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F-theory in Eight Dimensions: an Exceptional Field Theory and Heterotic String Perspective

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CHAPTER 1

Introduction

General Context

The twentieth century saw spectacular advances in our comprehension of the universe and its governing laws, from quantum mechanics to the standard model, and special relativity to general relativity. Special relativity, theorised in 1905 after prior observations from the electromagnetism theory of Maxwell completely rejected the notion of absolute time and absolute space. This led Einstein ten years later to the construction of general relativity in which the gravitational interaction is the geometry of space-time. Quantum mechanics on the other hand revolutionised our understanding of particle physics. Particles can no longer be interpreted as point-like objects or waves but rather both, described mathematically by a complex valued probability amplitude. Including the formalism of special relativity to quantum mechanics then contributed to the creation of quantum field theory and the standard model, a theoretical description of fundamental particles and their interactions: electromagnetic, weak and strong forces.

General relativity has been widely verified by various observations such as gravitational lensing, the perihelion advance of Mercury or more straightforwardly the direct measurement of time

shifts between atomic clocks. This culminated in the last few years with the detection of gravitational waves [1] as well as the direct observation of a black hole [2]. The formalism of quantum field theory on the other hand predicted successfully the existence of different particles such as the tau particle and the Brout-Englert-Higgs (BEH) boson [3]. Both general relativity and quantum field theory permitted the creation of a new generation of measuring equipment which strongly changed our ability to approach science in a vast variety of domains from physics and chemistry to medicine, geophysics, archaeology,... It is not out of place to say that the society as we currently know it, and our day-to-day life, would be strikingly different without technologies such as transistors and satellite positioning systems which were created thanks to these theories.

Unfortunately, there remain various observations for which general relativity and the standard model both fail to give a proper description. The observations of the oscillations of neutrinos [4] for example can only occur if at least two of them have a mass, which is not the case in the standard model. Maybe even more dramatically, the observation of large-scale phenomena such as the formation or rotations of galaxies and the expansion of the universe indicate either some deviation from general relativity or the inclusion of dark matter and dark energy. Neither gravitational interferometers nor particle accelerators were able to show any deviation from general relativity or were able to detect particles unpredicted by the standard model. Finally, one of the biggest conundrum of modern physics is to be able to describe phenomenon such as the big bang or black holes in which both general relativity and quantum mechanics play a role.

Black holes in general relativity are defined as a region of the universe from which nothing can escape, whether it be matter or light. They are among the possible ending scenarios of a dying star whose internal pressure is too weak to compensate its own gravitational force. In a sense, general relativity already fails to describe such objects, as it is possible for a black hole to have a region with gravitational singularity, in which the curvature of space-time becomes infinite. Hawking proved later that if one considers quantum field theory in a black hole background, the black hole behaves as a black body and therefore evaporates. This leads to the information loss puzzle. Resolving this issue has as a consequence rejecting at least one postulate from quan-

tum mechanics, or general relativity, namely the unitary transformations of quantum processes or the equivalence principle. All of these incompatibilities led the scientific community to seek for a quantum theory of gravity. Among current research concerning this unification is string theory, the formalism on which this thesis is based on.

String Theory

String theory was originally constructed as a possible explanation of the strong interaction, and was later dismissed in favour of quantum chromodynamics. It was then proposed as a possible theory of quantum gravity as well as a unifying formalism of all fundamental forces. Schematically, string theory can be seen as a generalisation of quantum field theory, describing one-dimensional objects called strings instead of point-like particles. In order to include fermionic states, one is forced to consider supersymmetric theories. In the end, one obtains five consistent superstring theories in ten dimensions: Type IIA, Type IIB, Type I and the $E_8 \times E_8$ and $SO(32)$ heterotic string theories. The issue raised by the dimension of an extended space-time is resolved by considering that additional space dimensions are compact. The four-dimensional manifestations or our universe are then low energy approximations of a more fundamental theory with more dimensions.

String theory is not only a theory of strings, but also of branes, which can extend over more than one spatial dimension. They are essential in type II and type I string theory in order to recover gauge groups compatible with the standard model and are the electric and magnetic charges of various fields in string theory.

The five ten dimensional superstring theories are in fact different facets of a unifying eleven dimensional one named M-theory. Various dualities relate each of them after compactifications and have been extensively studied in recent years. This thesis addresses such dualities in eight dimensions and focuses on F-theory [5]. F-theory is twelve dimensional and can be understood as type IIB string theory with 7-branes. It has various applications in non-perturbative quantum field theory, particle physics model building and makes the connection between physics and

algebraic geometry [6].

In this thesis, we first focus on possible links between F-theory and the so called exceptional field theory. Exceptional field theory is a quantum field theory for point particles that incorporates the stringy symmetries of string theory. In particular, it provides a low energy description of type IIB, and the string theory symmetries are manifest. We therefore discuss the possibility for exceptional field theory to describe aspects of F-theory.

Subsequently, we focus on the duality between F-theory and the heterotic string in eight dimensions. The heterotic string provides a vast variety of possible gauge groups after compactification, determined by its Wilson lines. In type IIB, the gauge groups appear from stacks of branes, and in F-theory this is due to the algebraic structure of the space on which one compactifies, in our case K3 surfaces. A convenient way to construct K3 surfaces is to use three-dimensional polytopes. The duality between K3 surfaces on the F-theory side and Wilson lines on the heterotic one is only well understood for two of the 4319 possible K3 surfaces constructed via polytopes [7, 8]. We thus focus on the dualities of F-theory with the heterotic string for other polytopes in the third part of this manuscript.

Organization of the thesis

This thesis contains three parts. Part I is an introduction to fundamental concepts of string theory. Part II focuses on Exceptional Generalised Geometry (EGG) and Exceptional Field Theory (EFT), in particular $E_{3(3)} \times \mathbb{R}^+$ EFT in eight dimensions and its link to F-theory. In Part III we discuss the duality between F-theory on elliptic K3 surfaces and the heterotic string theory compactified on a two torus. We provide new insight as to find an explicit map between reflexive polytopes defining K3 surfaces and Wilson lines.

Chapter 1 and 2 are the introductions (in English and French).

In chapter 3, we introduce the notion of compactification and discuss Kaluza-Klein and Scherk-Schwarz examples. We present the motivation to use Calabi-Yau compactifications in string

theory. We then briefly discuss T and S dualities. After a discussion on the link between electromagnetic duality with magnetic monopoles and strong/weak duality, we present the manifestation of the continuous $SL(2,\mathbb{R})$ symmetry of type IIB supergravity and its restriction to its discrete subgroup $SL(2,\mathbb{Z})$ in the quantum case.

In chapter 4, we look at the action of T and S dualities on branes. We start by S-duality in the case of Type IIB string theory which forces one to consider manifestly $SL(2,\mathbb{Z})$ branes. We then discuss the notion of monodromy with emphasise on (p,q) branes. We show that the S-duality transformations of the axio-dilaton is a modular invariance of an elliptic curve, which is the central notion of the construction of F-theory. We look at the consequences of T and more generally U-duality on branes. We conclude with a discussion on the web of dualities between different superstring theories.

In chapter 5, we present basic notions of Double Field Theory (DFT), Generalised Geometry (GG) and their U-dual extensions Exceptional Field Theory (EFT) and Exceptional Generalised Geometry (EGG). We start with a few reminders of the different steps necessary to obtain the Ricci tensor in Riemannian geometry. We then discuss how to incorporate a $O(d,d,\mathbb{R})$ symmetry into such a formalism, first by considering an extension of the fiber (GG), then by doubling the number of coordinates (DFT). We briefly describe the generalisations to U-duality.

In chapter 6, we present the results of [9]. We consider a non-trivial solution to the section condition in the context of $E_{3(3)} \times \mathbb{R}^+$ exceptional field theory. We show that allowing fields to depend on the additional stringy coordinates of the extended internal space permits to describe the monodromies of (p,q) 7-branes in the context of F-theory. General expressions of non-trivial fluxes with associated linear and quadratic constraints are obtained via a comparison to the embedding tensor of eight dimensional gauged maximal supergravity with gauged trombone symmetry. We write a generalised Christoffel symbol for $E_{3(3)} \times \mathbb{R}^+$ EFT and show that the equations of motion of F-theory, namely the vanishing of a 4 dimensional Ricci tensor with two of its dimensions fibered, can be obtained from a generalised Ricci tensor and an appropriate type IIB ansatz for the metric.

In chapter 7, we detail aspects of the compactifications of F-theory on elliptic K3 surfaces and the compactifications of the heterotic string theory on a two torus. We start by describing the heterotic string on T^2 and its moduli. We give some examples of gauge group enhancements and breaking in this context. Then, we present some general notions on the cohomology structure of Calabi-Yau surfaces. We discuss in more detail the homology and cohomology structure of K3 surfaces as well as their moduli space. We show how to construct K3 surfaces using reflexive polytopes, as well as their possible fibrations. Finally, we discuss the moduli structure of such parametrisations of K3 surfaces.

In chapter 8, we present the results of [10]. We show how to construct elliptically fibered K3 surfaces via Weierstrass models which can be parametrized in terms of Wilson lines in the dual heterotic string theory. We work with a subset of reflexive polyhedra that admit two fibrations and whose moduli spaces contain the ones of the $E_8 \times E_8$ or $\frac{Spin(32)}{\mathbb{Z}_2}$ heterotic theory compactified on a two-torus without Wilson lines. One can then interpret the additional moduli as a particular Wilson line in the heterotic string. A convenient way to find such polytopes is to use graphs of polytopes where links are related to inclusion relations of moduli spaces of different fibers. We are then able to map monomials in the defining equations of particular K3 surfaces to Wilson line moduli in the dual theories. We developed three Sagemath programs which permitted us to construct graphs giving the gauge group for a generic point in the moduli space, the Weierstrass model as well as basic enhancements of the generic gauge group, obtained by sending coefficients of the hypersurface equation defining the K3 surface to zero.

In chapter 9, we present preliminary results of an upcoming paper written in collaboration with Bernardo Fraiman [11]. We show that in the case of a specific polytope admitting five inequivalent fibration and two moduli, the generic gauge groups for each fibrations can be interpreted as coming from a $\mathbb{Z}_3$ shift vector. Different fibrations are obtained by splitting the shift vector differently between the two E_8 lattices and on the $SO(32)$ heterotic string. We then discuss other polytopes with two moduli which could possibly be described with $\mathbb{Z}_n$ shift vectors.

CHAPTER 2

Introduction en Français

Contexte Général

Le XXe siècle a vu des avancées spectaculaires vis-à-vis de notre compréhension de l'univers et des lois qui le gouvernent, de la mécanique quantique au modèle standard, et de la relativité restreinte à la relativité générale. La relativité restreinte, théorisée en 1905 après les observations antérieures de la théorie de l'électromagnétisme de Maxwell, a complètement rejeté la notion de temps et d'espace absolu. Cela a conduit Einstein dix ans plus tard à la construction de la relativité générale, dans laquelle l'interaction gravitationnelle est la géométrie de l'espace-temps. D'un autre côté, la mécanique quantique a révolutionné notre compréhension de la physique des particules. Celles-ci ne peuvent plus être interprétées comme des objets ponctuels, ou des ondes, mais les deux, et son décrites mathématiquement par une amplitude de probabilité à valeur complexe. L'intégration du formalisme de la relativité restreinte à la mécanique quantique a ensuite contribué à la création de la théorie quantique des champs et du modèle standard, une description théorique des particules fondamentales et de leurs interactions: forces électromagnétique, faible et forte.

La relativité générale a été largement vérifiée par diverses observations telles que les lentilles gravitationnelles, l'avance du périhélie de Mercure ou plus directement la mesure des décalages temporels entre horloges atomiques. Cela a culminé ces dernières années avec la détection d'ondes gravitationnelles [1] ainsi que l'observation directe d'un trou noir [2]. Le formalisme de la théorie quantique des champs, quant à lui, a prédit avec succès l'existence de différentes particules telles que la particule tau et le boson de Brout-Englert-Higgs (BEH) [3]. La relativité générale et la théorie quantique des champs ont permis de créer une nouvelle génération d'équipements de mesure qui a fortement modifié notre capacité à aborder la science dans une grande variété de domaines allant de la physique et de la chimie à la médecine, la géophysique, l'archéologie, ... Il n'est pas hors de propos de dire que la société telle que nous la connaissons actuellement, ainsi que notre vie quotidienne, seraient remarquablement différentes sans les technologies telles que les transistors et les systèmes de positionnement par satellite qui ont été créés grâce à ces théories.

Malheureusement, il reste diverses observations pour lesquelles la relativité générale et le modèle standard ne parviennent pas à donner d'explications. L'observation des oscillations des neutrinos [4], par exemple, ne peut se produire que si au moins deux d'entre eux ont une masse, ce qui n'est pas le cas dans le modèle standard. Peut-être encore plus dramatique, l'observation de phénomènes à grandes échelles tels que les formations ou les rotations de galaxies ainsi que l'expansion de l'univers indiquent soit une certaine déviation de la relativité générale, soit la nécessité d'inclure matière noire et d'énergie noire dans la théorie. Ni les interféromètres gravitationnels, ni les accélérateurs de particules, n'ont pu cependant montrer d'écart vis-à-vis à la relativité générale ou n'ont pu détecter des particules non prédites par le modèle standard. Enfin, l'une des plus grandes énigmes de la physique moderne est de pouvoir décrire des phénomènes tels le big bang ou les trous noirs, dans lesquels la relativité générale et la mécanique quantique jouent un rôle.

Les trous noirs en relativité générale sont définis comme une région de l'univers de laquelle rien ne peut échapper, que ce soit de la matière ou de la lumière. Ils font partie des possibles fin de vie d'une étoile dont la pression interne est trop faible pour compenser sa propre force gravi-

tationnelle. En un sens, la relativité générale ne parvient déjà pas à décrire de tels objets. Il est en effet possible pour un trou noir de présenter une singularité gravitationnelle dans laquelle la courbure de l'espace-temps devient infinie. Hawking a prouvé plus tard que si l'on considère la théorie quantique des champs sur un espace-temps décrivant un trou noir, il se comporte comme un corps noir et s'évapore. Cela conduit en particulier au paradoxe de l'information. Résoudre ce problème a pour conséquence de rejeter au moins un postulat de la mécanique quantique, ou de la relativité générale, à savoir les transformations unitaires des processus quantiques ou le principe d'équivalence. Toutes ces incompatibilités ont conduit la communauté scientifique à rechercher une théorie quantique de la gravité. Parmi les recherches actuelles concernant cette unification figure la théorie des cordes, le formalisme sur lequel cette thèse est basée.

Théorie des Cordes

La théorie des cordes a été construite à l'origine comme une explication possible de l'interaction forte, pour ensuite être rejetée en faveur de la chromodynamique quantique. Elle a ensuite été proposée comme une théorie de la gravité quantique ainsi qu'un formalisme d'unification de l'ensemble des forces fondamentales. Schématiquement, la théorie des cordes peut être considérée comme une généralisation de la théorie quantique des champs. Elle décrit des objets unidimensionnels appelés cordes au lieu de particules ponctuelles. Afin d'inclure des états fermioniques, il est nécessaire de considérer des théories supersymétriques. Finalement, on obtient cinq théories de supercordes cohérentes en dix dimensions : Type IIA, Type IIB, Type I et les théories hétérotiques $E_8 \times E_8$ et $SO(32)$. Le problème posé par la dimension d'un espace-temps étendu est résolu en considérant que les dimensions d'espace supplémentaires sont compactes. Les manifestations à quatre dimensions de notre univers sont alors une approximation à basse énergie d'une théorie plus fondamentale qui a plus de dimensions.

La théorie des cordes n'est pas seulement une théorie de cordes, mais aussi de branes, qui peuvent s'étendre sur plus d'une dimension spatiale. Elles sont essentielles en théorie des cordes de type II et de type I afin d'obtenir des théories de jauge compatibles avec le mod-

èle standard, et sont les charges électriques et magnétiques de divers champs en théorie des cordes.

Les cinq théories des supercordes à dix dimensions sont en fait différentes facettes d'une théorie unificatrice à onze dimensions appelée théorie M. Diverses dualités les relient après compactifications et ont été largement étudiées ces dernières années. Cette thèse aborde ces dualités en huit dimensions et est centrée sur la théorie F [5]. Celle-ci est douze dimensionnelle et peut être comprise comme la théorie des cordes de type IIB avec des 7-branes. Elle a diverses applications dans la théorie quantique des champs non-perturbatifs, la construction de modèles de physique des particules et fait le lien entre physique et géométrie algébrique [6].

Dans cette thèse, nous nous concentrons d'abord sur les liens possibles entre la théorie F et la théorie des champs exceptionnels. La théorie des champs exceptionnels est une théorie quantique des champs, ponctuelle, incorporant les symétries de la théorie des cordes. En particulier, elle fournit une description à basse énergie de la théorie des cordes de type IIB tout en présentant ses symétries de façon manifeste. Nous discutons donc de la possibilité pour la théorie des champs exceptionnels de décrire des aspects de la théorie F.

Dans un deuxième temps, nous nous concentrons sur la dualité entre la théorie F et la corde hétérotique en huit dimensions. La corde hétérotique fournit déjà une grande variété de groupes de jauges possibles après compactification, déterminées principalement par la structure de ses lignes de Wilson. Dans la théorie type IIB, les groupes de jauge apparaissent suite à un empilement de branes, et en F-théorie cela est dû à la structure algébrique de l'espace sur lequel on compactifie, dans notre cas une surface K3 elliptique. Un moyen concret de construire des surfaces K3 est d'utiliser des polyèdres tridimensionnels. La dualité entre ces surfaces, du côté de la théorie F, et les lignes de Wilson en corde hétérotique, n'est bien comprise que pour deux des 4319 surfaces K3 construites via des polyèdres [7, 8]. Nous nous intéressons donc aux dualités de la théorie F avec la corde hétérotique pour d'autres polyèdres dans la troisième partie de ce manuscrit.

Organisation de la thèse

Cette thèse comprend trois parties. La partie I est une introduction aux concepts fondamentaux de la théorie des cordes. La partie II se concentre sur la géométrie généralisée exceptionnelle et la théorie des champs exceptionnels, en particulier $E_{3(3)} \times \mathbb{R}^+$ EFT en huit dimensions et son lien avec la théorie F. Dans la partie III, nous discutons de la dualité entre la théorie F sur des surfaces elliptiques K3 et la théorie des cordes hétérotiques compactifiée sur un deux-tores. Nous fournissons de nouvelles perspectives qui permettent de construire une identification explicite entre des polyèdres réflexifs définissant des surfaces K3 et des lignes de Wilson.

Les chapitres 1 et 2 sont les introductions (en Anglais et en Français).

Dans le chapitre 3, nous introduisons la notion de compactification en traitant les exemples de Kaluza-Klein et Scherk-Schwarz. Nous présentons les motivations qui amène à considérer des compactifications de Calabi-Yau dans la théorie des cordes. Nous traitons ensuite des exemples basiques de la dualité T et S. Après avoir montré le lien entre la dualité électromagnétique avec des monopôles magnétiques, et la dualité forte/faible interaction, nous présentons la manifestation de la symétrie continue $SL(2, \mathbb{R})$ de la supergravité de type IIB et sa restriction à son sous-groupe discret $SL(2, \mathbb{Z})$ dans le cas quantique.

Dans le chapitre 4, nous examinons les actions des dualités T et S sur les branes. Nous commençons par la dualité S dans le cas de la théorie des cordes de type IIB qui force à considérer des branes présentant une symétrie $SL(2, \mathbb{Z})$ manifeste. Nous discutons ensuite la notion de monodromie en insistant sur le cas particulier des (p, q) branes. Nous montrons que les transformations associées à la dualité S de l'axio-dilaton correspond à une invariance modulaire d'une courbe elliptique, notion centrale vis-à-vis de la construction de la théorie F. Nous regardons ensuite les conséquences de la dualité T et plus généralement de la dualité U sur les branes. Nous concluons par une discussion sur le réseau de dualités entre les différentes théories des supercordes.

Dans le chapitre 5, nous présentons les notions de base de la théorie des champs doubles

(*Double Field Theory*, DFT), de la géométrie généralisée (GG) et de leurs extensions U-dual i.e. la théorie des champs exceptionnels (*Exceptional Field Theory*, EFT) et la géométrie exceptionnelle (*Exceptional Generalised Geometry*, EGG). Nous commençons par des rappels sur les étapes nécessaires à l'obtention du tenseur de Ricci en géométrie Riemannienne. Nous discutons ensuite comment incorporer une symétrie $O(d,d,\mathbb{R})$ dans un tel formalisme, d'abord en considérant une extension de la fibre (GG), puis en doublant le nombre de coordonnées (DFT). Enfin, nous décrivons plus brièvement les généralisations dues à la dualité U.

Dans le chapitre 6, nous présentons les résultats de [9]. Nous considérons une solution non triviale à la condition de section (*section condition*) dans le contexte de la théorie des champs exceptionnels $E_{3(3)} \times \mathbb{R}^+$. Nous montrons que permettre aux champs d'avoir une dépendance par rapport aux coordonnées supplémentaires de l'espace interne étendu permet de décrire les monodromies des (p,q) 7-branes dans le contexte de la théorie F. Nous obtenons des expressions générales de flux non triviaux avec contraintes linéaires et quadratiques par une comparaison avec le *embedding tensor* de la supergravité maximale jaugée à huit dimensions avec symétrie trombone jaugée. Nous déterminons un symbole de Christoffel généralisé pour la théorie des champs exceptionnelles $E_{3(3)} \times \mathbb{R}^+$ et montrons que les équations du mouvement de la théorie F, à savoir l'annulation d'un tenseur de Ricci à 4 dimensions ayant deux de ses dimensions fibrées, peuvent être obtenues à partir d'un tenseur de Ricci généralisé et d'un ansatz de type IIB pour la métrique.

Dans le chapitre 7, nous détaillons les aspects des compactifications de la théorie F sur les surfaces elliptiques K3 et les compactifications de la théorie des cordes hétérotiques sur des deux tores. Nous commençons par décrire la corde hétérotique sur T^2 ainsi que ses modules. Nous donnons quelques exemples d'extensions et de réductions des groupes de jauges dans ce contexte. Ensuite, nous présentons des notions générales sur la structure de cohomologie des espaces de Calabi-Yau. Nous discutons plus en détail la structure d'homologie et de cohomologie des surfaces K3, ainsi que l'espace des modules de celles-ci. Nous montrons comment construire des surfaces K3 ainsi que leurs éventuelles fibrations à l'aide de polyèdres réflexifs.

Enfin, nous discutons de la structure des modules pour de telles paramétrisations des surfaces K3.

Dans le chapitre 8, nous présentons les résultats de [10]. Nous montrons comment construire des surfaces K3 elliptiques via des modèles de Weierstrass qui peuvent être paramétrés en termes de lignes de Wilson dans la théorie des cordes hétérotiques dual. Nous travaillons avec un sous-ensemble de polyèdres réflexifs admettant deux fibrations, et dont les espaces de modules contiennent ceux obtenues après la compactification de la théorie hétérotique $E_8 \times E_8$ ou $\frac{Spin(32)}{\mathbb{Z}_2}$ sur des deux-tores et avec des lignes de Wilson nulles. On peut alors interpréter les modules supplémentaires comme des ceux associés à des lignes de Wilson particulières dans la corde hétérotique. Un moyen pratique de trouver de tels polyèdres consiste à utiliser des graphes de polyèdres où les liens sont liés aux relations d'inclusion des espaces de modules des différentes fibrations. Nous sommes ensuite en mesure d'identifier les monômes dans les équations définissant les surfaces K3 comme des modules des lignes de Wilson dans les théories duales. Nous avons construit ce genre de graphes en développant trois programmes Sagemath qui donnent: le groupe de jauge pour un point générique dans l'espace des modules, le modèle de Weierstrass, ainsi que les extensions du groupe de jauge générique, obtenues en envoyant à zéro les coefficients de l'équation définissant la surface K3.

Dans le chapitre 9, nous présentons des résultats préliminaires d'un prochain article écrit en collaboration avec Bernardo Fraiman [11]. Nous montrons que dans le cas d'un certain polyèdre présentant cinq fibrations inéquivalentes et deux modules, les groupes de jauge génériques pour chaque fibration peuvent être interprétés comme un *shift vector* $\mathbb{Z}_3$. Différentes fibrations sont obtenues en distribuant différement le *shift vector* entre les deux réseaux E_8 ainsi que sur la corde hétérotique $SO(32)$. Nous discutons enfin la possibilité pour d'autres polyèdres à deux modules d'être également décrits par des *shift vectors* $\mathbb{Z}_n$.

Part I

Introductory Concepts of String Theory

CHAPTER 3

Compactifications of String Theory and Dualities

One feature of string theory, which is probably one of the most popular aspect of the theory, is that it needs additional dimensions in order to be consistent. However peculiar at first, this condition leads to the concept of compactifications, theorised long time before string theory, and somehow forgotten for several decades. In this chapter, we introduce the notion of compactifications together with one closely related concept in string theory: dualities. We start by considering Kaluza and Klein's compactifications of general relativity in five dimensions on a circle S^1. This hundred-year-old example already displays the unification of different interactions, hence indicating interesting prospects as to use compactifications in the construction of a grand unified theory of fundamental forces. We briefly treat Scherk and Schwarz compactificatitions generalising our first example and discuss its major implications.

We then present compactifications on a circle in the context of the bosonic string in 26 dimensions. We show that a new feature appears due to the consideration of extended objects such as strings: T-duality. We then consider compactifications of the bosonic string on a n-torus T^n and show how T-duality can be understood in this case as a discrete $O(d, d, \mathbb{Z})$ symmetry at the quantum level.

Next, we focus on the weak/strong duality of string theory: S-duality. We begin by considering the electromagnetic duality of Maxwell's theory in vacuum that we generalise to the case of electrodynamic with magnetic monopole. We show schematically how the Dirac quantisation condition implies that the electromagnetic duality is also a weak/strong coupling duality. To conclude our discussion concerning S-duality we discuss it in the context of string theory by considering type IIB supergravity whose action has a $SL(2,\mathbb{R})$ symmetry.

Finally, we show how one is led to consider Calabi-Yau compactifications of string theory and discuss some of its basic aspects.

3.1 Dimensional reduction and compactifications

3.1.1 Kaluza and Klein's mechanism

Einstein's theory of gravity is a geometric description of the gravitational interaction. The equations of motion of the metric $g_{\mu\nu}$ of a Riemannian or semi-Riemannian manifold $\mathcal{M}$, with $\mu,\nu = 0,...,3$ can be obtained via the Einstein-Hilbert action in the vacuum and without cosmological constant

$$S_{4d} \sim \int_{\mathcal{M}} R\text{vol} = \int_{\mathcal{M}} R\sqrt{|g|}\mathrm{d}^4x \tag{3.1}$$

where R is the Ricci scalar. The Einstein's equation then reads

$$R_{\mu\nu} = 0 \tag{3.2}$$

with $R_{\mu\nu}$ the Ricci tensor. Now, one could ask if it is possible to geometrise other forces in a similar way. Kaluza had the idea to consider a five dimensional manifold instead of a four dimensional one in a paper published in 1921[1] and Klein later proposed that this fifth dimension be compact as a way to interpret it properly. Let us first discuss this interpretation. Consider a five dimensional manifold as the product of a four dimensional Riemann space and a circle $\mathcal{M} = M \times S^1$ as well as a scalar field Φ living on this space. We can choose to impose the

[1] This was however already treated in a paper by Gunnar Nordström in 1914.

following periodicity condition on the scalar field

$$\Phi(x^{\hat{\mu}}) = \Phi(x^{\mu}, x^4) = \Phi(x^{\mu}, x^4 + 2\pi R_0) \tag{3.3}$$

where R_0 is the radius of the fourth spatial direction, x^4 a local coordinate on the circle and x^{μ} local coordinates of M. This leads to the possible development of Φ as the Fourier expansion

$$\Phi(x^{\mu}, x^4) = \sum_{n\in\mathbb{Z}} \Phi_n(x^{\mu}) e^{\frac{inx^4}{R_0}}. \tag{3.4}$$

Now, let us apply the five dimensional Klein-Gordon equation

$$\partial_{\hat{\mu}}\partial^{\hat{\mu}}\Phi = \sum_{n\in\mathbb{Z}} e^{\frac{inx^4}{R_0}} \left[\partial_{\mu}\partial^{\mu} - \left(\frac{n}{R_0}\right)^2\right] \Phi_n(x^{\mu}) = 0. \tag{3.5}$$

This corresponds to an infinite tower of states with masses $M_n = |\frac{n}{R_0}|$. Generalising this to other particles we would find various towers of states associated to particles of spin $\frac{1}{2}, 1$,... No experiment as of yet however witnessed such towers of states which leads, if our assumption of considering an additional compact dimension is correct, to the conclusion that we witness at best the Kaluza-Klein zero modes with $n = 0$ of these tower of particles. We can thus ignore the various massive states which in practice can be done by taking the limit $R_0 \to 0$ or equivalently that no field has a dependency with respect to the compact coordinate. One should note an important distinction between what we call compactification and dimensional reduction: compactifying means taking the totality of the states into account, which could have various implications in the ultraviolet. If one considers no dependency with respect to x^4, one describes an effective theory in four dimensions where the dimension of the compact space is sent to zero: this is called *dimensional reduction*. The assumption of Klein that the fifth dimension is compact gives us therefore an interpretation of our four dimensional observations as our incapacity to witness states with masses of the order $\frac{1}{R_0}$.

Consider now general relativity on this five dimensional space. We write the metric $g_{\hat{\mu}\hat{\nu}}$ of the

total space-time $\mathcal{M}$ with coordinates $\hat{\mu}, \hat{\nu} = 0, ..., 4$ as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + e^{2\phi} A_\mu A_\nu & e^{2\phi} A_\mu \\ e^{2\phi} A_\nu & e^{2\phi} \end{pmatrix} \tag{3.6}$$

with line element

$$\mathsf{d}s^2 = g_{\mu\nu} \mathsf{d}x^\mu \mathsf{d}x^\nu + e^{2\phi} (A_\mu \mathsf{d}x^\mu + \mathsf{d}x^4)^2. \tag{3.7}$$

Let us assume that the five dimensional metric does not depend on the compact direction. We then write the five dimensional Einstein-Hilbert action equivalent to (3.1) with five dimensional Ricci tensor $\hat{R}$ associated to the metric $\hat{g}$ as

$$\begin{aligned} S_{5d} \sim \int_{\mathcal{M}\times S^1} \hat{R} \mathsf{vol} &= \int_{\mathcal{M}\times S^1} \hat{R} \sqrt{|\hat{g}|} \mathsf{d}^5 x \\ &= (2\pi R_0) e^\phi \int_{\mathcal{M}} \sqrt{|g|} \left[R - \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{4} \left(\partial_{[\mu} A_{\nu]} \right) \left(\partial^{[\mu} A^{\nu]} \right) \right] \mathsf{d}^4 x. \end{aligned} \tag{3.8}$$

Kaluza and Klein therefore showed that dimensional reduction of a purely five dimensional gravitational theory is a way to unify gravity and electrodynamics in four dimensions. Using this somehow basic example, we already see the inconveniences and the possibilities one can expect from dimensional reductions and more generally from compactifications. The first thing to point out is of course that considering a dimensional reduction leads here to the unification of gravity and $U(1)$ gauge field. To understand this, one has to remember that general relativity can be seen as a Yang-Mills theory whose curvature and connection are defined via the metric on the tangent bundle [12]. The Kaluza ansatz (3.7) decomposes the metric in five dimensions as a metric of a four dimensional subspace together with a scalar field and a four dimensional 1-form which hints to the gain of a $U(1)$ gauge field in four dimensions. This indicates possible generalisations to other Yang-Mills theories if one considers higher dimensional spaces. We however encountered also one of the main issue of compactifications and dimensional reductions in that the field ϕ is a *moduli*: it has no potential and thus cannot be stabilized. As the global factor appearing in the action (3.8) is proportional to the physical radius $R_{phys} = R_0 e^\phi$, the size of the extra dimension is thus unstable and a moduli.

3.1.2 Scherk and Schwarz examples

The example depicted before is restricted in two ways: the reduction to a one dimensional space as well as the periodicity condition considered for every field in the equation (3.3). As described by Scherk and Schwarz in [13, 14], one can consider general transformations of the fields which are consistent with the symmetries of the action. This can lead to two consequences in the reduced theory: giving masses to various fields and reducing the number of supersymmetry.

Let us consider again a complex scalar field $\Phi = \Phi(x^\mu, x^4)$ in a flat five dimensional manifold $\mathcal{M} = \mathbb{R}^{1,3} \times S^1$ for simplicity. With similar conventions for the space-time indices the action is

$$\mathcal{S} \sim \int_{\mathcal{M}} (\partial_{\hat{\mu}}\Phi) \left(\partial^{\hat{\mu}}\Phi^*\right) \mathrm{d}^5 x. \tag{3.9}$$

This is invariant by the global $U(1)$ phase transformation $\Phi \to e^{i\alpha}\Phi$. Now let us make the following ansatz

$$\Phi(x^\mu, x^4 + 2\pi R_0) = e^{2\pi i m R_0}\Phi(x^\mu, x^4) \tag{3.10}$$

for some $m \in \mathbb{R}$. With this ansatz we find the Fourier expansion

$$\Phi(x^\mu, x^4) = e^{imx^4} \sum_{n\in\mathbb{Z}} \Phi_n(x^\mu) e^{\frac{inx^4}{R_0}}. \tag{3.11}$$

Using again the Klein Gordon equation, we find that the effective field theory describes the dynamics of a field with mass $M_\phi = \min\left(|m + \frac{n}{R_0}| \text{ for } n \in Z\right)$ in the reduced theory. In the dimensional reduction limit where one takes $R_0 \to 0$ this evidently gives $M_\phi = |m|$ as the mass gap in the massive tower states is sent to infinity.

Now let us sketch how the use of the symmetries of the action can reduce the number of supersymmetries when one compactifies. To be more specific, compactifying on a torus does not change the number of supercharges, which therefore raises the number of supersymmetries of the compactified theory with respect to the uncompactified one. However, with an appro-

priate choice of boundary condition, or dependency of the fields with respect to the compact dimension, it is possible to reduce the number of supersymmetries of the compactified theory. In the original paper from Scherk and Schwarz [13], they consider $\mathcal{N} = 1$ supergravity in four dimensions. Compactification on a circle with no dependency of the fields with respect to the compact direction leads to $\mathcal{N} = 2$ supersymmetry due to the fact that the Majorana spinor Ψ can be decomposed as

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{3.12}$$

with Ψ_1 and Ψ_2 Majorana spinors in three dimensions. A solution to recover $\mathcal{N} = 1$ supergravity for the compactified theory is to start with the following dependency of the four dimensional spinor

$$\Psi_\mu(x, y) = e^{im\Gamma_5 y}\Psi_\mu(x) \tag{3.13}$$

with $(\mu = 0, 1, 2)$, x coordinates on the non compact space $\mathbb{R}^{1,2}$ and $y = x^3$ coordinate on the circle. With additional constraints one is then able to recover $\mathcal{N} = 1, d = 3$ supergravity.

3.1.3 Calabi-Yau compactifications

Now that we introduced basic notions of compactifications, let us focus on Calabi-Yau compactifications which are widely used in string theory and are central to part III of this thesis. As we discussed in the introduction, string theory incorporates supersymmetry in order to contain fermionic states. Type I and the $E_8 \times E_8$ and $SO(32)$ heterotic string both have $\mathcal{N} = 1$ supersymmetry i.e. 16 supercharges in ten dimensions while type II string theories have $\mathcal{N} = 2$ and 32 supercharges. Torus compactifications such as the one we described in the example of Kaluza-Klein preserve the number of supercharges therefore leading to either $\mathcal{N} = 4$ or $\mathcal{N} = 8$ depending on the original ten dimensional theory. These theories are non-chiral and are thus not acceptable. Ideally one would want to obtain a four dimensional theory with at most $\mathcal{N} = 1$ whether one wants to completely reject the notion of supersymmetry in four dimensions, or considers that its detection is not currently possible. Supersymmetry however could resolve various

unanswered issues of the standard model such as dark matter, the hierarchy problem or the construction of a grand unified theory. For these reasons it is therefore preferable to obtain $\mathcal{N} = 1$ supersymmetry in four dimensions. Here we consider a generalisation of torus compactifications that give the same amount of supersymmetries in the compactified theory: Calabi-Yau compactifications. For type II it therefore gives $\mathcal{N} = 2$ supersymmetry which still is not what we expect, but is a necessary step in our journey to an acceptable physical four dimensional theory.

To understand why one wants to consider Calabi-Yau compactifications let us write the equations of motion of the graviton for the bosonic string

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\eta\rho}H_\nu{}^{\eta\rho} = 0 \tag{3.14}$$

where $H = dB_2$. This equation typically appears in the effective field theory obtained from the massless modes of type II and heterotic superstring theories. In the case of type I the term B_2 is projected out. In all superstring theories however, if one removes the dilaton term and the fluxes terms similar to H^2 it becomes the following constraint on the Ricci tensor

$$R_{\mu\nu} = 0. \tag{3.15}$$

This is a necessary condition for a compact space to be Calabi-Yau, however not a sufficient one. Compact Calabi-Yau manifolds have several possible definitions which are equivalent, among which is a compact Kähler manifold of complex dimension n $(\mathcal{M}, J, g)$ with reduced holonomy Hol$(g) = SU(n)$ [15]. The holonomy group, to put it simply, gives information on how fields are transported along closed loops for a given connection. To be more precise take a vector field $V^\mu e_\mu \in T_p\mathcal{M}$ of some manifold $\mathcal{M}$ with $dim_{\mathbb{R}}(\mathcal{M}) = d$ in a neighbourhood of a point p, and parallel transport it around a closed loop. This gives a vector $V'^\mu = V^\nu G_\nu{}^\mu$ where $G \in GL(d)$. Given the natural group structure associated to loops, the holonomy group is therefore a subgroup of $GL(d)$. The reader should be careful to the condition that the holonomy group be exactly $SU(n)$ in the definition of Calabi-Yau manifolds. The Ricci-flat condition (3.15) is less restrictive and

implies only that Hol$(g) \subseteq SU(n)$ if the space is Kähler and simply connected[2].

Now, in the absence of fluxes it is possible to link the number or remaining supersymmetries to the holonomy group[3]. Compactifications of string theory (with n_{SC} = 32 or 16 supercharges) on a compact Kähler space $(\mathcal{M}, J, g)$ with $dim_{\mathbb{C}}(\mathcal{M}) = n$ and Hol$(g) = SU(k)$ ($k \leq n$) leads to a theory with $\frac{n_{SC}}{2^{k-1}}$ supercharges. With k maximal, Calabi-Yau spaces give therefore the minimal amount of supersymmetries among the Kähler Ricci-flat spaces.

3.2 Dualities

Now that we introduced the basic aspects of compactifications we present dualities, both inside and outside the scope of string theory. Dualities in physics are a vague and broad concept. One could define a duality between two a priori distinct theories simply as a non trivial or non apparent equivalence. Here we focus on two particular dualities of string theory. We first present T-duality in the context of the bosonic string compactified on a circle of radius R, dual to the same theory compactified on a circle of radius $R' = \frac{1}{R}$. This is then generalised to the case of the d-torus which gives an $O(d, d, \mathbb{Z})$ duality group.

We then discuss S-duality. It maps in particular the weak coupling regime of a quantum theory to the strong coupling regime of another. We begin by a discussion of electromagnetism with magnetic monopoles and show that this generalises the electromagnetic duality to a weak/strong duality. We then discuss Type IIB supergravity which is self-dual with respect to S-duality.

3.2.1 A first look at T-duality: compactifications of the bosonic string on a circle

Here we discuss the bosonic string in $D + 1$ dimensions, based on [17]. Let $X^{\mu}(\tau, \sigma)$ with $\mu = 0, .., D$ be the embedding function of a closed string in a $D + 1$ dimensional spacetime $\mathcal{M} = \mathbb{R}^{1,D-1} \times S^1$. Just as the case of a point particle presented in section 3.1.1, impulsion on the

[2] If the space is not simply connected the condition becomes Hol$^0(g) \subseteq SU(n)$ where Hol^0 is the restriction to the subgroup of Hol connexe to the identity.

[3] See [16] or [17]

circle of the closed string is quantized as $p_n^D = \frac{n}{R}$ with $n \in \mathbb{Z}$ because of the periodicity condition on the compact direction. In addition, the string can go around the compact direction. This leads to the relation

$$X^D(\tau, \sigma) = X^D(\tau, \sigma + 2\pi) = X^D(\tau, \sigma) + 2\pi w R \tag{3.16}$$

where $w \in \mathbb{Z}$ is called the winding number. We then write the following mode expansion decomposition of X^D

$$X^D(\tau, \sigma) = x^D + \alpha' p^D \tau + wR\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^D e^{-in(\tau - \sigma)} + \bar{\alpha}_n^D e^{-in(\tau + \sigma)} \right). \tag{3.17}$$

Replacing the value of the momentum in the previous equation, we now decompose it into its left and right moving parts as

$$\begin{aligned} X_R^D(\tau - \sigma) &= \frac{1}{2}\left(x^D - c\right) + \frac{\alpha'}{2}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right)(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^D e^{-in(\tau - \sigma)} \\ X_L^D(\tau + \sigma) &= \frac{1}{2}\left(x^D + c\right) + \frac{\alpha'}{2}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right)(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n^D e^{-in(\tau + \sigma)}. \end{aligned} \tag{3.18}$$

The mass equation is then

$$-\sum_{k=0}^{k=D-1} p_k p^k = M^2 = \left(\frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2}\right) + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right), \tag{3.19}$$

with N and $\tilde{N}$ the number of left and right oscillators. The modified level matching condition is

$$N - \tilde{N} + nw = 0. \tag{3.20}$$

Here we encounter the simplest case of duality which appears in string theory: interchanging the momentum n and winding m quantum numbers both in the mass equation (3.19) and (3.20), together with the exchange $R \leftrightarrow \frac{\alpha'}{R}$ leaves the zero mode unchanged. As the left and right momentum transform as $(p_L^D, p_R^D) \to (p_L^D, -p_R^D)$ we impose the following so that the full quantum

theory is invariant

$$\begin{gathered}\left(X_L^D, X_R^D\right) \to \left(X_L^D, -X_R^D\right) \\ c \leftrightarrow x^D.\end{gathered} \tag{3.21}$$

We showed schematically that compactifying the bosonic string on a circle of radius R is dual to the same theory compactified on a circle of radius $\frac{\alpha'}{R}$ by the exchange (3.21). In superstring theory, T-duality relates different string theories as one has to apply a transformation to the world-sheet fermions as well. In type II string theory compactified on a circle, T-duality corresponds to $X_R^9 \to -X_R^9$ together with $\Psi_R^9 \to -\Psi_R^9$ which relates type IIB to type IIA.

3.2.2 Generalisation to the torus compactifications

Now we extend the discussion of part 3.2.1 to the case of the compactification of a 10 dimensional bosonic string theory on a d dimensional torus T^d. We decompose the 10 dimensional space as $\mathcal{M} = \mathbb{R}^{1,9-d} \times T^d$. We write the 10 dimensional index as $\hat{\mu} = 0, ..., 9 = (\mu, a)$ with $a = 1, ..., d$ and μ index on the Minkowski space $\mathbb{R}^{1,9-d}$. Naturally, we generalise the periodicity conditions to the d dimensional torus as

$$x^a \sim x^a + 2\pi R_a. \tag{3.22}$$

The winding and momentum numbers in the direction x^a are written w^a and n_a respectively. They can be put into a single $2d$ dimensional vector as

$$N^A = \left(w^a, n_a\right). \tag{3.23}$$

The level matching condition in the torus case is then

$$N - \tilde{N} + N^A G_{AB} N^B = 0 \tag{3.24}$$

where η_{AB} is the $O(d,d)$ invariant metric

$$\eta_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.25}$$

and with

$$G_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \tag{3.26}$$

where g and B are the d dimensional metric and the two-form field on the torus respectively. As w^a and n_a are discrete quantum numbers, the level matching is thus invariant under an $O(d,d,\mathbb{Z})$ rotation. The mass equation is

$$M^2 = N^A G_{AB} N^B + \text{oscillator terms.} \tag{3.27}$$

It is clear from 3.27 that an $O(d,d,\mathbb{Z})$ transformation of the Kaluza-Klein double vector N^A can be compensated by considering appropriate redefinition of the metric g and two-form field B on the torus. As an example we can consider a T-duality applied to the b-th direction, assumed to be an isometry, which gives the transformations known as *Buscher rules* [18–20]

$$\begin{gathered} g_{bb} \to \frac{1}{g_{bb}} \quad , \quad g_{ba} \to \frac{B_{ba}}{g_{bb}} \quad , \quad g_{ac} \to g_{ac} - \frac{g_{ba}g_{bc} - B_{ba}B_{bc}}{g_{bb}} \\ B_{ba} \to \frac{g_{ba}}{g_{bb}} \quad , \quad B_{ac} \to B_{ac} - \frac{g_{ba}B_{bc} - B_{ba}g_{bc}}{g_{bb}}. \end{gathered} \tag{3.28}$$

Here we focused principally on the closed bosonic string and showed that it give rise to an $O(d,d,\mathbb{Z})$ duality when compactified on a d-dimensional torus. In the next chapter we will discuss T-duality focusing on open strings and show how it permits to predict the necessity of considering other extended objects in order for string theory to be fully invariant under $O(d,d,\mathbb{Z})$. First, we present the other kind of duality of string theory: *S-duality*.

3.3 S-duality

T-duality relates different theories via compactifications on different spaces that give a common lower dimensional quantum theory. Another possibility is to relate the strong coupling of one theory to the weak coupling of another. This is of particular importance as one can study the strong coupling limit of a quantum theory by studying the weak coupling limit of its dual theory. S-duality is of particular importance in type IIB string theory, as in this case it is a self-duality and give rise to *F-theory* which incorporates naturally non-perturbative objects of string theory.

We introduce S-duality via an analogy with electrodynamics with magnetic monopoles. We show that electromagnetic duality in this context is in fact a strong/weak coupling duality. We then present type IIB supergravity and discuss its $SL(2,\mathbb{R})$ symmetry which includes a strong/weak coupling invariance. At the quantum level the symmetry is rather a discrete subgroup of $SL(2,\mathbb{R})$ and usually one considers it to be $SL(2,\mathbb{Z})$. We show that starting with type IIB supergravity together with S-duality forces one to consider a dual extended object to the fundamental string in the full quantum theory: the $D1$-brane.

3.3.1 Electro-magnetic duality and magnetic monopoles

Here we want to show the relation between electromagnetic duality and strong/weak coupling starting with Maxwell's theory of electromagnetism, based on [21,22]. In the absence of particles Maxwell's equations are famously known to be dual by the transformation $\vec{E} \to \vec{B}$ together with $\vec{B} \to -\vec{E}$ or equivalently in tensorial formalism the hodge duality $F \to \tilde{F}$ and $\tilde{F} \to -F$. More generally it is easy to see that a rotation by an angle α of the vector $(\vec{E}, \vec{B})$ is also an invariant of the equations of motions. Here we consider the generalised version of the usual Maxwell equations by including magnetic charges

$$\begin{aligned} \nabla \cdot \vec{E} = \rho_e \quad , \quad \nabla \wedge \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}_e \qquad & \qquad *\mathsf{d} * \mathbf{F} = \mathbf{J_e} \\ \nabla \cdot \vec{B} = \rho_m \quad , \quad \nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{J}_m \qquad & \qquad *\mathsf{d} * \tilde{\mathbf{F}} = -\mathbf{J_m} \end{aligned} \tag{3.29}$$

where ρ_e and ρ_m are the electric and magnetic charge density respectively. $\vec{J}_e$ and $\vec{J}_m$ are the electric and magnetic currents and $\mathbf{J} = (\rho, \vec{J})$. This is again invariant via a rotation of angle α. Specifically we can render manifest this $SO(2)$ symmetry of the the theory with

$$\mathcal{F} = \begin{pmatrix} \mathbf{F} \\ \tilde{\mathbf{F}} \end{pmatrix} \quad , \quad \mathcal{J} = \begin{pmatrix} \mathbf{J_e} \\ -\mathbf{J_m} \end{pmatrix} . \tag{3.30}$$

A rotation $\mathcal{R}(\alpha)$ by an angle α on both $\mathcal{F}$ and $\mathcal{J}$ leaves the equations of motion $\mathsf{d} * \mathcal{F} = *\mathcal{J}$ unchanged, where the exterior derivative and Hodge dual are simply applied to each component of the $SO(2)$ vectors. Electric and magnetic charges are then given by the integration over a sphere S^2 in which lies the electrically or magnetically charged particle

$$e = \int_{S^2} *\mathbf{F} \quad , \quad g = \int_{S^2} - * \tilde{\mathbf{F}}. \tag{3.31}$$

Dirac showed in 1931 that an electrodynamic quantum theory in which one requires the presence of magnetic monopoles has to follow the Dirac quantisation condition[4] which using our conventions reads

$$e \cdot g = 2\pi n \quad , \quad n \in \mathbb{Z} \tag{3.32}$$

with e and g the fundamental electric and magnetic charge respectively, every other charges being their integer multiples. This leads to

$$\alpha_e \alpha_m = \frac{n^2}{4} \quad , \quad n \in \mathbb{Z} \tag{3.33}$$

[4]We are not considering dyons here, which implies the generalised Dirac-Schwinger quantization condition.

with α_e and α_g the electric and magnetic fine structure constants. Now if we consider a rotation by an angle $\frac{\pi}{2}$ of (3.30), the electric and magnetic charge are exchanged

$$e \to g \quad , \quad g \to -e. \tag{3.34}$$

Considering the Dirac quantisation this gives for $n = 1$

$$\alpha_e \to \frac{1}{4\alpha_e} \quad , \quad \alpha_m \to \frac{1}{4\alpha_m}. \tag{3.35}$$

We see here that electromagnetic duality in the case of Maxwell's equations in vacuum corresponds to a strong/weak coupling duality in a more general setting. In the next subsection we discuss such duality in type IIB string theory and more particularly its low energy limit.

3.3.2 Type IIB string theory and S-duality

Type IIB string theory is one of the five consistent ten dimensional superstring theories. It is obtained by the quantization of a ten dimensional fermionic string together with a choice of a Gliozzi-Scherk-Olive (GSO) projection. We do not write the details of this construction as they have been covered widely in the literature [5] and would not bring particularly interesting points to our construction. The bosonic field content of type IIB string theory includes a metric g, a two-form field B_2 and the dilaton ϕ from the $NSNS$ sector, while the RR sector contributes with p-form fields C_p with $p = 0, 2, 4$. In the low energy limit the type IIB supergravity theory bosonic action is[6]

$$\frac{1}{2\pi} S_{\text{IIB}} = \int \mathsf{d}^{10}x\sqrt{-g}\left(R - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2\,\mathsf{Im}(\tau)^2} - \frac{1}{2}\frac{|G_3|^2}{\mathsf{Im}(\tau)} - \frac{1}{4}|F_5|^2\right) + \frac{1}{4i}\int \frac{1}{\mathsf{Im}(\tau)} C_4 + G_3 \wedge \bar{G}_3 \tag{3.36}$$

[5] See for example [17, 22]

[6] One should note that self-duality of the five dimensional field strength $F_5 = *F_5$ is needed after one obtains the equation of motions.

with

$$\begin{aligned} &\tau = C_0 + ie^{-\phi} \quad , \quad G_3 = dC_2 - \tau dB_2 \\ &|F_p| = \frac{1}{p!} F_{\mu_1..\mu_p} F^{\mu_1..\mu_p} \quad , \quad F_5 = \mathsf{d}C_4 - \frac{1}{2} C_2 \wedge \mathsf{d}B_2 + \frac{1}{2} B_2 \wedge \mathsf{d}C_2 \end{aligned} \tag{3.37}$$

where τ is called the axio-dilaton. This action is invariant under the following transformations of the axio-dilaton and the two-form fields B_2 and C_2

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \quad C_4 \to C_4, \quad g_{\mu\nu} \to g_{\mu\nu} \tag{3.38}$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \tag{3.39}$$

Let us focus for now on the $SL(2, \mathbb{R})$ transormation with parameters $a = d = 0$ and $c = -b = 1$ which gives

$$\tau \to -\frac{1}{\tau}. \tag{3.40}$$

Now imposing $C_0 = 0$ we find

$$e^{\phi} \to e^{-\phi} \quad , \quad C_2 \to -B_2 \quad , \quad B_2 \to C_2. \tag{3.41}$$

As the string coupling constant is $g_s = e^{<\phi>}$, this is the string theory equivalent of the strong/weak duality we described in the previous section. However the $SL(2, \mathbb{R})$ continuous symmetry cannot be possible for the full type IIB string as it is the case for its supergravity limit. The full quantum symmetry of string theory is conjectured to be a discrete subgroup of the continuous group $SL(2, \mathbb{R})$. Indeed, we saw that a consistent theory of electrodynamic in which one considers magnetic monopoles leads to the Dirac quantization condition (3.32) and the notion of fundamental charges. The setting we are considering here is quite similar: the fundamental string is the fundamental electric charge for the field B_2. Now, applying a $SL(2, \mathbb{R})$ transformation (3.39)

gives an object carrying d fundamental electric charge with respect to B_2 implying $d \in \mathbb{Z}$. Rescaling C_2, the maximal subgroup of $SL(2,\mathbb{R})$ with $d \in \mathbb{Z}$ is then $SL(2,\mathbb{Z})$. The fields B_2 and C_2 play a symmetric role in the supergravity action. The fact that the fundamental string carries a fundamental electric charge for B_2 indicates that a similar object should carry an electric charge for the field C_2: the $D1$-brane.

CHAPTER 4

Branes, Dualities and Unifications

In chapter 3, we focused on general aspects of string theory such as compactifications and dualities. We however omitted to discuss other fundamental objects of string theory such as branes. They are in particular the electric and magnetic charges of the RR fields we encountered in type II supergravity and are necessary in order to obtain gauge theories.

This chapter aims to introduce basic facts about branes in the context of Type II string theory and some of the consequences of dualities of string theory on this matter. We start by a short introduction on D-branes and their charges. We then show that self S-duality of type IIB string theory implies the existence of more general branes related by dualities. We follow with a discussion on monodromies that characterise different branes and can therefore be considered as generalisation of charges. After a short introduction of some of the aspects of F-theory we discuss the consequences of T-duality and more generally U-duality on the brane content of string theory. We finally conclude by some remarks on heterotic string theory which does not contain D-branes.

4.1 D-branes

Dp-branes are originally defined as dynamical objects on which open strings with $(p+1)$ Neumann boundary conditions end. They can emit and absorb closed strings which led Polchinsky to the conclusion that their tension are equal to their charges with respect to the RR fields [23]. It is possible to write the action corresponding to Dp-branes as a Born-Infeld part

$$S_{BI} = -T_p \int \mathrm{d}^{p+1}\xi e^{-\phi}\sqrt{\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{4.1}$$

and a Chern-Simons part where the brane charge is indeed the brane tension T_p

$$S_{CS} = iT_p \int_{p+1} e^{2\pi\alpha' F} \wedge \mathbf{C} \tag{4.2}$$

with $T_p = a2\pi l_s^{-(p+1)}$ with $a = 1$ in type II and $a = \frac{1}{\sqrt{2}}$ in type I. ξ are coordinates on the world volume of the brane i.e. every field is pulled back to the world volume, and $\mathbf{C}$ is the sum of RR fields. Finally, the field strength F_{ab} of A_a corresponds to a $U(1)$ gauge field constrained to live on the brane world-volume. The gauge group living on the branes world-volume can be generalised by considering different configurations of branes and can lead to groups coherent with the standard model or grand unification theories.

Now let us analyse the two actions. First, we see that the physical string coupling τ_s behaves as $\sim \frac{1}{g_s}$. Dirichlet branes are therefore non-perturbative objects in the string coupling expansion. From the Chern-Simons equation (4.2) we can extract a term $iT_p \int_{p+1} C_{p+1}$. The Dp-brane is therefore an electric charge for the field C_{p+1} and a magnetic charge to C_{10-9-3}. In particular, if one splits the ten dimensional space-time with respect to the Dp-brane world-volume as $\mathbb{R}^{1,9} \sim \mathbb{R}^{1,p} \times \mathbb{R}^{10-p-1}$ and considers a sphere encircling the brane one finds

$$\int_{S^{8-p}} \mathrm{d}C_{7-p} = T_p. \tag{4.3}$$

This is analogous to the definition of the electric charge in electrodynamic (3.31). These terms

generate fluxes: a globally defined RR field C_{7-p} would lead by Stockes theorem to the integral (4.3) being zero. This therefore leads to compactifications on spaces which are not Calabi-Yau, as the fluxes terms cannot be identically zero.

Now the questions one could ask is: are Dirichlet branes and the fundamental string the only objects one needs to consider in string theory? Already we can see that there must be some missing pieces: let us consider type IIB string theory. As was explained in 3.3.2, there seem to be a fundamental discrete symmetry at the quantum level which has for a subgroup the inversion of the string coupling constant, therefore leading to weak/strong duality labelled S-duality. The fundamental string and the $D1$-brane are S-dual but one must introduce new types of branes that are S-dual to the other Dp-branes. In the following section we discuss the implications of enforcing S-duality to type IIB string theory with Dp-branes.

4.2 Branes, S-duality and F-theory

4.2.1 Branes and S-duality

We introduced before the action of the type IIB supergravity theory (3.36) as well as the action of Dp-branes decomposed into a Born-Infeld (4.1) and a Chern-Simons part (4.2). In a democratic formulation of type IIB we now consider the RR fields to be C_{2p} with $p = 0, .., 4, (5)$[1]. Together with the constraints $*F_{2p+1} = (-1)^p F_{9-p}$ we can write the democratic formulation of the type IIB action as

$$\begin{aligned}\frac{1}{2\pi} S_{IIB,dem} = &\int \mathrm{d}x^{10} e^{-2\phi}\sqrt{-g}\,(R + \partial_\mu\phi\partial^\mu\phi) - \frac{1}{2}\int e^{-2\phi} H_3 \wedge *H_3 \\ &- \frac{1}{4}\sum_{p=0}^{4} F_{2p+1}\wedge *F_{2p+1} - \frac{1}{2}\int C_4 \wedge H_3 \wedge F_3\end{aligned} \tag{4.4}$$

[1]The RR field C_{10} is non dynamical.

where the fields are defined in the same way as equation 3.37 with the additional definitions

$$F_1 = \mathsf{d}C_0 \quad , \quad H_3 = \mathsf{d}B_2. \tag{4.5}$$

The theory contains fundamental strings ($F1$) as well as $D1, D3, D5, D7, D9$ branes. Considering that S-duality is indeed a symmetry of string theory implies that one should be able to understand how strings and branes are related under such dualities. For example, the action of S-duality mixes the fields B_2 and C_2 whose electric charges are respectively the fundamental string and the D1-brane, indicating that they are S-dual. More generally, as S-duality is a weak/strong duality, a n dimensional object should be dual to another n dimensional one in order for the full action to be consistent. The $D5$-brane is an electric charge for C_8 and magnetic charge for C_2, which leads to consider its dual $F5$-brane, usually denoted $NS5$-brane in the literature, which is a magnetic charge for the two-form field B_2 and electric charge for a S-dual field of C_8 which we write B_8. In fact, one should be able to write the theory in terms of $SL(2,\mathbb{Z})$ representations: one needs to think of fundamental strings and $D1$-brane as a $SL(2,\mathbb{Z})$ doublet ($(1,0)$-string and $(0,1)$-string respectively), where a general (p,q)-string is a BPS bound state of p fundamental and q Dirichlet strings, with p and q coprime. The $D3$-branes are self-dual under S-duality due to the constraint $F_5 = *F_5$ one has to impose in type IIB supergravity and are therefore singlets. We later discuss with more details the case of S-duality with $D7$-branes. An important notion as to characterise such branes is the monodromy which we present now.

4.2.2 Monodromies

Monodromies can be considered as charges in the sense that they characterise objects such as branes by considering the impact of holonomies on the various fields of the theory. A well known example of this effect occurs in the path integral formulation of electrodynamics via the Aharonov-Bohm effect [12]. Here one considers an infinite solenoid extended in the z direction in $\mathbb{R}^3$ with cylindrical coordinates (r,θ,z). In an ideal setting, the magnetic field inside of the solenoid is constant given by $*B = B_0 dz$. It is null outside of the solenoid. The potential vector

which locally corresponds to a solution to $B = dA$ can then be written outside of the solenoid as

$$A = \frac{\Phi}{2\pi} d\theta. \tag{4.6}$$

The path integral formulation from a point A to B on the path γ includes a term of the form

$$\text{Phase shift } \sim e^{-iq\int_\gamma A} \tag{4.7}$$

for a particle of charge q. Integrating the vector potential (4.6) one finds that going around the solenoid gives

$$\oint_\gamma A = \Phi = \int_D B \tag{4.8}$$

where D is a surface whose boundary is γ. There is therefore a possible phase shift with respect to a particle going around the solenoid, which can contribute as destructive of constructive interferences. This phase shift depends on the "charge" Φ of the solenoide.

More generally, monodromies occur when one is considering codimension 2 charged objects[2]. As an example, let us first consider a ten dimensional Minkowski space with a $D7$-brane along $\mathbb{R}^{1,7} \subset \mathbb{R}^{1,9} \simeq \mathbb{R}^{1,7} \otimes \mathbb{C}$. The $D7$-brane is a magnetic charge for the axion C_0. One can show that, considering supersymmetry constraints, the axio-dilaton behaves as

$$\tau(z) = \frac{1}{2\pi i} \ln(z - z_0) + \text{terms regular at } z_0 \tag{4.9}$$

with z the complex coordinate of the normal space to the brane and z_0 the position of the brane. This implies in particular that the dilaton transforms as

$$\tau \to \tau + 1 \tag{4.10}$$

as one encircles the brane around z_0. It corresponds to the $SL(2,\mathbb{Z})$ transformation introduced

[2] See e.g. [24] for a general discussion on monodromies in the context of string theory.

before in equation (3.38) with matrix parameter

$$M_{\text{D7-brane}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.11}$$

This is quite similar to the example of Aharonov and Bohm where the one form A was shifted by the value Φ. If we mesure the monodromy charge in the case of a stack of N $D7$-branes we obtain

$$M_{N\times\text{D7-brane}} = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}. \tag{4.12}$$

Now, as we emphasized in the previous subsection the $SL(2,\mathbb{Z})$ duality of type IIB string theory predicts other types of branes which are dual to the $D7$-brane. If one considers the general $SL(2,\mathbb{Z})$ matrix U

$$U = \begin{pmatrix} s & r \\ q & p \end{pmatrix} \quad , \quad sp - rq = 1, \tag{4.13}$$

their associated charge becomes in the $SL(2,\mathbb{Z})$ dual description [24]

$$\tilde{M}_{(p,q)} = U^{-1} M_{\text{D7-brane}} U = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix} \tag{4.14}$$

with p and q coprime. This defines (p,q) 7-branes, on which (p,q) strings which couples to $pB_2 + qC_2$ can end. One important thing to note is that, locally, one can always recover a $D7$-brane from a (p,q) 7-brane by an $SL(2,\mathbb{Z})$ transformation. However, two mutually non-local brane, in the sense that their monodromy charge do not commute, cannot be describe as a purely $D7$-brane content in another frame. In some cases, it is possible to consider stacks of various (p,q)-branes in 8 dimensions that give rise to ADE groups in flat space. All possible gauge groups can be obtain using the following base of 7-branes

$$A = (1,0) \quad , \quad B = (3,1) \quad , \quad C = (1,1) \tag{4.15}$$

leading to

$$A^N \to SU(N) \quad , \quad A^N BC \to SO(2N) \quad , \quad A^{k-1}BC^2 \to E_k \quad (\text{for } k = 6, 7, 8). \tag{4.16}$$

In the next subsection we focus on a formulation of type IIB string theory which naturally incorporates 7-branes.

4.2.3 Basics of F-theory

F-theory was constructed as a 12 dimensional geometrisation of type IIB string theory with 7-branes [5, 25]. In D=10 type IIB supergravity we saw that the axion C_0 and the dilaton Φ can be arranged in a manifestly $SL(2,\mathbb{R})$ complex field τ named the axio-dilaton (see equation (3.37)). The quantum theory however has $SL(2,\mathbb{Z})$ symmetry due to non-perturbative effects. This corresponds exactly to the invariance of the complex parameter of an elliptic curve, i.e. a torus with a marked point [26]. To be more precise let us consider an elliptic curve with complex parameter τ defined as

$$\mathbb{E}_\tau = \mathbb{C}\backslash\Lambda_\tau = \{w \in \mathbb{C} : w \equiv w + (n + m\tau)\} \quad , \quad n, m \in \mathbb{Z}, \tag{4.17}$$

where τ is valued in the upper half plane and $w = 0$ is the origin. Applying an $SL(2,\mathbb{Z})$ transformation to the parameter τ as in (3.38) leaves the lattice Λ_τ invariant and thus describes the same elliptic curve. In F-theory one therefore considers the axio-dilaton as the complex parameter of such an elliptic curve fibered over the 10 dimensional original space of type IIB string theory. The variation of the value of the axio-dilaton on the 10 dimensional base space is then interpreted as a variation of the shape of the torus in the fiber. The elliptic curve $\mathbb{E}_\tau$ can be describe by a vanishing polynomial in $\mathbb{P}_{231}$ called the Weierstrass form of the elliptic curve. It is given by

$$y^2 - \left(x^3 + fxz^4 + gz^6\right) = 0 \tag{4.18}$$

together with the relation

$$(x, y, z) \in \mathbb{C}\backslash(0, 0, 0) \quad , \quad (x, y, z) \simeq (\lambda^2 x, \lambda^3 y, \lambda z) \quad , \quad \lambda \in \mathbb{C}^*. \tag{4.19}$$

One then recovers the complex parameter τ using the relation

$$j(\tau) = 4\frac{24^3 f^3}{\Delta} \quad , \quad \Delta = 4f^3 + 27g^2 \tag{4.20}$$

with j the Jacobi j-function. f and g verify

$$f \to (c\tau + d)^4 f \quad , \quad g \to (c\tau + d)^6 g \tag{4.21}$$

under an $SL(2, \mathbb{Z})$ transformation with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{4.22}$$

The fibration of the elliptic curve is then possible when one allows f and g to have a dependency with respect to some coordinates of the original ten dimensional space of type IIB string theory. It leads to a varying axio-dilaton with respect to the base as the shape of the torus described by its Weierstrass model varies. It describes the backreaction of the branes onto the geometry, and gives a strong coupling description of type IIB string theory with (p, q) branes [6, 26]. We will discuss with more detail the compactifications of F-theory to 8 dimensions in the third part of this manuscript.

4.3 Branes and T-duality

S-duality is obviously not the only quantum symmetry one would want to impose to string theory and additional branes need to be added to fundamental and Dirichlet ones in order to also be invariant under T-duality. To this end, let us consider the action of T-duality on open strings,

and more generally on branes. Under T-duality on S^1, the zero mode sector of the closed string corresponds to an exchange of the momentum number with the winding number as well as an inversion of the radius of the circle $R \to \frac{\alpha'}{R}$. This amounts more generally to the exchange of the left and right moving sector (3.21). We will now look at the implication of this transformation on an open string in the bosonic sector of string theory. We first consider an open string propagating on a circle with quantized momentum $p^D = \frac{n}{R}$ and Neumann boundary condition which gives

$$X^D(\tau,\sigma) = x_0 + 2\alpha'\frac{n}{R} + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^D e^{-in\tau}\cos(n\sigma). \tag{4.23}$$

Applying the T-duality transformation (3.21) this gives

$$X'^D(\tau,\sigma) = c + 2\alpha'\frac{n}{R}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^D e^{-in\tau}\sin(n\sigma). \tag{4.24}$$

Thus an open string with Neumann boundary condition and n quantity of fundamental momentum $\frac{1}{R}$ on a circle of radius R is T-dual to an open string with Dirichlet condition and winding n times around the dual circle of radius $\frac{\alpha'}{R}$. This can be generalised to the fermionic string and it relates Dp-branes with p Neumann boundary condition to $D(p+1)$-branes if the T-duality transformation is applied along the direction of the brane, and to $D(p-1)$ if is is applied on a transverse direction. This is a strong indication for the T-duality one expects between the two type II theories: type IIA with Dp-branes with p even, and type IIB with p odd. In fact one can already construct a simple diagram[3] of objects appearing in type IIA and type IIB string theories by exclusively starting with the fundamental string and S and T-duality, see Figure 4.1 [30]. This is quite remarkable: one

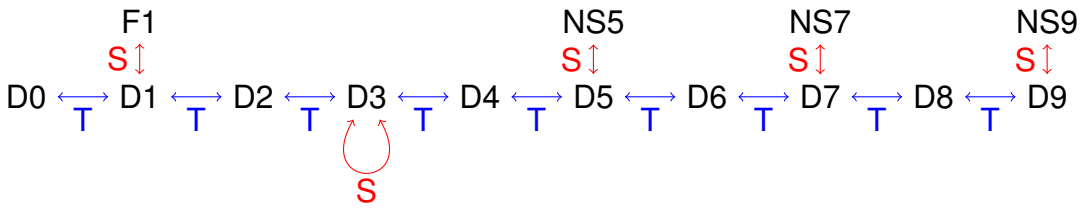


Figure 4.1: T and S duality relations between Type IIA and IIB branes.

[3]This is an illustration of more general diagrams presented in e.g. [27–29].

starts with consistent string theories in ten dimensions that leads to five different possibilities including type IIA and type IIB string theory. Looking at the zero modes of these theories we can construct type IIA and type IIB supergravities which have, due to the GSO projection, only odd and even k form fields respectively. The $SL(2,\mathbb{R})$ symmetry of type IIB supergravity conjectured to be restrained to a discrete subgroup at the quantum level leads to consider a dual to the fundamental string: the Dirichlet one brane. By considering several T-dualities one is then able to construct, starting with exclusively the fundamental string and consideration in the low energy limit of string theory, associated charged object of the RR form fields as depicted in Figure 4.1.

4.4 U-duality

4.4.1 U-Duality and exotic branes

Up to now we treated S and T duality separately but they obviously combine and the general duality of string theory is labeled U-duality. String theory then shows a particularly rich structure of discrete symmetries after toroidal compactification given by the discrete split real forms of exceptional groups $E_{d+1(d+1)}(\mathbb{Z})$, where d is the dimension of the torus on which one compactifies [31]. In its low energy limit, those discrete groups become continuous and 11 dimensional supergravity has a global $E_{d(d)}(\mathbb{R}) \equiv E_{d(d)}$ symmetry group, whereas for type II supergravity theories the symmetry groups are $E_{d+1(d+1)}$. The relevant discrete and continuous groups of string and supergravity compatifications are listed in Table 4.1. The construction we illustrated in the

d	Quantum Duality	Supergravity Symmetry	Max. Compact Subgroup
8	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{Z})$	$SL(5,\mathbb{R})$	$SO(5)$
6	$SO(5,5,\mathbb{Z})$	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	$E_{6(6)}(\mathbb{Z})$	$E_{6(6)}(\mathbb{R})$	$USp(8)$
4	$E_{7(7)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{R})$	$SU(8)$
3	$E_{8(8)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{R})$	$SO(16)$

Table 4.1: Discrete symmetry group of M theory and continuous symmetry of 11 dimensional supergravity after compactification to d dimension.

Figure 4.1 can then be generalised and involve a higher number of T and S-dualities. They can give non-pertubative objects called "exotic branes" which are low codimension branes (≤ 2) whose tensions scales as g_s^α with $\alpha < -2$. In the case of codimension-2 exotic branes they have in particular non-trivial monodromies as one encircles them [24, 27–29, 32–35]. To illustrate how such objects appear let us consider the example of type II string theory compactified on T^7 to 3 dimensions whose U-duality group is $E_{8(8)}(\mathbb{Z})$ as presented in [24]. A $D7$-brane wrapping the T^7 is a point in 3d and the mass of the apparent particle is then

$$M_{D7(3456789)} = \frac{R_3 R_4 ... R_9}{g_s l_s^8} \tag{4.25}$$

where $R_3, .., R_9$ are the radius on the seven torus. T-duality along the y direction and S-duality then transform the masses of such states as

$$T_y : R_y \to \frac{l_s^2}{R_y}, \quad g_s \to \frac{l_s}{R_y} g_s \qquad , \qquad S : g_s \to \frac{1}{g_s}, \quad l_s \to g_s^{\frac{1}{2}} l_s. \tag{4.26}$$

From the $D7$-brane U-duality orbit one gets 240 states which have to be interpreted as objects in type II or M theory before compactification. Among the 240 states are obviously all the branes with spatial dimension ≤ 7 of Figure 4.1 but others emerge and are necessary for U-duality to be a quantum symmetry of string theory.

4.4.2 Heterotic string

In this chapter we mainly focused on branes in string theory and some of the implications of U-duality on this matter. However not all superstring theories contain D-branes and in particular the heterotic string, which is central to the discussion of part III. The heterotic string is a closed string theory constructed from the combination of the 26 dimensional left-moving bosonic string together with the 10 dimensional type II right moving supersymmetric one. To recover a 10 dimensional theory one compactifies the additional 16 fields $X^I(\tau + \sigma), I = 1, .., 16$ on a T^{16} torus . The momentum on the torus are quantized and live in a 16 dimensional lattice Γ^{16}.

Modular invariance then forces this lattice to be Euclidian, even and self-dual. There exist only two possibilities in 16 dimensions which are the $\Gamma_{D_{16}}$ weight lattice of $\frac{Spin(32)}{\mathbb{Z}_2}$ and $\Gamma_{E_8} \otimes \Gamma_{E_8}$ where Γ_{E_8} is the root lattice of E_8 [17,36]. Heterotic string therefore naturally leads to $SO(32)$ and $E_8 \times E_8$ gauge theories. As we will see in part III, these gauge groups can be broken or enhanced after compactifications. We will focus on how the gauge structure arising from heterotic string theory compactifications on a two torus can be understood as a brane configuration in the context of elliptically fibered K3 compactifications of F-theory in 8 dimensions.

Part II

Exceptional Field Theory and F-theory

CHAPTER 5

Exceptional Field Theory and Exceptional Generalised Geometry

As we saw in the previous chapter, dualities play an important role in string theory. However, in their low energy limit the continuous symmetry listed in table 4.1 are not manifest. This has led to extensive work on constructing field theories that are manifestly invariant under these symmetries. In the case of T-duality this led to the construction of Double Field Theory (DFT) [37, 38] as well as Generalised Geometry (GG) [39, 40], which make manifest an $O(d,d)$ symmetry. DFT was constructed using a doubled space with additional "winding coordinates" [41–45]. They are later removed by a *section condition* to recover a physical theory. Generalised geometry on the other hand extends the tangent space T to the combination $T \oplus T^*$, thus describing both vectors and 1-forms in a unique fiber. Both theories are manifestly $O(d,d)$-covariant, and combine diffeomorphisms as well as B-field gauge transformations in a single object: double vectors in DFT and sections of the generalised fiber in generalised geometry. Extensions of those theories were constructed to consider the full U-duality and the expected $E_{d+1(d+1)}(\mathbb{R})$ symmetry one gets from string theory compactifications in the low energy limit: Exceptional Field Theory (EFT) [46–54] and Exceptional Generalised Geometry (EGG) [55, 56]. The group of symmetry is larger when one considers the S-duality in addition to T-duality. Thus, the space is no longer doubled for

exceptional field theories but is rather decomposed into an external space and an extended internal one. The geometric structure of this internal space is then constructed to be manifestly $E_{d(d)}$ covariant in order to render manifest the symmetries between the $NSNS$ and RR fields after compactification. Generalised vectors on this extended internal space and sections of the generalised fiber in the case of EGG then describe usual diffeomorphisms combined with $NSNS$ and RR gauge transformations [57–61].

In the following chapter we detail some of the aspects of double field theory, generalised geometry, exceptional field theory and exceptional generalised geometry. We start with basic mathematical notions on Riemannian spaces. Using this we first construct generalised geometry and double field theory and emphasize on the similarity with ordinary Riemannian geometry. We conclude with shorter descriptions of exceptional field theory and exceptional generalised geometry. We chose to emphasize first on the simpler formalisms of DFT and GG and then their extension. However, the reader can very well focus on DFT/EFT and GG/EGG without any consequences.

5.1 Elementary notions of Riemannian Geometry

We give here some basic notions of Riemannian geometry that will be central to construct generalised geometry and double field theory as well as their extensions exeptional field theory and exceptional generalised geometry. First let us consider a differentiable manifold $\mathcal{M}$ of real dimension d. One can then consider vector fields and 1-forms as sections of the tangent $T\mathcal{M}$ and cotangent bundle $T^*\mathcal{M}$, which by tensor products give (p,q)-tensors as section of $T\mathcal{M}^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q}$. This manifold is then considered to be Riemannian if there exist a section of $T^*\mathcal{M} \otimes T^*\mathcal{M}$ which corresponds to a symmetric-positive bilinear form locally[1]. In particular if $\mathrm{d}x^\mu$ is a local basis of the cotangent space one can express the metric locally as $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ with $g_{\mu\nu}$ symmetric positive and definite. Now we seek the equivalent of the potential vector of

[1] Pseudo-riemannian if it is a symetric bilinear form and non-degenerate.

electromagnetism or more generally a connection for Yang-Mills theory. Formally, let us first write the usual Lie derivative acting on a vector field as

$$\forall X, Y \in \Gamma(\mathcal{M}, T\mathcal{M}) \quad \mathbb{L}_X Y = [X, Y] \overset{loc}{\sim} \left(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu\right) \partial_\nu \tag{5.1}$$

with $\{\partial_\mu\}$ a local basis of $T\mathcal{M}$. The Lie derivative can therefore be decomposed locally as a transport term $X^\mu \partial_\mu Y^\nu$ together with a rotation of Y by a $gl(d)$ rotation $\partial_\mu X^\nu$ [61]. This reflects the diffeomorphism invariance of general relativity. The general definition of the covariant derivative is

$$\forall X, Y \in \Gamma(\mathcal{M}, T\mathcal{M}) \quad \nabla_X Y \overset{loc}{\sim} X^\mu \left(\partial_\mu Y^\nu + Y^\rho \Gamma_{\mu\rho}{}^\nu\right) \partial_\nu. \tag{5.2}$$

As we seek a connection on the tangent space there exists a unique one, called Levi-Civita connection, which is in particular torsion free i.e.

$$\begin{aligned} \forall X, Y \in \Gamma(\mathcal{M}, T\mathcal{M}) \quad T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] = 0 \\ &= \left(\mathbb{L}^\nabla_X - \mathbb{L}_X\right) Y \\ &\overset{loc}{\sim} \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho = 0 \end{aligned} \tag{5.3}$$

where $\mathbb{L}^\nabla$ is the usual Lie derivative with derivatives replaced by covariant ones. To obtain the Levi-Civita connection one has to further impose that the metric g is covariantly constant i.e. $\nabla_\mu g_{\nu\rho} = 0$. One then defines the Riemann tensor as a section of $T\mathcal{M} \otimes (T^*\mathcal{M})^{\otimes 3}$ via its action on three vector fields

$$\forall X, Y, Z \in \Gamma(\mathcal{M}, T\mathcal{M}) \quad R(X, Y, Z) \equiv R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5.4}$$

Considering the Riemann tensor associated to the Levi-Civita connection, we find the well known local formula for the Riemann tensor

$$R_{\mu\nu\rho}{}^\kappa = \partial_\mu \Gamma_{\nu\rho}{}^\kappa - \partial_\nu \Gamma_{\mu\rho}{}^\kappa + \Gamma^\kappa_{\mu\delta} \Gamma_{\nu\rho}{}^\delta - \Gamma_{\nu\delta}{}^\kappa \Gamma_{\mu\rho}{}^\delta \tag{5.5}$$

where the Levi-Civita connection is given locally by

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\delta} \left(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu} \right) . \tag{5.6}$$

One then defines the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$ and Ricci scalar $R = R_{\mu\nu} g^{\mu\nu}$ which we used in the definition of the Einstein-Hilbert action (3.1).

5.2 Generalised Geometry

Now that we wrote the mathematical notions necessary to the construction of general relativity, let us discuss how to implement symmetries of string theory in a somehow similar formulation. We saw in the compactification of the bosonic string on a torus in section 3.2.2 that both the mass equation (3.19) and the level matching condition (3.20) were written in terms of $O(d, d, \mathbb{Z})$ objects. In particular in the mass equation appears both the metric g and the two-form $NSNS$ field B_2 in a unified way. Specifically, one could consider a particular setting in which B_2 is null which would be equivalent by an $O(d, d, \mathbb{Z})$ rotation to another setting with B_2 non zero. At low energy this would imply that the diffeomorphism invariance of the first setting should translate into an invariance with respect to a transformation affecting both the metric g and the two-form field B_2. This is the premise on which is constructed generalised geometry. General relativity is based on diffeomorphism invariance encoded in the action of a Lie derivative with respect to a vector $X \in T\mathcal{M}$. To write a field theory which describes both the metric g and B_2, generalised geometry combines the diffeomorphism invariance of g together with the local gauge invariance $B_2 \sim B_2 + \mathrm{d}\Lambda$. One is then forced to consider a generalisation of the tangent bundle $T\mathcal{M}$ to a bundle which contains locally the cotangent bundle $T^*\mathcal{M}$. In order to do this let us consider the generalised tangent space E defined as

$$0 \to T^*\mathcal{M} \to E \to T\mathcal{M} \to 0 \tag{5.7}$$

which one can consider to be $T\mathcal{M} \oplus T^*\mathcal{M}$ locally. However, E is a priori a non-trivial fibration. If one considers an open cover $\{U_{(i)}\}$ of $\mathcal{M}$ the $NSNS$ field is only locally defined i.e. that in $U_i \cap U_j$ one has [60]

$$B_{2(i)} = B_{2(j)} + \mathsf{d}\Lambda_{(ij)} \tag{5.8}$$

with a one-form $\Lambda_{(ij)}$ veryfing

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\Lambda_{(ijk)} \tag{5.9}$$

over $U_i \cap U_j \cap U_k$. A section $V \in \Gamma(\mathcal{M}, E)$ can locally be written as $V_{(i)} = v_i + \lambda_i$ with $v_i \in \Gamma(TU_i)$ and $\lambda_i \in \Gamma(T^*U_i)$. This naturally defines a $O(d,d)$ metric by $< V, V >= i_v\lambda$. To properly describe the dilaton, it is necessary to extend the $O(d,d)$ structure to $O(d,d) \times \mathbb{R}^+$ and we thus consider $V \in (\mathsf{det}T^*\mathcal{M}) \times E$ as described in [60]. Following our comparison with general relativity one defines the generalised Lie derivative

$$\mathcal{L}_V W = \mathbb{L}_v w + \mathbb{L}_v \zeta - i_w \mathsf{d}\lambda \tag{5.10}$$

with $V = v + \lambda$ and $W = w + \zeta$. This expression explicitly shows the unification of diffeomorphisms and gauge transformations. However, it somehow misses to represent a $O(d,d)$ symmetry. In order to do that one decomposes V locally as $V^M = (v^\mu, \lambda_\mu)$ with $\mu = 1, .., d$, $v \in T\mathcal{M}$ and $\lambda \in T^*\mathcal{M}$. The Lie derivative can then be written as

$$\begin{aligned} \mathcal{L}_V W^M &= V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N + w(V)(\partial_N V^N) W^M \\ &= V^N \partial_N W^M - \left(\delta^M_Q \delta^P_N - \eta^{MP}\eta_{NQ}\right)(\partial_P V^Q) W^N + w(V)(\partial_N V^N) W^M \end{aligned} \tag{5.11}$$

together with $\partial_M = (\partial_\mu, 0)$ and $w(V)$ the conformal weight of V. The $gl(2d)$ term $\partial_N V^M$ of (5.1) has to change in generalised geometry so that the action of the Lie derivative on a vector in the fundamental representation of $O(d,d)$ stays into this representation. The transport and weight terms verify this automatically and the central term of (5.11) projects onto the adjoint of $O(d,d)$ thus leading to a coherent definition of generalised Lie derivative preserving the $O(d,d)$ symmetry. This generalised Lie derivative is not symmetric however and one usually defines the

Courant bracket

$$[|V,W|] = \frac{1}{2}\left(\mathcal{L}_V W - \mathcal{L}_W V\right). \tag{5.12}$$

There is no reason to consider a change in the definition of a generalised connection D and it would seem natural to define a generalised Ricci tensor as

$$R(U,V,W) = [D_U, D_V]W - D_{[|U,V|]}W. \tag{5.13}$$

However this expression is generally not a tensor. Going back to our comparison with general relativity there is still something missing: the main particularity of general relativity compared to Yang Mills theory is that the connection and the curvature are defined via the metric, which we did not define yet. We therefore consider the $\frac{O(d,d)}{O(d)\times O(d)}$ metric

$$G_{MN} = \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{5.14}$$

where g and B are usual d dimensional metric and two-form. One also defines a non vanishing section $\Phi \in \Gamma(\text{det}T^*\mathcal{M})$ which gives the generalised metric $(G,\Phi) \in \frac{O(d,d)\times\mathbb{R}^+}{O(d)\times O(d)}$. It is then always possible to find a generalised Levi-Civita connection which is generalised metric compatible $DG = D\Phi = 0$ and has a vanishing generalised torsion

$$\forall X, Y \in \Gamma(\mathcal{M}, \text{det}T^*\mathcal{M} \times E) \quad , \quad T(X,Y) = \left(\mathcal{L}^{\nabla}_X - \mathcal{L}_X\right) Y = 0. \tag{5.15}$$

In generalised geometry or exceptional generalised geometry, the generalised Levi-Civita connection is not necessarily unique. It is however possible to define a generalised Ricci tensor which, independently of the choice of generalised Levi-Civita connection, leads to the equations of motion of type II supergravity.

5.3 Double Field Theory

Let us now discuss another representation of T-duality in a field theory. Generalised geometry is constructed as a way to unify in a geometrical formalism the diffeomorphim invariance together with the gauge invariance of the two-form field appearing in the bosonic string. It is done by extending the tangent space one considers in Riemannian geometry to a fiber E which locally corresponds to $T\mathcal{M} \times T^*\mathcal{M}$. Double field theory on the other hand can be constructed starting with other observations which we detail below.

The momentum is quantized for both extended and punctual objects on a torus. The winding however will obviously not appear in a punctual field theory. What we already know is that exchanging winding and momentum should be one of the invariance of string theory and that one obtains the momentum quantum numbers from the d-dimensional torus as $x^m \leftrightarrow n_m$. A solution to double the quantum numbers is therefore to double the space so that winding coordinates are now understood as conjugate momentum of additional stringy coordinates $\tilde{x}_m \leftrightarrow w^m$. One then has to enforce the continuous $O(d,d)$ symmetry and apply consistency constraints in order for the theory to be coherent with its string theory perspective.

Instead of a fiber bundle $E \sim T\mathcal{M} \oplus T^*\mathcal{M}$ of a d dimensional manifold let us consider a $2d$ differentiable manifold whose coordinates[2] we write $X^M = (\tilde{x}_m, x^m)$ with $m = 1, .., d$. The various notions we discussed in the case of generalised geometry are very similar in this formulation: as the low energy limit of string theory has an $O(d,d)$ invariance one considers that the coordinates X^M live in its fundamental and therefore transforms in the same way under generalised diffeomorphisms (5.11). One can then choose to consider a generalised metric which decomposes as (5.14), and the condition that the derivative acts as $\partial_M = (0, \partial_m)$ on the various fields of the theory gives back a d dimensional theory. It is then possible to define a generalised Levi Civita connection which leads to generalised Ricci scalar and gives the equations of motion of the bosonic string compactified on a d dimensional torus[3].

Even though generalised geometry and double field theory lead to the equations of motion

[2] We only consider the compact space here.
[3] For more details see e.g. [20].

of the bosonic $NSNS$ sector of string theory it is not the only solution one can consider. It is however more subtle to treat in a coherent way these other solutions in generalised geometry than double field theory. In the later formalism, it is quite easy to formulate, at least schematically, how one can find other interesting solutions thus highlighting the perks of such T-dual formulation of a quantum field theory. One of the assumptions we made which breaks the $O(d,d)$ symmetry is to consider that $\partial_M = (0, \partial_m)$. More generally, it is possible to obtain a d dimensional solution coherent with the closure of the generalised Lie derivative (5.11) with

$$[\mathcal{L}_V, \mathcal{L}_W] = \mathcal{L}_{[|V,W)|]}. \tag{5.16}$$

This leads to the so called *section condition* which we write

$$\eta^{MN}\partial_M A \partial_N B = 0 \quad , \quad \eta^{MN}\partial_M\partial_N A = 0 \tag{5.17}$$

where A and B are any field of the theory and η_{MN} the $O(d,d)$ metric (3.25). The condition on the left is the *strong constraint* while the one on the right is called *weak constraint*. The strong constraint assures that the resulting fields depend on a d dimensional subspace of the original theory. It can however be relaxed as was done in [62] where one obtains a formulation of massive type II supergravity. In the next chapter we investigate the impact of considering fields that verify a non trivial solution of the section condition on the resulting theory.

5.4 Exceptional Generalised Geometry

In the previous sections we presented how to geometrize the $NSNS$ two-form field B_2 as to obtain two theories which manifestly incorporate a symmetry between diffeormorphism invariance and gauge transformations one obtains from T-duality considerations. Here we present succinctly how to incorporate in a similar way the RR fields in the context of type II string theory as to incorporate the full U-duality. For generality and as to consider a high number of fields on the internal space let us consider type IIB string theory compactified to 4 dimensions. The

continous U-duality group in this case is $E_{7(7)}(\mathbb{R})$ and its fundamental representation is **56**. This implies that the generalised fiber we considered in the previous section must be extended to 56 dimensions to fully describes a $E_{7(7)}(\mathbb{R})$ symmetry. This can be explained as follows: the gauge invariances of the various RR field of type IIB supergravity give a $\Lambda^{(odd)}T^*\mathcal{M}$ contribution in the fiber. One also has to consider the magnetic duals of both the $NSNS$ two-form field i.e. the six form field B_6 which gives a five form gauge parameter in $\Lambda^5 T^*\mathcal{M}$, as well as the dual of diffeomorphisms vectors in $T^*\mathcal{M} \otimes \Lambda^6 T^*\mathcal{M}$. In the end one finds that the generalised fiber E for the $E_{7(7)}$ exceptional generalised geometry is locally [63]

$$E_{\text{Type IIB}} \simeq T\mathcal{M} \oplus T^*\mathcal{M} \oplus \Lambda^5 T^*\mathcal{M} \oplus \left(T^*\mathcal{M} \otimes \Lambda^6 T^*\mathcal{M}\right) \oplus \Lambda^{(odd)}T^*\mathcal{M}. \tag{5.18}$$

Depending on the dimensions d of the compactification space, which is 6 in our example, one can remove different parts of this fiber as to construct $E_{d+1(d+1)}$ exceptional generalised geometry with $d < 7$ in the context of type IIB supergravity. In the case of type IIA supergravity one replaces the last term with odd forms gauge parameters by even forms [64]. Finally if one considers M-theory [55–57, 65] the dimension of the compact space is now seven instead of six which leads to the following fiber

$$E_{\text{M theory}} = T\mathcal{M} \oplus \Lambda^2 T^*\mathcal{M} \oplus \Lambda^5 T^*\mathcal{M} \oplus \left(T^*\mathcal{M} \otimes \Lambda^7 T^*\mathcal{M}\right). \tag{5.19}$$

Following a similar approach to what we described in section 5.2 one is then able to construct exceptional generalised geometries that incorporate a geometrization of the fields of supergravity theories.

5.5 Exceptional Field Theory

Let us conclude this chapter by discussing general aspects of exceptional field theory. Similarly to exceptional generalised geometry, one wants to describe the full U-duality which involves RR fields in addition to the $NSNS$ two-form field B_2. In order to do this construction let us con-

sider ten dimensional type IIB supergravity. Upon compactification on a d dimensional torus the resulting theory exhibits a $E_{d+1(d+1)}(\mathbb{R})$ symmetry. Let us separate the 10 dimensional coordinates which we write $X = (x^\mu, y^m)$ where $\mu = 0, .., 9-d$ and $m = 1, .., d$. In the case of DFT one doubles the coordinates on the compact space by introducing a dual winding coordinate $\tilde{y}_m$. However as we now wish to describe RR fields we need to consider the possibilities for branes to wrap around the compact space as well, which forces the dimension of the extended space to raise. Accounting for these brane wrapping contributions, exceptional field theory is constructed by considering an extended space whose internal coordinate Y^M lives in the lower dimensional fundamental representation of $E_{d+1(d+1)}$. A $\mathbb{R}^+$ factor is usually considered as to describe properly the dilaton. In the context of supergravity this conformal symmetry is called trombone symmetry and can be thought of as a generalisation of the rescaling symmetry of the metric in Einstein's theory of gravity [66, 67].

Again, the usual action of Riemannian Lie derivatives do not preserve the group structure of the theory. This leads to the introduction of a generalised Lie derivative which can be written for any exceptional geometry as [61, 65]

$$\mathcal{L}_\Lambda V^M = \mathbb{L}_\Lambda V^M + Z^{MN}{}_{PQ}\partial_N \Lambda^P V^Q + \left(\lambda(V) - \frac{1}{p}\right)\partial_N \Lambda^N V^M \tag{5.20}$$

where $\mathbb{L}$ is the usual Riemannian Lie derivative and $\lambda(V)$ the conformal weight of the vector V. The value of p and the tensor Z depend on which EFT one considers. The closure of the exceptional generalised Lie derivative gives the section condition[4]

$$Z^{MN}{}_{PQ}\partial_M A \partial_N B = 0 \quad , \quad Z^{MN}{}_{PQ}\partial_M \partial_N A = 0 \tag{5.21}$$

for every field A and B. This reduces the number of coordinates the fields are allowed to depend on. In fact, one is left with the possibility to obtain a ten or eleven dimensional theory. Using exceptional field theory, massless type II and eleven dimensional gauged supergravities are

[4]More generally, there are four constraints which in the case of the split forms of the exceptional groups $E_{d(d)}$ (d=2..7) are equivalent to the section condition [61].

obtained in a unified framework in various dimensions [52–54, 68–71]. Massive type IIA was then obtained using a violation of the section condition in double field theory [62] as well as deformation of the generalised Lie derivative structure in the context of EFT and EGG [72, 73].

CHAPTER 6

Geometry of $E_{3(3)} \times \mathbb{R}^+$ Exceptional Field Theory and F-Theory

In section 4.2.3, we saw that F-theory is a description of type IIB string theory with exotic 7-branes which has for consequence to mix the two-form field C_2 and B_2 under a monodromy (see equation (3.38)). Exceptional field theory on the other hand unifies the description of such fields geometrically. This chapter, based on [9], aims to provide insights on the relations between F-theory and $E_{3(3)} \times \mathbb{R}^+$ exceptional field theory. Such link between F-theory and EFT was first discussed for $\mathbb{R}^+ \times SL(2)$ EFT in [71].

Here we focus on the $E_{3(3)} \times \mathbb{R}^+ = SL(3) \times SL(2) \times \mathbb{R}^+$ exceptional field theory arising for compactifications of type IIB to 8 dimensions [68]. In the first part of this chapter we present a review of the basic results of $SL(3) \times SL(2) \times \mathbb{R}^+$ EFT and in particular the sufficient conditions needeed for a consistent theory. We then compute the fluxes of the theory, which we compare to the embedding tensor of the associated supergravity theory with gauged trombone symmetry. As the gauging of the trombone symmetry was only done for simple groups, we present a construction in the particular case where the original global group symmetry is $SL(3) \times SL(2)$. We then construct explicitly a generalised Christoffel symbol and remind the reader about the

construction of a generalised Ricci tensor done in [74]. We then focus on a non-standard solution to the section condition leading to the description of the monodromies of (p, q) 7-branes in F-theory. This is done by considering that the fields of the final theory have a dependency on 2 coordinates of the internal extended space, which are linear combination of both the usual coordinates and the stringy coordinates. This ensures that product and inverse of fields have a similar dependency on the generalised space and are therefore also solutions to the section condition. The description of the monodromies leads to the breaking of both gauge transformations of B_2 and C_2 which seem to be entirely constrained. This is however not a particular issue as the monodromies of (p, q) 7-brane are only constructed when the only non-zero field living on the brane is C_8, the dual field of the axion C_0. It is thus plausible that when one is describing the full backreaction of the brane with non-trivial $NSNS$ and RR fields living on its world volume, the gauge symmetry of these fields normal to the brane are broken. Finally, when one considers the standard solution to the section condition, we show that the generalised Ricci tensor gives the equations of motion of F-theory as a Ricci-flatness of a four dimensional space with two fibered directions.

6.1 Structure of $SL(3) \times SL(2)$ Exceptional Field Theory

Compactifying M-theory on a d-dimensional torus, or type II on a $d-1$ torus leads to an underlying U-duality symmetry given by the exceptional groups $E_{d(d)}(\mathbb{Z})$. In the low energy limit where we recover the eleven dimensional and massless type II supergravities, an underlying $E_{d(d)}(\mathbb{R})$ global symmetry appears. This symmetry can be made manifest in the context of exceptional field theory where the space is decomposed into an external space, and an internal extended space. Here we consider $d = 3$, corresponding to a 8-dimensional external space combined to a 6-dimensional internal extended space with a $E_{3(3)} = SL(3) \times SL(2)$ geometric structure. In fact, as mentioned in section 5.5, one can extend this duality group by considering the trombone symmetry appearing in supergravity theories. The duality group becomes therefore $\mathbb{R}^+ \times SL(3) \times SL(2)$. The extended internal space will be our main focus throughout this

paper as the tensor hierarchy of $E_{3(3)}$ exceptional field theory is done in [68]. We now present the basics of $\mathbb{R}^+ \times SL(3) \times SL(2)$ exceptional field theory which will be needed throughout this chapter.

We introduce a set of coordinates X^M, with $M, N, P = 1, .., 6$, of the 6-dimensional internal space which lives in the vector representation **(3,2)** of $SL(3) \times SL(2)$. We can decompose the index of the fundamental representation M into $M = m\gamma$ where all Latin letters $m, n, p, ... = 1, 2, 3$ and all Greek letters $\gamma, \eta, \rho, ... = 1, 2$ correspond respectively to the $SL(3)$ and $SL(2)$ part of $E_{3(3)}$. We will note $\partial_M = \partial_{m\gamma}$ the derivative with respect to $X^M = X^{m\gamma}$. We define the generalised Lie derivative as given by equation (5.20) where p is equal to 6 in the case of $E_{3(3)}$. The tensor Z encodes the deviation from Riemannian geometry and is given in terms of the invariants of the duality group, which in our case is

$$Z^{MN}{}_{PQ} = Z^{m\gamma n\eta}{}_{p\rho q\sigma} = \epsilon^{mnz}\epsilon_{pqz}\epsilon^{\gamma\eta}\epsilon_{\rho\sigma} \tag{6.1}$$

where ϵs are totally antisymmetric invariant tensors of $SL(3)$ and $SL(2)$. The invariant tensor verifies in particular $\mathcal{L}Z = 0$. Another expression for the generalised Lie derivatives which will be useful later to determine the fluxes of the extended space is

$$\begin{aligned}\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - 2(\mathbb{P}_{(\mathbf{8},\mathbf{1})})^M{}_N{}^P{}_Q \partial_P \Lambda^Q V^N - 3(\mathbb{P}_{(\mathbf{1},\mathbf{3})})^M{}_N{}^P{}_Q \partial_P \Lambda^Q V^N \\ +\lambda(V)\partial_N \Lambda^N V^M\end{aligned} \tag{6.2}$$

where $(\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ is the adjoint of $SL(3) \times SL(2)$ and the projections on each subspaces are given by

$$\begin{aligned}(\mathbb{P}_{(\mathbf{8},\mathbf{1})})^M{}_N{}^P{}_Q &= (\mathbb{P}_{(\mathbf{8},\mathbf{1})})^{m\gamma}{}_{n\eta}{}^{p\rho}{}_{q\sigma} = \frac{1}{2}\delta^\gamma_\eta \delta^\rho_\sigma \left(\delta^p_n \delta^m_q - \frac{1}{3}\delta^m_n \delta^p_q\right) = \frac{1}{2}\delta^\gamma_\eta \delta^\rho_\sigma \left(\mathbb{P}_{\mathbf{8}}\right)^m{}_n{}^p{}_q \\ (\mathbb{P}_{(\mathbf{1},\mathbf{3})})^M{}_N{}^P{}_Q &= (\mathbb{P}_{(\mathbf{1},\mathbf{3})})^{m\gamma}{}_{n\eta}{}^{p\rho}{}_{q\sigma} = \frac{1}{3}\delta^m_n \delta^p_q \left(\delta^\rho_\eta \delta^\gamma_\sigma - \frac{1}{2}\delta^\gamma_\eta \delta^\rho_\sigma\right) = \frac{1}{3}\delta^m_n \delta^p_q \left(\mathbb{P}_{\mathbf{3}}\right)^\gamma{}_\eta{}^\rho{}_\sigma\end{aligned} \tag{6.3}$$

with $\mathbb{P}_{\mathbf{8}}$ and $\mathbb{P}_{\mathbf{3}}$ the projectors onto the $SL(3)$ and $SL(2)$ adjoint respectively. The expressions of

the projectors onto the adjoint using the generators of $SL(3)$ and $SL(2)$ are detailed in Appendix A.1. Finally, using (6.3) we can write the generalised Lie derivative in terms of $SL(3)$ and $SL(2)$ indices

$$\mathcal{L}_\Lambda V^{m\gamma} = \Lambda^{n\eta}\partial_{n\eta}V^{m\gamma} - V^{m\eta}\partial_{n\eta}\Lambda^{n\gamma} - V^{n\gamma}\partial_{n\eta}\Lambda^{m\eta} + \left(\lambda(V) + \frac{5}{6}\right)\partial_{n\eta}\Lambda^{n\eta}V^{m\gamma}. \tag{6.4}$$

In order for the theory to be consistent, the algebra of the generalised Lie derivatives (5.20) has to close, i.e. it should satisfy

$$[\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_2}] = \mathcal{L}_{\Lambda_{12}} \tag{6.5}$$

where

$$\Lambda_{12} \equiv [\Lambda_1, \Lambda_2]_E = \frac{\mathcal{L}_{\Lambda_1}\Lambda_2 - \mathcal{L}_{\Lambda_2}\Lambda_1}{2} \tag{6.6}$$

is the analogy of the Courant bracket (5.12) introduced in generalised geometry but in the context of exceptional geometry [56, 65]. The closure of the algebra is however only achieved if one imposes the section condition (5.21) which in this case is

$$\begin{aligned} Z^{NK}{}_{PQ}\partial_N \otimes \partial_K &= \epsilon^{nkz}\epsilon_{pqz}\epsilon^{\eta\kappa}\epsilon_{\rho\delta}\partial_{n\eta} \otimes \partial_{k\kappa} = 0 \\ \Leftrightarrow \partial_{n\eta} \otimes \partial_{k\kappa} - \partial_{n\kappa} \otimes \partial_{k\eta} + \partial_{k\kappa} \otimes \partial_{n\eta} - \partial_{k\eta} \otimes \partial_{n\kappa} &= 0. \end{aligned} \tag{6.7}$$

The fields of the theory therefore can no longer depend arbitrarily on the 6-dimensional internal space, but in our case rather a 2 or 3 dimensional subspace. This allows one to describe in particular 8+3=11 dimensional supergravity or 8+2=10 dimensional type II supergravity respectively. We will consider the embedding of type IIB supergravity: we will focus on the solutions where the fields effectively depend on a two dimensional subspace of the six dimensional internal space. The usual way to do this is to consider that

$$\partial_{1\gamma}(A) = \partial_{2\gamma}(A) = 0 \tag{6.8}$$

for any field A. This leads to the breaking of $SL(3)$ into $SL(2) \times U(1)$. To make this breaking manifest we can split the index $M = m\gamma$ of the fundamental representation into $m\gamma = (\hat{m}\gamma, 3\gamma)$

where $\hat{m} = 1, 2$.

6.2 Fluxes

Compactifying string theory with fluxes leads, in the low energy limit, to gauged supergravity. They correspond to deformation of abelian supergravities where a subgroup G_0 of the global symmetry group G of the supergravity theory is promoted to a local symmetry. The embedding of the gauge group G_0 into the global symmetry group G can be described by an object called the embedding tensor, which corresponds exactly to the fluxes. Supersymmetry and gauge invariance of the embedding tensor then leads to a set of linear and quadratic constraint on the embedding tensor, which by extension should be verified by the fluxes of the corresponding low energy limit of string theory [75, 76].

Here we derive the expression of the generalised fluxes for the $SL(3) \times SL(2) \times \mathbb{R}^+$ exceptional field theory. They will have to verify both linear and quadratic constraints so that the corresponding 8 dimensional gauged maximal supergravity we obtain in the low energy limit after compactification with fluxes is consistent. Considering the warp factor in the duality group will lead us to consider gauged supergravity with a gauged trombone symmetry. The gauging of the trombone symmetry for $SL(3) \times SL(2)$ exceptional field theory has never been done before due to the group product structure of this particular theory. We construct it here similarly to what is done in [67] where the trombone symmetry is gauged for simple groups.

6.2.1 Embedding tensor structure of D=8 gauged maximal Supergravity with trombone symmetry

A way to describe the gauging of a subgroup of a global symmetry group G in supergravity theories is through the constant embedding tensor $\Theta_M{}^\Gamma$ [75, 76], where Γ is an index of $\mathbf{Adj(G)} = \mathbf{Adj}\left(\mathbf{SL(3)} \times \mathbf{SL(2)}\right)$ in our case and M corresponds to the fundamental representation $(\mathbf{3}, \mathbf{2})$. A consistent local gauging of the theory forces one to consider two constraints on

this embedding tensor: a linear one and a quadratic one. Let us recall the results already known for the particular case of $E_{3(3)}$ exceptional field theory, without the scale factor of the general extended group. A priori the embedding tensor $\Theta_M{}^\Gamma$ of the theory lives in

$$(\mathbf{3},\mathbf{2}) \times ((\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{3})) = [(\mathbf{3},\mathbf{2}) + (\mathbf{6},\mathbf{2}) + (\mathbf{15},\mathbf{2})] + [(\mathbf{3},\mathbf{2}) + (\mathbf{2},\mathbf{4})]], \tag{6.9}$$

but due to the linear and quadratic constraints, the embedding tensor only has $(\mathbf{6},\mathbf{2})$ and $(\mathbf{3},\mathbf{2})$ components. Using this linear constraint we can write the generators of the gauge group of the theory using the embedding tensor and the generators of the adjoint of the gauge group $\{t_\Gamma\}$

$$(X_M)_N{}^P = \Theta_M{}^\Gamma (t_\Gamma)_N{}^P = \Theta_{m\gamma,n}{}^p \delta_\eta^\rho + \Theta_{m\gamma,\eta}{}^\rho \delta_n^p \tag{6.10}$$

with

$$\begin{aligned} \Theta_{m\gamma,\eta}{}^\rho &= \xi_{m\eta}\delta_\gamma^\rho - \frac{1}{2}\delta_\eta^\rho \xi_{m\gamma} = \mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^\rho{}_\eta{}^\delta{}_\gamma \xi_{m\delta} \\ \Theta_{m\gamma,n}{}^p &= f_\gamma{}^{(pb)}\epsilon_{bmn} - \frac{3}{4}(\xi_{n\gamma}\delta_m^p - \frac{1}{3}\xi_{m\gamma}\delta_n^p) = f_\gamma{}^{(pb)}\epsilon_{bmn} - \frac{3}{4}\mathbb{P}_{(\mathbf{8})}{}^p{}_n{}^r{}_m \xi_{r\gamma} \end{aligned} \tag{6.11}$$

with $\Theta_{m\gamma,\eta}{}^\rho \delta_n^p \in (\mathbf{3},\mathbf{2})$ and $\Theta_{m\gamma,n}{}^p \delta_\eta^\rho \in (\mathbf{6},\mathbf{2})$[1]. To avoid confusion between the fundamental representation of $SL(3)$ and the adjoint of $SL(2)$ we write the later $(\mathbf{3_{SL(2)}})$.

This is not the more general setting of supergravity gauging however, as one can gauge the trombone symmetry [66, 67]. In order to do that we have to consider a more general ansatz than the one used in [67], as the global symmetry group is not simple in our case but a product of simple groups. Considering the $\mathbb{R}^+$ factor in the duality group leads to an additional generator $(t_0)_N{}^P = -\delta_N^P$ in equation (6.10), and a corresponding additional component of the embedding tensor $\Theta_M{}^0 \equiv K_M$. This component lives in the **(3,2)** representation, and we expect it to appear in the same way as the other **(3,2)** parameter ξ_M. This leads to the following ansatz for the

[1] $f_\gamma{}^{mn}$ and $\xi_{m\alpha}$ need to verify a set of quadratic constraints which can be found in [77].

generators of the gauge group

$$X_{MN}{}^{P} = \Theta_{m\gamma,n}{}^{p}\delta^{\rho}_{\eta} + \Theta_{m\gamma,\eta}{}^{\rho}\delta^{p}_{n} + \left(\zeta_1 \mathbb{P}_{(\mathbf{8})}{}^{p}{}_{n}{}^{k}{}_{m}\delta^{\rho}_{\eta}\delta^{\kappa}_{\gamma} + \zeta_2 \mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\rho}{}_{\eta}{}^{\kappa}{}_{\gamma}\delta^{p}_{n}\delta^{k}_{m} - \delta^{K}_{M}\delta^{P}_{N}\right) K_{k\kappa} \tag{6.12}$$

where ζ_1 and ζ_2 are two real parameters. The symmetric part of the generators of the gauge group, the intertwining tensor, should be in the same representation whether or not we consider an $\mathbb{R}^+$ gauging. This is necessary in order to preserve the two-form field content of the theory [67]. This is verified for $\zeta_1 = -\zeta_2 = 6$. The generators still have to verify a set of constraints which can be expressed in terms of the tensors introduced before as

$$0 = X_{MN}{}^{P}K_P + 6\mathbb{P}_{(\mathbf{8})}{}^{r}{}_{m}{}^{p}{}_{n}K_{r\gamma}K_{p\eta} - 6\mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\delta}{}_{\gamma}{}^{\rho}{}_{\eta}K_{m\delta}K_{n\rho} - K_{m\gamma}K_{n\eta} \tag{6.13}$$

$$\begin{aligned} 0 =& X_{PM}{}^{N}X_{NK}{}^{R} + X_{PK}{}^{N}X_{MN}{}^{R} - X_{PN}{}^{R}X_{MK}{}^{N} - K_P X_{MK}{}^{R} \\ &+ 6\left(\mathbb{P}_{(\mathbf{8})}{}^{q}{}_{p}{}^{n}{}_{m}\delta^{\sigma}_{\rho}\delta^{\eta}_{\gamma} - \mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\sigma}{}_{\rho}{}^{\eta}{}_{\gamma}\delta^{q}_{p}\delta^{n}_{m}\right) K_{q\sigma}X_{n\eta,k\kappa}{}^{r\delta} \\ &- 6\left(\mathbb{P}_{(\mathbf{8})}{}^{q}{}_{p}{}^{r}{}_{n}\delta^{\sigma}_{\rho}\delta^{\delta}_{\eta} - \mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\sigma}{}_{\rho}{}^{\delta}{}_{\eta}\delta^{q}_{p}\delta^{r}_{n}\right) K_{q\sigma}X_{m\gamma,k\kappa}{}^{n\eta} \\ &+ 6\left(\mathbb{P}_{(\mathbf{8})}{}^{q}{}_{p}{}^{n}{}_{k}\delta^{\sigma}_{\rho}\delta^{\eta}_{\kappa} - \mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\sigma}{}_{\rho}{}^{\eta}{}_{\kappa}\delta^{q}_{p}\delta^{n}_{k}\right) K_{q\sigma}X_{m\gamma,n\eta}{}^{r\delta}. \end{aligned} \tag{6.14}$$

6.2.2 Generalised Dynamical Fluxes

Now that we described the embedding tensor of maximal supergravity in 8 dimensions with a gauged trombone symmetry we look at the fluxes of $SL(3)\times SL(2)\times\mathbb{R}^+$ EFT. First let us consider the generalised metric of the extended space. We can define a generalised metric H living in the quotient $\frac{SL(3)}{SO(3)} \times \frac{SL(2)}{SO(2)} \times \mathbb{R}^+$ which transforms covariantly under $SL(3) \times SL(2) \times \mathbb{R}^+$ and is invariant under the maximal compact subgroup of $E_{3(3)}$ i.e $SO(3) \times SO(2)$. Due to the product structure of the group, we define a generalised bein which splits as

$$E_{\bar{A}}{}^{M} = e^{-\Delta}e_{\bar{a}}{}^{m}l_{\bar{\alpha}}{}^{\gamma} \tag{6.15}$$

where Δ is the $\mathbb{R}^+$ component of the metric, $e_{\bar{a}}{}^m$ and $l_{\bar{\alpha}}{}^\gamma$ the $SL(3)$ and $SL(2)$ beins respectively. $\bar{a}$ and $\bar{\alpha}$ are $SO(3)$ and $SO(2)$ planar indices respectively. The metric of the internal space is then

$$H^{MN} = E_{\bar{A}}{}^M E_{\bar{B}}{}^N \delta^{\bar{A}\bar{B}} = e^{-2\Delta} H^{mn} g^{\gamma\eta} \tag{6.16}$$

where

$$\begin{aligned} H^{mn} &= e_{\bar{a}}{}^m e_{\bar{b}}{}^n \delta^{\bar{a}\bar{b}} \\ g^{\gamma\eta} &= l_{\bar{\alpha}}{}^\gamma l_{\bar{\beta}}{}^\eta \delta^{\bar{\alpha}\bar{\beta}} \end{aligned} \tag{6.17}$$

correspond to an $SL(3)$ an $SL(2)$ metric respectively.

Having defined the generalised bein and a consistent generalised Lie derivative of the theory, one defines the generalised fluxes similarly to the fluxes in general relativity as[2]

$$\mathcal{L}_{E_{\bar{A}}} E_{\bar{B}} = F_{\bar{A}\bar{B}}{}^{\bar{C}} E_{\bar{C}}. \tag{6.18}$$

In a coordinate frame, we find the fluxes to be

$$F_{MN}{}^P = \Omega_{MN}{}^P - (2\mathbb{P}_{(\mathbf{8},\mathbf{1})}{}^P{}_N{}^R{}_S + 3\mathbb{P}_{(\mathbf{1},\mathbf{3})}{}^P{}_N{}^R{}_S)\Omega_{RM}{}^S + \frac{1}{6}\Omega_{RM}{}^R \delta_N^P \tag{6.19}$$

where

$$\Omega_{MN}{}^P = (E^{-1})_N{}^{\bar{A}} \partial_M E_{\bar{A}}{}^P \tag{6.20}$$

is the Weitzenböck connection[3]. Now, using the expressions of the bein (6.15) we obtain

$$\begin{aligned} \Omega_{MN}{}^P &= -\partial_M \Delta \delta_N^P + \delta_\eta^\rho (e^{-1})_n{}^{\bar{a}} \partial_{m\gamma}(e_{\bar{a}}{}^p) + \delta_n^p (l^{-1})_\eta{}^{\bar{\alpha}} \partial_{m\gamma}(l_{\bar{\alpha}}{}^\rho) \\ &= -\partial_M \Delta \delta_N^P + \delta_\eta^\rho \Omega_{m\gamma,n}{}^p + \delta_n^p \Omega_{m\gamma,\eta}{}^\rho \end{aligned} \tag{6.21}$$

[2]For more details see [74].

[3]We are abusing notation as the Weitzenböck connection should be globally defined, which is a priori not the case here.

where

$$\begin{aligned}\delta^{\rho}_{\eta}\Omega_{m\gamma,n}{}^{p} &= \delta^{\rho}_{\eta}(e^{-1})_{n}{}^{\bar{a}}\partial_{m\gamma}(e_{\bar{a}}{}^{p}) \in (\mathbf{3},\mathbf{2})\times(\mathbf{8},\mathbf{1})\\ \delta^{p}_{n}\Omega_{m\gamma,\eta}{}^{\rho} &= \delta^{p}_{n}(l^{-1})_{\eta}{}^{\bar{\alpha}}\partial_{m\gamma}(l_{\bar{\alpha}}{}^{\rho}) \in (\mathbf{3},\mathbf{2})\times(\mathbf{1},\mathbf{3}).\end{aligned} \tag{6.22}$$

The first term $-\partial_M\Delta\delta^P_N$ obviously lives in $(\mathbf{3},\mathbf{2})\times(\mathbf{1},\mathbf{1})=(\mathbf{3},\mathbf{2})$.

After some manipulations we find the following generalised fluxes

$$\begin{aligned}F_{MN}{}^{P} = &\left[f_{\gamma}{}^{pz}\epsilon_{zmn} - \frac{3}{4}\mathbb{P}_{(\mathbf{8})}{}^{r}{}_{m}{}^{p}{}_{n}\xi_{r\gamma}\right]\delta^{\rho}_{\eta} + \left[\mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\rho}{}_{\eta}{}^{\delta}{}_{\gamma}\xi_{m\delta}\right]\delta^{p}_{n}\\ &+\left(\frac{3}{2}-\frac{3}{4}\zeta\right)\mathbb{P}_{(\mathbf{8})}{}^{r}{}_{m}{}^{p}{}_{n}K_{r\gamma} + \zeta\mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\rho}{}_{\eta}{}^{\delta}{}_{\gamma}K_{m\delta} - K_{m\gamma}\delta^{p}_{n}\delta^{\rho}_{\eta}\end{aligned} \tag{6.23}$$

where

$$(\mathbf{6},\mathbf{2}) : f_{\gamma}{}^{pz} = \epsilon^{kq(z}\Omega_{k\gamma,q}{}^{p)} \tag{6.24}$$

$$(\mathbf{3},\mathbf{2}) : \begin{cases}\theta_{m\gamma} = \Omega_{r\gamma,m}{}^{r} - 4\partial_{m\gamma}\Delta\\ \tilde{\theta}_{m\gamma} = \Omega_{m\delta,\gamma}{}^{\delta} - 3\partial_{m\gamma}\Delta\\ K_{m\gamma} = -\frac{1}{6}(\theta_{m\gamma}+\tilde{\theta}_{m\gamma})\\ \xi_{m\gamma} = (\tilde{\theta}_{m\gamma}-\theta_{m\gamma}) - \zeta K_{m\gamma}\end{cases} \tag{6.25}$$

and ζ is only used to write the fluxes in a similar form compared to the gauge generators (6.12). Choosing $\zeta=-6$ gives us the the same expressions we found after considering the intertwining tensor constraint in the context of D=8 gauged maximal supergravity with gauged trombone symmetry. We thus have to consider the quadratic constraints (6.13) and (6.14) on K and f. We present simplified expressions of these constraints for the type IIB supergravity solution of the section condition in section 6.4.3, after choosing an appropriate ansatz of the generalised bein (6.15).

6.3 Equations of motion

We will now look at the equations of motion of the theory. General expressions of these equations were obtained in [57, 65] using the supersymmetric variations of the internal and external gravitino and a torsion-free/metric compatible connection. In fact it is specified that in our case, for $E_{3(3)}$ EFT, one can define a unique generalised Christoffel symbol. In this section, we find the expression of this generalised torsion free, metric compatible connection. We then find a generalised Ricci tensor following the construction of [74] for $E_{7(7)}$ EFT. In the last section we finally obtain the equations of motion of type IIB supergravity after a choice of an appropriate ansatz for the generalised $SL(3) \times SL(2) \times \mathbb{R}^+$ bein.

6.3.1 Generalised Christoffel symbol

Connections are defined to describe how a field is transported along curves on a manifold. Their definition can thus be chosen to be exactly the same as the one from Riemannian geometry

$$\nabla_M E_{\bar{A}}{}^N = \partial_M E_{\bar{A}}{}^N + \Gamma_{MK}{}^N E_{\bar{A}}{}^K. \tag{6.26}$$

The torsion however is defined using the Lie derivative and will differ from the usual Riemannian geometry [65]

$$\mathcal{T}_{\bar{A}\bar{B}}{}^{\bar{C}} = (E^{-1})_M{}^{\bar{C}} \left(\mathcal{L}^{\nabla}_{E_{\bar{A}}} - \mathcal{L}_{E_{\bar{A}}} \right) E_{\bar{B}}{}^M \tag{6.27}$$

with $\mathcal{L}^{\nabla}$ the generalised Lie derivative (5.20) where every derivative is replaced by a covariant one. Requiring that the generalised torsion is null we get from (6.26) that the generalised connection Γ verifies the following generalised torsion condition

$$\Gamma_{MN}{}^P = 2\mathbb{P}_{(\mathbf{8},\mathbf{1})}{}^P{}_N{}^D{}_Q \Gamma_{DM}{}^Q + 3\mathbb{P}_{(\mathbf{1},\mathbf{3})}{}^P{}_N{}^D{}_Q \Gamma_{DM}{}^Q - \frac{1}{6}\Gamma_{DM}{}^D \delta^P_N \tag{6.28}$$

which can also be written

$$2\Gamma_{[MN]}{}^P = -Z^P{}_N{}^R{}_K \Gamma_{RM}{}^K. \tag{6.29}$$

Using those expressions it is possible to seek a generalised Christoffel symbol of the form

$$\Gamma_{MN}{}^{P} = \Gamma_{m\gamma,n}{}^{p}\delta^{\rho}_{\eta} + \Gamma_{m\gamma,\eta}{}^{\rho}\delta^{p}_{n} + \text{trace terms}. \tag{6.30}$$

without loss of generality[4]. Now, considering the metric compatibility condition

$$0 = \nabla_{M} H^{NP} = \partial_{M} H^{NP} + \Gamma_{MR}{}^{N} H^{RP} + \Gamma_{MR}{}^{P} H^{RN} \tag{6.31}$$

and the splitting of the metric (6.16) the first two terms of the expression (6.30) are found to be

$$\Gamma_{m\gamma,n}{}^{p} = \frac{1}{2} H^{pr} \left(\partial_{m\gamma} H_{nr} + \partial_{n\gamma} H_{mr} - \partial_{r\gamma} H_{mn}\right) \tag{6.32}$$

$$\Gamma_{m\gamma,\eta}{}^{\rho} = \frac{1}{2} H^{\rho\delta} \left(\partial_{m\gamma} H_{\eta\delta} + \partial_{m\eta} H_{\gamma\delta} - \partial_{m\delta} H_{\gamma\eta}\right). \tag{6.33}$$

The first term (6.32) is just 2 copies of a three dimensional usual Riemannian Christoffel symbol (for each value of γ), and the second term (6.33) is 3 copies of a two dimensional one (for each value of m). Finally, using the torsion condition (6.28) we find the generalised Christoffel symbol, with vanishing generalised torsion and metric compatibility to be

$$\begin{aligned} \Gamma_{MN}{}^{P} = \Gamma_{m\gamma n\eta}{}^{p\rho} = \Gamma_{m\gamma,n}{}^{p}\delta^{\rho}_{\eta} + \Gamma_{m\gamma,\eta}{}^{\rho}\delta^{p}_{n} + 2\left(H^{pk} H_{mn} \partial_{k\gamma} \Delta \delta^{\rho}_{\eta} - \partial_{n\gamma}\Delta \delta^{p}_{m}\delta^{\rho}_{\eta}\right) \\ + 3\left(H^{\rho\kappa} H_{\gamma\eta} \partial_{m\kappa}\Delta\delta^{p}_{n} - \partial_{m\eta}\Delta\delta^{p}_{n}\delta^{\rho}_{\gamma}\right) + \partial_{M}\Delta\delta^{P}_{N} \end{aligned} \tag{6.34}$$

and whose traces are

$$\Gamma_{RM}{}^{R} = -\Gamma_{MR}{}^{R} = 6\partial_{M}\Delta. \tag{6.35}$$

This comes from the fact that the scalar that transforms properly under generalised diffeomorphisms is $e^{-6\Delta}$ for $SL(3) \times SL(2)$ i.e.

$$\delta_{\xi}(e^{-6\Delta}) = \partial_{P}\left(e^{-6\Delta}\xi^{P}\right). \tag{6.36}$$

[4]For more details see Appendix A.2.

We use the fact that the scalars of the theory should be of this particular form later in order to define a proper ansatz for the generalised metric and find the equations of motion one expects in F-theory.

6.3.2 Generalised Ricci tensor

A generalised Ricci tensor for the $E_{7(7)} \times \mathbb{R}^+$ EFT which transforms covariantly under generalised diffeormorphisms was proposed in [74]. It seems to hold for any exceptional field theory as it is written in terms of the tensor Z without need of its precise form. Here we show that for $SL(3) \times SL(2) \times \mathbb{R}^+$, it gives the expected equations of motion, thus confirming the proposed form of a generalised Ricci tensor in exceptional field theory. We review the main steps in order to construct a generalised Ricci tensor.

The usual Riemann tensor of a Riemannian space can be expressed as

$$R_{MNP}{}^{R} = \partial_M \Gamma_{NP}{}^{R} - \partial_N \Gamma_{MP}{}^{R} + \Gamma_{ML}{}^{R} \Gamma_{NP}{}^{L} - \Gamma_{NL}{}^{R} \Gamma_{MP}{}^{L}. \tag{6.37}$$

This object however does not transform properly under $SL(3) \times SL(2)$ generalised diffeomorphisms. Its non covariant variation is

$$\Delta_\xi R_{MNP}{}^{R} = 2 \Delta_\xi \Gamma_{[MN]}{}^{Q} \Gamma_{QP}{}^{R}$$

where $\Delta_\xi = \delta_\xi - \mathcal{L}_\xi$. If one considers the torsion condition of the generalised Christoffel symbol (6.29), the non covariant variation of the Riemann tensor is null if $Z = 0$ i.e. if the usual torsion condition $\Gamma_{[MN]}{}^{P}$ = 0 is satisfied. Now, the usual Ricci tensor should be

$$R_{MN} = R_{MRN}{}^{R} \tag{6.38}$$

but again this does not transform as a tensor under generalised diffeomorphisms. Its non covari-

ant variation is

$$\Delta_\xi R_{MP} = 2\Delta_\xi \Gamma_{[MR]}{}^{Q} \Gamma_{QP}{}^{R}. \tag{6.39}$$

One can then construct the following generalised Ricci tensor

$$\mathcal{R}_{MN} = \frac{1}{2}\left(R_{MN} + R_{NM} + \Gamma_{RM}{}^{P} Z^{RS}{}_{PQ} \Gamma_{SN}{}^{Q}\right) \tag{6.40}$$

which verifies

$$\Delta_\xi \mathcal{R}_{MN} = 0. \tag{6.41}$$

We will not detail here the expression of the generalised Ricci tensor obtained using our result on a generalised Christoffel symbol (6.34). This is described in section 6.4.3, where we consider a particular ansatz for the $SL(3) \times SL(2) \times \mathbb{R}^+$ bein in terms of the fields of type IIB supergravity.

6.4 Recovering F-theory

In this section, we use the results obtained before in order to relate $SL(3)\times SL(2)\times\mathbb{R}^+$ exceptional field theory to F-theory. We show that considering a non-trivial solution to the section condition allows us to describe the monodromies of (p,q) 7-branes appearing in F-theory. Finally we consider an ansatz for the $SL(3)\times SL(2)\times\mathbb{R}^+$ bein which leads to the type IIB equations of motion. We also show that the equations of motion obtained on the internal space using the generalised Ricci tensor and the generalised Christoffel symbol are equivalent to the Ricci-flatness of a 4 dimensional usual Ricci tensor, of which two of the dimensions are fibered as one expects from F-Theory.

6.4.1 Type IIB ansatz and generalised diffeomorphisms

In order to consider a type IIB solution of the $SL(3) \times SL(2) \times \mathbb{R}^+$ exceptional field theory the usual ansatz is (6.8) i.e. that the fields only depend on the coordinates $X^{3\gamma}$. This effectively

leads to an 8+2 dimensional theory where the fields have a dependency on the 8 dimensional external space-time, and two coordinates of the six dimensional internal extended space. Now, let us consider a particular choice of gauge for the generalised bein in terms of the fields of type IIB supergravity by breaking the $SL(3)$ subgroup into $SL(3) \to SL(2) \times U(1)$. The $SL(3)$ bein can be chosen to be

$$e_{\hat{\bar{a}}}{}^{\hat{m}} = \begin{pmatrix} e^{\frac{\phi+\Delta'}{2}} & e^{\frac{\phi+\Delta'}{2}} C_0 & 0 \\ 0 & e^{\frac{-\phi+\Delta'}{2}} & 0 \\ e^{-\Delta'} B & e^{-\Delta'} C & e^{-\Delta'} \end{pmatrix} \tag{6.42}$$

where ϕ and C_0 are the dilaton and axion respectively. B, C are defined properly below in terms of B_2 and C_2 and Δ' will be related to the scale factor Δ introduced in equation (6.16). In order to understand what are the fields B and C we look at the action of a generalised Lie derivative (6.2) of a generalised vector $V^M \equiv (V^{1\gamma}, V^{2\gamma}, v^\gamma)$ onto the $SL(3)$ bein (6.42) which in the most general case is[5]

$$\mathcal{L}_V \left(e_{\bar{a}}{}^m\right) = V^{k\gamma} \partial_{k\gamma} e_{\bar{a}}{}^m - e_{\bar{a}}{}^k \partial_{k\gamma} V^{m\gamma} + \left(\lambda \left(e_{\bar{a}}{}^m\right) + \frac{1}{3}\right) \partial_{k\gamma} V^{k\gamma} e_{\bar{a}}{}^m. \tag{6.43}$$

Now let us note $\mathcal{L}^0_V$ the action of a generalised Lie derivative with respect to the generalised vector V when the usual section condition (6.8) is verified. Inserting the expression of $\mathcal{L}^0_V(e_{\bar{3}}{}^3)$ in $\mathcal{L}^0_V(e_{\bar{3}}{}^1)$ and $\mathcal{L}^0_V(e_{\bar{3}}{}^2)$ and using the Leibniz rule property of generalised Lie derivatives leads to

$$\mathcal{L}^0_V(C) = v^\gamma \partial_{3\gamma} C + \partial_{3\gamma} v^\gamma C - \partial_{3\gamma} V^{2\gamma} \tag{6.44}$$

$$\mathcal{L}^0_V(B) = v^\gamma \partial_{3\gamma} B + \partial_{3\gamma} v^\gamma B - \partial_{3\gamma} V^{1\gamma}. \tag{6.45}$$

Moreover one can show that the antisymmetric 2 dimensional tensor $\epsilon^{\alpha\beta}$ is invariant under generalised diffeomorphisms. This allows to relate the **2** representation of $SL(2)$ to its dual $\bar{\mathbf{2}}$ using

$$V_\alpha = V^\beta \epsilon_{\beta\alpha}, \quad V^\alpha = \epsilon^{\alpha\beta} V_\beta. \tag{6.46}$$

[5]The expression of the action of generalised Lie derivatives onto the representation **(3,1)** can be found in [68].

The generalised Lie derivatives (6.44) and (6.45) are then

$$\mathcal{L}^0_V(C) = v^\gamma \partial_{3\gamma} C + \partial_{3\gamma} v^\gamma C - \partial_{3\gamma} V^2{}_\eta \epsilon^{\gamma\eta} \tag{6.47}$$

$$\mathcal{L}^0_V(B) = v^\gamma \partial_{3\gamma} B + \partial_{3\gamma} v^\gamma B - \partial_{3\gamma} V^1{}_\eta \epsilon^{\gamma\eta}. \tag{6.48}$$

To be more precise, we can define B and C to be the Hodge duals of the $NSNS$ (B_2) and RR (C_2) two-forms on the two dimensional space with metric $G_{\gamma\eta} \propto g_{\gamma\eta}$[6]

$$\begin{aligned} B &= \frac{1}{\sqrt{|G|}} \frac{\epsilon^{\gamma\eta}}{2} B_{\gamma\eta} \\ C &= \frac{1}{\sqrt{|G|}} \frac{\epsilon^{\gamma\eta}}{2} C_{\gamma\eta} \\ V^{1\gamma} &= V^1{}_\eta \epsilon^{\gamma\eta} = \frac{\lambda^B{}_\eta}{\sqrt{|G|}} \epsilon^{\gamma\eta} \\ V^{2\gamma} &= V^2{}_\eta \epsilon^{\gamma\eta} = \frac{\lambda^C{}_\eta}{\sqrt{|G|}} \epsilon^{\gamma\eta}. \end{aligned} \tag{6.49}$$

One then recovers the gauge transformations of B_2 and C_2

$$\begin{aligned} C_{\gamma\eta} &\to C_{\gamma\eta} + \partial_{[\gamma} \lambda^C{}_{\eta]} \\ B_{\gamma\eta} &\to B_{\gamma\eta} + \partial_{[\gamma} \lambda^B{}_{\eta]} \end{aligned} \tag{6.50}$$

with λ^C and λ^B the one-form parameters of the gauge transformations.

As expected from an exceptional field theory we see that using generalised diffeomorphisms, we can describe the usual diffeomorphisms of the two dimensional space (via v^γ), combined with two gauge transformations of the RR and $NSNS$ two-forms (via $V^{1\gamma}$ and $V^{2\gamma}$). In the next section, we look at the implication of a more general solution to the constraint, which describes the monodromies of exotic (p, q) 7-branes using generalised Lie derivatives.

[6] In the last section we find that $G_{\gamma\eta} = e^{-6\Delta} g_{\gamma\eta}$ in order to recover the equations of motion of type IIB supergravity.

6.4.2 F-theory as $\mathbb{R}^+ \times E_{3(3)}$ EFT with non standard solution to the section condition

Exotic branes have been extensively studied in the context of DFT and EFT where one considers the fields to depend on winding and wrapping coordinates [24, 27–29, 32–35]. In our case, one can solve the section condition (6.7) by requiring that the fields depend on two coordinates, but allow them to be a combination of the ordinary ones $X^{3\gamma}$ and the ones associated to winding and $D1$-brane wrapping $X^{\hat{m}\gamma}$. We propose the following solutions to the section condition of $\mathbb{R}^+ \times SL(3) \times SL(2)$ exceptional field theory

$$A = f(X^{3\gamma} + A_{\hat{m}} X^{\hat{m}\gamma}) \tag{6.51}$$

for any field A and with $\hat{m}$=1,2. In practice this can be seen as a $SL(3)$ rotation of the usual type IIB equations of motion and fluxes via the $SL(3)$ rotation of the generalised tangent space

$$\partial_{m\gamma} \to R_m{}^k \partial_{k\gamma} \tag{6.52}$$

with

$$R_m{}^k = \begin{pmatrix} 1 & 0 & A_1 \\ 0 & 1 & A_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.53}$$

and where one considers the fields to be usual solutions to the section condition (6.8). One has to note that it is equivalent to rotate the $SL(3)$ bein instead of the generalised tangent space

$$e_{\bar{a}}{}^m \to R_k{}^m e_{\bar{a}}{}^k. \tag{6.54}$$

The terms $A_{\hat{m}}$ are constant with respect to the 6-dimensional internal space, but can a priori depend on the 8-dimensional space-time coordinates. We recover the usual section condition for $A_{\hat{m}} = 0$. One should note that the structure of the ansatz (6.51) as a global function of the combined coordinates is necessary in order for products and inverts of fields to be well defined

i.e. so that they are also solutions to the section condition.

Let us see what happens when performing generalised diffeomorphisms. As we are only interested in the extra terms compared to the usual section condition solution (6.8), we will again denote by $\mathcal{L}_V^0$ the generalised Lie derivative with respect to a generalised vector V when $\partial_{1\alpha}(A) = \partial_{2\alpha}(A) = 0$ for any field A, which corresponds to the equations (6.44) and (6.45). Let us first look at the generalised Lie derivative of the fields ϕ and C_0 using the expression (6.43) and the Leibniz rule

$$\begin{aligned}\mathcal{L}_V(e^\phi) &= \mathcal{L}_V^0(e^\phi) + V^{\hat{k}\kappa}\partial_{\hat{k}\kappa}(e^\phi) \\ &\quad - e^\phi\partial_{1\kappa}V^{1\kappa} - e^\phi C_0\partial_{2\kappa}V^{1\kappa} + \left(\lambda(e^\phi) + \frac{2}{3}\right)\partial_{\hat{k}\kappa}V^{\hat{k}\kappa}e^\phi \\ \mathcal{L}_V(C_0) &= \mathcal{L}_V^0(C_0) + V^{\hat{k}\kappa}\partial_{\hat{k}\kappa}(C_0) - \partial_{1\kappa}V^{2\kappa} \\ &\quad - C_0\left(\partial_{2\kappa}V^{2\kappa} - \partial_{1\kappa}V^{1\kappa}\right) - C_0^2\partial_{2\kappa}V^{1\kappa} + \lambda(C_0)\partial_{\hat{k}\kappa}V^{\hat{k}\kappa}C_0.\end{aligned} \tag{6.55}$$

We consider generalised diffeomorphisms that satisfy $\partial_{\hat{k}\kappa}V^{\hat{k}\kappa} = V^{\hat{k}\kappa}\partial_{\hat{k}\kappa} = 0$ which can be achieved by considering that $V^{\hat{m}\gamma}$ verifies

$$A_2V^{2\kappa} = -A_1V^{1\kappa}. \tag{6.56}$$

This gives

$$\begin{aligned}\mathcal{L}_V(e^\phi) &= \mathcal{L}_V^0(e^\phi) - e^\phi\partial_{1\kappa}V^{1\kappa} - e^\phi C_0\partial_{2\kappa}V^{1\kappa} \\ \mathcal{L}_V(C_0) &= \mathcal{L}_V^0(C_0) - \partial_{1\kappa}V^{2\kappa} - C_0\left(\partial_{2\kappa}V^{2\kappa} - \partial_{1\kappa}V^{1\kappa}\right) - C_0^2\partial_{2\kappa}V^{1\kappa}.\end{aligned} \tag{6.57}$$

Here we see that the term $-\partial_{1\kappa}V^{2\kappa}$ is producing a shift of the axion, as is expected from a monodromy of a $(p,0)$ 7-brane given by the equation (3.38) and using the corresponding monodromy matrix (4.14). As the other terms are not particularly clear when one looks at the fields ϕ and C_0 let us consider the generalised Lie derivatives of the fields B and C. Considering the ansatz

(6.56) we obtain

$$\mathcal{L}_V(C) = \mathcal{L}^0_V(C) - B\partial_{1\kappa}V^{2\kappa} - C\partial_{2\kappa}V^{2\kappa} \tag{6.58}$$

$$\mathcal{L}_V(B) = \mathcal{L}^0_V(B) - B\partial_{1\kappa}V^{1\kappa} - C\partial_{2\kappa}V^{1\kappa}. \tag{6.59}$$

Now, to make sense of the two previous equation in terms of monodromies we take each component of V^M to be linear in its coordinates. Requiring the conditions (6.51) and (6.56) for $V^{\hat{m}\gamma}$ we are able to recover the monodromies of a general (p, q) 7-brane encoded into the generalised Lie derivatives of the exceptional field theory

$$\begin{aligned} \mathcal{L}_V(C) &= \mathcal{L}^0_V(C) + pqC + p^2B \\ \mathcal{L}_V(B) &= \mathcal{L}^0_V(B) - pqB - q^2C \end{aligned} \tag{6.60}$$

with the additional conditions

$$\begin{aligned} \partial_{1\kappa}V^{1\kappa} &= -\partial_{2\kappa}V^{2\kappa} = pq \\ \partial_{1\kappa}V^{2\kappa} &= -p^2 \\ \partial_{2\kappa}V^{1\kappa} &= q^2 \\ qA_1 &= pA_2. \end{aligned} \tag{6.61}$$

Now let us look at the particular case of a stack of p $D7$-branes, as an arbitrary (p', q') 7-brane can be mapped locally to a $(p, 0)$ one, using an $SL(2, \mathbb{Z})$ transformation. We can make the following ansatz for the dependency of the Lie derivative generalised vector parameter

$$V^M = (0, X^{3\gamma} - \frac{p^2}{2}X^{1\gamma}, 0) \tag{6.62}$$

where we put $V^{3\gamma}$ to zero to remove the diffeomorphisms component. Using this we obtain the

full transformations of the fields to be

$$\begin{aligned} \mathcal{L}_V(e^{\phi}) &= 0 \\ \mathcal{L}_V(C_0) &= p^2 \\ \mathcal{L}_V(C) &= p^2 B + 2 \\ \mathcal{L}_V(B) &= 0. \end{aligned} \tag{6.63}$$

The additional shift term in the action of the monodromy on C is coming from a breaking of the gauge symmetry invariance of this field, which is also the case for B. The gauge invariances of the fields B and C seem to be entirely constrained by the monodromies as one goes around a $D7$-brane. One could notice that breaking of the gauge invariances are expected when non perturbative effects of string theory are taken into account. This is however not an acceptable explanation in our case as we are in the perturbative regime. A more appropriate explanation would be that the monodromies we described before have an interpretation only when one is considering that the only term appearing in the Chern-Simons action of a $D7$-brane is the C_8 term dual to the axion C_0 [78]. The full Chern-Simons action is however

$$\int_{\mathcal{M}_8} \mathbf{C} \wedge e^{-B_2} \tag{6.64}$$

where $\mathcal{M}_8$ is the brane world volume and

$$\mathbf{C} = \sum_{p=0..4} C_{2p}. \tag{6.65}$$

This might break the gauge invariances of both B_2 and C_2.

6.4.3 Equations of motion via the generalised Ricci tensor

Here we write explicitly the equations of motion with usual section condition (6.8) using the generalised Ricci tensor (6.40) and with the help of the symbolic computer algebra system Cadabra [79, 80]. To begin with, let us consider the proposed ansatz (6.42) for the generalised

bein. As we showed that the scalars that transform properly are of the form $e^{6n\Delta}$ where $n \in \mathbb{Z}$, and as the $\mathbb{R}^+$ factor in the metric are of the form $e^{-2\Delta+\Delta'}$ and $e^{-2\Delta-2\Delta'}$, a plausible ansatz on the scalar Δ' is

$$\Delta' = -4\Delta. \tag{6.66}$$

Now according to [65], the equations of motions should live in the representation

$$(\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1}). \tag{6.67}$$

This leads to the following equations of motion

$$\begin{aligned} 0 &= \mathcal{R}_{m\gamma,n\eta}H^{mn} \in (\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1}) \\ 0 &= \mathcal{R}_{(m|\gamma,|n)\eta}g^{\gamma\eta} \in (\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{1}). \end{aligned} \tag{6.68}$$

Using the definition of the generalised Ricci tensor (6.40), the generalised Christoffel symbol (6.34), the ansatz on the bein (6.42) as well as the ansatz on Δ' (6.66) and the usual section condition (6.8) we obtain the equations of motion of type IIB supergravity in 2 dimensions[7]

$$R_{\gamma\rho}\left[e^{-6\Delta}g_{..}\right] + \frac{1}{2}\partial_\gamma\phi\partial_\rho\phi + \frac{1}{2}\partial_\gamma C_0\partial_\rho C_0 = 0 \tag{6.69}$$

$$g^{\gamma\rho}\left(\nabla_\gamma\nabla_\rho\phi - e^{2\phi}\nabla_\gamma C_0\nabla_\rho C_0\right) = 0 \tag{6.70}$$

$$g^{\gamma\rho}\left(\nabla_\gamma C_0\nabla_\rho C_0 + 2\nabla_\gamma C_0\nabla_\rho\phi\right) = 0 \tag{6.71}$$

where $\partial_\gamma \equiv \partial_{3\gamma}$ and $R_{\gamma\rho}[e^{-6\Delta}g_{..}]$ corresponds to the usual Ricci tensor associated to the metric $e^{-6\Delta}g_{\gamma\rho}$. ∇ is the covariant derivative whose connection is the usual two dimensional Christoffel symbol of the same metric. One should note that the only way to recover the equations of motions of type IIB supergravity is to combine the warp factor Δ with the $\mathbb{R}^+$ factor Δ' coming from the breaking of $SL(3)$ into $SL(2) \times U(1)$ as in (6.66). Finally, as we stated before, the fields should verify different constraints due to the quadratic conditions (6.13) and (6.14) which in the

[7]The expression (6.69) can differ from the literature by a minus sign due to the definition of the Riemann tensor (6.37).

end can be recast into

$$\tilde{\theta}_{3[\gamma|}\Omega_{3|\rho],k}{}^{r} = 0. \tag{6.72}$$

If we write the geometric fluxes of the two dimensional space with bein $\tilde{l}_{\bar{\alpha}}{}^{\gamma} = e^{3\Delta} l_{\bar{\alpha}}{}^{\gamma}$ as

$$w_{\gamma\eta}{}^{\rho} = 2(\tilde{l}^{-1})_{[\gamma|}{}^{\bar{\alpha}}\partial_{|\eta]}\tilde{l}_{\bar{\alpha}}{}^{\rho} \tag{6.73}$$

we find that the condition (6.72) is equivalent to

$$\begin{aligned} w_{[\gamma|\delta}{}^{\delta}\partial_{|\eta]}\Phi &= 0 \\ w_{[\gamma|\delta}{}^{\delta}\partial_{|\eta]}C_0 &= 0 \\ w_{[\gamma|\delta}{}^{\delta}\partial_{|\eta]}\Delta &= 0. \end{aligned} \tag{6.74}$$

These can be solved in particular if we consider the trace $w_{\gamma\delta}{}^{\delta}$ to be null, which is equivalent to $w = 0$ for a two dimensional space. This in particular ensures that the two dimensional internal space is compact [81].

To conclude we show that the equations of motion (6.68) are equivalent to the Ricci-flatness of a 4 dimensional space: a two torus with constant volume equal to one, fibered over a two dimensional Riemann space. To do that we consider a 4 dimensional space whose metric is

$$H_{MN} = \begin{pmatrix} H_{\hat{m}\hat{n}} & 0 \\ 0 & g_{\gamma\rho} \end{pmatrix} \tag{6.75}$$

and where $M = (\hat{m}, \gamma)$ with $\hat{m}$ and γ being 1 or 2. $H_{\hat{m}\hat{n}}$ is an $SL(2)$ metric while $g_{\gamma\rho}$ is a $GL(2)$ one. Now in order to describe a fibration we will consider that every field only depends on the two coordinates x^{γ}. Considering the usual Riemannian Ricci tensor of this four dimensional space with $\partial_{\hat{m}} = 0$ we have

$$R_{M=\gamma,P=\rho} = R_{\gamma\rho} - \frac{1}{4}\partial_{\gamma}H^{kr}\partial_{\rho}H_{kr} \tag{6.76}$$

where $R_{\gamma\rho}$ is the two dimensional usual Ricci tensor associated to the metric $g_{\gamma\rho}$. Assuming that

the $SL(2)$ metric is of the usual form

$$H_{\hat{m}\hat{n}} = \frac{1}{\mathsf{Im}(\tau)} \begin{pmatrix} |\tau|^2 & -\mathsf{Re}(\tau) \\ -\mathsf{Re}(\tau) & 1 \end{pmatrix} \tag{6.77}$$

with the axio-dilaton τ given by (3.37), we recover the expected equations of motion we derived before. The equations of motion for the dilaton (6.70) and the axion (6.71) are obtained by considering the other components of the Ricci tensor $R_{M=m,P=p}$.

Part III

New insights on F-theory and Heterotic String Duality in Eight Dimensions

CHAPTER 7

F-theory and Heterotic String Duality in Eight Dimensions

F-theory compactified on elliptically fibered K3 surfaces is believed to be dual at the quantum level to the heterotic string compactified on a two-torus with Wilson lines [5, 82–86]. In particular one should be able to relate the complex parameters of the moduli space on the F-theory side to the ones on the heterotic one as their moduli space are the same: the Narain space [87, 88].

In this chapter we present fundamental notions necessary to understand this duality. This will be necessary in the next chapter where we construct graphs of polytopes which give information on the map between moduli from F-theory to the ones in the heterotic string.

We begin with some of the aspects of the compactifications of string theory on a two-torus T^2. We show that non trivial Wilson lines break the $E_8 \times E_8$ and $SO(32)$ gauge symmetry. We then present a basic example which illustrates how one can enhance these gauge groups in a particular subspace of the moduli space. We then discuss an example which breaks the groups $E_8 \times E_8$ and $SO(32)$ to one of their subgroups.

We then focus on elliptically fibered K3 compactifications of F-theory. We begin with some aspects of cohomology and discuss the cohomological structure of K3 surfaces as well as its moduli space. We then define elliptic fibrations of such spaces. Finally, we discuss how to

construct K3 surfaces and elliptically fibered K3 surfaces using reflexive polyhedra and toric geometry.

7.1 Heterotic string theory compactifications on a two-torus

7.1.1 General aspects of the torus compactification

Here we introduce basic notions concerning the compactifications of the heterotic strings on a two-torus, based on [17, 36]. Let us consider the compactification of the heterotic string on a 2-dimensional torus with constant background metric $G_{mn} = e^a_m e^b_n \delta_{ab}$ with bein e and its inverse $\hat{e}$, two-form field B_{mn} and $U(1)^{16}$ gauge fields A^A_m with $m, n, a, b = 1, 2$ and $A = 1, .., 16$. It is then possible to decompose the momentum of the string $P = (p_{Ra}, p_{La}, p^A)$ as

$$p_{Ra} = \frac{\hat{e}^m_a}{\sqrt{2}} \left(n_m - (G_{mn} + B_{mn})\, w^n - \pi^A A^A_m - \frac{1}{2} A^A_n A^A_m w^n \right) \tag{7.1}$$

$$p_{La} = \frac{\hat{e}^m_a}{\sqrt{2}} \left(n_m + (G_{mn} - B_{mn})\, w^n - \pi^A A^A_m - \frac{1}{2} A^A_n A^A_m w^n \right) \tag{7.2}$$

$$p^A = \pi^A + w^m A^A_m \tag{7.3}$$

with $a = 1, 2$, w^m the winding numbers and n_m the momentum numbers on the internal torus, π^A belonging to the weight lattice $\Gamma_{D_{16}}$ of $\frac{Spin(32)}{\mathbb{Z}_2}$ or $\Gamma_{E_8} \otimes \Gamma_{E_8}$ with Γ_{E_8} the root lattice of E_8. This is once again forced by modular invariance. The twenty dimensional momentum P transforms as a vector under $O(2, 18, \mathbb{R})$ and verifies the relation

$$P^2 \equiv p_L^2 - p_R^2 = 2 w^m n_m + |\pi^A|^2 \in 2\mathbb{Z} \tag{7.4}$$

The momentum lattice has therefore signature $(2, 18)$, is even and self dual due to modular invariance and is called the Narain lattice. Now, considering the inequivalent lattices under the

action of $O(2,18,\mathbb{Z})$, which corresponds to T-duality, the moduli space is given by

$$\frac{O(2,18,\mathbb{R})}{O(2,\mathbb{R})\times O(18,\mathbb{R})\times O(2,18,\mathbb{Z})}. \tag{7.5}$$

A possible parametrisation of the moduli space in terms of the metric of the torus G, the two-form field B and the Wilson lines in the first and second direction on the torus A_1 and A_2 is [89]

$$\tau=\frac{G_{12}+i\sqrt{|G|}}{G_{11}} \quad , \quad \rho=B+i\sqrt{|G|}+\frac{1}{2}A_1^A A_1^A\tau-\frac{1}{2}A_1^A A_2^A \quad , \quad \xi^A=A_1^A\tau-A_2^A. \tag{7.6}$$

This can be put in parallel with the $\frac{O(2,2)}{O(2)\times O(2)}$ generalised metric of equation (3.26) of part I when we discussed T-duality in the context of the bosonic string. When the Wilson lines vanish, τ and ρ are then the complex and Kähler structure of the torus T^2.

Now, focusing on the massless sector of the heterotic string one finds that admissible states verify $p_R=0$ which leads to

$$|p_L|^2=2 \quad \text{with} \quad p_L=\left(\sqrt{2}\hat{e}_{am}w^m,\pi^A+w^m A_m^A\right) \tag{7.7}$$

$$n_m=\left(G_{mn}+B_{mn}\right)w^n+\pi^A A_m^A+\frac{1}{2}A_n^A A_m^A w^n\in\mathbb{Z}. \tag{7.8}$$

In fine, these are the conditions one has to check in order to understand the gauge group structure of the resulting theory. We now discuss some examples to illustrate how gauge groups are broken or enhanced for particular values of the moduli G, B and A_m.

7.1.2 Examples of gauge group enhancements

Let us first treat probably the simplest example where one considers $A_m^A=0$. In this case, for $n_m=w^m=0$ one gets the condition $|\pi^A|^2=2$. Therefore the root vectors of $E_8\times E_8$ and $SO(32)$ are possible states. On the other hand, one can consider states that verify $w^m G_{mn}w^n=1$ and $(G_{mn}+B_{mn})w^n\in\mathbb{Z}$ according to equations (7.7) and (7.8). This occurs for particular values of the background fields of the torus. There exist various possible enhancements in this case. Let us first consider that $G_{11}=1$ and $G_{12}=B_{12}$. In this case one finds that taking winding numbers

$(w^1, w^2) = (\pm 1, 0)$ gives $(n_1, n_2) = (\pm 1, 0)$ and thus verify the condition (7.8). Equation (7.7) is then $|p_L|^2 = |(\sqrt{2}\hat{e}_{am} w^m, 0^{16})|^2 = 2$. We can choose a convenient ansatz for the bein by fixing

$$e = \sqrt{G_{11}} \begin{pmatrix} 1 & \frac{G_{12}}{G_{11}} \\ 0 & \frac{\sqrt{G}}{G_{11}} \end{pmatrix}. \tag{7.9}$$

One finally finds the two states

$$p_L = \sqrt{2}(\pm 1, 0, 0^{16}) \tag{7.10}$$

generating an enhancement to $SU(2)$ for this particular background both for the $E_8 \times E_8$ and $SO(32)$ heterotic strings. The moduli in this case verify $\tau = \rho$.

The other possible possible enhancements are $SU(2) \times SU(2)$ and $SU(3)$. The first one occurs e.g. at $G_{12} = B_{12} = 0$, $G_{11} = G_{22} = 1$ i.e. $\tau = \rho = i$ and gives the states

$$p_L = \sqrt{2}(\pm 1, 0, 0^{16}) \quad , \quad p_L = \sqrt{2}(0, \pm 1, 0^{16}). \tag{7.11}$$

The SU(3) enhancement happens e.g. at $G_{12} = B_{12} = \frac{1}{2}$, $G_{11} = G_{22} = 1$ i.e. $\tau = \rho = e^{\frac{2\pi i}{3}}$ and gives

$$p_L = \sqrt{2}(\pm 1, 0) \quad , \quad p_L = \pm\sqrt{2}(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0^{16}) \quad , \quad p_L = \pm\sqrt{2}(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0^{16}). \tag{7.12}$$

7.1.3 Examples of gauge group breaking

Now let us detail an example where one considers a non trivial Wilson line. Take $A_1^A = (A, A, 0^{14})$ with $A \in \mathbb{R}$ and $A_2 = 0$ in the case of the $SO(32)$ heterotic string, the roots of SO(32) being

$$SO(32) : \left(\pm 1, \pm 1, 0^{14}\right) \text{ + permutations.} \tag{7.13}$$

Generic values of the two-form field B_{12}, the metric G and the Wilson line parameter A do not give rise to states with non-zero winding number. We thus focus on the case $(w^1, w^2) = (0, 0)$.

The conditions to obtain admissible states are now

$$|\pi^A|^2 = 2 \tag{7.14}$$

$$n_1 = \pi^A A_1^A \in \mathbb{Z}. \tag{7.15}$$

We are therefore left to find roots of $SO(32)$ which verify $\pi^A A_1^A \in \mathbb{Z}$ for a generic value of A. It is easy to show that the admissible states are

$$p_L = \left(0,0,0,0,\underline{\pm 1,\pm 1,0^{12}}\right) \to SO(28) \tag{7.16}$$

$$p_L = \left(0,0,\pm 1,\mp 1,0^{14}\right) \to SU(2). \tag{7.17}$$

We thus obtain the group $SO(28) \times SU(2)$ with this Wilson line. Doing the same anylisis for the $E_8 \times E_8$ heterotic string one finds that one of the E_8 is broken to E_7. More generally it is shown in [36] that considering one Wilson line $A_1^A = (A^k, 0^{16-k})$ gives $SO(32-2k) \times SU(k)$ gauge groups. Of course particular values of the Wilson lines, the metric and the two-form field lead to additional states. For example considering $A \in \mathbb{Z}$ automatically satisfy (7.15) and the group is $SO(32)$. More generally this happens when the Wilson line belongs to the weight lattice of $SO(32)$.

In Chapter 8 we will write the Wilson lines A_1 and A_2 in a complex form $A = A_1 + iA_2$. The main goal will be to be able to understand the duality map between heterotic string on T^2 and F-theory on elliptically fibered K3 surfaces by matching the gauge groups on each side. First we present in section 7.2 how one can construct such elliptically fibered K3 surfaces and how to identify the gauge groups in F-theory in 8 dimensions.

7.2 F-theory compactifications on elliptically fibered K3 surfaces

7.2.1 Cohomology classes and Hodge diamonds

Let us present a useful tool which gives information on the toplogy of Calabi-Yau spaces: the *Hodge diamond* [90,91]. Let us consider a complex manifold M with $dim_{\mathbb{C}}(M) = n$. We introduce holomorphic z^i and antiholomorphic local coordinates $\bar{z}^i$ $(i = 1, ..n)$ and then define (p, q) forms as

$$A_{p,q} = \frac{1}{p!\,q!} A_{i_1,..,i_p,\bar{j}_1,..,\bar{j}_q} \mathsf{d}z^{i_1} \wedge .. \wedge \mathsf{d}z^{i_p} \wedge \mathsf{d}z^{\bar{j}_i} \wedge .. \wedge \mathsf{d}z^{\bar{j}_q} \in \Omega^{(p,q)}. \tag{7.18}$$

with $\Omega^{(p,q)}$ the vector bundle of (p, q) forms. Then one can define an holomorphic and antiholomophic part of the exterior derivative which act as

$$\partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)} \quad , \quad \bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}. \tag{7.19}$$

As the operator $\bar{\partial}$ verifies

$$\bar{\partial}^2 = 0 \tag{7.20}$$

one can define an associated cohomology (Dolbeaut) with

$$H^{(p,q)}_{\bar{\partial}}(M) = H^{(p,q)}_{\bar{\partial}}(M, \mathbb{C}) = \frac{\mathsf{Ker}\left(\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}\right)}{\mathsf{Im}\left(\bar{\partial} : \Omega^{(p,q-1)} \to \Omega^{(p,q)}\right)}. \tag{7.21}$$

When M is a Kähler manifold, which is the case here as we consider Calabi-Yau manifold, there is a consistency between the deRham and Dolbeaut cohomology. Namely if one considers the operator

$$\mathsf{d} : \Omega^r \to \Omega^{r+1} \tag{7.22}$$

acting on r forms, the deRham cohomology reads

$$H^r_{\mathsf{d}}(M) = \bigoplus_{p+q=r} H^{(p,q)}_{\bar{\partial}}(M), \tag{7.23}$$

which, if we consider the restriction of r forms to its subpaces of (p,q) forms gives

$$H^{(p,q)}(\mathcal{M}) \equiv H^{(p,q)}_{\mathsf{d}}(\mathcal{M}) = H^{(p,q)}_{\partial} = H^{(p,q)}_{\bar{\partial}}. \tag{7.24}$$

Now we can define Hodge numbers as the dimension of these spaces $h^{p,q} = \text{dim}\left(H^{(p,q)}(M)\right)$ which are usually represented as a Hodge diamond shown in Figure 7.1. The Hodge numbers,

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & \cdots & & \cdots & & \\
h^{n,0} & & & & & & h^{0,n} \\
 & & \cdots & & \cdots & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}$$

Figure 7.1: Hodge diamond of a compact Kähler manifold.

in the case of a Calabi-Yau manifold verify

$$\begin{aligned}
&h^{p,q} = h^{q,p} = h^{n-p,n-q}, \\
&h^{0,k} = 0, 0 < k < n, \\
&h^{0,0} = h^{n,0} = 1.
\end{aligned} \tag{7.25}$$

The Hodge diamond plays a central role to understand the field content of the theory after compactifcation, and in particular the resulting moduli. It can be used to generalise the decomposition of the five dimensional metric we made in the much simpler case of the circle compactification in equation (3.6). In this case the space of one forms on S^1 is spanned by one generator given by dx^4 and gives one moduli in the four dimensional metric. If we take e.g. the compactification

of string theory on a CY3, the number of moduli exclusively obtained from the 10 dimensional metric is $2h^{2,1} + h^{1,1}$. Details on the field content of type IIA and IIB string theory compactified on CY3 can be found in [17]. Next, we focus on the topologically unique two dimensional Calabi-Yau manifold called K3.

7.2.2 K3 surfaces and their moduli space

Here we describe some properties of compact complex K3 surfaces which are necessary to understand the duality between heterotic string on T^2 and F-theory on elliptically fibered K3 surfaces. We discuss some of the key steps permitting to identify the moduli space of a K3 surfaces. Details of such construction can be found in [92–95].

A complex K3 surface M is the only Calabi-Yau manifold whose complex dimension is 2. Its Hodge diamond is therefore unique and given by

		1		
	0		0	
1		20		1
	0		0	
		1		

Table 7.1: Hodge Diamond of a complex K3 surface.

Its second cohomology class decomposes as

$$H^2(M,\mathbb{C}) = H^{(2,0)}(M,\mathbb{C}) \oplus H^{(1,1)}(M,\mathbb{C}) \oplus H^{(0,2)}(M,\mathbb{C}). \tag{7.26}$$

One can consider a Kähler manifold as a real manifold with second cohomology $H^2(M,\mathbb{R})$ and with product structure defined as

$$\begin{aligned} (\ ,\ ) \ :\ H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\to H^4(M,\mathbb{R}) \simeq \mathbb{R} \\ (w,v) &\mapsto \int_M w \wedge v. \end{aligned} \tag{7.27}$$

Due to $\star^2 = 1$, one can decompose the second cohomology class with respect to its eigenspaces

H^{2+} and H^{2-} with eigenvalues 1 and -1 [96]. Their dimensions are 3 and 19 respectively which leads to the identification

$$H^2(M,\mathbb{R}) \simeq \mathbb{R}^{3,19}. \tag{7.28}$$

Now let us consider the holomorphic two-form $\Omega \in H^{(2,0)}(M,\mathbb{C})$ of the K3 surfaces as well as the Kähler form $J \in H^{(1,1)}(M,\mathbb{C})$, orthogonal to Ω. One has

$$\Omega = x + iy \tag{7.29}$$

where $x, y \in H^2(M,\mathbb{R}) \simeq \mathbb{R}^{3,19}$. It verifies

$$\int_M \Omega \wedge \Omega = 0 \tag{7.30}$$

$$\int_M \Omega \wedge \bar{\Omega} > 0 \tag{7.31}$$

which means that x and y span a space like subspace of $H^2(M,\mathbb{R})$. Moreover J verifies

$$\int_M J \wedge J > 0 \tag{7.32}$$

and is orthogonal to Ω with respect to the product (7.27). Therefore, we have that J and Ω are determined by a space-like 3-plane in $\mathbb{R}^{3,19}$. The moduli space is then linked to the following Grassmaniann space [95]

$$G = \frac{O(3,19,\mathbb{R})}{O(3,\mathbb{R}) \times O(19,\mathbb{R})}. \tag{7.33}$$

Now, let us consider the integral homology structure $H_2(M,\mathbb{Z})$ together with its cup product, or intersection pairing such that

$$\forall \alpha, \beta \in H_2(M,\mathbb{Z}) \quad (\alpha,\beta) = N(\alpha \cap \beta) \tag{7.34}$$

with $N(\alpha \cap \beta)$ the number of oriented intersection. By Poincaré duality this defines an equivalent

structure in the integral second cohomology $H^2(M,\mathbb{Z})$

$$(\ ,\ ) : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}. \tag{7.35}$$

The lattice $H^2(M,\mathbb{Z})$ is even self dual of signature $(3,19)$ and is isomorphic to the lattice

$$\Lambda_{K3} = H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8) \tag{7.36}$$

where H is the hyperbolic plane and E_8 is the even, positive and definite unimodular lattice of rank 8. One can then define a marked K3 surface $(M,\Gamma^{3,19})$ by a choice of isometry $\Gamma^{3,19} : H^2(M,\mathbb{Z}) \to \Lambda_{K3}$. The moduli space associated to Ω and J of the marked K3 surface is then the Grassmanian space of equation (7.33) quotiented by the isometries of $\Gamma^{3,19}$

$$\mathcal{M} = O(\Gamma^{3,19})\backslash G. \tag{7.37}$$

Now, what will be of interest to us in the next chapter are K3 surfaces whose moduli space is restricted to a subspace of (7.37). This will be necessary to understand which Wilson lines in the heterotic string with n moduli on T^2 maps to the n moduli of F-theory on an elliptic K3 surface. To this end let us introduce the Picard lattice

$$Pic(M) = H^{(1,1)}(M) \cap H^2(M,\mathbb{Z}). \tag{7.38}$$

It has signature $(1,\rho-1)$ where ρ is the Picard number of the K3 surface. Demanding that the Picard lattice is preserved under the variation of the complex structure of the K3 surface reduces the number of moduli. This is in particular the case for elliptically fibered K3 surfaces, which have to preserve the fibration as we discuss next.

7.2.3 Elliptically fibered K3 surfaces

The moduli space we obtained in the previous section (7.37) is somehow close to the one we obtained in the case of compactifications of the heterotic string on T^2 in equation (7.5). It is however bigger and needs some restrictions as to understand the duality between the two theories after compactification. This is because in F-theory there exists a punctured 2-torus, or elliptic curve, fibered over the base space. In other words, we have to consider that the K3 surface M admits an elliptic fibration

$$\begin{array}{rl} \pi : \mathbb{E}_\tau \to & M \\ & \downarrow \\ & B \end{array} \tag{7.39}$$

with a section $\sigma_0 : B \to M$ which fixes the zero of the additive group[1] on the elliptic curve. In the case of K3 surfaces, the base space can always be taken to be $B = \mathbb{P}^1$. This additional structure that one imposes on the K3 surface reduces the moduli space. If one considers the class $[F]$ of the fiber F and $[B]$ of the base $B = \mathbb{P}^1$ in the Picard lattice one finds

$$([F],[F]) = 0 \quad , \quad ([F],[B]) = 1 \quad , \quad ([B],[B]) = -2. \tag{7.40}$$

The lattice defined by these elements is then isomorphic to the hyperplane H of signature $(1,1)$. Preserving the fibration structure of a general K3 surface therefore leads to the following moduli space of elliptically fibered K3 surfaces

$$\mathcal{M}_{\text{Elliptically fibered K3}} = \frac{O(2,18,\mathbb{R})}{O(2,\mathbb{R}) \times O(18,\mathbb{R}) \times O(2,18,\mathbb{Z})} \tag{7.41}$$

and matches the moduli space of the compactifications of heterotic string on T^2 of the equation (7.5).

In practice, every elliptically fibered K3 surfaces can be expressed as a Weierstrass model

[1] The additive group structure on the elliptic curves originates from the definition we gave in equation (4.17).

[26]

$$P_W = 0 = y^2 - \left(x^3 + f(s,t)xz^4 + g(s,t)z^6\right) \tag{7.42}$$

with $(s,t) \in \mathbb{P}^1$, f and g polynomial of degree 8 and 12 in (s,t) respectively and (x,y,z) are homogeneous coordinates on $\mathbb{P}^{(2,3,1)}$. The zero section is obtained for every point (s,t) in the base as

$$\sigma_0 : (s,t) \mapsto ((s,t),(x=1,y=1,z=0))\,. \tag{7.43}$$

f and g together account for 22 complex parameters. It is however possible to change the coordinate on the base by an $SL(2,\mathbb{C})$ transformation and scale f and g as $f \to \lambda^2 f$ and $g \to \lambda^3 g$ with $\lambda \in \mathbb{C}^*$. There remain therefore 18 complex parameters similarly to what we obtain in the case of the heterotic string in equation (7.6).

The complex parameter of the elliptic curve is now base dependent and can be extracted from the equation

$$j(\tau(s,t)) = 4\frac{24^3 f(s,t)^3}{\Delta} \quad , \quad \Delta(s,t) = 4f(s,t)^3 + 27g(s,t)^2. \tag{7.44}$$

The discriminant Δ is a degree 24 polynomial in (s,t) and has therefore 24 zeros with multiplicities. They correspond to the positions of (p,q) 7-branes in F-theory. We saw in section 4.2.2 that stacks of such branes give rise to gauge groups in the resulting 8 dimensional theory. Thanks to Kodaira and Néron, the gauge structure of the 8 dimensional theory can be known without having to consider explicitly the integral homology of the elliptically fibered K3 surface. One only needs to consider at which order f, g and Δ vanish at the zeros of the discriminant. These results are summarised in Table 7.2.

7.3 Elliptically fibered K3 surfaces from reflexive Polyhedra

Here we discuss the link between elliptically fibered K3 surfaces and reflexive polyhedra in three dimensions. We first detail how to construct toric spaces and give an example in the case of a

type	ord(f)	ord(g)	ord(Δ)	sing
I_0	≥ 0	≥ 0	0	-
I_1	0	0	1	-
II	≥ 1	1	2	-
III	1	≥ 2	3	A_1
IV	≥ 2	2	4	A_2
I_m	0	0	m	A_m
I_0^*	≥ 2	≥ 3	6	D_4
I_{2n-5}^*, $n \geq 3$	2	3	$2n+1$	D_{2n-1}
I_{2n-4}^*, $n \geq 3$	2	3	$2n+2$	D_{2n}
IV^*	≥ 3	4	8	E_6
III^*	3	≥ 5	9	E_7
II^*	≥ 4	5	10	E_8
non-min	≥ 4	≥ 6	≥ 12	non-can

Table 7.2: Kodaira and Néron table for singular fibers of Weierstrass models.

two dimensional reflexive polyhedron which results in the construction of $\mathbb{P}^{(2,3,1)}$. We then discuss how to obtain K3 surfaces and elliptically fibered K3 surfaces as hypersurfaces on toric spaces using reflexive polyhedra. Finally we discuss how to obtain moduli of such K3 surfaces.

7.3.1 Toric geometry and reflexive polyhedra

Here we introduce various notations about reflexive polyhedra and present briefly results about toric Fano varieties. Detailed constructions of toric Fano varieties have been widely discussed in the litterature (see e.g. [97, 98]). A pedagocical introduction to toric geometry can be found in [99].

Let us consider two dual lattices M (Monomials) and N (faN) in $\mathbb{Z}^n$ with real extension $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ and an associated product $< *, * >: M \times N \to \mathbb{Z}$. We note Δ an integral convex polytope whose vertices are in M and which contains only the origin as an interior point. We then define the dual of Δ as

$$\bigtriangledown \equiv \{v \in N_{\mathbb{R}} :< w, v > \geq -1 \text{ for all } w \in \Delta\}. \tag{7.45}$$

As usual we consider Δ to be reflexive, meaning that $\bigtriangledown$ is also convex, only contains the origin

and has its vertices $\{v_i, i = 1, ..., k\}$ in N. With this we define *strongly convex rational polyhedral cone*, which we simply call cone thereforth for simplicity, as well as *fans* [91, 99]. A cone $\sigma \in N_{\mathbb{R}}$ is a set

$$\sigma = \left\{ \sum_i a_i v_i \quad / \quad a_i \geq 0 \quad , \quad i \leq k \right\} \tag{7.46}$$

such that $\sigma \cap (-\sigma) = \{0\}$. A fan is then defined as a collection Σ of cones such that each face of a cone in Σ is also a cone in Σ and the intersection of two cones is a face of each. The one dimensional cones of a fan are usually called *rays*. The normal fan of the polytope Δ whose rays are the vertices of $\bigtriangledown$ then defines a projective toric variety P_Δ (which is Fano if and only if Δ is reflexive, which will be the case here). Explicitly, one associates a variable x_i to each of the vertices v_i of the polytope $\bigtriangledown$ in N which therefore defines $\mathbb{C}^k$. Then one has to remove the sets

$$Z_\Sigma = \bigcup_I \{(x_1, ..., x_k) \quad / \quad x_i = 0 \quad \forall i \in I\} \tag{7.47}$$

with I subsets of $[|1, k|]$ such that $\{v_i, i \in I\}$ is not included in a cone. Finally one has to quotient this space by an abelian group G as well as $(\mathbb{C}\backslash\{0\})^{k-n}$ acting as

$$(x_1, ..., x_k) \sim (\lambda^{q_j^1} x_1, ..., \lambda^{q_j^k} x_k) \quad \text{if} \quad \sum_{i=1}^{k} q_j^i v_j = 0 \quad , \quad \lambda \in \mathbb{C}. \tag{7.48}$$

j goes from 1 to $(k-n)$ as we can find $(k-n)$ independent relations such as these in the polytope N. Moreover, one chooses integer q_j^is such that for each linear relation one coefficient is equal to 1. P_Δ is then

$$P_\Delta = \frac{\mathbb{C}^k \backslash Z_\Sigma}{G \times \mathbb{C}^{*(k-n)}}. \tag{7.49}$$

Let us treat an example to illustrate the construction. Consider the two dimensional polytope, in the lattice we called N, defined by the three vertices

$$v_1 = (1, 0) \quad , \quad v_2 = (0, 1) \quad , \quad v_3 = (-2, -3), \tag{7.50}$$

illustrated in Figure 7.2. We define the fan associated to this polytope as the set containing

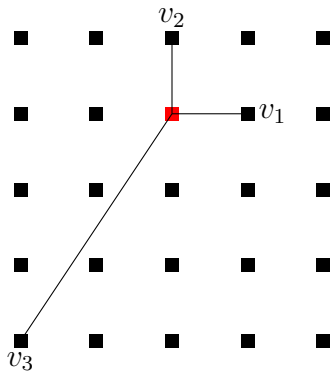


Figure 7.2: Vertices of the polytope defining $\mathbb{P}^{231}$.

the zero dimensional cone $\{0\}$, the three one dimensional cones $\mathbb{R}^+ v_i$ and the two dimensional ones $\{\mathbb{R}^+ v_1 + \mathbb{R}^+ v_2\}$, $\{\mathbb{R}^+ v_2 + \mathbb{R}^+ v_3\}$ and $\{\mathbb{R}^+ v_3 + \mathbb{R}^+ v_1\}$. To each of the rays we associate a coordinate x_i which thus defines $\mathbb{C}^3$. There is only one linear relation with one of the coefficient set to one which gives the following $\mathbb{C}^*$ action

$$2v_1 + 3v_2 + v_3 = 0 \quad \Rightarrow \quad (x_1, x_2, x_3) \sim (\lambda^2 x_1, \lambda^3 x_2, \lambda x_3) \quad , \quad \lambda \in \mathbb{C}^*. \tag{7.51}$$

The space defined by the polytope in the N lattice therefore correspond to the space $\mathbb{P}^{(2,3,1)}$ on which one defines a Weierstrass model. One should note that the space Z_Σ in this case only forbids to consider the point $(0, 0, 0)$. This is quite general: if there exist an equivalence between coordinates under the action of $\mathbb{C}^*$, the space Z_Σ remove the points for which all corresponding coordinates are set to zero simultaneously.

7.3.2 Elliptically fibered K3 surfaces from reflexive polyhedra

Now that we constructed the toric variety P_Δ using the polytope $\bigtriangledown$ it is possible when $n = 3$ to construct K3 surfaces as hypersurfaces in P_Δ by considering the dual polytope Δ. The K3 surface X_Δ can then be written as the locus in P_Δ of

$$\sum_{m \in \Delta \cap M} c_m \prod_{k=1}^{i} x_k^{<m, v_k> + 1} = 0 \tag{7.52}$$

with $c_m \in \mathbb{C}$.

We can then construct in some cases an elliptically fibered K3 as X_Δ together with a surjective morphism $\pi : X_\Delta \to \mathbb{P}_1$ such that generic fiber are genus one elliptic curves. They can be constructed by considering the K3 surface (7.52), as well as finding a subpolytope $\bigtriangledown^{(2)}$ of $\bigtriangledown$ in the N lattice. This two dimensional polytope plays the role of the fiber of the elliptic K3[2]. There are 16 reflexive polyhedra for $n = 2$ which we note F# using the notation of [100], and 4319 reflexive polytopes for $n = 3$ [101] which we note $M\#$ and correspond to the polytope $ReflexivePolytope(3, \#)$ in Sagemath. It is then possible to obtain Weierstrass models of elliptically fibered K3 surfaces upon a choice of a fan which contains as rays points of the fiber $\bigtriangledown^{(2)}$. For example, if the polytope $\bigtriangledown$ contains the subpolytope defined in Figure 7.2, the rays which are the vertices of the two dimensional polytope give coordinates (x, y, z) with the relation (7.51). This in turn identifies a torus in the hypersurface equation (7.52).

Quite amazingly, and upon a particular choice of a fan which will be described in section 8.1.2, the gauge groups associated to singularities of the elliptically fibered K3s can be read off directly once one chooses a particular subpolytope $\bigtriangledown^{(2)}$ [102]. Indeed, Candelas and Font noticed that the points located on both sides of the fiber of the polytope $\bigtriangledown$ in the N lattice are exactly the extended Dynkin diagrams which correspond to the gauge groups associated to singularities appearing in the Weierstrass model via the Kodaira and Néron classification [103, 104] (see Figure 8.1). This was later explained by Perevalov and Skarke in [105]. Depending on which of the 16 two dimensional reflexive polyhedra is the fiber, additional contribution coming from the Mordell-Weil group of rational sections of the elliptic fibration can occur [100, 106–109]. In particular the fibers F1, F2 or F4 give additional discrete symmetries $\mathbb{Z}_\#$ and fibers F13, F15 and F16 quotient by discrete symmetries $\frac{1}{\mathbb{Z}_\#}$[3]. Finally, additional contribution of $U(1)$s or $SU(\#)$ factors can appear, depending on how the polytope $\bigtriangledown^{(2)}$ intersects with $\bigtriangledown$.

[2]Such subpolytope cannot be found sometimes, in particular for small Picard numbers.
[3]Using the notations of [100]. The polytopes are i1,i2,i4 and i9,i7,i6 using notations of [110].

7.3.3 Invariant parameters of the moduli space

The number of complex moduli for a K3 surface with Picard number p is $20 - p$. Previously we defined an algebraic K3 as an hypersurface (7.52) in the toric variety P_Δ whose number of parameters is a priori given by the number of points in $\Delta \cap M$. However different sets of those parameters correspond to the same point in the moduli space. For example several of the coefficients can be put to 1 by a reparametrization of the coordinates in the projective space. In order to properly define complex parameters on the moduli space of the K3 surface we use the construction developed in [111]. It was shown there that monomials defined by points interior to facets in $\Delta \cap M$ can be removed by an appropriate change of coordinates for the different reflexive polyhedra they considered. We therefore restrict the hypersurface equation (7.52) to the integral points $m \in \text{Edges}\,(\Delta \cap M) \equiv \text{Edg}(\Delta)$ as well as the origin. The hypersurface equation can then be written as

$$H = -c_0 \prod_{k=1}^{n} x_k + \sum_{m \in \text{Edg}(\Delta)} c_m \prod_{k=1}^{n} x_k^{<m,v_k>+1} = 0 \tag{7.53}$$

with v_k rays of the normal fan P_Δ. Due to the link between the period map of K3 surfaces and their moduli spaces [112], one can seek for parameters of the moduli space by considering the fundamental period of the holomorphic two-form which can be written in our case as [113]

$$\bar{w}_{00} = -\frac{c_0}{(2\pi i)^n} \oint_{\mathcal{C}} \frac{\mathsf{d}x_1 \wedge ... \wedge \mathsf{d}x_n}{H} \tag{7.54}$$

with $\mathcal{C}$ a product of cycles that enclose the hypersurface defined by $x_i = 0$ [111]. This can be recast as

$$\bar{w}_{00} = \frac{1}{(2\pi i)^n} \oint_{\mathcal{C}} \frac{\mathsf{d}x_1 \wedge ... \wedge \mathsf{d}x_n}{\prod_{k=1}^{n} x_k} \sum_{l=0}^{\infty} \tilde{H}^l \tag{7.55}$$

with

$$\tilde{H} = \sum_{m \in \text{Edg}(\Delta)} \frac{c_m \prod_{k=1}^{n} x_k^{<m,v_k>+1}}{c_0 \prod_{k=1}^{n} x_k}. \tag{7.56}$$

The only non zero terms in (7.55) are the constant terms in the development of $\tilde{H}^l$ by the residue theorem. The fundamental period of the holomorphic two-form can therefore be parametrized by

the following invariants

$$\text{Moduli} \sim \frac{l!}{c_0^l} \prod_{m \in \text{Edg}(\Delta)} \frac{c_m^{l_m}}{l_m!} \tag{7.57}$$

such that

$$\sum_{m \in \text{Edg}(\Delta)} l_m = l \tag{7.58}$$

and

$$\forall k \sum_{m \in \text{Edg}(\Delta)} l_m \left(< m, v_k > +1\right) = l. \tag{7.59}$$

Taking the second equation, one can then simply look for inequivalent linear relations in the M lattice such that $\sum l_m \cdot m = (0,0,0)$ with l_ms positive and minimal. By a change of variables of these invariants, one can in fact look for inequivalent linear relations between points in the edges of Δ such that (7.58) and (7.59) are verified but this time with l_m in $\mathbb{Z}$ and $|l_m|$ minimal. The complex parameters can then be taken to have the following form

$$\text{Moduli} \sim \left(\prod_{m \in \text{Edg}(\Delta)} c_m^{l_m} \right) c_0^{-l}. \tag{7.60}$$

As an example let us take the polytope $M476$, with Picard number equal to 16 i.e. 4 moduli. Its vertices are given by

$$M476 : \underbrace{(1,\,0,\,0)}_{(1)_M}, \underbrace{(0,\,1,\,0)}_{(2)_M}, \underbrace{(0,\,0,\,1)}_{(3)_M}, \underbrace{(-4,\,-2,\,-1)}_{(4)_M}, \underbrace{(-5,\,-3,\,-1)}_{(5)_M}, \underbrace{(-1,\,-1,\,1)}_{(6)_M}. \tag{7.61}$$

An additional point, $(7)_M = (-3,-2,0)$, is situated on the edges of the polytope. We can thus consider four inequivalent linear relations between these points, a possibility being

$$(5)_M + (6)_M - 2 \cdot (7)_M \quad , \quad (7)_M + 2 \cdot (2)_M + 3 \cdot (1)_M \tag{7.62}$$

$$(3)_M - (1)_M - (2)_M - (6)_M \quad , \quad (4)_M - (1)_M - (2)_M - (5)_M \tag{7.63}$$

which leads using (7.60) to the complex parameters of the moduli space

$$M476 \text{ (4 Moduli): } \frac{c_5 c_6}{c_7^2}, \quad \frac{c_7 c_2^2 c_1^3}{c_0^6}, \quad \frac{c_3 c_0^2}{c_1 c_2 c_6}, \quad \frac{c_4 c_0^2}{c_1 c_2 c_5}. \tag{7.64}$$

CHAPTER 8

F-theory and Heterotic Duality, Weierstrass Models from Wilson Lines

The study of the full moduli space of the heterotic string on T^2 or equivalently F-theory on elliptically fibered K3 surfaces can be a difficult exercise and one wants to focus on subspaces with fewer complex modular parameters. In the heterotic string one can consider for example compactifications on a two torus with Wilson lines parametrized by few moduli. In F-theory, one can choose an algebraic K3 with a large Picard number p, as its modular space is parametrized by $20-p$ complex variables [112]. As we saw, a particularly interesting way to construct K3 surfaces is to consider reflexive polyhedra in 3 dimensions which define hypersurfaces in toric varieties. Thanks to Kreuzer and Skarke [101] it is possible to have a list of the totality of the 4319 different reflexive polyhedra in 3 dimensions and classify them with respect to their Picard number p.

The duality between F-theory and heterotic string has been written explicitly for only two of the 4319 different K3 surfaces one can construct via reflexive polytopes. First the duality between the parameters of a Weierstrass model presenting a particular $E_8 \times E_8$ singularity and the complex structure and Kähler moduli of the two torus on which the $E_8 \times E_8$ heterotic string is

compactified was constructed in [114]. Later it was found that a particular reflexive polyhedron admitting two fibrations has for gauge groups $E_8 \times E_8$ and $\frac{Spin(32)}{\mathbb{Z}_2}$ [7]. In a more general case with three moduli, Malmendier and Morrison showed that a particular polytope with again two fibers with gauge group $E_7 \times E_8$ and $\frac{Spin(28)\times SU(2)}{\mathbb{Z}_2}$ is related to compactifications of heterotic strings with one Wilson line modulus [8].

Here we show that if we focus on particular reflexive polyhedra that are linked in some way to the $E_8 \times E_8/\frac{\mathsf{Spin}(32)}{\mathbb{Z}_2}$ polytope, we can understand the Wilson line structure of the dual heterotic string. This is due to the fact that we can recover the torus on which we compactify the heterotic string theory as a particular subspace of the moduli spaces of the elliptically fibered K3s. To find these polytopes we construct graphs where a link between two polytopes $M+$ and $M-$ is drawn if, for every elliptically fibered K3 surface obtained via $M+$, there exist a limit in the moduli space where one obtains elliptically fibered K3 surfaces of the other polytope $M-$. In particular, we will consider the limit where one sends monomials of the hypersurface equation defining the K3 surface associated to $M+$ to zero, which is equivalent to removing a point in $M+$. This can be seen as an extension of the notion of chains presented by Kreuzer and Skarke in [101]. Focusing on polytopes which have two fibers, links between polytopes then correspond to inclusion relations between the moduli spaces of elliptically fibered K3s. Considering polytopes which are linked to $E_8 \times E_8/\frac{\mathsf{Spin}(32)}{\mathbb{Z}_2}$, we show that additional monomials in the hypersurface equation which defines the elliptically fibered K3s on which we compactify on correspond to additional Wilson line moduli in both the $E_8 \times E_8$ and $\frac{Spin(32)}{\mathbb{Z}_2}$ heterotic strings. Using this Wilson line/monomial duality we can construct Weierstrass models of elliptically fibered K3s which are not directly obtained from reflexive polyhedra. They can then be interpreted as a certain Wilson lines in the dual heterotic theories. Finally, we show that in some cases this notion of *Wilson line description of K3 surfaces* can be extended to polytopes with more than two fibers. This should be helpful to understand the duality between F-theory compactified on K3s and heterotic string on a two torus, and eventually in compactifications to lower dimensions involving K3 surfaces.

The chapter is organised as follows: in section 8.1, we present several computer programs

which we wrote and are helpful for constructing graphs of polytopes. They were written on SageMath and with the help of the package PALP [115–117]. The first program uses the extended Dynkin diagram structure of reflexive polyhedra with fibers in order to construct tables of gauge groups for each fibration of every reflexive polytope. The second program gives the corresponding Weierstrass model for every fiber of reflexive polytopes. The third one uses this Weierstrass model and finds the enhancements one can obtain by simply sending the coefficients which parametrize the hypersurface equation of the K3 in some toric varieties to zero. This can be particularly useful to construct graphs of polytopes and we show how one can link polytopes up to three moduli. In the Appendix B we present typical outputs of the programs and explain how to use them. The computer programs are available on GitHub at `https://github.com/lilianChabrol/Reflexivepolyhedras`. To summarize, here are the three SageMath programs available online

- Program 1 (Typical output in Appendix A): Gauge groups from the extended Dynkin diagram structure in the N lattice.
- Program 2 (Typical output in Appendix B): Determination of the Weierstass model of the corresponding elliptically fibered K3.
- Program 3 (Typical output in Appendix C): Possible enhancements of the gauge groups for each fibers by sending defining coefficients of the hypersurface to zero.

Finally in section 8.2 we present a *Wilson line description of K3 surfaces* by considering a particular graph of polytope which goes up to 6 moduli, or equivalently in this case four Wilson line moduli on the heterotic side. We then show how to construct Weierstrass models of elliptically fibered K3s which one can directly interpret in the dual theory as particular Wilson lines.

8.1 Obtaining data on elliptically fibered K3s

We now present the three computer programs that allow to obtain different information about elliptically fibered K3 surfaces automatically. Again, we writee $M\#$ the polytope Δ in the M

lattice corresponding to $ReflexivePolytope(3,\#)$ in SageMath[1].

8.1.1 Extended Dynkin diagram from polyhedra

As discussed in section 7.3.2, it is possible to obtain the gauge structure of an elliptically fibered K3 upon a choice of reflexive polytope $(\Delta, \bigtriangledown)$, and a choice of fiber $\bigtriangledown^{(2)}$. We now present a generic way to find the gauge group associated to each fiber of every reflexive polytope in the Kreuzer-Skarke classification of reflexive polyhedra in 3 dimensions.

We first find all two dimensional reflexive polyhedra $\bigtriangledown^{(2)}$ which are subpolytopes of $\bigtriangledown$ modulo $SL(3,\mathbb{Z})$ transformations in the N lattice[2]. Then we identify which of the 16 possible two dimensional reflexive polytope corresponds to each of the fibers $\bigtriangledown^{(2)}$. This permits in particular to know if the fiber contains product or quotient by discrete symmetry group [100]: F1, F2 and F3 contribute by a product by $\mathbb{Z}_3$, $\mathbb{Z}_2$ and $\mathbb{Z}_4$ respectively while fibers F13, F15 contribute by $\frac{1}{\mathbb{Z}_2}$ and F16 by $\frac{1}{\mathbb{Z}_3}$. We do not write the additional contributions of $U(1)$ factors coming from the Mordell Weil group as in the end the gauge group must be of rank $p-2$, where p is the Picard number of the K3 surface. We however look for additional $SU(\#)$ contribution from the fiber: if polytopes $\bigtriangledown^{(2)}$ and $\bigtriangledown$ have a common edge with n points, then there appears an additional $SU(n-1)$ part in the final gauge group[3]. Finally, the fiber $\bigtriangledown^{(2)}$ dividing $\bigtriangledown$ into two parts, we look at points "above" and "below" the fiber and read off the extended Dynkin diagrams.

As an example let us consider the polytope $M476$. In Figure 8.1 we represent the dual polytope $N476$ of $M476$ for the two inequivalent fibrations $\bigtriangledown^{(2)}$ it contains. On the left, one can read off two extended Dynkin diagram of E_7. On the right, there is a $SO(24)$ as well as $\frac{1}{\mathbb{Z}^2}$ comming from the fiber $F13$, and $SU(2)\times SU(2)$ contribution due to the intersection of $\bigtriangledown^{(2)}$ and $\bigtriangledown$ symbolised by red points.

[1] The vertices of each of the polytopes presented in this chapter are written in the Appendix B.4.

[2] This feature was already available on Sagemath.

[3] Equivalently the rank of the $SU(\#)$ can be seen by looking at the number of interior points in the common edge. See the red points in Figure 8.1.

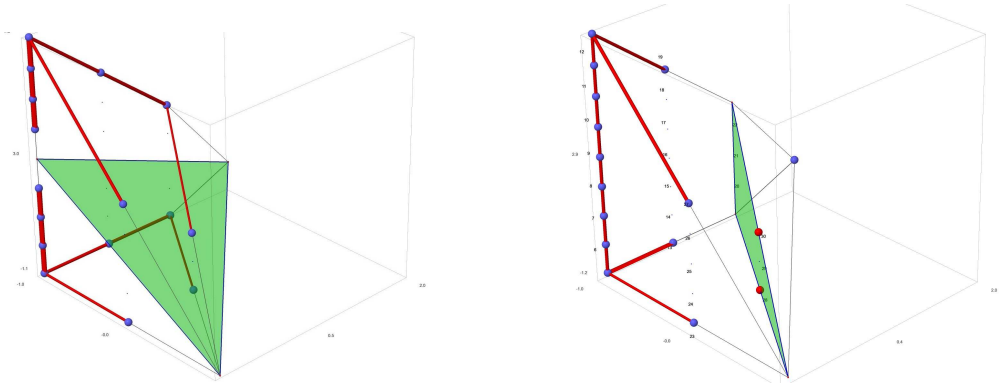


Figure 8.1: $E_7 \times E_7$ and $\frac{SO(24)\times SU(2)^2}{\mathbb{Z}_2}$ fiber of the polytope $M476$. The points in blue draw the extended Dynkin diagram of E_7s on the left, $SO(24)$ on the right. The contribution of $SU(2)$s from the fiber are symbolised by red points. The fiber being F13 there is an additional contribution of $\frac{1}{\mathbb{Z}_2}$.

The results for K3 surfaces with Picard number 19 and 18 i.e. one and two complex parameters respectively, are presented in the Tables 8.1 and 8.2. They were compared with results of an unpublished paper [118] presented at a seminar at CERN [119] as well as results from [111]. The result with Picard 17 and 3 complex parameters is presented in the Appendix B.1 (Table B.1). Tables with complex parameters up to 5 moduli are available on GitHub, and up to 10 moduli for elliptically fibered K3s admitting only two inequivalent fibrations.

M0	$\frac{SO(16)\times SO(16)}{Z_2}$	$\frac{SU(12)\times E_6}{Z_3}$	$E_8 \times E_8$	$\frac{E_7\times E_7\times SU(4)}{Z_2}$
M2	$\frac{E_7\times SO(20)}{Z_2}$	$\frac{SU(18)}{Z_3}$	$E_8 \times E_8 \times SU(2)$	

Table 8.1: Gauge groups for polytopes with Picard 19. Columns represent the inequivalent fibers $\bigtriangledown^{(2)}$ dividing the dual $N\#$ of $M\#$ into two parts. Additional $U(1)$s should be added so that the rank is 19.

8.1.2 Weierstrass model, gauge groups and basic enhancements

The computer program introduced in section 8.1 is particularly interesting to determine the gauge group at a generic point in the moduli space associated to any fiber of any reflexive polytope in three dimensions. It would however be interesting to get the Weierstrass model which correspond to these gauge groups in order to find their enhancements for particular values of the moduli. Some of the enhancements can then be found quite easily by removing points in the polytope

M3	$SO(14) \times E_7$	$SO(14) \times SU(9)$	$\frac{SU(12) \times SO(8)}{Z_2}$	$\frac{E_6 \times E_6 \times SU(3) SU(3)}{Z_3}$	$E_8 \times E_8 \times Z_3$		
M4	$E_8 \times E_8 \times Z_3$	$E_6 \times SO(14) \times SU(3)$	$E_7 \times E_7$	$\frac{SU(10) \times SO(12)}{Z_2}$	$\frac{SU(9) \times SU(9)}{Z_3}$		
M5	$E_7 \times E_7 \times SU(2)$	$SU(10) \times E_6$	$\frac{SO(16) \times SO(12) \times SU(2)}{Z_2}$	$\frac{E_7 \times SO(12) \times SU(4)}{Z_2}$	$E_8 \times E_7$	$\frac{SU(6) \times SU(12)}{Z_3}$	
M6	$E_6 \times E_7 \times SU(3)$	$E_7 \times E_8$	$E_8 \times E_8 \times Z_3$	$SO(14) \times SO(14)$	$SO(10) \times SU(11)$	$\frac{E_6 \times SU(9) \times SU(3)}{Z_3}$	$\frac{SU(8) \times SO(16)}{Z_2}$
M7	$E_7 \times E_8$	$\frac{SU(10) \times E_7}{Z_2}$	$\frac{SU(3) \times SU(15)}{Z_3}$	$E_6 \times SO(18)$			
M10	$\frac{SO(16) \times SO(16)}{Z_2}$	$\frac{E_7 \times E_7 \times SU(2) SU(2)}{Z_2}$	$E_8 \times E_8 \times Z_4$	$\frac{SU(16)}{Z_2}$			
M11	$\frac{SO(16) \times E_7 \times SU(2)}{Z_2}$	$E_8 \times E_8 \times Z_4$	$E_8 \times E_7 \times SU(2)$	$\frac{SO(12) \times SO(20)}{Z_2}$	$SU(16)$		
M16	$SO(18) \times E_6$	$\frac{SU(15) \times SU(3)}{Z_3}$	$E_7 \times E_8$	$\frac{SU(10) \times E_7}{Z_2}$			
M88	$E_8 \times E_8$	$\frac{SO(32)}{Z_2}$					

Table 8.2: Gauge groups for polytopes with Picard 18. Columns represent the inequivalent fibers $\bigtriangledown^{(2)}$ dividing the dual $N\#$ of $M\#$ into two parts. Additional $U(1)$s should be added so that the rank is 18.

Δ in the M lattice which amounts to sending to zero a coefficient in the hypersurface equation which defines the K3 surface. This is what the second and third program do: find the Weierstrass model, and the enhancements described above[4].

We first look at the polytope $\bigtriangledown$ in the N lattice. As explained in the introduction we then find inequivalent subpolytope $\bigtriangledown^{(2)}$ of dimension 2 in $\bigtriangledown$. For each of this 2 dimensional polytope we want to associate homogenous coordinates such that it describes the fiber. For most cases one can just associate one of them to each vertices of the subpolytope and obtain later the gauge groups expected from reading the extended Dynkin diagrams directly on the polytope $\bigtriangledown$. However in 3 cases (F13, F15 and F16) out of the 16 possible two dimensional reflexive polytopes, considering the vertices will not lead to these groups. This is due to the fact that for these particular polytopes the fibrations admit more than one section [7]. Using a similar construction to the one of [119] and in an upcoming paper [118], we then consider the homogeneous coordinates x_i of the fiber to be associated to the points as described in Figure 8.2. To define coordinates (s, t) on the base space $\mathbb{P}^{(1)}$ we seek for two vectors v_s and v_t, "above" and "below" the fiber. A fast way to obtain the appropriate Weierstrass model with correct ADE singularities, which correspond to the extended Dynkin diagrams seen in $\bigtriangledown$, is then to seek for the closest vectors to the fiber in $\bigtriangledown \cap N$.

Finally we write the hypersurface equation by considering the points on the edges of Δ and using equation (7.52). To each of these points corresponds a monomial in the hypersurface

[4] See Appendix B.2 and B.3 for the output of the programs.

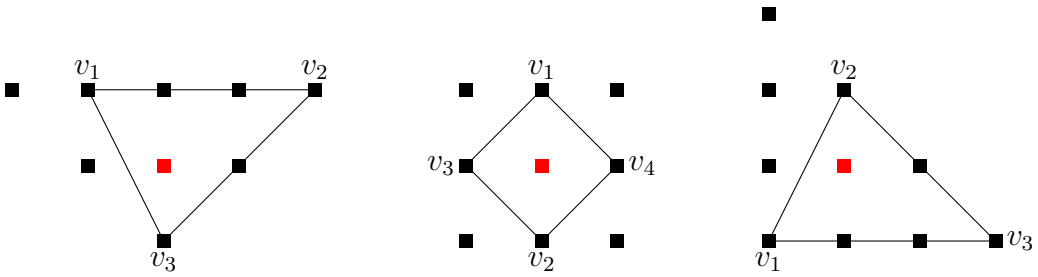


Figure 8.2: In order to obtain the groups associated to the extended Dynkin diagrams on the N lattice we consider the following rays when the two dimensional subpolytopes are F13, F15 and F16.

equation to which we associate a parameter $c_i \in \mathbb{C}$. Using SageMath we can finally recast this equation into the Weierstrass form

$$y^2 = x^3 + f(s,t)xz^4 + g(s,t)z^6 \tag{8.1}$$

where the homogeneous coordinates of the fiber are now (x, y, z) in $\mathbb{P}^{(2,3,1)}$, f and g are respectively polynomials of degree 8 and 12 in (s, t). The discriminant of (8.1) is then $\Delta_{(f,g)} = 4f^3 + 27g^2$ and vanishes at 24 points which are the locations of 7-branes.

Once one has the Weierstrass form of the elliptically fibered K3, one finds the ADE groups associated to the various singularities using Kodaira and Neron classification presented in Table 7.2 [103, 104]. Moreover, the moduli can be expressed via the parameters c_i as shown in section 7.3.3. Sending those parameters to zero, we can therefore find possible enhancements of the group associated to a generic point in the moduli space. The third SageMath program then gives all possible enhancements obtained by sending all possible combinations of parameters c_i to zero, when the hypersurface still defines an elliptically fibered K3.

8.1.3 Graphs of polytopes

Using this we construct *graphs* of K3 surfaces, generalising the "chains" defined by Kreuzer and Skarke in [101]. Nodes on a graph correspond to polytopes, or equivalently their associated

K3 surface. We then link two polytopes if, by sending the same coefficient c_i of (7.53) in every hypersurface equations for every possible fibration, we obtain the Weierstrass models of fibers of the other polytope[5]. Some of these graphs are represented in Figure 8.3, 8.4 and 8.5 and are discussed below. A less trivial case will be discussed in section 8.2.

Let us consider Figure 8.3: $M0$ is linked to both $M5$ and $M6$ by which we mean that if one removes a particular point in the polytopes $M5$ and $M6$, one recovers the Weierstrass models corresponding to fibers of $M0$. This means that the moduli spaces of elliptically fibered K3s corresponding to the fibrations of the polytopes $M5$ and $M6$ contains the moduli spaces of fibers of the polytope $M0$.

Figures 8.3, 8.4 and 8.5, combined with the polytopes $M15$, $M30$, $M38$, $M104$ and $M117$ with Picard 17 which, a priori, are not linked to any polytope with higher Picard number, describe all reflexive polyhedra up to 3 complex parameters.

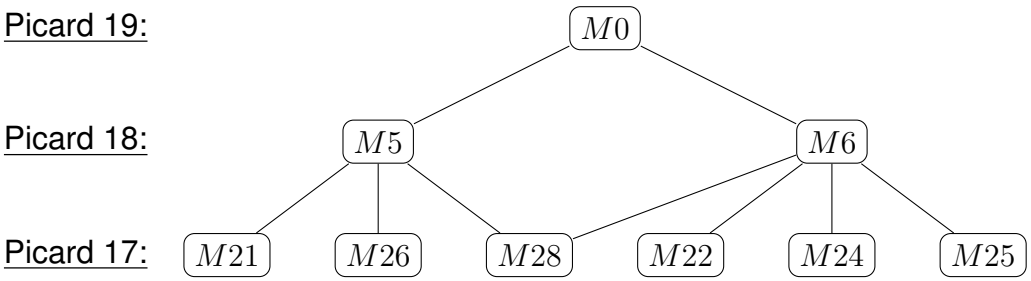


Figure 8.3: Polytopes up to 3 complex parameters that are linked to $M0$ by removing points in their M lattice (i.e. a monomial in the hypersurface equation).

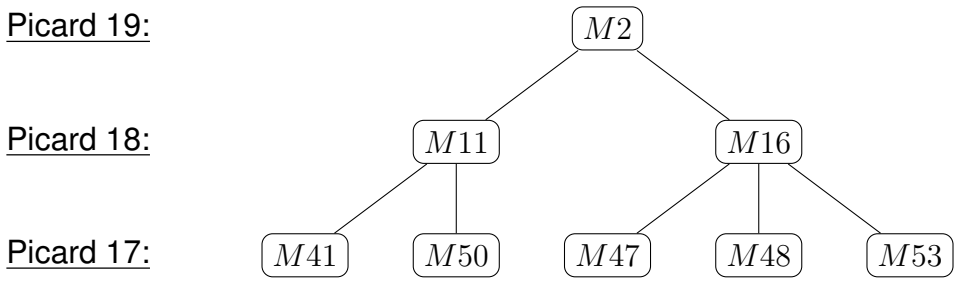


Figure 8.4: Polytopes up to 3 complex parameters that are linked to $M2$ by removing points in their M lattice (i.e. a monomial in the hypersurface equation).

[5]One does not necessarily obtain all the fibers of the polytope with fewer moduli. This is however the case for Figure 8.6.

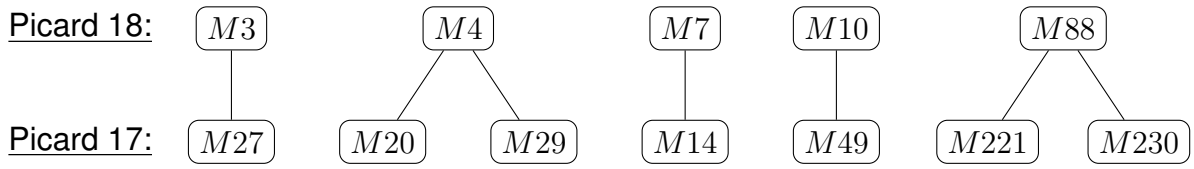


Figure 8.5: Links between polytopes with Picard 18 and 17. Going from Picard 17 to 18 amounts to removing a point in the polytope in the M lattice (i.e. a monomial in the hypersurface equation which defines the K3 surface).

8.2 F-theory/Heterotic string duality in 8 dimensions: Wilson lines from reflexive polyhedra?

8.2.1 Graphs of Polytopes: from Monomials to Wilson lines...

The duality map between F-theory on K3 and heterotic strings on a two torus has been explicitly written for two polytopes having each two inequivalent fibrations. The gauge groups associated to these fibers are amazingly $E_8 \times E_8$ and $SO(32)$ for the first polytope ($M88$ using our notation) and $E_7 \times E_8$ and $SO(28) \times SU(2)$ ($M221$). Using [36], we can see that adding a Wilson line of the form $A = (a_2, 0_{14})$ with $a \in \mathbb{C}$, using the notation of the section 7.1.3 breaks $E_8 \times E_8$ to $E_7 \times E_8$, and $SO(32)$ to $SO(28) \times SU(2)$ for a generic value of a. On the heterotic side one might be able to interpret the polytope $M221$ as a compactification on a two torus, together with one Wilson line of the form $(a_2, 0_{14})$. In fact considering this particular parametrization of Wilson line, the enhancements one finds on both heterotic strings and F-theory exactly match, as was presented by Anamaria Font at CERN [119] and studied with more details in an upcomming paper [118].

Now we want to see if we can make similar interpretations by considering polytopes which admit only two fibrations. Following the construction we presented in section 8.1.3 we seek a graph of polytopes with two fibers in its dual lattice and which contains $M88$. As an example let us consider the polytope $M1328$ which has Picard number 14. Its moduli space is parametrized by 6 complex parameters. We write the hypersurfaces equations $P_G = 0$ of its two fibers below,

where G is the group associated to its ADE singularities

$$P_{E_6\times SO(10)} = -\underline{c_0x_0x_1x_2x_3st} + \underline{c_1x_2x_3^2s} + c_2x_0^2x_3s + \underline{c_3x_0^3x_1st} + c_4x_0x_1^3x_2^2s^2t^2 \\ + c_5x_1^2x_2^2x_3t^3 + \underline{c_6x_1^4x_2^3s^2t^3} + c_7x_0x_1^3x_2^2t^4 + \underline{c_8x_1^4x_2^3t^5} + \underline{c_9x_1^4x_2^3st^4}, \tag{8.2}$$

$$P_{SU(11)\times SU(2)} = -\underline{c_0x_0x_1x_2x_3st} + \underline{c_1x_1x_2^2t} + c_2x_0x_1x_2x_3s^2 + \underline{c_3x_0^2x_1x_3^2s^3} \\ + c_4x_1^2x_3^3st^4 + c_5x_0^3x_2 + \underline{c_6x_1^2x_3^3t^5} + c_7x_0^4x_3s + \underline{c_8x_0^4x_3t} + \underline{c_9x_0^2x_1x_3^2t^3} \tag{8.3}$$

with x_i homogeneous coordinates of the fiber, and (s,t) coordinates on the base. These hypersurfaces can then be recast into a Weierstrass form where (s,t) correspond to coordinates on the base $\mathbb{P}_1$[6] . Considering the underlined monomials in the equations (8.2) and (8.3) gives the Weierstrass models associated to the polytope $M88$ and thus corresponds to the heterotic strings without Wilson lines. By this we mean that if one considers $c_2 = c_4 = c_5 = c_7 = 0$ then equation (8.2) in its Weierstrass form has for parameters

$$\begin{aligned} f &= \left(-\frac{1}{48}\right)\cdot t^4\cdot s^4\cdot c_0^4 \\ g &= \left(-\frac{1}{864}\right)\cdot t^5\cdot s^5\cdot (864c_1^3c_3^2c_6s^2 - c_0^6st + 864c_1^3c_3^2c_9st + 864c_1^3c_3^2c_8t^2) \\ \Delta_{(f,g)} &= \left(\frac{1}{16}\right)\cdot c_3^2\cdot c_1^3\cdot t^{10}\cdot s^{10}\cdot (c_6s^2 + c_9st + c_8t^2)(432c_1^3c_3^2c_6s^2 - c_0^6st + 432c_1^3c_3^2c_9st + 432c_1^3c_3^2c_8t^2) \end{aligned} \tag{8.4}$$

which has $E_8 \times E_8$ for singularities if one considers Table 7.2. In the case of equation (8.3) it

[6]We do not write the Weierstrass models due to the size of the parameters f, g and $\Delta_{(f,g)}$.

gives

$$\begin{aligned} f &= \left(-\frac{1}{48}\right)\cdot t^2\cdot(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t + c_0^4s^4t^2 + 32c_1^2c_3c_9s^3t^3 - 8c_0^2c_1c_9s^2t^4 \\ &\quad - 48c_1^2c_6c_8t^6 + 16c_1^2c_9^2t^6) \\ g &= \left(-\frac{1}{864}\right)\cdot t^3\cdot(4c_1c_3s^3 - c_0^2s^2t + 4c_1c_9t^3)(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t + c_0^4s^4t^2 \\ &\quad + 32c_1^2c_3c_9s^3t^3 - 8c_0^2c_1c_9s^2t^4 - 72c_1^2c_6c_8t^6 + 16c_1^2c_9^2t^6) \\ \Delta_{(f,g)} &= \left(-\frac{1}{16}\right)\cdot c_8^2\cdot c_6^2\cdot c_1^4\cdot t^{18}\cdot(16c_1^2c_3^2s^6 - 8c_0^2c_1c_3s^5t + c_0^4s^4t^2 \\ &\quad + 32c_1^2c_3c_9s^3t^3 - 8c_0^2c_1c_9s^2t^4 - 64c_1^2c_6c_8t^6 + 16c_1^2c_9^2t^6) \end{aligned} \tag{8.5}$$

i.e. a $SO(32)$ singularity. Now we define two moduli ξ and ρ

$$\xi = \frac{c_8c_6}{c_9^2} \quad , \quad \eta = \frac{c_9c_3^2c_1^3}{c_0^6}. \tag{8.6}$$

They parametrize the two dimensional moduli space and are found by considering linear relations on the edges of the polytope $M1328$ (see Equation (7.60))[7]. Now, we know that adding a monomial corresponds to adding complex parameters in the Wilson lines. As we have to add four complex parameters which correspond to the four additional monomials c_2, c_4, c_5 and c_7 in (8.2) and (8.3), we use the full graph which links $M1328$ to $M88$ represented in Figure 8.6[8]. Going down in the graph, we define four additional complex parameters as

$$A_{c_7} = \frac{c_7c_0^2}{c_1c_3c_8}, \quad A_{c_2} = \frac{c_2c_0}{c_1c_3} \quad , \quad A_{c_4} = \frac{c_4c_0^2}{c_1c_3c_6}, \quad A_{c_5} = \frac{c_5c_0^3}{c_3c_8c_1^2}. \tag{8.7}$$

We already know that the polytope $M221$ is obtained on the heterotic side by adding a Wilson line $a(1_2, 0_{14})$ therefore the monomial "c_7" is associated to this Wilson line. Looking at all the gauge groups in the graph and using results on the compactification of heterotic strings on a circle [36]

[7] They correspond to the parameter u and v in [111].

[8] This graph was found using the third program presented in this chapter applied to the polytope $M1328$.

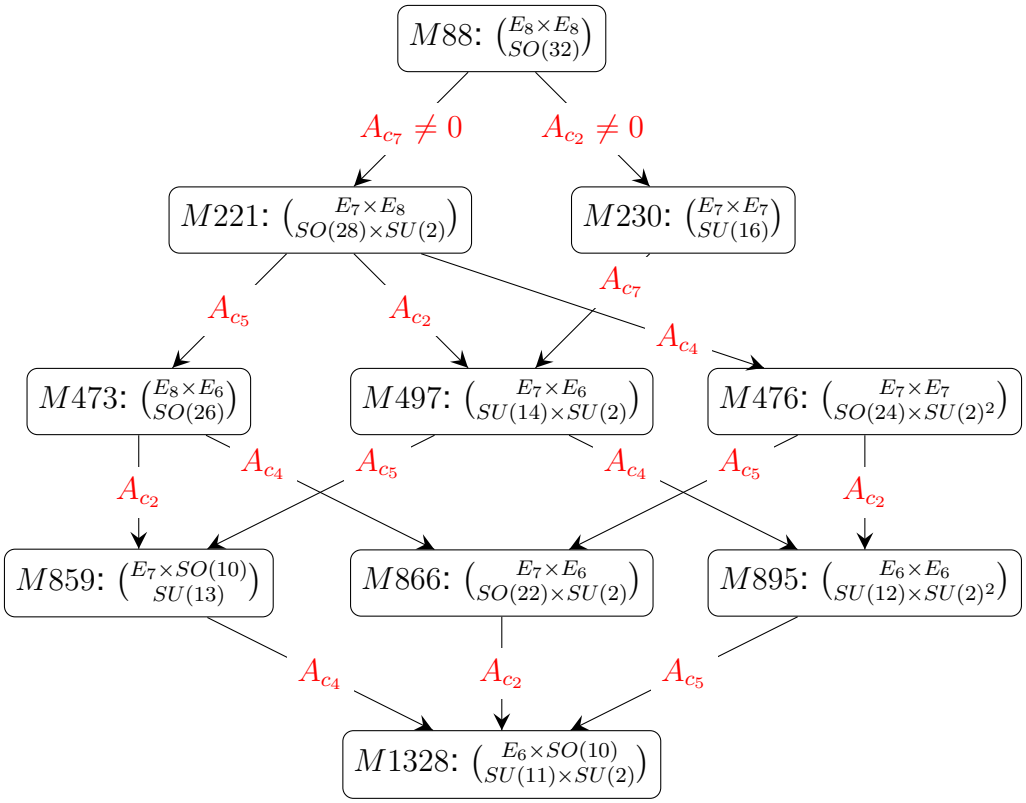


Figure 8.6: Links between various reflexive polyhedra. Going upward from $M1328$ amounts to removing points in the polytope $M1328$, or equivalently monomials in the hypersurface equations (8.2) and (8.3). Going downward corresponds to adding a complex modulus $A_{c_\#}$.

we find that a possibility for the Wilson lines associated to each monomial is

$$A_{c_7}\sim a(1_2,0_{14}) \quad , \quad A_{c_2}\sim b(1_{16}) \quad , \quad A_{c_4}\sim c(0_{14},1_2) \quad , \quad A_{c_5}\sim d\left[(1_2,0_{14})+i(0,1_2,0_{13})\right] \tag{8.8}$$

with a,b,c and d in $\mathbb{C}$ parametrizing the moduli on the heterotic side. We can see that A_{c_7} and A_{c_4} are linked to the same Wilson lines if it were not for the symmetry breaking of A_{c_5}[9]. Indeed, if one does not add the monomial c_5, or the Wilson line A_{c_5} in the dual theory, one can interchange c_7 and c_4 and obtain the same Weierstrass models obtained from $M221$. Moreover, due to the symmetry of the two parameters A_{c_7} and A_{c_4}, if $A_{c_7}=A_{c_4}$ i.e $a=c$ in (8.7), we obtain what we expect on the heterotic side, namely $SO(24)\times SU(2)^2\to SO(24)\times SU(4)$ for the polytope $M476$ while $E_7\times E_7$ is not enhanced.

[9] In the $E_8\times E_8$ heterotic string one can just interchange the E_8s.

8.2.2 ... and back to monomials

We are now able to describe K3s as parametrizations of Wilson lines of its dual theory (both for $E_8 \times E_8$ and $SO(32)$) for particular polytopes $M\#$ whose dual $N\#$ contain two fibers . Rather we linked *monomials* in the defining hypersurface equation of K3s to parameters in the Wilson lines. This means that we can construct Weierstrass models of elliptically fibered K3s which are not per say described by reflexive polyhedra, and directly interpret them as particular Wilson lines on the $E_8 \times E_8$ and $SO(32)$ heterotic strings. Indeed let us go back to the graph of Figure 8.6: adding the monomial c_4 to the underlined terms of (8.2) and (8.3) gives the Weierstrass models one gets from $M221$ as explained previously. Adding only c_5 however, we cannot obtain a polytope with 3 moduli which will give the same Weierstrass models. Thus let us write the parameters $(f, g, \Delta_{f,g})$ of the Weierstrass models of the polytope $M88$, together with the additional monomial c_5 in (8.2) and (8.3). For the first fiber we find

$$f = \left(-\frac{1}{48}\right) \cdot c_0 \cdot s^3 \cdot t^4 \cdot (c_0^3 s - 24 c_1 c_3 c_5 t) \tag{8.9}$$

$$g = \left(-\frac{1}{864}\right) \cdot s^4 \cdot t^5 \cdot (864 c_1^3 c_3^2 c_6 s^3 - c_0^6 s^2 t + 864 c_1^3 c_3^2 c_9 s^2 t + 36 c_0^3 c_1 c_3 c_5 s t^2 + 864 c_1^3 c_3^2 c_8 s t^2 - 216 c_1^2 c_3^2 c_5^2 t^3) \tag{8.10}$$

$$\begin{aligned}\Delta_{(f,g)} = \left(\frac{1}{16}\right) \cdot c_3^2 \cdot c_1^3 \cdot s^8 \cdot t^{10} \cdot (&432 c_1^3 c_3^2 c_6^2 s^6 - c_0^6 c_6 s^5 t + 864 c_1^3 c_3^2 c_6 c_9 s^5 t + 36 c_0^3 c_1 c_3 c_5 c_6 s^4 t^2 + 864 c_1^3 c_3^2 c_6 c_8 s^4 t^2 \\ &- c_0^6 c_9 s^4 t^2 + 432 c_1^3 c_3^2 c_9^2 s^4 t^2 - 216 c_1^2 c_3^2 c_5^2 c_6 s^3 t^3 - c_0^6 c_8 s^3 t^3 + 36 c_0^3 c_1 c_3 c_5 c_9 s^3 t^3 + 864 c_1^3 c_3^2 c_8 c_9 s^3 t^3 \\ &+ 36 c_0^3 c_1 c_3 c_5 c_8 s^2 t^4 + 432 c_1^3 c_3^2 c_8^2 s^2 t^4 - 216 c_1^2 c_3^2 c_5^2 c_9 s^2 t^4 - c_0^3 c_3 c_5^3 s t^5 - 216 c_1^2 c_3^2 c_5^2 c_8 s t^5 + 27 c_1 c_3^2 c_5^4 t^6)\end{aligned} \tag{8.11}$$

and for the second

$$\begin{aligned}f = \left(-\frac{1}{48}\right) \cdot t^2 \cdot (&16 c_1^2 c_3^2 s^6 - 8 c_0^2 c_1 c_3 s^5 t + c_0^4 s^4 t^2 + 32 c_1^2 c_3 c_9 s^3 t^3 - 8 c_0^2 c_1 c_9 s^2 t^4 - 24 c_0 c_1 c_5 c_6 s t^5 \\ &- 48 c_1^2 c_6 c_8 t^6 + 16 c_1^2 c_9^2 t^6)\end{aligned} \tag{8.12}$$

$$\begin{aligned}g = \left(-\frac{1}{864}\right) \cdot t^3 \cdot (&64 c_1^3 c_3^3 s^9 - 48 c_0^2 c_1^2 c_3^2 s^8 t + 12 c_0^4 c_1 c_3 s^7 t^2 - c_0^6 s^6 t^3 + 192 c_1^3 c_3^2 c_9 s^6 t^3 - 96 c_0^2 c_1^2 c_3 c_9 s^5 t^4 \\ &- 144 c_0 c_1^2 c_3 c_5 c_6 s^4 t^5 + 12 c_0^4 c_1 c_9 s^4 t^5 + 36 c_0^3 c_1 c_5 c_6 s^3 t^6 - 288 c_1^3 c_3 c_6 c_8 s^3 t^6 + 192 c_1^3 c_3 c_9^2 s^3 t^6 \\ &+ 72 c_0^2 c_1^2 c_6 c_8 s^2 t^7 - 48 c_0^2 c_1^2 c_9^2 s^2 t^7 - 144 c_0 c_1^2 c_5 c_6 c_9 s t^8 - 216 c_1^2 c_5^2 c_6^2 t^9 - 288 c_1^3 c_6 c_8 c_9 t^9 + 64 c_1^3 c_9^3 t^9)\end{aligned} \tag{8.13}$$

$$\Delta_{(f,g)} = \left(-\frac{1}{16}\right) \cdot c_6^2 \cdot c_1^3 \cdot t^{15} \cdot (16c_1^2c_3^3c_5^2s^9 - 8c_0^2c_1c_3^2c_5^2s^8t + c_0^4c_3c_5^2s^7t^2 + 16c_0c_1^2c_3^2c_5c_8s^7t^2 - 8c_0^3c_1c_3c_5c_8s^6t^3 + 16c_1^3c_3^2c_8^2s^6t^3 + 48c_1^2c_3^2c_5^2c_9s^6t^3 + c_0^5c_5c_8s^5t^4 - 8c_0^2c_1^2c_3c_8^2s^5t^4 - 16c_0^2c_1c_3c_5^2c_9s^5t^4 - 36c_0c_1c_3c_5^3c_6s^4t^5 + c_0^4c_1c_8^2s^4t^5 + c_0^4c_5^2c_9s^4t^5 + 32c_0c_1^2c_3c_5c_8c_9s^4t^5 + c_0^3c_5^3c_6s^3t^6 - 72c_1^2c_3c_5^2c_6c_8s^3t^6 - 8c_0^3c_1c_5c_8c_9s^3t^6 + 32c_1^3c_3c_8^2c_9s^3t^6 + 48c_1^2c_3c_5^2c_9^2s^3t^6 - 30c_0^2c_1c_5^2c_6c_8s^2t^7 - 8c_0^2c_1^2c_8^2c_9s^2t^7 - 8c_0^2c_1c_5^2c_9^2s^2t^7 - 96c_0c_1^2c_5c_6c_8^2st^8 - 36c_0c_1c_5^3c_6c_9st^8 + 16c_0c_1^2c_5c_8c_9^2st^8 - 27c_1c_5^4c_6^2t^9 - 64c_1^3c_6c_8^3t^9 - 72c_1^2c_5^2c_6c_8c_9t^9 + 16c_1^3c_8^2c_9^2t^9 + 16c_1^2c_5^2c_9^3t^9) \tag{8.14}$$

The gauge groups associated to the singularities of these Weierstrass models are $E_6 \times E_8$ and $SO(26)$ respectively. They are exactly what we expect from heterotic string theories with one Wilson line A_{c_5} in equation (8.8). This means that if we compactify F-theory on these elliptically fibered K3s, we know that the Wilson lines on the dual heterotic strings should be of a similar kind as A_{c_5}. Using this it is then possible to restrict the study of the duality map between the two theories to a three dimensional moduli space to verify that the enhancements on both F-theory and heterotic sides match.

8.2.3 Wilson line interpretation for polytope with more than two fibers

The Wilson line description of reflexive polyhedra can be extended to K3 surfaces which have more than two inequivalent elliptic fibrations. Indeed let us consider the polytope $M2$ with three fibers presented in the Figure 8.1. The fiber $E_8 \times E_8 \times SU(2)$ is obtained via the fiber $E_8 \times E_8$ of the polytope $M88$ with $\xi = \frac{1}{4}$ [111]. This in fact corresponds to taking the complex structure and Kähler moduli equal when compactifying on the two torus on the heterotic string. The two remaining fibers ($E_7 \times SO(20)$ and $SU(18)$) of $M2$ can be obtained by considering $M1328$ with $c_3 = c_4 = c_7 = c_8 = c_9 = 0$: $E_6 \times SO(10)$ is enhanced to $E_7 \times SO(20)$ while $SU(11) \times SU(2)$ to $SU(18)$. From our construction, the Wilson lines on the dual theory are therefore parametrized by A_{c_5} and A_{c_2}. The moduli spaces for the fibers $E_7 \times SO(20)$ and $SU(18)$ are thus contained in the moduli spaces of the heterotic strings $E_8 \times E_8$ and $SO(32)$ with this particular Wilson lines parametrization respectively.

CHAPTER 9

Fibration Structure of K3 Surfaces in F-theory as $\mathbb{Z}_n$ Shift Vectors in the Heterotic String

Up to now we considered polytopes for which we could interpret a subspace of the moduli space as the torus compactification of the heterotic string. We saw that a convenient way to do this is to consider polytopes which, in some limit, give the Weierstrass equations associated to the polytope $M88$ whose fibrations have $E_8 \times E_8$ and $SO(32)$ singularities. Here we want to discuss explicit map between the Wilson lines in the heterotic string and the fibration structure of K3 surfaces with more than two fibers. The results below are preliminary work of an upcoming paper with Bernardo Fraiman [11].

9.1 Wilson line structure of a polytope with more than two fibers: polytope $M3$

9.1.1 Gauge groups and maximal enhancement

Here we consider the polytope $M3$ with Picard 18 and 2 moduli. It is defined as the convex hull of the following points

$$M3 : (1,\, 0,\, 0)\,, (0,\, 1,\, 0)\,, (-1,\, -1,\, 0)\,, (0,\, 0,\, 1)\,, (-1,\, 0,\, -1)\,. \tag{9.1}$$

It's dual $N3$ has five fibrations whose gauge groups are given in Table 9.1.

Polytope	Fiber 1	Fiber 2	Fiber 3	Fiber 4	Fiber 5
M3	$SO(14) \times E_7$	$SO(14) \times SU(9)$	$\frac{SU(12)\times SO(8)}{Z_2}$	$\frac{(E_6\times SU(3))^2}{Z_3}$	$E_8 \times E_8 \times Z_3$

Table 9.1: Gauge groups of polytope $M3$. Additional $U(1)$s should be added so that the rank is 18.

The equation of the K3 surface can be written as

$$p = -c_0 x_0 x_1 x_2 x_3 x_4 + c_1 x_0^3 + c_2 x_1^3 x_2^3 + c_3 x_3^3 x_4^3 + c_4 x_1^3 x_4^3 + c_5 x_2^3 x_3^3 \tag{9.2}$$

with two moduli which can be taken to be

$$\xi = \frac{c_1 c_2 c_3}{c_0^3} \quad , \quad \eta = \frac{c_1 c_4 c_5}{c_0^3}. \tag{9.3}$$

The Weierstrass models of each of the five fibrations are written in appendix B.5. The groups $SO(14) \times U(1)$, $E_7 \times U(1)$, $E_6 \times SU(3)$ and $SU(9)$ typically appear in compactifications of the $E_8 \times E_8$ heterotic string on $\mathbb{Z}_3$ orbifolds [120–122]. This is also the case for the groups $SO(14) \times U(1) \times SU(9)$ and $SO(8) \times SU(12) \times U(1)$ in the $SO(32)$ heterotic string [123, 124]. In orbifold compactifications of the heterotic string these groups appear due to a so called shift vector of

the form

$$\frac{V}{n} \quad , \quad V \in \Gamma_{E_8} \otimes \Gamma_{E_8} \text{ or } \Gamma_{D_{16}} \tag{9.4}$$

for a $\mathbb{Z}_n$ orbifold, such that n times the shift vector is in the lattice $\Gamma_{D_{16}}$ or $\Gamma_{E_8} \otimes \Gamma_{E_8}$. In order to understand why the fibration structure of $M3$ selects the particular gauge groups of Table 9.1 let us consider such shifts as frozen Wilson lines. We consider the Wilson lines $A_2 = 0$ and

$$A_1^A = \left(\left(\frac{2}{3}\right)_{k_1}, 0_{8-k_1}, \left(\frac{2}{3}\right)_{k_2}, 0_{8-k_2} \right) \quad , \quad k_1, k_2 \in [|0, 8|]. \tag{9.5}$$

These choices are non-conventional in the literature on orbifold compactifications of the heterotic string. One can consider $A_1 \to A_1 + V$ with $V \in \Gamma_{D_{16}}$ or $\Gamma_{E_8} \otimes \Gamma_{E_8}$ so as to minimize the norm of A_1. However this form is more symmetric and necessary to compare our results to the paper [36] where they survey the compactifications of the heterotic string on a circle. We find that the following values of $[k_1, k_2]$[1] give for the $E_8 \times E_8$ lattice

- $SO(14) \times E_7$: [1,2], [1,8], [2,4], [2,7], [4,8], [7,8]

- $SO(14) \times SU(9)$: [1,5], [4,5], [5,7]

- $(E_6 \times SU(3))^2$: [3,3], [3,6], [6,6].

We see here that the Wilson line (9.5) give groups of the elliptic fibrations of the polytope $M3$ if $k = k_1 + k_2 \in 3\mathbb{Z}$. Moreover on the $SO(32)$ heterotic string one obtains the gauge groups

- $SO(14) \times SU(9)$ for $k = 9$

- $SU(12) \times SO(8)$ for $k = 12$.

We can therefore obtain the gauge groups appearing in the different fibrations of $M3$ with the Wilson lines (9.5). In order to verify that this description of the dual of the different fibers is coherent we look for a maximal enhancement. Let us focus on the fiber $(E_6 \times SU(3))^2$. We find

[1] One can exchange k_1 and k_2 without consequences.

a maximal enhancement on the F-theory side at

$$c_1 = c_2 = c_3 = c_4 = c_5 = 1 \quad , \quad c_0 = 0 \quad : \quad (E_6 \times SU(3))^2 \to E_6^3. \tag{9.6}$$

Now, considering the Wilson line with split $[3, 6]$ between the two E_8s, an enhancement to E_6^3 is found for a torus with parameters defined in equation (7.6)

$$\tau = -\frac{3}{2} + i\frac{\sqrt{3}}{2} \quad , \quad \rho_0 = B_{12} + i\sqrt{G} = -\frac{1}{2} + i\frac{\sqrt{3}}{6}. \tag{9.7}$$

Using equations (7.7) and (7.8), we get that the additional states on the heterotic side appear for the following values of winding numbers (w_1, w_2) as

- windings $\pm(0, 1)$, $\pm(3, 1)$ and $\pm(3, 2)$ give one additional states each

- windings $\pm(1, 0)$, $\pm(1, 1)$ and $\pm(2, 1)$ give nine additional states each.

If our comparison between the fibers and the Wilson lines is correct, the same point in the moduli space of the equation (9.6) for the other fibrations should match the enhancements of the other splits of Wilson lines with same moduli on the torus in the equation (9.7). This is indeed the case and we find the following enhancements on the F-theory side

M3 (generic point)	$SO(14) \times E_7$	$SO(14) \times SU(9)$	$\frac{SU(12)\times SO(8)}{Z_2}$	$\frac{E_6\times E_6\times SU(3)\times SU(3)}{Z_3}$
c1=c2=c3=c4=c5=1,c0=0	$SO(20) \times E_7$	$SO(14) \times SU(12)$	$SU(12) \times SO(14)$	E_6^3

Table 9.2: Enhancements for a particular point in the moduli space of $M3$ for different fibers.

On the heterotic part the splits $[1, 8]$, $[2, 7]$, $[4, 5]$ of the Wilson line (9.5) together with the moduli (9.7) for the torus give the same enhancements. The additional states come from the same values of the winding numbers, however with 15 additional states instead of 9 for the splits $[1, 8]$ and $[2, 7]$ giving $E_7 \times SO(20)$ gauge group. The Wilson lines with $k = k_1 + k_2$ in equation (9.5) equal to 9 or 12 on the $SO(32)$ heterotic string also match the enhancements from $SO(14) \times SU(9) \to SO(14) \times SU(12)$ and $SU(12) \times SO(8) \to SU(12) \times SO(14)$.

9.1.2 Interpretation of the Wilson line description of the polytope

Now that we discussed the gauge structure and the Wilson line associated to the various fibers in the dual heterotic string let us, discuss a possible interpretation. For the fiber $E_8 \times E_8 \times \mathbb{Z}_3$ in Table 9.1, the $\mathbb{Z}_3$ part indicates that there exists a section $\sigma : \mathbb{P}^1 \to M$ from the base to the K3 surface of order 3 on the elliptic curve i.e. such that $\sigma + \sigma + \sigma = \sigma_0$. The sum is understood in $\mathbb{E}_\tau = \frac{\mathbb{C}}{\Lambda_\tau}$ where τ is the complex parameter of the torus (see Equation (4.17)). Instead of considering the modular curve $\mathbb{E}_\tau$ we therefore have to consider the pair $(\mathbb{E}_\tau, p)$ where τ is the complex parameter of the elliptic curve and p a point of order 3. Such point of order n can always be written as [125]

$$p = \frac{c\tau + d}{n} + \Lambda_\tau \quad \text{for } c \text{ and } d \text{ such that} \quad \gcd(c, d, n) = 1. \tag{9.8}$$

Preserving this structure breaks the modular group $SL(2, \mathbb{Z})$ of the elliptic curve to a subgroup, given in this particular case by

$$\Gamma_1(n) = \left\{ \gamma \in SL(2, \mathbb{Z}) : \quad \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\text{mod } n) \right\}. \tag{9.9}$$

The moduli space is then $\mathcal{H} \backslash \Gamma_1(n)$ with $\mathcal{H}$ the upper half plane. This is the case when there is only one point of order 3. Other settings can break $SL(2, \mathbb{Z})$ to a different subgroup as discussed in [125].

Now, how does this translate into the Wilson lines we found for the other fibers of the polytope $M3$? In this case we considered one Wilson line as a shift vector of order 3 in equation (9.4) which we split differently between the two E_8s. One must therefore consider that the modular transformation of the parameters (7.6) preserve the shift vector in a similar way i.e. that in general one should seek to preserve the Wilson lines

$$A_1 = \frac{V_1}{n} \quad , \quad A_2 = \frac{V_2}{n} \quad , \quad V_1, V_2 \in \Gamma_{E_8} \otimes \Gamma_{E_8} \text{ or } \Gamma_{D_{16}} \tag{9.10}$$

It is not clear yet how this translates into the breaking of the $SL(2,\mathbb{Z})$ modular group into one of its subgroups for either τ or $\rho_0 = B_{12} + i\sqrt{G}$.

9.2 Other Picard 18 polytopes

In principle it is possible to do a similar analysis for other polytopes with 2 moduli. However, the dual Wilson lines on the heterotic side seems more subtle and we would like to emphasize some points. First, in the case of the polytope $M3$ it is not clear yet why (9.5) does not split as $[k_1, k_2] = [0, 3]$ or $[0, 6]$. This does not break one of the E_8 and therefore cannot be identified with one of the fibers.

Now, in Table 8.2 there are 3 polytopes which have a fiber with $E_8 \times E_8 \times \mathbb{Z}_3$ and 2 with $E_8 \times E_8 \times \mathbb{Z}_4$. According to our previous discussion in section 9.1.2 we therefore seek for shifts of order 3 and 4 respectively to describe these polytopes. This is what we discuss in the next subsections. We want to emphasize that the Wilson lines we choose, although giving a correct generic gauge group for the fibers of different polytopes, do not necessarily give correct enhancements. We believe that both Wilson lines should be considered non zero and hope to provide a coherent description in the near future.

9.2.1 $\mathbb{Z}_3$ Wilson lines shifts

Let us first discuss the case of $M4$ with $\mathbb{Z}_3$ shift. We find that the correct generic gauge group for each fiber with $k = k_1 + k_2 = 10$ splits differently between the E_8s. To be precise we find that the Wilson line (9.5) gives

- $E_7 \times E_7$: [2,8]
- $E_6 \times SU(3) \times SO(14)$: [3,7], [4,6]
- $SU(9) \times SU(9)$: [5,5]
- $SU(10) \times SO(12)$: $k = 10$ on $SO(32)$ heterotic string.

However this cannot be the correct answer: on the heterotic side with split $[5,5]$ i.e. $SU(9)\times SU(9)$ generic gauge group, there is enhancement to $SU(18)$ already on the compactification of the heterotic string on a circle and therefore necessarily on the torus. We thus believe that the case of $M3$ was unique in the sense that only one Wilson line is needed, and that others might need $\mathbb{Z}_3$ shifts on both A_1 and A_2.

The polytope $M6$ has 7 fibers including $E_8 \times E_8 \times \mathbb{Z}_3$. We find the following groups with $k = k_1 + k_2$ equal to 8 or 11 in equation (9.5)

- $E_8 \times E_7$: [0,8]
- $SO(14) \times SO(14)$: [1,7], [4,7], [4,4]
- $E_7 \times E_6 \times SU(3)$:[2,6], [3,8]
- $SU(9) \times E_6 \times SU(3)$: [3,5], [6,5]
- $SU(8) \times SO(16)$: $k = 8$ on $SO(32)$ heterotic string
- $SU(11) \times SO(10)$: $k = 11$ on $SO(32)$ heterotic string.

The groups obtained in these cases match what we expect from the fibrations of the polytope $M6$ in Table 8.2. We do not yet know if the enhancements on the F-theory and heterotic side match and one might need possibly two Wilson lines.

9.2.2 $\mathbb{Z}_4$ Wilson lines shifts

Now let us look at $\mathbb{Z}_4$ shifts of the Wilson lines. The polytope $M10$ has four fibers, including $E_8 \times E_8 \times \mathbb{Z}_4$. We find

- $SO(16) \times SO(16)$: $\left(\left(\frac{1}{2}\right)_4, 0_4, \left(\frac{1}{2}\right)_4, 0_4\right)$
- $E_7 \times E_7 \times (SU(2))^2$: $\left(\left(\frac{1}{2}\right)_6, 0_2, \left(\frac{1}{2}\right)_6, 0_2\right)$
- $SU(16)$: $\left(\left(\frac{1}{4}\right)_{16}\right)$ on $SO(32)$ heterotic string.

In the $SU(16)$ case we considered (1_{16}) in $\Gamma_{D_{16}}$.

The polytope $M11$ has five inequivalent fibers and we find

- $SO(16) \times E_7 \times SU(2)$: $\left(\left(\frac{1}{2}\right)_6, 0_2, \left(\frac{1}{2}\right)_4, 0_4\right)$
- $E_8 \times E_7 \times SU(2)$: $\left(\left(\frac{1}{2}\right)_6, 0_2, 0_8\right)$
- $SO(12) \times SO(20)$: $\left(\left(\frac{1}{2}\right)_6, 0_{10}\right)$ on $SO(32)$ heterotic string
- $SU(16)$: $\left(\left(\frac{1}{4}\right)_{16}\right)$ on $SO(32)$ heterotic string.

9.3 Final comment

In this chapter, we saw that gauge groups appearing in the compactifications of F-theory on elliptically fibered K3 surfaces with two moduli defined via reflexive polyhedra could be obtained on the heterotic side with $\mathbb{Z}_n$ shift vectors defined with (9.10). In the case of the polytope $M3$, the enhancements of the gauge group for each fibration correspond to the enhancements of the gauge group in the heterotic side with our choice of Wilson line (9.4) for particular values of k_1 and k_2. In the other cases we considered i.e. $M4, M6, M10$ and $M11$, gauge groups for a generic value of the moduli on the torus in equation (7.6) for our choices of Wilson lines are the ones corresponding to every fibration. However some of the enhancements do not match. We thus believe that for these particular polytopes, one has to consider both Wilson lines non-zero in the dual heterotic string theory.

Moreover, it would be interesting to understand how the $SL(2,\mathbb{Z})$ modular group of τ and ρ_0 in (7.6) are impacted by $\mathbb{Z}_n$ shift vectors in the heterotic string. Finally, the analysis considered in chapter 8 and represented in Figure 8.6, where we identify additional moduli using graphs of polytopes as Wilson lines moduli in the heterotic string, should be possible if we are able to understand properly the structure of the polytopes with two moduli.

APPENDIX A

Appendix of Part II

A.1 Projectors

Here we present the construction of the projectors on the useful representations we used along this paper. A detailed construction can be found in [76]: it is shown in the first appendix of this paper that for an arbitrary simple group G, with the exception of E_8, one can decompose the product of the fundamental representation of the group G ($\mathbf{D}(\mathbf{\Lambda})$) with its adjoint $\mathbf{Adj}(\mathbf{G})$ as

$$\mathbf{D}(\mathbf{\Lambda}) \times \mathbf{Adj}(\mathbf{G}) \rightarrow \mathbf{D}(\mathbf{\Lambda}) + \mathbf{D_1} + \mathbf{D_2} \tag{A.1}$$

where $\mathbf{D_1}$ and $\mathbf{D_2}$ are two other representations. The ones of interest for us are only the fundamental and the representation with the smaller dimension ($\mathbf{D_1}$), as they are the only representations allowed for the embedding tensor after one considers the linear constraint coming from supersymmetry consideration. If we note M the fundamental representation of G and $\{t^\alpha\}$ (α = 1...dim(G)) the generators of the adjoint of G, the projectors on those two representations can

be written[1]

$$\begin{aligned}\mathbb{P}_{(D(\Lambda))M}{}^{\alpha,N}{}_{\beta} &= A(t^{\alpha}t_{\beta})_{M}{}^{N} \\ \mathbb{P}_{(D_1)M}{}^{\alpha,N}{}_{\beta} &= a_1\delta^{\alpha}{}_{\beta}\delta_{M}{}^{N} + a_2(t_{\beta}t^{\alpha})_{M}{}^{N} + a_3(t^{\alpha}t_{\beta})_{M}{}^{N}\end{aligned} \tag{A.2}$$

with A, a_i constants which are given in [76] for every simple group.

Now let us look at the two simple groups of interest to us, $SL(3)$ and $SL(2)$, whose fundamental representations $(\mathbf{3})$ and $(\mathbf{2})$ are written m and γ respectively. For clarity we will note $\{t^{\alpha}\}$ (α = 1,...,8) the generators of the adjoint of $SL(3)$, and $\{s^{\tilde{\alpha}}\}$ ($\tilde{\alpha}$ = 1,2,3) the ones of $SL(2)$. With this we find the projectors onto the the fundamental representation of $SL(3)$ and $D_1 = (\mathbf{6})$ to be

$$\begin{aligned}\mathbb{P}_{(\mathbf{3}),m}{}^{\alpha,n}{}_{\beta} &= \frac{3}{8}(t^{\alpha}t_{\beta})_{m}{}^{n} \\ \mathbb{P}_{(\mathbf{6}),m}{}^{\alpha,n}{}_{\beta} &= \frac{1}{2}\delta^{\alpha}{}_{\beta}\delta_{m}{}^{n} - \frac{1}{2}(t_{\beta}t^{\alpha})_{m}{}^{n} - \frac{1}{4}(t^{\alpha}t_{\beta})_{m}{}^{n}.\end{aligned} \tag{A.3}$$

For the $SL(2)$ case, the result is a little peculiar as one has the following relation

$$\delta^{\tilde{\alpha}}_{\tilde{\beta}}\delta^{\eta}_{\gamma} - (s_{\tilde{\beta}}s^{\tilde{\alpha}})_{\gamma}{}^{\eta} - (s^{\tilde{\alpha}}s_{\tilde{\beta}})_{\gamma}{}^{\eta} = 0. \tag{A.4}$$

The only representations left are then $\mathbf{D}(\mathbf{\Lambda}) = (\mathbf{2})$ and $\mathbf{D_2} = (\mathbf{4})$. The projection onto the fundamental is

$$\begin{aligned}\mathbb{P}_{(\mathbf{2}),\gamma}{}^{\tilde{\alpha},\eta}{}_{\tilde{\beta}} &= \frac{2}{3}(s^{\tilde{\alpha}}s_{\tilde{\beta}})_{\gamma}{}^{\eta} \\ &= \frac{2}{3}(\delta^{\tilde{\alpha}}{}_{\tilde{\beta}}\delta^{\eta}_{\gamma} - (s_{\tilde{\beta}}s^{\tilde{\alpha}})_{\gamma}{}^{\eta}).\end{aligned} \tag{A.5}$$

We write these projectors in the fundamental representation of each groups, leading for $SL(3)$ to

$$\begin{aligned}\mathbb{P}_{(\mathbf{3})mn}{}^{p,ab}{}_{c} &= \frac{3}{8}(\delta^{p}_{m}\delta^{a}_{c}\delta^{b}_{n} - \frac{1}{3}\delta^{p}_{n}\delta^{a}_{c}\delta^{b}_{m} - \frac{1}{3}\delta^{p}_{m}\delta^{a}_{n}\delta^{b}_{c} + \frac{1}{9}\delta^{p}_{n}\delta^{a}_{m}\delta^{b}_{c}) \\ \mathbb{P}_{(\mathbf{6})mn}{}^{p,ab}{}_{c} &= \frac{1}{2}\epsilon_{mnr}\epsilon^{ab(r}\delta^{p)}_{c}\end{aligned} \tag{A.6}$$

[1]The adjoint indices are raised and lowered using the Cartan-Killing metric.

and for $SL(2)$

$$\mathbb{P}_{(\mathbf{2})\gamma\eta}{}^{\rho,\alpha\beta}{}_{\xi} = \frac{2}{3}(\delta^{\rho}_{\gamma}\delta^{\alpha}_{\xi}\delta^{\beta}_{\eta} - \frac{1}{2}\delta^{\rho}_{\eta}\delta^{\alpha}_{\xi}\delta^{\beta}_{\gamma} - \frac{1}{2}\delta^{\rho}_{\gamma}\delta^{\alpha}_{\eta}\delta^{\beta}_{\xi} + \frac{1}{4}\delta^{\rho}_{\eta}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\xi}). \tag{A.7}$$

These expressions are found using the projectors on the adjoint $(\mathbf{8})$ of $SL(3)$

$$\mathbb{P}_{(\mathbf{8})}{}^{m}{}_{n}{}^{p}{}_{q} = (t_{\alpha})_{n}{}^{m}(t^{\alpha})_{q}{}^{p} = \delta^{m}_{q}\delta^{p}_{n} - \frac{1}{3}\delta^{m}_{n}\delta^{p}_{q} \tag{A.8}$$

and the adjoint $(\mathbf{3_{SL(2)}})$ of $SL(2)$

$$\mathbb{P}_{(\mathbf{3_{SL(2)}})}{}^{\gamma}{}_{\eta}{}^{\rho}{}_{\delta} = (s_{\tilde{\alpha}})_{\eta}{}^{\gamma}(s^{\tilde{\alpha}})_{\delta}{}^{\rho} = \delta^{\gamma}_{\delta}\delta^{\rho}_{\eta} - \frac{1}{2}\delta^{\gamma}_{\eta}\delta^{\rho}_{\delta}. \tag{A.9}$$

A.2 Determination of Γ

The expression of the generalised Christoffel symbol (6.34) was hinted by a series of projections applied to the torsion condition (6.28). Here we detail the different relations that permitted in the end to look for a generalised Christoffel of the form (6.30).

First of all, one can relate the traces of the Christoffel symbol by taking the trace of the torsion condition

$$\Gamma_{MD}{}^{D} = -\Gamma_{DM}{}^{D}. \tag{A.10}$$

By taking the partial traces on the different subspaces it is also possible to write the following relations

$$\begin{aligned}\Gamma_{m\gamma,n\eta}{}^{n\rho} &= 3\Gamma_{n\eta,m\gamma}{}^{n\rho} - 2\Gamma_{r\delta,m\gamma}{}^{r\delta}\delta^{\rho}_{\eta}\\ \Gamma_{m\gamma,n\eta}{}^{p\eta} &= 2\Gamma_{n\eta,m\gamma}{}^{p\eta} - \Gamma_{r\delta,m\gamma}{}^{r\delta}\delta^{p}_{n}\end{aligned} \tag{A.11}$$

which can be recast into

$$\begin{aligned}\Gamma_{r\gamma,m\delta}{}^{r\delta} &= \Gamma_{r\delta,m\gamma}{}^{r\delta}\\ \Gamma_{m\delta,r\gamma}{}^{r\delta} &= \Gamma_{r\delta,m\gamma}{}^{r\delta}.\end{aligned} \tag{A.12}$$

Other useful relations are obtained by taking the projection of the torsion condition onto the representations $(\mathbf{8},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$

$$\begin{aligned}\mathbb{P}_{(8,1)}{}^{R}{}_{S}{}^{B}{}_{C}\Gamma_{AB}{}^{C} &= 2\mathbb{P}_{(8,1)}{}^{R}{}_{S}{}^{B}{}_{C}\Gamma_{BA}{}^{C}\\ \mathbb{P}_{(1,3)}{}^{R}{}_{S}{}^{B}{}_{C}\Gamma_{AB}{}^{C} &= 3\mathbb{P}_{(1,3)}{}^{R}{}_{S}{}^{B}{}_{C}\Gamma_{BA}{}^{C}.\end{aligned} \tag{A.13}$$

We also have to recall from (A.1) that

$$\mathbb{P}_{\mathbf{D}(\mathbf{\Lambda})} + \mathbb{P}_{\mathbf{D_1}} + \mathbb{P}_{\mathbf{D_2}} = \mathsf{Id}_{\mathbf{D}(\mathbf{\Lambda})\times\mathbf{Adj}(\mathbf{G})} \tag{A.14}$$

which for the groups $SL(2)$ and $SL(3)$ can be written[2]

$$\begin{aligned}\mathbb{P}_{(\mathbf{2})} + \mathbb{P}_{(\mathbf{4})} &= \mathsf{Id}_{(\mathbf{2})\times(\mathbf{3})}\\ \mathbb{P}_{(\mathbf{3})} + \mathbb{P}_{(\mathbf{6})} + \mathbb{P}_{(\mathbf{15})} &= \mathsf{Id}_{(\mathbf{3})\times(\mathbf{8})}\end{aligned} \tag{A.15}$$

where the traces on the spaces $(\mathbf{4})$ and $(\mathbf{15})$ are null. The torsion condition can then be recast as

$$\begin{aligned}\left(\tilde{\Gamma}_{(15,2)} + \tilde{\Gamma}_{(3,4)}\right)_{MN}{}^{P} = \Gamma_{MN}{}^{P} + \frac{1}{6}\Gamma_{DM}{}^{D}\delta^{P}_{N} - \left[\frac{7}{8}\mathbb{P}_{(8,1)}{}^{K}{}_{M}{}^{P}{}_{N} + \frac{7}{3}\mathbb{P}_{(1,3)}{}^{K}{}_{M}{}^{P}{}_{N}\right]\Gamma_{RK}{}^{R}\\ + \left[\frac{9}{4}\mathbb{P}_{(8,1)}{}^{K}{}_{M}{}^{P}{}_{N}\mathbb{P}_{(1,3)}{}^{R}{}_{K}{}^{S}{}_{T} + 4\mathbb{P}_{(1,3)}{}^{K}{}_{M}{}^{P}{}_{N}\mathbb{P}_{(8,1)}{}^{R}{}_{K}{}^{S}{}_{T}\right]\Gamma_{SR}{}^{T}.\end{aligned} \tag{A.16}$$

The relations

$$\begin{aligned}\frac{9}{4}\mathbb{P}_{(8,1)}{}^{K}{}_{M}{}^{P}{}_{N}\mathbb{P}_{(1,3)}{}^{R}{}_{K}{}^{S}{}_{T}\Gamma_{SR}{}^{T} &= \frac{3}{8}\mathbb{P}_{(8,1)}{}^{K}{}_{M}{}^{P}{}_{N}\Gamma_{SK}{}^{S}\\ 4\mathbb{P}_{(1,3)}{}^{K}{}_{M}{}^{P}{}_{N}\mathbb{P}_{(8,1)}{}^{R}{}_{K}{}^{S}{}_{T}\Gamma_{SR}{}^{T} &= \frac{4}{3}\mathbb{P}_{(1,3)}{}^{K}{}_{M}{}^{P}{}_{N}\Gamma_{SK}{}^{S}\end{aligned} \tag{A.17}$$

[2] In the case of $SL(2)$ the space D_1 is empty.

permit to write the last term between brackets into a trace part, which leads to

$$\Gamma_{MN}{}^{P} = \left(\tilde{\Gamma}_{(15,2)} + \tilde{\Gamma}_{(3,4)}\right)_{MN}{}^{P} + \text{trace terms.} \tag{A.18}$$

Using

$$\begin{aligned} \tilde{\Gamma}_{(15,2)m\gamma,n\eta}{}^{p\rho} &= \left[\mathbb{P}_{(15,2)}\Gamma\right]_{MN}{}^{P} = \left[\mathbb{P}_{(15)}\Gamma\right]_{m\gamma,n}{}^{p}\delta^{\rho}_{\eta} = \tilde{\Gamma}_{(15)m\gamma,n}{}^{p}\delta^{\rho}_{\eta} \\ \tilde{\Gamma}_{(3,4)m\gamma,n\eta}{}^{p\rho} &= \left[\mathbb{P}_{(3,4)}\Gamma\right]_{MN}{}^{P} = \left[\mathbb{P}_{(15)}\Gamma\right]_{m\gamma,\eta}{}^{\rho}\delta^{p}_{n} = \tilde{\Gamma}_{(4)m\gamma,\eta}{}^{\rho}\delta^{p}_{n} \end{aligned} \tag{A.19}$$

with partial traces of $\tilde{\Gamma}_{(\mathbf{15})}$ and $\tilde{\Gamma}_{(\mathbf{4})}$ null, we have

$$\Gamma_{MN}{}^{P} = \tilde{\Gamma}_{(\mathbf{15})m\gamma,n}{}^{p}\delta^{\rho}_{\eta} + \tilde{\Gamma}_{(\mathbf{4})m\gamma,\eta}{}^{\rho}\delta^{p}_{n} + \text{trace terms.} \tag{A.20}$$

Using this expression in the metric compatibility condition (6.31) and the torsion condition (6.29) leads to the solution (6.34).

APPENDIX B

Appendix of Part III

B.1 Program 1: Dynkin Diagram from Reflexive Polyhedra

Here we present how to use the first program. The first line is simply *reflexivePolytopes = []*. Just enter a list of number between 0 and 4318 to consider the reflexive polytopes *ReflexivePolytope(3,#)* of this list into Sagemath. The program then returns a table containing all gauge groups for all the fibers of any reflexive polytope. The table is written in latex format on a text file.

Figure B.1 shows the output of this first program where we gave as an entry the reflexive polytope with Picard number 17 (i.e. 3 complex parameters) and no correction term.

Table B.1: Picard 17

N14	$\frac{E_7\times SU(8)\times SU(2)}{Z_2}$	$SO(16)\times E_6$	$E_8\times E_7\times Z_4$	$SU(14)\times SU(2)$	$E_8\times E_6$	$SO(10)\times SO(18)$			
N15	$E_7\times E_6\times SU(2)$	$SO(14)\times SO(14)$	$\frac{SO(16)\times SU(8)}{Z_2}$	$E_7\times E_8\times Z_4$	$\frac{SU(14)\times SU(2)}{Z_2}$				
N20	$E_7\times E_8\times Z_3$	$E_6\times SO(14)$	$E_6\times SO(10)\times SU(3)SU(2)$	$SU(6)\times SO(14)\times SU(3)$	$E_7\times SO(12)$	$SO(12)\times SU(8)$	$\frac{SU(10)\times SO(8)\times SU(2)}{Z_2}$	$SU(9)\times SU(7)$	
N21	$E_7\times SO(12)\times SU(2)$	$SU(6)\times SU(10)$	$E_7\times E_7$	$\frac{SO(12)\times SO(12)\times SU(2)SU(2)}{Z_2}$	$E_7\times E_7\times Z_2$	$SU(8)\times E_6$	$\frac{SO(12)\times SO(12)\times SU(4)}{Z_2}$	$\frac{SO(16)\times SO(8)\times SU(2)SU(2)}{Z_2}$	
N22	$SO(10)\times SU(9)\times SU(2)$	$SO(14)\times SU(7)$	$E_8\times E_7\times Z_3$	$SO(14)\times SO(12)$	$E_6\times E_6\times SU(3)$	$SU(10)\times SO(8)$	$E_7\times SO(10)\times SU(2)$	$\frac{E_6\times SU(6)\times SU(3)SU(3)}{Z_3}$	
N23	$\frac{SU(10)\times SO(12)}{Z_2}$	$E_6\times SO(14)\times SU(2)$	$E_7\times SU(8)$	$SO(12)\times E_8$	$E_7\times E_7$	$SU(3)\times SU(13)$	$SO(18)\times SU(6)$		
N24	$E_7\times E_7$	$SO(10)\times SU(8)\times SU(3)$	$E_6\times E_6\times SU(2)$	$SO(12)\times SO(12)$	$\frac{SU(8)\times SU(8)}{Z_2}$				
N25	$SO(10)\times SO(14)\times SU(2)$	$SO(10)\times SU(9)$	$E_6\times E_7$	$SO(12)\times E_6\times SU(3)$	$E_7\times E_7$	$E_8\times E_7\times Z_3$	$SU(7)\times E_6\times SU(3)$	$\frac{SU(6)\times SU(9)\times SU(3)}{Z_3}$	$\frac{SU(8)\times SO(12)\times SU(2)}{Z_2}$
N26	$SU(9)\times E_6$	$E_6\times SU(9)$	$E_7\times E_7$	$SO(10)\times SO(16)$	$E_6\times E_8$	$\frac{SU(6)\times E_7\times SU(4)}{Z_2}$	$\frac{SU(3)\times SU(12)\times SU(3)}{Z_3}$		
N27	$SU(8)\times SO(14)$	$\frac{SU(4)\times SU(12)}{Z_2}$	$E_6\times E_6\times SU(2)SU(2)$	$E_7\times SO(12)$					
N28	$E_6\times E_7\times SU(2)$	$SU(10)\times SO(10)$	$SO(10)\times E_7\times SU(3)$	$SU(5)\times SU(11)$	$SO(14)\times SO(12)$	$E_6\times E_8$	$E_6\times SU(8)\times SU(2)$	$\frac{SU(6)\times SO(16)\times SU(2)}{Z_2}$	
N29	$E_7\times E_6$	$E_7\times E_7\times Z_2$	$\frac{SO(12)\times SU(8)\times SU(2)}{Z_2}$	$\frac{SU(6)\times SU(10)}{Z_2}$	$SO(12)\times E_6\times SU(2)$	$SO(10)\times SO(14)\times SU(3)$	$SU(8)\times SU(8)$		
N30	$E_6\times E_6$	$E_7\times E_7\times Z_2$	$\frac{SO(12)\times SO(12)\times SU(2)SU(2)}{Z_2}$	$\frac{SU(8)\times SU(8)}{Z_2}$					
N38	$E_7\times E_7$	$SO(14)\times E_8$	$\frac{SU(4)\times SO(24)}{Z_2}$	$\frac{SU(16)}{Z_2}$					
N41	$\frac{SO(12)\times SO(16)\times SU(2)}{Z_2}$	$E_7\times E_8\times Z_4$	$\frac{SO(12)\times E_7\times SU(2)SU(2)}{Z_2}$	$E_7\times E_7\times SU(2)$	$SU(14)$				
N47	$SO(18)\times SO(10)$	$E_8\times E_7\times Z_4$	$SU(14)\times SU(2)$	$E_8\times E_6$	$SO(16)\times E_6$	$\frac{SU(8)\times E_7\times SU(2)}{Z_2}$			
N48	$E_7\times E_7$	$SO(14)\times E_6\times SU(2)$	$SU(13)\times SU(3)$	$\frac{SU(10)\times SO(12)}{Z_2}$					
N49	$E_7\times E_6\times SU(2)$	$\frac{SU(8)\times SO(16)}{Z_2}$	$E_7\times E_8\times Z_4$	$SO(14)\times SO(14)$	$\frac{SU(14)\times SU(2)}{Z_2}$				
N50	$SU(15)$	$\frac{SU(6)\times SO(20)}{Z_2}$	$SO(14)\times E_7$	$E_8\times E_6\times SU(2)$					
N53	$E_6\times SU(9)$	$E_7\times E_7$	$SO(16)\times SO(10)$	$\frac{SU(12)\times SU(3)SU(3)}{Z_3}$					
N104	$E_7\times E_8$	$\frac{SO(28)\times SU(2)}{Z_2}$							
N117	$\frac{SO(24)\times SU(4)}{Z_2}$	$\frac{SU(16)}{Z_2}$	$E_7\times E_7$						
N221	$E_7\times E_8$	$\frac{SO(28)\times SU(2)}{Z_2}$							
N230	$\frac{SU(16)}{Z_2}$	$E_7\times E_7$							

B.2 Program 2: Weierstrass Models

Here we present again the typical output of the second computer program. Again on the first line one just specifies in a list the reflexive polytopes # (associated to *ReflexivePolytope(3,#)*. The output is the hypersurface equation for every fibration of the K3 surface as well as the corresponding Weierstrass models (upon a choice fiber described in Figure 8.2 for F13, F15 and F16. In another file are saved all the hypersurface equations in Sagemath form.
The following is the typical Latex output when putting as an input "[476]".

Polytope M476

Number of different Fiber is 2

Fiber 1

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0^2 + c_2x_2^3 + c_3x_1^4x_2s^3t^5 + c_4x_1^4x_2s^5t^3 + c_5x_1^6s^7t^5 + c_6x_1^6s^5t^7 + c_7x_1^6s^6t^6 + \quad \text{(B.1)}$$

Data of the Weierstrass model:

$$f = \left(\frac{1}{48}\right) \cdot t^3 \cdot s^3 \cdot (48c_1^2c_2c_4s^2 - c_0^4st + 48c_1^2c_2c_3t^2) \quad \text{(B.2)}$$

$$g = \left(-\frac{1}{864}\right) \cdot t^5 \cdot s^5 \cdot (72c_0^2c_1^2c_2c_4s^2 + 864c_1^3c_2^2c_5s^2 - c_0^6st + 864c_1^3c_2^2c_7st + 72c_0^2c_1^2c_2c_3t^2 + 864c_1^3c_2^2c_6t^2) \quad \text{(B.3)}$$

$$\begin{aligned}\Delta_{(f,g)} = \left(\frac{1}{16}\right) \cdot c_2^2 \cdot c_1^3 \cdot t^9 \cdot s^9 \cdot (&64c_1^3c_2c_4^3s^6 - c_0^4c_1c_4^2s^5t + 72c_0^2c_1^2c_2c_4c_5s^5t + 432c_1^3c_2^2c_5^2s^5t + 192c_1^3c_2c_3c_4^2s^4t^2 \\ &- c_0^6c_5s^4t^2 + 72c_0^2c_1^2c_2c_4c_7s^4t^2 + 864c_1^3c_2^2c_5c_7s^4t^2 - 2c_0^4c_1c_3c_4s^3t^3 + 72c_0^2c_1^2c_2c_3c_5s^3t^3 \\ &+ 72c_0^2c_1^2c_2c_4c_6s^3t^3 + 864c_1^3c_2^2c_5c_6s^3t^3 - c_0^6c_7s^3t^3 + 432c_1^3c_2^2c_7^2s^3t^3 + 192c_1^3c_2c_3^2c_4s^2t^4 - c_0^6c_6s^2t^4 \\ &+ 72c_0^2c_1^2c_2c_3c_7s^2t^4 + 864c_1^3c_2^2c_6c_7s^2t^4 - c_0^4c_1c_3^2st^5 + 72c_0^2c_1^2c_2c_3c_6st^5 + 432c_1^3c_2^2c_6^2st^5 + 64c_1^3c_2c_3^3t^6)\end{aligned} \quad \text{(B.4)}$$

Fiber 2

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0^2 + c_2x_1^2x_2^2st^3 + c_3x_1^4x_2t + c_4x_2^3s^6t + c_5x_2^3s^7 + c_6x_1^4x_2s + c_7x_1^2x_2^2s^4 + \quad \text{(B.5)}$$

Data of the Weierstrass model:

$$f = \left(-\frac{1}{48}\right) \cdot s^2 \cdot (-48c_1^2c_5c_6s^6 + 16c_1^2c_7^2s^6 - 48c_1^2c_3c_5s^5t - 48c_1^2c_4c_6s^5t - 48c_1^2c_3c_4s^4t^2 \quad \text{(B.6)}$$
$$- 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6)$$

$$g = \left(-\frac{1}{864}\right) \cdot s^3 \cdot (4c_1c_7s^3 - c_0^2st^2 + 4c_1c_2t^3) \cdot (-72c_1^2c_5c_6s^6 + 16c_1^2c_7^2s^6 - 72c_1^2c_3c_5s^5t - 72c_1^2c_4c_6s^5t \quad \text{(B.7)}$$
$$- 72c_1^2c_3c_4s^4t^2 - 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6)$$

$$\Delta_{(f,g)} = \left(-\frac{1}{16}\right) \cdot c_1^4 \cdot s^{14} \cdot (c_6s + c_3t)^2 \cdot (c_5s + c_4t)^2 \cdot (-64c_1^2c_5c_6s^6 + 16c_1^2c_7^2s^6 - 64c_1^2c_3c_5s^5t - 64c_1^2c_4c_6s^5t$$
$$- 64c_1^2c_3c_4s^4t^2 - 8c_0^2c_1c_7s^4t^2 + 32c_1^2c_2c_7s^3t^3 + c_0^4s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6) \quad \text{(B.8)}$$

B.3 Program 3: Finding Basic Enhancements and Constructing Graphs

In the following we see the enhancement for the input [476] for the third program. The numbers $\#$ between parenthesis correspond to the parameter $c_{\#}$ sent to zero in the definition of the Weierstrass model.

M476

fiber 1

()- E7xE7
(0_)- E7xE7
(3_)- E8xE7
(4_)- E7xE8
(5_)- E7xE7
(6_)- E7xE7
(7_)- E7xE7
(0_3_)- E8xE7
(0_4_)- E7xE8
(0_5_)- E7xE7
(0_6_)- E7xE7
(0_7_)- E7xE7
(3_4_)- E8xE8
(3_5_)- E8xE7
(3_7_)- E8xE7
(4_6_)- E7xE8
(4_7_)- E7xE8
(5_6_)- E7xE7
(5_7_)- E7xE7
(6_7_)- E7xE7
(0_3_4_)- E8xE8
(0_3_5_)- E8xE7
(0_3_7_)- E8xE7
(0_4_6_)- E7xE8
(0_4_7_)- E7xE8
(0_5_6_)- E7xE7
(0_5_7_)- E7xE7
(0_6_7_)- E7xE7
(3_4_7_)- E8xE8
(3_5_7_)- E8xE7
(4_6_7_)- E7xE8
(5_6_7_)- E7xE7
(0_3_4_7_)- E8xE8
(0_3_5_7_)- E8xE7
(0_4_6_7_)- E7xE8
(0_5_6_7_)- E7xE7

fiber 2

()- SO(24)xSU(2)xSU(2)

(0_)- SO(24)xSU(2)xSU(2)

(3_)- SO(28)xSU(2)

(4_)- SO(28)xSU(2)

(5_)- SO(24)xSU(2)xSU(2)

(6_)- SO(24)xSU(2)xSU(2)

(7_)- SO(24)xSU(2)xSU(2)

(0_3_)- SO(28)xSU(2)

(0_4_)- SO(28)xSU(2)

(0_5_)- SO(24)xSU(2)xSU(2)

(0_6_)- SO(24)xSU(2)xSU(2)

(0_7_)- SO(24)xSU(2)xSU(2)

(3_4_)- SO(32)

(3_5_)- SO(28)xSU(2)

(3_7_)- SO(28)xSU(2)

(4_6_)- SO(28)xSU(2)

(4_7_)- SO(28)xSU(2)

(5_6_)- SO(24)xSU(4)

(5_7_)- SO(24)xSU(2)xSU(2)

(6_7_)- SO(24)xSU(2)xSU(2)

(0_3_4_)- SO(32)

(0_3_5_)- SO(28)xSU(2)

(0_3_7_)- SO(28)xSU(2)

(0_4_6_)- SO(28)xSU(2)

(0_4_7_)- SO(28)xSU(2)

(0_5_6_)- SO(24)xSU(4)

(0_5_7_)- SO(24)xSU(2)xSU(2)

(0_6_7_)- SO(24)xSU(2)xSU(2)

(3_4_7_)- SO(32)

(3_5_7_)- SO(28)xSU(2)

(4_6_7_)- SO(28)xSU(2)

(5_6_7_)- SO(24)xSO(8)

(0_3_4_7_)- SO(32)

(0_3_5_7_)- SO(28)xSU(2)

(0_4_6_7_)- SO(28)xSU(2)

(0_5_6_7_) = 0: SO(24)xSO(8)

B.4 Vertices of the Polytopes presented in this paper

$M0$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(0,\,0,\,1)\,,(-1,\,-1,\,-1))$

$M2$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(0,\,0,\,1)\,,(-3,\,-1,\,-1))$

$M3$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(-1,\,-1,\,0)\,,(0,\,0,\,1)\,,(-1,\,0,\,-1))$

$M4$: $((1,\,0,\,0)\,,(-1,\,0,\,0)\,,(0,\,1,\,0)\,,(0,\,0,\,1)\,,(0,\,-1,\,-1))$

$M5$: $((1,\,0,\,0)\,,(-1,\,0,\,0)\,,(0,\,1,\,0)\,,(0,\,0,\,1)\,,(1,\,-1,\,-1))$

$M6$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(-1,\,-1,\,0)\,,(0,\,0,\,1)\,,(1,\,0,\,-1))$

$M7$: $((1,\,0,\,0)\,,(-1,\,0,\,0)\,,(0,\,1,\,0)\,,(0,\,0,\,1)\,,(2,\,-1,\,-1))$

$M10$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(-2,\,-1,\,0)\,,(0,\,0,\,1)\,,(-2,\,0,\,-1))$

$M11$: $((1,\,0,\,0)\,,(0,\,1,\,0)\,,(-2,\,-1,\,0)\,,(0,\,0,\,1)\,,(-1,\,1,\,-1))$

$M16$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-2,\,-1,\,-1),(-1,\,1,\,0))$

$M88$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-6,\,-4,\,-1))$

$M221$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-5,\,-3,\,-1),(-1,\,-1,\,1))$

$M230$: $((1,\,0,\,0),(0,\,1,\,0),(1,\,-1,\,0),(0,\,0,\,1),(-4,\,-2,\,-1))$

$M473$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-4,\,-3,\,-1),(-1,\,0,\,1),(-2,\,-1,\,1))$

$M476$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-4,\,-2,\,-1),(-5,\,-3,\,-1),(-1,\,-1,\,1))$

$M497$: $((1,\,0,\,0),(0,\,1,\,0),(-1,\,1,\,0),(0,\,0,\,1),(-2,\,-3,\,-1),(0,\,-1,\,1))$

$M859$: $((1,\,0,\,0),(0,\,1,\,0),(1,\,-1,\,0),(0,\,0,\,1),(-3,\,-1,\,-1),(0,\,-1,\,1),(-1,\,-1,\,1))$

$M866$: $((1,\,0,\,0),(0,\,1,\,0),(0,\,0,\,1),(-3,\,-2,\,-1),(-1,\,0,\,1),(-4,\,-3,\,-1),(-2,\,-1,\,1))$

$M895$: $((1,\,0,\,0),(0,\,1,\,0),(-1,\,1,\,0),(0,\,0,\,1),(-2,\,-2,\,-1),(-2,\,-3,\,-1),(0,\,-1,\,1))$

$M1328$: $((1,\,0,\,0),(0,\,1,\,0),(-1,\,1,\,0),(0,\,0,\,1),(-1,\,-2,\,-1),(0,\,-1,\,1),(-2,\,-2,\,-1),(-2,\,-3,\,-1))$

B.5 Weierstrass Models of the different fibrations of M3

Number of different Fiber is 5

Fiber 1

The hypersurface equation is:

$$p = -c_0x_0x_1x_2x_3x_4st + c_1x_0x_3^2x_4 + c_2x_0^2x_1^2x_3t^3 + c_3x_1x_2^3x_4^2s^3 + c_4x_0^2x_1^3x_2s^2t^3 + c_5x_2^2x_3x_4^2s+ \quad \text{(B.9)}$$

Data of the Weierstrass model:

$$f = \left(\frac{1}{48}\right)\cdot s^2\cdot t^3\cdot(48c_1^2c_3c_4s^3 - c_0^4s^2t + 24c_0c_1c_2c_3s^2t + 24c_0c_1c_4c_5s^2t + 8c_0^2c_2c_5st^2 - 16c_2^2c_5^2t^3) \quad \text{(B.10)}$$

$$g = \left(-\frac{1}{864}\right)\cdot s^3\cdot t^5\cdot(72c_0^2c_1^2c_3c_4s^4 - c_0^6s^3t + 36c_0^3c_1c_2c_3s^3t - 216c_1^2c_2^2c_3^2s^3t + 36c_0^3c_1c_4c_5s^3t + 144c_1^2c_2c_3c_4c_5s^3t$$
$$- 216c_1^2c_4^2c_5^2s^3t + 12c_0^4c_2c_5s^2t^2 - 144c_0c_1c_2^2c_3c_5s^2t^2 - 144c_0c_1c_2c_4c_5^2s^2t^2 - 48c_0^2c_2^2c_5^2st^3 + 64c_2^3c_5^3t^4) \quad \text{(B.11)}$$

$$\Delta = \left(\frac{1}{16}\right) \cdot c_1^2 \cdot t^9 \cdot s^9 \cdot (64c_1^4c_3^3c_4^3s^6 - c_0^4c_1^2c_3^2c_4^2s^5t + 96c_0c_1^3c_2c_3^3c_4^2s^5t + 96c_0c_1^3c_3^2c_4^3c_5s^5t - c_0^5c_1c_2c_3^2c_4s^4t^2 + 30c_0^2c_1^2c_2^2c_3^3c_4s^4t^2 - c_0^5c_1c_3c_4^2c_5s^4t^2 + 140c_0^2c_1^2c_2c_3^2c_4^2c_5s^4t^2 + 30c_0^2c_1^2c_3c_4^3c_5^2s^4t^2 - c_0^3c_1c_2^3c_3^3s^3t^3 + 27c_1^2c_2^4c_3^4s^3t^3 - c_0^6c_2c_3c_4c_5s^3t^3 + 41c_0^3c_1c_2^2c_3^2c_4c_5s^3t^3 - 36c_1^2c_2^3c_3^3c_4c_5s^3t^3 + 41c_0^3c_1c_2c_3c_4^2c_5^2s^3t^3 + 2c_1^2c_2^2c_3^2c_4^2c_5^2s^3t^3 - c_0^3c_1c_4^3c_5^3s^3t^3 - 36c_1^2c_2c_3c_4^3c_5^3s^3t^3 + 27c_1^2c_4^4c_5^4s^3t^3 - c_0^4c_2^3c_3^2c_5s^2t^4 + 36c_0c_1c_2^4c_3^3c_5s^2t^4 + 10c_0^4c_2^2c_3c_4c_5^2s^2t^4 - 52c_0c_1c_2^3c_3^2c_4c_5^2s^2t^4 - c_0^4c_2c_4^2c_5^3s^2t^4 - 52c_0c_1c_2^2c_3c_4^2c_5^3s^2t^4 + 36c_0c_1c_2c_4^3c_5^4s^2t^4 + 8c_0^2c_2^4c_3^2c_5^2st^5 - 32c_0^2c_2^3c_3c_4c_5^3st^5 + 8c_0^2c_2^2c_4^2c_5^4st^5 - 16c_2^5c_3^2c_5^3t^6 + 32c_2^4c_3c_4c_5^4t^6 - 16c_2^3c_4^2c_5^5t^6) \tag{B.12}$$

Fiber 2

The hypersurface equation is:

$$p = -c_0x_0x_1x_2x_3st + c_1x_0^2x_3^2t^3 + c_2x_1^2x_2^2s + c_3x_0x_1x_2x_3s^2 + c_4x_1^3x_3 + c_5x_0x_2^3s^3 + \tag{B.13}$$

Data of the Weierstrass model:

$$f = \left(-\frac{1}{48}\right) \cdot s^2 \cdot (c_3^4s^6 - 4c_0c_3^3s^5t + 6c_0^2c_3^2s^4t^2 - 4c_0^3c_3s^3t^3 - 8c_1c_2c_3^2s^3t^3 + 24c_1c_3c_4c_5s^3t^3 + c_0^4s^2t^4 + 16c_0c_1c_2c_3s^2t^4 - 24c_0c_1c_4c_5s^2t^4 - 8c_0^2c_1c_2st^5 + 16c_1^2c_2^2t^6) \tag{B.14}$$

$$g = \left(-\frac{1}{864}\right) \cdot s^3 \cdot (-c_3^6s^9 + 6c_0c_3^5s^8t - 15c_0^2c_3^4s^7t^2 + 20c_0^3c_3^3s^6t^3 + 12c_1c_2c_3^4s^6t^3 - 36c_1c_3^3c_4c_5s^6t^3 - 15c_0^4c_3^2s^5t^4 - 48c_0c_1c_2c_3^3s^5t^4 + 108c_0c_1c_3^2c_4c_5s^5t^4 + 6c_0^5c_3s^4t^5 + 72c_0^2c_1c_2c_3^2s^4t^5 - 108c_0^2c_1c_3c_4c_5s^4t^5 - c_0^6s^3t^6 - 48c_0^3c_1c_2c_3s^3t^6 - 48c_1^2c_2^2c_3^2s^3t^6 + 36c_0^3c_1c_4c_5s^3t^6 + 144c_1^2c_2c_3c_4c_5s^3t^6 - 216c_1^2c_4^2c_5^2s^3t^6 + 12c_0^4c_1c_2s^2t^7 + 96c_0c_1^2c_2^2c_3s^2t^7 - 144c_0c_1^2c_2c_4c_5s^2t^7 - 48c_0^2c_1^2c_2^2st^8 + 64c_1^3c_2^3t^9) \tag{B.15}$$

$$\Delta = \left(\frac{1}{16}\right) \cdot c_5^2 \cdot c_4^2 \cdot c_1^3 \cdot t^9 \cdot s^9 \cdot (-c_2c_3^4s^6 + c_3^3c_4c_5s^6 + 4c_0c_2c_3^3s^5t - 3c_0c_3^2c_4c_5s^5t - 6c_0^2c_2c_3^2s^4t^2 + 3c_0^2c_3c_4c_5s^4t^2 + 4c_0^3c_2c_3s^3t^3 + 8c_1c_2^2c_3^2s^3t^3 - c_0^3c_4c_5s^3t^3 - 36c_1c_2c_3c_4c_5s^3t^3 + 27c_1c_4^2c_5^2s^3t^3 - c_0^4c_2s^2t^4 - 16c_0c_1c_2^2c_3s^2t^4 + 36c_0c_1c_2c_4c_5s^2t^4 + 8c_0^2c_1c_2^2st^5 - 16c_1^2c_2^3t^6) \tag{B.16}$$

Fiber 3

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0x_1x_2t^2 + c_2x_0^2x_1^2s^3 + c_3x_2^2t + c_4x_0^2x_2s^3 + c_5x_1^2x_2t + \tag{B.17}$$

Data of the Weierstrass model:

$$f = \left(-\frac{1}{48}\right) \cdot t^2 \cdot (16c_2^2c_3^2s^6 - 16c_2c_3c_4c_5s^6 + 16c_4^2c_5^2s^6 - 8c_0^2c_2c_3s^5t - 8c_0^2c_4c_5s^5t + c_0^4s^4t^2 + 16c_0c_1c_2c_3s^4t^2 + 16c_0c_1c_4c_5s^4t^2 - 4c_0^3c_1s^3t^3 - 8c_1^2c_2c_3s^3t^3 - 8c_1^2c_4c_5s^3t^3 + 6c_0^2c_1^2s^2t^4 - 4c_0c_1^3st^5 + c_1^4t^6) \tag{B.18}$$

$$g = \left(-\frac{1}{864}\right) \cdot t^3 \cdot (4c_2c_3s^3 + 4c_4c_5s^3 - c_0^2s^2t + 2c_0c_1st^2 - c_1^2t^3) \cdot (16c_2^2c_3^2s^6 - 40c_2c_3c_4c_5s^6 + 16c_4^2c_5^2s^6 - 8c_0^2c_2c_3s^5t - 8c_0^2c_4c_5s^5t + c_0^4s^4t^2 + 16c_0c_1c_2c_3s^4t^2 + 16c_0c_1c_4c_5s^4t^2 - 4c_0^3c_1s^3t^3 - 8c_1^2c_2c_3s^3t^3 - 8c_1^2c_4c_5s^3t^3 + 6c_0^2c_1^2s^2t^4 - 4c_0c_1^3st^5 + c_1^4t^6) \tag{B.19}$$

$$\Delta = \left(-\frac{1}{16}\right) \cdot c_5^2 \cdot c_4^2 \cdot c_3^2 \cdot c_2^2 \cdot t^6 \cdot s^{12} \cdot (16c_2^2c_3^2s^6 - 32c_2c_3c_4c_5s^6 + 16c_4^2c_5^2s^6 - 8c_0^2c_2c_3s^5t - 8c_0^2c_4c_5s^5t + c_0^4s^4t^2 + 16c_0c_1c_2c_3s^4t^2 + 16c_0c_1c_4c_5s^4t^2 - 4c_0^3c_1s^3t^3 - 8c_1^2c_2c_3s^3t^3 - 8c_1^2c_4c_5s^3t^3 + 6c_0^2c_1^2s^2t^4 - 4c_0c_1^3st^5 + c_1^4t^6) \tag{B.20}$$

Fiber 4

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_0^3 + c_2x_1^3x_2t + c_3x_2^2s^3t^2 + c_4x_1^3x_2s + c_5x_2^2s^2t^3 + \tag{B.21}$$

Data of the Weierstrass model:

$$f = \left(\frac{1}{48}\right) \cdot c_0 \cdot t^3 \cdot s^3 \cdot (24c_1c_3c_4s^2 - c_0^3st + 24c_1c_2c_3st + 24c_1c_4c_5st + 24c_1c_2c_5t^2) \tag{B.22}$$

$$g = \left(\frac{1}{864}\right) \cdot t^4 \cdot s^4 \cdot (216c_1^2c_3^2c_4^2s^4 - 36c_0^3c_1c_3c_4s^3t + 432c_1^2c_2c_3^2c_4s^3t + 432c_1^2c_3c_4^2c_5s^3t + c_0^6s^2t^2 - 36c_0^3c_1c_2c_3s^2t^2 + 216c_1^2c_2^2c_3^2s^2t^2 - 36c_0^3c_1c_4c_5s^2t^2 + 864c_1^2c_2c_3c_4c_5s^2t^2 + 216c_1^2c_4^2c_5^2s^2t^2 - 36c_0^3c_1c_2c_5st^3 + 432c_1^2c_2^2c_3c_5st^3 + 432c_1^2c_2c_4c_5^2st^3 + 216c_1^2c_2^2c_5^2t^4) \tag{B.23}$$

$$\Delta = \left(\frac{1}{16}\right) \cdot c_1^3 \cdot t^8 \cdot s^8 \cdot (c_4s + c_2t)^3 \cdot (c_3s + c_5t)^3 \cdot (27c_1c_3c_4s^2 - c_0^3st + 27c_1c_2c_3st + 27c_1c_4c_5st + 27c_1c_2c_5t^2) \tag{B.24}$$

Fiber 5

The hypersurface equation is:

$$p = -c_0x_0x_1x_2st + c_1x_1^3 + c_2x_2^3st^2 + c_3x_0^3s^2t + c_4x_0^2x_2s^2t + c_5x_0x_2^2st^2 + \tag{B.25}$$

Data of the Weierstrass model:

$$f = \left(\frac{1}{48}\right) \cdot c_0 \cdot t^4 \cdot s^4 \cdot (-c_0^3 - 216c_1c_2c_3 + 24c_1c_4c_5) \tag{B.26}$$

$$g = \left(-\frac{1}{864}\right) \cdot t^5 \cdot s^5 \cdot (864c_1^2c_2c_4^3s^2 - c_0^6st + 540c_0^3c_1c_2c_3st + 5832c_1^2c_2^2c_3^2st + 36c_0^3c_1c_4c_5st - 3888c_1^2c_2c_3c_4c_5st - 216c_1^2c_4^2c_5^2st + 864c_1^2c_3c_5^3t^2) \tag{B.27}$$

$$\begin{aligned}\Delta = \left(\frac{1}{16}\right) \cdot c_1 \cdot t^{10} \cdot s^{10} \cdot (&432c_1^3c_2^2c_4^6s^4 - c_0^6c_1c_2c_4^3s^3t + 540c_0^3c_1^2c_2^2c_3c_4^3s^3t + 5832c_1^3c_2^3c_3^2c_4^3s^3t + 36c_0^3c_1^2c_2c_4^4c_5s^3t \\ &- 3888c_1^3c_2^2c_3c_4^4c_5s^3t - 216c_1^3c_2c_4^5c_5^2s^3t - c_0^9c_2c_3s^2t^2 + 81c_0^6c_1c_2^2c_3^2s^2t^2 - 2187c_0^3c_1^2c_2^3c_3^3s^2t^2 + 19683c_1^3c_2^4c_3^4s^2t^2 \\ &+ 45c_0^6c_1c_2c_3c_4c_5s^2t^2 - 243c_0^3c_1^2c_2^2c_3^2c_4c_5s^2t^2 - 26244c_1^3c_2^3c_3^3c_4c_5s^2t^2 - 513c_0^3c_1^2c_2c_3c_4^2c_5^2s^2t^2 \\ &+ 7290c_1^3c_2^2c_3^2c_4^2c_5^2s^2t^2 - c_0^3c_1^2c_4^3c_5^3s^2t^2 + 1836c_1^3c_2c_3c_4^3c_5^3s^2t^2 + 27c_1^3c_4^4c_5^4s^2t^2 - c_0^6c_1c_3c_5^3st^3 + 540c_0^3c_1^2c_2c_3^2c_5^3st^3 \\ &+ 5832c_1^3c_2^2c_3^3c_5^3st^3 + 36c_0^3c_1^2c_3c_4c_5^4st^3 - 3888c_1^3c_2c_3^2c_4c_5^4st^3 - 216c_1^3c_3c_4^2c_5^5st^3 + 432c_1^3c_3^2c_5^6t^4)\end{aligned} \tag{B.28}$$

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ÉCOLE DOCTORALE
Physique en île-de-France (EDPIF)

Titre: Théorie F en huit dimensions : une perspective de la Théorie des Champs Exceptionnels et de la Théorie des Cordes Hétérotique

Mots clés: Théorie F, Théorie des Champs Exceptionnels, Corde Hétérotique, Dualités

Résumé: L'une des théories les plus prometteuses qui vise à unifier la mécanique quantique et la relativité générale est actuellement la théorie des cordes. La recherche d'une formulation supersymétrique des cordes a conduit à cinq théories de supercordes cohérentes en dix dimensions qui ont ensuite été unifiées dans le cadre de la théorie M. L'aspect fondamental de cette unification est la découverte d'un réseau de dualités entre les cinq théories des supercordes et la supergravité à onze dimensions.

Cette thèse aborde les dualités dans le contexte de la théorie F en huit dimensions. La théorie F est douze dimensionnelle et fournit une formulation non perturbative de la supergravité de type IIB avec 7-branes. La théorie des champs exceptionnels, quant à elle, fournit une formulation U-dual de la supergravité de type IIB. Nous nous concentrons donc sur les liens possibles entre ces deux formulations. La théorie F est également supposée être duale à la théorie des cordes hétérotique en 8 dimensions. La structure des groupes de jauge apparaît radicalement différemment dans ces deux formulations. Dans la théorie F, elle est interprétée comme un choix particulier de structure algébrique d'une surface K3 elliptique, tandis que dans le cadre de la corde hétérotique, elle est principalement déterminée par les lignes de Wilson. Bien qu'étudiée dans le contexte de la théorie des cordes de type IIB, l'identification entre les modules de la théorie F et de la théorie des cordes hétérotique n'est que peu connue.

Dans la première partie de cette thèse, nous présentons les notions de base de la théorie des cordes, des compactifications, des branes et des dualités. Dans la seconde, nous montrons que la théorie des champs exceptionnels $\mathbb{R}^+ \times E_{3(3)}$ en huit dimensions présente des aspects de la théorie F dans un cadre spécifique, et permet en particulier de décrire les monodromies des (p, q) 7-branes. Enfin, dans la troisième partie, nous étudions la dualité entre la compactification de la théorie F sur une surface K3 elliptique et la corde hétérotique sur un deux-tore. Nous présentons comment construire des surfaces K3 elliptiques via des polyèdres réflexifs qui peuvent être interprétés en termes de lignes de Wilson dans la théorie des cordes hétérotique duale.

Title: F-theory in Eight Dimensions: an Exceptional Field Theory and Heterotic String Perspective

Keywords: F-theory, Exceptional Field Theory, Heterotic String, Dualities

Abstract: One of the most promising theories to unify quantum mechanics and general relativity is currently string theory. The search for a supersymmetric formulation of strings led to five consistent ten dimensional superstring theories which were later unified under the scope of M-theory. The fundamental aspect of this unification is the discovery of a web of dualities between the five superstring theories and eleven dimensional supergravity.

This thesis addresses dualities in the context of F-theory in eight dimensions. F-theory is twelve dimensional and provides a non-perturbative formulation of type IIB supergravity with 7-branes. On the other hand exceptional field theory provides a U-dual formulation of type IIB supergravity and we therefore focus on the possible links between these two formulations. F-theory is also conjectured to be dual to the heterotic string in 8 dimensions. The gauge group appears radically differently in these two formulations. In F-theory it is interpreted as a particular algebraic structure of an elliptically fibered K3 surface, while on the heterotic string it is principally determined the Wilson lines. Although studied in the context of type IIB string theory, the explicit map between the moduli in F-theory and its heterotic dual are still quite unknown.

In the first part of this thesis we present basic notions of string theory, compactifications, branes and dualities. In the second one, we show that $\mathbb{R}^+ \times E_{3(3)}$ exceptional field theory in eight dimensions can incorporate aspects of F-theory in a specific setting, and in particular describes the monodromies of (p, q) 7-branes. Finally, in the third part we study the duality between F-theory compactified on an elliptic K3 surface and the heterotic string on a two-torus. We present how to construct elliptically fibered K3 surfaces via reflexive polytopes which can be understood in terms of Wilson lines in the dual heterotic string theory.

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