Spectral theory of ultradifferentiable hyperbolic dynamics

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Thèse de doctorat

Discipline : Mathématiques

présentée par

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Théorie spectrale des dynamiques hyperboliques ultradifférentiables

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Soutenue le 9 décembre 2020 devant le jury composé de :

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I'll be waiting in the car
Don't forget the camera
If you wouldn't mind, could you find my yellow shoes ?
I think they are under the stairs

Ray Lamontagne, *To the sea*

Remerciements

Mes remerciements vont en premier lieu à ma directrice de thèse, Viviane Baladi. Elle a su m'enseigner un domaine des mathèmatiques passionant. J'ai beaucoup appris grâce à ses conseils, ses nombreuses relectures et son érudition, mais c'est avant tout pour la liberté qu'elle m'a laissée que je voudrais la remercier.

Je souhaite également remercier Hans Henrik Rugh et Masato Tsujii qui ont accepté de rapporter cette thèse. Hans Henrik m'a le premier initié aux opérateurs de transfert et Masato a donné au CIRM en mai 2019 un mini-cours sur ce sujet qui m'a particulièrement marqué. J'apporte d'autant plus de valeur à leur avis sur mon travail que celui-ci est nourri de leurs enseignements.

Mes remerciements vont ensuite aux autres membres du jury de cette thèse : Nalini Anantharaman, Frédéric Naud, Stéphane Nonnenmacher et Maciej Zworski.

Le point de départ de cette thèse est une question formulée par Semyon Dyatlov et Maciej Zworski. Les discussions que j'ai eu avec eux ont toujours été très enrichissantes, ma compréhension du sujet serait bien moindre sans celles-ci. Les conseils de Maciej ont également été fort utiles lorsque nous nous sommes intéressés avec Yannick Guedes Bonthonneau aux travaux de Helffer et Sjöstrand.

À ce propos, il est plus que temps de remercier Yannick. Lorsque que je vois aujourd'hui le résultat de notre travail, je comprends à quel point tu avais raison de ne pas laisser tes exigences céder à mon impatience.

Sébastien Gouëzel a un jour suggéré que la notion de régularité Gevrey pourrait être pertinente pour traiter des problèmes auxquels cette thèse s'intéresse. Sans cette suggestion, le contenu de ce manuscrit serait certainement très différent.

Je voudrais également remercier toutes les personnes qui m'ont offert l'opportunité de discuter de mathématiques lors de conférences, séminaires ou autres groupes de travail. Ma gratitude va tout particulièrement à la communauté microlocale travaillant sur les résonances de Ruelle, qui a su m'accueillir en son sein malgré une culture mathématique différente. Je pense notamment à Colin Guillarmou et aux membres du groupe de travail sur la théorie du scattering à l'université d'Orsay et en ligne lors du premier semestre 2020 (Thibault, Yann, Mihajlo, Benjamin, Victor, Nir, et même Hugo).

Tant qu'il est question de lui, je remercie Yann pour les conversations téléphoniques sur les familles à paramètre de projecteurs. Leur retour sempiternel a quelque chose de rassurant dans un monde au futur incertain.

Il me faut également remercier Nicolas Bergeron : j'ai beaucoup apprécié d'enseigner le sujet fascinant qu'est l'analyse complexe avec lui.

Mes pensées vont ensuite à toutes les personnes avec qui j'ai pu me lier d'amitié à l'IMJ-PRG puis au LPSM, et qui ont contribué à une ambiance de travail agréable. Je pense bien sûr en premier lieu à Vincent et Jean-Michel, et aux samedis après-midi passés à regarder *Survivor*. Il me semble important de remercier Hugo pour son accueil bienveillant dans le couloir des doctorants de l'IMJ. Enfin, je voudrais exprimer ma gratitude envers tous les doctorants avec qui j'ai pu partager un bureau. Je tiens en particulier à préciser à Xavier que je ne lui tiens pas rigueur d'avoir voulu faire croire à ma mort.

Il y a quelques personnes qui ne sont pas directement lié au cadre professionel qu'il me semble raisonnable de remercier. Tout d'abord, les Arcueillais.e.s (Gwenaël, Pierre-Louis, Alice et Jean) et tou.te.s les ami.e.s qui ont pu passer par chez nous. À sa demande, je remercie Louise pour « toutes [s]es contributions à [m]a thèse ». Je remercie Vincent pour son compagnonnage cinématographique.

Enfin, je remercie mes parents et mes frères pour leur soutien inconditionnel.

Résumé

Afin d'étudier la validité d'une formule de trace pour les flots d'Anosov $\mathcal{C}^\infty$ proposée par Dyatlov et Zworski, nous développons des outils qui permettent de comprendre le spectre de Ruelle des dynamiques hyperboliques infiniment dérivables. Un rôle central est joué dans cette étude par la notion d'ultradifférentiabilité (principalement *via* le langage des classes de Denjoy–Carleman). On donne ainsi un analogue ultradifférentiable des méthodes développées originellement par Ruelle, Rugh et Fried (basées sur les résultats de Grothendieck sur les opérateurs nucléaires) pour étudier les dynamiques hyperboliques analytiques. En particulier, une analyse détaillée des opérateurs de transfert associés aux applications dilatantes du cercle ultradifférentiables est menée. La formule de trace est ensuite démontrée pour une grande classe de flots d'Anosov ultradifférentiables. Enfin, une transformée de FBI analytique est utilisée pour prouver que l'ordre du déterminant dynamique associé à un flot d'Anosov de régularité Gevrey est fini.

Mots-clés

Dynamiques hyperboliques, spectre de Ruelle, formule de trace, déterminant dynamique, ultradifférentiabilité.

Abstract

In order to study the validity of a trace formula for $\mathcal{C}^\infty$ Anosov flows proposed by Dyatlov and Zworski, we develop tools that allow us to investigate the Ruelle spectrum of infinitely differentiable hyperbolic dynamics. The notion of ultradifferentiability (and in particular the language of Denjoy–Carleman classes) plays a central role in our study. We give an ultradifferentiable analogue of the methods originally developed by Ruelle, Rugh and Fried (based on Grothendieck's results on nuclear operators) to study analytic hyperbolic dynamics. In particular, a detailed analysis of transfer operators associated to ultradifferentiable expanding maps of the circle is performed. The trace formula is then proved for a large class of ultradifferentiable Anosov flow. Finally, an analytic FBI transform is used in order to establish that the order of the dynamical determinant associated to an Anosov flow of Gevrey regularity is finite.

Keywords

Hyperbolic dynamics, Ruelle spectrum, trace formula, dynamical determinant, ultradifferentiability.

Contents

Introduction

The interest in hyperbolic dynamics goes back at least to the work of Hadamard [Had98] on the geodesic flow on negatively curved surfaces. An important milestone in the history of the study of such systems is the definition by Anosov [Ano67] of the class of flows that now bear his name. The introduction of this notion by Anosov was motivated by the study of the dynamical properties of the geodesic flow on the unit tangent bundle of a Riemannian manifold with negative, *a priori* non-constant, sectional curvature. The seminal work of Anosov was the starting point of a systematic study of *Anosov flows*, and of their discrete-time counter parts, *Anosov diffeomorphisms*. This line of research has been very active since then, and, while many questions remain unanswered, the understanding of hyperbolic dynamics has greatly improved since the work of Anosov, due to the development of various tools to study those systems: Markov partitions, specification property, coupling arguments, Young towers, etc.

In this thesis, we focus on a particular way to deal with hyperbolic dynamical systems: the so-called functional approach. The basic idea behind this approach is simple: in order to study a dynamical system T, one may study the associated composition operator (also called *Koopman operator*): $v \mapsto v \circ T$. In particular, we expect that the spectral theory of the Koopman operator contains relevant information concerning the statistical properties of the system T. Koopman operator technique goes back to the work of Koopman and Von Neumann in the early 30's. However, in the particular case of uniformly hyperbolic systems, the work of Ruelle [Rue68, Rue78] plays a founding role that can hardly be overestimated.

We will see that the composition operator associated to a uniformly hyperbolic dynamical system may be used to define a relevant notion of spectrum that enables to describe the statistical properties of this system: the *Ruelle* or *Ruelle–Pollicott* spectrum. The main focus of this thesis is a fine property of this spectrum: a trace formula (TFF) for $\mathcal{C}^\infty$ Anosov flows conjectured by Dyatlov and Zworski in [DZ16]. We will develop the idea that the relevant notion to study this trace formula and related questions is the notion of *ultradifferentiability*, that is we will study hyperbolic dynamical systems that belong to classes of regularity that are intermediate between $\mathcal{C}^\infty$ and analytic. This approach originated in the suggestion by Sébastien Gouëzel that something could be said about trace formulae for *Gevrey* hyperbolic systems, and we give indeed in the last chapter of this thesis a detailed study of the spectral theory for Gevrey Anosov flows. However, we consider in this thesis much more general classes of regularity than Gevrey, using the language of *Denjoy–Carleman classes*. We give an analogue in this setting of the methods initiated by Ruelle [Rue76], Rugh [Rug92, Rug96] and Fried [Fri95] to study analytic hyperbolic dynamics. Their works was based on the use of strong functional analytic tools available in the real-analytic category, relying in particular on the theory of nuclear operators developed by Grothendieck [Gro55]. We will see that, going to larger classes of regularity, weaker versions of these tools may still be constructed. In particular, the trace formula (TFF) will be deduced for Anosov flow that satisfy a certain ultradifferentiability condition, while we do not expect (TFF) to hold for all $\mathcal{C}^\infty$ Anosov flows.

In the remainder of this introduction, we will recall needed facts about the functional approach to statistical properties of hyperbolic dynamical systems. After that, we will describe our main results and propose further lines of work. Most of the results from this thesis may be found in the articles [Jéz20a] (published in Journal of Spectral Theory), [Jéz19a] (submitted for publication),[Jéz20b] (published in Ergodic Theory and Dynamical Systems) and [BJ20] (submitted for publication).

Statistical properties of hyperbolic systems

Let us recall that a dynamical systems is said to be *hyperbolic* if it is expanding in a direction (the *unstable* direction) and contracting in a supplementary direction (the *stable* direction). Hyperbolic systems are notoriously *chaotic*: the long-time asymptotic of an orbit is highly sensitive to initial conditions. While the systems that we study in this thesis are deterministic, they tend to behave like random systems on large scales of times. The dynamics of a contracting map is well-known and far from chaotic. Consequently, the source of chaos in hyperbolic systems should be looked for in the unstable direction, and it seems natural to try to understand first statistical properties for expanding maps.

Statistical properties for expanding maps of the circle

Expanding maps of the circle are the simplest examples of hyperbolic systems that we consider in this thesis. They are indeed hyperbolic systems, but with trivial stable direction.

Definition 1 (Expanding map of the circle)**.** We say that a $\mathcal{C}^1$ map T from $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to itself is *expanding* if there is $\lambda > 1$ such that, for every $x \in \mathbb{S}^1$, we have $|T'(x)| \geq \lambda$. We say that λ is a *dilation constant* for T.

We are interested in these systems precisely because of their simplicity. Indeed, we will use expanding maps of the circle as a toy model in §2.2 to illustrate our methods to study transfer operators associated to ultradifferentiable maps. Considering expanding maps, we do not need to worry about the geometry of the stable and unstable directions, and we can consequently focus on how the regularity of the map and the dilatation contribute to produce rich statistical properties (and associated spectral theory). In order to understand how a chaotic behavior can arise from a deterministic system, it is interesting to consider the most basic example of expanding map.

Example 1. The *doubling map* $x \mapsto 2x \bmod 1$ is an expanding map of the circle.

If x and y are two points on $\mathbb{S}^1$ at distance $\sim 2^{-n}$, then after n iterations of the doubling map, the images of the points x and y are at distance ~ 1. If we were considering the doubling map on $\mathbb{R}$, the images of x and y would keep diverging under further iterations. However, since we are looking at the doubling map on the quotient $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, it is not possible for their images to keep diverging after n iterations since their distance has become similar to the diameter of the circle. Actually, after these first n iterations, the relative position of orbits of x and y could be almost anything: loosely speaking, they became independent.

These considerations suggest to apply methods from probability theory to study expanding maps – and more generally hyperbolic systems. The spectral theory of the transition matrix plays a key role in the study of Markov chains. When studying an expanding map T, the analogue of the transition matrix is the already mentioned *Koopman operator* $v \mapsto v \circ T$ acting on some space of functions or distributions on the circle (the choice of the space on which the Koopman operator acts is in fact of the utmost importance, as we will see later). In the particular case of an expanding map of the circle T, it is more convenient to consider its adjoint, the *transfer operator* or *Ruelle–Perron–Frobenius* operator:

$$\mathcal{L} = \mathcal{L}_T : v \mapsto \sum_{Ty=x} \frac{1}{|T'(y)|} v(y). \tag{1}$$

An advantage of the functional approach based on the study of the transfer operator is that we replaced a potentially complicated non-linear dynamics by a linear one: the iterations of the linear operator $\mathcal{L}$. It is natural to consider that the long-time asymptotics of such a linear system should be ruled by the spectral theory of the transfer operator. The price to pay for this simplification is to replace the finite-dimensional compact manifold $\mathbb{S}^1$ by an infinite-dimensional vector space, but we hope that the spectral properties of the transfer operator are reminiscent in some sense of the finite dimension. The best situation possible would be for the operator $\mathcal{L}$ to be compact. However, the operator $\mathcal{L}$ acting on $L^2\left(\mathbb{S}^1\right)$ or $\mathcal{C}^0\left(\mathbb{S}^1\right)$ is *not* compact.

This situation may be improved by changing the space on which $\mathcal{L}$ acts. The most comfortable situation is when the expanding map T is real-analytic. Indeed, it has been noticed by Ruelle [Rue76] that, in that case, the operator $\mathcal{L}$ acting on a space of holomorphic functions on a complex neighbourhood of $\mathbb{S}^1$ is not only compact but even *nuclear of order* 0. This very powerful property allowed him to use the theory developed by

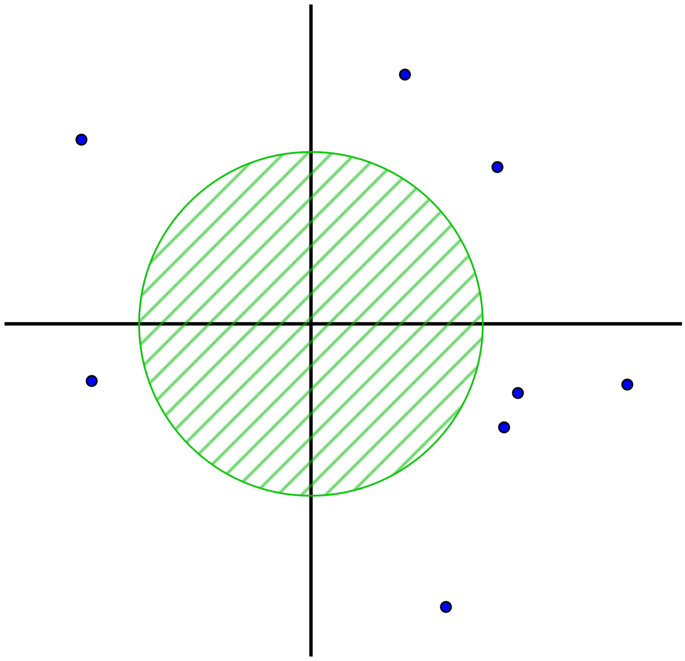

Figure 1: Example of spectrum of a quasi-compact operator. The radius of the dashed disc is the essential spectral radius. The blue dots are eigenvalues of finite multiplicities.

Grothendieck in [Gro55] to study the spectral theory for $\mathcal{L}$. To put it more concretely, due to Cauchy's Formula, the operator $\mathcal{L}$ acting on a space of holomorphic functions is morally an operator with smooth kernel (and hence a compact operator). Analytic expanding maps of the circle have been widely studied using refinements of this method recently [SBJ17, SBJ13, BJ08, Nau12, BJS17, BN19].

When T is only $\mathcal{C}^{1+k}$ for some $k > 0$, the example of real-analytic maps suggest to make $\mathcal{L}$ acts on a space of functions as smooth as possible. However, even acting on the space $\mathcal{C}^k(\mathbb{S}^1)$ of $\mathcal{C}^k$ functions from $\mathbb{S}^1$ to $\mathbb{C}$, the operator $\mathcal{L}$ is not compact (see for instance [CL99, Theorem 8.53] or [GL03]). In order to bypass this difficulty, one may introduce the notions of *essential spectral radius* and *quasi-compactness*.

Definition 2. Let $\mathcal{B}$ be a Banach space and $\mathcal{L}$ be a bounded linear operator from $\mathcal{B}$ to itself. The essential spectral radius $r_{\text{ess}}(\mathcal{L})$ of $\mathcal{L}$ is the infimum of the r's such that the intersection of the spectrum $\sigma(\mathcal{L})$ with $\{z \in \mathbb{C} : |z| > r\}$ is made of a finite number of eigenvalues with finite algebraic multiplicities. We say that $\mathcal{L}$ is quasi-compact if the essential spectral radius of $\mathcal{L}$ is strictly smaller than its spectral radius $r(\mathcal{L})$ – see Figure 1.

Using these notions, one can prove:

Theorem 1 ([Rue89]). *Let $k > 0$ and T be a $\mathcal{C}^{1+k}$ expanding map of the circle. Then, the transfer operator* (1) *induces a bounded operator $\mathcal{L}$ on the space $\mathcal{C}^k(\mathbb{S}^1)$. Moreover,* *the spectral radius of $\mathcal{L}$ is 1 and its essential spectral radius is less than λ^{-k} where λ is a dilation constant for T. In particular, $\mathcal{L}$ is quasi-compact.*

One of the modern proofs of this theorem is based on a so-called Lasota–Yorke inequality[1] combined with an argument due to Hennion [Hen93, Corollaire 1] based on Nussbaum's formula for the essential spectral radius [Nus70]. The intuition behind this proof is very simple: the transfer operator $\mathcal{L}$ acts like a contraction of factor λ^{-k} on the derivative of order k of a $\mathcal{C}^k$ function, while the action of the transfer operator on the other derivatives may be considered as a compact perturbation – that may consequently impact the spectral radius of $\mathcal{L}$ but not its essential spectral radius. This proof is in fact very general and can be adapted to prove that $\mathcal{L}$ is quasi-compact on any reasonable space of regular functions[2]. Here, the word regular can be understood in a very loose sense, including even some kind of discontinuities (for instance, the transfer operator $\mathcal{L}$ may be proved to be quasi-compact when acting on the space of functions with bounded variations on $\mathbb{S}^1$). See for instance [Bal18, Part I] and references therein for a comprehensive treatment of this topic.

From the topological mixing of T, we get further information on the spectrum of the operator $\mathcal{L}$ acting on $\mathcal{C}^k\left(\mathbb{S}^1\right)$. We find indeed that 1 is the only eigenvalue of $\mathcal{L}$ on the unit circle and that its algebraic multiplicity is 1. Moreover, $\mathcal{L}$ has an eigenvector ρ associated to the eigenvalue 1 which is everywhere positive. If we normalize ρ so that its mean (for the Lebesgue measure) is 1, then the probability measure $\mu = \rho \mathrm{d}x$ on $\mathbb{S}^1$ is invariant by the action of T – that is we have $\mu\left(T^{-1}A\right) = \mu\left(A\right)$ for every Borel subset A of $\mathbb{S}^1$. Moreover, it follows from the spectral decomposition for $\mathcal{L}$ that T is *exponentially mixing* for μ: if $f, g : \mathbb{S}^1 \to \mathbb{C}$ are Hölder functions then

$$\int_{\mathbb{S}^1} f \circ T^n . g \mathrm{d}\mu \underset{n \to +\infty}{\to} \left(\int_{\mathbb{S}^1} f . \mathrm{d}\mu\right)\left(\int_{\mathbb{S}^1} g . \mathrm{d}\mu\right) \tag{2}$$

at an exponential rate that only depends on f and g through their Hölder exponent – see [Rue89] for a proof of this fact. Going slightly further into the study of the transfer operator $\mathcal{L}$, we may establish even stronger statistical properties for the expanding map T such as the *central limit theorem* and the *almost sure invariance principle* (see for instance [HK82]).

Hence, considerations about the peripheral spectrum of the operator $\mathcal{L}$ led to a very fine understanding of statistical properties for the map T. One can go further by considering the deeper part of the spectrum. In particular, it is possible to give a finer asymptotics for the correlations than (2) in terms of spectral data for the operator $\mathcal{L}$. Indeed, if the

[1] Since their introduction by Lasota and Yorke to study the existence of absolutely continuous invariant measures for piecewise expanding map of the interval [LY73], Lasota–Yorke inequalities have become a very common tool in the study of statistical properties for hyperbolic dynamical systems.

[2] This is why we consider the transfer operator instead of its adjoint, the Koopman operator. Indeed, in order to make the Koopman operator quasi-compact, one has to make it act on a space of distributions of negative order.

expanding map T is $\mathcal{C}^{k+1}$ with dilation constant λ, and f, g are $\mathcal{C}^k$ observables, we can write for any $r > \lambda^{-k}$ the asymptotic expansion [Rue89, Proposition 5.3]

$$\int_{\mathbb{S}^1} f \circ T^n . g \mathrm{d}\mu \underset{n \to +\infty}{=} \sum_{\substack{z \in \sigma_{\mathcal{C}^k}(\mathcal{L}) \\ |z| > r}} \sum_{j=0}^{m(z)-1} c_{j,z}(f,g) n^j z^n + \mathcal{O}\left(r^n\right). \tag{3}$$

Here, $\sigma_{\mathcal{C}^k}(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$ acting on $\mathcal{C}^k$, and if z is an eigenvalue for $\mathcal{L}$, its multiplicity is denoted by $m(z)$. The $c_{j,z}$'s are continuous bilinear forms on $\mathcal{C}^k(\mathbb{S}^1)$ (the value of $c_{0,1}$ is given by (2)). Notice that, when T is $\mathcal{C}^\infty$, we get an asymptotic at any geometric rate r for the correlations of $\mathcal{C}^\infty$ functions. We see here the importance of the discrete spectrum of the operator $\mathcal{L}$, also called *Ruelle spectrum* or *Pollicott–Ruelle spectrum*.

In this thesis, we are mostly interested in infinitely differentiable hyperbolic dynamics. If T is a $\mathcal{C}^\infty$ expanding map of the circle, then Theorem 1 may be applied for any $k > 0$, making the essential spectral radius of $\mathcal{L}$ arbitrarily small. It happens that in that case the discrete spectrum of $\mathcal{L}$ acting on different spaces are coherent in the following sense: if $z \in \mathbb{C}$ belongs to the spectrum of $\mathcal{L}$ acting on $\mathcal{C}^k$ and $|z| > \max(r_{\mathrm{ess}}(\mathcal{L}_{|_{\mathcal{C}^k}}), r_{\mathrm{ess}}(\mathcal{L}_{|_{\mathcal{C}^{k'}}}))$, then z also belongs to the spectrum of $\mathcal{L}$ acting on $\mathcal{C}^{k'}$, and the associated generalized eigenspaces coincide (see Appendix B and in particular Lemma B.1 and Example B.2). Consequently, letting k go to infinity, we define a discrete spectrum for $\mathcal{L}$ that we will call *Ruelle spectrum.*

Definition 3 (Ruelle spectrum for expanding map)**.** Let T be a $\mathcal{C}^\infty$ expanding map[3] of the circle. We say that $z \in \mathbb{C}$ is a *Ruelle resonance* of multiplicity $m \in \mathbb{N}^*$ for T (or $\mathcal{L}$) if, for every k large enough, z is an eigenvalue of multiplicity m of the transfer operator (1) acting on $\mathcal{C}^k\left(\mathbb{S}^1\right)$. The associated generalized eigenvectors are called the *resonant states* for T or $\mathcal{L}$. The set of Ruelle resonances for $\mathcal{L}$ is called its Ruelle spectrum and is denoted by $\sigma_{\mathrm{R}}\left(\mathcal{L}\right)$.

In view of the importance of the Ruelle spectrum when investigating the statistical properties of T, it is legitimate to wonder if there is any efficient way to compute it. Following Ruelle [Rue76], one may tackle this question by introducing the *dynamical determinant* defined for $z \in \mathbb{C}$ small enough[4] by

$$d(z) = d_T(z) \coloneqq \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} \mathrm{tr}^\flat\left(\mathcal{L}^n\right) z^n\right), \tag{4}$$

[3] We are mainly interested in this thesis in dynamics that are *at least* $\mathcal{C}^\infty$, so that we only defined the Ruelle spectrum in that case. See [Bal18, Definition 1.1] for a more general definition.

[4] The convergence of the right-hand side in (4) for z small enough is ensured by classical bounds on the number of periodic orbits for expanding maps, see for instance [KH95, Theorem 2.46].

where the "flat trace" of the operator $\mathcal{L}^n$ is defined for $n \in \mathbb{N}^*$ by

$$\operatorname{tr}^\flat(\mathcal{L}^n) := \sum_{T^n x = x} \frac{1}{\left|(T^n)'(x) - 1\right|}. \tag{5}$$

Here, we take (5) as the definition of the flat trace of $\mathcal{L}^n$. This expression is motivated by a formal integration of the Schwartz kernel of $\mathcal{L}^n$ over the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$. However, it is possible to make this computation rigorous and then see (5) as a consequence of a general definition of the flat trace using distributional considerations[5]. The definition (4) for the dynamical determinant is motivated by the following computation, valid for any square matrix A and $z \in \mathbb{C}$ small enough:

$$\begin{aligned}\det(I - zA) &= \det(\exp(\ln(I - zA))) = \exp(\operatorname{tr}(\ln(I - zA))) \\ &= \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} \operatorname{tr}(A^n) z^n\right).\end{aligned}$$

Hence, formally we have "$d(z) = \det(I - z\mathcal{L})$".

We already mentioned that when T is real-analytic then the operator $\mathcal{L}$ acting on a space of holomorphic functions is nuclear of order 0. It implies in particular that $\mathcal{L}$ has a well-defined Fredholm determinant $\det(I - z\mathcal{L})$ and it happens that this determinant is given for z small enough by the expression (4). Consequently, the dynamical determinant $d(z)$ has a holomorphic extension to $\mathbb{C}$ whose zeros are the inverses of the Ruelle resonances for[6] $\mathcal{L}$. This fact can be generalized to $\mathcal{C}^\infty$ expanding map:

Theorem 2 ([Rue90]). *Let T be a $\mathcal{C}^\infty$ expanding map of the circle. Then the dynamical determinant $d(z)$ has a holomorphic extension to $\mathbb{C}$ whose zeros are exactly the inverses of the Ruelle resonances for T (counted with multiplicities).*

The proof of Theorem 2 is based on a precise analysis of the action of the transfer operator $\mathcal{L}$ on the spaces $\mathcal{C}^k$'s for $k > 0$. When T is real-analytic, the structure of nuclear operator for $\mathcal{L}$ implies much stronger results than Theorem 2 in that case. For instance, Ruelle [Rue76] proves the following bound on the growth of the dynamical determinant for analytic expanding maps of the circle: there is $C > 0$ such that, for every $z \in \mathbb{C}$, we have

$$|d(z)| \leq C \exp\left(C (\log(1 + |z|))^2\right). \tag{6}$$

The estimate (6) and Jensen's formula [Boa54, 1.2.1 p.2] implies an upper bound on the number of Ruelle resonances, that has been proven generically sharp by Bandtlow and

[5]Basically, since the multipliers associated to the fixed points of T^n are different from 1, we can check that it makes sense to integrate the Schwartz kernel of $\mathcal{L}^n$ on the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ using the general theory from [Hör03, §8.2]. Taking this integral as a definition of the flat trace, a computation yields (5).

[6]It follows from Lemma B.1 that if $\mathcal{L}$ is compact when acting on a reasonable space of holomorphic functions then its non-zero spectrum on that space coincides with its Ruelle spectrum.

Naud [BN19]. Another features of nuclear operator of order 0 is that they have a well-defined trace that coincides with the sum of their non-zero eigenvalues according to the work of Grothendieck [Gro55, Corollaire 4 p.18 chap.II] (working on a Hilbert space, one can just use Lidskii's Trace Theorem). It happens that, when T is real-analytic, for $n \in \mathbb{N}^*$, the trace of $\mathcal{L}^n$ coincides with its flat trace, so that we have the following *trace formula*:

$$\operatorname{tr}^\flat(\mathcal{L}^n) = \sum_{\lambda \in \sigma_{\mathrm{R}}(\mathcal{L})} \lambda^n. \tag{TFM}$$

This formula is the natural analogue for expanding maps of the trace formula (TFF) proposed by Dyatlov and Zworski for $\mathcal{C}^\infty$ Anosov flows. While the holomorphic extension to $\mathbb{C}$ of the dynamical determinant (4) remains valid for any $\mathcal{C}^\infty$ expanding map of the circle, it is not clear whether the same is true or not for the trace formula (TFM) or the bound (6). Notice for instance that there is no reason *a priori* for the right-hand side of (TFM) to converge, since no general bound on the number of Ruelle resonances for a $\mathcal{C}^\infty$ expanding map is known (and we no not expect that there is any). We will study these questions in Chapters 1 and 2.

Statistical properties for Anosov diffeomorphisms

Let us now explain how the picture from the last section is modified when we replace the expanding map T by a hyperbolic diffeomorphism, still denoted by T. For the sake of simplicity, we will assume in this introduction that T is a *transitive Anosov diffeomorphism.* We will need at some point to consider more general hyperbolic diffeomorphisms (including in particular Smale's horseshoe [Sma67, §I.5]), but let us ignore this technical subtleties for now. We start by recalling the definition of an Anosov diffeomorphism.

Definition 4. Let M be a compact manifold and $T : M \to M$ be a $\mathcal{C}^1$ diffeomorphism. We say that T is an Anosov diffeomorphism if, for every $x \in M$ there is a splitting of the tangent space

$$T_x M = E_x^u \oplus E_x^s,$$

and there are constants $C > 0, \lambda > 1$ and a smooth Riemannian metric on M such that

(i) for every $x \in M$ and $\sigma \in \{s, u\}$, we have $DT(x)(E_x^\sigma) = E_{Tx}^\sigma$;

(ii) for every $x \in M, v \in E_x^u$ and $n \in \mathbb{N}$ we have $|DT^{-n}(x)v| \leq C\lambda^{-n}|v|$;

(iii) for every $x \in M, v \in E_x^s$ and $n \in \mathbb{N}$ we have $|DT^n(x)v| \leq C\lambda^{-n}|v|$.

Example 2. The most classical examples of Anosov diffeomorphism are the so-called cat maps. These are the maps on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by matrices $A \in SL(2, \mathbb{Z})$ with no eigenvalue of modulus 1.

Remark 1. We did not make any regularity assumption on the stable and unstable directions (E^u and E^s) in Definition 4. It is a well-known fact though that the stable and unstable directions of a smooth uniformly hyperbolic dynamical systems are automatically Hölder-continuous, and that they integrate into Hölder foliations (called the stable and unstable foliations). However, these directions and the associated foliations do not satisfy *a priori* any better regularity hypothesis. This is an important feature of hyperbolic systems since the regularity of the foliations limit the applications of certain tools to study hyperbolic systems. One of the main advantages of the functional approach is that it bypasses this limitation (see Remark 2 for more details). The same remark applies to Anosov flows defined below (see Definition 6).

If T is a transitive Anosov diffeomorphism, it is convenient to introduce a smooth weight $g : M \to \mathbb{C}$ and then define the associated *weighted Koopman operator*[7] by

$$\mathcal{L} = \mathcal{L}_{T,g} : v \mapsto g.v \circ T. \tag{7}$$

In the invertible case, the Koopman operator and its adjoint, the transfer operator, are very much alike, so that there is no particular reason to consider one or the other.

As in the case of expanding maps, one wants to find a Banach space on which $\mathcal{L}$ defines a quasi-compact operator – and $L^2(M)$ or $\mathcal{C}^0(M)$ do not fulfill that requirement. However, the situation is slightly more complicated here due to the coexistence of the stable and unstable directions. In the expanding case, we saw that composition by a contracting map had good properties (such as quasi-compactness) as an operator acting on a space of regular functions. By duality considerations, this fact suggests that the operator of composition by an expanding map should have a good behavior on a space of distributions of negative order. Hence, to achieve quasi-compactness for the operator (7), we are looking for a space of distributions that are *smooth in the stable direction* and *dual of smooth in the unstable direction*. Such a space is now called a *space of anisotropic distributions* (see [Bal17, Dem18] for surveys on this topic).

The first appearance of spaces of anisotropic distributions in the dynamic literature is often considered to be in [BKL02]. However, this notion was already implicitly present in previous work by Rugh [Rug92, Rug96], Fried [Fri95] and Kitaev [Kit99b, Kit99a]. In [BKL02], a space on which the transfer operator (the adjoint of (7)) associated to a $\mathcal{C}^3$ Anosov map T is quasi-compact is constructed – no weight is considered though, which amounts to take $g = 1$ in (7). However, this construction is restricted by the regularity of the stable and unstable directions of T (see Remark 1), so that, even if T is $\mathcal{C}^\infty$, it does not provide a scale of spaces on which the essential spectral radius of the transfer operator is arbitrarily small – as it was the case for expanding maps in Theorem 1.

[7] We will also denote this operator by $\mathcal{L}$. The convention throughout this thesis is that the operator of most interest at some point of the text will always be called $\mathcal{L}$.

After the first construction in [BKL02], plenty of different spaces of anisotropic distributions have been introduced. In [GL06], Gouëzel and Liverani improved the geometric construction from [BKL02], lifting in particular the restriction due to the regularity of the stable and unstable directions. As a consequence, if T is a $\mathcal{C}^\infty$ Anosov map, the essential spectral radius of the transfer operator (without weight) can be made arbitrarily small on the spaces from [BKL02]. This fact is reproved by Baladi in [Bal05] using tools from micro-local analysis under the (very strong) assumption that the stable or unstable foliation of T is $\mathcal{C}^\infty$. The case of general hyperbolic basic sets (rather than Anosov diffeomorphisms, see §1.2.1) and of general weights for the Koopman operator (7) is dealt with by Baladi and Tsujii in [BT07]. As in [Bal05], the construction from [BT07] is based on tools from micro-local analysis. However, the analysis from [Bal05] relies on pseudo-differential calculus while [BT07] rather uses Paley–Littlewood decomposition – a more convenient tool if one wants to study map of finite differentiability. The case of weighted Koopman operators associated to hyperbolic sets can also be tackled by geometric constructions of spaces of anisotropic distributions, see [GL08]. These results have also been translated in the language of semi-classical analysis by Faure, Roy and Sjöstrand [FRS08]. From all these constructions, we have in particular the following result – the analogue of Theorem 1 for Anosov diffeomorphisms.

Theorem 3 ([BT07, GL08]). *Let $T : M \to M$ be a $\mathcal{C}^\infty$ transitive Anosov diffeomorphism and $g : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function. For every $\epsilon > 0$ there is a Banach space $\mathcal{B}$ with the following properties:*

(i) $\mathcal{B}$ is continuously contained in the space $\mathcal{D}'(M)$ of distributions on M;

(ii) the space $\mathcal{C}^\infty(M)$ of $\mathcal{C}^\infty$ functions on M is contained in $\mathcal{B}$, and the inclusion is continuous with dense image;

(iii) the Koopman operator (7) *induces a bounded operator on $\mathcal{B}$ with essential spectral radius less than ϵ.*

Hence, the essential spectral radius of the operator $\mathcal{L}$ defined by (7) can be made arbitrarily small. As in the case of expanding maps of the circle, one may see, using for instance Lemma B.1, that the intersection of the spectrum of $\mathcal{L}$ with $\{z \in \mathbb{C} : |z| > \epsilon\}$ does not depend on the choice of the space $\mathcal{B}$ in Theorem 3. In this context, the notion of Ruelle resonance may be defined as follow.

Definition 5 (Ruelle spectrum for Anosov diffeomorphism). Let $T : M \to M$ be a $\mathcal{C}^\infty$ transitive Anosov diffeomorphism and $g : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function. Let $z \in \mathbb{C}$ and $m \in \mathbb{N}^*$. We say that z is a Ruelle resonance of multiplicity m for (T, g) (or just for the operator $\mathcal{L}$), if there is[8] a Banach space $\mathcal{B}$ satisfying the points (i)–(iii) from Theorem 3

[8] Here, we could replace "there is" by "for any" without changing the definition.

for some $\epsilon \in]0, |z|[$ such that z is an eigenvalue of algebraic multiplicity m for the operator $\mathcal{L}$ defined by (7) acting on $\mathcal{B}$. The associated generalized eigenvectors are called *resonant states* for $\mathcal{L}$. The set of Ruelle resonances is called the Ruelle spectrum and denoted by $\sigma_{\mathrm{R}}(\mathcal{L})$.

As in the case of expanding maps, the understanding of the spectral theory for the Koopman (or transfer) operator gives a lot of information on the statistical properties of T, see for instance [BKL02, GL06, GL08]. In particular, we get a precise asymptotic for the correlations associated to Gibbs states for T, similar to (3) (see [GL08, Theorem 1.2]).

Remark 2. A lot was already known on statistical properties of expanding and hyperbolic maps before the functional approach was developed in the form that we exposed here (based on Theorems 1 and 3). For instance, the existence of an invariant measure absolutely continuous with respect to Lebesgue for smooth expanding maps was proved by Krzyżewski and Szlenk [KS69]. Concerning hyperbolic maps, many results were obtained first by means of Markov partitions and symbolic dynamics[9] (see for instance [Bow70, Rue78]). The main idea of this approach is to code a hyperbolic dynamical systems in order to replace it by a one-dimensional lattice that can then be studied using classical notions from statistical mechanics [Rue68].

However, the approach based on Markov partitions is restricted by the regularity of the coding, which is *a priori* only Hölder (this is due to the low regularity of the stable and unstable foliations, see Remark 1). Hence, it is very unlikely that an asymptotic expansion at any geometric order for correlations of smooth observables such as (3) may be obtained by the method of Markov partitions. Indeed, the size of the remainder in (3) is deeply related with the regularity of both the dynamics and the observables.

We also have a notion of dynamical determinant for the operator $\mathcal{L}$ defined by (7): the associated dynamical determinant $d(z) = d_{T,g}(z)$ may still be defined by (4) for $z \in \mathbb{C}$ small enough[10], but the flat traces of the powers of $\mathcal{L}$ are now defined for $n \in \mathbb{N}^*$ by

$$\operatorname{tr}^\flat(\mathcal{L}^n) := \sum_{T^n x = x} \frac{\prod_{k=0}^{n-1} g\left(T^k x\right)}{\left|\det\left(I - DT^n(x)\right)\right|}. \tag{8}$$

The analogue of Theorem 2 in this context reads:

Theorem 4 ([LT06, BT08])**.** *Let $T : M \to M$ be a $\mathcal{C}^\infty$ transitive Anosov diffeomorphism and $g : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function. Then the dynamical determinant $d(z)$ has a holomorphic extension to $\mathbb{C}$ whose zeros are exactly the inverses of the Ruelle resonances for the weighted Koopman operator $\mathcal{L}$ defined by* (7) *(counted with multiplicities).*

[9] Other tools are available though, such as the *specification property*, a notion developed by Bowen [Bow71, Bow72b, Bow74] that has been fruitfully revisited recently, see for instance [CT13, CT14, CT16].

[10] The convergence for z small enough is still ensured by classical bound on the number of periodic orbits, see for instance [Bow71, Theorem 4.5]

If T is a small real-analytic perturbation of a cat map, then there is a space [FR06] on which the operator $\mathcal{L}$ is nuclear of order 0. As in the case of expanding maps, the dynamical determinant $d(z)$ coincides then with the Fredholm determinant of $\mathcal{L}$, and a bound on the growth of the dynamical determinant follows. For the same reasons as in the case of expanding map, the trace formula (TFM) holds consequently for small real-analytic perturbations of cat maps. This fact has been used by Adam to prove the genericity of the existence of non-trivial resonances among analytic perturbations of a cat map [Ada17]. It follows from results of Rugh [Rug92, Rug96] that the trace formula (TFM) also holds for real-analytic Anosov diffeomorphisms in dimension 2. The question of the validity of (TFM) for more general Anosov diffeomorphisms will be discussed in Chapters 1 and 2. However, notice that, as in the case of expanding maps, there is no reason *a priori* for the right-hand side of (TFM) to converge when T is a general $\mathcal{C}^\infty$ Anosov diffeomorphism.

Statistical properties for Anosov flows

We turn now to the continuous-time analogues of the hyperbolic diffeomorphisms from the previous section: *Anosov flows*. As usual, let us start with a definition and an example.

Definition 6 (Anosov flows)**.** Let M be a $\mathcal{C}^\infty$ compact manifold and $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^1$ flow on M with zero-free generator X. We say that $(\phi_t)_{t\in\mathbb{R}}$ is Anosov if, for every $x \in M$, there is a splitting of the tangent space

$$T_xM = E_x^0 \oplus E_x^s \oplus E_x^u,$$

and there are constants $C > 0, \lambda > 1$ and a smooth Riemannian metric on M such that:

(i) for every $x \in M, t \in \mathbb{R}$ and $\sigma \in \{0, s, u\}$, we have $D\phi_t(x)\left(E_x^\sigma\right) = E_{\phi_t(x)}^\sigma$;

(ii) for every $x \in M$, the space E_x^0 is the span of $X(x)$;

(iii) for every $x \in M, v \in E_x^u$ and $t \in \mathbb{R}_+$ we have $|D\phi_{-t}(x)v| \leq C\lambda^{-t}|v|$;

(iv) for every $x \in M, v \in E_x^s$ and $t \in \mathbb{R}_+$ we have $|D\phi_t(x)v| \leq C\lambda^{-t}|v|$.

Example 3. As mentioned above, the example that motivated the definition of Anosov flows is the geodesic flow on the unit tangent bundle of a compact Riemannian manifold with negative sectional curvature. Other specimens of Anosov flows are obtained by considering suspensions of Anosov diffeomorphism. Notice also that any small $\mathcal{C}^1$ perturbation of an Anosov flow is also Anosov (the same is true for expanding maps and Anosov diffeomorphisms).

Let us consider a $\mathcal{C}^\infty$ Anosov flow $(\phi_t)_{t\in\mathbb{R}}$ on a compact manifold. We denote its generator by X and choose a $\mathcal{C}^\infty$ weight $V : M \to \mathbb{C}$. In this context, the Koopman operator defines a semi-group of operators:

$$\mathcal{L}_t = \mathcal{L}_t^{X,V} : v \mapsto \exp\left(\int_0^t V \circ \phi_\tau \mathrm{d}\tau\right) v \circ \phi_t \tag{9}$$

for $t \geq 0$. Notice that the generator of this semi-group is formally the differential operator $P := X + V$: whenever it makes sense we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}_t v\right) = P\left(\mathcal{L}_t v\right). \tag{10}$$

For instance, if $v \in \mathcal{C}^\infty(M)$ then (10) holds in $\mathcal{C}^\infty(M)$. As in the case of expanding maps or Anosov diffeomorphisms, we replaced the finite-dimensional non-linear dynamics $(\phi_t)_{t\in\mathbb{R}}$ by an infinite-dimensional linear dynamics. However, since $(\phi_t)_{t\in\mathbb{R}}$ is a continuous-time dynamical system, we are now studying the linear ODE (10) rather than the iterations of a single operator. Consequently, we want to understand the spectral theory of the differential operator P, hoping that it would allow us to describe the long-time asymptotic of the solutions of the linear ODE (10).

Once again, the (unbounded) operator P does not have discrete spectrum on $L^2(M)$ or $\mathcal{C}^0(M)$, and we need consequently to find new spaces on which the spectral theory of P is better-behaved – the analogues of the spaces from Theorem 3. Adapting ideas from [BKL02, GL06, GL08], Liverani in [Liv04] and then Butterley and Liverani in [BL07] (with necessary details provided in [BL13]) gave geometric constructions of spaces adapted to Anosov flows. Then, Faure and Sjöstrand adapted in [FS11] the semi-classical approach from [FRS08] to the continuous-time case. A slightly different point of view on this approach is presented in [DZ16] (using radial estimates and propagation of singularities). This construction has then been adapted to some open systems (including Axiom A systems) by Dyatlov and Guillarmou in [DG16, DG18] and to certain Morse–Smale flows by Dang and Rivière in [DR20a, DR20b]. The equivalent of Theorem 3 for Anosov flow reads:

Theorem 5 ([BL07, BL13, FS11])**.** *Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$ Anosov flow on a compact manifold M with generator X and $V : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function*[11]*. Then, for every $A > 0$, there is a Banach space $\mathcal{B}$ such that*

(i) $\mathcal{C}^\infty(M) \subseteq \mathcal{B} \subseteq \mathcal{D}'(M)$, both inclusions are continuous and the first one has dense image;

(ii) $(\mathcal{L}_t)_{t\geq 0}$, defined by (9), induces a strongly continuous semi-group of operator on $\mathcal{B}$ whose generator is $P = X + V$ with its natural domain

$$D(P) = \{u \in \mathcal{B} : Pu \in \mathcal{B}\};$$

[11] Actually, neither [BL07, BL13] nor [FS11] include the case of weighted Koopman operators. However, it is clear for specialists that their methods also apply in this case. The general case is dealt with in [DG16].

(iii) the intersection of the spectrum of P with $\{z \in \mathbb{C} : \operatorname{Re} z > -A\}$ is a discrete set of eigenvalues of finite (algebraic) multiplicity.

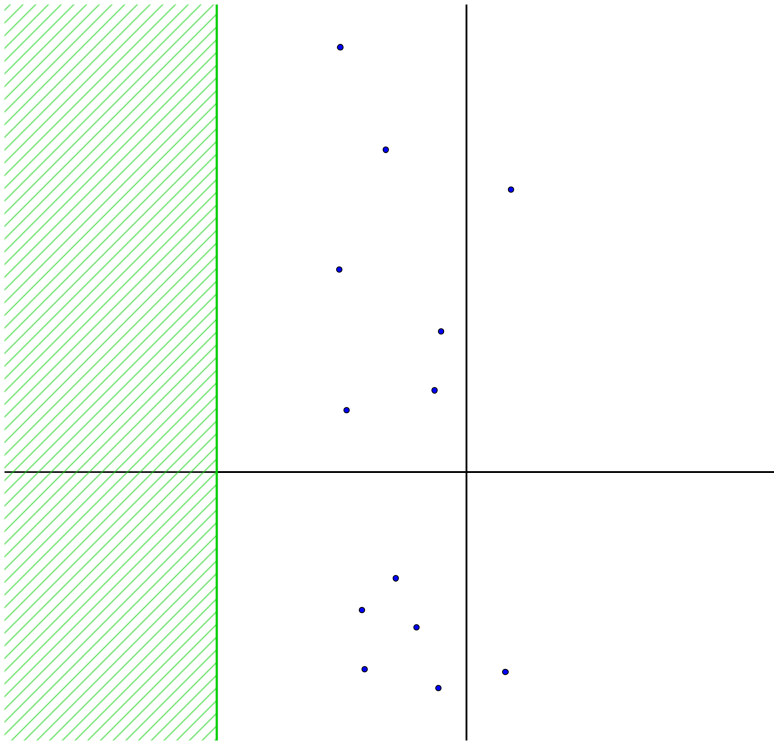

Figure 2: Spectrum of P acting on the space $\mathcal{B}$ from Theorem 5. The abscissa of the green line is $-A$. The blue dots depict eigenvalues of finite multiplicity. Their imaginary parts may be arbitrarily large but their real parts are bounded from above.

The statement of Theorem 5 deserves a little explanation. The point (i) is just a non-triviality condition: it ensures that the space $\mathcal{B}$ is made out of objects that "live" on M. This first condition can be mitigated in order to allow wilder objects in $\mathcal{B}$ than distributions – this fact will be crucial in Chapters 3 and 4. The point (ii) may seem technical but it is very important. The fact that $(\mathcal{L}_t)_{t \geq 0}$ is a strongly continuous semi-group makes it possible to link the spectral theory of P with the long-time asymptotic of $(\mathcal{L}_t)_{t \geq 0}$. It implies in particular that, if the real part of $z \in \mathbb{C}$ is positive and large enough, then the resolvent $(z - P)^{-1}$ of P at z coincides with the Laplace transform of the Koopman operator:

$$R(z) := \int_0^{+\infty} e^{-zt} \mathcal{L}_t \mathrm{d}t \tag{11}$$

where the integral converges in operator norm on $\mathcal{B}$. Finally, the point (iii) implies that the spectrum of P on $\mathcal{B}$ looks as depicted in Figure 2. As in the discrete-time case, the isolated eigenvalues of finite multiplicity on the right part of the complex plane in Figure 2 are intrinsically defined by X and V and called Ruelle resonances (see Lemma B.3). The definition of the Ruelle spectrum in this context is the following:

Definition 7 (Ruelle spectrum for Anosov flow)**.** Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$ Anosov flow on a compact manifold M with generator X and $V : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function. Let $z \in \mathbb{C}$ and $m \in \mathbb{N}^*$. We say that z is a Ruelle resonance for $(\phi_t)_{t\geq 0}$ and V (or just for $P = X + V$) of multiplicity m if there is[12] a Banach space $\mathcal{B}$ satisfying the points (i)-(iii) from Theorem 5 for some $A > -\operatorname{Re} z$ such that z is an eigenvalue of P acting on $\mathcal{B}$ with multiplicity m. The associated generalized eigenvectors are called *resonant states* for P. As usual, the Ruelle spectrum is defined as the set of Ruelle resonances, and is denoted by $\sigma_{\mathrm{R}}(P)$.

Remark 3. The notation $\sigma_{\mathrm{R}}(P)$ for the Ruelle spectrum of P may be slightly misleading: in general the Ruelle spectrum of $-P$ is **not** made of the $-\lambda$'s where λ runs over the Ruelle resonances of P. This is a matter of convention: the Ruelle resonances of $-P$ describe the statistical properties of $(\phi_{-t})_{t\in\mathbb{R}}$ when $t \to +\infty$ rather than those of $(\phi_t)_{t\in\mathbb{R}}$ when $t \to -\infty$, hence a sign flip.

Remark 4. Let us mention that there is another (equivalent) definition of the Ruelle spectrum that is very popular in this context – and maybe slightly easier to manipulate. If the real part of $z \in \mathbb{C}$ is positive and large enough, then formula (11) defines an operator $R(z)$ from $\mathcal{C}^\infty(M)$ to $\mathcal{D}'(M)$. We saw that $R(z)$ coincides with the resolvent of P acting on the space $\mathcal{B}$ from Theorem 5. Consequently, it follows from Theorem 6 that $R(z)$ has a meromorphic continuation to $\mathbb{C}$, as an operator from $\mathcal{C}^\infty(M)$ to $\mathcal{D}'(M)$, with residues of finite rank. The Ruelle resonances of P are then the poles of $R(z)$, and the multiplicity of a resonance is given by the rank of the residue of the corresponding poles of $R(z)$ (the image of this residue is the space of resonant states associated to the resonance).

When considering Anosov flows, the structure of the Ruelle spectrum is slightly more complicated than in the discrete-time case (it is not a bounded subset of $\mathbb{C}$ for instance). As a consequence, the study of the Ruelle spectrum, and hence of the statistical properties for the flow $(\phi_t)_{t\in\mathbb{R}}$, is slightly more involved. For instance, it is much trickier to prove exponential decay of correlations in this context. Consider for example the case $V = 0$ (that corresponds to the SRB measure), in that case 0 is a Ruelle resonance. Assuming that $(\phi_t)_{t\in\mathbb{R}}$ is transitive, then 0 is a simple eigenvalue and there is no other Ruelle resonance with non-negative real part. However, in order to prove exponential decay of correlation we need to establish a *spectral gap* for P: the existence of $A > 0$ such that the only Ruelle resonance for P with real part greater than $-A$ is 0 (some additional technical estimates are required, see [But16]). This is tricky because Ruelle resonances with large imaginary parts could accumulate on the vertical line $i\mathbb{R}$ without touching it.

Due to the complexity and the importance of the problem of spectral gaps for Anosov flows, there are a lot of works on that topic. Most proofs of spectral gaps for Anosov flows

[12] As in Definition 5, we could replace here "there is" by "for any".

are based on some version of Dolgopyat's argument [Dol98]. Liverani gave a functional analytic version of this argument [Liv04] that allowed him to prove exponential decay of correlations for $\mathcal{C}^4$ contact Anosov flows (for the measure induced by the contact form). This result has been reproved in the $\mathcal{C}^\infty$ case by Tsujii [Tsu10, Tsu12]. Exponential decay of correlations for $\mathcal{C}^\infty$ contact Anosov flows can also be seen as a consequence of a general result by Nonnenmacher and Zworski [NZ15] that gives a spectral gap of explicit size. Faure and Tsujii [FT13] went further than the spectral gap and stated a *band structure* for $\mathcal{C}^\infty$ contact Anosov flows. Giulietti, Liverani and Pollicott also proved exponential decay of correlations for the measure of maximal entropy of geodesic flows on Riemannian manifold with negative curvature satisfying a pinching condition [GLP13]. Exponential decay of correlations has been proved recently by Tsujii and Zhang for topologically mixing volume-preserving $\mathcal{C}^\infty$ Anosov flows [TZ20] (based on previous work by Tsujii [Tsu18]). Let us mention that, if the question of the existence of a spectral gap is crucial, there are many others interesting results regarding the distributions of Ruelle resonances, see for instance [FS11, DDZ14, JZ17, FT17].

Understanding the distribution of Ruelle resonances for P is consequently both difficult and very important. As in the discrete-time case, this question can be dealt with by the introduction of a dynamical determinant. Its definition is slightly more involved in that case, it is defined for $\operatorname{Re} z \gg 1$ by[13]

$$d(z) = d_{X,V}(z) \coloneqq \exp\left(-\sum_{\gamma} \frac{T_\gamma^{\#}}{T_\gamma} \frac{e^{\int_\gamma V}}{|\det(I - \mathcal{P}_\gamma)|} e^{-zT_\gamma}\right). \tag{12}$$

The sum in the right-hand side of (12) is over periodic orbits γ for the flow $(\phi_t)_{t\in\mathbb{R}}$. We write T_γ for the length of γ and $T_\gamma^{\#}$ for its primitive length (that is the shortest length of a periodic orbit for $(\phi_t)_{t\in\mathbb{R}}$ with the same image than γ). The integral of V along γ is defined by $\int_\gamma V \coloneqq \int_0^{T_\gamma} V(\phi_t(x))\,\mathrm{d}t$ for any x in the image of γ. Finally, $\mathcal{P}_\gamma$ denotes the linearized Poincaré map associated to γ, defined by $\mathcal{P}_\gamma \coloneqq D\phi_{T_\gamma}(x)\big|_{E_x^u\oplus E_x^s}$. The definition of $\mathcal{P}_\gamma$ depends on the choice of a point x in the image of γ. However, the conjugacy class of $\mathcal{P}_\gamma$ does not, and hence the determinant in the right-hand side of (12) is well-defined. The intuition behind the definition is that, formally, "$d(z) = \det(z - P)$". This heuristic understanding of $d(z)$ will be made rigorous in §4.2. The equivalent of Theorems 2 and 4 in this context is:

Theorem 6 ([GLP13, DZ16]). *Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$ Anosov flow, with generator X, on a compact manifold M. Let $V : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function*[14]*. Then the dynamical determinant*

[13] Once again, classical bounds on the number of periodic orbits justify the convergence for $\operatorname{Re} z \gg 1$, such as Margulis' bound [Mar04, Theorem 5] or the more elementary estimate [DZ16, Lemma 2.2].

[14] Actually, [GLP13, DZ16] do not consider the case of a general weight V. However, it is probably clear to specialists that their methods – at least those from [DZ16] – apply to the case of general weights – and even to the case of vector bundles. Theorem 6 is proven in a much more general setting in [DG16].

$d(z)$ defined by (12) *has a holomorphic extension to $\mathbb{C}$ whose zeros are exactly the Ruelle resonances for $P = X + V$ (counted with multiplicities).*

Remark 5. Dynamical determinants are often outshined by the closely related *Ruelle zeta functions.* The most simple example of Ruelle zeta function associated to an Anosov flow $(\phi_t)_{t\in\mathbb{R}}$ is given for $\operatorname{Re} z \gg 1$ by

$$\zeta(z) = \prod_{\gamma \text{ primitive}} \left(1 - e^{-zT_\gamma}\right), \tag{13}$$

where the product is over primitive periodic orbits of the flow $(\phi_t)_{t\in\mathbb{R}}$. An argument due to Ruelle [Rue76] allows to write the zeta function (13) as an alternated product of dynamical determinants, so that Theorem 6 implies that ζ has a meromorphic extension[15] to $\mathbb{C}$.

As in the discrete-time case, when studying dynamical determinants and zeta functions for Anosov flows, there are functional-analytic tools that are only available in the real-analytic category. These tools have been developed by Ruelle [Rue76] (under the very strong assumption that the stable or unstable foliation of the flow is real-analytic, see also [Fri86]), Rugh [Rug92, Rug96] (in dimension 3 but without any condition on the stable or unstable foliation) and Fried [Fri95] (in any dimension). They do not work directly with the Koopman operator $\mathcal{L}_t$ or its generator P, but rather with a family of auxiliary nuclear operators [Gro55] defined using symbolic dynamics. One of the consequences of their results is that if $(\phi_t)_{t\in\mathbb{R}}$ and V are real-analytic, then the dynamical determinant $d(z)$ has finite order (see Definition 1.2). It is then folklore (see Proposition 1.27) that the finite order for $d(z)$ implies a trace formula for the flow $(\phi_t)_{t\in\mathbb{R}}$ that reads, in the sense of distributions on $\mathbb{R}_+^*$,

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} e^{\lambda t} = \sum_\gamma T_\gamma^\# \frac{e^{\int_\gamma V}}{|\det(I - \mathcal{P}_\gamma)|}\delta_{T_\gamma}. \tag{TFF}$$

This means that for every $\mathcal{C}^\infty$ function $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}_+^*\right)$ compactly supported in $\mathbb{R}_+^*$ we have

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \operatorname{Lap}(\varphi)(-\lambda) = \sum_\gamma T_\gamma^\# \frac{e^{\int_\gamma V}}{|\det(I - \mathcal{P}_\gamma)|}\varphi(T_\gamma), \tag{14}$$

where $\operatorname{Lap}(\varphi)$ denotes the Laplace transform of φ:

$$\operatorname{Lap}(\varphi) : z \mapsto \int_0^{+\infty} e^{-zt}\varphi(t)\mathrm{d}t.$$

The notations on the right-hand side of (TFF) and (14) are the same as in (12) and we

[15] Actually, Ruelle arguments require an orientability condition on the stable and unstable directions of $(\phi_t)_{t\in\mathbb{R}}$. A strategy to overcome this condition may be found in [BS20], see [BT08] for the discrete-time analogue.

recall that $\sigma_{\mathrm{R}}(P)$ is our notation for the Ruelle spectrum of P. The trace formula (TFF) is a continuous-time analogue of (TFM). Indeed, according to Guillemin's trace formula, the right-hand side of (TFF) is a generalized trace for the Koopman operator (9) – see also [DZ16, Appendix B]. The formula (TFF) gives consequently a spectral interpretation for Guillemin's trace formula in the case of real-analytic Anosov flows.

In [DZ16], Dyatlov and Zworski gave a new proof of Theorem 6 (the first proof being in [GLP13]). One of their motivations was to investigate the validity of the trace formula (TFF) for $\mathcal{C}^\infty$ (rather than analytic) Anosov flows. As in the discrete-time case, notice that in the absence on a general upper bound on the number of Ruelle resonances, there is no reason for the left-hand side of (TFF) to converge in any sense. In [JZ17], Jin and Zworski proved a "local" version of (TFF) for $\mathcal{C}^\infty$ Anosov flow and used it to show that, for any $\mathcal{C}^\infty$ Anosov flow, there is a vertical strip that contains an infinite number of Ruelle resonances[16] – a bound that has been sharpened by Naud in an appendix to their paper. This is one of the reasons why trace formulae are interesting: they are one of the only tools available to prove *lower* bounds on the number of resonances. The already mentioned work of Adam [Ada17] also illustrates this idea.

The question asked by Dyatlov and Zworski of the validity of the trace formula (TFF) (and of its discrete-time analogue (TFM)) beyond real-analytic systems is the main thread of this thesis. Let us describe now our approach to this problem.

Main results

Maps: Trace formulae and counter-examples (Chapter 1)

Our first line of approach to the problem of the trace formula (TFF) for $\mathcal{C}^\infty$ Anosov flow was to consider the discrete-time analogue of this question (TFM), where $\mathcal{L}$ is either the transfer operator (1) associated to a $\mathcal{C}^\infty$ expanding map of the circle or a weighted Koopman operator (7) associated to an Anosov diffeomorphism. The advantage of considering a map rather than a flow is that the relationship between trace formulae and dynamical determinants is much simpler in that case:

Proposition 1 (See Theorem 1.3). *The trace formula* (TFM) *holds for n large enough with absolute convergence of the right-hand side if and only if the dynamical determinant* (4) *has finite order.*

We will prove a slightly more precise version of this result (Theorem 1.3) in §1.1. As mentioned above, it remains true in the continuous-time setting that finite order for the associated dynamical determinant (12) implies that the trace formula (TFF) holds, and we shall prove an analogue of Proposition 1 for Anosov flows (see Proposition 1.27). However,

[16] Their argument is written in the case $V = 0$ but it is not hard to see that it still applies as soon as V is real-valued.

the link between these two notions is much more explicit in the discrete-time case. For instance, in order to produce a counter-example to (TFM), one only needs to find a system for which the dynamical determinant d has infinite order.

In order to construct counter-examples to (TFM), we need to consider slightly more general systems than expanding maps or Anosov flows, namely hyperbolic basic sets. Indeed, there is much more flexibility when constructing open systems. The picture described above for transitive Anosov diffeomorphisms adapt without much change to basic hyperbolic sets: we have notions of Ruelle spectrum, flat trace, dynamical determinant, etc (see §1.2.1 for details). In §1.2, we will construct weights for a Smale's horseshoe [Sma67, §I.5] such that the dynamical determinant of the associated Koopman operator is explicit. From this construction, we will deduce for instance the following result:

Proposition 2 (See Corollary 1.19). *Let E be a subset of $\mathbb{N}^*$. Then there are a diffeomorphism $T : \mathbb{S}^4 \to \mathbb{S}^4$ with hyperbolic basic set K and a smooth function $g : \mathbb{S}^4 \to \mathbb{R}_+$, positive on K, such that, for every $n \in \mathbb{N}^*$, the trace formula* (TFM) *holds if and only if $n \in E$ (see* (1.18) *for the definition of the flat trace).*

The relationship between dynamical determinants and trace formulae in the discrete-time case is so explicit that we will construct counter-examples in §1.2 that are much more pathological than those of Proposition 2, see Corollary 1.20.

A question that is deeply related to the problem of trace formulae is the existence of an upper bound on the number of Ruelle resonances. The method that gives Proposition 2 also proves the following result, that asserts that there is no general upper bound on the number of resonances for a hyperbolic diffeomorphism.

Proposition 3 (Proof in §1.2.3). *Let $N_0 : \mathbb{R}_+^* \to \mathbb{R}_+$ be a locally bounded function. Then there are a diffeomorphism $T : \mathbb{S}^4 \to \mathbb{S}^4$ with hyperbolic basic set K and a smooth function $g : \mathbb{S}^4 \to \mathbb{R}_+$, positive[17] on K, such that, if we define for $r \in \mathbb{R}_+^*$*

$$N(r) = \# \left\{ z \in \mathbb{C} : |z| \geq r \text{ and } z \text{ is a Ruelle resonance associated to } (T, g) \right\},$$

then $N_0(r) \underset{r \to 0}{=} o\left(N(r)\right)$.

Denjoy–Carleman classes and compact transfer operator (Chapter 2)

There is a fundamental difference between the trace formulae (TFM) and (TFF) and, for instance, Theorems 2, 4 and 6: the formulae (TFF) and (TFM) involve the entire Ruelle spectrum at once, while Theorems 2, 4 and 6 are usually proved by analyzing the Ruelle spectrum little by little.

[17] The fact that g is positive on K implies that $\mathcal{L}$ may be used to describe the correlations for the Gibbs measure for T associated to the potential $\log g$. Consequently, we construct a Gibbs measure for which there are a lot of terms in the correlations asymptotics.

This fact is essential because the scales of spaces from Theorems 1, 3 and 5 only unveil the Ruelle spectrum gradually. Consequently, they are suited to prove statements like Theorems 2, 4 and 6, but they are not the most convenient tools in order to establish a global formula like (TFF) or (TFM). Hence, we want to construct a single space that unveils the entire Ruelle spectrum. As already mentioned, this had already been done in certain cases, but always for real-analytic dynamics. We will see that this phenomenon is not specific to the real-analytic case, and present tools that we developed to study $\mathcal{C}^\infty$ hyperbolic dynamics.

We will start with the simplest example and explain in Chapter 2 how one can construct a space fitted to study a given $\mathcal{C}^\infty$ expanding map of the circle T and its associated transfer operator (1). In §2.2, we will prove the following theorem, which is in some sense the $\mathcal{C}^\infty$ generalization of what Ruelle [Rue76] did for real-analytic expanding maps.

Theorem 7 (See Theorem 2.9)**.** *Let T be a $\mathcal{C}^\infty$ expanding map of the circle and denote by $\mathcal{L}$ the associated transfer operator* (1)*. Then, there is a Hilbert space $\mathcal{H}$ such that:*

(i) $\mathcal{H}$ is continuously contained in the space of $\mathcal{C}^\infty$ functions from $\mathbb{S}^1$ to $\mathbb{C}$;

(ii) $\mathcal{H}$ contains the trigonometric polynomials on $\mathbb{S}^1$ as a dense subspace;

(iii) $\mathcal{L}$ induces a compact operator on $\mathcal{H}$ whose non-zero spectrum is the Ruelle spectrum of $\mathcal{L}$.

Theorem 7 is not sufficient to prove the trace formula (TFM). Indeed, in order to prove a trace formula, it seems natural to try to make $\mathcal{L}$ trace class rather than compact. An important difference between these two notions is that being trace class is a quantitative statement (on the decay of the singular values of $\mathcal{L}$). Consequently, we need to make the fact that T is $\mathcal{C}^\infty$ quantitative. To do so, we will use the language of *Denjoy–Carleman classes* or *classes of ultradifferentiable functions*.

A short introduction to the topic of Denjoy–Carleman classes is given in §2.1 – the interested reader may refer to the more complete survey [KMR09] and references therein. Let us just say for now that this theory provides a general procedure to produce regularity classes intermediate between $\mathcal{C}^\infty$ and real-analytic. The most famous such classes are maybe the classes of Gevrey functions that have been introduced by Gevrey in his seminal paper [Gev18] to study the regularity of the solutions of certain PDE. Using the language of Denjoy–Carleman classes, we will give in §2.2 a quantitative version, Theorem 2.9, of Theorem 7 – we will bound the singular values of $\mathcal{L}$ acting on $\mathcal{H}$.

When $\mathcal{L}$ acting on $\mathcal{H}$ is trace class, we will see that the trace formula (TFM) holds. In particular, in that case the dynamical determinant $d(z)$ defined by (4) coincides with the Fredholm determinant $\det(I - z\mathcal{L})$ of $\mathcal{L}$ – see Proposition 2.19. When $\mathcal{L}$ is not trace class, we are still able to use the space $\mathcal{H}$ from Theorem 7 to study the dynamical determinant $d(z)$: the *nuclear power decomposition* from [BT08] may be implemented on $\mathcal{H}$. It allows us to rewrite the dynamical determinant $d(z)$ as a particular case of *Weinstein–Aronszajn determinant* (see [Kat66, IV.§6] and references therein).

Theorem 8 (See Proposition 2.18)**.** *Under the assumptions of Theorem 7, we may write* $\mathcal{L} = \mathcal{L}_b + \mathcal{L}_c$*, where* $\mathcal{L}_b$ *is a quasi-nilpotent operator and* $\mathcal{L}_c$ *is a nuclear operator of order* 0 *on* $\mathcal{H}$*. The dynamical determinant* (4) *may then be rewritten as*

$$d(z) = \det\left(I - z\left(I - z\mathcal{L}_b\right)^{-1}\mathcal{L}_c\right).$$

This statement can also be made quantitative using the language of Denjoy–Carleman classes – yielding in particular a bound on the growth of the dynamical determinant $d(z)$ (see Propositions 2.16, 2.17 and 2.18). Notice that a bound on the growth of the dynamical determinant immediately implies an upper bound on the number of Ruelle resonances by Jensen's inequality [Boa54, Theorem 1.2.1].

The proofs of Theorems 7 and 8 are given in §2.2. They are applications of our methods to work with transfer (or Koopman) operators associated to ultradifferentiable hyperbolic dynamics. We will also discuss the implication of Theorem 7 in terms of regularity of the resonant states for $\mathcal{L}$ and of the conjugacy problem (see Corollaries 2.11 and 2.12).

In §2.3, we recall and discuss Theorem 2.27, a result on Koopman operators associated to Gevrey hyperbolic diffeomorphisms that we obtained by similar methods in [Jéz20a] (before Theorems 7 and 8). In order to keep the page count reasonable, we will not give the proof of this result here.

Trace formulae for ultradifferentiable Anosov flows (Chapter 3)

We return to Anosov flows and the question of the validity of the trace formula (TFF) in Chapter 3. Our first idea when trying to prove trace formula (TFF) was to prove that the dynamical determinant (12) has finite order. Heuristic considerations (involving in particular Theorem 2.27, see Remark 2.28) let us think that Gevrey classes of regularity are a natural setting when it comes to finite order of the dynamical determinant. However, when applying the methods based on Paley–Littlewood decompositions exposed in Chapter 2, we were not able to get a good enough control of the Koopman operator (9) for small $t > 0$. Hence, we missed the finite order for the dynamical determinant and needed to find another way to prove the trace formula (TFF). The solution that we found was to consider operators of the form

$$\int_0^{+\infty} \varphi(t)\mathcal{L}_t \mathrm{d}t \tag{15}$$

for a test function $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}_+^*\right)$, instead of the resolvent (11). The intuition here is that (15) is "the Laplace transform of φ evaluated at $-P$" so that, if we can prove that this operator is trace class, we should be able to deduce the instance (14) of the trace formula from Lidskii's Trace Theorem. Indeed, we expect the trace of (15) to be equal to the right-hand side of (14) because of Guillemin's trace formula [Gui77].

The advantage of considering (15) is that, since φ is supported away from 0, we do not need to control the Koopman operator $\mathcal{L}_t$ for t near 0 (compare for instance with the expression (11)). It turned out that this strategy allows not only to retrieve the trace formula (TFF) using Paley–Littlewood like decompositions similar to those of Chapter 2, but also to work with ultradifferentiable classes that are much larger than Gevrey classes. Thus, we obtain in Chapter 3:

Theorem 9 (See Theorem 3.1 and Corollary 3.2)**.** *There is a class of regularity, larger than all Gevrey classes, such that the trace formula* (TFF) *holds for Anosov flow* $(\phi_t)_{t\in\mathbb{R}}$ *and weight* V *that belong to this class of regularity..*

The classes of Anosov flows and weights that are allowed in Theorem 9 will be discussed in §3.1, using the language of Denjoy–Carleman classes. We obtain in Proposition 3.3 an estimate on the number of Ruelle resonances (for the allowed ultradifferentiable flows) that implies that the left hand side of (TFF) defines a distribution on $\mathbb{R}_+^*$.

Finite order of dynamical determinants for Gevrey Anosov flows (Chapter 4)

After we established Theorem 9, the question of the finite order of dynamical determinants associated to Gevrey Anosov flows remained open. With Yannick Guedes Bonthonneau, we applied methods from analytic and Gevrey micro-local analysis to study this problem. The tool that turned out to be the most convenient for us is an analytic FBI transform in the spirit of Helffer and Sjöstrand [HS86, Sjö96]. Similar methods have been used recently by Galkowski and Zworski [GZ19a, GZ19b, GZ20a] to deal with certain PDEs problems in the real-analytic category. Our methods however are not restricted to the real-analytic regularity, but also apply in the Gevrey category. In particular, we finally prove that the order of the dynamical determinant associated to a Gevrey Anosov flow is finite.

Theorem 10 (Proof in §4.2)**.** *Let* $s \geq 1$*. Let* $(\phi_t)_{t\in\mathbb{R}}$ *be an* s*-Gevrey Anosov flow on an* s*-Gevrey manifold. Let* $V : M \to \mathbb{C}$ *be an* s*-Gevrey function. Denote by* $d(z)$ *the associated dynamical determinant* (12)*. Then there is a constant* $C > 0$ *such that, for every* $z \in \mathbb{C}$*, we have*

$$|d(z)| \leq C \exp\left(C |z|^{ns}\right),$$

where n *is the dimension of* M*. In particular,* $d(z)$ *has order less than* ns*.*

Gevrey classes are discussed in §4.1.1. Let us just mention for now that Gevrey classes are nice examples of Denjoy–Carleman classes depending on a parameter $s \in [1, +\infty[$.

When $s = 1$, we retrieve the class of real-analytic regularity, while for $s > 1$ the class of s-Gevrey functions is *non-quasianalytic*: it contains compactly supported functions.

The methods that we developed with Bonthonneau to study Gevrey Anosov flows are exposed in Chapter 4 in an abridged version. Indeed, the proof of Theorem 10 requires technical preparations that are not directly related to our topic. It was thus not possible to present here the full content of [BJ20] (available on the arXiv) without causing an inflation in the length of this thesis. Consequently, we give a short introduction to the FBI transform methods from [BJ20] in §4.1 focusing on the results that are needed for the proof of Theorem 10 – that we give in §4.2. Finally, we will mention other applications of our FBI transform methods to the spectral theory of Gevrey Anosov flows in §4.3.

Content of the appendices

Let us describe briefly the content of the appendices of this thesis. In Appendix A, we explain how the results from §2.2 (in particular Theorems 7 and 8) extend to more general transfer operators than (1). In Appendix B, we give the technical results that prove that the Ruelle resonances are intrinsically defined. In Appendix C, we establish a "Hadamard-like" factorization for the dynamical determinant (12) under the hypotheses of Theorem 9. In Appendix D, we discuss the optimality of the regularity hypothesis in Theorem 9. In Appendix E, we detail a heuristic computation that suggests that trace formulae such as (TFM) or (TFF) are rather exceptional phenomena in the $\mathcal{C}^\infty$ category.

Perspectives

This thesis is dedicated to the investigation of the trace formulae (TFF) and (TFM), and of the related question of the order of the dynamical determinants (4) and (12). Using the notion of ultradifferentiability, we design tools fitted to the study of infinitely differentiable uniformly hyperbolic dynamics. These tools allow us to prove *positive* results concerning trace formulae and order of dynamical determinants, such as Theorems 9 and 10. The main idea behind the proof of these results is that stronger assumptions of ultradifferentiability lead to better estimates on the singular values of the relevant operator acting on well-suited spaces. These estimates lead then naturally to bounds on the growth of the dynamical determinant, the number of Ruelle resonances, etc.

However, while we tend to think that phenomena such as the trace formula (TFF) are rather exceptional in the $\mathcal{C}^\infty$ category, we lack counter-examples that go beyond those of Chapter 1. Indeed, the counter-examples from Chapter 1 have at least two flaws in our opinion. First, these are open systems, it would be much more interesting to construct expanding maps or Anosov diffeomorphisms for which the trace formula (TFM) fails, or the dynamical determinant (4) has infinite order. Indeed, *a priori* there could be a topological obstruction for instance that would force the dynamical determinant of an expanding or

Anosov map to have finite order. The second issue with the counter-examples from Chapter 1 is that there are very particular. Indeed, we tend to think that failure of the trace formula should be a generic fact. It would be nice to be able to prove a result of the following form[18].

Conjecture 1. *Let M be a compact $\mathcal{C}^\infty$ manifold. Then the dynamical determinant associated to a generic $\mathcal{C}^\infty$ expanding map, Anosov diffeomorphism or Anosov flow on M has infinite order.*

Another reason why we ask for a generic result in Conjecture 1 is that most of the examples of hyperbolic systems for which we are able to compute more or less explicitly Ruelle resonances are real-analytic, so that their dynamical determinants have finite order. Since we are searching for objects with relatively low regularity, it would not be unusual to rely on Baire category or probabilistic[19] arguments. Moreover, there is a heuristic computation that suggests that the dynamical determinant associated to a generic perturbation of the doubling map Example 1 could have infinite order – see Appendix E.

One could propose stronger conjectures. For instance, the discussions from Chapter 2 (see in particular Remarks 2.5, 2.22 and 2.28) suggest that the $\mathcal{C}^\infty$ regularity could be replaced in Conjecture 1 either by any Denjoy–Carleman class that is not closed under differentiation (in the discrete-time case) or any Denjoy–Carleman class that is larger than all Gevrey classes (in the continuous-time case). Such a result would be particularly nice since it would imply that our results (in particular Theorem 10) are somehow sharp. One could also keep the $\mathcal{C}^\infty$ regularity in Conjecture 1 but replace the finiteness of the order by any reasonable bound on the growth of the dynamical determinant. This is also suggested by the computation from Appendix E.

However, Conjecture 1 is maybe too difficult for our present understanding of the topic, and it could be reasonable to work on an easier problem. For instance, one could wonder whether Theorem 10 is sharp in the analytic case.

Conjecture 2. *Let M be a compact real-analytic manifold of dimension n. Then the dynamical determinant associated to a generic real-analytic Anosov flow on M has order exactly n.*

Let us explain why Conjecture 2 could be easier to approach than Conjecture 1. First of all, a similar statement[20] has been proven by Bandtlow and Naud [BN19] in the case of real-analytic expanding map of the circle, and their methods could be a source of inspiration.

[18] We state here a conjecture about the order of the dynamical determinant rather than the trace formula since it is easier to state. Indeed, considering the trace formula, we always have the issue that the spectral size could be ill-defined, which makes harder to write down a precise conjecture.

[19] Considering the probabilistic approach, the work of Gossart [Gos20] could maybe be a source of inspiration, since it involves flat trace computations for random hyperbolic dynamical systems. However, the random systems from [Gos20] are almost surely not infinitely differentiable, so that his work cannot be used directly.

[20] Actually, they proved a lower bound on the number of Ruelle resonances, but it turns out that Conjecture 2 is equivalent to the fact that the bound on the number of Ruelle resonances from Theorem 4.38 is generically sharp.

Moreover, the real-analytic case is the one for which we have the strongest functional analytic tools: we construct in Chapter 4 a scale of spaces on which the generator of an Anosov flow behaves like an elliptic operator of order 1. One could then hope that our tools could be combined with recent methods that have been developed in order to establish Weyl's law for random perturbations of non self-adjoint elliptic operators (see [Sjö19] and references therein) in order to approach Conjecture 2.

We also hope that the tools that we developed with Bonthonneau in [BJ20] (briefly exposed in Chapter 4) can be used to understand other problems than those discussed in this thesis for which the Gevrey setting is relevant, maybe beyond dynamical systems. It seems natural to expect that our methods can be applied to some PDE problems. For instance, we could probably use our tools to discuss in the Gevrey setting the questions that have been dealt with by Galkowski and Zworski using similar tools in the real-analytic category [GZ19b, GZ20a]. Another possible applications of these FBI transform methods in the Gevrey setting is also mentioned in [GZ20b, Remark 4].

Résumé en français

Cette thèse est dédiée à l'étude de la théorie spectrale associée aux dynamiques hyperboliques $\mathcal{C}^\infty$. Notre approche se base sur les notions d'ultradifférentiabilité et de classe de Denjoy–Carleman. On s'intéresse principalement au problème de la formule de trace pour les flots d'Anosov, proposées par Dyatlov et Zworski dans [DZ16]. Avant de présenter cette question, rappelons les bases de l'approche fonctionnelle des propriétés statistiques des flots d'Anosov.

Soit $(\phi_t)_{t\in\mathbb{R}}$ un flot d'Anosov (voir la Définition 6) de régularité $\mathcal{C}^\infty$ sur une variété compacte M et $V : M \to \mathbb{C}$ une fonction $\mathcal{C}^\infty$. On note X le générateur de $(\phi_t)_{t\in\mathbb{R}}$ et on forme l'opérateur $P \coloneqq X + V$. On cherche à décrire l'asymptotique, lorsque t tend vers $+\infty$, de la famille d'opérateurs de Koopman définis par

$$\mathcal{L}_t : u \mapsto \exp\left(\int_0^t V \circ \phi_\tau \mathrm{d}\tau\right) u \circ \phi_t. \tag{16}$$

Ici u est par exemple une fonction $\mathcal{C}^\infty$ de M dans $\mathbb{C}$. Pour comprendre cette asymptotique, on introduit [BL07, BL13, FS11] une notion de spectre pour P, le *spectre de Ruelle* $\sigma_{\mathrm{R}}(P)$. Ce spectre peut être défini de la manière suivante. Pour $\operatorname{Re} z \gg 1$, on définit la *résolvante* de P par

$$R(z) \coloneqq \int_0^{+\infty} e^{-zt}\mathcal{L}_t \mathrm{d}t : \mathcal{C}^\infty(M) \to \mathcal{D}'(M).$$

On peut alors montrer que $R(z)$ s'étend en une famille méromorphe sur $\mathbb{C}$ d'opérateurs de $\mathcal{C}^\infty(M)$ dans $\mathcal{D}'(M)$, et que ses résidus sont de rang fini. Les pôles de $R(z)$ sont alors appelés *résonances de Ruelle* de P (la multiplicité de la résonance étant le rang du résidu). L'ensemble des résonances de Ruelle forment le spectre de Ruelle.

Les résonances de Ruelle contiennent de nombreuses informations sur les propriétés statistiques du flot $(\phi_t)_{t\in\mathbb{R}}$, il est donc important d'en comprendre la distribution. Pour cela, on peut introduire suivant Ruelle [Rue76] le *déterminant dynamique* défini pour $\operatorname{Re} z \gg 1$ par

$$d(z) = \exp\left(-\sum_\gamma \frac{T_\gamma^\sharp}{T_\gamma}\frac{e^{\int_\gamma V}}{|\det(I - \mathcal{P}_\gamma)|}e^{-zT_\gamma}\right). \tag{17}$$

La somme en argument de l'exponentielle parcourt les orbites périodiques γ du flot $(\phi_t)_{t\in\mathbb{R}}$. Pour une telle orbite, T_γ dénote sa longueur, $T_\gamma^\sharp$ sa longueur primitive (c'est-à-dire la longueur de la plus petite orbite périodique de $(\phi_t)_{t\in\mathbb{R}}$ de même image que γ). On a aussi posé $\int_\gamma V = \int_0^{T_\gamma} V(\phi_t(x))\,\mathrm{d}t$ pour x un point de l'image de γ, et $\mathcal{P}_\gamma = D\phi_{T_\gamma}(x)_{|E_x^u\oplus E_x^s}$ est l'application de Poincaré linéarisée associée à γ. Ici, E^u et E^s sont les directions respectivement stable et instable de $(\phi_t)_{t\in\mathbb{R}}$ et x est toujours un point de l'image de γ (l'application $\mathcal{P}_\gamma$ dépend du choix de x, mais pas sa classe de conjugaison, le déterminant dans (17) est donc bien défini). Un des intérêts du déterminant dynamique (17) est qu'il admet un prolongement holomorphe à $\mathbb{C}$ dont les zéros sont exactement les résonances de Ruelle de P [GLP13, DZ16]. Les déterminants dynamiques sont également des éléments essentiels de la théorie des *fonctions zêtas dynamiques*.

Lorsque le flot $(\phi_t)_{t\in\mathbb{R}}$ et le poids V sont analytiques réels, certains outils d'analyse fonctionnelle puissants peuvent être utilisés pour étudier les propriétés du déterminant dynamique (17). Ces outils, qui se basent sur les travaux de Grothendieck sur la théorie de Fredholm [Gro55], ont été utilisés d'abord par Ruelle [Rue76], puis par Rugh [Rug92, Rug96] et Fried [Fri95]. De ces travaux, il découle en particulier que lorsque $(\phi_t)_{t\in\mathbb{R}}$ et V sont analytiques réels, alors le déterminant dynamique (17) est *d'ordre fini* (voir la Définition 1.2). Il est classique (voir Proposition 1.27) que cette propriété du déterminant dynamique implique la *formule de trace*:

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} e^{\lambda t} = \sum_\gamma T_\gamma^\sharp \frac{e^{\int_\gamma V}}{|\det(I-\mathcal{P}_\gamma)|}\delta_{T_\gamma}. \tag{TFF}$$

Ici, les notations sont les mêmes que dans (17) et l'égalité est à comprendre au sens des distributions sur $\mathbb{R}_+^*$, c'est-à-dire que si φ est une fonction $\mathcal{C}^\infty$ à support compact sur $\mathbb{R}_+^*$ alors

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \mathrm{Lap}(\varphi)(-\lambda) = \sum_\gamma T_\gamma^\# \frac{e^{\int_\gamma V}}{|\det(I-\mathcal{P}_\gamma)|}\varphi(T_\gamma),$$

où $\mathrm{Lap}(\varphi)$ dénote la transformée de Laplace de φ définie pour $z\in\mathbb{C}$ par

$$\mathrm{Lap}(\varphi)(z) = \int_0^{+\infty} e^{-zt}\varphi(t)\mathrm{d}t.$$

Le membre de droite dans (TFF) correspond à une trace généralisée pour l'opérateur de Koopman (16), dont la valeur est donnée par la formule de trace de Guillemin [Gui77] – voir aussi [DZ16, Appendix B]. Le membre de gauche quant à lui joue le rôle de la somme des valeurs propres (le spectre de Ruelle étant ici la notion pertinente pour définir cette somme). Le formule de trace (TFF) donne donc une interprétation spectrale de la formule de trace de Guillemin.

Dans [DZ16], Dyatlov et Zworski s'interrogent sur la validité de la formule de trace (TFF) lorsque le flot $(\phi_t)_{t\in\mathbb{R}}$ et le poids V ne sont plus supposés analytiques (mais au moins $\mathcal{C}^\infty$), toujours dans le but de mieux comprendre la répartition des résonances de Ruelle. Cette question constitue le fil conducteur de notre thèse. Notre point de départ pour comprendre ce problème est la suggestion par Sébastien Gouëzel de s'y intéresser dans le contexte de la régularité Gevrey. Rappelons qu'il s'agit d'une hypothèse de régularité intermédiaire entre $\mathcal{C}^\infty$ et analytique, introduite par Gevrey pour étudier la régularité des solutions de certaines EDP [Gev18]. Cette suggestion nous a amené à développer des outils d'analyse fonctionnelle adaptés à l'étude de la théorie spectrale des dynamiques hyperboliques, non seulement de régularité Gevrey, mais aussi dans des classes de régularité plus générales: les *classes de Denjoy–Carleman* ou *classes de fonctions ultradifférentiables*. Il s'agit également de classes de régularité intermédiaire entre $\mathcal{C}^\infty$ et analytique, plus variées que les classes de Gevrey, ce qui nous permet d'énoncer des résultats plus précis sur la validité de la formule de trace (TFF). Ces outils que nous développons sont en quelque sorte la généralisation au cas ultradifférentiable des méthodes mentionnées ci-dessus dans le cas analytique.

Les quatre chapitres de ce document présentent l'essentiel de nos travaux autour de la formule de trace (TFF) et de questions proches liées à la théorie spectrale des dynamiques hyperboliques ultradifférentiables. Nous commençons par expliciter dans le Chapitre 1 les liens entre formules de traces et déterminants dynamiques. Cette étude est menée pour les flots d'Anosov, mais aussi pour des systèmes hyperboliques à temps discret (des analogues des questions que nous avons détaillées dans le cas continu se posent également à temps discret). Les principales relations entre déterminants dynamiques et formules de traces sont données dans la Proposition 1.27 (pour les flots d'Anosov) et dans la Proposition 1 (pour les systèmes à temps discret, voir aussi le Théorème 1.3). Nous utilisons ensuite ces liens pour construire *des exemples de difféomorphisme hyperboliques $\mathcal{C}^\infty$ pour lesquels la formule de trace est fausse* (voir par exemple les Propositions 2 et 3).

Dans le Chapitre 2, nous rappelons la définition des classes de Denjoy–Carleman et nous commençons à utiliser cette notion pour étudier la théorie spectrale des dynamiques hyperboliques. Afin de rendre le plus clair possible le fonctionnement de nos méthodes, nous nous intéressons tout d'abord à des systèmes hyperboliques parmi les plus simples: les applications dilatantes du cercle. On prouve en particulier le résultat suivant, connu auparavant uniquement dans le cas analytique [Rue76].

Théorème A (Voir le Théorème 2.9). *Soit T une application dilatante du cercle $\mathcal{C}^\infty$ (voir la Définition 1). Il existe un espace de Hilbert $\mathcal{H}$, continûment inclus dans l'espace des fonctions $\mathcal{C}^\infty$ sur le cercle, sur lequel l'opérateur de transfert*

$$\mathcal{L} : u \mapsto \left(x \mapsto \sum_{Ty=x} \frac{1}{|T'(y)|} u(y) \right) \tag{18}$$

définit un opérateur compact. De plus, les polynômes trigonométriques forment un sous-espace dense de $\mathcal{H}$.

L'opérateur (18) joue ici un rôle similaire à l'adjoint de (16) dans le cas des flots d'Anosov. L'espace $\mathcal{H}$ permet également d'étudier un déterminant dynamique associé à T, et de discuter la validité d'une formule de trace pour $\mathcal{L}$ (un analogue à temps discret de (TFF)). En effet, si $\mathcal{L}$ ne définit en général qu'un opérateur compact sur $\mathcal{H}$, on peut parfois montrer que $\mathcal{L}$ est en fait un opérateur à trace, et une formule de trace s'en déduit par le théorème de Lidskii. Tous ces résultats sont énoncés de manière quantitative en utilisant le langage des classes de Denjoy–Carleman (voir en particulier le Théorème 2.9). Nos méthodes nous permettent également d'établir un résultat sur le problème de la conjugaison pour les applications dilatantes ultradifférentiables (le Corollaire 2.12).

Enfin, nous clôturons le Chapitre 2 en énonçant un résultat sur la théorie spectrale de l'opérateur de Koopman de certains difféomorphismes hyperboliques de régularité Gevrey que nous avons obtenu par des méthodes similaires à celles détaillées pour les applications dilatantes du cercle (voir le Théorème 2.27).

Le Chapitre 3 est consacré à la preuve du résultat principal de cette thèse :

Théorème B (Voir le Théorème 3.1 et le Corollaire 3.2)**.** *Il existe une grande classe d'applications ultradifférentiables telle que si $M, (\phi_t)_{t\in\mathbb{R}}$ et V appartiennent à cette classe alors la formule de trace* (TFF) *est vérifiée.*

La classe de régularité en question est décrite en détail dans §3.1 en utilisant le langage des classes de Denjoy–Carleman introduit dans le chapitre précédent. Remarquons juste pour l'instant que cette classe de régularité est plus grande que toutes les classes de Gevrey. Nous prouvons également une borne sur le nombre de résonances de Ruelle sous les hypothèses du Théorème B (voir la Proposition 3.3) qui implique en particulier que le membre de gauche dans (TFF) définit bien une distribution sur $\mathbb{R}_+^*$. De manière notable, le Théorème B n'est pas obtenu en prouvant que le déterminant dynamique (17) est d'ordre fini (comme c'était le cas pour les flots analytiques). Nous ne nous attendons d'ailleurs pas à ce que les déterminants dynamiques associés aux flots considérés dans le Chapitre 3 soient d'ordre fini.

Si les résultats du Chapitre 3 sont relativement satisfaisants en ce qui concerne la formule des traces, la question de l'ordre du déterminant dynamique nous semblait également digne d'intérêt. Il s'agit du sujet du quatrième et dernier chapitre de cette thèse, dans lequel nous présentons une version abrégée d'un travail en collaboration avec Yannick Guedes Bonthonneau. Le principal outil technique dans le Chapitre 4 est une *transformée de FBI analytique* dans l'esprit de [HS86, Sjö96]. Afin que la taille de cette thèse reste raisonnable, nous admettrons les résultats techniques nécessaires afin d'utiliser cette transformée de FBI pour étudier les flots d'Anosov de régularité Gevrey (les preuves de ces résultats sont détaillées dans [BJ20] disponible sur arXiv). Des méthodes similaires ont été utilisées récemment par Galkowski et Zworski pour étudier des problèmes d'EDP en régularité analytique [GZ19a, GZ19b, GZ20a] (nos résultats s'appliquent à la fois en régularité Gevrey et analytique). Ces méthodes basées sur la transformée de FBI nous permettent alors de prouver la borne suivante sur le déterminant dynamique (17).

Théorème C (Voir le Théorème 10)**.** *Soit $s \geq 1$. Si $M, (\phi_t)_{t\in\mathbb{R}}$ et V sont s-Gevrey alors il existe $C > 0$ tel que pour tout $z \in \mathbb{C}$ on a*

$$|d(z)| \leq C \exp\left(C |z|^{ns}\right).$$

En particulier, l'ordre de d est plus petit que ns.

La définition des classes de Gevrey est rappelée dans §4.1.1. Rappelons que 1-Gevrey est un synonyme d'analytique réel, le Théorème C redonne ainsi dans le cas $s = 1$ une borne connue sur le déterminant dynamique, obtenue par les méthodes mentionnées ci-dessus [Rug92, Rug96, Fri95] (aucune borne n'était connue dans la cas $s > 1$). Notre preuve est cependant plus directe: nous travaillons directement avec l'opérateur de Koopman (16) et son générateur $P = X + V$. D'autres applications de la transformée de FBI à l'étude des flots d'Anosov sont également esquissées à la fin du Chapitre 4, nouvelles mêmes dans le cas $s = 1$.

Chapter 1

Trace formulae, dynamical determinants and counter-examples

In this chapter, we explain the complex analytic links between dynamical determinants and trace formulae for expanding and hyperbolic maps – in §1.1 – and for Anosov flows – in §1.3 –, proving in particular Proposition 1 (and its continuous-time analogue Proposition 1.27). We use this dictionary to give quite striking counter-examples to the trace formula (TFM) in §1.2, proving in particular Propositions 2 and 3.

The results from §1.1 and §1.2 may be found in [Jéz20a, §3 and 4]. The proofs from §1.3 were already exposed in [Jéz19a, Appendix E], up to some additional details that we give here.

1.1 Trace formulae and dynamical determinants for hyperbolic maps

We explain now the link between the trace formula (TFM) and the order of the dynamical determinant (4) – where $\mathcal{L}$ is either the transfer operator (1) associated to an expanding map of the circle or the weighted Koopman operator (7) associated to an Anosov diffeomorphism, and the flat trace $\operatorname{tr}^\flat(\mathcal{L}^n)$ is defined by (5) or (8) accordingly. It is convenient to introduce the following definition.

Definition 1.1. If f is an entire function, we say that $(z_m)_{m\geqslant 0}$ is an ordering of the zeros of f if $z_0, z_1, \dots, z_m, \dots$ are the zeros of f counted with multiplicities and the sequence $(|z_m|)_{m\geqslant 0}$ is non-decreasing.

In this section and in §1.2, we will always order the zeros of an entire function in this way. Recall the following definitions.

Definition 1.2. Let f be an entire function. The *order* of f may be defined as

$$\limsup_{r\to+\infty} \frac{\log_+ \log_+ \left(\sup_{|z|\leqslant r} |f(z)|\right)}{\log r}$$

where $\log_+ x = \log \max(1,x)$. If f is non-zero and has finite order, let p be the smallest natural integer such that

$$\sum_{m\geqslant 0} \frac{1}{1+|z_m|^{p+1}} < +\infty \tag{1.1}$$

where $(z_m)_{m\geqslant 0}$ is an ordering of the zeros of f (the integer p is well-defined thanks to Jensen's formula). By Hadamard's Factorization Theorem [Boa54, 2.7.1], if m_0 denotes the order of 0 as a zero of f, then there is a polynomial Q such that, for all $z \in \mathbb{C}$, we have

$$f(z) = z^{m_0} e^{Q(z)} \prod_{m\geqslant m_0} E\left(\frac{z}{z_m}, p\right) \tag{1.2}$$

where the function E is the Weierstrass primary factor defined by

$$E(u,p) = (1-u)\exp\left(\sum_{k=1}^{p} \frac{1}{k} u^k\right) = \exp\left(-\sum_{k=p+1}^{+\infty} \frac{1}{k} u^k\right), \tag{1.3}$$

where the last expression is only valid when $|u| < 1$. The *genus* of f is then defined as $\max(\deg Q, p)$. We will say that the genus of an entire function of infinite order is infinite.

As explained in Remark 1.4 below, the following theorem is an abstract way to express the link between trace formula and order of the dynamical determinant.

Theorem 1.3. *Let f be an entire function such that $f(0) = 1$. Let G be a holomorphic function defined on a neighbourhood of 0 such that $G(0) = 0$ and $f(z) = e^{G(z)}$ for z in a neighbourhood of zero. Write*

$$G(z) = -\sum_{n=1}^{+\infty} \frac{1}{n} a_n z^n \tag{1.4}$$

and denote by $(z_m)_{m\geqslant 0}$ an ordering of the zeros of f. Then for all $r > 0$ such that f has no zero of modulus r, we have

$$a_n \underset{n\to+\infty}{=} \sum_{|z_m|<r} \frac{1}{z_m^n} + \mathcal{O}\left(\frac{1}{r^n}\right). \tag{1.5}$$

Furthermore, the following properties are equivalent :

(i) the order of f is finite;

(ii) there is a natural integer n_0 such that for all integers $n \geqslant n_0 + 1$ the series

$$\sum_{m \geqslant 0} \frac{1}{z_m^n} \tag{1.6}$$

converges absolutely and its sum is a_n.

If (i) or (ii) holds then the minimal value of n_0 so that (ii) holds is the genus of f.

Remark 1.4. Taking $f = d$ the dynamical determinant (4), we have $a_n = \operatorname{tr}^\flat(\mathcal{L}^n)$, the flat trace of $\mathcal{L}$ defined by (5) or (8) according to the context. Recalling Theorems 2 and 4, it appears then that the trace formula (TFM) holds with absolute convergence of the right-hand side if and only if $n \geqslant n_0 + 1$, where n_0 denotes the genus of the dynamical determinant d. In particular, Proposition 1 follows immediately from Theorems 2, 4 and 1.3.

In §1.2, we will construct dynamical determinants with arbitrary (finite or infinite) genus, so that all the behaviours described in Theorem 1.3 may be realized by dynamical determinants. Notice also that, as in the continuous-time case, we have a "local" version of the trace formula: for every $r > 0$ such that there is no resonance of modulus r we have

$$\operatorname{tr}^\flat(\mathcal{L}^n) \underset{r \to 0}{=} \sum_{\substack{\lambda \text{ Ruelle resonance} \\ |\lambda| > r}} \lambda^n + \mathcal{O}(r^n). \tag{1.7}$$

From Theorem 1.3, the local trace formula (1.7) is in fact just a reformulation of Theorem 2 or 4 according to the context.

Remark 1.5. As we will see in Proposition 1.6 below, the absoluteness of the convergence in (ii) is essential to get an equivalence. This is quite unfortunate especially as we will realize the counter-examples from Proposition 1.6 below as dynamical determinants in §1.2. On the other hand, it is very easy to construct an example for which the series (1.6) converges to a sum different from a_n (for any chosen values of n): just multiply f by the exponential of an entire function.

Proof of Theorem 1.3.

- To prove (1.5), one only needs to notice that the holomorphic function

$$z \mapsto \frac{f(z)}{\prod_{\substack{i \in \mathbb{N} \\ |z_i| < r}} \left(1 - \frac{z}{z_i}\right)} = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n}\left(a_n - \sum_{|z_m| < r} \frac{1}{z_m^n}\right) z^n\right)$$

  does not vanish on a disc of center 0 and radius a little bigger than r, and so admits a holomorphic logarithm there.

- Suppose (i). Recall p from Definition 1.2 and notice that the series (1.6) converges absolutely for $n \geqslant p + 1$. Let r be a positive real number such that $r \leqslant |z_m|$ for all

m. Then define for $|z| \leqslant \frac{r}{2}$ and $m \geqslant 0$

$$f_m(z) = -\sum_{k=p+1}^{+\infty} \frac{1}{k}\left(\frac{z}{z_m}\right)^k, \tag{1.8}$$

and notice that $|f_m(z)| \leqslant \frac{r^{p+1}}{2^p}\frac{1}{|z_m|^{p+1}}$. Then, recalling (1.1), the series $\sum_{m\geqslant 0} f_m$ converges on the disc of center 0 and radius $\frac{r}{2}$ to a holomorphic function F and, recalling (1.2), we have for z close enough to 0

$$e^{G(z)} = f(z) = e^{Q(z)+F(z)}.$$

Thus we may identify the coefficients of order greater than $\deg Q$ in the expansions in power series of F and G, which ends the proof of (ii) recalling (1.4) and (1.8).

- Suppose (ii). Using the hypothesis for $n = n_0 + 1$, the infinite product

$$P(z) = \prod_{m\geqslant 0} E\left(\frac{z}{z_m}, n_0\right),$$

converges on $\mathbb{C}$ to a holomorphic function of finite order smaller than $n_0 + 1$ and genus n_0 (see [Boa54, Theorem 2.6.5]). But since $a_n = \sum_{m\geqslant 0} \frac{1}{z_m^n}$ for $n \geqslant n_0 + 1$, we have, recalling the definition (1.3) of E, for z close enough to 0,

$$P(z) = \exp\left(-\sum_{n=n_0+1}^{+\infty} \frac{1}{n} a_n z^n\right)$$

and consequently

$$f(z) = \exp\left(-\sum_{n=1}^{n_0} \frac{1}{n} a_n z^n\right) P(z).$$

Thus, f has finite order smaller than $n_0 + 1$ (and genus smaller than n_0).

□

We now give two counter-examples that highlight the necessity to ask for absolute convergence in (ii) in Theorem 1.3.

Proposition 1.6. *(a) There exists an entire function f with $f(0) = 1$ such that if $(z_m)_{m\geqslant 0}$ is an ordering of the zeros of f (as defined in Definition 1.1) then for all $n \geqslant 1$ the series $\sum_{m\geqslant 0} \frac{1}{z_m^n}$ converges with sum a_n (defined in (1.4)) but the convergence is not absolute.*

(b) There exists an entire function f with $f(0) = 1$, an ordering $(z_m)_{m\geqslant 0}$ of the zeros of f and a permutation σ of $\mathbb{N}$ such that $\left(z_{\sigma(n)}\right)_{n\in\mathbb{N}}$ is an ordering of the zeros of f

and, for all $n \geqslant 1$, the series $\sum_{m\geqslant 0} \frac{1}{z_m^n}$ converges with sum a_n, but $\sum_{m\geqslant 0} \frac{1}{z_{\sigma(m)}^n}$ does not converge.

Remark 1.7. Theorem 1.3 implies that the functions constructed by Proposition 1.6 have infinite order, while the associated "trace formula" (1.6) holds in some weak sense.

To prove Proposition 1.6, we will need the following lemma, whose proof is straightforward using an Abel transform.

Lemma 1.8. *Let $(b_m)_{m\geqslant 0}$ be a sequence of complex numbers such that there is a constant M such that, for all $\ell \in \mathbb{N}$, we have $\left|\sum_{m=0}^{\ell} b_m\right| \leqslant M$. Let $(c_m)_{m\geqslant 0}$ be a decreasing sequence of positive real numbers that converges to 0. Then the series $\Sigma_{m\geqslant 0} b_m c_m$ converges, and we have the estimates*

$$\left|\sum_{m=0}^{+\infty} b_m c_m\right| \leqslant 2Mc_0.$$

Proof of Proposition 1.6. We start by proving (a). Choose an irrational real number θ for which there is a constant $c > 0$ such, that for all $n \in \mathbb{N}^*$, we have $\left|1 - e^{2i\pi n\theta}\right| \geqslant \frac{c}{n^2}$ (almost any real number may be chosen thanks to Borel–Cantelli's lemma). For every integer n, set

$$a_n = \sum_{m=2}^{+\infty} \left(\frac{e^{2i\pi m\theta}}{\ln(m)}\right)^n,$$

which is well-defined thanks to Lemma 1.8, but the convergence is clearly not absolute. Furthermore, for all integers $m_0 \geqslant 2$, we have

$$\left|a_n - \sum_{m=2}^{m_0-1} \left(\frac{e^{2i\pi m\theta}}{\ln(m)}\right)^n\right| \leqslant \frac{4}{c}\frac{n^2}{\ln(m_0)^n} \tag{1.9}$$

(take $b_m = e^{2i\pi n(m+m_0)\theta}$ and $c_m = (\ln(m+m_0))^{-n}$ in Lemma 1.8). Now (1.9) with

$$\exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}\left(\sum_{m=2}^{m_0-1}\left(\frac{e^{2i\pi m\theta}}{\ln(m)}\right)^n\right)z^n\right) = \prod_{m=2}^{m_0-1}\left(1 - \frac{e^{2i\pi m\theta}}{\ln(m)}z\right)$$

implies that the function f defined by $f(z) = \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}a_n z^n\right)$, for z in a neighbourhood of zero, extends to an entire function whose zeros are exactly the $\left(\frac{e^{2i\pi m\theta}}{\ln(m)}\right)^{-1}$. Since there is only one way to order the zeros of f with increasing moduli, point (a) is proven.

We turn now to the proof of (b). Choose θ as above and denote by $(n_k)_{k\geqslant 0}$ the sequence of integers defined by $n_0 = 0$ and $n_k = k!$ for $k \geqslant 1$. Define $I_0 = \{0\}$ and $I_k = [\![n_k+1, n_{k+1}]\!]$ for $k \geqslant 1$. For every $n \in \mathbb{N}$, denote by $k(n)$ the unique integer such that $n \in I_{k(n)}$. Then set for all integers $n \geqslant 1$

$$a_n = \sum_{m=0}^{+\infty}\left(\frac{e^{2i\pi m\theta}}{\ln(k(m)+2)}\right)^n.$$

Then we use Lemma 1.8 as in the proof of (a) to show that $f(z) = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} a_n z^n\right)$ extends to an entire function whose zeros are exactly the $z_m = \left(\frac{e^{2i\pi m\theta}}{\ln(k(m)+2)}\right)^{-1}$ for $m \in \mathbb{N}$. We will see that there is another way to order the zeros of f, which breaks the convergence of the series (1.6) for all $n \geqslant 1$, but preserves the monotonicity of the sequence of moduli.

Choose $0 < \epsilon < 1$ such that for all $x \in [0, \epsilon]$ we have $\mathrm{Re}\left(e^{2i\pi x}\right) \geqslant \frac{1}{2}$. Then for every $k \in \mathbb{N}$ and $n \geqslant 1$, denote by $N_k^{(n)}$ the number of those $m \in I_k$ such that $mn\theta \in [0, \epsilon] \mod 1$, and choose a permutation $\sigma_k^{(n)}$ of I_k which puts these elements first. Equidistribution of the $mn\theta$, for n fixed and $m \geqslant 0$, implies that

$$\frac{\sum_{\ell=0}^{k} N_\ell^{(n)}}{n_{k+1}} \underset{k \to +\infty}{\to} \epsilon,$$

but $\sum_{\ell=0}^{k-1} N_\ell^{(n)} \leqslant n_k + 1 \underset{k \to +\infty}{=} o(n_{k+1})$, and thus

$$\frac{N_k^{(n)}}{\ln(k+2)} \underset{k \to +\infty}{\to} +\infty.$$

Now choose $\phi : \mathbb{N} \to \mathbb{N}^*$ such that for all $n \geqslant 1$ the reciprocal image $\phi^{-1}(\{n\})$ is infinite (for instance $1, 1, 2, 1, 2, 3, \dots$) and set

$$\sigma = \bigcup_{k=0}^{+\infty} \sigma_k^{(\phi(k))}.$$

If $n \geqslant 1$ the series $\sum_{m \geqslant 0} \frac{1}{z_{\sigma(m)}^n}$ does not converge. Indeed, for all k such that $\phi(k) = n$, we have

$$\mathrm{Re}\left(\tilde{S}_{n_{k-1}+N_k^{(n)}}\right) \geqslant \mathrm{Re}\left(S_{n_{k-1}}\right) + \frac{1}{2} \frac{N_k^{(n)}}{\ln(k+2)}, \tag{1.10}$$

where $(S_m)_{m \geqslant 0}$ is the sequence of partial sums of the series $\sum_{m \geqslant 0} \frac{1}{z_m^n}$, and $(\tilde{S}_m)_{m \geqslant 0}$ is the sequence of partial sums of the series $\sum_{m \geqslant 0} \frac{1}{z_{\sigma(m)}^n}$. We let k tend to $+\infty$ with $\phi(k) = n$, which is possible thanks to our choice of ϕ. We saw that the first term in the right-hand side of (1.10) converges but the second one tends to $+\infty$, and thus the left-hand side of (1.10) does not converge. □

In order to realize the counter-examples from Proposition 1.6 as dynamical determinants in §1.2, we will need the two following, merely technical, lemmas.

Lemma 1.9. *For every $\epsilon > 0$ and $\rho > 0$, the counter-examples from Proposition 1.6 may be realized as entire functions f of the form $f : z \mapsto 1 - 2z - z(1-z)h(z)$, where h is an entire function such that, for all $z \in \mathbb{C}$, we have $h(z) = \sum_{\ell=0}^{+\infty} \alpha_\ell z^\ell$, where $\alpha_\ell \in \left[-\frac{\epsilon}{\rho^\ell}, \frac{\epsilon}{\rho^\ell}\right]$ for all integers ℓ.*

Proof. For all $k \geqslant 2$ and $n \geqslant 1$ set either

$$a_n^{(k)} = \sum_{m=k}^{+\infty} \left(\frac{e^{2i\pi m\theta}}{\ln(m)} \right)^n$$

or

$$a_n^{(k)} = \sum_{m=k-2}^{+\infty} \left(\frac{e^{2i\pi m\theta}}{\ln(k(m)+2)} \right)^n,$$

depending on whether you want to get a counter-example of type (a) or (b). Then set for all $k \geqslant 1$

$$\tilde{f}_k(z) = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} \left(a_n^{(k)} + \bar{a}_n^{(k)} \right) z^n \right).$$

Estimate (1.9) (and its analogue for the case (b) of Proposition 1.6) implies that $\tilde{f}_k$ converges to 1 uniformly on all compact subsets of $\mathbb{C}$ as k goes to $+\infty$. Then set $f_k : z \to \left(1 - \frac{z}{\lambda_k}\right) \tilde{f}_k(z)$, where $\lambda_k = \frac{\tilde{f}_k(1)}{1+\tilde{f}_k(1)}$. Thus we have $f_k(0) = 1$, $f_k(1) = -1$, and it is easy to check that f_k is a counter-example of type (a) or (b), according to the way the $a_n^{(k)}$ have been defined. We will see that that for large enough k the function f_k satisfies the conditions of Lemma 1.9. Let h_k be the entire function defined by $h_k(z) = -\frac{f_k(z)-1+2z}{z(1-z)}$. We will also need the auxiliary function $H_k(z) = \frac{\tilde{f}_k(z)-1}{z} - (\tilde{f}_k(1) - 1)$ which vanishes at $z = 1$ and tends to 0 uniformly on all compact subsets of $\mathbb{C}$ when k tend to $+\infty$. Then notice that

$$-h_k(z) = \frac{H_k(z)}{1-z} \left(1 - \frac{z}{\lambda_k}\right) + \frac{\tilde{f}_k(1)^2 - 1}{\tilde{f}_k(1)}$$

and write

$$\frac{H_k(z)}{1-z} = \sum_{\ell=0}^{+\infty} \beta_\ell z^\ell$$

then we have

$$\alpha_0 = -\left(\beta_0 + \frac{\tilde{f}_k(1)^2 - 1}{\tilde{f}_k(1)} \right) \text{ and } \alpha_{\ell+1} = -\left(\beta_{\ell+1} + \frac{\beta_\ell}{\lambda_k} \right) \text{ if } \ell \geqslant 0.$$

But the sequence $(\lambda_k)_{k \geqslant 0}$ converges to $\frac{1}{2}$ and (we may suppose $\rho \geqslant 2$)

$$|\beta_\ell| = \frac{1}{2\pi} \left| \int_{D(0,\rho)} \frac{H_k(z)}{(1-z) z^{\ell+1}} \mathrm{d}z \right| \leqslant \frac{2}{\rho^\ell} \sup_{|z| \leqslant r} \left| \tilde{f}_k(z) - 1 \right|,$$

which ends the proof, recalling that $\tilde{f}_k$ converges to 1 uniformly on all compact subsets of $\mathbb{C}$ as k goes to $+\infty$. □

Lemma 1.10. *Let $f(z)$ be an entire function such that $f(0) = 1$, and $(c_k)_{k \geqslant 0}$ be a sequence* *of positive real numbers such that $\sum_{k\geqslant 0} c_k < +\infty$. Then the infinite product*

$$\prod_{k\geqslant 0} f(c_k z) \tag{1.11}$$

converges uniformly on all compact subsets of $\mathbb{C}$ to an entire function d that has same genus[1] than f. Furthermore, if f is one of the counter-example of type (a) or (b) constructed in Lemma 1.9, then d also satisfies point (a) or (b) respectively of Proposition 1.6.

Proof. If K is a compact subset of $\mathbb{C}$ then, since $f(0) = 1$, there is a constant $C > 0$ such that for all $z \in K$ we have $|f(z) - 1| \leqslant C|z|$. Thus for all $z \in K$ and $k \geqslant 0$, we have $|f(c_k z) - 1| \leqslant C|c_k||z|$. Hence, the infinite product (1.11) does converge uniformly on all compact subsets of $\mathbb{C}$ to an entire function d. That d has same genus as f is straightforward from the Definition 1.2 and Hadamard's factorization Theorem (we use the positivity of the c_k's to ensure that no unwanted cancellation happens). Let us point out that if $f(z) = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} a_n z^n\right)$ then $d(z) = \exp\left(-\sum_{n=1}^{+\infty} \frac{1}{n} a_n \left(\sum_{k=0}^{+\infty} c_k^n\right) z^n\right)$.

Suppose now that f is the counter-example of type (a) constructed in Lemma 1.9 and denote by $(z_m)_{m\geqslant 0}$ an ordering of its zeros. Let $(w_m)_{m\geqslant 0}$ be an ordering of the zeros of d , then there is a bijection $(\phi, \psi) : \mathbb{N} \to \mathbb{N}^2$ such that for all $m \in \mathbb{N}$ we have $w_m = \frac{z_{\phi(m)}}{c_{\psi(m)}}$, and for every $k \in \mathbb{N}$ the sequence $\left(z_{\phi(m)}\right)_{m\in\psi^{-1}(\{k\})}$ is an ordering of the zeros of f. Let $n \geqslant 1$. It is clear from our construction that f has no more than two zeros of a given modulus[2], and so there is a constant M such that for all $k \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ we have

$$\left| \sum_{\substack{m\leqslant m_0 \\ \psi(m)=k}} \frac{1}{z_{\phi(m)}^n} \right| \leqslant M. \tag{1.12}$$

Now, for $k \in \mathbb{N}$, let u_k be the sequence $\left(\sum_{\substack{m\leqslant m_0 \\ \psi(m)=k}} \frac{1}{w_m^n}\right)_{m_0\geqslant 0}$ whose limit is $c_k^n a_n$ by construction of f. From (1.12), the sup norm of u_k is smaller than $c_k^n M$, and thus the series $\sum_{k\geqslant 0} u_k$ converges in the space of converging sequences equipped with the sup norm. But its sum is clearly the sequence of partial sums of $\sum_{m\geqslant 0} \frac{1}{w_m^n}$. Thus this series converges, and its sum is $a_n \sum_{k=0}^{+\infty} c_k^n$, as wanted.

We suppose that f is a counter-example of type (b). There are two natural partitions of the zeros of d : the partition $Z_0, Z_1, \ldots, Z_k, \ldots$ by modulus (Z_0 contains the element of

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[1] If f has non integral order δ and $\sum c_k^\delta < +\infty$, then one may show using [Boa54, 2.9.1] that d has also same order than f.

[2] That's where we use that f is precisely the counter-example constructed above. We will not need it for the case (b)

minimal modulus, the following are in Z_1, etc) and the partition $Z'_1, Z'_2, \ldots$ defined by

$$Z'_k = \left\{ \frac{z}{c_k} : z \text{ is a zero of } f \right\}.$$

Both partitions are endowed with the natural notion of multiplicity. Now, we get an ordering for which (1.6) holds in the following way : we put first the elements of $Z_0 \cap Z'_0$ in the order which gave (1.6) for f, then we put the elements of $Z_0 \cap Z'_1$ (according to the same order), then $Z_0 \cap Z'_2$, etc, when we are done with Z_0 (which happens in a finite number of steps), we do the same with Z_1, then Z_2, etc. The proof that (1.6) holds in this case is as in case (a) (in fact a bit easier). To get an ordering for which there is divergence of the inverse of the zeros of d at any power, we do exactly the same, except that at each step we put the elements of Z'_0 in the order which gave the divergence for f. □

We end this section with the two following lemmas, that will be used to prove Propositions 2 and 3 in §1.2.

Lemma 1.11. *Let E be a subset of $\mathbb{N}^*$. Then there is an entire function Q such that $Q(0) = Q(1) = 0$ and if $Q : z \mapsto \sum_{n=1}^{+\infty} \beta_n z^n$ then $\beta_n = 0$ if and only if $n \in E$, and $\beta_n \in \mathbb{R}$ for all $n \in \mathbb{N}^*$. Moreover, for every $\epsilon > 0$ and $\rho > 0$, if $\alpha > 0$ is sufficiently small, then there is an entire function $h : z \mapsto \sum_{n=0}^{+\infty} \alpha_n z^n$ such that $(1-2z)\, e^{\alpha Q(z)} = 1 - 2z - z(1-z)\, h(z)$, for all $z \in \mathbb{C}$, and $\alpha_n \in \left[-\frac{\epsilon}{\rho^n}, \frac{\epsilon}{\rho^n}\right]$ for all $n \in \mathbb{N}$.*

Proof. We will construct Q of the form $Q : z \mapsto z(1-z) \sum_{n=0}^{+\infty} b_n z^n$. Then we have $\beta_1 = b_0$ and $\beta_{n+1} = b_n - b_{n-1}$, for all $n \geqslant 1$. If E contains a final segment of $\mathbb{N}^*$ then it is easy to see that there is a polynomial Q with real coefficients that satisfies the first part of Lemma 1.11. If E does not contain a final segment of $\mathbb{N}^*$ then the sequence $(b_n)_{n \in \mathbb{N}}$ may be recursively defined by

$$\begin{aligned} & b_0 = 1 \text{ if } 1 \notin E, 0 \text{ otherwise;} \\ & b_n = b_{n-1} \text{ if } n \geqslant 1 \text{ and } n+1 \in E; \\ & b_n = \frac{1}{\min\{\ell \geqslant n, \ell + 2 \notin E\}!} \text{ if } n \geqslant 1 \text{ and } n+1 \notin E. \end{aligned}$$

The second part of Lemma 1.11 may be proven in a similar way than Lemma 1.9. □

Lemma 1.12. *Let $N_0 : \mathbb{R}_+^* \to \mathbb{R}$ be a locally bounded function. Then for every $\epsilon > 0$ and $\rho > 0$ there is an entire function $h : z \mapsto \sum_{k=0}^{+\infty} \alpha_k z^k$ such that for all $k \in \mathbb{N}$ we have $\alpha_k \in \left[-\frac{\epsilon}{\rho^k}, \frac{\epsilon}{\rho^k}\right]$, and if $f : z \to 1 - 2z - z(1-z)\, h(z)$ and $(z_m)_{m \in \mathbb{N}}$ is an ordering of the zeros of f then*

$$N_0(r) \underset{r \to 0}{=} o\left(\# \left\{ m \in \mathbb{N} : |z_m| < \frac{1}{16r} \right\} \right). \tag{1.13}$$

Proof. Choose a sequence $(z_m)_{m\in\mathbb{N}}$ of non-zero positive real numbers such that $z_m \underset{m\to+\infty}{\to} +\infty$ and

$$N_0(r) \underset{r\to 0}{=} o\left(\#\left\{m\in\mathbb{N} : z_m \leqslant \frac{1}{16r}\right\}\right). \tag{1.14}$$

Such a sequence exists since N_0 is locally bounded. Notice that if $|z|\leqslant \frac{1}{2}$ and $p\in\mathbb{N}$ then

$$|E(z,p)-1| \leqslant \frac{1}{2}\sup_{|w|\leqslant\frac{1}{2}}\left|E'(w,p)\right| \leqslant \frac{1}{2^p}$$

where E is the Weierstrass primary factor from (1.3). Consequently, we can choose an increasing sequence of integer $(p_m)_{m\in\mathbb{N}}$ such that $p_m \underset{m\to+\infty}{\to} +\infty$ and the infinite product $\prod_{m\geqslant 0} E\left(\frac{z}{z_m}, p_m\right)$ converges uniformly on all compact subsets of $\mathbb{C}$. For m_0 large enough, define an entire function f_{m_0} by

$$\begin{aligned} f_{m_0}(z) &= (1-\lambda_{m_0} z)\prod_{m\geq m_0} E\left(\frac{z}{z_m}, p_m\right) \\ &= 1-2z-z(1-z)h_{m_0}(z) \end{aligned}$$

where

$$\lambda_{m_0} = 1 + \frac{1}{\prod_{m\geq m_0} E\left(\frac{1}{z_m}, p_m\right)}.$$

Using Cauchy's formula, it is easy to see that h_{m_0} converges to 1 uniformly on all compact subsets of $\mathbb{C}$ when $m_0\to+\infty$. Thus $h=h_{m_0}$ satisfy the first condition when m_0 is large enough. Moreover, we have for all $r>0$

$$\#\left\{m\in\mathbb{N} : z_m \leqslant \frac{1}{16r}\right\} = \#\left\{m\geqslant m_0 : z_m\leqslant\frac{1}{16r}\right\} + m_0, \tag{1.15}$$

and thus

$$N_0(r) \underset{r\to 0}{=} o\left(\#\left\{m\geqslant m_0 : z_m \leqslant \frac{1}{16r}\right\}\right)$$

with (1.14), and since the right-hand side of (1.15) tends to $+\infty$ when r tends to 0. This ends the proof because the z_m's are zeros of f. □

1.2 Counter-examples to trace formulae

In this section, we realize a wide class of entire functions as dynamical determinants. In particular, all the possibilities considered in Theorem 1.3 as well as the counter-examples from Proposition 1.6 will materialize. We will also construct dynamical determinants, associated with finitely differentiable weights, which cannot be holomorphically continued to the whole complex plane. Our construction will be based on a well-known example of zeta functions for hyperbolic flows which cannot be continued meromorphically to the whole complex plane (see [Bal92] and [PP90, Example 1 p.165]). The strategy is the following: we first construct a subshift of finite type and a weight for which the associated zeta function (1.19) is explicit, then we use Whitney's extension theorem [Whi34] as in [Bow72a] to get a hyperbolic dynamics on a manifold with the same dynamical zeta function, and finally we show that in this particular case the dynamical determinant may be obtained from the dynamical zeta function.

However, this construction produces an open system rather than an Anosov diffeomorphism, so that we explain first how the picture described in the introduction adapts to hyperbolic basic sets.

1.2.1 Hyperbolic basic sets

Let us recall the definition of a hyperbolic basic set.

Definition 1.13 (Hyperbolic basic set). Let M be a $\mathcal{C}^\infty$ manifold and K be a compact subset of M. Let T be a $\mathcal{C}^1$ diffeomorphism from M to itself. We say that K is *hyperbolic* for T if for every $x \in K$ there is a splitting of the tangent space

$$T_x M = E_x^u \oplus E_x^s,$$

and constants $C > 0, \lambda > 1$ and a smooth Riemannian metric on M such that:

(i) for every $x \in K$ and $\sigma \in \{u, s\}$, we have $DT(x)\left(E_x^\sigma\right) = E_{Tx}^\sigma$;

(ii) for every $x \in K, v \in E_x^u$ and $n \in \mathbb{N}$, we have $\left|DT^{-n}(x)v\right| \leq C\lambda^{-n}\left|v\right|$;

(iii) for every $x \in K, v \in E_x^s$ and $n \in \mathbb{N}$, we have $\left|DT^{n}(x)v\right| \leq C\lambda^{-n}\left|v\right|$.

We say that K is a hyperbolic basic set for T if in addition:

- K is T-invariant, that is $T^{-1}(K) = K$;
- K is *isolated*, i.e. there is an open neighbourhood U of K such that $K = \bigcap_{n\in\mathbb{Z}} T^n(U)$ (we say that U is an isolating neighbourhood for K);
- $T_{|K}$ is *transitive*, that is T has a dense orbit in K.

Example 1.14. If T is a transitive Anosov diffeomorphism on a compact manifold M, then M is a hyperbolic basic set for T. Another classical example of hyperbolic basic set is Smale's horseshoe [Sma67, §I.5]. In this section, we are mainly interested in this last example because the associated combinatorics is simple (it is exactly conjugated to a full shift), and it is "thin" (it has small Hausdorff dimension).

The functional approach of statistical properties for hyperbolic basic sets is very similar to the case of Anosov diffeomorphisms. Let $T : M \to M$ be a $\mathcal{C}^\infty$ diffeomorphism with hyperbolic basic set K. Let $g : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ weight. Here, we must assume in addition that g is supported in a small enough isolating neighbourhood U for K. As in the case of Anosov diffeomorphism, the associated Koopman operator is defined by

$$\mathcal{L} : v \mapsto g.v \circ T. \tag{1.16}$$

Then, as in Theorem 3, there are Banach spaces on which the essential spectral radius of $\mathcal{L}$ is arbitrarily small [BT07, GL08]. However, these spaces are not intermediate between $\mathcal{C}^\infty(M)$ and $\mathcal{D}'(M)$, but between $\mathcal{C}_c^\infty(U)$ and $\mathcal{D}_c'(U)$, the spaces respectively of $\mathcal{C}^\infty$ functions and of distributions that are compactly supported in U. Definition 5 of Ruelle resonances, Ruelle spectrum and resonants states adapt then naturally to this case. As in the case of Anosov diffeomorphisms the Ruelle resonances are the zeros of a dynamical determinant, the analytic continuation of

$$d(z) = \exp\left(-\sum_{n\geq 1} \frac{1}{n} \operatorname{tr}^\flat(\mathcal{L}^n) z^n\right), \tag{1.17}$$

where the flat traces of the powers of $\mathcal{L}$ are now defined by

$$\operatorname{tr}^\flat(\mathcal{L}^n) \coloneqq \sum_{\substack{T^n x = x \\ x\in K}} \frac{\prod_{k=0}^{n-1} g\left(T^k x\right)}{\left|\det\left(I - DT^n(x)\right)\right|} \tag{1.18}$$

for $n \in \mathbb{N}^*$. In particular, the Ruelle resonances of (T, g) only depend on g through its values on K. Beware that it is not the case of the associated resonant states. Notice that Proposition 1 also applies in that case, in view of Theorem 1.3.

1.2.2 Symbolic dynamics with explicit weighted zeta functions

Denote by (Σ, σ) the full (two-sided) shift on two symbols, that is

$$\Sigma = \{0,1\}^{\mathbb{Z}} \text{ and } \sigma : (x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}.$$

For $\theta \in]0,1[$, define a distance on Σ by $d_\theta(x,y) = \theta^k$, where k is the integer $\inf\{i \in \mathbb{N} : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$ (with the convention $\theta^\infty = 0$). Recall that if $G : \Sigma \to \mathbb{C}$ is a function, the weighted zeta function associated to (σ, G) is the formal power series defined by

$$\zeta_{\sigma,G}(z) = \exp\left(\sum_{n=1}^{+\infty} \frac{1}{n}\left(\sum_{\sigma^n x = x} \prod_{k=0}^{n-1} G\left(\sigma^k x\right)\right) z^n\right). \tag{1.19}$$

Notice that $\zeta_{\sigma,1}$ is the well-known Artin–Mazur zeta function, and that the radius of convergence of $\zeta_{\sigma,G}$ is non-zero as soon as G is bounded. We are going to construct weights G for which $\zeta_{\sigma,G}$ is given by (1.20), adapting a construction from [Bal92] and [PP90, Example 1 p.165].

Proposition 1.15. *Let h be a holomorphic function defined on a neighbourhood of* 0 *and whose expansions in power series at zero is $h(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$. Denote by ρ its convergence radius, and assume that for all $k \in \mathbb{N}$ we have $\alpha_k \neq -1$. Then there is a function $G : \Sigma \to \mathbb{C}$ such that*

$$\zeta_{\sigma,G}(z)^{-1} = 1 - 2z - z(1-z)h(z). \tag{1.20}$$

Moreover, for every $\theta \in]\frac{1}{\rho}, 1[$, the function G is Lipschitz for the distance d_θ and, if $\alpha_k \in]-1, +\infty[$ for all $k \in \mathbb{N}$, then G is strictly positive.

Proof. Set $\beta_m = \frac{1+\alpha_m}{1+\alpha_{m-1}}$ if $m \geqslant 1$ and $\beta_0 = 1 + \alpha_0$ and define $G : \Sigma \to \mathbb{C}$ by

$$G(x) = \begin{cases} \beta_m & \text{if } x_0 = \cdots = x_{m-1} = 0 \text{ and } x_m = 1 \\ 1 & \text{if } x_0 = \cdots = x_i = \cdots = 0, \end{cases}$$

where $x = (x_i)_{i \in \mathbb{Z}}$. An easy computation shows that G is Lipschitz for the distance d_θ provided that $\theta \in]\frac{1}{\rho}, 1[$. For $N > 0$ define a $(N+1) \times (N+1)$ matrix P_N by

$$\begin{cases} (P_N)_{0,i} = \beta_i & \text{if } 0 \leqslant i \leqslant N-1 \\ (P_N)_{0,N} = 1 & \\ (P_N)_{i+1,i} = \beta_i & \text{if } 0 \leqslant i \leqslant N-1 \\ (P_N)_{N,N} = 1 & \\ \text{the other entries are zero,} & \end{cases}$$

that is,

$$P_N = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{N-1} & 1 \\ \beta_0 & 0 & & \dots & & 0 \\ 0 & \beta_1 & 0 & \dots & & \dots \\ \dots & 0 & \beta_2 & \dots & & \\ & & & \dots & & 0 \\ & \dots & & 0 & \beta_{N-1} & 1 \end{bmatrix}.$$

Then an elementary graph-theoretic argument provides that, for all integers $k \geqslant 1$ and all $N > k$, we have

$$\sum_{\substack{x \in \Sigma \\ \sigma^k x = x}} \prod_{i=0}^{k-1} G(\sigma^i x) = \operatorname{tr}\left(P_N^k\right). \tag{1.21}$$

Using an argument of dominated convergence (it is easy to show that $\left|\operatorname{tr}\left(P_N^k\right)\right| \leqslant 2^k \|G\|_\infty^k$ by reducing to the positive case), one may then show that, for positive small enough z,

$$\zeta_{\sigma,G}(z)^{-1} = \lim_{N \to +\infty} \det(I - zP_N).$$

A computation provides

$$\begin{aligned}\det(I - zP_N) &= (1-z)\left(1 - \sum_{k=0}^{N-1}\left(\prod_{i=0}^{k}\beta_i\right)z^{k+1}\right) - z^{N+1}\prod_{i=0}^{N-1}\beta_i \\ &= (1-z)\left(1 - \sum_{k=0}^{N-1}(1+\alpha_k)z^{k+1}\right) - z^{N+1}(1+\alpha_{N-1})\end{aligned}$$

and thus

$$\zeta_{\sigma,G}(z)^{-1} = (1-z)\left(1 - \sum_{k=0}^{+\infty}(1+\alpha_k)z^{k+1}\right) = 1 - 2z - z(1-z)h(z).$$

□

Remark 1.16. We could get a more general expression for (1.20), for instance by allowing more than two symbols. However, we will not need this here.

1.2.3 Smooth hyperbolic dynamics with explicit dynamical determinants

We want now to conjugate our symbolic examples to smooth ones. To do so, we use a method due to Bowen [Bow72a] to conjugate a subshift of finite type to a piecewise affine horseshoe.

Proposition 1.17. *There are a $\mathcal{C}^\infty$ diffeomorphism T of the sphere $\mathbb{S}^4$ and a hyperbolic basic set K for T such that, if h is as in Proposition 1.15 with in addition that $\rho > 1$, then there is a function $g : \mathbb{S}^4 \to \mathbb{C}$ such that the dynamical determinant $d(z)$ defined by* (1.17) *and* (1.18) *is given by*[3]

$$d(z) = \prod_{k=0}^{+\infty}\left(\zeta_{\sigma,G}\left(\frac{z}{4^{k+2}}\right)^{-1}\right)^{\frac{(k+1)(k+2)(k+3)}{6}}, \tag{1.22}$$

where $\zeta_{\sigma,G}$ is from Proposition 1.15. Moreover, g is $\mathcal{C}^r$ for all integers r strictly smaller than $\frac{\ln\rho}{\ln 4}$, and, if $\alpha_k \in]-1,+\infty[$ for all integers k, then g is strictly positive on K.

Proof. Let $G : \Sigma \to \mathbb{C}$ be the function given by Proposition 1.15. We next recall a construction due to Bowen [Bow72a], in order to check that it has some extra properties that suit us. Let $(e_i)_{0 \leqslant i \leqslant 3}$ be the standard basis in $\mathbb{R}^4$. Set $R(k) = 0$ if $k \geqslant 0$ and $R(k) = 1$

[3]The infinite product converges for the same reason as in Lemma 1.10.

if $k < 0$. Then, for $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$, define

$$I(x) = \sum_{k\in\mathbb{Z}} 4^{-|k|} e_{2x_k + R(k)}.$$

Then one easily checks that for $x, y \in \Sigma$ we have

$$\frac{5}{6} d_{\frac{1}{4}}(x,y) \leqslant d(I(x), I(y)) \leqslant \frac{8}{3} d_{\frac{1}{4}}(x,y), \tag{1.23}$$

where d is the euclidean distance on $\mathbb{R}^4$. Thus I induces a homeomorphism on its image K, which is a compact subset of $\mathbb{R}^4$. Define then

$$V_i = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : 1 \leqslant x_{2i} \leqslant \frac{3}{2}, 0 \leqslant x_k \leqslant \frac{1}{2} \text{ for } k \neq 2i \right\}$$

and $F_i = I(\{x \in \Sigma : x_0 = i\})$ for $i = 0, 1$. It is easy to check that F_i is contained in V_i. Define

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

For $x \in V_i$, set $F_i x = Lx - 4e_{2i} + \frac{1}{4} e_{2i+1}$ (for $i = 0, 1$). Then, define F on $V_0 \cup V_1$ by $F|_{V_i} = F_i$. One easily checks that $F \circ I = I \circ \sigma$. Viewing $\mathbb{R}^4$ as embedded in $\mathbb{S}^4$, one may extend F to a diffeomorphism T of $\mathbb{S}^4$, that coincides with F_i on a neighbourhood U_i of V_i (see for instance [Pal60]). Setting $U = U_1 \cup U_2$ one has $\bigcap_{k\in\mathbb{Z}} T^k(U) = K$. Thus K is a hyperbolic basic set for T with isolating neighbourhood U.

Now, define $\tilde{g}$ on K by $\tilde{g} = G \circ I^{-1}$. Let r be an integer strictly smaller than $\frac{\ln \rho}{\ln 4}$. Choose $\theta \in]\frac{1}{\rho}, 4^{-r}[$. Next, recalling (1.23) and that G is Lipschitz for the distance d_θ, there exists a constant C such that for all $x, y \in K$ we have

$$\frac{|\tilde{g}(x) - \tilde{g}(y)|}{d(x,y)^r} \leqslant C \frac{d_\theta\left(I^{-1}(x), I^{-1}(y)\right)}{d_{\frac{1}{4}}\left(I^{-1}(x), I^{-1}(y)\right)^r} = C\left(4^r \theta\right)^{m(x,y)}$$

where $m(x,y)$ is the smallest integer such that $I^{-1}(x)$ and $I^{-1}(y)$ do not coincide at the position $m(x,y)$ or $-m(x,y)$. Using (1.23) again, one gets

$$m(x,y) = -\frac{\ln\left(d_{\frac{1}{4}}\left(I^{-1}(x), I^{-1}(y)\right)\right)}{\ln 4} \geqslant -\frac{\ln\left(\frac{6}{5} d(x,y)\right)}{\ln 4},$$

and thus

$$\frac{|\tilde{g}(x) - \tilde{g}(y)|}{d(x,y)^r} \leqslant \tilde{C} d(x,y)^{-\frac{\ln(4^r\theta)}{\ln 4}}.$$

Consequently, Whitney extension's theorem [Whi34] ensures that $\tilde{g}$ may be extended to a

$\mathcal{C}^r$ function g on $\mathbb{S}^4$. If $\rho = +\infty$, then g may be chosen $\mathcal{C}^\infty$. Moreover, up to multiplying g by a bump function, one may assume that g is supported in U and, if G is positive, that g is positive on K.

Since I conjugates (σ, G) and $(T|_K, g)$, one has for z small enough

$$1 - 2z - z(1-z)h(z) = \zeta_{\sigma,G}(z)^{-1} = \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}\left(\sum_{\substack{x\in K\\ T^n x = x}}\prod_{k=0}^{n-1} g\left(T^k x\right)\right)z^n\right).$$

Notice that for all $n \in \mathbb{N}^*$ we have

$$\frac{1}{|\det(I - L^n)|} = \frac{1}{16^n}\sum_{k=0}^{+\infty}(-1)^k\binom{-4}{k}\frac{1}{4^{nk}},$$

and recall that $(-1)^k\binom{-4}{k} = \frac{(k+1)(k+2)(k+3)}{6}$ is an integer. Fubini's theorem gives

$$\begin{aligned} d(z) &= \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}\sum_{\substack{x\in K\\ T^n x = x}}\frac{\prod_{k=0}^{n-1} g\left(T^k x\right)}{|\det(I - D_x T^n)|}z^n\right) \\ &= \exp\left(-\sum_{n=1}^{+\infty}\frac{1}{n}\frac{1}{\det(I - L^n)}\left(\sum_{\substack{x\in K\\ T^n x = x}}\prod_{k=0}^{n-1} g\left(T^k x\right)\right)z^n\right) \\ &= \exp\left(-\sum_{n=1}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^k\binom{-4}{k}}{n}\left(\sum_{\substack{x\in K\\ T^n x = x}}\prod_{k=0}^{n-1} g\left(T^k x\right)\right)\left(\frac{z}{4^{k+2}}\right)^n\right) \\ &= \prod_{k=0}^{+\infty}\left(\zeta_{\sigma,G}\left(\frac{z}{4^{k+2}}\right)^{-1}\right)^{\frac{(k+1)(k+2)(k+3)}{6}}. \end{aligned}$$

□

Remark 1.18. In Proposition 1.17, the dynamical determinant $d(z)$ is written as an infinite product involving the weighted zeta function $\zeta_{\sigma,G}(z)$ associated to the symbolic dynamical systems from the last section. As we mentioned it in Remark 5, zeta functions are usually written as alternated product of dynamical determinants. The fact that we can go here in the other way and deduce the dynamical determinant from the zeta function is quite exceptional and due to the affine structure of the horseshoe.

We use now the preparatory lemmas from §1.1 to deduce various corollaries from Proposition 1.17. From Proposition 1.17 and Lemma 1.11, we deduce the following corollary, which is a slightly more precise statement than Proposition 2.

Corollary 1.19. *Let E be a subset of $\mathbb{N}^*$. Then there are a $\mathcal{C}^\infty$ diffeomorphism T of $\mathbb{S}^4$, a hyperbolic basic set K for T, and a $\mathcal{C}^\infty$ function $g : \mathbb{S}^4 \to \mathbb{R}$, strictly positive on K, such that, for any ordering $(\lambda_m)_{m \geqslant 0}$ of the resonances of (T, g), and for all $n \in \mathbb{N}^*$, the series*

$$\sum_{m \geqslant 0} \lambda_m^n$$

converges absolutely, and its sum is $\mathrm{tr}^\flat \mathcal{L}^n$ (defined by (1.18)) if and only if $n \in E$.

Proposition 3 is an immediate consequence of Proposition 1.17 and Lemma 1.12. From Lemmas 1.9 and 1.10, Proposition 1.17 also gives:

Corollary 1.20. *The counter-examples from Proposition 1.6 may be produced as dynamical determinants. Namely :*

a) There are a $\mathcal{C}^\infty$ diffeomorphism T of $\mathbb{S}^4$, a hyperbolic basic set K for T, and a $\mathcal{C}^\infty$ function $g : \mathbb{S}^4 \to \mathbb{R}$, strictly positive on K, such that for any ordering $(\lambda_m)_{m \geqslant 0}$ of the resonances of (T, g) (see Definition 1.1) we have, for every $n \geqslant 1$, the trace formula

$$\mathrm{tr}^\flat (\mathcal{L}^n) = \sum_{\substack{T^n x = x \\ x \in K}} \frac{\prod_{k=0}^{n-1} g\left(T^k x\right)}{\left|\det\left(I - D_x T^n\right)\right|} = \sum_{m \geqslant 0} \lambda_m^n \tag{1.24}$$

but the convergence of the right-hand side is never absolute.

b) There are a $\mathcal{C}^\infty$ diffeomorphism T of $\mathbb{S}^4$, a hyperbolic basic set K for T, a $\mathcal{C}^\infty$ function $g : \mathbb{S}^4 \to \mathbb{R}$ strictly positive on K, an ordering $(\lambda_m)_{m \geqslant 0}$ of the resonances of (T, g), and a permutation σ of $\mathbb{N}$ such that $\left(\lambda_{\sigma(m)}\right)_{m \geqslant 0}$ is an ordering of the resonances of (T, g) and, for every $n \geqslant 1$, the trace formula (1.24) holds but the series $\sum_{m \geqslant 0} \lambda_{\sigma(m)}^n$ does not converge.

From Lemma 1.10 and Proposition 1.17, it also follows that:

Corollary 1.21. *The dynamical determinant (1.17) for a $\mathcal{C}^\infty$ diffeomorphism with $\mathcal{C}^\infty$ weight on a hyperbolic basic set may be of any (finite or infinite) genus*[4]*.*

Recall that Theorem 1.3 gives a characterization of the genus of the dynamical determinant in terms of the trace formula (TFM). Moreover from Corollary 1.21 and [Gro55, Corollary 1 p.17, second part of the book], we deduce that there are dynamical determinants that are not Fredholm determinants of any nuclear operators (and so there is no "good" Banach space on which the associated Koopman operators are nuclear). Finally, we notice that Proposition 1.17 can also be used to construct systems without any resonances.

Corollary 1.22. *There are a $\mathcal{C}^\infty$ diffeomorphism T of $\mathbb{S}^4$, a hyperbolic basic set K for T, and a $\mathcal{C}^\infty$ function $g : \mathbb{S}^4 \to \mathbb{C}$, such that the system (T, g) has no Ruelle resonances.*

[4] And even of any non-integral order according to footnote 1.

In Corollary 1.22, it is fundamental that g takes value in $\mathbb{C}$: if g was positive then $e^{P_{top}(T,\log g - J_u)}$ would be a resonance, where $P_{top}(T, \log g - J_u)$ is the topological pressure of $\log g - J_u$ (where J_u is the unstable Jacobian) with respect to the dynamics T (see Remark 1.25).

Proof of Corollary 1.22. The function $h : z \mapsto \frac{e^{i\pi z}-1+2z}{z(z-1)}$ continues holomorphically to $\mathbb{C}$ and may be written as $h(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$ with $\alpha_k = -\sum_{l=1}^{k+1} \frac{(i\pi)^l}{l!} - 2$. Thus $\alpha_k \neq -1$ for every $n \in \mathbb{N}$ (this is a consequence of the fact that π is transcendental). Thus applying Proposition 1.17, we find (T, g) such that the dynamical determinant

$$d(z) = \prod_{k=0}^{+\infty} \left(e^{\frac{i\pi z}{4^{k+2}}}\right)^{\frac{(k+1)(k+2)(k+3)}{6}}$$

does not vanish. Hence, (T, g) has no resonances. □

Remark 1.23. The weight g produced by Corollary 1.19, 1.20 or 1.21 being strictly positive on K, it is associated to some physically meaningful Gibbs measure μ_g (see [Bal18, Chapter 7] for details). For example if $g = 1$ or $g = |\det(DT|_{E^u})|$ ($= 16$ in our case), μ_g is respectively the physical measure or the measure of maximal entropy for $T|_K$ (for the T we constructed these measures coincide). It may be noticed that the weights produced by Corollaries 1.19, 1.20 and 1.21 may be chosen arbitrary close to 1 in the $\mathcal{C}^\infty$ topology on a neighbourhood of K. The proof of this relies on the fact that, according to Lemmas 1.9 and 1.11, the function h may be taken arbitrarily close to 0 in the topology of the uniform convergence on all compact subsets of $\mathbb{C}$ (but, to actually prove it, an investigation of a proof of Whitney's extension theorem is needed).

Remark 1.24. Proposition 1.17 realizes a lot of entire functions as dynamical determinants, thus we could have stated many variations on Corollaries 1.19, 1.20 and 1.21. For instance, one may construct a weight g for which the trace formula (1.24) always holds but the convergence is absolute only when n is bigger than some fixed integer (replace $\frac{1}{(\ln m)^n}$ in the expression of a_n in the proof of Proposition 1.6 by $\frac{1}{m^{\alpha n}}$ for some $\alpha > 0$ and then state analogues of Lemma 1.9 and Lemma 1.10).

Remark 1.25. If in Proposition 1.17 we take $h(z) = h_{a,\rho}(z) = a \ln\left(1 + \frac{z}{\rho}\right)$, where $\rho > 1$ and $a > 0$ is small, then we get weights $g = g_{a,\rho}$, strictly positive on K. From formulae (1.20) and (1.22), we know that the radius of convergence of the associated dynamical determinant $d(z) = d_{a,\rho}(z)$ is exactly[5] $\rho_{eff} = 16\rho$. Let $r \geqslant 2$ be an integer, and choose ρ such that $r < \frac{\ln \rho}{\ln 4}$, then [BT08, 1.5] predicted a radius of convergence greater than $\rho_{pred} = \exp(-P_{top}(\log g_{a,\rho} - \log 16))\, 4^{r-1}$ for $d_{a,\rho}$. However since g is strictly positive, [Bal18, Theorem 6.2] and [Bal18, Theorem 7.5] imply that $\exp(-P_{top}(\log g_{a,\rho} - \log 16))$

[5]The dynamical determinant $d_{a,\rho}(z)$ cannot even be continued meromorphically outside the disc of center 0 and radius 16ρ.

is the smallest zero of $d_{a,\rho}(z)$, which can be made arbitrary close to $\frac{1}{32}$ by taking a close enough to 0. On the other hand, we may chose ρ arbitrary close to 4^r. Thus, for all $\epsilon > 0$, there is a choice of a and ρ such that

$$\frac{\rho_{eff}}{\rho_{pred}} \leqslant 2048 + \epsilon.$$

This means that [BT08, Theorem 1.5] described accurately the way the radius of convergence of the dynamical determinant grows when the regularity of the weight grows (up to a bounded multiplicative constant that could probably be made smaller than 2048 by adding extra parameters to the construction from Proposition 1.17).

Remark 1.26. The Ruelle resonances of the systems constructed in Proposition 1.17 comes as infinite families. In particular, Proposition 1.17 does not allow to construct a system with a finite non-zero number of resonances. As far as we know, the only known examples of systems with finitely many resonances have either one or zero resonance.

1.3 Trace formulae and dynamical determinants for Anosov flows

We go back to the continuous-time setting and prove the following analogue of Proposition 1. The implication (i) $\Rightarrow$ (ii) in Proposition 1.27 is folklore, while the implication (ii) $\Rightarrow$ (i) seems to be new. Notice that the condition (1.25) implies that the left-hand side of (TFF) converges to a distribution on $\mathbb{R}_+^*$.

Proposition 1.27. *Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$ Anosov flow, with generator X, on a compact manifold M. Let $V : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function and set $P = X + V$. Then the following statements are equivalent:*

(i) the dynamical determinant d defined by (12) *has finite order;*

(ii) the trace formula (TFF) *holds and there is $\rho > 0$ such that*

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \frac{1}{1+|\lambda|^\rho} < +\infty. \tag{1.25}$$

We recall that $\sigma_{\mathrm{R}}(P)$ is our notation for the Ruelle spectrum of P. Moreover, when these statements hold, the infimum of the ρ's such that (1.25) *hold is the order of d.*

When trying to prove trace formula (TFF), one can certainly try to apply the implication (i) $\Rightarrow$ (ii) from Proposition 1.27. However, if on the contrary one tries to understand an Anosov flow $(\phi_t)_{t\in\mathbb{R}}$ for which the trace formula (TFF) is known to hold, while we do not know if the associated dynamical determinant (12) has finite order, then the implication (ii) $\Rightarrow$ (i) in Proposition 1.27 becomes interesting. For instance, using only tools from elementary complex analysis, the following result is obtained.

Corollary 1.28. *Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$ Anosov flow, with generator X, on a compact manifold M. Let $V : M \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function and set $P = X + V$. Then, if the trace formula* (TFF) *holds and if there is $\rho \in [0, 1[$ such that*

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \frac{1}{1+|\lambda|^\rho} < +\infty, \tag{1.26}$$

then the dynamical determinant $d(z)$ defined by (12) *is constant equal to one*[6] *(in particular, P has no resonance).*

We chose to present Corollary 1.28 here because its proof is very brief, illustrating the power of the trace formula. However, when the weight V is real-valued, [JZ17, Theorem 2] gives a better bound on the number of Ruelle resonances, using only the local trace formula from [JZ17]. The main interests of these results is that they give *lower bounds* on the number of Ruelle resonances, which are not very easy to obtain in general. Considering Ruelle resonances for a constant time suspension of a cat map, we see that the bounds from Corollary 1.28 and [JZ17, Theorem 2] are quite accurate as *general* lower bounds on Ruelle resonances. However, we expect that for a *generic* Anosov flows, there are much more Ruelle resonances than predicted by those results.

Proof of Proposition 1.27. Let us first recall the folklore proof of (i) $\Rightarrow$ (ii) – see [MPM15, Theorem 17] for another approach. Let α denote the order of the dynamical determinant d. The estimate (1.25) for any $\rho > \alpha$ is a consequence of Jensen's formula – see [Boa54, 2.5.12]. Let $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}_+^*\right)$ and recall that its Laplace transform is defined for $z \in \mathbb{C}$ by

$$\mathrm{Lap}\left(\varphi\right)(z) = \int_0^{+\infty} e^{-zt}\varphi(t)\mathrm{d}t. \tag{1.27}$$

Then, choose $x > 0$ large enough so that the series

$$\sum_\gamma T_\gamma^\# \frac{e^{-xT_\gamma}}{|\det\left(I - \mathcal{P}_\gamma\right)|} \exp\left(\int_\gamma V\right) \tag{1.28}$$

converges absolutely and $x > \mathrm{Re}\left(\lambda\right) + \epsilon$ for all the resonances λ of $P = X + V$ and some $\epsilon > 0$. It follows from standard integration by parts that for every $N > 0$ there is $C_N > 0$

[6] Notice that when V is real-valued, or when $\left(\phi^t\right)_{t\in\mathbb{R}}$ has a periodic orbit γ such that no other periodic orbit has the same length, then d is not constant.

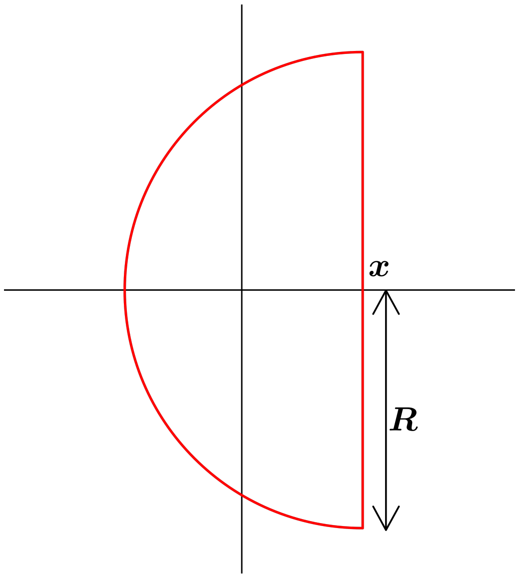

Figure 1.1: The contour Γ_R.

such that, for every $z \in \mathbb{C}$ such that $\operatorname{Re} z \geq -x$, we have

$$|\operatorname{Lap}(\varphi)(z)| \leq \frac{C_N}{1+|z|^N}. \tag{1.29}$$

Then, if γ is a periodic orbit for $(\phi_t)_{t\in\mathbb{R}}$, Laplace Inversion Formula writes

$$\varphi(T_\gamma) = \frac{1}{2i\pi}\int_{\{\operatorname{Re} z=-x\}} e^{zT_\gamma}\operatorname{Lap}(\varphi)(z)\mathrm{d}z = \frac{1}{2i\pi}\int_{\{\operatorname{Re} z=x\}} e^{-zT_\gamma}\operatorname{Lap}(\varphi)(-z)\mathrm{d}z.$$

Thanks to the bound (1.29) and the definition of x, we may apply Fubini's Theorem to find that

$$\sum_\gamma T_\gamma^{\#} e^{\int_\gamma V}\frac{\varphi(T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|} = \frac{1}{2i\pi}\int_{\{\operatorname{Re} z=x\}} h(z)\operatorname{Lap}(\varphi)(-z)\mathrm{d}z, \tag{1.30}$$

where h denotes the logarithmic derivative $h(z) = \frac{d'(z)}{d(z)}$ of the dynamical determinant d. Now, for $R>0$, define the contour Γ_R by

$$\Gamma_R = [x-iR, x+iR] \cup C_R,$$

where C_R denotes the half-circle

$$C_R = \{z \in \mathbb{C} : |z-x| = R \text{ and } \operatorname{Re} z \leq x\}.$$

By residue's formula, we have

$$\frac{1}{2i\pi}\int_{\Gamma_R} h(z)\mathrm{Lap}\,(\varphi)\,(-z) = \sum_{\lambda}\mathrm{Lap}\,(\varphi)\,(-\lambda),$$

where the sum on the right-hand side runs over Ruelle resonances of P that are enclosed by the contour Γ_R. By (1.30) and dominated convergence, we see that

$$\frac{1}{2i\pi}\int_{[x-iR,x+iR]} h(z)\mathrm{Lap}\,(\varphi)\,(-z)\mathrm{d}z \underset{R\to+\infty}{\to} \sum_{\gamma} T_\gamma^{\#} e^{\int_\gamma V}\frac{\varphi(T_\gamma)}{|\det\left(I-\mathcal{P}_\gamma\right)|}.$$

Consequently, we only need to prove that

$$\int_{C_{R_n}} h(z)\mathrm{Lap}\,(\varphi)\,(-z)\mathrm{d}z \underset{n\to+\infty}{\to} 0, \tag{1.31}$$

for a sequence of radii $(R_n)_{n\in\mathbb{N}}$ that tends to $+\infty$. Letting p be as in (1.1), we use Hadamard Factorization Theorem to write $d(z)$ as (1.2). We find then that

$$h(z) = Q'(z) + \frac{m_0}{z} - \sum_{m\geq m_0}\frac{1}{\lambda_m}\frac{\left(\frac{z}{\lambda_m}\right)^p}{1-\frac{z}{\lambda_m}},$$

where $(\lambda_m)_{m\in\mathbb{N}}$ denotes an ordering of the zeros of d (that is of the Ruelle resonances) and m_0 is the multiplicity of 0 as a Ruelle resonance. Then, let N be a large enough integer and write

$$h(z) = \widetilde{Q}(z) + \frac{m_0}{z} - \sum_{m\geq m_0}\frac{1}{\lambda_m^N}\frac{z^N}{\lambda_m - z}$$

where $\widetilde{Q}$ is the polynomial

$$\widetilde{Q}(z) = Q'(z) + \sum_{\ell=p}^{N-1}(-1)^{\ell+p}\left(\sum_{m\geq m_0}\frac{1}{\lambda_m^{\ell+1}}\right)z^\ell.$$

Thanks to (1.1), we may find a sequence $(R_n)_{n\in\mathbb{N}}$ that tends to $+\infty$ such that for every $n\in\mathbb{N}, z\in\mathcal{C}_{R_n}$ and $m\geq m_0$ we have $|z-\lambda_m|\geq|\lambda_m|^{-p-1}$. Then it follows that for $z\in\mathcal{C}_{R_n}$ we have

$$|h(z)| \leq \left|\widetilde{Q}(z)\right| + \frac{m_0}{|z|} + |z|^N\sum_{m\geq m_0}\frac{1}{|\lambda_m|^{N-p-1}},$$

and this last sum converges provided that N is large enough. Hence, we see that $|h(z)|$, for $z\in\mathcal{C}_{R_n}$, grows at most polynomially with R_n. Since $\mathcal{C}_{R_n}$ remains in the domain where (1.29) holds, (1.31) follows.

We prove now the implication (ii) $\Rightarrow$ (i). Let $x>0$ be as above. Write $k=\lceil\rho\rceil$ where ρ is from (1.25). Choose $z\in\mathbb{C}$ such that $\mathrm{Re}\,(z)>x$. Then, we can find a sequence $(\varphi_n)_{n\in\mathbb{N}}$

of $\mathcal{C}^\infty$ functions, compactly supported in $\mathbb{R}_+^*$ such that

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} e^{tx} \left|\varphi_n(t) - t^k e^{-zt}\right| = 0 \tag{1.32}$$

and

$$\sup_{\substack{n\in\mathbb{N}\\ t\in\mathbb{R}_+^*}} \left|e^{tx}\varphi_n^{(k)}(t)\right| < +\infty. \tag{1.33}$$

Then, since the series (1.28) converges absolutely, we find with (1.32) that

$$\sum_\gamma T_\gamma^\# \frac{\varphi_n(T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|} \exp\left(\int_\gamma V\right) \underset{n\to+\infty}{\to} \sum_\gamma T_\gamma^\# \frac{e^{-zT_\gamma} T_\gamma^k}{|\det(I-\mathcal{P}_\gamma)|} \exp\left(\int_\gamma V\right).$$

Now, since the trace formula holds (by assumption), we know that for all $n \in \mathbb{N}$ we have

$$\sum_\gamma T_\gamma^\# \frac{\varphi_n(T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|} \exp\left(\int_\gamma V\right) = \sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \mathrm{Lap}(\varphi_n)(-\lambda).$$

Using (1.32) and the definition (1.27) of the Laplace transform, we find that for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda < x$ we have

$$\mathrm{Lap}(\varphi_n)(-\lambda) \underset{n\to+\infty}{\to} \int_0^\infty t^k e^{-(z-\lambda)t} \mathrm{d}t = \frac{k!}{(z-\lambda)^{k+1}}.$$

Now, if $\lambda \in \mathbb{C}$ is non-zero, we have

$$\mathrm{Lap}(\varphi_n)(-\lambda) = \frac{(-1)^k}{\lambda^k} \int_0^{+\infty} e^{\lambda t} \varphi_n^{(k)}(t)\, \mathrm{d}t.$$

Thus, (1.33), the fact that $x > \operatorname{Re}(\lambda) + \epsilon$ for λ resonance, and the assumption (1.25) provide a domination of $\mathrm{Lap}(\varphi_n)(-\lambda)$, so that we have, using the dominated convergence theorem,

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \mathrm{Lap}(\varphi_n)(-\lambda) \underset{n\to+\infty}{\to} k! \sum_{\lambda \text{ resonances}} \frac{1}{(z-\lambda)^{k+1}}.$$

Finally we have (when $\operatorname{Re}(z) \gg 1$)

$$\begin{aligned} k! \sum_{\lambda\in\sigma_{\mathrm{P}}(P)} \frac{1}{(z-\lambda)^{k+1}} &= \sum_\gamma T_\gamma^\# \frac{e^{-zT_\gamma} T_\gamma^k}{|\det(I-\mathcal{P}_\gamma)|} \exp\left(\int_\gamma V\right) \\ &= (-1)^{k+1} (\ln d)^{(k+1)}(z), \end{aligned}$$

where d is the dynamical determinant defined by (12). Let F denote the canonical product of genus $k-1$ whose zeros are the Ruelle resonances of $P = X + V$ (well-defined by [Boa54, (2.6.4)] thanks to (1.25)). Then we see that, if z is not a Ruelle resonance for P, we have

$$(\ln F)^{(k)}(z) = (-1)^k (k-1)! \sum_{\lambda \in \sigma_{\mathrm{R}}(P)} \frac{1}{(z-\lambda)^k}. \tag{1.34}$$

It follows that $(\ln d)^{(k+1)} = (\ln F)^{(k+1)}$ and consequently there is a complex number a such that for every $z \in \mathbb{C}$ that is not a Ruelle resonance for X we have

$$(\ln F)^{(k)}(z) = (\ln d)^{(k)}(z) + a.$$

With assumption (1.25), (1.34) and dominated convergence we see that $(\ln F)^{(k)}(r) \underset{\substack{r \to +\infty \\ r \in \mathbb{R}}}{\to} 0$. By direct inspection, we see that $(\ln d)^{(k)}(r) \underset{\substack{r \to +\infty \\ r \in \mathbb{R}}}{\to} 0$, and consequently $a = 0$. Thus, there is a polynomial Q of degree at most $k - 1 \leq \rho$ such that, for every $z \in \mathbb{C}$, we have

$$d(z) = e^{Q(z)} F(z).$$

It follows that d has order less than ρ, since F has order less than ρ according to [Boa54, Theorem 2.6.5].

□

We end this section with the short proof of Corollary 1.28.

Proof of Corollary 1.28. Proposition 1.27 implies that the dynamical determinant d has order less than 1. But notice that d is bounded on a line (choose a line parallel to the imaginary axis corresponding to a large positive real part) and thus has to be constant by the Phragmén–Lindelöf Theorem [Boa54, Theorem 1.4.1]. Finally, it has to be constant equal to 1 since $d(z) \underset{\substack{z \to +\infty \\ z \in \mathbb{R}}}{\to} 1$. □

Chapter 2

Denjoy–Carleman classes and transfer operators

In this Chapter, we introduce the notion of *Denjoy–Carleman classes* in §2.1. Then, in §2.2, we explain our methods to study transfer or Koopman operators associated to ultradifferentiable hyperbolic dynamics through one of the most basic examples: expanding maps of the circle. In particular, the proofs of Theorems 7 and 8 may be found in §2.2. As we announced it in the introduction, we will in fact prove quantitative versions of these results (Theorem 2.9 and Propositions 2.16, 2.17 and 2.18) using the language of Denjoy–Carleman classes. Finally, in §2.3, we recall and discuss Theorem 2.27, one of the first result that we obtained by applications of our methods to hyperbolic diffeomorphisms.

Most of the contents of §2.1 and 2.2 may be found in [Jéz20b] (with the noticeable exceptions of Proposition 2.10 and Corollaries 2.11 and 2.12 that are new). Theorem 2.27 from §2.3 is proved in [Jéz20a].

2.1 Denjoy–Carleman classes

We give here a very brief introduction to the topic of Denjoy–Carleman classes. The interested reader may refer for instance to [KMR09] and references therein. For a more historical point of view on this topic, see [Bil82].

The ideas behind the notion of Denjoy–Carleman classes have first been introduced by Denjoy in [Den21] in order to understand the notion of *quasi-analyticity* that was proposed by Borel in [Bor12]. However, it seems that the notion of Denjoy–Carleman classes as we know it today has been introduced by Carleman in [Car23]. We give now this definition.

Let $A = (A_k)_{k \in \mathbb{N}}$ be an increasing and logarithmically convex sequence of positive real numbers such that $A_0 = 1$. Recall that the fact that A is logarithmically convex means that

$$\forall k \in \mathbb{N}^* : A_k^2 \leq A_{k-1}A_{k+1}. \tag{2.1}$$

Let us mention for later use that (2.1) implies that for every $k_1, \dots, k_n \in \mathbb{N}$ we have

$$\prod_{j=1}^{n} A_{k_j} \leq A_k, \tag{2.2}$$

where $k = k_1 + \dots + k_n$. This fact may be proved by induction using the fact that the sequence $(A_{k+1}/A_k)_{k\in\mathbb{N}}$ is increasing.

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{C}$ is a $\mathcal{C}^\infty$ function, we say that f belongs to the Denjoy–Carleman class $\mathcal{C}^A$ if, for every compact subset K of U, there are constants $C, R > 0$ such that for every $x \in K$ and $\alpha \in \mathbb{N}^n$ we have

$$|\partial^\alpha f(x)| \leq CR^{|\alpha|}\alpha! A_{|\alpha|}. \tag{2.3}$$

Replacing the modulus in the left-hand side of (2.3) by a norm, we define what it means for a function valued in a Banach space to be $\mathcal{C}^A$. Parallelizing the circle in the usual way, this definition is immediately adapted to the case of functions on $\mathbb{S}^1$: we say that a $\mathcal{C}^\infty$ function $f : \mathbb{S}^1 \to \mathbb{C}$ is in the Denjoy–Carleman class $\mathcal{C}^A$ if there are constants $C, R > 0$ such that for all $k \in \mathbb{N}$ and $x \in \mathbb{S}^1$ we have

$$\left| f^{(k)}(x) \right| \leq CR^k k! A_k. \tag{2.4}$$

We will say that a map $T : \mathbb{S}^1 \to \mathbb{S}^1$ belongs to the class $\mathcal{C}^A$ if its lift from $\mathbb{R}$ to $\mathbb{R}$ does. This definition will only be used in Corollary 2.12. We will rather assume when needed that the derivative $T' : \mathbb{S}^1 \to \mathbb{R}$ belongs to the class $\mathcal{C}^A$. When the class $\mathcal{C}^A$ is not closed under differentiation, it does not imply that T belongs to $\mathcal{C}^A$.

Example 2.1. Let $s \geq 1$. If $A = (k!^{s-1})_{k\in\mathbb{N}}$, then the class of regularity $\mathcal{C}^A$ is the class of s-Gevrey functions, that we will often denote by $\mathcal{G}^s$ in this thesis. We will give more detailed information on this class of regularity in §4.1.1. Notice that in the case $s = 1$, we retrieve the class of real-analytic functions (this is an easy consequence of Taylor's and Cauchy's Formulae).

Example 2.2. Let $\kappa > 0$ and $\upsilon \geq 1$. We denote by $\mathcal{C}^{\kappa,\upsilon}$ the class of regularity obtained by taking $A = (\exp(k^\upsilon/\kappa\upsilon))_{k\in\mathbb{N}}$. These are not standard classes of regularity. Actually, we are not aware of any appearance of these particular classes in the literature before our work, but the ultradifferentiable classes that are used in [TT17] look a bit alike. However, the classes $\mathcal{C}^{\kappa,\upsilon}$ are very interesting when considering hyperbolic dynamics: see §2.2.3 but especially Chapter 3. Indeed, when $\upsilon < 2$, the trace formula (TFF) holds for Anosov flow of regularity $\mathcal{C}^{\kappa,\nu}$. See §3.1 for a more advanced discussion on this classes of regularity – in particular, we will define what it means for a flow on a compact manifold to be of regularity $\mathcal{C}^{\kappa,\nu}$.

Our use of Denjoy–Carleman classes will be very basic, let us just recall a few well-known facts for the sake of contextualizing. It follows from the logarithmic convexity of $(A_k)_{k\in\mathbb{N}}$ that the class of functions $\mathcal{C}^A$ is closed under multiplication and composition – see [Rud87, Chapter 19]. We say that the class $\mathcal{C}^A$ is *quasi-analytic* if the following holds: for every function f of class $\mathcal{C}^A$ on a connected open subset U of $\mathbb{R}^n$, if there is $x_0 \in U$ such that all the derivatives of f vanish at x_0 then f is identically equal to 0. If the class $\mathcal{C}^A$ is not quasi-analytic, then there are compactly supported functions of regularity $\mathcal{C}^A$ and, consequently, $\mathcal{C}^A$ partitions of unity. The original motivation behind the introduction of Denjoy–Carleman classes is the *Denjoy–Carleman Theorem* that characterizes quasi-analytic classes among them.

Theorem 2.3 ([Den21, Car22, Car23]). *Let $A = (A_k)_{k\in\mathbb{N}}$ be an increasing, logarithmically convex sequence of positive real numbers such that $A_0 = 1$. Then the Denjoy–Carleman class $\mathcal{C}^A$ is quasi-analytic if and only if*

$$\sum_{k=0}^{+\infty} \frac{1}{(k!A_k)^{\frac{1}{k}}} = +\infty.$$

In Chapters 3 and 4, we will mostly be interested in the non-quasi-analytic case – with the exception of the class of real-analytic functions in Chapter 4. However, in the following section §2.2, we do not need to distinguish between quasi-analytic and non-quasi-analytic classes. This is possible thanks to the simple geometry of our context. We think that the following result can be enlightening considering the question of trace formulae.

Proposition 2.4. *Let $A = (A_k)_{k\in\mathbb{N}}$ be an increasing, logarithmically convex sequence of positive real numbers such that $A_0 = 1$. Then the Denjoy–Carleman class $\mathcal{C}^A$ is closed under differentiation if and only if*

$$\sup_{k\in\mathbb{N}^*} \left(\frac{A_{k+1}}{A_k}\right)^{\frac{1}{k}} < +\infty. \tag{2.5}$$

It is clear that (2.5) is a sufficient condition for $\mathcal{C}^A$ to be closed under differentiation. As noticed in [KMR09], it is also necessary as a consequence of [Thi08, Theorem 1].

Remark 2.5. Let us consider the classes $\mathcal{C}^{\kappa,\upsilon}$ from Example 2.2. For $\kappa > 0$ and $\upsilon \geq 1$, we deduce from Proposition 2.4 that the class $\mathcal{C}^{\kappa,\upsilon}$ is closed under differentiation if and only if $\upsilon \leq 2$. When $\upsilon < 2$, we have the estimate

$$\lim_{k\to+\infty} \left(\frac{A_{k+1}}{A_k}\right)^{\frac{1}{k}} = 1, \tag{2.6}$$

a stronger condition than (2.5). We will see in §2.2 below that, in the case of expanding maps of the circle, the condition (2.6) alone ensures that the trace formula (TFM) is satisfied by expanding maps of the circle of regularity $\mathcal{C}^A$. The case of Denjoy–Carleman classes for which the weaker condition (2.5) holds but not (2.6) is more involved but very interesting (see Remark 2.22 and §2.2.3).

We tend consequently to think that, when considering the question of trace formulae in Denjoy–Carleman classes, it is relevant to distinguish classes that are closed under differentiation or not (and, among those closed under differentiation, that satisfy (2.6) or not). We will argue in favor of this approach in Remark 2.22, see also the examples from §2.2.3 for a more concrete approach of this point. Actually, being closed under differentiation is a reasonable assumption to make on an ultradifferentiable class if someone wants to make differential geometry in that class (see [KMR09] for a discussion of that fact). Proposition 2.10 and Corollaries 2.11 and 2.12 also reinforce the idea that Denjoy–Carleman classes closed under differentiation are better-behaved than the others.

Let us now discuss some technical facts that are required for the next section. To the class $\mathcal{C}^A$, we associate the function $w = w^A$ on $\mathbb{R}_+^*$ defined by

$$\forall x \in \mathbb{R}_+^* : w(x) \coloneqq \inf_{k \in \mathbb{N}} x^k k! A_k. \tag{2.7}$$

The function w will play a fundamental role in estimates on singular values and norms of operators appearing in the nuclear power decomposition of the transfer operator. We are not aware of any reference introducing precisely the function w, but it seems common to introduce similar objects adapted to a particular problem (see for instance [FNRS20, (1.1)]). The following lemma lists basic properties of the function w.

Lemma 2.6. *Let $A = (A_k)_{k \in \mathbb{N}}$ be an increasing, logarithmically convex sequence of positive real numbers such that $A_0 = 1$. Then, the function $w = w^A$ is continuous and increasing from $\mathbb{R}_+^*$ to itself. Moreover, w vanishes at all orders in 0, i.e. for all $\alpha \in \mathbb{R}$ we have $x^\alpha w(x) \underset{x \to 0}{\to} 0$. If $\mu \in]0,1[$ then $w(\mu x)/w(x) \underset{x \to 0}{\to} 0$. If in addition $\gamma > 1$ is such that there is $C > 0$ such that for all $k \in \mathbb{N}$ we have*

$$(k+1)A_{k+1} \leq C\gamma^k A_k, \tag{2.8}$$

then, if $\mu \in]0,1[$, there is a constant C' such that for all $x > 0$ we have

$$\frac{w(\mu x)}{w(x)} \leq C' x^\delta, \tag{2.9}$$

where $\delta = -\log \mu / \log \gamma$.

Remark 2.7. Let us mention for further reference that it follows from Proposition 2.4

that there is $\gamma > 1$ such that the condition (2.8) is satisfied by the sequence $(A_k)_{k\in\mathbb{N}}$ for every $k \in \mathbb{N}$ if and only if the class $\mathcal{C}^A$ is closed under differentiation.

Proof of Lemma 2.6. Since w is defined as an infimum of increasing functions, w is increasing. Since $w(x)$ is smaller than $x^k k! A_k$ for all k, it is clear that w vanishes at all orders in 0. If $x \in \mathbb{R}_+^*$, since $x^k k! A_k \underset{k\to+\infty}{\to} +\infty$, the infimum in the definition of $w(x)$ is attained by a finite number of integers k. Denote by $k(x)$ the largest integer that realizes this infimum. Notice that if $\ell \leq m$ then the logarithmic derivative of $x \mapsto x^\ell \ell! A_\ell$ is smaller than that of $x \mapsto x^m m! A_m$. Consequently, the function $x \mapsto k(x)$ is decreasing. Thus if $x_0 > 0$ then for all $x > x_0$ since $k(x) \leq k(x_0)$ we have

$$w(x) = x^{k(x)} k(x)! A_{k(x)} = \min_{n=0,\dots,k(x_0)} x^n n! A_n,$$

and consequently w is continuous on $]x_0, +\infty[$. Since $x_0 > 0$ is arbitrary, w is continuous on $\mathbb{R}_+^*$.

Let μ be an element of $]0,1[$. Notice that for all $x > 0$ we have

$$\frac{w(\mu x)}{w(x)} = \frac{w(\mu x)}{x^{k(x)} k(x)! A_{k(x)}} \leq \frac{(\mu x)^{k(x)} k(x)! A_{k(x)}}{x^{k(x)} k(x)! A_{k(x)}} = \mu^{k(x)}, \tag{2.10}$$

and since it is clear that $k(x) \underset{x\to0}{\to} +\infty$, we get that $w(\mu x)/w(x) \underset{x\to0}{\to} 0$. Assume now that (2.8) holds. Notice that if $0 < x < C^{-1}$ then

$$\left(\frac{x}{\gamma}\right)^{k(x)+1} (k(x)+1)! A_{k(x)+1} \leq x^{k(x)} k(x)! A_{k(x)},$$

and thus we have

$$k\left(\frac{x}{\gamma}\right) \geq k(x) + 1.$$

Now, if $0 < x < C^{-1}$, letting n be the largest integer such that $\gamma^n x < C^{-1}$, we find that

$$k(x) = k\left(\frac{\gamma^n x}{\gamma^n}\right) \geq k(\gamma^n x) + n \geq n \geq -\frac{\log x}{\log \gamma} - a,$$

where $a = \log(\gamma C) / \log \gamma$. Thus by (2.10) we find that if $0 < x < C^{-1}$ then

$$\frac{w(\mu x)}{w(x)} \leq C' x^\delta,$$

where $C' = \mu^{-a}$. □

We end this section with a lemma that implies that every $\mathcal{C}^\infty$ function on the circle belongs to some Denjoy–Carleman class. It allows us to deduce Theorems 7 and 8 from their quantitative versions Theorem 2.9 and Propositions 2.16, 2.17 and 2.18. We omit the elementary proof.

Lemma 2.8. *Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of non-negative real numbers. Then there are a constant $C > 0$ and an increasing and logarithmically convex sequence $(B_k)_{k\in\mathbb{N}}$ of positive real numbers such that $B_0 = 1$ and*

$$\forall k \in \mathbb{N} : A_k \leq CB_k.$$

2.2 Transfer operator for ultradifferentiable expanding maps of the circle

We are now going to use the notion of Denjoy–Carleman classes in order to prove Theorems 7 and 8. Let thus T be a $\mathcal{C}^\infty$ expanding map of the circle (in the sense of Definition 1). Denote by $\lambda > 1$ a dilation constant for T. Moreover, let $A = (A_k)_{k\in\mathbb{N}}$ be an increasing, logarithmically convex sequence of positive real numbers such that $A_0 = 1$, and assume that $T' : \mathbb{S}^1 \to \mathbb{R}$ is of class $\mathcal{C}^A$. We recall that the transfer operator $\mathcal{L}$ associated to T is defined by

$$\mathcal{L} : u \mapsto \left(x \mapsto \sum_{Ty=x} \frac{1}{|T'(y)|} u(y) \right). \tag{2.11}$$

Using the function $w = w^A$ defined by (2.7), we can now state a quantitative version of Theorem 7.

Theorem 2.9. *For every $\theta \in \,]1,\lambda[$ there are constants $C, M > 0$ and a Hilbert space $\mathcal{H}$ continuously contained in $\mathcal{C}^\infty\left(\mathbb{S}^1\right)$ and containing trigonometric polynomials as a dense subspace, such that $\mathcal{L}$ defines a compact operator from $\mathcal{H}$ to itself. Moreover, if $(\sigma_k)_{k\in\mathbb{N}}$ is the sequence of singular values of $\mathcal{L}$ acting on $\mathcal{H}$ then we have*

$$\forall k \in \mathbb{N}^* : \sigma_k \leq C \sup_{0<x\leq\frac{1}{k}} \frac{w(Mx)}{w(\theta M x)}. \tag{2.12}$$

The proof of Theorem 2.9 is carried on in §2.2.1. In order to promote our idea that Denjoy–Carleman classes that are closed under differentiation are much better-behaved than the others, we will also prove the following:

Proposition 2.10. *Assume that the Denjoy–Carleman class $\mathcal{C}^A$ is closed under differentiation. Then the elements of the Hilbert space $\mathcal{H}$ from Theorem 2.9 belong to the Denjoy–Carleman class $\mathcal{C}^A$.*

If Proposition 2.10 does not look that impressing, it admits the two following corollaries that may be more interesting.

Corollary 2.11. *Assume that the Denjoy–Carleman class $\mathcal{C}^A$ is closed under differentiation. Then the resonant states of $\mathcal{L}$ are of class $\mathcal{C}^A$.*

Corollary 2.12. *Assume that the Denjoy–Carleman class $\mathcal{C}^A$ is closed under differentiation. Let T_1 and T_2 be expanding maps of the circle of regularity $\mathcal{C}^A$ and assume that they are $\mathcal{C}^1$ conjugated. Then the conjugacy is actually of regularity $\mathcal{C}^A$.*

Notice that Proposition 2.10 and Corollaries 2.11 and 2.12 are not from [Jéz20b]. Concerning Corollary 2.11, it is well-known that the resonant states of an expanding map of the circle T are $\mathcal{C}^\infty$ under the sole assumption that T is $\mathcal{C}^\infty$ (see Lemma B.1).

We will then prove Theorem 8 in §2.2.2. Actually, this proof will also come with quantitative estimates – see Propositions 2.16, 2.17 and 2.18. Finally, we will see how these results specify to the particular classes from Examples 2.1 and 2.2 in §2.2.3.

In Appendix A, we explain how the results from this section can be adapted to study weighted transfer operators that are more general than (2.11).

2.2.1 Compactness of the transfer operator

Let us start the proof of Theorem 2.9. Let $\theta \in]1, \lambda[$ be fixed once for all. If $n \in \mathbb{Z}$, we write e_n for the function on the circle $e_n : x \mapsto e^{2i\pi nx}$. Define the family $(\pi_n)_{n\in\mathbb{N}}$ of orthogonal projectors on $L^2\left(\mathbb{S}^1\right)$ by

$$\pi_n u = \begin{cases} \langle u, e_0\rangle_{L_2} e_0 & \text{if } n = 0 \\ \sum_{\theta^{n-1} \leq |k| < \theta^n} \langle u, e_k\rangle_{L^2} e_k & \text{otherwise} \end{cases}.$$

In order to give the definition of the space $\mathcal{H}$ from Theorem 2.9, we need to state a technical but fundamental result.

Lemma 2.13. *There are constants $C, R > 0$ such that for all $m, n \in \mathbb{N}$ and $u \in L^2\left(\mathbb{S}^1\right)$ such that $m \geq n$ we have*

$$\left\|\pi_m \mathcal{L} \pi_n u\right\|_{L^2} \leq C w\left(\frac{R}{\theta^m}\right) \theta^{\frac{m+n}{2}} \left\|\pi_n u\right\|_{L^2}.$$

We can now define $\mathcal{H} = \mathcal{H}_{\theta,R,A}$ as the space of $u \in L^2\left(\mathbb{S}^1\right)$ such that (R is the constant from Lemma 2.13):

$$\sum_{m\in\mathbb{N}} \lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2} \left\|\pi_m u\right\|_{L^2}^2 < +\infty. \tag{2.13}$$

It is easily seen that the square root of the quantity above defines a norm for which $\mathcal{H}$ is a Hilbert space. From Lemma 2.6, the quantity $\lambda^{-m} w\left(\frac{R}{\theta^{m-1}}\right)^{-1}$ tends to infinity faster than any geometric sequence when m tends to infinity. Consequently, the space $\mathcal{H}$ is continuously contained in $\mathcal{C}^\infty\left(\mathbb{S}^1\right)$. One can check easily that trigonometric polynomials form a dense subspace of $\mathcal{H}$.

Before proving Lemma 2.13, we need another technical result.

Lemma 2.14. *There are constants $C, R > 0$ such that for all $k, \ell \in \mathbb{Z}$ such that $|k| > \theta^{-1}|\ell|$ we have*

$$\left|\langle \mathcal{L} e_\ell, e_k \rangle_{L^2}\right| \leq C w\left(\frac{R}{|k|}\right).$$

Proof. Define the function $a_{k,\ell} : \mathbb{S}^1 \to \mathbb{C}$ by $a_{k,\ell}(x) = \frac{1}{2i\pi(kT'(x)-\ell)}$ and the differential operator

$$L_{a_{k,\ell}} : u \mapsto (a_{k,\ell} u)'.$$

Then for all $m \in \mathbb{N}$ we have

$$\langle \mathcal{L} e_\ell, e_k \rangle_{L^2} = \int_{\mathbb{S}^1} e^{2i\pi(\ell x - kT(x))} \mathrm{d}x = \int_{\mathbb{S}^1} e^{2i\pi(\ell x - kT(x))} L_{a_{k,\ell}}^m(1)(x) \mathrm{d}x,$$

so that

$$\left|\langle \mathcal{L} e_\ell, e_k \rangle_{L^2}\right| \leq \left\| L_{a_{k,\ell}}^m(1) \right\|_\infty.$$

In order to bound $L_{a_{k,\ell}}^m(1)$, we first investigate the derivatives of $a_{k,\ell}$. By Faa di Bruno's formula, for all $n \in \mathbb{N}$ and $x \in \mathbb{S}^1$ we have

$$\begin{aligned} a_{k,\ell}^{(n)}(x) = \frac{1}{2i\pi} \sum_{m_1 + 2m_2 + \dots + nm_n = n} & (-1)^{m_1 + \dots m_n} \frac{n!(m_1 + \dots + m_n)!}{m_1! \dots m_n!} \\ & \times \frac{1}{(kT'(x) - \ell)^{1 + m_1 + \dots + m_n}} \prod_{j=1}^n \left(\frac{kT^{(1+j)}(x)}{j!} \right)^{m_j}. \end{aligned}$$

Thus, since T' belongs to the class $\mathcal{C}^A$,

$$\begin{aligned} \left| a_{k,\ell}^{(n)}(x) \right| \leq \frac{1}{2\pi} \sum_{m_1 + 2m_2 + \dots + nm_n = n} & \frac{n!(m_1 + \dots + m_n)!}{m_1! \dots m_n!} \\ & \times \frac{1}{|kT'(x) - \ell|^{1 + m_1 + \dots + m_n}} \prod_{j=1}^n \left(|k| C R^j A_j \right)^{m_j}, \end{aligned}$$

where $C, R > 0$ are from the definition of $\mathcal{C}^A$. Recall (2.2) and notice that we have $|kT'(x) - \ell| \geq \lambda |k| - |\ell| > \delta |k|$, where $\delta = \lambda - \theta$. Thus, we find (assuming that $C > 1$ and

$\delta < 1$, which is true without loss of generality)

$$\left|a_{k,\ell}^{(n)}(x)\right| \leq \frac{A_n}{2\pi\delta\,|k|}\left(\frac{CR}{\delta}\right)^n \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!(m_1+\cdots+m_n)!}{m_1!\ldots m_n!}.$$

Now, notice that

$$\sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!(m_1+\cdots+m_n)!}{m_1!\ldots m_n!} = \begin{cases} 1 & \text{if } n=0 \\ 2^{n-1}n! & \text{otherwise} \end{cases}.$$

Indeed, as a consequence of Faa di Bruno's formula, the sum in the left hand side is the nth derivative at zero of the function

$$x \mapsto 1 + \frac{x}{1-2x} = \frac{1}{1-\frac{x}{1-x}}.$$

Notice then that for all $m \in \mathbb{N}$ there are natural integer coefficients that do not depend on $a_{k,\ell}$ such that

$$L_{a_{k,\ell}}^m 1 = \sum_{n_1+\cdots+n_m=m} c_{n_1,\ldots,n_m} \prod_{j=1}^m a_{k,\ell}^{(n_j)}. \tag{2.14}$$

Thus, using (2.2) again,

$$\begin{aligned}\left\|L_{a_{k,\ell}}^m(1)\right\|_\infty &\leq \sum_{n_1+\cdots+n_m=m} c_{n_1,\ldots,n_m} \prod_{j=1}^m \left(n_j!\frac{A_{n_j}}{2\pi\delta\,|k|}\left(\frac{2CR}{\delta}\right)^{n_j}\right) \\ &\leq A_m\left(\frac{CR}{\pi\delta^2\,|k|}\right)^m \sum_{n_1+\cdots+n_m=m} c_{n_1,\ldots,n_m}\prod_{j=1}^m n_j!.\end{aligned}$$

Now replacing $a_{k,\ell}$ by the function $a : x \mapsto \frac{1}{1-x}$ in (2.14) we have that (notice that $L_a^m(1) : x \mapsto \frac{(2m)!}{m!2^m}\frac{1}{(1-x)^{2m}}$, where L_a is the differential operator defined by $L_a(u) = (au)'$)

$$\sum_{n_1+\cdots+n_m=m} c_{n_1,\ldots,n_m}\prod_{j=1}^m n_j! = L_a^m(1)(0) = \frac{(2m)!}{m!2^m}.$$

Thus

$$\left\|L_{a_{k,\ell}}^m(1)\right\|_\infty \leq m!A_m\frac{(2m)!}{m!^2}\left(\frac{CR}{2\pi\delta^2\,|k|}\right)^m.$$

We only need to notice that $(2m)!/m!^2$ grows at most exponentially to end the proof (with different values of C and R of course). □

Proof of Lemma 2.13. We will only deal with the case $n \neq 0$, the case $n = 0$ is similar.

Let us compute (here $\mathcal{L}^*$ denotes the L^2-adjoint of $\mathcal{L}$, that is the Koopman operator):

$$\begin{aligned}
\|\pi_m \mathcal{L} \pi_n u\|_{L^2}^2 &= \sum_{\theta^{m-1} \leq |k| < \theta^m} |\langle \mathcal{L}\pi_n u, e_k \rangle_{L^2}|^2 = \sum_{\theta^{m-1} \leq |k| < \theta^m} |\langle \pi_n u, \mathcal{L}^* e_k \rangle_{L^2}|^2 \\
&\leq \|\pi_n u\|_{L^2}^2 \sum_{\theta^{m-1} \leq |k| < \theta^m} \|\pi_n \mathcal{L}^* e_k\|_{L^2}^2 \\
&\leq \|\pi_n u\|_{L^2}^2 \sum_{\theta^{m-1} \leq |k| < \theta^m} \sum_{\theta^{n-1} \leq |\ell| < \theta^n} |\langle \mathcal{L} e_\ell, e_k \rangle_{L^2}|^2 .
\end{aligned}$$

Now, if $\theta^{m-1} \leq |k| < \theta^m$ and $\theta^{n-1} \leq |\ell| < \theta^n$ then we have $|k| \geq \theta^{m-1} \geq \theta^{n-1} > \theta^{-1} |\ell|$ (since $m \geq n$) and thus by Lemma 2.14 we have (recall that w is increasing)

$$|\langle \mathcal{L} e_\ell, e_k \rangle_{L^2}| \leq C w \left(\frac{R}{|k|} \right) \leq C w \left(\frac{\theta R}{\theta^m} \right).$$

Consequently,

$$\|\pi_m \mathcal{L} \pi_n u\|_{L^2}^2 \leq 4C^2 \|\pi_n u\|_{L^2}^2 \left(\theta^m - \theta^{m-1} + 1\right) \left(\theta^n - \theta^{n-1} + 1\right) w \left(\frac{\theta R}{\theta^m} \right)^2$$

and the result follows. □

We will need another technical result to prove Theorem 2.9. For $N \in \mathbb{N}$, define the following finite rank operators on $\mathcal{H}$:

$$\mathcal{L}_{c,N} = \sum_{0 \leq n \leq m \leq N} \pi_m \mathcal{L} \pi_n \text{ and } \mathcal{L}_{b,N} = \sum_{0 \leq m < n \leq N} \pi_m \mathcal{L} \pi_n. \tag{2.15}$$

We will use these finite rank operators to approximate the transfer operator $\mathcal{L}$, to do so we need the following lemma.

Lemma 2.15. *There is a constant $C > 0$ such that for all $M \geq N \geq 0$ we have*

$$\|\mathcal{L}_{c,N} - \mathcal{L}_{c,M}\|_{L^2 \to \mathcal{H}} \leq C \sup_{m > N} \frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}$$

and

$$\|\mathcal{L}_{b,N} - \mathcal{L}_{b,M}\|_{\mathcal{H} \to \mathcal{H}} \leq C \sup_{m \geq N} \frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}.$$

Proof. If $u \in \mathcal{H}$ then we have

$$\left(\mathcal{L}_{c,M} - \mathcal{L}_{c,N}\right) u = \sum_{\substack{0 \leq n \leq m \leq M \\ N < m}} \pi_m \mathcal{L} \pi_n u$$

and thus

$$\left\|\left(\mathcal{L}_{c,M}-\mathcal{L}_{c,N}\right)u\right\|_{\mathcal{H}}^2=\sum_{N<m\leq M}\lambda^{-2m}w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\left\|\pi_m\mathcal{L}\sum_{n\leq m}\pi_n u\right\|_{L^2}^2.$$

But if $N<m\leq M$ we have with Lemma 2.13

$$\begin{aligned}\left\|\pi_m\mathcal{L}\sum_{n\leq m}\pi_n u\right\|_{L^2} &\leq \sum_{n\leq m}\left\|\pi_m\mathcal{L}\pi_n u\right\|_{L^2}\\ &\leq Cw\left(\frac{R}{\theta^m}\right)\theta^{\frac{m}{2}}\sum_{n\leq m}\theta^{\frac{n}{2}}\left\|\pi_n u\right\|_{L^2}\\ &\leq \widetilde{C}w\left(\frac{R}{\theta^m}\right)\theta^m\sqrt{\sum_{n\leq m}\left\|\pi_n u\right\|_{L^2}^2},\end{aligned}$$

and thus (for some new constant C that may change from one line to another)

$$\begin{aligned}\left\|\left(\mathcal{L}_{c,M}-\mathcal{L}_{c,N}\right)u\right\|_{\mathcal{H}}^2 &\leq C\sup_{m>N}\left(\frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}\right)^2\sum_{n\geq 0}\left(\sum_{m>N}\frac{\theta^{2m}}{\lambda^{2m}}\right)\left\|\pi_n u\right\|_{L^2}^2\\ &\leq C\sup_{m>N}\left(\frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}\right)^2\sum_{n\geq 0}\left\|\pi_n u\right\|_{L^2}^2\\ &\leq C\sup_{m>N}\left(\frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}\right)^2\left\|u\right\|_{L^2}^2.\end{aligned}$$

Before proving the second estimate, let us show that there is a constant $C>0$ such that for every integer n we have

$$\sum_{0\leq m<n}\lambda^{-2m}w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\leq C\lambda^{-2n}w\left(\frac{R}{\theta^{n-2}}\right)^{-2}. \tag{2.16}$$

To do so, recall the function $k(x)$ from the proof of Lemma 2.6 and choose m_0 large enough so that

$$\frac{\lambda^2}{\theta^{2k\left(\frac{R}{\theta^{m_0-1}}\right)}}<1.$$

Then, when n is large enough, we may split the sum in (2.16) between the sum over $0\leq m<m_0$ and the sum over $m_0\leq m<n$. The first sum is independent on n, and can consequently be ignored since the right-hand side of (2.16) tends to $+\infty$ when n tends to

$+\infty$, according to Lemma 2.6. To bound the second sum, recall (2.10) to see that

$$\begin{aligned}\lambda^{2n} w\left(\frac{R}{\theta^{n-2}}\right)^{2} \sum_{m_0\leq m<n} \lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2} &\leq \sum_{m_0\leq m<n} \lambda^{2(n-m)}\left(\theta^{m-n+1}\right)^{2k\left(\frac{R}{\theta^{m-1}}\right)}\\ &\leq \lambda^2 \sum_{\ell\geq 0}\left(\frac{\lambda^2}{\theta^{2k\left(\frac{R}{\theta^{m_0-1}}\right)}}\right)^{\ell} < +\infty.\end{aligned}$$

We turn now to the proof of the second estimate and write for $u \in \mathcal{H}$

$$(\mathcal{L}_{b,M} - \mathcal{L}_{b,N})\, u = \sum_{\substack{0\leq m<n\leq M\\ N<n}} \pi_m \mathcal{L} \pi_n u$$

from which we get (we use (2.16) on the fifth line and C may change from one line to another)

$$\begin{aligned}\|(\mathcal{L}_{b,M} - \mathcal{L}_{b,N})\, u\|_{\mathcal{H}}^2 &= \sum_{0\leq m<M} \lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\left\|\pi_m\mathcal{L}\sum_{n>\max(m,N)}\pi_n u\right\|_{L^2}^2\\ &\leq C\sum_{0\leq m<M} \lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\left\|\sum_{n>\max(m,N)}\pi_n u\right\|_{L^2}^2\\ &\leq C\sum_{0\leq m<M} \lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\sum_{n>\max(m,N)}\|\pi_n u\|_{L^2}^2\\ &\leq C\sum_{n>N}\left(\sum_{0\leq m<n}\lambda^{-2m} w\left(\frac{R}{\theta^{m-1}}\right)^{-2}\right)\|\pi_n u\|_{L^2}^2\\ &\leq C\sum_{n>N}\lambda^{-2n} w\left(\frac{R}{\theta^{n-2}}\right)^{-2}\|\pi_n u\|_{L^2}^2\\ &\leq C\sup_{n>N}\left(\frac{w\left(\frac{R}{\theta^{n-1}}\right)}{w\left(\frac{R}{\theta^{n-2}}\right)}\right)^2\|u\|_{\mathcal{H}}^2 .\end{aligned}$$

□

We are now in position to end the proof of Theorem 2.9.

Proof of Theorem 2.9. Lemma 2.15 implies in particular that the sequence $(\mathcal{L}_{c,N})_{N\in\mathbb{N}}$ is a Cauchy sequence of bounded operators from L^2 to $\mathcal{H}$ and thus converges to a bounded operator $\mathcal{L}_c : L^2 \to \mathcal{H}$. For the same reason, $(\mathcal{L}_{b,N})_{N\in\mathbb{N}}$ converges to a bounded operator $\mathcal{L}_b : \mathcal{H} \to \mathcal{H}$. By checking the identity on trigonometric polynomials, we see that

$$\mathcal{L} = \mathcal{L}_c + \mathcal{L}_b. \tag{2.17}$$

In particular, $\mathcal{L}$ is bounded (and even compact, as a limit of finite rank operators) from $\mathcal{H}$ to itself. The only thing that we still need to check is the bound (2.12) on singular values of the operator $\mathcal{L}$ acting on $\mathcal{H}$. If $N \in \mathbb{N}$, notice that the operator $\mathcal{L}_{c,N} + \mathcal{L}_{b,N}$ has rank at most $2\lceil \theta^N \rceil + 1$. From Lemma 2.15 (letting M tend to infinity), we deduce that

$$\left\| \mathcal{L} - (\mathcal{L}_{c,N} + \mathcal{L}_{b,N}) \right\|_{\mathcal{H} \to \mathcal{H}} \leq 2C \sup_{m \geq N} \frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}$$

and thus (see [GGK00, Theorem IV.2.5])

$$\sigma_{2\lceil \theta^N \rceil + 2} \leq 2C \sup_{m \geq N} \frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)}.$$

The result then follows from the fact that the sequence $(\sigma_k)_{k \in \mathbb{N}}$ is decreasing. □

We saw that $\mathcal{H}$ is continuously contained in $\mathcal{C}^\infty$. However, when the class $\mathcal{C}^A$ is closed under differentiation, we can improve this result and prove Proposition 2.10.

Proof of Proposition 2.10. Let $u \in \mathcal{H}$. Since u is $\mathcal{C}^\infty$, we may write it as the sum of its Fourier series

$$u = \sum_{k \in \mathbb{Z}} \langle u, e_k \rangle_{L^2} e_k. \tag{2.18}$$

From the definition of $\mathcal{H}$, we see that for $k \in \mathbb{Z}$ we have

$$\left| \langle u, e_k \rangle \right| \leq \lambda^m w\left(\frac{R}{\theta^{m-1}}\right) \|u\|_{\mathcal{H}}, \tag{2.19}$$

where m is such that $\theta^{m-1} \leq |k| < \theta^m$ if $k \neq 0$ and 0 otherwise. Since w is increasing, (2.19) implies that

$$\left| \langle u, e_k \rangle \right| \leq \lambda \left(|k| + 1\right)^{\frac{\log \lambda}{\log \theta}} w\left(\frac{\theta R}{\max(1, |k|)}\right) \|u\|_{\mathcal{H}}. \tag{2.20}$$

Now, choose $L > 0$ large and notice that if $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ then we have for every $x \in \mathbb{S}^1$:

$$\begin{aligned} \left| e_k^{(n)}(x) \right| &= (2\pi |k|)^n = \left(\frac{2\pi |k|}{L}\right)^n (A_n n!)^{-1} L^n A_n n! \\ &\leq w\left(\frac{L}{2\pi \max(1, |k|)}\right)^{-1} L^n A_n n!. \end{aligned} \tag{2.21}$$

From (2.18), (2.20) and (2.21), we find that for every $n \in \mathbb{N}$ and $x \in \mathbb{S}^1$

$$\left|u^{(n)}(x)\right| \leq \left(\sum_{k \in \mathbb{Z}} (|k|+1)^{\frac{\log \lambda}{\log \theta}} \frac{w\left(\frac{\theta R}{\max(1,|k|)}\right)}{w\left(\frac{L}{2\pi \max(1,|k|)}\right)} \right) L^n A_n n! \left\|u\right\|_{\mathcal{H}}.$$

Consequently, in order to prove that f belongs to $\mathcal{C}^A$, we only need to see that for L large enough we have

$$\sum_{k \in \mathbb{Z}} (|k|+1)^{\frac{\log \lambda}{\log \theta}} \frac{w\left(\frac{\theta R}{\max(1,|k|)}\right)}{w\left(\frac{L}{2\pi \max(1,|k|)}\right)} < +\infty.$$

Since $\mathcal{C}^A$ is closed under differentiation, it follows from Proposition 2.4 that there is $\gamma > 0$ such that (2.8) holds for every $k \in \mathbb{N}$. Hence, according to Lemma 2.6, there is $C > 0$ such that that for every $k \in \mathbb{Z}$ we have

$$\frac{w\left(\frac{\theta R}{\max(1,|k|)}\right)}{w\left(\frac{L}{2\pi \max(1,|k|)}\right)} \leq C \max(1,|k|)^{-\delta},$$

where $\delta = -\frac{\log(2\pi\theta R/L)}{\log \gamma}$. Taking L large enough so that

$$\log L > \log\left(2\pi\theta R \gamma\right) + \frac{\log \gamma \log \lambda}{\log \theta},$$

the lemma is proved. □

We end this section with the proof of Corollaries 2.11 and 2.12 of Proposition 2.10.

Proof of Corollary 2.11. Applying Lemma B.1 (as in Example B.2), we find that the eigenvectors associated to non-zero eigenvalues of $\mathcal{L}$ acting on $\mathcal{H}$ are in fact the resonant states for $\mathcal{L}$ in the sense of Definition 3. Hence, the resonant states for $\mathcal{L}$ belong to $\mathcal{H}$ and are thus of regularity $\mathcal{C}^A$ according to Proposition 2.10. □

Proof of Corollary 2.12. Any expanding map of the circle has at least one fixed point (this is for instance a consequence of the classical result [KH95, Theorem 2.4.6]). Hence, without loss of generality, we may assume that 0 is a common fixed point for T_1 and T_2. Then, we let ρ_1 and ρ_2 denote the density of the invariant probability measures absolutely continuous with respect to Lebesgue, respectively for T_1 and T_2. For $j = 1,2$, we define the map F_j from $\mathbb{S}^1$ to itself by

$$F_j(x) = \int_{[0,x]} \rho_j(y) \mathrm{d}y \bmod 1.$$

Since ρ_1 and ρ_2 are fully supported, we see that the F_j's are diffeomorphisms from $\mathbb{S}^1$ to itself. Moreover [Rue89, Corollary 5.2], ρ_1 and ρ_2 are resonant states associated to the resonance 1 respectively for T_1 and T_2. Hence, it follows from Corollary 2.11 that ρ_1 and ρ_2, and hence F_1 and F_2, are of class $\mathcal{C}^A$. Since the Inverse Function Theorem holds in regularity $\mathcal{C}^A$, the inverses of the diffeomorphisms F_1 and F_2 also belong to the class $\mathcal{C}^A$ (see [Kom79] or [BM04, Theorem 4.10]). In particular, the map $F_2^{-1} \circ (\pm F_1)$ is of regularity $\mathcal{C}^A$. However, it follows from the claim on the bottom of the second page of [Art94] that $F_2^{-1} \circ (\pm F_1)$ is the conjugacy map between T_1 and T_2 (recall that it is unique once its degree is fixed [KH95, Lemma 2.4.10]). □

2.2.2 Nuclear power decomposition

We saw in the proof of Theorem 2.9 that the transfer operator $\mathcal{L}$ may be written as the sum (2.17) of the operators $\mathcal{L}_b$ and $\mathcal{L}_c$. In this section, we show that this is a nuclear power decomposition in the spirit of [BT08], and we investigate the consequences of the existence of such a decomposition, in particular in terms of dynamical determinants (see Propositions 2.18 and 2.23). Thus, we will prove in particular Theorem 8.

We first investigate the operator $\mathcal{L}_b$. To do so, define the function $g : \mathbb{N} \mapsto \mathbb{R}_+^*$ by

$$g(N) = \sup_{m \geq N} \frac{w\left(\frac{R}{\theta^m}\right)}{w\left(\frac{R}{\theta^{m-1}}\right)} \tag{2.22}$$

and notice that $g(N) \underset{N \to +\infty}{\to} 0$ by Lemma 2.6. The operator $\mathcal{L}_b$ is morally strictly upper triangular, the following proposition uses the function g to quantify the fact that $\mathcal{L}_b$ is quasi-nilpotent.

Proposition 2.16. *There is a constant $C > 0$ such that for all $n \in \mathbb{N}^*$ we have*

$$\|\mathcal{L}_b^n\|_{\mathcal{H} \to \mathcal{H}} \leq C^n \prod_{k=0}^{n-1} g(k).$$

In particular, the spectral radius of $\mathcal{L}_b$ is zero (i.e. $\mathcal{L}_b$ is quasi-nilpotent).

Proof. Notice that if $k < N$ then from the definition of $\mathcal{L}_{b,N}$ it comes that

$$(\mathcal{L}_{b,N} - \mathcal{L}_{b,k})\,\mathcal{L}_{b,N} = (\mathcal{L}_{b,N} - \mathcal{L}_{b,k})\,(\mathcal{L}_{b,N} - \mathcal{L}_{b,k+1})\,.$$

Thus if $N \geq n-1$ we have

$$\mathcal{L}_{b,N}^n = \prod_{k=0}^{n-1} (\mathcal{L}_{b,N} - \mathcal{L}_{b,k})\,.$$

Letting N tends to infinity, we get that

$$\mathcal{L}_b^n = \prod_{k=0}^{n-1} (\mathcal{L}_b - \mathcal{L}_{b,k})$$

and the result follows from Lemma 2.15. □

Then we investigate the operator $\mathcal{L}_c$ (as an operator from $\mathcal{H}$ to itself).

Proposition 2.17. *There are constants $C, R' > 0$ such that, if we define the function f for $x > 0$ by*

$$f(x) = x^\alpha w \left(\frac{R'}{x} \right), \tag{2.23}$$

where $\alpha = \log \lambda / \log \theta$, and if $(s_\ell)_{\ell \in \mathbb{N}}$ denotes the sequence of singular values of $\mathcal{L}_c$ acting on $\mathcal{H}$, then, for all $\ell \geq 1$, we have

$$s_\ell \leq C f(\ell). \tag{2.24}$$

In particular, $\mathcal{L}_c$ is nuclear of order 0.

Proof. Since $\mathcal{L}_c$ is continuous from L^2 to $\mathcal{H}$, we have the following bound on its singular values as a compact operator from $\mathcal{H}$ to itself:

$$\forall m \in \mathbb{N} : s_{2\lceil \theta^m \rceil + 1} \leq C \sup_{p \geq m} \lambda^p w \left(\frac{R}{\theta^{p-1}} \right)$$

for some constant $C > 0$. Since f does not vanish, we only need to prove (2.24) for ℓ large. Thus, let ℓ be large and let m be the largest integer such that $\ell \geq 2\lceil \theta^m \rceil + 1$. Then, we have

$$s_\ell \leq s_{2\lceil \theta^m \rceil + 1} \leq C \sup_{y \geq \theta^m} y^\alpha w \left(\frac{\theta R}{y} \right). \tag{2.25}$$

Here, we performed the change of variables "$y = \theta^p$". Then, notice that $\theta^m \geq \frac{1}{2\theta}\ell - \frac{3}{2\theta} \geq \frac{1}{4\theta}\ell$ (provided that ℓ is large enough). Hence, we deduce from (2.25) that, for ℓ large enough, we have (with the change of variables "$x = 4\theta y$" and taking $R' = 4\theta^2 R$ in the definition of f)

$$s_\ell \leq C \sup_{x \geq \ell} f(x). \tag{2.26}$$

Recall the function $k(x)$ from the proof of Lemma 2.6 and use (2.10) to see that for all

$x \geq \ell$ we have

$$\frac{f(x)}{f(\ell)} \leq \left(\frac{x}{\ell}\right)^{\alpha - k\left(\frac{R'}{\ell}\right)}, \tag{2.27}$$

but if ℓ is large enough we have $k\left(\frac{R'}{\ell}\right) > \alpha$ and consequently $f(x) \leq f(\ell)$. Hence, for ℓ large enough we have

$$f(\ell) = \sup_{x \geq \ell} f(x), \tag{2.28}$$

and (2.24) follows from (2.26). To see that $\mathcal{L}_c$ is nuclear of order 0, recall from Lemma 2.6 that w vanishes at all orders in 0. Hence, f decays faster than the inverse of any polynomial and so does the sequence of singular values of $\mathcal{L}_c$. □

Now, following [BT08], we want to use the nuclear power decomposition in order to study the dynamical determinant d defined by (4) and (5). This is the point of Proposition 2.18, that completes the proof of Theorem 8.

Proposition 2.18. *If z is small enough then we have*

$$d(z) = \det\left(I - z\left(I - z\mathcal{L}_b\right)^{-1}\mathcal{L}_c\right), \tag{2.29}$$

where the dynamical determinant d is defined by (4) *and* (5).

Notice in particular that this proposition and Theorem 2 imply that the spectrum of $\mathcal{L}$ acting on $\mathcal{H}$ coincides with the Ruelle spectrum of $\mathcal{L}$ from Definition 3 (we already knew that fact, as this is a consequence of Lemma B.1).

Proof of Proposition 2.18. If $N \in \mathbb{N}$ then the operators $\mathcal{L}_{b,N}$ and $\mathcal{L}_c$ are trace class (recall that $\mathcal{L}_{b,N}$, defined by (2.15), has finite rank). Moreover, $\mathcal{L}_{b,N}$ is nilpotent and thus

$$\begin{aligned} \det\left(I - z\left(\mathcal{L}_{b,N} + \mathcal{L}_c\right)\right) &= \det\left(I - z\mathcal{L}_{b,N}\right)\det\left(I - z\left(I - z\mathcal{L}_{b,N}\right)^{-1}\mathcal{L}_c\right) \\ &= \det\left(I - z\left(I - z\mathcal{L}_{b,N}\right)^{-1}\mathcal{L}_c\right) \\ &\underset{N\to+\infty}{\to} \det\left(I - z\left(I - z\mathcal{L}_b\right)^{-1}\mathcal{L}_c\right), \end{aligned} \tag{2.30}$$

and the convergence holds uniformly on all compact subsets of $\mathbb{C}$. Denote by $h(z)$ the entire function on the right-hand side of (2.29). Since $h(0) = 1$, there is a sequence $(a_n)_{n\geq 1}$ of complex numbers such that for $|z|$ small enough we have

$$h(z) = \exp\left(-\sum_{n\geq 1}\frac{a_n}{n}z^n\right).$$

Then applying Cauchy's formula, we find for $n \geq 1$ (and for ϵ small enough)

$$a_n = -\frac{1}{2i\pi} \int_{\partial \mathbb{D}(0,\epsilon)} \frac{h'(z)}{h(z)} \frac{\mathrm{d}z}{z^n} = \lim_{N \to +\infty} \operatorname{tr}\left((\mathcal{L}_{b,N} + \mathcal{L}_c)^n \right).$$

Then notice that, since $\mathcal{L}_{b,N}$ is nilpotent we have

$$\operatorname{tr}\left((\mathcal{L}_{b,N} + \mathcal{L}_c)^n \right) = \operatorname{tr}\left((\mathcal{L}_{b,N} + \mathcal{L}_c)^n - \mathcal{L}_{b,N}^n \right)$$

but the operators $(\mathcal{L}_{b,N} + \mathcal{L}_c)^n - \mathcal{L}_{b,N}^n$ converge in trace class topology to the operator $\mathcal{L}^n - \mathcal{L}_b^n$. Thus we have

$$a_n = \operatorname{tr}\left(\mathcal{L}^n - \mathcal{L}_b^n \right) = \sum_{k \in \mathbb{Z}} \langle \left(\mathcal{L}^n - \mathcal{L}_b^n \right) e_k, e_k \rangle_{L^2}. \tag{2.31}$$

Now, notice that if $k \in \mathbb{Z}$ and $n \in \mathbb{N}^*$ we have $\langle \mathcal{L}_b^n e_k, e_k \rangle_{L^2} = 0$. Indeed, if $k \in \mathbb{Z}^*$ is such that $\theta^{m-1} \leq |k| < \theta^m$, then for every $N \in \mathbb{N}$, the image of e_k by $\mathcal{L}_{b,N}^n$ belongs to the span of the e_ℓ's such that $|\ell| < \theta^{m-n}$. In particular, $\mathcal{L}_{b,N}^n e_k$ is orthogonal to e_k. Since, we also have $\mathcal{L}_{b,N}^n e_0 = 0$, we find that, for every $k \in \mathbb{Z}$, we have $\langle \mathcal{L}_{b,N}^n e_k, e_k \rangle_{L^2} = 0$, and, letting N tends to infinity, that $\langle \mathcal{L}_b^n e_k, e_k \rangle_{L^2} = 0$. Hence, (2.31) gives

$$\begin{aligned} a_n &= \sum_{k \in \mathbb{Z}} \langle \mathcal{L}^n e_k, e_k \rangle_{L^2} = \sum_{k \in \mathbb{Z}} \int_{\mathbb{S}^1} e^{2ik\pi(x - T^n(x))} \mathrm{d}x \\ &= \lim_{m \to +\infty} \int_{\mathbb{S}^1} \frac{\sin\left((2m+1)\pi(x - T^n(x)) \right)}{\sin\left(\pi\left(x - T^n(x) \right) \right)} \mathrm{d}x. \end{aligned} \tag{2.32}$$

Finally, we use a partition of unity in the last integral and locally we perform the change of variable "$u = x - T^n(x)$". Then, we recognize the Dirichlet kernel and find $a_n = \operatorname{tr}^\flat\left(\mathcal{L}^n \right)$, where the flat trace is defined by (5). This proves that $h(z) = d(z)$.

The fact that the zeros of d are exactly the inverses of the non-zero eigenvalues of $\mathcal{L}$ counted with multiplicity follows from [Kur61, Theorem 3.1]. However, notice that in our case the situation is simpler than in the general theory of Weinstein–Aronszajn determinant, and that the correspondence between the zeros of d and the inverses of the non-zero eigenvalues of $\mathcal{L}$ may be deduced from the convergence (2.30). □

In some cases, it may happen that $\mathcal{L}$ acting on $\mathcal{H}$ is trace class, or in some Schatten class. In these cases, we may simplify Proposition 2.18 in the following way.

Proposition 2.19. *Assume that there is $p > 0$ such that $\mathcal{L}$ acting on $\mathcal{H}$ is in the Schatten class $\mathcal{S}_p$. Then, if m denotes the smallest integer larger than p, we have*

$$d(z) = \det_m\left(I - z\mathcal{L} \right) \exp\left(- \sum_{n=1}^{m-1} \frac{\operatorname{tr}^\flat\left(\mathcal{L}^n \right)}{n} z^n \right), \tag{2.33}$$

where the $\mathrm{tr}^\flat(\mathcal{L}^n)$*'s are defined by* (5) *and* $\det_m$ *denotes the regularized determinant of order* m *defined in [GGK00, IX] (this is the usual Fredholm determinant when* $m = 1$*). In particular, the order of* d *is less than* p *and the trace formula* (TFM) *holds for every* $n \geq p$*.*

Proof. We know that when $|z|$ is small enough we have

$$\det_m(I - z\mathcal{L}) = \exp\left(-\sum_{n\geq m} \frac{\mathrm{tr}(\mathcal{L}^n)}{n} z^n\right).$$

Then, the same computation (2.31)-(2.32) as in the proof of Proposition 2.18 ensures that for $n \geq m$ we have

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{k\in\mathbb{Z}} \langle \mathcal{L}^n e_k, e_k\rangle_{L^2} = \mathrm{tr}^\flat(\mathcal{L}^n), \tag{2.34}$$

and (2.33) follows. To see that d has order less than p, recall that the since $\mathcal{L}$ belongs to the Schatten class $\mathcal{S}_p$, its eigenvalues are p-summable, then apply Lidskii's Trace Theorem to recognize that $\det_m(I - z\mathcal{L})$ is a Weierstrass product and hence has order less than p thanks to [Boa54, Theorem 2.6.5]. The validity of the trace formula (TFM) follows then from Proposition 1. □

Remark 2.20. If the sequence A satisfies (2.8) for some $\gamma > 0$, then the estimates (2.9) and (2.12), respectively from Lemma 2.6 and Theorem 2.9 imply that the singular values of $\mathcal{L}$ acting on $\mathcal{H}$ satisfy

$$\sigma_k \underset{k\to+\infty}{=} \mathcal{O}\left(\frac{1}{k^\delta}\right), \tag{2.35}$$

where $\delta = \log\theta/\log\gamma$. Hence, $\mathcal{L}$ acting on $\mathcal{H}$ belongs to the Schatten class $\mathcal{S}_p$ for any $p > \delta^{-1}$ and Proposition 2.19 implies that the order of d is less than $\log\gamma/\log\theta$. Since θ may be chosen arbitrarily close to the expanding constant λ, the following result follows.

Corollary 2.21. *If there is* $\gamma > 0$ *such that the sequence* A *satisfies* (2.8)*, then the order of the dynamical determinant* d *is less than* $\log\gamma/\log\lambda$ *and the trace formula* (TFM) *holds for every* $n > \log\gamma/\log\lambda$*.*

Remark 2.22. Recall from Remark 2.7 that the validity of (2.8) (for some value of $\gamma > 0$) is equivalent to $\mathcal{C}^A$ being closed under differentiation. We advertised in Remark 2.5 that it was a relevant notion when investigating trace formula, we explain now why.

Notice that (2.8) implies that taking a derivative in the class $\mathcal{C}^A$ results in replacing R by γR in (2.4). Composing by a contraction of factor λ^{-1} (which is basically what $\mathcal{L}$ does) results in replacing R by $\lambda^{-1}R$. Thus $\mathcal{L}$ has morally the same regularizing effect in the class $\mathcal{C}^A$ as taking $\log\gamma/\log\lambda$ primitives (notice that this number is not necessarily an integer). To put it loosely, $\mathcal{L}$ is a smoothing operator from the point of view of the class $\mathcal{C}^A$. Consequently, the decay that we obtain on the singular values of $\mathcal{L}$, and ultimately the bound on the order of the dynamical determinant, is natural (considering for instance the case of Sobolev injections) for an operator that “makes a function gain $\log \gamma / \log \lambda$ derivatives”. We witness here an interesting phenomenon: the values of n for which Corollary 2.21 ensures that the trace formula (TFM) holds depend *a priori* on the dilation factor λ for T.

Now, if the sequence $(A_k)_{k\in\mathbb{N}}$ satisfies (2.6), we see that (2.8) holds for any $\gamma > 1$. Hence, we see from the discussion above that $\mathcal{L}$ is an operator that “makes a function gain an infinite number of derivatives” in the class $\mathcal{C}^A$. To put it more rigorously, Proposition 2.19 implies that $\mathcal{L}$ acting on $\mathcal{H}$ is nuclear of order 0, and following the dynamical determinant d has order zero and trace formula (TFM) always hold.

See the examples from §2.2.3 for further discussions of these phenomena.

Finally, we will use the nuclear power decomposition (2.17) with Propositions 2.16, 2.17 and 2.18 in order to bound the growth of the dynamical determinant d. To do so, define the entire functions F and G by

$$F(z) = (1+z)\prod_{m=1}^{+\infty}\left(1+f(m)z\right) \text{ and } G(z) = \sum_{n=0}^{+\infty}\left(\prod_{k=0}^{n-1} g(k)\right) z^n, \tag{2.36}$$

where we recall that f and g have been defined respectively in (2.23) and (2.22). Notice that F has genus zero. Thus if $n(r)$ denotes the number of integers m such that $f(m)^{-1} \leq r$ or $m = 0$, we have the following estimate [Boa54, Lemma 3.5.1] for $r > 0$:

$$\log F(r) \leq \int_0^r \frac{n(s)}{s}\mathrm{d}s + r\int_r^{+\infty}\frac{n(s)}{s^2}\mathrm{d}s. \tag{2.37}$$

This bound may be used with Proposition 2.23 to control the growth of the dynamical determinant, see §2.2.3 for examples.

Proposition 2.23. *There is a constant C such that for all $z \in \mathbb{C}$ we have*

$$|d(z)| \leq F\left(C\,|z|\,G\left(C\,|z|\right)\right).$$

Proof. Let $z \in \mathbb{C}$. Denote by $(c_k)_{k\in\mathbb{N}}$ the sequence of singular values of the operator $-z\left(I - z\mathcal{L}_b\right)^{-1}\mathcal{L}_c$ and by $(\lambda_k)_{k\in\mathbb{N}}$ the sequence of its eigenvalues. By Lidskii's Theorem we have

$$|d(z)| = \left|\prod_{k\in\mathbb{N}}\left(1+\lambda_k\right)\right| \leq 1 + \sum_{n\geq 1}\sum_{k_1<\cdots<k_n}\prod_{j=1}^{n}\left|\lambda_{k_j}\right|.$$

Then applying [GGK00, Theorem IV.3.1] we see that

$$|d(z)| \leq 1 + \sum_{n \geq 1} \sum_{k_1 < \cdots < k_n} \prod_{j=1}^{n} c_{k_j} = \prod_{k \geq 0} (1 + c_k).$$

Now if $(s_k)_{k \geq 0}$ denotes the sequence of singular values of $\mathcal{L}_c$ then we have for $k \geq 1$ (replace $f(k)$ by 1 in the case $k = 0$)

$$c_k \leq |z| \left\| (I - z\mathcal{L}_b)^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} s_k \leq C\,|z| \left\| (I - z\mathcal{L}_b)^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} f(k),$$

for some constant $C > 0$, and thus $|d(z)| \leq F\left(C\,|z| \left\| (I - z\mathcal{L}_b)^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} \right)$. But from Lemma 2.16, we get that, up to taking larger C, we have

$$\left\| (I - z\mathcal{L}_b)^{-1} \right\|_{\mathcal{H} \to \mathcal{H}} \leq G\left(C\,|z|\right),$$

which ends the proof of the proposition. □

Remark 2.24. Assume that the right-hand side of (2.12) in Theorem 2.9 is summable. Then we know that $\mathcal{L}$ acting on $\mathcal{H}$ is trace class and Proposition 2.19 implies that the order of the dynamical determinant d is less than 1. In particular, d has genus zero, and when the right-hand side of (2.12) decays very fast we may want to get a better bound on d. To do so, we may work as in the proof of Proposition 2.23 to find that there is a constant $C > 0$ such that for every $z \in \mathbb{C}$ we have

$$|d(z)| \leq (1 + C\,|z|) \prod_{k \in \mathbb{N}^*} \left(1 + C\,|z| \sup_{x \leq \frac{1}{k}} \frac{w\,(Mx)}{w\,(\theta M x)} \right). \tag{2.38}$$

The infinite product in the right-hand side of (2.38) may be bounded using [Boa54, Lemma 3.5.1] as we did for F: just replace $n(s)$ by the number of integer k such that $\sup_{x \leq \frac{1}{k}} \frac{w(Mx)}{w(\theta M x)} \geq s^{-1}$ in (2.37) or $k = 0$.

Remark 2.25. Notice that, using Jensen's formula [Boa54, 1.2.1 p.2], a bound on the growth of the dynamical determinant immediately gives an upper bound on the asymptotics of the number of Ruelle resonances outside of $\mathbb{D}(0, \epsilon)$, when ϵ tends to 0.

2.2.3 Examples

Gevrey and analytic dynamics

In this section we take $A_k = k!^{s-1}$ for some $s \geq 1$, that is we study the Gevrey classes of regularity from Example 2.1. We still denote by T an expanding map of the circle with expanding factor at least $\lambda > 1$, and we assume that T' is s-Gevrey. In this case, we see that for every $\gamma > 1$, we can find $C > 0$ such that for all $k \in \mathbb{N}$ the estimate (2.8) holds. Thus the dynamical determinant d has order 0. But we can of course get a better bound.

To do so, recall the function k from the proof of Lemma 2.6. Its definition implies that if $x > 0$ then

$$x^{k(x)} k(x)!^s < x^{k(x)+1} \left(k(x)+1\right)!^s$$

and thus

$$k(x) > x^{-\frac{1}{s}} - 1.$$

Then if $\theta \in]0, \lambda[$ and $M > 0$ is the constant from Theorem 2.9, we have when $m \geq 1$

$$\sigma_m \leq C \sup_{0 < x \leq \frac{1}{m}} \frac{w(Mx)}{w(\theta M x)} \leq \sup_{0 < x \leq \frac{1}{m}} \left(\frac{1}{\theta}\right)^{k(\theta M x)} \leq c \exp\left(-c^{-1} m^{\frac{1}{s}}\right) \tag{2.39}$$

for some constant $c > 0$. Thus, by [Jéz20a, Lemma 2.13] (or Remark 2.24), we have that for some constant $c > 0$ we have

$$|d(z)| \leq c \exp\left(c \left(\log_+ |z|\right)^{1+s}\right).$$

Notice that we retrieve the optimal result [BN19] when $s = 1$, which is also the result of Ruelle [Rue76].

Remark 2.26. The bound (2.39) on the singular values of $\mathcal{L}$ acting on $\mathcal{H}$ implies that the transfer operator $\mathcal{L}$ belongs to the exponential class of type (c^{-1}, σ^{-1}) defined in [Ban08]. Hence, we may apply the results from [Ban08] to transfer operators associated to Gevrey expanding maps of the circle. For instance, the resolvent estimates [Ban08, Theorem 3.13] may be used to derive a better (super-exponential) remainder in the asymptotics expansion for the correlations of Gevrey observables (see [GL08, Theorem 1.2] for the usual asymptotics of correlation in the case of hyperbolic diffeomorphisms). We could probably also use [Ban08, Theorem 4.2] to control globally the Ruelle spectrum of a perturbation of T in the Gevrey category.

The class $\mathcal{C}^{\kappa,\upsilon}$

We investigate now the classes from Example 2.2 and illustrate the discussion from Remark 2.22. Recall that these classes will be used in Chapter 3 to define a class of Anosov flow for which the trace formula (TFF). Hence, we choose $\kappa > 0$ and $\upsilon \geq 1$ and take $A_k = \exp(k^\upsilon / \kappa\upsilon)$. Recall that according to Proposition 2.4 the class $\mathcal{C}^{\kappa,\upsilon}$ is closed under differentiation if and only if $\upsilon \leq 2$. Moreover, the stronger condition (2.6) holds if and only if $\upsilon < 2$. We assume now that T' belongs to the class $\mathcal{C}^{\kappa,\upsilon}$.

Let us deal first with the case $1 < \upsilon < 2$ (the case in which (2.6) holds), then we see that for $0 < x < 1$ and some constants c depending on κ we have (k is still from the proof of Lemma 2.6)

$$k(x) \geq c^{-1} |\log x|^{\frac{1}{\upsilon - 1}} - c, \tag{2.40}$$

thus if $\mu \in]0,1[$, we have for some new constant $c > 0$ and small $x > 0$

$$\frac{w(\mu x)}{w(x)} \leq c \exp\left(-c^{-1} \; |\log x|^{\frac{1}{\upsilon-1}}\right). \tag{2.41}$$

Then, for some new constant $c > 0$ the estimates on the singular values of $\mathcal{L}$ from Theorem 2.9 becomes (for $k \geq 1$)

$$\sigma_k \leq c \exp\left(-c^{-1} (\log k)^{\frac{1}{\upsilon-1}}\right).$$

Once again, this gives, with Remark 2.24, that, up to taking larger c,

$$|d(z)| \leq \prod_{k \geq 1} \left(1 + c\,|z| \exp\left(-c^{-1} (\log k)^{\frac{1}{\upsilon-1}}\right)\right). \tag{2.42}$$

Using [Boa54, Lemma 3.5.1] to bound the right-hand side of (2.42) (that is using (2.37) with the modification described in Remark 2.24), we get that for some new constant $c > 0$ and all $z \in \mathbb{C}$ we have

$$\log_+ |d(z)| \leq c \exp\left(c \left(\log_+ |z|\right)^{\upsilon - 1}\right). \tag{2.43}$$

In particular, d has order zero, but this could have been seen as a consequence of Corollary 2.21 (see also Remark 2.22).

Now, if $\upsilon = 2$ (the case in which $\mathcal{C}^{\kappa,\upsilon}$ is closed under differentiation but (2.6) does not hold) then we have

$$\lim_{k \to +\infty} \left(\frac{(k+1)A_{k+1}}{A_k}\right)^{\frac{1}{k}} = e^{\frac{1}{\kappa}}.$$

Thus, by Corollary 2.21, the dynamical determinant d has order less than $(\kappa \log \lambda)^{-1}$. We have here a very interesting behaviour: the bound on the order of the dynamical determinant depends on the dilation factor (this implies in particular that trace formula (TFM) holds for large n according to Proposition 1). As pointed out in Remark 2.22, it is not surprising that this behaviour occurs for the value of υ that separates classes that are closed under differentiation and those that are not. As far as we know, it is the first time that such a behavior is proved and, consequently, it would be particularly interesting to know whether our result is sharp or not in that case.

Finally, we deal with the case $\upsilon > 2$, when the class $\mathcal{C}^{\kappa,\upsilon}$ is not closed under differentiation. The estimates (2.40), and thus (2.41), remain true. Thus, for some $c > 0$, we have for large N (recall that g is defined by (2.22))

$$g(N) \leq c \exp\left(-c^{-1} N^{\frac{1}{\upsilon-1}}\right)$$

and thus, changing the value of c,

$$\prod_{k=0}^{N-1} g(N) \leq c \exp\left(-c^{-1} N^{1+\frac{1}{\upsilon-1}}\right).$$

Then, in the definition (2.36) of G, we may split the sum between $n \leq \left(\frac{2\log r}{c}\right)^{\upsilon-1}$ and $n > \left(\frac{2\log r}{c}\right)^{\upsilon-1}$, to find that for some $c > 0$ and all $r > 0$

$$\log_+ G(r) \leq c\left(\left(\log_+ r\right)^{\upsilon} + 1\right)$$

An easy computation shows that for some $c > 0$ and all $m \geq 1$ we have

$$f(m) \leq c \exp\left(-c^{-1} \left(\log m\right)^{\frac{\upsilon}{\upsilon-1}}\right),$$

where f has been defined by (2.23). Thus reasoning as above in the case $\upsilon < 2$ (that is using [Boa54, Lemma 3.5.1], which has been stated as (2.37) in this case), we find that for some $c > 0$ and $r > 0$

$$\log_+ F(r) \leq c \exp\left(c\left(\log_+ r\right)^{\frac{\upsilon-1}{\upsilon}}\right).$$

And by Proposition 2.23, we see that there is still a new constant $c > 0$ such that for all $z \in \mathbb{C}$ we have

$$\log_+ |d(z)| \leq c \exp\left(c\left(\log_+ |z|\right)^{\upsilon-1}\right). \tag{2.44}$$

Notice that this is the same estimate than (2.43) that we established in the case $\upsilon < 2$, and that it is still true in the case $\upsilon = 2$ (but we have more precise information in this case). It is very interesting that the bound (2.44) is true regardless of the value of υ while there is a huge change in the structure of the transfer operator at $\upsilon = 2$. Hence, it seems that in most cases the nuclear power decomposition contains all the information that we need on the dynamical determinant. This is indeed a very versatile tool that allows also to deal with finitely differentiable map [Bal18], and as we have just seen, it does not seem that we lose much information by using this method in more favorable cases. Notice however that in some *very* favorable cases (such as Gevrey and analytic dynamics), the nuclear decomposition does not seem to give the best bound (this is because in this case, the bounds on the singular values of $\mathcal{L}$ and $\mathcal{L}_c$ are very similar).

2.3 Koopman operators for Gevrey hyperbolic maps

In [Jéz20a], we applied the strategy exposed in §2.2 to more complicated dynamics – hyperbolic basic sets for smooth diffeomorphisms, see §1.2.1. Hence, we were able to prove the following result. We recall that $\mathcal{G}^s$ is our abbreviation for s-Gevrey. The spaces $\mathcal{G}^s_c(V)$ and $\mathcal{U}^s_c(V)$ and their topologies will be introduced in §4.1.1. The latter is a space of linear functionals on Gevrey functions – or ultradistributions.

Theorem 2.27 ([Jéz20a]). *Let $s > 1$. Let M be a $\mathcal{G}^s$ manifold and $T : M \to M$ a $\mathcal{G}^s$ diffeomorphism. Let K be a hyperbolic basic set for T. Then there exist a compact isolating neighbourhood V for K and a separable Hilbert space $\mathcal{H}$ such that the following holds*

(i) *the Hilbert space $\mathcal{H}$ is contained in $\mathcal{U}^s_c(V)$ and the inclusion is continuous;*

(ii) *the Hilbert space $\mathcal{H}$ contains $\mathcal{G}^s_c(V)$ and the inclusion is continuous with dense image;*

(iii) *if $g : M \to \mathbb{C}$ is a $\mathcal{G}^s$ function supported in V, the weighted Koopman operator $\mathcal{L}$ defined by* (1.16) *extends to a trace class operator from $\mathcal{H}$ to itself;*

(iv) *for all $n \in \mathbb{N}^*$ we have $\operatorname{tr}(\mathcal{L}^n) = \operatorname{tr}^\flat(\mathcal{L}^n)$ – where the flat trace of $\mathcal{L}^n$ is defined by* (1.18)*–, in particular the dynamical determinant d, defined by* (1.17)*, is the Fredholm determinant of $\mathcal{L}$, and the trace formula* (TFM) *holds for every $n \in \mathbb{N}^*$;*

(v) *for all $\beta > 2 + (s+1)d$ we have*

$$\log_+ |d(z)| \underset{|z| \to +\infty}{=} \mathcal{O}\left((\log |z|)^{1+\beta} \right),$$

in particular d has order zero;

(vi) *if $N(r)$ is the number of Ruelle resonances for (T, g) outside of the closed disc of center 0 and radius r (counted with multiplicity) we have for all $\beta > 2 + (s+1)d$*

$$N(r) \underset{r \to 0}{=} \mathcal{O}\left(|\log r|^{1+\beta} \right).$$

Let us discuss a bit Theorem 2.27. First of all, the strategy to construct the space $\mathcal{H}$ is similar to the one exposed in §2.2. However, following the general philosophy exposed in the introduction, the space $\mathcal{H}$ from Theorem 2.27 has to be anisotropic. This means that the elements of $\mathcal{H}$ are indeed very smooth in the stable direction (as in §2.2) but very irregular in the unstable direction. This is why we end up with a space that is intermediate between Gevrey functions and ultradistributions (rather than between $\mathcal{C}^\infty$ function and distributions as in Theorem 3).

There are certain technical differences between the proofs of Theorems 2.9 and 2.27. There are indeed some additional geometric complications in the proof of Theorem 2.27 since we need to control the stable and unstable directions for the hyperbolic map T. This is slightly harder than in the $\mathcal{C}^\infty$ case, since we need to make the hyperbolicity of T effective after just one iteration of T in order to control the singular values of $\mathcal{L}$. We introduced the notion of "generalized cone-hyperbolicity" to do so (see [Jéz20a, Definition 7.1]). A similar notion will be used in Chapter 3 to deal with ultradifferentiable Anosov flows, see §3.3.

Another technical difference between the proofs of Theorems 2.9 and 2.27 is the kind of Paley–Littlewood decompositions that we are using. In §2.2, we introduced a Paley–Littlewood with blocks of exponential sizes adapted to the hyperbolicity of our map. On the contrary, in [Jéz20a] we used a Paley–Littlewood decomposition with blocks of polynomial size (depending on the regularity of the map). The construction from §2.2 seems to give slightly better results (see §2.2.3), and it is likely that they can be adapted to the case of hyperbolic maps in order to improve Theorem 2.27 (for instance, one could try to allow smaller values of β in (v) and (vi)). However, one have to keep in mind that the excessively simple geometry of the circle is one of the elements that allowed us to design the spaces from §2.2 more carefully than those of [Jéz20a].

Remark 2.28. Theorem 2.27 suggests that the dynamical determinant (12) associated to a Gevrey Anosov flow should have finite order. Let us explain why. In general, we expect an exponential scaling between results for Anosov diffeomorphisms and flows. For instance, if d denotes the dynamical determinant (4) associated to a transitive Anosov diffeomorphism T and $\tilde{d}$ the dynamical determinant (12) associated to the time 1 suspension of T, then an elementary computation ensures that $\tilde{d}(z) = d(e^z)$ for every $z \in \mathbb{C}$. Under this exponential scaling, the almost polynomial bound from (v) in Theorem 2.27 deteriorates into finite order for the dynamical determinant $\tilde{d}$.

This is one of the reasons why we think that Gevrey regularity is the good setting in order to establish finite order for the dynamical determinant (12) associated to an Anosov flow. Some heuristic computations based on the construction of Hilbert spaces of anisotropic distributions from [FS11] also support this idea.

Chapter 3

Trace formula for ultradifferentiable Anosov flows

This chapter is dedicated to the proof of Theorem 9. The contents of this chapter and of Appendices C and D may also be found in [Jéz19a]. Appendix B is written in a slightly more general way than the corresponding [Jéz19a, Appendix A] in order to cover the different types of systems that are considered in this thesis.

We will start by discussing a bit further the classes $\mathcal{C}^{\kappa,\upsilon}$ from Example 2.2 in §3.1. Using the notations from §3.1, we can get a precise version of Theorem 9.

Theorem 3.1. *Let $\kappa > 0$ and $\upsilon \in \left]1,2\right[$. Let M be a $\mathcal{C}^{\kappa,\upsilon}$ compact manifold, $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^{\kappa,\upsilon}$ Anosov flow on M with generator X and $V : M \to \mathbb{C}$ a $\mathcal{C}^{\kappa,\upsilon}$ function. Then for every $t_0 > 0$ there is a separable Hilbert space $\mathcal{H}$ such that*

(i) for every $\tilde{\upsilon} > \upsilon$ sufficiently close to υ, we have $\mathcal{C}^{\infty,\tilde{\upsilon}}(M) \subseteq \mathcal{H} \subseteq \mathcal{D}^{\tilde{\upsilon}}(M)'$, both inclusions are continuous, and the first one has dense image;

(ii) for every $t \in \mathbb{R}_+$, the operator $\mathcal{L}_t$ defined by (9) is bounded on $\mathcal{H}$;

(iii) $(\mathcal{L}_t)_{t\geq 0}$ defines a strongly continuous semi-group of operators on $\mathcal{H}$, whose generator coincides with $P = X + V$ on its domain, which is $\{u \in \mathcal{H} : Pu \in \mathcal{H}\}$;

(iv) the spectrum of P acting on $\mathcal{H}$ consists of isolated eigenvalues of finite multiplicity which coincide with the Ruelle resonances of P (multiplicity taken into account);

(v) if $h : \mathbb{R}_+^* \to \mathbb{C}$ is $\mathcal{C}^\infty$ and compactly supported in $[t_0, +\infty[$ then the operator

$$\int_0^{+\infty} h(t)\,\mathcal{L}_t \mathrm{d}t : \mathcal{H} \to \mathcal{H} \tag{3.1}$$

is trace class and its non-zero spectrum is the intersection of $\mathbb{C}\setminus\{0\}$ with the image of the spectrum of P by $\lambda \mapsto \mathrm{Lap}(h)(-\lambda)$ (multiplicity taken into account, $\mathrm{Lap}(h)$

denotes the Laplace transform of h). Moreover, the trace of the operator (3.1) *is given by*

$$\operatorname{tr}\left(\int_0^{+\infty} h(t)\mathcal{L}_t \mathrm{d}t\right) = \sum_\gamma T_\gamma^{\#} \frac{h\left(T_\gamma\right)}{\left|\det\left(I - \mathcal{P}_\gamma\right)\right|} \exp\left(\int_\gamma V\right),$$

where the sum on the right-hand side runs over periodic orbits γ of the flow $(\phi_t)_{t\in\mathbb{R}}$, the notations are as in (12).

This is indeed a more precise version of Theorem 9, since, with Lidskii's trace theorem [GGK00, Theorem 6.1 p.63], the last point of Theorem 3.1 implies the following Corollary.

Corollary 3.2 (Trace formula for ultradifferentiable Anosov flows)**.** *Under the assumptions of Theorem 3.1, the trace formula* (TFF) *holds. In particular, the right-hand side of* (TFF) *defines a distribution.*

As a by-product of the proof of Theorem 3.1, we get the following bound on the number of Ruelle resonances for ultradifferentiable Anosov flow. Notice that it implies that the right-hand side of (TFF) converges indeed to a distribution on $\mathbb{R}_+^*$.

Proposition 3.3. *Under the assumptions of Theorem 3.1, for all $\epsilon > 0$, we have*

$$\sum_{\lambda \in \sigma_{\mathrm{R}}(P)} \frac{e^{\epsilon \operatorname{Re}(\lambda)}}{1 + |\lambda|^{\dim M + \epsilon}} < +\infty,$$

where $P = X + V$.

The bound on the number of resonances given by Proposition 3.3 is not sufficient to apply Proposition 1.27 and get a Hadamard factorization [Boa54, Theorem 2.7.1] for the dynamical determinant d defined by (12). However, we will derive in Appendix C a "Hadamard-like" factorization for d.

Finally, although we need $\upsilon < 2$ to prove (TFF), most of the statements in Theorem 3.1 remain true when $\upsilon \geq 2$. We discuss in Appendix D the relevance and necessity of the condition $\upsilon < 2$ through the simplest possible example: the doubling map on the circle. Concerning this condition, we also refer to the discussion from Chapter 2 about Denjoy–Carleman classes that are closed under differentiation (see in particular Remarks 2.5 and 2.22 and the examples from §2.2.3).

Proposition 3.4. *If, in Theorem 3.1, we allow $\upsilon \geq 2$, then there is still a Hilbert space $\mathcal{H}$ satisfying (i),(ii),(iii) and (iv). Moreover, under the hypothesis of (v), the operator* (3.1) *is compact and its spectrum can be described as in Theorem 3.1 in terms of Ruelle resonances.*

This chapter is structured as follow. As already mentioned, §3.1 is dedicated to a discussion of the classes $\mathcal{C}^{\kappa,\upsilon}$ and associated generalized distributions. Then, in §3.2, we construct a "local" version of the space $\mathcal{H}$ from Theorem 3.1. In §3.3, we study the action of a local model for the Koopman operator (9) on these local spaces. In §3.4, we construct a first version of the space $\mathcal{H}$ from Theorem 3.1. However, the action of the Koopman operator $(\mathcal{L}_t)_{t\in\mathbb{R}}$ – from (9) – on this first space will not be controlled for small t's, a flaw that will be fixed in §3.5, proving Theorem 3.1 as well as Corollary 3.2 and Propositions 3.3 and 3.4.

3.1 The classes $\mathcal{C}^{\kappa,\upsilon}$

In §2.1, Example 2.2, we defined the classes of regularity $\mathcal{C}^{\kappa,\upsilon}$ for $\kappa > 0$ and $\upsilon > 1$. It is a well-established fact (see e.g. [KMR09] and references therein) that the condition (2.1) implies that the class $\mathcal{C}^{\kappa,\upsilon}$ is closed under multiplication, composition, the Inverse Function Theorem and solving ODEs. Notice that, according to Proposition 2.4, the class $\mathcal{C}^{\kappa,\upsilon}$ is closed under differentiation if and only if $\upsilon \leq 2$. Since $\mathcal{C}^{\kappa,\upsilon}$ is greater than any Gevrey class (we assume $\upsilon > 1$), it is non-quasi-analytic and thus contains partitions of unity.

The Fourier transform will be a key tool in this paper, it is thus natural to introduce a suitable class of rapidly decreasing functions and associated spaces of tempered generalized distributions. This is often done in the literature, in particular when dealing with Gevrey classes (see for instance [Pil88, CP92]). Notice that we will use the following convention for the Fourier transform: if $f \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$ we set

$$\mathbb{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,\mathrm{d}x.$$

For all $\kappa > 0$, $\upsilon > 1$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, define

$$\|f\|_{\kappa,\upsilon} = \sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{N}^n\\ m\in\mathbb{N}}} (1+|x|)^m |\partial^\alpha f(x)| \exp\left(-\frac{(m+|\alpha|)^\upsilon}{\kappa\upsilon}\right).$$

Then define, for $\upsilon > 1$,

$$\mathcal{S}^\upsilon = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n) : \forall \kappa \in \mathbb{R}_+^*, \|f\|_{\kappa,\upsilon} < +\infty \right\}, \tag{3.2}$$

which is a Fréchet space when endowed with the family of semi-norms $\|\cdot\|_{\kappa,\upsilon}$ for $\kappa > 0$. Notice that $\mathcal{S}^\upsilon$ is contained in the usual space of Schwartz functions and that the elements of $\mathcal{S}^\upsilon$ are in the Denjoy–Carleman class $\mathcal{C}^{\kappa,\upsilon}$ for every $\kappa > 0$. One may also check that $\mathcal{S}^\upsilon$ is closed under differentiation. We will denote by $(\mathcal{S}^\upsilon)'$ the space of continuous linear forms on $\mathcal{S}^\upsilon$ endowed with the weak-star topology. This space will play the role of tempered distributions in our context.

Proposition 3.5. *If $\upsilon > 1$, then the Fourier transform from $\mathcal{S}^\upsilon$ to itself is a continuous isomorphism.*

Proof. We start by proving that the Fourier transform is continuous from $\mathcal{S}^\upsilon$ to itself. Let $0<\kappa'<\kappa$. Let $f\in\mathcal{S}^\upsilon$ and recall that for all $\xi\in\mathbb{R}^n$ and $\alpha,\beta\in\mathbb{N}^n$ we have[1]

$$\begin{aligned}\xi^\alpha\partial^\beta\hat f(\xi) &= (-i)^{|\alpha|+|\beta|}\int_{\mathbb{R}^n} e^{-ix\xi}\partial^\alpha\left(x^\beta f(x)\right)\mathrm{d}x\\ &= (-i)^{|\alpha|+|\beta|}\sum_{\substack{\gamma_1+\gamma_2=\alpha\\ \gamma_2\preceq\beta}}\frac{\alpha!\beta!}{\gamma_1!\gamma_2!\left(\beta-\gamma_2\right)!}\int_{\mathbb{R}^d} e^{-ix\xi}x^{\beta-\gamma_2}\partial^{\gamma_1}f(x)\mathrm{d}x,\end{aligned}$$

where $\gamma_2\preceq\beta$ means that each coordinate of γ_2 is smaller than the corresponding coordinate of β. Then, notice that there is a constant $C>0$ such that, for every $\gamma_1,\gamma_2,\beta\in\mathbb{N}^d$ such that $\gamma_2\preceq\beta$, we have

$$\left|\int_{\mathbb{R}^n} e^{ix\xi}x^{\beta-\gamma_2}\partial^{\gamma_1}f(x)\mathrm{d}x\right|\le C\left\|f\right\|_{\kappa,\upsilon}\exp\left(\frac{\left(|\beta|-|\gamma_2|+|\gamma_1|+n+1\right)^\upsilon}{\kappa\upsilon}\right).$$

Moreover, up to making C larger we also have, for every $\gamma_2\in\mathbb{N}^n$,

$$\gamma_2!\le C\exp\left(\frac{|\gamma_2|^\upsilon}{\kappa\upsilon}\right).$$

Consequently, we find that for every $\xi\in\mathbb{R}^n$ and $\alpha,\beta\in\mathbb{N}^n$, the quantity $\left|\xi^\alpha\partial^\beta\hat f(\xi)\right|$ is smaller than

$$\begin{aligned}C^2\left\|f\right\|_{\kappa,\upsilon}&\sum_{\substack{\gamma_1+\gamma_2=\alpha\\ \gamma_2\preceq\beta}}\frac{\alpha!}{\gamma_1!\gamma_2!}\frac{\beta!}{\gamma_2!\left(\beta-\gamma_2\right)!}\exp\left(\frac{|\gamma_2|^\upsilon+\left(|\beta|-|\gamma_2|+|\gamma_1|+n+1\right)^\upsilon}{\kappa\upsilon}\right)\\ &\le C^2\left\|f\right\|_{\kappa,\upsilon}2^{|\alpha|+|\beta|}\exp\left(\frac{\left(|\alpha|+|\beta|+n+1\right)^\upsilon}{\kappa\upsilon}\right).\end{aligned}$$

Using the fact that for $\ell\in\mathbb{N}$

$$|\xi|^{2\ell}=\left(\sum_{j=1}^n|\xi_j|^2\right)^\ell=\sum_{|\alpha|=\ell}c\left(\alpha\right)\xi^{2\alpha},$$

where $\sum_{|\alpha|=\ell}c\left(\alpha\right)=n^\ell$, we see that, for some new constant $C>0$, we have for all $m\in\mathbb{N},\xi\in\mathbb{R}^n$ and $\alpha,\beta\in\mathbb{N}^n$:

$$\left(1+|\xi|\right)^m\left|\partial^\beta\hat f\left(\xi\right)\right|\le C\left\|f\right\|_{\kappa,\upsilon}\left(4\sqrt{n}\right)^m2^{|\beta|}\exp\left(\frac{\left(m+|\beta|+n+2\right)^\upsilon}{\kappa\upsilon}\right). \tag{3.3}$$

Indeed, we can deal first with m even and then argue that $\left(1+|\xi|\right)^m\le\left(1+|\xi|\right)^{m+1}$. Finally, since $\kappa'<\kappa$ and $\frac{(r+n+2)^\upsilon}{\kappa}-\frac{r^\upsilon}{\kappa'}\underset{r\to+\infty}{\sim}-\frac{\kappa-\kappa'}{\kappa\kappa'}r^\upsilon$, we see that, for some new constant

[1] There is an error in the expression for $\xi^\alpha\partial^\beta\hat f\left(\xi\right)$ in the proof of [Jéz20a, Proposition 5.3]. However, the proof is easily fixed by using the correct formula that we give here.

$C > 0$, we have

$$\left\|\hat{f}\right\|_{\kappa',\upsilon} \leq C \|f\|_{\kappa,\upsilon},$$

and the Fourier transform is indeed continuous from $\mathcal{S}^{\upsilon}$ to itself. The same argument gives that the inverse Fourier transform is also continuous from $\mathcal{S}^{\upsilon}$ to itself. Moreover, since $\mathcal{S}^{\upsilon}$ is included in the space of Schwartz function on $\mathbb{R}^n$, the elements of $\mathcal{S}^{\upsilon}$ satisfy the Fourier Inversion Formula. Hence, the Fourier transform is indeed a continuous automorphism of $\mathcal{S}^{\upsilon}$. □

Proposition 3.5 allows to define the Fourier transform on $(\mathcal{S}^{\upsilon})'$ by duality in the usual way. Since $\mathcal{S}^{\upsilon}$ is closed by multiplication, for $\psi \in \mathcal{S}^{\upsilon}$ we can define the Fourier multiplier $\psi(D) : (\mathcal{S}^{\upsilon})' \to (\mathcal{S}^{\upsilon})'$ by

$$\forall u \in (\mathcal{S}^{\upsilon})' : \psi(D)\, u = \mathbb{F}^{-1}(\psi.\hat{u}).$$

It is well-known that the Fourier transform of a $\mathcal{C}^{\infty}$ compactly supported function decays faster than the inverse of any polynomial. For functions in the class $\mathcal{C}^{\kappa,\upsilon}$ this statement is made quantitative in Proposition 3.6 below. This is the key point that will allow us in §3.2 to construct Sobolev-like spaces of anisotropic generalized distributions that are the pieces from which we will construct the space $\mathcal{H}$ from Theorem 3.1 in §3.4 and §3.5.

Proposition 3.6. *For every $R > 0$ and $\upsilon > 1$, there are constants $C > 0$ and $\kappa > 0$ such that, for all $f \in \mathcal{S}^{\upsilon}$ and $\xi \in \mathbb{R}^n$, we have*

$$\left|\hat{f}(\xi)\right| \leq C \|f\|_{\kappa,\upsilon} \exp\left(-R(\ln(1+|\xi|))^{\frac{\upsilon}{\upsilon-1}}\right). \tag{3.4}$$

Proof. Choose $\kappa > 0$ large enough so that

$$R' := \kappa^{\frac{1}{\upsilon-1}} \frac{\upsilon-1}{\upsilon} > R.$$

Then apply (3.3) from the proof of Proposition 3.5 with $\beta = 0$ to get a constant $C > 0$ such that, for all $\xi \in \mathbb{R}^n$ and $m \in \mathbb{N}$, we have

$$\left|\hat{f}(\xi)\right| \leq C \|f\|_{\kappa,\upsilon} \left(\frac{4\sqrt{n}}{1+|\xi|}\right)^m \exp\left(\frac{(m+n+2)^{\upsilon}}{\kappa\upsilon}\right).$$

When $|\xi|$ is small, we bound $\hat{f}(\xi)$ by taking $m = 0$. When $|\xi|$ is large enough so that the following expression makes sense and is non-negative, we take

$$m = \left\lfloor \left(-\ln\left(\frac{4\sqrt{n}}{1+|\xi|}\right)\right)^{\frac{1}{\upsilon-1}} \kappa^{\frac{1}{\upsilon-1}} - n - 2 \right\rfloor.$$

With this choice of m we have

$$\begin{aligned}
&\left(\frac{4\sqrt{n}}{1+|\xi|}\right)^{m}\exp\left(\frac{(m+n+2)^{\upsilon}}{\kappa\upsilon}\right)\\
&\quad\leq \exp\left(\left(\left|\ln\left(\frac{4\sqrt{n}}{1+|\xi|}\right)\right|^{\frac{1}{\upsilon-1}}\kappa^{\frac{1}{\upsilon-1}}-n-3\right)\ln\left(\frac{4\sqrt{n}}{1+|\xi|}\right)\right.\\
&\qquad\qquad\qquad\qquad\left.+\frac{\kappa^{\frac{1}{\upsilon-1}}}{\upsilon}\left|\ln\left(\frac{4\sqrt{n}}{1+|\xi|}\right)\right|^{\frac{\upsilon}{\upsilon-1}}\right)\\
&\quad\leq\left(\frac{1+|\xi|}{4\sqrt{n}}\right)^{n+3}\exp\left(\kappa^{\frac{1}{\upsilon-1}}\frac{1-\upsilon}{\upsilon}\left(\ln\left(\frac{1+|\xi|}{4\sqrt{n}}\right)\right)^{\frac{\upsilon}{\upsilon-1}}\right)\\
&\quad\leq\left(\frac{1+|\xi|}{4\sqrt{n}}\right)^{n+3}\exp\left(-R'\left(\ln\left(\frac{1+|\xi|}{4\sqrt{n}}\right)\right)^{\frac{\upsilon}{\upsilon-1}}\right),
\end{aligned}$$

and the result follows then from the fact that (recall that $R' > R$)

$$\left(\frac{1+r}{4\sqrt{n}}\right)^{n+3}\exp\left(R\left(\ln\left(1+r\right)\right)^{\frac{\upsilon}{\upsilon-1}}-R'\left(\ln\left(\frac{1+r}{4\sqrt{n}}\right)\right)^{\frac{\upsilon}{\upsilon-1}}\right)\underset{r\to+\infty}{\to}0.$$

□

We need to extend the notion of ultradifferentiability to more general objects than complex-valued functions in order to make sense of Theorem 3.1. For instance, we will define what a $\mathcal{C}^{\kappa,\upsilon}$ manifold is. To do it, we follow ideas that may be found in [KMR09], notice however that when $\upsilon > 2$ the sequence $(A_m)_{m\in\mathbb{N}}$ from Example 2.2 is not a DC-weight sequence in the sense of [KMR09], so that we cannot apply most of their results. Hopefully, it will be clear in the remaining of the section that, whereas the general theory of our ultradifferentiability classes may not be very satisfactory, this is of no harm in our pedestrian approach to the problem of the trace formula.

We already defined in §2.1 what it means for a map from an open subset of $\mathbb{R}^n$ to a Banach space to be $\mathcal{C}^{\kappa,\upsilon}$. We recall that this class of regularity is closed under composition and inversion. A $\mathcal{C}^{\kappa,\upsilon}$ manifold is then defined to be a Hausdorff topological space with countable basis endowed with a maximal $\mathcal{C}^{\kappa,\upsilon}$ atlas – i.e. a maximal atlas whose change of charts are $\mathcal{C}^{\kappa,\upsilon}$. As usual, a map $f : M \to N$ between two $\mathcal{C}^{\kappa,\upsilon}$ manifolds is said to be $\mathcal{C}^{\kappa,\upsilon}$ if it is $\mathcal{C}^{\kappa,\upsilon}$ "in charts".

We define now topological vector spaces associated to these classes of regularity. If M is a $\mathcal{C}^{\kappa,\upsilon}$ manifold for some $\kappa > 0$ and $\upsilon > 1$ then M has a natural $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ manifold structure for all $\kappa' > 0$ and $\tilde{\upsilon} > \upsilon$, so that we may define the class $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ of functions from M to $\mathbb{C}$ that are $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ for all $\kappa' > 0$. Notice that all $\mathcal{C}^{\kappa,\upsilon}$ functions from M to $\mathbb{C}$ belong to $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ if $\tilde{\upsilon} > \upsilon$.

Notice that if $\upsilon > 2$ then the class $\mathcal{C}^{\kappa,\upsilon}$ is not closed under differentiation and in particular in this case the tangent bundle TM has no natural $\mathcal{C}^{\kappa,\upsilon}$ structure. However, derivatives of $\mathcal{C}^{\kappa,\upsilon}$ functions are $\mathcal{C}^{\kappa',\upsilon}$ for all $0<\kappa'<\kappa$. Thus the tangent bundle TM may be endowed naturally with a $\mathcal{C}^{\kappa',\upsilon}$ structure, so that it makes sense to talk about a $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ vector field when $\tilde{\upsilon}>\upsilon$, or $\tilde{\upsilon}=\upsilon$ and $\kappa'<\kappa$. Integrating such a vector field gives rise to a $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ flow[2] $(\phi_t)_{t\in\mathbb{R}}$, see [KMR09, Kom80]. A consequence of this fact is that if X is a $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ vector field on M that does not vanish then X is locally conjugated *via* $\mathcal{C}^{\kappa',\tilde{\upsilon}}$ charts to a constant vector field on $\mathbb{R}^n$ where $n=\dim M$. This implies in particular that if $\upsilon'>\tilde{\upsilon}$ then $\mathcal{C}^{\infty,\upsilon'}$ is closed under differentiation with respect to X (this operation is even continuous with respect to the topology that we define below).

If M is compact, we endow $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ with a structure of Fréchet space in the following way: if U is an open subset of M and V is an open subset of $\mathbb{R}^n$, if $\psi:U\to V$ is a $\mathcal{C}^{\infty,\tilde{\upsilon}}$ chart, φ is an element of $\mathcal{C}^{\infty,\tilde{\upsilon}}$ supported in U and $\kappa>0$, define the semi-norm $\|\cdot\|_{\psi,\varphi,\kappa,\tilde{\upsilon}}$ by

$$\forall u\in\mathcal{C}^{\infty,\tilde{\upsilon}}:\|u\|_{\psi,\varphi,\kappa,\tilde{\upsilon}}=\sup_{\substack{\alpha\in\mathbb{N}^n\\ x\in V}}\left|\partial^\alpha\left(\left(\varphi u\right)\circ\psi^{-1}\right)(x)\right|\exp\left(-\frac{|\alpha|^{\tilde{\upsilon}}}{\kappa\tilde{\upsilon}}\right).$$

The topology of $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is generated by a countable family of these semi-norms: since M is compact we can cover M by a finite number of domain of charts and take a partition of unity subordinated to this cover, then we only need to let κ runs through the integers. The completeness of $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is easily verified. One can also check using Leibniz formula that pointwise multiplication $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)\times\mathcal{C}^{\infty,\tilde{\upsilon}}(M)\to\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is continuous. Notice also that if N is another $\mathcal{C}^{\kappa,\upsilon}$ manifold and $\psi:M\to N$ is a $\mathcal{C}^{\kappa,\upsilon}$ local diffeomorphism then the map $\mathcal{C}^{\infty,\tilde{\upsilon}}(N)\ni u\mapsto u\circ\psi\in\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is continuous.

We will also need the space $\mathcal{D}^{\tilde{\upsilon}}(M)$ of $\mathcal{C}^{\infty,\tilde{\upsilon}}$ densities on M: this is the space of complex measures of M which are absolutely continuous with respect to Lebesgue and whose density in any $\mathcal{C}^{\infty,\tilde{\upsilon}}$ chart is $\mathcal{C}^{\infty,\tilde{\upsilon}}$. We endow $\mathcal{D}^{\tilde{\upsilon}}(M)$ with a Fréchet structure as we did for[3] $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$. We will denote by $\mathcal{D}^{\tilde{\upsilon}}(M)'$ the space of continuous linear functionals on M on $\mathcal{D}^{\tilde{\upsilon}}(M)$, that we endow with the weak-star topology. Notice that if $u\in\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ then u defines an element of $\mathcal{D}^{\tilde{\upsilon}}(M)'$ that we also denote by u, by the formula

$$\forall\mu\in\mathcal{D}^{\tilde{\upsilon}}(M):\langle u,\mu\rangle=\int_M u\,\mathrm{d}\mu.$$

We define in this way an injection of $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ into $\mathcal{D}^{\tilde{\upsilon}}(M)'$ that can be shown to be continuous and to have dense image (by mollifying elements of $\mathcal{D}^{\tilde{\upsilon}}(M)'$, by convolution for instance).

Now, let M be a $\mathcal{C}^{\kappa,\upsilon}$ compact manifold for some $\kappa>0$ and $\upsilon>1$ and let $n=d+1$ denotes the dimension of M. Let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^{\kappa,\upsilon}$ flow on M (that is, the map $M\times\mathbb{R}\ni(x,t)\mapsto\phi_t(x)$ is $\mathcal{C}^{\kappa,\upsilon}$). Then the generator X of the flow $(\phi_t)_{t\in\mathbb{R}}$ is a $\mathcal{C}^{\kappa',\upsilon}$ vector field for

[2]That is, the map $(x,t)\mapsto\phi_t(x)$ is $\mathcal{C}^{\kappa',\tilde{\upsilon}}$.

[3]Notice that these two spaces may be identified by the choice of a particular element of $\mathcal{D}^{\tilde{\upsilon}}(M)$.

all $\kappa' < \kappa$. Choose $V : M \to \mathbb{C}$ a $\mathcal{C}^{\kappa,\upsilon}$ potential. Let $\tilde{\upsilon} > \upsilon$ and define for all $t \in \mathbb{R}$ the continuous operator $\mathcal{L}_t$ on $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ by

$$\forall u \in \mathcal{C}^{\infty,\tilde{\upsilon}}(M) : \forall x \in M : \mathcal{L}_t u(x) = \exp\left(\int_0^t V \circ \phi_\tau(x)\,\mathrm{d}\tau\right) u \circ \phi_t(x).$$

Here, let us notice that the prefactor in the definition of $\mathcal{L}_t$ is a $\mathcal{C}^{\kappa,\upsilon}$ function (since this class of regularity is closed under composition). It is convenient[4] to extend $\mathcal{L}_t$ and $P = X + V$ from $\mathcal{D}^{\tilde{\upsilon}}(M)'$ to itself. To do so, we need to compute their adjoints. Choose $\mu \in \mathcal{D}^{\tilde{\upsilon}}(M)$ positive and fully supported, it induces an isomorphism between $\mathcal{D}^{\tilde{\upsilon}}(M)$ and $\mathcal{C}^{\infty,\tilde{\upsilon}}(M), \nu \mapsto \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Then notice that $\frac{\mathrm{d}\left((\phi_t)_*\mu\right)}{\mathrm{d}\mu}$ satisfies for all $x \in M$ and $t, t' \in \mathbb{R}$ the cocycle equation

$$\frac{\mathrm{d}\left((\phi_{t+t'})_*\mu\right)}{\mathrm{d}\mu}(x) = \frac{\mathrm{d}\left((\phi_{t'})_*\mu\right)}{\mathrm{d}\mu}(x)\,\frac{\mathrm{d}\left((\phi_t)_*\mu\right)}{\mathrm{d}\mu}(\phi_{-t'}(x)),$$

so that we have

$$\forall x \in M : \forall t \in \mathbb{R} : \frac{\mathrm{d}\left((\phi_t)_*\mu\right)}{\mathrm{d}\mu}(x) = \exp\left(-\int_0^t \operatorname{div}(X) \circ \phi_{-\tau}(x)\,\mathrm{d}\tau\right),$$

where the divergence of X is defined by

$$\forall x \in M : \operatorname{div}(X)(x) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\left((\phi_t)_*\mu\right)}{\mathrm{d}\mu}(x)\right)\Big|_{t=0}.$$

Notice that $\operatorname{div}(X)$ is a $\mathcal{C}^{\kappa',\upsilon}$ function for all $\kappa' < \kappa$. Then the formal adjoint of $\mathcal{L}_t$ may be defined on $\mathcal{D}^{\tilde{\upsilon}}(M)$ by

$$(\mathcal{L}_t)^*\nu = \exp\left(\int_0^t (V - \operatorname{div}(X)) \circ \phi_{-\tau}\mathrm{d}\tau\right)\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \circ \phi_{-\tau}\mathrm{d}\mu$$

and the formal adjoint of P by

$$P^*\nu = (-X - \operatorname{div}(X) + V)\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu.$$

These two operators are continuous on $\mathcal{D}^{\tilde{\upsilon}}(M)$, so that P and $\mathcal{L}_t$ may be extended as continuous operators on $\mathcal{D}^{\tilde{\upsilon}}(M)'$. Notice that P and $\mathcal{L}_t$ commute.

We will need Lemmas 3.7 and 3.8 to prove Theorem 3.1.

Lemma 3.7. *With the notations above, we have:*

(i) If $u \in \mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ then the map $\mathbb{R} \ni t \mapsto \mathcal{L}_t u \in \mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is $\mathcal{C}^\infty$ and its derivative is $t \mapsto \mathcal{L}_t P u = P\mathcal{L}_t u$.

[4] It makes easier to apply the results from Appendix B or to define the norm $\|\cdot\|_{\mathcal{H}}$ in (3.50) for instance.

(ii) If $u \in \mathcal{D}^{\tilde{\upsilon}}(M)'$ then the map $\mathbb{R} \ni t \mapsto \mathcal{L}_t u \in \mathcal{D}^{\tilde{\upsilon}}(M)'$ is $\mathcal{C}^\infty$ and its derivative is $t \mapsto \mathcal{L}_t P u = P\mathcal{L}_t u$.

Proof. We only need to prove the first point: the same argument with $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ replaced by $\mathcal{D}^{\tilde{\upsilon}}(M)$, and $\mathcal{L}_t$ and P replaced by their formal adjoints gives the second point.

We start with the case $V = 0$. Using the group property of $(\mathcal{L}_t)_{t\in\mathbb{R}}$, we only need to prove differentiability at $t = 0$. Then we may cover M by flow boxes, and thus we only need to show that if $u \in \mathcal{S}^{\tilde{\upsilon}}$ is supported in a compact subset K of $\mathbb{R}^{d+1}$ then

$$\frac{u(\cdot + te_{d+1}) - u}{t} \underset{t\to 0}{\to} \partial_{x_{d+1}} u \text{ in } \mathcal{S}^{\tilde{\upsilon}}, \tag{3.5}$$

where e_{d+1} denotes the last vector of the canonical basis of $\mathbb{R}^{d+1}$. Up to enlarging K we may assume that for all $t \in [-1,1]$ the function $u(\cdot + te_{d+1})$ is supported in K. Then if $x \in K$, $\alpha \in \mathbb{N}^{d+1}$, and $t \in [-1,1]$ we have with Taylor's formula (for any $\kappa'' > 0$):

$$\begin{aligned}
&\left|\frac{\partial^\alpha u(x + te_{d+1}) - \partial^\alpha u(x)}{t} - \partial^\alpha \partial_{x_{d+1}} u(x)\right| \\
&\quad = \left|\frac{\partial^\alpha u(x + te_{d+1}) - \partial^\alpha u(x)}{t} - \partial_{x_{d+1}} \partial^\alpha u(x)\right| \\
&\quad \leq \frac{\left\|\partial^2_{x_{d+1}} \partial^\alpha u\right\|_\infty}{2} |t| \leq \frac{|t|}{2} \|u\|_{\kappa'',\tilde{\upsilon}} \exp\left(\frac{(|\alpha|+2)^{\tilde{\upsilon}}}{\kappa''\tilde{\upsilon}}\right).
\end{aligned}$$

Thus if $\kappa', \kappa'' > 0$ and for $R > 0$ depending only on K, we have for all $x \in \mathbb{R}^{d+1}, \alpha \in \mathbb{N}^{d+1}$ and $m \in \mathbb{N}$:

$$\begin{aligned}
(1+|x|)^m &\left|\frac{\partial^\alpha u(x + te_{d+1}) - \partial^\alpha u(x)}{t} - \partial^\alpha \partial_{x_{d+1}} u(x)\right| \exp\left(-\frac{(m+|\alpha|)^{\tilde{\upsilon}}}{\kappa'\tilde{\upsilon}}\right) \\
&\leq \frac{|t|}{2} \|u\|_{\kappa'',\tilde{\upsilon}} R^m \exp\left(\frac{(|\alpha|+2)^{\tilde{\upsilon}}}{\kappa''\tilde{\upsilon}} - \frac{(m+|\alpha|)^{\tilde{\upsilon}}}{\kappa'\tilde{\upsilon}}\right).
\end{aligned}$$

Thus if $\kappa' > 0$ and $\kappa'' > \kappa'$, then there is a constant $C > 0$ (that only depends on $K, \tilde{\upsilon}, \kappa'$, and κ'') such that for all $t \in [-1,1]$ we have

$$\left\|\frac{u(\cdot + te_{d+1}) - u}{t} - \partial_{x_{d+1}} u\right\|_{\kappa',\tilde{\upsilon}} \leq C|t| \|u\|_{\kappa'',\tilde{\upsilon}},$$

which implies (3.5) and thus ends the proof of the lemma in the case $V = 0$.

In order to deduce the result in the case of a general V from the case $V = 0$, we only need to prove that the map

$$t \mapsto \exp\left(\int_0^t V \circ \phi_\tau \mathrm{d}\tau\right) \tag{3.6}$$

is $\mathcal{C}^\infty$ from $\mathbb{R}$ to $\mathcal{C}^{\infty,\tilde{v}}(M)$. Indeed, the multiplication is continuous from the product $\mathcal{C}^{\infty,\tilde{v}}(M) \times \mathcal{C}^{\infty,\tilde{v}}(M)$ to $\mathcal{C}^{\infty,\tilde{v}}(M)$. The map (3.6) is easily seen to be $\mathcal{C}^\infty$ from $\mathbb{R}$ to $\mathcal{C}^0(M)$, and one may notice that its derivatives are valued in $\mathcal{C}^{\infty,\tilde{v}}$ (recall that the classes of regularity $\mathcal{C}^{\kappa,\tilde{v}}$, and hence $\mathcal{C}^{\infty,\tilde{v}}$, are closed by composition) with uniform bounds locally in t. Then, by successive applications of Taylor's formula at order 1 with integral remainder, one gets that the map (3.6) is $\mathcal{C}^\infty$ from $\mathbb{R}$ to $\mathcal{C}^{\infty,\tilde{v}}(M)$, ending the proof of the lemma (we use the exact formula for the remainder in order to bound it in $\mathcal{C}^{\infty,\tilde{v}}(M)$). □

Lemma 3.8. *Let $\mathcal{B}$ be a Banach space such that $\mathcal{B} \subseteq \mathcal{D}^{\tilde{v}}(M)'$, the inclusion being continuous. Assume that, for all $t \in \mathbb{R}_+$, the operator $\mathcal{L}_t$ is bounded from $\mathcal{B}$ to itself, and that $(\mathcal{L}_t)_{t\geq 0}$ is a strongly continuous semi-group of operator of $\mathcal{B}$. Then the generator of $(\mathcal{L}_t)_{t\geq 0}$ coincides with P on its domain which is*

$$\{u \in \mathcal{B} : Pu \in \mathcal{B}\}.$$

Proof. Denote for now the generator of $(\mathcal{L}_t)_{t\geq 0}$ by $\widetilde{X}$. Let $u \in \mathcal{B}$ be in the domain of $\widetilde{X}$, then the map $\mathbb{R}_+ \ni t \mapsto \mathcal{L}_t u \in \mathcal{B}$ is differentiable at 0 and its derivative at 0 is $\widetilde{X}u$ (by definition of $\widetilde{X}$). Since the inclusion $\mathcal{B} \subseteq \mathcal{D}^{\tilde{v}}(M)'$ is continuous, the same is true for the map $\mathbb{R}_+ \ni t \mapsto \mathcal{L}_t u \in \mathcal{D}^{\tilde{v}}(M)'$, whose derivative at 0 is Xu according to Lemma 3.7. Thus $\widetilde{X}u = Xu \in \mathcal{B}$.

Reciprocally, if $u \in \mathcal{B}$ is such that $Xu \in \mathcal{B}$, then we may define a $\mathcal{C}^1$ map $c : \mathbb{R}_+ \to \mathcal{B}$ by $c(t) = u + \int_0^t \mathcal{L}_\tau X u \mathrm{d}\tau$ for all $t \in \mathbb{R}_+$. Notice that $c'(0) = Xu$. Since the inclusion $\mathcal{B} \subseteq \mathcal{D}^{\tilde{v}}(M)'$ is continuous, the map c is still $\mathcal{C}^1$ when seen as a map from $\mathbb{R}_+$ to $\mathcal{D}^{\tilde{v}}(M)'$ and we have $c(0) = u$ and $c'(t) = \mathcal{L}_t X u$ for all $t \in \mathbb{R}_+$, so that $c(t) = \mathcal{L}_t u$ for all $t \in \mathbb{R}_+$, using Lemma 3.7. This proves that u belongs to the domain of $\widetilde{X}$. □

3.2 Local spaces

We define now "local" spaces $\mathcal{H}_{\Theta,\alpha}$ that will be the basic pieces to construct the space $\mathcal{H}$ from Theorem 3.1. These spaces will depend on the choice of a system of cones Θ: this system encodes the three distinguished directions from Definition 6 of an Anosov flow. These spaces are Sobolev-like spaces similar to the spaces from [Bal18, Definition 4.16] or from [BT07] (for discrete-time systems) or [Ada18b, Ada18a] (even though the approach is a bit different, spaces in [FS11] are also Sobolev-like spaces). As in [BT07, BT08, Bal18, Ada18b, Ada18a], we will use Paley–Littlewood decomposition to study these spaces and the action of Koopman operators on them. However, as in [Jéz20a], we cannot use the usual dyadic Paley–Littlewood decomposition since the weights that we use to define our Sobolev-like spaces have a growth faster than polynomial, so that we will introduce an adapted Paley–Littlewood-like decomposition. This approach is slightly different from the strategy exposed in §2.2 since the annuli that will appear in this decomposition are not of exponential size, but the ideas behind are the same (see the discussion in §2.3).

First of all, we need to define the systems of cones that we will use. As in [Jéz20a], we need to consider system of potentially a large number of cones, in order to deal with the low hyperbolicity of the flow for small times. The interior and the adherence of a subset Y of a topological space will be denoted respectively by $\overset{\circ}{Y}$ and $\overline{Y}$. If C and C' are two cones in an Euclidean space, we write $C \Subset C'$ for $\overline{C} \subseteq \overset{\circ}{C'} \cup \{0\}$. The dimension of a cone C in an Euclidean space E is by definition the maximum dimension of a linear subspace of E contained in C.

Definition 3.9 (System of cones)**.** Let $(E, \langle ., . \rangle)$ be an Euclidean vector space, $e \in E$ and $r \geq 2$ be an integer. A system of $r+2$ cones with respect to the direction e is a family $\Theta = (C_0, C_1, \dots, C_r, C_f)$ of non-empty closed cones in E such that

(i) $\overset{\circ}{C_0} \cup \overset{\circ}{C_1} \cup \overset{\circ}{C_f} = E \setminus \{0\}$;

(ii) C_f is one-dimensional and there is $c > 0$ such that for all $\xi \in C_f$ we have $|\langle \xi, e \rangle| \geq c\,|\xi|$;

(iii) there are integers d_u and d_s such that $d_u + d_s + 1 = \dim E$, C_0 is d_s-dimensional and, for $i \in \{1, \dots, r\}$, the cone C_i is d_u-dimensional;

(iv) if $i \in \{1, \dots, r-1\}$ then $C_{i+1} \Subset C_i$;

(v) $C_0 \cap C_2 = C_f \cap C_2 = \{0\}$.

Let us fix $d \in \mathbb{N}$. The vector space $\mathbb{R}^{d+1}$ will always be endowed with its canonical Euclidean structure and system of cones in $\mathbb{R}^{d+1}$ will always be with respect to the direction of $e_{d+1} = (0, \dots, 0, 1)$. We will mainly use Definition 3.9 with $E = \mathbb{R}^{d+1}$, however, it will be convenient in the proof of Lemma 3.29 to have at our disposal the definition of a system of cones in a general Euclidean space.

If $(C_0, \dots, C_r, C_f)$ is a system of $r+2$ cones in $\mathbb{R}^{d+1}$ (with respect to the direction e_{d+1}) then we can choose $(\varphi_0, \varphi_1, \dots, \varphi_{r-1}, \varphi_f)$ a Gevrey[5] partition of unity on $\mathbb{S}^d$ such that:

- for $i \in \{0, \dots, r-1, f\}$, the function φ_i is supported in the interior of $C_i \cap \mathbb{S}^d$;
- if $i \in \{1, \dots, r-2\}$ then φ_i vanishes on a neighborhood of $\mathbb{S}^d \cap C_{i+2}$.

Indeed, the interiors of $C_0 \cap \mathbb{S}^d$, $(C_f \setminus C_2) \cap \mathbb{S}^d$, $(C_1 \setminus C_3) \cap \mathbb{S}^d, \dots, (C_{r-2} \setminus C_r) \cap \mathbb{S}^d$ and $C_{r-1} \cap \mathbb{S}^d$ form an open cover of $\mathbb{S}^d$.

Fix $\alpha \in \,]0,1[$ for the remaining of the section. Choose a Gevrey function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(x) = 1$ if $x \leq \frac{1}{2}$ and $\chi(x) = 0$ if $x \geq 1$. Define for all $n \geq 1$ and $\xi \in \mathbb{R}^{d+1}$,

[5]This ensures that it is $\mathcal{C}^{\kappa,\upsilon}$ for any $\kappa > 0$ and $\upsilon > 1$, so that all the Fourier multipliers that appear later are automatically well-defined.

$\chi_n(\xi) = \chi(2^{-n}|\xi|)$ and $\chi_{\alpha,n}(\xi) = \chi\left(|\xi| - 2^{n^\alpha}\right)$, set also $\chi_n = \chi_{\alpha,n} = 0$ if $n \leq 0$. Then set for $n \in \mathbb{N}$, $\psi_n(\xi) = \chi_{n+1}(\xi) - \chi_n(\xi)$ and $\psi_{\alpha,n}(\xi) = \chi_{\alpha,n+1}(\xi) - \chi_{\alpha,n}(\xi)$. Thus we have for $n \geq 1$

$$\text{supp } \psi_n \subseteq \left\{\xi \in \mathbb{R}^{d+1} : 2^{n-1} \leq |\xi| \leq 2^{n+1}\right\}$$

and

$$\text{supp } \psi_{\alpha,n} \subseteq \left\{\xi \in \mathbb{R}^{d+1} : 2^{n^\alpha} \leq |\xi| \leq 2^{(n+1)^\alpha} + 1\right\}.$$

In addition, supp ψ_0 and supp $\psi_{\alpha,0}$ are contained in $\left\{\xi \in \mathbb{R}^{d+1} : |\xi| \leq 5\right\}$. Moreover, we have $\sum_{n\geq 0} \psi_n = \sum_{n \geq 0} \psi_{\alpha,n} = 1$. Set

$$\Gamma = \mathbb{N} \times \{0, \dots, r-1, f\}.$$

Define for $(n,i) \in \Gamma$ the function $\psi_{\Theta,n,i}$ by

$$\psi_{\Theta,n,i}(\xi) = \begin{cases} \psi_n(\xi)\,\varphi_i\left(\frac{\xi}{|\xi|}\right) & \text{if } n \geq 1, \\ \frac{\psi_0(\xi)}{r-1} & \text{if } n = 0, \end{cases}$$

if $i \in \{1, \dots, r-2, f\}$, and by

$$\psi_{\Theta,n,i}(\xi) = (1 - \psi_0(\xi))\,\psi_{\alpha,n}(\xi)\,\varphi_i\left(\frac{\xi}{|\xi|}\right)$$

if $i \in \{0, r-1\}$, so that we have

$$\sum_{(n,i)\in\Gamma} \psi_{\Theta,n,i} = 1.$$

We will give a Sobolev-like definition of the local space $\mathcal{H}_{\Theta,\alpha}$ (Definition 3.10) by mean of a weight $w_{\Theta,\alpha}$ (see (3.9)). If this description is convenient to prove the basic properties of $\mathcal{H}_{\Theta,\alpha}$ (see Proposition 3.11), we will rather use in the following sections a Paley–Littlewood-like description of the space $\mathcal{H}_{\Theta,\alpha}$ (see Proposition 3.13), for any $\upsilon \in \left]1, \frac{1}{1-\alpha}\right[$ we have:

$$\mathcal{H}_{\Theta,\alpha} = \left\{u \in (\mathcal{S}^\upsilon)' : \sum_{(n,i)\in\Gamma} \left(2^{n\beta_i} \left\|\psi_{\Theta,n,i}(D)\,u\right\|_2\right)^2 < +\infty\right\}$$

where

$$\beta_0 = d+2, \beta_{r-1} = -(d+2), \beta_f = -(d+2) \tag{3.7}$$

and

$$\beta_i = -(i+1)(d+2) \text{ for } i \in \{1, \dots, r-2\}. \tag{3.8}$$

The main idea behind the choice of the β_i is that the expected regularity of elements of $\mathcal{H}_{\Theta,\alpha}$ (measured *via* integrability of the Fourier transform) must decrease under the action of the linear model of the dynamics (the β_i play the role here of an analogue of the escape function from [FS11]). The particular choice has been made so that computations are as easy as possible. Our parameters have been designed in order to make the Paley–Littlewood description as simple as possible, at the cost of a definition of the weight $w_{\Theta,\alpha}$ that may seem a bit heavy. It is defined for $\xi \in \mathbb{R}^{d+1}$ by

$$w_{\Theta,\alpha}(\xi) = \psi_0(\xi) + (1-\psi_0(\xi))\left(\sum_{i\in\{0,r-1\}} \varphi_i\left(\frac{\xi}{|\xi|}\right)\exp\left(\frac{\beta_i \ln(1+|\xi|)^{\frac{1}{\alpha}}}{(\ln 2)^{\frac{1}{\alpha}-1}}\right)\right.$$
$$\left. + \sum_{i\in\{1,\dots,r-2,f\}} \varphi_i\left(\frac{\xi}{|\xi|}\right)\langle\xi\rangle^{\beta_i}\right), \tag{3.9}$$

where

$$\langle\xi\rangle = \sqrt{1+|\xi|^2} \text{ for } \xi \in \mathbb{R}^{d+1}.$$

Definition 3.10. Define the space (for any $\upsilon \in \left]1, \frac{1}{1-\alpha}\right[$)

$$\mathcal{H}_{\Theta,\alpha} = \left\{u \in (\mathcal{S}^{\upsilon})' : \hat{u} \in L^2_{\text{loc}} \text{ and } \int_{\mathbb{R}^{d+1}} |\hat{u}(\xi)|^2 w_{\Theta,\alpha}(\xi)^2 \mathrm{d}\xi < +\infty\right\}$$

endowed with the scalar product

$$\langle u, v\rangle_{\Theta,\alpha} = \int_{\mathbb{R}^d} \overline{\hat{u}(\xi)}\hat{v}(\xi) w_{\Theta,\alpha}(\xi)^2 \mathrm{d}\xi.$$

Recall (3.2) for the definition of $\mathcal{S}^{\upsilon}$ and (3.9) for the definition of $w_{\Theta,\alpha}$.

Proposition 3.11. *$\mathcal{H}_{\Theta,\alpha}$ is a separable Hilbert space that does not depend on the choice of υ. For all $1 < \upsilon < \frac{1}{1-\alpha}$, the space $\mathcal{S}^{\upsilon}$ is continuously contained and dense in $\mathcal{H}_{\Theta,\alpha}$, and $\mathcal{H}_{\Theta,\alpha}$ is continuously contained in $(\mathcal{S}^{\upsilon})'$.*

Proof. The map

$$\begin{array}{cccc} A : & \mathcal{H}_{\Theta,\alpha} & \to & L^2(\mathbb{R}^{d+1}) \\ & u & \mapsto & \hat{u}w_{\Theta,\alpha} \end{array}$$

is clearly an isometry. Choose $\upsilon < \frac{1}{1-\alpha}$, thanks to Propositions 3.5 and 3.6 (recall (3.4)), and since $\frac{1}{\alpha} < \frac{\upsilon}{\upsilon-1}$, the map $u \mapsto \widehat{u}.w_{\Theta,\alpha}^{-1}$ is continuous from $\mathcal{S}^{\upsilon}$ to $L^2(\mathbb{R}^{d+1})$. Thus the map $B : u \mapsto \mathbb{F}^{-1}\left(uw_{\Theta,\alpha}^{-1}\right)$ is continuous from $L^2(\mathbb{R}^{d+1})$ to $(\mathcal{S}^{\upsilon})'$. But if $u \in L^2(\mathbb{R}^{d+1})$ then it is clear that $Bu \in \mathcal{H}_{\Theta,\alpha}$ with $\|Bu\|_{\Theta,\alpha} = \|u\|_2$. Now, since A and B are inverses of each other, $\mathcal{H}_{\Theta,\alpha}$ is isometric to $L^2(\mathbb{R}^{d+1})$ and thus a separable Hilbert space.

Proposition 3.6 implies that $\mathcal{S}^{\upsilon}$ is continuously contained in $\mathcal{H}_{\Theta,\alpha}$ and that the inclusion of $\mathcal{H}_{\Theta,\alpha}$ in $(\mathcal{S}^{\upsilon})'$ is continuous. Let $u \in \mathcal{H}_{\Theta,\alpha}$ be in the orthogonal space to $\mathcal{S}^{\upsilon}$. If ρ is a compactly supported element of $\mathcal{S}^{\upsilon}$, then, for all $v \in \mathcal{S}^{\upsilon}$, we have

$$\int_{\mathbb{R}^{d+1}} \rho(\xi) \overline{\hat{u}(\xi)} w_{\Theta,\alpha}(\xi)^2 v(\xi) \,\mathrm{d}\xi = \left\langle u, \mathbb{F}^{-1}(\rho.v) \right\rangle_{\Theta,\alpha} = 0.$$

Thus the function $\rho\bar{\hat{u}}w_{\Theta,\alpha}^2 \in L^1\left(\mathbb{R}^{d+1}\right)$ vanishes (take for v a convolution kernel), and so does u. Consequently, $\mathcal{S}^{\upsilon}$ is dense in $\mathcal{H}_{\Theta,\alpha}$.

To see that $\mathcal{H}_{\Theta,\alpha}$ does not depend on the choice of υ, just notice that, if we use $\tilde{\upsilon} \in \left]\upsilon, \frac{1}{1-\alpha}\right[$ instead of υ in the definition of $\mathcal{H}_{\Theta,\alpha}$, then we obtain another Hilbert space $\widetilde{\mathcal{H}}_{\Theta,\alpha}$. But then $\widetilde{\mathcal{H}}_{\Theta,\alpha} \subseteq \mathcal{H}_{\Theta,\alpha}$, and the inclusion is isometric and has a dense image (because $\widetilde{\mathcal{H}}_{\Theta,\alpha}$ contains $\mathcal{S}^{\upsilon}$). Since $\widetilde{\mathcal{H}}_{\Theta,\alpha}$ and $\mathcal{H}_{\Theta,\alpha}$ are both Hilbert spaces, they must coincide. □

Remark 3.12. It is clear from the proof that in fact the elements of $\mathcal{S}^{\upsilon}$ whose Fourier transform is compactly supported form a dense subset of $\mathcal{H}_{\Theta,\alpha}$.

Proposition 3.13. *Let $1 < \upsilon < \frac{1}{1-\alpha}$ and $u \in (\mathcal{S}^{\upsilon})'$. Then $u \in \mathcal{H}_{\Theta,\alpha}$ if and only if*

$$\sum_{(n,i)\in\Gamma} \left(2^{n\beta_i} \left\|\psi_{\Theta,n,i}(D)\, u\right\|_2\right)^2 < +\infty. \tag{3.10}$$

Moreover, the square root of this quantity defines an equivalent (Hilbertian) norm on $\mathcal{H}_{\Theta,\alpha}$.

Proof. First, notice that there is $C > 0$ such that, if $n \in \mathbb{N}$, $i \in \{1, \ldots, r-2, f\}$ and $\xi \in \operatorname{supp} \psi_{\Theta,n,i}$, then

$$\frac{1}{C} 2^{n\beta_i} \leq \langle\xi\rangle^{\beta_i} \leq C 2^{n\beta_i}.$$

Up to enlarging C, it is also true that if $n \in \mathbb{N}$, $i \in \{0, r-1\}$ and $\xi \in \operatorname{supp} \psi_{\Theta,n,i}$ then

$$\frac{1}{C} 2^{n\beta_i} \leq \exp\left(\frac{\beta_i \ln(1+|\xi|)^{\frac{1}{\alpha}}}{(\ln 2)^{\frac{1}{\alpha}-1}}\right) \leq C 2^{n\beta_i}.$$

Now, using the fact that the intersection number of the support of the $\psi_{\Theta,n,i}$ for $(n,i) \in \Gamma$ is finite, we find another constant $C > 0$ such that for all $\xi \in \mathbb{R}^{d+1}$ we have

$$\frac{1}{C} w_{\Theta,\alpha}(\xi)^2 \leq \sum_{(n,i)\in\Gamma} \left(2^{n\beta_i} \psi_{\Theta,n,i}(\xi)\right)^2 \leq C w_{\Theta,\alpha}(\xi)^2. \tag{3.11}$$

From this, we get immediately that if $u \in \mathcal{H}_{\Theta,\alpha}$ then (3.10) holds. Reciprocally, if (3.10) holds, then $\hat{u}$ is in L^2_{loc} (the sum $\sum_{(n,i)\in\Gamma} \psi_{\Theta,n,i}$ is locally finite) and from (3.11) we get that $u \in \mathcal{H}_{\Theta,\alpha}$. The equivalence of norms is an immediate consequence of (3.11). □

Proposition 3.13 suggests to define the auxiliary Hilbert space

$$\mathcal{B} = \left\{ (u_{n,i})_{(n,i)\in\Gamma} \in \prod_{(n,i)\in\Gamma} L^2\left(\mathbb{R}^{d+1}\right) : \sum_{(n,i)\in\Gamma} \left(2^{n\beta_i} \|u_{n,i}\|_2\right)^2 < +\infty \right\}. \tag{3.12}$$

Define the map

$$\begin{array}{cccc} \mathcal{Q}_\Theta & : & \mathcal{H}_{\Theta,\alpha} & \to & \mathcal{B} \\ & & u & \mapsto & \left(\psi_{\Theta,n,i}(D)\, u\right)_{(n,i)\in\Gamma} \end{array}. \tag{3.13}$$

For $(n,i) \in \Gamma$ define also the natural projection and inclusion

$$\begin{array}{cccc} \pi_{n,i} & : & \mathcal{B} & \to & L^2\left(\mathbb{R}^{d+1}\right) \\ & & (u_{\ell,j})_{(\ell,j)\in\Gamma} & \mapsto & u_{n,i} \end{array}$$

and

$$\begin{array}{cccc} \iota_{n,i} & : & L^2\left(\mathbb{R}^{d+1}\right) & \to & \mathcal{B} \\ & & u & \mapsto & \left(u\delta_{(n,i)=(\ell,j)}\right)_{(\ell,j)\in\Gamma} \end{array}.$$

3.3 Local Koopman operator

We are now going to study a local model for the Koopman operator (9) associated to an Anosov flow $(\phi_t)_{t\in\mathbb{R}}$ on a $(d+1)$-dimensional manifold M. The main result of this section is Proposition 3.17 which is a local version of Theorem 3.1.

As a local model for a flow, we will consider a family $(\mathcal{T}_t)_{t\in\mathbb{R}}$ of diffeomorphisms of $\mathbb{R}^{d+1}$ such that if we define $F : \mathbb{R}^d \to \mathbb{R}^{d+1}$ by $x \mapsto \mathcal{T}_0(x,0)$ (here we make the identification $\mathbb{R}^{d+1} \simeq \mathbb{R}^d \times \mathbb{R}$) then we have

$$\forall t \in \mathbb{R} : \forall (x,y) \in \mathbb{R}^d \times \mathbb{R} \simeq \mathbb{R}^{d+1} : \mathcal{T}_t(x,y) = F(x) + y e_{d+1} + t e_{d+1}. \tag{3.14}$$

We will say that F is the map associated to the family of diffeomorphisms $(\mathcal{T}_t)_{t\in\mathbb{R}}$. Reciprocally, if $F : \mathbb{R}^d \to \mathbb{R}^{d+1}$ is an immersion, we define by (3.14) the associated family of diffeomorphisms $(\mathcal{T}_t)_{t\in\mathbb{R}}$ (provided they actually are diffeomorphisms).

Remark 3.14. Let us explain why we use such a family of diffeomorphisms as a local model for a flow. We want to study the flow $(\phi_t)_{t\in\mathbb{R}}$ in the neighbourhood of a fixed time $\tilde{t}_0$. To do it, we take charts κ and κ' for M and we study the family of diffeomorphisms $(\mathcal{T}_t)_{t\in\mathbb{R}}$ defined by the formula

$$\mathcal{T}_t = \kappa \circ \phi_{\tilde{t}_0+t} \circ \kappa'^{-1}.$$

Of course, this is not in general a family of diffeomorphisms from $\mathbb{R}^{d+1}$ to itself (*a priori* the domain of $\mathcal{T}_t$ depends on t). However, it is more convenient to deal with diffeomorphisms of the whole $\mathbb{R}^{d+1}$, and we will consequently provide extensions of the $\mathcal{T}_t$ to $\mathbb{R}^{d+1}$

when applying Proposition 3.17 in §3.4 (see Lemma 3.29). These extensions are far from canonical, but the use of a cutoff function will ensure that none of the objects that we consider in §3.4 depend on the choices we will make in a relevant way.

It is natural to ask for κ and κ' to be flow boxes, that is, if X is the generator of the flow $(\phi_t)_{t\in\mathbb{R}}$, we require $\kappa^*(e_{d+1}) = X$ and $\kappa'^*(e_{d+1}) = X$ (we identify e_{d+1} with the constant vector field with value e_{d+1}). This requirement implies (3.14) for small t and y, and, since we are only interested here in the behaviour of $(\phi_t)_{t\in\mathbb{R}}$ locally in both space and time, we may modify the definition of $\mathcal{T}_t$ for large t and design our extension to ensure that (3.14) holds (we refer to the proof of Lemma 3.29 for details). Once again, this will be of no harm in the global analysis thanks to the use of cutoff functions in both time and space.

In this section, we will study such a family with no reference to a particular Anosov flow. We will need further assumptions to do so. The first one is that F (or equivalently $\mathcal{T}_0$ or any $\mathcal{T}_t$ for $t \in \mathbb{R}$) is $\mathcal{C}^{\kappa,\upsilon}$ for some $\kappa > 0$ and $\upsilon > 1$. The second one is a condition of hyperbolicity that we will express using cones.

Let $r \geq 2$ be an integer and choose two systems of $r+2$ cones (with respect to the direction e_{d+1} as usual) $\Theta = (C_0, \dots, C_r, C_f)$ and $\Theta' = \left(C'_0, \dots, C'_r, C'_f\right)$. We assume that $(\mathcal{T}_t)_{t\in\mathbb{R}}$ is cone-hyperbolic from Θ' to Θ in the following sense:

(i) for all $x \in \mathbb{R}^{d+1}$, $i \in \{1, \dots, r\}$ and $t \in \mathbb{R}$ we have[6]

$$D_x\mathcal{T}_t^{\mathrm{tr}}(C_i) \subseteq C'_{\min(i+2,r)};$$

(ii) for all $x \in \mathbb{R}^{d+1}$ and $t \in \mathbb{R}$ we have

$$D_x\mathcal{T}_t^{\mathrm{tr}}(C_f) \cap C'_0 = \{0\};$$

(iii) there is $\Lambda > 1$ such that for all $x \in \mathbb{R}^{d+1}$, all $\xi \in C_{r-1}$, and all $t \in \mathbb{R}$ we have

$$\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\xi)\right| \geq \Lambda\,|\xi|;$$

(iv) for the same $\Lambda > 1$, for all $x \in \mathbb{R}^{d+1}$, all $\xi \in \mathbb{R}^d$, and all $t \in \mathbb{R}$ such that $D_x\mathcal{T}_t^{\mathrm{tr}}(\xi) \in C'_0$ we have[7]

$$\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\xi)\right| \leq \Lambda^{-1}\,|\xi|.$$

Remark 3.15. Notice that the definition of the cone-hyperbolicity of the family $(\mathcal{T}_t)_{t\in\mathbb{R}}$ only involves the derivatives $D_x\mathcal{T}_t^{\mathrm{tr}}$. However, these derivatives do not depend on t (this is a consequence of (3.14)). Consequently, one only needs to check that (i)-(iv) hold for

[6] Here, A^{tr} denotes the transpose of A.

[7] Notice that the condition $D_x\mathcal{T}_t^{\mathrm{tr}}(\xi) \in C'_0$ implies in particular that $\xi \in C_0$, as a consequence of (i) and (ii).

$t = 0$. This fact may be surprising since hyperbolicity is usually a phenomenon that can only be observed after a small amount of time, but recall Remark 3.14: in the application, the family $(\mathcal{T}_t)_{t\in\mathbb{R}}$ will only be used to describe the flow $(\phi_t)_{t\in\mathbb{R}}$ near some time $\tilde{t}_0$. Then, provided that $\tilde{t}_0 > 0$, the family $(\mathcal{T}_t)_{t\in\mathbb{R}}$ will be cone-hyperbolic (see §3.4 for the details).

Remark 3.16. Notice that if $\left(\mathcal{T}_t^1\right)_{t\in\mathbb{R}}$ and $\left(\mathcal{T}_t^2\right)_{t\in\mathbb{R}}$ are two families of diffeomorphisms as above, then their composition may naturally be defined as $\left(\mathcal{T}_t^1 \circ \mathcal{T}_0^2\right)_{t\in\mathbb{R}}$. Moreover, if there are systems of cones Θ, Θ' and Θ'' such that $\left(\mathcal{T}_t^1\right)_{t\in\mathbb{R}}$ is cone-hyperbolic from Θ' to Θ'' and $\left(\mathcal{T}_t^2\right)_{t\in\mathbb{R}}$ is cone-hyperbolic from Θ to Θ' then $\left(\mathcal{T}_t^1 \circ \mathcal{T}_0^2\right)_{t\in\mathbb{R}}$ is cone-hyperbolic from Θ to Θ''.

We will also consider a $\mathcal{C}^\infty$ family $(G_t)_{t\in\mathbb{R}}$ of $\mathcal{S}^\upsilon$ functions from $\mathbb{R}^{d+1}$ to $\mathbb{C}$, such that there is a compact subset K of $\mathbb{R}^{d+1}$ such that, if $x \in \mathbb{R}^{d+1} \setminus K$ and $t \in \mathbb{R}$, then $G_t(x) = 0$. In this section, we study the family $(\mathcal{L}_t)_{t\in\mathbb{R}}$ of local Koopman operators defined by

$$\mathcal{L}_t u = G_t \left(u \circ \mathcal{T}_t\right). \tag{3.15}$$

This definition makes sense for $u \in \mathcal{S}^{\tilde{\upsilon}}$ (for any $\tilde{\upsilon} > \upsilon$) and may be extended by duality to $u \in \left(\mathcal{S}^{\tilde{\upsilon}}\right)'$. The main result of this section is Proposition 3.17, which can be seen as a local version of Theorem 3.1.

Proposition 3.17. *Let $\alpha \in \left]\frac{\upsilon-1}{\upsilon}, 1\right[$. For every $t \in \mathbb{R}$, the operator $\mathcal{L}_t$ defined by* (3.15) *is bounded from $\mathcal{H}_{\Theta,\alpha}$ to $\mathcal{H}_{\Theta',\alpha}$. Moreover, the family $(\mathcal{L}_t)_{t\in\mathbb{R}}$ is strongly continuous (as a family of operators from $\mathcal{H}_{\Theta,\alpha}$ to $\mathcal{H}_{\Theta',\alpha}$), hence it is measurable.*

Moreover, if $\alpha < \frac{1}{2}$, if k is a non-negative integer and if $h : \mathbb{R} \to \mathbb{C}$ is a compactly supported kth time differentiable function whose kth derivative has bounded variation then the operator

$$\int_{\mathbb{R}} h(t)\, \mathcal{L}_t \mathrm{d}t : \mathcal{H}_{\Theta,\alpha} \to \mathcal{H}_{\Theta',\alpha} \tag{3.16}$$

is in the Schatten class S_q for all $q \geq 1$ such that $q > \frac{d+1}{k+1}$. Moreover, there is a constant $C > 0$, which depends on h only through its support, such that

$$\left\| \int_{\mathbb{R}} h(t)\, \mathcal{L}_t \mathrm{d}t \right\|_{S_q} \leq C \left(\|h\|_{\mathcal{C}^{k-1}} + \left\| h^{(k)} \right\|_{\mathrm{BV}} \right),$$

where $\|\cdot\|_{S_q}$ denotes the S_q Schatten class norm and $\|\cdot\|_{\mathrm{BV}}$ the bounded variation norm.

If $k+1 > d+1$ and $\Theta = \Theta'$ we have

$$\operatorname{tr}\left(\int_{\mathbb{R}} h(t)\, \mathcal{L}_t \mathrm{d}t \right) = \sum_{p \circ F(x) = x} \frac{h(T(x))}{\left|\det\left(I - p \circ D_x F\right)\right|} \int_{\mathbb{R}} G_{T(x)}(x, y)\, \mathrm{d}y,$$

where p is the orthogonal projection from $\mathbb{R}^{d+1}$ to $\mathbb{R}^d \simeq \mathbb{R}^d \times \{0\}$ and, for $x \in \mathbb{R}^d$, the number $T(x)$ is defined by $F(x) = p(F(x)) + (0, -T(x))$.

Without the hypothesis $\alpha < \frac{1}{2}$, it remains true that the operator (3.16) *is compact.*

Remark 3.18. Since $\alpha > \frac{\upsilon-1}{\upsilon}$, we may choose $\tilde{\upsilon} > \upsilon$ such that $\tilde{\upsilon} < \frac{1}{1-\alpha}$. Then $\mathcal{H}_{\Theta,\alpha} \subseteq \left(\mathcal{S}^{\tilde{\upsilon}}\right)'$ and thus $\mathcal{L}_t u$ is well-defined as an element of $\left(\mathcal{S}^{\tilde{\upsilon}}\right)'$ when $t \in \mathbb{R}$ and $u \in \mathcal{H}_{\Theta,\alpha}$.

Remark 3.19. Notice that the spaces $\mathcal{H}_{\Theta,\alpha}$ and $\mathcal{H}_{\Theta',\alpha}$ depend *a priori* not only on Θ and Θ' (and α) but also on the choice of partitions of unity $(\varphi_0, \dots, \varphi_{r-1}, \varphi_f)$ and $\left(\varphi_0', \dots, \varphi_{r-1}', \varphi_f'\right)$ on $\mathbb{S}^d$ as in §3.2. However, in view of Proposition 3.17, this choice is mostly irrelevant and the dependence on Θ and Θ' is the fundamental point.

The remainder of this section is devoted to the proof of Proposition 3.17. For this, we introduce in Lemma 3.20 a family of auxiliary operators $(\mathcal{M}_t)_{t\in\mathbb{R}}$ acting on the space $\mathcal{B}$ defined in (3.12). Then, we prove that the family $(\mathcal{M}_t)_{t\in\mathbb{R}}$ has the properties that we expect from $(\mathcal{L}_t)_{t\in\mathbb{R}}$: boundedness and strong continuity is proven in Lemma 3.20 (with the help of the preparatory Lemmas 3.21 and 3.22, see §3.3.1), that an operator similar to (3.16) is in a Schatten class is proven in Lemma 3.27 (with the help of Lemmas 3.23, 3.24 and 3.25, see §3.3.2) and the formula for the trace is given in Lemma 3.28 (see §3.3.3). Finally, we end the proof of Proposition 3.17 by showing that $(\mathcal{L}_t)_{t\in\mathbb{R}}$ inherits these properties from $(\mathcal{M}_t)_{t\in\mathbb{R}}$.

3.3.1 The auxiliary operators $\mathcal{M}_t$.

We will need smooth functions $\tilde{\varphi}_0, \dots, \tilde{\varphi}_{r-1}$, and $\tilde{\varphi}_f : \mathbb{S}^d \to [0,1]$ such that

- if $i \in \{0, \dots, r-1, f\}$ then $\tilde{\varphi}_i$ is supported in the interior of $C_i \cap \mathbb{S}^d$;
- if $i \in \{1, \dots, r-2\}$ then $\tilde{\varphi}_i$ vanishes on a neighborhood of $C_{i+2} \cap \mathbb{S}^d$;
- if $i \in \{0, \dots, r-1, f\}$, $x \in \mathbb{S}^d$, and $\varphi_i(x) \neq 0$ then $\tilde{\varphi}_i(x) = 1$.

Define then $\tilde{\psi}_n = \chi_{n+2} - \chi_{n-1}$ and $\tilde{\psi}_{\alpha,n} = \chi_{\alpha,n+b} - \chi_{\alpha,n-b}$ for $n \geq 0$, where b is chosen large enough so that for all $n \in \mathbb{N}^*$ we have

$$2^{(n+1)^\alpha} - 2^{(n+b)^\alpha} + 1 \leq \frac{1}{2} \text{ and } 2^{n^\alpha} - 2^{(n-b)^\alpha} \geq 1.$$

If $(n,i) \in \Gamma$ set

$$\tilde{\psi}_{\Theta,n,i}(\xi) = \begin{cases} \tilde{\psi}_n(\xi)\, \tilde{\varphi}_i\left(\frac{\xi}{|\xi|}\right) & \text{if } n \geq 1, \\ \tilde{\psi}_0(\xi) & \text{if } n = 0, \end{cases}$$

if $i \in \{1, \dots, r-2, f\}$, and

$$\tilde{\psi}_{\Theta,n,i}(\xi) = \begin{cases} \tilde{\psi}_{\alpha,n}(\xi)\, \tilde{\varphi}_i\left(\frac{\xi}{|\xi|}\right) & \text{if } n \geq 1, \\ \tilde{\psi}_{\alpha,0}(\xi) & \text{if } n = 0, \end{cases}$$

if $i \in \{0, r-1\}$. Thus $\psi_{\Theta,n,i}(\xi) \neq 0$ implies $\tilde{\psi}_{\Theta,n,i}(\xi) = 1$. Now if $(n,i), (\ell, j) \in \Gamma$, and $t \in \mathbb{R}$ define an operator $S_{t,n,i}^{\ell,j} : L^2(\mathbb{R}^{d+1}) \to L^2(\mathbb{R}^{d+1})$ by

$$S_{t,n,i}^{\ell,j} = \psi_{\Theta',n,i}(D) \circ \mathcal{L}_t \circ \tilde{\psi}_{\Theta,\ell,j}(D). \tag{3.17}$$

As announced above, we define in Lemma 3.20 a family of auxiliary operators whose study will take most of the remainder of this section.

Lemma 3.20. *For every $t \in \mathbb{R}$, the sum*

$$\sum_{(n,i),(\ell,j)\in\Gamma} \iota_{n,i} \circ S_{t,n,i}^{\ell,j} \circ \pi_{\ell,j} \tag{3.18}$$

converges in the strong operator topology to an operator $\mathcal{M}_t : \mathcal{B} \to \mathcal{B}$ that depends continuously on t in the strong operator topology.

The proof of Lemma 3.20 is based on Lemmas 3.21 and 3.22 below. In order to prove Lemma 3.20, we first define a relation $\hookrightarrow$ on Γ that indexes the transitions (in the frequency space) that would occur for a linear dynamics, in the spirit of [Ada18b, Ada18a]. Our local space has been designed so that it corresponds either to a transition from high regularity to low regularity (which makes this part of the action smoothing) or to a stationary frequency in the direction of the flow (we will integrate in this direction, so that it also corresponds to a smoothing operator). The other transitions do not happen in the linear case, and we will control this non-linearity using not only the hyperbolicity of the dynamics but also its high regularity. Choose $a > 0$ such that for all $x \in K$ and $t \in \mathbb{R}$ we have

$$a < \left\| \left(D_x \mathcal{T}_t^{tr}\right)^{-1} \right\|^{-1}.$$

Choose also ν such that $0 < \nu < \frac{\log_2 \Lambda}{\alpha}$. We define now the relation $\hookrightarrow$. For $(\ell, j), (n, i) \in \Gamma$, we say that $(\ell, j) \hookrightarrow (n, i)$ holds if either of the following conditions is satisfied:

- $i = j = 0$ and $\ell \geq n + \nu n^{1-\alpha}$;
- $i = j = r-1$ and $n \geq \ell + \nu \ell^{1-\alpha}$;
- $j = 0$ and $i \in \{1, \dots, r-1, f\}$;
- $j \in \{1, \dots, r-2, f\}, i = r-1$ and $\ell \leq n^\alpha + 4 - \log_2 a$;
- $j = f, i \in \{1, \dots, r-2\}$ and $n \geq \ell - 4 + \log_2 a$;
- $i, j \in \{1, \dots, r-2\}$ with $i \geq j+1$ and $n \geq \ell - 4 + \log_2 a$;
- $i = j = f$ and $|\ell - n| \leq 10 - \log_2 c$, where c is such that for all $\xi = (\xi_1, \dots, \xi_{d+1}) \in C_f \cup C'_f$ we have $|\xi_{d+1}| \geq c|\xi|$ (such a constant exists by our definition of a system of cones).

In all other cases, we say that $(\ell, j) \not\hookrightarrow (n,i)$. Let us list the cases in which $(\ell, j) \not\hookrightarrow (n,i)$ in prevision of the proof of Lemma 3.21:

- $i=j=0$ and $\ell < n+\nu n^{1-\alpha}$;
- $i=j=r-1$ and $n<\ell+\nu \ell^{1-\alpha}$;
- $i=r-1, j\in\{1,\dots,r-2,f\}$ and $\ell > n^{\alpha}+4-\log_2 a$;
- $i \in \{1,\dots,r-2\}, j=f$ and $n<\ell-4+\log_2 a$;
- $i,j\in\{1,\dots,r-2\}, i\geq j+1$ and $n<\ell-4+\log_2 a$;
- $i=j=f$ and $|\ell-n|>10-\log_2 c$;
- $j=f$ and $i=0$;
- $j=r-1$ and $i\neq r-1$;
- $j\in\{1,\dots,r-1\}$ and $i\in\{0,\dots,j,f\}$.

Lemma 3.21 is the main tool to use the hyperbolicity of the dynamics to rule out the transitions of frequencies that do not occur in the linear picture.

Lemma 3.21. *For $i\in\{0,\dots,r-1,f\}$, set $\alpha_i=\alpha$ if $i=0$ or $r-1$, and $\alpha_i=1$ otherwise. There are $c'>0$ and $N>0$ such that if $(\ell,j),(n,i)\in\Gamma$ we have: $(\ell,j)\hookrightarrow(n,i)$ or $\max(n,\ell)\leq N$ or, for all $x\in K$ and $t\in\mathbb{R}$,*

$$d\left(\text{supp }\psi_{\Theta',n,i}, D_x\mathcal{T}_t^{\mathrm{tr}}\left(\text{supp }\tilde{\psi}_{\Theta,\ell,j}\right)\right)\geq c'\max\left(2^{n^{\alpha_i}},2^{\ell^{\alpha_j}}\right).$$

Proof. We will make massive use of the following fact in this proof : if C_+ and C_- are two closed cones in $\mathbb{R}^{d+1}$ such that $C_+\cap C_-=\{0\}$ (we say that such cones are transverse) then for all $\xi\in C_+$ and $\eta\in C_-$ we have

$$d(\xi,\eta)\geq\mu\max(|\xi|,|\eta|) \tag{3.19}$$

where $\mu=\min\left(d\left(C_+\cap\mathbb{S}^d,C_-\right),d\left(C_-\cap\mathbb{S}^d,C_+\right)\right)>0$.

Assume that $(n,i),(\ell,j)\in\Gamma$ are such that $(\ell,j)\not\hookrightarrow(n,i)$ and $\max(n,l)>N$ for some large N, and take $\xi\in$ supp $\psi_{\Theta',n,i},\eta\in$ supp $\tilde{\psi}_{\Theta,\ell,j}$ and $t\in\mathbb{R}$. We go through the different cases in which $(\ell,j)\not\hookrightarrow(n,i)$ as listed above.

- If $i=j=0$ and $\ell<n+\nu n^{1-\alpha}$, there are two possibilities: either $D_x\mathcal{T}_t^{\mathrm{tr}}(\eta)\notin C_0'$, and we can conclude with (3.19) (since φ_0' is supported in the interior of C_0'), or

$D_x\mathcal{T}_t^{\mathrm{tr}}(\eta) \in C_0'$, and by cone-hyperbolicity we have

$$\begin{aligned}|\xi| - \left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta)\right| &\geq 2^{n^\alpha} - \Lambda^{-1}\left(2^{(\ell+b)^\alpha}+1\right)\\ &\geq 2^{n^\alpha} - \Lambda^{-1}\left(2^{\left(n+\nu n^{1-\alpha}+b\right)^\alpha}+1\right)\\ &\geq 2^{n^\alpha}\left(1-2^{\left(n+\nu n^{1-\alpha}+b\right)^\alpha-\log_2\Lambda-n^\alpha}-2^{-n^\alpha}\right).\end{aligned}$$

We can then conclude if N is large enough, since

$$\left(n+\nu n^{1-\alpha}+b\right)^\alpha - \log_2\Lambda - n^\alpha \underset{n\to+\infty}{\to} \alpha\nu - \log_2\Lambda < 0$$

and

$$2^{\left(n+\nu n^{1-\alpha}\right)^\alpha} \leq C2^{n^\alpha}, \tag{3.20}$$

for some constant $C>0$ that does not depend on n. We used here the asymptotic expansion $\left(n+\nu n^{1-\alpha}\right)^\alpha \underset{n\to+\infty}{=} n^\alpha + \alpha\nu + o(1)$.

- If $i=j=r-1$ and $n<\ell+\nu\ell^{1-\alpha}$ then

$$\begin{aligned}\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta)\right| - |\xi| &\geq \Lambda 2^{(\ell-b)^\alpha} - \left(2^{(n+1)^\alpha}+1\right)\\ &\geq \Lambda 2^{(\ell-b)^\alpha} - \left(2^{\left(\ell+\nu\ell^{1-\alpha}+1\right)^\alpha}+1\right)\\ &\geq 2^{\ell^\alpha}\left(\Lambda 2^{(\ell-b)^\alpha-\ell^\alpha} - 2^{\left(\ell+\nu\ell^{1-\alpha}+1\right)^\alpha-\ell^\alpha} - 2^{-\ell^\alpha}\right).\end{aligned}$$

We can conclude if N is large enough, since

$$\Lambda 2^{(\ell-b)^\alpha-\ell^\alpha} - 2^{\left(\ell+\nu\ell^{1-\alpha}+1\right)^\alpha-\ell^\alpha} - 2^{-\ell^\alpha} \underset{\ell\to+\infty}{\to} \Lambda - 2^{\alpha\nu} > 0,$$

and (3.20) still holds when n is replaced by ℓ.

- If $j\in\{1,\dots,r-2,f\}, i=r-1$ and $\ell>n^\alpha+4-\log_2 a$, then

$$\begin{aligned}\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta)\right| - |\xi| &\geq a2^{\ell-2} - \left(2^{(n+1)^\alpha}+1\right)\\ &\geq a2^{\ell-2} - 2^{n^\alpha+1} - 1\\ &\geq a2^{\ell-3} - 1.\end{aligned}$$

- If $i\in\{1,\dots,r-2\}, j=f$ and $n<\ell-4+\log_2 a$ then

$$\begin{aligned}\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta)\right| - |\xi| &\geq a2^{\ell-2} - 2^{n+1}\\ &\geq a2^{\ell-3}.\end{aligned}$$

- The case $i,j\in\{1,\dots,r-2\}, i\geq j+1$ and $n<\ell-4+\log_2 a$ is dealt as the previous one.

- If $i = j = f$ and $|\ell - n| > 10 - \log_2 c$, then just notice that the $d+1$th coordinate of $D_x\mathcal{T}_t^{\mathrm{tr}}(\eta) - \xi$ is $\eta_{d+1} - \xi_{d+1}$ and consequently

$$\left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta) - \xi\right| \geq |\eta_{d+1} - \xi_{d+1}| .$$

  Since in addition we have $|\xi_{d+1}| \geq c|\xi|$ and $|\eta_{d+1}| \geq c|\eta|$, we can conclude in this case (discussing whether $|\xi|$ or $|\eta|$ is larger).

- The three last cases are dealt with by cone hyperbolicity using (3.19) (the support of $\psi_{\Theta',n,i}$ and the image of the support of $\tilde{\psi}_{\Theta,\ell,j}$ by $D_x\mathcal{T}_t^{\mathrm{tr}}$ are contained in transverse cones).

□

We now use Lemma 3.21 to control transitions that do not happen in the linear picture.

Lemma 3.22. *There is $\delta > 1$ such that, for every bounded interval I of $\mathbb{R}$, there is $C > 0$ such that if $(\ell, j) \not\leftrightarrow (n,i)$ for $(n,i), (\ell, j) \in \Gamma$, then for all $t \in I$ we have, recalling* (3.17),

$$\left\|S_{t,n,i}^{\ell,j}\right\|_{L^2 \to L^2} \leq C \exp\left(-\frac{\max(n,\ell)^\delta}{C}\right).$$

Proof. First of all, notice that $\mathcal{L}_t$ is bounded from L^2 to L^2 (uniformly when $t \in I$) and, since for all $(n,i), (\ell,j) \in \Gamma$ and $t \in I$, we have

$$\left\|S_{t,n,i}^{\ell,j}\right\|_{L^2 \to L^2} \leq \|\mathcal{L}_t\|_{L^2 \to L^2},$$

the case of $\max(n,\ell) \leq N$ is dealt with by taking C large enough. We consider consequently $(n,i), (\ell,j) \in \Gamma$ and $t \in I$ such that $(\ell,j) \not\leftrightarrow (n,i)$ and $\max(n,\ell) > N$. If $u \in L^2\left(\mathbb{R}^{d+1}\right)$ then we have, using Plancherel's formula,

$$\begin{aligned} &(2\pi)^{2(d+1)} \left\|S_{t,n,i}^{\ell,j} u\right\|_2^2 \\ &\quad = \int_{\mathbb{R}^{d+1}} \psi_{\Theta',n,i}(\xi)^2 \left| \int_{\left(\mathbb{R}^{d+1}\right)^2} e^{-ix\xi} e^{i\mathcal{T}_t(x)\eta} \tilde{\psi}_{\Theta,\ell,j}(\eta)\, G_t(x)\, \hat{u}(\eta)\, \mathrm{d}x \mathrm{d}\eta \right|^2 \mathrm{d}\xi. \end{aligned} \tag{3.21}$$

We are going to bound the inner integral. To do so, define for all $x \in \mathbb{R}^{d+1}$ and $j \in \{1, \dots, d+1\}$ the linear form $l_j(x)$ on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ by $l_j(x)(\xi,\eta) = i\left(\partial_j \mathcal{T}_t(x)\eta - \xi_j\right)$. Define also for all $x \in \mathbb{R}^{d+1}$ the quadratic form $\Phi(x)$ on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ by $\Phi(x)(\xi,\eta) = \left|D_x\mathcal{T}_t^{\mathrm{tr}}(\eta) - \xi\right|^2$. Now for all $t \in I$ and $k \in \mathbb{N}$ we define a kernel $\mathcal{K}_{k,t} : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to$

$\mathbb{C}$ by induction: we set $\mathcal{K}_{0,t}(x,\xi,\eta) = G_t(x)$ and for all $k \in \mathbb{N}$

$$\mathcal{K}_{k+1,t}(x,\cdot,\cdot) = \sum_{j=1}^{d+1} \partial_{x_j} \left(\frac{l_j(x)\,\mathcal{K}_{k,t}(x,\cdot,\cdot)}{\Phi(x)} \right).$$

Integrating by parts in y we see that the inner integral of (3.21) is equal, for all $k \in \mathbb{N}, t \in I$ and $\xi \in \mathbb{R}^{d+1}$, to

$$\int_{(\mathbb{R}^{d+1})^2} e^{-ix\xi} e^{i\mathcal{T}_t(x)\eta} \tilde{\psi}_{\Theta,\ell,j}(\eta)\,\mathcal{K}_{k,t}(x,\xi,\eta)\,\hat{u}(\eta)\,\mathrm{d}x\mathrm{d}\eta. \tag{3.22}$$

To bound the kernel $\mathcal{K}_{k,t}$, we notice that it is the sum of at most $(5(d+1))^k\,k!$ terms of the form

$$\begin{aligned}(x,\xi,\eta) \mapsto \pm \frac{\partial^\sigma G_t(x)}{(\Phi(x)(\xi,\eta))^{k+m}} \partial^{\gamma_1} l_{j_1}(x)(\xi,\eta)\ldots\partial^{\gamma_k} l_{j_k}(x)(\xi,\eta) \\ \times \partial^{\mu_1}\Phi(x)(\xi,\eta)\ldots\partial^{\mu_m}\Phi(x)(\xi,\eta),\end{aligned} \tag{3.23}$$

where $m \leq k$ is an integer, $j_1,\ldots,j_k \in \{1,\ldots,d+1\}$, and $\sigma,\gamma_1,\ldots,\gamma_k,\mu_1,\ldots,\mu_m$ are elements of $\mathbb{N}^{d+1}$ such that $|\sigma|+|\gamma_1|+\cdots+|\gamma_k|+|\mu_1|+\cdots+|\mu_m| = k$ (all the derivatives are with respect to the variable x). In order to bound these terms, notice that Lemma 3.21 implies that if $x \in K$, if $\xi \in \operatorname{supp} \psi_{\Theta',n,i}$ and if $\eta \in \operatorname{supp} \tilde{\psi}_{\Theta,\ell,j}$ then

$$\begin{aligned}\Phi(x)(\xi,\eta) &\geq (c')^2 \left(\max\left(2^{n^{\alpha_i}}, 2^{n^{\alpha_j}}\right)\right)^2 \\ &\geq c_1 \max\left(2^{n^{\alpha_i}}, 2^{n^{\alpha_j}}\right) \max(|\xi|,|\eta|) \geq c_2 \max(|\xi|,|\eta|)^2,\end{aligned}$$

for some positive constants c_1 and c_2. Consequently, there is a constant $C > 0$ such that if l is a linear map from $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1} \to \mathbb{C}$ and if q is a quadratic map $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1} \to \mathbb{C}$ then we have, for all $x \in K, \xi \in \operatorname{supp} \psi_{\Theta',n,i}$ and $\eta \in \operatorname{supp} \tilde{\psi}_{\Theta,\ell,j}$

$$\left|\frac{l(\xi,\eta)}{\Phi(x)(\xi,\eta)}\right| \leq C \frac{\|l\|}{\max\left(2^{n^{\alpha_i}}, 2^{\ell^{\alpha_j}}\right)} \text{ and } \left|\frac{q(\xi,\eta)}{\Phi(x)(\xi,\eta)}\right| \leq C\,\|q\|.$$

The choice of the norms on the spaces of linear and quadratic maps $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1} \to \mathbb{C}$ is of course irrelevant. Thus for such x, ξ and η any term of the form (3.23) is bounded by

$$C^{2k}\left(\max\left(2^{n^{\alpha_i}}, 2^{\ell^{\alpha_j}}\right)\right)^{-k} \|\partial^\sigma G_t\|_\infty \|\partial^{\gamma_1} l_{j_1}\|_\infty \ldots \|\partial^{\gamma_k} l_{j_k}\| \, \|\partial^{\mu_1}\Phi\|_\infty \ldots \|\partial^{\mu_m}\Phi\|_\infty,$$

where $\|\cdot\|_\infty$ refers to the supremum of the corresponding norm on K. Now, notice that, since $\mathcal{T}_0$ is $\mathcal{C}^{\kappa,\upsilon}$ then for any $\kappa' < \kappa$ the maps $l_1,\ldots,l_{d+1}$ (valued in the space of linear maps from $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$ to $\mathbb{C}$) and Φ (valued in the space of quadratic maps from $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$ to $\mathbb{C}$) are $\mathcal{C}^{\kappa',\upsilon}$ (we can event take $\kappa' = \kappa$ if $\upsilon \leq 2$). Thus there are constants $M, R > 0$

such that for all $\mu \in \mathbb{N}^d$, we have

$$\|\partial^\mu \Phi\|_\infty \leq MR^{|\mu|} |\mu|! \exp\left(\frac{|\mu|^\upsilon}{\kappa' \upsilon}\right),$$

for all $\gamma \in \mathbb{N}^d$ and $j \in \{1, \ldots, d+1\}$, we have

$$\|\partial^\gamma l_j\|_\infty \leq MR^{|\gamma|} |\gamma|! \exp\left(\frac{|\gamma|^\upsilon}{\kappa' \upsilon}\right),$$

and for all $t \in I$ and $\sigma \in \mathbb{N}^d$, we have

$$\|\partial^\sigma G_t\|_\infty \leq MR^{|\sigma|} |\sigma|! \exp\left(\frac{|\sigma|^\upsilon}{\kappa' \upsilon}\right).$$

Thus each term of the form (3.23) is bounded by

$$C^{2k} M^{2k+1} R^k k^k \exp\left(\frac{k^\upsilon}{\kappa' \upsilon}\right) 2^{-k \max(n,\ell)^\alpha}$$

when $x \in K, \xi \in \text{ supp } \psi_{\Theta',n,i}, \eta \in \text{ supp } \tilde{\psi}_{\Theta,\ell,j}$ and $t \in I$. Consequently, for such x, ξ, η and t the kernel $\mathcal{K}_{k,t}(x, \xi, \eta)$ is bounded for all integers k by

$$2^{-k \max(n,\ell)^\alpha} (5(d+1))^k C^{2k} M^{2k+1} R^k k^{2k} \exp\left(\frac{k^\upsilon}{\kappa' \upsilon}\right). \tag{3.24}$$

Now, choose $\kappa'' > 0$ such that $\frac{1}{\kappa' \upsilon} + 2 \leq \frac{1}{\kappa''}$ and pick new values of the constants M and R so that (3.24) is now smaller than

$$M \left(\frac{R}{2^{\max(n,\ell)^\alpha}}\right)^k \exp\left(\frac{k^\upsilon}{\kappa''}\right).$$

Now, using this estimate and Cauchy–Schwarz in (3.22), we bound the inner integral in (3.21) by

$$\widetilde{C} \|u\|_2 2^{\frac{(d+1)\ell}{2}} \left(\frac{R}{2^{\max(n,\ell)^\alpha}}\right)^k \exp\left(\frac{k^\upsilon}{\kappa''}\right),$$

which gives

$$\left\|S^{\ell,j}_{t,n,i} u\right\|_2 \leq C' \|u\|_2 2^{\frac{(\ell+n)(d+1)}{2}} \left(\frac{R}{2^{\max(n,\ell)^\alpha}}\right)^k \exp\left(\frac{k^\upsilon}{\kappa''}\right).$$

Now take

$$k = \left\lfloor \left(\frac{-\kappa'' \ln\left(\frac{R}{2^{\max(n,\ell)^\alpha}}\right)}{\upsilon}\right)^{\frac{1}{\upsilon-1}} \right\rfloor$$

to get (with new constants and $\delta = \frac{\alpha \upsilon}{\upsilon - 1} > 1$, see the proof of Proposition 3.6 for a similar computation)

$$\left\| S_{t,n,i}^{\ell,j} u \right\|_2 \leq C \left\| u \right\|_2 2^{\frac{(\ell+n)(d+1)}{2}} \exp\left(- \frac{\max(n,\ell)^{\delta}}{C} \right).$$

Finally, we get rid of the factor $2^{\frac{(\ell+n)(d+1)}{2}}$ by taking larger C. □

We can now prove Lemma 3.20 about the family $(\mathcal{M}_t)_{t \in \mathbb{R}}$ of auxiliary operators.

Proof of Lemma 3.20. Let us split the sum (3.18) into

$$\sum_{\substack{(n,i),(\ell,j) \in \Gamma \\ (\ell,j) \not\hookrightarrow (n,i)}} \iota_{n,i} \circ S_{t,n,i}^{\ell,j} \circ \pi_{\ell,j} \text{ and } \sum_{\substack{(n,i),(\ell,j) \in \Gamma \\ (n,i) \hookrightarrow (\ell,j)}} \iota_{n,i} \circ S_{t,n,i}^{\ell,j} \circ \pi_{\ell,j}. \tag{3.25}$$

Thanks to Lemma 3.22, the first sum in (3.25) converges absolutely in norm operator topology. To deal with the second sum in (3.25), notice that there is some constant C depending on I such that, for all $t \in I$ and $(n,i),(\ell,j) \in \Gamma$, we have

$$\left\| \iota_{n,i} \circ S_{t,n,i}^{\ell,j} \circ \pi_{\ell,j} \right\|_{\mathcal{B} \to \mathcal{B}} \leq C 2^{n\beta_i} 2^{-\ell \beta_j}. \tag{3.26}$$

Then the second sum in (3.25) can be divided into seven sums that correspond to the different cases in the definition of $\hookrightarrow$. It is elementary, using (3.26), to see that the first six converge in norm operator topology. Consequently, we are left with the sum

$$\sum_{\substack{n,\ell \in \mathbb{N} \\ |n-\ell| \leq M}} \iota_{n,f} \circ S_{t,n,f}^{\ell,f} \circ \pi_{\ell,f} \tag{3.27}$$

for some $M > 0$. For all $N_1 \in \mathbb{N}$, define the operator

$$P_{N_1} = \sum_{\substack{0 \leq n,\ell \leq N_1 \\ |n-\ell| \leq M}} \iota_{n,f} \circ S_{t,n,f}^{\ell,f} \circ \pi_{\ell,f}.$$

Pick $u = (u_{m,k})_{(m,k) \in \Gamma} \in \mathcal{B}$. Then if $N_2 \geq N_1 \geq 0$, we have

$$\begin{aligned} \left\| (P_{N_2} - P_{N_1}) u \right\|_{\mathcal{B}}^2 \leq 2 \sum_{n=0}^{N_2} 2^{-2(d+2)n} & \left\| \sum_{\substack{N_1 < \ell \leq N_2 \\ |\ell - n| \leq M}} S_{t,n,f}^{\ell,f} u_{\ell,f} \right\|_2^2 \\ & + 2 \sum_{n=N_1+1}^{N_2} 2^{-2(d+2)n} \left\| \sum_{\substack{0 \leq \ell \leq N_1 \\ |\ell - n| \leq M}} S_{t,n,f}^{\ell,f} u_{\ell,f} \right\|_2^2. \end{aligned} \tag{3.28}$$

Next, we have by the triangle inequality,

$$\left\| \sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} S^{\ell,f}_{t,n,f} u_{\ell,f} \right\|_2^2 \leq \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} \left\| S^{\ell,f}_{t,n,f} u_{\ell,f} \right\|_2 \right)^2 \leq C \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} \|u_{\ell,f}\|_2 \right)^2 ,$$

for some constant $C > 0$. Then, from the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} \|u_{\ell,f}\|_2 \right)^2 &= \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} 2^{\ell(d+2)} 2^{-\ell(d+2)} \|u_{\ell,f}\|_2 \right)^2 \\ &\leq \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} 2^{2\ell(d+2)} \right) \left(\sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} 2^{-2\ell(d+2)} \|u_{\ell,f}\|_2^2 \right) \\ &\leq C' 2^{2n(d+2)} \sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} 2^{-2\ell(d+2)} \|u_{\ell,f}\|_2^2 \end{aligned}$$

for another constant $C' > 0$. Consequently, we can bound the first sum in (3.28)

$$\begin{aligned} \sum_{n=0}^{N_2} 2^{-2n(d+2)} \left\| \sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} S^{\ell,f}_{t,n,f} u_{\ell,f} \right\|_2^2 &\leq CC' \sum_{n=0}^{N_2} \sum_{\substack{N_1<\ell\leq N_2\\|\ell-n|\leq M}} 2^{-2\ell(d+2)} \|u_{\ell,f}\|_2^2 \\ &\leq \widetilde{C} \sum_{\ell>N_1} 2^{-2\ell(d+2)} \|u_{\ell,f}\|_2^2 , \end{aligned}$$

where in the last line we noticed that, when ℓ is fixed, there are at most $2M+1$ values of n for which $|\ell - n| \leq M$. Working similarly with the second sum, we see that there is a constant C such that

$$\|(P_{N_2} - P_{N_1})\, u\|_{\mathcal{B}}^2 \leq C \sum_{\ell \geq N_1 - M} 2^{-2(d+2)\ell} \|u_{\ell,f}\|_2^2 ,$$

and thus the sequence $(P_{N_1}u)_{N_1\geq 0}$ is Cauchy in $\mathcal{B}$. Consequently, the sequence $(P_N)_{N\geq 0}$ converges in strong operator topology, hence, so does the sum (3.27). It remains to prove that $\mathcal{M}_t$ depends continuously on t in the strong operator topology. To do so, just notice that when u is fixed the sum

$$\sum_{(n,i),(\ell,j)\in\Gamma} \iota_{n,i} \circ S^{\ell,j}_{t,n,i} \circ \pi_{\ell,j} u$$

converges uniformly (in $t \in I$) to $\mathcal{M}_t u$ and each of its terms is continuous with respect to t (to see this, notice that if $(n,i),(\ell,j) \in \Gamma$ then $S_{t,n,i}^{\ell,j}$ is locally uniformly bounded as an operator from L^2 to L^2, and the continuity is easily proven for smooth u). □

3.3.2 Schatten class properties

Now let $h : \mathbb{R}_+^* \to \mathbb{C}$ be a compactly supported function as in Proposition 3.17. If $(n,i),(\ell,j) \in \Gamma$, then write

$$H_{n,i}^{\ell,j} = \int_{\mathbb{R}} h(t)\, S_{t,n,i}^{\ell,j} \mathrm{d}t,$$

where we recall that $S_{t,n,i}^{\ell,j}$ is defined by (3.17). Notice that the sum

$$\sum_{(n,i),(\ell,j)\in\Gamma} \iota_{n,i} \circ H_{n,i}^{\ell,j} \circ \pi_{\ell,j}$$

converges in strong operator topology to $\int_{\mathbb{R}} h(t)\,\mathcal{M}_t \mathrm{d}t$, sincein Lemma 3.20 the convergence is uniform locally in t. To prove Proposition 3.17, we want now to prove that this operator is in a Schatten class (or at least compact), this is the point of Lemma 3.27. To do so we need first to establish a bunch of lemmas: Lemma 3.23 will be used to deal with the transition of frequencies corresponding to the linear model of the dynamics apart from the direction of the flow, Lemma 3.24 will settle the problem of frequency transitions corresponding to the non-linearity, and Lemmas 3.25 and 3.26 will be used to deal with stationary frequencies in the direction of the flow.

Lemma 3.23. *There is a constant $C > 0$ such that, for all $(n,i),(\ell,j) \in \Gamma$, the trace class operator norm of $H_{n,i}^{\ell,j} : L^2 \to L^2$ is bounded by $C 2^{\frac{(d+1)n^{\alpha_i}}{2}} 2^{\frac{(d+1)\ell^{\alpha_j}}{2}}$, where $\alpha_i = \alpha$ if $i = 0$ or $i = r-1$ and $\alpha_i = 1$ otherwise.*

Proof. Notice that if $u \in L^2$ then $\psi_{\Theta',n,i}(D)\,u = \mathbb{F}^{-1}\left(\psi_{\Theta',n,i}\right) * u$. Consequently, we have[8]

$$H_{n,i}^{\ell,j} = \int_K \mathbb{F}^{-1}\left(\psi_{\Theta',n,i}\right)(\cdot - y) \otimes \left(\int_{\mathbb{R}} h(t)\, G_t(y)\, \delta_{\mathcal{T}_t(y)} \circ \tilde{\psi}_{\Theta,\ell,j}(D)\, \mathrm{d}t\right) \mathrm{d}y. \tag{3.29}$$

And then the result follows from the fact that

$$\left\|\mathbb{F}^{-1}\left(\psi_{\Theta',n,i}\right)\right\|_2 = \frac{1}{\sqrt{2\pi}^{d+1}} \left\|\psi_{\Theta',n,i}\right\|_2 \leq C 2^{\frac{(d+1)n^{\alpha_i}}{2}}$$

and

$$\left\| \int_{\mathbb{R}} h(t)\, G_t(y)\, \delta_{\mathcal{T}_t(y)} \circ \tilde{\psi}_{\Theta,\ell,j}(D)\, \mathrm{d}t \right\|_{(L^2)^*} \leq C \left\|\tilde{\psi}_{\Theta,\ell,j}\right\|_2 \leq \tilde{C} 2^{\frac{(d+1)\ell^{\alpha_j}}{2}},$$

where $\|\cdot\|_{(L^2)^*}$ denotes the operator norm on the dual of $L^2\left(\mathbb{R}^{d+1}\right)$. □

[8] If E, F are Banach spaces, $e \in F$ and $l \in E'$, we denote by $e \otimes l$ the rank 1 operator defined by $e \otimes l(u) = l(u).e$ for $u \in E$.

Lemma 3.24. *There is a constant $C > 0$ and some $\delta > 1$ such that, if $(\ell, j) \not\twoheadrightarrow (n,i)$ for $(n,i), (\ell,j) \in \Gamma$, then the trace class operator norm of $H_{n,i}^{\ell,j} : L^2 \to L^2$ is bounded by $C \exp\left(-\frac{\max(n,\ell)^\delta}{C}\right)$.*

Proof. We may assume that $\max(n,\ell) > N$. Without loss of generality, we may assume that $K \subseteq]-\pi,\pi[^{d+1}$ and then, if $u \in L^2(\mathbb{R}^{d+1})$ write (the sum converges in L^2)

$$H_{n,i}^{\ell,j} u = \sum_{k \in \mathbb{Z}^{d+1}} c_k \left(\int_{\mathbb{R}} h(t) \mathcal{L}_t \tilde{\psi}_{\Theta,\ell,j}(D) u \mathrm{d}t \right) \psi_{\Theta',n,i}(D) \rho_k,$$

where ρ is a function supported in $]-\pi,\pi[^{d+1}$ that takes value 1 on K, the function ρ_k is defined by $\rho_k(x) = \rho(x) e^{ikx}$ and if v is supported in $]-\pi,\pi[^{d+1}$ and $k \in \mathbb{Z}^{d+1}$, its kth Fourier coefficient is denoted by $c_k(v)$:

$$c_k(v) = \frac{1}{(2\pi)^{d+1}} \int_{]-\pi,\pi[^{d+1}} e^{-ikx} v(x) \mathrm{d}x.$$

By requiring that ρ is s-Gevrey (for some $s > 1$), we may ensure as in [Jéz20a, Lemma 6.5] that (for some constant $C > 0$)

$$\left\| \psi_{\Theta',n,i}(D) \rho_k \right\|_2 \leq C 2^{\frac{(d+1)n^{\alpha_i}}{2}} \exp\left(- \frac{d\left(k, \operatorname{supp} \psi_{\Theta',n,i}\right)^{\frac{1}{s}}}{C} \right).$$

Now, if $k \in \mathbb{Z}^{d+1}$ and $(\ell,j) \in \Gamma$ define

$$\delta(k,\ell,j) = \sup_{x \in K} d\left(k, D_x \mathcal{T}_t^{\mathrm{tr}}\left(\operatorname{supp} \tilde{\psi}_{\Theta,\ell,j}\right)\right).$$

Then integrating by parts as in [Jéz20a, Lemma 6.7] or as in Lemma 3.22 we see that if $\delta(k,\ell,j) \geq \epsilon 2^{\ell^{\alpha_j}}$ (for some arbitrary fixed $\epsilon > 0$) then

$$\left\| c_k \circ \int_{\mathbb{R}} h(t) \mathcal{L}_t \mathrm{d}t \circ \tilde{\psi}_{\Theta,\ell,j}(D) \right\|_{(L^2)^*} \leq C 2^{(d+1)\ell^{\alpha_j}} \exp\left(- \frac{\ln\left(1 + \delta(k,\ell,j)\right)^{\frac{v}{v-1}}}{C} \right).$$

But now, if $(\ell,j) \not\twoheadrightarrow (n,i)$ and $\max(n,l) > N$, then, for all $k \in \mathbb{Z}^{d+1}$, either the distance $d\left(k, \operatorname{supp} \psi_{\Theta',n,i}\right)$ or the distance $\delta(k,\ell,j)$ is greater than $\frac{c'}{2} \max\left(2^{n^{\alpha_i}}, 2^{\ell^{\alpha_j}}\right)$, thanks to Lemma 3.21. Moreover, if $|k|$ is greater than $C 2^{\max(n,\ell)}$ (for some large $C > 0$), then we have $\delta(k,\ell,j) \geq \epsilon \max(2^{n^{\alpha_i}}, 2^{\ell^{\alpha_j}})$ and $d\left(k, \operatorname{supp} \psi_{\Theta',n,i}\right) \geq \epsilon |k|$. Thus, the sum

$$H_{n,i}^{\ell,j} = \sum_{k \in \mathbb{Z}^{d+1}} \left(\psi_{\Theta',n,i}(D) \rho_k \right) \otimes \left(c_k \circ \int_{\mathbb{R}} h(t) L_t \mathrm{d}t \circ \tilde{\psi}_{\Theta,\ell,j}(D) \right)$$

converges in trace class topology, and from the estimates above we see that the announced result holds with $\delta = \frac{\alpha v}{v-1}$. $\square$

Lemma 3.25. *Assume that h is kth times differentiable and that is kth derivative has bounded variation. Then there is a constant $C > 0$ such that for all $n, \ell \in \mathbb{N}$ we have*

$$\left\| H_{n,f}^{\ell,f} \right\|_{L^2 \to L^2} \leq C 2^{-(k+1)\ell}.$$

Proof. If $u \in L^2\left(\mathbb{R}^{d+1}\right)$ and $x \in \mathbb{R}^{d+1}$, then we have,

$$H_{n,f}^{\ell,f} u\left(x\right) = \int_{\mathbb{R}^{d+1}} V_{n,\ell}\left(x,\eta\right) \hat{u}(\eta) \mathrm{d}\eta,$$

where the kernel $V_{n,\ell}$ is defined by

$$\begin{aligned} V_{n,\ell}\left(x,\eta\right) = \frac{1}{\left(2\pi\right)^{2(d+1)}} \int_{\left(\mathbb{R}^{d+1}\right)^3 \times \mathbb{R}} & e^{i(x-z)\xi + i\mathcal{T}_0(z)\eta} e^{it\eta_{d+1}} \psi_{\Theta',n,f}\left(\xi\right) \\ & \times \tilde{\psi}_{\Theta,\ell,f}\left(\eta\right) h\left(t\right) G_t\left(z\right) \mathrm{d}z \mathrm{d}\xi \mathrm{d}t. \end{aligned} \tag{3.30}$$

We can assume that ℓ is large enough (the $H_{n,f}^{\ell,f}$'s are uniformly bounded on L^2), which ensures that η_{d+1} (the last coordinate of η) does not vanish on the support of $\tilde{\psi}_{\Theta,\ell,f}$. Consequently, we can perform $k+1$ integrations by parts in t in (3.30) to get

$$\begin{aligned} V_{n,\ell}\left(x,\eta\right) = \frac{i^{k+1}}{\left(2\pi\right)^{2(d+1)}} \int_{\left(\mathbb{R} \times \mathbb{R}^{d+1}\right)^3} & e^{i(x-z)\xi + i\mathcal{T}_0(z)\eta} e^{it\eta_{d+1}} \psi_{\Theta',n,f}\left(\xi\right) \\ & \times \frac{\tilde{\psi}_{\Theta,\ell,f}\left(\eta\right)}{\eta_{d+1}^{k+1}} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \left(h\left(t\right) G_t\left(z\right)\right) \mathrm{d}t \mathrm{d}z \mathrm{d}\xi. \end{aligned}$$

Using the Leibniz rule, we see that, if μ denotes the measure of total variation of $h^{(k+1)}$, the measure $\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\left(h\left(t\right) G_t\left(z\right)\right) \mathrm{d}t$ may be written as $f\left(t,z\right) \mathrm{d}\mu\left(t\right)$ for all $z \in \mathbb{R}^{d+1}$. Moreover, f has the following properties: it is measurable, $f\left(t,z\right) = 0$ if $z \in \mathbb{R}^{d+1} \setminus K$, and $\int_{\mathbb{R}} \sup_{z \in \mathbb{R}^{d+1}} \left|f\left(t,z\right)\right| \mathrm{d}\mu\left(t\right) < +\infty$. Then, define the function $\Psi_\ell : \mathbb{R}^{d+1} \to \mathbb{R}$ by $\Psi_\ell\left(\eta\right) = \frac{\tilde{\psi}_{\Theta,\ell,f}(\eta)}{\eta_{d+1}^{k+1}}$, the operator $L_t : L^2\left(\mathbb{R}^{d+1}\right) \to L^2\left(\mathbb{R}^{d+1}\right)$ by $L_t u\left(z\right) = f\left(t,z\right) . \left(u \circ \mathcal{T}_t\left(z\right)\right)$, and notice that we have

$$H_{n,f}^{\ell,f} = \psi_{\Theta',n,f}\left(D\right) \circ \int_{\mathbb{R}} L_t \mathrm{d}\mu\left(t\right) \circ \Psi_\ell\left(D\right).$$

Finally, notice that $\left\|\Psi_\ell\right\|_\infty \leq C 2^{-\ell(k+1)}$ to end the proof. $\square$

Lemma 3.26. *Let $s > 0$ and $\epsilon > 0$. Then there is a constant $C > 0$ such that for all $N > 0$ and $n \in \mathbb{N}$ with $n < N$ there is an operator $F_{n,N} : L^2\left(K\right) \to L^2\left(\mathbb{R}^{d+1}\right)$ of rank at* *most* $2^{(1+\epsilon)(d+1)N}$ *such that for all* $u \in L^2(K)$ *we have*

$$\left\|\psi_{\Theta',n,f}(D)\,u - F_{n,N}u\right\|_2 \leq C2^{-sN}.$$

Proof. The proof is similar to the proof of [BT08, Lemma 4.21]. □

We are now ready to prove Lemma 3.27.

Lemma 3.27. *Under the hypotheses of Lemma 3.25 and if in addition* $\alpha < \frac{1}{2}$*, the operator*

$$\int_{\mathbb{R}} h(t)\,\mathcal{M}_t \mathrm{d}t \tag{3.31}$$

belongs to the Schatten class S_p *for every* $p \geq 1$ *such that* $p > \frac{d+1}{k+1}$ *. Moreover, its norm in this Schatten class is bounded by* $C\left(\|h\|_{\mathcal{C}^{k-1}} + \left\|h^{(k)}\right\|_{BV}\right)$ *where* C *depends on* h *only through its support.*

Without the assumption that $\alpha < \frac{1}{2}$*, it remains true that the operator defined by* (3.31) *is compact.*

Proof. We know that

$$\int_{\mathbb{R}} h(t)\,\mathcal{M}_t \mathrm{d}t = \sum_{(n,i),(\ell,j)\in\Gamma} \iota_{n,i} \circ H_{n,i}^{\ell,j} \circ \pi_{\ell,j} \tag{3.32}$$

where the sum converges in the strong operator topology. From Lemma 3.24, it is clear that the sum

$$\sum_{\substack{(n,i),(\ell,j)\in\Gamma \\ (n,i)\not\hookrightarrow(\ell,j)}} \iota_{n,i} \circ H_{n,i}^{\ell,j} \circ \pi_{\ell,j}$$

converges in the trace class operator topology. We are left with the sum

$$\sum_{\substack{(n,i),(\ell,j)\in\Gamma \\ (n,i)\hookrightarrow(\ell,j)}} \iota_{n,i} \circ H_{n,i}^{\ell,j} \circ \pi_{\ell,j}$$

that we can divide, as in the proof of Lemma 3.20, into seven sums corresponding to the different cases in the definition of $\hookrightarrow$. The first six sums are dealt with by using Lemma 3.23. We will only detail the computation corresponding to the first case in the definition of $\hookrightarrow$ (i.e. the case $i = j = 0$, the case $i = j = r-1$ is dealt with in the same way and the others are easier), in order to highlight where the hypothesis $\alpha < \frac{1}{2}$ is used. If $n, \ell \in \mathbb{N}$, then the trace class operator norm of $\iota_{n,0} \circ H_{n,0}^{\ell,0} \circ \pi_{\ell,0}$ is smaller than $C2^{\frac{(d+1)n^\alpha}{2}}2^{\frac{(d+1)\ell^\alpha}{2}}2^{(d+2)n}2^{-(d+2)\ell}$. Thus, in order to deal with the sum corresponding with the case $i=j=0$ in the definition of $\hookrightarrow$, we only need to prove that the quantity

$$\sum_{\substack{\ell,n\in\mathbb{N}\\(\ell,0)\hookrightarrow(n,0)}} 2^{\frac{(d+1)n^\alpha}{2}}2^{\frac{(d+1)\ell^\alpha}{2}}2^{(d+2)n}2^{-(d+2)\ell} \tag{3.33}$$

is finite. Notice that

$$2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}} \underset{\ell\to+\infty}{\sim} \frac{1}{1-2^{-(d+2)}}\left(2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}}-2^{(d+2)(\ell+1)+\frac{(d+1)(\ell+1)^\alpha}{2}}\right)$$

so that

$$\sum_{\ell\geq\ell_0}2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}} \underset{\ell_0\to+\infty}{\sim} \frac{2^{-(d+2)\ell_0+\frac{(d+1)\ell_0^\alpha}{2}}}{1-2^{-(d+2)}}.$$

In particular, there is a constant $C>0$ such that, for all $\ell_0\in\mathbb{N}$. We have

$$\sum_{\ell\geq\ell_0}2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}} \leq C2^{-(d+2)\ell_0+\frac{(d+1)\ell_0^\alpha}{2}}.$$

Now if $n\in\mathbb{N}$, let ℓ_0 be the smallest integer such that $\ell_0\geq n+\nu n^{1-\alpha}$, we have then (notice that $\ell_0\leq Bn$ for some constant B that does not depend on n)

$$\begin{aligned}\sum_{\substack{\ell\in\mathbb{N}\\(\ell,0)\hookrightarrow(n,0)}} 2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}} &= \sum_{\ell\geq\ell_0}2^{-(d+2)\ell+\frac{(d+1)\ell^\alpha}{2}} \leq C2^{-(d+2)\ell_0+\frac{(d+1)\ell_0^\alpha}{2}}\\ &\leq C2^{-(d+2)n}2^{-(d+2)\nu n^{1-\alpha}}2^{\frac{(d+1)B^\alpha}{2}n^\alpha}.\end{aligned}$$

Thus, we have

$$\sum_{\substack{\ell,n\in\mathbb{N}\\(\ell,0)\hookrightarrow(n,0)}} 2^{\frac{(d+1)n^\alpha}{2}}2^{\frac{(d+1)\ell^\alpha}{2}}2^{(d+2)n}2^{-(d+2)\ell} \leq C\sum_{n\in\mathbb{N}}2^{-(d+2)\nu n^{1-\alpha}}2^{\frac{(d+1)\left(B^\alpha+1\right)}{2}n^\alpha}, \tag{3.34}$$

and this sum is finite since $\alpha<\frac{1}{2}$. As explained above, a similar argument allows to deal with the sum corresponding to any of the six first cases in the definition of $\hookrightarrow$. Consequently, we are left with the sum

$$P=\sum_{\substack{n,\ell\in\mathbb{N}\\|n-\ell|\leq M}}\iota_{n,f}\circ H_{n,f}^{\ell,f}\circ\pi_{\ell,f}.$$

Choose $s>k+1$ and $\epsilon>0$, and apply Lemma 3.26 to define for all $N>0$ the operator

$$P_N=\sum_{\substack{0\leq n,\ell<N\\|n-\ell|\leq M}}\iota_{n,f}\circ F_{n,N}\circ\int_{\mathbb{R}}h\left(t\right)\mathcal{L}_t\mathrm{d}t\circ\tilde{\psi}_{\Theta,\ell,j}\circ\pi_{\ell,f},$$

whose rank is at most $N^2 2^{(1+\epsilon)(d+1)N}$. Then notice, using Lemma 3.25, that we have

$$\begin{aligned} \|P_N - P\|_{\mathcal{B}\to\mathcal{B}} \leq C \sum_{\substack{n,\ell<N \\ |n-\ell|\leq M}} \Bigg(& \left\|F_{n,N} - \psi_{\Theta',n,f}(D)\right\|_{L^2(K)\to L^2\left(\mathbb{R}^{d+1}\right)} \\ & \times \left\| \int_{\mathbb{R}} h(t)\,\mathcal{L}_t \mathrm{d}t \circ \tilde{\psi}_{\Theta,\ell,f}(D)\right\|_{L^2\to L^2} \Bigg) \\ + C \sum_{\substack{n,\ell\geq N \\ |n-\ell|\leq M}} & \left\|H_{n,f}^{\ell,f}\right\|_{L^2\to L^2} \\ \leq \widetilde{C}\left(N^2 2^{-sN} + 2^{-(k+1)N}\right) & \leq C' 2^{-(k+1)N}, \end{aligned} \tag{3.35}$$

for some constants $C, \tilde{C}$ and C' that do not depend on N. Letting N tend to infinity, we see that P is compact. Moreover, if $(s_m)_{m\geq 0}$ denotes the sequence of singular values of P, we get from (3.35) and [GGK00, Theorem 2.5 p.51]

$$s_{N^2 2^{(1+\epsilon)(d+1)N}+1} \leq C' 2^{-(k+1)N}.$$

Thus, the sequence $(s_m)_{m\geq 0}$ is in ℓ^p for all $p > \frac{(1+\epsilon)(d+1)}{k+1}$ (the sequence $(s_m)_{m\geq 0}$ is decreasing). This ends the proof in the case $\alpha < \frac{1}{2}$ since $\epsilon > 0$ is arbitrary. Indeed, all the terms in the proof are controlled by the L^∞ norm of h, except the one that we bounded using Lemma 3.25 that is controlled by $\|h\|_{\mathcal{C}^{k-1}} + \left\|h^{(k)}\right\|_{BV}$.

In order to deal with the case $\alpha \geq \frac{1}{2}$, notice that we only used the assumption $\alpha < \frac{1}{2}$ to ensure that the series (3.34) converges. However, if we remove the factor $2^{\frac{(d+1)n^\alpha}{2}} 2^{\frac{(d+1)\ell^\alpha}{2}}$ from the sum (3.33), this new series converges, just like in the proof of Lemma 3.20. That is, if we consider the operator norm instead of the trace class norm, the sums corresponding to the first six cases in the definition of $\hookrightarrow$ converge, even if $\alpha \geq \frac{1}{2}$. Consequently, the right-hand side of (3.32) always converges in the operator norm topology, and the left-hand side of (3.32) is always compact. □

3.3.3 Trace of $\int_0^{+\infty} h(t)\,\mathcal{M}_t \mathrm{d}t$ and structure of the local Koopman operator.

Before proving that $(\mathcal{L}_t)_{t\in\mathbb{R}}$ inherits of the properties of $(\mathcal{M}_t)_{t\in\mathbb{R}}$, thus showing Proposition 3.17, we still need to prove that the operator $\int_{\mathbb{R}} h(t)\,\mathcal{M}_t dt$ has the expected trace, when it makes sense. This is the point of the following lemma.

Lemma 3.28. *Under the hypotheses of Proposition 3.17, if $\Theta = \Theta'$, if $\alpha < \frac{1}{2}$ and if $k+1 > d+1$ then*

$$\operatorname{tr}\left(\int_{\mathbb{R}} h(t)\,\mathcal{M}_t \mathrm{d}t\right) = \sum_{p\circ F(x)=x} \frac{h(T(x))}{|\det(I - p\circ D_x F)|} \int_{\mathbb{R}} G_{T(x)}(x,y)\,\mathrm{d}y,$$

where p denotes the orthogonal projection on $\mathbb{R}^d \simeq \mathbb{R}^d \times \{0\}$ and, for $x \in \mathbb{R}^d$, the number $T(x)$ is defined by $F(x) = p\,(F(x)) + (0, -T(x))$.

Proof. For all $N \in \mathbb{N}$ write

$$A_N = \sum_{\substack{(n,i),(\ell,j)\in\Gamma \\ 0\leq n,\ell\leq N}} \iota_{n,i} \circ H_{n,i}^{\ell,j} \circ \pi_{\ell,j}$$

and notice that [GGK00, Theorem 11.3 p.89] implies that

$$\operatorname{tr}\left(\int_{\mathbb{R}} h\,(t)\,\mathcal{M}_t \mathrm{d}t\right) = \lim_{N\to+\infty} \operatorname{tr}\,(A_N)\,.$$

Moreover, using Lidskii's trace theorem, we see that for all $N \in \mathbb{N}$ we have

$$\operatorname{tr}\,(A_N) = \sum_{\substack{(n,i)\in\Gamma \\ 0\leq n\leq N}} \operatorname{tr}\left(H_{n,i}^{n,i}\right).$$

Now, from (3.29), we see that

$$\begin{aligned}
\operatorname{tr}\left(H_{n,i}^{n,i}\right) &= \int_{\mathbb{R}}\int_K h\,(t)\,G_t\,(w)\,\tilde{\psi}_{\Theta,n,i}\,(D)\left(\mathbb{F}^{-1}\,(\psi_{\Theta,n,i})\,(\cdot - w)\right)(\mathcal{T}_t\,(w))\,\mathrm{d}w\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_K h\,(t)\,G_t\,(w)\,\mathbb{F}^{-1}\,(\psi_{\Theta,n,i})\,(\mathcal{T}_t\,(w) - w)\,\mathrm{d}w\mathrm{d}t.
\end{aligned}$$

We used in the second line that if $\psi_{\Theta,n,i}\,(\xi) \neq 0$ then $\tilde{\psi}_{\Theta,n,i}\,(\xi) = 1$. Now let M be such that $K \subseteq [-M, M]^{d+1}$ and h is supported in $[-M, M]$. Define the map $g : \mathbb{R}^{d+1} \simeq \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$ by $g\,(x,t) = F\,(x) - (x, -t)$. Notice that for all $(x,y) \in \mathbb{R}^{d+1}$ and $t \in \mathbb{R}$ we have

$$\mathcal{T}_t\,(x,y) - (x,y) = g\,(x,t)\,.$$

Cone-hyperbolicity implies that the Jacobian of g does not vanish. Consequently we can find a finite family $(\rho_a)_{a\in\mathcal{A}}$ of compactly supported $\mathcal{C}^\infty$ functions $\rho_a : \mathbb{R}^{d+1} \to [0,1]$ such that $\sum_{a\in\mathcal{A}} \rho_a\,(w) = 1$ for all $w \in [-M, M]^{d+1}$ and for all $a \in \mathcal{A}$ there is a $\mathcal{C}^\infty$ diffeomorphism $g_a : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ that coincides with g on a neighborhood of the support of ρ_a. Thus, we find that $\operatorname{tr}\left(H_{n,i}^{n,i}\right)$ is equal to

$$\begin{aligned}
&\sum_{a\in\mathcal{A}} \int_{[-M,M]^{d+2}} h\,(t)\,\rho_a\,(x,t)\,G_t\,(x,y)\,\mathbb{F}^{-1}\,(\psi_{\Theta,n,i})\,(g_a\,(x,t))\,\mathrm{d}x\mathrm{d}t\mathrm{d}y \\
&\qquad = \sum_{a\in\mathcal{A}} \int_{\mathbb{R}^{d+1}\times[-M,M]} h\,(t_a\,(z))\,\rho_a \circ g_a^{-1}\,(z)\,\frac{G_{t_a(z)}\,(x_a\,(z)\,,y)}{\left|\det D_{g_a^{-1}(z)} g_a\right|}\mathbb{F}^{-1}\,(\psi_{\Theta,n,i})\,(z)\,\mathrm{d}z\mathrm{d}y
\end{aligned}$$

where $w = (x,y)$ and $g_a^{-1}(z) = (x_a(z), t_a(z))$. Since $\sum_{(n,i)\in\Gamma} \psi_{\Theta,n,i} = 1$ we find that for all

$a \in \mathcal{A}$ we have

$$\lim_{N \to +\infty} \sum_{\substack{(n,i) \in \Gamma \\ 0 \leq n \leq N}} \int_{\mathbb{R}^{d+1} \times [-M,M]} h(t_a(z)) \rho_a \circ g_a^{-1}(z) \frac{G_{t_a(z)}(x_a(z), y)}{\left|\det D_{g_a^{-1}(z)} g_a\right|} \mathbb{F}^{-1}(\psi_{\Theta,n,i})(z) \, \mathrm{d}z \mathrm{d}y$$
$$= \int_{[-M,M]} h(t_a(0)) \rho_a\left(g_a^{-1}(0)\right) \frac{G_{t_a(0)}(x_a(0), y)}{\left|\det D_{g_a^{-1}(0)} g_a\right|} \mathrm{d}y.$$

And thus

$$\operatorname{tr}\left(\int_{\mathbb{R}} h(t) \mathcal{M}_t \mathrm{d}t\right) = \sum_{a \in \mathcal{A}} \int_{[-M,M]} h(t_a(0)) \rho_a\left(g_a^{-1}(0)\right) \frac{G_{t_a(0)}(x_a(0), y)}{\left|\det D_{g_a^{-1}(0)} g_a\right|} \mathrm{d}y.$$

Now, notice that $g(x,t) = 0$ if and only if $p \circ F(x) = x$ and $t = T(x)$, thus

$$\int_{[-M,M]} h(t_a(0)) \rho_a\left(g_a^{-1}(0)\right) \frac{G_{t_a(0)}(x_a(0), y)}{\left|\det D_{g_a^{-1}(0)} g_a\right|} \mathrm{d}y$$
$$= \sum_{p \circ F(x) = x} \int_{[-M,M]} \rho_a(x, T(x)) h(T(x)) \frac{G_{T(x)}(x,y)}{\left|\det(I - p \circ D_x F)\right|} \mathrm{d}y.$$

Here we noticed that the Jacobian of g do not depend on the last coordinate. Finally, summing over $a \in \mathcal{A}$ we get

$$\operatorname{tr}\left(\int_{\mathbb{R}} h(t) \mathcal{M}_t \mathrm{d}t\right) = \sum_{p \circ F(x) = x} \frac{h(T(x))}{\left|\det(I - p \circ D_x F)\right|} \int_{[-M,M]} G_{T(x)}(x,y) \, \mathrm{d}y.$$

□

We show Proposition 3.17 by proving that $(\mathcal{L}_t)_{t \in \mathbb{R}}$ also satisfies the properties established for $(\mathcal{M}_t)_{t \in \mathbb{R}}$ in Lemmas 3.20, 3.27 and 3.28.

Proof of Proposition 3.17. Recall that $\mathcal{Q}_\Theta$ (defined by (3.13)) induces an isomorphism between $\mathcal{H}_{\Theta,\alpha}$ and $\mathcal{Q}_\Theta(\mathcal{H}_{\Theta,\alpha})$, which is a closed subspace of $\mathcal{B}$. We denote by $\mathcal{Q}_\Theta^{-1}$ the inverse isomorphism (and similarly for $\mathcal{Q}_{\Theta'}$).

Now, if $(u_{n,i})_{(n,i) \in \Gamma}$ is finitely supported (i.e. there are finitely many $(n,i) \in \Gamma$ such that $u_{n,i} \neq 0$) and such that for all $(n,i) \in \Gamma$ we have $u_{n,i} \in S^{\tilde{v}}$ (for some $\tilde{v} \in \left]v, \frac{1}{1-\alpha}\right[$) we write $u = \sum_{(n,i) \in \Gamma} \tilde{\psi}_{\Theta,n,i}(D) u_{n,i}$ and notice that $\mathcal{L}_t u \in S^{\tilde{v}}$ for all $t \in \mathbb{R}$, and thus $\mathcal{L}_t u \in \mathcal{H}_{\Theta',\alpha}$. Consequently, $\mathcal{M}_t(u_{n,i})_{(n,i) \in \Gamma} = Q_{\Theta'} \mathcal{L}_t u$ is in $\mathcal{Q}_{\Theta'}(\mathcal{H}_{\Theta',\alpha})$. Since such elements are easily seen to be dense in $\mathcal{B}$, it appears that $\mathcal{M}_t$ sends $\mathcal{B}$ into $\mathcal{Q}_{\Theta'}(\mathcal{H}_{\Theta',\alpha})$. We can consequently define the operator $\mathcal{Q}_{\Theta'}^{-1} \circ \mathcal{M}_t \circ \mathcal{Q}_\Theta$.

The calculation above also implies that $\mathcal{L}_t$ and $\mathcal{Q}_{\Theta'}^{-1} \circ \mathcal{M}_t \circ \mathcal{Q}_\Theta$ coincides on $\mathcal{H}_{\Theta,\alpha}$ (since the element of $S^{\tilde{v}}$ whose Fourier transform is compactly supported are dense in $\mathcal{H}_{\Theta,\alpha}$, and $\mathcal{H}_{\Theta,\alpha}$ and $\mathcal{H}_{\Theta',\alpha}$ are continuously contained in $\left(\mathcal{S}^{\tilde{v}}\right)'$, see Remark 3.12). Now, since $\mathcal{L}_t : \mathcal{H}_{\Theta,\alpha} \to \mathcal{H}_{\Theta',\alpha}$ is conjugated to $\mathcal{M}_t : \mathcal{B} \to \mathcal{B}$ (the conjugacy being independent of t), it inherits of all the relevant properties of $\mathcal{M}_t$, which ends the proof of Proposition 3.17 with Lemmas 3.20, 3.27 and 3.28 (for the computation of the trace, use Lidskii's trace theorem and the fact that $\mathcal{M}_t$ sends $\mathcal{B}$ into $\mathcal{Q}_{\Theta'}\left(\mathcal{H}_{\Theta',\alpha}\right)$, and not only let this subspace stable). □

3.4 Global space: first step

We are now ready to start the proof of Theorem 3.1 using the tools from §3.2 and §3.3. So let M be a compact $d+1$-dimensional $\mathcal{C}^{\kappa,\upsilon}$ manifold, let $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^{\kappa,\upsilon}$ Anosov flow on M, and let $V : M \to \mathbb{C}$ a $\mathcal{C}^{\kappa,\upsilon}$ function. We denote by X the generator of $(\phi_t)_{t\in\mathbb{R}}$ and write $P = X + V$. We also denote by $(\mathcal{L}_t)_{t\in\mathbb{R}}$ the associated Koopman operator, defined by (9). We fix $t_0 > 0$ from now on.

We will construct in this section two auxiliary Hilbert spaces $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_0$. The space $\widetilde{\mathcal{H}}_0$ almost satisfies the conclusions of Theorem 3.1 (this is the point of Proposition 3.32) but the Koopman operator $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}_0$ to itself only for large values of t *a priori*. This problem will be settled in §3.5. The first thing that we need to do in order to construct the spaces $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_0$ is to show that, locally in space and for large times, the action of the flow $(\phi_t)_{t\in\mathbb{R}}$ behaves like the local model that we studied in §3.3, this is the point of Lemma 3.29. Indeed, we construct in Lemma 3.29 a system of admissible charts adapted to the dynamics of $(\phi_t)_{t\in\mathbb{R}}$ (this is a continuous-time analogue of [Jéz20a, Lemma 8.1]). We can then glue copies of the local spaces from §3.2 to define the global spaces $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_0$. Finally, we state and prove Proposition 3.32.

Lemma 3.29. *There are a finite set Ω, an integer r and $t_1 \in \left]0, t_0\right[$ such that:*

(i) there is no periodic orbit of $(\phi_t)_{t\in\mathbb{R}}$ of length less than $3t_1$;

(ii) for all $\omega \in \Omega$ there is a $\mathcal{C}^{\kappa,\upsilon}$ chart $\kappa_\omega : U_\omega \to V_\omega$, where U_ω is an open subset of M and V_ω an open subset of $\mathbb{R}^{d+1}$, such that $V_\omega = W_\omega \times \left]-t_1, t_1\right[$ for some open subset W_ω of $\mathbb{R}^d$, and for all $x \in U_\omega : D_x\kappa_\omega\left(X(x)\right) = e_{d+1}$;

(iii) $\bigcup_{\omega\in\Omega} U_\omega = M$;

(iv) for all $\omega \in \Omega$, there is a system of $r+2$ cones $\Theta_\omega = (C_{0,\omega}, \dots, C_{r,\omega}, C_{f,\omega})$ in $\mathbb{R}^{d+1}$ (with respect to the direction e_{d+1});

(v) for every $\omega, \omega' \in \Omega$ and $t \in [t_0, 3t_0]$ there is a $\mathcal{C}^{\kappa,\upsilon}$ immersion $F_{\omega,\omega',t} : \mathbb{R}^d \to \mathbb{R}^{d+1}$ such that the associated family $\left(\mathcal{T}_{t'}^{\omega,\omega',t}\right)_{t'\in\mathbb{R}}$ (defined by (3.14)) is indeed a family of diffeomorphisms and is cone-hyperbolic from Θ_ω to $\Theta_{\omega'}$;

(vi) for all $\omega, \omega' \in \Omega, t \in [t_0, 3t_0]$ and $t' \in \left]-t_1, t_1\right[$, if $x \in U_\omega$ is such that $\phi_{t+t'}(x) \in U_{\omega'}$ then $\mathcal{T}_{t'}^{\omega,\omega',t} \circ \kappa_\omega(x) = \kappa_{\omega'} \circ \phi_{t+t'}(x)$.

Proof. Choose a Mather metric $|\cdot|_x$ on M (see [Mat68]). This metric makes the splitting

$$T_xM = E_x^u \oplus E_x^s \oplus \mathbb{R}X(x) \tag{3.36}$$

orthogonal and is Hölder-continuous. Moreover, $|X(x)|_x = 1$ for all $x \in M$ and, for $t \geq 0$, we have $\left\| D_x\phi_t|_{E_x^s} \right\| \leq \lambda^{-t}$ and $\left\| D_x\phi_{-t}|_{E_x^u} \right\| \leq \lambda^{-t}$ (for the induced norm, $\lambda > 1$). Choose $\gamma > 0$ such that

$$\lambda^{-2t_0}\left(\gamma^2+1\right) < 1.$$

Then, choose $\gamma_1 \in]1/\gamma, \lambda^{\frac{t_0}{2}}/\gamma[$ and define for all $i \geq 2$ the number $\gamma_i = \lambda^{-\frac{t_0(i-1)}{2}}\gamma_1$. Now, let r be large enough so that

$$\frac{\lambda^{2t_0}}{1+\gamma_{r-1}^2} > 1.$$

Since $\gamma\gamma_1\lambda^{-\frac{5t_0}{8}} < 1$, we may choose $\tilde{\epsilon}_u > 0$ and $\tilde{\epsilon}_s > 0$ such that

$$\tilde{\epsilon}_u > \lambda^{-\frac{t_0}{2}}\gamma_1\tilde{\epsilon}_s \text{ and } \tilde{\epsilon}_s > \gamma\lambda^{-\frac{t_0}{8}}\tilde{\epsilon}_u,$$

and small enough so that

$$\lambda^{-2t_0}\left(\tilde{\epsilon}_s^2+\gamma^2+1\right) < 1$$

and

$$\frac{\lambda^{2t_0}}{1+\gamma_{r-1}^2+\tilde{\epsilon}_u^2} > 1.$$

Finally, set $\epsilon_u = \lambda^{-\frac{t_0}{8}}\tilde{\epsilon}_u$ and $\epsilon_s = \lambda^{-\frac{t_0}{2}}\tilde{\epsilon}_s$.

Now, for all $x \in M$, if $\xi \in T_xM$ write $\xi = \xi_u + \xi_s + \xi_0$ the decomposition of ξ with respect to (3.36), and define the cones $C_f(x)$ and $C_i(x)$, for $i \in \mathbb{N}$ by

$$\begin{aligned} C_0(x) &= \left\{\xi \in T_xM : |\xi_u|_x \leq \gamma\,|\xi_s|_x \text{ and } |\xi_0|_x \leq \tilde{\epsilon}_s\,|\xi_s|_x\right\}, \\ C_i(x) &= \left\{\xi \in T_xM : |\xi_s|_x \leq \gamma_i\,|\xi_u|_x \text{ and } |\xi_0|_x \leq \lambda^{-\frac{(i-1)t_0}{4}}\tilde{\epsilon}_u\,|\xi_u|_x\right\} \end{aligned} \tag{3.37}$$

for $i \in \mathbb{N}^*$ and

$$C_f(x) = \left\{\xi \in T_xM : |\xi_0|_x \geq \epsilon_s\,|\xi_s|_x \text{ and } |\xi_0|_x \geq \epsilon_u\,|\xi_u|_x\right\}.$$

Notice that all these cones depend Hölder-continuously on x. We will see that our choice of parameter ensures that if $x \in M$, then $\Theta(x) = (C_0(x), \dots, C_r(x),\ C_f(x))$ is a system of $r+2$ cones with respect to the direction $X(x)$. Indeed:

(i) if $\xi \in T_xM \setminus C_f(x)$, since $\gamma\gamma_1 > 1$, we have either $|\xi_u|_x < \gamma\,|\xi_s|_x$ or $|\xi_s|_x < \gamma_1\,|\xi_u|_x$. In the first case, either $|\xi_0|_x < \tilde{\epsilon}_s\,|\xi_s|_x$, in which case $\xi \in \overset{\circ}{C_0}(x)$, or $|\xi_0|_x \geq \tilde{\epsilon}_s\,|\xi_s|_x$, which implies $\xi \in \overset{\circ}{C_f}(x)$, since $\epsilon_s < \tilde{\epsilon}_s$ and $|\xi_0|_x > \frac{\tilde{\epsilon}_s}{\gamma}\,|\xi_u|_x > \epsilon_u\,|\xi_u|_x$. Similarly, we can see that in the second case either $\xi \in \overset{\circ}{C_1}(x)$ or $\xi \in \overset{\circ}{C_f}(x)$;

(ii) if $\xi \in C_f(x)$ then $|\xi_0|_x \geq \frac{1}{\sqrt{1+\epsilon_u^{-2}+\epsilon_s^{-2}}}|\xi|$, which implies that $C_f(x)$ is one dimensional;

(iii) if $\xi \in C_0(x)$ then $|\xi|_x \leq \sqrt{1+\gamma^2+\tilde{\epsilon}_s^2}\,|\xi_s|$ and thus $C_0(x)$ is d_s-dimensional, where d_s is the dimension of E_x^s, for the same reason $C_i(x)$ is d_u-dimensional for $i \in \{1,\dots,r\}$;

(iv) $C_{i+1}(x) \Subset C_i(x)$ for $i \in \{1,\dots,r-1\}$ because $\gamma_{i+1} < \gamma_i$ and $\lambda^{-\frac{it_0}{2}}\tilde{\epsilon}_u < \lambda^{-\frac{(i-1)t_0}{2}}\tilde{\epsilon}_u$;

(v) $C_0(x) \cap C_2(x) = \{0\}$ because $\gamma\gamma_2 < 1$ and $C_f(x) \cap C_2(x) = \{0\}$ because we have $\lambda^{\frac{t_0}{4}}\epsilon_u/\tilde{\epsilon}_u = \lambda^{\frac{t_0}{8}} > 1$.

Let us set $\Lambda := \lambda^{t_0}\min((\tilde{\epsilon}_u^2+\gamma_{r-1}^2+1)^{-\frac{1}{2}},(\tilde{\epsilon}_s^2+\gamma^2+1)^{-\frac{1}{2}}) > 1$. Our choice of parameter ensures that for $t \geq t_0$ and $x \in M$:

- for all $i \in \{1,\dots,r\}$ we have $(D_x\phi_t)^{\mathrm{tr}}(C_i(\phi_t(x))) \subseteq C_{i+4}(x)$ because $\lambda^{-2t}\gamma_i \leq \gamma_{i+4}$ and $\lambda^{-t}\lambda^{-\frac{(i-1)t_0}{4}}/\lambda^{-\frac{(i+3)t_0}{4}} < 1$;
- $(D_x\phi_t)^{\mathrm{tr}}(C_f(\phi_t(x))) \cap C_0(x) = \{0\}$ because $\epsilon_s\lambda^t > \tilde{\epsilon}_s$;
- for all $\xi \in C_{r-1}(\phi_t(x))$ we have $\left|(D_x\phi_t)^{\mathrm{tr}}(\xi)\right|_x \geq \Lambda\,|\xi|_{\phi_t(x)}$;
- if $\xi \in T_{\phi_t(x)}M$ and $(D_x\phi_t)^{\mathrm{tr}}(\xi) \in C_0(x)$ then $\left|(D_x\phi_t)^{\mathrm{tr}}(\xi)\right|_x \leq \Lambda^{-1}|\xi|_{\phi_t(x)}$ holds.

Then, for every $x \in M$, we may choose a $\mathcal{C}^{\kappa,\upsilon}$ chart $\kappa_x : V_x \to W_x = B(0,\delta_x)\times]-t_x,t_x[$ such that $\kappa_x(x) = 0$, the map $D_x\kappa_x : T_xM \to \mathbb{R}^{d+1}$ is an isometry and, for every $y \in V_x$, we have $D_y\kappa_x(X(y)) = e_{d+1}$ (we can require the last two points simultaneously because $|X(x)|_x = 1$). For every $x \in M$, choose a system of $r+2$ cones $\Theta_x = (C_{0,x},\dots,C_{r,x},C_{f,x})$ such that $D_x\kappa_x^{\mathrm{tr}}(C_{f,x}) \Subset C_f(x)$,$D_x\kappa_x^{\mathrm{tr}}(C_{0,x}) \Subset C_0(x)$, and, for every $i \in \{1,\dots,r\}$, we have $\left(D_x\kappa_x^{\mathrm{tr}}\right)^{-1}(C_{i+1}(x)) \Subset C_{i,x} \Subset \left(D_x\kappa_x^{\mathrm{tr}}\right)^{-1}(C_i(x))$. Here we recall that the definition (3.37) of $C_i(x)$ is valid for any $i \geq 1$. Up to making V_x smaller, we may ensure that for all $y \in V_x$ we have

$$D_y\kappa_x^{\mathrm{tr}}(C_{f,x}) \Subset C_f(y),\, D_y\kappa_x^{\mathrm{tr}}(C_{0,x}) \Subset C_0(y), \tag{3.38}$$

for all $i \in \{1,\dots,r\}$ we have

$$\left(D_y\kappa_x^{\mathrm{tr}}\right)^{-1}(C_{i+1}(y)) \Subset C_{i,x} \Subset \left(D_y\kappa_x^{\mathrm{tr}}\right)^{-1}(C_i(y)), \tag{3.39}$$

and, in addition,

$$\|D_y\kappa_x\| \leq 1+\epsilon \text{ and } \left\|(D_y\kappa_x)^{-1}\right\|^{-1} \geq 1-\epsilon, \tag{3.40}$$

where $\epsilon > 0$ is small enough so that

$$\frac{1-\epsilon}{1+\epsilon}\Lambda > 1.$$

By compactness of M, there are $x_1, \dots, x_n$ such that M is covered by the open sets $\kappa_{x_i}^{-1}\left(B\left(0, \frac{\delta_{x_i}}{2}\right) \times \left]-\frac{t_{x_i}}{100}, \frac{t_{x_i}}{100}\right[\right)$ for $i = 1, \dots, n$. Let $t_1 = \min_{i=1,\dots,n} \frac{t_{x_i}}{100}$. By cutting the charts into pieces and translating them, we may assume that for every $i = 1, \dots, n$ we have $t_{x_i} = 100 t_1$ (this could make us lose the fact that $D_{x_i}\kappa_{x_i}$ is an isometry, but this is of no harm since (3.40) remains true and that is all we need). Notice that for such a t_1 there is no periodic orbit of $(\phi_t)_{t \in \mathbb{R}}$ of length less than $3t_1$. If necessary, we reduce the value of t_1 so that $t_1 < t_0$. Set $t_2 = 30 t_1$ and let $N = \left\lceil \frac{2t_0}{t_2} \right\rceil$. Choose $\chi : \mathbb{R}^d \to [0,1]$ Gevrey, compactly supported and such that $\chi(y) = 1$ if $|y| \leq 1$.

If $i, j \in \{1, \dots, n\}$, if $k \in \{0, \dots, N\}$, and if $y \in \overline{B}\left(0, \frac{\delta_{x_i}}{2}\right)$ are such that the point $\phi_{t_0 + kt_2}\left(\kappa_{x_i}^{-1}(y, 0)\right)$ lies in $\kappa_{x_j}^{-1}\left(B\left(0, \frac{\delta_{x_j}}{2}\right)\right) \times [-t_2, t_2]$, and $\eta > 0$ is small enough define $F_{i,j,k,y,\eta} : \mathbb{R}^d \to \mathbb{R}^{d+1}$ by (here we see $\mathbb{R}^d \simeq \mathbb{R}^d \times \{0\}$ as a subset of $\mathbb{R}^{d+1}$)

$$\begin{aligned} F_{i,j,k,y,\eta}(z) &= \chi\left(\frac{z-y}{\eta}\right) \kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}(z) \\ &\quad + \left(1 - \chi\left(\frac{z-y}{\eta}\right)\right)\left(\kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}(y) + D_y\left(\kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}\right)(z-y)\right) \end{aligned}$$

Notice that $F_{i,j,k,y,\eta}$ coincides with $\kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}$ on $B(0,\eta)$, and that it can be made arbitrarily close in the $\mathcal{C}^1$ topology to the affine map $z \mapsto \kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}(y) + D_y\left(\kappa_{x_j} \circ \phi_{t_0+kt_2} \circ \kappa_{x_i}^{-1}\right)(z-y)$ by taking $\eta = \eta_{i,j,k,y}$ small enough. In particular, $F_{i,j,k,y,\eta}$ defines a cone-hyperbolic family of diffeomorphisms $\left(\mathcal{T}_{i,j,k,y,\eta,t'}\right)_{t' \in \mathbb{R}}$ from Θ_{x_i} to Θ_{x_j} (the cone-hyperbolicity follows from the properties of the differential of $\phi_{t_0+kt_2}$ proven above and the quasi-isometry property (3.40) of the charts, to see that the $\mathcal{T}_{i,j,k,y,\eta,t'}$'s are diffeomorphisms just notice that they are proper local diffeomorphism and hence covering of $\mathbb{R}^{d+1}$ by itself). Define $\tilde{\eta}_{i,y} = \min_{\substack{k=0,\dots,N \\ j=1,\dots,n}} \eta_{i,j,k,y}$ (if j is such that $\phi_{t_0+kt_2}\left(\kappa_{x_i}^{-1}(y,0)\right) \notin \kappa_{x_j}^{-1}\left(B\left(0, \frac{\delta_{x_j}}{2}\right)\right) \times [-t_2, t_2]$, i.e. there is no allowed transitions from i to j at the considered time, set $\eta_{i,j,k,y} = \infty$ and take for $F_{i,j,k,y,\tilde{\eta}_{i,y}}$ any $\mathcal{C}^{\kappa,\upsilon}$ map that defines a cone-hyperbolic family of diffeomorphisms[9] from Θ_{x_i} to Θ_{x_j}) .

Notice also that for all $(z, z') \in B(0, \tilde{\eta}_{i,y}) \times]-t_2, t_2[$ and all $t, t' \in]-t_2, t_2[$ we have

$$\begin{aligned} \kappa_{x_j} \circ \phi_{t_0+kt_2+t+t'} \circ \kappa_{x_i}^{-1}(z, z') &= \kappa_{x_j} \circ \phi_{t_0+kt_2+t+t'+z'} \circ \kappa_{x_i}^{-1}(z, 0) \\ &= F_{i,j,k,y,\tilde{\eta}_{i,y}}(z) + z' e_{d+1} + (t+t') e_{d+1} \\ &= \mathcal{T}_{t'}^{i,j,k,y,t}(z, z'), \end{aligned}$$

where $\left(\mathcal{T}_{t'}^{i,j,k,y,t}\right)_{t' \in \mathbb{R}}$ denotes the family of cone-hyperbolic diffeomorphisms associated with $F_{i,j,k,y,\tilde{\eta}_{i,y}} + t e_{d+1}$.

[9] There always is a linear such map.

By compactness of $\overline{B}\left(0, \frac{\delta_{x_i}}{2}\right)$, we may find $y_{i,1}, \ldots, y_{i,m_i} \in \overline{B}\left(0, \frac{\delta_{x_i}}{2}\right)$ such that

$$\overline{B}\left(0, \frac{\delta_{x_i}}{2}\right) \subseteq \bigcup_{\ell=1}^{m_i} B\left(y_{i,\ell}, \frac{\tilde{\eta}_{i,y_{i,\ell}}}{2}\right).$$

Finally, set

$$\Omega = \{(i,\ell) : i \in \{1, \ldots, n\}, \ell \in \{1, \ldots, m_i\}\},$$

and, for all $\omega = (i,\ell) \in \Omega$,

$$V_\omega = B\left(0, \frac{\tilde{\eta}_{i,y_{i,\ell}}}{2}\right) \times \left]-t_1, t_1\right[, \quad U_\omega = \kappa_{x_i}^{-1}\left(B\left(0, \frac{\tilde{\eta}_{i,y_{i,\ell}}}{2}\right) \times \left]-t_1, t_1\right[\right),$$
$$\kappa_\omega = \kappa_{x_i}|_{U_\omega}, \quad \Theta_\omega = \Theta_{x_i}.$$

If $\omega' = (j, \ell') \in \Omega$ and $t \in [t_0 + kt_2, t_0 + (k+1)t_2]$ let

$$F_{\omega,\omega',t} = F_{i,j,k,y_{i,\ell},\tilde{\eta}_{i,y_{i,\ell}}} + (t - kt_2)\, e_{d+1}.$$

From what is above, these data satisfy the statement of the lemma. □

Choose a Gevrey partition of unity $(\varphi_\omega)_{\omega\in\Omega}$ subordinated to the open cover $(U_\omega)_{\omega\in\Omega}$. Fix $\alpha \in \left]\frac{\upsilon-1}{\upsilon}, 1\right[$ (if $\upsilon < 2$, we choose $\alpha < \frac{1}{2}$) and choose $\tilde{\upsilon} \in \left]\upsilon, \frac{1}{1-\alpha}\right[$. Then define

$$\begin{array}{rccc} \Phi & : & \mathcal{D}^{\tilde{\upsilon}}(M)' & \rightarrow & \oplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)' \\ & & u & \mapsto & \left((\varphi_\omega u) \circ \kappa_w^{-1}\right) \end{array}$$

and

$$\begin{array}{rccc} S & : & \oplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)' & \rightarrow & \mathcal{D}^{\tilde{\upsilon}}(M)' \\ & & (u_\omega)_{\omega\in\Omega} & \mapsto & \sum_{\omega\in\Omega}(h_\omega u_\omega) \circ \kappa_\omega, \end{array}$$

where $h_\omega : \mathbb{R}^{d+1} \to [0,1]$ is Gevrey, supported in W_ω, and takes value 1 on $\kappa_\omega(\operatorname{supp} \varphi_\omega)$. Notice that $S \circ \Phi$ is the identity of $\mathcal{D}^{\tilde{\upsilon}}(M)'$. It can be verified that Φ and S are continuous.

We may now define the first version of the global Hilbert space (the final one will be introduced in §3.5). Define

$$\mathcal{H}_\Omega = \oplus_{\omega\in\Omega}\mathcal{H}_{\Theta_\omega,\alpha}$$

and

$$\widetilde{\mathcal{H}} = \left\{u \in \mathcal{D}^{\tilde{\upsilon}}(M)' : \Phi(u) \in \mathcal{H}_\Omega\right\},$$

endowed with the norm

$$\|u\|_{\widetilde{\mathcal{H}}} = \|\Phi(u)\|_{\mathcal{H}_\Omega} = \sqrt{\sum_{\omega\in\Omega}\left\|(\varphi_\omega u) \circ \kappa_\omega^{-1}\right\|^2_{\mathcal{H}_{\Theta_\omega,\alpha}}}.$$

Proposition 3.30. *$\widetilde{\mathcal{H}}$ is a separable Hilbert space (equivalently, $\Phi\left(\widetilde{\mathcal{H}}\right)$ is closed in $\mathcal{H}_\Omega$) that does not depend on the choice of $\tilde{\upsilon}$. The inclusion of $\widetilde{\mathcal{H}}$ in $\mathcal{D}^{\tilde{\upsilon}}(M)'$ is continuous, and $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is continuously contained in $\widetilde{\mathcal{H}}$.*

Proof. To see that $\Phi\left(\widetilde{\mathcal{H}}\right)$ is closed in $\mathcal{H}_\Omega$, just notice that

$$\Phi\left(\widetilde{\mathcal{H}}\right) = \left\{u \in \mathcal{H}_\Omega : \Phi S u = u\right\},$$

and that the inclusion of $\mathcal{H}_\Omega$ in $\bigoplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)'$ is continuous. The inclusion of $\widetilde{\mathcal{H}}$ in $\mathcal{D}^{\tilde{\upsilon}}(M)'$ may be written as the composition of Φ, the inclusion of $\mathcal{H}_\Omega$ in $\bigoplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)'$ and S. It is thus continuous. Finally, Φ sends $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ continuously into $\bigoplus_{\omega\in\Omega}\mathcal{S}^{\tilde{\upsilon}}$, which is continuously contained in $\mathcal{H}_\Omega$, thus $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ is contained in $\widetilde{\mathcal{H}}$, the inclusion being continuous. □

Let $\widetilde{\mathcal{H}}_0$ be the closure[10] of $\mathcal{C}^{\infty,\tilde{\upsilon}}(M)$ in $\widetilde{\mathcal{H}}$. Recall from §3.1 that for each $t \in \mathbb{R}$ we may define the operator $\mathcal{L}_t$ from (9) as an operator from $\mathcal{D}^{\tilde{\upsilon}}(M)'$ to itself. We start by proving that, for $t \geq t_0$, the operator $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{H}}_0$.

Proposition 3.31. *For all $t \in [t_0, +\infty[$ the operator $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{H}}_0$. Moreover, as an operator from $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{H}}_0$, the operator $\mathcal{L}_t$ depends continuously on $t \in [t_0, +\infty[$ in the strong operator topology.*

Proof. We only need to prove the result for $t \in [t_0, 3t_0]$, and then use the group property of $(\mathcal{L}_t)_{t\in\mathbb{R}}$. Recall indeed that $\widetilde{\mathcal{H}}_0$ is a closed subspace of $\widetilde{\mathcal{H}}$. For all $t \in \mathbb{R}$ define

$$\widetilde{\mathcal{L}}_t : \bigoplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)' \to \bigoplus_{\omega\in\Omega}\left(\mathcal{S}^{\tilde{\upsilon}}\right)'$$

by $\widetilde{\mathcal{L}}_t = \Phi \circ \mathcal{L}_t \circ S$. The operator $\widetilde{\mathcal{L}}_t$ may be described via a matrix of operators $\left(\widetilde{\mathcal{L}}_{\omega,\omega',t}\right)_{\omega,\omega'\in\Omega}$, that is, we have

$$\widetilde{\mathcal{L}}_t\left(u_\omega\right)_{\omega\in\Omega} = \left(\sum_{\omega'\in\Omega}\widetilde{\mathcal{L}}_{\omega,\omega',t}u_{\omega'}\right)_{\omega\in\Omega}. \tag{3.41}$$

Now, if $t \in [t_0, 3t_0]$ and $t' \in]-t_1, t_1[$, then the operator $\widetilde{\mathcal{L}}_{\omega,\omega',t+t'}$ for $\omega, \omega' \in \Omega$ may be described as

$$\begin{aligned}\widetilde{\mathcal{L}}_{\omega,\omega',t+t'}u(x) &= \varphi_\omega \circ \kappa_\omega^{-1}(x)\, e^{\int_0^{t+t'} V\circ\phi_\tau\left(\kappa_\omega^{-1}(x)\right)\mathrm{d}\tau} \\ &\qquad\times h_{\omega'} \circ \kappa_{\omega'} \circ \phi_{t+t'} \circ \kappa_\omega^{-1}(x)\, u \circ \kappa_{\omega'} \circ \phi_{t+t'} \circ \kappa_\omega^{-1}(x) \\ &= G_{\omega,\omega',t,t'}(x)\, u \circ \mathcal{T}_{t'}^{\omega',\omega,t}(x),\end{aligned} \tag{3.42}$$

[10]It could well be that $\widetilde{\mathcal{H}}_0 = \widetilde{\mathcal{H}}$, see Proposition 3.11, but we do not need this fact

where $\left(\mathcal{T}_{t'}^{\omega',\omega,t}\right)_{t'\in\mathbb{R}}$ is the family of diffeomorphisms associated to $F_{\omega',\omega,t}$ by (3.14), and

$$G_{\omega,\omega',t,t'}(x) = \varphi_\omega \circ \kappa_\omega^{-1}(x)\, e^{\int_0^{t+t'} V\circ\phi_\tau\left(\kappa_\omega^{-1}(x)\right)\mathrm{d}\tau} h_{\omega'} \circ \kappa_{\omega'} \circ \phi_{t+t'} \circ \kappa_\omega^{-1}(x)$$

properly extended by zero. We can then apply Proposition 3.17 to prove that $\widetilde{\mathcal{L}}_{\omega,\omega',t+t'}$ is bounded from $\mathcal{H}_{\Theta'_\omega,\alpha}$ to $\mathcal{H}_{\Theta_\omega,\alpha}$. Then $\widetilde{\mathcal{L}}_{t+t'}$ is bounded from $\mathcal{H}_\Omega$ to itself thanks to (3.41). Notice that if $u \in \bigoplus_{\omega\in\Omega} S^{\tilde{v}}$ then $\widetilde{\mathcal{L}}_{t+t'}u = \Phi\left(\mathcal{L}_{t+t'} \circ Su\right) \in \Phi\left(\widetilde{\mathcal{H}}_0\right)$. Thus, since $\bigoplus_{\omega\in\Omega} S^{\tilde{v}}$ is dense in $\mathcal{H}_\Omega$, the operator $\widetilde{\mathcal{L}}_{t+t'}$ sends $\mathcal{H}_\Omega$ into $\Phi\left(\widetilde{\mathcal{H}}_0\right)$. Denote by $\Psi : \Phi\left(\widetilde{\mathcal{H}}_0\right) \to \widetilde{\mathcal{H}}_0$ the inverse of the isomorphism induced by Φ, and notice that $\mathcal{L}_{t+t'}$ coincide on $\widetilde{\mathcal{H}}$ with $\Psi \circ \widetilde{\mathcal{L}}_{t+t'} \circ \Phi$, and is thus bounded from $\widetilde{\mathcal{H}}$ to $\widetilde{\mathcal{H}}_0$. Finally, from Proposition 3.17, we know that $\widetilde{\mathcal{L}}_{t+t'} : \mathcal{H}_\Omega \to \mathcal{H}_\Omega$ depends continuously on $t' \in \left]-t_1, t_1\right[$ in the strong operator topology, and consequently so does $\mathcal{L}_{t+t'} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}_0$. □

We want now to prove Schatten properties for operators defined in term of the $\mathcal{L}_t$'s for $t \geq t_0$. To do so, it is convenient to introduce $(\psi_\ell)_{\ell\in\mathbb{Z}}$, a $\frac{t_1}{3}\mathbb{Z}$ invariant smooth partition of unity on $\mathbb{R}$ (that is, we have $\psi_\ell = \psi_0\left(\cdot - \ell\frac{t_1}{3}\right)$) such that ψ_0 is supported in $\left]-\frac{t_1}{2}, \frac{t_1}{2}\right[$.

Proposition 3.32. *Assume $\upsilon < 2$. There is $\varpi \in \mathbb{R}$ with the following property: if $h : \mathbb{R}_+^* \to \mathbb{C}$ and $k \in \mathbb{N}$ satisfy*

(i) h is supported in $[t_0, +\infty[$;

(ii) h is kth time differentiable and its kth derivatives has bounded variations;

(iii) there is a constant $C > 0$ such that for every $\ell \in \mathbb{N}$ we have

$$\left\|\psi_0 h\left(\cdot + \ell\frac{t_1}{3}\right)\right\|_{\mathcal{C}^{k-1}} + \left\|\left(\psi_0 h\left(\cdot + \ell\frac{t_1}{3}\right)\right)^{(k)}\right\|_{BV} \leq C e^{-\varpi\ell};$$

then the operator

$$\int_0^{+\infty} h(t)\mathcal{L}_t \mathrm{d}t : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}_0 \tag{3.43}$$

is in the Schatten class S_p for every $p \geq 1$ such that $p > \frac{d+1}{k+1}$. Moreover, if $k > d$ and if we see (3.43) *as an operator from $\widetilde{\mathcal{H}}_0$ to itself, we have*

$$\operatorname{tr}\left(\int_{t_0}^{+\infty} h(t)\,\mathcal{L}_t \mathrm{d}t\right) = \sum_\gamma T_\gamma^{\#} \frac{h(T_\gamma)}{\left|\det\left(I - \mathcal{P}_\gamma\right)\right|} \exp\left(\int_\gamma V\right),$$

where the sum on the right-hand side runs over closed periodic orbits[11] *γ of the flow $(\phi_t)_{t\in\mathbb{R}}$.*

[11] Recall that T_γ is the length of γ, while $T_\gamma^{\#}$ denotes its primitive length and $\mathcal{P}_\gamma$ is a linearized Poincaré map. We will see during the proof of the proposition that this sum converges.

Finally, if $\upsilon \geq 2$, it remains true (under the same assumptions) that the operator (3.43) *is compact.*

Remark 3.33. Notice that if h is a $\mathcal{C}^{\infty}$ function supported in $[t_0, +\infty[$, then h clearly satisfies the conditions (i)-(iii) from Proposition 3.32. This will be the main application of Proposition 3.32 in order to prove the trace formula (see Lemma 3.44). However, we will also need to consider other functions h in the proof of Lemma 3.43 and in the Appendix C.

For the sake of the proof, we split Proposition 3.32 into Lemmas 3.34, 3.35 and 3.36.

Lemma 3.34. *Under the assumption of Proposition 3.32, the operator* (3.43) *is in the Schatten class S_p for every $p \geq 1$ such that $p > \frac{d+1}{k+1}$. If $\upsilon \geq 2$, it remains true that* (3.43) *is compact.*

Proof. Let $p \geq 1$ be such that $p > \frac{d+1}{k+1}$. Choose N large enough so that $Nt_1 \geq t_0$, and write for all $\ell \geq N$

$$\int_{\mathbb{R}} \psi_\ell (t) \, h (t) \, \mathcal{L}_t \mathrm{d}t = \mathcal{L}^q_{\frac{Nt_1}{3}} \int_{\mathbb{R}} \psi_0 (t) \, h \left(t + \ell \frac{t_1}{3} \right) \mathcal{L}_{\frac{rt_1}{3} + t} \mathrm{d}t,$$

where $\ell = qN + r$ with $q, r \in \mathbb{Z}$ and $N \leq r < 2N$ (notice that $q \geq 0$). Applying Proposition 3.17 as in the proof of Proposition 3.31, we see that the operator

$$\int_{\mathbb{R}} \psi_0 (t) \, h \left(t + \ell \frac{t_1}{3} \right) \mathcal{L}_{\frac{rt_1}{3} + t} \mathrm{d}t \tag{3.44}$$

is in the Schatten class S_p, with norm in this class an $\mathcal{O}\left(e^{-\varpi \ell}\right)$ (there is a finite number of possible values for r). Thus the sum

$$\sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}} \psi_\ell (t) \, h (t) \, \mathcal{L}_t \mathrm{d}t \tag{3.45}$$

converges in S_p provided that ϖ is large enough(there are a finite number of non-zero terms with $k < N$ that are also in S_p thanks to Proposition 3.17 since $h(t)$ vanishes for $t \leq t_0$). Now notice that, for every $t \in \mathbb{R}$, the sum

$$\sum_{\ell \in \mathbb{Z}} \psi_\ell (t) \, h (t) \, \mathcal{L}_t$$

converges in operator norm topology to $h(t) \mathcal{L}_t$, and the convergence is uniform in t (provided that ϖ is large enough), so that the sum (3.45) is in fact the operator

$$\int_0^{+\infty} h(t) \mathcal{L}_t \mathrm{d}t,$$

which is consequently in S_p.

Finally, when $\upsilon \geq 2$ it remains true that the operator (3.44) is compact according to Proposition 3.17, and the convergence in the operator norm topology ensures that (3.43) is compact. $\square$

We need now to compute the trace of this operator when $k > d$ and $\upsilon < 2$. We will deduce the global formula for the trace from the local formula from Proposition 3.17.

Lemma 3.35. *Under the assumptions of Proposition 3.32 and if $\ell \in \mathbb{Z}$ we have*

$$\operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \mathcal{L}_t \mathrm{d}t\right) = \sum_\gamma \psi_\ell(T_\gamma) \exp\left(\int_\gamma V\right) \frac{h(T_\gamma)\, T_\gamma^\#}{|\det(I - \mathcal{P}_\gamma)|}, \tag{3.46}$$

where the sum runs over periodic orbits γ of the flow $(\phi_t)_{t\in\mathbb{R}}$. Here, we recall that $(\psi_\ell)_{\ell\in\mathbb{Z}}$ is a $\frac{t_1}{3}\mathbb{Z}$ invariant smooth partition of unity on $\mathbb{R}$ such that ψ_0 is supported in $\left]-\frac{t_1}{2}, \frac{t_1}{2}\right[$.

Proof. If ℓ is such that $\frac{\ell t_1}{3} < t_0 - \frac{t_1}{2}$ then (3.46) is immediate: both sides vanish. Otherwise, choose an integer $m \geq 0$ such that $\frac{\ell t_1}{3} - m t_0 \in \left[t_0 - \frac{t_1}{2}, 2t_0\right]$ (one can for instance take m to be the largest integer such that $\frac{\ell t_1}{3} - m t_0 \geq t_0 - \frac{t_1}{2}$) and define $t_3 = \max\left(t_0, \frac{\ell t_1}{3} - m t_0\right)$. This ensures that the support of ψ_ℓ is contained in $m t_0 + t_3 +]-t_1, t_1[$ and that $t_3 \in [t_0, 2t_0]$. For all $\overrightarrow{\omega} = (\omega_1, \dots, \omega_m) \in \Omega^m$ define

$$U_{\overrightarrow{\omega}} = \bigcap_{j=1}^m \phi_{-jt_0}(U_{\omega_i}).$$

Then choose a refinement $\left(\widetilde{U}_{\overrightarrow{\omega},w}\right)_{(\overrightarrow{\omega},w)\in\Omega^m\times\mathcal{W}}$ of $(U_{\overrightarrow{\omega}})_{\omega\in\Omega^m}$ whose elements are small enough such that, if γ is a periodic orbit of $(\phi_t)_{t\in\mathbb{R}}$ of length T_γ less than $t_3 + m t_0 + t_1$, and $(\overrightarrow{\omega}, w) \in \Omega^m \times \mathcal{W}$, then the intersection of γ with $\widetilde{U}_{\overrightarrow{\omega},w}$ is an interval (i.e. connected, while possibly empty). This can be done because there are a finite number of such orbits. Choose a Gevrey partition of unity $\left(\theta_{\overrightarrow{\omega},w}\right)_{(\overrightarrow{\omega},w)\in\Omega^m\times\mathcal{W}}$ adapted to the open cover $\left(\widetilde{U}_{\overrightarrow{\omega},w}\right)_{(\overrightarrow{\omega},w)\in\Omega^m\times\mathcal{W}}$ of M. For $t \in t_3 + m t_0 +]-t_1, t_1[$, recall from the proof of Proposition 3.31 the operators $\widetilde{\mathcal{L}}_t = \Phi \circ \mathcal{L}_t \circ S$, and $\widetilde{\mathcal{L}}_{\omega,\omega',t}$, for $\omega, \omega' \in \Omega$, defined by the formula,

$$\widetilde{\mathcal{L}}_{\omega,\omega',t} u(x) = \varphi_\omega(\kappa_\omega^{-1}(x)) e^{\int_0^t V\circ\phi_\tau(\kappa_\omega^{-1}(x))\mathrm{d}\tau} h_{\omega'}(\kappa_{\omega'}\phi_t\kappa_\omega^{-1}(x)) u(\kappa_{\omega'}\phi_t\kappa_\omega^{-1}(x)).$$

Then write $\widetilde{\mathcal{L}}_{\omega,\omega',t}$ as a sum of operators

$$\widetilde{\mathcal{L}}_{\omega,\omega',t} = \sum_{(\overrightarrow{\omega},w)\in\Omega^m\times\mathcal{W}} A_{\omega,\omega',\overrightarrow{\omega},w,t}$$

where, for $\overrightarrow{\omega} = (\omega_0, \dots, \omega_{m-1})$ and $w \in \mathcal{W}$,

$$
\begin{aligned}
A_{\omega,\omega',\overrightarrow{\omega},w,t} u(x) &= \theta_{\overrightarrow{\omega},w}\left(\kappa_\omega^{-1}(x)\right) \widetilde{\mathcal{L}}_{\omega,\omega',t} u(x) \\
&= \left(\theta_{\overrightarrow{\omega},w}\varphi_\omega\right) \circ \kappa_\omega^{-1}(x)\, e^{\int_0^t V\circ\phi_\tau\left(\kappa_\omega^{-1}(x)\right)\mathrm{d}\tau} h_{\omega'} \circ \kappa_{\omega'} \circ \phi_t \circ \kappa_\omega^{-1}(x) \\
&\qquad \times u \circ \mathcal{T}_{t-t_3-mt_0}^{\omega_m,\omega',t_3} \circ \mathcal{T}_0^{\omega_{m-1},\omega_m,t_0} \circ \dots \circ \mathcal{T}_0^{\omega_1,\omega_2,t_0} \circ \mathcal{T}_0^{\omega,\omega_1,t_0}(x).
\end{aligned}
$$

Consequently, we can use Proposition 3.17 to see that $A_{\omega,\omega',\overrightarrow{\omega},w,t} : \mathcal{H}_{\Theta_{\omega'},\alpha} \to \mathcal{H}_{\Theta_\omega,\alpha}$ is bounded (here, we recall that α has been fixed after the proof of Lemma 3.29, when defining the space $\widetilde{\mathcal{H}}$). Then, working as in the proof of Lemma 3.34, the operator

$$
\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \widetilde{\mathcal{L}}_t \mathrm{d}t \tag{3.47}
$$

is trace class, sends $\mathcal{H}_\Omega$ into $\Phi\left(\widetilde{\mathcal{H}}_0\right)$ and the induced operator is conjugated to the operator defined by (3.47) without the tilde. Consequently, using Lidskii's trace theorem, we get

$$
\begin{aligned}
\operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \mathcal{L}_t \mathrm{d}t\right) &= \operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \widetilde{\mathcal{L}}_t \mathrm{d}t\right) = \sum_{\omega\in\Omega} \operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \widetilde{\mathcal{L}}_{\omega,\omega,t} \mathrm{d}t\right) \\
&= \sum_{\omega\in\Omega} \sum_{\left(\overrightarrow{\omega},w\right)\in\Omega^m\times\mathcal{W}} \operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, A_{\omega,\omega,\overrightarrow{\omega},w,t} \mathrm{d}t\right).
\end{aligned}
$$

Next, we fix ω and $(\overrightarrow{\omega}, w)$ and we compute

$$
\operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \mathcal{A}_{\omega,\omega,\overrightarrow{\omega},w,t} \mathrm{d}t\right)
$$

using Proposition 3.17. To do so, recall the family of cone-hyperbolic diffeomorphisms

$$
\left(\mathcal{T}_t^{\omega,\overrightarrow{\omega}}\right)_{t\in\mathbb{R}} := \left(\mathcal{T}_{t-t_3-mt_0}^{\omega_m,\omega',t_3} \circ \mathcal{T}_0^{\omega_{m-1},\omega_m,t_0} \circ \dots \circ \mathcal{T}_0^{\omega_1,\omega_2,t_0} \circ \mathcal{T}_0^{\omega,\omega_1,t_0}\right)_{t\in\mathbb{R}}
$$

and denote by $F_{\omega,\overrightarrow{\omega}} : \mathbb{R}^d \to \mathbb{R}^{d+1}$ the associated immersion. By Proposition 3.17, we have

$$
\begin{aligned}
&\operatorname{tr}\left(\int_{\mathbb{R}} \psi_\ell(t)\, h(t)\, \mathcal{A}_{\omega,\omega,\overrightarrow{\omega},w,t} \mathrm{d}t\right) \\
&\qquad = \sum_{p\circ F_{\omega,\overrightarrow{\omega}}(x)=x} \left(\frac{h\left(T_{\omega,\overrightarrow{\omega}}(x)\right) \psi_\ell\left(T_{\omega,\overrightarrow{\omega}}(x)\right)}{\left|\det\left(I - p\circ D_x F_{\omega,\overrightarrow{\omega}}\right)\right|} \int_{\mathbb{R}} G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(x,y)\,\mathrm{d}y\right),
\end{aligned}
$$

where, as in Proposition 3.17, if $x \in \mathbb{R}^d$, then $T_{\omega,\overrightarrow{\omega}}(x)$ denotes the opposite of the last coordinate of $F_{\omega,\overrightarrow{\omega}}(x)$, and

$$
G_{\omega,\overrightarrow{\omega},w,t}(x) = \left(\theta_{\overrightarrow{\omega},w}\varphi_\omega\right) \circ \kappa_\omega^{-1}(x)\, e^{\int_0^t V\circ\phi_\tau\left(\kappa_\omega^{-1}(x)\right)\mathrm{d}\tau} h_\omega \circ \kappa_\omega \circ \phi_t \circ \kappa_\omega^{-1}(x),
$$

properly extended by zero.

Now, denote by Q_0 the (finite) set of $x \in \mathbb{R}^d$ such that $p \circ F_{\omega,\overrightarrow{\omega}}(x) = x$ and

$$D(x) := \frac{h\left(T_{\omega,\overrightarrow{\omega}}(x)\right)\psi_\ell\left(T_{\omega,\overrightarrow{\omega}}(x)\right)}{\left|\det\left(I - p \circ D_x F_{\omega,\overrightarrow{\omega}}\right)\right|}\int_{\mathbb{R}} G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(x,y)\,\mathrm{d}y \neq 0,$$

and by Q_1 the (finite) set of periodic orbits γ for $(\phi_t)_{t\in\mathbb{R}}$ such that

$$E(\gamma) := \frac{h(T_\gamma)\psi_\ell(T_\gamma)}{\left|\det\left(I - \mathcal{P}_\gamma\right)\right|} e^{\int_\gamma g}\int_{\gamma^\#}\theta_{\overrightarrow{\omega},w}\varphi_\omega \neq 0.$$

We will construct a bijection $x \mapsto \gamma(x)$ between Q_0 and Q_1 such that, for all $x \in Q_0$, we have $D(x) = E(\gamma(x))$. This will immediately imply that

$$\operatorname{tr}\left(\int_{\mathbb{R}}\psi_\ell(t)\,h(t)\,\mathcal{A}_{\omega,\omega,\overrightarrow{\omega},w,t}\mathrm{d}t\right) = \sum_\gamma \frac{h(T_\gamma)\psi_\ell(T_\gamma)}{\left|\det\left(I - \mathcal{P}_\gamma\right)\right|} e^{\int_\gamma V}\int_{\gamma^\#}\theta_{\overrightarrow{\omega},w}\varphi_\omega$$

and we can then end the proof by summing over $\omega \in \Omega$ and $(\overrightarrow{\omega},w) \in \Omega^m \times \mathcal{W}$.

Let $x \in Q_0$. Since $D(x) \neq 0$, there is $\tilde{y} \in \mathbb{R}$ such that $G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(x,\tilde{y})$ is non-zero. Set $z = (x,\tilde{y})$, and notice that $z \in V_\omega$, so that $\kappa_\omega^{-1}(z)$ make sense. Moreover, since $G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(z) \neq 0$, we must have $\phi_{jt_0}(z) \in U_{\omega_j}$ for $j \in \{1,\dots,m\}$, and $\phi_{T_{\omega,\overrightarrow{\omega}}(x)}(z) \in U_\omega$. In addition, since $\psi_\ell\left(T_{\omega,\overrightarrow{\omega}}(x)\right) \neq 0$, we know that $T_{\omega,\overrightarrow{\omega}}(x) \in t_3 + mt_0 +]-t_1,t_1[$, and thus Lemma 3.29 ensures that

$$\begin{aligned}\kappa_\omega \circ \phi_{T_{\omega,\overrightarrow{\omega}}(x)} \circ \kappa_\omega^{-1}(z) &= \mathcal{T}_{t-t_3-mt_0}^{\omega_m,\omega',t_3} \circ \mathcal{T}_0^{\omega_{m-1},\omega_m,t_0} \circ \dots \circ \mathcal{T}_0^{\omega_1,\omega_2,t_0} \circ \mathcal{T}_0^{\omega,\omega_1,t_0}(z) \\ &= \mathcal{T}_{T_{\omega,\overrightarrow{\omega}}(x)}^{\omega,\overrightarrow{\omega}}(z) = F_{\omega,\overrightarrow{\omega}}(x) + T_{\omega,\overrightarrow{\omega}}(x)\,e_{d+1} + \tilde{y}_{d+1}e_{d+1} \\ &= p \circ F_{\omega,\overrightarrow{\omega}}(x) - T_{\omega,\overrightarrow{\omega}}(x)\,e_{d+1} + T_{\omega,\overrightarrow{\omega}}(x)\,e_{d+1} + \tilde{y}_{d+1}e_{d+1} = z.\end{aligned}$$

Consequently, there is a periodic orbit $\gamma(x)$ of length $T_{\gamma(x)} = T_{\omega,\overrightarrow{\omega}}(x)$ for $(\phi_t)_{t\in\mathbb{R}}$ passing through the point $\kappa_\omega^{-1}(z)$. Notice that, while the point $\kappa_\omega^{-1}(z)$ depends on the choice of $\tilde{y}$, the orbit $\gamma(x)$ does not (another choice of $\tilde{y}$ would only change $\kappa_\omega^{-1}(z)$ into another point of the orbit $\gamma(x)$). The map $D_{\kappa_\omega^{-1}(z)}\phi_{T_{\gamma(x)}}$ is conjugated via $D_{\kappa_\omega^{-1}(z)}\kappa_\omega$ to $D_z\mathcal{T}_{T_{\omega,\overrightarrow{\omega}}(x)}^{\omega,\overrightarrow{\omega}}$. However, in a base adapted to the decomposition of the tangent space into the stable and unstable directions and the direction of the flow, the matrix of the map $D_{\kappa_\omega^{-1}(z)}\phi_{T_{\gamma(x)}}$ is

$$\begin{bmatrix} \mathcal{P}_{\gamma(x)} & 0 \\ 0 & 1 \end{bmatrix},$$

while the matrix of $D_z\mathcal{T}_{T_{\omega,\overrightarrow{\omega}}(x)}^{\omega,\overrightarrow{\omega}}$ in the canonical basis of $\mathbb{R}^{d+1}$ is of the form

$$\begin{bmatrix} p \circ D_x F_{\omega,\overrightarrow{\omega}} & 0 \\ * & 1 \end{bmatrix}.$$

Thus, the linear maps $\mathcal{P}_{\gamma(x)}$ and $p \circ D_x F_{\omega,\overrightarrow{\omega}}$ have the same spectrum, which implies that $\det\left(I - \mathcal{P}_{\gamma(x)}\right) = \det\left(I - D_x F_{\omega,\overrightarrow{\omega}}\right)$. Denote by I_x the set of $y \in \mathbb{R}$ such that $G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(x,y) \neq 0$. Then for all $y \in I_x$, we have

$$\exp\left(\int_0^{T_{\omega,\overrightarrow{\omega}}(x)} V \circ \phi_\tau\left(\kappa_\omega^{-1}(x,y)\right)\mathrm{d}\tau\right) = \exp\left(\int_{\gamma(x)} g\right).$$

Moreover, the map $I_x \ni y \to \kappa_\omega^{-1}(x,y) = \phi_{y-\tilde{y}}\left(\kappa_\omega^{-1}(z)\right)$ is injective (the length of I_x is at most $2t_1$, and there is no periodic orbit of $(\phi_t)_{t\in\mathbb{R}}$ of length less than $3t_1$), and its image is $\gamma \cap \widetilde{U}_{\overrightarrow{\omega},w}$ (thanks to our assumption on the refinement), so that a change of variable gives

$$\int_{\mathbb{R}} G_{\omega,\overrightarrow{\omega},w,T_{\omega,\overrightarrow{\omega}}(x)}(x,y)\,\mathrm{d}y = \exp\left(\int_{\gamma(x)} V\right)\int_{\gamma^{\#}(x)} \theta_{\overrightarrow{\omega},w}\varphi_\omega,$$

and thus we have $E(\gamma(x)) = D(x) \neq 0$, in particular $\gamma(x) \in Q_1$. It remains to prove that $Q_0 \ni x \mapsto \gamma(x) \in Q_1$ is a bijection.

If $x \in Q_0$ then the intersection of $\gamma(x)$ with $\widetilde{U}_{\overrightarrow{\omega},w}$ is an interval, and thus the set $\kappa_\omega\left(\gamma(x) \cap \widetilde{U}_{\overrightarrow{\omega},w}\right)$ is contained in a line perpendicular to $\mathbb{R}^d \times \{0\}$ (recall that κ_ω is a flow box) and this line projects on $x \in \mathbb{R}^d$. Thus $\gamma(x)$ determines x and consequently the map $x \mapsto \gamma(x)$ is injective.

Reciprocally, if $\gamma \in Q_1$ then γ must intersect $\widetilde{U}_{\overrightarrow{\omega},w}$ on a non-empty interval that is sent by κ_ω into a line perpendicular to $\mathbb{R}^d \times \{0\}$, that projects on a point $x \in \mathbb{R}^d$. Choose $y \in \mathbb{R}$ such that (x,y) is the image by κ_ω of some point $\tilde{z} \in \gamma$ such that $\theta_{\overrightarrow{\omega},w}(\tilde{z})\varphi_\omega(\tilde{z}) \neq 0$. Working as in the other case, we see that $\mathcal{T}_{T_\gamma}^{\omega,\overrightarrow{\omega}}(x,y) = (x,y)$, and thus $p \circ F_{\omega,\overrightarrow{\omega}}(x) = x$ and $T_\gamma = T_{\omega,\overrightarrow{\omega}}(x)$. The same calculation as above implies that $D(x) = E(\gamma) \neq 0$, so that $x \in Q_0$. Finally, it is clear that $\gamma = \gamma(x)$ from the construction of $\gamma(x)$: these two periodic orbits pass through the point $\tilde{z}$. Thus, the map $x \mapsto \gamma(x)$ is surjective, and the proof is over. $\square$

Lemma 3.36. *Under the assumptions of Proposition 3.32, the series*

$$\sum_\gamma \frac{h(T_\gamma)\,T_\gamma^{\#}}{|\det(I - \mathcal{P}_\gamma)|} e^{\int_\gamma V} \tag{3.48}$$

converges absolutely and

$$\operatorname{tr}\left(\int_0^{+\infty} h(t)\mathcal{L}_t \mathrm{d}t\right) = \sum_\gamma \frac{h(T_\gamma)\,T_\gamma^{\#}}{|\det(I - \mathcal{P}_\gamma)|} e^{\int_\gamma V}.$$

Proof. First, use Lemma 3.35 and (the proof) of Lemma 3.34, with the weight V replaced by $\|\mathrm{Re}(V)\|_\infty$ h replaced by $1 + |h|^2$, to get that (this can also be seen using an estimates on the number of periodic orbit for $(\phi_t)_{t\in\mathbb{R}}$ such as [DZ16, Lemma 2.2]):

$$\sum_{\ell\in\mathbb{Z}}\sum_{\gamma}\psi_\ell\left(T_\gamma\right)\exp\left(T_\gamma\left\|\mathrm{Re}\left(V\right)\right\|_\infty\right)\frac{\left|h\left(T_\gamma\right)\right|T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}<+\infty.$$

We can then use the Fubini–Tonelli and monotone convergence theorems to get that

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}}\sum_{\gamma}\psi_\ell\left(T_\gamma\right)&\exp\left(T_\gamma\left\|\mathrm{Re}\left(V\right)\right\|_\infty\right)\frac{\left|h\left(T_\gamma\right)\right|T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}\\ &=\sum_{\gamma}\sum_{\ell\in\mathbb{Z}}\psi_\ell\left(T_\gamma\right)\exp\left(T_\gamma\left\|\mathrm{Re}\left(V\right)\right\|_\infty\right)\frac{\left|h\left(T_\gamma\right)\right|T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}\\ &=\sum_{\gamma}\exp\left(T_\gamma\left\|\mathrm{Re}\left(V\right)\right\|_\infty\right)\frac{\left|h\left(T_\gamma\right)\right|T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}<+\infty.\end{aligned}$$

This proves that the sum (3.48) converges absolutely and provides integrability and domination which allow us to apply Fubini's theorem and the dominated convergence theorem to get

$$\begin{aligned}\mathrm{tr}\left(\int_0^{+\infty}h(t)\mathcal{L}_t\mathrm{d}t\right)&=\sum_{\ell\in\mathbb{Z}}\mathrm{tr}\left(\int_0^{+\infty}\psi_\ell\left(t\right)h(t)\mathcal{L}_t\mathrm{d}t\right)\\ &=\sum_{\ell\in\mathbb{Z}}\sum_{\gamma}\psi_\ell\left(T_\gamma\right)\exp\left(\int_\gamma V\right)\frac{h\left(T_\gamma\right)T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}\\ &=\sum_{\gamma}\sum_{\ell\in\mathbb{Z}}\psi_\ell\left(T_\gamma\right)\exp\left(\int_\gamma V\right)\frac{h\left(T_\gamma\right)T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}\\ &=\sum_{\gamma}\exp\left(\int_\gamma V\right)\frac{h\left(T_\gamma\right)T_\gamma^\#}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}.\end{aligned}$$

□

We end this section with the proof of two merely technical lemmas that will be useful in the following section to construct and study the anisotropic Hilbert spaces from Theorem 3.1.

Lemma 3.37. *For all $u\in\mathcal{D}^{\tilde{v}}\left(M\right)'$, the map $\mathbb{R}\ni t\mapsto\left\|\mathcal{L}_t u\right\|_{\widetilde{\mathcal{H}}}$ is measurable (with the convention that $\left\|u\right\|_{\widetilde{\mathcal{H}}}=\infty$ if $u\notin\widetilde{\mathcal{H}}$).*

Proof. Let us prove first that the map $\mathcal{D}^{\tilde{v}}\left(M\right)'\ni u\mapsto\left\|u\right\|_{\widetilde{\mathcal{H}}}$ is measurable. Since the inclusion of $\widetilde{\mathcal{H}}$ in $\mathcal{D}^{\tilde{v}}\left(M\right)'$ is continuous (hence measurable) and $\left\|\cdot\right\|_{\widetilde{\mathcal{H}}}$ is continuous on $\widetilde{\mathcal{H}}$, we only need to check that $\widetilde{\mathcal{H}}$ is a measurable subset of $\mathcal{D}^{\tilde{v}}\left(M\right)'$. Keeping track of the different steps in the definition of $\widetilde{\mathcal{H}}$, we see that it is enough to prove that L^2_{loc} is a measurable subset of $\left(\mathcal{S}^{\tilde{v}}\right)'$, which is clear with the following characterization of L^2_{loc}:

$$
\begin{aligned}
L^2_{\text{loc}} = \{u \in \left(\mathcal{S}^{\tilde{v}}\right)' : \forall \text{ compact } K \subseteq \mathbb{R}^{d+1}, \exists C > 0,& \\
\forall \varphi \in \mathcal{S}^{\tilde{v}} \text{ supported in } K, |\langle u, \varphi\rangle| \leq C \left\|\varphi\right\|_2\}.&
\end{aligned}
$$

Finally, recall that, if $u \in \mathcal{D}^{\tilde{v}}(M)'$, the map $\mathbb{R} \ni t \mapsto \mathcal{L}_t u \in \mathcal{D}^{\tilde{v}}(M)'$ is measurable (and even $\mathcal{C}^\infty$) according to Lemma 3.7 to end the proof. □

Lemma 3.38. *There is a continuous semi-norm N on $\mathcal{C}^{\infty,\tilde{v}}(M)$ such that for all $u \in \mathcal{C}^{\infty,\tilde{v}}(M)$ and $t \in [-t_0, t_0]$ we have*

$$
\left\|\mathcal{L}_t u\right\|_{\widetilde{\mathcal{H}}} \leq N(u).
$$

The same is true replacing $\mathcal{L}_t$ by $(\mathcal{L}_{-t})^*$ and $\mathcal{C}^{\infty,\tilde{v}}(M)$ by $\mathcal{D}^{\tilde{v}}(M)$.

Proof. Since the inclusion of $\mathcal{C}^{\infty,\tilde{v}}(M)$ in $\widetilde{\mathcal{H}}$ is continuous and $(\mathcal{L}_t)_{t\in\mathbb{R}}$ is a group, we only need to prove that there is $\epsilon > 0$ such that for every continuous semi-norm N_1 on $\mathcal{C}^{\infty,\tilde{v}}(M)$ there is a continuous semi-norm N_2 on $\mathcal{C}^{\infty,\tilde{v}}(M)$ such that for all $u \in \mathcal{C}^{\infty,\tilde{v}}(M)$, and $t \in [-\epsilon, \epsilon]$ we have

$$
N_1(\mathcal{L}_t u) \leq N_2(u). \tag{3.49}
$$

In fact, we only need to achieve (3.49) for N_1 of the form $\left\|\cdot\right\|_{\kappa_\omega,\varphi_\omega,\kappa,\tilde{\nu}}$ for $\omega \in \Omega$ and $\kappa \in \mathbb{R}_+^*$ (because these semi-norms generate the topology of $\mathcal{C}^{\infty,\tilde{v}}(M)$). But then, it becomes clear that (3.49) can be achieved, since the κ_ω are flow boxes. The proof for the adjoint is similar. □

3.5 Global space: second step

In this section, we end the proof of Theorem 3.1. We keep using the notations that we introduced at the beginning of §3.4.

Given the spaces $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_0$ and Proposition 3.32 from the previous section, the proofs of Theorem 3.1, Proposition 3.3, and Proposition 3.4 are now reduced to functional analysis, and we deal with these proofs in this last section. These proofs are split into several lemmas as follow: as far as Theorem 3.1 and Proposition 3.4 are concerned, (i) is contained in Lemma 3.40, (ii) is in Lemma 3.41, (iii) is a consequence of Lemma 3.41 and Lemma 3.8, (iv) is in Lemma 3.43, and (v) is in Lemma 3.44 (with $2t_0$ instead of t_0). We end the section with the proof of Proposition 3.3. First of all, we define the space $\mathcal{H}$.

Definition 3.39. Thanks to Lemma 3.37, we may define for all $u \in \mathcal{D}^{\tilde{v}}(M)'$,

$$
\left\|u\right\|^2_{\mathcal{H}} = \int_0^{t_0} \left\|\mathcal{L}_t u\right\|^2_{\widetilde{\mathcal{H}}} \mathrm{d}t, \tag{3.50}
$$

and then define the space

$$\widehat{\mathcal{H}} = \left\{ u \in \mathcal{D}^{\tilde{v}}(M)' : \|u\|_{\mathcal{H}}^2 < \infty \right\}$$

endowed with the norm $\|\cdot\|_{\mathcal{H}}$. Let $\mathcal{H}$ be the closure of $\mathcal{C}^{\infty,\tilde{v}}(M)$ in $\widehat{\mathcal{H}}$ (for some $\tilde{v} \in \left]v, \frac{1}{1-\alpha}\right[$, where α has been defined in §3.4, we recall in particular that if $v < 2$ then $\alpha < \frac{1}{2}$).

Lemma 3.40. *$\mathcal{H}$ and $\widehat{\mathcal{H}}$ are separable Hilbert spaces. The inclusion of $\mathcal{H}$ and $\widehat{\mathcal{H}}$ in $\mathcal{D}^{\tilde{v}}(M)'$ are continuous, and $\mathcal{C}^{\infty,\tilde{v}}$ is contained in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, and the inclusion is continuous.*

Proof. We only need to prove the lemma for $\widehat{\mathcal{H}}$ (the statements for $\mathcal{H}$ immediately follows). Notice that the map

$$\begin{array}{ccc} \widehat{\mathcal{H}} & \rightarrow & L^2\left([0,t_0], \widetilde{\mathcal{H}}\right) \\ u & \mapsto & (\mathcal{L}_t u)_{0 \leq t \leq t_0} \end{array} \tag{3.51}$$

is an isometry. To show that $\widehat{\mathcal{H}}$ is a separable Hilbert space, we only need to prove that the image of the map (3.51) is closed . Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\widehat{\mathcal{H}}$ such that the sequence $\left((\mathcal{L}_t u_n)_{0\leq t\leq t_0}\right)_{n\in\mathbb{N}}$ converges to $(v(t))_{0\leq t\leq t_0}$ in $L^2\left([0,t_0], \widetilde{\mathcal{H}}\right)$. Then there is a subset A of $\mathbb{N}$ and a Borel subset B of full measure in $[0,t_0]$ such that, for all $t \in B$, the sequence $(\mathcal{L}_t u_n)_{n\in A}$ converges to $v(t)$ in $\widetilde{\mathcal{H}}$ (in particular, it converges in $\mathcal{D}^{\tilde{v}}(M)'$). Choose $t' \in B$ and set $u = \mathcal{L}_{-t'} v(t') \in \mathcal{D}^{\tilde{v}}(M)'$. Then, for all $t \in B$ and $n \in A$, we have

$$\mathcal{L}_t u_n = \mathcal{L}_{t-t'}\left(\mathcal{L}_{t'} u_n\right).$$

Letting n tend to infinity, we have

$$v(t) = \mathcal{L}_{t-t'} v\left(t'\right) = \mathcal{L}_t\left(\mathcal{L}_{-t'} v\left(t'\right)\right) = \mathcal{L}_t u.$$

Since $v \in L^2\left([0,t_0], \widetilde{\mathcal{H}}\right)$, this implies that $u \in \widehat{\mathcal{H}}$, and thus the image of $\widehat{\mathcal{H}}$ under the map (3.51) is closed, so that $\widehat{\mathcal{H}}$ is a Hilbert space.

To prove that the inclusion of $\widehat{\mathcal{H}}$ in $\mathcal{D}^{\tilde{v}}(M)'$ is continuous, just notice that if $\varphi \in \mathcal{C}^{\infty,\tilde{v}}(M)$ then

$$\langle u, \varphi \rangle = \frac{1}{t_0} \int_0^{t_0} \langle \mathcal{L}_t u, (\mathcal{L}_{-t})^* \varphi \rangle \mathrm{d}t,$$

and use Lemma 3.38. The continuous inclusion of $\mathcal{C}^{\infty,\tilde{v}}(M)$ in $\widehat{\mathcal{H}}$ is an immediate consequence of Lemma 3.38. $\square$

We now prove that $\mathcal{H}$ has the property that $\widetilde{\mathcal{H}}_0$ missed: the operator $\mathcal{L}_t$ for $t \geq 0$ is bounded from $\mathcal{H}$ to itself.

Lemma 3.41. *For every $t \geq 0$, the operator $\mathcal{L}_t$ is bounded from $\mathcal{H}$ to itself. Moreover, $(\mathcal{L}_t)_{t\geq 0}$ is a strongly continuous semi-group of operators on $\mathcal{H}$.*

Proof. If $u \in \widetilde{\mathcal{H}}$ and $t \geq t_0$ then we have

$$\|\mathcal{L}_t u\|_{\mathcal{H}}^2 = \int_0^{t_0} \|\mathcal{L}_\tau \mathcal{L}_t u\|_{\widetilde{\mathcal{H}}}^2 \mathrm{d}\tau = \int_0^{t_0} \|\mathcal{L}_t \mathcal{L}_\tau u\|_{\widetilde{\mathcal{H}}}^2 \mathrm{d}\tau \leq \|\mathcal{L}_t\|_{\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}}^2 \|u\|_{\mathcal{H}}^2 .$$

If $0 \leq t \leq t_0$ then we have

$$\begin{aligned}\|\mathcal{L}_t u\|_{\mathcal{H}}^2 &= \int_0^{t} \|\mathcal{L}_{t_0} \mathcal{L}_\tau u\|_{\widetilde{\mathcal{H}}}^2 \mathrm{d}\tau + \int_t^{t_0} \|\mathcal{L}_\tau u\|_{\widetilde{\mathcal{H}}}^2 \mathrm{d}\tau \\ &\leq \left(1 + \|\mathcal{L}_{t_0}\|_{\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}}^2\right) \|u\|_{\mathcal{H}}^2 .\end{aligned}$$

Thus $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}$ to itself, but since $\mathcal{L}_t$ sends $\mathcal{C}^{\infty,\tilde{v}}(M)$ into $\mathcal{C}^{\infty,\tilde{v}}(M)$ (and thus into $\mathcal{H}$), the operator $\mathcal{L}_t$ induces a bounded operator $\mathcal{L}_t : \mathcal{H} \to \mathcal{H}$. Since $(\mathcal{L}_t)_{t\geq 0}$ is locally uniformly bounded and $(\mathcal{L}_t u)_{t \geq 0}$ depends continuously on t as an element of $\mathcal{H}$ when $u \in \mathcal{C}^{\infty,\tilde{v}}(M)$ (see Lemma 3.7), the semi-group $(\mathcal{L}_t)_{t\geq 0}$ is strongly continuous. □

Notice that, according to Lemma 3.8, the generator of the semi-group $(\mathcal{L}_t)_{t\geq 0}$ is $P = X + V$. We prove now a lemma that allows us to go from $\mathcal{H}$ to $\widetilde{\mathcal{H}}_0$ and back, in order to prove that the properties that we stated for $\widetilde{\mathcal{H}}_0$ in Proposition 3.32 may be extended to $\mathcal{H}$.

Lemma 3.42. *For all $t \geq t_0$, the operator $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$. If $z \in \mathbb{C}$ is such that* $\mathrm{Re}(z) \gg 1$ *then $(z-P)^{-1}$ is bounded from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.*

Proof. Let $u \in \widetilde{\mathcal{H}}$ then

$$\|\mathcal{L}_t u\|_{\mathcal{H}}^2 \leq \sup_{\tau \in [t, t+t_0]} \|\mathcal{L}_\tau\|_{\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}}^2 \|u\|_{\widetilde{\mathcal{H}}}^2 .$$

Thus $\mathcal{L}_t$ is bounded from $\widetilde{\mathcal{H}}$ to $\widehat{\mathcal{H}}$. Since it sends $\mathcal{C}^{\infty,\tilde{v}}(M)$ into itself, $\mathcal{L}_t$ sends $\widetilde{\mathcal{H}}_0$ into $\mathcal{H}$. Now, recall [Kat66, Problem 1.15 p.487] that if $\mathrm{Re}(z) \gg 1$ and $u \in \mathcal{H}$ then

$$(z-P)^{-1} u = \int_0^{+\infty} e^{-zt} \mathcal{L}_t u \mathrm{d}t.$$

But recall that the norm of u in $\mathcal{H}$ is the norm of $(\mathcal{L}_t u)_{0 \leq t \leq t_0}$ in the space $L^2\left([0,t_0], \widetilde{\mathcal{H}}\right)$. Thus, for all $n \in \mathbb{N}$, the norm of $(\mathcal{L}_t u)_{nt_0 \leq t \leq (n+1)t_0}$ in the space $L^2\left([0,t_0], \widetilde{\mathcal{H}}\right)$ is smaller than $\|\mathcal{L}_{t_0}\|_{\mathcal{H}\to\mathcal{H}}^n \|u\|_{\mathcal{H}}$. Thus if $\mathrm{Re}(z) > \ln(\|\mathcal{L}_{t_0}\|_{\mathcal{H}\to\mathcal{H}})$, then, by Cauchy–Schwarz inequality, there is a constant $C > 0$ such that the L^1 norm of $t \mapsto e^{-zt}\mathcal{L}_t u$ is smaller than $C\|u\|_{\mathcal{H}\to\mathcal{H}}$. Thus $(z-P)^{-1}$ is bounded from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. □

We are now ready to prove that the spectrum of $P = X + V$ acting on $\mathcal{H}$ is discrete.

Lemma 3.43. *The spectrum of P acting on $\mathcal{H}$ is made of isolated eigenvalues of finite multiplicity and it coincides with the Ruelle spectrum of P (multiplicity taken into account).*

Proof. According to Lemmas B.3 and 3.41, it is enough to prove that the spectrum of P acting on $\mathcal{H}$ is made of isolated eigenvalues of finite multiplicity (recall Definition 7 of the Ruelle spectrum). Let $z \in \mathbb{C}$ be such that $\operatorname{Re} z \gg 1$. Since P is the generator of a strongly continuous semi-group, z belongs to the resolvent set of P. From [Kat66, Problem 6.16 p.177], we see that we only need to prove that the essential spectral radius of $(z-P)^{-1}$ is zero (see Definition 2).

To do so, let $\chi : \mathbb{R}_+^* \to [0,1]$ be a compactly supported $\mathcal{C}^\infty$ function such that $\chi(t) = 1$ if $t \leq 2t_0$. Then, according to [Kat66, Problem 1.15 p.487], for all $n \geq 1$ we have, provided that $\operatorname{Re} z \gg 1$:

$$\begin{aligned}(z-P)^{-n} &= \frac{1}{(n-1)!}\int_0^{+\infty} e^{-zt}t^{n-1}\mathcal{L}_t \mathrm{d}t \\ &= \frac{1}{(n-1)!}\int_0^{+\infty} \chi(t)e^{-zt}t^{n-1}\mathcal{L}_t \mathrm{d}t + \frac{1}{(n-1)!}\int_0^{+\infty} h_n(t)\mathcal{L}_t\mathrm{d}t,\end{aligned} \tag{3.52}$$

where the function $h_n : \mathbb{R}_+^* \to \mathbb{C}$ is defined by $h_n(t) = (1-\chi(t))\, e^{-zt}t^{n-1}$. Set also $\tilde{h}_n(t) = z h_n(t+t_0) + h_n'(t+t_0)$, so that for all $t \in \mathbb{R}_+^*$ we have

$$h_n(t+t_0) = e^{-zt}\int_0^t e^{z\tau}\tilde{h}_n(\tau)\mathrm{d}\tau.$$

Then, notice that

$$\begin{aligned}\mathcal{L}_{t_0} \circ \int_0^{+\infty} \tilde{h}_n(\tau)\mathcal{L}_\tau \mathrm{d}\tau \circ (z-P)^{-1} &= \int_0^{+\infty}\int_0^{+\infty} e^{-zt}\tilde{h}_n(\tau)\mathcal{L}_{t_0+t+\tau}\mathrm{d}t\mathrm{d}\tau \\ &= \int_0^{+\infty}\left(\int_\tau^{+\infty} e^{-zu}e^{z\tau}\tilde{h}_n(\tau)\mathcal{L}_{t_0+u}\mathrm{d}u\right)\mathrm{d}\tau \\ &= \int_0^{+\infty} e^{-zu}\left(\int_0^u e^{z\tau}\tilde{h}_n(\tau)\mathrm{d}\tau\right)\mathcal{L}_{t_0+u}\mathrm{d}u \\ &= \int_0^{+\infty} h_n(t_0+u)\mathcal{L}_{t_0+u}\mathrm{d}u \\ &= \int_0^{+\infty} h_n(t)\mathcal{L}_t\mathrm{d}t.\end{aligned} \tag{3.53}$$

Moreover, if $\operatorname{Re} z \gg 1$, then, for every $n \geq 1$, the function $\tilde{h}_n$ satisfies the assumptions of Proposition 3.32 and consequently the operator

$$\int_0^{+\infty} \tilde{h}_n(t)\mathrm{d}t : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$$

is compact. It follows then from (3.53) and Lemma 3.42 that the operator

$$\frac{1}{(n-1)!}\int_0^{+\infty} h_n(t)\mathcal{L}_t\mathrm{d}t : \mathcal{H} \to \mathcal{H}$$

is compact. On the other hand, we see that the operator norm of

$$\frac{1}{(n-1)!}\int_0^{+\infty}\chi(t)e^{-zt}t^{n-1}\mathcal{L}_t\mathrm{d}t : \mathcal{H}\to\mathcal{H}$$

is less than $\frac{C(2t_0)^n}{(n-1)!}$ for some constant $C>0$. With (3.52), it follows then from Hennion's argument [Hen93] based on Nussbaum formula [Nus70] that the essential spectral radius of $(z-P)^{-1}$ on $\mathcal{H}$ is less than

$$\liminf_{n\to+\infty}\left(\frac{C(2t_0)^n}{(n-1)!}\right)^{\frac{1}{n}}=0.$$

□

We can now give the proof of the most interesting property of the Hilbert space $\widetilde{\mathcal{H}}$.

Lemma 3.44. *Let h be a $\mathcal{C}^\infty$ function supported on a compact subset of $[2t_0,+\infty[$. Then the operator*

$$\int_0^{+\infty}h(t)\,\mathcal{L}_t\mathrm{d}t : \mathcal{H}\to\mathcal{H} \tag{3.54}$$

is compact. Its non-zero spectrum is the intersection of $\mathbb{C}\setminus\{0\}$ with the image of the spectrum of P by $\lambda\mapsto \mathrm{Lap}(h)(-\lambda)$, where Lap *denotes the Laplace transform.*

If $\upsilon<2$, the operator (3.54) *is trace class and*

$$\mathrm{tr}\left(\int_0^{+\infty}h(t)\,\mathcal{L}_t\mathrm{d}t\right)=\sum_\gamma\frac{h(T_\gamma)\,T_\gamma^{\#}}{|\det(I-\mathcal{P}_\gamma)|}\exp\left(\int_\gamma V\right).$$

Proof. As in the proof of Lemma 3.43, define the function $\tilde{h}$ on $\mathbb{R}_+^*$ by

$$\widetilde{h}(t)=zh(t+t_0)+h'(t+t_0).$$

Since $\tilde{h}$ is $\mathcal{C}^\infty$ and compactly supported in $[t_0,+\infty[$, it satisfies the assumption of Proposition 3.32 and, working as in the proof of Lemma 3.43, we see that the operator

$$\int_0^{+\infty}h(t)\,\mathcal{L}_t\mathrm{d}t=\mathcal{L}_{t_0}\circ\int_{t_0}^{+\infty}\widetilde{h}(t)\,\mathcal{L}_t\mathrm{d}t\circ(z-P)^{-1} : \mathcal{H}\to\mathcal{H}$$

is compact.

In order to identify the non-zero spectrum of (3.54), we denote by f the function defined by $f(z)=\mathrm{Lap}(h)(-z)$ and by A the operator (3.54). If $\lambda\in\mathbb{C}$ denotes by E_λ the generalized eigenspace of P associated to λ and, if $\lambda\neq 0$, by F_λ the generalized eigenspace of A associated to λ. We want to prove that for all $\mu \in \mathbb{C} \setminus \{0\}$ we have

$$F_\mu = \bigoplus_{\substack{\lambda \in \mathbb{C} \\ f(\lambda)=\mu}} E_\lambda,$$

which is a more precise statement that our claim on the eigenvalues of A. Let $\lambda \in \sigma(X)$ be such that $f(\lambda) \neq 0$. Since P commutes with $\mathcal{L}_t$ for $t \geq 0$, it commutes with A so that E_λ is stable by A. We denote by $\widetilde{P}$ and $\widetilde{A}$ the endomorphisms of E_λ induced respectively by P and A. Since E_λ is finite-dimensional (according to Lemma 3.43), the operator $\widetilde{P}$ is bounded, and we may define for $t \geq 0$ the operator $e^{t\widetilde{P}}$ on E_λ. Then, $e^{t\widetilde{P}}$ is nothing else than the operator induced by $\mathcal{L}_t$ on E_λ (they solve the same Cauchy problem). It follows that we have

$$\widetilde{A} = \int_0^{+\infty} h(t) e^{t\widetilde{P}} \mathrm{d}t = f\left(\widetilde{P}\right), \tag{3.55}$$

where $f(\widetilde{P})$ is meant in the sense of the holomorphic calculus of bounded operators (we may develop $e^{t\widetilde{P}}$ in power series). Since $\sigma(\widetilde{P}) = \{\lambda\}$ by definition of E_λ, it follows that $\sigma(\widetilde{A}) = \{f(\lambda)\}$, which gives

$$E_\lambda \subseteq F_{f(\lambda)}.$$

Reciprocally, let $\mu \in \sigma(A) \setminus \{0\}$. From the equality

$$PA = -\int_0^{+\infty} h'(t) \mathcal{L}_t \mathrm{d}t,$$

we see that the range of A is included in the domain of X. In particular, F_μ is contained in the domain of P and thus P induces a bounded operator on the finite dimensional space F_μ. Applying as above the holomorphic calculus of bounded operators, we get that

$$F_\mu \subseteq \bigoplus_{\substack{\lambda \in \mathbb{C} \\ f(\lambda)=\mu}} E_\lambda,$$

and (3.55) is proven.

If $\upsilon < 2$, we may replace "compact" by "trace class" in the argument above. Then, using Lemma B.1, we see that the operator

$$\int_0^{\infty} h(t) \mathcal{L}_t \mathrm{d}t$$

has the same non-zero spectrum when acting on $\mathcal{H}$ or on $\widetilde{\mathcal{H}}_0$ and thus, by Lidskii's trace theorem, the same trace. This ends the proof with Proposition 3.32. $\square$

Remark 3.45. As pointed out after the statement of Theorem 3.1, the point (v) of Theorem 3.1 proves trace formula (TFF) which was stated as an equality between distributions on $\mathbb{R}_+^*$. However, it is clear from the proof that the equality in fact holds in the dual of the space of compactly supported $\mathcal{C}^{d+2}$ functions on $\mathbb{R}_+^*$ whose $d+2$th derivative has bounded variation. In fact, using the same trick as in the proof of Proposition 3.3, we see that trace formula holds in the dual of the space of compactly supported $\mathcal{C}^{d+1}$ functions on $\mathbb{R}_+^*$ whose $d+1$th derivative has bounded variations.

We end this section with the proof of Proposition 3.3.

Proof of Proposition 3.3. First of all, we need to prove that, when $\operatorname{Re}(z) \gg 1$, the essential spectral radius (see Definition 2) of the operator

$$\mathcal{L}_{t_0}(z-P)^{-1} = \int_{t_0}^{+\infty} e^{-z(t-t_0)}\mathcal{L}_t \mathrm{d}t \tag{3.56}$$

acting on $\mathcal{H}$ is zero. From the proof of Lemma 3.43, we know that the essential spectral radius of $(z-P)^{-1}$ is zero. Then if $r > 0$ is such that $(z-P)^{-1}$ has no eigenvalue of modulus r we may define the spectral projection

$$\Pi_r = \frac{1}{2i\pi}\int_{\partial\mathbb{D}(0,r)} \left(w-(z-P)^{-1}\right)^{-1} \mathrm{d}w.$$

Then $I - \Pi_r$ has finite rank and the spectral radius of $(z-P)^{-1}\Pi_r$ is less than r. Since $\mathcal{L}_{t_0}$ commutes with $(z-P)^{-1}$, it also commutes with Π_r and thus the spectral radius of $\mathcal{L}_{t_0}(z-P)^{-1}\Pi_r$ is less than $\|\mathcal{L}_{t_0}\| r$. Then writing

$$\mathcal{L}_{t_0}(z-P)^{-1} = \mathcal{L}_{t_0}(z-P)^{-1}\Pi_r + \mathcal{L}_{t_0}(z-P)^{-1}(1-\Pi_r) \tag{3.57}$$

and using Hennion's argument [Hen93] as in the proof of Lemma 3.43 (notice that the second term of the right-hand side of (3.57) has finite rank), we see that the essential spectral radius of $\mathcal{L}_{t_0}(z-P)^{-1}$ is less than $\|\mathcal{L}_{t_0}\| r$. Since $r > 0$ may be chosen arbitrarily small, the essential spectral radius of $\mathcal{L}_{t_0}(z-P)^{-1}$ is zero. Consequently, using functional calculus in finite dimension as in the proof of Lemma 3.44, we may prove that the spectrum of $\mathcal{L}_{t_0}(z-P)^{-1}$ is made of the $\frac{e^{t_0\lambda}}{z-\lambda}$ where λ runs over the Ruelle spectrum of P.

On the other hand, according to Proposition 3.32 (with h the characteristic function of $[t_0,+\infty[$ and $k=0$), the right-hand side of (3.56) defines an operator on $\widetilde{\mathcal{H}}_0$ which is in the Schatten class S_p for any $p > d+1$ (in particular it is compact and has essential spectral radius zero). We may use Lemma B.1 to get that the spectrum of this operator is the same as the spectrum of the operator (3.56) acting on $\mathcal{H}$, that we just described. Consequently, for all $p > d+1$, since the operator acting on $\widetilde{\mathcal{H}}_0$ is in the Schatten class

S_p, its spectrum is in ℓ^p (see [GGK00, Corollary 3.4 p.54]), so that

$$\sum_{\lambda \in \sigma_{\mathrm{R}}(P)} \left| \frac{e^{\lambda t_0}}{z - \lambda} \right|^p < +\infty.$$

Since $t_0 > 0$ and $p > d + 1$ are arbitrary, Proposition 3.3 follows. $\square$

Chapter 4

Finite order of the dynamical determinant for Gevrey Anosov flows

This last chapter is dedicated to the proof of Theorem 10. In §4.1, we gather some facts about Gevrey classes and analytic FBI transforms. This is basically an abridged version of Chapters 1 and 2 from [BJ20]. In order to keep the length of this thesis reasonable, most of the results in §4.1 will be stated without proof. However, we will always give precise references to [BJ20] where the interested reader can find all the required proofs or references. The section 4.2 is dedicated to the detailed proof of Theorem 10 using the tools introduced in §4.1. Finally, in §4.3, we expose some other applications of FBI transforms method to the study of Anosov flows. For the same reasons as above, we will refer to [BJ20] for proofs. The content of §4.2 and 4.3 correspond to the third chapter of [BJ20].

Mind that in this chapter we will use, as it is common in the semi-classical analysis literature, an implicit parameter $h > 0$, thought of as tending to 0. Whenever needed, we will assume that h is small enough.

4.1 Crash course: Analytic FBI transform and Gevrey differential operators

In this first section, we recall the results from the first two chapters of [BJ20] that are needed for the proof of Theorem 10.

In §4.1.1, we recall some basic facts about Gevrey and analytic classes of regularity. These results may be found in §1.1 in [BJ20]. We will not need the other results from the first chapter of [BJ20] since we will not discuss Gevrey pseudo-differential operators here and because we will admit all the required results from the second chapter of [BJ20] – such as the existence of a global analytic FBI transform on a compact manifold.

In §4.1.2, we define a global analytic FBI transform on a compact analytic manifold. Admitting that such an object exists, we recall then its basic properties. We also investigate the notion of *complex Lagrangian deformation* which is central in [BJ20].

In §4.1.3, we explain how a global analytic FBI transform may be used to study differential operators with Gevrey coefficients, stating the results that will be fundamental in the proof of Theorem 10.

Remark 4.1. The FBI transform methods that we expose here are extensions of the work of Helffer and Sjöstrand [HS86, Sjö96] in the analytic category. These methods had already been investigated in the Gevrey category by Lascar and Lascar [LL97]. However, all these versions of this method contain limitations that we had to overcome in order to prove Theorem 10. The main problem was that they only consider compactly supported Lagrangian deformations (which amounts to take G compactly supported in Definition 4.14 below). Consequently, an important part of [BJ20] is dedicated to the extension of the methods of Helffer and Sjöstrand and to the investigation of certain micro-local techniques in the Gevrey category. For more details, see the introduction of [BJ20].

As we were working on this project, Galkowski and Zworski were elaborating a very similar extension of the work of Helffer and Sjöstrand [GZ19a, GZ19b]. However, their work is restricted to the real-analytic case and they only work on tori.

4.1.1 Gevrey functions and ultradistributions on manifolds

In §2.1, we introduced the notion of Gevrey regularity – see in particular Example 2.1. Let us now explore a bit further Gevrey classes.

Definitions

Let $s \geq 1$ be fixed. Let U be an open subset of $\mathbb{R}^n$. We recall from Example 2.1 that a function $f : U \to \mathbb{C}$ is said to be s-Gevrey (or, for short, $\mathcal{G}^s$) if f is $\mathcal{C}^\infty$ and if, for every compact subset K of U, there are constants $C, R > 0$ such that, for all $\alpha \in \mathbb{N}^n$ and $x \in K$, we have

$$|\partial^\alpha f(x)| \leq C R^{|\alpha|} \alpha!^s. \tag{4.1}$$

The constant R in (4.1) may be interpreted as the inverse of a (Gevrey) radius of convergence. Recall that when $s = 1$, this describes the class of real-analytic function on U. When $s > 1$, the class of $\mathcal{G}^s$ functions is non-quasi-analytic according to Theorem 2.3: it contains compactly supported functions. We will denote by $\mathcal{G}^s(U)$ the space of $\mathcal{G}^s$ functions on U and by $\mathcal{G}^s_c(U)$ the space of compactly supported elements of $\mathcal{G}^s(U)$. The non-quasi-analyticity of $\mathcal{G}^s$ when $s > 1$ implies in particular that there are $\mathcal{G}^s$ partitions of unity.

The definition above extends immediately to the case of Banach valued functions. A function f from U to some Banach space $\mathcal{B}$ is said to be $\mathcal{G}^s$ if (4.1) holds with the modulus replaced by the norm of $\mathcal{B}$. With this definition, the class of $\mathcal{G}^s$ functions is closed under composition, as was proved by Gevrey in his original paper [Gev18]. In fact, the class of $\mathcal{G}^s$ functions is very well-behaved and, when $s > 1$, quite flexible: for instance, it is Cartesian closed, closed under differentiation, solving ODEs, Implicit Function Theorem, etc, and there are versions of Borel's and Whitney's theorem for $\mathcal{G}^s$ functions (see [KMR09] and references therein for details).

Since the class $\mathcal{G}^s$ is stable by composition and inversion, we have a natural definition of a $\mathcal{G}^s$ structure on a manifold: a $\mathcal{G}^s$ manifold is a Hausdorff topological space with countable basis endowed with a maximal $\mathcal{G}^s$ atlas. Here, a $\mathcal{G}^s$ atlas is defined to be an atlas with $\mathcal{G}^s$ change of charts (notice that we retrieve the usual notion of real-analytic manifold when $s = 1$). As usual, if M and N are two $\mathcal{G}^s$ manifolds, then a map $f : M \to N$ is said to be $\mathcal{G}^s$ if it is $\mathcal{G}^s$ "in charts". Since the class $\mathcal{G}^s$ satisfies the Implicit Function Theorem, most elementary results from differential geometry are easily checked to be true in the $\mathcal{G}^s$ category. In particular, there is a well-defined notion of $\mathcal{G}^s$ (vector) bundle, and each usual bundle associated with M (tangent, cotangent, etc) admits a natural $\mathcal{G}^s$ structure. As a consequence, it makes sense to say that a vector field over M is $\mathcal{G}^s$.

Remark 4.2. Of course, a real-analytic manifold has a natural structure of $\mathcal{G}^s$ manifold, since real-analytic maps are $\mathcal{G}^s$. As pointed out in [LL97], all $\mathcal{G}^s$ manifolds may be described in this way. Indeed, there is a Gevrey version of the famous Whitney's Embedding Theorem [Whi36] : *every $\mathcal{G}^s$ compact manifold is $\mathcal{G}^s$-diffeomorphic to a real-analytic submanifold of an Euclidean space.* The adaptation of the proof of Whitney's Theorem to our setting is straightforward. Since the Inverse Function Theorem holds in the $\mathcal{G}^s$ category, it suffices to follow the lines of the proof of [Hir94, Theorem 7.1], replacing $\mathcal{C}^\infty$ by $\mathcal{G}^s$ at every step. This gives an analytic structure on a compact $\mathcal{G}^s$ manifold, compatible with the $\mathcal{G}^s$ structure.

We want now to define ultradistributions on a $\mathcal{G}^s$ manifold M. To do so, we need to give a structure of topological vector space to the space $\mathcal{G}^s_c(M)$ of compactly supported $\mathcal{G}^s$ functions on M. If (U, κ) is a $\mathcal{G}^s$ chart for M, and K a relatively compact subset of U, then we define for every $R > 0$ and function f, infinitely differentiable on a neighbourhood of K, the semi-norm (n denotes the dimension of M)

$$\|f\|_{s,R,K} \coloneqq \sup_{\substack{x\in\kappa(K)\\ \alpha\in\mathbb{N}^n}} \frac{\left|\partial^\alpha \left(f\circ\kappa^{-1}\right)(x)\right|}{R^{|\alpha|}\alpha!^s}. \tag{4.2}$$

We extend this definition to any relatively compact subset K of M by covering K by a finite number $K_1, \dots, K_N$ of compact sets included in some domains of charts and setting

$$\|f\|_{s,R,K} \coloneqq \sum_{j=1}^N \|f\|_{s,R,K_j}.$$

Then, if K is a relatively compact subset of M and $R > 0$, we define $E^{s,R}(K)$ to be the Banach space of functions $f \in \mathcal{C}^\infty(M)$, supported in K, such that $\|f\|_{s,R,K} < +\infty$, endowed with the norm $\|\cdot\|_{s,R,K}$. For $s = 1$, consider this definition as temporary, as we will give a more practical but equivalent scale of spaces of real analytic functions below.

Notice that these spaces heavily depend on the choices of charts that we made above. It is, however, not the case of the inductive limit, when U is an open subset of M,

$$\mathcal{G}_c^s(U) := \varinjlim_{K \Subset U} \varinjlim_{R>0} E^{s,R}(K).$$

Here, the first limit is taken over compact subsets K of U, and the inductive limit is taken in the category of locally convex topological vector spaces. Notice that the underlying set of this limit is indeed the set of compactly supported $\mathcal{G}^s$ functions on U. In particular, when $s = 1$, if U is not compact itself then $\mathcal{G}_c^s(U)$ is trivial. In the case that M itself is compact, we will write $\mathcal{G}^s(M)$ for $\mathcal{G}_c^s(M)$.

We define the space $\mathcal{U}^s(U)$ of ultradistributions on U to be the strong dual of $\mathcal{G}_c^s(U)$. Then, by [Gro50], we see that $\mathcal{U}^s(U)$ identifies with the projective limit

$$\mathcal{U}^s(U) = \varprojlim_{K \Subset U} \varprojlim_{R>0} \left(E^{s,R}(K)\right)'.$$

In particular, $\mathcal{U}^s(U)$ is a Fréchet space. It is a Roumieu-type space of ultradistributions. More details on Roumieu and Beurling spaces, can be found in [BBMT91]. The support of an ultradistribution is defined as in the case of usual distributions, and the spaces of compactly supported distributions on U is denoted by $\mathcal{U}_c^s(U)$. The topology of $\mathcal{U}_c^s(U)$ only matters for Theorem 2.27, and we can take for instance the weak-star topology. An element of $\mathcal{U}^1(M)$ or more generally of $\left(E^{1,R}(M)\right)'$ for some large $R \geq 1$ will be referred to as a hyperfunction.

Remark 4.3. The fact that the limit (4.1.1) does not depend on the choices of charts in the definition of the norms $\|\cdot\|_{s,R,K}$ follows from the stability by composition of the class $\mathcal{G}^s$. Indeed, if we denote by $(\|\cdot\|'_{s,R,K})_{R>0}$ the family of norms that is obtained with another choice of charts, one sees (applying Faa di Bruno's formula or adapting the original proof of Gevrey [Gev18]) that there is a constant $C > 0$ such that for every $R > 0$ we have

$$\|\cdot\|_{s,C\max(1,R),K} \leq C\,\|\cdot\|'_{s,R,K} \text{ and } \|\cdot\|'_{s,C\max(1,R),K} \leq C\,\|\cdot\|_{s,R,K}. \tag{4.3}$$

In particular, when the manifold M is an open subset of $\mathbb{R}^n$, we can (and will) choose the chart κ in the definition (4.2) to be the inclusion in $\mathbb{R}^n$.

Remark 4.4. Let $s \geq 1$. In order to discuss perturbations of $\mathcal{G}^s$ Anosov flows in §4.3, we need to define a topology on the space of $\mathcal{G}^s$ sections of a $\mathcal{G}^s$ vector bundle. Let $p : F \to M$ denote a real $\mathcal{G}^s$ vector bundle on M (the case of complex vector bundle is similar). Let

(U,κ) be a $\mathcal{G}^s$ chart for M and $(\kappa,\Psi) : p^{-1}(U) \to \kappa(U) \times \mathbb{R}^d$ be a trivialization for $p : F \to M$. Then, if K is a relatively compact subset of U and $R > 0$, we define for every $\mathcal{C}^\infty$ section f of F the semi-norm

$$\|f\|_{s,R,K} = \sup_{\substack{x\in\kappa(K)\\ \alpha\in\mathbb{N}^n}} \frac{\left\|\partial^\alpha \Psi\left(f\circ\kappa^{-1}\right)(x)\right\|}{R^{|\alpha|}\alpha!^s},$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^d$. Using this semi-norm to replace (4.2), the case of sections of the vector bundle F is dealt with as the case of the trivial line bundle over M. In particular, when M is a compact manifold, we have a definition the spaces $E^{s,R}(M;F)$, for $R > 0$, and a topology on the space $\mathcal{G}^s(M;F)$ of $\mathcal{G}^s$ sections of F.

The particular case of real-analytic functions

We want now to rewrite the definitions of the previous paragraph in a way that may be more intuitive, in the case $s = 1$. Indeed, it may seem more natural to describe real analytic functions as restrictions of holomorphic functions. If K is a compact subset of $\mathbb{R}^n$, then we see that a smooth function f defined on a neighbourhood of K such that $\|f\|_{1,R,K} < +\infty$ admits a holomorphic extension to a complex neighbourhood of K of size $(CR)^{-1}$ (for some $C > 0$ that does not depend on R), the L^∞ norm of this extension being bounded by $C\|f\|_{1,R,K}$. Reciprocally, if f admits a bounded holomorphic extension to a complex neighbourhood of size CR^{-1}, then $\|f\|_{1,R,K}$ is finite and controlled by the L^∞ norm of the extension (independently on R). We will now explain how this remark generalizes to the case of compact manifolds.

Let then M be a compact real-analytic manifold of dimension n. By a result of Bruhat and Whitney [WB59], the manifold M admits a complexification $\widetilde{M}$. That is, $\widetilde{M}$ is a holomorphic manifold of complex dimension n endowed with a real-analytic embedding $M \subseteq \widetilde{M}$, such that M is a totally real submanifold of $\widetilde{M}$. This means that at each $p \in M$, we have $T_pM \cap iT_pM = \{0\}$. It follows then that if N is a complex manifold and $f : M \to N$ is a real-analytic map, then f extends to a holomorphic map from a neighbourhood of M in $\widetilde{M}$ to N. In particular, if $\widetilde{M}'$ is another complexification for M then the identity of M extends to a biholomorphism between a neighbourhood of M in $\widetilde{M}$ and a neighbourhood of M in $\widetilde{M}'$.

Remark 4.5. If $\widetilde{M}$ is a complexification for M, let $B(\widetilde{M})$ denote the Banach space of bounded holomorphic functions on $\widetilde{M}$. Then, we may give a new definition of the space of real-analytic functions on M by

$$\mathcal{G}^1(M) = \varinjlim_{\widetilde{M}} B\big(\widetilde{M}\big).$$

Here, the inductive limit (in the category of locally convex topological vector spaces) is over all the complexifications[1] $\widetilde{M}$ of M. This coincides with (4.1.1) when $s = 1$. This definition may be quite appealing because it is very intrinsic. However, we will rather use a more concrete description of $\mathcal{G}^1(M)$, that boils down to choosing a basis of complex neighbourhoods for M.

We will use particular complexifications of M called Grauert tubes. The notion of Grauert tube first appeared in [Gra58], but we will rely on the exposition from [GS91]. First, according to [Mor58], there is a real-analytic embedding of M into an Euclidean space. Hence, we may choose a real-analytic Riemannian metric g on M. According to [WB59], there exists a complexification $\widetilde{M}$ of M endowed with an anti-holomorphic involution $z \mapsto \bar{z}$ such that M is the set of fixed point of $z \mapsto \bar{z}$. Then, since the square of the distance induced by g on M is real-analytic near the diagonal, it extends to a holomorphic function on a neighbourhood of the diagonal of $M \times M$ in $\widetilde{M} \times \widetilde{M}$. Following [GS91], we define ρ on $\widetilde{M}$ (up to taking $\widetilde{M}$ smaller) by

$$\rho(z) = -\frac{1}{4} d(z, \bar{z})^2.$$

From [GS91], we know that ρ defines a strictly plurisubharmonic function on $\widetilde{M}$ such that $M = \{z \in \widetilde{M} : \rho(z) = 0\}$. Then, if $\epsilon > 0$ is small, we define the Grauert tube $(M)_\epsilon$ as the sublevel set of ρ:

$$(M)_\epsilon := \left\{ z \in \widetilde{M} : \rho(z) < \epsilon^2 \right\}. \tag{4.4}$$

Notice that, since ρ is strictly plurisubharmonic, the Grauert tube $(M)_\epsilon$ is strictly pseudo-convex. Moreover, the real $(1,1)$-form $i\partial\bar{\partial}\rho$ is Kähler and the associated hermitian form coincides with g on M. We shall consequently still denote this hermitian form by g.

Using the notion of Grauert tube, we can replace the spaces $E^{1,R}(M)$ defined above with a more convenient scale. If $R > 0$ is large enough, denote by $E^{1,R}(M)$ the space of bounded holomorphic functions on $(M)_{1/R}$ (endowed with the sup norm). The spaces $E^{1,R}(M)$ defined in this way do not need to coincide with the previous version, but they give rise to the same inductive limit (in the category of locally convex topological vector spaces):

$$\mathcal{G}^1(M) = \varinjlim_{R \to +\infty} E^{1,R}(M).$$

Hence, we will always assume that we use this definition of the spaces $E^{1,R}(M)$ in the case $s = 1$. To see that we get the same topology on $\mathcal{G}^1(M)$ as previously, work as in Remark 4.3 (using Cauchy's and Taylor's formulae to prove an estimate similar to (4.3)).

[1] We get rid of the set-theoretic complications here in the usual way: the cardinal of a complexification for M is at most $2^{\aleph_0}$.

Let us point out that there is another way to describe Grauert tubes. Indeed, since the Riemannian metric g is real-analytic, so is its exponential map. Consequently, if x is a point in M, the exponential map $\exp_x : T_x M \to M$ extends to a holomorphic map, still denoted by $\exp_x$, from a neighbourhood of 0 in $T_x M \otimes \mathbb{C}$ to $\widetilde{M}$. Then, the map

$$(x, v) \mapsto \exp_x(iv) \tag{4.5}$$

defines a real analytic diffeomorphism between a neighbourhood of the zero section in TM and a neighbourhood of M in $\widetilde{M}$. For $\epsilon > 0$ small enough, under the map in (4.5), the Grauert tube $(M)_\epsilon$ is the image of

$$\left\{(x, v) \in TM : g(v, v) < \epsilon^2\right\}.$$

With this description of the Grauert tube $(M)_\epsilon$, we see that the projection $TM \to M$ induces a real-analytic projection from $(M)_\epsilon$ to M, that we shall denote by Re. We define also the function $|\mathrm{Im}| : (M)_\epsilon \to \mathbb{R}_+$ as the square root of ρ. We will sometimes, slightly abusively, write $|\mathrm{Im}\, z|$ instead of $|\mathrm{Im}|\,(z)$.

Since M is compact, we could have chosen any decreasing basis of neighbourhoods for M in $\widetilde{M}$ to define the spaces $E^{1,R}(M)$. However, we will need to consider real-analytic functions defined on T^*M (for instance symbols) or more generally on products of the type $M^{N_1} \times (T^*M)^{N_2}$. Since these manifolds are non-compact, the choice of a complex neighbourhood for T^*M becomes non-trivial. Since we want to consider symbols on T^*M, it seems natural to introduce the Grauert tubes for the Kohn–Nirenberg metric g_{KN}. Recall that g_{KN} is defined in the following way. The Levi–Civita connexion associated with g gives a splitting $TT^*M = V \oplus H$ into vertical and horizontal bundles, where both subbundles are identified with TM, so that we can define

$$g_{KN}(x, \xi) := g_H(x, \xi) + \frac{1}{1 + |\xi|_x^2} g_V(x, \xi).$$

In charts, it is uniformly equivalent to its flat version

$$g_{KN}^{\mathrm{flat}} = \mathrm{d}x^2 + \frac{1}{1 + |\xi|^2} \mathrm{d}\xi^2.$$

The curvature of g_{KN} is bounded, and so are all its covariant derivatives, and one can check that it admits Grauert tubes, that roughly looks like cones at infinity.

More precisely, the cotangent space $T^*\widetilde{M}$ of $\widetilde{M}$ is a complexification of T^*M. Notice that there is a natural inclusion of $T^*M \otimes \mathbb{C}$ into $T^*\widetilde{M}$ and that the anti-holomorphic involution that fixes T^*M is given on $T^*M \otimes \mathbb{C}$ by $(x, \xi) \mapsto (x, \bar{\xi})$. As above, we find a strictly plurisubharmonic function ρ_{KN} defined on a neighbourhood of T^*M in $T^*\widetilde{M}$.

Then, mimicking (4.4), we set for $\epsilon > 0$ small enough

$$(T^*M)_\epsilon := \left\{ \alpha \in T^*\widetilde{M} : \rho_{KN}(\alpha) < \epsilon^2 \right\}.$$

As in the compact case, $|\operatorname{Im}\alpha|$ will denote the square root of $\rho_{KN}(\alpha)$. To describe these tubes in more concrete terms, we may examine them in local coordinates. Given a real-analytic chart for M, it extends holomorphically to a chart for $\widetilde{M}$. It hence defines a holomorphic trivialization for $T^*\widetilde{M}$, mapping T^*M on $T^*\mathbb{R}^n$. If we denote these coordinates by $\tilde{x} = x + iy, \tilde{\xi} = \xi + i\eta$, then for a point $\alpha \in T^*\widetilde{M}$ that writes $(\tilde{x}, \tilde{\xi})$ in local coordinates, the quantity $|\operatorname{Im}\alpha|$ is uniformly equivalent to

$$|\operatorname{Im}\alpha| \asymp |y| + \frac{|\eta|}{\langle \xi \rangle}.$$

This gives a rough but tractable idea of the shape of Grauert tubes in local coordinates.

Let us discuss some others notations. Since $x \mapsto g_x$ is a Kähler metric, the map $\alpha \mapsto g_{\alpha_x}(\alpha_\xi, \alpha_\xi)$ is real analytic and non-negative. On the other hand, we can consider the holomorphic extension $\tilde{g}$ of g, so that $\alpha \mapsto \tilde{g}_{\alpha_x}(\alpha_\xi, \alpha_\xi)$ is a holomorphic map. With the determination of the square root positive on the reals, we define for $\alpha = (\alpha_x, \alpha_\xi)$ in $T^*\widetilde{M}$ the Japanese brackets

$$\langle \alpha \rangle = \sqrt{1 + \tilde{g}_{\alpha_x}(\alpha_\xi, \alpha_\xi)} \text{ and } \langle |\alpha| \rangle = \sqrt{1 + g_{\alpha_x}(\alpha_\xi, \alpha_\xi)}. \tag{4.6}$$

Hence, $\langle \alpha \rangle$ is holomorphic in α, while $\langle |\alpha| \rangle$ is not. However, notice that on a Grauert tube $(T^*M)_\epsilon$, for $\epsilon > 0$ small enough, the positive quantities $\langle |\alpha| \rangle, |\langle \alpha \rangle|$ and $\operatorname{Re}\langle \alpha \rangle$ are uniformly equivalent. Notice also that we may define a Kohn–Nirenberg metric on $T^*\widetilde{M}$ (since $T^*\widetilde{M}$ identifies with the cotangent bundle of $\widetilde{M}$ seen as a real-analytic manifold).

Almost analytic extensions

Let (M, g) be a compact real-analytic Riemannian manifold. To study deformations in the Grauert tube of M (or of T^*M), we will make extensive use of almost-analytic extensions of smooth functions on M. The notion of almost analytic extension was introduced by Hörmander [Hör69] and then by Nirenberg [Nir71]. It has become a very common notion in microlocal analysis, and are essential for instance in [MS75].

Recall that if f is a $\mathcal{C}^\infty$ function on M then an almost-analytic extension for f is a compactly supported $\mathcal{C}^\infty$ functions $\tilde{f}$ on some $(M)_\epsilon$ that coincides with f on M and such that $\bar{\partial}\tilde{f}$ vanishes at all orders on M. It is classical that such a $\tilde{f}$ exists [Zwo12, Theorem 3.6]. While this is hardly surprising, it will be crucial in our analysis that if f is $\mathcal{G}^s$ then $\tilde{f}$ may be chosen $\mathcal{G}^s$ as well. This will allow us to make the flatness of $\bar{\partial}\tilde{f}$ near M quantitative with Lemma 4.7 below.

It seems to be folklore that the fact that $\mathcal{G}^s$ functions admit a $\mathcal{G}^s$ almost analytic extension may be deduced from results of Carleson on universal moment problems [Car61]. Since we were not aware of any existing proof in the literature, we provided one in [BJ20, Lemma 1.1].

Lemma 4.6. *Given a Grauert tube $(M)_\epsilon \supset M$, for each $s > 1$, there exists a constant C_s and a compact subset $K \subseteq (M)_\epsilon$ so that for all $R > 0$, there exists a bounded map $f \mapsto \tilde{f}$ from $E^{s,R}(M)$ to $E^{s,C_s \max(1,R)}(K)$, such that $\tilde{f}$ is an almost-analytic extension for f.*

In order to apply Lemma 4.6, we need to investigate the way a Gevrey function can be flat. To do so, we can apply the "sommation au plus petit terme", a method for regularizing certain divergent series that is particularly well suited for Taylor series of Gevrey functions. The interested reader may refer to [Ram93] for details and historical references and to [BJ20, Lemma 1.3] for a proof of the following lemma.

Lemma 4.7. *Let U be an open subset of $\mathbb{R}^n$ and K a compact and convex subset of U. Then for every $s > 1$, there is a constant $C > 0$ such that for every $R \geq 1, x \in K$ and $f \in \mathcal{C}^\infty(U)$, if the quantity $\|f\|_{s,R,K}$ defined by (4.2) is finite, and all derivatives of f at x vanish, then for every $y \in K$ we have*

$$|f(y)| \leq C \|f\|_{s,R,K} \exp\left(-\frac{(R|x-y|)^{-\frac{1}{s-1}}}{C}\right). \tag{4.7}$$

Remark 4.8. Let us explain how Lemma 4.7 allows to control the size of an almost analytic extension of a Gevrey function. Let $s > 1$ and $R \geq 1$. Let M be a compact real-analytic manifold. According to Lemma 4.6, for some $\epsilon > 0, R' \geq 1$ and compact subset $K \subseteq (M)_\epsilon$, there is a continuous map $f \mapsto \tilde{f}$ from $E^{s,R}(M)$ to $E^{s,R'}(K)$ such that $\tilde{f}$ is an almost analytic extension for f.

Hence, if $f \in E^{s,R}(M)$ the 1-form $\bar{\partial}\tilde{f}$ vanishes at infinite order on M. Then, applying Lemma 4.7 in charts, we find that for $z \in (M)_\epsilon$ we have

$$\left|\bar{\partial}\tilde{f}(z)\right| \leq C_{s,R} \|f\|_{s,R,M} \exp\left(-\frac{1}{C_{s,R}|\operatorname{Im} z|^{\frac{1}{s-1}}}\right). \tag{4.8}$$

Recall that $|\operatorname{Im} z|$ has been defined on $(M)_\epsilon$. Here, we use any metric on the cotangent bundle of $(M)_\epsilon$ to measure the size of $\bar{\partial}\tilde{f}$. Moreover, since the derivatives of $\tilde{f}$ are Gevrey too, one can easily see that (4.8) can in fact be improved to a Gevrey estimates (in charts).

In $\mathcal{G}^s$ micro-local analysis, the error terms that are allowed are those that decay like $\exp(-1/(Ch)^{1/s})$, for some $C > 0$, when the small parameter h tends to 0. Hence, we see that it only makes sense to consider the value of the almost analytic extension of a $\mathcal{G}^s$ function at a distance at most $C^{-1}h^{1-1/s}$ of the real if we do not want to produce uncontrolled errors.

Analytic and Gevrey symbols, admissible phase

In order to define an analytic FBI transform in §4.1.2, we need the notion of analytic symbols on a compact real-analytic manifold M. To do so, we use the notations that we introduced above.

Definition 4.9. Let M be a compact real-analytic manifold. Let $m \in \mathbb{R}$. An *analytic symbol of order* m on T^*M is a real-analytic function $a : T^*M \to \mathbb{C}$ such that there are $\epsilon > 0$ and $C > 0$ such that a admits a holomorphic extension to $(T^*M)_\epsilon$ that satisfies for every $\alpha \in (T^*M)_\epsilon$

$$|a(\alpha)| \leq C \langle |\alpha| \rangle^m .$$

We denote by $S^{1,m}(T^*M)$ the set of analytic symbols of order m on T^*M. We say that a symbol $a \in S^{1,m}(T^*M)$ is elliptic in $S^{1,m}(T^*M)$ if there is $C > 0$ such that for every $\alpha \in T^*M$ we have

$$|a(\alpha)| \geq \frac{1}{C} \langle \alpha \rangle .$$

Definition 4.11 of real-analytic symbols extends *mutatis mutandis* to product of the form $(T^*M)^{N_1} \times M^{N_2}$, or reasonable subsets of such a product. Since we will not discuss here Gevrey pseudo-differential operators, we do not need to define the most general Gevrey symbols. We only need to consider symbols for Gevrey differential operators, that is polynomials on the fibers of T^*M with Gevrey coefficients. More concretely, such a symbol corresponds to a Gevrey section of the vector bundle

$$\bigoplus_{k=0}^{m} (TM \otimes \mathbb{C})^{\otimes k}$$

for some $m \in \mathbb{N}$, and it has a $\mathcal{G}^s$ almost analytic extension which is a section of the complex analytic vector bundle

$$\bigoplus_{k=0}^{m} T\widetilde{M}^{\otimes k}$$

where $\widetilde{M}$ is a complexification for M. Let us put it in more concrete terms in the case of differential operators of order 1, our principal interest here. If X is a $\mathcal{G}^s$ vector field on M and $V : M \to \mathbb{C}$ be a $\mathcal{G}^s$ function, let $P = X + V$. Then the principal symbol of the semi-classical differential operator hP may be defined by

$$p(\alpha) = i\alpha_\xi (X(\alpha_x))$$

for $\alpha = (\alpha_x, \alpha_\xi) \in T^*M$. Then X admits a $\mathcal{G}^s$ almost analytic extension $\widetilde{X}$ which is a section of $T\widetilde{M}$ (where we recall that $\widetilde{M}$ denotes a complexification for M). Then the almost analytic extension for p may be defined on $T^*\widetilde{M}$ by

$$\tilde{p}(\alpha) = i\alpha_\xi\left(\widetilde{X}(\alpha_x)\right).$$

This is a function on $T^*\widetilde{M}$, polynomial in every fiber, with $\mathcal{G}^s$ almost analytic coefficients. Before being able to define what an analytic FBI transform is, we need a last definition.

Definition 4.10. Let M be a compact real-analytic manifold. A *phase* on M is a holomorphic symbol Φ of order 1 defined for $(\alpha, y) \in (T^*M)_\epsilon \times (M)_\epsilon$ with $d(\alpha_x, y) < \delta$ (for some $\epsilon, \delta > 0$) such that

(i) if $(\alpha, y) \in T^*M \times M$ then the imaginary part of $\Phi(\alpha, y)$ is non-negative;

(ii) $\Phi(\alpha, \alpha_x) = 0$ for $\alpha = (\alpha_x, \alpha_\xi) \in T^*M$;

(iii) for $\alpha \in T^*M$, we have $\mathrm{d}_y\Phi(\alpha, \alpha_x) = -\alpha_\xi$.

We say that Φ is an *admissible phase* if it satisfies in addition the coercivity condition:

(iv) there is a constant $c > 0$ such that if α, y are real and $\Phi(\alpha, y)$ is defined, then $\operatorname{Im}\Phi(\alpha, y) \geq c\,\langle\alpha\rangle\, d\,(\alpha_x, y)^2$.

4.1.2 Global analytic FBI transform on a compact manifold

Definition of an FBI transform

We explain now what we mean by a global analytic FBI transform on a manifold and give a short introduction to the notion of complex Lagrangian deformation of the cotangent bundle. The interested reader may refer to [BJ20, §2.1] and to the introduction of the same paper for historical references.

We fix a compact real-analytic manifold M of dimension $n \in \mathbb{N}$. Without loss of generality, we may endow M with a Riemannian metric g, so that the machinery described in §4.1.1 is available. We denote by $\widetilde{M}$ a complexification for M. We define now the notion of analytic FBI transform.

Definition 4.11. An *analytic FBI transform* is an operator T, from $\mathcal{C}^\infty(M)$ to $\mathcal{D}'(T^*M)$, such that for some $C, \epsilon_0, \epsilon_1, \eta > 0$, the Schwartz kernel K_T of T is holomorphic in $(M \times T^*M)_{\epsilon_0}$ and for $(x, \alpha) = (x, (\alpha_x, \alpha_\xi))$ therein satisfies

(a) if $d(x, \alpha_x) > \epsilon_1/2$, then $|K_T(\alpha, x)| \leq Ce^{-\eta\frac{\langle|\alpha|\rangle}{h}}$.

(b) for $d(x, \alpha_x) \leq \epsilon_1$,

$$\left|K_T(\alpha, x) - e^{i\frac{\Phi_T(\alpha,x)}{h}}a(\alpha, x)\right| \leq Ce^{-\eta\frac{\langle|\alpha|\rangle}{h}}, \tag{4.9}$$

Φ_T being an admissible phase (as defined in Definition 4.10), and a being a semi-classical analytic symbol, elliptic in the symbol class $h^{-\frac{3n}{4}} S^{1,\frac{n}{4}}$ (with the subtlety that $a(\alpha, x)$ is maybe only defined for α_x and x close to each other). An *adjoint analytic FBI transform* is an operator $S : \mathcal{C}_c^\infty(T^*M) \to \mathcal{D}(M)'$ whose kernel $K_S(x, \beta)$ satisfies that $(\alpha, x) \mapsto \overline{K_S(x, \alpha)}$ is the kernel of an analytic FBI transform.

Remark 4.12. While it does not appear in the notation, the symbol a is allowed to depend on the small implicit parameter $h > 0$ (but it has to satisfy estimates uniformly in h). We say that Φ_T is the phase of T and that a is its symbol.

Recall that the fact that K_T is the kernel of T means that if u is a smooth function on M then Tu is defined by the formula

$$Tu : \alpha \mapsto \int_M K_T(\alpha, x) u(x) \mathrm{d}x.$$

Actually, it follows from the holomorphy of K_T that Tu is well-defined as a holomorphic function on $(T^*M)_{\epsilon_0}$ for $u \in \left(E^{1,R_0}(M)\right)'$ provided $\epsilon_0 > 0$ is small enough and R_0 is large enough.

Notice that if T is an analytic FBI transform, its adjoint T^* with respect to the L^2 spaces on M and T^*M is an adjoint analytic FBI transform, since the kernel of T^* is given by $(x, \alpha) \mapsto \overline{K_T(\alpha, x)}$ (the cotangent bundle T^*M is endowed with its canonical volume form).

Due to the holomorphy condition that we impose on the kernel of an FBI transform, it is not clear that such an object exists on a general compact real-analytic manifold. However in [BJ20, Lemma 2.3] we prove:

Proposition 4.13. *Let M be a compact real-analytic manifold. Then there is an analytic FBI transform on M.*

The proof of Proposition 4.13 is based on Hörmander's solution to the $\bar{\partial}$ equation, following a suggestion by Maciej Zworski.

Complex Lagrangian deformations

As noticed in Remark 4.12, if u is a hyperfunction on M and T an analytic FBI transform, then Tu defines not only a function on T^*M but also a holomorphic function on $(T^*M)_{\epsilon_0}$. At the heart of [HS86, Sjö96] is the idea to study the restriction of Tu to a well-chosen submanifold Λ of $(T^*M)_{\epsilon_0}$ – *a priori* different from T^*M. However, in order to make the analysis works, some assumptions on Λ are necessary.

In order to state the assumptions that we need to make on the manifold Λ , we need to recall a few facts on the symplectic geometry of $(T^*M)_{\epsilon_0}$. We let θ denotes the complex canonical 1-form on $(T^*M)_{\epsilon_0}$ and $\omega = \mathrm{d}\theta$ the associated symplectic form. Indeed, recall that $(T^*M)_{\epsilon_0}$ may be seen as an open subset in the cotangent bundle $T^*\widetilde{M}$ of a complexification $\widetilde{M}$ of M. Then set $\omega_R = \operatorname{Re}\omega$ and $\omega_I = \operatorname{Im}\omega$. Notice that ω_R and ω_I are *real* symplectic forms on $(T^*M)_{\epsilon_0}$. In local charts with $\tilde{x} = x + iy$, $\tilde{\xi} = \xi + i\eta$, the expression for ω is given by

$$\omega = \underbrace{\mathrm{d}\xi \wedge \mathrm{d}x - \mathrm{d}\eta \wedge \mathrm{d}y}_{=\omega_R} + i\underbrace{(\mathrm{d}\eta \wedge \mathrm{d}x + \mathrm{d}\xi \wedge \mathrm{d}y)}_{=\omega_I}.$$

We can also express the Liouville 1-form:

$$\theta = \xi \cdot \mathrm{d}x - \eta \cdot \mathrm{d}y + i(\xi \cdot \mathrm{d}y + \eta \cdot \mathrm{d}x).$$

Following Sjöstrand, we will denote with an I objects of symplectic geometric defined through the use of ω_I. For example, the I-Hamiltonian (i.e. w.r.t. ω_I) vector field of a $\mathcal{C}^1$ function f is given in the coordinates above by

$$H_f^{\omega_I} = \nabla_\eta f \cdot \frac{\partial}{\partial x} - \nabla_x f \cdot \frac{\partial}{\partial \eta} + \nabla_\xi f \cdot \frac{\partial}{\partial y} - \nabla_y f \cdot \frac{\partial}{\partial \xi}, \tag{4.10}$$

so that $\mathrm{d}f = \omega_I(\cdot, H_f^{\omega_I})$.

One finds directly that T^*M is a I-Lagrangian submanifold of $(T^*M)_{\epsilon_0}$. The idea of [HS86] is to replace it by another I-Lagrangian submanifold of $(T^*M)_{\epsilon_0}$. However, we will not work with all I-Lagrangian subspaces of $(T^*M)_{\epsilon_0}$, we shall only consider adapted I-Lagrangians as we define now.

Definition 4.14. Let $s \geq 1$ and $\tau_0 \geq 0$. Let Λ be an I-Lagrangian in $(T^*M)_{\epsilon_0}$. We say that Λ is a (τ_0, s)-*adapted Lagrangian* if it takes the form

$$\Lambda = e^{H_G^{\omega_I}}\left(T^*M\right). \tag{4.11}$$

Here, we assume that G is a real-valued function so that $G_0 := h^{-1+1/s} G$ is a symbol (in the usual Kohn–Nirenberg class of symbol) on $(T^*M)_{\epsilon_0}$ of order $1/s$, supported in some $(T^*M)_{\epsilon_1}$ with $\epsilon_1 < \epsilon_0$. Additionally, we require that (we use the covariant derivatives associated to the Kohn–Nirenberg metric to measure the derivatives of G_0)

$$\sup_{\substack{\alpha \in (T^*M)_{\epsilon_0} \\ k \leq 3}} \frac{\left\|\nabla^k G_0(\alpha)\right\|_{KN}}{\langle |\alpha| \rangle^{\frac{1}{s}}} \leq \tau_0. \tag{4.12}$$

Remark 4.15. If G is as in Definition 4.14, then one easily sees that the vector field $H_G^{\omega_I}$ is complete, so that we can define a (τ_0, s)-adapted Lagrangian Λ by the formula (4.11) (the manifold Λ is then I-Lagrangian since $\exp(H_G^{\omega_I})$ is an I-symplectomorphism). Notice also that the assumptions on the symbol G impose that it depends on h, but in a uniform fashion as $h \to 0$ (in particular, Λ is uniformly smooth with respect to the Kohn–

Nirenberg metric when h tends to 0). In the applications, the dependence of G on Λ will be fairly explicit since G will be of the form $\tau_0 h^{1-1/s} G_0$ with G_0 of order $1/s$ satisfying the assumptions of Definition 4.14 with $\tau_0 = 1$.

One may notice that if $\tau_0 \geq \tau_1$ and $s \geq s'$ then any (τ_0, s)-adapted Lagrangian is also (τ_1, s')-adapted.

Remark 4.16. Let $m \in \mathbb{R}$ and Ω be a manifold on which there is a notion of Kohn–Nirenberg metric and of Japanese bracket (the main examples are $T^*M, (\widetilde{T^*M})_{\epsilon_0}$ and an adapted Lagrangian Λ), then we define as usual the Kohn–Nirenberg class of symbol $S^m_{KN}(\Omega)$ as the space of $\mathcal{C}^\infty$ functions $a : \Omega \to \mathbb{C}$ such that for every $k \in \mathbb{N}$ we have (using the covariant derivative associated to the Kohn–Nirenberg metric):

$$\sup_{\alpha \in \Omega} \frac{\left\| \nabla^k a(\alpha) \right\|_{KN}}{\langle |\alpha| \rangle^m} < +\infty.$$

For instance, in Definition 4.14 we ask for $G_0 \in S^{1/s}_{KN}\left((T^*M)_{\epsilon_0} \right)$.

Since the adapted Lagrangians are uniformly smooth submanifolds with respect to the Kohn–Nirenberg metric, and image of T^*M under a uniformly smooth flow, the symbol class $S^m_{KN}(\Lambda)$ is well defined, and to check that $a \in S^m_{KN}(\Lambda)$, we can compute the derivatives either directly on Λ, or through the pullback by $\exp(H_G^{\omega_I})$, with covariant derivatives or with partial derivatives in coordinates.

The notion of (τ_0, s)-adapted Lagrangian is tailored so that it makes sense to restrict the almost analytic extension of the principal symbol of a differential operator with Gevrey coefficients (as defined in 4.1.1, see also Remark 4.8) to a (τ_0, s)-adapted Lagrangians when τ_0 is small. More precisely, if G is as in Definition 4.14, it follows from the local expression 4.10 for $H_G^{\omega_I}$ that the norm of $H_G^{\omega_I}$ for the Kohn–Nirenberg metric is $\mathcal{O}\big(\tau_0 h^{1-1/s} \langle |\alpha| \rangle^{1/s-1}\big)$. This essentially proves

Lemma 4.17. *Let $s \geq 1$ and $T \geq 0$. There is a constant $C > 0$ such that, for every $\tau_0 \in [0, T]$, if Λ is a (τ_0, s)-adapted Lagrangian and $\alpha \in \Lambda$ then*

$$|\operatorname{Im} \alpha| \leq C \tau_0 h^{1-\frac{1}{s}} \langle |\alpha| \rangle^{\frac{1}{s}-1}$$

In particular, for every $\epsilon > 0$, there is a $\tau_1 > 0$ such that, for every $\tau_0 \in [0, \tau_1]$, any (τ_0, s)-adapted Lagrangian is contained in $(T^*M)_\epsilon$.

In particular, we see that if T is an analytic FBI transform and Λ is a (τ_0, s)-adapted Lagrangian manifold with τ_0 small enough, then, according to Remark 4.12, if u is a hyperfunction on M then Tu is defined on Λ. We define then the transform T_Λ associated by restriction: $T_\Lambda u = (Tu)_{|\Lambda}$.

In order to define weighted L^2 spaces on adapted Lagrangians, we need to define a volume form on such Lagrangians. To do so, we use the following lemma, which is proven by a perturbative argument (see [BJ20, Lemma 2.2])

Lemma 4.18. *Let $s \geq 1$. There exists $T > 0$ and a constant $C > 0$ such that, for every $\tau_0 \in [0, T]$, if Λ is a (τ_0, s)-adapted Lagrangian then the restriction of ω_R to Λ (that we shall also denotes by ω_R) is symplectic.*

From now on, if Λ is an adapted Lagrangian, we will just denote by $\mathrm{d}\alpha$ the $2n$-form $\omega_R^n/n!$, which induces a volume form on Λ. We will denote as usual the corresponding duality pairing

$$\langle f, g\rangle_\Lambda = \int_\Lambda fg \,\mathrm{d}\alpha. \tag{4.13}$$

Using the volume form $\mathrm{d}\alpha$, we may also define adjoint analytic FBI transforms associated to Λ. If S is an adjoint analytic transform with kernel K_S and Λ is a (τ_0, s)-adapted Lagrangian manifold with τ_0 small enough, then we define the operator S_Λ by

$$S_\Lambda v(x) = \int_\Lambda K_T(x, \alpha) v(\alpha) \mathrm{d}\alpha,$$

where $x \in M$ and v is a measurable function on Λ that decays fast enough – we will extend later this definition to more general v.

The natural space in our setting will not be $L^2(\Lambda, \mathrm{d}\alpha)$ but rather $L^2(\Lambda, e^{-2H/h}\mathrm{d}\alpha)$, where H is an action associated with Λ, solving

$$\mathrm{d}H = -\operatorname{Im}\theta_{|\Lambda}. \tag{4.14}$$

Since Λ is I-Lagrangian, we deduce that there are local solutions to this equation. However, since Λ is assumed to be of the form (4.11), we can find an explicit *global* solution, given by

$$H := \int_0^1 \left(e^{(\tau-1)H_G^{\omega_I}}\right)^* (G - \operatorname{Im}\theta(H_G^{\omega_I}))\mathrm{d}\tau. \tag{4.15}$$

Analytic FBI transform and Gevrey regularity

Since an FBI transform is some kind of generalization of the Fourier transform, we expect that regularity of a hyperfunction u on M may be described in terms of decay properties of the transform Tu. This is indeed true and we explain it now.

Let T be an analytic FBI transform, S be an adjoint analytic transform and Λ be a $(\tau_0, 1)$-adapted Lagrangian with τ_0 small enough. We recall the associated transforms T_Λ and S_Λ that we introduced above. We need spaces to describe the decay or growth of a function on Λ. Let $s \geq 1$ and $r \in \mathbb{R}$. If f is a function from Λ to $\mathbb{C}$, let

$$\|f\|_{\Lambda,s,r} := \sup_{\alpha\in\Lambda} |f(\alpha)|\, e^{-r\langle|\alpha|\rangle^{\frac{1}{s}}} \in \mathbb{R}_+ \cup \{+\infty\}.$$

Then we define $F^{s,r}(\Lambda)$ to be the Banach space of continuous function $f : \Lambda \to \mathbb{C}$ such that $\|f\|_{\Lambda,s,r} < +\infty$.

We define then the following spaces of functions on the Lagrangian Λ. For $s \geq 1$ introduce the space of functions decaying at least exponentially

$$\mathfrak{G}^s(\Lambda) := \bigcup_{r<0} F^{s,r}(\Lambda),$$

and the space of functions diverging slower than any exponential

$$\mathfrak{U}^s(\Lambda) := \bigcap_{r>0} F^{s,r}(\Lambda).$$

These spaces are endowed respectively with the inductive and projective limit structure (in the category of locally convex topological vector spaces). Notice that $\mathfrak{G}^s(\Lambda)$ is dense in $\mathfrak{U}^s(\Lambda)$ (just multiply by a bump function) and that the L^2 pairing (4.13) in Λ gives a natural duality bracket between $\mathfrak{U}^s(\Lambda)$ and $\mathfrak{G}^s(\Lambda)$. The spaces $\mathfrak{G}^s(\Lambda)$ and $\mathfrak{U}^s(\Lambda)$ are natural analogues of $\mathcal{G}^s(M)$ and $\mathcal{U}^s(M)$ on the FBI side. Indeed, we have the following results.

Proposition 4.19 (Proposition 2.1 in [BJ20]). *Let $s' > s \geq 1$ and assume that Λ is a (τ_0, s')-Gevrey adapted Lagrangian with τ_0 small enough. Then, the transform T_Λ is continuous from $\mathcal{G}^s(M)$ to $\mathfrak{G}^s(\Lambda)$ and from $\mathcal{U}^s(M)$ to $\mathfrak{U}^s(\Lambda)$.*

Proposition 4.20 (Proposition 2.2 in [BJ20]). *Let $s' > s \geq 1$ and assume again that Λ is a (τ_0, s')-Gevrey adapted Lagrangian with τ_0 small enough. Then, the transform S_Λ is continuous from $\mathfrak{G}^s(\Lambda)$ from $\mathcal{G}^s(M)$ and admits a continuous extension from $\mathfrak{U}^s(\Lambda)$ to $\mathcal{U}^s(M)$.*

If Propositions 4.19 and 4.20 give a good idea of the link between regularity on M and decay on the FBI side, they are not precise enough for what we intend to do. In particular, when dealing with $\mathcal{G}^s$ Anosov flows, we want to consider (τ_0, s)-adapted Lagrangians, in order to get the best results possible. Thus, we need more precise estimates that explains how (τ_0, s)-adapted Lagrangians relate with $\mathcal{G}^s$ functions and associated ultradistributions. We can use for instance the following lemmas.

Lemma 4.21. *Let $s \geq 1$ and $R \geq 1$. Then there is $C > 0$ such that, if Λ is a (τ_0, s)-adapted Lagrangian with τ_0 small enough and $0 \leq r \leq C^{-1}h^{-1/s}$, then T_Λ is bounded from $E^{s,R}(M)$ to $F^{s,-r}(\Lambda)$ and S_Λ extends continuously to an operator from $F^{s,r}(\Lambda)$ to $\left(E^{s,R}(M)\right)'$.*

Lemma 4.22. *Let $s \geq 1$ and $\epsilon > 0$. Then there is $R > 0$ such that, if Λ is a (τ_0, s)-adapted Lagrangian with τ_0 small enough and $r \geq \epsilon h^{-1/s}$, then T_Λ is bounded from $\left(E^{s,R}(M)\right)'$ to $F^{s,r}(\Lambda)$ and S_Λ is bounded from $F^{s,-r}(\Lambda)$ to $E^{s,R}(\Lambda)$.*

While the statements of Lemmas 4.21 and 4.22 may seem technical, their proofs are relatively easy. Lemma 4.22 follows from bounds on the growth of Gevrey norms of $y \mapsto K_T(\alpha, y)$ and $y \mapsto K_S(y, \alpha)$ when α tends to $+\infty$ along Λ. These bounds may be obtained for instance by applying Cauchy's Formula on a polydisc with shrinking radius (see [BJ20, Lemmas 2.5 and 2.6]). The first point in Lemma 4.21 may be proved by an application of Gevrey non-stationary phase method and the second point follows by a duality argument (see [BJ20, Lemma 2.4 and Proposition 2.2]).

Inversion formula

It follows from Lemmas 4.21 and 4.22 that if T is an analytic FBI transform and S is an adjoint analytic FBI transform, then the composition[2] ST makes sense as an operator from $\left(E^{1,R_0}(M)\right)'$ to $\left(E^{1,R_1}(M)\right)'$ for any R_1, and R_0 large enough depending on R_1. It would be useful to be able to retrieve an ultradistribution u from its FBI transform Tu – if possible in an explicit way. We will use the following result to do so.

Proposition 4.23 (Theorem 6 in [BJ20]). *Let M be a compact real-analytic manifold and $h > 0$ be small enough. Then there is an analytic FBI transform T on M such that T^*T is the identity*[3].

The proof of Proposition 4.23 is given in [BJ20, §2.1.3]. Let us give a short glimpse of the ideas behind this demonstration. First of all, from Proposition 4.13, we know that there is an analytic FBI transform T_0 on M. Then, from an application of the stationary phase method, we may prove that the self-adjoint operator $T_0^*T_0$ is an elliptic analytic pseudo-differential operator of order 0 (in a sense defined in [BJ20, §1.3]). Then, we apply a suited functional calculus to construct an inverse square root for $T_0^*T_0$: this is a self-adjoint analytic pseudo-differential operator P that satisfies $PT_0^*T_0P = I$ (the operator P is *a priori* only defined for h small enough). It remains to see by another application of the holomorphic stationary phase method that $T = T_0P$ is an analytic FBI transform to end the proof of Proposition 4.23.

From now on, we fix a compact real-analytic manifold M and let T be an analytic FBI transform given by Proposition 4.23 – assuming that $h > 0$ is small enough. We will also denote by S the adjoint T^* of T. Using Stokes' Formula to shift a contour, we may generalize Proposition 4.23 to FBI transforms associated to Lagrangian deformations of T^*M.

Proposition 4.24 (Lemma 2.7 in [BJ20]). *Let Λ be a $(\tau_0, 1)$-adapted Lagrangian with τ_0 small enough. Then, for every R_1 large enough, for R_0 large enough, the composition*

[2] In order to lighten notations, we sometimes drop the index Λ when $\Lambda = T^*M$. That is we use the short hand $T = T_{T^*M}$ and $S = S_{T^*M}$.

[3] With the notations above, it would be more precise to say that T^*T is the inclusion of $\left(E^{1,R_0}(M)\right)'$ in $\left(E^{1,R_1}(M)\right)'$.

$S_\Lambda T_\Lambda$ is well-defined from $\left(E^{1,R_0}(M)\right)'$ to $\left(E^{1,R_1}(M)\right)'$ and is the inclusion between these spaces.

Remark 4.25. Beware that S_Λ is only a left inverse for T_Λ. Indeed, the image of T_Λ only contains smooth function and the operator $\Pi_\Lambda := T_\Lambda S_\Lambda$ is a projector on the image of T_Λ. The study of this projector and of related operators is the object of [BJ20, §2.2].

Spaces associated to Lagrangian deformation

Let us fix a $(\tau_0, 1)$-adapted Lagrangian Λ with τ_0 small enough. We want to associate a scale of "Sobolev-like" spaces to Λ. In §4.2, we will see that for a relevant choice of Λ these spaces are suited to the spectral analysis of Gevrey Anosov flows. First of all, recalling the action H on Λ given by (4.15) and satisfying the equation (4.14), we define a scale of weighted L^2 spaces on Λ by

$$L^2_k(\Lambda) := L^2\left(\Lambda, \langle|\alpha|\rangle^{2k} e^{-\frac{2H}{h}} \mathrm{d}\alpha\right),$$

where $k \in \mathbb{R}$ and we recall that $\mathrm{d}\alpha$ denotes the volume form $\omega_R^n/n!$ on Λ. Then we define the associated scale of spaces of hyperfunctions on the manifold by

$$\mathcal{H}^k_\Lambda := \left\{u \in \left(E^{1,R}(M)\right)' : T_\Lambda u \in L^2_k(\Lambda)\right\}, \tag{4.16}$$

where $k \in \mathbb{R}$, and $R \geq 1$ is large enough so that Proposition 4.24 applies. The space $\mathcal{H}^k_\Lambda$ is endowed with the norm

$$\|u\|_{\mathcal{H}^k_\Lambda} := \|T_\Lambda u\|_{L^2_k(\Lambda)}.$$

Finally, it is also useful to introduce the notations

$$\mathcal{H}^\infty_\Lambda = \bigcap_{k\in\mathbb{R}} \mathcal{H}^k_\Lambda \text{ and } \mathcal{H}^\infty_{\Lambda,\mathrm{FBI}} = \bigcap_{k\in\mathbb{R}} \mathcal{H}^k_{\Lambda,\mathrm{FBI}}.$$

Let us now list the basic properties of these spaces.

Proposition 4.26. *Let Λ be a $(\tau_0, 1)$-adapted Lagrangian with τ_0 small enough. Then, for every $k \in \mathbb{R}$, the space $\mathcal{H}^k_\Lambda$ is a separable Hilbert space and (equivalently) $\mathcal{H}^k_{\Lambda,\mathrm{FBI}}$ is a closed subspace of $L^2_k(\Lambda)$.*

Proposition 4.27. *Let $s \geq 1$ and R_0 be large enough. Then, if τ_0 is small enough – depending on R_0 – and Λ is a (τ_0, s)-adapted Lagrangian, we have for every $k \in \mathbb{R}$*

$$E^{s,R_0} \subseteq \mathcal{H}^k_\Lambda \subseteq (E^{s,R_0})'. \tag{4.17}$$

Under the same assumptions, for $1 \leq s' < s$ and every $k \in \mathbb{R}$, we have

$$\mathcal{G}^{s'}(M) \subseteq \mathcal{H}_{\Lambda}^{k} \subseteq \mathcal{U}^{s'}(M).$$

All these inclusions are continuous.

Propositions 4.26 and 4.27 are consequences of Proposition 4.24 (see [BJ20, Corollary 2.2]). Further analysis leads to a density statement.

Proposition 4.28 (Corollary 2.3 in [BJ20]). *Assume that τ_0 is small enough and that Λ is a $(\tau_0, 1)$-adapted Lagrangian. Then, assuming that $h > 0$ is small enough, there is $R > 0$ such that, for all $k \in \mathbb{R}$, the space $E^{1,R}(M)$ is dense in $\mathcal{H}_{\Lambda}^{k}$. Moreover, if $u \in L^2(M) \cap \mathcal{H}_{\Lambda}^{k}$, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $E^{1,R}(M)$ such that $(u_n)_{n\in\mathbb{N}}$ converges to u both in $L^2(M)$ and in $\mathcal{H}_{\Lambda}^{k}$.*

It may also be insightful to notice that in the absence of deformations, we just gave an equivalent definition of the usual (semi-classical) Sobolev spaces.

Proposition 4.29 (Corollary 2.4 in [BJ20]). *Let $h > 0$ be small enough. Then, for every $k \in \mathbb{R}$, the space $\mathcal{H}_{T^*M}^{k}$ is the usual semi-classical Sobolev space of order k on M, with uniformly equivalent norms as h tends to 0.*

Some properties of the undeformed case remain true for general Λ's, for instance we have:

Proposition 4.30 (Lemma 2.24 in [BJ20]). *Let Λ be a $(\tau_0, 1)$-adapted Lagrangian. Assume that τ_0 and h are small enough and let $k \in \mathbb{R}$. Then, if $u \in \mathcal{H}_{\Lambda}^{-k}$ and $v \in \mathcal{H}_{\Lambda}^{k}$ the pairing*

$$\langle u, v \rangle := \langle T_\Lambda u, T_\Lambda v \rangle_{L_0^2(\Lambda)} \tag{4.18}$$

is well-defined and induces an (anti-linear) identification between $\mathcal{H}_{\Lambda}^{-k}$ and the dual of $\mathcal{H}_{\Lambda}^{k}$ (inducing equivalent, but a priori *not equal norms).*

Estimates on singular values of certain operators will be crucial in the proof of Theorem 10. To this end, we will use the following result – which is also reminiscent of the undeformed case.

Proposition 4.31 (Proposition 2.13 in [BJ20]). *Assume that Λ is a $(\tau_0, 1)$-adapted Lagrangian with τ_0 small enough. Assume that $h > 0$ is small enough. Let $m > 0$ and $q \in \mathbb{R}$, then the inclusion of $\mathcal{H}_{\Lambda}^{m+q}$ into $\mathcal{H}_{\Lambda}^{q}$ is compact. In addition, if $(\mu_k)_{k\in\mathbb{N}}$ denotes the singular values of this inclusion then we have*

$$\mu_k \underset{k\to+\infty}{=} \mathcal{O}\left(\frac{1}{k^{\frac{m}{n}}}\right).$$

In particular, the inclusion of $\mathcal{H}_\Lambda^{m+q}$ into $\mathcal{H}_\Lambda^q$ belongs to the Schatten class S_p for every $p > n/m$.

4.1.3 Analytic FBI transform and Gevrey differential operators

It is now time to explain how differential operator with Gevrey coefficients acts on the spaces $\mathcal{H}_\Lambda^k$'s that we just define, which is the object of [BJ20, §2.2 and §2.3]. We start by stating a boundedness result.

Proposition 4.32 (Proposition 2.4 in [BJ20])**.** *Let $s \geq 1$. Let P be a (semi-classical) differential operator of order $m \in \mathbb{N}$ with $\mathcal{G}^s$ coefficients. Let Λ be a (τ_0, s)-adapted Lagrangian with τ_0 small enough – depending on P. Assume that $h > 0$ is small enough. Then, for every $k \in \mathbb{R}$ the operator P is bounded from $\mathcal{H}_\Lambda^k$ to $\mathcal{H}_\Lambda^{k-m}$ – with uniform bound when h tends to 0.*

Actually, we will rather consider differential operators as unbounded operators on $\mathcal{H}_\Lambda^0$, so that the following result is more suited to our needs. This is a straightforward generalization of [BJ20, Lemmas 3.2 and 3.4]

Proposition 4.33. *Let $s \geq 1$. Let P be a differential operator of order 1 with $\mathcal{G}^s$ coefficients. Let Λ be a (τ_0, s)-adapted Lagrangian with τ_0 small enough and assume that $h > 0$ is small enough. Then the operator P acting on $\mathcal{H}_\Lambda^0$ with domain*

$$D(P) = \left\{ u \in \mathcal{H}_\Lambda^0 : Pu \in \mathcal{H}_\Lambda^0 \right\}$$

is a closed operator. Moreover, if $u \in D(P)$, there is a sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{H}_\Lambda^\infty$ such that $(u_n)_{n \in \mathbb{N}}$ and $(Pu_n)_{n \in \mathbb{N}}$ converge respectively to u and Pu in $\mathcal{H}_\Lambda^0$. The same approximation property holds when P is replaced by its adjoint[4] P^*.

Finally, the main tool that we will use to study Anosov flows is the following "multiplication formula" [BJ20, Proposition 2.11].

Proposition 4.34 (Multiplication formula)**.** *Let $s \geq 1$ and P be a differential operator of order m with $\mathcal{G}^s$ coefficients. Assume that Λ is a (τ_0, s)-Gevrey adapted Lagrangian with τ_0 small enough. Let p_Λ denotes the restriction to Λ of a $\mathcal{G}^s$ almost analytic extension of the principal symbol of P. Assume that h is small enough. Let $f \in S_{KN}^{m'}(\Lambda)$ be a symbol of order m' on Λ, uniformly in h. Then, if $m_1, m_2 \in \mathbb{R}$ are such that $m_1 + m_2 = m + m' - 1$, there is a constant $C > 0$ such that for any $u, v \in \mathcal{H}_\Lambda^\infty$, we have (the scalar product is in $L_0^2(\Lambda)$)*

$$\left| \langle f T_\Lambda P u, T_\Lambda v \rangle - \langle f p_\Lambda T_\Lambda u, T_\Lambda v \rangle \right| \leq C h \|u\|_{\mathcal{H}_\Lambda^{m_1}} \|v\|_{\mathcal{H}_\Lambda^{m_2}}. \tag{4.19}$$

[4]Here, we consider the adjoint of P with respect to the Hilbert structure on $\mathcal{H}_\Lambda^0$ and not its formal adjoint.

We recall that the notion of almost analytic extension for the principal symbol of a differential operators with $\mathcal{G}^s$ coefficients has been defined in §4.1.1 and that the class of symbols $S^{m'}_{KN}(\Lambda)$ is defined in Remark 4.16.

Let us discuss briefly the proof of Proposition 4.34, considering that this is one of the main results from [BJ20] and that it will be a key tool in the proof of Theorem 10 (see in particular the proof of Lemma 4.37). What we expose here rapidly is exposed in details in [BJ20, §2.2 and 2.3]. Recalling the inversion formula Proposition 4.24, in order to understand the action of a differential operator P on the spaces $\mathcal{H}^k_\Lambda$, one only needs to understand the action of the operator $T_\Lambda P S_\Lambda$ on the spaces $L^2_k(\Lambda)$. Using the Gevrey stationary phase method, we are able to identify the kernel of the operator $T_\Lambda P S_\Lambda$. Thanks to the introduction of the action H in the definition of the space $L^2_0(\Lambda)$, we see that the reduced kernel of $T_\Lambda P S_\Lambda$ associated to its action on this space is the kernel of a certain Fourier Integral Operator with complex phase with positive imaginary part. We can then use Melin–Sjöstrand's version[5] of the stationary phase method with complex phase [MS75] to study a certain class of Fourier Integral Operators that contains in particular $T_\Lambda P S_\Lambda$. We prove in particular that the orthogonal projector B_Λ on the image of T_Λ in $L^2_0(\Lambda)$ is one of these Fourier Integral Operators. The understanding of the algebraic properties of these operators allow then to prove a formula of the form

$$B_\Lambda f T_\Lambda P S_\Lambda B_\Lambda \simeq B_\Lambda \sigma B_\Lambda \tag{4.20}$$

up to negligible operators (this is some kind of Toeplitz representation for the operator P). Here, σ is a symbol on Λ that coincides at first order with $f p_\Lambda$. Proposition 4.34 follows then from the representation formula (4.20).

4.2 Finite order of the dynamical determinant

We turn now to the proof of Theorem 10. From now on, $s \in [1, +\infty[$ is fixed, X is a $\mathcal{G}^s$ vector field on a compact $\mathcal{G}^s$ manifold M that generates an Anosov flow $(\phi_t)_{t\in\mathbb{R}}$, and $V : M \to \mathbb{C}$ is a $\mathcal{G}^s$ function. We define the differential operator $P = X + V$, and the associated Koopman operator is given by (9). Without loss of generality, we may assume that M is endowed with a structure of real-analytic Riemannian manifold (coherent with its $\mathcal{G}^s$ structure, see Remark 4.2).

The machinery from §4.1.1 is then available, in particular we denote by $\widetilde{M}$ a complex neighbourhood for M. According to Proposition 4.23, there is an analytic FBI transform T on M such that $T^*T = I$. As above we set $S = T^*$. In order to apply the results from §4.1 to the operator P, we need first to find a suitable (τ_0, s)-adapted Lagrangian Λ.

[5] The Gevrey regularity assumption on P is only used in order to justify the applications of the stationary phase method mentioned above. After that, we only need to do $\mathcal{C}^\infty$ micro-local analysis on Λ.

The Lagrangian Λ will be defined by (4.11) where the symbol G is defined by $G = \tau G_0$ where $\tau \ll h^{1-\frac{1}{s}}$ and G_0 is a so-called *escape function*. The section §4.2.1 is devoted to the construction of G_0 (see Lemma 4.35). In §4.2.2, we will then describe the spectral theory of P on the related spaces defined in §4.1.2. Finally, the proof of Theorem 10 is given in §4.2.3.

4.2.1 Constructing an escape function

Recall that the decomposition $TM = E^0 \oplus E^u \oplus E^s$ of the tangent bundle from Definition 6 induces a dual decomposition $T^*M = E_0^* \oplus E_u^* \oplus E_s^*$ of the cotangent bundle. Here, $E_0^* = (E^u \oplus E^s)^\perp$, $E_u^* = (E^0 \oplus E^u)^\perp$ and $E_s^* = (E^0 \oplus E^s)^\perp$. We denote by $p : T^*M \to \mathbb{C}$ the principal symbol of the semi-classical differential operator hP. We recall that for $\alpha = (\alpha_x, \alpha_\xi) \in T^*M$,

$$p(\alpha) = i\alpha_\xi \left(X(\alpha_x) \right). \tag{4.21}$$

To apply the machinery presented in the previous part, we will need an almost analytic extension for p. We construct it as in §4.1.1: we take a $\mathcal{G}^s$ almost analytic extension $\widetilde{X}$ for X, given by Lemma 4.6 if $s > 1$ (we just take $\widetilde{X} = X$ if $s = 1$), and then we set for $\alpha = (\alpha_x, \alpha_\xi) \in (T^*M)_{\epsilon_0}$ (for some small $\epsilon_0 > 0$)

$$\tilde{p}(\alpha) = i\alpha_\xi \left(\widetilde{X}\left(\alpha_x\right) \right). \tag{4.22}$$

It will be important when constructing the escape function G_0 that this almost analytic extension is linear in α_ξ. We are now ready to construct G_0.

Lemma 4.35. *Let $\mathscr{C}^0$ be a conical neighbourhood of E_0^* in T^*M. Let $\delta \geq 0$. Then there are arbitrarily small $\epsilon_0 > \epsilon_1 > 0$ and a symbol G_0 of order δ on $(T^*M)_{\epsilon_0}$, supported in $(T^*M)_{\epsilon_1}$ with the following properties:*

(i) the restriction of $\{G_0, \operatorname{Re}\tilde{p}\}$ to T^*M is negative and classically elliptic of order δ outside of $\mathscr{C}^0$, that is, for some $C > 0$ and $\alpha \in T^*M \setminus \mathscr{C}^0$ large enough, we have

$$\{G_0, \operatorname{Re}\tilde{p}\}(\alpha) \leq -C \left\langle |\alpha| \right\rangle^\delta;$$

(ii) there are $C, \epsilon_2 > 0$ such that $\{G_0, \operatorname{Re}\tilde{p}\} \leq C$ on $(T^*M)_{\epsilon_2}$.

Here, the Poisson Bracket $\{G_0, \operatorname{Re}\tilde{p}\}$ is the one associated with the real symplectic form ω_I on $(T^*M)_{\epsilon_0}$.

Notice that Lemma 4.35 is slightly simpler than the corresponding result in [BJ20] (Lemma 3.1 from the mentioned paper). This is because we focus on Theorem 10. In

[BJ20], we prove additionally a statement on the Gevrey wave front set of resonant states (that we recall in §4.3, see Proposition 4.41) and we need additional properties on the escape function G_0 to do so.

Before proving Lemma 4.35, let us explain why we need our escape function to satisfy these properties. The point (i) will be used in the proof of Lemma 4.37 to deduce the hypoellipticity of the operator P acting on $\mathcal{H}_\Lambda^0$ from the multiplication formula, Proposition 4.34. The point (ii) will be used in the proof of Proposition 4.36 to show that the Koopman operator (9) defines a continuous semi-group on $\mathcal{H}_\Lambda^0$. This property is what ensures that the spectrum of P on our spaces has a dynamical meaning. Even though point (i) seems to be the most crucial one, since it is the one that allows us to enter the world of Schatten operators and finally prove Theorem 10, the importance of (ii) could not be overestimated. Finally, notice that, in order to apply Proposition 4.34 in the most favorable case for us, it will be natural in the following to choose $\delta = 1/s$.

Proof of Lemma 4.35. We want to understand $\{G_0, \operatorname{Re}\tilde{p}\}$ in order to control how the real part of $\tilde{p}$ evolves under the flow of $H_{G_0}^{\omega_I}$. However, since $\{G_0, \operatorname{Re}\tilde{p}\} = -\{\operatorname{Re}\tilde{p}, G_0\}$, we may understand $\{G_0, \operatorname{Re}\tilde{p}\}$ by controlling how G_0 evolves under the flow of $-H_{\operatorname{Re}\tilde{p}}^{\omega_I}$. Hence, we need to understand the dynamics of this flow. To do so, we may multiply $\widetilde{X}$ by a bump function identically equals to 1 near M (since we only claim properties for G_0 near T^*M and we will not use the high regularity of $\widetilde{X}$ in this proof). Then, it follows from the formula (4.10) that the flow of $-H_{\operatorname{Re}\tilde{p}}^{\omega_I}$ is complete. We denote this flow by $(\Theta_t)_{t\in\mathbb{R}}$ and write

$$\Theta_t(\alpha) = (\Theta_{t,x}(\alpha), \Theta_{t,\xi}(\alpha)) .$$

Using (4.10), we see that $H_{\operatorname{Re}\tilde{p}}^{\omega_I}$ is given in coordinates $(x+iy, \xi+i\eta)$ by

$$\begin{aligned} -H_{\operatorname{Re}\tilde{p}}^{\omega_I} = \sum_{j=1}^n \operatorname{Re}\widetilde{X}_j \frac{\partial}{\partial x_j} + \operatorname{Im}\widetilde{X}_j \frac{\partial}{\partial y_j} - \left(\xi\left(\frac{\partial \operatorname{Im}\widetilde{X}}{\partial y_j}\right) + \eta\left(\frac{\partial \operatorname{Re}\widetilde{X}}{\partial y_j}\right)\right)\frac{\partial}{\partial \xi_j} \\ - \left(\xi\left(\frac{\partial \operatorname{Im}\widetilde{X}}{\partial x_j}\right) + \eta\left(\frac{\partial \operatorname{Re}\widetilde{X}}{\partial x_j}\right)\right)\frac{\partial}{\partial \eta_j}. \end{aligned} \tag{4.23}$$

Indeed, it follows from (4.21) that in such coordinates we have

$$\operatorname{Re}\tilde{p} = -\xi\left(\operatorname{Im}\widetilde{X}\right) - \eta\left(\operatorname{Re}\widetilde{X}\right).$$

From (4.23), we see that the projection $\Theta_{t,x}$ of Θ_t is in fact given by the formula

$$\Theta_{t,x}(\alpha) = \tilde{\phi}_t(\alpha_x),$$

where $(\tilde{\phi}_t)_{t\in\mathbb{R}}$ denotes the flow of $\widetilde{X}$ (in particular, the restriction of $\tilde{\phi}_t$ to M is ϕ_t). Then, we notice that in (4.23), the component of $-H^{\omega_I}_{\mathrm{Re}\,\tilde{p}}$ along $\partial/\partial\xi$ and $\partial/\partial\eta$ is linear in (ξ,η). It implies that

$$\Theta_{t,\xi}(\alpha) = L_t(\alpha_x)(\alpha_\xi),$$

where $L_t(\alpha_x)$ is a $\mathbb{R}$-linear application from $T^*_{\alpha_x}\widetilde{M}$ to $T^*_{\tilde{\phi}_t(\alpha_x)}\widetilde{M}$ (that depends smoothly on t and α_x). Now, since $\widetilde{X}$ satisfies the Cauchy–Riemann equations and is tangent to M on M, we find that, for $y=0$, in the same system of coordinates than (4.23), we have

$$\frac{\partial \operatorname{Re}\widetilde{X}}{\partial y} = -\frac{\partial \operatorname{Im}\widetilde{X}}{\partial x} = 0 \text{ and } \frac{\partial \operatorname{Im}\widetilde{X}}{\partial y} = \frac{\partial \operatorname{Re}\widetilde{X}}{\partial x} = \frac{\partial X}{\partial x}. \tag{4.24}$$

By uniqueness in the Cauchy–Lipschitz Theorem, we find by plugging (4.24) in (4.23) that, for $x \in M$ and $t \in \mathbb{R}$ we have[6]

$$L_t(x) = \left(D\phi_t(x)^{-1}\right)^{\mathrm{tr}}. \tag{4.25}$$

Hence, the hyperbolicity of $(\phi_t)_{t\in\mathbb{R}}$ will have important consequences on the dynamics of $(\Theta_t)_{t\in\mathbb{R}}$. Let us "complexify" the bundles $E^*_{0,u,s}$. For $x \in M$, we denote by $E_0^{\mathbb{C},*}, E_u^{\mathbb{C},*}$ and $E_s^{\mathbb{C},*}$ the complexification of E_0^*, E_u^* and E_s^*, considering linear forms valued in $\mathbb{C}$ instead of $\mathbb{R}$. For instance, for $x \in M$, we write $E_{0,x}^{\mathbb{C},*}$ for the subspace of $T_x^*M \otimes \mathbb{C}$ consisting of $\mathbb{R}$-linear maps from T_xM to $\mathbb{C}$ that vanish on $E^u \oplus E^s$ (or, under a natural identification, of $\mathbb{C}$-linear forms on $T_x^*\widetilde{M}$ that vanish on $E^u \oplus E^s$). From the fact that $T_xM = E_x^0 \oplus E_x^u \oplus E_x^s$ is a totally real subspace of maximal dimension of $T_x\widetilde{M}$, we deduce that $T_x^*\widetilde{M} = E_{0,x}^{\mathbb{C},*} \oplus E_{u,x}^{\mathbb{C},*} \oplus E_{s,x}^{\mathbb{C},*}$. Since E_x^0, E_x^u and E_x^s depends in a Hölder-continuous fashion on $x \in M$, so does $E_{0,x}^{\mathbb{C},*}, E_{u,x}^{\mathbb{C},*}$ and $E_{s,x}^{\mathbb{C},*}$. Consequently, we may extend continuously $E_0^{\mathbb{C},*}, E_u^{\mathbb{C},*}$ and $E_s^{\mathbb{C},*}$ to $\widetilde{M}$. Then, if $\widetilde{M}$ is small enough, we have $T_x\widetilde{M} = E_{0,x}^{\mathbb{C},*} \oplus E_{u,x}^{\mathbb{C},*} \oplus E_{s,x}^{\mathbb{C},*}$ for all $x \in \widetilde{M}$. *A priori*, this decomposition is only invariant under L_t for $t \in \mathbb{R}$ when x is real. If $\sigma \in T_x^*\widetilde{M}$ then we write $\sigma = \sigma_0 + \sigma_u + \sigma_s$ for the decomposition of σ under $T_x\widetilde{M} = E_{0,x}^{\mathbb{C},*} \oplus E_{u,x}^{\mathbb{C},*} \oplus E_{s,x}^{\mathbb{C},*}$. Then, we put a real Riemannian metric on $\widetilde{M}$ and define for $\gamma > 0$ the cone fields $\mathcal{C}_u^\gamma$ and $\mathcal{C}_s^\gamma$ by setting for $x \in \widetilde{M}$

$$C_u^\gamma(x) = \left\{\sigma \in T_x^*\widetilde{M} : |\sigma_0| + |\sigma_s| \leq \gamma\,|\sigma_u|\right\}$$

and

$$C_s^\gamma(x) = \left\{\sigma \in T_x^*\widetilde{M} : |\sigma_0| + |\sigma_u| \leq \gamma\,|\sigma_s|\right\}.$$

[6] We recall that A^{tr} is our notation for the transpose of A.

Without loss of generality, we may assume that $\mathscr{C}^0$ is closed and does not intersect $E_u^* \oplus E_s^* \setminus \{0\}$. We may also assume that there is a closed conic neighbourhood $\mathscr{C}^{\mathbb{C},0}$ of $E_0^{\mathbb{C},*}$ in $T^*\widetilde{M}$ such that $\mathscr{C}^{\mathbb{C},0} \cap T^*M = \mathscr{C}^0$. Choose then a small closed conic neighbourhood of $\mathcal{C}^{0s}$ of $E_s^* \oplus E_0^*$ in T^*M.

From (4.25) and standard arguments in hyperbolic dynamic, there are a large $T_0 > 0$, small $0 < \gamma' < \gamma$, some $\lambda > 1$ and a constant $c > 0$ such that if $\alpha = (\alpha_x, \alpha_\xi) \in T^*\widetilde{M}$, and $T_1 \geq T_0$ then:

a. either $\Theta_{T_1,\xi}(\alpha) \in C_u^{\gamma'}(\tilde{\phi}_{T_1}(\alpha_x))$ or $\Theta_{-T_0,\xi}(\alpha) \in C_s^{\gamma'}(\tilde{\phi}_{-T_0}(\alpha_x))$ or $\alpha \in \mathscr{C}^{\mathbb{C},0}$;

b. if $\Theta_{-T_0,\xi}(\alpha) \in C_u^{\gamma}(\tilde{\phi}_{-T_0}(\alpha_x))$ then $\Theta_{T_1,\xi}(\alpha) \in C_u^{\gamma'}(\tilde{\phi}_{T_1}(\alpha_x))$ and we have the bound $|\Theta_{T_1,\xi}(\alpha)| \geq \lambda \, |\Theta_{-T_0,\xi}(\alpha)|$;

c. if $\Theta_{T_1,\xi}(\alpha) \in C_s^{\gamma}(\tilde{\phi}_{T_1}(\alpha_x))$ then $\Theta_{-T_0,\xi}(\alpha) \in C_s^{\gamma'}(\tilde{\phi}_{-T_0}(\alpha_x))$ and we have the bound $|\Theta_{T_1,\xi}(\alpha)| \leq \lambda^{-1} \, |\Theta_{-T_0,\xi}(\alpha)|$;

d. if $\alpha \in T^*M$ does not belong to $\mathscr{C}^{0s}$, then, for $t \geq 0$, we have that $\Theta_{t,\xi}(\alpha)$ does not belong to $C_s^{\gamma}(\tilde{\phi}_t(\alpha_x))$, and, for $t \geq T_0$, we have $\Theta_{t,\xi}(\alpha) \in C_u^{\gamma'}(\tilde{\phi}_t(\alpha_x))$ and $|\Theta_{t,\xi}(\alpha)| \geq c \, |\alpha_\xi|$.

Since we ask here for $x \in M$, these are consequences of the hyperbolicity of $(\phi_t)_{t\in\mathbb{R}}$, that is it only relies on the dynamic on M. We want to apply a perturbation argument to show that b and c remain true on a small complex neighbourhood of M, but we need first to fix the value of T_1. Hence, we fix the value of T_1, large enough such that we have

$$\sup_{\alpha \in T^*M \setminus \{0\}} \frac{1}{|\alpha_\xi|^\delta} \int_{-T_0}^{0} |\Theta_{t,\xi}(\alpha)|^\delta \, \mathrm{d}t < \frac{c^\delta (T_1 - T_0)}{2}. \tag{4.26}$$

Now that T_1 is fixed, it follows from a perturbation argument that, up to taking a smaller λ, a smaller γ, a larger γ' and a smaller c, the properties b and c above remain true when $\alpha \in T^*(M)_{\epsilon_0}$, for some small $\epsilon_0 > 0$ (and (4.26) remains true since we asked for a strict inequality). Then, we choose a symbol $m \in S_{KN}^0\left(T^*(M)_{\epsilon_0}\right)$ of order 0 on $T^*(M)_{\epsilon_0}$, valued in $[-1, 1]$, with the following properties:

- if $x \in (M)_{\epsilon_0}$ and $\sigma \in T_x^*\widetilde{M} \setminus (C_u^{\gamma}(x) \cup C_s^{\gamma}(x))$ or σ is near 0 then $m(x,\sigma) = 0$;
- there is $C > 0$ such that if $x \in (M)_{\epsilon_0}$ and $\sigma \in C_s^{\gamma'}(x)$ satisfies $|\sigma| \geq C$ then $m(x,\sigma) = 1$;
- there is $C > 0$ such that if $x \in (M)_{\epsilon_0}$ and $\sigma \in C_u^{\gamma'}(x)$ satisfies $|\sigma| \geq C$ then $m(x,\sigma) = -1$;
- m is non-positive on $\mathcal{C}_\gamma^u$ and non-negative on $\mathcal{C}_\gamma^s$.

Then, we may define G_0 near $T^*M \otimes \mathbb{C}$ by the formula

$$G_0(\alpha) = \int_{T_0}^{T_1} m(\Theta_t(\alpha)) \left|\Theta_{t,\xi}(\alpha)\right|^\delta \mathrm{d}t. \tag{4.27}$$

We multiply G_0 by a bump function to satisfy the claim on the support. It does not change the value of G_0 near T^*M and hence it will not interfere with the properties (i) and (ii). We may consequently use the formula (4.27) to prove (i) and (ii).

We start by proving (i). To do so we compute

$$\begin{aligned} \{G_0, \operatorname{Re}\tilde{p}\} &= -\{\operatorname{Re}\tilde{p}, G_0\} = -H^{\omega_I}_{\operatorname{Re}\tilde{p}} G_0 \\ &= m(\Theta_{T_1}(\alpha)) \left|\Theta_{T_1,\xi}(\alpha)\right|^\delta - m(\Theta_{-T_0}(\alpha)) \left|\Theta_{-T_0,\xi}(\alpha)\right|^\delta \end{aligned} \tag{4.28}$$

Assuming that α does not belong to $\mathscr{C}^0$, we know that $\Theta_{T_1,\xi}(\alpha)$ belongs to $\mathcal{C}_u^{\gamma'}$ or $\Theta_{-T_0,\xi}(\alpha)$ belongs to $\mathcal{C}_s^{\gamma'}$. Let us assume for instance that $\Theta_{T_1,\xi}(\alpha)$ belongs to $\mathcal{C}^u_{\gamma'}$ (the other case is symmetric). Then again there are two possibilities: either $\Theta_{-T_0,\xi}(\alpha)$ belongs to $\mathcal{C}^u_\gamma$ or it does not. If it does then (for $|\alpha_\xi|$ large enough)

$$\begin{aligned} & m(\Theta_{T_1}(\alpha)) \left|\Theta_{T_1,\xi}(\alpha)\right|^\delta - m(\Theta_{-T_0}(\alpha)) \left|\Theta_{-T_0,\xi}(\alpha)\right|^\delta \\ &\qquad \le -\left|\Theta_{T_1,\xi}(\alpha)\right|^\delta + \left|\Theta_{-T_0,\xi}(\alpha)\right|^\delta \\ &\qquad \le -\left(\lambda^\delta - 1\right) \left|\Theta_{-T_0,\xi}(\alpha)\right|^\delta \\ &\qquad \le -\frac{1}{C} \left|\alpha_\xi\right|^\delta, \end{aligned}$$

for some $C > 0$. If $\Theta_{-T_0,\xi}(\alpha)$ does not belong to $\mathcal{C}^u_\gamma$, then the situation is even simpler since the term $-m(\Theta_{-T_0}(\alpha)) \left|\Theta_{-T_0,\xi}(\alpha)\right|^\delta$ is non-positive. Hence, the right-hand side in (4.28) is negative and elliptic of order δ outside of $\mathscr{C}^0$.

It remains to prove (ii). The analysis is based on (4.28) again. Let $\alpha \in (T^*M)_{\epsilon_2}$. If $\Theta_{T_1,\xi}(\alpha) \in \mathcal{C}_u^{\gamma'}$ or $\Theta_{-T_0,\xi}(\alpha) \in \mathcal{C}_s^{\gamma'}$, then the analysis from the proof of (ii) applies, and we see that the right-hand side in (4.28) is non-positive for α_ξ large enough. Otherwise, $\Theta_{-T_0,\xi}(\alpha) \notin \mathcal{C}_u^\gamma$ and $\Theta_{T_1,\xi}(\alpha) \notin \mathcal{C}_s^\gamma$ and both terms in the right-hand side of (4.28) are non-positive. Thus, the right-hand side of (4.28) is always non-positive when $|\alpha_\xi|$ is large enough. Hence, by compactness, the right hand side of (4.28) is bounded from above, proving (ii). □

Now, that we are equipped with a good escape function, we are in position to apply the tools from §4.1 to study the spectral theory of $P = X + V$.

4.2.2 Spectral theory for the generator of the flow

With the notations of the previous section, we set $\delta = 1/s$ and let G_0 be an escape function given by Lemma 4.35 (for arbitrary $\mathscr{C}^0$, we only assume that $\mathscr{C}^0$ is closed an does not intersect $E_u^* \oplus E_s^* \setminus \{0\}$). Then we define $G = \tau G_0$ and $\Lambda = e^{H_G^{\omega_I}} T^*M$ with $\tau = c\tau_0 h^{1-\frac{1}{s}}$, where c and τ_0 are small. Notice that if c is small enough, then Λ is a (τ_0, s)-adapted Lagrangian (in the sense of Definition 4.14), and hence the results from §4.1 will apply to the $\mathcal{G}^s$ semi-classical pseudor hP. We will not consider the asymptotic $h \to 0$, and hence we shall assume that h is fixed, small enough so that the results from §4.1 apply. We shall also assume that τ_0 is small enough for the same reason.

We want now to study the spectral theory of the operator $P = X + V$ on the space $\mathcal{H}_\Lambda^0$ defined by (4.16). Notice that, according to Proposition 4.32, if h and τ_0 are small enough, then the operator hP (and hence P) is bounded from $\mathcal{H}_\Lambda^k$ to $\mathcal{H}_\Lambda^{k-1}$ for every $k \in \mathbb{R}$. From Proposition 4.33, we see that P defines a closed operator on $\mathcal{H}_\Lambda^0$. We will start by proving that P is the generator of a semi-group.

Proposition 4.36. *The operator P is the generator of a strongly continuous semi-group $(\mathcal{L}_t)_{t\in\mathbb{R}}$ on $\mathcal{H}_\Lambda^0$. Moreover, if $t \geq 0$ and $u \in \mathcal{H}_\Lambda^0 \cap L^2(M)$ then $\mathcal{L}_t u$ is given by the expression (9).*

We will then prove the following key lemma that will be used with Proposition 4.31 to prove that the resolvent of P is in a Schatten class.

Lemma 4.37 (Hypo-ellipticity of P)**.** *There is a constant $C > 0$ such that for every $u \in D(P)$ we have $u \in \mathcal{H}_\Lambda^\delta$ and*

$$\|u\|_{\mathcal{H}_\Lambda^\delta} \leq C \left(\|u\|_{\mathcal{H}_\Lambda^0} + \|Pu\|_{\mathcal{H}_\Lambda^0} \right),$$

where we recall that we set $\delta = 1/s$.

With Proposition 4.31, we deduce then from Lemma 4.37 that P has a good spectral theory on $\mathcal{H}_\Lambda^0$. More precisely, we have:

Theorem 4.38. *If z is any element in the resolvent set of P, then the resolvent $(z-P)^{-1} : \mathcal{H}_\Lambda^0 \to \mathcal{H}_\Lambda^0$ is compact and if $(\sigma_k)_{k\geq 0}$ denotes the sequence of its singular values, we have*

$$\sigma_k \underset{k\to+\infty}{=} \mathcal{O}\left(k^{-\frac{1}{sn}}\right).$$

In particular, the operator $(z-P)^{-1}$ is in the Schatten class S_p for any $p > ns$. Consequently, P has discrete spectrum on $\mathcal{H}_\Lambda^0$, and this spectrum is the Ruelle spectrum of P. The eigenvectors of P acting on $\mathcal{H}_\Lambda^0$ are also the resonant states for P. If $N(R)$ denotes the number of Ruelle resonances of modulus less than R, we have

$$N(R) \underset{R\to+\infty}{=} \mathcal{O}\left(R^{ns}\right).$$

Theorem 10 will then be proved in the following section as a corollary of Theorem 4.38.

The key tool in the proof of these results will be the multiplication formula Proposition 4.34. Let us start by proving that P is the generator of a semi-group.

Proof of Proposition 4.36. We will apply Hille–Yosida Theorem to prove that P is the generator of a strongly continuous semi-group. We denote by p_Λ the restriction to Λ of the almost analytic extension $\tilde{p}$ of the principal symbol of hP given by (4.22). It follows from (ii) in Lemma 4.35 that there is a constant $C > 0$ such that $\operatorname{Re} p_\Lambda \leq C$. Indeed, since $\operatorname{Re} \tilde{p}$ vanishes on T^*M, the value of $\operatorname{Re} p_\Lambda$ is obtained by integrating $\{G_0, \operatorname{Re} \tilde{p}\}$ on an orbit of time τ of the flow of $H_{G_0}^{\omega_I}$.

We see that by Proposition 4.34, up to making C larger (depending on h, that we recall is fixed), we have for $u \in \mathcal{H}_\Lambda^\infty$

$$\begin{aligned}\operatorname{Re}\langle -Pu, u\rangle_{\mathcal{H}_\Lambda^0} &\geq -\frac{1}{h}\langle \operatorname{Re} p_\Lambda T_\Lambda P u, T_\Lambda u\rangle - C\left\|u\right\|_{\mathcal{H}_\Lambda^0}^2 \\ &\geq -2C\left\|u\right\|_{\mathcal{H}_\Lambda^0}^2 .\end{aligned}$$

Hence, if $z \in \mathbb{C}$, we have

$$\operatorname{Re}\langle (z-P)\,u, u\rangle_{\mathcal{H}_\Lambda^0} \geq (\operatorname{Re} z - 2C)\left\|u\right\|_{\mathcal{H}_\Lambda^0}^2 .$$

By Cauchy–Schwarz, we find that

$$\begin{aligned}\left\|(z-P)\,u\right\|_{\mathcal{H}_\Lambda^0} &= \frac{\left\|(z-P)\,u\right\|_{\mathcal{H}_\Lambda^0}\left\|u\right\|_{\mathcal{H}_\Lambda^0}}{\left\|u\right\|_{\mathcal{H}_\Lambda^0}} \geq \frac{\left|\langle (z-P)\,u, u\rangle_{\mathcal{H}_\Lambda^0}\right|}{\left\|u\right\|_{\mathcal{H}_\Lambda^0}} \\ &\geq \frac{\operatorname{Re}\langle (z-P)\,u, u\rangle_{\mathcal{H}_\Lambda^0}}{\left\|u\right\|_{\mathcal{H}_\Lambda^0}} \geq (\operatorname{Re} z - 2C)\left\|u\right\|_{\mathcal{H}_\Lambda^0} ,\end{aligned} \tag{4.29}$$

for $u \in \mathcal{H}_\Lambda^\infty$. By Proposition 4.33, this estimate remains true when $u \in D(P)$. This proves that if $\operatorname{Re} z > 2C$, then the operator $z - P$ is injective and its image is closed. To prove that the image of $z - P$ is dense, notice that if $u \in \mathcal{H}_\Lambda^\infty$ then

$$\operatorname{Re}\langle (z-P)^* u, u\rangle_{\mathcal{H}_\Lambda^0} = \operatorname{Re}\langle (z-P) u, u\rangle_{\mathcal{H}_\Lambda^0} ,$$

and consequently (4.29) still holds when $z - P$ is replaced by $(z-P)^*$ (for $u \in \mathcal{H}_\Lambda^\infty$, but it implies the same result for $u \in D(P^*)$ by Proposition 4.33). Hence, $(z-P)^*$ is injective, and thus the image of $z - P$ is closed.

Thus, $z - P$ is invertible and from (4.29), we see that

$$\left\|(z-P)^{-1}\right\| \leq \frac{1}{\operatorname{Re} z - 2C},$$

for the operator norm on $\mathcal{H}_\Lambda^0$. Hence, the Hille–Yosida Theorem applies (the domain of

P is dense since it contains E^{1,R_0}), and we know that P is the generator of a strongly continuous semi-group.

Denote by $(\widetilde{\mathcal{L}}_t)_{t\geq 0}$ the semi-group generated by P on $\mathcal{H}_\Lambda^0$ and $(\mathcal{L}_t)_{t\geq 0}$ the semi-group on $L^2(M)$ defined by (9). We want to prove that for $t \geq 0$ and $u \in \mathcal{H}_\Lambda^0 \cap L^2(M)$ we have $\mathcal{L}_t u = \widetilde{\mathcal{L}}_t u$. Thanks to the semi-group property, we only need to prove it for $t \in [0, t_0]$ for some small $t_0 > 0$. Then, since elements of $L^2(M) \cap \mathcal{H}_\Lambda^0$ may be simultaneously approximated in $L^2(M)$ and in $\mathcal{H}_\Lambda^0$ by elements of E^{1,R_0} (according to Proposition 4.28), we only need to prove the equality for $u \in E^{1,R_0}$. Now, there is a $t_0 > 0$ and a $R_1 > 0$ such that for $u \in E^{1,R_0}$, the curve $\gamma : [0,t_0] \ni t \mapsto \mathcal{L}_t u$ is $\mathcal{C}^1$ in E^{s,R_1} with $\gamma'(t) = P\gamma(t)$. Provided that τ_0 is small enough, E^{s,R_1} is continuously included in $\mathcal{H}_\Lambda^0$ (see Proposition 4.26) and hence the curve γ has the same property in $\mathcal{H}_\Lambda^0$. Consequently, we have $\gamma(t) = \widetilde{\mathcal{L}}_t u$ for $t \in [0, t_0]$, according to [ABHN11, Proposition 3.1.11], ending the proof of the proposition. □

We turn now to the proof of the hypo-ellipticity of the operator P.

Proof of Lemma 4.37. Assume first that $u \in \mathcal{H}_\Lambda^\infty$. Let χ_+, χ_- and χ_0 be $\mathcal{C}^\infty$ functions from $\mathbb{R} \to [0,1]$ such that $\chi_+ + \chi_0 + \chi_- = 1$, and, for some small $\eta > 0$, the function χ_0 is supported in $[-\eta, \eta]$, the function χ_- is supported in $\left]-\infty, -\frac{\eta}{2}\right]$ and the function χ_+ is supported in $\left[\frac{\eta}{2}, +\infty\right[$. Then define for $\sigma \in \{+,-,0\}$ the symbol f_σ on Λ by

$$f_\sigma(\alpha) = \chi_\sigma\left(-i\frac{p\left(e^{-\tau H_{G_0}^{\omega_I}}(\alpha)\right)}{\langle|\alpha|\rangle}\right).$$

Then notice that if α is in the support of f_+ then we have

$$\begin{aligned}\operatorname{Im} p_\Lambda(\alpha) &= \operatorname{Im} p\left(e^{-\tau H_{G_0}^{\omega_I}}(\alpha)\right) + \mathcal{O}\left(\tau \langle|\alpha|\rangle\right) \\ &\geq \frac{\eta}{2}\langle|\alpha|\rangle,\end{aligned} \tag{4.30}$$

provided that τ is small enough (depending on η). And similarly, if α belongs to the support of f_- we have

$$\operatorname{Im} p_\Lambda(\alpha) \leq -\frac{\eta}{2}\langle|\alpha|\rangle. \tag{4.31}$$

If α belongs to the support of f_0 then we have

$$\left|p\left(e^{-\tau H_{G_0}}(\alpha)\right)\right| \leq \eta\langle|\alpha|\rangle \leq C\eta\left\langle\left|e^{-\tau H_{G_0}}(\alpha)\right|\right\rangle.$$

Hence, either $e^{-\tau H_{G_0}}(\alpha)$ is small, either it does not belong to $\mathscr{C}^0$ (provided that η is small enough, we use here the assumption that $\mathscr{C}^0$ does not intersect $E_u^* \oplus E_s^*$). In the second case, we may apply (i) in Lemma 4.35 to find that

$$\begin{aligned} \operatorname{Re} p_\Lambda(\alpha) &= \operatorname{Re} p\left(e^{-\tau H_{G_0}^{\omega_I}}(\alpha)\right) + \tau\left\{G_0, \operatorname{Re}\tilde{p}\right\} + \mathcal{O}\left(\tau_0^2 \left\langle|\alpha|\right\rangle^{2\delta-1}\right) \\ &\leq -\frac{1}{C}\left\langle|\alpha|\right\rangle^\delta + C, \end{aligned} \tag{4.32}$$

provided that τ is small enough. Here, we added the constant C so that (4.32) remains true for any α in the support f_0. Now, Proposition 4.34 and (4.32) give that (the constant C may vary from one line to another)

$$\begin{aligned} -\operatorname{Re}\langle\left\langle|\alpha|\right\rangle^\delta f_0 T_\Lambda P u, T_\Lambda u\rangle &\geq -\frac{1}{h}\int_\Lambda \left\langle|\alpha|\right\rangle^\delta f_0(\alpha) \operatorname{Re} p_\Lambda(\alpha) \left|T_\Lambda u(\alpha)\right|^2 \mathrm{d}\alpha - C\left\|u\right\|_{\mathcal{H}_\Lambda^0}\left\|u\right\|_{\mathcal{H}_\Lambda^\delta} \\ &\geq \frac{1}{C}\int_\Lambda f_0(\alpha)\left\langle|\alpha|\right\rangle^{2\delta}\left|T_\Lambda u(\alpha)\right|^2 \mathrm{d}\alpha - C\left\|u\right\|_{\mathcal{H}_\Lambda^0}\left\|u\right\|_{\mathcal{H}_\Lambda^\delta}. \end{aligned}$$

Applying Cauchy–Schwarz formula, we find then that

$$\int_\Lambda f_0(\alpha)\left\langle|\alpha|\right\rangle^{2\delta}\left|T_\Lambda u(\alpha)\right|^2 \mathrm{d}\alpha \leq C\left\|u\right\|_{\mathcal{H}_\Lambda^\delta}\left(\left\|Pu\right\|_{\mathcal{H}_\Lambda^0} + \left\|u\right\|_{\mathcal{H}_\Lambda^0}\right). \tag{4.33}$$

Working similarly with (4.32) replaced by the better estimates (4.30) and (4.31), we find that (4.33) still holds when f_0 is replaced by f_+ or f_-. Summing these three estimates, we get

$$\left\|u\right\|_{\mathcal{H}_\Lambda^\delta}^2 \leq C\left\|u\right\|_{\mathcal{H}_\Lambda^\delta}\left(\left\|Pu\right\|_{\mathcal{H}_\Lambda^0} + \left\|u\right\|_{\mathcal{H}_\Lambda^0}\right). \tag{4.34}$$

Since the result is trivial when $u = 0$, we may divide by $\left\|u\right\|_{\mathcal{H}_\Lambda^\delta}$ in (4.34) to end the proof of the lemma when $u \in \mathcal{H}_\Lambda^\infty$.

We deal now with the general case $u \in D(P)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{H}_\Lambda^\infty$ as in Proposition 4.33. Since we already dealt with the case of elements of $\mathcal{H}_\Lambda^\infty$ we know that for some $C > 0$ and all $n \in \mathbb{N}$ we have

$$\left\|u_n\right\|_{\mathcal{H}_\Lambda^\delta} \leq C\left(\left\|u\right\|_{\mathcal{H}_\Lambda^0} + \left\|Pu\right\|_{\mathcal{H}_\Lambda^0}\right).$$

In addition, $T_\Lambda u_n$ converges pointwise to $T_\Lambda u$ and hence the result follows by Fatou's Lemma. $\square$

We are ready to prove Theorem 4.38.

Proof of Theorem 4.38. Let z be any element of the resolvent set of P. If $u \in \mathcal{H}_\Lambda^0$ then we have that

$$P(z-P)^{-1}u = z(z-P)^{-1}u - u.$$

Hence, $(z-P)^{-1}$ and $P(z-P)^{-1}$ are both bounded from $\mathcal{H}_\Lambda^0$ to itself and, consequently, Lemma 4.37 implies that $(z-P)^{-1}$ is bounded from $\mathcal{H}_\Lambda^0$ to $\mathcal{H}_\Lambda^\delta$. Hence, Proposition 4.31 implies that $(z-P)^{-1}$, as an operator from $\mathcal{H}_\Lambda^0$ to itself, is compact, with the announced estimates on its singular values.

We prove the estimates on the number of eigenvalues of P before proving that these eigenvalues are indeed the Ruelle resonances. Let $z \in \mathbb{C}$ be any point in the resolvent set of P and denote by $\widetilde{N}(R)$ the number of eigenvalues of $(z-P)^{-1}$ of modulus larger than R^{-1}. Then let $(\mu_k)_{k\in\mathbb{N}}$ denote the sequence of eigenvalues of $(z-P)^{-1}$ and $(\sigma_k)_{k\in\mathbb{N}}$ the sequence of its singular values, and choose $p>0$ such that $\delta p/n < 1$. According to [GGK00, Corollary IV.3.4], we have then for every $R>0$ that

$$\begin{aligned}\frac{\widetilde{N}(R)}{R^p} &\leq \sum_{k=0}^{\widetilde{N}(R)-1} |\mu_k|^p \leq \sum_{k=0}^{\widetilde{N}(R)-1} \sigma_k^p \\ &\leq C \sum_{k=0}^{\widetilde{N}(R)-1} (1+k)^{-\frac{\delta p}{n}} \\ &\leq C\widetilde{N}(R)^{1-\frac{\delta p}{n}}.\end{aligned}$$

Here, we applied the estimates on singular values that we just proved, and C may vary from one line to another. It follows that $\widetilde{N}(R) \leq C^{\frac{n}{\delta p}} R^{\frac{n}{\delta}}$. The estimates on $N(R)$ follows since, if $(\lambda_k)_{k\in\mathbb{N}}$ denotes the sequence of eigenvalues of P, we have the relation $\mu_k = \frac{1}{z-\lambda_k}$ (up to reordering, and recall that $\delta = 1/s$).

It remains to prove that the eigenvalues of P acting on $\mathcal{H}_\Lambda^0$ are indeed the Ruelle resonances of P. The situation here is a bit complicated due to the use of very irregular hyperfunctions, so that we cannot rely directly on Lemma B.3. We detail here the argument, but the basic ideas are the same than those that we explain in Appendix B. Let $R_0 > 0$ be large enough, and assume that τ_0 is small enough, so that (4.17) holds and that E^{1,R_0} is dense in $\mathcal{H}_\Lambda^0$ (see Proposition 4.28). We also assume that R_0 is large enough so that $E^{1,R_0}(M)$ is dense[7] in $\mathcal{C}^\infty(M)$. Denote by i_0 the inclusion of E^{1,R_0} in $\mathcal{H}_\Lambda^0$ and by i_1 the inclusion of $\mathcal{H}_\Lambda^0$ in $\left(E^{1,R_0}\right)'$. Then, we define

$$\widetilde{R}(z) = i_1 \circ (z-P)^{-1} \circ i_0 : E^{1,R_0} \to \left(E^{1,R_0}\right)',$$

where $(z-P)^{-1}$ denotes the resolvent of P on the space $\mathcal{H}_\Lambda^0$. We just saw that it is a meromorphic family of operator on $\mathbb{C}$.

Then, if we denote by i the inclusion of E^{1,R_0} into $\mathcal{C}^\infty(M)$ and by j the inclusion of

[7] This is possible since, if Δ is an analytic Laplacian on M, then $E^{1,R_0}(M)$ contains all the eigenvectors for Δ when R_0 is large enough. See [Zel17, §14.5] and references therein. One can also refer to [BJ20]: the density statement is proved there under the name of Corollary 2.1.

$\mathcal{D}'(M)$ into $\left(E^{1,R_0}\right)'$, we see that

$$\widetilde{R}(z) = j \circ R(z) \circ i, \tag{4.35}$$

where $R(z) : \mathcal{C}^\infty(M) \to \mathcal{D}'(M)$ is defined by (11), and admits an analytic continuation by the usual theory of Banach spaces of anisotropic distributions (see Remark 4). Indeed, when $\operatorname{Re} z$ is large (4.35) follows from Proposition 4.36 since $\widetilde{R}(z)$ and $R(z)$ are both obtained as the Laplace transform of the family of operators (9). The equality (4.35) then follows for any z by analytic continuation. Integrating on small circles, we see that (4.35) is also satisfied by the residues of $\widetilde{R}(z)$ and $R(z)$. Since these residues have finite rank and since E^{1,R_0} is dense in $\mathcal{C}^\infty(M)$, it follows that the eigenvalues of P on $\mathcal{H}_\Lambda^0$ (the poles of $\widetilde{R}(z)$) are the Ruelle resonances of P (the poles of $R(z)$) counted with multiplicity (the rank of the associated residues). For the same reason, the resonant spaces (the images of the residues) also coincide. □

4.2.3 Proof of Theorem 10

In order to show that the dynamical determinant $d(z)$ given by (12) has finite order (under our Gevrey assumption), we shall relate it to a regularized determinant associated with the resolvent of P. This will be based on the following version of the Guillemin trace formula:

Lemma 4.39. *If the real part of z is large enough and m is an integer such that $m > sn$, then the operator $(z-P)^{-m}$ acting on $\mathcal{H}_\Lambda^0$ is trace class and*

$$\operatorname{tr}\left((z-P)^{-m}\right) = \frac{1}{(m-1)!} \sum_\gamma \frac{T_\gamma^\# e^{\int_\gamma V}}{\left|\det\left(I - \mathcal{P}_\gamma\right)\right|} T_\gamma^{m-1} e^{-zT_\gamma}. \tag{4.36}$$

Here, we use the notations from (12) *and* (TFF)*.*

The proof of Lemma 4.39 that we give here differs from the one from [BJ20, Lemma 3.5]. Indeed, in [BJ20], we privileged a proof based on Guillemin's trace formula because it was in principle more general. We give here a proof based on the trace formula (TFF) for ultradifferentiable Anosov flows, Theorem 9, that we proved in the previous chapter of this thesis.

Proof. Let $b \in \mathbb{R}$ and $B^{m,b}$ denote the Banach space of $\mathcal{C}^m$ function $f : \mathbb{R}_+ \to \mathbb{C}$ such that $f(0) = \cdots = f^{(m-1)}(0) = 0$ and

$$\|f\|_{B^{m,b}} := \sup_{\substack{k \in \{0,\dots,m\} \\ x \in \mathbb{R}}} e^{bx} \left|f^{(k)}(x)\right| < +\infty.$$

If $f \in B^{m,b}$ then the Laplace transform

$$\mathrm{Lap}(f)(z) = \int_0^{+\infty} e^{-zt} f(t)$$

is well-defined for $\mathrm{Re}\, z > -b$. Moreover, successive integration by part ensures that

$$|\mathrm{Lap}(f)(z)| = \left| \frac{1}{z^m} \mathrm{Lap}(f^{(m)})(z) \right| \leq \frac{1}{|z|^m} \frac{\|f\|_{B^{m,b}}}{b + \mathrm{Re}\, z}.$$

Hence, if b is greater than the real part of all Ruelle resonances for P, then, according to the counting bound in Theorem 4.38, we may define a continuous linear form on $B^{m,b}$ by

$$l : f \mapsto \sum_{\lambda \in \sigma(P)} \mathrm{Lap}(f)(-\lambda).$$

However, if $\mathrm{Re}\, z > b$, the function $f_z : t \mapsto \frac{e^{-zt} t^{m+1}}{(m+1)!}$ belongs to $B^{m,b}$ and, by Lidskii's Trace Theorem,

$$l(f_z) = \sum_{\lambda \in \sigma(P)} \frac{1}{(z-\lambda)^{m+2}} = \mathrm{tr}\left((z - X)^{-m-2} \right).$$

Then, since X and V are Gevrey, the global trace formula (TFF) holds according to Theorem 9: for every $f \in \mathcal{C}_c^\infty \left(\mathbb{R}_+^* \right)$ we have

$$l(f) = \sum_\gamma \frac{T_\gamma^\# e^{\int_\gamma V}}{|\det (I - \mathcal{P}_\gamma)|} f(T_\gamma). \tag{4.37}$$

Notice that the right-hand side in (4.37) defines a continuous linear form on $B^{m,b}$, provided that b is large enough (apply for instance Margulis' bound). Then, if $\mathrm{Re}\, z > b$, the function f_z belongs to the adherence of $\mathcal{C}_c^\infty \left(\mathbb{R}_+^* \right)$ in $B^{m,b}$ (use the fact that $f_z^{(m)}(0) = 0$ to see so). Hence, an approximation argument provides that, for $\mathrm{Re}\, z > b$,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(\frac{1}{m(m+1)} \mathrm{tr}\left((z - X)^{-m} \right) \right) &= \mathrm{tr}\left((z - X)^{-m-2} \right) \\
&= \frac{1}{(m+1)!} \sum_\gamma \frac{T_\gamma^\# e^{\int_\gamma V}}{|\det (I - \mathcal{P}_\gamma)|} e^{-zT_\gamma} T_\gamma^{m+1} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(\frac{1}{(m+1)!} \sum_\gamma \frac{T_\gamma^\# e^{\int_\gamma V}}{|\det (I - \mathcal{P}_\gamma)|} e^{-zT_\gamma} T_\gamma^{m-1} \right).
\end{aligned}$$

Consequently, there are constants $a_1, a_2 \in \mathbb{C}$ such that for $\mathrm{Re}\, z \gg 1$ we have

$$\mathrm{tr}\left((z - X)^{-m} \right) = \frac{1}{(m-1)!} \sum_\gamma \frac{T_\gamma^\# e^{\int_\gamma V}}{|\det (I - \mathcal{P}_\gamma)|} e^{-zT_\gamma} T_\gamma^{m-1} + a_1 + a_2 z. \tag{4.38}$$

Thus, we only need to prove that $a_1 = a_2 = 0$ to end the proof of the lemma. By dominated convergence, when z is real and tends to $+\infty$ the first term in the right-hand side of (4.38) tends to 0. Hence, we only need to prove that the trace of $(z - X)^{-m}$ tends to 0 when z is real and tends to $+\infty$. However, this fact follows from Lidskii's Trace Formula

$$\operatorname{tr}\left((z - X)^{-m}\right) = \sum_{\lambda \in \sigma(P)} \frac{1}{(z - \lambda)^m},$$

the counting bound in Theorem 4.38 and dominated convergence. $\square$

With Lemma 4.39, we are ready to relate the dynamical determinant $d(z)$ defined by (12) with a regularized determinant. See [GGK00, Chapter XI] for the general theory of regularized determinants.

Lemma 4.40. *Let z be a complex number with large real part and m be the smallest integer strictly larger than sn. Let Q_z be the polynomial of order at most $m - 1$*

$$Q_z(\lambda) = -\sum_{\ell=0}^{m-1} \left(\sum_{\gamma} \frac{T_\gamma^{\#} e^{\int_\gamma V} e^{-zT_\gamma} T_\gamma^{\ell-1}}{|\det(I - \mathcal{P}_\gamma)|} \right) \frac{(z - \lambda)^\ell}{\ell!}.$$

Then for every $\lambda \in \mathbb{C}$ we have

$$d(\lambda) = \det_m \left(I + (\lambda - z)(z - P)^{-1} \right) \exp\left(Q_z(\lambda)\right),$$

where $\det_m$ denotes the regularized determinant of order m.

Proof. By analytic continuation principle, we only need to prove this result for λ close to z. For such a λ the regularized determinant is defined by

$$\begin{aligned}
\det_m \left(I + (\lambda - z)(z - P)^{-1} \right) &= \exp\left(-\sum_{\ell \geq m} \frac{(z - \lambda)^\ell}{\ell} \operatorname{tr}\left((z - P)^{-\ell} \right) \right) \\
&= \exp\left(-\sum_{\ell \geq m} \frac{(z - \lambda)^\ell}{\ell!} \sum_{\gamma} \frac{T_\gamma^{\#} e^{\int_\gamma V}}{|\det(I - \mathcal{P}_\gamma)|} T_\gamma^{\ell-1} e^{-zT_\gamma} \right) \\
&= \exp\left(-\sum_{\gamma} \frac{T_\gamma^{\#}}{T_\gamma} \frac{e^{\int_\gamma V} e^{-zT_\gamma}}{|\det(I - \mathcal{P}_\gamma)|} \sum_{\ell \geq m} \frac{((z - \lambda)T_\gamma)^\ell}{\ell!} \right) \\
&= \exp\left(-\sum_{\gamma} \frac{T_\gamma^{\#}}{T_\gamma} \frac{e^{\int_\gamma V} e^{-zT_\gamma}}{|\det(I - \mathcal{P}_\gamma)|} \left(e^{(z-\lambda)T_\gamma} - \sum_{\ell=0}^{m-1} \frac{((z - \lambda)T_\gamma)^\ell}{\ell!} \right) \right) \\
&= d(\lambda) e^{-Q_z(\lambda)}.
\end{aligned}$$

The applications of Fubini Theorem are justified when $\operatorname{Re} z \gg 1$ and $|z - \lambda|$ is small enough by Margulis' bound. $\square$

We are now ready to prove Theorem 10.

Proof of Theorem 10. Let m be as in Lemma 4.40. Recall the Weierstrass primary factor (1.3)

$$E(\lambda, m-1) = (1-\lambda)\exp\left(\sum_{\ell=1}^{m-1} \frac{1}{\ell}\lambda^{\ell}\right) = \exp\left(-\sum_{\ell=m}^{+\infty} \frac{1}{\ell}\lambda^{\ell}\right),$$

the second expression being valid when $|\lambda| < 1$. It follows from Lidskii's Trace Theorem that for $\operatorname{Re} z \gg 1$ and $\lambda \in \mathbb{C}$ we have

$$\det_m \left(I - (z-\lambda)(z-P)^{-1}\right) = \prod_{k=0}^{+\infty} E\left(\frac{\lambda - z}{\lambda_k - z}, m-1\right), \tag{4.39}$$

where $(\lambda_k)_{k\in\mathbb{N}}$ denotes the sequence of Ruelle resonances of P. We want to use this expression with Lemma 4.40 in order to prove Theorem 10, but let us make an observation first. If λ is a complex number such that

$$\left|\frac{\lambda}{|\lambda|} - 1\right| \leq \frac{1}{2} \tag{4.40}$$

then we have $\operatorname{Re}\lambda \geq |\lambda|/2$. Hence, using the expression (12) and dominated convergence, we see that when $|\lambda|$ tends to $+\infty$ while satisfying (4.40), the function $d(\lambda)$ tends to 1. In particular, $d(\lambda)$ remains bounded when λ satisfies (4.40). Hence, we may assume in the following that

$$\left|\frac{\lambda}{|\lambda|} - 1\right| \geq \frac{1}{2}, \tag{4.41}$$

and that $|\lambda|$ is large of course. When $|\lambda|$ is large enough, we may apply Lemma 4.40 with $z = |\lambda|$. Then, notice that $Q_{|\lambda|}(\lambda)$ tends to 0 when $|\lambda|$ tends to $+\infty$ and thus we may ignore the factor $\exp(Q_z(\lambda))$ from Lemma 4.40. The other factor is given by (4.39) (with $z = |\lambda|$).

Notice that if $|\lambda_k| \geq 5|\lambda|$ then $\left|\frac{\lambda - |\lambda|}{\lambda_k - |\lambda|}\right| \leq \frac{1}{2}$ and hence

$$\begin{aligned} \log\left|E\left(\frac{\lambda - |\lambda|}{\lambda_k - |\lambda|}, m-1\right)\right| &\leq 2\left|\frac{\lambda - |\lambda|}{\lambda_k - |\lambda|}\right|^m \leq 2^{m+1}\left(\frac{|\lambda|}{|\lambda_k| - |\lambda|}\right)^m \\ &\leq 2\left(\frac{5}{2}\right)^m \left|\frac{\lambda}{\lambda_k}\right|^m. \end{aligned} \tag{4.42}$$

On the other hand if $|\lambda_k| < 5|\lambda|$, we have, since we assume (4.41),

$$\left|\frac{\lambda - |\lambda|}{\lambda_k - |\lambda|}\right| \geq \frac{1}{2}\frac{1}{\left|\frac{\lambda_k}{|\lambda|} - 1\right|} \geq \frac{1}{12}.$$

Hence, we have

$$\begin{aligned}\log\left|E\left(\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|},m-1\right)\right| &\leq \log\left|1-\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|+\sum_{\ell=1}^{m-1}\frac{1}{\ell}\left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|^{\ell}\\ &\leq \left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|+\sum_{\ell=1}^{m-1}\left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|^{\ell}\\ &\leq \left(\sum_{\ell=0}^{m-1}12^{m-1-\ell}\right)\left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|^{m-1}\\ &\leq 12^m\left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right|^{m-1}.\end{aligned} \tag{4.43}$$

Then, introduce a constant C such that $\operatorname{Re}\lambda_k \leq C$ for all $k \in \mathbb{N}$ (such a constant exists because P is the generator of a strongly continuous semi-group) and notice that for $|\lambda|$ large enough, we have

$$\left|\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|}\right| \leq \frac{2}{\left|\frac{\lambda_k-C}{|\lambda|}-1+\frac{C}{|\lambda|}\right|} \leq \frac{2}{1-\frac{C}{|\lambda|}} \leq 4.$$

And thus, (4.43) becomes:

$$\log\left|E\left(\frac{\lambda-|\lambda|}{\lambda_k-|\lambda|},m-1\right)\right| \leq \frac{48^m}{4}. \tag{4.44}$$

Now, gathering (4.42) and (4.44), that are valid respectively when $|\lambda_k| \geq 5|\lambda|$ and $|\lambda_k| < 5|\lambda|$, we find that

$$\begin{aligned}&\log\left|\det_m\left(I-(|\lambda|-\lambda)(|\lambda|-P)^{-1}\right)\right|\\ &\quad\leq 2\times\left(\frac{5}{2}\right)^m|\lambda|^m\sum_{|\lambda_k|\geq 5|\lambda|}|\lambda_k|^{-m}+\frac{48^m}{4}\#\left\{k\in\mathbb{N}:|\lambda_k|<5|\lambda|\right\}.\end{aligned} \tag{4.45}$$

Then, from the counting bound in Theorem 4.38, we see that

$$\#\left\{k\in\mathbb{N}:|\lambda_k|<5|\lambda|\right\}=\mathcal{O}\left(|\lambda|^{sn}\right), \tag{4.46}$$

and that,

$$\sum_{|\lambda_k|\geq 5|\lambda|}|\lambda_k|^{-m}=\mathcal{O}\left(|\lambda|^{sn-m}\right). \tag{4.47}$$

We end the proof of Theorem 10 by plugging (4.46) and (4.47) in (4.45). □

4.3 Further applications of the FBI transform

In order to lighten the exposition, we focused in this chapter on the proof of Theorem 10. However, [BJ20] contains other results on Anosov flows that we sum up now. Let consequently $s \geq 1$ and $(\phi_t)_{t\in\mathbb{R}}$ be a $\mathcal{G}^s$ Anosov flow on a $\mathcal{G}^s$ compact manifold M of dimension n. Let also $V : M \to \mathbb{C}$ be a $\mathcal{G}^s$ function. It is classical [FS11] that the $\mathcal{C}^\infty$ wave front sets of the resonant states for $P = X + V$ are contained in the stable[8] direction E_s^* defined in §4.2.1. In [BJ20, Definition 2.3], we define a Gevrey wavefront set for an ultradistribution, using an analytic FBI transform. With this definition, we are able to prove the following.

Proposition 4.41 (Proposition 3.2 in [BJ20]). *The $\mathcal{G}^s$ wave front sets of the resonant states for $P = X + V$ are contained in the stable direction E_s^*.*

Certain perturbations of the operator P are also studied in [BJ20]. The first kind of perturbations that we consider are of the form (for $\epsilon \geq 0$ small)

$$P_\epsilon = P + \epsilon\Delta, \tag{4.48}$$

where Δ is a self-adjoint elliptic non-positive differential[9] operator of order $m > 1$ with $\mathcal{G}^s$ coefficients – for instance the Laplace–Beltrami operator associated to a $\mathcal{G}^s$ Riemannian metric. On a dynamical level, the operator (4.48) appears when studying stochastic perturbations of the flow $(\phi_t)_{t\in\mathbb{R}}$. We see that when $\epsilon > 0$ the operator P_ϵ is elliptic of order m and has bounded from above real part, so that it has discrete spectrum on $L^2(M)$. We will denote by $\sigma_{L^2}(P_\epsilon)$ this spectrum. It has been proven by Dyatlov and Zworski [DZ15, Theorem 1] that $\sigma_{L^2}(P_\epsilon)$ converges to the Ruelle spectrum of $P = P_0$ when ϵ tends to 0. In our $\mathcal{G}^s$ context, we are able to give a "global" version of this result. To do so, we need to introduce a new distance to compare spectrum. If $z \in \mathbb{C}$, we define the distance d_z on $\mathbb{C} \cup \{\infty\} \setminus \{z\}$ by

$$d_z(x, y) = \left| \frac{1}{z - x} - \frac{1}{z - y} \right|.$$

Then our result on stochastic perturbations of $\mathcal{G}^s$ Anosov flow reads:

Proposition 4.42 (Theorem 9 in [BJ20]). *Let $p > ns$ and $z \in \mathbb{R}_+$ be large enough. Then*

[8] Beware that most of the references in the literature, in particular [FS11], do not work with the Koopman operator (9) but with its adjoint, the transfer operator. It amounts to reverse the direction of the time for the flow $(\phi_t)_{t\in\mathbb{R}}$, and consequently the stable and unstable direction are inverted. That is why it is often stated that the wave front sets of resonant states are contained in the unstable rather than stable direction.

[9] Actually, certain pseudo-differential operators are allowed in [BJ20].

there is $C > 0$ such that for $\epsilon > 0$ small enough we have

$$d_{z,H}\left(\sigma_{\mathrm{R}}\left(P\right) \cup \{\infty\}, \sigma_{L^2}\left(P_\epsilon\right) \cup \{\infty\}\right) \leq C \left|\ln \epsilon\right|^{-\frac{1}{p}}.$$

Here, $d_{z,H}$ denotes the Hausdorff distance associated to the distance d_z.

The proof of Proposition 4.42 is based on an investigation of the Schatten properties of the resolvent of the operator (4.48) and on resolvent bounds from [Ban04]. The convergence in Proposition 4.42 seems to be very weak. However, we think that it is not reasonable to expect too fast a convergence in such a global result. Indeed, when we add the differential operator Δ to P in order to form P_ϵ, since Δ has higher order, we can expect that the spectrum of P_ϵ looks globally like the spectrum of Δ, rather than like the Ruelle spectrum of P. Indeed, the higher order operator will be predominant at higher frequencies – this fact can be made rigorous looking for instance at the symbol of P_ϵ and using the ellipticity of Δ. Furthermore, the spectrum of Δ is contained in $\mathbb{R}_-$ while we expect some kind of vertical structure for the Ruelle spectrum of P (see for instance [JZ17, FT13]). Hence, we may expect $\sigma_{L^2}\left(P_\epsilon\right)$ to be some kind of "flattened" version of the Ruelle spectrum of P, and its global structure is thus very different from the actual Ruelle spectrum of P.

We also consider in [BJ20] deterministic perturbations of the Anosov flow $\left(\phi_t\right)_{t\in\mathbb{R}}$ – or *linear response*. To do so, we consider a perturbation

$$\epsilon \mapsto X_\epsilon$$

of our vector field $X = X_0$. Here, the perturbation is defined for ϵ in a neighbourhood of 0 and is assumed to be (at least) $\mathcal{C}^\infty$ from this neighbourhood of zero to a space of $\mathcal{G}^s$ sections of the tangent bundle of M (see Remark 4.4). We can also consider a perturbation $\epsilon \mapsto V_\epsilon$ with the same features and then form for ϵ near zero the operator

$$P_\epsilon = X_\epsilon + V_\epsilon. \tag{4.49}$$

Let us consider the most simple example: in the $\mathcal{C}^\infty$ case, if P_0 has a simple resonances λ_0, then it will extend to a $\mathcal{C}^\infty$ family of simple resonances $\epsilon \mapsto \lambda_\epsilon$ (provided that the perturbation is $\mathcal{C}^\infty$). However, even if the perturbation $\epsilon \mapsto X_\epsilon$ is real-analytic in the $\mathcal{C}^\infty$ category, then we do not know that the family $\epsilon \mapsto \lambda_\epsilon$ is real-analytic (in fact, it is reasonable to expect that it is not). We shall see below that if the perturbation $\epsilon \mapsto X_\epsilon$ is real-analytic in the real-analytic category, then the family $\epsilon \mapsto \lambda_\epsilon$ is real-analytic (this is an immediate consequence of Theorem 4.44). Using the notations from §4.2, we are able to prove the following result.

Proposition 4.43 (Proposition 3.4 in [BJ20]). *Assume that $\delta > 1/2$ (i.e. $s < 2$). Let $\ell \in \mathbb{R}_+ \setminus \mathbb{N}$. Assume that h and τ are small enough. Then for ϵ small enough, the spectrum* *of P_ϵ acting on $\mathcal{H}_\Lambda^0$ is the Ruelle spectrum of P_ϵ. Moreover, if $k = (\ell+1)\delta - \ell$ and r is a large enough positive real number, then the the map*

$$\epsilon \mapsto (r - P_\epsilon)^{-1} \in \mathcal{L}\left(\mathcal{H}_\Lambda^0, \mathcal{H}_\Lambda^k\right)$$

is $\mathcal{C}^\ell$ on a neighbourhood of 0.

The interesting point in Proposition 4.43 is that the loss of regularity when differentiating the resolvent of P_ϵ is mitigated by its smoothing property (given by Lemma 4.37). This new feature of our high regularity setting is even more striking when $s = 1$.

Theorem 4.44 (Theorem 11 in [BJ20]). *Assume that $s = 1$ and that the perturbations $\epsilon \mapsto X_\epsilon$ and $\epsilon \mapsto V_\epsilon$ are real-analytic. Assume that h and τ are small enough. Then, for $r \in \mathbb{R}_+$ large enough, the map*

$$\epsilon \mapsto (r - P_\epsilon)^{-1} \in \mathcal{L}\left(\mathcal{H}_\Lambda^0, \mathcal{H}_\Lambda^1\right)$$

is real-analytic on a neighbourhood of zero.

Theorem 4.44 allows to apply Kato theory on analytic perturbations of operators [Kat66]. We can prove in particular the following result, which is hardly surprising, but we are not aware of any proof in the literature devoted to this subject (see [KKPW89, Corollary 1] however for a related statement).

Theorem 4.45 (Theorem 2 in [BJ20]). *Let $\epsilon \mapsto X_\epsilon$ be a real-analytic family of real-analytic vector fields on a real-analytic manifold M, defined for ϵ near* 0. *We assume that X_0 generates an Anosov flow that admits a unique SRB measure*[10] *μ_0. Let μ_ϵ denote the unique SRB measure of the Anosov flow generated by X_ϵ, for ϵ near zero. Then the map*

$$\epsilon \mapsto \mu_\epsilon \in \mathcal{U}^1(M)$$

is real-analytic on a neighbourhood of zero.

Remark 4.46. In Proposition 4.43, we need to make the additional assumption $s < 2$. This is for technical, but definitely not anecdotal, reasons. A natural first step in the proof of Proposition 4.43 would be to prove that Proposition 4.36 and Lemma 4.37 still holds for ϵ near 0. This is true for Lemma 4.37 but *a priori* not for Proposition 4.36. This is because unfortunately the property (ii), from Lemma 4.35, of the escape function G_0 is not stable under small perturbations of the vector field X. This is quite an issue since Proposition 4.36 is what ensures that the resolvent set of P acting on $\mathcal{H}_\Lambda^0$ is non-empty and that its spectrum coincides with the Ruelle spectrum. To prove that the resolvent set

[10] See for instance [You02] for definition and discussion of the notion of SRB measure.

of P_ϵ remains non-empty when ϵ is near 0, we use another construction of the resolvent of P, based on an alternative version of Proposition 4.34 (namely [BJ20, Proposition 2.5]). However, this construction is restricted to the case $s < 2$, hence the additional assumption in Proposition 4.43. To prove that the spectrum of P_ϵ on $\mathcal{H}^0_\Lambda$ coincides with its Ruelle spectrum when ϵ is near 0, we use the characterization of the Ruelle spectrum as limit of stochastic perturbations [DZ15, Theorem 1].

Appendix A

Weighted transfer operators for ultradifferentiable expanding maps of the circle

When studying an expanding map of the circle T, it is sometimes useful to consider more general transfer operators that the one defined by (1). We explain here how the methods from §2.2 may be adapted to deal with weighted transfer operators.

If $\psi : \mathbb{S}^1 \to \mathbb{C}$ is a function we may define the weighted transfer operator $\mathcal{L}_\psi$ by

$$\mathcal{L}_\psi \varphi : x \mapsto \sum_{y:Ty=x} \frac{\psi(y)}{|T'(y)|} \varphi(y).$$

We will assume in the following that ψ is in $\mathcal{C}^A$, the Denjoy–Carleman class from §2.2. It is then easy to see that the analysis above remains true for the operator $\mathcal{L}_\psi$, so that we can state:

Proposition A.1. *Theorem 2.9 remains true when $\mathcal{L}$ is replaced by $\mathcal{L}_\psi$. Moreover, we may also define in this case the nuclear power decomposition* (2.17)*. This decomposition satisfies Propositions 2.16 and 2.17. Propositions 2.18 and 2.23 and Corollary 2.21 remain true as well if we replace the dynamical determinant d, from* (4)*, by d_ψ which is obtained from* (4) *by replacing* $\operatorname{tr}^\flat(\mathcal{L}^n)$ *by*

$$\operatorname{tr}^\flat\left(\mathcal{L}_\psi^n\right) = \sum_{x:T^n x = x} \frac{\prod_{k=0}^{n-1} \psi(T^k x)}{\left|1 - (T^n)'(x)\right|}.$$

To prove Proposition A.1, notice that the actual definition of $\mathcal{L}$ is only used in the proofs of Lemma 2.14 and Proposition 2.18 in the analysis from §2.2. The computation that gives Proposition 2.18 can still be carried out and will give the formula that we announced for the flat trace of the weighted transfer operator. Thus we will only explain how we can replace the operator $\mathcal{L}$ by $\mathcal{L}_\psi$ in the proof of Lemma 2.14.

Lemma A.2. *Lemma 2.14 remains true when $\mathcal{L}$ is replaced by $\mathcal{L}_\psi$.*

Proof. Recall the differential operator $L_{a_{k,\ell}}$ introduced in the proof of Lemma 2.14 and notice that for all $m \in \mathbb{N}$ we have

$$\langle \mathcal{L}_\psi e_\ell, e_k \rangle_{L^2} = \int_{\mathbb{S}^1} e^{2i\pi(\ell x - kT(x))} \psi(x) \mathrm{d}x = \int_{\mathbb{S}^1} e^{2i\pi(\ell x - kT(x)} L^m_{a_{k,\ell}}(\psi)(x) \mathrm{d}x,$$

and thus we want to bound $\left\| L^m_{a_{k,\ell}}(\psi) \right\|_\infty$ instead of $\left\| L^m_{a_{k,\ell}}(1) \right\|_\infty$. As in the proof of Lemma 2.14, we notice that there are natural integer coefficients that do not depend on $a_{k,\ell}$ nor ψ such that

$$L^m_{a_{k,\ell}}(\psi) = \sum_{n_1+\dots+n_m+k=m} c_{n_1,\dots,n_m,k} \psi^{(k)} \prod_{j=1}^m a_{k,\ell}^{(n_j)}. \tag{A.1}$$

Then, working as in the proof of Lemma 2.14 and using the fact that ψ is of class $\mathcal{C}^A$, we find constants $C, R > 0$ that do not depend on m, k or ℓ such that

$$\left\| L^m_{a_{k,\ell}}(\psi) \right\|_\infty \leq C \left(\frac{R}{|k|} \right)^m A_m \sum_{n_1+\dots+n_m+k=m} c_{n_1,\dots,n_m,k} k! \prod_{j=1}^m n_j!.$$

As in the proof of Lemma 2.14, we introduce now the operator L_a obtained by replacing the function $a_{k,\ell}$ in the definition of $L_{a_{k,\ell}}$ by $a : x \mapsto \frac{1}{1-x}$. Since the coefficients in (A.1) do not depend on $a_{k,\ell}$ nor ψ we have

$$L^m_a(a)(0) = \sum_{n_1+\dots+n_m+k=m} c_{n_1,\dots,n_m,k} k! \prod_{j=1}^m n_j!$$

but direct computation shows that $L^m_a(a) : x \mapsto \frac{2^m m!}{(1-x)^{2m+1}}$, and this ends the proof. $\square$

Appendix B

Ruelle resonances are intrinsic

As pointed out in the introduction, the Banach spaces that appear in Theorems 1, 3 and 5 are highly non-canonical. Hence, in order to see that the Ruelle spectrum (see Definitions 3, 5 and 7) is a well-defined object, we need a result to compare discrete spectrum of an operator acting on different spaces. For the discrete time case, we can for instance use the following lemma (which is slightly more general than [Bal18, Lemma A.3] or [BT08, Lemma A.1]).

Lemma B.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Let $\mathcal{B}$ be a Hausdorff topological vector space. Assume that there are continuous inclusions of $\mathcal{B}_1$ and $\mathcal{B}_2$ into $\mathcal{B}$, and that the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ is dense in both $\mathcal{B}_1$ and $\mathcal{B}_2$. For $i = 1, 2$, let $\mathcal{L}_i$ denotes a bounded operator on $\mathcal{B}_i$. Assume that the restrictions of $\mathcal{L}_1$ and $\mathcal{L}_2$ to $\mathcal{B}_1 \cap \mathcal{B}_2$ coincide. Let U be an unbounded connected open subset of $\mathbb{C}$. Assume that the intersections of the spectra of $\mathcal{L}_1$ and $\mathcal{L}_2$ with U only consist of isolated eigenvalues of finite algebraic multiplicities.*

Then, the intersection of the spectra of $\mathcal{L}_1$ and $\mathcal{L}_2$ with U coincide and so do the associated generalized eigenspaces (that are consequently included in $\mathcal{B}_1 \cap \mathcal{B}_2$).

Proof. Define a norm on $\mathcal{B}_1 + \mathcal{B}_2$ by

$$\|u\|_{\mathcal{B}_1+\mathcal{B}_2} \coloneqq \inf_{\substack{u=v+w \\ v\in\mathcal{B}_1, w\in\mathcal{B}_2}} \|v\|_{\mathcal{B}_1} + \|w\|_{\mathcal{B}_2}.$$

It follows from the fact that $\mathcal{B}$ is a Hausdorff topological space that this formula indeed defines a norm on $\mathcal{B}_1 + \mathcal{B}_2$. Moreover, $\mathcal{B}_1 + \mathcal{B}_2$ with this norm identifies with the quotient of $\mathcal{B}_1 \times \mathcal{B}_2$ by the closed subspace $\{\{u, v\} \in \mathcal{B}_1 \times \mathcal{B}_2 : u = v\}$, so that this norm makes $\mathcal{B}_1 + \mathcal{B}_2$ a Banach space. We may consequently replace $\mathcal{B}$ by $\mathcal{B}_1 + \mathcal{B}_2$ and assume that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ is a Banach space.

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide on the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$, we may define an operator $\mathcal{L}$ on $\mathcal{B}$ by setting $\mathcal{L}u = \mathcal{L}_1 v + \mathcal{L}_2 w$ if $u = v + w$ (this definition is independent of the choice of v and w). One easily checks that $\mathcal{L}$ is a bounded operator on $\mathcal{B}$, and induces a bounded operator on $\mathcal{B}_1 \cap \mathcal{B}_2$ that we endow with the complete norm

$$\|u\|_{\mathcal{B}_1 \cap \mathcal{B}_2} \coloneqq \|u\|_{\mathcal{B}_1} + \|u\|_{\mathcal{B}_2}.$$

It follows from the discussion in [Kat66, Chapter III, §5] that for $i = 1, 2$ the resolvent $(z - \mathcal{L}_i)^{-1}$ define a meromorphic family of bounded operators from $\mathcal{B}_i$ to itself on U, with residues of finite rank. We define then a meromorphic family on U of operators from $\mathcal{B}_1 \cap \mathcal{B}_2$ to $\mathcal{B}$ by

$$R_i(z) = j_{\mathcal{B}_i \hookrightarrow \mathcal{B}} \circ (z - \mathcal{L}_i)^{-1} \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_i},$$

where we denote by the letter j inclusion maps. Now, if $z \in U$ is such that $|z| > \max(\|\mathcal{L}\|, \|\mathcal{L}_1\|, \|\mathcal{L}_2\|)$, we can write

$$\begin{aligned} R_1(z) &= j_{\mathcal{B}_1 \hookrightarrow \mathcal{B}} \circ (z - \mathcal{L}_1)^{-1} \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_1} = j_{\mathcal{B}_1 \hookrightarrow \mathcal{B}} \circ \sum_{n \geq 0} z^{-n-1} \mathcal{L}_1^n \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_1} \\ &= \sum_{n \geq 0} z^{-n-1} j_{\mathcal{B}_1 \hookrightarrow \mathcal{B}} \circ \mathcal{L}_1^n \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_1} = \sum_{n \geq 0} z^{-n-1} \mathcal{L}^n \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}} \\ &= j_{\mathcal{B}_2 \hookrightarrow \mathcal{B}} \circ \sum_{n \geq 0} z^{-n-1} \mathcal{L}_2^n \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_2} = R_2(z). \end{aligned}$$

Since U is unbounded and connected, it follows then from the analytic continuation principle that the meromorphic map R_1 and R_2 coincides on U. Now, let $\lambda \in U$ be an element of the spectrum of $\mathcal{L}_1$. By assumption, this is an isolated eigenvalue of finite multiplicity, so that the associated spectral projector is given by

$$P_{\lambda,1} = \frac{1}{2i\pi} \int_\gamma (z - \mathcal{L}_1)^{-1} \, \mathrm{d}z, \tag{B.1}$$

where γ is a small enough circle around λ. We also know that $P_{\lambda,1}$ has finite rank. Replacing 1 by 2 in (B.1), we define similarly $P_{\lambda,2}$, which is the spectral projector associated to λ for $\mathcal{L}_2$ if λ belongs to the spectrum of $\mathcal{L}_2$ and 0 otherwise. Since $R_1 = R_2$, we find that

$$\begin{aligned} j_{\mathcal{B}_1 \hookrightarrow \mathcal{B}} \circ P_{\lambda,1} \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_1} &= \frac{1}{2i\pi} \int_\gamma R_1(z) \mathrm{d}z = \frac{1}{2i\pi} \int_\gamma R_2(z) \mathrm{d}z \\ &= j_{\mathcal{B}_2 \hookrightarrow \mathcal{B}} \circ P_{\lambda,2} \circ j_{\mathcal{B}_1 \cap \mathcal{B}_2 \hookrightarrow \mathcal{B}_2}. \end{aligned} \tag{B.2}$$

Now, for $i = 1, 2$, let $E_{\lambda,i}$ denotes the generalized eigenspace associated to λ for $\mathcal{L}_i$, that is the image of $P_{\lambda,i}$. Since this is a finite-dimensional space, its topology is unambiguously defined. By assumption, $\mathcal{B}_1 \cap \mathcal{B}_2$ is dense in $\mathcal{B}_i$, so that $P_{\lambda,i}(\mathcal{B}_1 \cap \mathcal{B}_2)$ is dense in $E_{\lambda,i}$. However, $P_{\lambda,i}(\mathcal{B}_1 \cap \mathcal{B}_2)$ is a subspace of the finite-dimensional vector space $E_{\lambda,i}$ and is thus closed in $E_{\lambda,i}$, so that

$$E_{\lambda,i} = P_{\lambda,i} \left(\mathcal{B}_1 \cap \mathcal{B}_2 \right). \tag{B.3}$$

The inclusion of $\mathcal{B}_1$ in $\mathcal{B}$ induce an inclusion of $E_{\lambda,i}$ in $\mathcal{B}$. Recalling (B.2), we deduce from (B.3) that

$$E_{\lambda,1} = P_{\lambda,1} \left(\mathcal{B}_1 \cap \mathcal{B}_2 \right) = P_{\lambda,2} \left(\mathcal{B}_1 \cap \mathcal{B}_2 \right) = E_{\lambda,2}.$$

Consequently, the spectrum of $\mathcal{L}_1$ in U is contained in the spectrum of $\mathcal{L}_2$ and the associated eigenspaces coincide. Since the roles of $\mathcal{L}_1$ and $\mathcal{L}_2$ are symmetric here, the lemma is proven. □

Example B.2. Let T be a $\mathcal{C}^\infty$ expanding map of the circle and $\mathcal{L}$ denote the associated transfer operator (1). Let $z \in \mathbb{C}^*$ and take $k, k' > 0$ such that $|z| > \max(\lambda^{-k}, \lambda^{-k'})$, where λ is a dilation constant for T. Recalling Theorem 1, we can then apply Lemma B.1 with $\mathcal{B}_1 = \mathcal{C}^k(\mathbb{S}^1)$, $\mathcal{B}_2 = \mathcal{C}^{k'}(\mathbb{S}^1)$, $\mathcal{B} = \mathcal{C}^0(\mathbb{S}^1)$ and $U = \{w \in \mathbb{C} : |w| > \max(\lambda^{-k}, \lambda^{-k'})\}$. It implies that z is an eigenvalue of $\mathcal{L}$ acting on $\mathcal{C}^k(\mathbb{S}^1)$ if and only if it is an eigenvalue of $\mathcal{L}$ acting on $\mathcal{C}^{k'}(\mathbb{S}^1)$. Moreover, when z is actually an eigenvalue the associated generalized eigenspaces coincide. This argument justifies Definition 3 and proves that the resonant states associated to a $\mathcal{C}^\infty$ expanding map of the circle are $\mathcal{C}^\infty$. A similar reasoning can be made in the case of Anosov diffeomorphism to justify Definition 5 (in that case we will take a space of distributions for $\mathcal{B}$).

We give now the results that we will use to identify Ruelle resonances for Anosov flows. This is just an abstract version of the argument based on the meromorphic extension of the resolvent presented in the introduction. However, since we will be working with non-standard spaces of ultradifferentiable functions and generalized distributions, it will be useful to have a general result at our disposal that do not make reference to a particular class of regularity.

Lemma B.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Let $\mathcal{B}$ be a Hausdorff topological vector space. Assume that there are continuous inclusions of $\mathcal{B}_1$ and $\mathcal{B}_2$ into $\mathcal{B}$, and that the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ is dense in both $\mathcal{B}_1$ and $\mathcal{B}_2$. For $i = 1, 2$, let $\left(\mathcal{L}_{i,t}\right)_{t \geq 0}$ be a strongly continuous semi-group of operator on $\mathcal{B}_i$ whose generator is denoted by X_i. Assume that for every $t \geq 0$ the restrictions of $\mathcal{L}_{1,t}$ and $\mathcal{L}_{2,t}$ to $\mathcal{B}_1 \cap \mathcal{B}_2$ coincide. Let U be a connected open subset of $\mathbb{C}$ that contain points with arbitrarily large positive real parts. Assume that the intersections of the spectra of X_1 and X_2 with U only contain isolated eigenvalues of finite algebraic multiplicities.*

Then, the intersections of the spectra of X_1 and X_2 with U coincide and so do the associated generalized eigenspaces (that are consequently contained in $\mathcal{B}_1 \cap \mathcal{B}_2$).

Proof. The proof is basically the same as for Lemma B.1. The only difference is that we replace the representation of the resolvent as a power series by the representation as a Laplace transform

$$(z - X_i)^{-1} = \int_0^{+\infty} e^{-zt}\mathcal{L}_{i,t}\mathrm{d}t$$

which is valid for $i = 1, 2$ when $\mathrm{Re}\, z \gg 1$. $\square$

Appendix C

Factorization of the dynamical determinant

We establish here, under the hypotheses of Theorem 3.1, a Hadamard-like factorization (C.3) for the dynamical determinant $d(z)$ defined by (12). We use the same notations as in §3.4 and §3.5. Let $t_0 > 0$ be shorter than any periodic orbit of $(\phi_t)_{t\in\mathbb{R}}$. Then, working as in the proof of Proposition 3.3, we see that, for $\operatorname{Re} z \gg 1$, the essential spectral radius of

$$\mathcal{L}_{t_0}(z-P)^{-(d+2)} = \frac{1}{(d+1)!}\int_{t_0}^{+\infty} e^{-z(t-t_0)}(t-t_0)^{d+1}\mathcal{L}_t \mathrm{d}t : \mathcal{H} \to \mathcal{H} \tag{C.1}$$

is zero. Here, as in §3.4 and 3.5, the operator $P = X + V$ denotes the generator of the semi-group $(\mathcal{L}_t)_{t\geq 0}$ defined by (9). Then, applying holomorphic functional calculus in finite dimension as in the proof of Lemma 3.44, we see that the spectrum of (C.1) is made of the $\frac{e^{\lambda t_0}}{(z-\lambda)^{d+2}}$ for λ in the spectrum of P. Then, for $\operatorname{Re} z \gg 1$, Proposition 3.32 implies that the right-hand side of (C.1) defines a trace class operator on $\widetilde{\mathcal{H}}_0$. From Lemma B.1, we see that the spectrum of (C.1) is the same when acting on $\mathcal{H}$ or on $\tilde{h}_0$. Then, using Lidskii's Trace Theorem and Proposition 3.32, we see that[1],

$$\sum_{\lambda\in\sigma_{\mathrm{R}}(P)} \frac{e^{\lambda t_0}}{(z-\lambda)^{d+2}} = \frac{1}{(d+1)!}\sum_{\gamma} T_\gamma^{\#} \exp\left(\int_\gamma V\right)(T_\gamma - t_0)^{d+1}\frac{e^{-z(T_\gamma - t_0)}}{\left|\det\left(I-\mathcal{P}_\gamma\right)\right|}.$$

For all $\lambda \in \mathbb{C}\setminus\{0\}$ notice that the meromorphic map

$$z \mapsto -\sum_{n\geq d+1}\frac{z^n}{\lambda^{n+1}}e^{-(z-\lambda)t_0} = \frac{e^{-(z-\lambda)t_0}}{z-\lambda} + \sum_{n=0}^{d}\frac{z^n}{\lambda^{n+1}}e^{-(z-\lambda)t_0}$$

[1] Notice that the trace formula (TFF) may be deduced from this equality using residue's formula as in the proof of Proposition 1.27.

has a unique pole at λ whose order is 1 and whose residue is 1. Thus there is an entire function G_{λ,t_0} such that for all $z \in \mathbb{C}$

$$\frac{G'_{\lambda,t_0}(z)}{G_{\lambda,t_0}(z)} = -\sum_{n\geq d+1} \frac{z^n}{\lambda^{n+1}} e^{-(z-\lambda)t_0} = \frac{e^{-(z-\lambda)t_0}}{z-\lambda} + \sum_{n=0}^{d} \frac{z^n}{\lambda^{n+1}} e^{-(z-\lambda)t_0}$$

and $G_{\lambda,t_0}(0) = 1$. Choose for G_{0,t_0} any logarithmic primitive of $z \mapsto \frac{e^{-t_0 z}}{z}$. Now, let $R > 0$ and assume that $\lambda \in \sigma_{\mathrm{R}}(P)$ is such that $|\lambda| \geq 2R$. Notice that for all $z \in \mathbb{D}(0,R)$ we have

$$\left| \frac{G'_{\lambda,t_0}(z)}{G_{\lambda,t_0}(z)} \right| \leq 2e^{Rt_0} R^{d+1} \frac{e^{\mathrm{Re}(\lambda)t_0}}{|\lambda|^{d+2}}$$

and using the fact that G_{λ,t_0} has a logarithm on $\mathbb{D}(0,R)$ that vanishes at 0 (since G_{λ,t_0} vanishes only at λ) we get that, for some constant C depending only on R and all $z \in \mathbb{D}(0,R)$,

$$|1 - G_{\lambda,t_0}(z)| \leq C \frac{e^{\mathrm{Re}(\lambda)t_0}}{|\lambda|^{d+2}}.$$

Using Proposition 3.3, this implies that the infinite product

$$\widetilde{d}(z) = \prod_{\lambda \in \sigma_{\mathrm{R}}(P)} G_{\lambda,t_0}(z)$$

converges uniformly on all compact subsets of $\mathbb{C}$. Notice that the zeros of $\widetilde{d}(z)$ are precisely the Ruelle resonances of P. Now, we find that

$$\left(e^{zt_0} \left(\ln G_{\lambda,t_0}(z)\right)'\right)^{(d+1)} = (-1)^{d+1} (d+1)! \frac{e^{\lambda t_0}}{(z-\lambda)^{d+2}}$$

and thus, for $\mathrm{Re}\, z \gg 1$,

$$\begin{aligned} \left(e^{zt_0} \left(\ln \widetilde{d}(z)\right)'\right)^{(d+1)} &= (-1)^{d+1} (d+1)! \sum_{\lambda \in \sigma_{\mathrm{R}}(P)} \frac{e^{\lambda t_0}}{(z-\lambda)^{d+2}} \\ &= (-1)^{d+1} \sum_{\gamma} T_\gamma^{\#} \exp\left(\int_\gamma V\right) (T_\gamma - t_0)^{d+1} \frac{e^{-z(T_\gamma - t_0)}}{|\det(I - \mathcal{P}_\gamma)|} \\ &= \left(e^{zt_0} \sum_\gamma T_\gamma^{\#} \exp\left(\int_\gamma V\right) \frac{e^{-zT_\gamma}}{|\det(I - \mathcal{P}_\gamma)|}\right)^{(d+1)} \\ &= \left(e^{zt_0} (\ln d(z))'\right)^{(d+1)}, \end{aligned} \tag{C.2}$$

where $d(z)$ is the usual dynamical determinant defined by (12). From (C.2), we deduce that there are a polynomial F of degree at most d and $\mu \in \mathbb{C}$ such that, for all $z \in \mathbb{C}$, we have the Hadamard-like factorization

$$d(z) = \mu \exp\left(F(z)\, e^{-t_0 z}\right) \prod_{\lambda \in \sigma_{\mathrm{R}}(P)} G_{\lambda,t_0}(z). \tag{C.3}$$

In order to make this factorization more explicit, let us describe the G_{λ,t_0}'s. For all $\lambda \in \mathbb{C} \setminus \{0\}$, define the polynomial

$$Q_{\lambda,t_0} = -\sum_{k=0}^{d} \left(\sum_{n=k}^{d} \frac{k!}{n!} \frac{(t_0 - \lambda)^{n-k-1}}{\lambda^{k+1}} \right) X^k,$$

and notice that

$$\left(Q_{\lambda,t_0}(z)\, e^{-z(t_0-\lambda)}\right)' = \sum_{n=0}^{d} \frac{z^n}{\lambda^{n+1}} e^{-(z-\lambda)t_0}.$$

Thus we have for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$

$$\begin{aligned} G_{\lambda,t_0}(z) = \left(1 - \frac{z}{\lambda}\right) \exp\left(Q_{\lambda,t_0}(z)\, e^{-(z-\lambda)t_0} - Q_{\lambda,t_0}(0) e^{\lambda t_0}\right) \\ \times \exp\left(z \int_0^1 \frac{e^{-(zu-\lambda)t_0} - 1}{zu - \lambda} \mathrm{d}u\right). \end{aligned}$$

The last factor is a logarithmic primitive of $z \mapsto \frac{e^{-(z-\lambda)t_0} - 1}{z - \lambda}$.

Appendix D

About the condition $\upsilon < 2$

In order to discuss the condition $\upsilon < 2$ in Theorem 3.1 and Corollary 3.2, let us get back to the simplest example of hyperbolic systems: the doubling map from Example 1. The reader may also refer to the discussion about Denjoy–Carleman classes that are closed under differentiation in Chapter 2, see in particular Remarks 2.5 and 2.22, and the examples from §2.2.3. Replacing an Anosov flow by the doubling map, an analogue of the space $\mathcal{H}$ from Theorem 3.1 would then be an isotropic space of the type (here $(\hat{f}(n))_{n\in\mathbb{Z}}$ denotes the sequence of Fourier coefficient of a function f)

$$\mathcal{H}_{\alpha,\beta} = \left\{ f \in \mathcal{C}^{\infty}\left(\mathbb{S}^1, \mathbb{C}\right) : \sum_{n \in \mathbb{Z}} \left| \hat{f}\left(n\right) \right|^2 e^{2\beta \ln(1+|n|)^{\frac{1}{\alpha}}} < +\infty \right\},$$

where $\beta > 0$ and $\alpha \in \left]\frac{\upsilon-1}{\upsilon}, 1\right[$ (this is the same condition as in Proposition 3.17), endowed with the norm

$$\|f\|_{\alpha,\beta} = \sqrt{\sum_{n\in\mathbb{Z}} \left|\hat{f}\left(n\right)\right|^2 e^{2\beta \ln(1+|n|)^{\frac{1}{\alpha}}}}.$$

Then the transfer operator

$$\mathcal{L} : f \mapsto \frac{f\left(\frac{\cdot}{2}\right) + f\left(\frac{\cdot+1}{2}\right)}{2}$$

associated to the doubling map[1] may be written as

$$\mathcal{L} = \sum_{n\in\mathbb{Z}} \langle \cdot, e_{2n} \rangle_{L^2} e_n,$$

where $e_n : x \mapsto e^{2i\pi n x}$ (the sum converges in strong operator topology on the space of continuous endomorphisms of $\mathcal{H}_{\alpha,\beta}$). Thus, the singular values of $\mathcal{L}$ acting on $\mathcal{H}_{\alpha,\beta}$ are the

[1]This is just the transfer operator from (1) when T is the doubling map from Example 1.

$e^{\beta\left(\ln(1+|n|)^{\frac{1}{\alpha}}-\ln(1+2|n|)^{\frac{1}{\alpha}}\right)}$ for $n \in \mathbb{Z}$. Using the fact that

$$\ln\left(1+|n|\right)^{\frac{1}{\alpha}} - \ln\left(1+2\left|n\right|\right)^{\frac{1}{\alpha}} \underset{|n|\to+\infty}{=} -\frac{\ln 2}{\alpha}\ln\left(1+|n|\right)^{\frac{1}{\alpha}-1} + O\left(\ln\left(1+|n|\right)^{\frac{1}{\alpha}-2}\right)$$

we see that $\mathcal{L}$ acting on $\mathcal{H}_{\alpha,\beta}$ is trace class when $\alpha < \frac{1}{2}$ and is not trace class when $\alpha > \frac{1}{2}$ (in the case $\alpha = \frac{1}{2}$ it depends on the value of β, it corresponds to the case $\upsilon = 2$ in §2.2.3). Thus, we need to chose $\alpha < \frac{1}{2}$ if we want $\mathcal{L}$ to be nuclear. For general maps, this choice is possible only when $\upsilon < 2$ (see the condition in Proposition 3.17).

Consequently, using our method to prove the trace formula (TFF) for $\mathcal{C}^{\kappa,\upsilon}$ Anosov flows would require to construct Hilbert spaces in a totally different way, if $\upsilon > 2$. The case $\upsilon = 2$ also seems to be tricky, but the results from §2.2, and in particular 2.2.3, suggest that maybe something interesting can be proven in that case.

Appendix E

Heuristic argument

We detail here an argument that suggests that the dynamical determinant (defined by (4) and (5)) associated to a generic perturbation of the doubling map, Example 1, has infinite order (hence supporting Conjecture 1). We denote by T the doubling map and by $t \mapsto T_t$ a perturbation of T in the $\mathcal{C}^\infty$ topology (such that $T_0 = T$) . We also assume that this perturbation is $\mathcal{C}^\infty$ in t. We denote by $\mathcal{L}_t$ the transfer operator (1) associated to T and by d_t the associated dynamical determinant (4). Let us write for $x \in \mathbb{S}^1$

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left(T_t'(x)\right)_{|t=0}, \tag{E.1}$$

and notice that for $n \in \mathbb{N}$ and $x \in \mathbb{S}^1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(T_t^n\right)'(x)\right)_{|t=0} = 2^{n-1}\sum_{k=0}^{n-1} f \circ T^k.$$

Using the Implicit Function Theorem to follow periodic point of T_t, we find then that for $n \in \mathbb{N}^*$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{tr}^\flat\left(\mathcal{L}_t^n\right)\right)_{|t=0} &= -\sum_{T^n x = x} \frac{\left(T^n\right)''(x)\frac{\mathrm{d}x}{\mathrm{d}t} + 2^{n-1}\sum_{k=0}^{n-1} f \circ T^k(x)}{\left(\left(T^n\right)'(x) - 1\right)^2} \\
&= -\frac{2^{n-1}}{\left(2^n - 1\right)^2}\sum_{T^n x = x}\sum_{k=0}^{n-1} f \circ T^k(x) \\
&= -\frac{n 2^{n-1}}{\left(2^n - 1\right)^2}\sum_{T^n x = x} f(x) = -\frac{n 2^{n-1}}{2^n - 1} a_n(f),
\end{aligned}$$

where we introduced for $n \in \mathbb{N}^*$ the linear form a_n defined by

$$a_n(f) = \frac{1}{2^n - 1}\sum_{k=0}^{2^n - 2} f\left(\frac{k}{2^n - 1}\right).$$

Consequently, we find that for $z \in \mathbb{C}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(d_t(z)\right)_{|t=0} = (1-z)\underbrace{\sum_{n\geq 1}\frac{2^{n-1}}{2^n-1}a_n(f)}_{:=h_f(z)}. \tag{E.2}$$

We denote by E the space of $\mathcal{C}^\infty$ functions from $\mathbb{S}^1$ to $\mathbb{R}$ with zero average. We endow E with the $\mathcal{C}^\infty$ topology. Notice that f belongs to E and that all functions in E may be realized as (E.1) by choosing the perturbation $t \mapsto T_t$. We will prove that the derivative (E.2) is an entire function of the variable z that grows arbitrarily fast for generic $f \in E$. Since $d_0(z) = 1 - z$, this suggests that for t near 0 the dynamical determinant $d_t(z)$ grows arbitrarily fast for generic X. Indeed, we have

$$d_t(z) \underset{t\to 0}{=} (1-z)\left(1 + t h_f(z) + \mathcal{O}(t^2)\right). \tag{E.3}$$

This asymptotic expansion *a priori* only holds[1] uniformly on all compact subsets of $\mathbb{C}$, so that it cannot be used to investigate the growth of $d_t(z)$, but it can certainly guide our intuition.

It follows from Euler–MacLaurin's formula [Kre98, Corollary 9.27] that for every $f \in E$ the sequence $(a_n(f))_{n\in\mathbb{N}}$ decreases faster than any geometric sequence, so that h_f is an entire function. In order to see that any bound on the growth of h_f fails generically, we only need to prove the following – due to Cauchy's formula.

Lemma E.1. *Let $(b_n)_{n\geq 1}$ be a sequence of strictly positive real numbers that decays faster than any geometric sequence. Then there is a G_δ dense subset A of E such that for every $f \in A$ we have*

$$|a_n(f)| \geq b_n$$

for infinitely many n's.

Proof. We start by constructing a particular element $g \in E$. Let $\mathcal{P} \subseteq \mathbb{N}^*$ denotes the set of prime integers. For every $p \in \mathcal{P}$ define the function

$$e_p : x \mapsto \exp\left(2i\pi\left(2^p - 1\right)x\right).$$

Then, notice that for every $p, q \in \mathcal{P}$, we have

$$a_{2^p-1}\left(e_q\right) = \delta_{p,q}.$$

[1]The estimate (E.3) may be established using the methods from [Jéz19b]. We asserted there that there was an issue in the presence of non-simple resonances, but we realized since then that it was not the case.

Indeed, if $p \neq q$, then $2^p - 1$ and $2^q - 1$ are relatively prime[2]. Then let

$$g = \sum_{p \in \mathcal{P}} \sqrt{b_{2^p-1}} e_p.$$

Since $\left(\sqrt{b_n}\right)_{n \geq 1}$ decays faster than any geometric sequence, we see that g belongs to E. Moreover, for every $p \in \mathcal{P}$, we have

$$a_{2^p-1}(g) = \sqrt{b_{2^p-1}}.$$

Now, for every $m \in \mathbb{N}^*$, we define

$$B_m = \{f \in E : \forall n \geq m : |a_n(f)| \leq b_n\}.$$

The B_m's are closed subsets of E. Letting $A = E \setminus \cup_{m \geq 0} B_m$, we only need to show that the B_m's have empty interior. The lemma then follows by an application of Baire category theorem. Let consequently $m \in \mathbb{N}^*$ and $f \in B_m$. Then, for p a prime integer greater than m and $\epsilon > 0$ we have

$$|a_{2^p-1}(f + \epsilon g)| \geq \epsilon |a_{2^p-1}(g)| - |a_{2^p-1}(f)| \geq \epsilon \sqrt{b_{2^p-1}} - b_{2^p-1},$$

and if p is large enough, we have $\sqrt{b_{2^p-1}} > 2 b_{2^p-1}/\epsilon$, since $(b_n)_{n \geq 1}$ tends to 0. Hence, for every $\epsilon > 0$ we have $f + \epsilon g \notin B_m$, so that B_m has empty interior. $\square$

[2] More generally, if $m, n \in \mathbb{N}^*$, then the greatest common divisor of $2^m - 1$ and $2^n - 1$ is $2^{m \wedge n} - 1$, where $m \wedge n$ denotes the greatest common divisor of m and n.

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