Rapid stabilization of Burgers equations and of Korteweg-de Vries equations

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[1]'观书有感', poème de *Xi Zhu*. Traduction : "Pourquoi est-il tant limpide,cet étang ? –De la source vient de l'eau fraîche, continuellement."

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[2]–Ernest Hemmingway
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à mes parents
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Contents

Chapter 1

Introduction

Contents

1.1 Basic definitions

Control theory is a subject devoted to the study of systems whose dynamics can be modified by using suitable controls. The major issues of control theory are controllability problems and stabilization problems [Lio88, Cor07a]. A major goal of this thesis is to understand global controllability and stabilization of two equations used in fluid mechanics (PDE models) with nonlinear phenomena, namely the Korteweg-de Vries equation and the Burgers equation.

The controllability problem consists in looking for controls allowing to move the control system from a given state to a given desired target. Moreover, does there exist an optimal control [Tr5]?

Another central problem is the stabilization issue, especially the relation between controllability and stabilization. In controllability problems, the control depends on the initial data, time and the desired target. Such controls are called open-loop controls. In stabilization problems, the control does not depend anymore on the initial data. Instead it depends, at each time t, on the state at this time t. Such controls are called feedback laws. They very important since they are much more robust to perturbations than open-loop controls. A natural example for this is the problem of balancing an upturned broomstick on the end of one's finger.

1.1.1 Two simple evolution equations as example

Before stating definitions and results, let us have a look at two simple equations, keeping in mind that controllability means to move the state to a given target, and that stabilization means to make the system more stable.

1.1.1.1 The equation $\dot{x} = x$

The evolution of $\dot{x} = x$ is quite clear, as we can directly get the solution. Of course this system is neither controllable nor stable. We investigate control problems by adding a control term,

$$\dot{x} = x + f, x \in \mathbb{R} \tag{1.1.1}$$

where f is the control (or feedback law for stabilization problems).

A simple calculation yields the controllability of (1.1.1).

As for the stabilization, we define a feedback law $f(x) := -2x$ and get

$$\dot{x} = -x.$$

The above system is exponentially stable with exponential decay rate 1. Moreover, if we replace $f := -2x$ by $f := -nx$ with n large enough, we get rapid stabilization (exponential stabilization with an arbitrary large exponential decay rate).
However, these feedback laws do not lead to finite time stabilization (the state does not go to zero in finite time). To get stabilization in finite time one may consider either *time-varying* feedback laws or *nonlinear* feedback laws.
Because the term x can be absorbed by the source term f, we consider the stabilization of $\dot{x} = f$ from now on.

- Time-varying feedback law

As we can see from the above example, $f := -nx$ is a stationary feedback law which "only" leads to rapid stabilization. We try to stabilize the system by means of time-varying feedback laws.
An idea is to consider piecewise stationary feedback law, for example,

$$f := -2^{n+1}x, \text{ for } t \in [1 - 2^{-n}, 1 - 2^{-(n+1)}).$$

With the help of (1.1.2), the state x goes to zero in finite time. By using the same approach we can even let the state x go to zero in any small time (small-time stabilization).

- A local aspect:

As linear stationary feedback law does not lead to finite time stabilization, what about nonlinear (stationary) feedback laws? If we consider the evolution of

$$\dot{x} = -x^{1/3},$$

then we have

$$\frac{3}{2}\mathrm{d}x^{\frac{2}{3}} = x^{-\frac{1}{3}}\mathrm{d}x = -\mathrm{d}t,\ \text{ if } x > 0.$$

Formally

$$x^{\frac{2}{3}}(T) = x^{\frac{2}{3}}(0) - 3T/2,$$

hence, $x(T) = 0$ if $x(0)$ is small enough.

By considering a feedback law as $f := -Ax^{1/3}$, we are able to get small-time stabilization of the system.

- A global aspect:

A way of getting global stabilization is the so called universal bound. For example, we consider

$$\dot{x} = -x^3,$$

then we have

$$\frac{1}{2}\mathrm{d}x^{-2} = -x^{-3}\mathrm{d}x = \mathrm{d}t,\ \text{ if } x > 0.$$

Hence

$$x(T) = (2T + x^{-2}(0))^{-1/2} \leqslant (2T)^{-1/2}.$$

Combining both local and global aspects, we can construct a nonlinear stationary feedback law which stabilizes the system in finite time,

$$f := -x^3 - x^{1/3}.$$

1.1.1.2 The one dimemsional heat equation

The heat equation is a very good example for the control of PDEs, not only because the *Laplace operator is the simplest and the best operator*[1] but also because *it is a goose that laid the golden eggs*.

Let us consider the following controlled heat equation

$$y_t - y_{xx} = 0,\ \text{ in } [0,T]\times(0,1), \tag{1.1.2}$$
$$y_x(t,0) = 0,\ \ y_x(t,1) = u(t),\ \text{ in } [0,T], \tag{1.1.3}$$

with $u(t) \in \mathbb{R}$ as control. The first controllability result was proved in the 70's by Fattorini and Russell [FR71]: in dimension one, the heat equation with boundary control is controllable. The multiple dimensional heat equation is also controllable. It was proved by Lebeau and Robbiano [LR95a, LR95b], where they introduced the Lebeau-Robbiano strategy. At the same time, an alternative method was given by Fursikov and Imanuvilov [FI96]. This was the first time that global Carleman estimates were used to prove observability inequalities. They have become one of the most commonly used methods.

Concerning stabilization issues, the exponential stabilization of the heat equation is well-known.

[1] Nalini Anantharaman's plenary talk at ICM 2018.

For simplicity we consider the heat equation with a Dirichlet control

$$y_t - y_{xx} = 0 \text{ in } [0,T]\times(0,1), \tag{1.1.4}$$
$$y(t,0)=0, \ \ y(t,1)=u(t), \text{ in } [0,T]. \tag{1.1.5}$$

If we set $u(0)$ as zero, then Laplace operator and Poincare's inequality lead to a natural exponential dissipation of the energy.

We also know a method for rapid stabilization: the backstepping approach as introduced by Krstic and his co-authors [LK00]. An excellent book to enter inside this method is [KS08a] by Krstic and Smyshlyaev. In the framework of our heat equation it consists in looking for linear maps which transform our heat equation into a new heat equation having a stronger dissipation. More precisely, for every $\lambda > 1$, there exist a stationary feedback law $U_\lambda(y)$ and a Volterra transformation of the secund kind $L_\lambda : L^2(0,1) \longrightarrow L^2(0,1)$ (which is therefore invertible) such that if y is the solution of equation (1.1.4) with feedback law $U_\lambda(y)$, then $z := L_\lambda y$ satisfies

$$z_t - z_{xx} + \lambda z = 0 \text{ in } [0,T]\times(0,1),$$
$$z(t,0)=0, \ \ z(t,1)=0, \text{ in } [0,T].$$

Hence

$$\|y(t)\|_{L^2}\leqslant C_\lambda e^{-\lambda t}\|y(0)\|_{L^2}, \forall t\geqslant 0. \tag{1.1.6}$$

There are many other important works for rapid stabilization of the heat equation [Bar18, CT04]. However, the small-time stabilization of the one dimensional heat equation remained open until Coron and Nguyen's paper [CN17], where they introduced a piecewise backstepping approach and pointed out that

$$C_\lambda := \|L_\lambda\|\|L_\lambda^{-1}\|\leqslant e^{c\sqrt{\lambda}}, \tag{1.1.7}$$

where the constant c is independent of $\lambda > 1$. One can actually get some intuition from Section 1.1.1.1, where we introduce a piecewise feedback law to achieve finite-time stabilization. We look for sequences

$$\lambda_n \longrightarrow +\infty \text{ and } t_n \longrightarrow 1^-,$$

and a piecewise feedback law

$$U(t,y) := U_{\lambda_n}(y), t\in[t_n, t_{n+1}),$$

such that $y(t)$ tends to 0 thanks to inequality (1.1.7).

1.1.2 Well-posedness, stability, and also controllability, stabilization

Let

$$\dot{x}(t) = f(t, x(t)). \tag{1.1.8}$$

be an evolution equation or a discrete dynamical system. Many important models can be written by this general form: ODE, evolution PDE (the heat equation, Navier-Stokes equation, Euler equation *etc.*).

The most frequently asked questions concerning system (1.1.8) are the following:

- *Existence and uniqueness of solutions*: does equation (1.1.8) have solutions? Strong solution, weak solution, or mild solution? If yes, is the solution unique?
- *Regularity*: if equation (1.1.8) has solutions, then what is the regularity of these solutions?
- *Stability*: if equation (1.1.8) has solutions, then will the solutions converge to equilibrium

points ($f(t, x_e) = 0$). If yes, what is the convergence rate? Asymptotical, exponential, or in finite time?

Definition 1 (Uniform asymptotic stability)**.** *Suppose that* 0 *is an equilibrium point of system* (1.1.8)*. One says that 0 is locally asymptotically stable if*

(i) (uniform stability) for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $s \in \mathbb{R}$ and for every $\tau \geqslant s$,

$$(\dot{x} = f(t, x(t)), |x(s)| < \delta) \Rightarrow (|x(\tau)| < \varepsilon), \tag{1.1.9}$$

(ii) (asymptotic decay) there exists $a > 0$ such that, for every $\varepsilon > 0$, there exists M_ε such that, for every $s \in \mathbb{R}$,

$$(\dot{x} = f(t, x(t)), |x(s)| < a) \Rightarrow (|x(\tau)| < \varepsilon, \forall \tau > s + M_\varepsilon). \tag{1.1.10}$$

Note that to simplify notations in the above definition, $|x|$ may represent any norm (or topology).

- *Blow up, soliton, limiting problem, stochastic etc.*

Different from the above problems which are based on properties of given solutions, in control theory we change the dynamics of the solutions thanks to flexible control terms. We add an extra control term to system (1.1.8), and study

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1.1.11}$$

with $x(t) \in H$ the state and $u(t) \in U$ the control.

- *Controllability:* for any given $x_0, x_1 \in H$, does there exist a trajectory which connects them in a suitable sense (the sense of the solution and the sense of controllability)? For example,

  Definition 2 (Exact controllability)**.** *We say that system* (1.1.11) *is exact controllable in time $T > 0$ if and only if, for any $x_0 \in H$ and $x_1 \in H$, there exists a control $t \in [0, T] \mapsto u(t) \in U$ such that the solution, $x(t)$, of* (1.1.11) *with $x(0) = x_0$ satisfies $x(T) = x_1$.*

- *Stabilization:* beyond the study of stability of system (1.1.8), with the help of feedback laws are we able to stabilize a system which is not stable (Section 1.1.1.1 provides a good example), or make a system "more stable" (see Section 1.1.1.2)? More precisely, by adding a feedback law $U(t, x(t))$ we study the stability of the system

  $$\dot{x}(t) = f(t, x(t), U(t, x(t))).$$

  It is equivalent to investigate the stability of

  $$\dot{x}(t) = \tilde{f}(t, x(t)) \text{ with } \tilde{f}(t, x(t)) := f(t, x(t), U(t, x(t))).$$

In control theory we also care about the well-posedness issue, as it ensures the existence of the objects we investigate. Concerning the well-posedness of control systems, a very classical framework was built by Jacques-Louis Lions, the so called *solution in the transposition sense.* An excellent introduction of this theory is presented by the book of Tucsnak and Weiss [TW09]. For simplicity of presentation, we only consider linear control systems to explain the idea.

For $T > 0$, we are interested in the linear system

$$\begin{cases} \dot{x} = Ax + Bu, \forall t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1.12}$$

where A and B are two operators, which can be bounded or unbounded. Like always we consider $x \in H$ as state and $u(t) \in \mathcal{U} := L^2((0,T);U)$ as control, with H and U two Hilbert spaces.
Let $S(t)$ be the semigroup of continuous linear operators on H generated by A. If $x_0 \in \mathcal{D}(A)$ and $B \in \mathcal{L}(U;\mathcal{D}(A))$, then thanks to Duhamel's formula, system (1.1.12) with $u \in L^2((0,T);U)$ has a unique strong solution in the space $C([0,T];\mathcal{D}(A)) \cap C^1([0,T];H)$

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds, \forall t \in [0,T]. \tag{1.1.13}$$

Furthermore, if $x_0 \in H$ (rather than $\mathcal{D}(A)$) and $B \in \mathcal{L}(U;H)$, then formula (1.1.13) presents the unique weak solution. However, what if $x_0 \in H$ and $B \in \mathcal{L}(U;\mathcal{D}(A^*)')$? We notice that Bu is not well-defined for some $u \in H$, thus it seems that the natural solution space $C(H) := C([0,T];H)$ is not compatible with operator B. Here we need to consider solutions in the sense of transposition:

$$\begin{aligned}(x(t),y)_H &= (x_0, S(t)^*y)_H \\ &\quad + \int_0^t (u(s), B^*S(t-s)^*y)_U ds, \quad \forall y \in \mathcal{D}(\mathcal{A}^*), \forall t \in [0,T].\end{aligned} \tag{1.1.14}$$

Moreover, for any given $x_0 \in H$ the solution "map" is continuous: $\mathcal{U} \to C(H)$. Transposition solutions are widely used in control theory especially for boundary control problems. That is because under this definition we are able to include solutions whose boundaries are not regular enough. For example, in Section 1.1.1.2 we study the controllability of the heat equation in $L^2(0,1)$ space where $y_x(t,1)$ is not well-defined for some elements from $L^2(0,1)$.

1.2 Controllability methods

1.2.1 Linear model: Hilbert Uniqueness Method

Exact controllability of a linear system can be proved by observability inequalities and the duality between controllability and observability [DR77]. The rest of this section is devoted to Hilbert Uniqueness Method due to Lions [Lio88]. We study the problem of the exact controllability of system (1.1.12) in 3 steps.

Step 1. Transform the control problem into an application problem.
As we have seen for the definition of transposition sense solution, for every $x_0 \in H$ we are able to define a continuous application from $\mathcal{U}$ to H

$$\mathcal{F}_{x_0} : \mathcal{U} \longrightarrow H,$$

with $\mathcal{F}_{x_0}(u) = x(T)$.
Therefore, it suffices to prove that $\mathcal{F}_{x_0}(u)$ is a surjection.

Step 2. Exact controllability $\Longleftrightarrow$ Exact controllability from 0.

Definition 3. *We say that system* (1.1.12) *is exact controllable from 0 in time $T > 0$ if and only if, for any $x_1 \in H$, there exists a control $u(t) \in \mathcal{U}$ such that the solution $x(t)$ of* (1.1.12) *with $x(0) = 0$ satisfies $x(T) = x_1$.*

It is obvious that exact controllability implies exact controllability from 0. The converse assertion can be proved by decomposing $x(t)$ into two parts, $x(t) = x_1(t) + x_2(t)$, where x_1 is the solution of (1.1.12) with $x_0 = 0$, and x_2 is the solution with $u = 0$.

Step 3. The duality between controllability and observability.
Our goal is to show that $\mathcal{F}_0$ is surjective. For the image of an operator between two Hilbert spaces, we first recall the following classical result from functional analysis.

Theorem 1. *Let E and F be two Hilbert spaces. Let $L : \mathcal{D}(L) \subset E \to F$ be a closed operator with dense domain. Then L is onto if and only if there exists $C > 0$ such that*

$$\|f\|_F \leqslant C\|L^*(f)\|_E, \quad \forall f \in F. \tag{1.2.1}$$

The following theorem is a direct application of Theorem 1.

Theorem 2. *System (1.1.12) is*

– *exact controllable if and only if, there exists a constant $C_T > 0$ such that*

$$\|x\|_H \leqslant C_T\|\mathcal{F}^* x\|_E, \forall x \in H; \tag{1.2.2}$$

– *approximate controllable if and only if*

$$(\mathcal{F}^* x = 0, x \in H) \Longrightarrow (x = 0). \tag{1.2.3}$$

Inequality (1.2.2) is the so called observability inequality, and property (1.2.3) is the unique continuation principle.

Different methods were introduced to prove the observability inequalities: multiplier method [Ho86], moments theory [Rus72, Bea05], defect measure [BLR92], global Carleman estimates [FI96].

1.2.2 Nonlinear model: return method *etc.*

In many cases one can obtain local controllability of a nonlinear system whose linearized system around the equilibrium point is controllable, by using standard fixed point arguments. However, this procedure does not work for many important systems. Consider, for example, the two dimensional Euler equation

$$\begin{gathered} v_t + (v \cdot \nabla)v + \nabla p = 0 \text{ in } [0,T] \times \Omega, \\ \text{div } v = 0 \text{ in } [0,T] \times \Omega, \\ v(t,\cdot) \text{ satisfies some boundary conditions on } \Gamma \setminus \Gamma_0, \end{gathered}$$

where $\Gamma = \partial\Omega$, and controls act on the boundary $\Gamma_0 \subset \Gamma$. We observe that $(\bar{v}, \bar{p}) = (0,0)$ is a solution. But the linearized system around this equilibrium point is not controllable. Indeed this linearized system is

$$\begin{gathered} z_t + \nabla q = 0 \text{ in } [0,T] \times \Omega, \\ \text{div } z = 0 \text{ in } [0,T] \times \Omega, \\ z(t,\cdot) \text{ satisfies some boundary conditions on } \Gamma \setminus \Gamma_0, \end{gathered}$$

and, therefore, the vorticity $\omega := \text{curl } v$ satisfies

$$\omega_t(t,x) = 0,$$

which shows that this system is not controllable (the vorticity can not be modified whatever is the control). In this situation the linear part does not lead to the controllability, a natural idea is to ask whether the nonlinear part could help to get the controllability: we need to develop *nonlinear* methods. One of the most commonly used method is the so called “return method” introduced by Coron [Cor93, Cor96]:

• (Jean-Michel Coron) *If the linearized system around an equilibrium point is not controllable, then consider the control systems around other trajectories which start from the equilibrium point and end at the same point, such that the linearized system around these trajectories are controllable.*

Following the idea of the return method, we are looking for another equilibrium $\bar{y} := \nabla\theta$, with θ in $C^\infty(\bar{\Omega})$ satisfies

$$\Delta\theta = 0, \nabla\theta \neq 0, \text{ in } \bar{\Omega}, \tag{1.2.4}$$

$$\frac{\partial\theta}{\partial n} = 0, \text{ on } \Gamma \setminus \Gamma_0. \tag{1.2.5}$$

The vorticity of the linearized system around this equilibrium point satisfies

$$\omega_t + (\bar{y} \cdot \nabla)\omega = 0,$$

which is a transport type equation. Conditions (1.2.4)–(1.2.5) ensures that every point x inside Ω can be transported outside of Ω by passing through Γ_0 where the control acts. Therefore, Euler system is locally controllable around equilibrium $\bar{y}$. Then we construct a trajectory $a(t)\nabla\theta$ satisfying $a(0) = a(T) = 0$, $a(t) = 1\ \forall t \in (1/3, 2/3)$. We can prove local controllability around this trajectory. Due to the pressure term ∇p and the scaling invariant, we can further get global controllability of Euler equation.

In control theory we surprisingly find that many models whose linearized systems are not controllable can be controlled thanks to nonlinear terms, e.g., many ODE models (as the unicycle or the baby stroller), Navier-Stokes equations with reduced forcing terms ([CL14a] with a new algebraic method inspired by the works by Gromov [Gro86, Gro72]), Navier-Stokes equations with finite dimensional controls [AS05, AS06], Euler equations with boundary control ([Cor96] "return method"), KdV equations ([CC04] "power series expansion"), and Schrödinger equations ([Bea05] with the use of Nash-Moser method [Nas56, Mos61, Mos66, H8̈5]).

1.3 Stabilization is different from controllability

1.3.1 Periodic time-varying feedback laws

The stabilization issue is linked with both stability and controllability. It is natural to ask for the best possible type of stabilization we can achieve given a controllability result. Can we stabilize a system which is controllable?

For finite dimensional linear stationary systems, it is shown that the invertibility of the controllability Gramian (on a finite interval) is equivalent to the pole-shifting property (and also to the controllability). Concerning stationary feedback laws, it is proved by Coron and Praly in [CP91] that every equilibrium which has a controllable linearized system around it can be stabilized in small time by means of continuous stationary feedback laws. However, as it was proved by Brockett in [Bro83] (see also [Cor90], there are control systems which are small-time locally controllable at some equilibrium such that this equilibrium cannot be asymptotically stabilized by means of continuous stationary feedback laws. More precisely, Brockett proved the following necessary condition for asymptotic stabilizability by means of continuous stationary feedback laws.

Theorem 3 (Brockett [Bro83])**.** *A necessary condition for the control system $\dot{x} = f(x, u)$ to be locally asymptotically stabilizable at the equilibrium point $(x_0, 0)$ by a continuous stationary feedback law vanishing at x_0 is that the image by f of any neighborhood of $(x_0, 0)$ is a neighborhood of 0.*

Let us point out that there are control system $\dot{x} = f(x, u)$ which are small-time locally controllable at the equilibrium point $(x_0, 0)$ which does not satisfy the above necessary Brockett condition. Moreover, it was previously pointed out in [Sus79] that a system which is globally controllable may not be globally asymptotically stabilizable by means of continuous stationary feedback laws. To overcome this obstruction a strategy is to use, instead of continuous stationary feedback laws, continuous (with respect to the state) time-varying feedback laws. This seems to be the "right" class of feedback laws for stabilization issues since it is shown by Coron in [Cor95] that, in finite dimension, many powerful sufficient conditions for small-time local controllability imply the existence of feedback laws which locally stabilize the system in small time.

Another challenging issue of stabilization problems is that the starting time could be any time, *i.e.* see Definition 1 for uniform stability. We seek for periodic time-varying feedback laws, under which the starting time belong to an interval $[0, T)$ rather than $\mathbb{R}$.

What about nonlinear systems? What about (linear and nonlinear) PDE models? It seems difficult to get a general theory to the one given in [Cor95] for finite dimensional control system. In order to build a bridge between controllability and stabilization especially in PDE level, it is reasonable to consider the following three important cases.

1.3.2 Three important but less studied problems

Concerning stabilization of control systems in infinite dimension (PDE models), the classical linearization technique works well around equilibrium points. For example, if we are able to construct a linear feedback law such that the linearized system is asymptotically stable with this feedback law, one may hope that the same linear feedback law is going to stabilize asymptotically the non linear system. Sometimes it can be proved by a Lyapunov approach: a Lyapunov function for the closed-loop linear system is sometimes also a Lyapunov function for the closed-loop nonlinear system. See, for example, [BCHS17, BC16] for 1-D quasilinear hyperbolic systems on an interval. However, since this technique is based on the perturbation and the linearized system, it is clear that this method does not work for the following important cases.

(1) *Stabilization of nonlinear systems whose linearized system are not asymptotically stabilizable.*
Many important models fall in this class. For example, every nonlinear system whose linearized form is not controllable, which includes Euler equations, Schrödinger equations, Saint-Venant equations *etc.*

(2) *Small-time local stabilization of linear controllable systems.*
Small-time local controllability is well studied (see Section 1.2.1), different methods are introduced for this issue. But results on related small-time stabilization problems are very limited. As we have seen in Section 1.1.1.2, small-time local stabilization of the one dimensional heat equation was solved very recently. It is possible that time-varying feedback laws can always allow to get small-time stabilization for this kind of systems. It might seem strange to seek a local result for a linear system. That is because of a starting time problem which does not exist in controllability problems. We refer to Section 1.3.3.4 and Section1.3.3.5 for details on it.

(3) *Small-time global stabilization of nonlinear controllable systems.*
Small-time global controllability of nonlinear systems is an interesting and important subject. Further studies are related to boundary layer problems, such as for Navier-Stokes equations.

This thesis is devoted to the study of stabilization of three typical models where standard linear perturbation theory can not be applied. More precisely, Chapter 2 deals with a KdV control system with a Neumann boundary control, which is included in Case (1); Chapter 3–4 deals with small-time local stabilization of a KdV system with Dirichlet boundary control, which is included in Case (2); and Chapter 5 handles a typical example for Case (3), the viscous Burgers equation. The problems we meet for these models are representative. And the methods that we present could be applied to other models. It helps for the understanding of the links between controllability and time-varying feedback stabilization.

1.3.3 Beyond *a priori* estimates, several typical problems and our solutions

Well-posedness of closed-loop systems

Well-posedness is a significant issue in the study of PDEs. Control theory consists in studying the dynamic of the solution, thus we also need to care about well-posedness. Controllability problems are based on open-loop systems, for which we can normally use regular controls (even smooth controls) so that several known well-posedness results can be used. For instance, if the initial boundary value problem is well investigated, then we let the control on the boundary be regular enough such that there exists a unique solution.
However, stabilization is the action of designing feedback terms to make the origin be more stable. In other words, we study the closed-loop system

$$\dot{x}(t) = f(t, x(t), U(t, x(t))) = \tilde{f}(t, x(t)), \tag{1.3.1}$$

which is clearly different from the original one. As soon as a feedback law $U(t, x(t))$ is chosen, the required stability of system (1.3.1) is fulfilled by *a prior* estimates. It suffices to prove the existence of solutions to make sure that the feedback law is "well-designed".

1.3.3.1 Non-Lipschitz feedback laws

Even though $\tilde{f}$ is different from f, classical methods can still be adapted for well-posedness problems: energy methods, Galerkin iteration, fixed point arguments *etc.*
However, in many situations we have to construct non-Lipschitz feedback laws. Let us recall Section 1.1.1.1 as example, where we constructed a feedback law $-x^{1/3}$ which is not Lipschitz around 0. Generally, a lack of Lipschitz condition may result in non-uniqueness of solutions. From the definition of stabilization and the engineering point of view this is not a problem. However, from the mathematical point of view, this prevents the application of powerful Banach fixed point arguments to show the existence of solutions.

How can we solve equation (1.3.1) when f (or $\tilde{f}$) are not Lipschitz? We can borrow some ideas from ODE theory. Actually, there is a strategy introduced by Carathéodory to solve ordinary differential equations $\dot{x} = f(t, x)$ when f is not smooth. Roughly speaking it consists in solving $\dot{x} = f(t, x(t-h))$ where h is a positive time-delay (the solution can be obtained by integration), and then pass the limit by letting h tend to 0.
We try to adapt this strategy to PDE models. Here we do not put the time-delay on x: it does not seem to be possible, as in this case we break essential semi-group structures of operators in PDE theory. Based on the fact that the well-posedness of original system $\dot{x} = f(t, x)$ and of open-loop control system $\dot{x} = f(t, x, u)$ are always known (if not, the first purpose of this model is well-posedness), we study $\tilde{f}$ by regarding it as $f(t, x, U(t, x))$. In such a case, we can benefit from known theories. The idea is to put a time delay on the feedback law: $U(t, x(t))$ is replaced by $U(t, x(t-h))$.

Step 1. *Carathéodory setting.* We say that $U(t, x) : \mathbb{R} \times H \to \mathbb{R}$ is a Carathéodory map if it satisfies the three following properties

$$\forall R > 0, \exists\, C_B(R) > 0 \text{ such that } \left(\|x\|_H \leqslant R \Rightarrow |U(t, x)| \leqslant C_B(R), \;\; \forall t \in \mathbb{R}\right), \tag{1.3.2}$$

$$\forall x \in H, \text{ the function } t \in \mathbb{R} \mapsto U(t, x) \in \mathbb{R} \text{ is measurable}, \tag{1.3.3}$$

$$\text{for almost every } t \in \mathbb{R}, \text{ the function } x \in H \mapsto U(t, x) \in \mathbb{R} \text{ is continuous}. \tag{1.3.4}$$

Step 2. *Time-decay system.* Let $h > 0$. Let us set $x(t) := x_0, \forall t \in [0, h]$. For $t \in (h, 2h]$, the system $\dot{x} = f(t, x, U(t, x(t-h)))$ is equivalent to the in-homogeneous open-loop system $f(t, x, \tilde{u})$ with $\tilde{u}$ given. Hence we are able to solve $x(t)$ on $(h, 2h]$. Then we continue this procedure to get solutions on $[0, nh]$.

Step 3. *A priori estimates and pass the limit.* This part is standard. Some key points are *a priori* estimates and Carathéodory setting.

This method was first introduced in paper [CRX17] which is also Chapter 2 of this thesis. As we can see, this method is generic, one can apply it on different problems of well-posedness for closed-loop systems.

1.3.3.2 Global solutions: maximal solutions

Focusing on stabilization issues, we need to prove the existence of global solutions (with respect to time). If there is a unique solution, we can simply call this solution a flow of the system. But when the solutions are not unique, how can we tell different solutions apart? One way of distinguishing them is the maximal solution.
Let us suppose that the initial time is s and the initial data is x_0. For system

$$\dot{x} = f(t, x, U(t, x)), \tag{1.3.5}$$

and for the Cauchy problem

$$\dot{x} = f(t, x, U(t, x)) \text{ with } x(s) = x_0, t \in (s, +\infty), \tag{1.3.6}$$

we give the following definitions.

Definition 4. *Let I be an interval of $\mathbb{R}$ with a nonempty interior. A function x is a solution of (1.3.5) on I if $x \in C^0(I; H)$ is such that, for every $[T_1, T_2] \subset I$ with $-\infty < T_1 < T_2 < +\infty$, the restriction of x to $[T_1, T_2] \times (0, L)$ is a solution of (1.3.5). A function x is a solution to the Cauchy problem (1.3.6) if there exists an interval I with a nonempty interior satisfying $I \cap (-\infty, s] = \{s\}$ such that $x \in C^0(I; H)$ is a solution of (1.3.5) on I and satisfies the initial condition $x(s) = x_0$. The interval I is denoted by $D(x)$. We say that a solution y to the Cauchy problem (1.3.6) is maximal if, for every solution y to the Cauchy problem (1.3.6) such that*

$$D(x) \subset D(y), \tag{1.3.7}$$
$$x(t) = y(t) \textit{ for every } t \textit{ in } D(y), \tag{1.3.8}$$

one has

$$D(x) = D(y). \tag{1.3.9}$$

This is a general definition (or rather a general idea) of maximal solutions, one can of course modify spaces according to specific settings. This idea was first introduced in [Cor99], and it is also used in our paper [CRX17] which is Chapter 2 of this thesis.

1.3.3.3 Flow and proper feedback laws

Flow is one of the most important ideas in mathematics and physics, *e.g.* Anosov flow, heat flow, Ricci flow. It can be generally regarded as a continuous motion of the state over time. In ODE theory local existence of flow is always due to Lipschitz-continuous of the vector field. It is difficult to show that a flow is globally defined, one criterion is that the vector field is compactly supported. In the PDE setting, for example the Laplace operator generates a flow which is global thanks to some compactness arguments. We call feedback laws proper if the corresponding closed-loop system has a global flow.

Definition 5. *A periodic feedback law $U(t, x)$ is called proper if, for any s and any x_0 the Cauchy problem (1.3.6) has a unique global solution, and this solution is continuous with respect to time, i.e. $x \in C^0([s, +\infty); H)$.*

This generic definition was introduced in paper [CX18] which is Chapter 5 of this thesis.

Uniform stability
Another important issue of stabilization is uniform stability, as we have seen in Definition 1. Let us illustrate it with the example in Section 1.1.1.2. Actually, the piecewise feedback law that we construct in Section 1.1.1.2 is not a good one.
In fact, if the starting time is $s = 0$, then there exists a constant C such that

$$\|y(t)\|_{L^2} \leqslant C\|y(0)\|_{L^2}, \forall t \in [0, +\infty).$$

However, there is a problem if the starting time is not 0. Let $s := t_n$ for instance. Since the feedback law is defined as U_{λ_n} on time interval $[t_n, t_{n+1})$, we have

$$\|y(t)\|_{L^2} \leqslant e^{c\sqrt{\lambda_n}}\|y(t_n)\|_{L^2}, \forall t \in [t_n, t_{n+1}). \tag{1.3.10}$$

In such a case, uniform stability is not satisfied, because of the cost term in (1.3.10). The cost term comes from the fact that $|U(t_n, y)| \sim e^{c\sqrt{\lambda_n}}|y|$. In order to avoid this problem, an idea is to set a uniform bound on the feedback law. However, if we set such a bound on the feedback law, then another problem concerning the well-posedness issue will appear.

Lemma 1 (Lions-Magenes [LM73])**.** *Let $y_0 \in L^2$. The Cauchy problem (1.1.4)–(1.1.5) with $y(0) = y_0$ has a unique solution $y \in C^0([0, T]; L^2) \cap L^2([0, T]; H^1)$ if $u \in H^{1/4}$.*

As we can see from the above theorem, we even have a problem upon the existence of solutions when the control is L^∞.

1.3.3.4 The maximum principle

In [CN17] Coron and Nguyen find that the maximum principle can solve this problem. They proved that if $|U(t, y)| \leqslant \sqrt{y}$, the linear heat equation still has a unique solution in $C^0([0, T]; L^2)$. Their proof relies on some explicit calculations.

In Chapter 5, we improve this result. We find that $L^\infty L^2 \cap L^2 L^\infty$ space is the suitable space for bounded control terms (trace) instead of $C^0 L^2 \cap L^2 H^1$ space proposed by Lions-Magenes theory. We show that the maximum principle leads to solutions to the nonlinear heat equations considered in this chapter, namely the viscous Burgers equations.

1.3.3.5 Add an integration term

The maximum principle only works for elliptic equations and parabolic equations. If we are dealing with KdV equations or Schrödinger equations, the maximum principle can no longer be applied. We think that the technique of "add an integration term" provides a systematic way of solving such trace problems.
Let us illustrate the idea with a simple example,

$$\dot{x}_1 = x_2. \tag{1.3.11}$$

We want x_2 to be C^1 (with respect to time) so that the solution x_1 is C^2. But due to some problems from closed-loop systems and feedback laws, x_2 is C^0 instead of being C^1. In other words, we can only guarantee a C^0 "input" x_2. The idea is to regard x_2 as a new state term

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \tag{1.3.12}$$

with $x_3 \in C^0$ as input (control, feedback law). An easy integration shows that $x_2 \in C^1$ and $x_1 \in C^2$. Of course system (1.3.11) is slightly different from system (1.3.12). However we are able to perform

such dynamic control techniques in reality.
Let us come back and apply this technique to the heat equation (1.1.4)–(1.1.5),

$$\frac{d}{dt}\int_0^1 y^2(x)dx = -2\int_0^1 y_x^2(x)dx + 2y(1)y_x(1).$$

Basic energy estimates do not lead to well-posedness if $y(1) \in L^\infty$. Lions-Magenes method gives the well-posedness if $y(1) \in H^{1/4}$. We add an integration term on $y(1)$, the system becomes

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0, \\ y(t,0) = 0, y(t,1) = u(t), \\ u_t(t) = v(t). \end{cases} \tag{1.3.13}$$

Therefore, at least formally, if the input $v \in L^2$, then $u \in H^1 \subset H^{1/4}$. The energy estimate becomes

$$\frac{d}{dt}\Big(\int_0^1 y^2(x)dx + u^2\Big) = -2\int_0^1 y_x^2(x)dx + 2uy_x(1) + 2uv.$$

We replace v by $v - y_x(1)$, and consider the system

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0, \\ y(t,0) = 0, y(t,1) = u(t), \\ u_t(t) = v(t) - y_x(t,1). \end{cases} \tag{1.3.14}$$

Now system (1.3.14) is well-posed with a L^2 control $v(t)$.

In Chapter 4 we find that the "add an integration term" technique is an ideal way to deal with low regularity trace stabilization problems. And we applied this method for the stabilization of a KdV equation.

Statement: In the rest of this introduction, we only care about *a priori* estimates, as the *well-posedness* and the *uniform stability* issues can be solved by the above techniques. Though we keep the same notations (such as spaces) that are used in Chapter 2–5 for consistency, readers may simply regard those norms as $\|\cdot\|$ and focus on ideas.

1.4 Local exponential stabilization of the KdV equation with a Neumann boundary control

Let $L \in (0,+\infty)$. We consider the stabilization of the following control Korteweg-de Vries (KdV) system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t,x) \in (s,+\infty)\times(0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{1.4.1}$$

where $s \in \mathbb{R}$ and where, at time $t \in [s,+\infty)$, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $u(t) \in \mathbb{R}$. Following the statement at the end of last section, the solution space that we use throughout this section is $y \in C^0([0,T];L^2) \cap L^2([0,T];H^1)$, and we do not state any well-posedness result.

1.4.1 The linearized system is not controllable

Let us study at first the controllability of the linearized system.

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{1.4.2}$$

Lionel Rosier showed in [Ros97] that for this model L^2 space can be decomposed by

$$L^2 = H \oplus M,$$

where

$$H := \text{ controllable states,}$$

$$M := \text{ uncontrollable states, and } \dim M < +\infty.$$

Thus system (1.4.2) is controllable if and only if $L \notin \mathcal{N}$, where $\mathcal{N}$ is called the set of critical lengths, and is defined by

$$\mathcal{N} := \left\{ 2\pi\sqrt{\frac{l^2 + lk + k^2}{3}};\, l,k \in \mathbb{N}^* \right\}. \tag{1.4.3}$$

Moreover, the dimension of M is the number of different pairs of positive integers (l_j, k_j) satisfying (1.4.3).

1.4.2 The nonlinear system is controllable: power series expansion

Although the existence of M for the linearized system, KdV system (1.4.1) is still controllable thanks to the nonlinear term.

Theorem 4. *KdV system (1.4.1) is (locally) controllable, if*

- *(Coron and Crépeau [CC13]) dim $M = 1$, $\forall T > 0$;*
- *(Cerpa [Cer07]) dim $M = 2$, for T large enough;*
- *(Cerpa and Crépeau [CC09a]) dim $M > 2$, for T large enough.*

Their proofs rely on the "power series method" which will also be used in our stabilization problem. An heuristic proof, the details of which we omit, suggests to make the following power series expansion

$$y := \varepsilon y^1 + \varepsilon^2 y^2 + \varepsilon^3 y^3 + ...,$$
$$u := \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 + ...$$

It follows that the linearized system (1.4.2) is actually the first order,

$$\begin{cases} y^1_t + y^1_{xxx} + y^1_x = 0, \\ y^1(t,0) = y^1(t,L) = 0, \\ y^1_x(t,L) = u^1(t), \end{cases} \tag{1.4.4}$$

Rosier [Ros97] tells us that, if $y^1(t,0) = 0$, y^1 can not reach the set $M \setminus \{0\}$. A natural idea is to go further on higher orders to see if y^2, which satisfies the following equation, can reach uncontrollable space M. The dynamics of y^2 is given by

$$\begin{cases} y^2_t + y^2_{xxx} + y^2_x = -y^1 y^1_x, \\ y^2(t,0) = y^2(t,L) = 0, \\ y^2_x(t,L) = u^2(t). \end{cases} \tag{1.4.5}$$

If the dimension of M is even, one can get the controllability with the help of y^2. If the dimension of M is odd, one need to consider third order terms. Let us point out that for the moment one can only prove local controllability in large time when dimension of M is greater than 2: One has the following open problem

Open problem 1. *What about the small-time local controllability when* $\dim M \geqslant 2$*?*

To our knowledge, the only global controllability result concerning KdV systems is due to Chapouly [Cha09a], where she used the return method with three scalar controls. Is it possible to get global controllability with less control terms?

Open problem 2. *What about the global controllability of KdV system* (1.4.1) *in small time.*

1.4.3 Quadratic structure and exponential stabilization

1.4.3.1 Known stabilization results

Let us start by introducing some stability results of KdV systems. Let $S(t)$ be the semi-group generated by the linear operator $\mathcal{A} := -\partial_x - \partial_{xxx}$ with domain $\mathcal{D}(\mathcal{A}) := \{y \in H^3(0,L); y(0) = y(L) = y_x(L) = 0\}$.

- (Zuazua et al. [PMVZ02]) Consider system (1.4.2) with $u = 0$. If $y_0 \in H$, then there exists $c > 0$ such that
$$\|S(t)y_0\|_{L^2} \leq e^{-ct}\|y_0\|_{L^2}. \tag{1.4.6}$$
- (Rosier [Ros97]) Consider system (1.4.2) with $u = 0$. If $y_0 \in M$, then
$$\|S(t)y_0\|_{L^2} = \|y_0\|_{L^2}. \tag{1.4.7}$$
Actually, the trajectory is a rotation in $\mathbb{S}^1 \times ... \times \mathbb{S}^1$.
- (Zuazua et al. [PMVZ02]) Consider system (1.4.1) with $u = 0$. If $\dim M = 0$, then there exists $r > 0$ and $c > 0$ such that
$$\|S(t)y_0\|_{L^2} \leq e^{-ct}\|y_0\|_{L^2}, \forall \|y_0\|_{L^2} \leqslant r.$$
- (Coron et al. [CCS15, TCSC16]) Consider system (1.4.1) with $u = 0$. If $\dim M = 1$ or 2, system (1.4.1) is asymptotically stable. The proofs rely on the center manifold method: there is a center manifold which is invariant under the action of the nonlinear semi-group, the flows of elements on this manifold decay polynomially, while the flows of elements outside of this manifold converge exponentially to this manifold.

Concerning the stabilization of system (1.4.1), the only result [CL14b] is the rapid stabilization for the case when $\dim M = 0$. Their proof relies on a general backstepping type of transformation and the controllability of system (1.4.1).

1.4.3.2 Our result

What about the stabilization for cases when $\dim M \neq 0$? Under this situation we need to take care of the uncontrollable part of the linearized system. It is natural to split the system into a coupled system $(y_1, y_2) := (P_H(y), P_M(y))$, and consider the stabilization of

$$\left\{\begin{array}{l}\left\{\begin{array}{l} y_{1t} + y_{1x} + y_{1xxx} + P_H\big((y_1+y_2)(y_1+y_2)_x\big) = 0,\\ y_1(t,0) = y_1(t,L) = 0,\\ y_{1x}(t,L) = u(t, y_1+y_2),\\ y_1(0,\cdot) = P_H(y_0),\end{array}\right.\\ \left\{\begin{array}{l} y_{2t} + y_{2x} + y_{2xxx} + P_M\big((y_1+y_2)(y_1+y_2)_x\big) = 0,\\ y_2(t,0) = y_2(t,L) = 0,\\ y_{2x}(t,L) = 0,\\ y_2(0,\cdot) = P_M(y_0).\end{array}\right.\end{array}\right. \tag{1.4.8}$$

Let us remark that the control only has influence on y_1. That is because for every $y_2 \in M$, $y_{2x}(L) = 0$. For this system, we are able to prove the following local exponential stabilization result.

Theorem 5. *If $\dim M = 2n$ with $n \in \mathbb{N}^*$, then there exist a periodic time-varying feedback law $u(t,y)$, $C > 0$, $\lambda > 0$, and $r > 0$ such that, for every $s \in \mathbb{R}$ and for every $y_0 \in L^2(0,L)$ such that $\|y_0\|_{L^2_L} < r$, the system (1.4.1) has at least one solution in $C^0([s,+\infty); L^2(0,L)) \cap L^2_{loc}([s,+\infty); H^1(0,L))$ and every solution y satisfies*

$$\|P_H(y(t))\|_{L^2_L} + \|P_M(y(t))\|^{\frac{1}{2}}_{L^2_L} \leqslant Ce^{-\lambda(t-s)}\big(\|P_H(y_0)\|_{L^2_L} + \|P_M(y_0)\|^{\frac{1}{2}}_{L^2_L}\big), \forall t \geqslant s. \tag{1.4.9}$$

1.4.3.3 Quadratic structure and power series expansion

The stabilization of a class of general ODE coupled systems is studied in [CR17],

$$\dot{x} = Ax + R_1(x,y) + Bu, \quad \dot{y} = Ly + Q(x,x) + R_2(x,y), \tag{1.4.10}$$

where A, B, and L are matrices, Q is a quadratic map, R_1, R_2 are polynomials and u is the control. Here we directly consider our coupled KdV system (1.4.8).

If we follow the idea of power series expansion,

$$\begin{aligned} y_1 &:= \varepsilon y_1^1 + \varepsilon^2 y_1^2 + \varepsilon^3 y_1^3 + ...,\\ y_2 &:= \varepsilon y_2^1 + \varepsilon^2 y_2^2 + \varepsilon^3 y_2^3 + ...,\\ u &:= \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 + ..., \end{aligned}$$

and consider the first order with $u^1 := 0$, we will recover linearized system stability results (1.4.6) and (1.4.7). Our idea is to mix the decay of y_1 and the conservation of y_2 by considering second order terms (y_1^2, y_2^2). The construction of the required feedback law is rather technical, we simply present some key steps here.

Step 1. *A Lyapunov function*
We consider the potential Lyapunov function $\|P_H y\|^2_{L^2} + \|P_M y\|_{L^2}$ and try to stabilize KdV system by decreasing this function along the flow. The intuition of selecting this function is due to the quadratic structure.

Step 2. *Drop the small terms*
Because we have $y_1^2 \simeq y_2$, nonlinear terms $(y_1 y_2)_x$ and $(y_2^2)_x$ become small in a local sense. Hence

it suffices to stabilize the following coupled system,

$$\left\{\begin{array}{l}\left\{\begin{array}{l}y_{1t}+y_{1x}+y_{1xxx}=0,\\ y_1(t,0)=y_1(t,L)=0,\\ y_{1x}(t,L)=u(t,y_1+y_2),\end{array}\right.\\ \left\{\begin{array}{l}y_{2t}+y_{2x}+y_{2xxx}+P_M\big(y_1y_{1x}\big)=0,\\ y_2(t,0)=y_2(t,L)=0,\\ y_{2x}(t,L)=0.\end{array}\right.\end{array}\right. \tag{1.4.11}$$

Step 3. *Compare $y_1(0)$ and $y_2(0)$*
We need to compare projections of the initial state on H and on M.

(i) if $\|y_1(0)\|^2_{L^2}> \varepsilon^{4/3}\|y_2(0)\|_{L^2}$, then the leading term is y_1. Inequality (1.4.6) shows that y_1 has a strong dissipation, thus there is a weaker dissipation for y.

(ii) if $\|y_1(0)\|^2_{L^2}\leq \varepsilon^{4/3}\|y_2(0)\|_{L^2}$, then the leading term becomes y_2. However, as we know that y_1 is controllable, we are able to create a trajectory of $y_1(t)$ with the same scaling as $y_2^{1/2}(t)$ such that it influences y_2. The following steps are mostly devoted to this situation. In order to simplify the calculation we only focus on the case when $|y_1(0)|\ll|y_2(0)|$, hence $y_1(0)\simeq 0$. The other cases need more precise asymptotic calculations.

Step 4. *Study on y_1*
For ease of notations, let us define

$$T_{\mathcal{A}}: y\mapsto y_t+y_x+y_{xxx}.$$

As y_1 satisfies a linear equation, $y_1=S(t)(y_1(0))+\tilde{y}_1\simeq\tilde{y}_1$, with $\tilde{y}_1$ satisfies

$$\left\{\begin{array}{l}T_{\mathcal{A}}\tilde{y}_1=0,\\ \tilde{y}_1(t,0)=\tilde{y}_1(t,L)=0,\\ \tilde{y}_{1x}(t,L)=u,\\ \tilde{y_1}(0)=0.\end{array}\right. \tag{1.4.12}$$

Step 5. *Study on y_2*
If we define a quadratic operator Q by

$$Q(f,g):=P_M(f\cdot g_x).$$

We observe that y_2 satisfies

$$\begin{aligned}0=&T_{\mathcal{A}}y_2+Q(S(t)(y_1(0))+\tilde{y}_1,S(t)(y_1(0))+\tilde{y}_1)\\ \simeq&T_{\mathcal{A}}y_2+Q(\tilde{y}_1,\tilde{y}_1)+o(\varepsilon^2).\end{aligned}$$

Hence, $y_2(t)=S(t)(y_2(0))+\tilde{y}_2(t)+o(\varepsilon^2)$, where

$$\left\{\begin{array}{l}T_{\mathcal{A}}\tilde{y}_2=-Q(\tilde{y}_1,\tilde{y}_1),\\ \tilde{y}_2(t,0)=\tilde{y}_2(t,L)=0,\\ \tilde{y}_{2x}(t,L)=0,\\ \tilde{y}_2(0)=0.\end{array}\right. \tag{1.4.13}$$

Step 6. *Make $|y_2(T)|<|y_2(0)|$.*
We observe that the coupled system (1.4.12)–(1.4.13) is controllable, thus it is possible to make $|y_2(T)|<|y_2(0)|$ and $y_1(T)\simeq\tilde{y}_1(T)=0$.

1.4.4 Further questions

In this proof, we mix the natural dissipation of y_1 and the rotation of y_2 to obtain a decay which is slightly weaker than the natural decay of y_1. In fact, as we know that y_1 is controllable, with the help of feedback laws y_1 can decay very fast. Is that possible to get rapid stabilization for this KdV model instead of an exponential stabilization?
As we can see that our proof relies on second order expansion, and only works for the cases when dim $M = 2n$. It is natural to ask if it is possible to get exponential stabilization when dim $M = 2n + 1$ by using higher order expansions.
It also sounds interesting to study the stability of system (1.4.1) without control, especially for the case when dim $M > 2$.

1.5 Small-time local stabilization of a KdV equation with a Dirichlet boundary control

Let $L \in (0, +\infty)$. We consider the controlled KdV system with Dirichlet boundary controls,

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, & \text{for } (t,x) \in (s, +\infty) \times (0, L), \\ y(t,0) = a(t), & \text{for } t \in (s, +\infty), \\ y(t,L) = b(t), & \text{for } t \in (s, +\infty), \\ y_x(t,L) = 0, & \text{for } t \in (s, +\infty), \end{cases} \tag{1.5.1}$$

where $s \in \mathbb{R}$ and where, at time $t \in [s, +\infty)$, the state is $y(t, \cdot) \in L^2(0, L)$ and the controls are $a(t), b(t) \in \mathbb{R}$.

In Chapter 3 we are able to prove the following null controllability result.

Theorem 6. *Let $b(t) = 0$. For any given $T > 0$, the control system (1.5.1) is locally null controllable in time T by using backstepping approach with some piecewise continuous controls.*

Actually, in Chapter 4 we only proved the null controllability of the linearized system. Then the Kato smoothing effect allows us to treat the nonlinear term as a perturbation, and to get local null controllability of the nonlinear system.

Since the backstepping is a typical stabilization tool, we can more or less get small-time stabilization by applying the same feedback law. Indeed if the starting time is 0, then the solution of the closed-loop system will become 0 at time T. However, as we have seen in Section 1.3.3.5 that there is a uniform stability problem for the small-time stabilization problem of the heat equation, the same problem appears in our case. Luckily, we can use the "add an integrator" technique to solve this low regularity problem, which combined with Theorem 6 leads to the following stabilization result.

Theorem 7. *Let $b(t) = 0$. The KdV system (1.5.1) is locally small-time stabilizable. (Remark: due to the "add an integrator" technique, we actually need to change system (1.5.1) a little.)*

1.5.1 Controllability

The local controllability of system (1.5.1) is well studied.

- (Glass and Guerrero [GG10]) Let $a(t) = 0$. The linearized KdV system is uncontrollable if and only if L belongs to a countable critical length set. (Carleman estimates)
- (Rosier [Ros04]) Let $b(t) = 0$. The linearized KdV system is null controllable, but not exactly controllable. (Carleman estimates)

There is a very interesting problem concerning the global controllability of system (1.5.1) with both $a(t)$ and $b(t)$.

Open problem 3. *What about the global controllability of system* (1.5.1) *with two Dirichlet controls?*

One may get some intuitions from similar results for a viscous Burgers system

$$\begin{cases} y_t - y_{xx} + yy_x = 0, \\ y(t,0) = u(t), \\ y(t,L) = v(t). \end{cases} \tag{1.5.2}$$

- (Fursikov and Imanuvilov [FI96]) Let $u(t) = 0$ or $v(t) = 0$. For any $T > 0$, system (1.5.2) is locally controllable, thanks to Carleman estimates.
- (Guerrero and Imanuvilov [GI07]) System (1.5.2) is not globally controllable for any $T > 0$.

It is natural to ask whether we can prove an analogue result of Guerrero-Imanuvilov [GI07] or not. However, their proof relies on the Hopf-Core transformation and the maximum principle, both of them do not hold on our KdV equation.
As a dispersive equation the KdV equation allows to have solitons which do not appear in Burgers equation, could those solitons lead to the global controllability?

1.5.2 Backstepping and rapid stabilization

For the linearized system of system (1.5.1), it is well-known that its energy is dissipating,

$$\frac{d}{dt}\|y(t,\cdot)\|_{L^2}\leqslant 0.$$

However, as the decay rate of the energy is bounded, it can not have rapid decay without feedback laws. If we further consider the energy of

$$\begin{cases} z_t + z_{xxx} + z_x + \lambda z = 0 & \text{for } (t,x) \in (s,+\infty)\times(0,L), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,L) = 0 & \text{for } t \in (s,+\infty), \\ z_x(t,L) = 0 & \text{for } t \in (s,+\infty), \end{cases} \tag{1.5.3}$$

then it will decay faster than y,

$$\frac{d}{dt}\|z(t,\cdot)\|_{L^2}\leqslant -\lambda\|z(t,\cdot)\|_{L^2}.$$

Is it possible to pass from y to z so that y decays as fast as z does? The answer is yes, and it is called the backstepping method.

The main idea of this method is to find a bounded linear invertible transformation

$$\Pi_\lambda : L^2_y \to L^2_z,$$

such that the flow of y (the solution of linearized KdV equation with a precise feedback law) is mapped into a flow of z.
Therefore,

$$\|y(t,\cdot)\|_{L^2}\leqslant e^{-\lambda t}\|\Pi_\lambda^{-1}\|_{L^2\to L^2}\|\Pi_\lambda\|_{L^2\to L^2}\|y(0,\cdot)\|_{L^2}. \tag{1.5.4}$$

More precisely, in [CC13] the transformation and the feedback law are given by

$$z(x) = \Pi_\lambda(y(x)) := y(x) - \int_x^L k_\lambda(x,r)y(r)dr, \tag{1.5.5}$$

$$a_\lambda(t) := \int_0^L k_\lambda(0,x)y(t,x)dx, \tag{1.5.6}$$

where the kernel k_λ satisfies a third order partial differential equation

$$\begin{cases} (k_\lambda)_{xxx} + (k_\lambda)_{yyy} + (k_\lambda)_x + (k_\lambda)_y + \lambda k_\lambda = 0 & \text{in } \mathcal{T}, \\ k_\lambda(x,L) = 0 & \text{on } [0,L], \\ k_\lambda(x,x) = 0 & \text{on } [0,L], \\ (k_\lambda)_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L]. \end{cases} \tag{1.5.7}$$

1.5.3 Piecewise backstepping and null controllability

We are now in position to prove the null controllability by backstepping approach. For any $\lambda > 0$, thanks to inequality (1.5.4), there exists $\tilde{t}$ such that the solution of (1.5.2) with feedback law a_λ satisfies

$$\|y(\tilde{t},\cdot)\|_{L^2} \leqslant \frac{1}{2}\|y(0,\cdot)\|_{L^2}, \forall y(0) \in L^2.$$

If the value of $\|\Pi_{\lambda_n}\|$ and $\|\Pi_{\lambda_n}^{-1}\|$ are "well controlled" such that the value of $\tilde{t}$ can be as small as we want (when λ tends to $+\infty$), then we are able to find a sequence $\{\lambda_n\}_n$ such that

$$\sum_n \tilde{t}_n < T < +\infty.$$

Hence, we get null controllability in time T by using piecewise backstepping control. It only remains to study the kernel equation (1.5.7), and to prove the well-posedness and "well controlled" estimates.

1.5.3.1 Uniqueness of solutions

Since our estimates on k_λ are based on a constructed solution, we need to prove the uniqueness of the solution of (1.5.7). Because equation (1.5.7) is linear, it suffices to prove that 0 is the only solution $h \in H^3([0,L]\times[0,L])$ of the equation

$$\begin{cases} h_{xxx} + h_{yyy} + h_x + h_y = 0 & \text{in } [0,L]\times[0,L], \\ h(x,0) = 0 & \text{on } [0,L], \\ h(x,L) = h_y(x,L) = h_{yy}(x,L) = 0 & \text{on } [0,L], \\ h(0,y) = h_x(0,y) = h_{xx}(0,y) = 0 & \text{on } [0,L]. \end{cases} \tag{1.5.8}$$

We can imagine that h behaves like a wave equation: regard x as t and study (h, h_x, h_{xx}). In order to study h, one of the most natural idea is to consider eigenfunctions.

Riesz basis
Let us define

$$\mathcal{A}_y : \mathcal{D}(\mathcal{A}_y) \subset L^2(0,L) \to L^2(0,L),$$
$$\mathcal{D}(\mathcal{A}_y) := \{f \in H^3(0,L); f(0) = f(L) = f_y(L) = 0\},$$
$$\mathcal{A}_y f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y).$$

If the eigenfunctions $\{\varphi_n(y)\}_n$ form a Riesz basis of $L^2(0,L)$ space, then the Fourier series decomposition

$$h(x,y) = \sum_n \phi_n(x) \cdot \varphi_n(y)$$

easily leads the required uniqueness argument. But it does not seem to be the case.

Theorem 8 (Papanicolaou [Pap11])**.** *Eigenfunctions $\{\varphi_n(y)\}_n$ do not form a Riesz basis of L^2 space.*

Wavelet and completeness
Another idea is to investigate the completeness of eigenfunctions, $\{\psi(y)_n\}_n$, of the adjoint operator $\mathcal{A}_y^*$. One can write the equation as

$$(\partial_{xxx} + \partial_x - \lambda_n)\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0. \tag{1.5.9}$$

Since

$$\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2} = \partial_x \langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2} = \partial_{xx}\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2} = 0,$$

we obtain

$$\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0,L]. \tag{1.5.10}$$

If $\{\psi_n(y)\}_n$ is complete in $L^2(0,L)$, then $h(x,\cdot)$ is 0. However, we do not know the completeness of eigenfunctions $\{\psi_n(y)\}_n$.

Gel'fand's idea: generalized eigenfunctions
• (Gel'fand) *In non-self-adjoint cases, it is not always possible to expand a function as the sum of eigenfunctions. In order to avoid this problem, one uses different generalizations of eigenfunctions: eigenfunctionals, generalized eigenfunctions etc.*
One of the most commonly used generalized eigenfunction space is

$$\bigcup \mathcal{N}((\lambda_i I - \mathcal{L})^{m_i}), \textit{ union for all } m_i \in \mathbb{N}, \textit{ and } \lambda_i \textit{ eigenvalues},$$

where $\mathcal{L}$ denotes the operator, $\mathcal{N}$ denotes the kernel.

Theorem 9 (Locker [Loc08])**.** *Let $L > 0$, let a be a constant. For differential operator $\mathcal{L}f := f_{xxx} + af_x$ with boundary conditions*

$$f(0) = f(L) = 0,$$
$$f_x(0) + \beta f_x(L) = 0,$$

the generalized eigenfunction space $\mathcal{E}_{\mathcal{G}}$ is complete in $L^2(0,L)$ space iff $\beta \neq 0$.

Augmented Eigenfunctions (Fokas [Fok08])
Suppose that Φ is a function space defined on the closure of a real interval I with sufficient smoothness and decay conditions, that $\mathcal{L}$ is a linear operator defined on Φ. Let γ be an oriented contour in $\mathbb{C}$ and let $\mathbf{E} = \{\mathbf{E}_\lambda : \lambda \in \gamma\}$ be a family of functionals. Then the corresponding remainder functionals $\mathbf{R}_\lambda \in \Phi'$ with respect to eigenvalues λ is

$$\mathbf{R}_\lambda(\phi) := \lambda^n \mathbf{E}_\lambda(\phi) - \mathbf{E}_\lambda(\mathcal{L}\phi), \quad \forall \phi \in \Phi, \forall \lambda \in \gamma.$$

As we can see above, the study of augmented eigenfunctions involves complicated asymptotic calculations. So far, this method is just applied on evolution equations.

Uniqueness: e.a.f. (Naimark [Nai67, Naĭ68])
Eigenfunctions and associated functions ($e.a.f.$) is a kind of generalized eigenfunctions which is more

general than $\mathcal{E}_{\mathcal{G}}$ but less complicated than augmented eigenfunctions. We find that e.a.f. is complete and can lead to the uniqueness of solutions.

Theorem 10 (Shkalikov [Ška76]). *The eigenfunctions and associated functions of the boundary-value problem generated by an ordinary differential equation with separated boundary conditions*

$$l(y) - \lambda^n y = y^{(n)} + p_{n-2}(x)y^{(n-2)} + ... + p_0(x)y - \lambda^n y = 0,$$
$$U_j(y) = \sum_{k=0}^{n-1} \alpha_{jk} y^{(k)}(0) = 0, \text{ with } j = 1, 2, ..., l,$$
$$U_j(y) = \sum_{k=0}^{n-1} \beta_{jk} y^{(k)}(L) = 0, \text{ with } j = 1, 2, ..., n-l,$$

form a complete system in L^2.

1.5.3.2 Existence of solutions and "well controlled" estimates

The existence of solutions is proved by a successive construction proposed in [CC13]. Because we proved the uniqueness of solutions in the previous section, this constructed solution is the unique solution. The main difficulty is to give estimates on this solution, which is done in the paper [Xia19].

Lemma 2. *Let $\lambda > 2$, the unique solution $k_\lambda \in C^3(\mathcal{T})$ of (1.5.7) satisfies*

$$\|k_\lambda\|_{C^0(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}}. \tag{1.5.11}$$

The proof of this lemma is rather technical, we refer to Chapter 3 for details.

1.5.4 Further questions

We observed that small-time stabilization of the KdV equation and of the heat equation deeply rely on kernel estimates of $e^{C\sqrt{\lambda}}$ type which are obtained from a "global" point of view. On the other hand, the proof of controllability of the heat equation by Lebeau and Robbiano was also based on some $e^{C\sqrt{\lambda}}$ type estimates, though they come from "microlocal" approach. I believe it is not a coincidence. As we also know that Lebeau-Robbiano strategy is highly related to Carleman estimates, a very interesting problem is to study the connection between the backstepping approach, Lebeau-Robbiano strategy, global Carleman estimates (hence small-time controllability), and small-time (local) stabilization.

1.6 Small-time global stabilization of a viscous Burgers equation

We consider the stabilization of the following controlled viscous Burgers system

$$\begin{cases} y_t - y_{xx} + yy_x = \alpha(t) & \text{for } (t,x) \in (s,+\infty)\times(0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (s,+\infty), \\ y(t,1) = \gamma(t) & \text{for } t \in (s,+\infty), \\ a_t(t) = \alpha(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{1.6.1}$$

where the state is $(y(t,\cdot), a(t)) \in L^2(0,1)\times\mathbb{R}$ and the control is $(\alpha(t), \beta(t), \gamma(t)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$. The main result of this section is the following one.

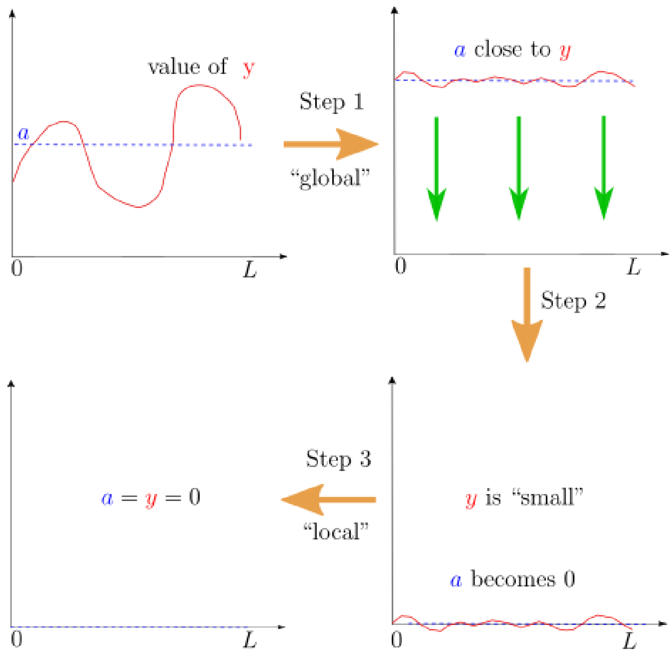

Figure 1.1: Small-time global stabilization by steps.

Theorem 11 (Coron & Xiang [CX18]). *The viscous Burgers system* (1.6.1) *is small-time globally stabilizable, i.e.,* $\forall T > 0$, *there exists a* $T-$*periodic feedback law* (α, β, γ) *such that* $(0,0) \in L^2(0,1) \times \mathbb{R}$ *is (uniformly) stable for the closed-loop system and*

$$\Phi(s + 2T, s; y_0, a_0) = 0, \forall (y_0, a_0) \in L^2 \times \mathbb{R}, \forall s \in \mathbb{R},$$

where Φ *denotes the flow of the closed-loop system.*

There is no critical length set for the Burgers equation. Hence we can replace the interval $(0, 1)$ by $(0, L)$.

1.6.1 An interesting problem and our strategy

Concerning small-time stabilization, there is a very interesting natural problem.

Open problem 4. *Build systematic methods to solve the small-time (global) stabilization problem of many systems which are small-time (global) null controllable.*

It is straightforward that a small-time stabilizable system is small-time null controllable: we simply define the control by the value of the feedback law along the flow. But the converse is much more difficult. For example, as we have seen in Section 1.1.1.2, the small-time local stabilization of the one dimensional heat equation was solved very recently. Actually, Section 1.5, the small-time local stabilization of a KdV equation is also devoted to this subject.

In [CX18], we gave the first small-time global stabilization result based on PDE models. It provides a strategy to solve the small-time global stabilization problems with two main stages: *global approximate stabilization* and *small-time local stabilization.*

More precisely, for our Burgers equation we stabilize (y, a) by three steps.

Step 1. Global approximate stabilization of "$y - a$"
Whatever the initial value (y_0, a_0) of (y, a) at the beginning of this step is, the value (y_1, a_1) of (y, a) at the end of this step satisfies $\mid y_1 - a_1 \mid < \varepsilon$. A property that we write as follows

$$\forall (y_0, a_0) \Longrightarrow (y_1, a_1) \text{ s.t. } \mid y_1 - a_1 \mid < \varepsilon.$$

Step 2. Global stabilization of "a"

$$\forall (y_1, a_1) \text{ s.t. } \mid y_1 - a_1 \mid < \varepsilon \Longrightarrow (y_2, a_2) \text{ s.t. } \mid y_2 - a_2 \mid < 2\varepsilon \text{ and } a_2 = 0.$$

Step 3. Local stabilization of "y"

$$\forall (y_2, 0) \text{ s.t. } \mid y_2 \mid < 2\varepsilon \Longrightarrow (0, 0).$$

Following our strategy of *global approximate stabilization* and *small-time local stabilization*, the first two steps are devoted to *global approximate stabilization* stage. As the small-time local stabilization is highly related to Section 1.5, we focus on the first step in this introduction.

1.6.2 Small-time global approximate stabilization

In this stage, we try to construct feedback laws which steer the control system in a small neighborhood of the origin in short time. The idea is to use the nonlinear transportation term yy_x to get global stabilization. Thanks to the two controls on the boundary, $y(t, 0) = u_1(t)$, $y(t, 1) = u_2(t)$, we do not have any boundary layer difficulties.

Let us consider the two dimensional Navier-Stokes equation in a domain,

$$v_t + (v \cdot \nabla)v - \nu \Delta v + \nabla p = 0 \text{ in } [0, T] \times \Omega, \tag{1.6.2}$$

$$\text{div } v = 0 \text{ in } [0, T] \times \Omega, \tag{1.6.3}$$

with Dirichlet boundary condition $v = 0$ on $\partial\Omega$. At least formally, when viscosity $\nu \to 0$, the Navier-Stokes equation converge to the Euler equation. Mathematically this convergence is known for manifolds without boundary. However, near the boundary the situation becomes rather complicated: as the "order" of these two equations are different, we can not put the same boundary conditions. (Moreover it is not known if this lack of convergence near the boundary can also create a lack of convergence far away from the boundary.) In order to explain this phenomenon, Prandtl proposed the idea of studying Prandtl equation [Pra04]. The convergence to Euler equation and the study of Prandtl equation are central problems in mathematics (fluid dynamic) and physics, see also [OS99].

The boundary layer problem naturally appears in control theory: when replacing the Dirichlet boundary condition "$v = 0$ on $\partial\Omega$" by the Dirichlet boundary control condition "$v = 0$ on $\partial\Omega \setminus \Gamma_0$" (the control acts on Γ_0). One of the most outstanding open problems in control theory states as follows.

Open problem 5 (Jacques-Louis Lions' problem)**.** *Is system (1.6.2)–(1.6.3) with Dirichlet boundary control condition small-time global controllable?*

The difficulty is that on the boundary where there is no control $\partial\Omega \setminus \Gamma_0$, we need to study the boundary layer. This problem has been studied for decades from different aspects. Recently, there is a breakthrough made by Coron, Marbach, Sueur and Zhang [CMS16, CMSZ18]. In [CMS16] Navier slip-with-friction boundary conditions are considered and the global null controllability is obtained. The Dirichlet boundary control condition is considered in [CMSZ18] with a domain which is a rectangle. However in this case the global null controllability is obtained on the extra assumption that can also use a force which can be arbitrary small but has a support distributed on the full domain.

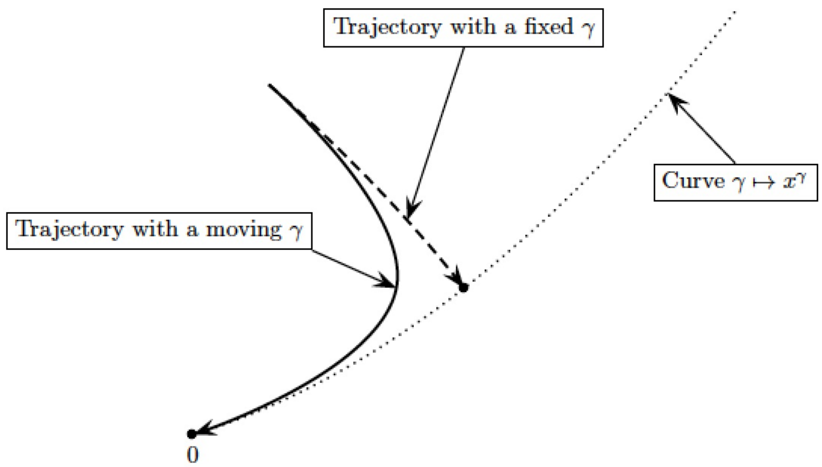

Figure 1.2: Phantom Tracking, picture from [Cor13]

In order to develop our strategy for stabilizing systems, we also need to face boundary layer problems.

1.6.2.1 General idea of the phantom tracking method

The phantom tracking method was introduced in [Cor99] for asymptotic stabilization of two dimensional Euler equations. Then it has been adapted to several models, one can refer to the survey paper [Cor13] on this method. Our goal is to stabilize

$$(X) \qquad \dot{x} = f(x, u).$$

Step 1. We find that $(0, 0)$ is an equilibrium point of system (X). But we do not know how to stabilize this system, which is quite possible for nonlinear systems, the two dimensional Euler equation for example.

Step 2. In many situations though we do not know how to stabilize system (X) around $(0, 0)$, we are able to stabilize the system around (x_0, u_0) which is another equilibrium point of system (X).

Step 3. Then we find a sequence of stabilizable equilibrium points, $(x(\gamma), u(\gamma))$, with feedback laws U_γ such that

$$(x(\gamma), u(\gamma)) \text{ is an equilibrium point of } (X), \forall \gamma \in (0, 1],$$
$$(x(\gamma), u(\gamma)) \xrightarrow{\gamma \to 0} 0,$$
$$(x(\gamma), u(\gamma)) \text{ is asymptotically stable for } (X) \text{ with feedback law } U_\gamma.$$

Step 4. We construct a feedback law U^Γ such that

$$0 \text{ is asymptotically stable for } (X) \text{ with feedback law } U^\Gamma.$$

1.6.2.2 A toy model

The phantom tracking method is more a philosophy than an explicit algorithm. In order to stabilize our Burgers equation we propose a toy model, on which we are able to design the feedback law and explain the idea. Moreover, the idea can be adapted on the Burgers equation and probably on some other nonlinear equations. Let us consider the following hyperbolic equation,

$$y_t(t,x) + yy_x(t,x) = \alpha(t) + g(t,x), \text{ in } [0,T]\times\mathbb{T}, \tag{1.6.4}$$

with a scalar control $\alpha(t)$, and an internal control $g(t,x)$ which is supported on $[b,c] \subsetneqq \mathbb{T}$. It seems that we add many controls, but the system is not small-time globally controllable if $\alpha(t) = 0$ or $g(t,x) = 0$. We are able to get controllability with less controls, but this is not our goal for introducing this toy model. Let us consider L^2 state space for control problems, even though y is not well-posed in this space.

It is easy to check that the linearized system around 0

$$y_t(t,x) = \alpha(t) + g(t,x), \text{ in } [0,T]\times\mathbb{T}, \tag{1.6.5}$$

is not controllable. Indeed, for the solutions of (1.6.5), $y(t,x_1) - y(t,x_2)$ does not change with respect to time if $x_1, x_2 \in \mathbb{T}\setminus[b,c]$.

The return method tells us that in such a case we can consider the controllability around a trajectory. For simplicity of presentation, we consider another equilibrium point. Notice that $\bar{y} = A \neq 0$ is a solution of stationary equation $yy_x = 0$, its linearized system becomes

$$y_t(t,x) + Ay_x(t,x) = g(t,x), \text{ in } [0,T]\times\mathbb{T}. \tag{1.6.6}$$

This is equivalent to a transport equation with boundary control term. Following the characteristic line, the system is controllable if and only if $|A| \geqslant 1/T$. The next step is to construct a trajectory with $|A|$ big enough,

$$\tilde{y}(t,x) = l(t)A, \text{ with } l(0) = l(T) = 0 \text{ and } l(t) = 1 \text{ in } [T/3, 2T/3].$$

Thanks to the scalar control term $\alpha(t)$, $\tilde{y}$ is a solution of (1.6.4), thus a trajectory. From this trajectory going from 0 to 0 one can deduce the small-time global null controllability: see [Cha09b].

What about stabilization problems? Of course we can not simply consider the linearized system (1.6.5). The idea is to stabilize it around some phantom trajectories, which is divided into several steps.

• Add an observer

What about the linearized system around a trajectory? Let $y := z + A$, then

$$z_t + Az_x + zz_x = \alpha + g. \tag{1.6.7}$$

For a given A if we stabilize z, then y will converge to A (even if the initial state $y_0 = 0$). In this case the uniform stability is not satisfied (see Section 1.3.3).

$$\text{“stabilization of } y\text{”} \not\Leftrightarrow \text{ “stabilization of } z\text{”}.$$

How can we benefit on other equilibrium points and stabilize z instead of y? At least we need $y = 0 \Leftrightarrow z = 0$. An idea is to set $A \approx y/2$, but we do not know how to control the term A. The answer is to add a scalar observer, a, which plays the role of A. We consider the coupled system of

$(y, a) \in L^2 \times \mathbb{R}$,

$$y_t(t,x) + yy_x(t,x) = \alpha(t) + g(t,x),$$
$$a_t = \alpha(t).$$

Let $z := y - a$, then

$$z_t + az_x + zz_x = g, \tag{1.6.8}$$
$$a_t = \alpha. \tag{1.6.9}$$

Hence

$$\text{“stabilization of } (y,a)\text{”} \Leftrightarrow \text{ “stabilization of } (z,a)\text{”}.$$

- Cascade structure and backstepping

How can we stabilize the cascade system of (z, a)? We observe that α controls state a, and state a further controls z indirectly. What if we regard a as a direct control, and consider the following system

$$z_t + az_x + zz_x = g? \tag{1.6.10}$$

Let us emphasis that system (1.6.10) is different from system (1.6.7). Because a is a control term (feedback law) which can be as large as we want so that the nonlinear term zz_x can be absorbed by az_x. It suffices to stabilize

$$z_t + az_x = g, \tag{1.6.11}$$

with feedback law a, for which some details will be given in the next step.
The question is that whether

$$\text{“stabilization of } (z,a) \text{ with } \alpha\text{”} \Leftrightarrow \text{ “stabilization of } z \text{ with } a\text{”?}$$

The answer is called backstepping. This method was introduced independently by Byrnes and Isidori in [BI89], Koditschek in [Kod87] and Tsinias in [Tsi89] for the stabilization of cascade systems, for the stabilization of coupled system $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$ with control term $u \in \mathbb{R}^k$,

$$\dot{x}_1 = f_1(x_1, x_2), \tag{1.6.12}$$
$$\dot{x}_2 = u. \tag{1.6.13}$$

At first we consider the stabilization of $\dot{x}_1 = f(x_1, v)$. Suppose that there exists a feedback law $v(x_1)$ which stabilizes the state x_1 with respect to a Lyapunov function V_1,

$$\dot{V}_1(x_1) = f(x-1, v(x_1)) \cdot \nabla V(x_1) < 0.$$

Then we consider the Lyapunov function

$$V_2(x_1, x_2) := V_1(x_1) + (x_2 - v(x_1))^2,$$

for system (1.6.12)–(1.6.13). Then, at least formally under a good choice of u, we have

$$\dot{V}_2(x_1, x_2) = f(x_1, x_2) \cdot \nabla V(x_1) - 2(x_2 - v(x_1)) \cdot (v'(x_1) f(x_1, x_2) - u) < 0.$$

Let us illustrate it with an easy example,

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = u,$$

with $x_1, x_2, u \in \mathbb{R}$. Clearly $v := -x_1$ is a stabilizing feedback law for system $\dot{x}_1 = v$ with the Lyapunov function $V_1 := x_1^2$. Let

$$V_2 := V_1 + (x_2 + x_1)^2,$$

then

$$\dot{V}_2 = -2x_1^2 + 2(x_1 + x_2)(u + x_2 + x_1).$$

Let us choose $u := -2x_1 - 2x_2$, then

$$\dot{V}_2 = -2V_2.$$

Following the idea of backstepping, we know that

"stabilization of (z, a) with α" $\Leftrightarrow$ "stabilization of z with a".

• Lyapunov function

It only remains to stabilize the transport equation (1.6.11). We consider at first the case when $a := 1$, thus

$$z_t + z_x = g. \tag{1.6.14}$$

If $g := 0$, then the energy is conserved. Actually, in any case the energy satisfies

$$\frac{d}{dt}\int_{\mathbb{T}} z^2 dx = 2\int_{\mathbb{T}} zg dx.$$

From the above equality, we do not know whether the energy will decay as fast as we want. In fact, we know that the solution follows the characteristic line, thus it could seem better to construct some weighted energy.

Without loss of generality, we assume that $[b, c] := [b, 1]$. Let us define a weighted energy as

$$V(z) := \int_{\mathbb{T}} z^2(x) f(x) dx,$$

with some $f > 0$ to be chosen later. Hence,

$$\frac{d}{dt}\int_{\mathbb{T}} z^2 f dx = \int_{\mathbb{T}} 2zgf + z^2 f_x dx = \int_{[0,b]} z^2 f_x dx + \int_{(b,1)} z(2gf + zf_x) dx$$

We define $f(x)$ as e^{-x} on $[0, b]$, and then extend it to $\mathbb{T}$ such that $f(x) \geq e^{-1}$. Let us denote this function f by f_1. With a good choice of g, we have an exponential decay of the energy,

$$\frac{d}{dt}V(z) \leqslant -V(z).$$

Furthermore, if we define $f(x)$ as $e^{-\lambda x}$ on $[0, b]$, then the energy will decay rapidly. But due to the finite speed of propagation, we are not able to get small-time global approximate stabilization of system (1.6.14) by using this kind of control. Recently, Zhang [Zha18b, Zha18a] proved the finite time stabilization of (1.6.14) with one scalar control, $g(t, x) = h(t)\Omega(x)$ where $h(t)$ is the control.

• Small-time global approximate stabilization

What about small-time global stabilization of (1.6.11)? A similar calculation on the weighted energy shows that

$$\frac{d}{dt}\int_{\mathbb{T}} z^2 f_1 dx = \int_{\mathbb{T}} 2zgf_1 + az^2 f_{1x} dx = -\int_{[0,b]} az^2 f_1 dx + \int_{(b,1)} z(2gf_1 + azf_{1x}) dx.$$

For a good choice of g, we have

$$\frac{d}{dt}V(z) \leqslant -aV(z).$$

With an intuition from the feedback law designed in Section 1.1.1.1, we let $a(t) := MV(z(t))$ and get

$$\frac{d}{dt}V(z) \leqslant -MV(z)^2,$$

which implies the small-time global approximate stabilization.

1.6.2.3 Burgers equations

The above toy model is an analogue of the viscous Burgers equation that we want to stabilize, for which some new difficulties appear. The way that we treat these difficulties will not be detailed here, but can be found in Chapter 5.

1.6.3 Further questions

In order to develop our strategy for small-time stabilization [CX18], it would be interesting to investigate the following crucial problems:

- Trace problem from the classical Lions-Magenes method [LM73].
- Backstepping method in higher dimension.
- The relation between backstepping, Lebeau-Robbiano strategy, and global Carleman estimates.
- Generalization of "phantom tracking method".
- Boundary layer difficulties.

Chapter 2

Local exponential stabilization of the KdV equation with a Neumann boundary control

Contents

2.1 Introduction

Let $L \in (0, +\infty)$. We consider the stabilization of the following controlled Korteweg-de Vries (KdV) system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{2.1.1}$$

where $s \in \mathbb{R}$ and where, at time $t \in [s, +\infty)$, the state is $y(t, \cdot) \in L^2(0, L)$ and the control is $u(t) \in \mathbb{R}$.

Boussinesq in [Bou77], and Korteweg and de Vries in [KdV95] introduced the KdV equations for describing the propagation of small amplitude long water waves. For better understanding of KdV, one can see Whitham's book [Whi99], in which different mathematical models of water waves are deduced. These equations have turned out to be good models not only for water waves but also to describe other physical phenomena. For mathematical studies on these equations, let us mention the following [BS75, CS88, CKS92, Tem69] and the references therein as well as the discovery of solitons and the inverse scattering method [GGKM67, Mur78] to solve these equations. We also refer here to

[BSZ03, BSZ09, CC04, RUZ11, Zha99] for well-posedness results of initial-boundary-value problems of our KdV equation (2.1.1) or for other equations which are similar to (2.1.1). Finally, let us refer to [Cer14, RZ09] for reviews on recent progresses on the control of various KdV equations.

The controllability research on (2.1.1) began in 1997 when Lionel Rosier showed in [Ros97] that the linearized KdV control system (around 0 in $L^2(0,L)$)

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = y(t,L) = 0 & \text{on } (0,L), \\ y_x(t,L) = u(t) & \text{on } (0,T), \end{cases} \tag{2.1.2}$$

is controllable if and only if $L \notin \mathcal{N}$, where $\mathcal{N}$ is called the set of critical lengths and is defined by

$$\mathcal{N} := \Big\{ 2\pi\sqrt{\frac{l^2+lk+k^2}{3}};\, l,k \in \mathbb{N}^* \Big\}. \tag{2.1.3}$$

From this controllability result Lionel Rosier, in the same article, deduced that the nonlinear KdV equations (2.1.1) are locally controllable (around 0 in $L^2(0,L)$) if $L \notin \mathcal{N}$. His work also shows that the $L^2(0,L)$ space can be decomposed as $H \oplus M$, where M is the “uncontrollable” part for the linearized KdV control systems (2.1.2), and H is the “controllable” part. Moreover, M is of finite dimension, a dimension which is strongly depending on some number theory property of the length L. More precisely, the dimension of M is the number of different pairs of positive integers (l_j,k_j) satisfying

$$L = 2\pi\sqrt{\frac{l_j^2+l_jk_j+k_j^2}{3}}. \tag{2.1.4}$$

For each such pair of (l_j,k_j) with $l_j \geqslant k_j$, we can find two nonzero real valued functions φ_1^j and φ_2^j such that $\varphi^j := \varphi_1^j + i\varphi_2^j$ is a solution of

$$\begin{cases} -i\omega(l_j,k_j)\varphi^j + {\varphi^j}' + {\varphi^j}''' = 0, \\ \varphi^j(0) = \varphi^j(L) = 0, \\ {\varphi^j}'(0) = {\varphi^j}'(L) = 0, \end{cases} \tag{2.1.5}$$

where $\varphi_1^j, \varphi_2^j \in C^\infty([0,L])$ and $\omega(l_j,k_j)$ is defined by

$$\omega(l_j,k_j) := \frac{(2l_j+k_j)(l_j-k_j)(2k_j+l_j)}{3\sqrt{3}(l_j^2+l_jk_j+k_j^2)^{3/2}}. \tag{2.1.6}$$

When $l_j > k_j$, the functions φ_1^j, φ_2^j are linearly independent, but when $l_j = k_j$ then $\omega(l_j,k_j) = 0$ and φ_1^j, φ_2^j are linearly dependent. It is also proved in [Ros97] that

$$M = \operatorname{Span}\{\varphi_1^1, \varphi_2^1, ..., \varphi_1^n, \varphi_2^n\}. \tag{2.1.7}$$

Multiplying (2.1.2) by φ^j, integrating on $(0,L)$, performing integrations by parts and combining with (2.1.5), we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi^j(x)dx\right) = i\omega(l_j,k_j)\int_0^L y(t,x)\varphi^j(x)dx,$$

which shows that M is included in the “uncontrollable” part of (2.1.2). Let us point out that there exists at most one pair of (l_j,k_j) such that $l_j = k_j$. Hence we can classify $L \in \mathbb{R}^+$ in 5 different cases and therefore divide $\mathbb{R}^+$ into five disjoint subsets of $(0,+\infty)$, which are defined as follows:

1. $\mathcal{C} := \mathbb{R}^+ \setminus \mathcal{N}$. Then $M = \{0\}$.
2. $\mathcal{N}_1 := \{L \in \mathcal{N}$; there exists one and only one ordered pair (l_j, k_j) satisfying (2.1.4) and one has $l_j = k_j\}$. Then the dimension of M is 1.
3. $\mathcal{N}_2 := \{L \in \mathcal{N}$; there exists one and only one ordered pair (l_j, k_j) satisfying (2.1.4) and one has $l_j > k_j\}$. Then the dimension of M is 2.
4. $\mathcal{N}_3 := \{L \in \mathcal{N}$; there exist $n \geqslant 2$ different ordered pairs (l_j, k_j) satisfying (2.1.4), and none of them satisfies $l_j = k_j\}$. Then the dimension of M is $2n$.
5. $\mathcal{N}_4 := \{L \in \mathcal{N}$; there exist $n \geqslant 2$ different ordered pairs (l_j, k_j) satisfying (2.1.4), and one of them satisfies $l_j = k_j$ $\}$. Then the dimension of M is $2n - 1$.

The five sets $\mathcal{C}$, $\{\mathcal{N}_i\}_{i=1}^4$ are pairwise disjoint and

$$\mathbb{R}^+ = \mathcal{C} \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \cup \mathcal{N}_4, \, \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \cup \mathcal{N}_4.$$

Additionally, Eduardo Cerpa proved that each of these five sets has infinite number of elements: see [Cer07, Lemma 2.5]; see also [Cor07a, Proposition 8.3] for the case of $\mathcal{N}_1$.

Let us point out that $L \notin \mathcal{N}$ is equivalent to $M = \{0\}$. Hence, Lionel Rosier solved the (local) controllability problem of the nonlinear KdV equations for $L \in \mathcal{C}$. Later on Jean-Michel Coron and Emmanuelle Crépeau proved in [CC04] the small-time local controllability of nonlinear KdV equations for the second case $L \in \mathcal{N}_1$, by "power series expansion" method, the nonlinear term yy_x gives this controllability. Later on, in 2007, Eduardo Cerpa proved the local controllability in large time for the third case $L \in \mathcal{N}_2$ [Cer07], still by using the "power series expansion" method. In this case, an expansion to the order 2 is sufficient and the local controllability in small time remains open. Finally Eduardo Cerpa and Emmanuelle Crépeau in [CC09a] concluded the study by proving the local controllability in large time of (2.1.1) for the two remaining critical cases (for which dim $M \geqslant 3$). The proof of all these results rely on the "power series expansion" method, a method introduced in [CC04]. This method has also been used to prove controllability results for Schrödinger equations [Bea05, BC06, BM14, Mor14] and for rapid asymptotic stability of a Navier-Stokes control system in [CE17]. In this article we use it to get exponential stabilization of (2.1.1). For studies on the controllability of other KdV control systems problems, let us refer to [CFPR15, Gag15, GG10, GS07, Ros04, Zha99] and the references therein.

The asymptotic stability of 0 without control (control term equal to 0) has been studied for years, see, in particular, [CC13, GS07, JZ12, MPMP07, Paz05, PMVZ02, RZ06, RZ95, RZ96]. Among which, for example, the local exponential stability for our KdV equation if $L \notin \mathcal{N}$ was proved in [PMVZ02]. Let also point out here that in [DN14], the authors give the existence of (large) stationary solutions which ensures that the exponential stability result in [PMVZ02] is only local.

Concerning the stabilization by means of feedback laws, the locally exponentially stabilization with arbitrary decay rate (rapid stabilization) with some linear feedback law was obtained by Eduardo Cerpa and Emmanuelle Crépeau in [CC09b] for the linear KdV equation (2.1.2). For the nonlinear case, the first rapid stabilization for Korteweg-de Vries equations was obtained in [LRZ10] by Camille Laurent, Lionel Rosier and Bing-Yu Zhang in the case of localized distributed control on a periodic domain. In that case the linearized control system, let us write it $\dot{y} = Ay + Bu$, is controllable. These authors used an approach due to Marshall Slemrod [Sle74] to construct linear feedback laws leading to the rapid stabilization of $\dot{y} = Ay + Bu$ and then proved that the same feedback laws give the rapid stabilization of the nonlinear Korteweg de Vries equation. In the case of distributed control the operator B is bounded. For boundary control the operator B is unbounded. The Slemrod approach has been modified to handle this case by Vilmos Komornik in [Kom97] and by Jose Urquiza in [Urq05]; and [CC09b] precisely uses the modification presented in [Urq05]. However, in contrast with the case of distributed control, it leads to unbounded linear feedback laws and one does not know for the moment if these linear feedback laws lead to asymptotic stabilization for the nonlinear Korteweg de Vries equation. One does not even know if the closed system is well posed for this nonlinear equation. The first rapid stabilization result in the nonlinear case and with boundary controls was obtained by Eduardo Cerpa and Jean-Michel Coron in [CC13]. Their approach relies on the backstepping method/transformation (see [KS08b] for an excellent starting point to get inside this method due to Miroslav Krstic and his collaborators). When $L \notin \mathcal{N}$, by using a more general transformation and the controllability of (2.1.2) , Jean-Michel Coron and Qi Lü proved in [CL14b] the rapid stabilization of our KdV control system. Their method can be applied to many other equations, like Schrödinger equations [CGM16] and Kuramoto-Sivashinsky equations [CL15]. When $L \in \mathcal{N}$, as mentioned above, the linearized control system (2.1.2) is not controllable, but the control system (2.1.1) is controllable. Let us recall that for the finite dimensional case, the controllability doesn't imply the existence of a (continuous) stationary feedback law which stabilizes (asymptotically, exponentially etc.) the control system, see [Bro83, Cor90]. However the controllability in general implies the existence of (continuous) *time-varying* feedback laws which asymptotically (and even in finite time) stabilize the control system; see [Cor95]. Hence it is natural to look for time-varying feedback laws $u(t, y(t, \cdot))$ such that 0 is (locally) asymptotically stable for the closed-loop system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t, y(t,\cdot)) & \text{for } t \in (s,+\infty). \end{cases} \tag{2.1.8}$$

Let us also point out that in [LRZ10], as in [CR94] by Jean-Michel Coron and Lionel Rosier which was dealing with finite dimensional control systems, time-varying feedback laws were used in order to combine two different feedback laws to get rapid *global* asymptotic stability of the closed loop system. Let us emphasize that $u = 0$ leads to (local) asymptotic stability when $L \in \mathcal{N}_1$ [CCS15] and $L \in \mathcal{N}_2$ [TCSC16]. However, in both cases, the convergence is not exponential. It is then natural to ask if we can get exponential convergence to 0 with the help of some suitable time-varying feedback laws $u(t, y(t, \cdot))$. The aim of this paper is to prove that it is indeed possible in the case where

$$L \text{ is in } \mathcal{N}_2 \text{ or in } \mathcal{N}_3. \tag{2.1.9}$$

Let us denote by $P_H : L^2(0,L) \to H$ and $P_M : L^2(0,L) \to M$ the orthogonal projection (for the L^2-scalar product) on H and M respectively. Our main result is the following one, where the precise definition of a solution of (2.1.10) is given in Section 2.2.

Theorem 12. *Assume that (2.1.9) holds. Then there exists a periodic time-varying feedback law u, $C > 0$, $\lambda > 0$ and $r > 0$ such that, for every $s \in \mathbb{R}$ and for every $\|y_0\|_{L^2_L} < r$, the Cauchy problem*

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t, y(t,\cdot)) & \text{for } t \in (s,+\infty), \\ y(s,\cdot) = y_0 & \text{for } x \in (0,L), \end{cases} \tag{2.1.10}$$

has at least one solution in $C^0([s,+\infty); L^2(0,L)) \cap L^2_{loc}([s,+\infty); H^1(0,L))$ and every solution y of (2.1.10) is defined on $[s,+\infty)$ and satisfies, for every $t \in [s,+\infty)$,

$$\|P_H(y(t))\|_{L^2_L} + \|P_M(y(t))\|_{L^2_L}^{\frac{1}{2}} \leqslant Ce^{-\lambda(t-s)} \big(\|P_H(y_0)\|_{L^2_L} + \|P_M(y_0)\|_{L^2_L}^{\frac{1}{2}} \big). \tag{2.1.11}$$

In order to simplify the notations, in this paper we sometimes simply denote $y(t,\cdot)$ by $y(t)$, if there is no misunderstanding, sometimes we also simply denote $L^2(0,L)$ (resp. $L^2(0,T)$) by L^2_L (resp. L^2_T). Let us explain briefly an important ingredient of our proof of Theorem 12. Taking into account the uncontrollability of the linearized system, it is natural to split the KdV system into a coupled system for $(P_H(y), P_M(y))$. Then the finite dimensional analogue of our KdV control system is

$$\dot{x} = Ax + R_1(x,y) + Bu, \quad \dot{y} = Ly + Q(x,x) + R_2(x,y), \tag{2.1.12}$$

where A, B, and L are matrices, Q is a quadratic map, R_1, R_2 are polynomials and u is the control. The state variable x plays the role of $P_H(y)$, while y plays the role of $P_M(y)$. The two polynomials R_1 and R_2 are quadratic and $R_2(x,y)$ vanishes for $y = 0$. For this ODE system, in many cases the Brockett condition [Bro83] and the Coron condition [Cor07a] for the existence of continuous stationary stabilizing feedback laws do not hold. However, as shown in [CR17], many physical systems of form (2.1.12) can be exponentially stabilized by means of time-varying feedback laws. We follow the construction of these time-varying feedback laws given in this article. However, due to the fact that H is of infinite dimension, many parts of the proof have to be modified compared to those given in [CR17]. In particular we do not know how to use a Lyapunov approach, in contrast to what is done in [CR17].

This article is organized as follows. In Section 2.2, we recall some classical results and definitions about (2.1.1) and (2.1.2). In Section 2.3, we study the existence and uniqueness of solutions to the closed-loop system (2.1.10) with time-varying feedback laws u which are not smooth. In Section 2.4, we construct our time-varying feedback laws. In Section 2.5, we prove two estimates for solutions to the closed-loop system (2.1.10) (Propositions 4 and 5) which imply Theorem 12. The article ends with three appendices where proofs of propositions used in the main parts of the article are given.

2.2 Preliminaries

We first recall some results on KdV equations and give the definition of a solution to the Cauchy problem (2.1.10). Let us start with the nonhomogeneous linear Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x = \tilde{h} & \text{in } (T_1, T_2) \times (0, L), \\ y(t,0) = y(t,L) = 0 & \text{on } (T_1, T_2), \\ y_x(t,L) = h(t) & \text{on } (T_1, T_2), \\ y(T_1, x) = y_0(x) & \text{on } (0, L), \end{cases} \tag{2.2.1}$$

for

$$-\infty < T_1 < T_2 < +\infty, \tag{2.2.2}$$

$$y_0 \in L^2(0,L), \tag{2.2.3}$$

$$\tilde{h} \in L^1(T_1, T_2; L^2(0,L)), \tag{2.2.4}$$

$$h \in L^2(T_1, T_2). \tag{2.2.5}$$

Let us now give the definition of a solution to (2.2.1).

Definition 6. *A solution to the Cauchy problem (2.2.1) is a function $y \in L^1(T_1, T_2; L^2(0,L))$ such that, for almost every $\tau \in [T_1, T_2]$ the following holds: for every $\phi \in C^3([T_1, \tau] \times [0, L])$ such that*

$$\phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \ \ \forall t \in [T_1, \tau], \tag{2.2.6}$$

one has

$$-\int_{T_1}^{\tau}\int_0^L(\phi_t+\phi_x+\phi_{xxx})ydxdt-\int_{T_1}^{\tau}h(t)\phi_x(t,L)dt-\int_{T_1}^{\tau}\int_0^L\phi\tilde{h}dxdt$$
$$+\int_0^L y(\tau,x)\phi(\tau,x)dx-\int_0^L y_0\phi(T_1,x)dx=0. \tag{2.2.7}$$

For T_1 and T_2 satisfying (2.2.2), let us define the linear space $\mathcal{B}_{T_1,T_2}$ by

$$\mathcal{B}_{T_1,T_2}:=C^0([T_1,T_2];L^2(0,L))\cap L^2(T_1,T_2;H^1(0,L)). \tag{2.2.8}$$

This linear space $\mathcal{B}_{T_1,T_2}$ is equipped with the following norm

$$\|y\|_{\mathcal{B}_{T_1,T_2}}:=\max\{\|y(t)\|_{L^2_L};\, t\in[T_1,T_2]\}+\left(\int_{T_1}^{T_2}\|y_x(t)\|^2_{L^2_L}dt\right)^{1/2}. \tag{2.2.9}$$

With this norm, $\mathcal{B}_{T_1,T_2}$ is a Banach space.

Let $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset L^2(0,L)\to L^2(0,L)$ be the linear operator defined by

$$\mathcal{D}(\mathcal{A}):=\left\{\phi\in H^3(0,L);\,\phi(0)=\phi(L)=\phi_x(L)=0\right\}, \tag{2.2.10}$$
$$\mathcal{A}\phi:=-\phi_x-\phi_{xxx},\quad\forall\phi\in\mathcal{D}(\mathcal{A}). \tag{2.2.11}$$

It is known that both $\mathcal{A}$ and $\mathcal{A}^*$ are closed and dissipative (see e.g. [Cor07a, page 39]), and therefore $\mathcal{A}$ generates a strongly continuous semigroup of contractions $S(t)$, $t\in[0,+\infty)$ on $L^2(0,L)$.

In [Ros97], Lionel Rosier using the above properties of $\mathcal{A}$ together with multiplier techniques proved the following existence and uniqueness result for the Cauchy problem (2.2.1).

Lemma 3. *The Cauchy problem* (2.2.1) *has one and only one solution. This solution is in* $\mathcal{B}_{T_1,T_2}$ *and there exists a constant* $C_2>0$ *depending only on* T_2-T_1 *such that*

$$\|y\|_{\mathcal{B}_{T_1,T_2}}\leqslant C_2\left(\|y_0\|_{L^2_L}+\|h\|_{L^2(T_1,T_2)}+\|\tilde{h}\|_{L^1(T_1,T_2;L^2(0,L))}\right). \tag{2.2.12}$$

In fact the notion of solution to the Cauchy problem (2.2.1) considered in [Ros97] is a priori stronger than the one we consider here (it is required to be in $C^0([T_1,T_2];L^2(0,L))$. However the uniqueness of the solution in the sense of Definition 6 still follows from classical arguments; see, for example, [Cor07a, Proof of Theorem 2.37, page 53].

Let us now turn to the nonlinear KdV equation

$$\begin{cases} y_t+y_{xxx}+y_x+yy_x=\tilde{H} & \text{in } (T_1,T_2)\times(0,L),\\ y(t,0)=y(t,L)=0 & \text{on } (T_1,T_2),\\ y_x(t,L)=H(t) & \text{on } (T_1,T_2),\\ y(T_1,x)=y_0(x) & \text{on } (0,L).\end{cases} \tag{2.2.13}$$

Inspired by Lemma 3, we adopt the following definition.

Definition 7. *A solution to* (2.2.13) *is a function* $y\in\mathcal{B}_{T_1,T_2}$ *which is a solution of* (2.2.1) *for* $\tilde{h}:=\tilde{H}-yy_x\in L^1(T_1,T_2;L^2(0,L))$ *and* $h:=H$.

Throughout this article we will use similar definitions without giving them precisely. As an example, it will be the case for system (2.3.15).

In [CC04], Jean-Michel Coron and Emmanuelle Crépeau proved the following lemma on the well-posedness of the Cauchy problem (2.2.13) for small initial data.

Lemma 4. *There exist $\eta > 0$ and $C_3 > 0$ depending on L and $T_2 - T_1$ such that, for every $y_0 \in L^2(0,L)$, every $H \in L^2(T_1,T_2)$ and every $\tilde{H} \in L^1(T_1,T_2;L^2(0,L))$ satisfying*

$$\|y_0\|_{L^2_L} + \|H\|_{L^2(T_1,T_2)} + \|\tilde{H}\|_{L^1(T_1,T_2;L^2(0,L))} \leqslant \eta, \tag{2.2.14}$$

the Cauchy problem (2.2.13) has a unique solution and this solution satisfies

$$\|y\|_{\mathcal{B}_{T_1,T_2}} \leqslant C_3\big(\|y_0\|_{L^2_L} + \|H\|_{L^2(T_1,T_2)} + \|\tilde{H}\|_{L^1(T_1,T_2;L^2(0,L))}\big). \tag{2.2.15}$$

2.3 Time-varying feedback laws and well-posedness of the associated closed-loop system

Throughout this section u denotes a time-varying feedback law: it is a map from $\mathbb{R} \times L^2(0,L)$ with values into $\mathbb{R}$. We assume that this map is a Carathéodory map, i.e. it satisfies the three following properties

$$\forall R > 0, \exists\, C_B(R) > 0 \text{ such that } \Big(\|y\|_{L^2_L} \leqslant R \Rightarrow |u(t,y)| \leqslant C_B(R), \ \ \forall t \in \mathbb{R}\Big), \tag{2.3.1}$$

$$\forall y \in L^2(0,L), \text{ the function } t \in \mathbb{R} \mapsto u(t,y) \in \mathbb{R} \text{ is measurable}, \tag{2.3.2}$$

$$\text{for almost every } t \in \mathbb{R}, \text{ the function } y \in L^2(0,L) \mapsto u(t,y) \in \mathbb{R} \text{ is continuous}. \tag{2.3.3}$$

In this article we always assume that

$$C_B(R) \geq 1, \ \ \forall R \in [0,+\infty), \tag{2.3.4}$$

$$R \in [0,+\infty) \mapsto C_B(R) \in \mathbb{R} \text{ is a non-decreasing function}. \tag{2.3.5}$$

Let $s \in \mathbb{R}$ and let $y_0 \in L^2(0,L)$. We start by giving the definition of a solution to

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } \ t \in \mathbb{R},\, x \in (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } \ t \in \mathbb{R}, \\ y_x(t,L) = u(t,y(t,\cdot)) & \text{for } \ t \in \mathbb{R}, \end{cases} \tag{2.3.6}$$

and to the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } \ t > s,\, x \in (0,L), \\ y(t,0) = y(t,L) = 0 & \text{for } \ t > s, \\ y_x(t,L) = u(t,y(t,\cdot)) & \text{for } \ t > s, \\ y(s,x) = y_0(x) & \text{for } \ x \in (0,L), \end{cases} \tag{2.3.7}$$

where y_0 is a given function in $L^2(0,L)$ and s is a given real number.

Definition 8. *Let I be an interval of $\mathbb{R}$ with a nonempty interior. A function y is a solution of (2.3.6) on I if $y \in C^0(I;L^2(0,L))$ is such that, for every $[T_1,T_2] \subset I$ with $-\infty < T_1 < T_2 < +\infty$, the restriction of y to $[T_1,T_2] \times (0,L)$ is a solution of (2.2.13) with $\tilde{H} := 0$, $H(t) := u(t,y(t))$ and $y_0 := y(T_1)$. A function y is a solution to the Cauchy problem (2.3.7) if there exists an interval I with a nonempty interior satisfying $I \cap (-\infty, s] = \{s\}$ such that $y \in C^0(I;L^2(0,L))$ is a solution of (2.3.6) on I and satisfies the initial condition $y(s) = y_0$ in $L^2(0,L)$. The interval I is denoted by $D(y)$. We say that a solution y to the Cauchy problem (2.3.7) is maximal if, for every solution z to* *the Cauchy problem* (2.3.7) *such that*

$$D(y) \subset D(z), \tag{2.3.8}$$

$$y(t) = z(t) \text{ for every } t \text{ in } D(y), \tag{2.3.9}$$

one has

$$D(y) = D(z). \tag{2.3.10}$$

Let us now state our theorems concerning the Cauchy problem (2.3.7).

Theorem 13. *Assume that u is a Carathéodory function and that, for every $R > 0$, there exists $K(R) > 0$ such that*

$$\left(\|y\|_{L^2_L} \leqslant R \text{ and } \|z\|_{L^2_L} \leqslant R\right) \Rightarrow \left(|u(t,y) - u(t,z)| \leqslant K(R)\|y - z\|_{L^2_L}, \quad \forall t \in \mathbb{R}\right). \tag{2.3.11}$$

Then, for every $s \in \mathbb{R}$ and for every $y_0 \in L^2(0,L)$, the Cauchy problem (2.3.7) *has one and only one maximal solution y. If $D(y)$ is not equal to $[s, +\infty)$, there exists $\tau \in \mathbb{R}$ such that $D(y) = [s, \tau)$ and one has*

$$\lim_{t \to \tau^-} \|y(t)\|_{L^2_L} = +\infty. \tag{2.3.12}$$

Moreover, if $C_B(R)$ satisfies

$$\int_0^{+\infty} \frac{R}{(C_B(R))^2} dR = +\infty, \tag{2.3.13}$$

then

$$D(y) = [s, +\infty). \tag{2.3.14}$$

Theorem 14. *Assume that u is a Carathéodory function which satisfies condition* (2.3.13)*. Then, for every $s \in \mathbb{R}$ and for every $y_0 \in L^2(0,L)$, the Cauchy problem* (2.3.7) *has at least one maximal solution y such that $D(y) = [s, +\infty)$.*

The proofs of Theorem 13 and Theorem 14 will be given in Appendix 2.7.

We end up this section with the following proposition which gives the expected connection between the evolution of $P_M(y)$ and $P_H(y)$ and the fact that y is a solution to (2.3.6).

Proposition 1. *Let $u : \mathbb{R} \times L^2(0,L) \to \mathbb{R}$ be a Carathéodory feedback law. Let $-\infty < s < T < +\infty$, let $y \in \mathcal{B}_{s,T}$ and let $y_0 \in L^2(0,L)$. Let us denote $P_H(y)$ (resp. $P_M(y)$) by y_1 (resp. y_2). Then y is a solution to the Cauchy problem* (2.3.7) *if and only if*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} y_{1t} + y_{1x} + y_{1xxx} + P_H\big((y_1 + y_2)(y_1 + y_2)_x\big) = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1x}(t,L) = u(t, y_1 + y_2), \\ y_1(0,\cdot) = P_H(y_0), \end{array} \right. \\ \left\{ \begin{array}{l} y_{2t} + y_{2x} + y_{2xxx} + P_M\big((y_1 + y_2)(y_1 + y_2)_x\big) = 0, \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2x}(t,L) = 0, \\ y_2(0,\cdot) = P_M(y_0). \end{array} \right. \end{array} \right. \tag{2.3.15}$$

The proof of this proposition is given in Appendix 2.6.

2.4 Construction of time-varying feedback laws

In this section, we construct feedback laws which will lead to the local exponential stability stated in Theorem 12. Let us denote by $M_{\mathbf{1}}$ the set of elements in M having a L^2-norm equal to 1:

$$M_{\mathbf{1}} := \big\{ y \in M;\ \|y\|_{L^2_L} = 1 \big\}. \tag{2.4.1}$$

Let M^j be the linear space generated by φ_1^j and φ_2^j for every $j \in \{1, 2, ..., n\}$:

$$M^j := \operatorname{Span}\{\varphi_1^j, \varphi_2^j\}. \tag{2.4.2}$$

The construction of our feedback laws relies on the following proposition.

Proposition 2. *There exist $T > 0$ and $v \in L^\infty\big([0,T] \times M_{\mathbf{1}}; \mathbb{R}\big)$ such that the following three properties hold.*

($\mathcal{P}_1$) There exists $\rho_1 \in (0,1)$ such that

$$\|S(T)y_0\|^2_{L^2(0,L)} \leqslant \rho_1 \|y_0\|^2_{L^2(0,L)}, \quad \text{for every } y_0 \in H.$$

($\mathcal{P}_2$) For every $y_0 \in M$,

$$\|S(T)y_0\|^2_{L^2(0,L)} = \|y_0\|^2_{L^2(0,L)}.$$

($\mathcal{P}_3$) There exists $C_0 > 0$ such that

$$|\ v(t,y) - v(t,z)\ | \leqslant C_0 \|y - z\|_{L^2(0,L)}, \quad \forall t \in [0,T],\ \forall y, z \in M_{\mathbf{1}}. \tag{2.4.3}$$

Moreover, there exists $\delta > 0$ such that, for every $z \in M_{\mathbf{1}}$, the solution (y_1, y_2) to the following equation

$$\begin{cases} y_{1t} + y_{1x} + y_{1xxx} = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1x}(t,L) = v(t,z), \\ y_1(0,x) = 0, \\ y_{2t} + y_{2x} + y_{2xxx} + P_M\big(y_1 y_{1x}\big) = 0, \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2x}(t,L) = 0, \\ y_2(0,x) = 0, \end{cases} \tag{2.4.4}$$

satisfy

$$y_1(T) = 0 \quad \text{and} \quad \big\langle y_2(T), S(T)z \big\rangle_{L^2(0,L)} < -2\delta. \tag{2.4.5}$$

Proof of Proposition 2. Property ($\mathcal{P}_2$) is given in [Ros97], one can also see (2.4.14) and (2.4.44). Property ($\mathcal{P}_1$) follows from the dissipativity of $\mathcal{A}$ and the controllability of (2.1.2) in H (see also [PMVZ02]). Indeed, integrations by parts (and simple density arguments) show that, in the distribution sense in $(0, +\infty)$,

$$\frac{d}{dt}\|S(t)y_0\|^2_{L^2_L} = -y_x^2(t,0). \tag{2.4.6}$$

Moreover, as Lionel Rosier proved in [Ros97], for every $T > 0$, there exists $c > 1$ such that, for every $y_0 \in H$,

$$\|y_0\|^2_{L^2_L} \leqslant c \|y_x(t,0)\|^2_{L^2(0,T)}. \tag{2.4.7}$$

Integration of identity (2.4.6) on $(0,T)$ and the use of (2.4.7) give

$$\|S(T)y_0\|^2_{L^2_L} \leqslant \frac{c-1}{c}\|y_0\|^2_{L^2_L}. \tag{2.4.8}$$

Hence $\rho_1 := (c-1)/c \in (0,1)$ satisfies the required properties.

Our concern now is to deal with $(\mathcal{P}_3)$. Let us first recall a result on the controllability of the linear control system

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = y(t,L) = 0 & \text{on } (0,L), \\ y_x(t,L) = u(t) & \text{on } (0,T), \end{cases} \tag{2.4.9}$$

where, at time $t \in [0,T]$ the state is $y(t,\cdot) \in L^2(0,L)$. Our goal is to investigate the cases where $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, but in order to explain more clearly our construction of v, we first deal with the case where

$$L = 2\pi\sqrt{\frac{1^2+1\times 2+2^2}{3}} = 2\pi\sqrt{\frac{7}{3}}, \tag{2.4.10}$$

which corresponds to $l=1$ and $k=2$ in (2.1.3). In that case the uncontrollable subspace M is a two dimensional vector subspace of $L^2(0,L)$ generated by

$$\varphi_1(x) = C\left(\cos\left(\frac{5}{\sqrt{21}}x\right) - 3\cos\left(\frac{1}{\sqrt{21}}x\right) + 2\cos\left(\frac{4}{\sqrt{21}}x\right)\right),$$
$$\varphi_2(x) = C\left(-\sin\left(\frac{5}{\sqrt{21}}x\right) - 3\sin\left(\frac{1}{\sqrt{21}}x\right) + 2\sin\left(\frac{4}{\sqrt{21}}x\right)\right),$$

where C is a positive constant such that $\|\varphi_1\|_{L^2_L} = \|\varphi_2\|_{L^2_L} = 1$. They satisfy

$$\begin{cases} \varphi_1' + \varphi_1''' = -\dfrac{2\pi}{p}\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{cases} \tag{2.4.11}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = \dfrac{2\pi}{p}\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0, \end{cases} \tag{2.4.12}$$

with (see [Cer07])

$$p := \frac{441\pi}{10\sqrt{21}}. \tag{2.4.13}$$

For every $t > 0$, one has

$$S(t)M \subset M \text{ and } S(t) \text{ restricted to } M \text{ is the rotation of angle } \frac{2\pi t}{p}, \tag{2.4.14}$$

if the orientation on M is chosen so that (φ_1, φ_2) is a direct basis, a choice which is done from now on. Moreover the control u has no action on M for the linear control system (2.1.2): for every initial data $y_0 \in M$, whatever is $u \in L^2(0,T)$, the solution y of (2.1.2) with $y(0) = y_0$ satisfies $P_M(y(t)) = S(t)y_0$, for every $t \in [0,+\infty)$. Let us denote by H the orthogonal in $L^2(0,L)$ of M

for the L^2-scalar product $H := M^{\perp}$. This linear space is left invariant by the linear control system (2.1.2): for every initial data $y_0 \in H$, whatever is $u \in L^2(0,T)$, the solution y of (2.1.2) satisfying $y(0) = y_0$ is such that $y(t) \in H$, for every $t \in [0,+\infty)$. Moreover, as it is proved by Lionel Rosier in [Ros97], the linear control system (2.1.2) is controllable in H in small-time. More precisely, he proved the following lemma.

Lemma 5. *Let $T > 0$. There exists $C > 0$ depending only on T such that, for every y_0, $y_1 \in H$, there exists a control $u \in L^2(0,T)$ satisfying*

$$\|u\|_{L^2_T} \leqslant C(\|y_0\|_{L^2_L} + \|y_1\|_{L^2_L}), \tag{2.4.15}$$

such that the solution y of the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = y(t,L) = 0 & \text{on } (0,T), \\ y_x(t,L) = u(t) & \text{on } (0,T), \\ y(0,x) = y_0(x) & \text{on } (0,L), \end{cases}$$

satisfies $y(T,\cdot) = y_1$.

A key ingredient of our construction of v is the following proposition.

Proposition 3. *Let $T > 0$. For every $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, for every $j \in \{1, 2, ..., n\}$, there exists $u^j \in H^1(0,T)$ such that*

$$\alpha(T,\cdot) = 0 \quad \text{and} \quad P_{M^j}(\beta(T,\cdot)) \neq 0,$$

where α and β are the solution of

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = u^j(t), \\ \alpha(0,x) = 0, \\ \beta_t + \beta_x + \beta_{xxx} + \alpha\alpha_x = 0, \\ \beta(t,0) = \beta(t,L) = 0, \\ \beta_x(t,L) = 0, \\ \beta(0,x) = 0. \end{cases} \tag{2.4.16}$$

Proposition 3 is due to Eduardo Cerpa and Emmanuelle Crépeau if one requires only u to be in $L^2(0,T)$ instead of being in $H^1(0,T)$: see [Cer07, Proposition 3.1] and [CC09a, Proposition 3.1]. We explain in Appendix 2.8 how to modify the proof of [Cer07, Proposition 3.1] (as well as [CC09a, Proposition 3.1]) in order to get Proposition 3.

We decompose β by $\beta = \beta_1 + \beta_2$, where $\beta_1 := P_H(\beta)$ and $\beta_2 := P_M(\beta)$. Hence, similarly to Proposition 1, we get

$$\begin{cases} \beta_{2t} + \beta_{2x} + \beta_{2xxx} + P_M\big(\alpha\alpha_x\big) = 0, \\ \beta_2(t,0) = \beta_2(t,L) = 0, \\ \beta_{2x}(t,L) = 0, \\ \beta_2(0,x) = 0, \end{cases} \tag{2.4.17}$$

where $\beta_2(T,\cdot) = P_M(\beta(T,\cdot)) \neq 0$. In particular, $P_{M^j}(\beta_2(T,\cdot)) = P_{M^j}(\beta(T,\cdot)) \neq 0$.

Combining (2.4.16) and (2.4.17), we get:

Corollary 1. *For every $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, for every $T_0 > 0$, for every $j \in \{1, 2, ..., n\}$, there exists $u_0^j \in L^\infty(0, T_0)$ such that the solution (y_1, y_2) to equation (2.4.4) with $v(t, z) := u_0^j(t)$ satisfies*

$$y_1(T_0) = 0 \quad \textit{and} \quad P_{M^j}(y_2(T_0)) \neq 0. \tag{2.4.18}$$

Now we come back to the case when (2.4.10) holds. Let us fix $T_0 > 0$ such that

$$T_0 < \frac{p}{4}. \tag{2.4.19}$$

Let

$$q := \frac{p}{4}. \tag{2.4.20}$$

Let u_0 be as in Corollary 1. We denote by

$$Y_1(t) := y_1(t), \ \ Y_2(t) := y_2(t), \ \text{for } t \in [0, T_0], \tag{2.4.21}$$

and

$$\psi_1 := Y_2(T_0) \in M \setminus \{0\}. \tag{2.4.22}$$

Let

$$\psi_2 = S(q)\psi_1 \in M, \quad \psi_3 = S(2q)\psi_1 \in M, \quad \psi_4 = S(3q)\psi_1 \in M, \tag{2.4.23}$$

$$T := 3q + T_0, \tag{2.4.24}$$

$$K_1 := [3q, 3q + T_0], \tag{2.4.25}$$

$$K_2 := [2q, 2q + T_0]\,, \tag{2.4.26}$$

$$K_3 := [q, q + T_0]\,, \tag{2.4.27}$$

$$K_4 := [0, T_0]\,. \tag{2.4.28}$$

Note that (2.4.19) implies that

$$K_1,\ K_2,\ K_3 \text{ and } K_4 \text{ are pairwise disjoint.} \tag{2.4.29}$$

Let us define four functions $[0, T] \to \mathbb{R}$: u_1, u_2, u_3 and u_4 by requiring that, for every $i \in \{1, 2, 3, 4\}$,

$$u_i := \begin{cases} 0 & \text{on } [0, T] \setminus K_i, \\ u_0(\cdot - \tau_i) & \text{on } K_i, \end{cases} \tag{2.4.30}$$

with

$$\tau_1 = 3q,\ \tau_2 = 2q,\ \tau_3 = q, \tau_4 = 0. \tag{2.4.31}$$

One can easily verify that, for every $i \in \{1, 2, 3, 4\}$, the solution of (2.4.4) for $v = u_i$ is given explicitly by

$$y_{i,1}(t) = \begin{cases} 0 & \text{on } [0, T] \setminus K_i, \\ Y_1(\cdot - \tau_i) & \text{on } K_i, \end{cases} \tag{2.4.32}$$

and

$$y_{i,2}(t) = \begin{cases} 0 & \text{on } [0, \tau_i], \\ Y_2(\cdot - \tau_i) & \text{on } K_i, \\ S(\cdot - \tau_i - T_0)\psi_1 & \text{on } [\tau_i + T_0, T]. \end{cases} \tag{2.4.33}$$

For $z \in M_1$, let α_1, α_2, α_3 and α_4 in $[0, +\infty)$ be such that

$$-S(T)z = \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + \alpha_4\psi_4, \tag{2.4.34}$$

$$\alpha_1\alpha_3 = 0, \ \alpha_2\alpha_4 = 0. \tag{2.4.35}$$

Let us define

$$v(t, z) := \alpha_1 u_1(t) + \alpha_2 u_2(t) + \alpha_3 u_3(t) + \alpha_4 u_4(t). \tag{2.4.36}$$

We notice that

$$(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)\|\psi_1\|^2_{L^2_L} = 1, \tag{2.4.37}$$

which, together with (2.4.36), implies that

$$v \in L^\infty([0, T] \times M_1; \mathbb{R}). \tag{2.4.38}$$

Moreover, using the above construction (and in particular (2.4.29)), one easily checks that the solution of (2.4.4) satisfies

$$y_1(t) = \alpha_1 y_{1,1}(t) + \alpha_2 y_{2,1}(t) + \alpha_3 y_{3,1}(t) + \alpha_4 y_{4,1}(t), \text{ for } t \in [0, T], \tag{2.4.39}$$

$$y_2(t) = \alpha_1^2 y_{1,2}(t) + \alpha_2^2 y_{2,2}(t) + \alpha_3^2 y_{3,2}(t) + \alpha_4^2 y_{4,2}(t), \text{ for } t \in [0, T]. \tag{2.4.40}$$

In particular

$$y_1(T) = 0, \tag{2.4.41}$$

$$y_2(T) = \alpha_1^2\psi_1 + \alpha_2^2\psi_2 + \alpha_3^2\psi_3 + \alpha_4^2\psi_4. \tag{2.4.42}$$

From (2.4.34), (2.4.37) and (2.4.42), we can find that (2.4.5) holds if $\delta > 0$ is small enough. It is easy to check that the Lipschitz condition (2.4.3) is also satisfied. This completes the construction of $v(t, z)$ such that $(\mathcal{P}_3)$ holds and also the proof of Proposition 2 if (2.4.10) holds.

For other values of $L \in \mathcal{N}_2$, only the values of φ_1, φ_2 and p have to be modified. For $L \in \mathcal{N}_3$, as mentioned in the introduction, M is now of dimension $2n$ where n is the number of ordered pairs. It is proved in [CC09a] that (compare with (2.4.11)–(2.4.14)), by a good choice of order on $\{\varphi^j\}$ one can assume

$$0 < p^1 < p^2 < ... < p^n, \tag{2.4.43}$$

where $p^j := 2\pi/\omega^j$. For every $t > 0$, one has

$$S(t)M^j \subset M^j \text{ and } S(t) \text{ restricted to } M^j \text{ is the rotation of angle } \frac{2\pi t}{p^j}. \tag{2.4.44}$$

From (2.4.43), (2.4.44) and Corollary 1, one can get the following corollary (see also [CC09a, Proposition 3.3]):

Corollary 2. *For every $L \in \mathcal{N}_3$, there exists $T_L > 0$ such that, for every $j \in \{1, 2, ..., n\}$, there exists $u_0^j \in L^\infty(0, T_L)$ such that the solution (y_1, y_2) to equation (2.4.4) with $v(t, z) := u_0^j(t)$ satisfies*

$$y_1(T_L) = 0 \quad \textit{and} \quad y_2(T_L) = \varphi_1^j. \tag{2.4.45}$$

Let us define

$$\psi_1^j := \varphi_1^j, \quad \psi_2^j := S(q^j)\varphi_1^j, \quad \psi_3^j := S(2q^j)\varphi_1^j, \quad \psi_4^j := S(3q^j)\varphi_1^j, \tag{2.4.46}$$

where $q^j := p^j/4$.

Compare with (2.4.22)–(2.4.33), we can find $T > T_L$ and closed interval sets $\{K_i^j\}$, where

$i \in \{1,2,3,4\}$ and $j \in \{1,2,...,n\}$, such that

$$K_i^j \subset [0,T], \tag{2.4.47}$$

$$\{K_i^j\} \text{ are pairwise disjoint.} \tag{2.4.48}$$

We can also find functions $\{u_i^j\} \in L^\infty([0,T];\mathbb{R})$, with

$$u_i^j(t) \text{ supports on } K_i^j, \tag{2.4.49}$$

such that when we define the control as u_i^j, we get the solution of (2.4.4) satisfies

$$y_{i,1}^j(t) \text{ supports on } K_i^j, \tag{2.4.50}$$

$$y_{i,1}^j(T) = 0, \tag{2.4.51}$$

$$y_{i,2}^j(T) = \psi_i^j. \tag{2.4.52}$$

Then for $z \in M_1$, let α_i^j in $[0,+\infty)$ be such that (where $i \in \{1,2,3,4\}$ and $j \in \{1,2,...,n\}$)

$$-S(T)z = \sum_{i,j} \alpha_i^j \psi_i^j, \tag{2.4.53}$$

$$\alpha_1^j \alpha_3^j = 0, \quad \alpha_2^j \alpha_4^j = 0, \quad \sum_{i,j} (\alpha_i^j)^2 = 1. \tag{2.4.54}$$

Let us define

$$v(t,z) := \sum_{i,j} \alpha_i^j u_i^j(t). \tag{2.4.55}$$

Then the solution of (2.4.4) with control defined as $v(t,z)$ satisfies

$$y_1(T) = 0, \tag{2.4.56}$$

$$y_2(T) = \sum_{i,j} (\alpha_i^j)^2 \psi_i^j. \tag{2.4.57}$$

One can easily verify that condition (2.4.5) holds when $\delta > 0$ is small enough, and that Lipschitz condition (2.4.3) also holds. This completes the construction of $v(t,z)$ and the proof of Proposition 2.
□

We are now able to define the periodic time-varying feedback laws $u_\varepsilon : \mathbb{R} \times L^2(0,L) \to \mathbb{R}$, which will lead to the exponential stabilization of (2.1.1). For $\varepsilon > 0$, we define u_ε by

$$u_\varepsilon\big|_{[0,T)\times L^2_L}(t,y) := \begin{cases} 0 & \text{if } \|y^M\|_{L^2_L} = 0, \\ \varepsilon\sqrt{\|y^M\|_{L^2_L}}\, v\big(t, \frac{S(-t)y^M}{\|y^M\|_{L^2_L}}\big) & \text{if } 0 < \|y^M\|_{L^2_L} \leqslant 1, \\ \varepsilon v\big(t, \frac{S(-t)y^M}{\|y^M\|_{L^2_L}}\big) & \text{if } \|y^M\|_{L^2_L} > 1, \end{cases} \tag{2.4.58}$$

with $y^M := P_M(y)$, and

$$u_\varepsilon(t,y) := u_\varepsilon\big|_{[0,T)\times L^2_L}(t - [\frac{t}{T}]T, y), \quad \forall t \in \mathbb{R}, \quad \forall y \in L^2(0,L). \tag{2.4.59}$$

2.5 Proof of Theorem 12

Let us first point out that Theorem 12 is a consequence of the following two propositions.

Proposition 4. *There exist $\varepsilon_1 > 0$, $r_1 > 0$ and C_1 such that, for every Carathéodory feedback law u satisfying*

$$|u(t,z)| \leqslant \varepsilon_1 min\{1, \sqrt{\|P_M(z)\|_{L^2_L}}\}, \quad \forall t \in \mathbb{R}, \forall z \in L^2(0,L), \tag{2.5.1}$$

for every $s \in \mathbb{R}$ and for every maximal solution y of (2.3.6) defined at time s and satisfying $\|y(s)\|_{L^2_L} < r_1$, y is well-defined on $[s, s+T]$ and one has

$$\|P_H(y)\|^2_{\mathcal{B}_{s,s+T}} + \|P_M(y)\|_{\mathcal{B}_{s,s+T}} \leqslant C_1(\|P_H(y(s))\|^2_{L^2_L} + \|P_M(y(s))\|_{L^2_L}). \tag{2.5.2}$$

Proposition 5. *For ρ_1 as in Proposition 2, let $\rho_2 > \rho_1$. There exists $\varepsilon_0 \in (0,1)$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists $r_\varepsilon > 0$ such that, for every solution y to (2.3.6) on $[0,T]$, for the feedback law $u := u_\varepsilon$ defined in (2.4.58) and (2.4.59), and satisfying $\|y(0)\|_{L^2_L} < r_\varepsilon$, one has*

$$\|P_H(y(T))\|^2_{L^2_L} + \varepsilon \|P_M(y(T))\|_{L^2_L} \leqslant \rho_2 \|P_H(y(0))\|^2_{L^2_L} + \varepsilon(1 - \delta\varepsilon^2)\|P_M(y(0))\|_{L^2_L}. \tag{2.5.3}$$

Indeed, it suffices to choose $\rho_2 \in (\rho_1, 1)$, $\varepsilon \in (0, \varepsilon_0)$ and $u := u_\varepsilon$ defined in (2.4.58) and (2.4.59). Then, using the T-periodicity of u with respect to time, Proposition 4 and Proposition 5, one checks that inequality (2.1.11) holds with $\lambda :=$ min $\{-(\ln(\rho_2))/(2T), -(\ln(1-\delta\varepsilon^2))/(2T)\}$ provided that C is large enough and that r is small enough. We now prove Proposition 4 and Proposition 5 successively.

Proof of Proposition 4. Performing a time-translation if necessary, we may assume without loss of generality that $s = 0$. The fact that the maximal solution y is at least defined on $[0,T]$ follows from Theorem 14 and (2.5.1). We choose ε_1 and r_1 small enough so that

$$r_1 + \varepsilon_1 T^{\frac{1}{2}} \leqslant \eta, \tag{2.5.4}$$

where $\eta > 0$ is as in Lemma 4. From (2.5.1) and (2.5.4), we have

$$\|y(0)\|_{L^2_L} + \|u(t, y(t))\|_{L^2_T} \leqslant \eta, \tag{2.5.5}$$

which allows to apply Lemma 4 with $H(t) := u(t, y(t))$, $\tilde{H} := 0$. Then, using (2.5.1) once more, we get

$$\begin{aligned}
\|y\|_{\mathcal{B}} &\leqslant C_3\big(\|y_0\|_{L^2_L} + \|u(t,y(t))\|_{L^2_T}\big) \\
&\leqslant C_3\big(r_1 + \varepsilon_1\sqrt{T\|P_M(y)\|_{C^0 L^2_L}}\big) \\
&\leqslant C_3\big(r_1 + \varepsilon_1^2 T C_3 + \frac{1}{4C_3}\|y\|_{\mathcal{B}}\big),
\end{aligned}$$

which implies that

$$\|y\|_{\mathcal{B}} \leqslant 2C_3\big(r_1 + \varepsilon_1^2 T C_3\big). \tag{2.5.6}$$

In the above inequalities and until the end of the proof of Proposition 5, $\mathcal{B} := \mathcal{B}_{0,T}$.

We have the following lemma, see the proof of [Ros97, Proposition 4.1 and (4.14)] or [PMVZ02, page 121].

Lemma 6. *If $y \in L^2(0,T;H^1(0,L))$, then $yy_x \in L^1(0,T;L^2(0,L))$. Moreover, there exists $c_4 > 0$, which is independent of T, such that, for every $T > 0$ and for every $y, z \in L^2(0,T;H^1(0,L))$, we* *have*

$$\|yy_x - zz_x\|_{L^1_T L^2_L} \leqslant c_4 T^{\frac{1}{4}} \big(\|y\|_{\mathcal{B}} + \|z\|_{\mathcal{B}}\big) \|y - z\|_{\mathcal{B}}. \tag{2.5.7}$$

Let us define $C_4 := c_4 T^{\frac{1}{4}}$. To simplify the notations, until the end of this section, we write y_1 and y_2 for $P_H(y)$ and $P_M(y)$ respectively. From (2.5.1), (2.5.6), Lemma 3, Lemma 6 and Proposition 1, we get

$$\begin{aligned}
\|y_1\|_{\mathcal{B}} &\leqslant C_2 \big(\|y_0^H\|_{L^2_L} + \|u(t, y_1 + y_2)\|_{L^2_T} + \|P_H\big((y_1+y_2)(y_1+y_2)_x\big)\|_{L^1_T L^2_L}\big) \\
&\leqslant C_2 \big(\|y_0^H\|_{L^2_L} + \varepsilon_1 \|\sqrt{\|y_2\|_{L^2_L}}\|_{L^2_T} + \|(y_1+y_2)(y_1+y_2)_x\|_{L^1_T L^2_L}\big) \\
&\leqslant C_2 \big(\|y_0^H\|_{L^2_L} + \varepsilon_1 \|y_2\|^{\frac{1}{2}}_{L^1_T L^2_L} + C_4 \|y_1 + y_2\|^2_{L^2_T H^1_L}\big),
\end{aligned} \tag{2.5.8}$$

and

$$\begin{aligned}
\|y_2\|_{\mathcal{B}} &\leqslant C_2 \big(\|y_0^M\|_{L^2_L} + \|P_M\big((y_1+y_2)(y_1+y_2)_x\big)\|_{L^1_T L^2_L}\big) \\
&\leqslant C_2 \big(\|y_0^M\|_{L^2_L} + \|(y_1+y_2)(y_1+y_2)_x\|_{L^1_T L^2_L}\big) \\
&\leqslant C_2 \big(\|y_0^M\|_{L^2_L} + C_4 \|y_1+y_2\|^2_{L^2_T H^1_L}\big) \\
&\leqslant 2C_2 \big(\|y_0^M\|_{L^2_L} + C_4 \|y_1\|^2_{\mathcal{B}} + C_4 \|y_2\|^2_{\mathcal{B}}\big).
\end{aligned} \tag{2.5.9}$$

Since M is a finite dimensional subspace of $H^1(0, L)$, there exists $C_5 > 0$ such that

$$\|f\|_{H^1(0,L)} \leqslant C_5 \|f\|_{L^2_L}, \quad \text{for every } f \in M. \tag{2.5.10}$$

Hence

$$\begin{aligned}
\|y_2\|_{\mathcal{B}} &= \|y_2\|_{L^\infty_T L^2_L} + \|y_2\|_{L^2_T H^1_L} \\
&\leqslant \|y_2\|_{L^\infty_T L^2_L} + C_5 \sqrt{T} \|y_2\|_{L^\infty_T L^2_L}.
\end{aligned} \tag{2.5.11}$$

Since $y_2(t)$ is the L^2-orthogonal projection on M of $y(t)$, we have

$$\|y_2\|_{L^\infty_T L^2_L} \leqslant \|y\|_{L^\infty_T L^2_L} \leqslant \|y\|_{\mathcal{B}},$$

which, together with (2.5.6) and (2.5.11), implies that

$$\|y_2\|_{\mathcal{B}} \leqslant (1 + C_5\sqrt{T}) \|y\|_{\mathcal{B}} \leqslant 2(1 + C_5 \sqrt{T}) C_3 \big(r_1 + \varepsilon_1^2 T C_3\big). \tag{2.5.12}$$

Decreasing if necessary r_1 and ε_1, we may assume that

$$4C_2 C_4 (1 + C_5 \sqrt{T}) C_3 \big(r_1 + \varepsilon_1^2 T C_3\big) < \frac{1}{2}. \tag{2.5.13}$$

From estimation (2.5.9) and condition (2.5.13), we get that

$$\|y_2\|_{\mathcal{B}} \leqslant 4C_2 \big(\|y_0^M\|_{L^2_L} + C_4 \|y_1\|^2_{\mathcal{B}}\big). \tag{2.5.14}$$

From (2.5.6), (2.5.8), (2.5.12) and (2.5.14), we deduce that

$$
\begin{aligned}
\|y_1\|^2_{\mathcal{B}} &\leqslant 3C_2^2\big(\|y_0^H\|^2_{L^2_L} + \varepsilon_1^2\|y_2\|_{L^1_T L^2_L} + C_4^2\|y_1+y_2\|^4_{L^2_T H^1_L}\big)\\
&\leqslant 3C_2^2\big(\|y_0^H\|^2_{L^2_L} + \varepsilon_1^2 T\|y_2\|_{L^\infty_T L^2_L} + 2C_4^2\|y\|^2_{\mathcal{B}}(\|y_1\|^2_{\mathcal{B}} + \|y_2\|^2_{\mathcal{B}})\big)\\
&\leqslant 3C_2^2\|y_0^H\|^2_{L^2_L} + 3C_2^2\big(\varepsilon_1^2 T + 16C_4^2(1+C_5\sqrt{T})C_3^3(r_1+\varepsilon_1^2 TC_3)^3\big)\|y_2\|_{\mathcal{B}}\\
&\qquad + 24C_2^2C_4^2C_3^2(r_1+\varepsilon_1^2TC_3)^2\|y_1\|^2_{\mathcal{B}}\\
&\leqslant 3C_2^2\|y_0^H\|^2_{L^2_L} + 12C_2^3\big(\varepsilon_1^2 T + 16C_4^2(1+C_5\sqrt{T})C_3^3(r_1+\varepsilon_1^2 TC_3)^3\big)\|y_0^M\|_{L^2_L}\\
&+ \Big(12C_2^3C_4\big(\varepsilon_1^2 T + 16C_4^2(1+C_5\sqrt{T})C_3^3(r_1+\varepsilon_1^2TC_3)^3\big) + 24C_2^2C_4^2C_3^2(r_1+\varepsilon_1^2TC_3)^2\Big)\|y_1\|^2_{\mathcal{B}}.
\end{aligned}
\tag{2.5.15}
$$

Again, decreasing if necessary r_1 and ε_1, we may assume that

$$
12C_2^3C_4\big(\varepsilon_1^2 T + 16C_4^2(1+C_5\sqrt{T})C_3^3(r_1+\varepsilon_1^2TC_3)^3\big) + 24C_2^2C_4^2C_3^2(r_1+\varepsilon_1^2TC_3)^2 < \frac{1}{2}. \tag{2.5.16}
$$

From (2.5.15) and (2.5.16), we get

$$
\begin{aligned}
\|y_1\|^2_{\mathcal{B}} &\leqslant 6C_2^2\|y_0^H\|^2_{L^2_L} + 24C_2^3\big(\varepsilon_1^2 T + 16C_4^2(1+C_5\sqrt{T})C_3^3(r_1+\varepsilon_1^2TC_3)^3\big)\|y_0^M\|_{L^2_L}\\
&\leqslant 6C_2^2\|y_0^H\|^2_{L^2_L} + C_4^{-1}\|y_0^M\|_{L^2_L},
\end{aligned}
$$

which, combined with (2.5.14), gives the existence of $C_1 > 0$ independent of y such that

$$
\|y_1\|^2_{\mathcal{B}} + \|y_2\|_{\mathcal{B}} \leqslant C_1\big(\|y_0^H\|^2_{L^2_L} + \|y_0^M\|_{L^2_L}\big). \tag{2.5.17}
$$

This completes the proof of Proposition 4. □

Proof of Proposition 5. To simplify the notations, from now on we denote by C various constants which vary from place to place but do not depend on ε and r.

By Lemma 3 applied with $y := y_1(t) - S(t)y_0^H$, $h(t) := u_\varepsilon(t, y(t))$ and $\tilde{h} := (y_1+y_2)(y_1+y_2)_x$ and by Proposition 4, we have

$$
\begin{aligned}
\|y_1(t) - S(t)y_0^H\|_{\mathcal{B}} &\leqslant C\big(\|u_\varepsilon\|_{L^2_T} + \|P_H((y_1+y_2)(y_1+y_2)_x)\|_{L^1_T L^2_L}\big)\\
&\leqslant C\big(\varepsilon\|y_2\|^{\frac{1}{2}}_{L^1_T L^2_L} + \|y_1+y_2\|^2_{\mathcal{B}}\big)\\
&\leqslant C\big(\varepsilon\|y_2\|^{\frac{1}{2}}_{\mathcal{B}} + \|y_1\|^2_{\mathcal{B}} + \|y_2\|^2_{\mathcal{B}}\big)\\
&\leqslant C(\varepsilon + \sqrt{r})\big(\|y_0^H\|^2_{L^2_L} + \|y_0^M\|_{L^2_L}\big)^{\frac{1}{2}},
\end{aligned}
\tag{2.5.18}
$$

where $r := \|y_0\|_{L^2_L} < r_\varepsilon < 1$. On r_ε, we impose that

$$
r_\varepsilon < \varepsilon^{12}. \tag{2.5.19}
$$

From (2.5.18) and (2.5.19), we have

$$
\|y_1(t) - S(t)y_0^H\|_{\mathcal{B}} \leqslant C\varepsilon\big(\|y_0^H\|^2_{L^2_L} + \|y_0^M\|_{L^2_L}\big)^{\frac{1}{2}}. \tag{2.5.20}
$$

Notice that, by Lemma 3, we have

$$\|S(t)y_0^M\|_{\mathcal{B}} \leqslant C\|y_0^M\|_{L_L^2}, \tag{2.5.21}$$
$$\|S(t)y_0^H\|_{\mathcal{B}} \leqslant C\|y_0^H\|_{L_L^2}. \tag{2.5.22}$$

Proceeding as in the proof of (2.5.20), we have

$$\begin{aligned}\|y_2(t) - S(t)y_0^M\|_{\mathcal{B}} &\leqslant C\|P_M((y_1+y_2)(y_1+y_2)_x)\|_{L_T^1 L_L^2}\\ &\leqslant C\|y_1+y_2\|_{\mathcal{B}}^2\\ &\leqslant C\big(\|y_2\|_{\mathcal{B}}+\|S(t)y_0^H\|_{\mathcal{B}}+\varepsilon\big(\|y_0^H\|_{L_L^2}^2+\|y_0^M\|_{L_L^2}\big)^{\frac{1}{2}}\big)^2\\ &\leqslant C\big((r+\varepsilon^2)(\|y_0^H\|_{L_L^2}^2+\|y_0^M\|_{L_L^2})+\|y_0^H\|_{L_L^2}^2\big)\\ &\leqslant C\big(\varepsilon^2\|y_0^M\|_{L_L^2}+\|y_0^H\|_{L_L^2}^2\big).\end{aligned} \tag{2.5.23}$$

Let us now study successively the two following cases

$$\|y_0^H\|_{L_L^2} \geqslant \varepsilon^{\frac{2}{3}}\sqrt{\|y_0^M\|_{L_L^2}}, \tag{2.5.24}$$
$$\|y_0^H\|_{L_L^2} < \varepsilon^{\frac{2}{3}}\sqrt{\|y_0^M\|_{L_L^2}}. \tag{2.5.25}$$

We start with the case where (2.5.24) holds. From $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, (2.5.20), (2.5.23) and (2.5.24), we get the existence of $\varepsilon_2 \in (0,\varepsilon_1)$ such that, for every $\varepsilon \in (0,\varepsilon_2)$,

$$\begin{aligned}&\|y_1(T)\|_{L_L^2}^2+\varepsilon\|y_2(T)\|_{L_L^2}\\ \leqslant{}& \big(C\varepsilon\big(\|y_0^H\|_{L_L^2}^2+\|y_0^M\|_{L_L^2}\big)^{\frac{1}{2}}+\|S(T)y_0^H\|_{L_L^2}\big)^2+\varepsilon\big(C\big(\varepsilon^2\|y_0^M\|_{L_L^2}+\|y_0^H\|_{L_L^2}^2\big)+\|S(T)y_0^M\|_{L_L^2}\big)\\ \leqslant{}& (\rho_1\rho_2)^{\frac{1}{2}}\|y_0^H\|_{L_L^2}^2+C\varepsilon^2\big(\|y_0^H\|_{L_L^2}^2+\|y_0^M\|_{L_L^2}\big)+C\varepsilon\|y_0^H\|_{L_L^2}^2+(\varepsilon+C\varepsilon^3)\|y_0^M\|_{L_L^2}\\ \leqslant{}& \rho_2\|y_0^H\|_{L_L^2}^2+\varepsilon(1-\delta\varepsilon^2)\|y_0^M\|_{L_L^2}.\end{aligned} \tag{2.5.26}$$

Let us now study the case where (2.5.25) holds. Let us define

$$b := y_0^M. \tag{2.5.27}$$

Then, from (2.5.20), (2.5.22), (2.5.23) and (2.5.25), we get

$$\begin{aligned}\|y_1(t)\|_{\mathcal{B}} &\leqslant \|S(t)y_0^H\|_{\mathcal{B}}+C\varepsilon\big(\|y_0^H\|_{L_L^2}^2+\|y_0^M\|_{L_L^2}\big)^{\frac{1}{2}}\\ &\leqslant C\varepsilon\sqrt{\|b\|_{L_L^2}}+C\|y_0^H\|_{L_L^2}\\ &\leqslant C\varepsilon^{\frac{2}{3}}\sqrt{\|b\|_{L_L^2}},\end{aligned} \tag{2.5.28}$$

and

$$\|y_2(t)-S(t)y_0^M\|_{\mathcal{B}} \leqslant \varepsilon^{\frac{4}{3}}\|b\|_{L_L^2}, \tag{2.5.29}$$

which shows that $y_2(\cdot)$ is close to $S(\cdot)y_0^M$. Let $z:[0,T]\to L^2(0,L)$ be the solution to the Cauchy problem

$$\begin{cases} z_{1t} + z_{1xxx} + z_{1x} = 0 & \text{in } (0,T)\times(0,L), \\ z_1(t,0) = z_1(t,L) = 0 & \text{on } (0,T), \\ z_{1x}(t,L) = v(t, \dfrac{b}{\|b\|_{L^2_L}}) & \text{on } (0,T), \\ z_1(0,x) = 0 & \text{on } (0,L). \end{cases} \tag{2.5.30}$$

From $(\mathcal{P}_3)$, we know that $z_1(T) = 0$. Moreover, Lemma 3 tells us that

$$\|z_1(t)\|_{\mathcal{B}} \leqslant C\|v(t, \frac{b}{\|b\|_{L^2_L}})\|_{L^2_T} \leqslant C. \tag{2.5.31}$$

Let us define w_1 by

$$w_1 := y_1 - S(t)y_0^H - \varepsilon\|b\|_{L^2_L}^{\frac{1}{2}} z_1. \tag{2.5.32}$$

Then w_1 is the solution to the Cauchy problem

$$\begin{cases} w_{1t} + w_{1xxx} + w_{1x} + P_H\big((y_1+y_2)(y_1+y_2)_x\big) = 0, \\ w_1(t,0) = w_1(t,L) = 0, \\ w_{1x}(t,L) = \varepsilon\left(\|y_2(t)\|_{L^2_L}^{\frac{1}{2}} v(t, \dfrac{S(-t)y_2(t)}{\|y_2(t)\|_{L^2_L}}) - \|b\|_{L^2_L}^{\frac{1}{2}} v(t, \dfrac{b}{\|b\|_{L^2_L}})\right), \\ w_1(0,x) = 0. \end{cases} \tag{2.5.33}$$

By Lemma 3, we get

$$\begin{aligned} \|w_1\|_{\mathcal{B}} \leqslant & C\|P_H\big((y_1+y_2)(y_1+y_2)_x\big)\|_{L^1_T L^2_L} \\ & + \varepsilon C\|\big(\|y_2(t)\|_{L^2_L}^{\frac{1}{2}} v(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L^2_L}}) - \|b\|_{L^2_L}^{\frac{1}{2}} v(t, \frac{b}{\|b\|_{L^2_L}})\big)\|_{L^2_T}. \end{aligned} \tag{2.5.34}$$

Note that (2.5.29) insures that the right hand side of (2.5.34) is of order ε^2. Indeed, for the first term of the right-hand side of inequality (2.5.34), we have, using (2.5.19), (2.5.28) and (2.5.29),

$$\begin{aligned} C\|P_H\big((y_1+y_2)(y_1+y_2)_x\big)\|_{L^1_T L^2_L} & \leqslant C\|y_1+y_2\|_{\mathcal{B}}^2 \\ & \leqslant C\varepsilon^{\frac{4}{3}}\|b\|_{L^2_L} + C\|b\|_{L^2_L} \\ & \leqslant C\|b\|_{L^2_L}^{\frac{1}{2}}\|b\|_{L^2_L}^{\frac{1}{2}} \\ & \leqslant C\varepsilon^6\|b\|_{L^2_L}^{\frac{1}{2}}. \end{aligned} \tag{2.5.35}$$

For the second term of the right hand side of inequality (2.5.34), by (2.4.14), the Lipschitz condition (2.4.3) on v and (2.5.29), we get, for every $t \in [0,T]$,

$$\begin{aligned} & \Big|\, \|b\|_{L^2_L}^{\frac{1}{2}}\big(v(t, \frac{b}{\|b\|_{L^2_L}}) - v(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L^2_L}})\big) \,\Big| \\ \leqslant & C\|b\|_{L^2_L}^{\frac{1}{2}}\|(\frac{b}{\|b\|_{L^2_L}} - \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L^2_L}})\|_{L^2_L} \\ \leqslant & C\|b\|_{L^2_L}^{-\frac{1}{2}}\|y_2(t)\|_{L^2_L}^{-1}\big(\|y_2(t)\|_{L^2_L}\|b - S(-t)y_2(t)\|_{L^2_L} + \|S(-t)y_2(t)\|_{L^2_L}\big|\|y_2(t)\|_{L^2_L} - \|b\|_{L^2_L}\big|\big) \\ \leqslant & C\varepsilon^{\frac{4}{3}}\|b\|_{L^2_L}^{\frac{1}{2}}, \end{aligned} \tag{2.5.36}$$

and

$$|(\|y_2(t)\|_{L^2_L}^{\frac{1}{2}} - \|b\|_{L^2_L}^{\frac{1}{2}})v(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L^2_L}})| \leqslant C\varepsilon^{\frac{4}{3}}\|b\|_{L^2_L}^{\frac{1}{2}}. \tag{2.5.37}$$

Combining (2.5.35), (2.5.36) and (2.5.37), we obtain the following estimate on w_1

$$\|w_1\|_{\mathcal{B}} \leqslant C\varepsilon^2\|b\|_{L^2_L}^{\frac{1}{2}}. \tag{2.5.38}$$

We fix

$$\rho_3 \in (\rho_1, \rho_2). \tag{2.5.39}$$

Then, by (2.5.32), $(\mathcal{P}_1)$ as well as the fact that $z_1(T) = 0$, we get

$$\|y_1(T)\|_{L^2_L}^2 \leqslant \rho_3\|y_0^H\|_{L^2_L}^2 + C\varepsilon^4\|b\|_{L^2_L}. \tag{2.5.40}$$

We then come to the estimate of y_2. Let $\tau_1(t) := S(t)y_0^H$ and let $\tau_2 : [0, T] \to L^2(0, L)$ and $z_2 : [0, T] \to L^2(0, L)$ be the solutions to the Cauchy problems

$$\begin{cases} \tau_{2t} + \tau_{2xxx} + \tau_{2x} + P_M(\tau_1 y_{1x} + \tau_{1x} y_1) - P_M(\tau_1\tau_{1x}) = 0, \\ \tau_2(t, 0) = \tau_2(t, L) = 0, \\ \tau_{2x}(t, L) = 0, \\ \tau_2(0, x) = 0, \end{cases} \tag{2.5.41}$$

and

$$\begin{cases} z_{2t} + z_{2xxx} + z_{2x} + P_M(z_1 z_{1x}) = 0, \\ z_2(t, 0) = z_2(t, L) = 0, \\ z_{2x}(t, L) = 0, \\ z_2(0, x) = 0. \end{cases} \tag{2.5.42}$$

Lemma 3, Lemma 6, (2.5.25) and (2.5.28) show us that

$$\begin{aligned} \|\tau_2\|_{\mathcal{B}} &\leqslant C\|P_M(\tau_1 y_{1x} + \tau_{1x} y_1 - \tau_1\tau_{1x})\|_{L^1_T L^2_L} \\ &\leqslant C\|\tau_1\|_{\mathcal{B}}(\|y_1\|_{\mathcal{B}} + \|\tau_1\|_{\mathcal{B}}) \\ &\leqslant C\varepsilon^{\frac{2}{3}}\|b\|_{L^2_L}^{\frac{1}{2}}\|y_0^H\|_{L^2_L}, \end{aligned} \tag{2.5.43}$$

and

$$\|z_2\|_{\mathcal{B}} \leqslant \|z_1\|_{\mathcal{B}}^2 \leqslant C. \tag{2.5.44}$$

From $(\mathcal{P}_3)$, (2.5.30) and (2.5.42), we get

$$\left\langle z_2(T), S(T)b \right\rangle_{(L^2_L, L^2_L)} < -2\delta\|b\|_{L^2_L}. \tag{2.5.45}$$

Hence

$$\begin{aligned} \|S(T)b + \varepsilon^2\|b\|_{L^2_L} z_2(T)\|_{L^2_L} &= \left(\left\langle S(T)b + \varepsilon^2\|b\|_{L^2_L} z_2(T), S(T)b + \varepsilon^2\|b\|_{L^2_L} z_2(T)\right\rangle_{(L^2_L, L^2_L)}\right)^{\frac{1}{2}} \\ &\leqslant \left(\|b\|_{L^2_L}^2 + \varepsilon^4\|b\|_{L^2_L}^2 C - 4\delta\varepsilon^2\|b\|_{L^2_L}^2\right)^{\frac{1}{2}} \\ &\leqslant \|b\|_{L^2_L}(1 - 2\delta\varepsilon^2 + C\varepsilon^4). \end{aligned} \tag{2.5.46}$$

Let us define $w_2 : [0,T] \to L^2(0,L)$ by

$$w_2 := y_2 - \tau_2 - \varepsilon^2 \|b\|_{L^2_L} z_2 - S(t)b. \tag{2.5.47}$$

Then, from (2.3.15), (2.5.41) and (2.5.42), we get that

$$\begin{aligned}
w_{2t} &= y_{2t} - \tau_{2t} - \varepsilon^2 \|b\|_{L^2_L} z_{2t} - (S(t)b)_t \\
&= -w_{2x} - w_{2xxx} - P_M\big((y_1+y_2)(y_1+y_2)_x\big) + P_M\big(\tau_1 y_{1x} + \tau_{1x} y_1\big) \\
&\quad - P_M\big(\tau_1 \tau_{1x}\big) + \varepsilon^2 \|b\|_{L^2_L} P_M\big(z_1 z_{1x}\big) \\
&= -w_{2x} - w_{2xxx} - \varepsilon \|b\|^{\frac{1}{2}}_{L^2_L} P_M\big(w_1 z_{1x} + w_{1x} z_1\big) - P_M\big(w_1 w_{1x}\big) \\
&\quad - P_M\big(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x}\big).
\end{aligned}$$

Hence, w_2 is the solution to the Cauchy problem

$$\begin{cases}
w_{2t} + w_{2xxx} + w_{2x} + \varepsilon \|b\|^{\frac{1}{2}}_{L^2_L} P_M\big(w_1 z_{1x} + w_{1x} z_1\big) + P_M\big(w_1 w_{1x}\big) \\
\qquad\qquad + P_M\big(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x}\big) = 0, \\
w_2(t,0) = w_2(t,L) = 0, \\
w_{2x}(t,L) = 0, \\
w_2(0,x) = 0.
\end{cases} \tag{2.5.48}$$

From Lemma 3, Lemma 6, Proposition 4, (2.5.19), (2.5.25) and (2.5.38), we get

$$\begin{aligned}
\|w_2\|_{\mathcal{B}} &\leqslant C\varepsilon \|b\|^{\frac{1}{2}}_{L^2_L} \|P_M\big(w_1 z_{1x} + w_{1x} z_1\big)\|_{L^1_T L^2_L} + C\|P_M\big(w_1 w_{1x}\big)\|_{L^1_T L^2_L} \\
&\qquad + C\|P_M\big(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x}\big)\|_{L^1_T L^2_L} \\
&\leqslant C\varepsilon \|b\|^{\frac{1}{2}}_{L^2_L} \varepsilon^2 \|b\|^{\frac{1}{2}}_{L^2_L} + C\varepsilon^4 \|b\|_{L^2_L} + C\big(\|y_0^H\|^2_{L^2_L} + \|y_0^M\|_{L^2_L}\big)^{\frac{3}{2}} \\
&\leqslant C\varepsilon^3 \|b\|_{L^2_L}.
\end{aligned} \tag{2.5.49}$$

We can now estimate $y_2(T)$ from (2.5.43), (2.5.46), (2.5.47) and (2.5.49):

$$\begin{aligned}
\|y_2(T)\|_{L^2_L} &= \|w_2(T) + \tau_2(T) + \varepsilon^2 \|b\|_{L^2_L} z_2(T) + S(T)b\|_{L^2_L} \\
&\leqslant \|b\|_{L^2_L}\big(C\varepsilon^3 + 1 - 2\delta\varepsilon^2 + C\varepsilon^4\big) + C\varepsilon^{\frac{2}{3}} \|b\|^{\frac{1}{2}}_{L^2_L} \|y_0^H\|_{L^2_L}.
\end{aligned} \tag{2.5.50}$$

Combining(2.5.27), (2.5.39), (2.5.40) and (2.5.50), we get existence of $\varepsilon_3 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_3]$, we have

$$\begin{aligned}
&\|y_1(T)\|^2_{L^2_L} + \varepsilon \|y_2(T)\|_{L^2_L} \\
&\quad \leqslant \rho_3 \|y_0^H\|^2_{L^2_L} + C\varepsilon^4 \|b\|_{L^2_L} + \varepsilon\left(\|b\|_{L^2_L}\big(C\varepsilon^3 + 1 - 2\delta\varepsilon^2 + C\varepsilon^4\big) + C\varepsilon^{\frac{2}{3}} \|b\|^{\frac{1}{2}}_{L^2_L} \|y_0^H\|_{L^2_L}\right) \\
&\quad \leqslant \rho_2 \|y_0^H\|^2_{L^2_L} + \varepsilon(1 - \delta\varepsilon^2)\|y_0^M\|_{L^2_L}.
\end{aligned} \tag{2.5.51}$$

This concludes the proof of Proposition 5. □

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2.6 Appendix A: Proof of Proposition 1

Proof of Proposition 1. It is clear that, if (y_1, y_2) is a solution to (2.3.15), then y is solution to (2.3.7). Let us assume that y is a solution to the Cauchy problem (2.3.7). Then, by Definition 7, for every $\tau \in [s, T]$ and for every $\phi \in C^3([s,\tau] \times [0, L])$ satisfying

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0, \quad \forall t \in [s, \tau], \tag{2.6.1}$$

we have

$$\begin{aligned} -\int_s^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y dx dt - \int_s^\tau u(t, y(t,\cdot))\phi_x(t, L)dt + \int_s^\tau \int_0^L \phi y y_x dx dt \\ + \int_0^L y(\tau, x)\phi(\tau, x)dx - \int_0^L y_0 \phi(s, x)dx = 0. \end{aligned} \tag{2.6.2}$$

Let us denote by ϕ_1 and ϕ_2 the projection of ϕ on H and M respectively: $\phi_1 := P_H(\phi)$, $\phi_2 := P_M(\phi)$. Because M is spanned by φ_1^j and φ_2^j, $j \in \{1, \dots, n\}$, which are of class C^∞ and satisfy

$$\varphi_1^j(0) = \varphi_1^j(L) = \varphi_{1x}^j(0) = \varphi_{1x}^j(L) = 0,$$
$$\varphi_2^j(0) = \varphi_2^j(L) = \varphi_{2x}^j(0) = \varphi_{2x}^j(L) = 0,$$

the functions $\phi_1, \phi_2 \in C^3([s, \tau] \times [0, L])$ and satisfy

$$\phi_1(t, 0) = \phi_1(t, L) = \phi_{1x}(t, 0) = 0, \forall t \in [s, \tau], \tag{2.6.3}$$
$$\phi_2(t, 0) = \phi_2(t, L) = \phi_{2x}(t, 0) = \phi_{2x}(t, L) = 0, \forall t \in [s, \tau]. \tag{2.6.4}$$

Using (2.6.2) for $\phi = \phi_2$ in (2.6.2) together with (2.6.4), we get

$$\begin{aligned} -\int_s^\tau \int_0^L (\phi_{2t} + \phi_{2x} + \phi_{2xxx}) y dx dt + \int_s^\tau \int_0^L \phi_2 y y_x dx dt \\ + \int_0^L y(\tau, x)\phi_2(\tau, x)dx - \int_0^L y_0 \phi_2(s, x)dx = 0, \end{aligned} \tag{2.6.5}$$

which, combined with the fact that $\phi_{2t} + \phi_{2x} + \phi_{2xxx} \in M$, gives

$$\begin{aligned} -\int_s^\tau \int_0^L (\phi_{2t} + \phi_{2x} + \phi_{2xxx}) y_2 dx dt + \int_s^\tau \int_0^L \phi_2 P_M(y y_x) dx dt \\ + \int_0^L y_2(\tau, x)\phi_2(\tau, x)dx - \int_0^L P_M(y_0) \phi_2(s, x)dx = 0. \end{aligned} \tag{2.6.6}$$

Simple integrations by parts show that $\phi_{1x} + \phi_{1xxx} \in M^\perp = H$. Since, ϕ_1 and ϕ_{1t} are also in H, we get from (2.6.6) that

$$\begin{aligned} -\int_s^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y_2 dx dt + \int_s^\tau \int_0^L \phi P_M(y y_x) dx dt \\ + \int_0^L y_2(\tau, x)\phi(\tau, x)dx - \int_0^L P_M(y_0) \phi(s, x)dx = 0, \end{aligned} \tag{2.6.7}$$

which is exactly the definition of a solution of the second part of the linear KdV system (2.3.15).

We then combine (2.6.2) and (2.6.7) to get

$$-\int_s^\tau\int_0^L(\phi_t+\phi_x+\phi_{xxx})y_1dxdt-\int_s^\tau u(t,y(t,\cdot))\phi_x(t,L)dt+\int_s^\tau\int_0^L\phi P_H(yy_x)dxdt$$
$$+\int_0^L y_1(\tau,x)\phi(\tau,x)dx-\int_0^L P_H(y_0)\phi(0,x)dx=0, \tag{2.6.8}$$

and we get the definition of a solution to the first part of the linear KdV system (2.3.15). This concludes the proof of Proposition 1. □

2.7 Appendix B: Proofs of Theorem 13 and Theorem 14

Our strategy to prove Theorem 13 is to prove first the existence of a solution for small times and then to use some a priori estimates to control the L^2_L-norm of the solution with which we can extend the solution to a longer time, and to continue until the solution blows up. We start by proving the following lemma.

Lemma 7. *Let $C_2>0$ be as in Lemma 3 for $T_2-T_1=1$. Assume that u is a Carathéodory function and that, for every $R>0$, there exists $K(R)>0$ such that*

$$\Big(\|y\|_{L^2_L}\leqslant R \text{ and } \|z\|_{L^2_L}\leqslant R\Big)\Rightarrow\Big(|u(t,y)-u(t,z)|\leqslant K(R)\|y-z\|_{L^2_L},\quad \forall t\in\mathbb{R}\Big). \tag{2.7.1}$$

Then, for every $R\in(0,+\infty)$, there exists a time $T(R)>0$ such that, for every $s\in\mathbb{R}$ and for every $y_0\in L^2(0,L)$ with $\|y_0\|_{L^2_L}\leqslant R$, the Cauchy problem (2.3.7) has one and only one solution y on $[s,s+T(R)]$. Moreover, this solution satisfies

$$\|y\|_{\mathcal{B}_{s,s+T(R)}}\leqslant C_R:=3C_2R. \tag{2.7.2}$$

Proof of Lemma 7. Let us first point out that it follows from our choice of C_2 and Lemma 3 that, for every $-\infty<T_1<T_2<+\infty$ such that $T_2-T_1\leqslant 1$, for every solution y of problem (2.2.1), estimation (2.2.12) holds.

Let $y_0\in L^2(0,L)$ be such that

$$\|y_0\|_{L^2_L}\leqslant R. \tag{2.7.3}$$

Let us define $\mathcal{B}_1$ by

$$\mathcal{B}_1:=\big\{y\in\mathcal{B}_{s,s+T(R)};\ \|y\|_{\mathcal{B}_{s,s+T(R)}}\leqslant C_R\big\}.$$

The set $\mathcal{B}_1$ is a closed subset of $\mathcal{B}_{s,s+T(R)}$. For every $y\in\mathcal{B}_1$, we define $\Psi(y)$ as the solution of (2.2.1) with $\tilde h:=-yy_x$, $h(t):=u(t,y(t,\cdot))$ and $y_0:=y_0$. Let us prove that, for $T(R)$ small enough, the smallness being independent of y_0 provided that it satisfies (2.7.3), we have

$$\Psi(\mathcal{B}_1)\subset\mathcal{B}_1. \tag{2.7.4}$$

Indeed for $y\in\mathcal{B}_1$, by Lemma 3 and Lemma 6, we have, if $T(R)\leqslant 1$,

$$\begin{aligned}\|\Psi(y)\|_{\mathcal{B}}&\leqslant C_2\big(\|y_0\|_{L^2_L}+\|h\|_{L^2_T}+\|\tilde h\|_{L^1(0,T;L^2(0,L))}\big)\\&\leqslant C_2\big(\|y_0\|_{L^2_L}+\|u(t,y(t,\cdot))\|_{L^2_T}+\|-yy_x\|_{L^1(s,s+T(R);L^2(0,L))}\big)\\&\leqslant C_2\big(R+C_B(C_R)T(R)^{\frac12}+c_4T(R)^{\frac14}\|y\|^2_{\mathcal{B}}\big).\end{aligned} \tag{2.7.5}$$

In (2.7.5) and until the end of the proof of Lemma 7, for ease of notation, we simply write $\|\cdot\|_{\mathcal{B}}$ for

$\|\cdot\|_{\mathcal{B}_{s,s+T(R)}}$. From (2.7.5), we get that, if

$$T(R) \leqslant \min\left\{ \left(\frac{R}{C_B(C_R)}\right)^2, \left(\frac{1}{9c_4 C_2^2 R}\right)^4, 1 \right\}, \tag{2.7.6}$$

then (2.7.4) holds. From now on, we assume that (2.7.6) holds.

Note that every $y \in \mathcal{B}_1$ such that $\Psi(y) = y$ is a solution of (2.3.7). In order to use the Banach fixed point theorem, it remains to estimate $\|\Psi(y) - \Psi(z)\|_{\mathcal{B}}$. We know that $\Psi(y) - \Psi(z)$ is the solution of equation (2.2.1) with $T_1 := s$, $T_2 = s + T(R)$, $\tilde{h} := -yy_x + zz_x$, $h(t) := u(t, y(t,\cdot)) - u(t, z(t,\cdot))$ and $y_0 := 0$. Hence, from Lemma 3, Lemma 6 and (2.7.1), we get that

$$\begin{aligned}
\|\Psi(y) - \Psi(z)\|_{\mathcal{B}} &\leqslant C_2\big(\|y_0\|_{L^2_L} + \|h\|_{L^2_T} + \|\tilde{h}\|_{L^1(0,T;L^2(0,L))}\big) \\
&\leqslant C_2\big(0 + T(R)^{\frac{1}{2}} K(C_R)\|y - z\|_{\mathcal{B}} + c_4 T(R)^{\frac{1}{4}}\|y - z\|_{\mathcal{B}}(\|y\|_{\mathcal{B}} + \|z\|_{\mathcal{B}})\big) \\
&\leqslant C_2\|y - z\|_{\mathcal{B}}\big(T(R)^{\frac{1}{2}} K(C_R) + 2c_4 T(R)^{\frac{1}{4}} C_R\big),
\end{aligned}$$

which shows that, if

$$T(R) \leqslant \min\left\{ \left(\frac{1}{12c_4 C_2^2 R}\right)^4, \left(\frac{1}{4C_2 K(3C_2 R)}\right)^2 \right\}, \tag{2.7.7}$$

then,

$$\|\Psi(y) - \Psi(z)\|_{\mathcal{B}} \leqslant \frac{3}{4}\|y - z\|_{\mathcal{B}}.$$

Hence, by the Banach fixed point theorem, there exists $y \in \mathcal{B}_1$ such that $\Psi(y) = y$, which is the solution that we are looking for. We define $T(R)$ as

$$T(R) := \min\left\{ \left(\frac{R}{C_B(3C_2 R)}\right)^2, \left(\frac{1}{12c_4 C_2^2 R}\right)^4, \left(\frac{1}{4C_2 K(3C_2 R)}\right)^2, 1 \right\}. \tag{2.7.8}$$

It only remains to prove the uniqueness of the solution to the Cauchy problem (2.3.7) (the above proof gives only the uniqueness in the set $\mathcal{B}_1$). Clearly it suffices to prove that two solutions to (2.3.6) which are equal at a time τ are equal in a neighborhood of τ in $[\tau, +\infty)$. This property follows from the above proof and from the fact that, for every solution $y : [\tau, \tau_1] \to L^2(0, L)$ of (2.3.7), then, if $T > 0$ is small enough (the smallness depending on y),

$$\|y\|_{\mathcal{B}_{\tau,\tau+T}} \leqslant 3C_2\|y(\tau)\|_{L^2_L}. \tag{2.7.9}$$

□

Proceeding similarly as in the above proof of Lemma 7, one can get the following lemma concerning the Cauchy problem (2.2.13).

Lemma 8. *Let $C_2 > 0$ be as in Lemma 3 for $T_2 - T_1 = 1$. Given $R, M > 0$, there exists $T(R, M) > 0$ such that, for every $s \in \mathbb{R}$, for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2_L} \leqslant R$, and for every measurable $H : (s, s + T(R, M)) \to \mathbb{R}$ such that $|H(t)| \leqslant M$ for every $t \in (s, s + T(R, M))$, the Cauchy problem*

$$\begin{cases}
y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s, s + T(R, M)) \times (0, L), \\
y(t, 0) = y(t, L) = 0 & \text{on } (s, s + T(R, M)), \\
y_x(t, L) = H(t) & \text{on } (s, s + T(R, M)), \\
y(s, x) = y_0(x) & \text{on } (0, L),
\end{cases} \tag{2.7.10}$$

has one and only one solution y on $[s, s + T(R, M)]$. Moreover, this solution satisfies

$$\|y\|_{\mathcal{B}_{s,s+T(R,M)}} \leqslant 3C_2 R. \tag{2.7.11}$$

We are now in position to prove Theorem 13.

Proof of Theorem 13. The uniqueness follows from the proof of the uniqueness part of Lemma 7. Let us give the proof of the existence. Let $y_0 \in L^2(0,L)$, let $s \in \mathbb{R}$ and let $T_0 := T(\|y_0\|_{L^2_L})$. By Lemma 7, there exists a solution $y \in \mathcal{B}_{s,s+T_0}$ to the Cauchy problem (2.3.7). Hence, together with the uniqueness of the solution, we can find a maximal solution $y : D(y) \to L^2(0,L)$ with $[s, s+T_0] \subset D(y)$. By the maximality of the solution y and Lemma 7, there exists $\tau \in [s+T_0, +\infty)$ such that $D(y) = [s, \tau)$. Let us assume that $\tau < +\infty$ and that (2.3.12) does not hold. Then there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers in (s,τ) and $R \in (0, +\infty)$ such that

$$\lim_{n\to+\infty} t_n = \tau, \tag{2.7.12}$$

$$\|y(t_n)\|_{L^2_L} \leq R, \quad \forall n \in \mathbb{N}. \tag{2.7.13}$$

By (2.7.12), there exists $n_0 \in \mathbb{N}$ such that

$$t_{n_0} \geq \tau - T(R)/2. \tag{2.7.14}$$

From Lemma 7, there is a solution $z : [t_{n_0}, t_{n_0} + T(R)] \to L^2(0,L)$ of (2.3.7) for the initial time $s := t_{n_0}$ and the initial data $z(t_{n_0}) := y(t_{n_0})$. Let us then define $\tilde{y} : [s, t_{n_0} + T(R)] \to L^2(0,L)$ by

$$\tilde{y}(t) := y(t), \quad \forall t \in [s, t_{n_0}], \tag{2.7.15}$$

$$\tilde{y}(t) := z(t), \quad \forall t \in [t_{n_0}, t_{n_0} + T(R)]. \tag{2.7.16}$$

Then $\tilde{y}$ is also a solution to the Cauchy problem (2.3.7). By the uniqueness of this solution, we have $y = \tilde{y}$ on $D(y) \cap D(\tilde{y})$. However, from (2.7.14), we have that $D(y) \subsetneqq D(\tilde{y})$, in contradiction with the maximality of y.

Finally, we prove that, if $C(R)$ satisfies (2.3.13), then, for the maximal solution y to (2.3.7), we have $D(y) = [s, +\infty)$. We argue by contradiction and therefore assume that the maximal solution y is such that $D(y) = [s, \tau)$ with $\tau < +\infty$. Then (2.3.12) holds. Let us estimate $\|y(t)\|_{L^2_L}$ when t tends to τ^-. We define the energy $E : [s, \tau) \to [0, +\infty)$ by

$$E(t) := \int_0^L |y(t,x)|^2 dx. \tag{2.7.17}$$

Then $E \in C^0([s,\tau))$ and, in the distribution sense, it satisfies

$$\frac{dE}{dt} \leqslant |u(t, y(t,\cdot))|^2 \leqslant C_B^2(\sqrt{E}). \tag{2.7.18}$$

(We get such estimate first in the classical sense for regular initial data and regular boundary conditions $y_x(t,L) = \varphi(t)$ with the related compatibility conditions; the general case then follows from this special case by smoothing the initial data and the boundary conditions, by passing to the limit, and by using the uniqueness of the solution.) From (2.3.12) and (2.7.18), we get that

$$\frac{1}{2}\int_0^{+\infty} \frac{1}{C_B^2(\sqrt{E})} dE < +\infty. \tag{2.7.19}$$

However the left hand side of (2.7.19) is equal to the left hand side of (2.3.13). Hence (2.3.13) and (2.7.19) are in contradiction. This completes the proof of Theorem 13. $\square$

The proof of Theorem 14 is more difficult. For this proof, we adapt a strategy introduced by Carathéodory to solve ordinary differential equations $\dot{y} = f(t,y)$ when f is not smooth. Roughly speaking it consists in solving $\dot{y} = f(t, y(t-h))$ where h is a positive time-delay and then let h tend to 0. Here we do not put the time-delay on y (it does not seem to be possible) but only on the feedback law: $u(t, y(t))$ is replaced by $u(t, y(t-h))$.

Proof of Theorem 14. Let us define $H : [0, +\infty) \to [0, +\infty)$ by

$$H(a) := \int_0^a \frac{1}{(C_B(\sqrt{E}))^2} dE = 2 \int_0^{\sqrt{a}} \frac{R}{(C_B(R))^2} dR. \tag{2.7.20}$$

From (2.3.13), we know that H is a bijection from $[0, +\infty)$ into $[0, +\infty)$. We denote by H^{-1} : $[0, +\infty) \to [0, +\infty)$ the inverse of this map.

For given $y_0 \in L^2(0, L)$ and $s \in \mathbb{R}$, let us prove that there exists a solution y defined on $[s, +\infty)$ to the Cauchy problem (2.3.7), which also satisfies

$$\|y(t)\|^2_{L^2(0,L)} \leqslant H^{-1}\left(H\left(\|y(s)\|^2_{L^2_L}\right) + (t-s)\right) < +\infty, \quad \forall t \in [s, +\infty). \tag{2.7.21}$$

Let $n \in \mathbb{N}^*$. Let us consider the following Cauchy system on $[s, s + 1/n]$

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s, s + (1/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s, s + (1/n)), \\ y_x(t, L) = u(t, y_0) & \text{on } (s, s + (1/n)), \\ y(s, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{2.7.22}$$

By Theorem 13 applied with the feedback law $(t, y) \mapsto u(t, y_0)$ (a measurable bounded feedback law which now does not depend on y and therefore satisfies (2.3.11)), the Cauchy problem (2.7.22) has one and only one solution y. Let us now consider the following Cauchy problem on $[s + (1/n), s + (2/n)]$

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s + (1/n), s + (2/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s + (1/n), s + (2/n)), \\ y_x(t, L) = u(t, y(t - (1/n))) & \text{on } (s + (1/n), s + (2/n)), \\ y(s, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{2.7.23}$$

As for (2.7.22), this Cauchy problem has one and only one solution, that we still denote by y. We keep going and, by induction on the integer i, define $y \in C^0([s, +\infty); L^2(0, L))$ so that, on $[s + (i/n), s + ((i+1)/n)]$, $i \in \mathbb{N} \setminus \{0\}$, y is the solution to the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s + (i/n), s + ((i+1)/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s + (i/n), s + ((i+1)/n)), \\ y_x(t, L) = u(t, y(t - (1/n))) & \text{on } (s + (i/n), s + ((i+1)/n)), \\ y(s + (i/n)) = y(s + (i/n) - 0) & \text{on } (0, L), \end{cases} \tag{2.7.24}$$

where, in the last equation, we mean that the initial value, i.e. the value at time $(s + (i/n))$, is the value at time $(s + (i/n))$ of the y defined previously on $[(s + ((i-1)/n)), s + (i/n)]$.

Again, we let, for $t \in [s, +\infty)$,

$$E(t) := \int_0^L |y(t, x)|^2 dx. \tag{2.7.25}$$

Then $E \in C^0([s,+\infty))$ and, in the distribution sense, it satisfies (compare with (2.7.18))

$$\frac{dE}{dt} \leqslant |u(t,y_0)|^2 \leqslant C_B^2(\sqrt{E(s)}),\ t \in (s, s+(1/n)), \tag{2.7.26}$$

$$\frac{dE}{dt} \leqslant |u(t,y(t-(1/n))|^2 \leqslant C_B^2(\sqrt{E(t-(1/n))}),\ t \in (s+(i/n), s+((i+1)/n)), i>0. \tag{2.7.27}$$

Let $\varphi : [0,+\infty) \to [0,+\infty)$ be the solution of

$$\frac{d\varphi}{dt} = C_B^2\left(\sqrt{\varphi(t)}\right),\ \varphi(s) = E(s). \tag{2.7.28}$$

Using (2.7.26), (2.7.27), (2.7.28) and simple comparaison arguments, one gets that

$$E(t) \leqslant \varphi(t),\quad \forall t \in [s,+\infty), \tag{2.7.29}$$

i.e.

$$E(t) \leqslant H^{-1}\left(H\left(E(s)\right) + (t-s)\right),\quad \forall t \in [s,+\infty). \tag{2.7.30}$$

We now want to let $n \to +\infty$. In order to show the dependance on n, we write y^n instead of y. In particular (2.7.30) becomes

$$\|y^n(t)\|^2_{L^2(0,L)} \leqslant H^{-1}\left(H(\|y_0(s)\|^2_{L^2_L}) + (t-s)\right),\quad \forall t \in [s,+\infty). \tag{2.7.31}$$

From Lemma 8, (2.7.31) and the construction of y^n, we get that, for every $T > s$, there exists $M(T) > 0$ such that

$$\|y^n\|_{\mathcal{B}_{s,T}} \leqslant M(T),\quad \forall n \in \mathbb{N}. \tag{2.7.32}$$

Hence, upon extracting a subsequence of $(y^n)_n$ that we still denote by $(y^n)_n$, there exists

$$y \in L^\infty_{\text{loc}}([s,+\infty); L^2(0,L)) \cap L^2_{\text{loc}}([s,+\infty); H^1(0,L)), \tag{2.7.33}$$

such that, for every $T > s$,

$$y^n \rightharpoonup y \text{ in } L^\infty(s,T;L^2(0,L)) \text{ weak } * \text{ as } n \to +\infty, \tag{2.7.34}$$

$$y^n \rightharpoonup y \text{ in } L^2(s,T;H^1(0,L)) \text{ weak as } n \to +\infty. \tag{2.7.35}$$

Let us define $z^n : [s, s+\infty) \times (0,L) \to \mathbb{R}$ and $\gamma^n : [s,+\infty) \to \mathbb{R}$ by

$$z^n(t) := y_0,\quad \forall t \in [s, s+(1/n)], \tag{2.7.36}$$

$$z^n(t) := y^n(t-(1/n)),\quad \forall t \in (s+(1/n), +\infty), \tag{2.7.37}$$

$$\gamma^n(t) := u(t, z^n),\quad \forall t \in [s,+\infty). \tag{2.7.38}$$

Note that y^n is the solution to the Cauchy problem

$$\begin{cases} y^n_t + y^n_{xxx} + y^n_x + y^n y^n_x = 0 & \text{in } (s,+\infty) \times (0,L), \\ y^n(t,0) = y^n(t,L) = 0 & \text{on } (s,+\infty), \\ y^n_x(t,L) = \gamma^n(t) & \text{on } (s,+\infty), \\ y^n(s,x) = y_0(x) & \text{on } (0,L). \end{cases} \tag{2.7.39}$$

From (2.7.32) and the first line of (2.7.39), we get that

$$\forall T > 0,\ \left(\frac{d}{dt} y^n\right)_{n \in \mathbb{N}} \text{ is bounded in } L^2(s, s+T; H^{-2}(0,L)). \tag{2.7.40}$$

From (2.7.34), (2.7.35), (2.7.40) and the Aubin-Lions Lemma [Aub63], we get that

$$y^n \to y \text{ in } L^2(s,T;L^2(0,L)) \text{ as } n \to +\infty, \ \forall T > s. \tag{2.7.41}$$

From (2.7.41) we know that, upon extracting a subsequence if necessary, a subsequence still denoted by $(y^n)_n$,

$$\lim_{n\to+\infty} \|y^n(t) - y(t)\|_{L^2_L} = 0, \text{ for almost every } t \in (s,+\infty). \tag{2.7.42}$$

Letting $n \to +\infty$ in inequality (2.7.30) for y^n and using (2.7.42), we get that

$$\|y(t)\|^2_{L^2(0,L)} \leqslant H^{-1}\left(H(\|y_0\|^2_{L^2_L}) + (t-s)\right), \text{ for almost every } t \in (0,+\infty). \tag{2.7.43}$$

Note that, for every $T > s$,

$$\begin{aligned} \|z^n - y\|_{L^2((s,T);L^2_L)} \leq\ & (1/\sqrt{n})\|y_0\|_{L^2_L} + \|y^n(\cdot - (1/n)) - y(\cdot - (1/n))\|_{L^2(s+(1/n),T;L^2(0,L))} \\ & + \|y(\cdot - (1/n)) - y(\cdot)\|_{L^2(s+(1/n),T;L^2(0,L))} + \|y\|_{L^2(s,s+(1/n);L^2(0,L))} \\ \leq\ & (1/\sqrt{n})\|y_0\|_{L^2_L} + \|y^n - y\|_{L^2(s,T;L^2(0,L))} \\ & + \|y(\cdot - (1/n)) - y(\cdot)\|_{L^2(s+(1/n),T;L^2(0,L))} + \|y(\cdot)\|_{L^2(s,s+(1/n);L^2(0,L))}. \end{aligned} \tag{2.7.44}$$

From (2.7.36), (2.7.37), (2.7.41) and (2.7.44), we get that

$$z^n \to y \text{ in } L^2(s,T;L^2(0,L)) \text{ as } n \to +\infty, \ \forall T > s. \tag{2.7.45}$$

Extracting, if necessary, from the sequence $(z^n)_n$ a subsequence, a subsequence still denoted $(z^n)_n$, and using (2.7.45), we have

$$\lim_{n\to+\infty} \|z^n(t) - y(t)\|_{L^2_L} = 0, \text{ for almost every } t \in (s,+\infty). \tag{2.7.46}$$

From (2.3.1), (2.3.2), (2.3.3), (2.7.32), (2.7.36), (2.7.37) and (2.7.46), extracting a subsequence from the sequence $(\gamma^n)_n$ if necessary, a subsequence still denoted $(\gamma^n)_n$, we may assume that

$$\gamma^n \rightharpoonup \gamma(t) := u(t,y(t)) \text{ in } L^\infty(s,T) \text{ weak } * \text{ as } n \to +\infty, \ \forall T > s. \tag{2.7.47}$$

Let us now check that

$$y \text{ is a solution to the Cauchy problem (2.3.7).} \tag{2.7.48}$$

Let $\tau \in [s,+\infty)$ and let $\phi \in C^3([s,\tau]\times[0,L])$ be such that

$$\phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \ \forall t \in [T_1,\tau]. \tag{2.7.49}$$

From (2.7.39), one has, for every $n \in \mathbb{N}$,

$$\begin{aligned} -\int_{T_1}^{\tau}\int_0^L (\phi_t + \phi_x + \phi_{xxx})y^n dxdt - \int_{T_1}^{\tau} \gamma^n \phi_x(t,L)dt + \int_{T_1}^{\tau}\int_0^L \phi y^n y^n_x dxdt \\ + \int_0^L y(\tau,x)\phi(\tau,x)dx - \int_0^L y_0\phi(s,x)dx = 0. \end{aligned} \tag{2.7.50}$$

Let τ be such that

$$\lim_{n\to+\infty} \|y^n(\tau) - y(\tau)\|_{L^2_L} = 0. \tag{2.7.51}$$

Let us recall that, by (2.7.42), (2.7.51) holds for almost every $\tau \in [s,+\infty)$. Using (2.7.35), (2.7.41), (2.7.47), (2.7.51) and letting $n \to +\infty$ in (2.7.50), we get

$$-\int_{T_1}^{\tau}\int_0^L (\phi_t + \phi_x + \phi_{xxx}) y dx dt - \int_{T_1}^{\tau} u(t, y(t))\phi_x(t, L) dt + \int_{T_1}^{\tau}\int_0^L \phi y y_x dx dt$$
$$+ \int_0^L y(\tau, x)\phi(\tau, x) dx - \int_0^L y_0 \phi(s, x) dx = 0. \tag{2.7.52}$$

This shows that y is a solution to (2.2.1), with $T_1 := s$, T_2 arbitrary in $(s, +\infty)$, $\tilde{h} := -yy_x \in L^1([s, T_2]; L^2(0, L))$ and $h = u(\cdot, y(\cdot)) \in L^2(s, T_2)$. Let us emphasize that, by Lemma 3, it also implies that $y \in \mathcal{B}_{s,T}$ for every $T \in (s, +\infty)$. This concludes the proof of (2.7.48) and of Theorem 14. □

2.8 Appendix C: Proof of Proposition 3

Let us first recall that Proposition 3 is due to Eduardo Cerpa if one requires only u to be in $L^2(0, T)$ instead of being in $H^1(0, T)$: see [Cer07, Proposition 3.1] and [CC09a, Proposition 3.1]. In his proof, Eduardo Cerpa uses Lemma 5, the controllability in H with controls $u \in L^2$. Actually, the only place in Eduardo Cerpa's proof where the controllability in H is used is in page 887 of [Cer07] for the construction of α_1, where, with the notations of [Cer07] $\Re(y_\lambda)$, $\Im(y_\lambda) \in H$. We notice that $\Re(y_\lambda)$, $\Im(y_\lambda)$ share more regularity and better boundary conditions. Indeed, one has

$$\begin{cases} \lambda y_\lambda + y_\lambda' + y_\lambda''' = 0, \\ y_\lambda(0) = y_\lambda(L) = 0, \end{cases}$$

which implies that

$$\Re(y_\lambda), \Im(y_\lambda) \in \mathcal{H}^3,$$

where

$$\mathcal{H}^3 := H \cap \{\omega \in H^3(0, L); \omega(0) = \omega(L) = 0\}. \tag{2.8.1}$$

In order to adapt Eduardo Cerpa's proof in the framework of $u \in H^1(0, T)$, it is sufficient to prove the following controllability result in $\mathcal{H}^3$ with control $u \in H^1(0, T)$.

Proposition 6. *For every y_0, $y_1 \in \mathcal{H}^3$ and for every $T > 0$, there exists a control $u \in H^1(0, T)$ such that the solution $y \in \mathcal{B}$ to the Cauchy problem*

$$\begin{cases} y_t + y_{xxx} + y_x = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = u(t), \\ y(0, \cdot) = y_0, \end{cases}$$

satisfies $y(T, \cdot) = y_1$.

The proof of Proposition 3 is the same as the one of [Cer07, Proposition 3.1], with the only difference that one uses Proposition 6 instead of Lemma 5.

Proof of Proposition 6. Let us first point out that 0 is not an eigenvalue of the operator $\mathcal{A}$. Indeed this follows from Property $(\mathcal{P}_2)$, (2.1.5) and (2.1.6). Using Lemma 5 and [TW09, Proposition 10.3.4] with $\beta = 0$, it suffices to check that

$$\text{for every } f \in H, \text{ there exists } y \in \mathcal{H}^3 \text{ such that } -y_{xxx} - y_x = f. \tag{2.8.2}$$

Let $f \in H$. We know that there exists $y \in H^3(0,L)$ such that

$$-y_{xxx} - y_x = f, \tag{2.8.3}$$

$$y(0) = y(L) = y_x(L) = 0. \tag{2.8.4}$$

Simple integrations by parts, together with (2.4.11), (2.4.12), (2.8.3) and (2.8.4), show that, with $\varphi := \varphi_1 + i\varphi_2$,

$$0 = \int_0^L f\varphi dx = \int_0^L (-y_{xxx} - y_x)\varphi dx = \int_0^L y(\varphi_{xxx} + \varphi_x)dx = i\frac{2\pi}{p}\int_0^L y\varphi dx, \tag{2.8.5}$$

which, together with (2.8.4), implies that $y \in \mathcal{H}^3$. This concludes the proof of (2.8.2) as well as the proof of Proposition 6 and of Proposition 3.

□

Chapter 3

Null controllability of a KdV equation with a Dirichlet boundary control

Contents

3.1 Introduction

We consider the null controllability of the following linearized KdV control system,

$$\begin{cases} u_t(t,x) + u_{xxx}(t,x) + u_x(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ u(t,L) = u_x(t,L) = 0 & \text{on } (0,+\infty), \\ u(t,0) = \kappa(t) & \text{on } (0,+\infty), \end{cases} \tag{3.1.1}$$

where $\kappa(t) \in \mathbb{R}$ is a scalar control.

In [Ros97] Rosier introduced the KdV system with a right boundary Neumann control. One surprisingly finds that controllability depends on the length of the interval, which never happens for the linear finite-dimensional system. More precisely, the system is controllable if and only if

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{l^2+lk+k^2}{3}};\, l,k \in \mathbb{N}^* \right\}. \tag{3.1.2}$$

This model has been studied for years, in both controllability [Cer07, Cer14, CC09a, Cor07a, CC04, Ros97, RZ09] and stabilization [CCS15, CL14b, CRX17, PMVZ02, TCSC16].

Concerning the system studied in this paper, we use the left boundary Dirichlet control. For system (3.1.1), Rosier (see [Ros04]) proved that controllability does not depend on the length of the interval. This system was then further studied in [CC13, GG08].

When we study the well-posedness of the control system by using the classical Lions-Magenes method (see [LM73]), a $H^{1/3}$ regularity on the control (with respect to time) is required. Such a problem appears for many boundary control systems, the heat equation and the Burgers equation for example. However, since most control problems are based on evolution models, Sobolev type controls are less preferred than piecewise continuous controls (or even L^p type conditions), especially for stabilization problems. In [CC13], Coron and Cerpa proved rapid stabilization of the system (3.1.1) by using the backstepping method. But since they used some stationary feedback laws, this boundary condition problem is avoided. Recently, by using the (piecewise) backstepping approach, Coron and Nguyen proved the null controllability and semi-global finite time stabilization for a class of heat equations (see [CN17]). They showed how the use of the maximum principle leads to the well-posedness of the closed-loop system. Their method turns out to be a potential way to solve the local (or even semi-global) finite time stabilization problem for systems which can be rapidly stabilized by means of backstepping methods. At the same time, this method provides a visible way to get null controllability directly instead of using observability inequalities and the duality between controllability and observability.

Initially the backstepping is a method to design stabilizing feedback laws in a recursive manner for systems having a triangular structure. See, for example, [Cor07a, Section 12.5]. It was first introduced to deal with finite-dimensional control systems. But it can also be used for control systems modeled by means of partial differential equations (PDEs) as shown first in [CdN98]. For linear partial differential equations, a major innovation is due to Krstic and his collaborators. They observed that, when applied to the classical discretization of these systems, the backstepping leads, at the PDEs level (as the mesh size tends to 0), to the transformation of the initial system into a new target system which can be easily stabilized. This transformation is accomplished by means of a Volterra transform of the second kind. An excellent introduction to this method is presented in [KS08b]. Krstic's innovation has been shown to be very useful for many PDEs control systems as, in particular, heat equations [Liu03, BBK03, CN17], wave equations [SCK10], hyperbolic systems [CVKB13, DMVK13, HDMVK16, CHO17] [BC16, Chapter 7], Korteweg de Vries equations [CC13, CL14b], and Kuramoto–Sivashinsky equations [LK01, CL15]. It was observed later on that for some PDEs more general transforms than Volterra transforms of the second kind have to be considered: see [CGM16, CL14b, CL15]). Recently, the backstepping method has been adapted to coupled systems, for example the Boussinesq system of KdV-KdV type [CFG18]. For the case of finite dimensional control system and Krstic's backstepping, see [Cor15].

Krstic's backstepping requires solving a kernel equation. In the case of the heat equation, the kernel equation is a wave equation; however, in this paper the kernel equation turns out to be a third-order equation, which generates new difficulties both for the well-posedness of the closed-loop system and for important estimation issues.

In this paper, we prove that the method developed by Coron and Nguyen can be used to get the null controllability of (3.1.1).

Theorem 15. *For any given $T > 0$, the control system (3.1.1) is null controllable in time T by using some piecewise continuous controls.*

Remark 1. *Let us recall that the exact controllability of (3.1.1) fails, which is proved in [Ros04].*

Remark 2. *We study in detail the well-posedness of the system. The approach and tools introduced for this study do not rely on precise structures. In particular the control is not given by a stationary feedback law (compare to [CC13]) and no maximum principle is used (compare to [CN17]). Hence, the well-posedness arguments, as well as a priori estimates, and procedure could easily be adapted to many other partial differential equations.*

This paper is organized as follows. Section 3.2 is a preliminary part including the well-posedness of the systems and the rapid stabilization obtained in [CC13]. In Section 3.3, we design the control and provide some estimates which will lead to the null controllability. In Section 3.4, we prove the null controllability. We put some further comments in Section 3.5. It ends with Appendix 3.6 (Proposition 7): the proof of the uniqueness of the solution to the kernel equation, which is essential to this paper.

3.2 Preliminary

3.2.1 Well-posedness of the control system

We start with the non-homogeneous linear Cauchy problem

$$\begin{cases} u_t + u_{xxx} + u_x = \tilde{h} & \text{in } (T_1, T_2) \times (0, L), \\ u_x(t, L) = u(t, L) = 0 & \text{on } (T_1, T_2), \\ u(t, 0) = \kappa(t) & \text{on } (T_1, T_2), \\ u(0, x) = u_0(x) & \text{on } (0, L), \end{cases} \tag{3.2.1}$$

for

$$-\infty < T_1 < T_2 < +\infty, \tag{3.2.2}$$

$$u_0 \in L^2(0, L), \tag{3.2.3}$$

$$\tilde{h} \in L^1(T_1, T_2; L^2(0, L)), \tag{3.2.4}$$

$$\kappa \in L^2(T_1, T_2). \tag{3.2.5}$$

Definition 9. *A solution to the Cauchy problem (3.2.1)–(3.2.5) is a function $u \in C^0([T_1, T_2]; L^2(0, L))$ such that, for every $\tau \in [T_1, T_2]$ and for every $\phi \in C^3([T_1, \tau] \times [0, L])$ satisfying*

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0, \forall t \in [T_1, \tau], \tag{3.2.6}$$

one has

$$\begin{aligned} -\int_{T_1}^{\tau}\int_0^L (\phi_t + \phi_x + \phi_{xxx}) u dx dt - \int_{T_1}^{\tau} \kappa(t)\phi_{xx}(t, 0) dt - \int_{T_1}^{\tau}\int_0^L \phi\tilde{h} dx dt \\ + \int_0^L u(\tau, x)\phi(\tau, x) dx - \int_0^L u_0 \phi(T_1, x) dx = 0. \end{aligned} \tag{3.2.7}$$

The uniqueness of the solution to the Cauchy problem (3.2.1)–(3.2.5) is straightforward, one can get details from the book by Coron [Cor07a]. For the existence of the solution, in [BSZ03], Bona, Sun, and Zhang proved the following result.

Lemma 9. *If $h \in H^{1/3}(T_1, T_2)$, then the Cauchy problem (3.2.1) has one and only one solution. This solution is in $C^0([T_1, T_2]; L^2(0, L)) \cap L^2(T_1, T_2; H^1(0, L))$. There exists a constant $c_1 > 0$ depending on $(T_2 - T_1)$ such that*

$$\begin{aligned} \|u\|_{C^0([T_1,T_2];L^2(0,L))} + \|u\|_{L^2(T_1,T_2;H^1(0,L))} + \sup_{x\in[0,L]} \|u_x(\cdot, x)\|_{L^2(T_1,T_2)} \\ \leqslant c_1 \left(\|u_0\|_{L^2(0,L)} + \|\kappa\|_{H^{1/3}(T_1,T_2)} + \|\tilde{h}\|_{L^1(T_1,T_2;L^2(0,L))} \right). \end{aligned} \tag{3.2.8}$$

3.2.2 Rapid stabilization of (3.1.1)

We recall some results given in [CC13]. Given a positive parameter $\lambda > 1$, we consider the following equations in the triangle $\mathcal{T} := \{(x,y) : x \in [0,L], y \in [x,L]\}$,

$$\begin{cases} k_{xxx} + k_{yyy} + k_x + k_y + \lambda k = 0 & \text{in } \mathcal{T}, \\ k(x,L) = 0 & \text{on } [0,L], \\ k(x,x) = 0 & \text{on } [0,L], \\ k_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L], \end{cases} \tag{3.2.9}$$

and

$$\begin{cases} l_{xxx} + l_{yyy} + l_x + l_y - \lambda l = 0 & \text{in } \mathcal{T}, \\ l(x,L) = 0 & \text{on } [0,L], \\ l(x,x) = 0 & \text{on } [0,L], \\ l_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L]. \end{cases} \tag{3.2.10}$$

In [CC13], it is noted that both (3.2.9) and (3.2.10) have solutions in $C^3(\mathcal{T})$. These solutions are further studied in Section 3.3, where we provide some estimates on $\|k\|_{C^0(\mathcal{T})}$ with respect to λ. Actually, the solutions of equation (3.2.9) and of (3.2.10) satisfy the following conditions

$$k_{xy}(x,x) = -\frac{\lambda}{3}, \tag{3.2.11}$$

$$l_{xy}(x,x) = -\frac{\lambda}{3}, \tag{3.2.12}$$

respectively. The Properties (3.2.11) and (3.2.12) can be checked as follows: we perform the change of variables,

$$t = y - x, \quad s = x + y, \tag{3.2.13}$$

and define

$$G(s,t) := k(x,y). \tag{3.2.14}$$

Then equation (3.2.9) of k becomes the following equation of G,

$$\begin{cases} 6G_{tts} + 2G_{sss} + 2G_s + \lambda G = 0 & \text{in } \mathcal{T}_0, \\ G(s, 2L-s) = 0 & \text{on } [L,2L], \\ G(s,0) = 0 & \text{on } [0,2L], \\ G_t(s,0) = \frac{\lambda}{6}(s-2L) & \text{on } [0,2L], \end{cases} \tag{3.2.15}$$

where $\mathcal{T}_0 := \{(s,t); t \in [0,L], s \in [t, 2L-t]\}$. From (3.2.15), one easily gets

$$G_{tts}(s,0) = 0 \text{ in } [0,2L]. \tag{3.2.16}$$

Hence, $G_{tt}(s,0) = G_{tt}(2L,0)$. In order to calculate $G_{tt}(2L,0)$, one observes from (3.2.15) that

$$G_{tt}(2L,0) = 2G_{ts}(2L,0) = \frac{\lambda}{3}. \tag{3.2.17}$$

Direct calculations show that

$$k_{xy}(x,x) = -G_{tt}(s,0) = -\frac{\lambda}{3}, \tag{3.2.18}$$

which concludes (3.2.11). The proof of (3.2.12) is similar.

Now, let us define a continuous transformation $\Pi_\lambda : L^2(0,L) \to L^2(0,L)$ by

$$\omega(x) = \Pi_\lambda(u(x)) := u(x) - \int_x^L k(x,y)u(y)dy. \tag{3.2.19}$$

Moreover, its inverse is given by (let us denote by Π_λ^{-1})

$$u(x) = \Pi_\lambda^{-1}(\omega(x)) := \omega(x) + \int_x^L l(x,y)\omega(y)dy. \tag{3.2.20}$$

That is because $k(x,y)$ and $l(x,y)$ are related by the formula

$$l(x,y) - k(x,y) = \int_x^y k(x,\eta)l(\eta,y)d\eta. \tag{3.2.21}$$

Actually, one can define

$$\tilde{l}(x,y) := k(x,y) + \int_x^y k(x,\eta)l(\eta,y)d\eta. \tag{3.2.22}$$

Hence one only needs to prove $l = \tilde{l}$ to get (3.2.21). Direct calculations show that $\tilde{l}$ satisfies

$$\begin{cases} \tilde{l}_{xxx} + \tilde{l}_{yyy} + \tilde{l}_x + \tilde{l}_y - \lambda l = 0 & \text{in } \mathcal{T}, \\ \tilde{l}(x,L) = 0 & \text{on } [0,L], \\ \tilde{l}(x,x) = 0 & \text{on } [0,L], \\ \tilde{l}_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L], \\ \tilde{l}_{xy}(x,x) = -\frac{\lambda}{3} & \text{on } [0,L]. \end{cases} \tag{3.2.23}$$

Let us define $l_0 := l - \tilde{l}$. From (3.2.23), (3.2.10), and (3.2.12), one knows that

$$\begin{cases} (l_0)_{xxx} + (l_0)_{yyy} + (l_0)_x + (l_0)_y = 0 & \text{in } \mathcal{T}, \\ (l_0)(x,L) = 0 & \text{on } [0,L], \\ (l_0)(x,x) = 0 & \text{on } [0,L], \\ (l_0)_x(x,x) = 0 & \text{on } [0,L], \\ (l_0)_{xy}(x,x) = 0 & \text{on } [0,L]. \end{cases} \tag{3.2.24}$$

Regarding to the Cauchy problem (3.2.24), we have the following proposition (hence $l = \tilde{l}$), whose proof is given in Appendix 3.6.

Proposition 7. *The equation (3.2.24) has a unique solution in $C^3(\mathcal{T})$. More precisely, this solution is $l_0 = 0$.*

Remark 3. *This proposition is important to this paper. In the following section we construct precisely a solution to equation (3.2.9)(and of (3.2.10) respectively), the proof of Theorem 15 relies on some estimates of this solution. Proposition 7 ensures the solution that we construct satisfies (3.2.21) (hence (3.2.20)).*

We find that by using the transformation Π_λ, the solution of (3.1.1) with control

$$\kappa(t) = \int_0^L k(0,y)u(t,y)dy, \tag{3.2.25}$$

is mapped to a solution of the system

$$\begin{cases} \omega_t + \omega_{xxx} + \omega_x + \lambda\omega = 0 & \text{in } (0,+\infty)\times(0,L), \\ \omega(t,L) = \omega_x(t,L) = 0 & \text{on } (0,+\infty), \\ \omega(t,0) = 0 & \text{on } (0,+\infty). \end{cases} \tag{3.2.26}$$

For system (3.2.26), one can easily obtain exponential decay of the solution

$$\|\omega(t,\cdot)\|_{L^2(0,L)} \leqslant e^{-\lambda t}\|\omega(0,\cdot)\|_{L^2(0,L)}. \tag{3.2.27}$$

Hence the solution of (3.1.1) with feedback law (3.2.25) satisfies

$$\begin{aligned} \|u(t,\cdot)\|_{L^2(0,L)} &\leqslant \|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\omega(t,\cdot)\|_{L^2(0,L)} \\ &\leqslant e^{-\lambda t}\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\omega(0,\cdot)\|_{L^2(0,L)} \\ &\leqslant e^{-\lambda t}\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\Pi_\lambda\|_{L^2(0,L)\to L^2(0,L)}\|u(0,\cdot)\|_{L^2(0,L)}. \end{aligned} \tag{3.2.28}$$

From now on, we simply denote $\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}$ by $\|\Pi_\lambda^{-1}\|$ to simplify the notations.

3.2.3 Well-posedness of system (3.2.26)

For a positive parameter $\lambda > 0$, we consider the following linear operator $\mathcal{A}_\lambda : \mathcal{D}(\mathcal{A}_\lambda) \subset L^2(0,L) \to L^2(0,L)$ with

$$\mathcal{D}(\mathcal{A}_\lambda) := \{f \in H^3(0,L); f(0) = f(L) = f_x(L) = 0\}, \tag{3.2.29}$$

$$\mathcal{A}_\lambda := -f_x - f_{xxx} - \lambda f, \forall f \in \mathcal{D}(\mathcal{A}_\lambda). \tag{3.2.30}$$

Similarly, for the case where $\lambda = 0$ (see [Cor07a, page 38–43]), the following properties also hold.

$$\mathcal{D}(\mathcal{A}_\lambda) \text{ is dense in } L^2(0,L), \tag{3.2.31}$$

$$\mathcal{A}_\lambda \text{ is closed}, \tag{3.2.32}$$

$$\mathcal{A}_\lambda \text{ and } A^* \text{ are dissipative}. \tag{3.2.33}$$

Hence, $\mathcal{A}_\lambda$ generates a strongly continuous semigroup of linear operator $\{S_\lambda(t)\}_{t\geqslant 0}$ on $L^2(0,L)$. Furthermore, for every initial data $\omega_0 \in \mathcal{D}(\mathcal{A}_\lambda)$, system (3.2.26) has one and only one solution $\omega(t,x) \in C^0([0,+\infty); L^2(0,L))$. This solution also satisfies

$$\omega \in C^1([0,+\infty); L^2(0,L)) \cap C^0([0,+\infty); \mathcal{D}(\mathcal{A}_\lambda)), \tag{3.2.34}$$

$$\|\omega_x(\cdot,0)\|_{L^2(0,T)} \leqslant \|\omega_0\|_{L^2(0,L)}, \forall T > 0, \tag{3.2.35}$$

$$\|\omega\|_{L^2(0,T;H^1(0,L))} \leqslant C_T\|\omega_0\|_{L^2(0,L)}, \text{ where } C_T \text{ only depends on T} > 0. \tag{3.2.36}$$

By standard approximation arguments, it follows that when $\omega_0 \in L^2(0,L)$ equation (3.2.26) has a unique solution $\omega(t,x) \in C^0([0,+\infty); L^2(0,L))$. This solution also satisfies (3.2.35) and (3.2.36). For more details on the results and proofs of this subsection, one can refer to Coron's book [Cor07a, page 38–43 and page 374-377]. Although the book only describes the case when $\lambda = 0$, the general case $\lambda > 0$ follows by considering $e^{\lambda t}\omega$.

Remark 4. *Inequality (3.2.35) is a hidden inequality. It was first found by Rosier in [Ros97]. Inequality (3.2.36) is the Kato smoothing effect.*

3.3 Control design

Inspired by the work of Coron and Nguyen in [CN17], we construct a piecewise control such that on each piece, the solution of (3.1.1) can be transformed to a solution of (3.2.26). More precisely, we select

$$\{\lambda_n\}_{n\in\mathbb{N}},\ \text{increasing positive numbers that tends to infinity,} \tag{3.3.1}$$

$$\{t_n\}_{n\in\mathbb{N}},\ \text{increasing numbers with } t_0=0 \text{ that tends to } T \text{ as } n \text{ tends to infinity.} \tag{3.3.2}$$

First we define

$$u(0):=u_0 \text{ and } u(T):=0. \tag{3.3.3}$$

Then, for $t_n<t\leqslant t_{n+1}$, we successively define

$$u(t):=\Pi_{\lambda_n}^{-1}S_{\lambda_n}(t-t_n)\Pi_{\lambda_n}u(t_n), \tag{3.3.4}$$

$$\kappa(t):=\int_0^L k_{\lambda_n}(0,y)u(t,y)dy, \tag{3.3.5}$$

where S_{λ_n} is the semigroup given in Section 3.2.3.

One has the following lemma, whose proof is given at the end of this section.

Lemma 10. *As defined in (3.3.3)–(3.3.4), $u(t)|_{t_n\leqslant t\leqslant t_{n+1}}$ is a solution of (3.2.1) with $T_1=t_n, T_2=t_{n+1}, \tilde{h}=0$, and $\kappa(t)$ given by (3.3.5).*

Let us define

$$s_0:=0 \text{ and } s_n:=\sum_{k=0}^{n-1}\lambda_k(t_{k+1}-t_k) \text{ for } n\geqslant 1, \tag{3.3.6}$$

thanks to (3.3.4) and (3.2.28), we get

$$\|u(t,\cdot)\|_{L^2(0,L)}\leqslant e^{-s_n}\|u_0\|_{L^2(0,L)}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right), \tag{3.3.7}$$

$$|\kappa(t)|\leqslant e^{-s_n}\|u_0\|_{L^2(0,L)}\|k_{\lambda_n}(0,\cdot)\|_{L^2(0,L)}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right) \tag{3.3.8}$$

for $t\in[t_n,t_{n+1}]$.

Hence, if we have a good estimation on k_λ, it will be possible to get $u(t)\to 0$ when t tends to T. Actually, we have the following estimates.

Lemma 11. *Let $\lambda>2$, then (3.2.9) has a unique solution $k_\lambda\in C^3(\mathcal{T})$ (respectively (3.2.10) has a unique solution $l_\lambda\in C^3(\mathcal{T})$). Those solutions also satisfy*

$$\|k_\lambda\|_{C^0(\mathcal{T})}\leqslant e^{(1+L)^2\sqrt{\lambda}} \textit{ and } \|l_\lambda\|_{C^0(\mathcal{T})}\leqslant e^{(1+L)^2\sqrt{\lambda}}. \tag{3.3.9}$$

Proof of Lemma 11. The existence of solution to (3.2.9) is given in [CC13]. The uniqueness of the solution is proved in Appendix 3.6. Here we focus on the C^0 norm estimate (3.3.9).
Take the following change of variable,

$$t=y-x,\quad s=x+y \tag{3.3.10}$$

and define

$$G(s,t):=k(x,y). \tag{3.3.11}$$

Then we transform (3.2.9) into an integral equation of $G(s,t)$ (see formula (21) in [CC13]),

$$\begin{aligned} G(s,t) &= -\frac{\lambda t}{6}(2L-t-s) \\ &+ \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau (2G_{sss}+2G_s+\lambda G)(\eta,\xi)d\xi d\tau d\eta, \end{aligned} \tag{3.3.12}$$

in $\mathcal{T}_0 := \{(s,t); t\in[0,L], s\in[t,2L-t]\}$.

We use a successive approximation to give a solution of the equation (3.3.12). Thanks to Proposition 7, this solution is the unique solution of (3.3.12). Let us take

$$G^1(s,t) := -\frac{\lambda t}{6}(2L-t-s) \tag{3.3.13}$$

and define

$$G^{n+1}(s,t) = \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau (2G^n_{sss}+2G^n_s+\lambda G^n)(\eta,\xi)d\xi d\tau d\eta. \tag{3.3.14}$$

For instance,

$$G^2(s,t) = \frac{1}{108}\Big\{t^3\big(\lambda-\lambda^2 L+\frac{\lambda^2 t}{4}\big)(2L-t-s)+\frac{t^3\lambda^2}{4}[(2L-t)^2-s^2]\Big\}. \tag{3.3.15}$$

But unfortunately, we can not perform such explicit calculation by hand each time. We try to estimate $G^n(s,t)$ from another way. Notice that if $f(s,t) := g(s)h(t)$, then

$$\begin{aligned} &\int_s^{2L-t}\int_0^t\int_0^\tau \frac{\partial^m}{\partial s^m}f(\eta,\xi)d\xi d\tau d\eta \\ =&\int_s^{2L-t}\int_0^t\int_0^\tau \frac{\partial^m}{\partial s^m}g(\eta)h(\xi)d\xi d\tau d\eta \\ =&\int_s^{2L-t}\frac{\partial^m}{\partial s^m}g(\eta)d\eta\int_0^t\int_0^\tau h(\xi)d\xi d\tau. \end{aligned} \tag{3.3.16}$$

Let $\mathbb{P}$ be the space of polynomials of one variable on $\mathbb{R}$. We define operator $\mathbf{T}$ by

$$\begin{aligned} &\mathbf{T}:\mathbb{P}\otimes\mathbb{P}\to\mathbb{P}\otimes\mathbb{P} \\ &g(s)h(t)\mapsto\frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau\left(2\frac{\partial^3}{\partial s^3}+2\frac{\partial}{\partial s}+\lambda Id\right)(g\cdot h)(\eta,\xi)d\xi d\tau d\eta. \end{aligned} \tag{3.3.17}$$

Equality (3.3.16) shows that (3.3.17) is well defined. In fact

$$\mathbf{T}\big(g(s)h(t)\big) = g_{\mathbf{T}}(s,t)h_{\mathbf{T}}(t), \tag{3.3.18}$$

where $g_{\mathbf{T}}(s,t)$ and $h_{\mathbf{T}}(t)$ are given by

$$g_{\mathbf{T}}(s,t) := \frac{1}{6}\int_s^{2L-t}\left(2\frac{\partial^3}{\partial s^3}+2\frac{\partial}{\partial s}+\lambda Id\right)(g)(\eta)d\eta, \tag{3.3.19}$$

$$h_{\mathbf{T}}(t) := \int_0^t\int_0^\tau h(\xi)d\xi d\tau. \tag{3.3.20}$$

Observe that, if

$$|h(t)|\leqslant t^n, \forall\, 0\leqslant t\leqslant L, \tag{3.3.21}$$

then

$$|h_{\mathbf{T}}(t)| = |\int_0^t \int_0^\tau h(\xi) d\xi d\tau| \leqslant \frac{1}{(n+1)(n+2)} t^{n+2}, \quad \text{for } t \in [0, L]. \tag{3.3.22}$$

As for $g_{\mathbf{T}}(s,t)$, notice that if $g(s) = s^m$ with $m \geqslant 3$, then

$$\begin{aligned} g_{\mathbf{T}}(s,t) =& \frac{1}{6}\Big(2m(m-1)(2L-t)^{m-2} - 2m(m-1)s^{m-2} \\ &+ 2(2L-t)^m - 2s^m + \frac{\lambda(2L-t)^{m+1}}{m+1} - \frac{\lambda s^{m+1}}{m+1}\Big). \end{aligned} \tag{3.3.23}$$

This inspires us to separate $\mathbf{T}$ into the following 6 linear operators $\{\mathbf{T}_i\}_{1\leqslant i\leqslant 6}$ from $\mathbb{P}\otimes\mathbb{P}$ to $\mathbb{P}\otimes\mathbb{P}$.

$$\mathbf{T}_1 : s^m h(t) \mapsto \begin{cases} \frac{1}{6}\big(2m(m-1)(2L-t)^{m-2}\big) h_{\mathbf{T}}(t), & \text{when } m \geqslant 3, \\ 0, & \text{when } 0 \leqslant m \leqslant 2, \end{cases} \tag{3.3.24}$$

$$\mathbf{T}_2 : s^m h(t) \mapsto \begin{cases} -\frac{1}{6}\big(2m(m-1)s^{m-2}\big) h_{\mathbf{T}}(t), & \text{when } m \geqslant 3, \\ 0, & \text{when } 0 \leqslant m \leqslant 2, \end{cases} \tag{3.3.25}$$

$$\mathbf{T}_3 : s^m h(t) \mapsto \begin{cases} \frac{1}{6}\big(2(2L-t)^m\big) h_{\mathbf{T}}(t), & \text{when } m \geqslant 1, \\ 0, & \text{when } m = 0, \end{cases} \tag{3.3.26}$$

$$\mathbf{T}_4 : s^m h(t) \mapsto \begin{cases} -\frac{1}{6}\big(2s^m\big) h_{\mathbf{T}}(t), & \text{when } m \geqslant 1, \\ 0, & \text{when } m = 0, \end{cases} \tag{3.3.27}$$

$$\mathbf{T}_5 : s^m h(t) \mapsto \frac{\lambda}{6} \frac{(2L-t)^{m+1}}{m+1} h_{\mathbf{T}}(t), \text{ when } m \geqslant 0, \tag{3.3.28}$$

$$\mathbf{T}_6 : s^m h(t) \mapsto -\frac{\lambda}{6} \frac{s^{m+1}}{m+1} h_{\mathbf{T}}(t), \text{ when } m \geqslant 0. \tag{3.3.29}$$

Since $\mathbf{T}_i$ is linear, we easily find that

$$\mathbf{T}_i(0) = 0. \tag{3.3.30}$$

From (3.3.17)–(3.3.29), we know that

$$\mathbf{T} = \sum_{i=1}^{6} \mathbf{T}_i, \text{ on } \mathbb{P}\otimes\mathbb{P}. \tag{3.3.31}$$

Hence,

$$\begin{aligned} G^{n+1}(s,t) &= \mathbf{T} G^n(s,t) \\ &= \Big(\sum_{i=1}^{6} \mathbf{T}_i\Big) G^n(s,t) \\ &= \Big(\sum_{i=1}^{6} \mathbf{T}_i\Big)^n G^1(s,t). \end{aligned}$$

By (3.3.13), we get

$$G^1(s,t) = -\frac{\lambda}{6}t(2L-t) + \frac{\lambda}{6}(st) = I(s,t) + J(s,t), \tag{3.3.32}$$

where

$$I(s,t) := -\frac{\lambda}{6}t(2L-t) \text{ and } J(s,t) := \frac{\lambda}{6}(st). \tag{3.3.33}$$

Let us define

$$\mathbb{A}_n := \Big\{(x_1, x_2, ..., x_n); x_i \in \{1,2,3,4,5,6\}, \forall\, 1 \leqslant i \leqslant n\Big\}, \forall\, n \geqslant 1. \tag{3.3.34}$$

For any $n \in \mathbb{N}^*$, for any $a = (a(1), a(2), ..., a(n)) \in \mathbb{A}_n$, we define the operator

$$\mathbf{T}_a := \mathbf{T}_{a(n)}\mathbf{T}_{a(n-1)}...\mathbf{T}_{a(1)}. \tag{3.3.35}$$

We define additionally $\mathbb{A} := \{a_0\}$ and $\mathbf{T}_{a_0} := Id$ (identity operator on $\mathbb{P}$).
Hence for any $n \in \mathbb{N}$, we have

$$\begin{aligned} G^{n+1}(s,t) &= \mathbf{T}^n\Big(I + J\Big) \\ &= \sum_{a\in\mathbb{A}_n} (\mathbf{T}_a I) + \sum_{a\in\mathbb{A}_n} (\mathbf{T}_a J). \end{aligned} \tag{3.3.36}$$

Now we use mathematical induction to conclude the following lemma.

Lemma 12. *For every $\lambda > 2$, for every $n \in \mathbb{N}$ and for every $a \in \mathbb{A}_n$, $\mathbf{T}_a I$ and $\mathbf{T}_a J$ are of the form $s^l h(t)$. They also satisfy*

$$|h(t)| \leqslant (\frac{\lambda}{6})^{n+1}\frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1-l}\frac{1}{l!},\ \ t \in [0,L]. \tag{3.3.37}$$

Proof of Lemma 12. When $n = 0$, one can check that Lemma 12 holds. Let us suppose that Lemma 12 holds when $n = k \geqslant 0$. Then we can check in the rest of the proof that Lemma 12 holds when $n = k+1$.

For any $n \geqslant 1$, and for any $a := (a(1), a(2), ..., a(n+1)) \in \mathbb{A}_{n+1}$, let us define

$$\varrho(a) := (a(1), a(2), ..., a(n)). \tag{3.3.38}$$

For any $a \in \mathbb{A}_1$, let us define

$$\varrho(a) := a_0. \tag{3.3.39}$$

Hence for any $a := (a(1), a(2), ..., a(k+1)) \in \mathbb{A}_{k+1}$, we have

$$\mathbf{T}_a = \mathbf{T}_{a(k+1)}\mathbf{T}_{\varrho(a)}. \tag{3.3.40}$$

From the assumption, we know that

$$\mathbf{T}_{\varrho(a)} I = s^l h(t). \tag{3.3.41}$$

If $\mathbf{T}_{\varrho(a)} I = 0$, then we conclude the proof.

If $\mathbf{T}_{\varrho(a)} I = s^l h(t)$, then we know from (3.3.37) that

$$|h(t)| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+1}\frac{t^{2k+1}}{(2k+1)!}(2L+1)^{1+k-l}\frac{1}{l!},\ \ t \in [0,L]. \tag{3.3.42}$$

Let us first consider $\mathbf{T}_1$. We know that $\mathbf{T}_1(s^l h(t)) = 0$ if $l \leqslant 2$. Therefore, it suffices to prove the

case when $l \geqslant 3$.
From (3.3.20)–(3.3.22), (3.3.24), and (3.3.42), we know that

$$\begin{aligned}|\mathbf{T}_1(s^l h(t))| &= \Big|\frac{2}{6}\big(l(l-1)(2L-t)^{l-2}\big)h_{\mathbf{T}}(t)\Big| \\ &\leqslant \frac{2}{6}\big(l(l-1)(2L-t)^{l-2}\big)\Big(\frac{\lambda}{6}\Big)^{1+k}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{1}{l!} \\ &\leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{k-1}\frac{1}{(l-2)!}. \end{aligned} \tag{3.3.43}$$

Notice that $\mathbf{T}_1(s^l h(t))$ can be written as $s^0 g(t)$. Thus, it can be seen from (3.3.43) that (3.3.37) is satisfied.
By using the same procedure, we can check that

$$|\mathbf{T}_2(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^{l-2}}{(l-2)!}, \tag{3.3.44}$$

$$|\mathbf{T}_3(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k}\frac{1}{l!}, \tag{3.3.45}$$

$$|\mathbf{T}_4(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^l}{l!}, \tag{3.3.46}$$

$$|\mathbf{T}_5(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{k+2}\frac{1}{(l+1)!}, \tag{3.3.47}$$

$$|\mathbf{T}_6(s^l h(t))| \leqslant \Big(\frac{\lambda}{6}\Big)^{k+2}\frac{t^{2k+3}}{(2k+3)!}(2L+1)^{1+k-l}\frac{s^{l+1}}{(l+1)!}. \tag{3.3.48}$$

Hence, we complete the proof. □

By the same idea of partition and Lemma 12, we can further obtain the following estimate.

Lemma 13. *For every $\lambda > 2$, for every $n \in \mathbb{N}$, and for every $a \in \mathbb{A}_n$, $\mathbf{T}_a I$ and $\mathbf{T}_a J$ are of the form $s^l h(t)$. They also satisfy*

$$|\partial_t(h(t))| \leqslant 2\Big(\frac{\lambda}{6}\Big)^{n+1}\frac{t^{2n}}{(2n)!}(2L+1)^{n+3-l}\frac{1}{l!},\ \ t \in [0,L]. \tag{3.3.49}$$

Remark 5. *One can get similar estimates for C^2-norm or even C^n-norm. However, since in this paper we do not need to use such estimates, this part is omitted. Actually, C^2-norm estimates can be obtained directly by Lemma 12 (as the way of getting Lemma 13), but the C^n-norm (with $n \geq 3$) is more complicated. Furthermore, it can be seen from [KS08b] that for the heat equation the kernel is analytic in the triangle. It is of interest to know if the kernel we obtained in this article is also analytic.*

We come back to the estimate (3.3.36). From Lemma 12, we know that, for every $n \in \mathbb{N}$, for every $a \in \mathbb{A}_n$, for every $m \in \mathbb{N}$, and for $t \in [0,L]$ we have

$$\Big|\frac{\partial^m}{\partial s^m}\Big(\mathbf{T}_a I\Big)(s,t)\Big| \leqslant \Big(\frac{\lambda}{6}\Big)^{1+n}\frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1}, \tag{3.3.50}$$

$$\Big|\frac{\partial^m}{\partial s^m}\Big(\mathbf{T}_a J\Big)(s,t)\Big| \leqslant \Big(\frac{\lambda}{6}\Big)^{1+n}\frac{t^{2n+1}}{(2n+1)!}(2L+1)^{n+1}, \tag{3.3.51}$$

These together with (3.3.36) imply that

$$|\frac{\partial^m}{\partial s^m} G^{n+1}(s,t)| \leqslant \frac{\lambda^{1+n}}{3} \frac{t^{2n+1}}{(2n+1)!} (2L+1)^{n+1}, \tag{3.3.52}$$

hence

$$\sum_{n=1}^{+\infty} \frac{\partial^m}{\partial s^m} G^n(s,t) \text{ is uniformly convergent in } \mathcal{T}_0. \tag{3.3.53}$$

The same approach shows that the series

$$\sum_{n=1}^{+\infty} \frac{\partial^{m+1}}{\partial t \partial s^m} G^n(s,t) \text{ is uniformly convergent in } \mathcal{T}_0. \tag{3.3.54}$$

We define

$$G(s,t) := \sum_{n=1}^{+\infty} G^n(s,t), \tag{3.3.55}$$

which is the solution of (3.3.12) (see [CC13, page 1691]). First, we estimate $|G|$ from (3.3.52), (3.3.53), and (3.3.55):

$$|G(s,t)| \leqslant \frac{\sqrt{(2L+1)\lambda}}{3} e^{\sqrt{(2L+1)\lambda} t}, \text{ in } \mathcal{T}_0. \tag{3.3.56}$$

Hence

$$|G(s,t)| \leqslant \frac{e^{(1+L)\sqrt{(2L+1)\lambda}}}{3} \leqslant e^{(1+L)^2\sqrt{\lambda}}, \text{ in } \mathcal{T}_0. \tag{3.3.57}$$

It only remains to prove that $G(s,t) \in C^3(\mathcal{T}_0)$. Actually, from (3.3.53) and (3.3.54) we know that it suffices to prove $G_{tt}, G_{tts}, G_{ttt} \in C^0(\mathcal{T}_0)$. We know from (3.2.15) that $G_{tts} \in C^0(\mathcal{T}_0)$. As for G_{tt} and G_{ttt}, thanks to (3.3.12), we get

$$\begin{aligned}
6G_t(s,t) = &-\lambda(2L-s-2t) - \int_0^t \int_0^\tau (2G_{sss} + 2G_s + \lambda G)(2L-t,\xi) d\xi d\tau \\
&+ \int_s^{2L-t} \int_0^t (2G_{sss} + 2G_s + \lambda G)(\eta,\xi) d\xi d\eta, \\
6G_{tt}(s,t) = &\, 2\lambda + \int_0^t \int_0^\tau (2G_{ssss} + 2G_{ss} + \lambda G_s)(2L-t,\xi) d\xi d\tau \\
&- \int_0^t (2G_{sss} + 2G_s + \lambda G)(2L-t,\xi) d\xi - \int_0^t (2G_{sss} + 2G_s + \lambda G)(2L-t,\xi) d\xi \\
&+ \int_s^{2L-t} (2G_{sss} + 2G_s + \lambda G)(\eta,t) d\eta, \\
6G_{ttt}(s,t) = &-\int_0^t \int_0^\tau (2G_{sssss} + 2G_{sss} + \lambda G_{ss})(2L-t,\xi) d\xi d\tau + \int_0^t (2G_{ssss} + 2G_{ss} + \lambda G_s)(2L-t,\xi) d\xi \\
&- 2(2G_{sss} + 2G_s + \lambda G)(2L-t,t) d\xi + 2\int_0^t (2G_{ssss} + 2G_{ss} + \lambda G_s)(2L-t,\xi) d\xi \\
&+ \int_s^{2L-t} (2G_{ssst} + 2G_{st} + \lambda G_t)(\eta,t) d\eta - (2G_{sss} + 2G_s + \lambda G)(2L-t,t).
\end{aligned}$$

The above formulas together with (3.3.53)–(3.3.54) give the continuity of G_{tt} and of G_{ttt}. Hence we complete the proof of Lemma 11. ☐

Remark 6. *As we can see from [CN17, KS08b] for the heat equation, the L^∞-norm of the kernel*

k_λ is of the form $e^{C\sqrt{\lambda}}$. One may naturally ask if the sharp estimate on the L^∞-norm of kernel k_λ is of the form $e^{C\sqrt[3]{\lambda}}$ for the KdV case, as KdV is of order 3. However, we do not know how to get such estimates.

At last, it remains to give the proof of Lemma 10.

Proof of Lemma 10. It is equivalent to prove the following statement:

$$(S)\qquad \begin{aligned} &\text{Given } u_0 \in L^2(0,L), \lambda > 0, s > 0, \text{ one has}\\ &\kappa(t) := \int_0^L k_\lambda(0,\cdot)\Big(\Pi_\lambda^{-1} S_\lambda(t)\Pi_\lambda u_0\Big)(\cdot)dy \in L^2(0,s), \text{ and}\\ &u(t) := \Pi_\lambda^{-1} S_\lambda(t)\Pi_\lambda u_0 \text{ is the solution of (3.2.1) on } [0,s] \text{ with } \tilde{h} = 0.\end{aligned}$$

We only need to prove the case where $u_0 \in \Pi_\lambda^{-1}\mathcal{D}(\mathcal{A}_\lambda)$, since standard approximation methods then lead to the general case of (S). From Section 3.2.3, we know that

$$\omega(t) := S_\lambda(t)\Pi_\lambda u_0 \in C^0([0,s];\mathcal{D}(\mathcal{A}_\lambda)) \cap C^1([0,s];L^2(0,L)), \tag{3.3.58}$$

which shows that $\kappa(t) \in C^0([0,s])$. Direct calculations, based on (3.2.10), show that (similar to page 1690 in [CC13])

$$\begin{aligned} u_t = \omega_t &+ \int_x^L (l_y + l_{yyy} - \lambda l)(x,z)\omega(z)dz \\ &- l_y(x,x)\omega_x(x) + l_{yy}(x,x)\omega(x), \end{aligned} \tag{3.3.59}$$

$$u_x = \omega_x(x) + \int_x^L l_x(x,z)\omega(z)dz, \tag{3.3.60}$$

$$u_{xx} = \omega_{xx}(x) - l_x(x,x)\omega(x) + \int_x^L l_{xx}(x,z)\omega(z)dz, \tag{3.3.61}$$

$$\begin{aligned} u_{xxx} = \omega_{xxx} &- \Big(l_{xx}(x,x) + l_{xy}(x,x)\Big)\omega(x) \\ &- l_x(x,x)\omega_x(x) - l_{xx}(x,x)\omega(x) + \int_x^L l_{xxx}(x,z)\omega(x)dz, \end{aligned} \tag{3.3.62}$$

all these calculations hold on $C^0([0,T];L^2(0,L))$. From (3.3.59)–(3.3.62) and (3.2.10), we know that

$$u(t,x) \in C^1([0,s];L^2(0,L)) \cap C^0([0,s];H^3(0,L)), \tag{3.3.63}$$

$$u_t + u_x + u_{xxx} = 0, \text{ in } L^2(0,L), \tag{3.3.64}$$

$$u(t,L) = u_x(t,L) = 0, \tag{3.3.65}$$

$$u(t,0) = \kappa(t), \tag{3.3.66}$$

which show that u satisfies Definition 9. □

Remark 7. *In fact, by using (3.2.9) and the hidden inequality (3.2.35), we can also prove that $\kappa(t) \in H^1(0,s)$ with its norm controlled by $\|u_0\|_{L^2(0,L)}$.*

3.4 Null controllability

Finally, we are able to prove the null controllability (Theorem 15) by constructing a piecewise continuous bounded control. The construction is explained in Section 3.3. Thanks to Lemma 10 and (3.3.1)–(3.3.8), we only need to find good sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to+\infty} e^{-s_n}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right)=0, \tag{3.4.1}$$

$$\lim_{n\to+\infty} e^{-s_n}\|k_{\lambda_n}(0,\cdot)\|_{L^2(0,L)}\prod_{k=0}^{n}\left(\|\Pi_{\lambda_k}^{-1}\|\|\Pi_{\lambda_k}\|\right)=0, \tag{3.4.2}$$

and that

$$u(t)|_{0\leqslant t\leqslant T} \text{ is a solution of (3.2.1) with } \tilde{h}=0, \kappa(t) \text{ given by (3.3.5)}. \tag{3.4.3}$$

Thanks to Lemma 10, from Definition 9, (3.3.7), (3.3.8), (3.4.1), and (3.4.2), one can easily deduce that $u(t)|_{0\leqslant t\leqslant T}$ is the solution of (3.1.1). It remains to prove that (3.4.1)–(3.4.2) hold.

From the definition of Π_λ and Π_λ^{-1}, (3.2.19)–(3.2.20), we know that

$$\|\Pi_\lambda\|_{L^2(0,L)\to L^2(0,L)}\leqslant (1+L\|k_\lambda\|_{C^0(\mathcal{T})})\leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{3.4.4}$$

$$\|\Pi_\lambda^{-1}\|_{L^2(0,L)\to L^2(0,L)}\leqslant (1+L\|l_\lambda\|_{C^0(\mathcal{T})})\leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{3.4.5}$$

$$\|k_\lambda(0,\cdot)\|_{L^2(0,L)}\leqslant \sqrt{L}\|l_\lambda\|_{C^0(\mathcal{T})}\leqslant e^{2(1+L)^2\sqrt{\lambda}}, \tag{3.4.6}$$

where Lemma 11 is used. Hence it suffices to select $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ such that

$$e^{-s_n}\prod_{k=0}^{n} e^{6(1+L)^2\sqrt{\lambda_k}}\longrightarrow 0. \tag{3.4.7}$$

Inspired by the choices given by Coron and Nguyen in [CN17, Proposition 1], we select $t_n := T-1/n^2$ and $\lambda_n := 2n^8$. One easily verifies that (3.4.7) holds, which completes the proof.

Remark 8. *To deal with the heat equations (by using backstepping approach), one needs to study the wave equation instead, which is already well investigated. In this article, we study the KdV system which has an order of 3. Hence the kernel system (see (3.6.1)) becomes a third order "wave-like" equation. For this reason, we encountered some difficulties: Lemma 11 for estimation and Proposition 7 for uniqueness. We believe that the Coron-Nguyen method, as well as the techniques introduced in this paper, could be used for other systems whose order is greater than 3. As we know, the backstepping method is well used on the rapid stablization problem of first-order hyperbolic systems (see [KS08a]). Unfortunately, as they are of order one, we are not sure if some good estimates could be obtained for the null controllability or even the finite time stabilization. However, looking for [CVKB13, HVDMK15, Li94], this might be possible for quasilinear hyperbolic systems.*

3.5 Further comments

The above procedure has the following advantages.

- The null controllability is precisely obtained by an explicit piecewise continuous (actually piecewise H^1) bounded control instead of some unknown $H^{1/3}$ control.
- The well-posedness results and the estimates investigated in this paper should allow small-time stabilization to be proven (instead of rapid stabilization in [CC13]).
- The backstepping approach as used in [CN17] together with the techniques introduced in this paper could be applied to more models, as the backstepping method was widely used in the last 20 years for different models.

We also want to point out a list of open problems which could be further studied.

- The (global or local) controllability and the (global or local) small-time stabilization of the nonlinear KdV equations.
  Based on the linear result and the Kato smoothing effect, we may expect the local controllability by standard perturbation. Actually, as it is shown in [CFG18, CC13, CL14b, MC18] that the backstepping method can be directly used to treat the nonlinear case. But as the main purpose of this paper is to extend the new method found by Coron and Nguyen to more general models (as we stated in the Introduction), we do not consider nonlinear cases here.
  However, since there is only one scalar control on the boundary, the global controllability in small time is a real challenge and might be false, as it is the case for the Burgers equations [GI07]. If there are more controls there are several global controllability results in small time for the nonlinear KdV equations [Cha09a] and for the Burgers equations [Cha09b, CX18, FCG07, Gag15, GG07, Mar, Mar14]. Note that, using the backstepping approach, [CX18] allows to recover the global controllability result of [Cha09b] obtained by means of the return method. It would be interesting to see if [CX18] can be adapted to nonlinear KdV equations.

- In [LR95a, LR95b], using a "microlocal" approach and Carleman's inequalities, Lebeau and Robbiano proved some $e^{C\sqrt{\lambda}}$ type estimates and then deduced from these estimates the controllability of the heat equation. From a "global" point of view, we also obtain the $e^{C\sqrt{\lambda}}$ type estimates (Lemma 11), and also get the controllability.
  An important and interesting question: is there any connection between the backstepping method, Lebeau-Robbiano's strategy, Carleman inequalities, and small-time (local) stabilization?

- Let us also recall open problems raised in Remark 5 about the analytic regularity of the kernel and Remark 6 about sharp kernel estimates. For the sharp L^∞-norm (or even C^n-norm) estimates, considering [CCST18], perhaps it would be more natural to consider Gevrey class regularity instead of the analytic regularity.

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3.6 Appendix: Proof of Proposition 7

In this part, we give the proof of the uniqueness of the solution to equation (3.2.24). As the function is defined in the triangle $\mathcal{T}$, we extend l_0 by 0 in the lower triangle $[0,L]\times[0,L]\setminus\mathcal{T}$, and denote by h the extended function. Since on the diagonal $x=y$, $C^3(\mathcal{T})$ function l_0 satisfies

$$l_{0x}=l_{0y}=l_{0xx}=l_{0xy}=l_{0yy}=0,$$

the extended function is $H^3([0,L]\times[0,L])$. Moreover, h satisfies

$$\begin{cases} h_{xxx}+h_{yyy}+h_x+h_y=0 & \text{in } [0,L]\times[0,L],\\ h(x,L)=0 & \text{on } [0,L],\\ h(x,0)=h_y(x,0)=h_{yy}(x,0)=0 & \text{on } [0,L],\\ h(L,y)=h_x(L,y)=h_{xx}(L,y)=0 & \text{on } [0,L]. \end{cases} \tag{3.6.1}$$

By simple change of variables, $\tilde{x} = L - x$ and $\tilde{y} = L - y$, it suffices to prove that the solution $h \in H^3([0,L] \times [0,L])$ of

$$\begin{cases} h_{xxx} + h_{yyy} + h_x + h_y = 0 & \text{in } [0,L] \times [0,L], \\ h(x,0) = 0 & \text{on } [0,L], \\ h(x,L) = h_y(x,L) = h_{yy}(x,L) = 0 & \text{on } [0,L], \\ h(0,y) = h_x(0,y) = h_{xx}(0,y) = 0 & \text{on } [0,L]. \end{cases} \tag{3.6.2}$$

is 0.

As (3.6.2) is similar to the wave equation, it is natural to consider eigenfunctions of the operator (with respect to y variable),

$$\mathcal{A}_y : \mathcal{D}(\mathcal{A}_y) \subset L^2(0,L) \to L^2(0,L), \tag{3.6.3}$$
$$\mathcal{D}(\mathcal{A}_y) := \{f \in H^3(0,L); f(0) = f(L) = f_y(L) = 0\}, \tag{3.6.4}$$
$$\mathcal{A}_y f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y). \tag{3.6.5}$$

If the eigenfunctions, $\{\varphi_n(y)\}_n$, form a Riesz basis of $L^2(0,L)$, then the Fourier series decomposition

$$h(x,y) = \sum_n \phi_n(x) \cdot \varphi_n(y) \tag{3.6.6}$$

easily infers the uniqueness required. Unfortunately, this operator is a non-self-adjoint operator and eigenfunctions do not form a Riesz basis, see [Pap11].

Another idea is to investigate the completeness of eigenfunctions, $\{\psi(y)_n\}_n$, of the adjoint operator $\mathcal{A}_y^*$,

$$\mathcal{A}_y^* : \mathcal{D}(\mathcal{A}_y^*) \subset L^2(0,L) \to L^2(0,L), \tag{3.6.7}$$
$$\mathcal{D}(\mathcal{A}_y^*) := \{f \in H^3(0,L); f(0) = f(L) = f_y(0) = 0\}, \tag{3.6.8}$$
$$\mathcal{A}_y^* f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y^*). \tag{3.6.9}$$

Actually, suppose that $\{\psi_n(y)\}_n$ is an eigenfunction of the adjoint operator $\mathcal{A}_y^*$, then from (3.6.2) as well as the boundary conditions of h and ψ one can deduce that

$$(\partial_{xxx} + \partial_x - \lambda_n)\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0. \tag{3.6.10}$$

Combine (3.6.10) with the fact that

$$\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = \partial_x\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = \partial_{xx}\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L^2(0,L)} = 0, \tag{3.6.11}$$

we obtain

$$\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0,L]. \tag{3.6.12}$$

If $\{\psi_n(y)\}_n$ is complete in $L^2(0,L)$, then $h(x,0)$ is 0. However, we don't know the completeness of the eigenfunctions $\{\psi_n(y)\}_n$.

More generally, one could consider eigenfunctionals or even generalized eigenfunctions, following Gel'fand and the coauthors [GS16, GV16]. More precisely, in the non-self-adjoint cases it is not always possible to expand a function as the sum of eigenfunctions. In order to avoid this problem, one uses different generalisations of eigenfunctions.

For example, the generalisation introduced by Fokas, augmented eigenfunctions, which is itself a generalisation of Gel'fand's eigenfunctions (allow the appearance of remainder functionals). This generalisation turns out to be a powerful tool to investigate the initial-boundary value problem (IBVP). One can find an almost complete investigation from the papers [Loc00, Loc08, SF]. In general, let Φ be a function space defined on the closure of a real interval I with sufficient smoothness and decay conditions, $\mathcal{L}$ be a linear operator defined on Φ. Let γ be an oriented contour in $\mathbb{C}$, and let $\mathbf{E} = \{\mathbf{E}_\lambda : \lambda \in \gamma\}$ be a family of functionals (imagine as a family of eigenfunction when γ is only defined on a discrete set). Then the corresponding remainder functionals $\mathbf{R}_\lambda \in \Phi'$ with respect to eigenvalues λ is

$$\mathbf{R}_\lambda(\phi) := \lambda^n \mathbf{E}_\lambda(\phi) - \mathbf{E}_\lambda(\mathcal{L}\phi), \quad \forall \phi \in \Phi, \forall \lambda \in \gamma. \tag{3.6.13}$$

One is interested in the cases in which one of the following two conditions is satisfied,

$$\int_\lambda e^{i\lambda x} \mathbf{R}_\lambda(\phi) d\lambda = 0, \quad \forall \phi \in \Phi, \forall x \in I, \tag{3.6.14}$$

or

$$\int_\lambda \frac{e^{i\lambda x}}{\lambda^n} \mathbf{R}_\lambda(\phi) d\lambda = 0, \quad \forall \phi \in \Phi, \forall x \in I, \tag{3.6.15}$$

where (3.6.14) (*resp.* (3.6.15)) is called the type I (*resp.* type II) condition of augmented eigenfunctions of $\mathcal{L}$ up to the integration along γ.

As we can see above, the study of augmented eigenfunctions involves complicated asymptotic calculations. In Fokas' work this method is only used to study the evolution equations based on a good transform pair, which does not seem to be a good (easy) option to our problem (3.6.2). Instead of augmented eigenfunctions, Locker [Loc08] also considered the generalized eigenspace $\mathcal{E}_\mathcal{G}$ given by

$$\bigcup \mathcal{N}((\lambda_i I - \mathcal{L})^{m_i}), \ \textit{union for all } m_i \in \mathbb{N}, \ \textit{and } \lambda_i \ \textit{eigenvalues},$$

where $\mathcal{L}$ denotes the operator, $\mathcal{N}$ denotes the kernel. More precisely, to the linearized KdV operator he proved the following.

Theorem 16. *Let $L > 0$. For the differential operator $\mathcal{L}f := f_{xxx} + af_x$ with boundary conditions*

$$f(0) = f(L) = 0, \tag{3.6.16}$$
$$f_x(0) + \beta f_x(L) = 0, \tag{3.6.17}$$

the generalized eigenfunction space $\mathcal{E}_\mathcal{G}$ is complete in $L^2(0, L)$ space when $\beta \neq 0$.

Remark 9. *When $\beta = 0$, it does not seem to be known whether generalized eigenfunction space $\mathcal{E}_\mathcal{G}$ is complete in $L^2(0, L)$. This is one of the reasons that much more complicated augmented eigenfunctions are introduced (the other reasons are about the regularities and some more general boundary conditions). Actually, this case can be regarded as a limit of the cases when the coupling constant β approaches 0.*
In fact, in [Loc08] Locker only considered the operator $\mathcal{L}f := f_{xxx}$. One can easily verify with the same proof that the same result holds when there is an additional f_x in the operator.

In order to solve our problem, we use another kind of generalized eigenfunctions, which is more general than $\mathcal{E}_\mathcal{G}$ but is less general than augmented eigenfunctions, namely eigenfunctions and associated functions (*e.a.f.*). The definition of *e.a.f.*, which is defined on equations with λ as parameter, is rather complicated. One can see [Nai67, chapter 1] and [Naĭ68] for precise description on this subject.

With eigenfunctions and associated functions, Shkalikov in [Ška76] proved the following theorem.

Theorem 17. *The eigenfunctions and associated functions of the boundary-value problem generated* *by an ordinary differential equation with separated boundary conditions*

$$l(y) - \lambda^n y = y^{(n)} + p_{n-2}(x)y^{(n-2)} + ... + p_0(x)y - \lambda^n y = 0, \tag{3.6.18}$$

$$U_j(y) = \sum_{k=0}^{n-1} \alpha_{jk} y^{(k)}(0) = 0, \ \text{with } j = 1, 2, ..., l, \tag{3.6.19}$$

$$U_j(y) = \sum_{k=0}^{n-1} \beta_{jk} y^{(k)}(L) = 0, \ \text{with } j = 1, 2, ..., n-l, \tag{3.6.20}$$

form a complete system in the space $L^2[0, L]$, where $p_i(x)$ are arbitrary summable functions, and $l > n - l > 0$.

Applying Theorem 17 to our case (linearised KdV), we get

Corollary 3. *For the ordinary differential equation with separated boundary conditions*

$$\mu(f) = l(f) - \lambda^3 f = f_{yyy} + f_y - \lambda^3 f = 0, \tag{3.6.21}$$
$$U_1(f) = f(0) = 0, \tag{3.6.22}$$
$$U_2(f) = f_y(0) = 0, \tag{3.6.23}$$
$$U_3(f) = f(L) = 0, \tag{3.6.24}$$

the eigenfunctions and associated functions form a complete system in the space $L^2[0, L]$.

Finally, we are able to prove Proposition 7.

Proof of Proposition 7. Let us consider the boundary-value problem (3.6.21)–(3.6.24). Let λ_0 be an eigenvalue, and let $\varphi_0(y) = \varphi(y)$ an eigenfunction for the eigenvalue λ_0. The associated functions associated with the eigenfunction $\varphi(y)$ are given by the functions

$$\varphi_1(y), \varphi_2(y), ..., \varphi_k(y).$$

These functions satisfy (the boundary conditions)

$$U_1(\varphi_i) = U_2(\varphi_i) = U_3(\varphi_i) = 0, \quad \forall i = 0, 1, ..., k, \tag{3.6.25}$$

and, for $\lambda = \lambda_0$, the following relations

$$\mu(\varphi_i) + \frac{1}{1!}\frac{\partial}{\partial \lambda}\mu(\varphi_{i-1}) + ... + \frac{1}{i!}\frac{\partial^i}{\partial \lambda^i}\mu(\varphi_0) = 0, \quad \forall i = 0, 1, ..., k. \tag{3.6.26}$$

Now we prove that for all those functions (*e.a.f.*), we have

$$\langle h(x, \cdot), \varphi_i(\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0, L], \quad \forall i = 0, 1, ..., k. \tag{3.6.27}$$

At first, for φ_0, as what we have done in (3.6.12), clearly

$$\langle h(x, \cdot), \varphi_0(\cdot)\rangle_{L^2(0,L)} = 0, \quad \forall x \in [0, L]. \tag{3.6.28}$$

For φ_1, equation (3.6.26) shows that

$$(\varphi_1)_{yyy} + (\varphi_1)_y - \lambda_0^3 \varphi_1 - 3\lambda_0^2 \varphi_0 = 0. \tag{3.6.29}$$

Hence from (3.6.2), (3.6.25), (3.6.28), and (3.6.29) we get

$$\begin{aligned} 0 &= \langle (\partial_x^3 + \partial_x + \partial_y^3 + \partial_y) h(x,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x) \langle h(x,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} - \langle h(x,\cdot), (\partial_y^3 + \partial_y) \varphi_1(\cdot) \rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x) \langle h(x,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} - \langle h(x,\cdot), -\lambda_0^3 \varphi_1(\cdot) - 3\lambda_0^2 \varphi_0(\cdot) \rangle_{L^2(0,L)} \\ &= (\partial_x^3 + \partial_x + \lambda_0^3) \langle h(x,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)}. \end{aligned} \tag{3.6.30}$$

By using the the fact that

$$\langle h(0,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} = \partial_x \langle h(0,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} = \partial_{xx} \langle h(0,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} = 0, \tag{3.6.31}$$

we get

$$\langle h(x,\cdot), \varphi_1(\cdot) \rangle_{L^2(0,L)} = 0, \quad \forall x \in [0, L]. \tag{3.6.32}$$

Repeating this procedure we get (3.6.27), which combined with Corollary 3 shows that

$$h(x,\cdot) = 0, \quad \forall x \in [0, L]. \tag{3.6.33}$$

Hence the proof of Proposition 7 is completed. □

Remark 10. *For the y-variable, we only used 3 boundary conditions in the proof to deduce the uniqueness of the solution h: $h(x, L) = h_y(x, L) = h(x, 0) = 0$. This is natural, since once we consider 4 boundary conditions (for a third order differential operator), the eigenfunctions could never become a basis.*

We may also wonder, if we can get the uniqueness of h by using the other 3 boundary conditions in y-variable: $h(x, L) = h_y(x, L) = h_{yy}(x, L) = 0$? Unfortunately, Theorem 17 can not be applied for these 3 boundary conditions: we observe from (3.6.19) and (3.6.20) that there should be boundary conditions on both side. Hence, it is difficult to get the uniqueness of h by using the Carleman estimate, see [ABBG+12, Chapter 4], [Car39] and [Hö97], though the Carleman estimate is a standard way to solve the unique continuation problem.

Chapter 4

Small-time local stabilization of a KdV equation with a Dirichlet boundary control

Contents

4.1 Introduction

We consider the stabilization problem of the Korteweg-de Vries equation

$$y_t + y_x + y_{xxx} + yy_x = 0 \tag{4.1.1}$$

posed on a bounded domain $[0, L]$. This system requires three boundary conditions including both left and right end-point (see [Xia19], the system fails to be well-posed when all boundary conditions are given at only one end-point), among which the most studied case is

$$y(t,0),\ y(t,L),\ y_x(t,L). \tag{4.1.2}$$

When there is only one control term y_x, *i.e.* $y(t,0) = y(t,L) = 0$, the phenomenon becomes quite mysterious: starting from the linearized KdV equation, Lionel Rosier [Ros97] found that the system is controllable if and only if the length of the interval satisfies

$$L \notin \mathcal{N} := \left\{2\pi\sqrt{\frac{l^2+lk+k^2}{3}};\ l,k \in \mathbb{N}^*\right\}. \tag{4.1.3}$$

It allows us to decompose the $L^2(0,L)$ space into the controllable state and the uncontrollable state (for the linearized KdV equation). To get the controllability of the KdV equations, “Power Series Expansion” method was introduced in [Cer07, CC09a, CC04], which turned out to be a classical example of getting controllability by using nonlinear terms. The stabilization problem is even more interesting, as we need to investigate a closed-loop system which involves more difficulties (even for the well-posedness). Several works [CCST18, CCS15, CL14b, CRX17, PMVZ02, TCSC16] have been made in this special KdV control system, here we refer [CRX17] where the authors used the nonlinear term (and also uncontrollable part) to find a time-varying feedback law which stabilize the system exponentially. As we can see, in this model we really used the nonlinear term, by "fixing" the uncontrollable part, to reach the goal of controlling and stabilizing. However, at the same time, those results are only local, and the exponential decay rate is small (compare to rapid stabilization or small-time stabilization). To get the same results in global sense, is a challenging and interesting problem, since one may need to find other techniques for the nonlinear term.

In this paper, we focus on the control acting on $y(t,0)$, with $y(t,L) = y_x(t,L) = 0$. This system has an advantage of being locally controllable, see [CC13, GG08, Ros04] for discussions on this system. From the stabilization point of view, the final aim should be (local) small-time stabilization, especially by means of time-varying feedback laws (inspired by [Cor95] for the finite dimensional case). Recently in [CN17], Jean-Michel Coron and Hoai-Minh Nguyen made a first step, they used a piece-wise backstepping control to get the null controllability and the semi-global small-time stabilization for the heat equation. Here we refer to [Cor15, CdN98, KS08b, LK00] for the history, explanation and development of backstepping method. As the backstepping method has already been used for a rapid stabilization for this KdV system, one may naturally expect the small-time stabilization. In the recent paper [Xia19], the author used this technique to give a new proof of null controllability of the linearized KdV equation. However, when one considers the stabilization problems, there came a difficulty of lacking regularity on the control (feedback) term $y(t,0)$. When we consider the well-posedness of the KdV system, *a priori* the $H^{1/3}$ regularity on $y(t,0)$ is needed. But one can hardly ensure the feedback to be more than C^0 with respect to time.

A technique called "adding an integrator" solves our (regularity) problem. Usually used to avoid the offset in the stabilization problem, this technique also has the advantage of gaining regularity. Indeed, if we "add" another term, $a(t)$, as

$$y(t,0) = a(t), \quad a_t(t) = u(t), \tag{4.1.4}$$

where $u(t)$ is the control, then we have $a(t) \in H^1(0,T)$ if $u(t) \in L^2(0,T)$. One can see the paper of Jean-Michel Coron [Cor99], where this method is used for the stabilization of Euler equations. This technique has been widely used in different cases, for example one can see [Krs09, TX11]. Let us also point out that, the controllability and stabilizability of the control system with additional integral term are related but may be different from the ones of the original control system; see, in particular [Cor07a, Proposition 3.30 and Section 12.5] and [CP91] for finite dimensional control systems.

We consider in this paper the following system:

$$y_t + y_x + y_{xxx} + yy_x = 0, \tag{4.1.5}$$

$$y(t,L) = y_x(t,L) = 0, \tag{4.1.6}$$

$$y(t,0) = a(t), \tag{4.1.7}$$

$$a_t(t) + y_{xx}(t,0) + \frac{1}{2}a(t) + \frac{1}{3}a^2(t) = u(t), \tag{4.1.8}$$

in the interval $[0,1]$ (we only consider only the case when $L = 1$ to simplify the notations). We notice the extra $y_{xx}(t,0)$ term in (4.1.8). It naturally comes from (4.1.7) and helps to ensure the well-posedness of our new system (see the energy estimate (4.1.10)). As for terms $(1/2)a$ and $(1/3)a^2$, which could be put in the control term, we let them here to make the dissipative nature visible (see the energy estimate (4.1.10)). It is a control system where the state is $(y(x), a)$, but with only one control u. Let us set

$$V := L^2(0,1) \times \mathbb{R} \text{ and } \|(y,a)\|_V^2 := \|y\|_{L^2(0,1)}^2 + a^2. \tag{4.1.9}$$

Then easy calculations show that the "flow" of system (4.1.5)–(4.1.8) satisfies:

$$\begin{aligned}\frac{d}{dt}\|(y,a)\|_V^2 &= 2\langle y_t, y\rangle_{L^2} + 2a_t a \\ &= 2\langle -y_x - y_{xxx} - yy_x, y\rangle_{L^2} + 2a_t a \\ &= a^2 + \frac{2}{3}a^3 - y_x(0)^2 + 2ay_{xx}(0) + 2a_t a \\ &\leqslant 2au. \end{aligned} \tag{4.1.10}$$

In order to get the well-posedness of the nonlinear system, we may need some (Kato type) smoothing effects. We first consider the linearized system of (4.1.5)–(4.1.8). By multiplying xy the linearized part of (4.1.5), we get the Kato smoothing effect $y \in L^2(0,T;H^1(0,1))$. This together with (4.1.10) and some fixed point argument, allows us to get the well-posedness of the nonlinear system (4.1.5)–(4.1.8) in the transposition sense (with initial data $(y_0, a_0) \in V$ and control $u(t) \in L^1(0,T)$), the solution being in

$$\big(C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1(0,1))\big) \times C^0([0,T];\mathbb{R}).$$

Here, we are not going to reconstruct the whole theory of transposition solutions, which is already well explained in the book [Cor07a] (one can also see similar cases in [CN17, CRX17]). Based on the method introduced in [CN17] and the estimates given in [Xia19], we are able to stabilize system (4.1.5)–(4.1.8) in small time. More precisely, for every $T > 0$, we construct time-varying feedback laws $U(t;y,a) : \mathbb{R} \times L^2(0,1) \times \mathbb{R} \mapsto \mathbb{R}$, satisfying properties $(\mathcal{P}_1)$–$(\mathcal{P}_5)$ (see Section 4.3 for details), which stabilize the system in small time. From now on, let us consider the Cauchy problem of the closed-loop system (4.1.5)–(4.1.8)

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,1) = y_x(t,1) = 0, \\ y(t,0) = a(t), \\ a_t + y_{xx}(0) + \frac{1}{2}a + \frac{1}{3}a^2 = u, \\ y(s,x) = y_0, \\ a(s) = a_0, \\ u := U(t;y,a) \end{cases} \tag{4.1.11}$$

with $(t,x) \in (s,+\infty) \times (0,1)$. For this Cauchy problem, from properties $(\mathcal{P}_1)$–$(\mathcal{P}_3)$ we have the existence and uniqueness of solution in small time. A solution $(y_1,a_1) : [s,\tau_1) \to V$ to the Cauchy problem is maximal, if there is no solution $(y_2,a_2) : [s,\tau_2) \to V$ such that $\tau_2 > \tau_1$ and $(y_1,a_1) = (y_2,a_2)$ in $[s,\tau_1]$. From the uniqueness of solution, let us denote $\Phi(t,s;y_0,a_0)$ with $t \in [s,\iota(s;y_0,a_0)]$ the unique maximal solution with initial data (y_0,a_0), we call this solution the flow of the Cauchy system (4.1.11). Properties $(\mathcal{P}_4)$–$(\mathcal{P}_5)$ let every maximal solution to be defined on $[s,+\infty)$, *i.e.* $\iota(s;y_0,a_0) = +\infty$.

The main purpose of this paper is to prove the following theorem:

Theorem 18. *Let $T > 0$. There exists $\varepsilon > 0$ and a time-varying feedback law $U(t;y,a)$ satisfying properties $(\mathcal{P}_1)$–$(\mathcal{P}_5)$ such that following properties hold:*

(i) $\iota(s;y_0,a_0) = +\infty$, *for every* $(s;y_0,a_0) \in \mathbb{R} \times V$.

(ii) $\Phi(t+2T,t;y_0,a_0) = 0$, *if* $\|(y_0,a_0)\|_V \leqslant \varepsilon$.

(iii) (Uniform stability property) For $\forall \delta > 0, \exists \eta > 0$ *such that*

$$(\|(y_0,a_0)\|_V \leqslant \eta) \Rightarrow (\|\Phi(t,t';y_0,a_0)\|_V \leqslant \delta, \forall t \geqslant t'). \tag{4.1.12}$$

This paper is organized as follows. In Section 4.2, we give a stationary feedback law F_λ which can locally exponentially stabilize the system with decay rate λ. Section 4.3 contains the construction of the time-varying feedback law, which leads to the local small-time stabilization that we will prove in Section 4.4.

4.2 Rapid stabilization

This section is based on the rapid stabilization of a KdV system proved in [CC13] and estimates (Lemma 15) given in [Xia19]. Let us start by considering the linearized system

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,1) = y_x(t,1) = 0, \\ y(t,0) = u(t). \end{cases} \tag{4.2.1}$$

It is proved in [CC13] that for any given positive λ, there is a kernel k_λ defined in the triangle $\mathcal{T} := \{(x,v) : x \in (0,1), v \in (x,1)\}$ such that if we perform the transformation $\Pi_\lambda : L^2(0,1) \longrightarrow L^2(0,1)$

$$z(x) = \Pi_\lambda(y(x)) := y(x) - \int_x^1 k_\lambda(x,v)y(v)dv, \tag{4.2.2}$$

then the solution y of system (4.2.2) with feedback law

$$u(t) := \int_0^1 k_\lambda(0,v)y(t,v)dv \tag{4.2.3}$$

is mapped to the solution z of the system

$$\begin{cases} z_t + z_x + z_{xxx} + \lambda z = 0, \\ z(t,1) = z_x(t,1) = 0, \\ z(t,0) = 0. \end{cases} \tag{4.2.4}$$

Therefore we have the exponential stabilization:

$$\|z(t)\|_{L^2(0,1)} \leqslant e^{-\lambda t}\|z(0)\|_{L^2(0,1)}, \tag{4.2.5}$$

hence exponentially decay for the solution $y(t,\cdot)$ thanks to the invertibility of the transformation Π_λ.

As for the kernel, the following result is given in [CC13]:

Lemma 14.

(1) The kernel k_λ satisfies equation

$$\begin{cases} k_{xxx} + k_{vvv} + k_x + k_v + \lambda k = 0 & \text{in } \mathcal{T}, \\ k(x,1) = 0 & \text{in } [0,1], \\ k(x,x) = 0 & \text{in } [0,1], \\ k_x(x,x) = \frac{\lambda}{3}(1-x) & \text{in } [0,1]. \end{cases} \tag{4.2.6}$$

(2) The inverse of transformation Π_λ, Π_λ^{-1}, is given by

$$y(x) = \Pi_\lambda^{-1}(z(x)) := z(x) + \int_x^1 l_\lambda(x,v)z(v)dv. \tag{4.2.7}$$

And the kernel l_λ satisfies

$$\begin{cases} l_{xxx} + l_{vvv} + l_x + l_v - \lambda l = 0 & \text{in } \mathcal{T}, \\ l(x,1) = 0 & \text{in } [0,1], \\ l(x,x) = 0 & \text{in } [0,1], \\ l_x(x,x) = \frac{\lambda}{3}(1-x) & \text{in } [0,1]. \end{cases} \tag{4.2.8}$$

Later, in [Xia19], more information on k_λ is given:

Lemma 15. *The equation (4.2.6) has a unique solution. There exists a constant C_1, which is independent of $\lambda > 1$, such that*

$$\|k_\lambda\|_{C^3(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}, \text{ and } \|l_\lambda\|_{C^3(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}. \tag{4.2.9}$$

Remark 11. *In [Xia19], the estimate is only given for the C^1-norm, but one can easily get similar C^3 estimates by using the same method. As stated in [Xia19], it will be a challenging and interesting problem to know whether the right hand side of (4.2.9) can be replaced by $exp(C\lambda^{1/3})$.*

Now, we consider the linearized system of (4.1.11):

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,1) = y_x(t,1) = 0, \\ y(t,0) = a(t), \\ a_t + y_{xx}(0) + \frac{1}{2}a = u. \end{cases} \tag{4.2.10}$$

We look for a transformation Ξ_λ with $(z,b) = \Xi_\lambda((y,a))$ of the form:

$$\begin{cases} z := \Pi_\lambda(y), \\ b := a + F_\lambda(y), \end{cases} \tag{4.2.11}$$

and a feedback law of the form

$$u(t) := K_\lambda(y) + L_\lambda a. \tag{4.2.12}$$

We want (z,b) to satisfy the following target system:

$$\begin{cases} z_t + z_x + z_{xxx} + \lambda z = 0, \\ z(t,1) = z_x(t,1) = 0, \\ z(t,0) = b(t), \\ b_t + z_{xx}(0) + \frac{1}{2}b + \mu b = w, \end{cases} \tag{4.2.13}$$

with μ and w to be chosen later.

Actually, performing the same calculation as in [CC13, page 1690] with (4.2.6), we get

$$z_t + z_x + z_{xxx} + \lambda z = 0. \tag{4.2.14}$$

Besides, we have

$$z(t,1) = y(t,1) = 0, \tag{4.2.15}$$

$$z_x(t,1) = y_x(t,1) + k_\lambda(1,1)y(t,1) = 0, \tag{4.2.16}$$

$$z(t,0) = a - \int_0^1 k_\lambda(0,v)y(t,v)dv. \tag{4.2.17}$$

Hence, $F_\lambda(y)$ should be

$$F_\lambda(y) := -\int_0^1 k_\lambda(0,v)y(v)dv. \tag{4.2.18}$$

At last, let us calculate w:

$$\begin{aligned} w &= b_t + z_{xx}(0) + \frac{1}{2}b + \mu b \\ &= (a + F_\lambda(y))_t + (\mu + \frac{1}{2})(a + F_\lambda(y)) + y_{xx}(0) \\ &\quad - \Big(\int_x^1 k_\lambda(x,v)y(v)dv\Big)_{xx}(0) \\ &= \Big(a_t + y_{xx}(0) + \frac{1}{2}a\Big) + (F_\lambda(y))_t + \frac{1}{2}F_\lambda(y) \\ &\quad - \Big(\int_0^1 k_{\lambda,xx}(0,v)y(v)dv - k_{\lambda,x}(0,0)a\Big) + \mu(a + F_\lambda(y)). \end{aligned} \tag{4.2.19}$$

Since

$$\begin{aligned} (F_\lambda(y))_t &= -\int_0^1 k_\lambda(0,v)y_t(t,v)dv \\ &= \int_0^1 k_\lambda(0,v)\big(y_x(t,v) + y_{xxx}(t,v)\big)dv \\ &= -\int_0^1 k_{\lambda,v}(0,v)y(t,v)dv + k_\lambda(0,v)y(t,v)\Big|_0^1 \\ &\quad - \int_0^1 k_{\lambda,vvv}(0,v)y(t,v)dv + k_\lambda(0,v)y_{xx}(t,v)\Big|_0^1 \\ &\quad - k_{\lambda,v}(0,v)y_x(t,v)\Big|_0^1 + k_{\lambda,vv}(0,v)y(t,v)\Big|_0^1 \\ &= -\int_0^1 \big(k_{\lambda,v}(0,v) + k_{\lambda,vvv}(0,v)\big)y(t,v)dv \\ &\quad + k_{\lambda,v}(0,0)y_x(t,0) - k_{\lambda,vv}(0,0)a, \end{aligned} \tag{4.2.20}$$

from (4.2.10)–(4.2.20) we get

$$\begin{aligned} w = u &- \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\ &\qquad + \mu k_\lambda + k_{\lambda,xx}\Big)(0,v)y(t,v)dv \\ &- \big(-k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu\big)a + k_{\lambda,v}(0,0)y_x(t,0) \\ = u &- \big(-k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu\big)a + k_{\lambda,v}(0,0)z_x(t,0) \\ &- \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\ &\qquad + \mu k_\lambda + k_{\lambda,xx} + \frac{\lambda}{3}k_{\lambda,x}\Big)(0,v)y(t,v)dv, \end{aligned}$$

where we used the fact that

$$z_x(t,0) - y_x(t,0) = -\int_0^1 k_{\lambda,x}(0,v)y(v)dv. \tag{4.2.21}$$

Hence we define the feedback, $u(t) = K_\lambda(y) + L_\lambda a$, by

$$\begin{cases} K_\lambda(y) := \int_0^1 \Big(k_{\lambda,v} + k_{\lambda,vvv} + \frac{1}{2}k_\lambda \\ \qquad\qquad +\mu k_\lambda + k_{\lambda,xx} + \frac{\lambda}{3}k_{\lambda,x}\Big)(0,v)y(v)dv, \\ L_\lambda := -k_{\lambda,x}(0,0) + k_{\lambda,vv}(0,0) - \mu, \end{cases} \tag{4.2.22}$$

which leads to

$$w = k_{\lambda,v}(0,0)z_x(t,0) = -\frac{\lambda}{3}z_x(t,0). \tag{4.2.23}$$

Let us choose

$$\mu := \lambda^2 + \lambda. \tag{4.2.24}$$

From (4.2.13), (4.2.23) and (4.2.24), we get

$$\begin{aligned} \frac{d}{dt}\|(z,b)\|_V^2 &= -z_x(0)^2 - 2\lambda\|z\|_{L^2}^2 - 2(\lambda^2+\lambda)b^2 + 2wb \\ &\leqslant -2\lambda\|(z,b)\|_V^2, \end{aligned} \tag{4.2.25}$$

which leads to the (global) exponential decay with rate λ to the target system (4.2.13). In order to get exponential decay to the system (4.2.10), we need to point that both $\Xi_\lambda, \Xi_\lambda^{-1} : V \to V$:

$$\Xi_\lambda : \begin{pmatrix} y \\ a \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_\lambda & 0 \\ F_\lambda & 1 \end{pmatrix} \begin{pmatrix} y \\ a \end{pmatrix},$$
$$\Xi_\lambda^{-1} : \begin{pmatrix} z \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_\lambda^{-1} & 0 \\ -F_\lambda\Pi_\lambda^{-1} & 1 \end{pmatrix} \begin{pmatrix} z \\ b \end{pmatrix},$$

are bounded.

From (4.2.2), (4.2.18), (4.2.22) and Lemma 15, there exists C_2 independent of $\lambda > 1$ such that following estimates on the norm of operators hold

$$|L_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \ \ |K_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \ \ |F_\lambda| \leqslant e^{C_2\sqrt{\lambda}}, \tag{4.2.26}$$

$$|\Pi_\lambda| \leqslant e^{C_2\sqrt{\lambda}} \text{ and } \left|\Pi_\lambda^{-1}\right| \leqslant e^{C_2\sqrt{\lambda}}. \tag{4.2.27}$$

Hence

$$|\Xi_\lambda| \leqslant 2e^{2C_2\sqrt{\lambda}} \text{ and } \left|\Xi_\lambda^{-1}\right| \leqslant 2e^{2C_2\sqrt{\lambda}}. \tag{4.2.28}$$

Let us consider now the stability of nonlinear system (4.1.11) with feedback law u given by (4.2.12) and (4.2.22). Suppose that $(y,a)(t)$ is a solution of (4.1.11) with (4.2.12), then $(z,b) := \Xi_\lambda(y,a)$ satisfies

$$\begin{aligned} &z_t(t,x) + z_x(t,x) + z_{xxx}(t,x) + \lambda z(t,x) \\ &= -\left(z(t,x) + \int_x^1 l_\lambda(x,v)z(t,v)dv\right) \\ &\qquad \cdot \left(z_x(t,x) + \int_x^1 l_{\lambda,x}(x,v)z(t,v)dv\right) \\ &\quad - \frac{1}{2}\int_x^1 k_{\lambda,v}(x,v)\big(\Pi_\lambda^{-1}z\big)^2(t,v)dv \ = I, \end{aligned} \tag{4.2.29}$$

$$z(t,1) = 0, z_x(t,1) = 0, z(t,0) = b, \tag{4.2.30}$$

$$
\begin{aligned}
&b_t + z_{xx}(0) + \frac{1}{2}b + \mu b + \frac{\lambda}{3} z_x(t,0)\\
=& -\frac{1}{3}a^2 - \frac{1}{2}\int_0^1 k_{\lambda,v}(0,v) y^2(t,v)dv\\
=& -\frac{1}{3}(F_\lambda \Pi_\lambda^{-1} z - b)^2 - \frac{1}{2}\int_0^1 k_{\lambda,v}(0,v)\big(\Pi_\lambda^{-1} z\big)^2(t,v)dv\\
=& J.
\end{aligned}
\tag{4.2.31}
$$

Hence, together with (4.2.24), the flow (z,b) satisfy

$$
\begin{aligned}
\frac{d}{dt}\|(z,b)\|_V^2 = & -z_x(0)^2 - 2\lambda\|z\|_{L^2}^2 - 2(\lambda^2+\lambda)b^2 - \frac{2\lambda}{3} z_x(0)b\\
& + 2\langle z, I\rangle_{L^2} + 2bJ\\
\leqslant & -2\lambda\|(z,b)\|_V^2 + 2\langle z, I\rangle_{L^2} + 2bJ.
\end{aligned}
\tag{4.2.32}
$$

Performing the same calculation as in [CC13, page 1692], we get

$$
|2\langle z, I\rangle_{L^2}| + 2|bJ| \leqslant e^{C_3\sqrt{\lambda}}\|(z,b)\|_V^3,
\tag{4.2.33}
$$

with $C_3 > 3C_2$ independent of $\lambda > 1$.

Hence if the initial state (z_0, b_0) satisfies

$$
\|(z_0,b_0)\|_V \leqslant e^{-C_3\sqrt{\lambda}} (i.e.\ \ \|(y_0,a_0)\|_V \leqslant e^{-2C_3\sqrt{\lambda}}),
\tag{4.2.34}
$$

the solution (z,b) has the exponential decay

$$
\|(z,b)(t)\|_V \leqslant e^{-\frac{\lambda}{2}t}\|(z_0,b_0)\|_V.
\tag{4.2.35}
$$

4.3 Control design

This section is devoted to the construction of time-varying feedback laws satisfying following properties.

$(\mathcal{P}_1)$ The feedback law U is T-periodic with respect to time:

$$
U(t;y,a) = U(T+t;y,a).
\tag{4.3.1}
$$

$(\mathcal{P}_2)$ There exists an increasing sequence $\{t_n\}$ of real numbers such that

$$
t_0 = 0, \tag{4.3.2}
$$

$$
\lim_{n\to\infty} t_n = T, \tag{4.3.3}
$$

$$
\text{U is of class } C^1 \text{ in } [t_n, t_{n+1}) \times L^2(0,1)\times\mathbb{R}. \tag{4.3.4}
$$

$(\mathcal{P}_3)$ The feedback law U vanishes on $\mathbb{R}\times\{0\}\times\{0\}$. There exists a continuous function $M : [0,T) \to [0,+\infty)$ such that

$$
|U(t;y_1,a_1)| - |U(t;y_2,a_2)| \tag{4.3.5}
$$

$$
\leqslant M(t)(\|y_1-y_2\|_{L^2} + |a_1-a_2|), \tag{4.3.6}
$$

for $\forall t\in[0,T)$.

($\mathcal{P}_4$) For all $(t; y, a) \in \mathbb{R} \times L^2(0,1) \times \mathbb{R}$, we have

$$| U(t; y, a) | < \min\{1, \sqrt{\|(y, a)\|_V}\}. \tag{4.3.7}$$

($\mathcal{P}_5$) $\|(y, a)\|_V \geqslant 1 \Rightarrow U(t; y, a) = 0$, for $\forall t \in \mathbb{R}$.

As what is done in [Xia19], we can find a piecewise continuous feedback law in time $[0, T)$ such that properties ($\mathcal{P}_2$)–($\mathcal{P}_5$) holds. Once the feedback law on $[0, T)$ is chosen, we can prolong this feedback law periodically to get a feedback law fulfills ($\mathcal{P}_1$)–($\mathcal{P}_5$). Since the feedback law (4.2.12) given in Section 4.2 is Lipschitz in V, it is not difficult to design such piecewise feedback laws.

Actually, for each "piece" (on time $[t_n, t_{n+1})$), the feedback law given by $\Big(K_{\lambda_n}(y) + L_{\lambda_n} a\Big)$ locally exponentially stabilizes the system. Hence, if we multiply $\Big(K_{\lambda_n}(y) + L_{\lambda_n} a\Big)$ by a cutoff function φ_{λ_n} , the obtained feedback law still locally exponentially stabilizes the system. Moreover, with a good choice of φ_{λ_n}, this new feedback law can satisfies ($\mathcal{P}_3$)–($\mathcal{P}_5$). More precisely, we define

$$u(t; y, a)\Big|_{t \in [t_n, t_{n+1})} := \varphi_{\lambda_n}(\|(y, a)\|_V)\Big(K_{\lambda_n}(y) + L_{\lambda_n} a\Big), \tag{4.3.8}$$

where $\varphi_{\lambda_n} := \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$\begin{cases} 1, & \text{if } x \in [0, e^{-C_2\sqrt{\lambda_n}}/5], \\ 2 - 5e^{C_2\sqrt{\lambda_n}} x, & \text{if } x \in [e^{-C_2\sqrt{\lambda_n}}/5, 2e^{-C_2\sqrt{\lambda_n}}/5] \\ 0, & \text{if } x \in [2e^{-C_2\sqrt{\lambda_n}}/5, +\infty). \end{cases}$$

One can easily verify that properties ($\mathcal{P}_2$)–($\mathcal{P}_5$) hold for proper choice of $\{t_n\}$.

The difficult part is to choose $\{\lambda_n\}$ (increasing positive numbers that tend to infinity) and $\{t_n\}$ (increasing numbers with $\mathbf{t_0 = 0}$ that tend to $\mathbf{T}$ as $\mathbf{n}$ tends to infinity), such that Theorem 18 holds.

Let us directly choose

$$t_n := 0, \lambda_n := 0 \text{ for } n < n_0 := 1 + [\frac{2}{\sqrt{T}}], \tag{4.3.9}$$
$$t_n := T - 1/n^2, \lambda_n := 2n^8, \text{ for } n \geqslant n_0 := 1 + [\frac{2}{\sqrt{T}}].$$

In the next section, we can see that such feedback law stabilizes the system in small time.

4.4 Small-time stabilization

The proof of Theorem 18 is divided into three parts:

(1) The solution exists in arbitrary time.

(2) There exists $\varepsilon > 0$ such that, $\Phi(T, 0; y_0, a_0) = 0$, if $\|(y_0, a_0)\|_V \leqslant \varepsilon$.

(3) Uniform stability property, see (4.1.12).

In fact, (1) equals to (i), (3) equals to (iii), and (2)–(3) imply (ii).

Let us start by (1). By classical fixed point argument, for every $R > 0$, we know the existence of T_R such that for every initial state $\|(y, a)\|_V < R$, the solution exists on $(0, T_R)$. We only need to verify that the solution will never blow-up. Following the simple calculation in (4.1.10) with the help of ($\mathcal{P}_4$)–($\mathcal{P}_5$), we can control the V-norm of the solution in arbitrary time. As the time-varying feedback law is bounded at every time except $t = T$, we also need to prove that for $\forall s \in [0, T)$, following limit

$$\lim_{t \to T^-} \Phi(t, s; y_0, a_0) \tag{4.4.1}$$

exists. This can be proved by using the same method given in [CN17, page 22]. Briefly, by using the standard Banach fixed point argument, one can find that (for $s \in [0, T)$)

$$\Phi(t, s; y_0, a) \in C^0([s, T); L^2(0,1)) \cap L^2(s, T; H^1(0,1)). \tag{4.4.2}$$

Moreover, there exists C_0 such that

$$\|\Phi(t, s; y_0, a_0)\|_{C^0([s,T);L^2(0,1))} \leqslant C_0, \tag{4.4.3}$$

$$\|\Phi(t, s; y_0, a_0)\|_{L^2(s,T;H^1(0,1))} \leqslant C_0. \tag{4.4.4}$$

Let us denote by $\tilde{\Phi}(t, s; y_0, a_0)$ the unique solution of the Cauchy problem (4.1.11) with $U(t; y, a) \equiv 0$. Direct calculation shows that, $\forall \varepsilon > 0$, there exists $t_\varepsilon \in [s, T)$ such that

$$\|(\Phi - \tilde{\Phi})(t, t_\varepsilon; y, a)\|_{L^2(0,1)} \leqslant \varepsilon, \forall t \in [t_\varepsilon, T), \forall \|(y, a)\|_V \leqslant C_0. \tag{4.4.5}$$

Since $\tilde{\Phi}(t, t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) \in C^0([t_\varepsilon, T]; L^2(0,1))$, there exists $t'_\varepsilon \in [t_\varepsilon, T)$ such that

$$\|\tilde{\Phi}(t, t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) - \tilde{\Phi}(T, t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0))\|_{L^2(0,1)} \leqslant \varepsilon, \tag{4.4.6}$$

for $\forall t \in [t_{\varepsilon', T}]$. Hence, for $\forall t'_\varepsilon < t' < t'' < T$, we have

$$\begin{aligned}
&\|\Phi(t', s; y_0, a_0) - \Phi(t'', s; y_0, a_0)\|_{L^2(0,1)} \\
=&\|\Phi(t', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) - \Phi(t'', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0))\|_{L^2(0,1)} \\
\leqslant&\|\Phi(t', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) - \tilde{\Phi}(t', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0))\|_{L^2(0,1)} \\
&+ \|\Phi(t'', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) - \tilde{\Phi}(t'', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0))\|_{L^2(0,1)} \\
&+ \|\tilde{\Phi}(t', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0)) - \tilde{\Phi}(t'', t_\varepsilon; \Phi(t_\varepsilon, s; y_0, a_0))\|_{L^2(0,1)} \\
\leqslant&4\varepsilon,
\end{aligned}$$

which implies (4.4.1).

The next and the most important step is to prove (2). On time $[t_n, t_{n+1})$, the feedback u is given by

$$\varphi_{\lambda_n}(\|(y, a)\|_V)\Big(K_{\lambda_n}(y) + L_{\lambda_n} a\Big).$$

We observe that if $\varphi_{\lambda_n} \neq 1$, the exponential decay no longer holds. The idea is to prove that

$$\varphi_{\lambda_n}\big(\|(y, a)(t))\|_V\big) \equiv 1, \text{ for } t \in [t_n, t_{n+1})$$

which is equivalent to have that

$$\|(y, a)(t))\|_V \leqslant e^{-C_2\sqrt{\lambda_n}/5}, \text{ for } t \in [t_n, t_{n+1}). \tag{4.4.7}$$

As we have seen in Section 4.2, in order to get exponential stabilization of our nonlinear system (4.1.11), the following condition on the "initial state"

$$\|(y, a)(t_n)\|_V \leqslant e^{-2C_3\sqrt{\lambda_n}} \tag{4.4.8}$$

is sufficient. One can simply verify the following lemma:

Lemma 16. *For every $n \geqslant 1$. For every*

$$\|(y, a)(t_n)\|_V \leqslant e^{-4C_3\sqrt{\lambda_n}}, \tag{4.4.9}$$

conditions (4.4.7)–(4.4.8) hold for $t \in [t_n, t_{n+1})$.

If both (4.4.7) and (4.4.9) are fulfilled, the solution $(y,a)(t)$ is controlled by the following estimate:
For $t \in [0, T - 1/n_0^2]$, we have $\|(y,a)(t)\|_V \leqslant \|(y_0,a_0)\|_V$.
For $t \in [t_n, t_{n+1})$ with $n \geqslant n_0$, we have

$$\begin{aligned}
&\|(y,a)(t)\|_V / \|(y_0,a_0)\|_V \\
\leqslant &|\Xi_{\lambda_n}||\Xi_{\lambda_n}^{-1}| \prod_{k=n_0}^{n-1} \left(|\Xi_{\lambda_k}||\Xi_{\lambda_k}^{-1}| e^{-(t_{k+1}-t_k)\lambda_k/2} \right) \\
\leqslant &\left(\prod_{k=n_0}^{n-1} e^{-ck^5} \right) \left(\prod_{k=n_0}^{n} e^{5C_2 k^4} \right).
\end{aligned}$$

In order to ensure the conditions (4.4.7) and (4.4.9), and to get the stabilization to 0 on time T, we only need to find $\varepsilon > 0$ such that

$$\varepsilon \left(\prod_{k=n_0}^{n-1} e^{-ck^5} \right) \left(\prod_{k=n_0}^{n} e^{5C_2 k^4} \right) \leqslant e^{-4(C_2+C_3)n^4} \tag{4.4.10}$$

for all $n \geqslant n_0$. Such ε obviously exists.

At last, it remains to prove (3), the uniform stability property. On the one hand, observe from (4.1.10) and $(\mathcal{P}_4)$ that, for $\forall \delta_0 > 0$, there exists $T_0 \in [0, T)$ such that

$$\begin{aligned}
&(\|(y_0,a_0)\|_V \leqslant \delta_0/2, t_0 \in [T_0, T)) \\
\Rightarrow &(\|\Phi(t, t_0; y_0, a_0)\|_V \leqslant \delta_0, \forall t \in [t_0, T)).
\end{aligned} \tag{4.4.11}$$

On the other hand, from $(\mathcal{P}_3)$ we can find a M such that

$$u(t; y, a) \leqslant M \|(y,a)\|_V, \text{ for } t \in [0, T_0], \tag{4.4.12}$$

which concludes the existence of C such that

$$\|\Phi(t, s; y_0, a_0)\|_V \leqslant C \|(y_0,a_0)\|_V, \forall 0 \leqslant s \leqslant t \leqslant T_0. \tag{4.4.13}$$

Estimates (4.4.11) and (4.4.13) together with (2) give the uniform stability property (3), which completes the proof.

Remark 12. *As we have seen, the main idea is to use the "kernel" (linear part), which forces our results to be local. From the controllability point of view, one can use the return method to get global control results (even in small time), see [Cha09a, Cor92, Cor96]. From local stabilization to some global result, there still exists a big gap, especially for small time.*

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Chapter 5

Small-time global stabilization of a viscous Burgers equation

Contents

5.1 Introduction

A very classical problem for controllable system is the asymptotic stabilization issue. Stabilization is an action to render a system more stable than it is naturally: with the help of some well-designed feedback laws the closed-loop system has stability properties that does not have the system without feedback laws as, for example, asymptotically stability, exponentially stability, or even better. One of the strongest stabilization property that one may ask is the small-time stabilization, which means that for any positive time T there exist feedback laws such that the closed-loop system is stable and the solutions of the closed-loop system are zero after any interval of time of length at least equal to T. (Other definitions are in fact possible; the one given here is the one directly inspired by the classical small-time local controllability; see, for example, [Cor07a, Definition 3.2].) However, due to some special properties of partial differential equations, for example finite speed of propagation, one can sometimes only achieve finite-time stabilization: there exist a positive time T and feedback laws such that the closed-loop system is stable and the solutions of the closed-loop systems are zero after an interval of time of length at least equal to T.

Let us first recall some results concerning systems in finite dimension. It was first pointed out in [Sus79] that a system which is globally controllable may not be globally asymptotically stabilizable by means of continuous stationary feedback laws. In [Bro83] a necessary condition for asymptotic stabilizability by means of continuous stationary feedback laws is established. See also [Cor90]. There are controllable systems which do not satisfy this necessary condition. In order to overcome this problem two main strategies have been introduced, namely the use of discontinuous stationary feedback laws and the use of continuous (with respect to the state) time-varying feedback laws. For the first strategy, let us mention in particular [Sus79] and [CLSS97]. Concerning the second strategy, which was introduced in [SS80] and [Sam91], it is proved in [Cor95] that many powerful sufficient conditions for small-time local controllability imply the existence of feedback laws which stabilize locally the system in small time.

Concerning control systems modelled by means of partial differential equations much less is known. The classical approach for local results is to first consider the linearized control system around the equilibrium of interest. If this linear system can be asymptotically stabilized by a linear feedback law one may expect that the same feedback law is going to stabilize asymptotically the initial nonlinear control system. This approach has been successfully applied to many control systems. Let us, for example, mention [Bad09, BT11, Bar11, BLT06a, BLT06b, Ray06, Ray07], which are dealing with the stabilization of the Navier-Stokes equations of incompressible fluids, equations which are close to the one we study here, i.e. the viscous Burgers equation. However this strategy does not work in two important cases, namely the case where the linearized system is not asymptotically stabilizable and the case where one is looking for a global result. In both cases one expects that the construction of (globally or locally) asymptotically stabilizing feedback laws heavily depends on the methods allowing to use the nonlinearity in order to prove the associated controllability property (global or local controllability). For the local controllability one of this method is the "power series expansion" method. See in particular [Cer07, CC09a, CC04] where an expansion to the order 2 and 3 is used in order to prove the local controllability of Korteweg-de Vries equations. This method can be indeed adapted to construct stabilizing feedback laws: see [CR17] for control systems in finite dimension and [CRX17] for a Korteweg-de Vries control system and [CE17] for a Navier-Stokes equation.

Concerning the second case (global stabilization), even less is known. It is natural to expect that the construction of globally asymptotically stabilizable feedback laws depends strongly on the arguments allowing to prove this controllability. One of these arguments is the use of the return method together with scaling arguments (and, in some cases, a local controllability result) as introduced in [Cor96, Cor96]. These arguments have been used to get global controllability results for

- The Euler equations of incompressible fluids in [Cor96, Gla00],
- The Navier-Stokes equations of incompressible fluids in [Cha09c, Cor96, CF96, CMS16, FI99],
- Burgers equations in [Cha09b, Mar14],
- The Vlasov-Poisson system in [Gla03, GHK12].

In some of these cases the "phantom tracking" method gives a possibility to get global stabilization. This method was introduced in [Cor99] for the asymptotic stabilization of the Euler systems, then it has been used in various models [BCMR07, Gla05]. One can find a tutorial introduction to this method in [Cor13]. However, it is not clear how to get finite-time stabilization with this method.

Concerning the stabilization in small time or even in finite time of partial differential equations very little is known. Let us mention

- The use of Krstic's backstepping method [KS08b] to get stabilization in finite time of linear hyperbolic systems; see in particular [ADM16, CHO17, CVKB13, DMVK13, HDMVK16],
- The small-time stabilization of $1-D$ parabolic equations [CN17],
- The small-time local stabilization of Korteweg-de Vries equations [Xia18].

In this paper, we give the first small-time global stabilization result in a case where the global null controllability is achieved by using the return method together with scaling arguments and a local controllability result. We investigate the Burgers equation

$$y_t - y_{xx} + yy_x = a(t),\ y(t,0) = u_1(t),\ y(t,1) = u_2(t), \tag{5.1.1}$$

where, at time t, the state is $y(t,\cdot)$ and the controls are $a(t) \in \mathbb{R}$, $u_1(t) \in \mathbb{R}$, and $u_2(t) \in \mathbb{R}$. The Burgers equation has been very much studied for its important similarities with the Navier-Stokes equation as the appearance of boundary layers and the balance between the linear viscous term and the quadratic transport term.

Let us briefly recall some controllability results on the Burgers control system (5.1.1). When $a = 0$ and $u_1 = 0$, the small-time local null controllability is proved in [FI95] . When $a = 0$, it is proved in [GI07] that the small-time global null controllability does not hold. Before and after this, many related results were given in [AM98, Cor07b, Dia96, FCG07, GG07, Hor98, Per12]. In [Cha09b] the return method and scaling arguments are used as in [Cor96, CF96] to prove that (5.1.1) is globally null controllable. The global null controllability in small time also holds if $u_2 = 0$ as proved in [Mar14], even if in this case boundary layers appear when applying the return method. Moreover, it is proved in [Mar] that the small-time local controllability fails when $u_1 = u_2 = 0$.

This article is dealing with the small-time global stabilization of (5.1.1). To overcome some regularity issues we add an integration on the control variable a : now $a_t = \alpha(t)$ and $a(t)$ is part of the state. In other words, we consider a dynamical extension of (5.1.1) -see for example [Cor07a, p. 292]- with an extension with a variable of dimension only 1. Dynamical extensions are usually considered to handle output regulations. It can also be used to handle obstructions to full state stabilization for nonlinear systems even in finite dimension: see [CP91, Proposition 1]. In this paper, we therefore consider the following viscous Burgers controlled system:

$$\begin{cases} y_t - y_{xx} + yy_x = \alpha(t) & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = u_1(t) & \text{for } t \in (s,+\infty), \\ y(t,1) = u_2(t) & \text{for } t \in (s,+\infty), \\ a_t = \alpha(t) & \text{for } t \in (s,+\infty), \end{cases} \tag{5.1.2}$$

where, at time t, the state is $(y(t,\cdot), a(t)) \in L^2(0,1) \times \mathbb{R}$, and the control is $(\alpha(t), u_1(t), u_2(t)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. (We could have considered $a_t = \beta(t)$ where $\beta(t)$ is a new control; however it turns out that one can just take $\beta(t) = \alpha(t)$.)

Before stating our results on stabilization, let us introduce the notion of feedback law, closed-loop system, proper feedback law, and flow associated to a proper feedback law. A feedback law is an application F

$$\begin{cases} F : D(F) \subset \mathbb{R} \times L^2(0,1) \times \mathbb{R} & \to & \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ (t; y, a) & \mapsto & F(t; y, a) = (A(t; y, a), U_1(t; y, a), U_2(t; y, a)). \end{cases} \tag{5.1.3}$$

The closed-loop system associated to such a feedback law F is the evolution equation

$$\begin{cases} y_t - y_{xx} + yy_x = A(t;y,a) & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = U_1(t;y,a) & \text{for } t \in (s,+\infty), \\ y(t,1) = U_2(t;y,a) & \text{for } t \in (s,+\infty), \\ a_t = A(t;y,a) & \text{for } t \in (s,+\infty). \end{cases} \tag{5.1.4}$$

The feedback law F is called *proper* if the Cauchy problem associated to the closed-loop system (5.1.4) is well posed for every $s \in \mathbb{R}$ and for every initial data $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ at time s; see Definition 13 for the precise definition of a solution to this Cauchy problem and see Definition 14 for the precise definition of proper. For a proper feedback law, one can define the flow $\Phi : \Delta \times (L^2(0,1) \times \mathbb{R}) \to (L^2(0,1) \times \mathbb{R})$, with $\Delta := \{(t,s);\, t > s\}$ associated to this feedback law: $\Phi(t,s;y_0,a_0)$ is the value at $t > s$ of the solution (y,a) to the closed-loop system (5.1.4) which is equal to (y_0,a_0) at time s.

Let

$$V := L^2(0,1) \times \mathbb{R} \text{ with } \|(y,a)\|_V := \|y\|_{L^2} + |a|. \tag{5.1.5}$$

Our main result is the following small-time global stabilization result.

Theorem 19. *Let $T > 0$. There exists a proper $2T$-periodic time-varying feedback law for system (5.1.2) such that*

(i) $\Phi(4T + t, t; y_0, a_0) = 0, \quad \forall t \in \mathbb{R}, \quad \forall y_0 \in L^2(0,1), \quad \forall a \in \mathbb{R}.$

(ii) *(Uniform stability property.) For every $\delta > 0$, there exists $\eta > 0$ such that*

$$\left(\|(y_0,a_0)\|_V \leqslant \eta\right) \Rightarrow \left(\|\Phi(t,t';y_0,a_0)\|_V \leqslant \delta, \quad \forall t' \in \mathbb{R}, \quad \forall t \in (t',+\infty)\right). \tag{5.1.6}$$

Our strategy to prove Theorem 19 is to decompose the small-time global stabilization into two stages:

- Stage 1: Global "approximate stabilization", i.e., the feedback law steers the control system in a small neighborhood of the origin,
- Stage 2: Small-time local stabilization.

In the remaining part of this introduction, we heuristically describe these two stages (see Figure 5.1).

5.1.1 Global approximate stabilization

In this part we use the transport term yy_x and the "phantom tracking" strategy to get global approximate stabilization in small time, i.e. to get, for a given $\varepsilon > 0$, $\|y(t)\|_{L^2} + |a(t)| \leqslant \varepsilon$ for t larger than a given time. For this issue, let us perform the following change of variable

$$z := y - a. \tag{5.1.7}$$

Then (5.1.2) becomes

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = u_1(t) - a(t) & \text{for } t \in (s,+\infty), \\ z(t,1) = u_2(t) - a(t) & \text{for } t \in (s,+\infty), \\ a_t = \alpha(t) & \text{for } t \in (s,+\infty). \end{cases} \tag{5.1.8}$$

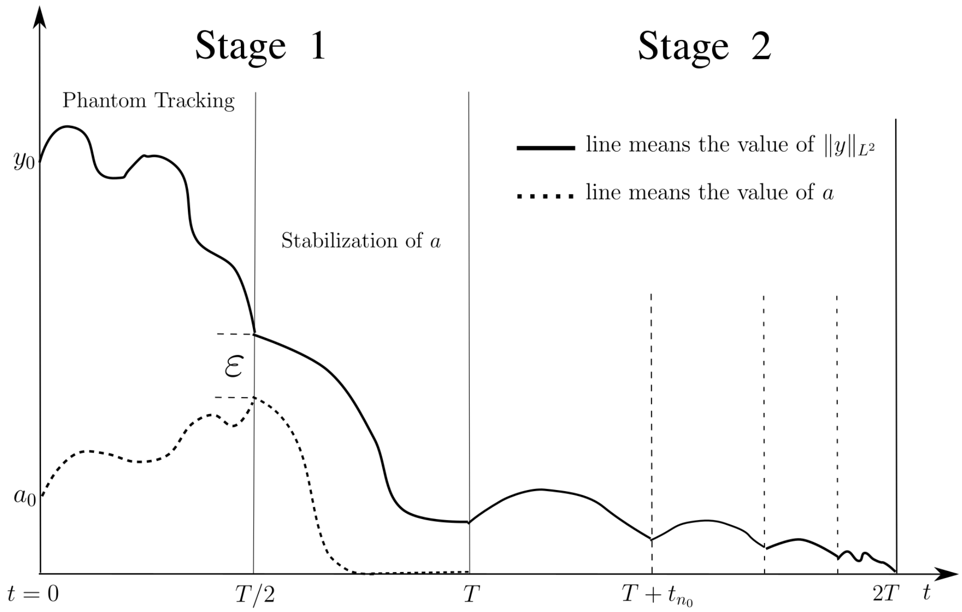

Figure 5.1: Small-time global stabilization of (y, a).

In this stage, we will always set

$$U_1(t; y, a) = U_2(t; y, a) = a. \tag{5.1.9}$$

Then the energy (i.e. the square of the L^2-norm) is dissipating:

$$\frac{d}{dt}\|z\|_{L^2}^2 \leqslant 0. \tag{5.1.10}$$

As we know, the "transport term" $a(t)z_x$ can lead to a small value for $\|z(T)\|_{L^2}$. For example letting $a(t) = C\|z\|_{L^2}$, one can expect that $\|z(T)\|_{L^2} \leqslant \varepsilon$ for $T > 0$ given, whatever the initial data is. However $|a(t)|$ can become larger. Thanks to the control of $a(t)$ (see (5.1.8)) and the dissipation of z (see (5.1.10)), a, as we will see, can be stabilized later on. In order to stabilize z only, we will try to find suitable feedback laws for system (5.1.8).

Using this strategy, we will get the following theorem, the proof of which is given in Section 5.3.

Theorem 20. *Let $T > 0, \varepsilon > 0$. There exists*

$$A : \mathbb{R} \times L^2(0, 1) \times \mathbb{R} \to \bar{\mathbb{R}}, \ (t; y, a) \mapsto A(t; y, a), \tag{5.1.11}$$

such that the associated feedback law F_1 (see (5.1.3) and (5.1.9)) is proper for system (5.1.2) and such that the following properties hold, where Φ_1 denotes the flow associated to F_1,

($\mathcal{Q}_1$) The feedback law A is T-periodic with respect to time:

$$A(t; y, a) = A(T + t; y, a), \quad \forall (t, y, a) \in \mathbb{R} \times L^2(0, 1) \times \mathbb{R}, \tag{5.1.12}$$

($\mathcal{Q}_2$) There exists a stationary feedback law $A_0 : L^2(0, 1) \times \mathbb{R} \to \bar{\mathbb{R}}$, $(y, a) \mapsto A_0(y, a)$, such that

$$A(t; y, a) = A_0(y, a), \ \ \forall t \in [0, T/2), \ \ \forall (y, a) \in D(A_0), \tag{5.1.13}$$

$(\mathcal{Q}_3)$ *There exists a stationary feedback law* $A_1 : \mathbb{R} \to \mathbb{R}$, $a \mapsto A_1(a)$, *such that*

$$A(t; y, a) = A_1(a), \ \ \forall (t; y, a) \in [T/2, T) \times L^2(0,1) \times \mathbb{R}, \tag{5.1.14}$$

$(\mathcal{Q}_4)$ *(Local uniform stability property.) For every* $\delta > 0$, *there exists* $\eta > 0$ *such that*

$$(\|(y_0, a_0)\|_V \leqslant \eta) \Rightarrow (\|\Phi_1(t, t'; y_0, a_0)\|_V \leqslant \delta, \ \ \forall 0 \leqslant t' \leqslant t \leqslant T), \tag{5.1.15}$$

$(\mathcal{Q}_5)$ *For every* y_0 *in* $L^2(0,1)$ *and for every* $a_0 \in \mathbb{R}$,

$$\Phi_1(T, 0; y_0, a_0) = (y(T), 0) \ \textit{with} \ \|y(T)\|_{L^2(0,1)} \leqslant \varepsilon. \tag{5.1.16}$$

Theorem 20 is not a stabilization result, since we only get that $y(T)$ is "close to 0". For this reason we name this stage "global approximate stabilization".

5.1.2 Small-time local stabilization

Thanks to the first stage we now only need to get the small-time local stabilization. Since we already have $\Phi_1(T, 0; y_0, a_0) = (y(T), 0)$ with $\|y(T)\|_{L^2} \leqslant \varepsilon$, we can set $\alpha \equiv 0$. Inspired by the piecewise backstepping approach introduced in [CN17], we also set $u_1 = 0$. Hence the system becomes

$$\begin{cases} y_t - y_{xx} + y y_x = 0 & \text{for } \ (t, x) \in (s, +\infty) \times (0,1), \\ y(t, 0) = 0 & \text{for } \ t \in (s, +\infty), \\ y(t, 1) = u_2(t) & \text{for } \ t \in (s, +\infty). \end{cases} \tag{5.1.17}$$

We do not care about a since it does not change. In [CN17] the authors get small-time semi-global stabilization for the heat equation. Since we only need small-time local stabilization, the nonlinear term $y y_x$ could naturally be regarded as a small perturbation. However, by classical Lions–Magenes method, in order to have a $C^0([0,T]; L^2(0,1))$ solution (to the system (5.1.17)), a $H^{1/4}(0,T)$ regularity of the control term is needed. For the control problem with the open-loop systems, the regularity condition on the control term is not a big obstacle. But when we consider the closed-loop system, it is hard to expect our feedback law will lead to a control in $H^{1/4}(0,T)$, especially when the feedback laws are given by some unbounded operators. Actually this problem also appears for the KdV system [Xia18], where based on the special structure of KdV (leading to the Kato hidden regularity of $y_x(t,0)$), the "adding an integrator" method (i.e. the control is no longer u_2 but $\dot{u}_2$ in the framework of (5.1.17)) solved this problem. Nevertheless, this idea does not work for our case, since there is no such hidden regularity.

However, instead of the hidden regularity, we have now the maximum principle. With this principle we get that a control in $C^0([0,T])$ leads to a solution in $C^0([0,T]; L^2(0,1))$. Hence we get a solution in $C^0([0,T]; L^2(0,1))$ for the closed-loop system. We look for $U_2 : \mathbb{R} \times L^2(0,1) \to \mathbb{R}$ satisfying the following properties

$(\mathcal{P}_1)$ The feedback law U_2 is T-periodic with respect to time:

$$U_2(t; y) = U_2(T + t; y), \tag{5.1.18}$$

($\mathcal{P}_2$) There exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that

$$t_0 = 0, \tag{5.1.19}$$

$$\lim_{n\to+\infty} t_n = T, \tag{5.1.20}$$

$$U_2 \text{ is of class } C^1 \text{ in } [t_n, t_{n+1}) \times L^2(0,1), \tag{5.1.21}$$

($\mathcal{P}_3$) The feedback law U vanishes on $\mathbb{R}\times\{0\}$ and there exists a continuous function $M : [0,T) \to [0,+\infty)$ such that, for every $(t, y_1, y_2) \in [0,T) \times L^2(0,1) \times L^2(0,1)$,

$$|U(t; y_1) - U(t; y_2)| \leqslant M(t)(\|y_1 - y_2\|_{L^2}), \tag{5.1.22}$$

($\mathcal{P}_4$) For every $(t, y) \in \mathbb{R} \times L^2(0,1)$, we have

$$|U(t; y)| \leqslant \min\{1, \sqrt{\|y\|_{L^2}}\}, \tag{5.1.23}$$

($\mathcal{P}_5$) If $\|y\|_{L^2(0,1)} \geqslant 1$, then, for every $t\in\mathbb{R}$, $U(t; y) = 0$,

and leading to the small-time local stability for the y variable if the feedback law $F = F_2$ is defined by

$$F_2(t; y, a) = (0, 0, U_2(t, y)). \tag{5.1.24}$$

More precisely, one has the following theorem.

Theorem 21. *Let $T > 0$. There exists $\varepsilon > 0$ and $U_2 : \mathbb{R}\times L^2(0,1) \to \mathbb{R}$ satisfying properties ($\mathcal{P}_1$)–($\mathcal{P}_5$), such that the feedback law F_2 defined by (5.1.24) is proper and, if the flow for the closed-loop system is denoted by Φ_2,*

(i) For every $y_0 \in L^2(0,1)$ and for every $a_0 \in \mathbb{R}$,

$$\Phi_2(T, 0; y_0, a_0) = (0, a_0) \text{ if } \|y_0\|_{L^2} \leqslant \varepsilon, \tag{5.1.25}$$

(ii) (Local uniform stability property.) For every $\delta > 0$, there exists $\eta > 0$ such that

$$(\|(y_0, a_0)\|_V \leqslant \eta) \Rightarrow (\|\Phi_2(t, t'; y_0, a_0)\|_V \leqslant \delta, \ \ \forall 0 \leqslant t' \leqslant t \leqslant T). \tag{5.1.26}$$

This paper is organized as follows. Section 5.2 is dealing with the well-posedness of various Cauchy problems and the definition of proper feedback laws. Section 5.3 and Section 5.4 are on the global approximate stabilization and the small-time local stabilization. Then we define our time-varying feedback laws in Section 5.5. These feedbacks law lead to Theorem 19, which will be proved in Section 5.6. In the appendices, we prove some well-posedness results (for both open-loop systems and closed-loop systems), namely Proposition 9, Proposition 10, Theorem 21, Lemma 18, Lemma 23, and Lemma 24.

5.2 Well-posedness of the open-loop system (5.1.2) and proper feedback laws

In this section we briefly review results on the well-posedness of the open-loop system (5.1.2). Then we establish our new estimates which will be used for the well-posedness of the closed-loop systems. Finally we define *proper* feedback laws, i.e. feedback laws such that the closed-loop systems are well-posed in the context of our notion of solutions to (5.1.2).

Let us start with the linear Cauchy problem:

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = f(t,x) & \text{for } (t,x) \in (t_1,t_2) \times (0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (t_1,t_2), \\ y(t,1) = \gamma(t) & \text{for } t \in (t_1,t_2), \\ y(t_1,\cdot) = y_0. \end{cases} \tag{5.2.1}$$

We use the following definition of solutions to the Cauchy problem (5.2.1) (solution in a transposition sense; see [Cor07a, CN17, LM73]).

Definition 10. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be such that $t_1 < t_2$. Let $y_0 \in H^{-1}(0,1), \beta$ and $\gamma \in L^2(t_1,t_2)$, and $f \in L^1(t_1,t_2;H^{-1}(0,1))$. A solution to the Cauchy problem (5.2.1) is a function y in $C^0([t_1,t_2];H^{-1}(0,1))$ such that*

$$\begin{aligned} - \langle y_0, u(t_1,\cdot)\rangle_{H^{-1},H_0^1} + \langle y(s,\cdot), u(s,\cdot)\rangle_{H^{-1},H_0^1} + \int_{t_1}^{s} \gamma(t) u_x(t,1)dt \\ - \int_{t_1}^{s} \beta(t) u_x(t,0)dt - \int_{t_1}^{s} \langle f(t,x), u(t,x)\rangle_{H^{-1},H_0^1} dt = 0, \end{aligned} \tag{5.2.2}$$

for every $s \in [t_1,t_2]$, for every $u \in L^2(t_1,t_2;H^2(0,1)) \cap H^1(t_1,t_2;H_0^1(0,1))$ such that

$$u_t(t,x) + u_{xx}(t,x) = 0 \text{ in } L^2((t_1,t_2)\times(0,1)). \tag{5.2.3}$$

This definition ensures the uniqueness (there exists at most one solution), but is not sufficient to get the existence of solutions. Concerning this existence of solutions, and therefore the well-posedness of the Cauchy problem (5.2.1), one has the following proposition.

Proposition 8. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2 < t_1 + 1$.*

(1) If $f = 0$, $\beta = \gamma = 0$, then, for every y_0 in $H^{-1}(0,1)$), the Cauchy problem (5.2.1) has a unique solution $y \in C^0([t_1,t_2];H^{-1}(0,1))$. Moreover, when $y_0 \in L^2(0,1)$, this solution is in

$$C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;H_0^1(0,1)), \tag{5.2.4}$$

and satisfies

$$\|y\|_{C^0L^2} \leqslant \|y_0\|_{L^2} \text{ and } \|y\|_{L^2H_0^1} \leqslant \|y_0\|_{L^2}. \tag{5.2.5}$$

(2) If $y_0 = 0, \beta = \gamma = 0$, and $f \in L^1(t_1,t_2;L^2(0,1)) \cup L^2(t_1,t_2;H^{-1}(0,1))$, the Cauchy problem (5.2.1) has a unique solution y. Moreover

$$\|y\|_{C^0L^2} \leqslant \|f\|_{L^1L^2} \text{ and } \|y\|_{L^2H_0^1} \leqslant \|f\|_{L^1L^2} \tag{5.2.6}$$

and there exists $C_1 \geqslant 1$ (which is independent of $0 < t_2 - t_1 < 1$) such that

$$\|y\|_{C^0L^2 \cap L^2H_0^1} \leqslant C_1 \|f\|_{L^2H^{-1}}. \tag{5.2.7}$$

(3) If $y_0 = 0$, $f = 0$, β, and $\gamma \in L^2(t_1,t_2)$, the Cauchy problem (5.2.1) has a unique solution y. If in addition β and $\gamma \in H^{3/4}(t_1,t_2)$, this solution is also in $C^0H^1 \cap L^2H^2$.

In this proposition and in the following, in order to simplify the notations, when there is no possible misunderstanding on the the time interval, C^0L^2 denotes the space $C^0([t_1,t_2];L^2(0,1))$, L^2L^∞ denotes the space $L^2(t_1,t_2;L^2(0,1))$ etc.

Properties (1) and (3) follow from classical arguments; see, for example, [Cor07a, Sections 2.3.1 and 2.7.1], [GI07, Mar14]. Property (2) follows from direct calculations and one can find similar results in [GG08]. Since we want to investigate the well-posedness of closed-loop systems, (3) is difficult to use. For that reason, we investigate the well-posedness with lower regularities on β and γ. For the heat equation, we have the maximum principle:

Lemma 17 (Maximum principle: linear case)**.** *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in H^{-1}$, $\beta \in L^2(t_1, t_2)$, $\gamma \in L^2(t_1, t_2)$, and $f \in L^2(t_1, t_2; H^{-1})$. Let $y \in C^0([0, T]; H^{-1})$ be the solution of the Cauchy problem*

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = f & \text{for } (t,x) \in (t_1,t_2) \times (0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (t_1,t_2), \\ y(t,1) = \gamma(t) & \text{for } t \in (t_1,t_2), \\ y(t_1,\cdot) = y_0. \end{cases} \tag{5.2.8}$$

If

$$y_0 \geqslant 0, f \geqslant 0, \beta \geqslant 0, \text{ and } \gamma \geqslant 0, \tag{5.2.9}$$

then

$$y(t,\cdot) \geqslant 0, \ \ \forall t \in [t_1, t_2]. \tag{5.2.10}$$

Thanks to the maximum principle, we get a new version of the well-posedness of system (5.2.1), the proof of which is given in Appendix 5.7.

Proposition 9. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. If $f = 0, y_0 = 0$, β and $\gamma \in L^\infty(t_1, t_2)$, the unique solution y of the Cauchy problem (5.2.1) is in $L^\infty(t_1, t_2; L^2(0,1)) \cap L^2(t_1, t_2; L^\infty(0,1))$ and this solution is also in $C^0([t_1, t_2]; L^2(0,1))$ provided that β and γ are in $C^0([t_1, t_2])$. Moreover, for every $T_0 > 0$, and for every $\eta > 0$, there exists a constant $C_{T_0,\eta} > 0$ such that, for every $t_1 \in \mathbb{R}$ and for every $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \leq t_1 + T_0$, for every β and for every $\gamma \in L^\infty(t_1, t_2)$, and for every $t \in (t_1, t_2]$,*

$$\|y\|_{L^\infty(t_1,t;L^2) \cap L^2(t_1,t;L^\infty)} \leqslant \left(\eta + C_{T_0,\eta}(t-t_1)^{1/2}\right)\left(\|\beta\|_{L^\infty(t_1,t)} + \|\gamma\|_{L^\infty(t_1,t)}\right). \tag{5.2.11}$$

Let us now turn to the nonlinear Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) + yy_x = f & \text{for } (t,x) \in (t_1,t_2) \times (0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (t_1,t_2), \\ y(t,1) = \gamma(t) & \text{for } t \in (t_1,t_2), \\ y(t_1,\cdot) = y_0. \end{cases} \tag{5.2.12}$$

The idea is to regard, in (5.2.12), $-yy_x = -(y^2)_x/2$ as a force term. Hence we adopt the following definition.

Definition 11. *Let $y_0 \in H^{-1}(0,1)$, β and $\gamma \in L^2(t_1, t_2)$, and $f \in L^1(t_1, t_2; H^{-1}(0,1))$. A solution to the Cauchy problem (5.2.12) is a function*

$$y \in L^\infty(t_1, t_2; L^2(0,1)) \cap L^2(t_1, t_2; L^\infty(0,1)) \tag{5.2.13}$$

which, in the sense of Definition 10, is a solution of (5.2.1) with

$$f := -(y^2)_x/2 + f \in L^1(t_1, t_2; H^{-1}(0,1)). \tag{5.2.14}$$

Remark 13. *Let us point out that it would be better to write in (5.2.12) $(y^2)_x/2$ instead of yy_x. However, for the sake of better readability, we keep yy_x instead of $(y^2)_x/2$ here and in the following.).*

For this nonlinear system, thanks to Proposition 9, the classical well-posedness results, stability results, and the maximum principle on the Cauchy problem (5.2.12) can be modified into the following ones, which are more suitable for the stabilization problem and which are also proved in Appendix 5.7.

Proposition 10. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in L^2(0,1)$, β and $\gamma \in L^\infty(t_1,t_2)$. If β and γ are piecewise continuous the Cauchy problem (5.2.12) with $f = 0$ has one and only one solution. This solution is in $C^0([t_1,t_2];L^2(0,1))$.*

Moreover, for every $R > 0$, $r > 0$, and $\varepsilon > 0$, there exists $T^\varepsilon_{R,r} > 0$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \leq t_1 + T^\varepsilon_{R,r}$ and for every $y_0 \in L^2(0,1)$, β and $\gamma \in L^\infty(t_1,t_2)$ (not necessary to be piecewise continuous) such that

$$\|y_0\|_{L^2}\leqslant R \text{ and } \|\beta\|_{L^\infty}+\|\gamma\|_{L^\infty}\leqslant r, \tag{5.2.15}$$

the Cauchy problem (5.2.12) with $f = 0$ has one and only one solution and this solution satisfies

$$\|y\|_{L^\infty(t_1,t_2;L^2(0,1))}\leqslant 2R, \tag{5.2.16}$$
$$\|y\|_{L^2(t_1,t_2;L^\infty(0,1))}\leqslant \varepsilon R. \tag{5.2.17}$$

Remark 14. *The conditions on β and γ are for the existence of solutions: one can get the uniqueness of the solution with less regularity on β and γ.*

Lemma 18 (Maximum principle: nonlinear case)**.** *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0^\pm \in L^2(0,1)$, $\beta^\pm \in L^\infty(t_1,t_2)$ be piecewise continuous, and $\gamma^\pm \in L^\infty(t_1,t_2)$ be piecewise continuous. Let $y^\pm \in C^0([t_1,t_2];H^{-1}(0,1)) \cap C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;L^\infty(0,1))$ be solutions to the Cauchy problem*

$$\begin{cases} y^\pm_t(t,x) - y^\pm_{xx}(t,x) + y^\pm y^\pm_x = 0 & \text{for } (t,x) \in (t_1,t_2)\times(0,1),\\ y^\pm(t,0) = \beta^\pm(t) & \text{for } t \in (t_1,t_2),\\ y^\pm(t,1) = \gamma^\pm(t) & \text{for } t \in (t_1,t_2),\\ y^\pm(t_1,\cdot) = y_0^\pm. \end{cases} \tag{5.2.18}$$

If

$$y_0^- \leqslant y_0^+,\ \beta^- \leqslant \beta^+, \text{ and } \gamma^- \leqslant \gamma^+, \tag{5.2.19}$$

then

$$y^-(t,\cdot) \leqslant y^+(t,\cdot),\ \ \forall t \in [t_1,t_2]. \tag{5.2.20}$$

Lemma 19. *For every $R > 0$, $r > 0$, and $\tau > 0$, there exists $C(R,r,\tau) > 0$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 \leq t_1 + \tau$, and for every $y_0^\pm \in L^2(0,1)$, $\beta^\pm \in L^\infty(t_1,t_2)$ piecewise continuous, and $\gamma^\pm \in L^\infty(t_1,t_2)$ piecewise continuous such that*

$$\|y_0^\pm\|_{L^2}\leqslant R \text{ and } \|\beta^\pm\|_{L^\infty}+\|\gamma^\pm\|_{L^\infty}\leqslant r, \tag{5.2.21}$$

the solution to the Cauchy problem (5.2.12) with $f = 0$ satisfies

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1))}\leqslant C(R,r,\tau)\left(\|y_0^+ - y_0^-\|_{L^2(0,1)}+\|\beta^+ - \beta^-\|_{L^\infty}+\|\gamma^+ - \gamma^-\|_{L^\infty}\right). \tag{5.2.22}$$

Let us now come back to system (5.1.2). We start with the definition of a solution to the Cauchy problem associated to (5.1.2).

Definition 12. *Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2$. Let $y_0 \in L^2(0,1)$, $a_0 \in \mathbb{R}$, $\alpha \in L^1(t_1,t_2)$, u_1 and $u_2 \in L^\infty(t_1,t_2)$. A solution (y,a) to (5.1.2) with initial data (y_0,a_0) at time*

t_1 is a (y,a) satisfying

$$y \in C^0([t_1,t_2];L^2(0,1)) \cap L^2(t_1,t_2;L^\infty(0,1)), \tag{5.2.23}$$

$$a \in C^0([t_1,t_2]),\ a_t = \alpha \text{ in the distribution sense, and } a(t_1) = a_0, \tag{5.2.24}$$

$$\text{(5.2.1) holds in the sense of Definition 10, with } f := (-yy_x) + a(t),\ \beta := u_1,\ \gamma := u_2. \tag{5.2.25}$$

Remark 15. *Let us point out that, with Definition 12, Proposition 10 does not imply the existence of a solution to the Cauchy problem (5.1.2) since, in Definition 12, u_1 and u_2 are assumed to be only in $L^\infty(t_1,t_2)$ and not necessarily in $C^0([t_1,t_2])$. However this proposition implies this existence if u_1 and u_2 are only piecewise continuous. We choose L^∞ condition for u_1 and u_2 precisely to cover this case, which will be useful in the framework of the well-posedness of the closed-loop systems that we are going to consider.*

Definition 12 allows to define the notion of solution to the Cauchy problem associated to the closed-loop system (5.1.4) as follows.

Definition 13. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. Let*

$$\begin{array}{rcl} F : [s_1,s_2] \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)). \end{array}$$

Let $t_1 \in [s_1,s_2]$, $t_2 \in (t_1,s_2]$, $a_0 \in \mathbb{R}$, and $y_0 \in L^2(0,1)$. A solution on $[t_1,t_2]$ to the Cauchy problem associated to the the closed-loop system (5.1.4) with initial data (y_0,a_0) at time t_1 is a couple $(y,a) : [t_1,t_2] \to L^\infty(0,1) \times \mathbb{R}$ such that

$$t \in (t_1,t_2) \mapsto a(t) := A(t;y(t,\cdot),a(t)) \in L^1(t_1,t_2), \tag{5.2.26}$$

$$t \in (t_1,t_2) \mapsto u_1(t) := U_1(t;y(t,\cdot),a(t)) \in L^\infty(t_1,t_2), \tag{5.2.27}$$

$$t \in (t_1,t_2) \mapsto u_2(t) := U_2(t;y(t,\cdot),a(t)) \in L^\infty(t_1,t_2), \tag{5.2.28}$$

$$(y,a) \text{ is a solution (see Definition 12) of (5.1.2) with initial data } (y_0,a_0) \text{ at time } t_1. \tag{5.2.29}$$

We can now define feedback laws such that the closed-loop system has a unique solution in the sense of Definition 12. These feedback laws are called *proper* and are defined as follows.

Definition 14. *Let $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ be given such that $s_1 < s_2$. A proper feedback law on $[s_1,s_2]$ is an application*

$$\begin{array}{rcl} F : [s_1,s_2] \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)) \end{array}$$

such that, for every $t_1 \in [s_1,s_2]$, for every $t_2 \in (t_1,s_2]$, for every $a_0 \in \mathbb{R}$, and for every $y_0 \in L^2(0,1)$, there exists a unique solution on $[t_1,t_2]$ to the Cauchy problem associated to the the closed-loop system (5.1.4) with initial data (y_0,a_0) at time t_1 (see Definition 13).

A proper feedback law is an application F

$$\begin{array}{rcl} F : (-\infty,\infty) \times L^2(0,1) \times \mathbb{R} & \to & \bar{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \\ (t;y,a) & \mapsto & F(t;y,a) = (A(t;y,a), U_1(t;y,a), U_2(t;y,a)) \end{array}$$

such that, for every $s_1 \in \mathbb{R}$ and for every $s_2 \in \mathbb{R}$ such that $s_1 < s_2$, the feedback law restricted to $[s_1,s_2] \times L^2(0,1) \times \mathbb{R}$ is a proper feedback law on $[s_1,s_2]$.

5.3 Global approximate stabilization

Let $T > 0$ be given. As explained in Section 5.1.1, throughout this section we work with (z, a) instead of (y, a), where z is defined by (5.1.7). The equation satisfied by (z, a) is

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \\ a_t = \alpha(t) & \text{for } t \in (0,T). \end{cases} \tag{5.3.1}$$

The idea is to use the "transport term" $a(t)z_x$. Following the idea of backstepping (see e.g.[Cor07a, Section 12.5]), we first regard the term $a(t)$ as a control term: we consider the system

$$\begin{cases} z_t - z_{xx} + zz_x + a(t)z_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \end{cases} \tag{5.3.2}$$

where, at time $t \in [0,T]$, the state is $z(t,\cdot) \in L^2(0,1)$ and the control is $a(t) \in \mathbb{R}$. Inequality (5.1.10) shows that the L^2-norm of the state decays whatever is the control. However it does not provide any information on the decay rate of this L^2-norm. In order to get information on this decay rate, we consider the weighted energy (see e.g. [BC16, Chapter 2], [CBdN08])

$$V_1(z) := \int_0^1 z^2 e^{-x} dx. \tag{5.3.3}$$

With a slight abuse of notations, let $V_1(t) := V_1(z(t))$. Then, at least if

$$z \in C^1([0,T]; H^{-1}(0,1)) \cap C^0([0,T]; H_0^1(0,1)) \text{ and } a \in C^0([0,T]), \tag{5.3.4}$$

$V_1 \in C^1([0,T])$ and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}V_1 &= \langle z_t, ze^{-x} \rangle_{H^{-1},H_0^1} \\ &= \langle z_{xx} - (z^2/2)_x - a(t)z_x, ze^{-x} \rangle_{H^{-1},H_0^1} \\ &= -\int_0^1 z_x^2 e^{-x} dx + \left(\frac{1}{2} - \frac{a}{2}\right) V_1(z) - \frac{1}{3}\int_0^1 z^3 e^{-x} dx. \end{aligned} \tag{5.3.5}$$

In fact, as one easily sees, (5.3.5) holds in the distribution sense in $L^1(0,T)$ if

$$z \in H^1(0,T; H^{-1}(0,1)) \cap L^2(0,T; H_0^1(0,1)) \text{ and } a \in L^\infty(0,T). \tag{5.3.6}$$

From now on we assume that (5.3.6) holds. Since

$$\|z\|_{L^\infty} \leqslant 2 \left(\int_0^1 z_x^2 e^{-x} dx \right)^{1/2}, \tag{5.3.7}$$

(5.3.5) leads to

$$\frac{d}{dt}V_1 \leqslant -\int_0^1 z_x^2 e^{-x} dx + V_1^2 + (1-a)V_1. \tag{5.3.8}$$

We choose $a := (k+1)V_1$, then

$$\frac{d}{dt}V_1 \leqslant V_1 - kV_1^2. \tag{5.3.9}$$

The positive "equilibrium" point (of V_1) of the right hand side of (5.3.9) is $1/k$. Hence if k large enough we have

$$V_1(T) \leqslant 2/\sqrt{k}, \tag{5.3.10}$$

whatever is the initial data as shown by the following lemma.

Lemma 20. *Let $T > 0$. There exists $k_T \in \mathbb{N}$ such that, for every $k \geqslant k_T$ and for every $V_1 \in C^0([0,T];[0,+\infty))$ satisfying (5.3.9) in the distribution sense in $(0,T)$,*

$$V_1(T) \leqslant 2/\sqrt{k}. \tag{5.3.11}$$

Proof of Lemma 20. It is easy to observe that, if for some time $t_0 \in [0,T]$, $V_1(t_0) \leqslant 2/\sqrt{k}$, then $V_1(t) \leqslant 2/\sqrt{k}$, for every $t \in [t_0,T]$. So, arguing by contradiction, we may assume that

$$V_1(t) > 2/\sqrt{k},\ \ \forall t \in [0,T]. \tag{5.3.12}$$

Then

$$\dot{V}_1(t) \leqslant -\frac{k}{2}V_1^2(t), \tag{5.3.13}$$

which implies that

$$-\frac{1}{V_1(T)} + \frac{1}{V_1(0)} \leq -\frac{kT}{2}. \tag{5.3.14}$$

From (5.3.12) and (5.3.14), we get

$$\frac{k}{2}T \leqslant \frac{1}{V_1(t)} \leqslant \frac{\sqrt{k}}{2}, \tag{5.3.15}$$

which implies that

$$\sqrt{k}T \leqslant 1. \tag{5.3.16}$$

□

5.3.1 Construction of feedback laws

5.3.1.1 Phantom tracking stage

Let us come back to the system (5.3.1). For any T given, we consider the following Lyapunov function generated from the phantom tracking idea:

$$V_2(z,a) := V_1(z) + (a - \lambda V_1(z))^2, \tag{5.3.17}$$

with λ to be chosen later. The idea is to penalize $a \neq \lambda V_1(z)$; see [Cor13]. Again, with a slight abuse of notations, we define $V_2(t) := V_2(z(t),a(t))$. Then, at least if z is in $C^1([0,T];H^{-1}(0,1)) \cap C^0([0,T];H_0^1(0,1))$ and $a \in C^1([0,T])$, V_2 is of class C^1 and

$$\begin{aligned}\frac{d}{dt}V_2 &= \frac{d}{dt}V_1 + 2(a - \lambda V_1(z))(\alpha - \lambda\frac{d}{dt}V_1)\\ &\leqslant -\int_0^1 z_x^2 e^{-x}dx + V_1^2 + V_1 - \lambda V_1^2 + 2\left(a - \lambda V_1(z)\right)\left(\alpha - \lambda\frac{d}{dt}V_1 - \frac{V_1}{2}\right).\end{aligned}$$

We choose

$$\begin{aligned}\alpha :=&\lambda \frac{d}{dt}V_1 + \frac{V_1}{2} - \frac{1}{2}\lambda(a-\lambda V_1(z))^3\\ =&\lambda\left(-2\int_0^1 z_x^2 e^{-x}dx + (1-a)\,V_1(z) - \frac{2}{3}\int_0^1 z^3e^{-x}dx\right) + \frac{V_1}{2} - \frac{1}{2}\lambda(a-\lambda V_1(z))^3. \end{aligned}\tag{5.3.18}$$

Then, at least if z is in $C^1([0,T];H^{-1}(0,1))\cap C^0([0,T];H_0^1(0,1))$ and $\alpha\in C^0([0,T])$

$$\begin{aligned}\frac{d}{dt}V_2 &\leqslant V_1^2 + V_1 - \lambda V_1^2 - \lambda(a-\lambda V_1(z))^4\\ &\leqslant V_1 - \frac{\lambda-1}{2}V_2^2\\ &\leqslant V_2 - \frac{\lambda-1}{2}V_2^2.\end{aligned}\tag{5.3.19}$$

In fact, as one easily sees, (5.3.19) holds in the distribution sense in $L^1(0,T)$ if

$$z\in H^1(0,T;H^{-1}(0,1))\cap L^2(0,T;H_0^1(0,1)) \text{ and } \alpha\in L^\infty(0,T).\tag{5.3.20}$$

Let $\varepsilon>0$. Using Lemma 20 and (5.3.19), one gets the existence of $\lambda_0>1$, independent of (z,a) satisfying (5.3.20), such that, for every $\lambda\in[\lambda_0,+\infty)$,

$$|V_2(T/2)|\leqslant \frac{3}{\sqrt{\lambda-1}}\leqslant\varepsilon.\tag{5.3.21}$$

Meanwhile, there exists a constant C_ε such that $|a(T/2)|\leqslant C_\varepsilon$.

We denote this stationary feedback law by A_0, i.e., $A_0: L^\infty(0,1)\times\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is defined by

$$A_0(y,a):=\alpha \text{ with } \alpha \text{ given in (5.3.18), where } z \text{ is defined by (5.1.7)},\tag{5.3.22}$$

with the natural convention that $A_0(y,a)=-\infty$ if $y\notin H^1(0,1)$. This convention is justified by the fact that, by (5.3.7), there exists $C>0$

$$\left|\int_0^1 z^3e^{-x}dx\right|\leqslant \|z\|_{L^\infty}\|z\|_{L^2}^2\leqslant \frac{3}{2}\int_0^1 z_x^2e^{-x}dx + C\|z\|_{L^2}^4,\quad \forall z\in H_0^1(0,1).\tag{5.3.23}$$

Remark 16. *Let us point out that A_0 is an unbounded operator on the state space, which is $L^2(0,1)\times\mathbb{R}$. The set where it takes finite value is $H^1(0,1)\times\mathbb{R}\subsetneq L^2(0,1)\times\mathbb{R}$. However, as we will see in Subsection 5.3.2, the feedback law*

$$F(t;y,a):=(A_0(y,a),a,a),\quad \forall(t,y,a)\in D(F):=\mathbb{R}\times H^1(0,1)\times\mathbb{R}\tag{5.3.24}$$

is proper.

From (5.3.21) we see that $V_1(T/2)$ is small thanks to λ. However, at the same time, because of λ, a will approach to $\lambda V_1(T/2)$. This is bounded by some constant, and unfortunately we can not expect more precise uniform bounds than the above one. To solve this problem, in the next phase we construct a (stationary) feedback law which makes a decay to 0, but keeps V_1 small.

Remark 17. *Similar a priori estimates could be obtained for L^p-norm cases. Even more, one could further get L^∞ type estimates by using the technique introduced in [BC16, Chapter 4] and [CB15].*

5.3.1.2 Small-time global stabilization of the variable a

In this section we construct the stationary feedback law A_1 (see $(\mathcal{Q}_3)$). Since L^2-norm of z decays whatever is the control α, we only need to find a feedback law which stabilizes "a". In this section, we give a feedback law which stabilizes "a" in small time. For that it suffices to define A_1 by

$$A_1(a) := -\mu(a^2 + \sqrt{|a|}) \cdot \mathrm{sgn(a)}. \tag{5.3.25}$$

Indeed, with this A_1, one can easily verify that there exists $\mu_T > 0$ such that, whatever is $a(T/2)$, $a(T) = 0$ if $\mu \geq \mu_T$ and $\dot{a} = A_1(a)$.

Remark 18. *The feedback law A_1 is continuous but not Lipschitz. However, for every $t_1 \in \mathbb{R}$, for every $t_2 \in [t_1, +\infty)$, and for every $a_0 \in \mathbb{R}$, the ordinary differential equation $a_t = A_1(a)$ has a unique solution on $[t_1, t_2]$ such that $a(t_1) = a_0$.*

Remark 19. *For our Burgers equation, thanks to the energy dissipation (5.1.10), we do not need to care of z during the interval of time $[T/2, T]$. For some other partial differential equations, such decay phenomenon may not hold. However, the same strategy would also work. Indeed, we can stabilize a in very small time so that the change of z keeps small.*

Remark 20. *Another idea to stabilize a in finite time is to design a time-varying feedback law of the form $A_1(a) = -\mu_n a$ for $t \in [t_n, t_{n+1})$. However, if for every solution of $\dot{a} = A_1(a)$ on $[T/2, T]$, one has $a(T) = 0$ whatever is $a(T/2)$, this feedback law has to be unbounded on $[T/2, T) \times (-\delta, \delta)$ for every $\delta > 0$, which is not the case of A_1 defined by (5.3.25).*

In this section, our feedback law $F = F_1$ is defined by $(\mathcal{Q}_1)$, (5.1.9), (5.1.14), (5.3.22), and (5.3.25). Let us point out that it satisfies $(\mathcal{Q}_2)$–$(\mathcal{Q}_3)$.

5.3.2 Well-posedness and properties of the flow

This part is devoted to the properness of the feedback law F_1. From the definition of $F_1(t; y, a)$ for $t \in [T/2, T]$ (see (5.1.9), (5.1.14), and (5.3.25)), it follows from Proposition 10 that F_1 is proper on the interval of time $[T/2, T]$.

By the T-periodicity of F_1 it just remains to prove the properness of F_1 on the interval of time $[0, T/2]$. This properness is a consequence of the following lemma, the proof of which is given in Appendix 5.8

Lemma 21. *For every $T \in (0, +\infty)$, every $z_0 \in L^2(0, 1)$, and every $a_0 \in \mathbb{R}$, there exists one and only one (z, a) satisfying*

$$z \in L^2(0, T; H_0^1(0, 1)), \tag{5.3.26}$$

such that $(y, a) := (z + a, a)$ is a solution to (5.1.2) (see Definition 12) with initial data $(z_0 + a_0, a_0)$ at time 0 with

$$\alpha(t) := A_0(y(t), a(t)), \text{ for almost every } t \in (0, T), \tag{5.3.27}$$

$$\beta(t) = a(t), \gamma(t) = a(t), \quad \forall t \in [0, T]. \tag{5.3.28}$$

At a first sight it seems that (5.3.26) is too strong compared to what is imposed by (5.2.26) for the properness of F_1. Indeed, (5.2.26) just impose that $z \in L^1(0, T; H^1(0, 1))$. However, it follows from (1) and (2) of Proposition 8 that, if (y, a) is as in Lemma 21 with $y \in L^1(0, T; H^1(0, 1))$, then $z := y - a \in L^2(0, T; H^1(0, 1))$.

5.3.3 Proof of Theorem 20

It only remains to give the proof of Properties $(\mathcal{Q}_4)$ and $(\mathcal{Q}_5)$ of Theorem 20.

We first look at $(\mathcal{Q}_5)$. Let $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$. Note that (5.3.20) holds, and therefore (5.3.21) also holds:

$$|V_2(z(T/2), a(T/2))| \leqslant \frac{3}{\sqrt{\lambda - 1}} \leqslant \varepsilon. \tag{5.3.29}$$

From (5.3.29) one gets the existence of $\bar{\lambda} > 0$ independent of $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ such that

$$\|z(T/2)\|_{L^2(0,1)} + (a(T/2) - \lambda V_1(z(T/2)))^2 \leqslant \varepsilon \tag{5.3.30}$$

when $\lambda \geq \bar{\lambda}$.

Then the next stage (i.e. the evolution during the interval of time [T/2,T]; see Section 5.3.1.2) gives

$$\|z(T)\|_{L^2(0,1)} \leqslant \varepsilon \text{ and } a(T) = 0, \tag{5.3.31}$$

which concludes the proof of $(\mathcal{Q}_5)$.

It only remains to prove Property $(\mathcal{Q}_4)$. If $T/2 \leqslant t \leqslant T$, this property clearly holds, since both $\|z\|_{L^2}$ and $|a|$ decay as time is increasing. If $0 \leqslant t \leqslant T/2$, we only need to care about the case where $0 \leqslant t' \leqslant t \leqslant T/2$, thanks to the first case. Since (5.3.20) holds, (5.3.19) holds in $L^1(t', t)$. This shows that

$$\dot{V}_2 \leqslant V_2 \text{ on } [t', t] \tag{5.3.32}$$

if λ is larger than 1. Then, using (5.3.17),

$$\begin{aligned}\left(V_1(z(t) + (a(t) - \lambda V_1(z(t)))^2\right) &= V_2(t) \\ &\leqslant e^{T/2} V_2(t') = e^{T/2}\left(V_1(z(t') + (a(t') - \lambda V_1(z(t')))^2\right),\end{aligned} \tag{5.3.33}$$

which concludes the proof of Property $(\mathcal{Q}_4)$.

5.4 Small-time local stabilization

The aim of this section is to get the small-time local stabilization (for the y variable). The small-time local (and even semi-global) stabilization of the heat equation is given in [CN17]. Here we follow [CN17] and regard yy_x term as a small force term (as in [Xia18] for a KdV equation). Throughout this section we define $\alpha := 0$ and $u_1 := 0$ in (5.1.2), hence it is sufficient to consider

$$\begin{cases} y_t - y_{xx} + yy_x = 0 & \text{for } (t,x) \in (s, +\infty) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (s, +\infty), \\ y(t,1) = u_2(t) & \text{for } t \in (s, +\infty). \end{cases} \tag{5.4.1}$$

We construct a time-varying feedback law satisfying $(\mathcal{P}_1)$–$(\mathcal{P}_5)$ leading to the small-time local stabilization of system (5.4.1).

5.4.1 Local rapid stabilization

At first, let us briefly recall (see [Liu03], [KS08b, Chapter 4] or [CN17]) how the backstepping approach is used to get rapid stabilization for the following heat equation:

$$\begin{cases} y_t - y_{xx} = 0 & \text{for } (t,x) \in (s, +\infty) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (s, +\infty), \\ y(t,1) = u_2(t) & \text{for } t \in (s, +\infty). \end{cases} \tag{5.4.2}$$

Let $\lambda > 1$ given. Since Volterra type transformations will be considered, let us define

$$D := \{(x,v) \in [0,1]^2; v \leqslant x\}. \tag{5.4.3}$$

We define the feedback law by

$$u_2(t) := K_\lambda y = \int_0^1 k_\lambda(1,v) y(t,v) dv, \tag{5.4.4}$$

where the kernel function k_λ is the unique solution of

$$\begin{cases} k_{xx} - k_{vv} - \lambda k = 0 & \text{in } D, \\ k(x,0) = 0 & \text{in } [0,1], \\ k(x,x) = -\frac{\lambda x}{2} & \text{in } [0,1]. \end{cases} \tag{5.4.5}$$

Let us perform the following transformation $\Pi_\lambda : L^2(0,1) \to L^2(0,1)$, $y \mapsto z$,

$$z(x) = \Pi_\lambda(y(x)) := y(x) - \int_0^x k_\lambda(x,v) y(v) dv. \tag{5.4.6}$$

The kernel function k_λ is of class C^2 in D and satisfies the following estimate (see [CN17, Lemma 1]).

Lemma 22. *There exists a constant C_1 which is independent of $\lambda > 1$, such that*

$$\|k_\lambda\|_{C^2(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}. \tag{5.4.7}$$

Remark 21. *In fact [CN17, Lemma 1] is dealing with the H^1-norm (for more general equations). However, the proof can easily be adapted to get Lemma 22. Moreover in the case of (5.4.2), the kernel can be expressed in terms of the Bessel function:*

$$K(x,v) = -v \frac{I_1\left(\sqrt{\lambda(x^2 - v^2)}\right)}{\sqrt{\lambda(x^2 - v^2)}}, \tag{5.4.8}$$

where I_1 is the first order modified Bessel function of the first kind; see [KS08a, (4.33)]. This explicit formula allows also to prove Lemma 22. Inequality (5.4.7) is related to the estimate given in [LR95a, Proposition 1] by Lebeau and Robbiano. See also [LR95b]. With no difficulty, the C^2-estimate can be generalized to C^n-estimates, $n \geqslant 3$, and one can prove the analyticity of the solution, which also follows from (5.4.8). For similar estimates for a Korteweg-de Vries equation, see [Xia18, Lemma 2] and [Xia19, Lemma 3].

In particular the transformation $\Pi_\lambda : L^2(0,1) \to L^2(0,1)$ is a bounded linear operator. This operator is also invertible. The inverse transformation, $\Pi_\lambda^{-1} : L^2(0,1) \to L^2(0,1)$, is given by

$$y(x) = \Pi_\lambda^{-1}(z(x)) := z(x) + \int_0^x l_\lambda(x,v) z(v) dv, \tag{5.4.9}$$

with the kernel l_λ satisfies

$$\begin{cases} l_{xx} - l_{vv} + \lambda l = 0 & \text{in } D, \\ l(x,0) = 0 & \text{in } [0,1], \\ l(x,x) = -\frac{\lambda x}{2} & \text{in } [0,1]. \end{cases} \tag{5.4.10}$$

The same estimate as (5.4.7) holds for l,

$$\|l_\lambda\|_{C^2(\mathcal{T})} \leqslant e^{C_1\sqrt{\lambda}}. \tag{5.4.11}$$

In fact one can even get better estimates than (5.4.11) (see [CN17, Corollary 2]). Let us denote $z := \Pi_\lambda y$ by z to simplify the notations. From (5.4.6) and (5.4.9), we know that

$$\|y\|_{L^2} \leqslant e^{3/2C_1\sqrt{\lambda}}\|z\|_{L^2} \text{ and } \|z\|_{L^2} \leqslant e^{3/2C_1\sqrt{\lambda}}\|y\|_{L^2}, \tag{5.4.12}$$

$$\|y\|_{H^1} \leqslant \|z\|_{H_0^1} + C\|z\|_{L^2} \text{ and } \|z\|_{H_0^1} \leqslant \|y\|_{H^1} + C\|y\|_{L^2}. \tag{5.4.13}$$

Then, following (5.4.2), (5.4.4), and (5.4.6), the solution y of (5.4.2) with (5.4.4), is transformed (via Π_λ) into a solution of

$$\begin{cases} z_t - z_{xx} + \lambda z = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,1) = 0 & \text{for } t \in (s,+\infty), \end{cases} \tag{5.4.14}$$

from which we get exponential decay of the energy of z with an exponential decay rate at least equal to 2λ.

Let us now consider the local rapid stabilization of the Burgers equation (5.4.1). The idea is to construct a stationary continuous locally supported feedback law which is given by (5.4.4) near the equilibrium point.

Suppose that y is a solution of (5.4.1) with feedback law (5.4.4), *i.e.* y is a solution of the Cauchy problem

$$\begin{cases} y_t - y_{xx} + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (s,+\infty), \\ y(t,1) = K_\lambda y = \int_0^1 k_\lambda(1,v) y(t,v) dv & \text{for } t \in (s,+\infty), \\ y(0,\cdot) = y_0, \end{cases} \tag{5.4.15}$$

with $y_0 \in L^2(0,1)$. Then $z := \Pi_\lambda(y)$ satisfies

$$\begin{cases} z_t - z_{xx} + \lambda z = I & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,1) = 0 & \text{for } t \in (s,+\infty), \\ z(0,\cdot) = z_0, \end{cases} \tag{5.4.16}$$

with

$$z_0 := \Pi_\lambda(y), \tag{5.4.17}$$

$$I := -\Pi_\lambda^{-1}(z) \left(\Pi_\lambda^{-1}(z)\right)_x + \int_0^x k_\lambda(x,v)(yy_x)(v)dv. \tag{5.4.18}$$

For the Cauchy problem (5.4.15) and (5.4.16), we have the following lemma, whose proof is given in Appendix 5.10.

Lemma 23. *Let $\lambda > 1$, $R > 0$, and $s \in \mathbb{R}$. There exists $0 < T_R^{tr} < 1$ such that, for every $y_0 \in L^2(0,1)$ such that*

$$\|y_0\|_{L^2} \leqslant R, \tag{5.4.19}$$

the Cauchy problem (5.4.15) has one and only one solution. This solution is also in $C^0([s, s+$

$T_R^{tr}]; L^2(0,1)) \cap L^2(s, s+T_R^{tr}; H^1(0,1))$. *Moreover, this solution satisfies*

$$\|y\|_{C^0L^2\cap L^2\dot{H}^1} \leqslant 3e^{3C_1\sqrt{\lambda}}R. \tag{5.4.20}$$

By Lemma 23, for any $z_0 \in L^2(0,1)$, there exists $\tilde{T} > 0$ such that, the Cauchy problem (5.4.16) has a unique solution on $t[s, s+\tilde{T}]$ and this solution is also in $C^0([s, s+\tilde{T}]; L^2(0,1)) \cap L^2(s, s+\tilde{T}; H_0^1(0,1))$.
Since

$$\|w^2\|^2_{L^2L^2} = \|w^4\|_{L^1L^1} \leqslant \|w\|^2_{L^\infty L^2}\|w\|^2_{L^2L^\infty} \leqslant C\|w\|^2_{C^0L^2}\|w\|^2_{L^2H^1}, \tag{5.4.21}$$

we know form direct calculations that $I \in L^2(s, s+\tilde{T}; H^{-1}(0,1))$ and that

$$I = -\frac{1}{2}\Big(\big(\Pi_\lambda^{-1}(z)\big)^2\Big)_x - \frac{1}{2}\int_0^x k_{\lambda,v}(x,v)y^2(v)dv - \frac{\lambda x}{4}y^2(x). \tag{5.4.22}$$

Note that, since $z \in C^0([s, s+\tilde{T}]; L^2(0,1)) \cap L^2(s, s+\tilde{T}; H_0^1(0,1))$ and $I \in L^2(s, s+\tilde{T}; H^{-1}(0,1))$, we have

$$\left\langle z, \Big(\big(\Pi_\lambda^{-1}(z)\big)^2\Big)_x\right\rangle_{H_0^1, H^{-1}} = \left\langle z_x, \big(\Pi_\lambda^{-1}(z)\big)^2\right\rangle_{L^2,L^2}, \tag{5.4.23}$$

and

$$\frac{d}{dt}\|z\|^2_{L^2} = -2\int_0^1 z_x^2(x)dx - 2\lambda\|z\|^2_{L^2} + 2\langle z, I\rangle_{L^2,L^2} \text{ in } L^1(s, s+\tilde{T}). \tag{5.4.24}$$

Thanks to (5.4.7), (5.4.22) and (5.4.23), there exists $C_0 > 1$, $C_2 > 2C_1$ and $C_3 > C_2$, independent of $\lambda > 1$ and z, such that

$$\begin{aligned}2|\langle z, I\rangle_{L^2,L^2}| &\leqslant \|z_x\|_{L^2}\|\big(\Pi_\lambda^{-1}(z)\big)^2\|_{L^2} + C_0e^{2C_1\sqrt{\lambda}}\|y\|^2_{L^2}\|z\|_{L^2} + C_0e^{2C_1\sqrt{\lambda}}\|y\|^2_{L^2}\|z\|_{L^\infty}\\ &\leqslant e^{C_2\sqrt{\lambda}}\|z_x\|_{L^2}\left(\|z\|^2_{L^2} + \|z\|^2_{L^4}\right)\\ &\leqslant \|z_x\|^2_{L^2} + e^{C_3\sqrt{\lambda}}\left(\|z\|^4_{L^2} + \|z\|^6_{L^2}\right).\end{aligned} \tag{5.4.25}$$

Here, we used the estimate

$$\|z_x\|_{L^2}\|z\|^2_{L^4} \leqslant \|z_x\|_{L^2}\|z\|_{L^2}\|z\|_{L^\infty} \leqslant \|z_x\|^{3/2}_{L^2}\|z\|^{3/2}_{L^2} \leqslant \|z_x\|^2_{L^2} + C\|z\|^6_{L^2}. \tag{5.4.26}$$

Therefore

$$\frac{d}{dt}\|z\|^2_{L^2} \leqslant -2\lambda\|z\|^2_{L^2} + e^{C_3\sqrt{\lambda}}\left(\|z\|^4_{L^2} + \|z\|^6_{L^2}\right) \text{ in } L^1(s, s+\tilde{T}). \tag{5.4.27}$$

If the initial energy $\|z_0\|_{L^2}$ is smaller than $e^{-C_3\sqrt{\lambda}}$ (this is not a sharp bound), we then have an exponential decay of the energy

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \quad \forall t \in [s, s+\tilde{T}]. \tag{5.4.28}$$

Since the energy of z decays, we can continue to use Lemma 23 in order to get that the solution z of (5.4.16) is in $C^0([s, s+2\tilde{T}]; L^2(0,1)) \cap L^2(s, s+2\tilde{T}; H_0^1(0,1))$, and that

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \quad \forall t \in [s, s+2\tilde{T}]. \tag{5.4.29}$$

We continue such procedure as time goes to infinity to get that the unique solution z satisfies

$$z \in C^0([s, +\infty); L^2(0,1)) \cap L^2_{loc}([s, +\infty); H_0^1(0,1)), \tag{5.4.30}$$

$$\|z(t)\|_{L^2} \leqslant e^{-\frac{\lambda(t-s)}{2}}\|z_0\|_{L^2}, \quad \forall t \in [s, +\infty). \tag{5.4.31}$$

This solves the local rapid stabilization problem. More precisely, we have the following theorem.

Theorem 22. *Let $\lambda > 1$ and $s \in \mathbb{R}$. For every $y_0 \in L^2(0,1)$ such that*

$$\|y_0\|_{L^2} \leqslant e^{-2C_3\sqrt{\lambda}}, \tag{5.4.32}$$

the Cauchy problem (5.4.15) has one and only one solution.
This solution is also in $C^0([s,+\infty); L^2(0,1)) \cap L^2_{loc}([s,+\infty); H^1(0,1))$. Moreover, this solution satisfies

$$\|y(t-s)\|_{L^2} \leqslant e^{3C_1\sqrt{\lambda}} e^{-\frac{\lambda(t-s)}{2}} \|y_0\|_{L^2}. \tag{5.4.33}$$

However, one also needs the feedback law to be proper. As it will be seen later on, it suffices to multiply the former feedback law by a suitable cut-off function (see, in particular, Lemma 24).

5.4.2 Construction of feedback laws: piecewise backstepping

Inspired by [CN17], we construct a piecewise continuous feedback law on $[0,T)$ such that properties $(\mathcal{P}_2)$–$(\mathcal{P}_5)$ hold.

Let us choose

$$n_0 := 1 + [\frac{2}{\sqrt{T}}], \tag{5.4.34}$$

$$t_n := 0, \lambda_n := 0 \text{ for } n \in \{0, 1, \dots, n_0 - 1\}, \tag{5.4.35}$$

$$t_n := T - 1/n^2, \lambda_n := 2n^8 \text{ for } n \in \{n_0, n_0 + 1, \dots\}. \tag{5.4.36}$$

It is tempting to define $U_2 : (-\infty, +\infty) \times L^2(0,1)$ by

$$U_2(t; y) = K_{\lambda_n}(t, y),\ n \in \{n_0 - 1, n_0, \dots\},\ t \in [t_n, t_{n+1}),\ y \in L^2(0,1), \tag{5.4.37}$$

$$U_2(t + T; y) = U_2(t; y),\ t \in \mathbb{R},\ y \in L^2(0,1). \tag{5.4.38}$$

However with this definition U_2 is not locally bounded in a neighborhood of $[0,T) \times \{0\}$, which is a drawback for robustness issue with respect to measurement errors. In order to handle this problem, we introduce a Lipschitz cutoff function $\varphi_\lambda : \mathbb{R}^+ \to \mathbb{R}^+$:

$$\varphi_\lambda(x) := \begin{cases} 1, & \text{if } x \in [0, e^{-C_3\sqrt{\lambda}}/5], \\ 2 - 5e^{C_3\sqrt{\lambda}}x, & \text{if } x \in (e^{-C_3\sqrt{\lambda}}/5, 2e^{-C_3\sqrt{\lambda}}/5), \\ 0, & \text{if } x \in [2e^{-C_3\sqrt{\lambda}}/5, +\infty), \end{cases} \tag{5.4.39}$$

and replace (5.4.37) by

$$U_2(t, y) = \mathcal{K}_{\lambda_n}(t, y),\ n \in \{n_0 - 1, n_0, \dots\},\ t \in [t_n, t_{n+1}),\ y \in L^2(0,1), \tag{5.4.40}$$

where, for $\lambda \in (1, +\infty)$, $\mathcal{K}_\lambda : L^2(0,1) \to L^2(0,1)$ is defined by

$$\mathcal{K}_\lambda(y) := \varphi_\lambda(\|y\|_{L^2}) K_\lambda y,\ y \in L^2(0,1). \tag{5.4.41}$$

From (5.4.7), (5.4.4), (5.4.39), (5.4.40), and (5.4.41), one can easily verify that

$$|U_2(t, y)| \leqslant \min\{1, \sqrt{\|y\|_{L^2}}\},\ t \in [0,T],\ y \in L^2(0,1). \tag{5.4.42}$$

In particular $(\mathcal{P}_4)$ holds.

5.4.3 Proof of Theorem 21

Let us start this proof by stating a lemma, whose proof is given in Appendix 5.9, giving a properness result on stationary feedback laws.

Lemma 24. *Let $M > 0$ and $G : L^2(0,1) \to \mathbb{R}$ be a (stationary) feedback law satisfying*

$$|G(y) - G(z)| \leqslant M\|y - z\|_{L^2(0,1)}, \ \ \forall y \in L^2(0,1), \ \ \forall z \in L^2(0,1) \text{ and } G(0) = 0, \tag{5.4.43}$$

$$|G(y)| \leqslant M, \ \ \forall y \in L^2(0,1). \tag{5.4.44}$$

Then, for every $y^0 \in L^2(0,1)$ and for every $T > 0$, the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) + yy_x = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = G(y(t,\cdot)) & \text{for } t \in (0,T), \\ y(0,\cdot) = y_0. \end{cases} \tag{5.4.45}$$

has a unique solution (in the sense of Definition 13 with $A = 0$ and $a_0 = a = 0$).

Similar to Lemma 19, we also have the following stability result, whose proof is omitted since it is quite similar to the proof of Lemma 19.

Lemma 25. *Let $R > 0$, $M > 0$, and $T > 0$. There exists $C_S(R,M,T)$ such that, for every $G : L^2(0,1) \to \mathbb{R}$ a (stationary) feedback law satisfying*

$$|G(y) - G(z)| \leqslant M\|y - z\|_{L^2(0,1)}, \ \ \forall y \in L^2(0,1), \ \ \forall z \in L^2(0,1) \text{ and } G(0) = 0, \tag{5.4.46}$$

$$|G(y)| \leqslant M, \ \ \forall y \in L^2(0,1), \tag{5.4.47}$$

for every $y_0^\pm \in L^2(0,1)$ satisfying

$$\|y_0^\pm\|_{L^2(0,1)} \leqslant R, \tag{5.4.48}$$

the solutions $y^\pm$ to the Cauchy problem (5.4.45) satisfy

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant C_S(R,M,T)\|y_0^+ - y_0^-\|_{L^2(0,1)}. \tag{5.4.49}$$

Until the end of the proof of Theorem 21 our feedback law F is defined by $A := 0$, $U_1 := 0$, and (5.4.38)-(5.4.40). Let us recall that the time-varying feedback law in Section 5.4.2 is piecewisely (with respect to time) given by the stationary feedback laws (5.4.41), where K_λ is designed in Section 5.4.1. Let us point out that, for every $\lambda \in [1, +\infty)$,

$$|\mathcal{K}_\lambda(y)| \leqslant 1, \ \ \forall y \in L^2(0,1), \text{ and } \mathcal{K}_\lambda(0) = 0 \tag{5.4.50}$$

and there exists $M_\lambda > 0$ such that

$$|\mathcal{K}_\lambda(y^1) - \mathcal{K}_\lambda(y^2)| \leqslant M_\lambda\|y^1 - y^2\|_{L^2(0,1)}, \ \ \forall (y^1, y^2) \in L^2(0,1) \times L^2(0,1). \tag{5.4.51}$$

Hence, by Lemma 24, these stationary feedback laws are proper on $(-\infty, +\infty)$. In particular, the time-varying feedback law F is proper on $[0, s_2]$ for every $s_2 \in (0,T)$. Hence, for every $(y_0, a_0) \in L^2(0,1) \times \mathbb{R}$ and $t_1 \in [0,T)$ we get the existence and the uniqueness of $y : [t_1, T) \to L^2(0,1)$ and $a : [t_1, T) \to \mathbb{R}$ such that, for every $t_2 \in (t_1, T)$ the restriction of (y, a) to $[t_1, t_2]$ is the solution on $[t_1, t_2]$ to the Cauchy problem of the closed-loop system (5.1.4) with initial data (y_0, a_0) at time t_1 (in the sense of Definition 13).

In order to get the properness of the feedback law F_2 (defined in (5.1.24)), it suffices to show that

$$\lim_{t\to T^-} y(t) \text{ exists in } L^2(0,1). \tag{5.4.52}$$

In order to prove (5.4.52), we check that $\{y(t)\}(t \to T^-)$ is a Cauchy sequence in $L^2(0,1)$. Let us point out that

$$y \in L^\infty(t_1, T; L^2(0,1)) \cap L^2(t_1, T; L^\infty(0,1)). \tag{5.4.53}$$

Indeed, (5.4.53) follows from the maximum principle (Lemma 18), Proposition 10 (applied with $f = 0$, $\beta = 0$, $\gamma = \pm 1$), and (5.4.42). Let

$$f := -(1/2)(y^2)_x. \tag{5.4.54}$$

By (5.4.53) and (5.4.54),

$$f \in L^2(t_1, T; H^{-1}(0,1)). \tag{5.4.55}$$

Let $t_2 \in (t_1, T)$. Let $y^\pm_{t_2}$ be the solutions of

$$\begin{cases} (y^\pm_{t_2})_t - (y^\pm_{t_2})_{xx} = f & \text{for } (t,x) \in (t_2, T) \times (0,1), \\ (y^\pm_{t_2})(t,0) = 0 & \text{for } t \in (t_2, T), \\ (y^\pm_{t_2})(t,1) = \pm 1 & \text{for } t \in (t_2, T), \\ (y^\pm_{t_2})(t_2, \cdot) = y(t_2). \end{cases} \tag{5.4.56}$$

Let us define $w_{t_2} := (y_{t_2})^+ - (y_{t_2})^-$. Then

$$\begin{cases} (w_{t_2})_t - (w_{t_2})_{xx} = 0 & \text{for } (t,x) \in (t_2, T) \times (0,1), \\ (w_{t_2})(t,0) = 0 & \text{for } t \in (t_2, T), \\ (w_{t_2})(t,1) = 2 & \text{for } t \in (t_2, T), \\ (w_{t_2})(t_2, \cdot) = 0. \end{cases} \tag{5.4.57}$$

Let $\varepsilon > 0$. From Proposition 9, (5.4.54), and (5.4.57), there exists $t_2 \in (t_1, T)$ such that

$$\|w_{t_2}\|_{C^0([t_2,T];L^2(0,1))} \leqslant \varepsilon/4. \tag{5.4.58}$$

Moreover, from the maximum principle in the linear case (see Proposition 17), (5.4.42), (5.4.45), and (5.4.56), we know that

$$y^-_{t_2}(t) \leqslant y(t) \leqslant y^+_{t_2}(t), \quad \forall t \in [t_2, T), \tag{5.4.59}$$

which, together with (5.4.58), implies that

$$\|y^+_{t_2} - y\|_{C^0([t_2,T];L^2(0,1)} \leqslant \varepsilon/4. \tag{5.4.60}$$

Since $y^+_{t_2}$ is in $C^0([t_2,T];L^2(0,1))$, there exists $\tilde{t}_2 \in [t_2, T)$ such that

$$\|y^+_{t_2}(t) - y^+_{t_2}(T)\|_{L^2} \leqslant \varepsilon/4, \quad \forall t \in [\tilde{t}_2, T]. \tag{5.4.61}$$

From (5.4.60) and (5.4.61),

$$\|y(t) - y(t')\|_{L^2} \leqslant \varepsilon, \quad \forall t, t' \in [\tilde{t}_2, T). \tag{5.4.62}$$

This implies (5.4.52) and concludes the proof of the properness of the feedback law F_2.

Now we are ready to prove Theorem 21. Since "a" does not change (see (5.1.24)), it suffices to only consider y. About property *(i)*, as we saw in Section 5.4.1, $\|y(t)\|_{L^2}$ decays rapidly on $[t_n, t_{n+1})$ provided that $y(t_n)$ is small enough in $L^2(0,1)$. Our idea is to set $\|y(0)\|_{L^2}$ sufficiently small so that the flow will decay exponentially (in $L^2(0,1)$) with rate $\lambda_0/2$ on $[t_0,t_1)$; then at time t_1, the energy $y(t_1)$ is already small enough to have an exponential decay with rate $\lambda_1/2$ on $[t_1,t_2)$. Continuing this way one may expect that, at the end, $y(T)=0$. In order to have an exponential decay with rate $\lambda_n/2$ on $[t_n,t_{n+1})$, it is sufficient to have

$$\|y(t_n)\|_{L^2}\leqslant e^{-2C_3\sqrt{\lambda_n}}. \tag{5.4.63}$$

These exponential decay rates on $[t_n,t_{n+1})$ for every $n\in\mathbb{N}$ can be achieved for $\|y(0)\|_{L^2}$ sufficiently small if there exists $c>0$ such that

$$c|\Pi_{\lambda_n}||\Pi_{\lambda_n}^{-1}|\prod_{k=n_0}^{n-1}\left(|\Pi_{\lambda_k}||\Pi_{\lambda_k}^{-1}|e^{-(t_{k+1}-t_k)\lambda_k/2}\right)\leqslant e^{-2C_3n^4},\ \text{ for all } n\in\mathbb{N}. \tag{5.4.64}$$

Hence, it suffices to find $c>0$ such that

$$c\left(\prod_{k=n_0}^{n-1}e^{-k^5}\right)\left(\prod_{k=n_0}^{n}e^{3C_1k^4}\right)\leqslant e^{-2C_3n^4} \text{ holds for every } n\in\mathbb{N}. \tag{5.4.65}$$

Such c obviously exist, and one can find similar computations in [Xia18].

Actually, the above proof also shows the following lemma.

Lemma 26. *Let $\varepsilon>0$. Let $0<T_0<T$. There exists a constant $\eta>0$ such that*

$$(\|y_0,a_0)\|_V\leqslant\eta)\Rightarrow(\|\Phi_2(t,t';y_0,a_0)\|_V\leqslant\varepsilon,\ \ \forall 0\leqslant t'\leqslant t\leqslant T_0)\,. \tag{5.4.66}$$

The second part, *(ii)*, of Theorem 21 is then a consequence of the following lemma.

Lemma 27. *Let $\varepsilon>0$. There exists $0<T_1<T$ such that*

$$(\|y_0,a_0)\|_V\leqslant\varepsilon)\Rightarrow(\|\Phi_2(t,t';y_0,a_0)\|_V\leqslant 2\varepsilon,\ \ \forall T_1\leqslant t'\leqslant t\leqslant T)\,. \tag{5.4.67}$$

Property (5.4.67) is a consequence of Proposition 10 and (5.4.42). This completes the proof of Theorem 21.

5.5 Proper feedback laws for system (5.1.2)

Finally, we are now in position to define our proper feedback law $F=(A,U_1,U_2)$ for system (5.1.2). We define a $2T$-periodic feedback law which leads to the approximate stabilization in the first stage ($[0,T]$) and then "stabilizes" (y,a) to 0 in the second stage ($[T,2T)$). Our feedback law F is defined as follows.

$$A(t;y,a):=\left\{\begin{array}{cc} A_0(y,a), & \text{if } t\in[0,T/2),\\ A_1(a), & \text{if } t\in[T/2,T),\\ 0, & \text{if } t\in[T,2T),\end{array}\right. \tag{5.5.1}$$

$$U_1(t;y,a):=\left\{\begin{array}{cc} a, & \text{if } t\in[0,T/2),\\ a, & \text{if } t\in[T/2,T),\\ 0, & \text{if } t\in[T,2T),\end{array}\right. \tag{5.5.2}$$

$$U_2(t;y,a) := \begin{cases} a, & \text{if } t \in [0,T/2), \\ a, & \text{if } t \in [T/2,T), \\ \mathcal{K}_{\lambda_n}(y), & \text{if } t \in [T+t_n, T+t_{n+1}), \end{cases} \tag{5.5.3}$$

where λ_n and t_n are defined in (5.4.35) and (5.4.36), $\mathcal{K}_\lambda$ is defined in (5.4.41), A_0 is defined in (5.3.22), and A_1 is defined in (5.3.25).

Thanks to Section 5.3.2 and Section 5.4.3, the feedback law (5.5.2)–(5.5.1) is proper (in the sense of Definition 14).

5.6 Small-time global stabilization

The small-time global stabilization (Theorem 19) contains two parts, *(i)* and *(ii)*. Let us first consider *(i)*. Let us denote by Φ the flow associated to the feedback law F. From (5.1.16) and (5.1.25) we get that

$$\Phi(2T, 0; y, a) = (0,0). \tag{5.6.1}$$

Let $t \in [0, 2T)$. Then

$$\Phi(4T, t; y, a) = \Phi(4T, 2T; \Phi(2T, t; y, a)) = (0,0), \tag{5.6.2}$$

which shows that *(i)* holds. Property *(ii)* follows directly from (5.1.15), (5.1.26), and (5.6.1). This completes the proof of Theorem 19.

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5.7 Appendix A: Proofs of Proposition 9, of Proposition 10, of Lemma 18, and of Lemma 19

This appendix is devoted to the proof of two propositions and of two lemma that we stated in Section 5.2: Proposition 9, Proposition 10, Lemma 18, and Lemma 19.

Let us start with the proof of Proposition 9. Without loss of generality we may assume that $t_1 = 0$. Let $T_0 > 0$ and let $t_2 = T \leq T_0$. We consider the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ y(0,\cdot) = 0. \end{cases} \tag{5.7.1}$$

If $\gamma \in L^\infty(0,T)$, then the solution is in $C^0([0,T]; H^{-1}(0,1))$. By the maximum principle (Lemma 17), one knows that y is also in $L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; L^\infty(0,1))$. Let us now assume that $\gamma \in C^1$. Then that solution is in $C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1(0,1))$. In order to give estimates on y in that space, let us define

$$z := y - x^n \gamma \text{ with } n \in \mathbb{N} \setminus \{0,1\} \text{ to be chosen later.} \tag{5.7.2}$$

Hence

$$\begin{cases} z_t - z_{xx} = -x^n\gamma_t + n(n-1)x^{n-2}\gamma & \text{for } (t,x)\in(0,T)\times(0,1), \\ z(t,0)=0 & \text{for } t\in(0,T), \\ z(t,1)=0 & \text{for } t\in(0,T), \\ z(0,\cdot) = -x^n\gamma(0). \end{cases} \tag{5.7.3}$$

Then, by Proposition 8, we have

$$\begin{aligned} \|z\|_{C^0L^2\cap L^2H_0^1} &\leqslant 2C_1\|-x^n\gamma_t + n(n-1)x^{n-2}\gamma\|_{L^1L^2} + 2|\gamma(0)|\|x^n\|_{L^2} \\ &\leqslant 2C_1\left((\|\gamma_t\|_{L^1} + 2|\gamma(0)|)\|x^n\|_{L^2} + n(n-1)\|\gamma\|_{L^1}\|x^{n-2}\|_{L^2}\right). \end{aligned} \tag{5.7.4}$$

For $y\in H^1(0,1)$ such that $y(0)=0$, let us define the $\dot{H}(0,1)$-norm of y by

$$\|y\|_{\dot{H}(0,1)} := \|y_x\|_{L^2(0,1)}. \tag{5.7.5}$$

By direct calculations, we know that

$$\|x^n\gamma\|_{C^0L^2\cap L^2\dot{H}^1} \leqslant \|x^n\|_{L^2}\|\gamma\|_{C^0} + n\|x^{n-1}\|_{L^2}\|\gamma\|_{L^2}. \tag{5.7.6}$$

Let $\eta\in(0,1/2)$. Taking n large enough, we get the existence of $C_\eta>0$, which is independent of $T\leqslant T_0$ and of γ, such that

$$\|y\|_{C^0L^2\cap L^2\dot{H}^1} \leqslant \eta(\|\gamma_t\|_{L^1} + \|\gamma\|_{C^0}) + C_\eta\|\gamma\|_{L^2}. \tag{5.7.7}$$

Now, suppose that $\gamma\in L^\infty$. Let us consider the solution $y^\pm$ of

$$\begin{cases} y_t^\pm - y_{xx}^\pm = 0 & \text{for } (t,x)\in(0,T)\times(0,1), \\ y^\pm(t,0)=0 & \text{for } t\in(0,T), \\ y^\pm(t,1)=\pm v & \text{for } t\in(0,T), \\ y^\pm(0,\cdot)=0, \end{cases} \tag{5.7.8}$$

with $v := \|\gamma\|_{L^\infty}\in[0,+\infty)$. Thanks to (5.7.7), we have

$$\|y^\pm\|_{C^0L^2\cap L^2\dot{H}^1} \leqslant \eta v + C_\eta T^{1/2}v = (\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{5.7.9}$$

By direct computations, there exists $C>0$ such that, for every $\varphi\in H^1(0,1)$ with $\varphi(0)=0$,

$$\|\varphi\|_{L^\infty(0,1)} \leqslant C\|\varphi\|_{L^2(0,1)}^{1/2}\|\varphi_x\|_{L^2(0,1)}^{1/2}, \tag{5.7.10}$$

Actually, since $\varphi(0)=0$, we have

$$\varphi^2(x) = 2\int_0^x \varphi(s)\varphi'(s)ds \tag{5.7.11}$$

which leads to inequality (5.7.10). It is also a simple case of Gagliardo–Nirenberg interpolation inequality. From (5.7.10) one gets that, for every $T>0$ and for every $\varphi\in L^\infty(0,T;L^2(0,1))\cap L^2(0,T;\dot{H}^1(0,1))$ such that $\varphi(\cdot,0)=0\in L^2(0,T)$,

$$\|\varphi\|_{L^2(0,T;L^\infty(0,1))} \leqslant CT^{1/4}\|\varphi\|_{L^\infty(0,T;L^2(0,1))}^{1/2}\|\varphi\|_{L^2(0,T;\dot{H}^1(0,1))}^{1/2}. \tag{5.7.12}$$

Hence we have,

$$\|y^{\pm}\|_{C^0L^2}\leqslant (\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{5.7.13}$$
$$\|y^{\pm}\|_{L^2L^\infty}\leqslant CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{5.7.14}$$

Since $-v \leqslant \gamma \leqslant +v$, by the maximum principle (Lemma 17), we have

$$y^- \leqslant y \leqslant y^+, \text{ for all } t \in [0,T], \tag{5.7.15}$$

which, together with (5.7.13) and (5.7.14), implies that

$$\|y\|_{L^\infty L^2}\leqslant 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{5.7.16}$$
$$\|y\|_{L^2L^\infty}\leqslant 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{5.7.17}$$

Let us now prove that if $\gamma \in C^0([0,T])$ then the solution y is in $C^0([0,T];L^2(0,1))\cap L^2(0,T;L^\infty(0,1))$. Suppose that $\gamma \in C^0([0,T])$ is given, then there exists

$$\{\gamma_n\}_{n\in\mathbb{N}}, \text{ a sequence of } C^1([0,T]) \text{ functions which uniformly converges to } \gamma. \tag{5.7.18}$$

Let us denote by $\{y_n\}_{n\in\mathbb{N}}$ the sequence of solutions of (5.7.1) with controls given by $\{\gamma_n\}_{n\in\mathbb{N}}$. Thanks to (5.7.18), for any $\varepsilon > 0$, there exists N such that when $m, n > N$, we have

$$\|\gamma_m - \gamma_n\|_{C^0([0,T])}\leqslant \varepsilon. \tag{5.7.19}$$

Hence, by (5.7.16) and (5.7.17),

$$\|y_m - y_n\|_{L^2L^\infty\cap L^\infty L^2}\leqslant C(\eta + C_\eta T^{1/2})\|\gamma_m - \gamma_n\|_{L^\infty}. \tag{5.7.20}$$

Since $\gamma_n \in C^1([0,T])$, from Proposition 8 we have $y_n \in C^0([0,T];L^2(0,1))$. Hence

$$\|y_m - y_n\|_{L^2L^\infty\cap C^0L^2}\leqslant C(\eta + C_\eta T^{1/2})\|\gamma_m - \gamma_n\|_{L^\infty}, \tag{5.7.21}$$

which means that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T];L^2(0,1))$. Hence

$$y \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;L^\infty(0,1)). \tag{5.7.22}$$

Letting also n to infinity in (5.7.16) and (5.7.17) for y_n and γ_n, we get again (5.7.16) and (5.7.17).

Let us finally consider the case where $\gamma \in L^\infty(0,T)$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}^*}$ of functions in $C^0([0,T])$ such that

$$\|\gamma_n\|_{L^\infty(0,T)}\leq \|\gamma\|_{L^\infty(0,T)}, \quad \forall n \in \mathbb{N}^*, \tag{5.7.23}$$
$$\lim_{n\to+\infty} \gamma_n(t) = \gamma(t) \text{ for almost every t } \in (0,T). \tag{5.7.24}$$

One can, for example, take

$$\gamma_n(t) := \frac{1}{n}\int_{\max(0,t-(1/n))}^t \gamma(s)ds. \tag{5.7.25}$$

Let us denote by $\{y_n\}_{n\in\mathbb{N}^*}$ the sequence of solutions of (5.7.1) with control given by $\{\gamma_n\}_{n\in\mathbb{N}^*}$. Then $\{y_n\}_{n\in\mathbb{N}^*}$ is bounded in $L^2((0,T)\times(0,1))$. Then there exists a subsequence of the $\{y_n\}_{n\in\mathbb{N}^*}$, that one also denotes by $\{y_n\}_{n\in\mathbb{N}^*}$, and $y \in L^2((0,T)\times(0,1)$ such that

$$y_n \rightharpoonup y \text{ weakly in } L^2((0,T)\times(0,1)). \tag{5.7.26}$$

Then one easily checks that y is a solution of (5.7.1) with control given by γ and that y satisfies

(5.7.16) and (5.7.17).

All these calculations are based on the assumption $\beta = 0$. If $\gamma = 0$ and $\beta \neq 0$, similar estimates hold. By linearity, one then gets Proposition 9.

Remark 22. *One can also get the continuity of $y : [0,T] \to L^2(0,1)$ when γ is in $BV([0,1]) \cap L^\infty(0,T)$. The idea is to use directly (5.7.7).*

Remark 23. *Multiplying (5.7.1) by $(1-x)y$ and integration by parts show that $(1-x)y_x^2 \in L^1L^1$ if $\beta = 0$.*

We are now ready to prove Proposition 10 and Lemma 18, the proof is given by 4 steps.

Step 1. Local existence and uniqueness of the solution,

Step 2. Continuity of the solution with respect to the initial data and the boundary conditions,

Step 3. Maximum principle (Lemma 18),

Step 4. Global existence of the solution.

Step 1. Local existence and uniqueness of the solution. In this step we prove the second part of the statement of Proposition 10. Again, for simplicity we only treat the case where $\beta = 0$. To simplify the notations we let $t_1 = 0$ and $T := t_2$, i.e. we consider the Cauchy problem

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) + yy_x(t,x) = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0 & \text{for } t \in (0,T), \\ y(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ y(0,\cdot) = y_0. \end{cases} \tag{5.7.27}$$

We use the standard Banach fixed point theorem to get the local existence and uniqueness. We consider the space

$$X := L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;L^\infty(0,1)) \tag{5.7.28}$$

with the norm, adapted to deal with (5.2.16) and (5.2.17),

$$\|y\|_{X_\mu} := \|y\|_{L^\infty L^2} + \frac{1}{\mu}\|y\|_{L^2L^\infty}, \tag{5.7.29}$$

with $\mu > 0$ to be chosen later. We denote by X_μ the space X with the norm $\|\cdot\|_{X_\mu}$, which is a Banach space. The choice of the norm $\|\cdot\|_{X_\mu}$ is based on the observation that $\|y\|_{L^2L^\infty}$ can be sufficiently small once we set time small enough.

We consider the following map $\Gamma : X_\mu \to X_\mu$, where $\Gamma(y)$ is the unique solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -\frac{1}{2}(y^2)_x(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = \gamma(t) & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0. \end{cases} \tag{5.7.30}$$

This map is well defined thanks to Proposition 8, (5.7.16) and (5.7.17). A function y is solution to (5.7.27) if and only if it is a fixed point of Γ. The function $\Gamma(y)$ can be decomposed as follows

$$\Gamma(y) = z^1 + z^2 + z^3, \tag{5.7.31}$$

where z^1, z^2, and z^3 are the solutions to the following Cauchy problems

$$\begin{cases} z^1_t(t,x) - z^1_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^1(t,0) = 0 & \text{for } t \in (0,T),\\ z^1(t,1) = 0 & \text{for } t \in (0,T),\\ z^1(0,\cdot) = y_0 \end{cases} \tag{5.7.32}$$

$$\begin{cases} z^2_t(t,x) - z^2_{xx}(t,x) = -\frac{1}{2}(y^2)_x(t,x) & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^2(t,0) = 0 & \text{for } t \in (0,T),\\ z^2(t,1) = 0 & \text{for } t \in (0,T),\\ z^2(0,\cdot) = 0, \end{cases} \tag{5.7.33}$$

$$\begin{cases} z^3_t(t,x) - z^3_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,T)\times(0,1),\\ z^3(t,0) = 0 & \text{for } t \in (0,T),\\ z^3(t,1) = \gamma(t) & \text{for } t \in (0,T),\\ z^3(0,\cdot) = 0. \end{cases} \tag{5.7.34}$$

From Proposition 8, (5.7.16), and (5.7.17), one gets

$$\|z^1\|_{C^0L^2} \leqslant \|y_0\|_{L^2} \text{ and } \|z^1\|_{L^2H^1_0} \leqslant \|y_0\|_{L^2}, \tag{5.7.35}$$

$$\|z^2\|_{C^0L^2} + \|z^2\|_{L^2H^1} \leqslant C_1\|yy_x\|_{L^2H^{-1}} \leqslant \frac{C_1}{2}\|y^2\|_{L^2L^2} \leqslant \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}, \tag{5.7.36}$$

$$\|z^3\|_{L^\infty L^2} \leqslant 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}, \tag{5.7.37}$$

$$\|z^3\|_{L^2L^\infty} \leqslant 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}. \tag{5.7.38}$$

From (5.7.36), (5.7.10), and (5.7.12), we get further

$$\|z^1\|_{L^2L^\infty} \leqslant CT^{1/4}\|y_0\|_{L^2}, \tag{5.7.39}$$

$$\|z^2\|_{C^0L^2} \leqslant \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}, \tag{5.7.40}$$

$$\|z^2\|_{L^2L^\infty} \leqslant CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}. \tag{5.7.41}$$

Since $\|y_0\|_{L^2} \leqslant R$ and $\|\gamma\|_{L^\infty} \leqslant r$, let us choose the ball

$$B_R := \{y \in X : \|y\|_{X_\mu} \leqslant 2R\}. \tag{5.7.42}$$

Then, from (5.7.29) and (5.7.31)–(5.7.42), one knows that

$$\begin{aligned} \|\Gamma(y)\|_{L^\infty L^2} &\leqslant \|y_0\|_{L^2} + \frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2} + 2(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}\\ &\leqslant \|y_0\|_{L^2} + 2C_1\mu R^2 + 2(\eta + C_\eta T^{1/2})r, \end{aligned} \tag{5.7.43}$$

and

$$\begin{aligned} \|\Gamma(y)\|_{L^2L^\infty} &\leqslant CT^{1/4}\|y_0\|_{L^2} + CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2} + 2CT^{1/4}(\eta + C_\eta T^{1/2})\|\gamma\|_{L^\infty}\\ &\leqslant CT^{1/4}\|y_0\|_{L^2} + 2CT^{1/4}C_1\mu R^2 + 2CT^{1/4}(\eta + C_\eta T^{1/2})r. \end{aligned} \tag{5.7.44}$$

Hence

$$\|\Gamma(y)\|_{X_\mu} \leqslant (1+CT^{1/4}/\mu)R + 2C_1\mu(1+CT^{1/4}/\mu)R^2 + 2(1+CT^{1/4}/\mu)(\eta + C_\eta T^{1/2})r, \quad (5.7.45)$$

and we can successively choose η, μ, and T such that

$$\|\Gamma(y)\|_{X_\mu} \leqslant (3/2)R. \quad (5.7.46)$$

Hence

$$\Gamma(B_R) \subset B_R. \quad (5.7.47)$$

Let us now prove that Γ is a contraction in B_R. We perform similar computations. Let us assume that

$$y_1 \text{ and } y_2 \in B_R. \quad (5.7.48)$$

Then $w := \Gamma(y_1) - \Gamma(y_2)$ is the solution of

$$\begin{cases} w_t(t,x) - w_{xx}(t,x) = -\frac{1}{2}\left((y_1^2)_x - (y_2^2)_x\right)(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ w(t,0) = 0 & \text{for } t \in (0,T), \\ w(t,1) = 0 & \text{for } t \in (0,T), \\ w(0,\cdot) = 0. \end{cases} \quad (5.7.49)$$

Hence, by Proposition 8,

$$\begin{aligned} \|w\|_{C^0L^2 \cap L^2H_0^1} &\leqslant \frac{C_1}{2}\|y_1 + y_2\|_{L^2L^\infty}\|y_1 - y_2\|_{L^\infty L^2} \\ &\leqslant C_1 R\mu \|y_1 - y_2\|_{L^\infty L^2}. \end{aligned} \quad (5.7.50)$$

Thus

$$\|w\|_{L^2L^\infty} \leqslant C_1 C T^{1/4} R\mu \|y_1 - y_2\|_{L^\infty L^2}. \quad (5.7.51)$$

When μ and T are small enough, we have

$$\|\Gamma(y_1) - \Gamma(y_2)\|_{X_\mu} \leqslant (1/2)\|y_1 - y_2\|_{X_\mu}. \quad (5.7.52)$$

Hence we get the existence of a unique solution in B_R. Let us now prove the uniqueness of solution in X_μ. It suffice to show the uniqueness of solution in X_μ for small time. Let $\|y_0\|_{L^2} \leqslant R$, $\|\gamma\|_{L^\infty} \leqslant r$. Let $y \in X_\mu$ be a solution to (5.7.27). One can always find $0 < T_s < T$ such that

$$\|y\|_{L^\infty(0,T_s;L^2(0,1))} + \|y\|_{L^2(0,T_s;L^\infty(0,1))} \leqslant 2R, \quad (5.7.53)$$

which implies the uniqueness of the solution in time $(0, T_s)$.

The above proof gives the local existence and uniqueness of $L^\infty L^2 \cap L^2 L^\infty$ solution. When $\gamma \in C^0$, Proposition 9 shows that the solution is also in $C^0L^2 \cap L^2L^\infty$.

Step 2. Continuity of the solution with respect to the data y_0, β and γ. More precisely, in this step, we prove the following lemma.

Lemma 28. *For every $R > 0$, $r > 0$, and $\varepsilon > 0$ such that $4\varepsilon R C_1 < 1$, there exists $0 < \tilde{T}^\varepsilon_{R,r} < T^\varepsilon_{R,r}$ such that, for every $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 < t_2 < t_1 + \tilde{T}^\varepsilon_{R,r}$, for every $y_0^\pm \in L^2(0,1)$, for every $\beta^\pm \in L^\infty(t_1,t_2)$, and for every $\gamma^\pm \in L^\infty(t_1,t_2)$ such that*

$$\|y_0^\pm\|_{L^2} \leqslant R \text{ and } \|\beta^\pm\|_{L^\infty} + \|\gamma^\pm\|_{L^\infty} \leqslant r, \quad (5.7.54)$$

the solutions $y^{\pm}$ to the Cauchy problem

$$\begin{cases} y_t^{\pm}(t,x) - y_{xx}^{\pm}(t,x) + y^{\pm}y_x^{\pm} = 0 & \text{for } (t,x) \in (t_1,t_2)\times(0,1), \\ y^{\pm}(t,0) = \beta^{\pm}(t) & \text{for } t \in (t_1,t_2), \\ y^{\pm}(t,1) = \gamma^{\pm}(t) & \text{for } t \in (t_1,t_2), \\ y^{\pm}(t_1,\cdot) = y_0^{\pm}, \end{cases} \tag{5.7.55}$$

satisfy

$$\|y^+ - y^-\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant 2(\|y_0^+ - y_0^-\|_{L^2} + \|\beta^+ - \beta^-\|_{L^\infty} + \|\gamma^+ - \gamma^-\|_{L^\infty}). \tag{5.7.56}$$

Proof of Lemma 28. Let us first point out that the existence of $y^{\pm}$ (on $[t_1,t_2]$) follows from Step 1 and (5.7.54). We also only treat the case where $\beta = 0$ in order to simplify the notations. From Step 1 we also know that

$$\|y^{\pm}\|_{L^\infty(t_1,t_2;L^2(0,1))} \leqslant 2R, \tag{5.7.57}$$

$$\|y^{\pm}\|_{L^2(t_1,t_2;L^\infty(0,1))} \leqslant \varepsilon R. \tag{5.7.58}$$

Let us denote $z := y^+ - y^-$ as the solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -\frac{1}{2}\left(\left((y^+ - y^-)(y^+ + y^1)\right)_x\right)(t,x) & \text{for } (t,x) \in (0,T)\times(0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = \gamma^+ - \gamma^- & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0^+ - y_0^-. \end{cases} \tag{5.7.59}$$

Hence by using the same estimates as in Step 1, we get

$$\begin{aligned} \|z\|_{L^\infty L^2} &\leqslant C_1\|z(y^+ + y^-)\|_{L^2L^2} + \|y_0^+ - y_0^-\|_{L^2} + (\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ &\leqslant 2\varepsilon C_1 R\|z\|_{L^\infty L^2} + \|y_0^+ - y_0^-\|_{L^2} + (\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ &\leqslant 1/2\|z\|_{L^\infty L^2} + \|y_0^+ - y_0^-\|_{L^2} + (\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty}, \end{aligned} \tag{5.7.60}$$

where $T := t_2 - t_1$. Hence

$$\begin{aligned} \|z\|_{L^\infty L^2} &\leqslant 2\|y_0^+ - y_0^-\|_{L^2} + 2(\eta + C_\eta T^{1/2})\|\gamma^+ - \gamma^-\|_{L^\infty} \\ & 2\left(\|y_0^+ - y_0^-\|_{L^2} + \|\gamma^+ - \gamma^-\|_{L^\infty}\right), \end{aligned} \tag{5.7.61}$$

by choosing $0 < \tilde{T}^\varepsilon_{R,r} < T^\varepsilon_{R,r}$ small enough such that $\eta + C_\eta(\tilde{T}^\varepsilon_{R,r})^{1/2} < 1$, which concludes the proof of Lemma 28. □

Remark 24. *We observe from (5.7.61) that*

$$\|y^+ - y^-\|_{L^\infty L^2} \to 0, \tag{5.7.62}$$

if

$$\|y_0^+ - y_0^-\|_{L^2} \to 0,\ \|\beta^+ - \beta^-\|_{L^\infty} \to 0,\ \text{and } \|\gamma^+ - \gamma^-\|_{L^\infty} \to 0. \tag{5.7.63}$$

Step 3. Maximum principle: nonlinear case (Lemma 18). Let us first point out that Lemma 18 is a consequence of the following local version of Lemma 18.

Lemma 29 (Local maximum principle: nonlinear case)**.** *Let $R > 0$, $r > 0$, and $\varepsilon > 0$ be given such that $4\varepsilon RC_1 < 1$. Let $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ be given such that $t_1 < t_2 < t_1 + \tilde{T}^\varepsilon_{R,r}$. Let $y_0^{\pm} \in L^2(0,1)$,*

$\beta^{\pm} \in L^{\infty}(t_1, t_3)$ *be piecewise continuous, and* $\gamma^{\pm} \in L^{\infty}(t_1, t_2)$ *be piecewise continuous such that* (5.2.19) *holds and*

$$\|y_0^{\pm}\|_{L^2} \leq R \text{ and } \|\beta^{\pm}\|_{L^{\infty}} + \|\gamma^{\pm}\|_{L^{\infty}} \leq r. \tag{5.7.64}$$

Then the solutions $y^{\pm}$ *to the Cauchy problem* (5.7.55) *satisfy* (5.2.20).

Indeed, using this lemma and arguing by contradiction, one easily gets that, under the assumptions of Lemma 18,

$$\max\{\tau \in [0, T];\ y^{-}(t, \cdot) \leq y^{+}(t, \cdot),\ \ \forall t \in [0, \tau]\} = T. \tag{5.7.65}$$

Proof of Lemma 29. Under the extra assumption that β and γ are in $H^{1/4}(t_1, t_2)$, property (5.2.20) follows from [Mar14, Lemma 1]. The general case follows from this special case by using a density argument and Lemma 28 (or Remark 24). Indeed, using the fact that $\beta^{\pm}$ and $\gamma^{\pm}$ are piecewise continuous, there are sequences $\beta^{n\pm} \in H^{1/4}(t_1, t_2) \cap L^{\infty}(t_1, t_2)$ and $\gamma^{n\pm} \in H^{1/4}(t_1, t_2) \cap L^{\infty}(t_1, t_2)$ such that

$$\beta^{n\pm} \to \beta^{\pm} \text{ in } L^{\infty}(t_1, t_2) \text{ and } \gamma^{n\pm} \to \gamma^{\pm} \text{ in } L^{\infty}(t_1, t_2). \tag{5.7.66}$$

Moreover, using (5.2.19) and (5.7.64), we may also impose that

$$\|\beta^{n\pm}\|_{L^{\infty}} + \|\gamma^{n\pm}\|_{L^{\infty}} \leq r, \ \ \forall n \in \mathbb{N}, \tag{5.7.67}$$

$$\beta^{n-} \leqslant \beta^{n+} \text{ and } \gamma^{n-} \leqslant \gamma^{n+}, \ \ \forall n \in \mathbb{N}. \tag{5.7.68}$$

Let $y^{n\pm}$ be the solutions to the Cauchy problem (5.7.55) for $\beta^{\pm} := \beta^{n\pm}$ and $\gamma^{\pm} := \gamma^{n\pm}$. From [Mar14, Lemma 1]

$$y^{n-}(t, \cdot) \leqslant y^{n+}(t, \cdot), \ \ \forall t \in [t_1, t_2]. \tag{5.7.69}$$

By Lemma 28 and (5.7.66),

$$\|y^{n+} - y^{+}\|_{L^{\infty}L^2} \to 0 \text{ and } \|y^{n-} - y^{-}\|_{L^{\infty}L^2} \to 0 \text{ as } n \to +\infty. \tag{5.7.70}$$

Property (5.2.20) follows from (5.7.69) and (5.7.70). This concludes the proof of Lemma 29. □

Step 4. It only remains to prove the global existence of the solution to (5.2.12) with $f = 0$. Let

$$B := \|\beta\|_{L^{\infty}(0,T)} + \|\gamma\|_{L^{\infty}(0,T)}, \tag{5.7.71}$$

$$R_M := 2\|y_0\|_{L^2} + 4B,\ r_M := B,\ \varepsilon_M := \frac{1}{8C_1 R_M}, \text{ and } T_M := T^{\varepsilon_M}_{R_M, r_M}. \tag{5.7.72}$$

Note that

$$\|y_0\|_{L^2} \leq R_M \|\beta\|_{L^{\infty}(0,T)} + \|\gamma\|_{L^{\infty}(0,T)} \leq r_M. \tag{5.7.73}$$

By (5.7.73) and the second part of Proposition 10 (Step 1) the solution y of (5.7.27) is defined at least on $[0, \min\{T, T_M\}]$. Hence we may assume that $T > T_M$.

Let $y^{\pm} : (0, \tau_{\pm}) \to L^2(0,1)$ be the (maximal) solution to the Cauchy problems

$$\begin{cases} y_t^{\pm}(t,x) - y_{xx}^{\pm}(t,x) + y^{\pm} y_x^{\pm}(t,x) = 0 & \text{for } (t,x) \in (0, \tau_{\pm}) \times (0,1), \\ y^{\pm}(t,0) = \pm B & \text{for } t \in (0, \tau_{\pm}), \\ y^{\pm}(t,1) = \pm B & \text{for } t \in (0, \tau_{\pm}), \\ y^{\pm}(0, \cdot) = y_0. \end{cases} \tag{5.7.74}$$

As for y, we have $\tau_\pm \le T_M$. Let $z^\pm : [0, T_M] \to L^2(0,1)$ be defined by

$$z^\pm := y^\pm \mp B. \tag{5.7.75}$$

Then $z^\pm$ is a solution of

$$\begin{cases} z_t^\pm(t,x) - z_{xx}^\pm(t,x) \pm B z_x^\pm(t,x) + z^\pm z_x^\pm(t,x) = 0 & \text{for } (t,x) \in (0,T_M)\times(0,1), \\ z^\pm(t,0) = 0 & \text{for } t \in (0,T_M), \\ z^\pm(t,1) = 0 & \text{for } t \in (0,T_M), \\ z^\pm(0,\cdot) = y_0 \mp B, \end{cases} \tag{5.7.76}$$

from which we get that

$$\frac{d}{dt}\int_0^1 (z^\pm)^2 dx \le 0 \text{ in } \mathcal{D}'(0,T_M). \tag{5.7.77}$$

Hence

$$\|y^\pm(t,\cdot)\|_{L^2(0,1)} \leqslant \|y_0\|_{L^2} + 2B. \tag{5.7.78}$$

Moreover, thanks to the maximum principle (Lemma 18), we have

$$y^- \le y \le y^+, \ \ \forall t \in [0, T_M]. \tag{5.7.79}$$

In particular, using (5.7.78),

$$\|y(T_M)\|_{L^2} \le R_M,\ \|y^-(T_M)\|_{L^2} \le R_M, \ \text{ and } \|y^+(T_M)\|_{L^2} \le R_M. \tag{5.7.80}$$

This allows to redo the above procedure with the initial time T_M and the initial data $y(T_M)$, $y^-(T_M)$, and $y^+(T_M)$. Let us emphasize that the initial data for $y^\pm$ at time T_M is not $y(T_M)$ but $y^\pm(T_M)$ (which is given by the definition of $y^\pm$ on $[0,T_M]$). In particular y, y^-, and y^+ are defined on $[0, \min\{T, 2T_M\}]$. So we may assume that $T > 2T_M$. Moreover, using once more the maximum principle,

$$y^-(t) \le y(t) \le y^+(t), \ \ \forall t \in [T_M, 2T_M]. \tag{5.7.81}$$

Property (5.7.77) and therefore also (5.7.78) hold on $[T_M, 2T_M]$. Together with (5.7.81) this implies that

$$\|y(2T_M)\|_{L^2} \le R_M,\ \|y^-(2T_M)\|_{L^2} \le R_M, \ \text{ and } \|y^+(2T_M)\|_{L^2} \le R_M. \tag{5.7.82}$$

We keep going and using an induction argument get that, for every integer $n > 0$, y is defined on $[0, \min\{nT_M, T\}]$. This concludes the proof of Proposition 10.

At last, Lemma 19 follows directly from Lemma 28 and Step 4.

5.8 Appendix B: Proof of Lemma 21

Let us start the proof of Lemma 21 by proving the following lemma, which deals with the well-posedness for small time of the Cauchy problem

$$\begin{cases} z_t - z_{xx} + z z_x + a(t) z_x = 0 & \text{for } (t,x) \in (0,T)\times(0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = 0 & \text{for } t \in (0,T), \\ a_t = \tilde{A}_0(z,a) & \text{for } t \in (0,T), \\ z(0,\cdot) = z_0 \quad \text{and} \quad a(0) = a_0, \end{cases} \tag{5.8.1}$$

with

$$\tilde{A}_0(z,a) = A_0(z+a,a). \tag{5.8.2}$$

Lemma 30. *Let $A_0(y,a)$ be given by (5.3.22). Let $R>0$. There exists $T_R>0$ such that, for every $(z,a)\in L^2(0,1)\times\mathbb{R}$ such that $\|(z,a)\|_V\leqslant R$, there exists one and only one $z:[0,T_R]\times(0,L)$ such that*

$$z\in C^0([0,T_R];L^2(0,1))\cap L^2(0,T_R;H^1(0,1)), \tag{5.8.3}$$

$$a\in C^0([0,T_R]),\ a_t=\alpha \text{ in the distribution sense, and } a(0)=a_0, \tag{5.8.4}$$

and such that $y:=z$ is a solution on $[0,T_R]$ to the Cauchy problem (5.2.1), in the sense of Definition 10, for $f:=-zz_x-a(t)z_x$, $\beta:=0$, $\gamma:=0$, and $y_0:=z_0$.

Let us define

$$W:=\{(z,a): z\in C^0([0,T];L^2(0,1))\cap L^2(0,T;H^1_0(0,1)), a\in C^0([0,T])\}. \tag{5.8.5}$$

We introduce the W_η-norm on W by

$$\|(z,a)\|_{W_\eta}:=\|z\|_{C^0L^2\cap L^2H^1_0}+\eta\|a\|_{C^0}, \tag{5.8.6}$$

with $\eta<1$. Hence (z,a) is as requested in Lemma 30 if and only if it is a fixed point of $\Lambda: W\to W$, $(z,a)\mapsto W(z,a)=:(w,b)$ where w is the unique solution to the Cauchy problem (in the sense of Definition 10)

$$\begin{cases} w_t-w_{xx}=-zz_x-a(t)z_x & \text{for } (t,x)\in(0,T)\times(0,1),\\ w(t,0)=0 & \text{for } t\in(0,T),\\ w(t,1)=0 & \text{for } t\in(0,T), w(0,\cdot)=z_0 \text{ and } b(0)=a_0 \end{cases} \tag{5.8.7}$$

and

$$b(t):=a_0+\int_0^t \tilde{A}_0(z(\tau),a(\tau))d\tau \text{ for } t\in[0,T]. \tag{5.8.8}$$

It follows from Proposition 8 that Λ is well defined. For $\|(z_0,a_0)\|_V\leqslant R$, we try to find a fixed point of Λ on

$$\tilde{B}_R:=\{(z,a)\in W: \|(z,a)\|_{W_\eta}\leqslant 3R\}. \tag{5.8.9}$$

For every $(z,a)\in\tilde{B}_R$, by Proposition 8 and (5.7.10)–(5.7.12), we have

$$\begin{aligned}\|w\|_{C^0L^2\cap L^2H^1_0} &\leqslant 2\|zz_x+az_x\|_{L^1L^2}+2\|z_0\|_{L^2}\\ &\leqslant CT^{1/4}\|z\|^2_{C^0L^2\cap L^2H^1_0}+2T^{1/2}\|a\|_{C^0}\|z\|_{L^2H^1_0}+2R\\ &\leqslant 2R+9CR^2T^{1/4}+9R^2T^{1/2}/\eta.\end{aligned} \tag{5.8.10}$$

Moreover

$$
\begin{aligned}
\|b\|_{C^0} &\leqslant |a_0| + \|\tilde{A}_0(z,a)\|_{L^1} \\
&\leqslant R + \lambda\| -2\int_0^1 z_x^2 e^{-x}dx + (1-a)\,V_1(z) - \frac{2}{3}\int_0^1 z^3 e^{-x}dx\|_{L^1} \\
&\quad + \|\frac{V_1}{2} - \frac{1}{2}\lambda(a-\lambda V_1(z))^3\|_{L^1} \\
&\leqslant R + 2\lambda\|z\|^2_{L^2H_0^1} + \lambda T\|z\|^2_{C^0L^2} + \lambda T\|a\|_{C^0}\|z\|^2_{C^0L^2} + \lambda T^{1/2}\|z\|_{L^2H_0^1}\|z\|^2_{C^0L^2} \\
&\quad + T\|z\|^2_{C^0L^2} + \lambda T(\|a\|_{C^0} + \lambda\|z\|^2_{C^0L^2})^3 \\
&\leqslant R + 18\lambda R^2 + 9\lambda TR^2 + 27\lambda TR^3/\eta + 27\lambda T^{1/2}R^3 + 9TR^2 + \lambda T(3R/\eta + 9\lambda R^2)^3.
\end{aligned} \tag{5.8.11}
$$

Hence

$$
\begin{aligned}
\|(w,b)\|_{W_\eta} &\leqslant 2R + 9CR^2T^{1/4} + 9R^2T^{1/2}/\eta + \eta(R + 18\lambda R^2) \\
&\quad + \eta(9\lambda TR^2 + 27\lambda TR^3/\eta + 27\lambda T^{1/2}R^3 + 9TR^2 + \lambda T(3R/\eta + 9\lambda R^2)^3).
\end{aligned} \tag{5.8.12}
$$

We can successively choose η and T so that the right hand side of (5.8.12) is less or equal than $3R$, leading to

$$\|(w,b)\|_{W_\eta} \leqslant 3R, \tag{5.8.13}$$

which implies that

$$\Lambda(\tilde{B}_R) \subset \tilde{B}_R. \tag{5.8.14}$$

It remains to get the contraction property. Suppose that $(w_i,b_i) := \Lambda((z_i,a_i))$ with $i \in \{1,2\}$, then by using Proposition 8 one gets

$$
\begin{aligned}
\|w_1 - w_2\|_{C^0L^2\cap L^2H_0^1} &\leqslant 2\|z_1(z_1)_x - z_2(z_2)_x + a_1(z_1)_x - a_2(z_2)_x\|_{L^1L^2} \\
&\leqslant 2CT^{1/4}\left(\|z_1\|_{C^0L^2\cap L^2H_0^1} + \|z_2\|_{C^0L^2\cap L^2H_0^1}\right)\|z_1 - z_2\|_{C^0L^2\cap L^2H_0^1} \\
&\quad + 2T^{1/2}\|a_1 - a_2\|_{C^0}\|z_2\|_{L^2H^1} + 2T^{1/2}\|a_1\|_{C^0}\|z_1 - z_2\|_{L^2H^1} \\
&\leqslant \left(12CRT^{1/4} + 12RT^{1/2}/\eta\right)\|(z_1,a_1) - (z_2,a_2)\|_{W_\eta}
\end{aligned} \tag{5.8.15}
$$

and

$$
\begin{aligned}
\|b_1 - b_2\|_{C^0} &\leqslant \|\tilde{A}_0(z_1,a_1) - \tilde{A}_0(z_2,a_2)\|_{L^1} \\
&\leqslant \lambda\| -2\int_0^1 \left((z_1)_x^2 - (z_2)_x^2\right)e^{-x}dx - \frac{2}{3}\int_0^1 (z_1^3 - z_2^3)e^{-x}dx\|_{L^1} + \|\frac{V_1(z_1) - V_1(z_2)}{2}\|_{L^1} \\
&\quad + \frac{1}{2}\lambda\|(a_1 - \lambda V_1(z_1))^3 - (a_2 - V_1(z_2))^3\|_{L^1} + \lambda\|(1-a_1)\,V_1(z_1) - (1-a_2)\,V_1(z_2)\|_{L^1} \\
&\leqslant 12\lambda R\|z_1 - z_2\|_{L^2H_0^1} + 24\lambda R^2T^{1/2}\|z_1 - z_2\|_{L^2H_0^1} \\
&\quad + T(3R + 6\lambda R)\|z_1 - z_2\|_{C^0L^2} + (27TR^2/\eta)\|(z_1,a_1) - (z_2,a_2)\|_{W_\eta} \\
&\quad + \lambda T\big(\|a_1 - a_2\|_{C^0} + \lambda\|z_1 - z_2\|_{C^0L^2}\|z_1 + z_2\|_{C^0L^2}\big)\cdot 4(3R/\eta + \lambda 9R^2)^2 \\
&\leqslant \big(12\lambda R + 24\lambda R^2T^{1/2} + 3RT + 6\lambda RT + 27TR^2/\eta + 36\lambda TR^2(1/\eta + 6\lambda R)^3\big) \\
&\qquad \|(z_1,a_1) - (z_2,a_2)\|_{W_\eta}.
\end{aligned} \tag{5.8.16}
$$

Hence one gets the contraction property of the map Λ on $\tilde{B}_R$ when η and T are well chosen, which implies the existence of a unique solution in $\tilde{B}_R$. Then, proceeding as in the proof of uniqueness part of Proposition 10, we can further get the uniqueness of solution in W_η. This completes the proof of Lemma 30.

In order to end the proof of Lemma 21, it only remains to prove the existence of the solution (z, a) for large time. For this existence in large time, it suffices to check that $\|z(t)\|_{L^2}$ remains bounded. This can be done by using the maximum principle for the nonlinear Burgers equation (Lemma 18) as in the proof of Proposition 10 (Step 4).

5.9 Appendix C: Proof of Lemma 24

This section is devoted to the proof of Lemma 24. Let us start our proof of this lemma with a proof of the following lemma.

Lemma 31. *Let $M > 0$ and let $G : L^2(0,1) \to \mathbb{R}$ be a (stationary) feedback law satisfying (5.4.43). Let $R > 0$ and $\varepsilon > 0$. There exists $T_R^\varepsilon > 0$ such that, for every $y_0 \in L^2(0,1)$ satisfying*

$$\|y_0\|_{L^2} \leqslant R, \tag{5.9.1}$$

the Cauchy problem (5.4.45) has a unique solution

$$y \in C^0([0, T_R^\varepsilon]; L^2(0,1)) \cap L^2(0, T_R^\varepsilon; L^\infty(0,1)), \tag{5.9.2}$$

and moreover this solution satisfies

$$\|y\|_{C^0([0,T_R^\varepsilon];L^2(0,1))} \leqslant 2R, \tag{5.9.3}$$

$$\|y\|_{L^2(0,T_R^\varepsilon;L^\infty(0,1))} \leqslant \varepsilon R. \tag{5.9.4}$$

This lemma is quite similar to Proposition 10. Therefore we use the same strategy to get the proof of this lemma. Let us consider the space

$$Y := C^0([0,T]; L^2(0,1)) \cap L^2(0,T; L^\infty(0,1)) \tag{5.9.5}$$

and the norm

$$\|y\|_{Y_\mu} := \|y\|_{C^0 L^2} + \frac{1}{\mu}\|y\|_{L^2 L^\infty}, \tag{5.9.6}$$

with $\mu > 0$ to be chosen later. Then we consider the following map $\Gamma : Y_\mu \to Y_\mu$, where $\Gamma(y)$ is the unique solution of

$$\begin{cases} z_t(t,x) - z_{xx}(t,x) = -(yy_x)(t,x) & \text{for } (t,x) \in (0,T) \times (0,1), \\ z(t,0) = 0 & \text{for } t \in (0,T), \\ z(t,1) = G(y) & \text{for } t \in (0,T), \\ z(0,\cdot) = y_0. \end{cases} \tag{5.9.7}$$

Again, this map is well defined (one only need to notice the $L^\infty L^2$ estimate can be replaced by $C^0 L^2$ estimate since $G(y(t))$ is continuous). As in the proof of Proposition10, it suffices to find the unique fixed point in the following ball

$$B_R^1 := \{y \in Y : \|y\|_{Y_\mu} \leqslant 2R\}. \tag{5.9.8}$$

From (5.7.31)–(5.7.41) and (5.7.43)–(5.7.44), we get

$$\begin{aligned} \|\Gamma(y)\|_{C^0 L^2} &\leqslant \|y_0\|_{L^2} + \frac{C_1}{2}\|y\|_{L^2 L^\infty}\|y\|_{L^\infty L^2} + 2(\eta + C_\eta T^{1/2})\|G(y)\|_{C^0} \\ &\leqslant \|y_0\|_{L^2} + 2C_1 \mu R^2 + 4(\eta + C_\eta T^{1/2}) MR, \end{aligned} \tag{5.9.9}$$

and

$$\begin{aligned}\|\Gamma(y)\|_{L^2L^\infty} &\leqslant CT^{1/4}\|y_0\|_{L^2}+CT^{1/4}\frac{C_1}{2}\|y\|_{L^2L^\infty}\|y\|_{L^\infty L^2}+2CT^{1/4}(\eta+C_\eta T^{1/2})\|G(y)\|_{C^0}\\ &\leqslant CT^{1/4}\|y_0\|_{L^2}+2CT^{1/4}C_1\mu R^2+4CT^{1/4}(\eta+C_\eta T^{1/2})MR. \qquad (5.9.10)\end{aligned}$$

With a good choice of η,μ and T, Γ is from B^1_R to B^1_R. By using similar estimates (see also the proofs of (5.7.50) and (5.7.51)), we have

$$\begin{aligned}\|\Gamma(y_1)-\Gamma(y_2)\|_{C^0L^2} &\leqslant \frac{C_1}{2}\|y_1+y_2\|_{L^2L^\infty}\|y_1-y_2\|_{L^\infty L^2}+2(\eta+C_\eta T^{1/2})\|G(y_1)-G(y_2)\|_{C^0}\\ &\leqslant C_1R\mu\|y_1-y_2\|_{C^0L^2}+2(\eta+C_\eta T^{1/2})M\|y_1-y_2\|_{C^0L^2},\end{aligned}$$

and

$$\begin{aligned}\|\Gamma(y_1)-\Gamma(y_2)\|_{L^2L^\infty} &\leqslant CT^{1/4}\frac{C_1}{2}\|y_1+y_2\|_{L^2L^\infty}\|y_1-y_2\|_{L^\infty L^2}\\ &\quad+2CT^{1/4}(\eta+C_\eta T^{1/2})\|G(y_1)-G(y_2)\|_{C^0}\\ &\leqslant CT^{1/4}C_1R\mu\|y_1-y_2\|_{C^0L^2}+2CT^{1/4}(\eta+C_\eta T^{1/2})M\|y_1-y_2\|_{C^0L^2}.\end{aligned}$$

Hence a good choice of η, μ, and T, makes Γ a contraction map. This concludes the proof of Lemma 31.

Remark 25. *If we replace C^0L^2 by $L^\infty L^2$, we get the local well-posedness in $L^\infty L^2\cap L^2L^\infty$.*

So far, we get the local existence and uniqueness of the solution of (5.4.45). In order to get the global existence statement of Lemma 24 it suffices to control the L^2-norm of $y(t)$. This control follows from (5.4.44), which leads to (5.7.78) with $B:=M$. This concludes the proof of Lemma 24.

5.10 Appendix D: Proof of Lemma 23

The proof is to consider an equation of $z(x):=\Pi_\lambda(y(x))$ instead of equation (5.4.15) (see (5.4.16)), this gives the advantage that $z(t,0)=z(t,1)=0$. The local existence and uniqueness of the solution z is given by a standard procedure (by considering the nonlinear term I as a force term and using Banach fixed point theorem).

Proof of Lemma 23. In this proof, the constant C may change from line to line, but it is independent of $0<T<1$ and of R. From (5.4.16) and (5.4.18)

$$I(z)=-yy_x+\int_0^x k_\lambda(x,v)(yy_x)(v)dv. \qquad (5.10.1)$$

We notice that

$$\|I(z)\|_{L^2(0,1)}\leqslant C\|yy_x\|_{L^2}\leqslant C\|y\|_{L^\infty}\|y_x\|_{L^2}\leqslant C\|y\|^{1/2}_{L^2}\|y\|^{3/2}_{H^1_0}\leqslant C\|z\|^{1/2}_{L^2}\|z\|^{3/2}_{H^1_0}, \qquad (5.10.2)$$

and that

$$\begin{aligned}
\|I(z_1)-I(z_2)\|_{L^2(0,1)} &\leqslant C\|y_1-y_2\|_{L^\infty}\|y_{1x}+y_{2x}\|_{L^2}+C\|y_1+y_2\|_{L^\infty}\|y_{1x}-y_{2x}\|_{L^2}\\
&\leqslant C\|y_1-y_2\|_{L^2}^{1/2}\|y_1-y_2\|_{H_0^1}^{1/2}\|y_1+y_2\|_{H_0^1}\\
&\quad + C\|y_1+y_2\|_{L^2}^{1/2}\|y_1+y_2\|_{H_0^1}^{1/2}\|y_1-y_2\|_{H_0^1}\\
&\leqslant C\|z_1-z_2\|_{L^2}^{1/2}\left(\|z_1-z_2\|_{H_0^1}^{1/2}+\|z_1-z_2\|_{L^2}^{1/2}\right)\left(\|z_1+z_2\|_{H_0^1}+\|z_1+z_2\|_{L^2}\right)\\
&\quad + C\|z_1+z_2\|_{L^2}^{1/2}\left(\|z_1+z_2\|_{H_0^1}^{1/2}+\|z_1+z_2\|_{L^2}^{1/2}\right)\left(\|z_1-z_2\|_{H_0^1}+\|z_1-z_2\|_{L^2}\right)\\
&\leqslant C\|z_1-z_2\|_{L^2}^{1/2}\|z_1-z_2\|_{H_0^1}^{1/2}\|z_1+z_2\|_{H_0^1}\\
&\quad + C\|z_1+z_2\|_{L^2}^{1/2}\|z_1+z_2\|_{H_0^1}^{1/2}\|z_1-z_2\|_{H_0^1}. \qquad (5.10.3)
\end{aligned}$$

Regarding the linear Cauchy problem

$$\begin{cases}
z_t-z_{xx}+\lambda z=f & \text{for } (t,x)\in(s,s+T)\times(0,1),\\
z(t,0)=0 & \text{for } t\in(s,s+T),\\
z(t,1)=0 & \text{for } t\in(s,s+T),\\
z(0,\cdot)=z_0,
\end{cases} \qquad (5.10.4)$$

similar to Proposition 8, we have

$$\|z\|_{C^0L^2}\leqslant \|z_0\|_{L^2}+\|f\|_{L^1L^2}, \qquad (5.10.5)$$

$$\|z\|_{L^2H_0^1}\leqslant \|z_0\|_{L^2}+\|f\|_{L^1L^2}. \qquad (5.10.6)$$

As normal, let us denote the space $C^0([s,s+T];L^2(0,1))\cap L^2(s,s+T;H_0^1(0,1))$ endowed with norm $\|\cdot\|_{C^0L^2}+\|\cdot\|_{L^2H_0^1}$ by $\mathcal{H}$ (or $\mathcal{H}_T$ if necessary).

For y_0 with $\|y_0\|_{L^2}\leqslant R$ given, we have $\|z_0\|_{L^2}\leqslant e^{3/2C_1\sqrt{\lambda}}\|y_0\|_{L^2}\leqslant e^{3/2C_1\sqrt{\lambda}}R$. Let us define

$$\mathcal{B}:=\{z\in\mathcal{H}:\|z\|_{\mathcal{H}}\leqslant 3e^{3/2C_1\sqrt{\lambda}}R\}. \qquad (5.10.7)$$

We consider the map $\Gamma:\mathcal{H}\to\mathcal{H}$, $z\mapsto w$ where w is the unique solution of

$$\begin{cases}
w_t-w_{xx}+\lambda w=I(z) & \text{for } (t,x)\in(s,s+T)\times(0,1),\\
w(t,0)=0 & \text{for } t\in(s,s+T),\\
w(t,1)=0 & \text{for } t\in(s,s+T),\\
w(0,\cdot)=z_0.
\end{cases} \qquad (5.10.8)$$

From (5.4.12), (5.10.5), and (5.10.6), we know that

$$\|\Gamma(z)\|_{\mathcal{H}}\leqslant 2e^{3/2C_1\sqrt{\lambda}}R+2\|I(z)\|_{L^1L^2}. \qquad (5.10.9)$$

Hence, for every $z \in \mathcal{B}$, by (5.10.2) we have

$$\begin{aligned}
\|\Gamma(z)\|_{\mathcal{H}} &\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + 2\|I(z)\|_{L^1L^2} \\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + C\Big\|\|z\|_{L^2}^{1/2}\|z\|_{H_0^1}^{3/2}\Big\|_{L^1(s,s+T)} \\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + CT^{1/4}\|z\|_{C^0L^2}^{1/2}\|z\|_{L^2H_0^1}^{3/2} \\
&\leqslant 2e^{3/2C_1\sqrt{\lambda}}R + 9CT^{1/4}e^{3C_1\sqrt{\lambda}}R^2. \qquad (5.10.10)
\end{aligned}$$

For every z_1 and $z_2 \in \mathcal{B}$, we have

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{H}} \leqslant 2\|I(z_1) - I(z_2)\|_{L^1L^2}, \qquad (5.10.11)$$

Above estimate together with (5.10.3) give

$$\begin{aligned}
\|\Gamma(z_1) - \Gamma(z_2)\|_{\mathcal{H}} &\leqslant C\|z_1 - z_2\|_{L^2}^{1/2}\|z_1 - z_2\|_{H_0^1}^{1/2}\|z_1 + z_2\|_{H_0^1} \\
&\quad + C\|z_1 + z_2\|_{L^2}^{1/2}\|z_1 + z_2\|_{H_0^1}^{1/2}\|z_1 - z_2\|_{H_0^1} \\
&\leqslant CT^{1/4}\|z_1 - z_2\|_{C^0L^2}^{1/2}\|z_1 - z_2\|_{L^2H_0^1}^{1/2}\|z_1 + z_2\|_{L^2H_0^1} \\
&\quad + CT^{1/4}\|z_1 + z_2\|_{C^0L^2}^{1/2}\|z_1 + z_2\|_{L^2H_0^1}^{1/2}\|z_1 - z_2\|_{L^2H_0^1} \\
&\leqslant T^{1/4}3e^{3/2C_1\sqrt{\lambda}}RC\|z_1 - z_2\|_{\mathcal{H}}. \qquad (5.10.12)
\end{aligned}$$

From (5.10.10) and (5.10.12), we get the existence of T_R^{tr} which completes the proof. $\square$

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Rapid stabilization of Burgers equations and of Korteweg-de Vries equations

Abstract :
This thesis is devoted to the study of stabilization of partial differential equations by nonlinear feedbacks. We are interested in the cases where classical linearization and stationary feedback law do not work for stabilization problems, for example KdV equations and Burgers equations. More precisely, it includes three important cases : stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable ; small-time local stabilization of linear controllable systems ; small-time global stabilization of nonlinear controllable systems. We find a strategy for the small-time global stabilization of the viscous Burgers equation : small-time global approximate stabilization and small-time local stabilization. Moreover, using a quadratic structure, we prove that the KdV system is exponentially stabilizable even in the case of critical lengths.
Keywords : controllability, stabilization, nonlinear feedback, Burgers, KdV, partial differential equations, small-time.

Stabilisation rapide d'équations de Burgers et de Korteweg-de Vries

Résumé :
Cette thèse est consacrée à l'étude de la stabilisation d'équations aux dérivées partielles par feedbacks non linéaires. Nous nous intéressons aux cas où la technique de linéarisation et l'utilisation de feedback stationnaire ne fonctionnent pas pour des problèmes de stabilisation, par exemple des équations de Korteweg-de Vries (KdV) et des équations de Burgers. Plus précisément, nous traitons trois cas importants : la stabilisation de systèmes non linéaires dont les systèmes linéarisés ne sont pas stabilisables asymptotiquement ; la stabilisation locale en temps petit de systèmes contrôlables linéaires ; la stabilisation globale en temps petit de systèmes contrôlables non linéaires. En particulier, nous trouvons une stratégie pour la stabilisation globale en temps petit de l'équation de Burgers visqueuse. Elle repose sur la stabilisation globale approchée en temps petit et sur la stabilisation locale en temps petit. De plus, nous prouvons que le système de KdV même pour des longueurs critiques est stabilisable de manière exponentielle. Nous utilisons pour cela une structure quadratique de la dynamique de la partie dont le linéarisé n'est pas contrôlable.
Mot clés : contrôlabilité, stabilisation, feedback non linéaire, Burgers, KdV, équations aux dérivées partielles, rapide.