamma)$, it thus suffices to show that the Jacobian of M does not vanish in Q_Γ. For all $(r,s)\in Q_\Gamma$, using $\gamma'=T$ and $N'=-KT$ again, we compute

$$\mathrm{Jac}_M(r,s)=det\Big(\frac{\partial M}{\partial r},\frac{\partial M}{\partial s}\Big)=det\Big(N,(1-Kr)T\Big)=K(s)r-1<0,$$

since $r<R(s)=1/K(s)$, and the conclusion follows. □

4 Preliminary results II: Nondegeneracy and localization of GBU and corner lemma

In this section we give three preliminary results that we use in the proofs of Theorems 2.3 and 2.5. We start with the following nondegeneracy lemma, proved in [22], which implies that, at any gradient blow-up point, the estimate (1.0.2) is essentially optimal in the normal direction to the boundary.

Lemma 4.1. *Let $\Omega \in \mathbb{R}^2$ be a smoothly bounded domain and $x_0 \in \partial\Omega$. There exists $c_0 = c_0(p)$ such that, if*

$$u \le c_0 \delta^{(p-2)/(p-1)}(x,y) \quad \text{in } (B_\rho(x,0) \cap \Omega) \times [0,T),$$

for some $\rho > 0$, then x_0 is not a gradient blow-up point.

We observe that, as a consequence of this Lemma, if $x_0 \in \partial\Omega$ is a gradient blow-up point, then we must have

$$\limsup_{x \to x_0, t \to T} u(x,y,t)\delta^{-(p-2)/(p-1)}(x,y) \ge c_0(p),$$

In view of (1.0.2), it follows in particular that

$$\limsup_{x \to x_0, t \to T} u_\nu(x,y,t)\delta^{1/(p-1)}(x,y) \in (0,\infty).$$

where u_ν is the derivative of u in the outward normal direction to the boundary.

The second preliminary result is the following proposition, which provides a sufficient condition on the initial data u_0 under which the solution blows up, with GBU set concentrated near an arbitrary given point. The idea of proof is based on that of [22, Theorem 1.1], where a more particular example of initial data was given.

Proposition 4.2. *Let $p > 2$, $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain and let $x_0 \in \partial\Omega$ and $\rho > 0$. There exist constants $C_1(p) > 0$ and $C_2(p,\Omega,\rho) > 0$ with the following property:*

If for some $\varepsilon > 0$ such that $\tilde{B}_\varepsilon := B(x_0 + \varepsilon\nu(x_0), \varepsilon) \subset \Omega$, $u_0 \in X_+$ satisfies

$$\operatorname{supp}(u_0) \subset \overline{\Omega} \cap \overline{B}(x_0, \rho/2), \tag{4.0.1}$$

$$\|u_0\|_\infty \le C_2, \tag{4.0.2}$$

$$\inf_{\tilde{B}_{\varepsilon/2}} u_0 \ge C_1 \varepsilon^k, \qquad \text{with } \tilde{B}_{\varepsilon/2} := B(x_0 + \varepsilon\nu(x_0), \varepsilon/2), \tag{4.0.3}$$

where $k = (p-2)/(p-1)$, then

$$T(u_0) < \infty \quad \text{and} \quad GBUS(u_0) \subset B_\rho(x_0) \cap \partial\Omega.$$

Proof. We divide the proof into two steps.

Step 1: ∇u blows up in finite time. The idea here is to use the auxiliary function introduced in [22] as subsolution. Let $\varphi \in C^\infty([0,\infty))$ be a function satisfying

$$\varphi' \le 0, \qquad \varphi(r) = 1, \text{ for } r \le 1/4, \qquad \varphi(r) = 0, \text{ for } r \ge 1/2.$$

Consider the following problem:

$$\begin{aligned} v_t - \Delta v &= |\nabla v|^p, && x \in B_1(0),\ t > 0, \\ v(x,t) &= 0, && x \in \partial B_1(0),\ t > 0, \\ v(x,0) = \phi(x) &:= C_1 \varphi(|x|), && x \in B_1(0). \end{aligned} \tag{4.0.4}$$

By [29, Thm 4.2] (see also [33, Prop. 7.1]), there exists $C_0 = C_0(p)$ such that, if $\|\phi\|_1 \ge C_0$, then $T(\phi) < \infty$. Therefore, we have $T(\phi) < \infty$ whenever C_1 is bigger than some constant depending on p. We now use the scale invariance of the equation. Namely we consider the rescaled function

$$v_\varepsilon(x,t) = \varepsilon^k v\left(\varepsilon^{-1}|x - \tilde{x}_0|, \varepsilon^{-2}t\right),$$

where $\tilde{x}_0 = x_0 + \varepsilon\nu(x_0)$. Then v_ε solves (4.0.4) in $\tilde{B}_\varepsilon \subset \Omega$.

Since we have

$$v_\varepsilon(x,0) = \varepsilon^k C_1 \varphi(\varepsilon^{-1}|x-\tilde{x}_0|) \le \varepsilon^k C_1 \text{ in } \tilde{B}_{\varepsilon/2},$$

and $v_\varepsilon(x,0) = 0$ in $\tilde{B}_\varepsilon \setminus \tilde{B}_{\varepsilon/2}$, we can use (4.0.3), together with the comparison principle to get

$$u \ge v_\varepsilon \quad \text{in } \tilde{B}_\varepsilon \times (0,\tilde{T}), \quad \text{where } \tilde{T} = \min(T(u_0), T_\varepsilon) \text{ and } T_\varepsilon = \varepsilon^2 T(\phi).$$

Now we observe that $\tilde{B}_\varepsilon$ is tangent to $\partial\Omega$ at x_0, so we deduce

$$-\frac{\partial u}{\partial \nu}(x_0,t) \ge -\frac{\partial v_\varepsilon}{\partial \nu}(x_0,t), \qquad 0 < t < \tilde{T}.$$

On the other hand, as a consequence of the maximum principle applied to ∇v (see e.g. [29, Prop. 40.3]), we know that

$$\max_{t\in[0,\tau]} \|\nabla v(\cdot,t)\|_\infty = \max\left(\|\nabla v(\cdot,0)\|_\infty, \max_{\partial B_1(0)\times[0,\tau]} \left(-\frac{\partial v}{\partial \nu}\right)\right), \qquad 0 < \tau < T(\phi).$$

Since v is radially symmetric, it follow that

$$\limsup_{t\to T_\varepsilon} \frac{\partial v_\varepsilon}{\partial \nu}(x_0,t) = \infty,$$

hence $T(u_0) \le T_\varepsilon < \infty$.

Step 2: No GBU on $\partial\Omega \setminus B_\rho(x_0)$. For $\rho > 0$, consider a cut-off function $h \in C^\infty([0,\infty))$ satisfying

$$h' \le 0, \qquad h(r) = 1, \quad \text{for } r \le \rho/2, \qquad h(r) = 0, \quad \text{for } r \ge 3\rho/4.$$

Now, let h_{x_0} be the function in $\overline{\Omega}$ defined by

$$h_{x_0}(x) := h(|x-x_0|).$$

Let $\psi = \psi_{x_0}$ be the unique classical solution of the linear elliptic problem

$$\begin{cases} -\Delta\psi(x) = 1, & x \in \Omega, \\ \psi(x) = h_{x_0}(x), & x \in \partial\Omega. \end{cases} \tag{4.0.5}$$

We claim that there exists $c_1 > 0$, independent of x_0, satisfying

$$\psi(x) \ge c_1, \qquad \text{for all } x \in \Omega \cap B(x_0,\rho/2).$$

We can prove this claim by using a contradiction and compactness argument. Suppose there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset \partial\Omega$ such that

$$\underline{\min_{\Omega\cap B(x_i,\rho/2)}} \psi_{x_i}(x) \to 0, \quad \text{as } i \to \infty, \tag{4.0.6}$$

where ψ_{x_i} is the solution of (4.0.5) with boundary data h_{x_i}. Since $\partial\Omega$ is compact, we can suppose, by extracting a subsequence, that x_i converges to some $x_\infty \in \partial\Omega$.

Now fix some $\alpha \in (0,1)$ and observe that, by the construction of h_{x_0} above, there exists $C > 0$, independent of i, such that $\|h_{x_i}\|_{C^{2+\alpha}(\overline{\Omega})} \le C$, and therefore $\|\psi_{x_i}\|_{C^{2+\alpha}(\overline{\Omega})} \le$

$C'(C,\Omega)$ by interior-boundary elliptic Schauder estimates (see Theorem 47.2 (ii) in [29]). Hence, as h_{x_i} converges to h_{x_∞} in $C^{2+\alpha}(\overline{\Omega})$, by compact embeddings and uniqueness for problem (4.0.5), we can deduce that ψ_{x_i} converges to ψ_{x_∞} in $C^2(\overline{\Omega})$. It then follows from (4.0.6) that ψ_{x_∞} vanishes somewhere in $\overline{\Omega \cap B(x_\infty,\rho/2)}$.

Since $h_{x_\infty}(x)=1$ in $\overline{B}(x_\infty,\rho/2)$, and then $\psi_{x_\infty}(x)=1$ in $\partial\Omega\cap\overline{B}(x_\infty,\rho/2)$, we deduce that ψ_{x_∞} vanishes somewhere in the interior of Ω, contradicting the strong maximum principle. The claim is then proved.

On the other hand, applying elliptic estimates again, there exists $\tilde{C}=\tilde{C}(\rho,\Omega)>0$ such that $\|\nabla\psi\|_\infty \le \tilde{C}$. Choosing $c_2=\tilde{C}^{-p/(p-1)}$, we then have $\|\nabla\psi\|_\infty^{-p/(p-1)} \ge c_2$, hence

$$-\Delta(c_2\psi)=c_2 \ge |\nabla(c_2\psi)|^p, \qquad \text{in } \Omega.$$

And by (4.0.2) with $C_2=c_1c_2$, we have

$$c_2\psi \ge c_2c_1 \ge u_0, \qquad \text{in } B(x_0,\rho/2),$$

hence, using (4.0.1), we get $c_2\psi \ge u_0$ in Ω. By the comparison principle, it follows that $u \le c_2\psi$ in $\Omega\times(0,T(u_0))$. Therefore, since $\psi=0$ on $\partial\Omega\setminus B_{3\rho/4}(x_0)$, we have

$$0 \le -\frac{\partial u}{\partial\nu} \le -c_2\frac{\partial\psi}{\partial\nu} \le C, \quad \text{on } (\partial\Omega\setminus B_{3\rho/4}(x_0))\times(0,T(u_0)).$$

The conclusion then follows from Lemma 4.1. □

We conclude this section with a parabolic version of "Serrin's corner Lemma", adapted to our parabolic problem and domain.

Lemma 4.3. *Let $p>2$ and $u_0\in X_+$, let Ω,Γ,γ,M be as in Notation 2.1. and assume* (2.0.4) *and* (2.0.5)*. Suppose that there exist $t_0\in(0,T)$, $s_1\in(0,s_0)$, $r_0>0$ and $c_1>0$ such that*

$$\omega_0 := M((0,r_0)\times(0,s_0)) \subset \Omega\cap D_\Gamma,$$
$$u_x<0 \quad \text{in } \omega_0\times(t_0,T) \tag{4.0.7}$$

and

$$u_x \le -c_1 r \quad \text{on } (0,r_0)\times\{s_1\}\times(t_0,T). \tag{4.0.8}$$

Then, for any fixed $r_1\in(0,r_0)$ and $t_1\in(t_0,T)$, there exists $\tilde{c}_1>0$ such that

$$u_x(r,s,t_1) \le -\tilde{c}_1 rs \quad \text{in } \omega_1,$$

where $\omega_1 := M((0,r_1)\times(0,s_1))$.

Proof. We fix a nontrivial smooth function $\phi\ge 0$ on $[0,r_0]$, with $\operatorname{supp}(\phi)\subset\subset(0,r_0)$ and another smooth function ψ on $[0,s_1]$ such that

$$\psi=0 \quad \text{on } \left[0,\frac{s_1}{2}\right], \quad \psi(s_1)=1, \quad \psi',\psi''\ge 0.$$

Fix a constant $M>0$ such that

$$M \ge \frac{K}{1-rK}+p|\nabla u|^{p-1}, \quad M \ge \frac{rK'}{(1-rK)^3}+\frac{p|\nabla u|^{p-1}}{1-rK}, \quad \text{in } \omega_0\times(t_0,t_1]. \tag{4.0.9}$$

Next, fix $t_2\in(t_0,t_1)$ and let v,V be the respective global solutions of

$$\begin{aligned}
&v_t-v_{rr}=-M|v_r|, && r\in(0,r_0),\ t>t_2,\\
&v(0,t)=v(r_0,t)=0, && t>t_2,\\
&v(r,t_2)=\phi(r), && r\in[0,r_0],
\end{aligned}$$

and

$$\begin{array}{ll} V_t - V_{ss} = -MV_s, & s \in (0,s_1),\ t > t_2, \\ V(0,t) = 0,\ V(s_1,t) = 1, & t > t_2, \\ V(s,t_2) = \psi(s), & s \in [0,s_1]. \end{array} \tag{4.0.10}$$

By the maximum principle we have $v \geq 0$, $0 \leq V \leq 1$, and $V_s \geq 0$. Also, by (4.0.10), we deduce that $V_{ss}(s,t) \geq 0$, for $s \in \{0, s_1\}$ and $t > t_2$. Since $\psi'' \geq 0$, it follows from the maximum principle that $V_{ss} \geq 0$, for $s \in (0,s_1)$, $t > t_2$. Moreover, by Hopf's lemma, for some $c_0 > 0$, we have

$$v(r,t_1) \geq c_0 r \text{ in } (0,r_1), \qquad V(s,t_1) \geq c_0 s \text{ in } (0,s_1). \tag{4.0.11}$$

Let then $z(r,s,t) = v(r,t)V(s,t)$. We compute

$$\begin{aligned} z_t - z_{rr} - \frac{1}{(1-rK)^2} z_{ss} &= V(v_t - v_{rr}) + v\left(V_t - \frac{1}{(1-rK)^2} V_{ss}\right) \\ &\leq -M|z_r| - M|z_s|. \end{aligned}$$

Hence, using (3.0.5), Proposition 3.2 and the choice of M in (4.0.9), we obtain

$$\begin{aligned} z_t - \Delta z &= z_t - z_{rr} + \frac{K}{1-rK} z_r - \frac{1}{(1-rK)^2} z_{ss} - \frac{rK'}{(1-rK)^3} z_s \\ &\leq -\left(M - \frac{K}{1-rK}\right)|z_r| - \left(M - \frac{rK'}{(1-rK)^3}\right)|z_s| \\ &\leq p|\nabla u|^{p-2} \nabla u \cdot \nabla z. \end{aligned} \tag{4.0.12}$$

On the other hand,$W := -u_x$ satisfies

$$W_t - \Delta W = p|\nabla u|^{p-2} \nabla u \cdot \nabla W. \tag{4.0.13}$$

For $\mu \in (0,1)$ small enough, due to (4.0.7), together with $\mathrm{supp}(\phi) \subset\subset (0,r_0)$ and $\psi \equiv 0$ in $[0, s_1/2]$, we have

$$-u_x(r,s,t_2) \geq \mu\phi(r)\psi(s) = \mu z(r,s,t_2) \quad \text{in } \omega_1.$$

Moreover, for possibly smaller $\mu > 0$, using (4.0.8), we see that

$$-u_x(r,s_1,t) \geq c_1 r \geq \mu v(r,t) = \mu z(r,s_1,t), \quad r \in (0,r_0),\ t \in [t_2,t_1].$$

Since $z = 0$ on the rest of the lateral boundary of $\omega_1 \times [t_2,t_1]$ (i.e. for $r \in \{r_0, 1\}$ or $s = 0$), it follows from (4.0.12), (4.0.13), the comparison principle and (4.0.11) that

$$-u_x(r,s,t_1) \geq \mu v(r,t_1)V(s,t_1) \geq \tilde{c}_1 rs \quad \text{in } \omega_1,$$

with $\tilde{c}_1 = \mu c_0^2$. □

5 Proof of Theorem 2.3

Theorem 2.3 will be proved by using the techniques introduced in [22], that we here have to modify in a nontrivial way in order to adapt the method to the boundary with non constant curvature. These techniques are based on a Friedman-McLeod-type argument [13], which is very useful for solutions which are monotone in some sense. In our case, this monotonicity follows from the hypothesis (2.0.10).

Since the proof of this theorem is rather long and similar arguments are sometimes applied to each auxiliary function J and $\bar{J}$, defined in (5.1.1) and (5.1.2) respectively, we give here an outline of the proof, that can be useful to make clear our strategy.

Outline of the proof:

i. The basic line of the proof is given in Step 1 of section 5.2: if one can show that the first auxiliary function J is non-positive in a neighborhood of the boundary arbitrarily close to the origin (that is, for any $s > 0$), then an integration over the coordinate curves parallel to the boundary (i.e. integration in the s variable), gives an upper estimate which is sufficient to apply Lemma 4.1 for any $s > 0$, proving that gradient blow-up can only take place at the origin. The rest of the proof consists in proving, by a maximum principle argument, that J is non-positive in a suitable neighborhood of the boundary near the origin.

ii. In step 2, we use Lemma 5.2 to prove that, after a suitable choice of the parameters γ and r_1, the auxiliary function J satisfies $\mathcal{P}J \leq 0$, provided the second auxiliary function $\bar{J}$ is non-positive (see (5.2.13)), where the parabolic operator $\mathcal{P}$ is defined in (5.1.5). The initial and boundary conditions needed to conclude the maximum principle argument are postponed to step 4.

iii. In step 3 we prove that, for $k > 0$ sufficiently small, we have $\bar{J} \leq 0$, and then we can deduce the parabolic inequality $\mathcal{P}J \leq 0$. To prove $\bar{J} \leq 0$, we use a maximum principle argument similar to the one used to prove $J \leq 0$. First, we also apply a parabolic inequality from Lemma 5.2, this time for the parabolic operator $\bar{\mathcal{P}}$ defined in 5.1.6. The main difference is that, since the derivative in the $x-$direction commutes with the Laplace operator, the computations are simpler and the sign of $\bar{\mathcal{P}}\bar{J}$ does not depend on other directional derivatives. For the initial and boundary conditions, we use a parabolic version of Serrin's corner Lemma (Proposition 4.3) that we prove in section 4.

iv. In step 4, we conclude the proof of the theorem by proving that, for k possibly smaller and depending on s, the initial and boundary conditions to apply the maximum principle argument to J are satisfied. In this step, we use Lemma 5.1, which gives the parabolic equation satisfied by the first part of J. This result is also used to establish the tangential monotonicity of solutions(see the proof of Theorem 2.5 in section 6) and as a part of the computations in the proof of Lemma 5.2.

5.1 Auxiliary parabolic inequalities

Recall Notation 3.1 and (3.0.4). Let $\sigma \in \big(0, \frac{1}{2(p-1)}\big)$ be fixed. For given $\eta \in (0, s_0/2)$, we consider the auxiliary functions

$$J = \frac{u_s}{1 - rK(s)} + c(s)d(r)F(u) \tag{5.1.1}$$

and

$$\bar{J} = u_x + \bar{c}(s)d(r)F(u), \tag{5.1.2}$$

defined in $(D_\Gamma \cap \Omega) \times (0, T)$, where D_Γ is given in (2.0.7) and

$$\begin{aligned}
&F(u) = u^q, && 1 < q < 2,\\
&d(r) = r^{-\gamma}, && \gamma = (1 - 2\sigma)(q - 1),\\
&c(s) = k(s - \eta), && k \in (0, 1),\\
&\bar{c}(s) = ks,
\end{aligned} \tag{5.1.3}$$

where k, γ will be taken small (i.e., q close to 1).

We start with a Lemma giving the equation satisfied by the first part of J.

Lemma 5.1. *Let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume* (2.0.4),(2.0.5). *Then, the function $w = \dfrac{u_s}{1-rK}$ satisfies*

$$w_t - \Delta w = a_w w + b_w \cdot \nabla w + \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x \qquad \text{in } (D_\Gamma \cap \{s>0\} \cap \Omega) \times (0,T), \quad (5.1.4)$$

with

$$\begin{aligned} a_w &= \frac{K^2}{(1-rK)^2} - \frac{pK}{1-rK}|\nabla u|^{p-2}u_r - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'}, \\ b_w &= p|\nabla u|^{p-2}\nabla u - \frac{2K}{1-rK}N(s). \end{aligned}$$

The following lemma contains the key inequalities that enable one to apply the maximum principle to the auxiliary functions J and $\bar{J}$.

Lemma 5.2. *Let $\Omega, \Gamma, \gamma, M$ be as in Notation 2.1 and assume* (2.0.4),(2.0.5). *Let $J, \bar{J}$ be the functions defined in* (5.1.1), (5.1.2) *and define the parabolic operators*

$$\mathcal{P}J := J_t - \Delta J - aJ - b \cdot \nabla J \quad (5.1.5)$$

and

$$\bar{\mathcal{P}}\bar{J} := \bar{J}_t - \Delta \bar{J} - \bar{a}\bar{J} - \bar{b} \cdot \nabla \bar{J}, \quad (5.1.6)$$

with

$$\begin{aligned} a &= -\frac{pK}{1-rK}|\nabla u|^{p-2}u_r - \frac{p}{1-rK}c'dF|\nabla u|^{p-2} + \frac{K^2}{(1-rK)^2} \\ &\quad -\frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'} - \frac{2}{1-rK}c'dF', \\ b &= p|\nabla u|^{p-2}\nabla u - \frac{2K}{1-rK}N(s), \\ \bar{a} &= -\alpha'\frac{p}{1-rK}\bar{c}'dF|\nabla u|^{p-2} - \frac{2\alpha'}{1-rK}\bar{c}'dF', \\ \bar{b} &= p|\nabla u|^{p-2}\nabla u. \end{aligned}$$

Then we have,

$$\frac{\mathcal{P}J}{cdF} \le \Theta(A) + \frac{K'}{(1-rK)^3}\frac{1}{\beta' cdF}\bar{J}, \qquad \text{in } (D_\Gamma \cap \{s>\eta\} \cap \Omega) \times (0,T), \quad (5.1.7)$$

and

$$\frac{\bar{\mathcal{P}}\bar{J}}{\bar{c}dF} \le \Theta(\bar{A}), \qquad \text{in } (D_\Gamma \cap \{s>0\} \cap \Omega) \times (0,T), \quad (5.1.8)$$

with

$$\begin{aligned} \Theta(A) &= -(p-1)q\frac{|\nabla u|^p}{u} + \frac{pk}{1-rK}\frac{u^q|\nabla u|^{p-2}}{r^\gamma} + p\frac{|\nabla u|^{p-1}}{r}A \\ &\quad -q(q-1)\frac{|\nabla u|^2}{u^2} + \frac{2q}{r}\frac{|\nabla u|}{u}A + \frac{2qk}{1-rK}\frac{u^{q-1}}{r^\gamma} - \frac{\gamma(\gamma+1)}{r^2}, \end{aligned} \quad (5.1.9)$$

and $A = A(r,s) = \gamma + \dfrac{rK}{1-rK}$, $\bar{A} = \bar{A}(r,s) = \gamma + \dfrac{\tau r}{1-rK}$, for some $\tau = \tau(\Omega) > 0$.

In addition, there exists a constant $L = L(p, \Omega, \|u_0\|_{C^1}) > 0$ such that, for all real numbers $X > 0$, we have

$$\begin{aligned}\Theta(X) \quad &\leq \quad \left[kB\left(pL^{q+p-2} + 2qL^{q-1}\right) + \frac{q}{q-1}X^2 + \frac{\sigma}{2}X - \gamma(\gamma+1)\right]\frac{1}{r^2} \\ &\quad + \left(\frac{p^2X}{2\sigma}L^{p-1} - (p-1)q\right)\frac{|\nabla u|^p}{u},\end{aligned} \tag{5.1.10}$$

where $B = B(r,s) = \dfrac{r^{(q-1)(2\sigma - \frac{1}{p-1})+2}}{1 - rK(s)}$.

Since the proofs of these two Lemmas require long computations, we postpone them after the proof of Theorem 2.3.

5.2 Proof of Theorem 2.3

Step 1: Preparations. Fix any $\eta \in (0, s_0/2)$ and recall the definition of the auxiliary function J given in (5.1.1)

$$J = \frac{u_s}{1-rK} + c(s)d(r)F(u) \ = \frac{u_s}{1-rK} + k(s-\eta)r^{-\gamma}u^q \qquad \text{in } \omega_0 \times (t_0, T),$$

with $1 < q < 2$, $\gamma = (1-2\sigma)(q-1) > 0$, where $\sigma \in \left(0, \frac{1}{2(p-1)}\right)$ is fixed, and $k \in (0,1)$ and γ will be taken small (i.e., q close to 1). Without loss of generality, by taking $r_0 > 0$ possibly smaller, we may assume that

$$\omega_0 = M((0, r_0) \times (0, s_0)) \subset \Omega,$$

where M is the coordinate map defined in (2.0.3).

Observe that, for each $t_0 < T' < T$, we have

$$u \leq Cr \qquad \text{in } \omega_0 \times [t_0, T'], \tag{5.2.1}$$

for some $C = C(T') > 0$. Since $\gamma < q$, we have in particular

$$J \in C(\overline{\omega_0} \times [0,T)) \cap C^{2,1}(\omega_0 \times (0,T)). \tag{5.2.2}$$

Fix $t_1 = \frac{t_0+T}{2}$, $s_1 = \frac{3}{4}s_0$ and set $K_1 = \max\limits_{[0,s_0]} K(s)$. Our aim is to use the maximum principle to prove that

$$J \leq 0 \quad \text{in } \omega_{1,\eta} \times (t_1, T), \tag{5.2.3}$$

where

$$\omega_{1,\eta} := M((0, r_1) \times (\eta, s_1)),$$

for $r_1 \in \left(0, \min(r_0, \frac{1}{2K_1})\right)$ to be chosen below.

Note that since $1 - rK \geq 1/2$ in $\omega_{1,\eta}$, inequality (5.2.3) implies

$$u_s \leq -(1-rK)cdF \leq -\tfrac{k}{2}(s-\eta)r^{-(1-2\sigma)(q-1)}u^q. \tag{5.2.4}$$

Hence, if (5.2.3) is proved, then integrating (5.2.4) over the curve

$$\{\gamma(\theta) + rN(\theta);\ \theta \in [\eta, s)\}$$

for $\eta < s < s_1$, $0 < r < r_1$ and $t_1 < t < T$, we get

$$u \leq C(s-\eta)^{-\frac{2}{q-1}} r^{1-2\sigma} \leq C(s-\eta)^{-\frac{2}{q-1}} \delta^{1-2\sigma}(x,y) \qquad \text{in } \omega_{1,\eta} \times (t_1, T),$$

for some constant $C = C(\eta) > 0$. Then, since $1-2\sigma > (p-2)/(p-1)$, it will follow from Lemma 4.1 and symmetry that $GBUS(u_0) \subset \gamma\big([-\eta,\eta]\big)$. Since η is arbitrarily small, we will conclude that $GBUS(u_0) = \{(0,0)\}$.

Step 2: Parabolic inequality for J.

It follows from (5.1.7) and (5.1.10) in Lemma 5.2 that, for the parabolic operator $\mathcal{P}$ defined in (5.1.5), we have

$$\begin{aligned} \frac{\mathcal{P}J}{cdF} \quad \leq \quad & \left[kB(pL^{q+p-2} + 2qL^{q-1}) + \frac{q}{q-1}A^2 + \frac{\sigma}{2}A - \gamma(\gamma+1)\right]\frac{1}{r^2} \\ & + \left(\frac{p^2 A}{2\sigma}L^{p-1} - (p-1)q\right)\frac{|\nabla u|^p}{u} + \frac{K'}{(1-rK)^3}\frac{1}{\beta' cdF}\bar{J}, \quad \text{in } \omega_{1,\eta} \times (t_0, T), \end{aligned} \tag{5.2.5}$$

with $L = L(p, \Omega, \|u_0\|_{C^1}) > 0$. At this point we fix γ and r_1 satisfying

$$0 < \gamma < \sigma \min\Big(\frac{1}{4}, \frac{1}{p^2 L^{p-1}}\Big) < 1 \tag{5.2.6}$$

and

$$0 < r_1 < \min\Big[r_0, 1, \frac{\gamma^2}{2K_1}, \frac{\gamma^2}{2\tau}, \frac{3\gamma^2}{2(pL^{q+p-2} + 2qL^{q-1})}\Big], \tag{5.2.7}$$

where $\tau = \tau(\Omega) > 0$ is given by Lemma 5.2 (some of the conditions in (5.2.6), (5.2.7) will be used only in Step 3), and we set

$$\omega_1 := M((0, r_1) \times (0, s_1)).$$

It follows, from $r_1 < \frac{1}{2K_1}$, that

$$1 - rK \geq 1/2 \quad \text{in } \omega_1, \tag{5.2.8}$$

hence

$$A \;=\; \gamma + \frac{rK}{1-rK} \leq \gamma(\gamma+1) \quad \text{in } \omega_1, \tag{5.2.9}$$

$$B \;=\; \frac{r^{(q-1)(2\sigma - \frac{1}{p-1})+2}}{1-rK} \leq 2r_1 \quad \text{in } \omega_1, \tag{5.2.10}$$

where we used $(q-1)\left(2\sigma - \frac{1}{p-1}\right) + 2 \geq 1$, which follows from $1 < q < p$. As a consequence of (5.2.6) and (5.2.9), using $p > 2$ and $q > 1$, we first get

$$\frac{p^2 A}{2\sigma}L^{p-1} - (p-1)q \leq \frac{p^2\gamma}{\sigma}L^{p-1} - 1 \leq 0 \quad \text{in } \omega_1. \tag{5.2.11}$$

Next, since $\gamma = (1-2\sigma)(q-1)$, we deduce from (5.2.6) and (5.2.9) that

$$\begin{aligned} \frac{q}{q-1}A^2 + \frac{\sigma}{2}A - \gamma(\gamma+1) \quad & \leq \gamma(\gamma+1)\left((1-2\sigma+\gamma)(\gamma+1) + \frac{\sigma}{2} - 1\right) \\ & = \gamma(\gamma+1)\left([\gamma + 2(1-\sigma)]\gamma - \frac{3\sigma}{2}\right) \\ & \leq 3\gamma(\gamma+1)\left(\gamma - \frac{\sigma}{2}\right) \leq -3\gamma^2 \quad \text{in } \omega_1. \end{aligned}$$

In view of (5.2.7), (5.2.10), and recalling $k \in (0,1)$, we obtain

$$\begin{aligned} kB(pL^{q+p-2} + 2qL^{q-1}) + \frac{q}{q-1}A^2 + \frac{\sigma}{2}A - \gamma(\gamma+1) \\ \le 2r_1(pL^{q+p-2} + 2qL^{q-1}) - 3\gamma^2 \le 0 \quad \text{ in } \omega_1. \end{aligned} \tag{5.2.12}$$

It follows from (5.2.5), (5.2.11), (5.2.12) that, for all $k \in (0,1)$,

$$\mathcal{P}J \le \frac{K'}{\beta'(1-rK)^3}\bar{J} \qquad \text{in } \omega_{1,\eta} \times (t_0,T). \tag{5.2.13}$$

Moreover, in view of (2.0.4), (2.0.5), (5.2.1) and (5.2.8), the coefficient a in $\mathcal{P}$ satisfies

$$\sup_{\omega_{1,\eta}\times(t_0,T')} |a| < \infty, \qquad \text{for any } T' < T. \tag{5.2.14}$$

Step 3: Control of $\bar{J}$.

We claim that under assumptions (5.2.6), (5.2.7), there exists $\tilde{k} \in (0,1)$ such that, for all $k \in (0,\tilde{k}]$,

$$\bar{J} = u_x + \bar{c}dF \ = u_x + ksr^{-\gamma}u^q \le 0 \quad \text{in } \omega_1 \times (t_1,T), \tag{5.2.15}$$

hence

$$\mathcal{P}J \le 0 \quad \text{in } \omega_{1,\eta} \times (t_1,T). \tag{5.2.16}$$

By (5.1.8) and (5.1.10) in Lemma 5.2, we have the following inequality for the parabolic operator $\bar{\mathcal{P}}$ defined in (5.1.6):

$$\begin{aligned} \frac{\bar{\mathcal{P}}\bar{J}}{\bar{c}dF} \ \le \ & \left[kB\left(pL^{q+p-2} + 2qL^{q-1}\right) + \frac{q}{q-1}\bar{A}^2 + \frac{\sigma}{2}\bar{A} - \gamma(\gamma+1)\right]\frac{1}{r^2} \\ & + \left(\frac{p^2\bar{A}}{2\sigma}L^{p-1} - (p-1)q\right)\frac{|\nabla u|^p}{u}, \qquad \text{in } \omega_1 \times (t_0,T), \end{aligned}$$

where

$$\bar{A} = \gamma + \frac{\tau(\Omega)r}{1-rK} \quad \text{ and } \quad B = \frac{r^{(q-1)(2\sigma-\frac{1}{p-1})+2}}{1-rK}.$$

Moreover, under assumptions (5.2.6), (5.2.7) (which in particular guarantee $\bar{A} \le \gamma(\gamma+1)$ in ω_1), the argument leading to (5.2.11), (5.2.12) yields:

$$\frac{p^2\bar{A}}{2\sigma}L^{p-1} - (p-1)q \le 0 \quad \text{ in } \omega_1,$$

and

$$kB(pL^{q+p-2} + 2qL^{q-1}) + \frac{q}{q-1}\bar{A}^2 + \frac{\sigma}{2}\bar{A} - \gamma(\gamma+1) \le 0 \quad \text{ in } \omega_1.$$

For any $k \in (0,1)$, we thus obtain

$$\bar{\mathcal{P}}\bar{J} \le 0, \qquad \text{in } \omega_1 \times (t_0,T). \tag{5.2.17}$$

By (2.0.9), there exists a constant $C > 0$ such that

$$|\nabla u| \le C \quad \text{in} \quad (\omega_0 \setminus M\left((0,r_1/2)\times(0,\theta_1)\right)) \times (t_0,T),$$

for $\theta_1 \in \left(\frac{s_0}{2}, s_1\right)$. Consequently, by parabolic estimates, u can be extended to a function such that

$$u, \nabla u \in C^{2,1}(\tilde{\mathcal{Q}}) \quad \text{where } \tilde{\mathcal{Q}} = \left(\overline{\omega_0} \setminus M\left(\left(0, \tfrac{3r_1}{4}\right) \times (0, \theta_2)\right)\right) \times (t_0, T], \tag{5.2.18}$$

with $\theta_2 \in (\theta_1, s_1)$. Fix any $t_2 \in (t_0, t_1)$ and $r_2 \in \left(r_1, \min(r_0, \frac{1}{2K_1})\right)$. Since $w = u_x$ satisfies

$$w_t - \Delta w = p|\nabla u|^{p-2} \nabla u \cdot \nabla w \quad \text{in } \omega_0 \times (t_0, T),$$

by Hopf's Lemma, (5.2.18) and (2.0.10), there exist $c_1, c_2 > 0$ such that

$$u_x \le -c_1 r \quad \text{on } (0, r_2) \times \{s_1\} \times (t_2, T), \tag{5.2.19}$$

$$u_x \le -c_1 s \quad \text{on } \{r_1\} \times (0, s_1) \times (t_2, T), \tag{5.2.20}$$

$$u \le c_2 r \quad \text{on } (0, r_1) \times \{s_1\} \times (t_2, T). \tag{5.2.21}$$

Moreover, in view of (2.0.10), (5.2.19), and since $M\left((0, r_2) \times (0, s_0)\right) \subset \Omega$, we can apply Lemma 4.3 to deduce the existence of $\tilde{c}_1 > 0$ such that

$$u_x(r, s, t_1) \le -\tilde{c}_1 rs \quad \text{in } (0, r_1) \times (0, s_1). \tag{5.2.22}$$

Now, on the lateral boundary of $\omega_1 \times (t_1, T)$, we have

$$\bar{J}(0, s, t) = 0 \quad \text{on } \{0\} \times (0, s_1) \times (t_1, T), \tag{5.2.23}$$

$$\bar{J}(r, 0, t) \le 0 \quad \text{on } (0, r_1) \times \{0\} \times (t_1, T), \tag{5.2.24}$$

$$\bar{J}(r_1, s, t) \le -c_1 s + ksr_1^{-\gamma} \|u_0\|_\infty^q \le 0 \quad \text{on } \{r_1\} \times (0, s_1) \times (t_1, T), \tag{5.2.25}$$

$$\bar{J}(r, s_1, t) \le -c_1 r + ks_1 c_2^q r^{q-\gamma} \le 0 \quad \text{on } (0, r_1) \times \{s_1\} \times (t_1, T), \tag{5.2.26}$$

for any $0 < k \le \tilde{k}$ with $\tilde{k} > 0$ sufficiently small, where we used $q > \gamma + 1$. And at the initial time $t = t_1$, for any $0 < k \le \tilde{k}$ with possibly smaller $\tilde{k} > 0$, inequality (5.2.22) guarantees

$$\bar{J}(r, s, t_1) \le -\tilde{c}_1 rs + ksc_2^q r^{q-\gamma} \le 0 \quad \text{in } (0, r_1) \times (0, s_1). \tag{5.2.27}$$

Moreover, owing to (5.2.1) and (5.2.8), we have

$$\sup_{\omega_1 \times (t_0, T')} \bar{a} < \infty, \qquad \text{for any } T' < T. \tag{5.2.28}$$

Then, for any $0 < k \le \tilde{k}$, claim (5.2.15) follows from (5.2.17), (5.2.23)–(5.2.28) and the maximum principle applied to $\bar{J}$ in $\omega_1 \times (t_1, T)$ (see Proposition 52.4 in [29]). Note that the use of the maximum principle is justified in view of the regularity property (5.2.2), which obviously also applies for $\bar{J}$. Finally, (5.2.16) follows from (2.0.4), (2.0.5), (5.2.13) and (5.2.15).

Step 4: Initial and boundary conditions for J.

Let $w = \dfrac{u_s}{1 - rK}$. In view of Lemma 5.1, (2.0.4), (2.0.5) and (2.0.10), it follows that

$$w_t - \Delta w \ - a_w w - b_w \cdot \nabla w = \frac{K'}{(1 - rK)^3} \frac{1}{\beta'} u_x \le 0, \qquad \text{in } \omega_0 \times (t_0, T),$$

with

$$a_w = \frac{K^2}{(1 - rK)^2} - \frac{pK}{1 - rK} |\nabla u|^{p-2} u_r - \frac{K'}{(1 - rK)^3} \frac{\alpha'}{\beta'},$$

$$b_w = p|\nabla u|^{p-2} \nabla u - \frac{2K}{1 - rK} N(s).$$

Note in particular that $\beta'(s)$ and $1-rK$ are uniformly positive for $s \in [\eta, s_1]$ by (2.0.5) and (5.2.8). In view of (2.0.10) and (5.2.18), we may thus apply the strong maximum principle and Hopf's Lemma to deduce the existence of $c_3, c_4, c_5 > 0$ (possibly depending on η) such that

$$u_s \le -c_3 r \quad \text{on } (0,r_1)\times\{s_1\}\times(t_1,T), \tag{5.2.29}$$
$$u_s \le -c_4 \quad \text{on } \{r_1\}\times(\eta,s_1)\times(t_1,T), \tag{5.2.30}$$
$$u_s \le -c_3 r \quad \text{in } \omega_{1,\eta}\times\{t_1\}, \tag{5.2.31}$$

as well as

$$u \le c_5 r, \quad \text{on } (0,r_1)\times\{s_1\}\times(t_1,T).$$

Consequently, we may choose $\tilde{k} > 0$ small enough (possibly depending on η) such that, for any $0 < k \le \tilde{k}$, on the lateral boundary of $\omega_{1,\eta}\times(t_1,T)$, we have

$$J(0,s,t) = 0 \quad \text{on } \{0\}\times(\eta,s_1)\times(t_1,T), \tag{5.2.32}$$
$$J(r,\eta,t) \le 0 \quad \text{on } (0,r_1)\times\{\eta\}\times(t_1,T), \tag{5.2.33}$$
$$J(r_1,s,t) \le -c_4 + ks_1 r_1^{-\gamma}\|u_0\|_\infty^q \le 0 \quad \text{on } \{r_1\}\times(\eta,s_1)\times(t_1,T), \tag{5.2.34}$$
$$J(r,s_1,t) \le -c_3 r + ks_1 c_5^q r^{q-\gamma} \le 0 \quad \text{on } (0,r_1)\times\{s_1\}\times(t_1,T), \tag{5.2.35}$$

where we used $q > \gamma+1$, and at the initial time $t = t_1$,

$$J(r,s,t_1) \le -c_3 r + ks_1 c_4^q r^{q-\gamma} \le 0, \quad \text{in } \omega_{1,\eta}. \tag{5.2.36}$$

Then (5.2.3) follows from (5.2.14), (5.2.16), (5.2.32)–(5.2.36) and the maximum principle applied to J in $\omega_{1,\eta}\times(t_1,T)$ (see Proposition 52.4 in [29]). Note that the use of the maximum principle is justified in view of (5.2.2).

In view of Step 1, this concludes the proof of the Theorem. □

5.3 Proof of auxiliary parabolic inequalities (Lemmas 5.1 and 5.2)

Proof of Lemma 5.1. Let $w = \dfrac{u_s}{1-rK}$, and compute, in $(D_\Gamma \cap \{s>0\}\cap\Omega)\times(0,T)$,

$$\begin{aligned}
w_r &= \frac{u_{rs}}{1-rK} + \frac{K}{(1-rK)^2}u_s,\\
w_{rr} &= \frac{u_{rrs}}{1-rK} + 2\frac{Ku_{rs}}{(1-rK)^2} + 2\frac{K^2}{(1-rK)^3}u_s,\\
w_s &= \frac{u_{ss}}{1-rK} + \frac{rK'}{(1-rK)^2}u_s,\\
w_{ss} &= \frac{u_{sss}}{1-rK} + 2\frac{rK'}{(1-rK)^2}u_{ss} + 2\frac{r^2K'^2}{(1-rK)^3}u_s + \frac{rK''}{(1-rK)^2}u_s.
\end{aligned}$$

Then, using Proposition 3.2 we get

$$\begin{aligned}
\Delta w &= w_{rr} - \frac{K}{1-rK}w_r + \frac{1}{(1-rK)^2}w_{ss} + \frac{rK'}{(1-rK)^3}w_s\\
&= \frac{1}{1-rK}u_{rrs} + \frac{K}{(1-rK)^2}u_{rs} + \frac{1}{(1-rK)^3}u_{sss} + \frac{K^2}{(1-rK)^3}u_s\\
&\quad +3\frac{rK'}{(1-rK)^4}u_{ss} + \frac{rK''}{(1-rK)^4}u_s + 3\frac{r^2K'^2}{(1-rK)^5}u_s
\end{aligned} \tag{5.3.1}$$

and also

$$\begin{aligned}
\frac{(\Delta u)_s}{1-rK} &= \frac{1}{1-rK}u_{rrs} - \frac{K}{(1-rK)^2}u_{rs} + \frac{1}{(1-rK)^3}u_{sss} + 3\frac{rK'}{(1-rK)^4}u_{ss} \\
&\quad + \frac{rK''}{(1-rK)^4}u_s + 3\frac{r^2K'^2}{(1-rK)^5}u_s - \frac{K'}{(1-rK)^2}u_r - \frac{rKK'}{(1-rK)^3}u_r \\
&= \Delta w - \frac{1}{(1-rK)^2}\left(\frac{K'}{1-rK}u_r + 2Ku_{rs} + \frac{K^2}{1-rK}u_s\right),
\end{aligned}$$

and replacing $u_{rs} = (1-rK)w_r - \frac{K}{1-rK}u_s$ and using identity (3.0.10), we obtain

$$\begin{aligned}
\Delta w &= \frac{(\Delta u)_s}{1-rK} + \frac{2K}{1-rK}w_r - \left(\frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'}\right)w \\
&\quad - \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x.
\end{aligned} \tag{5.3.2}$$

Note that the use of (3.0.10) is justified since $s > 0$, and then $\beta' > 0$. Then we get

$$\begin{aligned}
w_t - \Delta w &= \frac{(|\nabla u|^p)_s}{1-rK} - \frac{2K}{1-rK}w_r + \left(\frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'}\right)w \\
&\quad + \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x.
\end{aligned} \tag{5.3.3}$$

Now, we write

$$(|\nabla u|^p)_s = p|\nabla u|^{p-2}\nabla u \cdot (\nabla u)_s, \tag{5.3.4}$$

and using (3.0.6), we obtain

$$\begin{aligned}
\nabla u \cdot (\nabla u)_s &= \left(u_r N(s) + \frac{u_s}{1-rK}T(s)\right) \cdot \left(u_{rs}N(s) + \frac{u_{ss}}{1-rK}T(s) + u_r(N(s))_s\right. \\
&\quad \left. + \frac{rK'}{(1-rK)^2}u_s T(s) + \frac{u_s}{1-rK}(T(s))_s\right),
\end{aligned}$$

where $T(s)$ and $N(s)$ are defined in Notation 2.1.

We observe that

$$\begin{aligned}
N(s) \cdot (N(s))_s = T(s) \cdot (T(s))_s = 0, \\
N(s) \cdot (T(s))_s + T(s) \cdot (N(s))_s = (N(s) \cdot T(s))_s = 0,
\end{aligned}$$

so we have

$$\begin{aligned}
\nabla u \cdot (\nabla u)_s &= \nabla u \cdot \nabla(u_s) + \frac{rK'}{(1-rK)^3}u_s^2 \\
&= (1-rK)\nabla u \cdot \nabla w - w\nabla u \cdot \nabla(rK) + w\frac{rK'}{(1-rK)^2}u_s \\
&= (1-rK)\nabla u \cdot \nabla w - Ku_r w.
\end{aligned}$$

Plugging this in (5.3.4), we obtain

$$\frac{(|\nabla u|^p)_s}{1-rK} = p|\nabla u|^{p-2}\nabla u \cdot \nabla w - \frac{pK}{1-rK}|\nabla u|^{p-2}u_r w, \tag{5.3.5}$$

and combining this with (5.3.3), we obtain (5.1.4). $\square$

Proof of Lemma 5.2.

Proof of inequality (5.1.7)*:* Using Proposition 3.2 and (5.3.2), we compute, in $(D_\Gamma \cap \{s > \eta\} \cap \Omega) \times (0, T)$,

$$\begin{aligned}
J_t &= \frac{u_{ts}}{1-rK} + cdF'u_t, \\
\Delta J &= \Delta w + cdF'\Delta u + cdF''|\nabla u|^2 + \frac{2}{1-rK}c'dF'w \\
&\quad +2cd'F'u_r + cd''F - \frac{K}{1-rK}cd'F + \frac{rK'}{(1-rK)^3}c'dF \\
&= \frac{(\Delta u)_s}{1-rK} - \left(\frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'} - \frac{2}{1-rK}c'dF'\right)w \\
&\quad +\frac{2K}{1-rK}w_r + cdF'\Delta u + cdF''|\nabla u|^2 - \frac{K}{1-rK}cd'F \\
&\quad +cd''F + \frac{rK'}{(1-rK)^3}c'dF + 2cd'F'u_r - \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
J_t - \Delta J &= \frac{(|\nabla u|^p)_s}{1-rK} + \left(\frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'} - \frac{2}{1-rK}c'dF'\right)w \\
&\quad -\frac{2K}{1-rK}w_r + cdF'|\nabla u|^p - cdF''|\nabla u|^2 + \frac{K}{1-rK}cd'F \\
&\quad -cd''F - \frac{rK'}{(1-rK)^3}c'dF - 2cd'F'u_r + \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x,
\end{aligned}$$

and plugging (5.3.5) here, we get

$$\begin{aligned}
J_t - \Delta J &= \left(-\frac{pK}{1-rK}|\nabla u|^{p-2}u_r + \frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3}\frac{\alpha'}{\beta'} - \frac{2}{1-rK}c'dF'\right)w \\
&\quad +p|\nabla u|^{p-2}\nabla u \cdot \nabla w - \frac{2K}{1-rK}w_r + cdF'|\nabla u|^p - cdF''|\nabla u|^2 - 2cd'F'u_r \\
&\quad +\frac{K}{1-rK}cd'F - cd''F - \frac{rK'}{(1-rK)^3}c'dF + \frac{K'}{(1-rK)^3}\frac{1}{\beta'}u_x.
\end{aligned}$$

Now, we use the following identities:

$$\begin{aligned}
w &= J - cdF, \\
w_r &= N(s)\cdot\nabla J - cdF'u_r - cd'F, \\
\nabla u \cdot \nabla w &= \nabla u \cdot \nabla J - \nabla u \cdot \nabla(cdF),
\end{aligned}$$

and

$$\begin{aligned}
\nabla u \cdot \nabla(cdF) &= u_r(cd'F + cdF'u_r) + \frac{u_s}{(1-Kr)^2}(c'dF + cdF'u_s) \\
&= cdF'|\nabla u|^2 + cd'Fu_r + c'dF\frac{u_s}{(1-rK)^2},
\end{aligned} \tag{5.3.6}$$

hence

$$\nabla u \cdot \nabla w = \nabla u \cdot \nabla J - \frac{1}{1-rK}c'dFJ - cdF'|\nabla u|^2 - cd'Fu_r + \frac{1}{1-rK}cc'd^2F^2,$$

to obtain

$$
\begin{aligned}
J_t - \Delta J \quad = \quad & \left(-\frac{pK}{1-rK} |\nabla u|^{p-2} u_r + \frac{K^2}{(1-rK)^2} - \frac{K'}{(1-rK)^3} \frac{\alpha'}{\beta'} - \frac{2}{1-rK} c' dF' \right. \\
& \left. - \frac{p}{1-rK} c' dF |\nabla u|^{p-2} \right) J + \left(p |\nabla u|^{p-2} \nabla u - \frac{2K}{1-rK} N(s) \right) \cdot \nabla J \\
& -(p-1) c dF' |\nabla u|^p + \frac{p}{1-rK} c c' d^2 F^2 |\nabla u|^{p-2} \\
& -p c d' F |\nabla u|^{p-2} u_r + \frac{pK}{1-rK} c dF |\nabla u|^{p-2} u_r - c dF'' |\nabla u|^2 \\
& + \frac{2K}{1-rK} c dF' u_r - 2 c d' F' u_r + \frac{3K}{1-rK} c d' F - c d'' F \\
& + \frac{2}{1-rK} c c' d^2 F F' - \frac{rK'}{(1-rK)^3} c' dF - \frac{K^2}{(1-rK)^2} c dF \\
& + \frac{K'}{(1-rK)^3} \frac{1}{\beta'} \left(u_x + \alpha' c dF \right).
\end{aligned}
$$

Let $\mathcal{P}J := J_t - \Delta J - aJ - b \cdot \nabla J$, where a, b are defined in the statement of the Lemma. Using $K, K' \geq 0$ and the definitions of c, d, F, along with $\beta' > 0$, $0 < \alpha' \leq 1$ and $0 < c \leq \bar{c}$, we then have, in $(D_\Gamma \cap \{s > \eta\} \cap \Omega) \times (0, T)$:

$$
\begin{aligned}
\frac{\mathcal{P}J}{c dF} \quad = \quad & -(p-1) \frac{F'}{F} |\nabla u|^p + \frac{p}{1-rK} c' dF |\nabla u|^{p-2} - p \frac{d'}{d} |\nabla u|^{p-2} u_r \\
& + \frac{pK}{1-rK} |\nabla u|^{p-2} u_r - \frac{F''}{F} |\nabla u|^2 + \frac{2K}{1-rK} \frac{F'}{F} u_r - 2 \frac{d' F'}{dF} u_r \\
& + \frac{3K}{1-rK} \frac{d'}{d} - \frac{d''}{d} + \frac{2}{1-rK} c' dF' - \frac{rK'}{(1-rK)^3} \frac{c'}{c} - \frac{K^2}{(1-rK)^2} \\
& + \frac{K'}{(1-rK)^3} \frac{1}{\beta' c dF} (u_x + \alpha' c dF) \\
\leq \quad & -(p-1) q \frac{|\nabla u|^p}{u} + \frac{pk}{1-rK} \frac{u^q |\nabla u|^{p-2}}{r^\gamma} + p \frac{|\nabla u|^{p-1}}{r} \left(\gamma + \frac{rK}{1-rK} \right) \\
& -q(q-1) \frac{|\nabla u|^2}{u^2} + \frac{2q}{r} \frac{|\nabla u|}{u} \left(\gamma + \frac{rK}{1-rK} \right) + \frac{2qk}{1-rK} \frac{u^{q-1}}{r^\gamma} - \frac{\gamma(\gamma+1)}{r^2} \\
& + \frac{K'}{(1-rK)^3} \frac{1}{\beta' c dF} (u_x + \bar{c} dF)
\end{aligned}
$$

that is, (5.1.7).

Proof of inequality (5.1.8):

In a similar but simpler way as in the computation for J and using (3.0.9), we compute, in $(D_\Gamma \cap \{s > 0\} \cap \Omega) \times (0, T)$,

$$\begin{aligned}
\bar{J}_t &= u_{tx} + \bar{c} d F' u_t, \\
\Delta \bar{J} &= (\Delta u)_x + \bar{c} d F'' |\nabla u|^2 + \bar{c} F \left(d'' - \frac{K}{1 - rK} d' \right) + \frac{rK'}{(1 - rK)^3} \bar{c}' d F \\
&\quad + 2 \bar{c} d' F' u_r + \frac{2}{(1 - rK)^2} \bar{c}' d F' u_s + \bar{c} d F' \Delta u \\
&= (\Delta u)_x + \bar{c} d F'' |\nabla u|^2 + \bar{c} F \left(d'' - \frac{K}{1 - rK} d' \right) + \frac{rK'}{(1 - rK)^3} \bar{c}' d F \\
&\quad + \frac{2\alpha'}{1 - rK} \bar{c}' d F' u_x + 2 \bar{c} d' F' \left(u_r + \beta' \frac{u_y}{1 - rK} \frac{\bar{c}' d}{\bar{c} d'} \right) + \bar{c} d F' \Delta u.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\bar{J}_t - \Delta \bar{J} &= (|\nabla u|^p)_x + \bar{c} d F' |\nabla u|^p - \frac{2}{1 - rK} \alpha' \bar{c}' d F' u_x - \bar{c} d F'' |\nabla u|^2 \\
&\quad - 2 \bar{c} d' F' \left(u_r + \beta' \frac{u_y}{1 - rK} \frac{\bar{c}' d}{\bar{c} d'} \right) - \frac{rK'}{(1 - rK)^3} \bar{c}' d F - \bar{c} F \left(d'' - \frac{K}{1 - rK} d' \right).
\end{aligned}$$

In view of $u_x = \bar{J} - \bar{c} d F$, and using (3.0.9) and (5.3.6), we compute

$$\begin{aligned}
(|\nabla u|^p)_x &= p |\nabla u|^{p-2} \nabla u \cdot \nabla u_x \\
&= p |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{J} - p |\nabla u|^{p-2} \nabla u \cdot \nabla (\bar{c} d F) \\
&= p |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{J} - p \bar{c} d F' |\nabla u|^p - \alpha' \frac{p}{1 - rK} \bar{c}' d F |\nabla u|^{p-2} u_x \\
&\quad - p \bar{c} d' F |\nabla u|^{p-2} \left(u_r + \beta' \frac{u_y}{1 - rK} \frac{\bar{c}' d}{\bar{c} d'} \right).
\end{aligned}$$

It then follows that

$$\begin{aligned}
\bar{J}_t - \Delta \bar{J} &= \bar{a} \bar{J} + \bar{b} \cdot \nabla \bar{J} - (p - 1) \bar{c} d F' |\nabla u|^p + \alpha' \frac{p}{1 - rK} \bar{c} \bar{c}' d^2 F^2 |\nabla u|^{p-2} \\
&\quad - p \bar{c} d' F |\nabla u|^{p-2} \left(u_r + \beta' \frac{u_y}{1 - rK} \frac{\bar{c}' d}{\bar{c} d'} \right) - \bar{c} d F'' |\nabla u|^2 \\
&\quad - 2 \bar{c} d' F' \left(u_r + \beta' \frac{u_y}{1 - rK} \frac{\bar{c}' d}{\bar{c} d'} \right) + \frac{2\alpha'}{1 - rK} \bar{c} \bar{c}' d^2 F F' \\
&\quad - \frac{rK'}{(1 - rK)^3} \bar{c}' d F - \bar{c} F \left(d'' - \frac{K}{1 - rK} d' \right),
\end{aligned} \tag{5.3.7}$$

where

$$\bar{a} = -\alpha' \frac{p}{1 - rK} \bar{c}' d F |\nabla u|^{p-2} - \frac{2\alpha'}{1 - rK} \bar{c}' d F', \quad \bar{b} = p |\nabla u|^{p-2} \nabla u.$$

In view of the symmetry of Ω and Γ (assumption (2.0.4)), we have $\beta'(0) = 0$. By the regularity of $\partial\Omega$, it follows that there exists $\tau = \tau(\Omega) > 0$ such that

$$\beta'(s) \leq \tau s, \qquad \forall s \in [0, s_0].$$

Let $\bar{\mathcal{P}} \bar{J} := \bar{J}_t - \Delta \bar{J} - \bar{a} \bar{J} - \bar{b} \cdot \nabla \bar{J}$, where $\bar{a}, \bar{b}$ are defined in the statement of the Lemma. Plugging the definitions of $\bar{c}, d, F$ in the expression (5.3.7), and using the above inequality, $K' \geq 0$ and $\alpha' \leq 1$, we obtain

$$\begin{aligned}
\frac{\bar{\mathcal{P}}\bar{J}}{\bar{c}dF} &= -(p-1)\frac{F'}{F}|\nabla u|^p + \alpha'\frac{p}{1-rK}\bar{c}'dF|\nabla u|^{p-2} \\
&\quad -p\frac{d'}{d}|\nabla u|^{p-2}\left(u_r + \beta'\frac{u_y}{1-rK}\frac{\bar{c}'d}{\bar{c}d'}\right) - \frac{F''}{F}|\nabla u|^2 \\
&\quad -2\frac{d'F'}{dF}\left(u_r + \beta'\frac{u_y}{1-rK}\frac{\bar{c}'d}{\bar{c}d'}\right) + \frac{2\alpha'}{1-rK}\bar{c}'dF' \\
&\quad -\frac{rK'}{(1-rK)^3}\bar{c}'dF - \frac{d''}{d} + \frac{K}{1-rK}\frac{d'}{d} \\
&\leq -(p-1)q\frac{|\nabla u|^p}{u} + \frac{pk}{1-rK}\frac{u^q|\nabla u|^{p-2}}{r^\gamma} + p\frac{|\nabla u|^{p-1}}{r}\left(\gamma + \frac{\tau r}{1-rK}\right) \\
&\quad -q(q-1)\frac{|\nabla u|^2}{u^2} + \frac{2q}{r}\frac{|\nabla u|}{u}\left(\gamma + \frac{\tau r}{1-rK}\right) + \frac{2qk}{1-rK}\frac{u^{q-1}}{r^\gamma} - \frac{\gamma(\gamma+1)}{r^2}.
\end{aligned}$$

Proof of inequality (5.1.10)*:*

Using Young's inequality we obtain, for any $X > 0$,

$$\frac{2q}{r}\frac{|\nabla u|}{u}X \leq q(q-1)\frac{|\nabla u|^2}{u^2} + \frac{q}{q-1}\frac{X^2}{r^2},$$

and

$$p\frac{|\nabla u|^{p-1}}{r}X \leq \frac{\sigma}{2r^2}X + \frac{p^2}{2\sigma}X|\nabla u|^{2p-2},$$

hence,

$$\frac{2q}{r}\frac{|\nabla u|}{u}X - q(q-1)\frac{|\nabla u|^2}{u^2} - \frac{\gamma(\gamma+1)}{r^2} \leq \left(\frac{q}{q-1}X^2 - \gamma(\gamma+1)\right)\frac{1}{r^2}, \tag{5.3.8}$$

and

$$\begin{aligned}
&-(p-1)q\frac{|\nabla u|^p}{u} + p\frac{|\nabla u|^{p-1}}{r}X \\
&\leq \left(\frac{p^2X}{2\sigma}u|\nabla u|^{p-2} - (p-1)q\right)\frac{|\nabla u|^p}{u} + \frac{\sigma}{2r^2}X.
\end{aligned} \tag{5.3.9}$$

Using (1.0.2), we obtain the following estimates

$$\frac{u^q|\nabla u|^{p-2}}{r^\gamma} \leq L^{q+p-2}r^{(q-1)\frac{p-2}{p-1}-\gamma} = L^{q+p-2}r^{(q-1)(2\sigma-\frac{1}{p-1})}, \tag{5.3.10}$$

$$\frac{u^{q-1}}{r^\gamma} \leq L^{q-1}r^{(q-1)\frac{p-2}{p-1}-\gamma} = L^{q-1}r^{(q-1)(2\sigma-\frac{1}{p-1})}, \tag{5.3.11}$$

$$u|\nabla u|^{p-2} \leq L^{p-1}, \tag{5.3.12}$$

where $L = L(p, \Omega, \|u_0\|_{C^1}) > 0$. Combining (5.3.8)-(5.3.12), we obtain

$$\begin{aligned}
\Theta(X) \leq &\left[k\left(pL^{q+p-2} + 2qL^{q-1}\right)\frac{r^{(q-1)(2\sigma-\frac{1}{p-1})+2}}{1-rK} + \frac{q}{q-1}X^2 + \frac{\sigma}{2}X - \gamma(\gamma+1)\right]\frac{1}{r^2} \\
&+ \left(\frac{p^2X}{2\sigma}L^{p-1} - (p-1)q\right)\frac{|\nabla u|^p}{u},
\end{aligned}$$

hence (5.1.10). $\square$

6 Proof of Theorem 2.5

Proof. (i) We shall produce suitable initial data by means of Proposition 4.2. Fix $\phi \in C^\infty([0,\infty))$ such that $\phi = 1$ on $[0,1]$, $\phi = 0$ on $[3/2,\infty)$ and $\phi' \le 0$. Take $\rho > 0$ so small that

$$B_\rho(0,0) \cap \partial\Omega \subset \gamma\Big(-\frac{s_0}{2}, \frac{s_0}{2}\Big). \tag{6.0.1}$$

Let C_1, C_2 be given by Proposition 4.2, pick any $\varepsilon \in (0,\rho/4)$ such that $C_1\varepsilon^k < C_2$ and set

$$u_0(x,y) = C_2\phi\bigg(\frac{\sqrt{x^2+(y-\varepsilon)^2)}}{\varepsilon/2}\bigg).$$

Then we immediately have (4.0.2) and $\mathrm{supp}(u_0) \subset B_\varepsilon(0,\varepsilon) \subset B_{\rho/2}(0,0)$. Also, by taking $\varepsilon > 0$ possibly smaller, we get $B_\varepsilon(0,\varepsilon) \subset \Omega$, hence (4.0.1) and (4.0.3). It thus follows from Proposition 4.2 that $T(u_0) < \infty$ and, in view of (6.0.1), that condition (2.0.20) is satisfied.

On the other hand, (2.0.16), and then $u_{0,x} \le 0$ in (2.0.10), are clearly satisfied. Moreover, by considering $\varepsilon > 0$ possibly smaller, the reflection properties (2.0.18) and (2.0.19) hold trivially. In order to prove $u_{0,s} \le 0$ in (2.0.10), we can use formula (3.0.9) to obtain

$$\frac{u_{0,s}}{1-rK} = \frac{2C_2}{\varepsilon}\phi'\left(\frac{\sqrt{x^2+(y-\varepsilon)^2}}{\varepsilon/2}\right)\frac{\alpha' x + \beta'(y-\varepsilon)}{\sqrt{x^2+(y-\varepsilon)^2}}.$$

Then, in view of $\phi' \le 0$ and the definition of the change of coordinates map $(x,y) = M(r,s) \ = \gamma(s) + rN(s)$, it suffices to check that $(\gamma', \gamma + rN - \varepsilon e_2) \ge 0$ for all sufficiently small $\varepsilon, s > 0$. To do this, let us write the Taylor expansions

$$\gamma'(s) = e_1 + sR_1(s), \quad \gamma(s) = se_1 + s^2R_2(s), \qquad \text{for all } s > 0 \text{ small},$$

where $|R_1|, |R_2| \le C_3$ for some constant $C_3 > 0$. Using also $N \perp \gamma'$, it follows that

$$(\gamma', \gamma + rN - \varepsilon e_2) = \big(e_1 + sR_1(s), s(e_1 + sR_2(s)) - \varepsilon e_2\big) \ge s(1 - C_3\varepsilon - 2C_3 s - C_3^2 s^2) \ge 0$$

for all sufficiently small $\varepsilon, s > 0$.

(ii) The assertion will be derived as a consequence of Theorem 2.3. For this it suffices to establish the monotonicity properties (2.0.10). The proof is done in two steps.

Step 1: Parabolic inequality. Consider the auxiliary function

$$w = \frac{u_s}{1-rK} \qquad \text{in } \mathcal{Q}_T := \omega_0 \times [0,T).$$

In view of (5.1.4), w satisfies

$$w_t - \Delta w = a_w w + b_w \cdot \nabla w + \frac{K'}{(1-rK)^3}\frac{1}{\beta'(s)}u_x, \tag{6.0.2}$$

with

$$\begin{aligned} a_w &= \frac{K^2}{(1-rK)^2} - \frac{pK}{1-rK}|\nabla u|^{p-2}u_r - \frac{K'}{(1-rK)^3}\frac{\alpha'(s)}{\beta'(s)}, \\ b_w &= p|\nabla u|^{p-2}\nabla u - \frac{2K}{1-rK}N(s). \end{aligned}$$

For any $T' \in (0,T)$, we have $\sup_{\mathcal{Q}_{T'}} |\nabla u| < \infty$. Also, by hypothesis (2.0.11), $1 - rK$ is bounded away from 0 in ω_0. This, together with $K' \ge 0, \alpha', \beta' > 0$, implies

$$\sup_{\mathcal{Q}_{T'}} a_w < \infty. \tag{6.0.3}$$

Since $u_0 \geq 0$ in Ω, by the strong maximum principle we have $u > 0$ in $\Omega \times (0,T)$. Therefore, by Hopf's lemma we get

$$u_\nu < 0 \quad \text{on } \partial\Omega \times (0,T), \tag{6.0.4}$$

where u_ν is the derivative of u in the outward normal direction to the boundary. As consequence, by (2.0.12), we have

$$u_x = \nu_x u_\nu \leq 0 \quad \text{on } (\partial\Omega \cap \{x > 0\}) \times (0,T).$$

By the symmetry of u_0 and Ω, we also have

$$u_x = 0 \quad \text{on } [\Omega \cap \{x = 0\}] \times (0,T).$$

Now, we see that $v = u_x$ satisfies

$$v_t - \Delta v = p|\nabla u|^{p-2}\nabla u \cdot \nabla v \quad \text{in } [\Omega \cap \{x > 0\}] \times (0,T).$$

Then, after hypothesis (2.0.17) and the strong maximum principle, we have

$$u_x < 0 \quad \text{in } [\Omega \cap \{x > 0\}] \times (0,T). \tag{6.0.5}$$

Since $\omega_0 \subset \Omega \cap \{x > 0\}$, it follows from (6.0.2), (6.0.5) and $K' \geq 0, \beta' > 0$ that

$$w_t - \Delta w - a_w w - b_w \cdot \nabla w \leq 0 \qquad \text{in } \mathcal{Q}_T. \tag{6.0.6}$$

Step 2: Boundary conditions and conclusion. We split the boundary of ω_0 in five parts:

$$\begin{aligned}
\Gamma_1 &= \{(\alpha(s), \beta(s)); \quad 0 < s < s_0\},\\
\Gamma_2 &= \Omega \cap \{x = 0\},\\
\Gamma_3 &= \partial\Omega \cap \partial\omega_0 \cap \{r > 0\},\\
\Gamma_4 &= \Omega \cap \Lambda_{s_0},\\
\Gamma_5 &= \Omega \cap D_\Gamma \cap \{y = y_0\}.
\end{aligned}$$

See Figures 4.3 and 4.4 for illustrations of such partitions. Note that Γ_3 and/or Γ_5 may be empty. In that case, we need not to care about them.

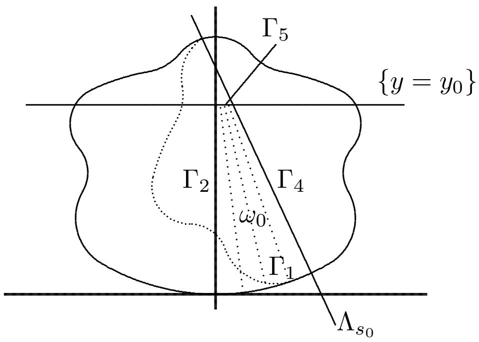

Figure 4.3 – Illustration of the partition of $\partial\omega_0$. In this case $\Gamma_3 = \emptyset$.

Since $u = 0$ on $\partial\Omega$, we have

$$u_s = 0 \quad \text{on } \Gamma_1 \times [0,T). \tag{6.0.7}$$

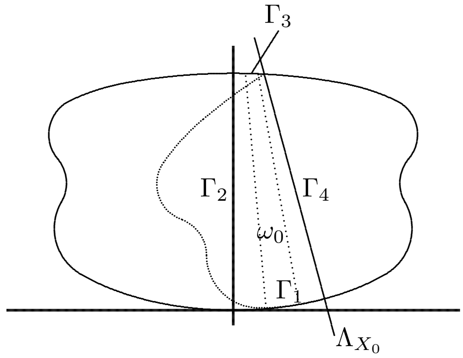

Figure 4.4 – Illustration of the partition of $\partial\omega_0$ when $y_0 = \infty$. In this case $\Gamma_5 = \emptyset$.

By the symmetry of the domain, and using (3.0.9), we have $u_s = (1-rK)u_x$ on Γ_2, and by (2.0.16), we deduce

$$u_s = 0 \quad \text{on } \Gamma_2 \times [0, T). \tag{6.0.8}$$

On the other hand, as consequence of (6.0.4), (2.0.12) and (2.0.13), we have

$$u_x = \nu_x u_\nu \leq 0, \qquad u_y = \nu_y u_\nu \leq 0 \qquad \text{on } \Gamma_3 \times [0, T).$$

Now, we recall from (3.0.9)

$$u_s = (1 - rK)(\alpha'(s)u_x + \beta'(s)u_y) \tag{6.0.9}$$

Then it follows from (2.0.5) that

$$u_s \leq 0 \qquad \text{on } \Gamma_3 \times [0, T). \tag{6.0.10}$$

Next, we shall prove by a moving planes argument that

$$u_s \leq 0 \quad \text{on } \Gamma_4 \times [0, T). \tag{6.0.11}$$

We define in $\Omega_{s_0} \times (0, T)$ the functions

$$u_1(x, y, t) = u(x, y, t), \qquad u_2(x, y, t) = u(\mathcal{T}_{s_0}(x, y), t),$$

where Ω_{s_0} and $\mathcal{T}_{s_0}$ are defined in Notation 2.4. We note that u_2 is well defined since by condition (2.0.14), $\mathcal{T}_{s_0}(x, y) \in \Omega$, for all $(x, y) \in \Omega_{s_0}$. Both functions u_1, u_2 satisfy the equation

$$u_{i,t} - \Delta u_i = |\nabla u_i|^p \qquad \text{in } \Omega_{s_0} \times (0, T),$$

for $i = 1, 2$. By condition (2.0.18), we have

$$u_1(x, y, 0) \leq u_2(x, y, 0) \qquad \text{in } \Omega_{s_0}.$$

The boundary of Ω_{s_0} is composed of two parts:

$$\Gamma_1^{s_0} := \partial\Omega \cap \partial\Omega_{s_0}, \qquad \Gamma_2^{s_0} := \Lambda_{s_0} \cap \Omega.$$

On $\Gamma_1^{s_0}$ we have $u_1(x, y, t) = 0$ and $u_2(x, y, t) \geq 0$ since $u \geq 0$ in $\Omega \times (0, T)$. On $\Gamma_2^{s_0}$ we have $u_1(x, y, t) = u_2(x, y, t)$, since $\mathcal{T}_{s_0}(x, y) = (x, y)$, for all $(x, y) \in \Lambda_{s_0}$. So we conclude that $u_1(x, y, t) \leq u_2(x, y, t)$ on $\partial\Omega_{s_0} \times [0, T)$. As a consequence of the comparison principle, we get $u_1 \le u_2$ in $\Omega_{s_0} \times [0,T)$. Letting (x,y) go to Λ_{s_0} in the normal direction to Λ_{s_0}, we deduce (6.0.11).

In order to show that
$$u_s \le 0 \quad \text{ on } \Gamma_5 \times [0,T), \tag{6.0.12}$$
we observe that, as a consequence of (2.0.15), (2.0.19) and of a similar moving planes argument as in the case of Γ_4, we have
$$u_y \le 0 \quad \text{ in } (\Omega \cap \{y = y_0\}) \times [0,T). \tag{6.0.13}$$
Property (6.0.12) then follows from (6.0.5), (6.0.9), (6.0.13) and (2.0.5).

Then, since $\partial\omega_0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$, it follows from (6.0.7), (6.0.8), (6.0.10)-(6.0.12) that
$$u_s \le 0 \qquad \text{on } \partial\omega_0 \times (0,T). \tag{6.0.14}$$
In view of (6.0.3), (6.0.6), (6.0.14), (2.0.17), it follows from the strong maximum principle that
$$u_s < 0 \qquad \text{in } \omega_0 \times (0,T).$$
This, together with (6.0.5) and (2.0.20), allows us to apply Theorem 2.3, and the conclusion follows. □

7 Proof of Theorems 1.1 and 1.2

Here we give the proofs of Theorems 1.1 and 1.2 as consequence of Theorem 2.5. We shall verify that the hypotheses of Theorem 2.5 hold for ellipses and for the domains satisfying the assumptions of Theorem 1.2.

Proof of Theorem 1.1. We only give the proof for ellipses with positive eccentricity. For disks, see [22]. Without loss of generality, we assume that the minor axis of the ellipse is on the half-line $\{x = 0;\ y \ge 0\}$ and that the lower co-vertex is at the origin. Then, assumption (2.0.12) holds. If we consider Γ a connected boundary piece containing the origin and symmetric with respect to $x = 0$, we can use Notation 2.1.

Now, take $y_0 > 0$ such that the major axis of the ellipse is on the line $y = y_0$. Hypothesis (2.0.15) is then satisfied. Moreover, in view of the position of the ellipse, it is well known that the center of curvature at any point of $\partial\Omega \cap \{y < y_0\}$ lies in the half-plane $\{y > y_0\}$. Considering $s_0 > 0$ small enough so that $\Gamma \subset \{y < y_0\}$, it follows that conditions (2.0.4), (2.0.5), (2.0.11) and (2.0.13) are satisfied.

Now let us verify that (2.0.14) also holds for this choice of Γ. Here, we recall the definitions of H_{s_0} and Λ_{s_0} in Notation 2.4. We shall prove that the symmetric of $\partial\Omega \cap H_{s_0}$ with respect to Λ_{s_0} lies in Ω, which guarantees (2.0.14) by convexity.

Let $\partial\Omega$ be the original ellipse and $\mathcal{T}_{s_0}(\partial\Omega)$ its symmetric with respect to the line Λ_{s_0}. We observe that the two ellipses intersect in at least two points, which are the two intersection points of $\partial\Omega$ with Λ_{s_0}. We also know that any two ellipses intersect in at most four points, counting the multiplicity. Since Λ_{s_0} is normal to $\partial\Omega$ at $\gamma(s_0)$, the two ellipses $\partial\Omega$ and $\mathcal{T}_{s_0}(\partial\Omega)$ are tangent to each other at that point, which is then an intersection point of multiplicity at least 2.

Therefore, there can be at most one other intersection point between the two ellipses. By convexity, it cannot be on the segment $\Lambda_{s_0} \cap \Omega$, and by symmetry with respect to the line Λ_{s_0}, if there is an intersection point on one side of Λ_{s_0}, there must be another one on the other side. Hence, the two ellipses only intersect in two points.

Finally, since the curvature of $\partial\Omega$ increases from the origin up to the right vertex, near $\gamma(s_0)$, the symmetric of $\partial\Omega \cap H_{s_0}$ lies in Ω. As we have seen, it does not intersect again the boundary of Ω until the other intersection of Λ_{s_0} with $\partial\Omega$. Therefore, we conclude that $\mathcal{T}_{s_0}(\partial\Omega \cap H_{s_0}) \subset \Omega$. Hence, Ω satisfies all the hypothesis of Theorem 2.5, and the conclusion follows. □

Proof of Theorem 1.2. We give the proof for the case when Ω is not locally flat at the origin. That is, in assumption (1.0.4) $\partial\Omega$ only touches $y = 0$ at the origin. For locally flat domains, see [22].

As in the proof of Theorem 1.1, we shall verify that all the hypotheses of Theorem 2.5 hold. In view of assumptions (1.0.3) and (1.0.4), and considering a suitable boundary piece, we can use Notation 2.1, and hypothesis (2.0.12) is satisfied. By taking a smaller Γ if necessary, hypotheses (2.0.4), (2.0.5) and (2.0.14) are guaranteed by assumptions (1.0.5) and (1.0.6).

The assumption $\overline{\Omega} \subset \{y < R(0)\}$ in (1.0.5), implies that the center of curvature of the boundary at the origin is at positive distance of Ω (possibly at infinity). Since the curvature is a continuous function due to the regularity of the boundary, considering a smaller Γ if necessary, the evolute of Γ is also at positive distance of Ω. Therefore, hypothesis (2.0.11) is satisfied with $y_0 = +\infty$, and then (2.0.15) is trivial.

As for hypothesis (2.0.13), we note that in view of (1.0.3) and since Ω is smooth and connected, $\Omega \cap \{x = \eta\}$ is a segment for all $\eta > 0$ small. Therefore, (2.0.13) holds by considering a possibly smaller Γ. □

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Chapter 5

The evolution problem associated with eigenvalues of the Hessian

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Abstract. In this paper we study the evolution problem

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = g(x,t), & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (that verifies a suitable geometric condition on its boundary) and $\lambda_j(D^2u)$ stands for the $j-$th eigenvalue of the Hessian matrix D^2u. We assume that u_0 and g are continuous functions with the compatibility condition $u_0(x) = g(x,0)$, $x \in \partial\Omega$.

We show that the (unique) solution to this problem exists in the viscosity sense and can be approximated by the value function of a two-player zero-sum game as the parameter measuring the size of the step that we move in each round of the game goes to zero.

In addition, when the boundary datum is independent of time, $g(x,t) = g(x)$, we show that viscosity solutions to this evolution problem stabilize and converge exponentially fast to the unique stationary solution as $t \to \infty$. For $j = 1$, the limit profile is just the convex envelope inside Ω of the boundary datum g, while for $j = N$, it is the concave envelope. We obtain this result with two different techniques: with PDE tools and with game theoretical arguments. Moreover, in some special cases (for affine boundary data) we can show that solutions coincide with the stationary solution in finite time (that depends only on Ω and not on the initial condition u_0).

1 Introduction

Consider the problem

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = g(x,t), & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1.0.1}$$

here, Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 1$ and $\lambda_j(D^2u)$ stands for the $j-$th eigenvalue of $D^2u = (\partial^2_{x_i,x_j}u)_{ij}$, which is the hessian matrix of u. We will assume from now on that u_0 and g are continuous functions with the compatibility condition $u_0(x) = g(x, 0)$, $x \in \partial\Omega$.

Problem (1.0.1) is the evolution version of the elliptic problem

$$\begin{cases} \lambda_j(D^2 z(x)) = 0, & \text{in } \Omega, \\ z(x) = g(x), & \text{on } \partial\Omega, \end{cases} \tag{1.0.2}$$

which was extensively studied in [1, 6, 4, 5, 7, 8, 12, 13, 23, 24]. In particular, for $j = 1$ and $j = N$, problem (1.0.2) is the equation for the convex and concave envelope of g in Ω, respectively, i.e., the solution z is the biggest convex (smallest concave) function u, satisfying $u \leq g$ ($u \geq g$) on $\partial\Omega$, see [23, 24].

In [7], existence and uniqueness of a continuous solution for (1.0.2) is proved under a hypothesis on the geometry of the domain. Moreover, from the results in [12], a comparison principle holds for viscosity sub and supersolutions of (1.0.2). Using this comparison principle, together with the connection with concave/convex envelopes of the boundary datum g for solutions to (1.0.2), the geometric condition introduced in [7] turns out to be necessary and sufficient for the well posedness of this problem in the viscosity sense. In our parabolic setting, using classical ideas from [10], one can show that there is also a comparison principle. Hence, uniqueness of a viscosity solution follows. Existence of solutions to (1.0.2) was shown in [12] using Perron's method. A different existence proof was given in [7], where the authors introduce a two-player zero-sum game whose value function approximates the solution of the PDE as the size of the game step goes to zero.

For our parabolic problem, in order to show existence of a continuous viscosity solution it seems natural to try to use Perron's method relying on the comparison principle. However, we prefer to take a different approach. We provide an existence proof using an approximation based on game theory (this approach will be very useful since it allows us to gain some intuition that will be used when dealing with the asymptotic behaviour of the solutions). For references concerning games (Tug-of-War games) and fully nonlinear PDEs we refer to [3, 9, 14, 15, 17, 18, 20, 21, 22, 25, 26, 27] and to [11, 19] for parabolic versions. Here we propose a parabolic version of the game introduced in [7] in order to show existence of a viscosity solution to (1.0.1). Like for the elliptic problem, it is a two-player zero-sum game. The initial position of the game is determined by a token placed at some point $x_0 \in \Omega$ and at some time $t_0 > 0$. Player I, who wants to minimize the final payoff, chooses a subspace S of dimension j in $\mathbb{R}^N$ and then, Player II, who wants to maximize the final payoff, chooses a unitary vector $v \in S$. Then, for a fixed $\varepsilon > 0$, the position of the token is moved to $(x_0 + \varepsilon v, t_0 - \varepsilon^2/2)$ or to $(x_0 - \varepsilon v, t_0 - \varepsilon^2/2)$ with equal probabilities. After the first round, the game continues from the new position (x_1, t_1) according to the same rules. Notice that we take $t_1 = t_0 - \varepsilon^2/2$, but $x_1 = x_0 \pm \varepsilon v$ depends on a coin toss. The game ends when the token leaves the domain $\Omega \times (0, T]$.

A continuous function h is defined outside the domain. For our purposes, we choose h to be such that $h(x, t) = g(x, t)$ for $x \in \partial\Omega$ and $t > 0$, and $h(x, 0) = u_0(x)$ for $x \in \Omega$. That is, h is a continuous extension of the boundary data. We denote by (x_τ, t_τ) the point where the token leaves the domain, that is, either $x_\tau \notin \Omega$ with $t_\tau > 0$, or $t_\tau \leq 0$. At this point the game ends and the final payoff is given by $h(x_\tau, t_\tau)$. That is, Player I pays Player 2 the amount given by $h(x_\tau, t_\tau)$.

For Player I, we denote by S_I a strategy, which is a collection of measurable mappings $S_I = \{S_k\}_{k=0}^{\infty}$, where each mapping has the form

$$\begin{array}{rccc} S_k: & \Omega^{k+1} \times \left(k\varepsilon^2/2, +\infty\right) & \longrightarrow & Gr(j, \mathbb{R}^N) \\ & (x_0, \ldots, x_k, t_0) & \longmapsto & S, \end{array}$$

where S is a subspace of dimension j (here $Gr(j,\mathbb{R}^N)$ denotes the $j-$Grasmanian, that is the collection of all subspaces of dimension j). For Player II, a strategy S_{II} is a collection of measurable mappings $S_{II}=\{S_k\}_{k=0}^{\infty}$, where each mapping has the form

$$\begin{array}{rlcl} S_k: & \Omega^{k+1}\times Gr(j,\mathbb{R}^N)\times\big(k\varepsilon^2/2,+\infty\big) & \longrightarrow & S \\ & (x_0,\dots,x_k,S,t_0) & \longmapsto & v, \end{array}$$

where v is a unitary vector in S.

Once both players have chosen their strategies, we can compute the expected value for the final payoff, which we denote by

$$\mathbb{E}_{S_I,S_{II}}^{x_0,t_0}[h(x_\tau,t_\tau)].$$

The value of the game for each player is the best expected value of the final payoff using one of their respective strategies. Since Player I wants to minimize the final payoff and Player II wants to maximize it, we can write the value of the game for each player as follows.

$$u_I^\varepsilon(x_0,t_0)=\inf_{S_I}\sup_{S_{II}}\mathbb{E}_{S_I,S_{II}}^{x_0,t_0}[h(x_\tau,t_\tau)],\qquad u_{II}^\varepsilon(x_0,t_0)=\sup_{S_{II}}\inf_{S_I}\mathbb{E}_{S_I,S_{II}}^{x_0,t_0}[h(x_\tau,t_\tau)].$$

Observe that the expectations above are well defined since the number of steps of the game is at most $\lceil 2t_0/\varepsilon^2\rceil$, and therefore, the game ends in a finite number of steps with probability 1. For this game it holds that $u_I^\varepsilon(x_0,t_0)=u_{II}^\varepsilon(x_0,t_0)$. Then, we define the value of the game as

$$u^\varepsilon(x_0,t_0)=u_I^\varepsilon(x_0,t_0)=u_{II}^\varepsilon(x_0,t_0).$$

In section 2, we prove that the game has a value $u^\varepsilon(x,t)$ that verifies an equation (called the Dynamic Programming Principle in the literature) and that $u^\varepsilon(x,t)$ converges uniformly in $\Omega\times[0,T]$ for every $T>0$ to a function $u(x,t)$, which is continuous and is the unique viscosity solution of the problem (1.0.1). This is the content of our first result, see Theorem 1.1 below. For the convergence of $u^\varepsilon(x,t)$ we need to assume a condition on the domain that we impose from now on and reads as follows: For every $y\in\partial\Omega$, we assume that there exists $r>0$ such that for every $\delta>0$ there exists $T\subset\mathbb{R}^N$ a subspace of dimension j, $w\in\mathbb{R}^N$ of norm 1, $\lambda>0$ and $\theta>0$ such that

$$\{x\in\Omega\cap B_r(y)\cap T_\lambda:\langle w,x-y\rangle<\theta\}\subset B_\delta(y) \tag{F_j}$$

where

$$T_\lambda=\{x\in\mathbb{R}^N: d(x-y,T)<\lambda\}.$$

As in [7], for our game with a given j, we will assume that Ω satisfies both (F_j) and (F_{N-j+1}). Notice that a uniformly convex domain verifies this condition for every $j\in\{1,..,N\}$, but more general domains also satisfy this hypothesis, see [7].

Theorem 1.1. *There is a value function for the game described before, u^ε. This value function can be characterized as being the unique solution to the Dinamic Programing Principle (DPP)*

$$\begin{cases} u^\varepsilon(x,t)=\displaystyle\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\left\{\frac12 u^\varepsilon(x+\varepsilon v,t-\frac{\varepsilon^2}{2})+\frac12 u^\varepsilon(x-\varepsilon v,t-\frac{\varepsilon^2}{2})\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if\ x\in\Omega,\ t>0, \\ u^\varepsilon(x,t)=h(x,t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad if\ x\notin\Omega,\ ou\ t\le 0. \end{cases}$$

Moreover, if Ω satisfies conditions (F_j) and (F_{N-j+1}), there exists a function $u \in C(\overline{\Omega} \times [0,+\infty)$ such that

$$u^\varepsilon \to u \qquad \textit{uniformly in} \quad \overline{\Omega} \times [0,T],$$

as $\varepsilon \to 0$ for every $T > 0$. This limit u is the unique viscosity solution to

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \textit{in } \Omega \times (0,+\infty), \\ u(x,t) = g(x,t), & \textit{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(0), & \textit{in } \Omega. \end{cases}$$

Once we have proven existence and uniqueness of solutions, we focus on their asymptotic behaviour as $t \to \infty$. We restrict our attention to the case where the boundary datum does not depend on t, that is,

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = g(x), & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.0.3}$$

where u_0 is a continuous function defined on $\overline{\Omega}$ and $g = u_0|_{\partial\Omega}$.

Using PDE techniques, a comparison argument with super and sub solutions constructed using an associated eigenvalue problem, we can show that $u(x,t)$ converges exponentially fast to the stationary solution. In the special case of $j = 1$ (or $j = N$) this result provides us with an approximation of the convex envelope (or the concave envelope) of a boundary datum by solutions to a parabolic problem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, and let u_0 be a continuous function defined on $\overline{\Omega}$ and $g = u_0|_{\partial\Omega}$. Then, there exist two positive constants, $\mu > 0$ (depending on Ω) and C (depending on the initial condition u_0) such that the unique viscosity solution u of* (1.0.3) *verifies*

$$\|u(\cdot,t) - z(\cdot)\|_\infty \le Ce^{-\mu t}, \tag{1.0.4}$$

where z is the unique viscosity solution of (1.0.2).

In addition, we also describe an interesting behavior of the solutions. Let us present our ideas in the simplest case and consider the special case $j = 1$ with $g \equiv 0$, that is, we deal with the problem

$$\begin{cases} u_t(x,t) - \lambda_1(D^2u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0, & \text{in } \Omega. \end{cases}$$

Notice that in this case $z \equiv 0$ and from Theorem 1.2 we have that $u(x,t) \to 0$ exponentially fast, $-Ce^{-\mu t} \le u(x,t) \le Ce^{-\mu t}$. In this scenario we can improve the upper bound. We show that there exists a finite time $T > 0$ depending only on Ω, such that the solution satisfies $u(x,t) \le 0$, for any $t > T$. This is a consequence of the fact that the eigenvalue problem

$$-\lambda_1(D^2\varphi(x)) = \mu\varphi(x), \quad \text{in } \Omega, \tag{1.0.5}$$

admits a positive solution for any $\mu > 0$ whenever Ω is bounded. In particular, this result says that, for $g \equiv 0$ and $j = 1$, there exists $T > 0$ such that the solution of (1.0.1) is below the convex envelope of g in Ω for any time beyond T. In fact, the same situation occurs when g is an affine function (we just apply the same argument to $\tilde{u} = u - g$).

When we consider $j = N$ and an affine function g, we have the analogous behavior, i.e. there exists $T > 0$ such that the solution of (1.0.1) is above the concave envelope of g in Ω for any time beyond T. However, the situation is different when one considers $1 < j < N$. In this case, equation (1.0.5) admits a positive and a negative solution for any $\mu > 0$, and therefore, it is possible to prove the existence of $T > 0$, depending only on Ω, such that $u(x,t) = z(x)$, for any $t > T$, where z is the solution of (1.0.2). We sum up these results in the following theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be an open bounded domain. Let g be the restriction of an affine function to $\partial\Omega$ and u_0 a continuous function in Ω. If $u(x,t)$ is the viscosity solution of* (1.0.3) *and $z(x)$ is the affine function (that turns out to be the viscosity solution of* (1.0.2)*), then there exists $T > 0$, depending only on Ω, such that*

i. If $j = 1$, then $u(x,t) \leq z(x)$, for any $t > T$.

ii. If $j = N$, then $u(x,t) \geq z(x)$, for any $t > T$.

iii. If $1 < j < N$, then $u(x,t) = z(x)$, for any $t > T$.

Notice that (iii) says that for $1 < j < N$ and an affine boundary datum we have convergence to the stationary solution *in finite time*.

Although this result implies that for some situations the exponential decay given in Theorem 1.2 is not sharp, we will also describe some other situations (with boundary data that are not affine functions) for which the solution $u(x,t)$ does not fall below or above the convex or concave envelope in finite time (see Proposition 3.5).

In the final part of the paper, we also look at the asymptotic behaviour of the values of the game described above and show that there exists $\mu > 0$, a constant depending only on Ω, and C independent of ε such that

$$\|u^\varepsilon(\cdot,t) - z^\varepsilon(\cdot)\|_\infty \leq Ce^{-\mu t},$$

being u^ε the value function for the game and z^ε a stationary solution to (DPP). Note that from here, we can provide a different proof (using games) of Theorem 1.2. We also provide a new proof of Theorem 1.3 using game theoretical arguments. With these techniques we can obtain a similar result in the case that g coincides with an affine function in a half-space.

Moreover, thanks to the game theoretical approach, we can show a more bizarre behaviour in a simple configuration of the data. Consider the equation $u_t = \lambda_{N-1}(D^2u)$, for $N \geq 2$. Let Ω be a ball centered at the origin, $\Omega = B_R \subset \mathbb{R}^N$, and call $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Assume that the boundary datum is given by two affine functions (for example, take $g(x', x_N) = |x_N|$, for $(x', x_N) \in \mathbb{R}^N \setminus \Omega$) and the initial condition is strictly positive inside Ω, $u_0 > 0$. For this choice of g, we have that the stationary solution satisfies $z(x', x_N) = 0$ in $\Omega \cap \{x_N = 0\}$. In this configuration of the data we have that for every point x_0 in the section $\{x_N = 0\} \cap \Omega$, we have

$$u(x_0, t) > 0 = z(x_0)$$

for every $t > 0$. However, for any point x_0 outside the section $\{x'' = 0\} \cap \Omega$, there exists a finite time t_0 (depending on x_0) such that

$$u(x_0, t) = z(x_0)$$

for every $t > t_0$.

That is, the solution to the evolution problem eventually coincides with the stationary solution outside the section $\{x_N = 0\} \cap \Omega$, but this fact does not happen in this section.

The paper is organized as follows: in section 2 we collect some preliminaries and we use the game theoretical approach to obtain existence of solutions; finally, in section 3 we deal with the asymptotic behaviour of the solutions.

2 Games

2.1 Preliminaries on viscosity solutions and a comparison principle

We begin by stating the usual definition of a viscosity solution to (1.0.1).

Definition 2.1. *A function $u : \Omega_T := \Omega \times (0, +\infty) \to \mathbb{R}$ verifies*

$$u_t - \lambda_j(D^2 u) = 0$$

in the viscosity sense *if the lower and upper semicontinuous envelopes of u given respectively by*

$$u_*(x,t) = \sup_{r>0} \inf\{u(y,s); \quad y \in B_r(x),\ |s-t| < r\},$$
$$u^*(x,t) = \inf_{r>0} \sup\{u(y,s); \quad y \in B_r(x),\ |s-t| < r\},$$

satisfy

i. for every $\phi \in C^2(\Omega_T)$ such that $u_* - \phi$ has a strict minimum at the point $(x,t) \in \Omega_T$ with $u_*(x,t) = \phi(x,t)$, we have

$$\phi_t(x,t) - \lambda_j(D^2\phi(x,t)) \geq 0.$$

ii. for every $\psi \in C^2(\Omega_T)$ such that $u^* - \psi$ has a strict maximum at the point $(x,t) \in \Omega_T$ with $u^*(x,t) = \psi(x,t)$, we have

$$\psi_t(x,t) - \lambda_j(D^2\psi(x,t)) \leq 0.$$

From our results we will obtain a solution that is continuous in the whole $\overline{\Omega_T}$ and hence we can avoid the use of u^* and u_* in what follows.

Comparison holds for our equation, see Theorem 8.2 from [10]. Let $\overline{u}$ be a supersolution, that is, it verifies

$$\begin{cases} \overline{u}_t(x,t) - \lambda_j(D^2_x\overline{u}(x,t)) \geq 0, & \text{in } \Omega \times (0,+\infty), \\ \overline{u}(x,t) \geq g(x,t), & \text{on } \partial\Omega \times (0,+\infty), \\ \overline{u}(x,0) \geq u_0(x), & \text{in } \Omega, \end{cases} \tag{2.1.1}$$

and $\underline{u}$ be a subsolution, that is,

$$\begin{cases} \underline{u}_t(x,t) - \lambda_j(D^2_x\underline{u}(x,t)) \leq 0, & \text{in } \Omega \times (0,+\infty), \\ \underline{u}(x,t) \leq g(x,t), & \text{on } \partial\Omega \times (0,+\infty), \\ \underline{u}(x,0) \leq u_0(x), & \text{in } \Omega. \end{cases} \tag{2.1.2}$$

Notice that the inequalities $\overline{u}_t(x,t) - \lambda_j(D^2_x\overline{u}(x,t)) \geq 0$ and $\underline{u}_t(x,t) - \lambda_j(D^2_x\underline{u}(x,t)) \leq 0$ are understood in the viscosity sense (see Definition 2.1), while the other inequalities (that involve boundary/initial data) are understood in a pointwise sense.

Lemma 2.2. *Let $\overline{u}$ and $\underline{u}$ verify* (2.1.1) *and* (2.1.2) *respectively, then*

$$\overline{u}(x,t) \geq \underline{u}(x,t)$$

for every $(x,t) \in \Omega \times (0,+\infty)$.

As an immediate consequence of this result uniqueness of continuous viscosity solutions to our problem (1.0.1) follows.

2.2 Existence using games

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $T > 0$. We define $\Omega_T = \Omega \times (0,T]$. Two values, $\varepsilon > 0$ and $j \in \{1,\dots,N\}$, are given. The game under consideration is a two-player zero-sum game that is played in the domain Ω_T. Initially, a token is placed at some point $(x_0,t_0) \in \Omega_T$. Player I chooses a subspace S of dimension j and then Player II chooses a unitary vector v in the subspace S. Then the position of the token is moved to $(x_0 \pm \varepsilon v, t_0 - \frac{\varepsilon^2}{2})$ with equal probabilities. After the first round, the game continues from (x_1,t_1) according to the same rules. This procedure yields a sequence of game states

$$(x_0,t_0),(x_1,t_1),\dots$$

where every x_k is a random variable. The game ends when the token leaves Ω_T. At this point, the token will be in the parabolic boundary strip of width ε given by

$$\Gamma_T^\varepsilon = \left(\Gamma^\varepsilon \times \left[-\frac{\varepsilon^2}{2}, T\right]\right) \cup \left(\Omega \times \left[-\frac{\varepsilon^2}{2}, 0\right]\right)$$

where

$$\Gamma^\varepsilon = \{x \in \mathbb{R}^N \setminus \Omega \, : \, \mathrm{dist}(x,\partial\Omega) \leq \varepsilon\}.$$

We denote by $(x_\tau,t_\tau) \in \Gamma_T^\varepsilon$ the first point in the sequence of game states that lies in Γ_T^ε, so that τ refers to the first time we hit Γ_T^ε. At this time the game ends with the final payoff given by $h(x_\tau,t_\tau)$, where $h : \Gamma_T^\varepsilon \to \mathbb{R}$ is a given continuous function that we call *payoff function*. Player I earns $-h(x_\tau,t_\tau)$ while Player II earns $h(x_\tau,t_\tau)$ (recall that this game is a zero-sum game). For our purposes we choose

$$h(x,t) = \begin{cases} g(x,t), & x \in \partial\Omega, t > 0, \\ u_0(x), & x \in \Omega, t = 0. \end{cases} \tag{2.2.1}$$

A strategy S_{I} for Player I, the player seeking to minimize the final payoff, is a function defined on the partial histories that at every step of the game gives a $j-$dimensional subspace S

$$S_{\mathrm{I}}(t_0,x_0,x_1,\dots,x_k) = S \in Gr(j,\mathbb{R}^N).$$

A strategy S_{II} for Player II, who seeks to maximize the final payoff, is a function defined on the partial histories that at every step of the game gives a unitary vector in a prescribed $j-$dimensional subspace S

$$S_{\mathrm{II}}(t_0,x_0,x_1,\dots,x_k,S) = v \in S.$$

When the two players fix their strategies S_I and S_{II} we can compute the expected outcome as follows: given the sequence $(x_0,t_0),(x_1,t_1),\dots,(x_k,t_k)$ in Ω_T, the next game position is distributed according to the probability

$$\pi_{S_{\mathrm{I}},S_{\mathrm{II}}}((x_0,t_0),(x_1,t_1),\dots,(x_k,t_k),A) = \frac{1}{2}\delta_{(x_k+\varepsilon v, t_k - \frac{\varepsilon^2}{2})}(A) + \frac{1}{2}\delta_{(x_k-\varepsilon v, t_k - \frac{\varepsilon^2}{2})}(A),$$

for all $A \subset \Omega_T \cup \Gamma_T^\varepsilon$, where $v = S_{\mathrm{II}}(t_0, x_0, x_1, \dots, x_k, S_{\mathrm{I}}(t_0, x_0, x_1, \dots, x_k))$. By using the one step transition probabilities and Kolmogorov's extension theorem, we can build a probability measure $\mathbb{P}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}$ on the game sequences for which the initial position is (x_0, t_0), that we call H^∞. The expected payoff, when starting from (x_0, t_0) and using the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$, is then computed according to this probability as

$$\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\left[h(x_\tau, t_\tau)\right] = \int_{H^\infty} h(x_\tau, t_\tau)\, d\mathbb{P}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}. \tag{2.2.2}$$

The *value of the game for Player I* is defined as

$$u^\varepsilon_{\mathrm{I}}(x_0, t_0) = \inf_{S_{\mathrm{I}}} \sup_{S_{\mathrm{II}}} \mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\left[h(x_\tau, t_\tau)\right]$$

while the *value of the game for Player II* as

$$u^\varepsilon_{\mathrm{II}}(x_0, t_0) = \sup_{S_{\mathrm{II}}} \inf_{S_{\mathrm{I}}} \mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\left[h(x_\tau, t_\tau)\right].$$

Intuitively, the values $u^\varepsilon_{\mathrm{I}}(x_0, t_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0, t_0)$ are the best expected outcome that each player can expect when the game starts at (x_0, t_0). If these two values coincide, $u^\varepsilon_{\mathrm{I}} = u^\varepsilon_{\mathrm{II}}$, we say that the game has a value.

Let us observe that the game ends after at most a finite number of steps, in fact, we have

$$\tau \le \left\lceil \frac{2T}{\varepsilon^2} \right\rceil.$$

Hence, the expected value computed in (2.2.2) is well defined.

To see that the game has a value, we can consider u^ε, a function that satisfies the Dynamic Programing Principle (DPP) associated with this game, that is given by

$$\begin{cases} u^\varepsilon(x,t) = \displaystyle\inf_{dim(S)=j} \sup_{v\in S, |v|=1} \left\{ \frac{1}{2} u^\varepsilon(x+\varepsilon v, t - \frac{\varepsilon^2}{2}) + \frac{1}{2} u^\varepsilon(x - \varepsilon v, t - \frac{\varepsilon^2}{2}) \right\}, & \\ & \text{if } (x,t) \in \Omega_T, \\ u^\varepsilon(x,t) = h(x,t), & \text{if } (x,t) \notin \Omega_T. \end{cases}$$

The existence of such a function can be seen defining the function backwards in time. In fact, given $h(x, t)$ we can compute $u^\varepsilon(x, t)$ for $0 < t < \frac{\varepsilon^2}{2}$ using the (DPP) and then continue with u^ε for $\frac{\varepsilon^2}{2} \le t < 2\frac{\varepsilon^2}{2}$, etc.

Now, we want to prove that a function that verifies the (DPP) u^ε, is in fact the value of the game, that is, it holds that $u^\varepsilon = u^\varepsilon_{\mathrm{I}} = u^\varepsilon_{\mathrm{II}}$. We know that $u^\varepsilon_{\mathrm{II}} \le u^\varepsilon_{\mathrm{I}}$, to obtain the equality, we will show that $u^\varepsilon \le u^\varepsilon_{\mathrm{II}}$ and $u^\varepsilon_{\mathrm{I}} \le u^\varepsilon$.

Given u^ε a function that verifies the (DPP) and $\eta > 0$, we can consider the strategy S^0_{II} for Player II that at every step almost maximize

$$u^\varepsilon(x_k + \varepsilon v, t_k - \frac{\varepsilon^2}{2}) + u^\varepsilon(x_k - \varepsilon v, t_k - \frac{\varepsilon^2}{2}).$$

That is,

$$S^0_{\mathrm{II}}(t_0, x_0, x_1, \dots, x_k, S) = w \in S$$

such that

$$\begin{aligned} &\frac{1}{2} u^\varepsilon(x_k + \varepsilon w, t_k - \frac{\varepsilon^2}{2}) + \frac{1}{2} u^\varepsilon(x_k - \varepsilon w, t_k - \frac{\varepsilon^2}{2}) \ge \\ &\qquad \sup_{v\in S, |v|=1} \left\{ \frac{1}{2} u^\varepsilon(x_k + \varepsilon v, t_k - \frac{\varepsilon^2}{2}) + \frac{1}{2} u^\varepsilon(x_k - \varepsilon v, t_k - \frac{\varepsilon^2}{2}) \right\} - \eta 2^{-(k+1)} \end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[u^\varepsilon(x_{k+1},t_{k+1})-\eta 2^{-(k+1)}|\,x_0,\dots,x_k]\\
&\qquad\geq \inf_{S,dim(S)=j}\sup_{v\in S,|v|=1}\left\{\frac{1}{2}u^\varepsilon(x_k+\varepsilon v,t_k-\frac{\varepsilon^2}{2})+\frac{1}{2}u^\varepsilon(x_k-\varepsilon v,t_k-\frac{\varepsilon^2}{2})\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\eta 2^{-(k+1)}-\eta 2^{-(k+1)}\\
&\qquad\geq u^\varepsilon(x_k,t_k)-\eta 2^{-k},
\end{aligned}$$

where we have estimated the strategy of Player I by inf and used that u^ε satisfies the (DPP). Thus

$$M_k = u^\varepsilon(x_k,t_k)-\eta 2^{-k}$$

is a submartingale. Now, we have

$$\begin{aligned}
u^\varepsilon_{\mathrm{II}}(x_0,t_0)&=\sup_{S_{\mathrm{II}}}\inf_{S_{\mathrm{I}}}\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[h(x_\tau,t_\tau)]\\
&\geq \inf_{S_{\mathrm{I}}}\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[h(x_\tau,t_\tau)]\\
&\geq \inf_{S_{\mathrm{I}}}\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[M_\tau]\\
&\geq \inf_{S_{\mathrm{I}}}\mathbb{E}^{x_0,t_0}_{S_{\mathrm{I}},S^0_{\mathrm{II}}}[M_0]=u^\varepsilon(x_0,t_0)-\eta,
\end{aligned}$$

where we used the optional stopping theorem for M_k. Since η is arbitrary small, this proves that $u^\varepsilon_{\mathrm{II}}\geq u^\varepsilon$. Analogously, we can consider a strategy S^0_I for Player I to prove that $u^\varepsilon\geq u^\varepsilon_{\mathrm{I}}$. This shows that the game has a value that can be characterized as the solution to the (DPP).

Our next aim is to pass to the limit in the values of the game

$$u^\varepsilon\to u$$

as $\varepsilon\to 0$ and obtain in this limit process a viscosity solution to (1.0.1).

We will use the following Arzela-Ascoli type lemma, to obtain a convergent subsequence $u^\varepsilon\to u$. For its proof we refer to Lemma 4.2 from [21].

Lemma 2.3. *Let $\{u^\varepsilon:\overline{\Omega}\times[0,T]\to\mathbb{R},\ \varepsilon>0\}$ be a set of functions such that*

i. there exists $C>0$ such that $|u^\varepsilon(x,t)|<C$ for every $\varepsilon>0$ and every $(x,t)\in\overline{\Omega}\times[0,T]$,

ii. given $\eta>0$ there are constants r_0 and ε_0 such that for every $\varepsilon<\varepsilon_0$ and any $x,y\in\overline{\Omega}$ with $|x-y|<r_0$ and for every $t,s\in[0,T]$ with $|t-s|<r_0$ it holds

$$|u^\varepsilon(x,t)-u^\varepsilon(y,s)|<\eta.$$

Then, there exists a uniformly continuous function $u:\overline{\Omega}\times[0,T]\to\mathbb{R}$ and a subsequence still denoted by $\{u^\varepsilon\}$ such that

$$u^\varepsilon\to u\qquad\text{uniformly in}\quad\overline{\Omega}\times[0,T],$$

as $\varepsilon\to 0$.

So, our goal now is to show that the family u^ε satisfies the hypotheses of the previous lemma. First, let us observe that

$$\min h \le u^\varepsilon(x,t) \le \max h$$

for every $(x,t) \in \overline{\Omega} \times [0,T]$. To prove that u^ε satisfies the second condition in Lemma 2.3 we will have to make some geometric assumptions on the domain. As in [7], given $y \in \partial\Omega$ we assume that there exists $r > 0$ such that for every $\delta > 0$ there exists $T \subset \mathbb{R}^N$ a subspace of dimension j, $w \in \mathbb{R}^N$ of norm 1, $\lambda > 0$ and $\theta > 0$ such that

$$\{x \in \Omega \cap B_r(y) \cap T_\lambda : \langle w, x-y\rangle < \theta\} \subset B_\delta(y) \tag{F_j}$$

where

$$T_\lambda = \{x \in \mathbb{R}^N : d(x-y,T) < \lambda\}.$$

For our game with a fixed j we will assume that Ω satisfies both (F_j) and (F_{N-j+1}). As we mentioned in the introduction, observe that every strictly convex domain verifies (F_j) for any $1 \le j \le N$.

The key point to obtain the asymptotic equicontinuity required in the second condition in Lemma 2.3 is to obtain the bound for $(x,t) \in \Omega_T$ and $(y,s) \in \Gamma_T^\varepsilon$. For the case $(x,t),(y,s) \in \Gamma_T^\varepsilon$ the bound follows from the uniform continuity of h in Γ_T^ε. For the case $(x,t),(y,s) \in \Omega_T$ we argue as follows. We fix the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$ for the game starting at (x,t). We define a virtual game starting at (y,s) using the same random steps as the game starting at (x,t). Furthermore, the players adopt their strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$ from the game starting at (x,t), that is, when the game position is (y_k,s_k) a player make the choices that would have taken at (x_k,t_k) while playing the game starting at (x,t). We proceed in this way until for the first time one of the positions leave the parabolic domain, that is, until $(x_k,t_k) \in \Gamma_T^\varepsilon$ or $(y_k,s_k) \in \Gamma_T^\varepsilon$. At that point we have $|(x_k,t_k)-(y_k,s_k)| = |(x,t)-(y,s)|$, and the desired estimate follows from the one for $x_k, y_k \in \Gamma_\varepsilon$ (in the case that both positions leave the domain at the same turn, k) or $x_k \in \Omega$, $y_k \in \Gamma_\varepsilon$ (if only one have leaved the domain).

Thus, we can concentrate on the case $(x,t) \in \Omega_T$ and $(y,s) \in \Gamma_T^\varepsilon$. We can assume that $(y,s) \in \partial_P\Omega_T$. If we have the bound for those points we can obtain a bound for a point $(y,s) \in \Gamma_T^\varepsilon$ just by considering $(z,u) \in \partial_P\Omega_T$ close to (x,t) and (y,s). If $s<0$, we can consider the point $(x,0)$ and for $y \notin \Omega$ we can consider (z,t) with $z \in \partial\Omega$ a point in the line segment that joins x and y.

Hence, we have to handle two cases. In the first one we have to prove that $|u^\varepsilon(x,t) - u^\varepsilon(x,0)| < \eta$ for $x \in \Omega$ and $0 < t < r_0$. In the second one we have to prove that $|u^\varepsilon(x,t) - u^\varepsilon(y,t)| < \eta$ for $x \in \Omega$, $y \in \partial\Omega$ such that $|x-y| < r_0$ and $0 < t \le T$.

In the first case we have

$$u^\varepsilon(x,0) = u_0(x),$$

we have to show that the game starting at (x,t) will not end too far a way from $(x,0)$. We have $-\frac{\varepsilon^2}{2} < t_\tau < t$, so we have to obtain a bound for $|x - x_\tau|$. To this end we consider $M_k = |x_k - x|^2 - \varepsilon^2 k$. We have

$$\begin{aligned}
&\mathbb{E}^{x,t}_{S_I,S_{II}}[|x_{k+1}-x|^2 - \varepsilon^2(k+1)|x,x_1,\dots,x_k] \\
&\quad = \frac{|x_k+\varepsilon v_k - x|^2 + |x_k - \varepsilon v_k - x|^2}{2} - \varepsilon^2(k+1) \\
&\quad = |x_k - x|^2 + \varepsilon^2|v_k|^2 - \varepsilon^2(k+1) \\
&\quad = M_k.
\end{aligned} \tag{2.2.3}$$

Hence, M_k is a martingale. By applying the optional stopping theorem, we obtain

$$\mathbb{E}^{x,t}_{S_I,S_{II}}[|x_\tau - x|^2] = \varepsilon^2 \mathbb{E}^{x,t}_{S_I,S_{II}}[\tau] \le \varepsilon^2 \left\lceil \frac{2t}{\varepsilon^2} \right\rceil \le \varepsilon^2 \left\lceil \frac{2r_0}{\varepsilon^2} \right\rceil \le \varepsilon^2 + 2r_0 \le \varepsilon_0^2 + 2r_0 \quad (2.2.4)$$

Hence, using

$$\mathbb{E}^{x,t}_{S_I,S_{II}}[|x_\tau - x|^2] \ge \mathbb{P}(|(x_\tau, t_\tau) - (x,0)| \ge \delta)\delta^2,$$

we obtain

$$\mathbb{P}(|(x_\tau, t_\tau) - (x,0)| \ge \delta) \le \frac{\varepsilon_0^2 + 2r_0}{\delta^2}.$$

With this bound, we can obtain the desired result as follows:

$$\begin{aligned} |u_\varepsilon(x,t) - h(x,0)| &\le \mathbb{P}(|(x_\tau, t_\tau) - (x,0)| < \delta) \times \sup_{(x_\tau,t_\tau)\in B_\delta(x,0)} |h(x_\tau,t_\tau) - h(x,0)| \\ &\quad + \mathbb{P}(|(x_\tau, t_\tau) - (x,0)| \ge \delta)) 2\max|h| \\ &\le \sup_{(x_\tau,t_\tau)\in B_\delta(x,0)} |h(x_\tau,t_\tau) - h(x,0)| + \frac{(\varepsilon_0^2 + 2r_0)2\max|h|}{\delta^2} < \eta \end{aligned} \quad (2.2.5)$$

if δ, ε_0 and r_0 are small enough.

Now we move to the second case, we have $u^\varepsilon(y,s) = g(y,s)$, Here, we need to make the geometric assumptions (F_j) and (F_{N-j+1}) on $\partial\Omega$. In this parabolic game we have an extra difficulty compared with the elliptic case treated in [7], we have to make an extra effort to bound the amount of time that it takes for the game to end.

We start with the case $j = 1$, in this case we assume (F_N). This condition reads as follows: For every $y \in \partial\Omega$ we assume that there exists $r > 0$ such that for every $\delta > 0$ there exists $w \in \mathbb{R}^N$ of norm 1 and $\theta > 0$ such that

$$\{x \in \Omega \cap B_r(y) : \langle w, x - y\rangle < \theta\} \subset B_\delta(y). \quad (2.2.6)$$

Let us observe that for any possible choice of the direction v at every step we have that the projection of the position of the game, x_n, in the direction of a fixed unitary vector w, that is,

$$\langle x_n - y, w\rangle,$$

is a martingale. We fix $r > 0$ and consider $\tilde\tau$, the first time x leaves Ω or $B_r(y)$. Hence

$$\mathbb{E}\langle x_{\tilde\tau} - y, w\rangle \le \langle x - y, w\rangle \le d(x,y) < r_0. \quad (2.2.7)$$

We consider the vector w given by the geometric assumption on Ω, we have that

$$\langle x_n - y, w\rangle \ge -\varepsilon.$$

Therefore, (2.2.7) implies that

$$\mathbb{P}\left(\langle x_{\tilde\tau} - y, w\rangle > r_0^{1/2}\right) r_0^{1/2} - \left(1 - \mathbb{P}\left(\langle x_{\tilde\tau} - y, w\rangle > r_0^{1/2}\right)\right)\varepsilon < r_0.$$

Hence, we have (for every $\varepsilon > \varepsilon_0$ small enough)

$$\mathbb{P}\left(\langle x_{\tilde\tau} - y, w\rangle > r_0^{1/2}\right) < 2r_0^{1/2}.$$

Then, (2.2.6) implies that given $\delta > 0$ we can conclude that

$$\mathbb{P}(d(x_{\tilde{\tau}}, y) > \delta) < 2r_0^{1/2}.$$

by taking r_0 small enough and an appropriate w.

Hence, $d(x_{\tilde{\tau}}, y) \leq \delta$ with probability close to one, and in this case the point $x_{\tilde{\tau}}$ is actually the point where the process has leaved Ω, that is $\tilde{\tau} = \tau$. When $d(x_\tau, y) \leq \delta$, by the same martingale argument used in (2.2.4), we obtain

$$\mathbb{E}[t - t_\tau] = \mathbb{E}\left[\frac{\varepsilon^2}{2}\tau\right] = \frac{\mathbb{E}[|x_\tau - x|^2]}{2} \leq \frac{\delta^2}{2}.$$

Hence,

$$\mathbb{P}(t - t_\tau > \delta) \leq \frac{\delta}{2}$$

and the bound follows as in (2.2.5).

In the general case, for any value of j, we can proceed in the same way. In order to be able to use condition (F_j), we have to argue that the points x_n involved in our argument belong to T_λ. For $r_0 < \lambda$ we have that $x \in T_\lambda$, so if we ensure that at every move $v \in T$ we will have that the game sequence will be contained in $x + T \subset T_\lambda$.

Recall that here we are assuming that both (F_j) and (F_{N-j+1}) are satisfied. We can separate the argument into two parts. We will prove on the one hand that $u_\varepsilon(x,t) - g(y,s) < \eta$ and on the other that $g(y,s) - u_\varepsilon(x,t) < \eta$. For the first inequality we can make choices for the strategy for Player I, and for the second one we can do the same for strategies of Player II.

Since Ω satisfies condition (F_j), Player I can make sure that at every move the vector v belongs to T by selecting $S = T$. This proves the upper bound $u_\varepsilon(x,t) - g(y,s) < \eta$. On the other hand, since Ω satisfies (F_{N-j+1}), Player II will be able to select v in a space S of dimension j and hence he can always choose $v \in S \cap T$ since

$$\dim(T) + \dim(S) = N - j + 1 + j = N + 1 > N.$$

This shows the lower bound $g(y,s) - u_\varepsilon(x,t) < \eta$.

We have shown that the hypotheses of the Arzela-Ascoli type lemma, Lemma 2.3, are satisfied. Hence, we have obtained uniform convergence of a subsequence of u^ε.

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying conditions (F_j) and (F_{N-j+1}). Then there exists a subsequence of u^ε that converges uniformly. That is,*

$$u^{\varepsilon_j} \to u, \qquad \text{as } \varepsilon_j \to 0,$$

uniformly in $\overline{\Omega} \times [0,T]$, where u is a uniformly continuous function.

Now, let us prove that any possible uniform limit of u^ε is a viscosity solution to the limit PDE problem. This result shows existence of a continuous up to the boundary solution defined in $\overline{\Omega} \times [0,T]$ for every $T > 0$. Uniqueness of this viscosity solution follows from the comparison principle stated in Lemma 2.2.

Theorem 2.5. *Let u be a uniform limit of the values of the game u^ε. Then u is a viscosity solution to* (1.0.1).

Proof. First, we observe that since $u^\varepsilon = g$ on $\partial\Omega\times(0,T)$ and $u^\varepsilon(x,0)=u_0(x)$ for $x\in\Omega$, we obtain, from the uniform convergence, that $u=g$ on $\partial\Omega\times(0,T)$ and $u(x,0)=u_0(x)$ for $x\in\Omega$. Also, notice that Lemma 2.3 gives that the limit function is continuous.

To check that u is a viscosity supersolution to $\lambda_j(D^2u)=0$ in Ω, let $\phi\in C^2(\Omega_T)$ be such that $u-\phi$ has a strict minimum at the point $(x,t)\in\Omega_T$ with $u(x,t)=\phi(x,t)$. We need to check that

$$\phi_t(x,t)-\lambda_j(D^2\phi(x,t))\geq 0.$$

As $u^\varepsilon\to u$ uniformly in $\overline{\Omega}\times[0,T]$ we have the existence of two sequences x_ε, t_ε such that $x_\varepsilon\to x$, $t_\varepsilon\to t$ as $\varepsilon\to 0$ and

$$u^\varepsilon(z,s)-\phi(z,s)\geq u^\varepsilon(x_\varepsilon,t_\varepsilon)-\phi(x_\varepsilon,t_\varepsilon)-\varepsilon^3$$

(remark that u^ε is not continuous in general). Since u^ε is a solution to

$$u^\varepsilon(x,t)=\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\left\{\frac{1}{2}u^\varepsilon\Big(x+\varepsilon v,t-\frac{\varepsilon^2}{2}\Big)+\frac{1}{2}u^\varepsilon\Big(x-\varepsilon v,t-\frac{\varepsilon^2}{2}\Big)\right\}$$

we obtain that ϕ verifies

$$\begin{aligned}\phi(x_\varepsilon,t_\varepsilon)-\phi\Big(x_\varepsilon,t_\varepsilon-\tfrac{\varepsilon^2}{2}\Big)\geq \inf_{dim(S)=j}\sup_{v\in S,|v|=1}&\left\{\frac{1}{2}\phi\Big(x_\varepsilon+\varepsilon v,t_\varepsilon-\frac{\varepsilon^2}{2}\Big)+\right.\\ +\frac{1}{2}\phi\Big(x_\varepsilon-\varepsilon v,t_\varepsilon-\tfrac{\varepsilon^2}{2}\Big)-\phi\Big(x_\varepsilon,t_\varepsilon-\tfrac{\varepsilon^2}{2}\Big)&\left.\vphantom{\frac12}\right\}-\varepsilon^3.\end{aligned}\tag{2.2.8}$$

Now, consider the second order Taylor expansion of ϕ (to simplify the notation we omit the dependence of t here)

$$\phi(y)=\phi(x)+\nabla\phi(x)\cdot(y-x)+\frac{1}{2}\langle D^2\phi(x)(y-x),(y-x)\rangle+o(|y-x|^2)$$

as $|y-x|\to 0$. Hence, we have

$$\phi(x+\varepsilon v)=\phi(x)+\varepsilon\nabla\phi(x)\cdot v+\varepsilon^2\frac{1}{2}\langle D^2\phi(x)v,v\rangle+o(\varepsilon^2)$$

and

$$\phi(x-\varepsilon v)=\phi(x)-\varepsilon\nabla\phi(x)\cdot v+\varepsilon^2\frac{1}{2}\langle D^2\phi(x)v,v\rangle+o(\varepsilon^2).$$

Using these expansions we get

$$\frac{1}{2}\phi(x_\varepsilon+\varepsilon v)+\frac{1}{2}\phi(x_\varepsilon-\varepsilon v)-\phi(x_\varepsilon)=\frac{\varepsilon^2}{2}\langle D^2\phi(x_\varepsilon)v,v\rangle+o(\varepsilon^2).$$

Plugging this into (2.2.8) and dividing by $\varepsilon^2/2$, we obtain

$$\frac{\phi(x_\varepsilon,t_\varepsilon)-\phi\left(x_\varepsilon,t_\varepsilon-\dfrac{\varepsilon^2}{2}\right)}{\varepsilon^2/2}\geq\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\left\{\langle D^2\phi(x_\varepsilon,t_\varepsilon-\varepsilon^2/2)v,v\rangle+2\frac{o(\varepsilon^2)}{\varepsilon^2}\right\}-2\varepsilon.$$

Therefore, passing to the limit as $\varepsilon\to 0$ in (2.2.8) we conclude that

$$\phi_t(x,t)\geq\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\Big\{\langle D^2\phi(x,t)v,v\rangle\Big\}.$$

which is equivalent to

$$\phi_t(x,t) \geq \lambda_j(D^2\phi(x))$$

as we wanted to prove.

When we consider a smooth function ψ that touches u from above, we can obtain the reverse inequality in a similar way. □

Remark 2.6. From the uniqueness of viscosity solutions to the limit problem (recall that a comparison principle holds) we obtain that the convergence of the whole family u^ε. That is,

$$u^\varepsilon \to u$$

uniformly as $\varepsilon \to 0$ (not only along subsequences). Hence, we have completed the proof of Theorem 1.1.

3 Asymptotic behaviour

Along this section we restrict our attention to the case where the boundary condition does not depend on the time, that is,

$$\begin{cases} u_t(x,t) - \lambda_j(D_x^2 u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = g(x), & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(0), & \text{in } \Omega. \end{cases}$$

where u_0 is a continuous function defined on $\overline{\Omega}$ and $g = u_0|_{\partial\Omega}$.

We want to study the asymptotic behaviour as $t \to \infty$ of the solution to this parabolic equation. We deal with the problem with two different techniques, on the one hand we use pure PDE methods (comparison arguments) and on the other hand we use our game theoretical approach.

3.1 PDE arguments

We will use the eigenvalue problem associated with $-\lambda_N(D^2u)$. For every strictly convex domain there is a positive eigenvalue μ_1, with an eigenfunction ψ_1 that is positive inside Ω and continuous up to the boundary with $\psi_1|_{\partial\Omega} = 0$ such that

$$\begin{cases} -\lambda_N(D^2\psi_1) = \mu_1\psi_1, & \text{in } \Omega, \\ \psi_1 = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1.1}$$

This eigenvalue problem was studied in [4, 5]. Notice that $\varphi_1 = -\psi_1$ is a negative solution to

$$\begin{cases} -\lambda_1(D^2\varphi_1) = \mu_1\varphi_1, & \text{in } \Omega, \\ \varphi_1 = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1.2}$$

We will use the following lemma.

Lemma 3.1. *For any two symmetric matrices A, B, it holds that*

$$\lambda_1(A) + \lambda_j(B) \leq \lambda_j(A+B) \leq \lambda_N(A) + \lambda_j(B). \tag{3.1.3}$$

Proof. Given a subspace S of dimension j, we have

$$\sup_{v\in S,|v|=1}\langle Bv,v\rangle+\inf_{|v|=1}\langle Av,v\rangle\le\sup_{v\in S,|v|=1}\langle (A+B)v,v\rangle\le\sup_{v\in S,|v|=1}\langle Bv,v\rangle+\sup_{|v|=1}\langle Av,v\rangle.$$

Hence, the first inequality follows from

$$\begin{aligned}\lambda_j(A+B)&=\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\langle (A+B)v,v\rangle\\&\le\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\langle Bv,v\rangle+\sup_{|v|=1}\langle Av,v\rangle\\&=\lambda_N(A)+\lambda_j(B)\end{aligned}$$

and the second one from

$$\begin{aligned}\lambda_j(A+B)&=\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\langle (A+B)v,v\rangle\\&\ge\inf_{dim(S)=j}\sup_{v\in S,|v|=1}\langle Bv,v\rangle+\inf_{|v|=1}\langle Av,v\rangle\\&=\lambda_1(A)+\lambda_j(B).\end{aligned}$$

This ends the proof. □

Theorem 3.2. *Let u_0 be continuous with $u_0|_{\partial\Omega}=g$. Let ψ_R and φ_R be the eigenfunctions associated with μ_R the first eigenvalue for* (3.1.1) *and* (3.1.2) *in a large strictly convex domain Ω_R such that $\Omega\subset\subset\Omega_R$. Then, there exist two positive constants C_1,C_2, depending on the initial condition u_0, such that*

$$z(x)+C_1e^{-\mu_R t}\varphi_R(x)\le u(x,t)\le z(x)+C_2e^{-\mu_R t}\psi_R(x),\tag{3.1.4}$$

where z is the unique stationary solution.

Proof. We just observe that $\underline{u}(x,t)=z(x)+C_1e^{-\mu_R t}\varphi_R(x)$ with C_1 large enough is a subsolution to our evolution problem in Ω. In fact, we have

$$\underline{u}_t(x,t)=-\mu_R C_1e^{-\mu t}\varphi_R(x)$$

and

$$\begin{aligned}\lambda_j(D^2\underline{u}(x,t))&=\lambda_j(D^2z(x)+C_1e^{-\mu_R t}D^2\varphi_R(x))\\&\ge\lambda_j(D^2z(x))+C_1e^{-\mu_R t}\lambda_1(D^2\varphi_R(x))=-\mu C_1e^{-\mu_R t}\varphi_R(x).\end{aligned}$$

An analogous computation shows that $\overline{u}(x,t)=z(x)+C_2e^{-\mu_R t}\psi_R(x)$ is a supersolution.

In addition, we have that

$$\underline{u}(x,t)\le g(x)\le\overline{u}(x,t),\qquad x\in\partial\Omega,t>0,$$

and for C_1, C_2 large enough (depending on u_0)

$$\underline{u}(x,0)=z(x)+C_1\varphi_R(x)\le u_0(x)\le\overline{u}(x,0)=z(x)+C_2\psi_R(x),\qquad x\in\Omega.$$

Notice that here we are using that φ_R and ψ_R are strictly negative and strictly positive respectively inside Ω_R.

Finally we apply the comparison principle in Ω to obtain the desired conclusion

$$z(x)+C_1e^{-\mu_R t}\varphi_R(x)\le u(x,t)\le z(x)+C_2e^{-\mu_R t}\psi_R(x).$$

□

As an immediate consequence of this result, we obtain that solutions to our evolution problem converge uniformly to the unique stationary solution. This proves Theorem 1.2.

Notice that in the previous result μ is the first eigenvalue for $-\lambda_N(D^2u)$ in the *larger* domain Ω_R. Now, our aim is to obtain a sharper bound (involving μ_1 the first eigenvalue in Ω). To this aim we have to assume that u_0 is $C^1(\overline{\Omega})$ with $u_0|_{\partial\Omega} = g$ and that the solution z of (1.0.2) is $C^1(\overline{\Omega})$. This regularity of the solution of (1.0.2) up to the boundary is not included in [23] (there only interior regularity for the convex envelope is shown). Under these hypotheses on u_0 and z the difference $u_0 - z$ is $C^1(\overline{\Omega})$ and vanishes on $\partial\Omega$. Notice that we do not know if there is a regularizing effect for our evolution problem. That is, we do not know if for a smooth boundary datum and a continuous initial condition the solution is smooth in $\overline{\Omega}$ for any positive time t (as happens with solutions to the heat equation).

As a previous step in our arguments, we need to show that the eigenfunctions have a "negative normal derivative". Notice that the existence of such eigenfunction is proved in [4] for strictly convex domains. Although this hypothesis is sufficient but not necessary (see [5] for construction of eigenfunctions in rectangles), we shall assume it here since the optimal hypotheses for existence of eigenfunctions are unknown (as far as we know). In the next two results we need to assume that the domain Ω has some extra regularity (it has an interior tangent ball at every boundary point).

Lemma 3.3. *Assume that Ω is strictly convex and has a uniform interior tangent ball at every point of its boundary. Let φ_1 and ψ_1 be the eigenfunctions associated with μ_1 the first eigenvalue for* (3.1.1) *and* (3.1.2) *in Ω. Assume that they are normalized with $\|\psi\|_\infty = \|\varphi\|_\infty = 1$. Then, there exists $C > 0$ such that*

$$\psi_1(x) \geq C \operatorname{dist}(x, \partial\Omega) \qquad \text{and} \qquad \varphi_1(x) \leq -C \operatorname{dist}(x, \partial\Omega),$$

for $x \in \Omega$.

Proof. Take $x_0 \in \partial\Omega$. Let $B_r(y)$ be a ball inside Ω, tangent to $\partial\Omega$ at x_0. In $B_{r/2}(y)$ the eigenfunction ψ_1 is strictly positive and then we obtain that there exists a constant c such that

$$\mu_1\psi_1(x) \geq c, \qquad x \in B_{r/2}(y).$$

Now, we take $a(x)$ the solution to

$$\begin{cases} -\lambda_N(D^2a(x)) = c\chi_{B_{r/2}(y)}(x), & \text{in } B_r(y), \\ a(x) = 0, & \text{on } \partial B_r(y). \end{cases} \tag{3.1.5}$$

This function a is radial $a(x) = a(|x-y|)$ and can be explicitly computed. In fact,

$$a(x) = \begin{cases} c_1(r - |x-y|), & \text{in } B_r(y) \setminus B_{r/2}(y), \\ c_2 - \dfrac{c}{2}|x-y|^2, & \text{in } B_{r/2}(y) \end{cases}$$

with c_1, c_2 such that $c_1 = cr/2$ (continuity of the derivative at $r/2$) and $c_1 r/2 = c_2 - c/2(r/2)^2$ (continuity of the function at $r/2$).

To conclude we use the comparison argument for (3.1.5) to obtain that

$$a(x) \leq \psi_1(x) \qquad x \in B_r(y).$$

This implies that

$$\psi_1(x) \geq C \operatorname{dist}(x, \partial\Omega).$$

A similar argument shows that

$$\varphi_1(x) \le -C\,\mathrm{dist}(x, \partial\Omega).$$

□

Theorem 3.4. *Assume that Ω is strictly convex and has a uniform interior tangent ball at every point of its boundary. Let g be such that the solution z of (1.0.2) is $C^1(\overline{\Omega})$ and let u_0 be $C^1(\overline{\Omega})$ with $u_0|_{\partial\Omega} = g$ and let μ_1 the first eigenvalue for (3.1.1) and (3.1.2) in Ω. Then, there exist two positive constants (depending on the initial condition u_0) such that*

$$z(x) + C_1 e^{-\mu_1 t}\varphi_1(x) \le u(x,t) \le z(x) + C_2 e^{-\mu_1 t}\psi_1(x). \tag{3.1.6}$$

Proof. We just observe that the arguments used in the proof of Theorem 3.2 also work here since we can find two constants C_1 and C_2 such that

$$z(x) + C_1\varphi_1(x) \le u_0(x) \le z(x) + C_2\psi_1(x), \qquad x \in \Omega. \tag{3.1.7}$$

Here, we are using that $u_0 - z$ is $C^1(\overline{\Omega})$ with $(u_0 - z)|_{\partial\Omega} = 0$ to get that there is a constant C such that

$$-C\,\mathrm{dist}(x, \partial\Omega) \le (u_0 - z)(x) \le C\,\mathrm{dist}(x, \partial\Omega),$$

and observe that from our previous Lemma 3.3 we obtain (3.1.7). □

We next give the proof of Theorem 1.3, which is a refined description of the asymptotic behavior of the solution to (1.0.1) when the boundary datum g comes from the restriction of an affine function to $\partial\Omega$. For instance, if we consider the case $j = 1$, it shows that there exists a finite time $T > 0$ beyond which the upper estimate in (3.1.4) can be reduced to $z(x)$, the λ_j−envelope of g inside Ω.

Proof of Theorem 1.3. We assume that there is an affine function (a plane if we are in the case $N = 2$) π such that $g = \pi|_{\partial\Omega}$. In this case the λ_j−envelope z of g inside Ω is given by

$$z(x) = \pi(x).$$

Hence, let us consider

$$\hat{u}(x,t) = u(x,t) - z(x) = u(x,t) - \pi(x).$$

This function $\hat{u}$ is the viscosity solution to

$$\begin{cases} \hat{u}_t - \lambda_j(D^2\hat{u}) = 0, & \text{in } \Omega \times (0, +\infty), \\ \hat{u} = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ \hat{u}(x,0) = u_0(x) - z(x), & \text{in } \Omega. \end{cases} \tag{3.1.8}$$

For $1 \le j \le N-1$, we consider a large ball B_R with $\Omega \subset B_R$. Inside this ball we take

$$w(x,t) = e^{R^2\mu} e^{-\mu t} e^{-\mu\frac{r^2}{2}}.$$

Here and in what follows $r = |x|$. For large μ, this function w verifies

$$\begin{cases} w_t - \lambda_j(D^2 w) = -\mu e^{R^2\mu} e^{-\mu t} e^{-\mu\frac{r^2}{2}} + \mu e^{R^2\mu} e^{-\mu t} e^{-\mu\frac{r^2}{2}} = 0, & \text{in } \Omega \times (0, +\infty), \\ w > 0, & \text{on } \partial\Omega \times (0, +\infty), \\ w(x,0) = e^{R^2\mu} e^{-\mu\frac{r^2}{2}} \ge u_0(x) - z(x), & \text{in } \Omega. \end{cases} \tag{3.1.9}$$

Hence, w is a supersolution to (3.1.8) and then, by the comparison principle, we get

$$\hat{u}(x,t) \le w(x,t) = e^{R^2\mu} e^{-\mu t} e^{-\mu \frac{r^2}{2}},$$

for every μ large enough. Then, for every $t > T = R^2/2$ we get

$$\hat{u}(x,t) \le \lim_{\mu\to\infty} e^{R^2\mu} e^{-\mu t} e^{-\mu \frac{r^2}{2}} = 0.$$

Hence, we have shown that when the boundary condition is the restriction of an affine function to the boundary, then there exists a finite time T such that the solution to the evolution problem lies below the stationary solution z, regardless the initial condition u_0, that is, it holds that

$$u(x,t) \le z(x)$$

for every $x \in \Omega$ and every $t < T$ for $1 \le j \le N-1$.

For $2 \le j \le N$ the same argument proves that there exists a finite time T such that the solution to the evolution problem lies above the stationary solution. Hence for $2 \le j \le N-1$ there exists a finite time T such that the solution to the evolution problem coincides with the stationary solution. This proves Theorem 1.3. $\square$

Observe that for $j = 1$, $u(x,t) = e^{-\mu_1 t}\varphi_1(x)$ is a solution to the problem that do not become zero in finite time. The same holds for $u(x,t) = e^{-\mu_1 t}\psi_1(x)$ for $j = N$. Our next result shows that, in general, we can not expect that all solutions lie below z in the whole $\overline{\Omega}$ in finite time.

Theorem 3.5. *Let Ω be an open bounded domain in $\mathbb{R}^N$, and let $1 \le j \le N$. For any $x_0 \in \Omega$, there exist g and u_0 continuous in $\partial\Omega$ and $\overline{\Omega}$ respectively, with $u_0|_{\partial\Omega} = g$, such that the solution of problem* (1.0.1) *satisfies*

$$u(x_0,t) \ge z(x_0) + ke^{-\mu_1 t}, \qquad \textit{for all } t > 0,$$

where $\mu_1, k > 0$ are two constants and z is the solution of (1.0.2).

We can obtain the analogous result for the inequality

$$u(x_0,t) \le z(x_0) - ke^{-\mu_1 t}.$$

Proof. Consider, without loss of generality, that $x_0 \in \Omega$ is the origin. Take $r > 0$ small enough such that the ball B_r of radius r and center at the origin satisfies $B_r \subset\subset \Omega$.

In the rest of the proof we will denote $\mathbb{R}^N = \mathbb{R}^j \times \mathbb{R}^{N-j}$, and we will write any point in $\mathbb{R}^N$ as $x = (x', x'') \in \mathbb{R}^j \times \mathbb{R}^{N-j}$.

Consider $B_r^j = B_r \cap \{x'' = 0\}$. We observe that B_r^j is a $j-$dimensional ball. Therefore, as it is proven in [4], there exists a positive eigenvalue μ_1, with an eigenfunction ψ_1 which is continuous up to the boundary, such that

$$\begin{cases} -\lambda_j(D^2\psi_1) = \mu_1\psi_1 & \text{in } B_r^j, \\ \psi_1 = 0 & \text{on } \partial B_r^j, \\ \psi_1 > 0 & \text{in } B_r^j. \end{cases}$$

Consider g a nonnegative continuous function defined on $\partial\Omega$ such that

$$g(x', x'') \ge \psi_1(x'), \quad \text{for all } (x', x'') \in \partial\Omega, \text{ with } x' \in B_r^j, \tag{3.1.10}$$

and

$$g(x',0) = 0 \quad \text{for all } (x',0) \in \partial\Omega \cap \{x''=0\}. \tag{3.1.11}$$

We note that this choice of g is always possible since, if $x' \in B_r^j$, then $(x',0) \in B_r$, and since have considered $B_r \subset\subset \Omega$, we deduce $(x',0) \notin \partial\Omega$.

For this choice of g, we claim that the solution of problem (1.0.2) satisfies

$$z(x',x'') = 0, \qquad \text{in} \quad \Omega \cap \{x''=0\}.$$

In order to prove this claim, we use the geometric interpretation of solutions to problem (1.0.2) given in [7]. Consider the $j-$dimensional subspace $\{x''=0\}$, and the $j-$dimensional domain $D := \Omega \cap \{x''=0\}$. Following the ideas of [7], the solution z of (1.0.2) must satisfy

$$z \le z_D, \qquad \text{in} \quad D,$$

where z_D is the concave envelope of g in $D = \Omega \cap \{x''=0\}$. By the choice of g, using (3.1.11), it follows that $z_D \equiv 0$. The claim then follows from the maximum principle, since $g \ge 0$ in $\partial\Omega$. In particular, we have

$$z(0) = 0.$$

Now, take u_0 a nonnegative continuous function in $\overline{\Omega}$ satisfying $u_0|_{\partial\Omega} = g$ and

$$u_0(x',x'') \ge \psi_1(x') \quad \text{for all } (x',x'') \in \Omega, \text{ with } x' \in B_r^j. \tag{3.1.12}$$

Consider the following function defined in the subdomain $\mathcal{Q} := \Big(\Omega \cap \{x' \in B_r^j\}\Big) \times [0,+\infty)$:

$$\underline{u}(x',x'',t) := \psi_1(x')e^{-\mu_1 t}.$$

We have

$$\begin{aligned} \underline{u}_t(x',x'',t) \quad &= -\mu_1\psi_1(x')e^{-\mu_1 t}, \\ \lambda_j(D^2\underline{u}(x',x'',t)) \quad &= -\mu_1\psi_1(x')e^{-\mu_1 t}, \\ \underline{u}(x',x'',t) \quad &\le \psi_1(x'), \end{aligned}$$

in $\mathcal{Q}$. By (3.1.10) and (3.1.12), together with the comparison principle, we get

$$\underline{u}(x',x'',t) \le u(x',x'',t), \qquad \text{in } \mathcal{Q},$$

and since $z(0)=0$, we have

$$u(0,t) \ge z(0) + \psi_1(0)e^{-\mu_1 t}, \qquad \text{for all } t > 0.$$

□

3.2 Probabilistic arguments

Here we will argue relating the value for our game and the value for the game *a random walk for* λ_j introduced in [7]. We call $z^\varepsilon(x_0)$ the value of the game for the elliptic case (see [7]) considering the initial position x_0 and a length step of ε. This game is the same as the one described in Section 2 but now we do not take into account the time, that is, we do not stop when $t_k < 0$ (and therefore we do not have that the number of plays is a priori bounded by $\lceil 2T/\varepsilon^2 \rceil$). We will call $x_\tau \notin \Omega$ the final position of the token. In what follows we will refer to the game described in Section 2 as the parabolic game while when we disregard time we refer to the elliptic game. Notice that the elliptic DPP is given by

$$\begin{cases} v^\varepsilon(x) = \inf_{dim(S)=j} \sup_{v\in S,|v|=1} \left\{ \frac{1}{2} v^\varepsilon\Big(x+\varepsilon v\Big) + \frac{1}{2} v^\varepsilon\Big(x-\varepsilon v\Big) \right\} & x \in \Omega, \\ v^\varepsilon(x) = g(x) & x \notin \Omega. \end{cases}$$

Solutions to this DPP are stationary solutions (solutions independent of time) for the DPP that correspond to the parabolic game. Let us recall it here,

$$\begin{cases} u^\varepsilon(x,t) = \inf_{dim(S)=j} \sup_{v\in S,|v|=1} \left\{ \frac{1}{2} u^\varepsilon(x+\varepsilon v, t-\frac{\varepsilon^2}{2}) + \frac{1}{2} u^\varepsilon(x-\varepsilon v, t-\frac{\varepsilon^2}{2}) \right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x,t)\in\Omega_T, \\ u^\varepsilon(x,t) = h(x,t), \qquad\qquad\qquad\qquad\qquad \text{if } (x,t)\notin\Omega_T. \end{cases}$$

Here we choose h in such a way that it does not depend on t (we can do this since we are assuming that g does not depend on t).

Our goal will be to show that there exist two positive constants μ, depending only on Ω, and C, depending on u_0, but both independent of ε, such that

$$\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot)\|_\infty \leq Ce^{-\mu t}.$$

For the elliptic game, the strategies are denoted by $\tilde{S}_{\mathrm{I}}$ and $\tilde{S}_{\mathrm{II}}$. Given two strategies for the elliptic game, we can play the parabolic game according to those strategies by considering, for all $t_0 > 0$,

$$\begin{aligned} S_{\mathrm{I}}(t_0, x_0, x_1, \dots, x_k) &= \tilde{S}_{\mathrm{I}}(x_0, x_1, \dots, x_k) \\ S_{\mathrm{II}}(t_0, x_0, x_1, \dots, x_k, S) &= \tilde{S}_{\mathrm{II}}(x_0, x_1, \dots, x_k, S). \end{aligned} \tag{3.2.1}$$

When we attempt to do the analogous construction, building a strategy for the elliptic game given one for the parabolic game, we require that the game sequences are not too long since the strategies for the parabolic game are only defined for $t_k > 0$ (when $t_k \leq 0$ the parabolic game ends). However, for any $t > 0$, if we suppose that the game ends in less than $\lceil 2t/\varepsilon^2 \rceil$ steps, i.e. $\tau < \lceil 2t/\varepsilon^2 \rceil$, then we have a bijection between strategies for the two games that have the same probability distribution for the game histories $(x_0, x_1, \dots, x_\tau)$.

The next lemma ensures that, in the parabolic game, the probability of the final payoff being given by the initial data goes to 0 exponentially fast when $t \to +\infty$. In addition, we also prove that in the elliptic game, trajectories that take too long to exit the domain have exponentially small probability.

Lemma 3.6. *Let Ω be a bounded domain, S_I, S_{II} two strategies for the parabolic game and $\tilde{S}_I, \tilde{S}_{II}$ two strategies for the elliptic game. We have, for any $t > 0$,*

$$\mathbb{P}^{x_0,t}_{S_I,S_{II}}[t_\tau \leq 0] \leq Ce^{-\mu t} \quad \text{and} \quad \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] \leq Ce^{-\mu t}$$

where $\mu > 0$ is a constant depending only on Ω and C is another constant independent on the size of the steps, ε. We recall that τ denotes the number of steps until the game ends.

Proof. Take $B_R(x)$ such that $\Omega \subset B_R(x)$. We start by proving the estimate for the elliptic game. Let $\tilde{S}_I, \tilde{S}_{II}$ be two strategies for this game. As computed in (2.2.3),

$$M_k = |x_k - x|^2 - \varepsilon^2 k$$

is a martingale. By applying the optional stopping theorem, we obtain

$$\varepsilon^2 \mathbb{E}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}[\tau] = \mathbb{E}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}[|x_\tau - x|^2] \leq R^2.$$

Hence, we get

$$\mathbb{E}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2}\right] \leq \frac{R^2}{2}$$

and we can show the bound

$$\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] \leq \frac{R^2}{2t}.$$

For $n \in \mathbb{N}$ by considering the martingale starting after n steps, we can obtain

$$\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n + t \Big| \frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n\right] \leq \frac{R^2}{2t}.$$

Hence, for $n, k \in \mathbb{N}$, applying this bound multiple times we obtain

$$\begin{aligned}
\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}nk\right] &= \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}nk \Big| \frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n(k-1)\right] \\
&\quad \times \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n(k-1) \Big| \frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n(k-2)\right] \\
&\quad \times \cdots \times \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}n\right] \\
&\leq \left(\frac{R^2}{2(\frac{\varepsilon^2 n}{2})}\right)^k.
\end{aligned}$$

For $\varepsilon < \varepsilon_0 = 1$ we consider

$$\delta = \frac{R^2}{2e^{-1}} + \frac{1}{2}.$$

We have

$$\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] \leq \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq \frac{\varepsilon^2}{2}\left\lfloor\frac{\delta 2}{\varepsilon^2}\right\rfloor\left\lfloor\frac{t}{\delta}\right\rfloor\right].$$

By the above argument we obtain

$$\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] \leq \left(\frac{R^2}{2\left\lfloor\frac{\delta 2}{\varepsilon^2}\right\rfloor\frac{\varepsilon^2}{2}}\right)^{\lfloor\frac{t}{\delta}\rfloor} \leq \left(\frac{R^2}{2(\delta - \frac{\varepsilon_0^2}{2})}\right)^{\frac{t}{\delta}-1} = e^{-\frac{t}{\delta}+1}.$$

We have shown

$$\mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] \leq Ce^{-\mu t}$$

for $C = e$ and $\mu = \frac{1}{\delta}$. The same bound holds for the parabolic game, using the relation between the strategies given in (3.2.1). That is,

$$\begin{aligned}
\mathbb{P}^{x_0,t}_{S_I,S_{\mathrm{II}}}[t_\tau \leq 0] &= \mathbb{P}^{x_0,t}_{S_I,S_{II}}\left[\frac{\varepsilon^2\tau}{2} \geq t\right] = 1 - \mathbb{P}^{x_0,t}_{S_I,S_{II}}\left[\tau < \frac{2t}{\varepsilon^2}\right] \\
&= 1 - \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{II}}\left[\tau < \frac{2t}{\varepsilon^2}\right] \leq Ce^{-\mu t}.
\end{aligned}$$

The use of the equivalence (3.2.1) between strategies of the two games is justified because we are computing the probability of the number of steps being less than $\lceil 2t/\varepsilon^2 \rceil$. □

Using Lemma 3.6, we are able to prove that, as happens for the evolution PDE (see the previous subsection), also in the game formulation, the asymptotic behaviour of the value function as t goes to infinity is given by the value of the elliptic game (that is, by the stationary solution of the game). Notice that in the probabilistic approach we obtain a bound for $\|u(\cdot,t)-z(\cdot)\|_\infty$ of the form $C\|u_0\|_\infty e^{-\mu t}$. However, we do not have that μ comes from an eigenvalue problem but from the exponential bounds obtained in Lemma 3.6.

Proposition 3.7. *There exists $\mu>0$, a constant depending only on Ω, and $C>0$ depending on u_0, such that*

$$\|u^\varepsilon(\cdot,t)-v^\varepsilon(\cdot)\|_\infty \le Ce^{-\mu t},$$

where u^ε and v^ε are the value functions for the parabolic and the elliptic game, respectively.

Moreover, as a consequence of this exponential decay, we obtain that the solution u of the problem (1.0.1) *and the convex envelope $z(x)$ of g in Ω satisfy*

$$\|u(\cdot,t)-z(\cdot)\|_\infty \le Ce^{-\mu t}.$$

Proof. Recall the payoff function h defined in (2.2.1), here do not depend on t. For any $(x_0,t_0)\in\Omega\times(0,+\infty)$ fixed, we have

$$\begin{aligned} u^\varepsilon(x_0,t_0) &= \inf_{S_I}\sup_{S_{\mathrm{II}}}\mathbb{E}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}\big[h(x_\tau,t_\tau)\big] \\ &= \inf_{S_I}\sup_{S_{\mathrm{II}}}\Big\{\mathbb{E}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}\big[g(x_\tau)|t_\tau>0\big]\mathbb{P}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}(t_\tau>0) \\ &\quad+\mathbb{E}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}\big[u_0(x_\tau)|t_\tau\le 0\big]\mathbb{P}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}(t_\tau\le 0)\Big\} \\ &\le \inf_{S_I}\sup_{S_{\mathrm{II}}}\mathbb{E}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}\big[g(x_\tau)|t_\tau>0\big]+(\|g\|_\infty+\|u_0\|_\infty)\sup_{S_{\mathrm{I}},S_{\mathrm{II}}}\mathbb{P}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}(t_\tau\le 0) \end{aligned} \tag{3.2.2}$$

and

$$u^\varepsilon(x_0,t_0) \ge \inf_{S_I}\sup_{S_{\mathrm{II}}}\mathbb{E}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}\big[g(x_\tau)|t_\tau>0\big]-(\|g\|_\infty+\|u_0\|_\infty)\sup_{S_{\mathrm{I}},S_{\mathrm{II}}}\mathbb{P}^{x_0,t_0}_{S_I,S_{\mathrm{II}}}(t_\tau\le 0). \tag{3.2.3}$$

Now, let $z^\varepsilon(x_0)$ be the value of the elliptic game considering as payoff function the same function g as before. We have

$$\begin{aligned} z^\varepsilon(x_0) &= \inf_{\tilde S_I}\sup_{\tilde S_{\mathrm{II}}}\Big\{\mathbb{E}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}\big[g(x_\tau)|\tau<2t_0/\varepsilon^2\big]\mathbb{P}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}(\tau<2t_0/\varepsilon^2) \\ &\quad+\mathbb{E}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}\big[g(x_\tau)|\tau\ge 2t_0/\varepsilon^2\big]\mathbb{P}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}(\tau\ge 2t_0/\varepsilon^2)\Big\} \\ &\le \inf_{\tilde S_I}\sup_{\tilde S_{\mathrm{II}}}\mathbb{E}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}\big[g(x_\tau)|\tau<2t_0/\varepsilon^2\big]+\|g\|_\infty\sup_{\tilde S_I,\tilde S_{\mathrm{II}}}\mathbb{P}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}(\tau\ge t_0/\varepsilon^2). \end{aligned} \tag{3.2.4}$$

and

$$z^\varepsilon(x_0) \ge \inf_{\tilde S_I}\sup_{\tilde S_{\mathrm{II}}}\mathbb{E}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}\big[g(x_\tau)|\tau<2t_0/\varepsilon^2\big]-\|g\|_\infty\sup_{\tilde S_I,\tilde S_{\mathrm{II}}}\mathbb{P}^{x_0}_{\tilde S_I,\tilde S_{\mathrm{II}}}(\tau\ge t_0/\varepsilon^2). \tag{3.2.5}$$

Given $t_0>0$ in the parabolic game, if we suppose that $\tau<2t_0/\varepsilon^2$ in both games, we have an equivalence between the strategies of both games, regardless what happens after step $\lfloor 2t_0/\varepsilon^2 \rfloor$. That is,

$$\inf_{\tilde{S}_I} \sup_{\tilde{S}_{\rm II}} \mathbb{E}^{x_0}_{\tilde{S}_I,\tilde{S}_{\rm II}}\big[g(x_\tau)|\tau < 2t_0/\varepsilon^2\big] = \inf_{S_I} \sup_{S_{\rm II}} \mathbb{E}^{x_0,t_0}_{S_I,S_{\rm II}}\big[g(x_\tau)|t_\tau > 0\big].$$

Now, combining (3.2.2), (3.2.3), (3.2.4) and (3.2.5), we obtain

$$|u^\varepsilon(x_0,t_0) - z^\varepsilon(x_0)| \le 2\|u_0\|_\infty \left(\sup_{\tilde{S}_I,\tilde{S}_{\rm II}} \mathbb{P}^{x_0}_{\tilde{S}_I,\tilde{S}_{\rm II}}(\tau \ge 2t_0/\varepsilon^2) + \sup_{S_{\rm I},S_{\rm II}} \mathbb{P}^{x_0,t_0}_{S_I,S_{\rm II}}(t_\tau \le 0) \right). \tag{3.2.6}$$

Applying Lemma 3.6, for $\varepsilon < \varepsilon_0 = 1$, we have

$$|u^\varepsilon(x_0,t_0) - z^\varepsilon(x_0)| \le 4\|u_0\|_\infty C e^{-\mu t_0},$$

for some μ depending only on Ω. Letting $\varepsilon \to 0$ and using the uniform convergence of $u^\varepsilon(x_0,t_0)$ and $z^\varepsilon(x_0)$ to $u(x_0,t_0)$ and $z(x_0)$, respectively, we obtain

$$|u(x_0,t_0) - z(x_0)| \le 4\|u_0\|_\infty C e^{-\mu t_0}.$$

This completes the proof. □

Now, assume that there is an affine function π such that $g = \pi$ for $x \notin \Omega$. In this case, we have that $\pi(x_k)$ is a martingale. Hence, under a strategy that forces the game to end outside Ω, we obtain that $\mathbb{E}^{x_0,t_0}_{S_I,S_{\rm II}}[h(x_\tau,t_\tau)] = \pi(x_0)$.

Suppose $1 \le j \le N-1$, $\Omega \subset B_R(x)$ and $g \equiv \pi$. Player I can choose S at every step in such a way that it is normal to $x - x_k$, hence $v \in S$ is normal to $x - x_k$, we have

$$|x - x_{k+1}|^2 = |x - x_k - v\varepsilon|^2 \ge |v\varepsilon|^2 + |x - x_k|^2 = \varepsilon^2 + |x - x_k|^2.$$

If Player I plays with this strategy, we obtain $|x - x_k|^2 = k\varepsilon^2 + |x - x_0|^2$. Since $\Omega \subset B_R(x)$, $|x - x_k|^2 \le R^2$ for every $x_k \in \Omega$, and hence the game ends after at most

$$\frac{R^2 - |x - x_0|^2}{\varepsilon^2}$$

turns. Hence, it holds that

$$u(x,t) \le \pi(x)$$

for every $x \in \Omega$ and every $t > T = 2R^2$.

Analogously, if $2 \le j \le N$, Player II can choose $v \in S$ such that v is normal to $x - x_k$ (because the intersection of S and the $N-1$ dimensional normal space to $x - x_k$ is not empty). By the same arguments used before, we can show that

$$u(x,t) \ge \pi(x)$$

for every $x \in \Omega$ and every $t > T = 2R^2$.

Hence, we have shown that, for $2 \le j \le N-1$

$$u(x,t) = \pi(x)$$

for every $x \in \Omega$ and every $t > T = 2R^2$. Note that this argument can be considered as a proof of Theorem 1.3 based on the game strategies.

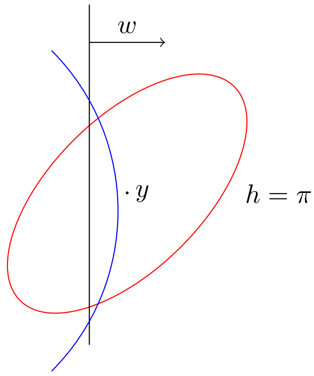

Figure 5.1 – Here $\partial\Omega$ is in red and $\partial B_r(\xi)$ in blue.

We can obtain a similar result when $g = \pi$ in a half-space. Suppose that $h = \pi$ for every $x \in \{x \in \Omega^c : x \cdot w > \theta\}$ for a given $w \in \mathbb{R}^N$ of norm 1 and $\theta \in \mathbb{R}$. Given $y \in \{x \in \Omega : x \cdot w > \theta\}$ we can choose $\xi \in \mathbb{R}^N$ and $r > 0$ such that $\{x \in \Omega : x \cdot w \leq \theta\} \subset B_r(\xi)$ and $y \notin B_r(\xi)$ as depicted in Figure 5.1.

Now, arguing in the same way as before, we can consider the strategies that give a vector v normal to $x_k - \xi$. Hence, in the case $1 \leq j \leq N-1$ we can prove that

$$u(x,t) \leq \pi(x)$$

for every $y \in \{x \in \Omega : x \cdot w > \theta\}$ and every t large enough (for instance we can take $t > 2r^2$ where r is the radius of the ball described before, that depends on x). Note that the closer is y to the hyperplane $x \cdot w = \theta$, the longer we will have to wait for having the above inequality.

In the case $2 \leq j \leq N$, with analogous arguments, we can also show that we have the reverse inequality, that is,

$$u(x,t) \geq \pi(x)$$

for every $x \in \{x \in \Omega : x \cdot w > \theta\}$ and every t large enough.

Next, we present an example where the solution exhibits the behaviour of Theorem 3.5 in the intersection of an hyperplane with Ω (i.e. $u(x,t) > z(x)$ for all $t > 0$), while in the two half-spaces separated by this hyperplane, the behaviour of the solution is the one described above (i.e. $u(x,t) = z(x)$ in finite time).

Example 3.8. *Consider the parabolic game for λ_{N-1} in a ball B_R centered at the origin, and take as initial and boundary data two functions $u_0(x', x_N)$ and $g(x', x_N)$, with $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, such that*

$$u_0 > 0, \quad \textit{in } \Omega, \quad \textit{and} \quad g(x', x_N) = |x_N|, \quad \textit{for all } (x', x_N) \in \mathbb{R}^N \setminus \Omega.$$

For this choice of g, we claim that the solution of problem (1.0.2) *satisfies*

$$z(x', x_N) = 0, \qquad \textit{in} \quad \Omega \cap \{x_N = 0\}.$$

In order to prove this claim, we use the geometric interpretation of solutions to problem (1.0.2) *given in [7]. Consider the $(N-1)$−dimensional subspace $\{x_N = 0\}$, and the $(N-1)$−dimensional domain $D := \Omega \cap \{x_N = 0\}$. Following the ideas of [7], the solution z of* (1.0.2) *must satisfy*

$$z \leq z_D, \qquad \textit{in} \quad D,$$

where z_D is the concave envelope of g in $D = \Omega \cap \{x_N = 0\}$. By the choice of g, it follows that $z_D \equiv 0$. The claim then follows from the maximum principle, since $g \geq 0$ on $\partial\Omega$.

Now, let us prove that for any $x_0 \in \Omega \cap \{x_N = 0\}$ and $t_0 > 0$, we have

$$u^\varepsilon(x_0, t_0) = \inf_{S_I} \sup_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0, t_0} \left[h(x_\tau, t_\tau) \right] > 0.$$

Let $x_0 \in \Omega \cap \{x_N = 0\}$. Since $u_0 \geq 0$, if $u^\varepsilon(x_0, t_0) = 0$, Player I should have a strategy such that whatever Player II does, the final payoff is 0 with probability 1. Since u_0 vanishes only on $\partial\Omega \cap \{x_N = 0\}$, Player I needs to make sure that x_k reaches this set before the game comes to end.

We claim that the only strategy Player I can follow is to choose the $(N-1)-$dimensional subspace $\{x_N = 0\}$ at every step. Indeed, if at some step, x_k leaves this subspace, the probability of never coming back, and then the final payoff being non-zero, is positive.

Once Player I has fixed this only possible strategy to obtain zero as final payoff, Player II can choose any unitary vector in the subspace $\{x_N = 0\}$, and play always with the same vector. Playing with these strategies, the game is reduced to a random walk in a segment, and it is well known that for this process, the probability of not reaching the extremes of the segment in less than $\lceil 2t_0/\varepsilon^2 \rceil$ steps is strictly positive for any $t_0 > 0$. Since the initial condition verifies $u_0 > 0$ in Ω, we conclude that the value of the game is also strictly positive at (x_0, t_0), for any $x_0 \in \Omega \cap \{x_N = 0\}$ and $t_0 > 0$. Then, $u^\varepsilon(x_0, t_0)$, and hence its limit as $\varepsilon \to 0$, $u(x_0, t_0)$, does not lie below the stationary solution z in finite time.

Finally, notice that from our previous arguments, we have that for any point $x_0 \in \Omega \setminus \{x_N = 0\}$ there is a finite time t_0 (that depends on x_0) such that $u^\varepsilon(x_0, t) = z(x_0)$ for every $t \geq t_0$.

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Appendix A

Scripts for numerical lower estimates for the optimization problems in Chapter 3

In this appendix we include the Matlab scripts that we have used to compute the numerical lower estimates given in the Table 1.1 of the Introduction and the other examples given in Chapter 3. An explanation of the strategy and schemes used for each optimization problem can be found in Chapter 3, section 6.

1 Scripts for theorems 1.1 and 1.2

We start with the exploration of the parameter set $\mathcal{A}$ in order to find a suitable triplet of parameters (τ, β, K) to evaluate the RHS of (1.3.4) in Chapter 3. See Chapter 3, section 6 for details on the first exploration, second finer exploration and the stopping conditions.

Script for the exploration:

```
%%% Exploration procedure for Theorems 1.1 and 1.2
function [tauopt,betaopt,Kopt,rho_opt] = explore(mu,fmax,d,d0,p)

rho_opt = 0;
mu1 = p^p*pi^2/(4*(p+1)^(p+1));
tau0 = mu/(2*mu-mu1);

%%% First exploration %%%

tau_step = (1-tau0)/10;
tau_list = tau0:tau_step:1;

% Stopping parameters for increasing and decreasing beta respectively.
% It will keep iterating as far as it equals 0.
stop_beta0 = 0;
stop_beta1 = 0;
```

```
% In the first iteration beta_list has length 1.
% Then, we increase and decrease it and it has length two
% until any of the stopping parameters becomes 1.
beta_list = min(1+d,(d0+d)/2);

while stop_beta0*stop_beta1 == 0
  for beta = beta_list
    t0 = @(x) (1-x.^(p+1))/((p+1)*fmax);
    H_tau =  @(x) H_low_est(t0(x),beta,20,0,p);
    K = max(p/(mu*beta^2)-1/(p+1),0);
    stop_K1 = 0;
    while stop_K1 == 0
      L = 1+(p+1)*K;
      Gamma = sqrt((p+1)*L/(p*K*mu*beta^2));
      A = atan(Gamma);
      delta = A*(1+K)*sqrt((p+1)/(p*L*K*mu));
      if delta<=1
        for tau = tau_list
          t = t0(tau);
          H = H_tau(tau);
          St0 = exp(-4*pi^2*t/(d0+1)^2)*(1-exp(-d0*(d0-beta)/t));
          G = G_low_est(t,mu,beta,K,20,0,p);
          rho = 0.5*((beta-d)/beta)^(p+1)*St0*min(H,G)/(K+tau^(-p));
          if rho>rho_opt
              rho_opt = rho;
              Kopt = K;
              betaopt = beta;
              tauopt = tau;
          end
        end
      end
      if K > 0.5/rho_opt - 1
        stop_K1 = 1;
      else
        K=K+0.1;
      end
    end
  end
  if min(beta_list)<=d/(1-(2*rho_opt)^(1/(p+1))) || min(beta_list)-0.1<d
    stop_beta0 =1;
  end
  %%% Extra stopping condition for beta.
  %%% We can also use only the stopping condition beta > d0.
  H_list = [];
  for tau = tau_list
    H_list = [H_list,H_tau(tau)];
  end
  %%%%
  if 0.5*max(H_list) <= rho_opt || beta+0.1 > d0
    stop_beta1 = 1;
  end
```

```
  if stop_beta0 + stop_beta1 == 0
    beta_list = [min(beta_list)-0.1, max(beta_list)+0.1];
  elseif stop_beta0 == 0
    beta_list = min(beta_list)-0.1;
  elseif stop_beta1 == 0
    beta_list = max(beta_list)+0.1;
  end
end

%%% Second exploration %%%

%%% We iterate beta around the optimal value obtained in the first
%%% exploration

beta0 = max(betaopt - 0.1,d);
beta1 = min(betaopt + 0.1,d0);
beta_step = (beta1-beta0)/10;

%%% The lower bound of the exploration interval for K is 0.1 close to the
%%% optimal value obtained in the first exploration.

K0 = Kopt - 0.1;

%%% We iterate tau around the optimal value obtained in the first
%%% exploration

tau0 = max(tauopt - tau_step,tau0);
tau1 = min(tauopt + tau_step,1);
tau_step = (tau1-tau0)/10;

for beta = beta0:beta_step:beta1
  t0 = @(x) (1-x.^(p+1))/((p+1)*fmax);
  H_tau =  @(x) H_low_est(t0(x),beta,20,0,p);
  Kmin = max(p/(mu*beta^2)-1/(p+1),0);
  %%% We make sure that the lower bound for the exploration interval for
  %%% K satisfies the restriction given in the enunciate of the Theorem.
  K00 = max(Kmin,K0);
  %%% We explore K in a interval of length 0.2 since in the
  %%% first exploration
  %%% we chose a step of 0.1.
  K11 = K00+0.2;
  K_step = (K11-K00)/10;
  for K = K00:K_step:K11
    L = 1+(p+1)*K;
    Gamma = sqrt((p+1)*L/(p*K*mu*beta^2));
    A = atan(Gamma);
    delta = A*(1+K)*sqrt((p+1)/(p*L*K*mu));
    if delta<=1
      for tau = tau0:tau_step:tau1
        t = t0(tau);
        H = H_tau(tau);
        St0 = exp(-4*pi^2*t/(d0+1)^2)*(1-exp(-d0*(d0-beta)/t));
```

```
        G = G_low_est(t,mu,beta,K,20,0,p);
        rho = 0.5*((beta-d)/beta)^(p+1)*St0*min(H,G)/(K+tau^(-p));
        if rho>rho_opt
            rho_opt = rho;
            Kopt = K;
            betaopt = beta;
            tauopt = tau;
        end
      end
    end
  end
end
```

The following two algorithms give a numerical approximation for the functions H and G respectively. This approximation can be a safer lower estimate if we choose the input `lower = 1`. It is the case when we evaluate ρ in the final step. Here, we divide the interval $[0, 1]$ in n subintervals, with n big enough, ensuring an error of order no larger than 10^{-4}. In practice we used `n=2000` to compute H and `n=50000` to compute G. See Chapter 3, section 6 for the details on the monotone discretization of the integrals for the lower estimate. However, during the exploration procedure we cannot use such a fine partition of $[0, 1]$ since we need to evaluate H and G many times. In practice, we used a partition in `n=20` subintervals for H and G. For such a coarse partition of the interval, it happens to be better not to use lower estimates, which give bigger errors.

Script for H:

```
%%% Computes a numerical approximation or a lower estimate of H for
%%% Theorems 1.1 and 1.2.
function [H,errorH] = H_low_est(t,beta,n,lower,p)
%%% lower = 1 for a lower estimate of H and G.
%%% lower != 1 for a better approximation.

errorH = 1;

numH = @(x) erf((1+beta*x/2)./sqrt(t))-erf(beta*x./(2*sqrt(t)));
denH = @(x) (1-x).^(p+1);

if lower == 1
    % Let's divide the interval [0,1] in n intervals.
    % x1 is the final point of each interval.
    % x2 is the initial point of each interval.
    x2 = 0:1/n:1-1/n;
    % numH and denH are both decreasing in [0,1]
    H = 10;
    for x = x2
        x1 = x + 1/n;
        Hx = numH(x1)/denH(x);
        if Hx < H
            H = Hx;
            errorH = (numH(x)-numH(x1))/denH(x);
        end
    end
```

```
else
    x = 0:1/n:1;
    H = min(numH(x)./denH(x));
end
```

Script for G:

```
%%% Computes a numerical approximation or a lower estimate of G for
%%% Theorems 1.1 and 1.2.

function [G,errorG] = G_low_est(t,mu,beta,K,n,lower,p)
%%% lower = 1 for a lower estimate of H and G.
%%% lower != 1 for a better approximation.

errorG = 1;

L = 1+(p+1)*K;
Gamma = sqrt((p+1)*L/(p*K*mu*beta^2));
A = atan(Gamma);
alpha = 1+ p/L;
delta = A*(1+K)*sqrt((p+1)/(p*L*K*mu));

numG = @(x) erf((1-delta*(1-x)/2)/sqrt(t))+erf((1-x)*delta/(2*sqrt(t)));
denG = @(x) (cos(A*x)).^(alpha);

if lower == 1
    % Let's divide the interval [0,1] in n intervals.
    % x1 is the final point of each interval.
    % x2 is the initial point of each interval.
    x2 = 0:1/n:1-1/n;
    % numG and denG are both decreasing in [0,1]
    G = 10;
    for x = x2
        x1 = x + 1/n;
        Gx = (Gamma^2+1)^(-alpha/2)*numG(x1)/denG(x);
        if Gx < G
            G = Gx;
            errorG = (Gamma^2+1)^(-alpha/2)*(numG(x) - numG(x1))./denG(x);
        end
    end
else
    x = 0:1/n:1;
    G = (Gamma^2+1)^(-alpha/2)*min(numG(x)./denG(x));
end
```

The following script is used to compute the lower estimates for the threshold ratio ρ for the chosen values of $\mu, \|f\|_\infty, d, d_0$ in each example. In the output table, that we call `B`, we include, a part of the parameters τ, β, K used to evaluate the RHS of formula (1.3.4) in Chapter 3, the lower estimates of the functions H and G, as well as an estimate for the error. We have not included this error in the Table 3.3 of Chapter 3 because it is always of order no larger than 10^{-4}.

Script for the examples in Chapter 3, section 6

```
%%%%%% TABLE OF EXAMPLES for Theorems 1.1 and 1.2 %%%%
clear
p=2;

mu1 = p^p*pi^2/(2*(p+1)^(p+1));

%%% Each vector contains the values [mu; fmax; d; d_0] for each example.

v1 = [1;1.1;0.1;5];
v2 = [1.25;1.3;0.1;3];
v3 = [2;2.25;0.1;4];
v4 = [2;2.25;0.05;4];
v5 = [3;3.5;0.01;5];
v6 = [4;4.1;0.05;5];
v7 = [4;4.1;0.01;5];
v8 = [4;7;0.01;5];
v9 = [6;6.2;0.01;10];
v10 = [10;10;0.005;10];

V = [v1,v2,v3,v4,v5,v6,v7,v8,v9,v10];

B=[];

for v=V
    [tau,beta,K] = explore(v(1),v(2),v(3),v(4),p);
    t = (1-tau^(p+1))/((p+1)*v(2));
    [H,errorH] =  H_low_est(t,beta,50000,1,p);
    [G,errorG] = G_low_est(t,v(1),beta,K,2000,1,p);
    St0 = exp(-4*pi^2*t/(v(4)+1)^2)*(1-exp(-v(4)*(v(4)-beta)/t));
    rho = 0.5*((beta-v(3))/beta)^(p+1)*St0*min(H,G)/(K+tau^(-p));
    B = [B;[v(1),v(2),v(3),v(4),tau,beta,K,H,errorH,G,errorG,St0,rho]];
end

B
```

2 Scripts for theorems 2.1 and 2.2

Script for the explorations:

```
%%% Exploration procedure for Theorems 2.1 and 2.2
function [tauopt,betaopt,Kopt,lambda_opt,rho_opt] = explore(mu,fmax,d,d0,p)

rho_opt = 1e-6;

%%% First exploration %%%

tau_step = 1/10;
```

```
tau_list = 0.1:tau_step:1;

for tau = tau_list
  t0 = (1-tau^(p+1))/((p+1)*fmax);
  beta = d+0.1;
  while beta < d0 && beta < 2
    S_t_beta = exp(-pi^2*t0/(4*(d0+1)^2))*(1-exp(-d0*(d0-beta)/t0));
    K = 0.1;
    while K <= min(p*(p+2)/(p*mu*beta^2+1),p)
      %%% The condition delta1+delta2 < 1 is checked in the
      %%% function rho_explore.
      [rho,lambda] = rho_explore(t0,tau,beta,K,mu,fmax,d,d0,S_t_beta,...
                                 ...p,rho_opt,10,20);
      if rho>rho_opt
        rho_opt = rho;
        Kopt = K;
        betaopt = beta;
        tauopt = tau;
        lambda_opt = lambda;
      end
      K = K + 0.1;
    end
    beta = beta + 0.1;
  end
end

%%%%% Second exploration %%%%%

%%% We iterate beta around the optimal value obtained in the first
%%% exploration.

beta0 = max(betaopt - 0.1,d);
beta1 = min(betaopt + 0.1,d0);
beta_step = (beta1-beta0)/10;

%%% The upper limit of the exploration interval for K is 0.1 close to the
%%% optimal value obtained in the first exploration and smaller than p.

K1 = min(Kopt + 0.1,p);

%%% We iterate tau around the optimal value obtained in the first
%%% exploration

tau0 = max(tauopt - tau_step,0.1);
tau1 = min(tauopt + tau_step,1);
tau_step = (tau1-tau0)/10;

for tau = tau0:tau_step:tau1
  t0 = (1-tau^(p+1))/((p+1)*fmax);
  for beta = beta0:beta_step:beta1
    S_t_beta = exp(-pi^2*t0/(4*(d0+1)^2))*(1-exp(-d0*(d0-beta)/t0));
    %%% We make sure that the upper limit of the interval for K
```

```
    %%% satisfies the restriction.
    K11 = min(K1,p*(p+2)/(p*mu*beta^2+1));
    %%% We iterate K in a interval of length 0.2 since the value
    %%% obtained in the first exploration is 0.1 close to the real
    %%% optimum.
    K10 = max(K11-0.2,0.1);
    K_step = (K11-K10)/10;
    for K = K10:K_step:K11
      %%% The condition delta1+delta2 < 1 is checked in the
      %%% function rho_explore.
      [rho,lambda] = rho_explore(t0,tau,beta,K,mu,fmax,d,d0,S_t_beta,...
                                   ...p,rho_opt,10,20);
      if rho>rho_opt
          rho_opt = rho;
          Kopt = K;
          betaopt = beta;
          tauopt = tau;
          lambda_opt = lambda;
      end
    end
  end
end
```

Script for the numerical approximation of ρ during the exploration procedure:

```
%%% For (tau, beta, K) fixed, this function computes a numerical
%%% approximation of rho and lambda by iterating lambda, from
%%% an unattainable initial value for rho and decreasing it.
%%% We also input rho_opt so that we stop the iteration
%%% of lambda if it is smaller than rho_opt obtained in previous
%%% explorations for other parameters (tau, beta, K).
%%% Here, n_t is the number of subintervals for the numerical time
%%% integral to compute Lambda(t0,r).
%%% nr is the number of points where we evaluate the function G
%%% in each subinterval [r0,r1], [r1,1] and [1,1+beta].
%%% Script for Theorems 2.1 and 2.2.

function [rho,lambda] = rho_explore(t0,tau,beta,K,mu,fmax,d,d0,S_t_beta,...
                                     ...p,rho_opt,n_t,nr)

%%% Initial lambda. We can increase this value for the cases p>2 since
%%% it is possible to reach values for rho bigger than 0.3.

lambda = 0.3;

cp = (p+1)^(p+1)/p^p;

mu0 = pi^2/(4*cp);
T = 1/((p+1)*(mu-mu0));

rho = 0;
```

```
A0 = sqrt((p*(1+K)+(p*(p+2)-K)*K)/(p*(1+K)^2));
A1 = atan(sqrt(p*(1+K)/(p*(p+2)-K)+K));
A2 = atan(sqrt(p/(p*(p+2)-K)));
A3 = atan(1/sqrt(K*beta^2*mu));
delta1 = A1/(A0*sqrt(K*mu));
delta2 = (A3-A2)/sqrt(K*mu);

%%% We first check if the condition delta1 + delta2 > 1 is satisfied
if delta1 + delta2 > 1
  rho = 0;
else
  r0 = 1-delta1-delta2;
  r1 = 1-delta2;

  %%% Since Lambda(r) does not depend on lambda, we compute it first in
  %%% the three subintervals and stock it in vectors that we will use
  %%% later.

  r_int = r0:(r1-r0)/nr:r1;
  Lambda_r_int = [];
  for r = r_int
    Lambda_r_int = [Lambda_r_int, Lambda_Simpson(t0,r,n_t,mu,d0,p)];
  end

  r_mid = r1:(1-r1)/nr:1;
  Lambda_r_mid = [];
  for r = r_mid
    Lambda_r_mid = [Lambda_r_mid, Lambda_Simpson(t0,r,n_t,mu,d0,p)];
  end

  r_ext = 1:beta/nr:1+beta;
  Lambda_r_ext = [];
  for r = r_ext
    Lambda_r_ext = [Lambda_r_ext, Lambda_Simpson(t0,r,n_t,mu,d0,p)];
  end

  alpha = (p+1)/((1+K)*A0^2);

  D11 = sqrt(1+K+p*(1+K)/(p*(p+2)-K));
  D12 = sqrt(1+p/(p*(p+2)-K));
  D2 = (1+1/(K*mu*beta^2))^((p+1)/2);
  D1 = D2*D11^alpha/D12^(p+1);

  a1 = @(x) D1*cos(A0*sqrt(K*mu)*(x-r0)).^alpha;
  a2 = @(x) D2*cos(sqrt(K*mu)*(x-1)+A3)^(p+1);
  a3 = @(x) (1-(x-1)/beta).^(p+1);

  %%% G does not depend on lambda in the interval [0,1], so we first
  %%% compute Gint, the minimum of G in [0,1].
  Gint = 10;
  j = 1;
```

```
  for rj = r_int
    num1 = 1+p*mu*S_t_beta*Lambda_r_int(j);
    num2 = erf((rj+1)/sqrt(4*t0))+erf((1-rj)/sqrt(4*t0));
    Gint = min( Gint, num1*num2/((K*tau + tau^(-p))*a1(rj)));
    j = j+1;
  end

  j = 1;

  for rj = r_mid
    num1 = 1+p*mu*S_t_beta*Lambda_r_mid(j);
    num2 = erf((rj+1)/sqrt(4*t0))+erf((1-rj)/sqrt(4*t0));
    Gint = min( Gint, num1*num2/((K*tau + tau^(-p))*a2(rj)));
    j = j+1;
  end

  %%% We iterate lambda. For each lambda, we compute G in the interval
  %%% [1,1+beta] and retain the minimum of this and Gint.

  while rho < lambda && lambda > rho_opt
    G = Gint;
    u_tilde = @(x) lambda*mu/fmax + ...
    ...+ (1-lambda*mu/fmax)/cosh(sqrt(cp*fmax)*(x-1-d));
    W = @(x) (x<=1+d).*(K*tau + tau^(-p)) + ...
    ...+ (x>1+d).*(K*(1-(1-tau)*u_tilde(x)) + (1-(1-tau)*u_tilde(x))^(-p));
    j=1;
    for rj = r_ext
      num1 = 1+p*mu*S_t_beta*Lambda_r_ext(j);
      num2 = erf((rj+1)/sqrt(4*t0))+erf((1-rj)/sqrt(4*t0));
      G = min(G, num1*num2/(W(rj)*a3(rj)));
      j=j+1;
    end
    rho = min([0.5*((beta-d)/beta)^(p+1)*S_t_beta*G, ...
              ...tau^(p+1)/((p+1)*(T-t0)*mu),lambda]);
    lambda = lambda - 0.01;
  end
  %%% We add 0.01 since we have subtracted it at the end of the loop.
  lambda = lambda + 0.01;
end
```

Script for the numerical approximation of $\Lambda(t, r)$ during the exploration procedure:

```
%%% Numerical approximation of Lambda using the Simpson method and a
%%% partition of the time interval in n_t subintervals.
%%% In each subinterval, we evaluate Lambda at the initial and final
%%% points, and at the midpoint.
%%% For Theorems 2.1 and 2.2
function Lambda = Lambda_Simpson(t,r,n_t,mu,d0,p)

a=-p/(p+1)-1;
S_t_0 = exp(-pi^2*t/(4*(d0+1)^2)).*(1-exp(-d0^2./t));
```

```
Y = @(x) 0.5*(p+1)*mu*S_t_0 * erf(1/sqrt(x)).*x;

tmin = 0.0001*t;
tmax = 0.9999*t;
t_step = (tmax-tmin)/n_t;
t_mesh = tmin+t_step:t_step:tmax;
Lambda = 0;
%%%%Lambda en t0
tj = tmin;
Lambda1 = 0.5*(1-Y(tj))^a *(erf((r+1)/(2*sqrt(t-tj)))+...
                             ...+ erf((1-r)/(2*sqrt(t-tj))));
for tj = t_mesh
  tj12 = tj-t_step/2;
  Lambda2 = 0.5*(1-Y(tj12))^a *(erf((r+1)/(2*sqrt(t-tj12)))+...
                                 ...+ erf((1-r)/(2*sqrt(t-tj12))));
  Lambda3 = 0.5*(1-Y(tj))^a *(erf((r+1)/(2*sqrt(t-tj)))+...
                              ...+ erf((1-r)/(2*sqrt(t-tj))));
  Lambda = Lambda + t_step*(Lambda1 + 4*Lambda2 + Lambda3)/6;
  Lambda1 = Lambda3;
end
```

Script for the numerical lower estimate of ρ in the final step:

```
%%% For (tau, beta, K,lambda), this function computes a lower
%%% estimate of rho.
%%% Script for Theorems 2.1 and 2.2.
%%% It also plots the function Gstar.

function [rho,Gstar,r0] = rho_final(t0,tau,beta,K,lambda,mu,fmax,d,d0,...
                                    ...S_t_beta,p)
%%% Discretization of the time integral in Lambda(t0,r)
n_t = 200;

%%% Partition of the interval [r0,1+beta]. We use monotonicity
%%% properties to compute a lower estimate of Gstar in each subinterval.
nr = 5000;

cp = (p+1)^(p+1)/p^p;

mu0 = pi^2/(4*cp);
T = 1/((p+1)*(mu-mu0));

A0 = sqrt((p*(1+K)+(p*(p+2)-K)*K)/(p*(1+K)^2));
A1 = atan(sqrt(p*(1+K)/(p*(p+2)-K)+K));
A2 = atan(sqrt(p/(p*(p+2)-K)));
A3 = atan(1/sqrt(K*beta^2*mu));
delta1 = A1/(A0*sqrt(K*mu));
delta2 = (A3-A2)/sqrt(K*mu);

r0 = 1-delta1-delta2;
r1 = 1-delta2;
```

```
alpha = (p+1)/((1+K)*A0^2);

D11 = sqrt(1+K+p*(1+K)/(p*(p+2)-K));
D12 = sqrt(1+p/(p*(p+2)-K));
D2 = (1+1/(K*mu*beta^2))^((p+1)/2);
D1 = D2*D11^alpha/D12^(p+1);

a1 = @(x) D1*cos(A0*sqrt(K*mu)*(x-r0)).^alpha;
a2 = @(x) D2*cos(sqrt(K*mu)*(x-1)+A3).^(p+1);
a3 = @(x) (1-(x-1)/beta).^(p+1);

a = @(x) (x<r0) + (x>=r0).*(x<r1).*a1(x) + (x>=r1).*(x<1).*a2(x) + ...
         ... + (x>=1).*(x<=1+beta).*a3(x);

r_step = (1+beta-r0)/nr;
r_mesh = r0:r_step:1+beta-r_step;

Gstar = 10;

u_tilde=@(x)lambda*mu/fmax+(1-lambda*mu/fmax)/cosh(sqrt(cp*fmax)*(x-1-d));
W=@(x) (x<=1+d)*(K*tau+tau^(-p))+(x>1+d)*(K*(1-(1-tau)*u_tilde(x)) +...
         ...+ (1-(1-tau)*u_tilde(x))^(-p));

%%% The vector to plot G
Gstarr= [];

for rj = r_mesh
  %%% Numerator and denominator are both decrasing with r.
  %%% rj is the lower limit of each subinterval, and rj1 the upper limit.
  rj1 = rj+r_step;
  Lambda = Lambda_est(t0,rj1,n_t,mu,d0,p);
  num1 = 1 + p*mu*S_t_beta*Lambda;
  num2 = erf((rj1+1)/sqrt(4*t0)) + erf((1-rj1)/sqrt(4*t0));
  Gstar = min(Gstar, num1*num2/(W(rj)*a(rj)));
  Gstarr = [Gstarr,num1*num2/(W(rj)*a(rj))];
end

%%% We do not plot the 1000 last points since G becomes too big
plot(r_mesh(1:nr-1000), Gstarr(1:nr-1000))
rho = min([0.5*((beta-d)/beta)^(p+1)*S_t_beta*Gstar, ...
          ... tau^(p+1)/((p+1)*(T-t0)*mu), lambda]);
```

Script for the numerical lower estimate of $\Lambda(r,s)$ in the final step:

```
%%% Lower estimate of Lambda using the rectangles rule with a
%%% discretization of the time interval in n_t subintervals. We use the
%%% monotonicity properties of each term.
%%% For Theorems 2.1 and 2.2
function Lambda = Lambda_est(t,r,n_t,mu,d0,p)

a=-p/(p+1)-1;
```

```
S_t_0 = exp(-pi^2*t/(4*(d0+1)^2)).*(1-exp(-d0^2./t));

Y = @(x) 0.5*(p+1)*mu*S_t_0 * erf(1/sqrt(x)).*x;

%%% We cut the integral in order to avoid singularities
tmin = 0.0001*t;
tmax = 0.9999*t;
t_step = (tmax-tmin)/n_t;
t_mesh = tmin:t_step:tmax-t_step;

Lambda = 0;

for tj = t_mesh
  %%% The second term in L is decreasing with s when r>1,
  %%% and increasing else.
  tj1 = tj + t_step*(r>1);
  L = erf((r+1)/(2*sqrt(t-tj))) + erf((1-r)./(2*sqrt(t-tj1)));
  Lambda = Lambda + 0.5*t_step*(1-Y(tj))^a*L.*(L>0);
end
```

Script for the examples in Chapter 3, section 6

```
%%%%%% TABLE OF EXAMPLES SECTION 6 for Theorem 2.1 and 2.2 %%%%
clear
p=2;
mu0 = p^p*pi^2/(4*(p+1)^(p+1));

%%% Each vector contains the values [mu; fmax; d; d_0] for each example.
v1 = [1;1.1;0.1;5];
v2 = [1.25;1.3;0.1;3];
v3 = [2;2.25;0.1;4];
v4 = [2;2.25;0.05;4];
v5 = [3;3.5;0.01;5];
v6 = [4;4.1;0.05;5];
v7 = [4;4.1;0.01;5];
v8 = [4;7;0.01;5];
v9 = [6;6.2;0.01;10];
v10 = [10;10;0.005;10];

V = [v1,v2,v3,v4,v5,v6,v7,v8,v9,v10];

B=[];

for v=V
  [tauopt,betaopt,Kopt,lambda_opt,rho_explored] = ...
     ... = explore(v(1),v(2),v(3),v(4),p);
  t0 = (1-tauopt^(p+1))/((p+1)*v(2));
  T = 1/((p+1)*(v(1)-mu0));
  S_t_beta = exp(-pi^2*t0/(4*(v(4)+1)^2))*...
     ...*(1-exp(-v(4)*(v(4)-betaopt)/t0));
  [rho,Gstar] = rho_final(t0,tauopt,betaopt,Kopt,lambda_opt,...
                          ...v(1),v(2),v(3),v(4),S_t_beta,p);
```

```
  rho2 = tauopt^(p+1)/((p+1)*(T-t0)*v(1));
  B = [B;[v(1),v(2),v(3),v(4),tauopt,betaopt,Kopt,Gstar,S_t_beta,rho2,...
          ...lambda_opt,rho_explored,rho]];
end
B
```

3 Scripts for Theorem 2.3

Script for the exploration:

```
%%% Exploration procedure for Theorem 2.3.
function [tauopt,betaopt,Kopt,etaopt,rho_opt] = explore(mu,fmax,d,d0,p)

rho_opt = 0;
mu0 = p^p*pi^2/(4*(p+1)^(p+1));
T = 1/((p+1)*(mu-mu0));

%%% First exploration %%%

tau_step = 1/10;
tau_list = 0.1:tau_step:1;

eta_step = 1/10;
eta_list = 0.1:eta_step:1;

stop_beta0 = 0;
stop_beta1 = 0;

beta_list = min(1+d,(d0+d)/2);

while stop_beta0*stop_beta1 == 0
  for beta = beta_list
    t0 = @(x) (1-x.^(p+1))/((p+1)*fmax);
    H_tau =  @(x) H_low_est(t0(x),beta,20,0,p);
    for eta = eta_list
      K = max(p*eta/(mu*beta^2)-1/((p+1)*eta^p),0);
      stop_K1 = 0;
      while stop_K1 == 0
        L = 1+(p+1)*K*eta^p;
        Gamma = sqrt((p+1)*L*eta/(p*K*mu*beta^2));
        A = atan(Gamma);
        delta = A*(1+K*eta^p)*sqrt((p+1)*eta/(p*L*K*mu));
        if delta<=1
          for tau = tau_list
            t = t0(tau);
            H = H_tau(tau);
            St0 = exp(-4*pi^2*t/(d0+1)^2)*(1-exp(-d0*(d0-beta)/t));
            ST = exp(-4*pi^2*T/(d0+1)^2)*(1-exp(-d0^2/T));
            Gt0 = G_low_est(t,mu,beta,K,eta,20,0,p);
```

```
            GT = G_low_est(T,mu,beta,K,eta,20,0,p);
            rho1 = 0.5*((beta-d)/beta)^(p+1)*min(ST*GT/(K+eta^(-p)),...
                                 ...St0*min(H,Gt0)/(K+tau^(-p)));
            rho = min(rho1 , tau^(p+1)/((p+1)*(T-t)*mu));
            if rho>rho_opt
                rho_opt = rho;
                Kopt = K;
                betaopt = beta;
                tauopt = tau;
                etaopt = eta;
            end
          end
        end
        if K > 0.5/rho_opt - 1
            stop_K1 = 1;
        else
            K=K+0.5;
        end
      end
    end
  end
  if min(beta_list)<=d/(1-(2*rho_opt)^(1/(p+1))) || min(beta_list)-0.2<d
      stop_beta0 = 1;
  end
  %%% Extra stopping condition for beta
  %%% We can just use the stopping condition beta > d0.
  H_list = [];
  for tau = tau_list
      H_list = [H_list,H_tau(tau)];
  end
  if 0.5*max(H_list) <= rho_opt || beta+0.5 > d0
      stop_beta1 = 1;
  end
  if stop_beta0 + stop_beta1 == 0
      beta_list = [min(beta_list)-0.2, max(beta_list)+0.5];
  elseif stop_beta0 == 0
      beta_list = min(beta_list)-0.2;
  elseif stop_beta1 == 0
      beta_list = max(beta_list)+0.5;
  end
end

%%%%% Second exploration %%%%%

%%% We iterate beta around the optimal value obtained in the first
%%% exploration

beta0 = max(betaopt - 0.5,d);
beta1 = min(betaopt + 0.5,d0);
beta_step = (beta1-beta0)/10;

%%% The lower limit of the exploration interval for K is 0.1 close
```

```
%%% to the optimal value obtained in the first exploration.

K0 = Kopt - 0.5;

%%% We iterate tau around the optimal value obtained in the first
%%% exploration

tau0 = max(tauopt - tau_step,0.1);
tau1 = min(tauopt + tau_step,1);
tau_step = (tau1-tau0)/10;

%%% We iterate eta around the optimal value obtained in the first
%%% exploration

eta0 = max(etaopt - eta_step,0.1);
eta1 = min(etaopt + eta_step,1);
eta_step = (eta1-eta0)/10;

for beta = beta0:beta_step:beta1
  t0 = @(x) (1-x.^(p+1))/((p+1)*fmax);
  H_tau =  @(x) H_low_est(t0(x),beta,20,0,p);
  for eta = eta0:eta_step:eta1
    %%% We make sure that the initial K satisfies the restriction.
    Kmin = max(p*eta/(mu*beta^2)-1/((p+1)*eta^p),0);
    K00 = max(Kmin,K0);
    %%% We explore K in a interval of length 1 since we are 1 colse to
    %%% the real optimal value.
    K11 = K00+1;
    K_step = (K11-K00)/10;
    for K = K00:K_step:K11
      L = 1+(p+1)*K*eta^p;
      Gamma = sqrt((p+1)*L*eta/(p*K*mu*beta^2));
      A = atan(Gamma);
      delta = A*(1+K*eta^p)*sqrt((p+1)*eta/(p*L*K*mu));
      if delta<=1
        for tau = tau0:tau_step:tau1
          t = t0(tau);
          H = H_tau(tau);
          St0 = exp(-4*pi^2*t/(d0+1)^2)*(1-exp(-d0*(d0-beta)/t));
          ST = exp(-4*pi^2*T/(d0+1)^2)*(1-exp(-d0^2/T));
          Gt0 = G_low_est(t,mu,beta,K,eta,20,0,p);
          GT = G_low_est(T,mu,beta,K,eta,20,0,p);
          rho1 = 0.5*((beta-d)/beta)^(p+1)*min(ST*GT/(K+eta^(-p)),...
                                      ... St0*min(H,Gt0)/(K+tau^(-p)));
          rho = min(rho1 , tau^(p+1)/((p+1)*(T-t)*mu));
          if rho>rho_opt
              rho_opt = rho;
              Kopt = K;
              betaopt = beta;
              tauopt = tau;
              etaopt = eta;
          end
```

```
        end
      end
    end
  end
end
```

Script for H:

```
%%% Computes a numerical approximation or a lower estimate of H for
%%% Script for Theorem 2.3.
function [H,errorH] = H_low_est(t,beta,n,lower,p)
%%% lower = 1 for a lower estimate of H and G.
%%% lower != 1 for an approximation.

errorH = 1;

numH = @(x) erf((1+beta*x/2)./sqrt(t))-erf(beta*x./(2*sqrt(t)));
denH = @(x) (1-x).^(p+1);

if lower == 1
    % Let's divide the interval [0,1] in n intervals.
    % x1 is the final point of each interval.
    % x2 is the initial point of each interval.
    x2 = 0:1/n:1-1/n;
    % numH and denH are both decreasing in [0,1]
    H = 10;
    for x = x2
        x1 = x + 1/n;
        Hx = numH(x1)/denH(x);
        if Hx < H
            H = Hx;
            errorH = (numH(x)-numH(x1))/denH(x);
        end
    end
else
    x = 0:1/n:1;
    H = min(numH(x)./denH(x));
end
```

Script for G:

```
%%% Computes a numerical approximation or a lower estimate of G for
%%% Theorem 2.3.
function [G,errorG] = G_low_est(t,mu,beta,K,eta,n,lower,p)
%%% lower = 1 for a lower estimate of H and G.
%%% lower != 1 for an approximation.

errorG = 1;

L = 1+(p+1)*K*eta^p;
Gamma = sqrt((p+1)*L*eta/(p*K*mu*beta^2));
```

```
A = atan(Gamma);
alpha = 1+ p/L;
delta = A*(1+K*eta^p)*sqrt((p+1)*eta/(p*L*K*mu));

numG = @(x) erf((1-delta*(1-x)/2)/sqrt(t))+erf((1-x)*delta/(2*sqrt(t)));
denG = @(x) (cos(A*x)).^(alpha);

if lower == 1
    % Let's divide the interval [0,1] in n intervals.
    % x1 is the final point of each interval.
    % x2 is the initial point of each interval.
    x2 = 0:1/n:1-1/n;
    % numG and denG are both decreasing in [0,1]
    G = 10;
    for x = x2
        x1 = x + 1/n;
        Gx = (Gamma^2+1)^(-alpha/2)*numG(x1)/denG(x);
        if Gx < G
          G = Gx;
          errorG = (Gamma^2+1)^(-alpha/2)*(numG(x) - numG(x1))./denG(x);
        end
    end
else
    x = 0:1/n:1;
    G = (Gamma^2+1)^(-alpha/2)*min(numG(x)./denG(x));
end
```

Script for the examples in Chapter 3, section 6

```
%%%%%% TABLE OF EXAMPLES SECTION 6 for Theorems 2.3 %%%%
clear
p=2;

mu0 = p^p*pi^2/(4*(p+1)^(p+1));

%%% Each vector contains the values [mu; fmax; d; d_0] for each example.
v1 = [0.7;0.8;0.01;8];
v2 = [0.6;0.65;0.05;10];
v3 = [0.5;0.6;0.001;6];
v4 = [0.5;0.5;0.01;7];

V = [v1,v2,v3,v4];
B=[];
error = [];

for v=V
  [tau,beta,K,eta] = explore(v(1),v(2),v(3),v(4),p);
  t = (1-tau^(p+1))/((p+1)*v(2));
  T = 1/((p+1)*(v(1)-mu0));
  [H,errorH] =  H_low_est(t,beta,50000,1,p);
  [Gt0,errorGt0] = G_low_est(t,v(1),beta,K,eta,2000,1,p);
  [GT,errorGT] = G_low_est(T,v(1),beta,K,eta,2000,1,p);
```

```
  St0 = exp(-4*pi^2*t/(v(4)+1)^2)*(1-exp(-v(4)*(v(4)-beta)/t));
  ST = exp(-4*pi^2*T/(v(4)+1)^2)*(1-exp(-v(4)^2/T));
  rho1 = 0.5*((beta-v(3))/beta)^(p+1)*min(ST*GT/(K+eta^(-p)),...
         ... St0*min(H,Gt0)/(K+tau^(-p)));
  rho2 = tau^(p+1)/((p+1)*(T-t)*v(1));
  rho = min(rho1 , rho2);
  B = [B;[v(1),v(2),v(3),v(4),tau,eta,beta,K,GT,H,Gt0,ST,St0,rho2,rho]];
  error = [error;[v(1),v(2),v(3),v(4),errorGt0,errorH,errorGT]];
end
B
error
```