A quest for exactness: machines, algebra and geometry for tractional constructions of differential equations

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A QUEST FOR EXACTNESS : machines, algebra and geometry for tractional constructions of differential equations

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Abstract

In *La Géométrie*, Descartes proposed a "balance" between geometric constructions and symbolic manipulation with the introduction of suitable ideal machines. In particular, Cartesian tools were polynomial algebra (analysis) and a class of diagrammatic constructions (synthesis). This setting provided a classification of curves, according to which only the algebraic ones were considered "purely geometrical." This limit was overcome with a general method by Newton and Leibniz introducing the infinity in the analytical part, whereas the synthetic perspective gradually lost importance with respect to the analytical one—geometry became a mean of visualization, no longer of construction.

Descartes's foundational approach (analysis without infinitary objects and synthesis with diagrammatic constructions) has, however, been extended beyond algebraic limits, albeit in two different periods. In the late 17th century, the synthetic aspect was extended by "tractional motion" (construction of transcendental curves with idealized machines). In the first half of the 20th century, the analytical part was extended by "differential algebra," now a branch of computer algebra.

This thesis seeks to prove that it is possible to obtain a new balance between these synthetic and analytical extensions of Cartesian tools for a class of transcendental problems. In other words, there is a possibility of a new convergence of machines, algebra, and geometry that gives scope for a foundation of (a part of) infinitesimal calculus without the conceptual need of infinity.

The peculiarity of this work lies in the attention to the constructive role of geometry as idealization of machines for foundational purposes. This approach, after the "de-geometrization" of mathematics, is far removed from the mainstream discussions of mathematics, especially regarding foundations. However, though forgotten these days, the problem of defining appropriate canons of construction was very important in the early modern era, and had a lot of influence on the definition of mathematical objects and methods. According to the definition of Bos [2001], these are "exactness problems" for geometry.

Such problems about exactness involve philosophical and psychological interpretations, which is why they are usually considered external to mathematics. However, even though lacking any final answer, I propose in conclusion a very primitive algorithmic approach to such problems, which I hope to explore further in future research.

From a cognitive perspective, this approach to calculus does not require infinity and, thanks to idealized machines, can be set with suitable "grounding metaphors" (according to the terminology of Lakoff and Núñez [2000]). This concreteness can have useful fallouts for math education, thanks to the use of both physical and digital artifacts (this part will be treated only marginally).

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Riassunto

Ne *La Géométrie* del 1637 Descartes ha proposto un "equilibrio" tra costruzioni geometriche e manipolazioni simboliche con l'introduzione di opportune macchine ideali. In particolare gli strumenti di Descartes erano l'algebra polinomiale (analisi) e una classe di costruzioni diagrammatiche (sintesi). Questa impostazione implica una classificazione delle curve, secondo cui solo quelle algebriche possono essere considerate "puramente geometriche". Questo limite è stato superato con un metodo generale da Newton e Leibniz introducendo l'infinito nella parte analitica, mentre la prospettiva sintetica ha gradualmente sempre più perso importanza rispetto a quella analitica (la geometria diventa un mezzo di visualizzazione e non più di costruzione).

L'approccio fondazionale di Descartes (analisi con oggetti finiti e sintesi con costruzioni diagrammatiche) è stato comunque esteso oltre i limiti delle curve algebriche, anche se in due periodi distinti. Nel tardo XVII secolo la parte sintetica è stata estesa con il "movimento trazionale" (costruzione di curve trascendenti con macchine idealizzate), e nella prima metà del XX secolo la parte analitica è stata estesa con la "algebra differenziale" (oggigiorno considerata una branca dell'algebra computazionale).

L'obiettivo di questa tesi è di provare come sia possibile ottenere un nuovo equilibrio tra queste estensioni (sintentica e analitica) degli strumenti Cartesiani, un equilibrio che superi il limite delle curve algebriche e permetta di trattare una classe di problemi trascendenti. In altre parole, l'obiettivo è di evidenziare come sia possibile una nuova convergenza di macchine, algebra e geometria che permetta una fondazione di (parte della) analisi infinitesimale senza il bisogno concettuale dell'infinito.

La caratteristica di questo lavoro è l'attenzione al ruolo costruttivo della geometria (come idealizzazione del comportamento di opportune macchine) per fini fondazionali. Questo approccio, dopo la "de-geometrizzazione" della matematica, è molto distante dal filone principale delle discussioni sulla matematica, specie dal punto di vista fondazionale. Comunque, anche se oggigiorno caduto in oblio, il problema di definire degli appropriati canoni di costruzioni era molto sentito nel periodo della prima età moderna, ed ha avuto profonde influenze sul modo in cui sono stati definiti gli oggetti e i metodi matematici dell'epoca. Secondo la definizione di Bos [2001], questi sono i "problemi di esattezza" per la geometria.

Questi problemi di esattezza riguardano interpretazioni filosofiche e psicologiche, pertanto sono solitamente considerati esterni alla matematica. Comunque, anche se senza una risposta esaustiva, nelle conclusioni propongo un approccio algoritmico (molto primitivo) per inquadrare tali problemi, che spero di approfondire in lavori futuri.

Dalla prospettiva delle scienze cognitive, questo approccio all'analisi infinitesimale non richiede l'infinito e, grazie alle macchine idealizzate, può essere concepito con opportune "metafore fondanti" (secondo la terminologia di Lakoff and Núñez [2000]). Questa concretezza può avere utili ricadute in didattica della matematica, grazie all'uso di artefatti sia fisici che digitali (questa parte sarà trattata solo marginalmente).

Esattezza delle costruzioni

É possibile stabilire un canone di costruzioni geometriche per alcune classi di curve oltre quelle algebriche? E può questo canone estendere in un qualche modo l'intuizione classica della geometria senza il bisogno dell'introduzione dell'infinito nella parte analitica?

Per impostare queste domande dobbiamo andare indietro alla seconda metà del XVII secolo, quando La Géométrie di Descartes era influente, l'algebra era stata generalmente accettata come strumento analitico per problemi geometrici, e le curve "geometriche" avevano raggiunto una legittimazione ontologica largamente condivisa. Il lavoro di Descartes aveva suggerito un canone per distinguere tra le curve "geometriche" (intuitive e analiticamente trattabili) da quelle "meccaniche" (tutte le altre): in termini moderni, ciò corrisponde alla distinzione tra curve algebriche e trascendenti. Prima algebra e geometria, anche se profondamente correlate, avevano dei ruoli chiaramente distinti: l'algebra era semplicemente un "metodo di manipolazione simbolica" per aiutare a rispondere a domande circa entità geometriche o aritmetiche, entità la cui esistenza era (teoricamente) legittimata in modo indipendente, per esempio con costruzioni geometriche.

L'uso di curve non algebriche ha sollevato il problema della "legittimazione delle curve trascendenti" [1] con certi strumenti geometrici che estendono quelli ammessi da Descartes: per esempio con "costruzioni trazionali".

Questo costituisce il primo punto su cui soffermarci: Leibniz e Newton, considerato il fallimento dell'algebra polinomiale nel trattare problemi trascendenti, hanno sviluppato nuovi metodi analitici introducendo non solo entità finite, ma anche infinite o infinitesime, in contrasto con l'originale prospettiva finitista dell'algebra di Descartes. D'altro canto il movimento trazionale, pur non rispettando il paradigma cartesiano, può essere considerato un'estensione degli strumenti geometrici di Descartes, anche se in un modo da precisare.

L'analisi dei fondamenti del movimento trazionale è durata una sessantina di anni ma, a differenza di quanto accaduto con la geometria cartesiana, queste ricerche sono terminate senza la definizione di un canone di costruzioni ampiamente accettato. Questa mancanza è stata probabilmente favorita dal cambio di paradigma dominante, da costruzioni geometriche a manipolazioni simboliche. In fatti, la possibilità di rappresentare una curva semplicemente con una formula (anche se richiamando entità non finite) era molto più conveniente (per la manipolazione) della rappresentazione con procedure geometriche. Pertanto, passo dopo passo, la legittimazione ontologica delle curve algebriche con costruzioni perdeva sempre più di importanza, arrivando al risultato pratico che un oggetto matematico si considera definito completamente da una formula. Questo cambio di paradigma, prima semplicemente adottato in pratica, ha raggiunto nella

1. Cf. Bos [1988].

seconda metà del XIX secolo la sua fondazione nella "aritmetizzazione dell'analisi". Ciò ha segnato il completamento del passaggio da una prospettiva di fondazione della matematica "geometrica" (gli oggetti sono costruiti con costruzioni geometriche e il simbolismo è utile per analizzarli o caratterizzarli) a una fondazione "aritmetica" (gli oggetti sono ottenuti come insiemi di numeri reali e la geometria è utile per visualizzarli).

Riassumendo, l'eredità di Descartes ha favorito storicamente due cambi fondamentali:

- in analisi il passaggio da strumenti finiti a infiniti;
- il passaggio da fondazione geometrica ad aritmetica.

Se il primo punto era considerato necessario per gestire i problemi trascendenti, il secondo è stato essenziale per dare rigore a una fondazione geometrica intuitiva e non precisa, e per generalizzare ed estendere i metodi geometrici (che rimanevano inadeguati a studiare le nuove curve).

Alla luce di queste osservazioni, le domande all'inizio del paragrafo possono essere meglio riformulate: E' possibile definire un nuovo canone di costruzioni per estendere in un modo conservativo (geometrico e finito) la geometria cartesiana fino alla geometria differenziale? E per quanto riguarda gli strumenti analitici necessari per queste costruzioni, è possibile evitare enti non finiti e usare un'estensione dell'algebra polinomiale? Questa tesi vuole rispondere positivamente ad entrambe le domande.

Secondo Bos, il problema di definire i canoni delle costruzioni geometriche è detto "problema dell'esattezza": questo problema è stato fondamentale durante l'inizio dell'età moderna, ma è caduto totalmente nell'oblio una volta che la comunità di matematici ha diffusamente accettato le formule come la migliore rappresentazione di oggetti matematici. In particolare, in questa tesi propongo un canone di costruzioni basato su una definizione specifica di alcune macchine ottenute assemblando opportuni componenti (queste macchine sono una reinterpretazione degli strumenti del "movimento trazionale").

Inoltre, come il canone di Descartes ha prodotto un equilibrio tra macchine (sistemi articolati), strumenti analitici (algebra) e curve geometriche (quelle algebriche), in questa tesi propongo un nuovo equilibrio tra una nuova classe di macchine (che ho chiamato "macchine differenziali"), l'estensione degli strumenti analitici (algebra differenziale) e una classe di funzioni (soluzioni di equazioni differenziali algebriche), dove ognuno di questi domini si può vedere come un'estensione conservativa dei domini cartesiani.

Schema della tesi

Nel capitolo 2 introduco alcune considerazioni storico-filosofiche relative alla definizione dei canoni di costruzione geometrica: questa parte non contiene contenuti originali. In particolare, dopo un primo sguardo alle costruzioni di Euclide e alcune estensioni del periodo classico, mi focalizzo sul canone cartesiano e sui successivi tentativi di superarlo con il movimento trazionale.

Il centro della tesi è costituito di capitoli centrali, dove introduce l'approccio basato sulle macchine, formalizzato tramite lo "approccio comportamentale" della modellizzazione matematica (sezione 3.1). Utilizzando le macchine introdotte nella parte storica, introduco gli strumenti adottati come un'opportuna astrazione degli strumenti trazionali. Con questa impostazione arrivo a definire analiticamente i limiti di questa reinterpretazione del movimento trazionale. Vedremo come questo modello possa essere considerato un'estensione delle geometrie di Euclide e di Descartes. Inoltre, con l'approccio comportamentale, evidenzio la relazione profonda tra macchine, costruzioni geometriche e teorie di manipolazione di simboli in modo da rispondere a domande circa l'uguaglianza tra macchine. Diversamente dall'approccio di Descartes, al momento queste macchine non hanno una ben definita giustificazione filosofica della limitazione delle componenti introdotte, quindi spero che in futuro il paradigma delle costruzioni geometriche effettuate con opportune macchine possa essere esteso oltre i limiti delle macchine di questa tesi, pur sempre continuando a soddisfare i requisiti di profonda correlazione tra macchine idealizzate e strumenti analitici finiti.

Più specificamente, in questo lavoro traccio un parallelismo tra macchine, algebra e geometria in un'estensione a tre step: inizio dalla geometria del piano di Euclide nella sezione 3.2, imposto la geometria cartesiana nella sezione 3.3 e finalmente la estendo a oggetti differenziali nel capitolo 4. Introduco appropriati modelli di macchine per definire queste tre geometrie, e le chiamo rispettivamente macchine "classiche", "algebriche" e "differenziali". Inoltre, per vedere alcune applicazioni delle macchine differenziali, approfondisco il loro uso per risolvere equazioni differenziali complesse nel capitolo 5, e propongo opportuni esempi per chiarire e meglio esplorare il modello con macchine concrete nel capitolo 6, abbozzando anche alcune proposte didattiche nella sezione 6.3.

Finalmente, nel capitolo 7 ci sono le conclusioni. In esse spiego in che senso l'equilibrio tra macchine, algebra e geometria oltre Descartes possa essere considerato una "estensione conservativa" del programma cartesiano. Inoltre mi soffermo su alcune riflessioni fondazionali circa la possibilità di evitare oggetti non finiti per trattare parte dell'analisi infinitesimale (sia da un punto di vista cognitivo che computazionale). Inoltre, anche se costituisce un tentativo molto naif, concludo proponendo una possibile impostazione dell'esattezza non come un problema meta-matematico ma come uno algoritmico.

Résumé

Préface

Dans *La Géométrie* de 1637, Descartes a trouvé un "équilibre" entre constructions géométriques et manipulation symbolique au moyen de l'introduction d'opportunes machines idéales. En particulier, les instruments de Descartes étaient l'algèbre polynomiale (analyse) et une classe de constructions diagrammatiques (synthèse). Cette approche implique une classification des courbes, suivant laquelle les courbes algébriques peuvent être considérées comme "purement géométriques". Cette limite a été dépassée à l'aide d'une méthode générale par Newton et Leibniz, en introduisant l'infini dans la partie analytique, tandis que la perspective synthétique a graduellement et de plus en plus perdu de son importance par rapport à la perspective analytique (la géométrie devient un moyen de visualisation et cesse d'être un moyen de construction).

L'approche fondationnelle de Descartes (analyse par éléments finis et synthèse par constructions diagrammatiques) a été tout de même étendue au-delà des limites des courbes algébriques, bien qu'en deux périodes distinctes. Vers la fin du XVII siècle la partie synthétique a été étendue avec le "mouvement tractionnel" (construction de courbes transcendantes à l'aide de machines idéalisées) et vers le début du XX siècle la partie analytique a été étendue avec l'"algèbre différentielle" (de nos jours considérée comme une branche de l'algèbre computationnelle).

L'objectif de cette thèse est de prouver comment il est possible d'obtenir un nouvel équilibre entre ces extensions (synthétique et analytique) des instruments cartésiens, un équilibre dépassant la limite des courbes algébriques et permettant de traiter une classe de problèmes transcendants. En d'autres termes, le but est de mettre en exergue comment une nouvelle convergence de machines, algèbre et géométrie est possible, permettant une fondation d'une partie de l'analyse infinitésimale sans exigence conceptuelle de l'infini.

Ce travail se caractérise par l'attention qui est portée sur le rôle constructif de la géométrie (en tant qu'idéalisation du comportement de machines opportunes) à des fins fondationnelles. Cette approche, suite à la "dé-géométrisation" des mathématiques, se détache fortement du courant principal des discussions sur les mathématiques, notamment du point de vue fondationnel. Toutefois, même si aujourd'hui cette question est tombée dans l'oubli, le problème de définir des critères de constructions appropriés, très débattu à l'âge classique, a eu de profondes influences sur la façon dont les objets et les méthodes mathématiques de l'époque ont été définis. D'après la définition de Bos [2001], ce sont là les "problèmes d'exactitude" de la géométrie.

Ces problèmes d'exactitude ont trait aux interprétations philosophiques et psychologiques, c'est pourquoi ils sont normalement considérés comme externes aux mathématiques. Toutefois, même si je ne vais pas apporter de réponse exhaustive, dans mes conclusions je propose une approche algorithmique (très primitive) pour cerner ces problèmes, que j'espère pouvoir approfondir dans des travaux à venir.

Depuis la perspective des sciences cognitives, cette approche par rapport à l'analyse infinitésimale ne demande pas l'infini et, grâce aux machines idéalisées, peut être conçue au travers d'opportunes "métaphores fondatrices" (selon la terminologie de Lakoff and Núñez [2000]). Ce caractère concret peut avoir des retombées utiles dans la didactique des mathématiques, grâce à l'usage d'artefacts tant physiques que numériques (cette partie ne sera abordée que de façon marginale).

Exactitude des constructions

Est-il possible d'établir des critères de constructions géométriques pour certaines classes de courbes en plus que de constructions algébriques ? Et puis, est-ce que ces critères peuvent étendre d'une façon ou d'une autre l'intuition classique de la géométrie sans besoin d'introduire l'infini dans la partie analytique ?

Pour poser ces questions, il faut revenir en arrière vers la seconde moitié du XVII siècle, lorsque *La Géométrie* de Descartes était influente, l'algèbre avait en général été acceptée comme instrument analytique pour des problèmes géométriques et les courbes "géométriques" étaient parvenues à une légitimation ontologique largement partagée. Le travail de Descartes avait suggéré des critères susceptibles de distinguer entre courbes "géométriques" (intuitives et abordables analytiquement) et courbes "mécaniques" (toutes les autres) : en termes modernes, cela correspond à la distinction entre courbes algébriques et transcendantes. Avant, l'algèbre et la géométrie, bien que profondément reliées entre elles, jouaient des rôles nettement distingués : l'algèbre était simplement une "méthode de manipulation symbolique" pour aider à répondre à des questions concernant des entités géométriques ou arithmétiques, dont l'existence était (théoriquement) légitimée de manière indépendante, par exemple à l'aide de constructions géométriques.

L'emploi de courbes non algébriques a soulevé le problème de la "légitimation des courbes transcendantes" [2] avec certains instruments géométriques qui étendent les instruments admis par Descartes : par exemple avec des "constructions tractionnelles".

Voilà le premier point sur lequel s'attarder : étant donné l'échec de l'algèbre polynomiale dans le traitement de problèmes transcendants, Leibniz et Newton ont développé de nouvelles méthodes analytiques, en introduisant non seulement des entités finies mais également des entités infinies ou infinitésimales, en contraste avec la perspective finitiste originale de l'algèbre de Descartes. D'ailleurs, le mouvement tractionnel, tout en ne respectant pas le paradigme cartésien, peut être considéré comme une extension des instruments géométriques de Descartes, bien que d'une façon qui demande à être précisée.

2. Cf. Bos [1988].

L'analyse des fondements du mouvement tractionnel a duré une soixantaine d'années mais, contrairement à ce qu'il s'est passé pour la géométrie cartésienne, ces recherches sont terminées sans aboutir à une définition de critères de constructions amplement acceptés. Ce manque a sans doute été exacerbé par le changement de paradigme dominant, des constructions géométriques aux manipulations symboliques. En effet, la possibilité de représenter une courbe par une simple formule (bien qu'en évoquant des entités non finies) convenait bien plus (pour la manipulation) que la représentation par des procédures géométriques. Peu à peu, la légitimation ontologique des courbes algébriques par des constructions perdait donc de plus en plus d'importance, parvenant au résultat pratique selon lequel un objet mathématique ne peut être considéré comme complètement défini que par une formule. Ce changement de paradigme, auparavant adopté simplement dans la pratique, a atteint vers la seconde moitié du XIX siècle son fondement dans l'"arithmétisation de l'analyse". Cela a marqué l'achèvement du passage d'une perspective de fondation "géométrique" des mathématiques (les objets sont construits à partir de constructions géométriques et le symbolisme s'avère utile afin de les analyser ou de les caractériser) à une fondation "arithmétique" (les objets sont obtenus à partir d'ensembles de nombres réels et la géométrie s'avère utile afin de les visualiser).

En résumé, l'héritage de Descartes a historiquement favorisé deux changements fondamentaux :

- en analyse, le passage des instruments finis à des instruments infinis ;
- le passage d'une fondation géométrique à une fondation arithmétique.

Si le premier point était considéré comme nécessaire pour maîtriser les problèmes transcendants, le second a été essentiel pour donner de la rigueur à une fondation intuitive et pas précise de la géométrie ainsi que pour généraliser et étendre les méthodes géométriques (qui demeuraient inadéquates pour l'étude des nouvelles courbes).

À la lumière de ces observations, les questions du début de ce paragraphe peuvent être mieux reformulées : est-il possible de définir de nouveaux critères de constructions pour étendre d'une manière conservative (géométrique et finie) la géométrie cartésienne jusqu'à la géométrie différentielle ? Et concernant les instruments analytiques nécessaires pour ces constructions, est-il possible d'éviter des entités non finies et d'utiliser une extension de l'algèbre polynomiale ? Cette thèse souhaite répondre positivement à ces deux questions.

D'après Bos, le problème de la définition des critères des constructions géométriques est appelé "problème de l'exactitude" : ce problème a été fondamental à l'âge classique mais il est tombé dans l'oubli le plus complet au moment où la communauté de mathématiciens a diffusément accepté les formules comme la meilleure représentation d'objets mathématiques. En particulier, dans cette thèse ce que je propose c'est des critères de constructions fondés sur une définition spécifique de certaines machines obtenues en assemblant des composants opportuns (ces machines sont une réinterprétation des instruments du "mouvement tractionnel").

De plus, tout comme les règles de Descartes ont produit un équilibre entre machines (systèmes articulés), instruments analytiques (algèbre) et courbes géométriques (courbes algébriques), dans cette thèse je propose un nouvel équilibre entre une nouvelle classe de machines (que j'ai appelées "machines différentielles"), l'extension des instruments analytiques (algèbre différentielle) et une classe de fonctions (solutions d'équations différentielles algébriques), où chacun de ces domaines peut être considéré comme une extension conservative des domaines cartésiens.

Schéma de la thèse

Dans le chapitre 2 j'introduis quelques considérations historiques et philosophiques concernant la définition des critères de construction géométrique : cette partie ne comporte pas de contenus originaux. En particulier, après un premier regard sur les constructions d'Euclide et sur certaines extensions de l'Antiquité, je me focalise sur les règles cartésiennes et sur les tentatives ultérieures par le "mouvement tractionnel" de les dépasser.

C'est dans les chapitres centraux qui constituent le coeur de ma thèse que j'introduis l'approche fondée sur les machines, formalisée au moyen de l'"approche comportementale" de la modélisation mathématique (section 3.1). Et encore, c'est en utilisant les machines introduites dans la partie historique que j'introduis les instruments adoptés comme une opportune abstraction des instruments tractionnels. Et c'est justement par cette approche que j'en arrive à la définition analytique des limites de cette réinterprétation du mouvement tractionnel. On verra comment ce modèle peut être considéré comme une extension des géométries d'Euclide et de Descartes. De plus, par l'approche comportementale, je mets en évidence le rapport profond existant entre machines, constructions géométriques et théories de manipulation de symboles, de façon à pouvoir répondre aux questions concernant l'égalité entre machines. Contrairement à l'approche de Descartes, ces machines n'ont à présent pas de justification philosophique bien définie eu égard à la limitation des composantes introduites ; j'espère donc qu'à l'avenir le paradigme des constructions géométriques effectuées à l'aide de machines appropriées pourra être étendu au-delà des limites des machines faisant l'objet de cette thèse, tout en continuant de satisfaire les conditions requises quant à une corrélation profonde entre machines idéalisées et instruments analytiques finis.

Plus précisément, dans cet ouvrage je trace un parallélisme entre machines, algèbre et géométrie dans une extension à trois étapes : je commence par la géométrie du plan d'Euclide dans la section 3.2, je me penche sur la géométrie cartésienne dans la section 3.3, pour l'étendre enfin à des objets différentiels dans le chapitre 4. J'introduis des modèles appropriés de machines en vue de définir ces trois géométries et je les appelle respectivement machines "classiques", machines "algébriques" et machines "différentielles". De plus, afin de voir certaines applications des machines différentielles, j'approfondis leur usage pour résoudre des équations différentielles complexes dans le chapitre 5, et je propose des exemples appropriés, aptes à clarifier et à mieux explorer le modèle avec des machines concrètes dans le chapitre 6, en ébauchant même quelques propositions didactiques dans la section 6.3.

Enfin, le chapitre 7 est consacré aux conclusions. Là, j'explique comment l'équilibre entre machines, algèbre et géométrie au-delà de Descartes peut être considéré comme une "extension conservative" du programme cartésien. Et encore, je m'attarde sur quelques réflexions fondationnelles à propos de la possibilité d'éviter des objets non finis pour traiter une partie de l'analyse infinitésimale (aussi bien d'un point de vue cognitif que computationnel). En outre, bien que cela puisse paraître une tentative fort naïve, je termine en proposant une possible approche de l'exactitude non pas comme un problème métamathématique mais comme un problème algorithmique.

Introduction Historique

Constructions géométriques dans l'Antiquité

La géométrie grecque classique a vu que certains problèmes, comme doubler un cube, la trisection d'un angle, la quadrature d'un cercle et construire certains polygones réguliers, ne semblait pas possible utilisant un compas et une règle seuls. Cependant, les problèmes, quant à eux, sont solubles, et les grecques savaient comment les résoudre, sans la contrainte de travailler uniquement avec des règles et compas.

Archimède savait que deux signes sur une règle était suffisantes pour effectuer la trisection d'un angle et la duplication du cube (les constructions *neusis*). La littérature grecque classique amène divers autres exemples d'outils permettant des constructions autrement impossible. La majorité d'entre eux, comme la règle avec signes, a permis la construction de racines cubiques. D'autres étaient plus efficace et plus poussés ; la spirale d'Archimède permet la construction de n sections de n'importe quel angle et donc la construction de n'importe quel polygone ; et la quadratique de Hippias permet la quadrature du cercle.

Ainsi, les limites de la géométrie d'Euclide peuvent être contournées avec d'autres outils. Cependant, quand peuvent être accepté les nouvelles méthodes en géométrie ? Et plus généralement, que peut une construction géométrique être considéré légitime ? Ces questions sont désormais considérées comme avoir pour sujet "l'exactitude géométrique". Avec un saut chronologique, le besoin d'un nouveau canon pour constructions est devenu plus fort quand les outils algébriques ont fourni une différente façon de résoudre les problèmes, une méthode analytique demandant une légitimité dans un paradigme géométrique.

L'âge classique

Parmi les historiens de les mathématiques, les ouvres de Henk Bos sur la géométrie analytique de Descartes et le calcul différentiel de Leibniz sont le point de départ pour tous ceux qui ont essayé de comprendre les développements conceptuels des mathématiques dans le période cruciale entre la Renaissance et la période des Lumières. Je suggère l'interprétation de Bos comme le point de départ de cette thèse.

La particularité de Bos [2001] est la perspective. Souvent, les problèmes concernant le changement des concepts géométriques des constructions, typique du début de l'âge classique, sont assimilés à la perspective future de la géométrie analytique. Au contraire, Bos se concentre sur un ensemble de problèmes qui avaient une signification particulière et importante pour les mathématiciens de cette période de révolutions mathématiques. Ces mathématiciens étaient troublés par des questions telles que : quelle construction peut être considérée comme légitime ? Laquelle est moins complexe ? Et concernant les entités mathématiques, quand est-ce que celles-ci peuvent être considérées comme achevées ? Quelle est la signification pour un problème d'être résolu and ses solutions d'être découverte ?

Dans ce contexte les concepts d'exactitude, certitude, précision étaient fréquemment utilisés, et ont fait particulièrement l'objet de discussions à cause des problèmes conceptuels et méthodologiques en résultante de l'algèbre comme un outil analytique pour les solutions géométriques : comment les solutions algébriques peuvent être considérées comme légitimes, à l'intérieur d'un paradigme universellement accepté ? L'interprétation géométrique des nouvelles techniques algébriques posa de nombreux problèmes, à savoir que Viète et Descartes passèrent de nombreux moments sur ces problèmes. Tous ces sujets ont disparu au 18ème siècle à cause de l'affirmation générale des procédures symboliques comme autonomes et indépendantes de la géométrie. Même si historiquement oubliés, la réflexion sur certains sujets est indispensable afin de comprendre l'évolution des mathématiques au cours de cette période révolutionnaire.

Dans le livre de Bos, le rôle principal est joué par Descartes, où les questionnements sur les constructions sont un sujet principal. Depuis cette perspective, différente des futures interprétations sur la réduction des études des courbes aux leurs équations, Bos soutient le fait que pour Descartes, l'équation est seulement un élément de la définition d'une courbe (analyse), car, si l'on souhaite pratiquer la géométrique, il est nécessaire de produire la construction géométrique (synthèse). Ainsi, même en introduisant le pouvoir de l'algèbre dans la géométrie, cela ne se réfère qu'à la partie analytique de la chose, tandis que la partie synthétique reste nécessaire. Suivant ce point de vue, Descartes n'a pas dévié de la vision ancienne de considérer une solution connue uniquement si provenant d'une construction avec des éléments et outils géométriques.

Mon objectif est d'adhérer aux anciens paradigmes de constructions géométriques et les étendre vers des objets différentiels suivant une ligne de conduite différente de celle devenue historiquement dominante (l'introduction d'entités infinies dans l'analyse).

Au delà des outils cartésiens

Malgré quelques exceptions (surtout au Royaume-Uni), peu après la révolution géométrique de Descartes, il fut soudainement accepté la partie analytique de son programme (une introduction bien cadrée de l'algèbre dans la géométrie), pendant que l'intérêt porté aux constructions géométriques demeura vivant seulement pour justifier les courbes transcendantes (aucunement soluble avec l'algèbre polynomiale). En particulier, même si quelques courbes non-algébriques étaient connues par Descartes (exemples de courbes mécaniques étaient les quadratiques, la spirale d'Archimède, le cycloïde), c'était le "problème inverse de la tangente" que a généré une famille générale de courbes pour lesquelles les outils de Descartes n'étaient pas assez avancés. Ainsi, il était temps de dépasser les critères cartésiens à travers une extension des "machines à tracer" afin de perpétuer le paradigme des constructions géométriques.

Pendant que le raisonnement de Descartes était orienté vers un ensemble d'objets constructibles, le nouvel essai était orienté d'avantage vers la liberté mathématique. Dans cette vision, le rôle de Leibniz fût très important pour son opposition aux restrictions cartésiennes. Derrière l'effondrement du paradigme géométrique devant le pouvoir de la partie analytique, il y a le passage des entités “finies” aux entités “infinies”, inatteignable avec des instruments de constructions géométriques. Suivant cette thèse j’exhume le paradigme des constructions géométriques afin d’éviter l’utilisation d’entité et procédures plus ou moins impliquée dans le rappel de l’infinité, pour étendre la balance entre machines, géométrie et algèbre.

Comme déterminé dans les reconstructions de Bos [1988, 1989], Tournès [2007, 2009], le problème d’étendre la géométrie au delà des limites cartésiennes fut dominant dans les année 1650 – 1750. Si le problème direct de la tangente est présent depuis l’Antiquité, ce n’était seulement que pendant la seconde moitié du 17ème siècle que le problème inverse de la tangent est apparu. La différence principale entre le problème inverse et direct c’est le rôle de la courbe ; dans le cas direct, elle est donnée, tandis que dans le second cas, la courbe est la solution. Cependant, au delà des outils admis par la géométrie cartésienne, il y a eu l’introduction de certaines machines comme instrument théorique et géométrique afin de tracer telle courbes. Les premières courbes enregistrées fussent construites sous des conditions de tangente grâce à la traction d’une ficelle attachée à un poids soumise à friction, d’où l’appellation “mouvement tractionnel”. Pendant cette période, les mathématiciens tel que Huygens commencèrent à prendre en compte des instruments, tel que le guidon de vélo, pour guider la tangente d’une courbe (avec des termes de mécanique analytique, ils ont introduit des contraintes “non holonome”) : cela était le signal de l’ascension du mouvement tractionnel. Le mouvement tractionnel a donné la possibilité de construire courbes en imposant des conditions tangentiels, généralisant (d’une façon non-cartésienne) l’idée d’objets géométriques avec des nouveaux objets, non seulement les courbes algébriques, mais également quelques une transcendante (considéré comme solution aux équations différentielles). Durant cette période, le développement d’idées géométriques correspondait souvent à la pratique (ou du moins la conception) de machines mécaniques capable de représenter les propriétés théoriques et ainsi tracer les courbes.

Pendant que les questions de l’exactitude dans les constructions géométriques étaient si importantes dans l’âge classique, celles-ci ont disparues pendant le 18ème siècle à cause de l’affirmation générale des procédures symboliques, considérées dans le futur comme autonomes et indépendantes de la géométrie. Cependant, à la différence de ce qu’il c’est produit pour les courbes algébriques, le mouvement tractionnel n’a pas atteint un critère de constructions largement partagé. De plus, à cause du changement au sein du paradigme, les idées géométriques-mécaniques derrières les machines à mouvement tractionnel furent oubliées durant des siècles, même à des fins pratiques, et furent réinventées indépendamment dans la seconde moitié 19ème siècle, quand elles ont été utilisées pour la construction d’instruments d’intégration grapho-mecaniques (intégraphes) afin de résoudre des problèmes non solubles symboliquement.

Machines, d’Euclide à Descartes

L’objectif de cette thèse est de suggérer une définition précise et convenable des machines utilisées pour les constructions effectuées sous mouvement tractionnel. Pour ce faire, j’ai commencé par une réinterprétation des machines utilisées lors des constructions d’Euclide et Descartes. Avant cela, je dois introduire quelques notions et notes essentielles afin de les analyser correctement, l' "approche comportementale" de la modélisation mathématique.

La modélisation mathématique : une approche comportementale

La modélisation est une activité cognitive suivant laquelle un individu pense et modélise afin de déterminer le comportement des objets et des appareils. Il y existe plusieurs descriptions pour les objets et les appareils : je suis intéressé par l'utilisation du langage mathématique pour crée des modèles de machines dans un plan, mais avant je nécessite des concepts de base permettant de modéliser n'importe quel phénomène.

Selon [Polderman and Willems, 1998, pp. 1–8], un modèle mathématique exprime le fait que certaines choses peuvent arriver, qu'elles sont possibles, tandis que d'autres ne le sont pas, déclarées impossibles. Ainsi, Kepler dis que l'orbite planétaire qui ne répond pas à ses trois lois est impossible. Dans cette perspective, une modélisation mathématique agît en tant que loi d'exclusion.

Ces idées peuvent être formulées en citant qu'une modélisation mathématique sélectionne un sous-ensemble parmi une infinité de possibilités. Ce sous-ensemble concentre le nombre d'évènements accepté par le modèle, qu'il déclare possible. Je me réfère au sous-ensemble comme le "comportement" du modèle mathématique. La grande différence entre le comportement et l'approche des données d'entrée/de sortie est que dans l'approche comportementale les variables sont définies sans besoin de distinction *a priori* entre les données d'entrée et les données de sortie. L'avantage de ne pas nécessiter cette distinction vient du fait que, considérant l'interconnexion entre les éléments (appelée "feedback"), généralement il n'est pas possible de comprendre facilement quelles variables sont les données d'entrée et de sortie.

Souvent, les modèles mathématiques sont vus comme des équations, car les équations peuvent être considérées comme des lois excluant l'évènement de certains résultats, des combinaisons de variables pour lesquelles l'équation n'est pas satisfaite. De cette manière, les équations définissent un comportement. Je parle donc d'équations comportementales quand les équations mathématiques ont pour intention de modéliser un phénomène. Il est important de souligner que les "équations comportementales" fournissent de manière efficace, mais également très aléatoire, une façon de spécifier un comportement. Différentes équations peuvent définir le même modèle mathématique, donc un travail important est de déterminer si différentes représentations définissent un même comportement.

Désormais, je peux introduire dans le langage une distinction concernant le type de variable. On peut penser aux variables comme variable "manifeste" (ou "externe") : elles sont des attributs sur lesquels on porte l'attention. Cependant, afin d'aboutir à une modélisation mathématique d'un phénomène, on doit quelques fois considérer des variables auxiliaires. Je me réfère à elles sous le préfixe de variable "latente" (ou "interne"). Celles-ci peuvent être introduites pour exprimer de manière précise et convenable les lois gouvernant un modèle mathématique.

La structure principale de ce langage de modélisation est donné par trois piliers : le comportement, les équations comportementales et les variables (manifestes et latentes). Concernant la notation et l'écriture, le comportement est noté '$\mathfrak{B}$', et se réfère seulement aux variables externes. Le comportement dit "total" (à savoir prenant en compte les variables externes et internes) est noté '$\mathfrak{B}_f$'.

Pour tout le reste de la thèse, le phénomène correspond à l'analyse et l'inspection de machines dans le plan.

Les machines classiques

Dans cette partie, je souhaite donner à la théorie de la géométrie du plan d'Euclide une base qui dans les parties suivant est étendue vers le géométrie cartésienne et au-delà. Même si la théorie d'Euclide était basée sur des cercles et des lignes, mon point de départ s'effectue avec des instruments (règle et compas). Ainsi, afin de choisir les éléments, j'analyse les opérations caractérisant l'usage pratique de règles et compas pour tracer des segments et des cercles. Puis, je propose une solution légèrement différente, qui permet d'obtenir les mêmes points constructibles par la géométrie classique. J'appelle cettes machines "les machines classiques".

Pour introduire les machines dites classiques, je dois commencer par les objets primitifs autorisés. En particuliers, j'introduis les éléments suivants :

- des **tiges** infiniment extensibles. Elles sont des règles idéales (elles possèdent un alignement parfaitement droit et une largeur négligeable), et elles sont différentes des lignes droites Euclidiennes car elles ne sont pas des objets statiquement tracés, mais des corps rigides pleins (entités physique avec trois degrés de liberté, deux caractérisant la position d'un point spécifique et le troisième identifiant la pente par rapport à une ligne fixe).
- afin de permettre le mouvement le long d'une tige (par example quand on trace avec un crayon un segment en utilisant une règle), il est possible de mettre des **chariots** sur une tige, chacun utilisant la tige comme binaire. Un chariot a un degrés de liberté une fois qu'il est placé sur une tige (le chariot peut seulement se déplacer vers le haut ou le bas de la tige).

A différence de l'approche classique avec les règles et compas, je ne considère pas les machines classiques comme des outils traçant des diagrammes, mais comme des mécanismes assemblés en mouvement sur le plan.

Sans des courbes tracées, les points définis comme intersections de lignes et de cercles peuvent être construits avec des contraintes mécaniques. Cela peut être accompli considérant le chariot non seulement comme un élément supplémentaire sur la tige (introduisant un nouveau point en mouvement sur la tige), mais comme une chose pouvant contraindre un point introduit préalablement à rester sur une tige (à savoir un chariot peut contraindre un point spécifique à se positionner sur une tige).

En particulier, comme dans le setting d'Euclide, le commencement d'une construction est composé de points sur un plan, et récursivement on construit de plus en plus ces points. J'appelle tels points *fixes sur le plan*, et leur construction s'étend de manière récursive par la construction de points *fixes sur une tige*. Quand cela ne génère pas de confusion, dans "points fixes sur le plan" ou "points fixes sur une tige", j'omet l'adjectif "fixe". Comme je vais le décrire, les points sur les tiges peuvent être construits en marquant sur un tige quelques points qui coincident avec un point du plan quand la tige présente une certaine inclinaison.

J'ai introduit comme objets les tiges, les chariots, les points sur un plan, et les points sur les tiges. Maintenant je dois spécifier comment ils peuvent être utilisés afin de satisfaire les conditions instrumentales des constructions avec règles et compas.

Concernant les tiges, une règle est introduite si elle est contrainte à être articulée à un point sur le plan, ainsi je peux introduire le premièr postulat :

> R1. Une tige r est introduite après avoir été contrainte à pivoter autour d'un point sur le plan P (à travers lequel r doit passer). J'affirme que la tige r est articulée en P.

Je dois préciser quels sont les points qui pouvant être considérés sur une tige donnée. Selon les conditions pour tracer des cercles, on peut marquer un point sur une règle rotative si, pour certaines pentes, ce point rotatif coïncide avec un autre point du plan :

> R2. Sur une tige r articulé dans P, on peut considérer un point A (en gardant la distance PA constante) tel que, pour certaines pentes de la tige, A coïncide avec le point Q sur le plan. Je dis que le point A sur r est superposable sur Q.

La rotation de r force A à bouger le long du cercle. Je dois faire une remarque que, donné une tige r articulée dans P et un autre point sur le plan Q, le point A sur r superposable sur Q n'est pas défini exclusivement : en fait, si Q est différent de P, il y a deux point sur r (en respect symétrique à P) que la rotation peut coïncider avec Q.

Comme introduit, un chariot contraint un point A de rester sur une tige r. Premièrement, le point A est de type différent que les points observés précédemment, car A n'est pas fixé ni sur le plan ni sur la tige. Je ne suis pas intéressé par l'utilisation des ces points libres en général : je veux spécifier comment nous pouvons construire des points fixes sur un plan. Afin de restreindre l'utilisation de points libres, j'ai besoin de certaines définitions.

Je dis que une tige r articulée dans P est une *tige fixe* si, donné un point sur le plan Q distinct de P, on impose le point Q de rester sur r utilisant la contrainte du chariot (à savoir la tige dois passer par un point fixe sur le plan). Je nomme *tiges rotatives* lesquelles n'étant pas fixes.

J'ai déjà affirmé que je voulais distinguer les points généraux (par example contraint par un chariot) et les points constructibles (ceux ayant une position définie, à savoir les points fixes sur le plan). Tout ce dont j'ai besoin c'est de définir les conditions pour obtenir des nouveaux points constructibles. Spécifiquement, on peut obtenir de nouveaux points sur les tiges si ils construits de nouveaux points sur ce plan. Ainsi, je peux définir quelques postulats afin de construire de nouveaux points sur le plan :

> R3. Donné deux tiges fixes r, s non parallèles, A un point contraint sur r avec un chariot. Si A est aussi contraint avec un chariot de rester sur s, alors A est un point fixe sur le plan ;

R4. Donné un point A fixe sur une tige rotative r, et donné une tige fixe s. Si A est aussi contraint avec un chariot de rester sur s, alors A est un point fixe sur le plan ;

R5. Donné les tiges rotative r articulée en P et s articulée en Q (distinct de P), si les points A fixé sur r et B fixé sur s sont contraints à assumer la même position [3], alors A est un point fixe sur le plan [4].

Il est facilement possible d'observer que les points sur le plan atteignables avec des machines classiques sont exactement les points constructibles avec règles et compas (constructions classiques). Similairement au cas des constructions classiques quand il n'y pas d'intersection, dans certaines configurations il n'est pas possible d'imposer certaines contraintes.

Utilisant l'approche comportementale, on peut considérer les machines classiques comme les appareils devant être décrit et les points construits (fixés sur un plan) comme les résultats. Considérant les coordonnées des points construits dans un plan cartésien, $\mathfrak{B}$ est un sous-ensemble fini de $\mathbb{R}^2$. Dans l'approche comportementale deux machines sont équivalentes si leurs comportements sont les mêmes : dans ce cas, les points construits par la première machine coïncident avec les points construits par la seconde.

Donné un point obtenu avec n'importe quelle construction, ses coordonnées peuvent être écrites comme combinaison des quatre opérations arithmétiques et de la racine carrée. Ainsi, en revenant au problème de tester les équivalences de deux points construits (obtenu avec deux machines différentes), il peut apparaître que pour le résoudre, on doit seulement vérifier si les ordonnées et les abscisses des points construits sont égaux. Mais ce n'est pas facilement déductible, car on doit comparer de nombres réels qui, même si égaux, peuvent être représentés de différentes manières : considérez l'exemple qu'on doit comparer si $\sqrt{2}+\sqrt{3}$ et égal à $\sqrt{5+2\sqrt{6}}$. On a besoin d'une méthode générale afin de décider si deux représentations pointent à la même nombre réelle : se faisant, je regarde pour une forme normale pour une représentation, obtenant que les représentations symboliques sont égales si et seulement si les valeurs réelles sont les mêmes. Une "représentation normale" pour les nombres constructibles (ceux pouvant être obtenus avec les constructions avec règles et compas) peuvent être trouvées dans [Bouhineau, 1996, sections 4,5]. Cela signifie qu'il est possible de tester formellement si deux machines classiques sont équivalentes ou non.

Le problème de décider si deux représentations différentes définissent le même objet est généralement appelé "le test d'équivalence". Sa résolubilité n'est pas toujours possible : par exemple, considérant non pas seulement des nombres constructibles mais ceux calculables (un nombre réel est calculable si on peut en calculer une approximation aussi précise que l'on veut), le test d'équivalence n'est pas résoluble algorithmiquement.

3. Afin de contraindre A pour assumer la même position que B, on peut contraindre avec des chariots B à rester sur r et A à rester sur s. Cela impose instrumentalement que A et B assument la même position.

4. Evidemment, A et B coïncide, alors aussi B est un point fixe sur le plan.

Les machines algébriques

Le pouvoir constructif de machines classiques est équivalent à ceux des constructions d'Euclide. En particulier, je me focalise sur le rôle de points fixés sur le plan, qui, étant équivalents aux points constructibles avec la règle et le compa, je nomme tout de même "points constructibles". Avec les outils d'Euclide, il est impossible de résoudre les trois fameux problèmes de la géométrie classique. Cependant, avec des outils mieux élaborés, au moins depuis Archimède, les mathématiciens concevaient des constructions géométriques résolvant ces problèmes. Cette quête pour des extensions (le problème de l' "exactitude géométrique") me fait introduire dans cette partie, un nouveau groupe de machines étendant les machines classiques. En particulier, considérant le canon de Descartes sur l'acceptation des constructions *neusis* et non celles des "courbes mécaniques" (comme le quadratix), je commence à revoir l'extension nécessaire des machines classiques afin d'introduire les construction neusis. J'ai pensé à l'introduction de ces machines comme un canon pour les machines utilisées dans la géométrie de Descartes (tel que son compas proportionnel).

Une fois étendue le principe constructif derrière les machines classiques, je vais explorer le potentiel des nouvelles machines se focalisant sur leur équivalents algébriques. En particulier, au lieu de considérer des points constructibles, je suis intéressé par les positions que les points en mouvement peuvent atteindre. Ainsi, le rôle des postulats est différent. Dans les machines classiques, je ne suis pas intéressé par la position de chaque point spécifique, mais seulement par la position des points fixes sur le plan (ainsi les postulats aident à savoir quand un point spécifique est fixe sur un plan). Dans les machines algébriques, au contraire, je suis intéressé par la position de toutes les sortes de point.

Analytiquement, je caractérise le comportement des machines algébriques : considérant comme variables certaines coordonnées de ces points, le comportement correspond à un ensemble semi-algébrique réel de dimension n.

Afin d'étendre les constructions au delà de celles autorisées par les machines classiques, je peux modifier R1, demandant à ce que la jonction soit non seulement dans un point fixe du plan, mais dans n'importe quel point spécifique [5]. Ainsi, je remplace R1 avec

> R1′. Une tige r est introduite après ayant été contrainte de pivoter autour d'un point *spécifique* P (à travers lequel r doit passer). Je dis que la tige r est articulée en P.

Ainsi, je peux distinguer entre les tiges *non-flottantes* (articulées autour des points fixes sur le plan, à savoir les tiges fixes et rotatives autorisées dans les machines classiques) et les tiges *flottantes* (articulées autour de n'importe quel type d'autre point). Il est à rappeler qu'afin de définir des points constructibles (postulats R3, R4, R5), je dois me restreindre aux tiges non-flottantes, mais cela

5. Il faut rappeler que dans les machines j'ai introduit des types de point différents : des points fixes sur un plan, des points fixe sur une tige, des points contraints par un chariot à rester sur la tige. Chaque point de ce types est introduit avec une procédure bien définie, il est possible de s'y référer exactement, ainsi je les appelle points "spécifiques". Les points non-spécifiques sont des points génériques, à savoir des points non-définis par une procédure mais considérés comme partie générale d'une tige ou simplement sur le plan. De ma part, les points génériques ne peuvent être introduits dans les constructions.

était déjà spécifié car dans ces postulats des machines classiques je considère explicitement les tiges fixes et rotatives.

Concernant R2, afin de faire face aux constructions neusis, je dois généraliser cela aux tiges flottantes :

> R2′. Donné deux points P, Q fixes sur un plan et r la tige articulée dans R, il est possible de considérer un point S fixe sur r tel que le segment RS ait la même longueur que PQ.

Notons qu'il est possible de transférer des distances seulement entre des points fixes sur un plan (et non entre n'importe quels points spécifiques).

Historiquement, le canon géométrique le plus accepté étendant les constructions d'Euclide était le canon cartésien. Si Euclide avait basé ses settings sur des lignes et des cercles, les objets de Descartes étaient des courbes (algébriques). Dans le setting mécanique des machines algébriques, l'objet principal n'est pas des courbes, mais des machines. La différence avec le canon cartésien est subtile ; les machines avaient un rôle important dans *La Géométrie*, mais elles ne servaient qu'à tracer des courbes (ainsi, après le traçage, les machines n'étaient plus nécessaires). Au contraire, dans mon approche les objets principaux sont des machines, et les courbes (encore défini comme *loci* de points en mouvement des machines) ne sont pas utilisé pour des constructions. Cette distinction est encore plus visible quand on construit de manière récursive de nouveaux objets : dans la perspective cartésienne, on a besoin des machines afin de tracer des courbes, et ces courbes sont utilisées pour definir, avec intersection quand elles se déplacent, des nouvelles courbes [6], ainsi on a besoin de machines au départ, et par la suite des courbes purement géométriques. Au contraire, de ma perspective purement instrumentale, je n'ai pas besoin d'introduire deux outils générateur (machines et courbes) : afin de tracer des courbes de plus en plus complexes, je considère simplement des machines plus complexes, ayant comme sous-éléments des machines plus simples. Les avantages de cette perspective sont qu'on peut éviter toutes les référence au rôle constructif des courbes (chaque objet est défini par des machines qui doivent satisfaire des postulats de constructions spécifiques), et aussi, que même avec des machines en mouvement sur un plan, on peut générer espaces de dimension n (pour n'importe quelle valeur positive d'un nombre entier n). Au contraire, dans l'approche par les courbes on nécessite de se référer aux machines comme source primaire afin de tracer les courbes, et doit se restreindre à des objets de dimension au plus 2 (parce que sous-ensemble du plan).

En assumant l'approche par machines, on peut observer la différence entre les machines classiques et algébriques. En particulier les objets construits selon les postulats des machines classique sont un ensemble fini de points constructibles (ainsi le comportement est un sous-ensemble fini de $\mathbb{R}^2$), tandis que les objets des machines algébriques ne sont pas seulement des points fixe sur un plan, mais en général, des points spécifiques satisfaisant certaines contraintes : la position mutuel de tels points définie les configurations admissibles des machines construites. Ainsi, les machines algébriques définissent un ensemble (généralement infini) de $\mathbb{R}^n$.

6. Cf. [Panza, 2011, pp. 78–89].

Considérant une machine algébrique $\mathcal{M}$, à savoir un assemblage de tiges et chariots donné des points $P_0, \ldots, P_n$ (fixes sur un plan) selon R1′, R2′ [7]. Afin de caractériser analytiquement ce qui peut être obtenu avec $\mathcal{M}$, j'introduis un système de coordonnées cartésiennes. Différemment des machines classiques, qui construisent des points fixes sur un plan (à savoir couple statique de nombres), $\mathcal{M}$ peut contraindre en général des points spécifiques de bouger selon certaines trajectoires en relation avec la position d'autres points : ainsi, pour decrire analytiquement $\mathcal{M}$, il n'est pas assez de donner un vecteur fini de nombres. Les objets définissant $\mathcal{M}$ sont les points spécifiques et les tiges, mais on peut noter que la configuration de $\mathcal{M}$ dépend seulement des points spécifiques.

Afin d'exprimer la configuration de la machine M dans le comportement total $\mathfrak{B}_f$ (définissant k points spécifiques) je peux utiliser le vecteur $(a_1, \ldots, a_k)$ (avec $a_i \in \mathbb{R}^2$) tel que a_i est le couple des coordonnées de l'$i-$ème point spécifique. Si on écrit $a_i = (x_{2i-1}, x_{2i})$, une configuration est donnée par le vecteur $(x_1, \ldots, x_{2k})$, ainsi $\mathfrak{B}_f$ est un sous-espace de $\mathbb{R}^{2k}$.

Maintenant je peux introduire l'idée de comportement (restreint, non total). J'utilise les termes *manifest* et *latent* pour respectivement noter les variables que je considérer ou pas dans le comportement. Il est à noter qu'il y a un léger abus de notation : je me restreint à quelques variables à cause de mon choix (par exemple, en considérant l'orbite d'un point), non parce que leur rôle est différent du rôle des variables latents.

Des $2k-$variables réelles on peut considérer seulement certains : appelé $J = \{1, \ldots, 2k\}$, pour chaque ensemble de "variables manifests" $I \subset J$, je défini $\mathfrak{B}_I$ (ou, quand il n'y a pas de confusion, juste $\mathfrak{B}$) la restriction de $\mathfrak{B}_f$ en les variables avec index dans I. La définition du comportement d'une machine est importante non seulement pour la caractériser analytiquement, mais également pour définir l'équivalence entre les machines.

Un autre concept concernant le comportement d'une machine est la "configuration accessible". Donné une configuration initiale M_0, l'ensemble de configurations (restreint aux variables avec index dans I) accessible de M_0 est un sous-ensemble de $\mathfrak{B}_I$, mais ce n'est pas nécessairement le même : en effet, pour atteindre une configuration physiquement, il doit y avoir un passage connectant la configuration initiale à la configuration finale. En particulier l'ensemble des configurations accessibles (ou "espace accessible") est la partie connectée de $\mathfrak{B}_I$ contenant M_0.

Revenant sur la distinction entre l'approche basée sur la machine et celle basée sur la courbe, dans l'approche par courbe, je considère l'orbite atteignable par un point spécifique de la machine (en prenant comme variables manifestes l'abscisse et l'ordonnée du point), tandis que en général, on peut considérer comme variables les positions relatives des points, se déplaçant de la courbe plane à une variété de dimension arbitraire. En particulier, je montre quelques caractérisation analytique : tout comportement total $\mathfrak{B}_f$ est une variété algébrique, ainsi, considérant la restriction aux éléments dans I, $\mathfrak{B}_I$ est un ensemble semi-algébrique réel. Il y a également l'inverse : donné un ensemble semi-algébrique réel S, on peut construire une machine tel que son comportement coïncide avec S. De plus, considérant l'espace accessible d'une configura-

7. Il est à noter que dans R2′ je rappelle les points constructibles, ainsi, les machines classiques ont besoin des postulats pour les machines classiques.

tion initiale M_0, il est un élément connecté d'un ensemble algébrique, à savoir un ensemble semi-algébrique réel.

Les machines classiques sont analytiquement définies par les nombres constructibles, et il est possible d'effectuer des opérations arithmétiques entre les longueurs de segments statiques [8]. Maintenant je dois étendre ces opérations entre des longueurs "dynamiquement changeantes". Plus précisément, afin de faire de telles opérations avec des machines algébriques, je dois utiliser des *variables* : une variable (donné un système de coordonnées cartésiens), est l'abscisse ou l'ordonnée d'un point spécifique de la machine. Je montre comment on peut faire les opérations arithmétiques avec de telles variables : donnée deux variables (les coordonnées d'un point seul de deux éléments de deux points différents), il est possible de contraindre un point spécifique d'avoir une de ses coordonnées équivalent au résultat de n'importe quelle opération arithmétique entre le deux variables.

Donné deux variables, a, b (qui peuvent être considéré comme les points $(a, 0)$ et $(b, 0)$ dans un plan cartésien), on peut construire avec des machines algébriques $a + b$ et $a \cdot b$ (à savoir les points $(a + b, 0)$ et $(a \cdot b, 0)$). De plus, il est à noter qu'il est possible de contraindre une variable à être nul simplement en contraignant le point $(a, 0)$ à coïncider avec l'origine $(0, 0)$. Cela signifie qu'il est possible d'établir des équations polynomiales avec des machines algébriques.

La géométrie algébrique sur un anneau $\mathbb{K}$ étudie les ensembles algébriques dans $\mathbb{K}^n$, à savoir les ensembles de forme $\{x \in \mathbb{K}^n | P_1(x) = \ldots = P_k(x) = 0\}$ où P_i sont des polynômes avec des coefficients dans $\mathbb{K}$. Pour les machines algébriques, j'utilise des variables réels, et ainsi, j'ai restreint l'attention vers les sous-ensembles de $\mathbb{R}^n$, comme visible dans Bochnak et al. [1998], Basu et al. [2006]. Une des difficultés quand on étudie les ensembles algébriques réels est que $\mathbb{R}$ n'est pas algébriquement fermé, par example le nombre de zéros (compté avec multiplicité) d'un polynôme réel peut être inférieur à son degré. Et même, les ensembles algébriques réels sont fermés sous des unions et intersections finies, mais ne sont pas fermés sous le complémentaire. De plus, en général, des images d'ensembles algébriques réels obtenues par des fonctions polynômiales et leurs éléments connectés ne sont pas des ensembles algébriques. Par exemple, l'équation $xy - 1 = 0$ définie une hyperbole dans $\mathbb{R}^2$ consistant en deux parties connectées, et l'image sous un projection sur la coordonnée x est donné par l'union des deux intervalles disjoints de valeurs négatives et positives (seulement la valeur nulle n'appartient pas à la projection). Les éléments projetés sont donnés par des équations et inégalités, et en général, ils ne peuvent pas être donnés par des équations seules. En particulier, les ensembles définis par les combinaisons booléennes d'égalités et inégalité sont appelé "ensembles semi-algébriques", et ce type d'ensemble est stable sous la projection (le théorème de Tarski-Seidenberg). De plus, un ensemble semi-algébrique possède seulement un nombre fini des parties connectées, et chaque partie est également semi-

8. Je considère les opérations binaires à partir de couples de segments à un seul segment de longueur. L'arithmétique avec des objets géométriques fût introduite depuis Euclide, mais la multiplication était donnée par la construction de rectangles bidimensionnels. Pour une multiplication *interne*, nous avons besoin de l'introduction d'une unité de longueur de taille arbitraire : ces constructions internes avec segments sont visibles au début de *La Géométrie* de Descartes.

algébrique (théorème de Łojasiewicz).

Une méthode constructive utile pour prouver les deux théorèmes est la "décomposition cylindrique algébrique" (abrévié DCA) introduit en Collins [1975] avec un algorithme. Donné un ensemble S de polynôme dans $\mathbb{R}^n$, une DCA est la décomposition de $\mathbb{R}^n$ dans des ensembles semi-algébriques appelés "cellules", sur lesquelles chaque polynôme a un signe constant $+, -$ ou 0.

Le comportement d'une machine algébrique est défini par les coordonnées de ses points spécifiques. En particulier, ces points peuvent être :

1. des points donnés sur un plan ("point donnés") ;
2. des points constructibles sur un plan ("points constructibles") ;
3. des points fixes sur une tige (par R2$'$, utilisant les points constructibles) ;
4. des points libres sur une tige (introduit avec un chariot).

Si une machine contient n points spécifiques, le comportement total est un ensemble algébrique réel sur $\mathbb{R}^{2n}$.

De plus, soit S un ensemble algébrique dans $\mathbb{R}^n$. Je donne une méthode générale afin de construire une machine ayant comportement externe sur n variable exactement S.

Concernant les machines classiques, j'étais intéressé de tester si deux différents points constructibles sont équivalents. Concernant les machines algébriques, je dois considérer l'équivalence entre les différentes configurations. Selon l'approche comportementale, l'égalité est facilement définie : deux machines sont équivalentes (ou "équivalentes de manières externes") si leurs comportements externes sont équivalents, à savoir elles définissent le même ensemble. Comme dans les machines classiques, aussi pour les machines algébriques l'équivalence est algorithmiquement testable. En particulier, un ensemble semi-algébrique étant le comportement d'une machine algébrique, on doit simplement tester l'équivalence entre des ensembles semi-algébriques réels.

Cette définition d'équivalence considère les machines comme des ensembles de configurations avec contraintes satisfaisantes sur certaines variables, mais cela ne traite pas le problème de accessibilité. Pour l'interprétation de l'espace accessible, la machine est definie par un ensemble d'opérations assemblées plus une valeur initiale. Les variables introduites, en tant que coordonnées de points physiques, ne peuvent physiquement pas changer de manière discontinues (en fonction du temps), ainsi l'espace obtenu de la configuration accessible doit être connecté. Il est possible de montrer qu'on peut tester si deux espaces accessibles sont égaux.

La discussion sur les objets algébriques [9] est conclue avec quelques remarques sur les constructions avec des outils différentes de les machines algébriques. J'étudie superficiellement des machines acceptant des chaines (même de façon restreinte), des courbes construites comme *loci* obtenues par règle et compas, et des machines algébriques assemblée sans chariots. J'ai choisi le problème des chaines (restreintes) et celui des loci par règles et compas car dans la géométrie

9. Ici, avec "objets algébriques", je souhaite indiquer génériquement une famille d'objets qui peut être analysée avec de l'algèbre polynomial.

de Descartes, ils étaient introduits comme constructions autorisées de courbes "acceptables" ; J'ai aussi examiné les machines algébriques sans chariots car c'était un problème important sur les liens mécaniques au 18ème siècle.

Les machines différentielles

Au delà des machines algébriques

J'ai introduit les machines classiques en les positionnant d'une manière différente des constructions d'Euclide, et je les ai étendues aux machines algébriques, à savoir des machines capables de définir des ensemble semi-algébriques réels. Dans cette partie, qui est le coeur centrale de la thèse, j'étends les machines algébrique aux "machines différentielles", des machines pour des problèmes transcendantaux. C'est un groupe bien formalisé de machines capable de convertir les problèmes historiques de "mouvement tractionnel".

Afin d'étendre les machines algébriques, j'introduis un nouvel élement en posant des contraintes non-holonome : la "roulette". La roulette sur un point S d'un tige r évite S de se déplacer perpendiculairement à r (considérant le mouvement de S sur le plan) [10]. La roulette pose une contrainte non-holonome car son application implique l'introduction d'une dérivée dans la partie analytique. De plus, d'un point de vue mécanique, la contrainte de la roulette est non-holonome car elle pose une condition sur l'orbite du point sur lequel nous plaçons la roulette. Je peux donc introduire le principe suivant :

> **Principe de la roulette**. Donné une tige r et un point S fixe sur r, on peut placer une roulette à S qui empêche S lui-même de bouger perpendiculairement à r (considérant le mouvement de S sur le plan).

Techniquement, la roulette fonctionne si l'on pose (orientée comme r) sur S, et elle est en rotation sans glisser du plan. L'évitement du mouvement latéral est fortement lié à la tangente. Si l'on considère la roulette comme un disque roulant perpendiculairement à la base du plan, la projection du disque est toujours tangentielle à la courbe décrite par le point de contact du disque avec le plan (quand le disque est en rotation). Ainsi, la tige est la tangente à l'orbite du point de la roulette, ayant la même direction que la roulette.

Si l'on considère même des machines algébriques complexes, et sur n'importe quel point S on pose une tige r et l'on place une roulette sur r dans S, quand S bouge r doit être la tangente de l'orbite de S. Cela signifie que la roulette résout le "problème directe de la tangente". Mais le problème directe de la tangente était déjà soluble avec des machines algébriques : la roulette est particulièrement utile pour le problème inverse. En effet, on peut construire de nouvelles courbes avec leurs propriétés tangentielles imposant des conditions sur la tige où la roulette est placée. Le concept de "champ de vecteurs dynamique" peut être utilisé pour évincer le rôle de la roulette : étant la roulette posée sur un point S d'une tige r, on peut considérer la pente de r en fonction de la position du point S, définissant une champ de vecteurs dynamique. La construction de

10. Même si j'introduis la roulette, la même contrainte peut être placée utilisant différentes solutions pratiques. Par exemple, au lieu d'avoir une roulette en rotation sans glisser, on peut considérer une lame, ainsi évitant l'idée de rotation.

champs de vecteurs était déjà disponible avec les machines algébriques, mais la contrainte de la roulette permet d'obtenir la courbe de solution, étant donné une position initiale de S. Ainsi, la roulette peut être considérée comme outil mécanique résolvant le champ de vecteurs dynamique défini par la direction de r en fonction de la position de S.

Je peux définir le groupe de machine s'étendant aux machines algébrique : je nomme "machines différentielles" les machines construites selon les postulats de machines algébriques étendues avec le "principe de roulette". Ainsi, une machine différentielle est obtenue en additionnant à une machine algébrique n'importe quel nombre de roulettes (sur des points fixes sur une tige). Les machines différentielles peuvent néanmoins être considérées comme une formalisation des machines avec mouvement tractionnel.

Après avoir défini les machines différentielles, je dois les cadrer dans une approche comportementale. Le comportement des machines algébriques est un sous-ensemble de $\mathbb{R}^n$: pour des machines différentielles, ayant des contraintes non-holonome, le comportement est composé de familles de courbes, non seulement des familles de points.

Dans l'antiquité, certaines courbes ont été introduit en utilisant l'approche synthétique. La géométrie différentielle prend une autre direction : les courbes sont représentées dans une forme paramétrée comme groupe d'équivalence sur des fonction à valeur vectorielle. Revenant sur les machines différentielles, je peux continuer l'interprétation des variables comme coordonnées de points spécifiques de machines (comme effectué avec les machines algébriques). Cependant, de manière différente que précédemment, il n'est désormais pas assez de considérer les variables comme nombres réels, mais, avec l'introduction de contraintes non-holonomes, je dois considérer ces variables comme des fonctions réelles ($\mathbb{R} \to \mathbb{R}$), où le paramètre représente le temps [11]. Etant une idéalisation de machines physiques, je peux considérer ces fonctions d'être C^∞ (on se réfère aux fonction C^∞ comme des fonction *lisses* ou *régulières.*).

Pour le comportement je dois considérer des ensembles de courbes : mais les courbes peuvent être définies comme des groupes d'équivalence selon des fonction à valeur vectorielle. Ainsi, pour simplifier mathématiquement la définition, pour les machines différentielles je considère "une variété différentielle de fonctions C^∞". En particulier, considérant n variables, ces fonctions doivent être $\mathbb{R} \to \mathbb{R}^n$.

Une machine algébrique est également une machine différentielle, donc je peux réinterpréter son comportement (un ensemble S semi-algébrique réel) comme une variété différentielle. Une machine algébrique accepte tout mouvement et déplacement à l'intérieur de l'ensemble $S \subset \mathbb{R}^n$, ainsi la variété différentielle est composé de toutes les fonctions C^∞ ayant leur image à l'intérieur de S.

Donné un systeme Σ d'équations polynômiales différentielles, nous pouvons construire une machine ayant pour comportement l'ensemble des solutions de Σ

11. Il est difficile de considérer l'introduction géométrique du temps. Plus précisément, étant les courbes des classes d'équivalence sur la paramétrage, elles ne sont pas dépendants du paramètre : similairement, la relation avec les paramètre (le temps) n'est pas nécessaire, mais c'est utile de l'utiliser pour caractériser analytiquement le comportement des machines.

(il est à noter que dans ce cas précis, je me réfère au comportement externe, et non le comportement total). Premièrement, travaillant sur des valeurs réelles, nous pouvons convertir le système Σ en un seul polynôme [12]. Les machines différentielles étant l'extension des machines algébriques, elles sont capables de calculer des sommes et des multiplications, et il est possible de poser la condition qu'une certaine variable est égale à 0. Ainsi, tout ce que je dois démontrer, c'est que nous pouvons construire la dérivée de variables $x_1, \ldots, x_n$. Une construction est fournie [13].

Pour le comportement total d'une machine différentielle, les contraintes de direction ainsi que les conditions algébriques sont transposables dans les polynômes différentiels. Considérant l'ensemble Σ de toutes les équations de polynômes différentiels réels obtenues par les contraintes de la machine, le comportement total est l'ensemble de toutes les fonctions $\mathbb{R} \to \mathbb{R}^{2k}$ satisfaisant l'ensemble Σ.

Il est à noter que je donne une forme analytique au comportement total mais, en général, pour un comportement (externe, non total) j'ai besoin d' "éliminer" les variables non souhaitée du comportement total (comme effectué pour les machines algébrique lors de l'introduction d'ensembles semi-algébriques). Pour cela, je dois introduire certains outils d' "algèbre différentiel" (plus spécifiquement "élimination différentielles") afin de nous permettre de répondre à certaines questions sur ces machines.

Outils analytiques

La grande force de la géométrie cartésienne est la fusion mathématique de algèbre et géométrie à travers l'utilisation de machines. En particulier, l'algèbre polynômial est utilisé comme outil d'analyse, tandis que les constructions géométriques composent la partie synthétique. Cette fusion permet de caractériser les courbes avec des outils algébriques.

Dans l'extension différentielle proposée, je souhaite substituer des polynômes avec des polynômes différentiels, des machines pour constructions algébriques (machines algébriques) avec des machines différentielles, et des courbes algébriques avec des variétés différentielles. Dans cette partie, je veux approfondir les outils analytiques des machines différentielles, l' "algèbre différentiel", et plus particulièrement "l'élimination différentielle". La particularité de cette approche c'est qu'elle est mise en oeuvre de façon algorithmique : sa manipulation ne nécessite pas de référence au concept des objets non finis (contrairement au calcul différentiel). Ces outils algébriques donnent des réponses utiles à certaines questions sur les machines différentielles.

L'histoire de l'algèbre différentielle commença avec Ritt [1932], où Ritt introduit des outils algébriques pour les équations différentielles. Ces résultats ont ensuite été reformulés dans Ritt [1950] avec grand effort, afin d'arriver à un language plus proche à l'algèbre, mais pour compléter ce passage il faudra attendre Kolchin [1973].

12. Le système de polynômes différentiels réel $p_1 = \ldots = p_l = 0$ est équivalent à $(p_1)^2 + \ldots + (p_l)^2 = 0$.
13. Le premier travail a été Milici [2012a], où j'ai exprimé la possibilité pour de telles machines à résoudre les problèmes polynomiaux de Cauchy.

Sous Ritt et Kolchin, l'algèbre différentielle fut développé d'un point de vue constructif et les fondations qu'ils ont construites ont été améliorées afin d'être applicable efficacement par les ordinateurs, notamment grâce au passage des ancienne méthodes de construction (l'algorithme de Ritt-Seidenberg dans Seidenberg [1956]) à les méthodes actueles d'optimisation plus complexes avec les bases de Gröbner. Ce qui a apporté, par exemple, le développement du progiciel *DifferentialAlgebra* dans le logiciel *Maple*.

Actuellement, il existe beaucoup d'intérêts pour l'algèbre différentielle. En effet, elle est utilisé afin de résoudre des problèmes dans les domaines de la robotique, du contrôle et des systèmes dynamiques. Une autre application importante de l'algèbre différentielle est la preuve mécanique des théorèmes dans la géométrie différentielle.

L'objectif de l'algèbre différentielle est d'apporter un théorie algébrique pour les équations différentielles ordinaires et avec dérivés partielles. En particuliers ses outils et ses notations sont une extension de l'algèbre commutative. Pour donner une introduction courte à l'algèbre différentielle, je remémore [Boulier, 2007, pp. 112–116] pour la clarté et la superposition avec mes objectifs (en basant sur le type de contrainte obtenue avec les machines différentielles, je suis seulement intéressé dans les équations différentielles ordinaires). Pour commencer je me dois de donner quelques définitions.

Un *anneau différentiel* (respectivement *champ*) est un anneau (resp. champ) R doté d'une dérivation. Une *dérivation* est une fonction de type $D : R \to R$ distributive par rapport à l'addition [14] qui doit obéir la règle du produit (aussi appelé "la règle de Leibniz") :

$$D(ab) = D(a)b + aD(b).$$

Il est à noter que tout anneau standard (resp. champ) est différentiel avec la dérivée trivial $D(a) = 0$ (pour chaque $a \in R$), et dans ce cas tous les éléments du anneau (resp. champ) peuvent être considérés comme constants. Un exemple pas trivial de champs différentiels est le champ de les fonctions méromorphes $f(z)$ sur le plan complexe. Pour mon but, nous pouvons considerer le champ (différentiel) des nombres rationnels.

Similairement à l'algèbre géométrique classique, nous pouvons considérer l'anneau polynomial différentiel $K\{U\}$ où K est un champ différentiel de coefficients et U est l'ensemble fini des indéterminées différentiels. Les éléments de $K\{U\}$, les polynômes différentiels, sont des polynômes dans un ensemble fini de toutes les dérivées des indéterminées différentiels, notées ΘU. Pour mon but, les indéterminées différentiels peuvent être simplement considérés comme des fonctions dépendantes uniquement de la variable t, qui peut être défini comme le temps. Les indéterminées différentiels sont aussi appelés *variables dépendantes*, et t est la *variable indépendante*.

Dans l'algèbre géométrique, il est connu qu'un ensemble de polynômes qui disparaît sous les solutions d'un polynôme donné forme un idéal, et même un

14. Pour chaque $a, b \in R$, $D(a + b) = D(a) + D(b)$.

idéal radical [15]. Cherchant quelque chose similaire pour les équations différentielles, j'introduis l'extension différentielle de ces concepts.

Dans un anneau différentiel R, un idéal I est différentiel s'il est stable sous dérivation ($a' \in I$ pour tout $a \in I$). De plus, un idéal différentiel I est radical si $a^p \in I$ implique $a \in I$ pour n'importe quel nombre entier $p > 0$.

L'ensemble de toutes les "conséquence algébriques et différentielles" des polynômes différentiels dans un système Σ est l'idéal différentiel radical généré par Σ, que je note par $\sqrt{[\Sigma]}$. En général, donné un système différentiel Σ (un système de polynômes différentiels), au lieu d'étudier directement les solutions de $\Sigma = 0$ nous devrions inspecter $\sqrt{[\Sigma]}$.

L'analogue du théorème d'Hilbert pour les anneaux polynomiaux est donné par le théorème de Ritt-Raudenbush :

Theorem (Ritt-Raudenbush)**.** *Si J est n'importe quel idéal différentiel radical dans $K\{U\}$, il existe un sous-ensemble fini Σ de $K\{U\}$ tel que $J = \sqrt{[\Sigma]}$.*

A noter que le résultat est bon pour les idéals différentiels radicaux, mais généralement, il n'est pas bon pour les idéals différentiels.

Mais je ne suis pas intéressé par la preuve de l'existence, je cherche des algorithmes destinés à être appliqués dans l'analyse de systèmes différentiels (et un retour sur les machines différentielles). Ainsi, une fois associé un système différentiel Σ à une machine différentielle, mon but est de donner une représentation adéquate de son idéal différentiel radical $\sqrt{[\Sigma]}$, a fin de répondre aux questions sur Σ.

L'élimination différentielle est une application importante de l'algèbre différentielle. C'est un processus qui prend en entrée un système d'équation différentielles (ordinaires ou avec dérivées partielles) et un classement. L'élimination reformule le système en un système équivalent (ou un ensemble de famille équivalent quand la séparation de certains éléments est nécessaire). Le classement permet de contrôler le processus d'élimination, indiquant ce qui devrait être éliminé. L'algorithme principal de l'élimination différentielle est appelé *Rosenfeld-Gröbner*; il est important pour décider de l'appartenance dans un idéal différentiel radical [16]. Même si ces algorithmes sont très inefficaces dans le pire des cas, en principe l'élimination différentielle est toujours possible.

Pour un système d'équations différentielles $\Sigma = 0$ et un choix approprié du classement, avec l'aide d'algorithmes symboliques on peut résoudre questions comme :

- est-ce qu'une équation différentielle (non apparente dans $\Sigma = 0$) est satisfaite par toutes les solution du système $\Sigma = 0$?

15. Un idéal I est un sous-ensemble d'un anneau R formant un groupe et a la propriété que, $\forall x \in R, \forall y \in I$, le produit $xy \in I$. Un idéal I est dit radical si $a \in I$ à chaque fois qu'il existe des nombres entier non négatifs p tel que $a^p \in I$.

16. Dans l'algèbre polynomial, le test d'appartenance à un idéal est réussi dans la réduction de Gröbner, comme visible dans Buchberger [1985]. D'un autre côté le lemma de Rosenfeld (apparu dans Rosenfeld [1959]) était un lien entre l'algèbre différentielle et l'algèbre polynomial, et ainsi, la clé dans les algorithmes d'algèbre différentielle : l'algorithme de Rosenfeld-Gröbner introduit en Boulier [1994] et Boulier et al. [1995], combine le lemma de Rosenfeld et les bases de Gröbner.

- quelles sont les équations différentielles satisfaites par les solutions de $\Sigma = 0$ dans un sous-ensemble de variables dépendantes ? Si Σ est un systéme différentiel dans les fonctions inconnues $x_1, \ldots, x_n$, nous pouvons être intéressé par l'équation qui contrôle le comportement de la fonction x_1 indépendamment des autres.

Toutes ces questions requièrent, d'une manière ou d'une autre, un test d'appartenance pour les idéals différentiels radicaux générés par le système de polynômes différentiels Σ. Pour ce but, nous pouvons représenter les idéals différentiels radicaux comme des intersections de chaines différentielles qui peuvent être obtenues avec l'algorithme de Rosenfeld-Gröbner [17].

Concernant les problèmes d'équations différentielles avec des conditions initiales [18], il reste beaucoup du travail : malgré Pritchard and Sit [2007] et l'approche proposée par Markus Rosenkranz sur les méthodes symboliques pour problèmes linéaires (par example Rosenkranz et al. [2012]), à ma connaissance une solution générale des problèmes avec conditions initiales est loin d'être trouvée.

Résolution de problèmes

Avec l'introduction de machines différentielles, je ai étendue la géométrie cartésienne et, grâce à l'algèbre différentielle, j'ai également apporté un langage bien défini et un ensemble d'algorithmes pour l'analyse. Il est temps d'utiliser l'algèbre différentielle afin de définir les comportements externes et l'égalité entre les machines.

Similairement à la méthode géométrique cartésienne, mes étapes de résolution de problème sont les suivantes :

1. commencer d'un problème concernant les machines différentielles,
2. convertir ce problème en des équations différentielles,
3. résoudre le problème avec des algorithmes d'algèbre différentielle,
4. quand demandé, après la simplification, construire la solution graphique avec une machine différentielle.

Donné une machine différentielle avec des variables manifestes et latentes, le comportement peut être décrit par un systéme Σ d'équations polynomiales différentielles dans les variables manifestes mais aussi latentes. Puis, considérant tout classement éliminant les variables latentes, nous pouvons obtenir une représentation du comportement externe donné par l'intersection d'une famille de systèmes différentiels (chacun obtenu avec des équation et inéquations) grâce à l'algorithme de Rosenfeld-Gröbner.

17. Cf. [Hubert, 2003, pp. 41–42].

18. Par exemple, concernant les machines différentielles, je suis intéressé par le problème suivant : donné deux machines différentielles avec leur configurations initiales, sont leur comportements égaux ? Analytiquement, cela devient : donné deux systèmes d'équations différentielles avec des conditions initiales, leur solutions sont-ils équivalent ? Je cherche un algorithme afin de résoudre symboliquement ce problème. L'algèbre différentielle de base ne permet pas d'évoquer ce problème car il requiert de rendre explicit la relation entre les variables dépendantes et la variable indépendante (pour poser la condition initiale).

Ainsi, le problème de caractérisation du comportement externe peut être considéré comme résolu en adoptant les outils de bases de l'algèbre différentielle. En effet, donnée un idéal radical défini comme l'intersection d'un nombre fini de systèmes différentiels (avec équations et inéquations), nous pouvons construire une machine qui résout exactement les équations et inéquations voulues.

Il est également possible de définir la nature de les fonctions que ces machines caractérisent : les fonction construites sont exactement celles appelées "différentielles algébriques" (DA) [19]. Cela est important car cela signifie que les machines différentielles génèrent un nouveau dualisme au delà de algébrique ou transcendantal (et cette fois concernant les fonctions, et non courbes ou variétés comme fait avec les machines algébriques). Il est à noter cependant, qu'une machine peut construire des fonctions qui ne sont globalement pas DA [20], mais localement chacune doit être DA.

Toutes les fonctions élementaire sont DA, et même la plupart des fonctions transcendantales que nous trouvons dans les ouvrages : historiquement le premier exemple d'une fonction non DA était le Γ de Euler, comme prouvé dans Hölder [1886], ainsi son graphique ne peut être tracé par des machines différentielles.

Concernant l'équivalence entre les machines différentielles, il faut considérer deux d'entre elles : $\mathcal{M}$ et $\mathcal{N}$. Considérant le comportement total de ces machines comme les solutions de deux systèmes d'équations polynômiales différentiels, le comportement externe est la restriction des variétés différentielles sur certaines variables. Pour commencer le test d'équivalence on suppose que les variables du comportement externe dans $\mathcal{M}$ et $\mathcal{N}$ sont en nombre égal.

Pour vérifier l'équivalance entre deux comportements totals (à savoir entre idéals différentiels radicaux donné par un ensemble fini de générateurs), on peut fixer un certain classement et calculer les chaines différentielles en utilisant l'algorithme de Rosenfeld-Gröbner, et avec celles tester si tous les générateurs du premier idéal appartiennent au second et vice-versa.

La même procédure n'est pas aussi facilement applicable aux comportements externes. En effet, ces comportements sont obtenus en éliminant certaines variables : ainsi, les représentations sont données par l'intersection de familles de chaines différentielles, et il n'existe aucun algorithme connu à ce jour pour passer d'une représentation de famille de chaines différentielles à une liste de générateurs. Il est a noter que ça est toujours théoriquement possible, selon le théorème de Ritt-Raudenbush, mais il n'y a aucune méthode générale connue.

Une approche différente pour vérifier l'équivalance peut être introduite en utilisant les représentations canoniques. Fixé un classement, il existe un algorithme fournissant une "décomposition canonique" ayant pour entrée une décomposition prime (généralement non unique) d'un idéal différentiel radical [21]. Ainsi, considérant un idéal défini comme un comportement externe, cet algo-

19. Une fonction y est différentielle algébrique si elle satisfait une équation différentielle algébrique (EDA), à savoir une équation différentielle de la forme $P(t, y, y', \dots, y^{(n)})$ ou P est un polynôme non-trivial dans $n+2$ variables (cf. Rubel [1989]).

20. Cette propriété était visible depuis l'introduction des machines différentielles : même si elles étaient appelées "tractional motion machines", leur première apparition dans Milici [2012a] a concerné la construction d'une machine traçant une courbe qui globalement n'était pas DA (le cycloïde, considéré comme le graphique d'une fonction $y = f(x)$).

21. Voir [Golubitsky et al., 2009, section 3.2, pp. 519–520].

rithme fourni une méthode de calculer une nouvelle décomposition qui est indépendant des représentations initiales de l'idéal. Cela veut dire que, étant donnée les machines $\mathcal{M}$ et $\mathcal{N}$, nous pouvons trouver une représentation canonique de leurs comportements externes. Ainsi les deux machines sont équivalentes si leurs comportements définissent le même idéal différentiel, à savoir si les idéals possèdent la même représentation canonique. Ainsi, même s'il n'existe pas d'algorithme pour tester l'inclusion entre les idéals radicaux (le problème ouvert de Ritt), il est possible de tester leur équivalance.

Concernant les machines algébriques nous avons observé que nous pouvons considérer différentes interprétations du concept d'équivalance entre les machines, selon le rôle des machines. Une machine algébrique peut être vue comme un ensemble de contraintes ou comme la configuration accessible d'un point initial : différentes interprétations conduisent à différents comportements et ainsi différents tests d'équivalance. La possibilité de formuler des conditions initiales est également présente dans le cas de différentiel, mais à ma connaissance le problème d'équivalence dans ce cas reste encore ouvert. Concernant des résultats positive, on peut se référer à Buchberger and Rosenkranz [2012], qui fourni un algorithme pour la solution symbolique du problème linéaire avec valeurs initiales, passant de l'algèbre differentielle à l' "algèbre integro-différentiel" (opérateurs de Green).

Pour résumer, dans cette partie, j'ai fourni une définition de machines différentielles, et, grâce à l'algèbre différentielle, j'ai exploré le comportement des machines. En particulier, j'ai eu du succès pour la caractérisation du comportement externe des machines, qui est defini comme intersection de systèmes différentiels (chacun composé d'équations et inéquations polynômiales différentialles). J'ai également observé que, considérant comme indéterminées des fonctions lisses, les solutions constructibles sont exactement les fonctions différentielles algébriques : cela peut être considéré comme une extension du dualisme cartésien entre les objets algébriques et les objets transcendantaux.

De plus, comme dans le cas algébrique, il existe un algorithme pour vérifier l'équivalance entre deux machines considérées comme ensembles de contraintes (analytiquement, vérifiant l'égalité entre les idéals différentiels radicaux). Le problème est encore ouvert si on considère l'équivalence entre les machines différentielles avec les conditions de valeur initiale. Je pense que le test d'équivalence joue un rôle crucial dans la définition précise de l'exactitude, ainsi la possibilité de tester l'équivalance entre les idéals différentiels radicaux implique également des conséquences philosophiques sur la possibilité de définir l'exactitude de machines différentielles. Differemment d'autres théories pour le calcul infinitésimale (par exemple l'analyse "calculable"), il est aussi important qu'il n'a pas été prouvé l'indécidabilité du test de l'équivalence entre idéals differentiels radicaux avec conditions de valeurs initiales [22].

22. Il est à noter que des problèmes d'algèbre différentiels, comme le problème de l'appartenance dans un idéal différentiel (non radical), sont prouvés être non-décidables (cf. Gallo et al. [1991]).

Machines pour les équations différentielles complexes

Nous avons observé comment les machines différentielles peuvent être utilisées afin de résoudre des systèmes différentiels polynomiaux. De plus, avec le plan d'Argand-Gauss, nous pouvons considérer n'importe quel point du plan comme un nombre complexe. Ainsi, il est assez naturel de nous poser la question d'utiliser ces machines pour résoudre des équations différentielles complexes données par des polynômes différentiels.

Même si les fonctions complexes nécessitent un espace à 4 dimensions pour être statiquement représentées, nous pouvons représenter celles-ci par un transformation dans le plane fusionnant l'ensemble de départ et l'ensemble d'arrivée dans le même espace 2D. Ainsi la fonction est donnée par une corrélation point à point liant la motion du point d'entrée avec la motion du point de sortie. Pour représenter la valeur complexe de la fonction f on peut considérer, donné un point d'entrée z libre de mouvement sur le plan de Argand-Gauss, le point de sortie $w = z + f(z)$: ainsi (pour chaque z) $f(z)$ peut être vu comme le vecteur de différence entre w et z.

Donné la variable dépendante complexe z_j, avec z'_j je note $\frac{dz_j}{dz}$ où z est la variable complexe indépendante. Ainsi, la variable dépendante $z_j = z_j(z)$ est une fonction complexe $\mathbb{C} \to \mathbb{C}$. Etant la variable indépendante z une variable complexe, il est naturel de considérer cela comme un point libre sur le plan. Cependant, si l'on considère le mouvement du point z en fonction du temps, on peut considérer la fonction $z : \mathbb{R} \to \mathbb{C}$, tel que $z(t)$ est la position de z dans le temps t. D'un autre point de vue, on peut considérer la courbe tracée par z, ainsi $z(t)$ est un paramètrization. Je note $\frac{dz}{dt}$ d'être non nul (éventuellement en changeant le paramètrization). L'introduction du temps t pour z est nécessaire car les machines différentielles ne sont pas directement capable de déterminer la dérivée d'une variable complexe ; c'est obtenu en utilisant la dérivée de la fonction composée $\frac{dz_j}{dt} = \frac{dz_j}{dz}\frac{dz}{dt}$.

Afin de résoudre les polynômes différentiels, nous avons tout d'abord besoin de savoir comment effectuer des opérations algébriques complexes (somme et produit) avec des machines différentielles [23]. Même s'il y a des travaux qui résolvent des opérations complexes avec des machines spécifiques, nous pouvons contourner ce problème facilement en utilisant l'algèbre. En effet, nous savons qu'avec les machines algébriques, nous pouvons effectuer les opérations réelles et pouvons trouver les coordonnées cartésiennes d'un point : ainsi nous pouvons construire une machine pour $(a + ib) + (c + id) = (a + c) + i(b + d)$ et une autre pour $(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$.

Maintenant, le problème est de contrôler la dérivée complexe z'_j. Considérant $w = z + z_j(z)$ et $z(t)$, la tangente de la courbe tracée par w prend une direction complexe

$$\frac{dw}{dt} = \frac{dz}{dt}\left(1 + \frac{dz_j}{dz}\right) = \frac{dz}{dt}(1 + z'_j).$$

23. Voir par exemple Emch [1902], où l'auteur a montré les transformations algébriques de variables complexes utilisant les liens de Kempe. Des machines plus modernes (et théoriques) pour les opérations complexes peuvent être vues dans Kapovich and Millson [2002].

Il est facile de poser la condition tangentielle, mais différemment du cas réel, telle condition n'est pas assez pour contrôler le mouvement de z_j. En effet, la dérivée étant donné par un nombre complexe, elle possède deux dimensions, ainsi le changement de fonction ne peut être contrôlé par une simple condition tangentielle. Dans le cas réel, le point d'entrée $(t, 0)$ imposé l'abscisse du point de sortie : dans le cas complexe, ni l'abscisse ni l'ordonnée du point de sortie sont contraints a être celui du point d'entrée z. Ainsi, à première vue, pour la dérivée complexe, il semble que nous devons imposer non seulement une condition de direction, mais également une condition sur le module : je donne une méthode de contrôle du module avec des conditions tangentielles, sans ayant recours à de nouveaux outils mécaniques (par exemple incluent non seulement la direction mais également la rapidité de rotation des roulettes). Il est temps d'introduire le "point de sortie auxiliaire" w_c : donné n'importe quelle constante complexe c, nous pouvons construire $w_c = z + cz_j$. Calculant $\frac{dw_c}{dt}$, nous obtenons que l'argument doit être $\arg\left(\frac{dz}{dt}\right) + \arg(1 + cz_j')$, ainsi nous pouvons poser les conditions tangentielles dans w_c. L'idée est d'obtenir de nouvelles conditions sans augmenter le degré de liberté [24].

Je dois préciser l'usage des conditions tangentielles sur les points auxiliaires afin de poser des contraintes sur z_j'. En particulier, je donne des conditions sous lesquelles, la valeur de z_j' est uniquement déterminée données les tangentes de z et de deux points de sortie.

Considérant les points auxiliaires $w_1 = z + c_1 z_j$ et $w_2 = z + c_2 z_j$ (où c_1, c_2 sont des constantes complexes), nous devons imposer dans les deux la condition tangentielle $\arg\left(\frac{dz}{dt}\right) + \arg(1 + c_i z_j')$ avec une roulette $(i = 1, 2)$. Afin d'éviter tout problème de non-unicité pour n'importe quelle valeur $z, \frac{dz}{dt}, z_j, z_j'$, il suffit de prendre c_1, c_2 non-parallèles. Cela signifie que, comme dans le cas réel, nous sommes capables de diriger des dérivées dans le cas complexe avec des machines différentielles. Une application est la machine différentielle pour la fonction exponentielle complexe.

L'introduction de machines complexes peut être nécessaire afin d'évincer quelques propriétés. Par exemple, dans la machine exponentielle complexe, j'utilise le point $z - 1$ afin de construire de manière simple la condition tangentielle dans un point de sortie. Ce point à la propriété que, quand $\arg\left(\frac{dz}{dt}\right) = 0$, les tangentes aux points de sortie passent par la. Je généralise cette propriété à n'importe quelle fonction complexe à dérivé continue f : je nomme "pivot" le point $p = z - \frac{f(z)}{f'(z)}$ [25]. J'explore les usages possibles et les propriétés de tels points pour quelques constructions cinématiques et géométriques.

24. Il est implicitement voulu que $c \neq 0$. En effet, cherchant des conditions tangentielles supplémentaires, le cas $w_0 = z$ est inutile car cela ne donne pas de nouvelle conditions tangentielles (je considère donné la direction du point z).

25. Considérant le dénominateur $f'(z)$, le pivot est un point fini si et seulement si $f'(z) \neq 0$. Il est à noter que le point pivot est la généralisation complexe du point cartésien $\left(t - \frac{f(t)}{f'(t)}, 0\right)$ dans le cas des fonctions réelles. Ce point est l'intersection de la tangente au graphique avec l'abscisse, et c'est utilisé dans la méthode de Newton pour approximer les zéros de $f(t)$.

Les machines différentielles comme appareils physiques

J'ai introduit les machines différentielles comme des instruments théoriques : dans cette partie je considère leurs variantes concrètes. En particulier, je commence par créer une interrelation entre les machines différentielles et le mouvement tractionnel. Spécifiquement, au lieu de l'approche théorique de Riccati au 18ème siècle [26], je préfère la classification plus pratique des machines grapho-mécaniques pour l'intégration d'équations différentielles faites en Pascal [1914] [27]. Je veux montrer que telles machines peuvent être obtenues avec les machines différentielles.

Par la suite, j'explore les constructions graphiques disponibles grâce à une unique machine différentielle étendant et unifiant la règle et le compas, le "compas logarithmique", évinçant quelques retombées fondationnelles.

Je conclu cette partie en donnant quelques retombées didactiques pour les machines différentielles. Biensûr, le but de cette thèse n'est pas de suggérer un usage pratique de telles machines afin de résoudre des équations différentielles, mais de faire face à des problèmes de calcul infinitésimale avec l'idéalisation d'outils concrets. Dans ma perspective, cette quête pour une fondation instrumentale at finie peut être utile en didactique pour rendre les mathématiques moins abstraites et plus abordables. L'utilisation de telles machines au moment est suggérée avec l'utilisation et la manipulation concrète d'objets, mais dans le futur peut être étendue à l'utilisation dans certains logiciels géométriques. Une autre perspective future est d'approfondir le potentiel des machines différentielles pour une nouvelle approche pédagogique au calcul infinitésimale (avec l'algèbre différentielle).

Machines différentielles et intégraphes

Afin d'évincer la relation entre les machines différentielles et le mouvement tractionnel, je traduis les machines de Pascal [1914] (probablement la classification d'intégraphes la plus complète) en machines différentielles.

En dehors des roulettes, les intégraphes utilisent des éléments droits et des parties glissantes (qui correspondent respectivement à tiges et chariots), mais pas seulement : nous pouvons trouver des outils comme des barres incurvés ou des ressorts. Je démontre que les comportements des intégraphes peuvent être obtenus avec des machines différentielles seulement. D'une perspective analytique, cet essai peut paraître inutile car les tous les intégraphes connus arrivent à résoudre des équations différentielles algébriques (EDA). Cependant, je considère qu'il est intéressant de regarder cela de plus près, car il aurait pu y avoir des machines qui, utilisant la méthode d'autres intégraphes mais avec des éléments légèrement différents, résolvent quelque équation qui n'est pas différentielle algébrique. Mais cela n'arrive pas car j'évince dans cette section que toutes les

26. Dans Riccati [1752] les constructions tractionnelles étaient autorisées même dans le cas où il n'y avait pas d'instrumentalisation dans la réalisation, comme dans "tractorias avec directrix variables", où les directrix peut changer de forme.

27. Comme visible dans [Tournès, 2009, Chap. 9] et introduit dans notre partie historique, les méthodes d'intégration graphique des équation différentielles développées jusqu'au milieu du 18ème siècle fussent oubliées, puis réincarnées au 19ème siècle avec la même famille de concepts et outils. Les machines de la période du 19ème étaient appelées "intégraphes".

idées pratiques derrière les intégraphes peuvent être traduites en des machines différentielles.

Les intégraphes de E. Pascal ont deux éléments fondamentaux, le "chariot différentiel"et le "chariot integral" [28]. La roulette est sur le chariot intégral, et trace la courbe intégrale, tandis que le chariot différentiel est couplé rigidement à un pic que l'utilisateur doit bouger le long de la courbe qui doit être intégrée. Une distinction générale de telles machines concerne le système de coordonnées du plan dans lequel nous voulons interpréter la courbe intégrale. Si les points sont considérés dans des coordonnées cartésiennes nous avons des "intégraphes cartésiens", tandis que dans les coordonnées polaires nous avons "intégraphes polaires". La configuration de base pour les intégraphes cartésiens est un rectangle glissant en ligne droite, tandis que pour les intégraphes polaires c'est une zone circulaire en rotation. Sur ces configurations de base il existe des guides pour les chariots différentiels et intégrales.

Le compas logarithmique

Probablement l'intégraphe polaire non-algébrique le plus élémentaire est celui traçant des spirales logarithmiques. Même si l'intégraphe polaire de Pascal est capable, autre que beaucoup d'autres usages, de tracer de telles spirales, je suis intéressé par des questions géométriques fondationnelles. La caractérisation de Wantzel des nombres constructibles et la preuve de Lindemann sur la transcendance de π ont prouvée l'impossibilité, utilisant un compas et une règle non-marquée, de résoudre les problèmes géométriques grecques tel que doubler un cube, tri-sectionner un angle, quadrer un cercle, et construire certains polygones réguliers.

Dans cette partie, j'introduis un instrument qui unifie et étend les pouvoirs constructibles des compas et des règles, un instrument que j'appelle le "compas logarithmique" (ou "compas equiangulaire"). Il peut tracer une spirale logarithmique depuis n'importe quel centre donné, à travers n'importe quel point, avec n'importe quelle tangente donné à ce point [29]. Au moins deux problèmes de géométrie classique sur trois sont résolubles avec. J'introduis également d'autres problèmes qui sont résolus (et d'autres qui ne le sont pas) utilisant de tel appareil.

Application dans la didactique des mathématiques

Je pense qu'une application très importante de ma thèse peut être de fournir une nouvelle approche pédagogique au calcul infinitésimal. En fait, depuis la formalisation rigoureuse de Cauchy, le concept principal derrière les objets différentiels est le concept de limit. Cela amène l'idée que les processus infinis sont à la base du calcul différentiel, avec les problèmes épistémologiques lié à cela dans l'apprentissage. Au contraire, l'algèbre différentielle permet de manipuler les polynômes différentiels de manière déterministe sans ayant nécessairement besoin du concept d'infini. L'algèbre différentielle seul, cependant,

28. Il est à noter que, même si ils sont similaires à mes chariots, ils sont différentes car ils ne doivent pas uniquement glisser en ligne droite.
29. Tous les résultats et images de cette partie sont apparus dans Milici and Dawson [2012].

manque de la possibilité synthétique de montrer la solution des équations différentielles et, pour son introduction dans l'enseignement, il est nécessaire d'avoir en avant une connaissance du calcul infinitésimal [30]. Dans cette perspective, les machines différentielles peuvent introduire fonctions et dérivés sans nécessiter du calculs infinitésimal, et elles peuvent être considérées comme des méthodes synthétiques finies pour résoudre des problèmes d'algèbre différentiels avec des machines idéales. Ainsi, à première vue, il me semble que le mouvement tractionnel peut être utilisé afin d'introduire le calcul infinitésimal [31]. Didactiquement, les machines différentielles et l'algèbre différentielle peuvent aider les étudiants ayant une approche plus concrète de ces sujets. De plus, l'introduction de l'algèbre différentielle peut donner continuité entre les problèmes algébriques et différentiels.

Pensant à une restructuration possible de la manière dont le calcul inifinitésimal est enseigné, bien-sûr l'approche traditionnelle avec limites doit être introduite (essentiel pour les méthodes numériques), mais pas comme la seule approche possible. Evidemment, il faudra tester empiriquement si l'approche avec les machines et l'algèbre différentielle peut être util dans la didactique des mathématiques.

Même si encore sans aucun support de résultats expérimentaux, je suggère une introduction hypothétique du calcul infinitésimal aux étudiants divisée en les quatres étapes citées ci-dessous.

1. Introduction préliminaire de l'algèbre avec des machines (concrètes et numeriques). Cettes machines (algébriques ou similaires) jouent un rôle important dans les constructions géométriques, et regroupe énormément de connaissance techniques et mathématiques. L'utilisation de telles machines dans les classes a été approfondie dans nombreuses travailles expérimentaux.

2. Introduction de certaines machines différentielles physiques à être manipulées et étudiées par les étudiants durant des activités en laboratoire. Même si les étudiants n'ont pas encore étudié le calcul infinitésimal, considérer cettes machines est util afin de poser les bases pour la leur traduction mathématique [32].

3. Traduction des machines physiques dans leurs variantes digitales. Même si moins concrète [33], cette étape est importante afin de donner aux utilisateurs la possibilité non seulement d'explorer, mais aussi de construire des

30. L'algèbre différentielle manipule les polynômes différentiels, ainsi elle utilise des fonctions lisses (et leur dérivées) comme variables qui, pour être définies précisément, nécessitent une connaissance du calcul infinitésimal. De plus, si un individu est intéressé par l'évaluation d'une fonction défini comme solution d'un système différentiel avec des valeurs initiales données, l'algèbre différentielle seul n'est pas capable de fournir une réponse.

31. Historiquement Giovanni Poleni (1683–1761, université de Padua, Italie) et Ernesto Pascal (1865–1940, université de Naples, Italie) ont introduit le mouvement tractionnel dans la didactique des mathématiques, car ils ont conçus des instruments tractionnels pour leurs étudiants (ils ont créé des laboratoires mathématiques dans leurs universités). Pour des informations complémentaire, voir Tournès [2009].

32. Comme suggéré dans *Theory of the Semiotic Mediation* (cf. Bartolini Bussi and Mariotti [1999]), qui se focalise sur l'usage d'artefacts afin de transmettre la connaissance des mathématiques.

33. Les simulations par ordinateur empêchent d'atteindre les mécanismes physiques sousjacents aux comportements simulés.

machines différentielles (trop difficile à réaliser physiquement). Le problème d'une telle étape, c'est qu'à ma connaissance, aucun logiciel didactique pour la géométrie dynamique résout les problems inverses de la tangente [34]. Dans cette perspective, une solution optimale peut être le développement d'un progiciel pour un logiciel de géométrique dynamique (par example *GeoGebra* ou *Cabri Géomètre*), essayant d'intégrer la possibilité de gérer les problèmes inverses de tangente comme conditions de roulettes (pour les construction à mouvement tractionnel) et comme équations différentielles.

4. Passage d'unification théorique. Introduction du formalisme mathématique, premièrement comme manipulation symbolique (algèbre différentielle de base) et ensuite egalement comme calcul infinitésimal classique. Cela pourrait réaliser la convergence de machines, algèbre, et géométrie au delà des frontières cartésiennes.

De ce point de vue, le but de ma thèse est de suggérer une ligne de conduite pour la recherche dans la didactique des mathématiques sur le calcul infinitésimal (ces étapes sont laissé à des études futures). Cependant, je suggère quelques calquages d'activités possibles en laboratoire concernant l'étape 2. En particulier, je considère l'utilisation d'une machine différentielle, originalement conçue et physiquement réalisée, afin de permettre un double usage : une première approche d'exploration (de la machine aux mathématiques), et une seconde approche constructive (de la formule à la machine).

Conclusions et perspectives futures

Dans cette thèse, j'ai traité certains des plus importantes approches en géométrie : synthétique (machines classiques), analytique (machines algébriques) et différentielle (machines différentielles). Pour chacune, j'ai donné le composants mécaniques et la caractérisation analytique.

Dans ces conclusions, je me focalise sur comment l'équilibre entre les machines, l'algèbre et la géométrie est centrale pour une vue multi-perspective du même objet. D'un point de vue historique, l'introduction du mouvement tractionnel peut être considérée comme une "extension conservative" du programme de Descartes (analyse finie et synthèse avec des constructions géométriques). Cependant, contrairement à Descart, je propose que les limites de la connaissance exacte ne sont pas données par les objets algébriques, mais par l'équilibre entre les machines, l'algèbre et la géométrie. Ainsi, même la limite des machines différentielles est temporaire, attendant pour une nuovelle extension.

Par la suite, j'ajoute quelques réflexions concernant mon setting, et plus généralement, le calcul infinitésimal. Premièrement je considère le point de vu des sciences cognitives sur les mathématiques. Dans cette perspective, principalement instaurée par Lakoff and Núñez [2000], les idées mathématiques sont analysées par la perspective des connaissances incarnées. Concernant le calcul

34. Nous devons noter que, pour mes objectifs, la géométrie est un outil non seulement pour la visualisation, mais également pour les constructions dynamiques. Il existe beaucoup de logiciels décrivant des solutions d'équations différentielles, mais pas dans une perspective de géométrie dynamique.

infinitésimal, depuis Newton et Leibniz, nous savons que le concept au coeur est le rôle constructif de méthodes incluant l'infini. Au contraire, le setting mécanique proposé et l'analyse avec l'algèbre différentielle suggèrent qu'il est possible de prendre en compte le calcul infinitésimal (du moins la partie traitant des polynômes différentiels) sans le besoin de l'infini, mais avec la métaphore [35] "la direction de la roulette est la tangente". Cette métaphore, étant très concrète, peut être considérée comme une métaphore "fondatrice".

Les machines peuvent être considérées non seulement comme des instruments idéaux, mais également comme des outils de calcul. L'introduction des approximations infinies dans les constructions est possible si l'on considère des méthodes récursives. D'un point de vue computational, la récursivité est l'outil principal du calcul "numérique" (symbolique) ; les machines différentielles peuvent éviter l'infini car elles ne sont pas basées sur la récursivité, étant des machines "analogiques". La perspective du calcul analogique est brièvement explorée.

Le rôle de l'infini comme concept qui sous-tend le calcul infinitésimal n'est pas quelque chose qui réside dans les mathématique, mais au delà. En effet, d'un perspective formaliste, les mathématiques impliquent simplement les manipulations finies de formules finie selon des règles déductibles. De ce point de vue, les problèmes comme celui de Bos sur l'exactitude des constructions sont clairement meta-mathématiques. Cependant, je pense qu'une approche algorithmique peut être utile afin de caractériser le problème traitant l'exactitude : en particulier, je propose l'idée (encore à approfondir) que l'exactitude peut être reconsidérée dans une théorie algorithmique en relation avec la solution du "test d'équivalence". Ainsi, l'existence d'une méthode afin de vérifier l'équivalence entre deux objets avec des représentations différentes peut être un outil pour distinguer les théories exacts et approximatifs.

Depuis cette perspective, les approches informatiques usuelles du calcul infinitésimal ne sont pas exactes (par example l'analyse calculable) car, intuitivement, elles ont besoin d'approximations, et formellement, il n'existe pas d'algorithme pour le test d'équivalence. Au contraire, le setting proposé au calcul différentiel avec des machines idéales et outils anaytiques finis peut être considéré exact, car le test d'équivalence est disponible pour les machines différentielles [36].

Concernant les perspectives futures, les sujets principaux nécessitant approfondissement sont cités ci-dessous :

- Exactitude comme solution d'un test d'équivalence : y at-il un argument fort soutenant cette idée ? Quel est le rôle du finitism ? (Spécialement concernant le calcul infinitésimal.)

- Est-il possible d'étendre l'équilibre entre les machines, l'algèbre et la géométrie afin de construire la fonction Γ de Euler, inclure le calcul fractionnaire et résoudre des polynômes différentiels avec dérivées partielles ?

- Les machines algébriques peuvent elles être considérées comme des modèles des machines de Descartes ?

35. Métaphore doit être considérée comme la "métaphore conceptuelle" de Lakoff and Núñez [2000].

36. Du moins, concernant les systèmes de polynômes différentiels sans le problème des valeurs initiales. Cependant, il n'a pas été prouvé que la relation d'équivalence ne est pas calculable avec conditions initiales.

- Concernant les machines différentielles : peuvent elles être considérées comme une “extension conservatrice” des machines algébriques ? [37]
- Est ce que l’équivalence entre les machines différentielles est calculable considérant les problèmes de valeur initiale ?

De plus, concernant une approche plus construtrive et concrète dans la didactique des mathématiques, les buts et sujets futurs sont tournés vers :

- les machines différentielles concrètes pour les activités de laboratoire ;
- le développement d’un progiciel de géométrie dynamique ;
- une restructuration du calcul infinitésimal avec machines et algèbre différentielles.

En particulier, la possibilité d’une restructuration du calcul infinitésimal est intéressante d’un point de vue instrumental, visuel, algébrique, fondationnel et cognitif.

37. J’ai introduit les machines différentielles comme une extension des machines algébriques afin de résoudre le problème inverse de la tangente : mais pourquoi ai-je choisi précisément ce problème ci ? Y-a-t-il une justification logique ou algorithmique ?

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Finally, not respecting the chronological order, I owe special thanks to one person who, after some conferences at which we met, really changed the way of my research, someone who is able to see through my thoughts even though they lie hidden inside my deepest confusions, someone who is able not only to share his vision, but also to clarify and orient mine. It was his supervision that improved my way of thinking from purely visionary/intuitive to somehow rational. For all my ignorance, your help gave this work a shape: Thank you Marco!

Chapter 1

Preface

Theories of the known, which are described by different physical ideas, may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in one's attempt to understand what is not yet understood.

Richard P. Feynman, *acceptance speech of the Nobel Prize in Physics 1965* (printed in Feynman [1966])

In *La Géométrie*, Descartes proposed a "balance" between geometric constructions and symbolic manipulation with the introduction of suitable ideal machines. In particular, Cartesian tools were polynomial algebra (analysis) and a class of diagrammatic constructions (synthesis). This setting provided a classification of curves, according to which only the algebraic ones were considered "purely geometrical." This limit was overcome with a general method by Newton and Leibniz introducing the infinity in the analytical part, whereas the synthetic perspective gradually lost importance with respect to the analytical one—geometry became a mean of visualization, no longer of construction.

Descartes's foundational approach (analysis without infinitary objects and synthesis with diagrammatic constructions) has, however, been extended beyond algebraic limits, albeit in two different periods. In the late 17th century, the synthetic aspect was extended by "tractional motion" (construction of transcendental curves with idealized machines). In the first half of the 20th century, the analytical part was extended by "differential algebra," now a branch of computer algebra.

This thesis seeks to prove that it is possible to obtain a new balance between these synthetic and analytical extensions of Cartesian tools for a class of transcendental problems. In other words, there is a possibility of a new convergence of machines, algebra, and geometry that gives scope for a foundation of (a part of) infinitesimal calculus without the conceptual need of infinity.

The peculiarity of this work lies in the attention to the constructive role of geometry as idealization of machines for foundational purposes. This approach, after the "de-geometrization" of mathematics, is far removed from the mainstream discussions of mathematics, especially regarding foundations. However, though forgotten these days, the problem of defining appropriate canons of construction was very important in the early modern era, and had a lot of influence on the definition of mathematical objects and methods. According to Bos's definition [1], these are "exactness problems" for geometry.

Such problems about exactness involve philosophical and psychological interpretations, which is why they are usually considered external to mathematics. However, even though lacking any final answer, I propose in conclusion a very primitive algorithmic approach to such problems, which I hope to explore further in future research.

From a cognitive perspective, this approach to calculus does not require infinity and, thanks to idealized machines, can be set with suitable "grounding metaphors" (according to the terminology of Lakoff and Núñez [2000]). This concreteness can have useful fallouts for math education, thanks to the use of both physical and digital artifacts (this part will be treated only marginally).

1.1 Exactness of constructions

Is it possible to fix a canon of geometric construction for some classes of curves beyond the algebraic ones? Moreover, can this canon extend in a certain way the classical insight of geometry without the need of introducing the infinite in the analytical counterpart?

To set these questions, let us go back to the second half of the 17th century when Descartes's *La Géométrie* was influent, algebra was introduced as an analytical tool for geometric problems, and "geometric" curves reached a widely recognized ontological legitimacy. Descartes's work suggested a canon to distinguish between these curves (intuitive and analytically treatable objects) and "mechanical" ones (all the other curves): in modern terms, this corresponds to the distinction between algebraic and transcendental curves. Previously, algebra and geometry, even if deeply interrelated, had very different roles—algebra was just a "symbolic manipulation method" to help answering questions about geometric or arithmetic entities, entities whose existence was (theoretically) legitimized independently, such as by geometric constructions.

The use of curves that could not be considered geometric from a Cartesian perspective raised the question of the "legitimation of the transcendental curves" [2] by certain geometric tools extending the ones allowed by Descartes—for example with "tractional constructions," which will cover a substantial role in this thesis.

This is the first point we have to focus on. In contrast to the original finitistic perspective of Descartes's algebra, Leibniz and Newton, given the failure of polynomial algebra to deal with transcendental problems, developed new methods introducing not just finite, but also infinite and infinitesimal entities, On the other side, the tractional motion, even if not respecting the Cartesian paradigm, was not—or at least can be considered not—substantially stranger to it, and, even if in a to-be-defined way, these new constructions can be considered an extension of Cartesian geometric tools.

1. The main reference about this topic is Bos [2001].
2. Cf. Bos [1988].

The foundational inspection of tractional motion lasted about 60 years [3], and, contrary to what happened to Cartesian geometry, these researches ended without any widely accepted canon of constructions. This lack was probably favored by the change of paradigm from geometry to symbolic manipulation. In fact, the possibility of representing a curve just by a formula (even if recalling infinitary entities [4]) was much more convenient (for manipulation) than a representation by geometric constructive procedures. Thus, the ontological legitimation of algebraic curves through constructions gradually came to lose its importance, giving rise to the practical result that a mathematical object was considered completely defined just by a formula. This change of paradigm, from geometric to symbolic representation, firstly just adopted in practice, found its basis in the second half of the 19th century in the "arithmetization of the analysis." This marked the completion of the passage from a "geometric" foundational perspective of mathematics—objects are constructed by geometric constructions and symbolism is useful to analyze or characterize them—to an "arithmetic" foundation—objects are obtained as sets of real numbers and geometry is useful to visualize them.

To sum up, Descartes's legacy favored two fundamental changes:

1. the passage from finite to infinitary tools in analysis;

2. the passage from geometric to arithmetic foundation.

If the first one was something deemed necessary to manage problems beyond algebraic ones, the second was pursued to give rigor to an intuitive and imprecise geometric foundation, and to generalize and extend geometric methods that remained inadequate to study new curves.

In light of these remarks, the questions at the beginning of this section can be better rephrased: Is it possible to define a new canon of constructions in order to extend in a conservative way (geometric and finite) Cartesian geometry to differential geometry? With regard to the analytical tools necessary for these constructions, is it possible to avoid infinitary entities and use a finitist conservative extension of polynomial algebra? I claim to provide positive answers to both questions.

According to Bos, the problem of defining the allowed canon of geometric constructions is called "exactness problem." This problem was deeply present during the early modern period, but fell into total oblivion after formulas were widely accepted by the mathematical community as the best representation for mathematical objects. In particular, I will propose a canon of constructions based on a specific definition of some machines obtained by assembling suitable components (these machines are a reinterpretation of the "tractional motion" tools), and this canon will be legitimized by a suitable interpretation of the role of machines in algebra and geometry. Furthermore, as Cartesian canon

3. Tractional instruments were studied mostly between 1692 and 1752, and there were many mathematicians interested in: Huygens, Leibniz, Jean and Jacques Bernoulli, L'Hospital, Varignon, Fontenelle, Bomie, Fontaine, Jean-Baptiste Clairaut and his son Alexis-Claude, Maupertuis, Euler and, in Italy, Vincenzo Riccati, Giovanni Poleni and Giambatista Suardi (cf. Tournès [2007]).

4. Henceforth, I will not use different words to distinguish between infinite or infinitesimal entities, because I just want to focus on the introduction of actual infinite in constructions: so "infinitary entities" will denote both of them.

produced a "balance"[5] between allowed machines (geometric linkages), analytical tools (algebra), and geometric curves (algebraic ones), I also claim that the proposed machines, analytical tools, and geometric curves—all of them seen as conservative extensions of the Cartesian ones—will determine a new balance between the objects generated in these different domains.

1.2 The role of machines

The aim of this work is to show that it is possible to go beyond the limit of Descartes's geometry while still remaining within a genuinely geometric and finitistic perspective[6]. To develop my argument, I want to extend the Cartesian canon still maintaining its general perspective. In particular, I will move on from polynomial algebra to differential algebra[7] and from algebraic curves to differential ones. However, the most important difference is in the role of machines, which I will explain shortly.

In Euclid's works, there is no reference to compass and ruler, because the exactness of his planar geometry is given by the possibility of constructing lines and circles given two points, without any need of references to physical or idealized tools necessary to concretely realize a certain diagrammatic construction. This suggests thinking of Euclid's geometry as something static. In other words, to consider Euclid's geometry there is no need to introduce machines, which, during their motions, trace segments or circles—these machines will just rely on the practical construction of diagrams. On the contrary, to legitimize algebraic curves, Descartes needed to introduce a class of tracing machines (geometric linkages) to construct curves beyond lines and circles. So, even if the objects of Cartesian geometry are static curves, there is the need of defining the allowed class of tracing machines so as to distinguish the curves that can be considered geometric from the others. In particular, Descartes started off from some theoretical machines and constructed the elementary curves that can be recursively used to construct new curves. In contrast, I propose that curves (or more generally varieties) are just the loci of the possible configurations that suitable machines can reach (in a sense specified below). This implies that I am less interested in geometric constructions obtained through the use of tracing machines and traced curves, and more directly interested in the components of the machines themselves. Thus, when introducing the acceptable tools in subsequent chapters, I will start specifying the idealized physical components of the allowed machines. In particular, I will first put forward the components defining real algebraic varieties, and then extend them by the introduction of one more component to enter into the differential landscape.

Furthermore, to give a precise definition of what a machine is and what

5. Here "balance" denotes the possibility of converting objects from a certain domain A to another B. In particular, every object obtainable in A has to correspond to an object obtainable in B and vice versa.

6. Note that also Descartes's foundational process can be seen as "conservative extension" of Euclid's planar geometry (cf. Panza [2011]).

7. The term "differential algebra" refers to the area of mathematics comprising the study of rings, fields, and algebras equipped with an operation of derivation, which is a linear unary function that satisfies the Leibnizian product rule. In particular, these tools are used for an algebraic study of the differential equations. Differential algebra has essentially been introduced by Ritt [1932].

generates, I will adopt a "behavioral approach" of mathematical modeling (part of the mathematical theory of systems and controls).

1.3 Schema of the work

This journey begins in chapter 2, which provides some historical and philosophical considerations related to the definition of geometric construction canons (the "exactness problem"). This part is essentially compilatory and based on secondary literature but is necessary to introduce the specific problems as they historically developed. In particular, after a first overview of Euclid's constructions and some extensions of the Classical period, I focus on Descartes's canon of constructions and the successive geometric attempt to overcome Cartesian constructions through the "tractional motion."

The core of the work is made up of the central chapters, where I introduce the machine-based approach and its precise setting through a "behavioral approach" of mathematical modeling (section 3.1). According to the machines explored in the historical part, I introduce the allowed components as a suitable abstraction of tractional instruments. Using such a setting, I move on to analytically define the limits of tractional constructions (which was an open problem). We will see how the proposed reinterpretation of tractional motion can be considered as an extension of Euclid's and Descartes's geometries. Furthermore, with the behavioral approach, I evince the deep relation between machines, geometric constructions, and a finite symbolic manipulation theory (differential algebra) in order to answer suitable questions about equality of different machines, or about general constructions. In contrast to Descartes's setting, my machines presently do not have a precise philosophical justification of the allowed components, so I hope that in the future the paradigm of geometric constructions based on machines will be extended even beyond the limits of this thesis, but still satisfying the requirements of deep correlation with idealized machines and finite analytical tools.

More specifically, in this work I trace a parallelism between machines, algebra, and geometry [8] in a three-step extension: I begin with Euclid's planar geometry in section 3.2, set Cartesian geometry in section 3.3, and finally extend it to differential objects in chapter 4 (remaining on finite analysis). I introduce appropriate machine models to define these three geometries, calling them respectively "classical," "algebraic," and "differential" machines. Additionally, in order to see some applications of differential machines, I deepen their use to solve complex differential equations in chapter 5, and propose suitable examples to clarify and better explore the model about concrete machines in chapter 6, along with some sketched proposals in math education.

Finally, in chapter 7, I give some conclusions. I explain how the balance between machines, algebra, and geometry beyond Descartes can be considered

8. It is necessary to clarify something about the meaning of these fields because they assumed very different meanings owing to their historical evolution. "Algebra" for us denotes the study of symbolic manipulations (modern computer algebra) and "geometry" is about geometric constructions (even if with instruments beyond the classical ones). Therefore, I am interested in a procedural approach and not in an abstract one about structures as the mainstream of modern mathematics.

as a “conservative extension” of Cartesian program. I also focus on some foundational reflections on the possibility of avoiding infinite/infinitesimal objects to treat a part of calculus, from both cognitive and computational standpoints. Furthermore, even though only as a preliminary attempt, I propose a possible setting of exactness not as a meta-mathematical problem but as a computational one.

Chapter 2

Historical introduction

This journey begins with some preliminary historical and philosophical considerations related to the definition of geometric construction canons (the "exactness problem"). The present chapter on the historical foundation of this thesis is essentially unoriginal and based on secondary literature. Following an initial overview of Euclid's constructions and some extensions of the classical period, it deals with Descartes's canon of constructions and with the successive geometrical attempt to overcome Cartesian limits with the "tractional motion."

2.1 Geometric constructions in Classical and Hellenic period

If we take a wide look at the history of mathematics, we find two great traditions of operative approaches: the "geometric" (or constructive) and the "algebraic" (or computational) [1]. If today the mainstream vision of mathematics is algebraic, the historically most lasting perspective was the geometric one, mainly thanks to Euclid's *Elements* [2].

2.1.1 Compass and straightedge constructions

Euclid's plane geometry constructions involve the use of lines and circles to recursively generate points in their intersections. According to the practical drawing of lines and circles on a sheet, this construction are usually called "compass and straightedge constructions," and are probably the idealization of the ancient "peg and cord" constructions. The reason for the choice of such tools was lost in the past, but many reconstructions have emerged ever since (about philosophical, epistemological, and religious standpoints [3]). More concretely, there are also technical reasons. In fact, these constructions were particularly useful because they were quite precise in practice and general in application [4].

1. Cf. Seidenberg [1978].
2. For a translation, see Heath et al. [1956].
3. Cf. Seidenberg [1961].
4. "La soluzione ottenuta con riga e compasso aveva due caratteristiche che la rendevano particolarmente utile: innanzitutto aveva un errore relativo molto piccolo (dell'ordine del rapporto tra spessore e lungheza di una linea disegnata) e nessuna applicazione tecnica poteva aspirare a una precisione maggiore; inoltre era facilmente riproducibile per risolvere problemi

Thus, the theory developed in Euclid's *Elements* provides a mathematical model of the activities available with these tools, a model well set in rigorous scientific canons: In particular, the existence of geometric objects is provided by their constructability.

The constructive power of these tools is captured by the following axioms:

(R) Given two distinct points A, B, it is possible to construct the line through A and B.

(C) Given two distinct points A, B, it is possible to construct the circle with center A and radius AB.

Given any two (distinct) elements that intersect, their points of intersection are tacitly supposed to be constructed as well. The straightedge is not allowed to be marked, and the compass is not allowed to be used as a divider. That is, compasses have to be set according to postulate (C) every time they are used. The second and third propositions of the first book of the *Elements* show that the straightedge and "classical compass" can be used to simulate what is sometimes called the "modern compass," which can perform the following operation that is more general:

(MC) Given two distinct points B, C, it is possible to construct the circle with center A and radius congruent to BC.

Thus, the adoption of a "modern compass" in place of a "classical one," even though simplifying some constructions, does not extend the class of solvable problems. As I am going to note, other generalizations of the allowed tools will imply an extension of constructions.

2.1.2 Neusis constructions

Classical Greek geometry recognized that certain problems, such as doubling a cube, trisecting an angle, squaring a circle, and constructing certain regular polygons, did not appear to be possible using a compass and an unmarked straightedge alone [5]. The problems themselves, however, are solvable, and the Greeks knew how to solve them, without the constraint of working only with straightedge and compass.

Archimedes knew [6] that the addition of two marks on the straightedge was enough to make the trisection of the angle and duplication of the cube possible. The classical Greek literature provides several other examples of tools permitting otherwise impossible constructions. The majority of them, such as the marked straightedge, permitted the construction of cube roots, hence the solution of all cubics (and quartics). Others were more powerful; the Archimedean spiral permits the n-section of any angle [7] and thus the construction of any polygon; and the quadratrix of Hippias allows the circle to be squared. In this subsection,

eguali con dati numerici diversi. [...] L'efficienza dell'algebra geometrica basata sulla riga e il compasso era strettamente connessa alla possibilità di effettuare precisi disegni su fogli di papiro." Russo [2001].

5. Cf. Heath [1981].

6. Cf. Archimedes' *The Book of Lemmas*, Prop. 8 (translated and reprinted in Hutchins [1952]).

7. This is implicit in Archimedes' *On Spirals*, Prop. 14 (translated and reprinted in Hutchins [1952]).

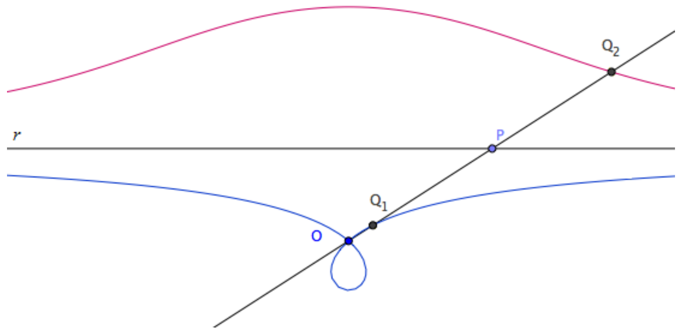

Figure 2.1: Consider a point O, a line r and a distance d. On a generic line through O that intersects r at P consider the points Q_1, Q_2 which are distant d from P. The loci of the points Q_1, Q_2 are two "conchoids."

I will focus on the extension of Euclid's tools with the marked straightedge, the so-called *neusis* constructions.

For example, the use of a markable ruler permits the following construction. Given a line segment, two lines, and a point, one can draw a line that passes through the given point and intersects both lines so that the distance between the points of intersection equals the given segment. The Greeks called this "neusis" because the new line tends to the point [8]. In this expanded scheme, any distance whose ratio to an existing distance is the solution of a cubic or a quartic equation is constructible. It follows that, if markable rulers and neusis are permitted, the trisection of the angle and the duplication of the cube can be achieved; the quadrature of the circle is still impossible.

Moreover, using such tools it is possible to draw curves that are different from lines and circles: As evident from Fig. 2.1, neusis is at the core of the construction of the conchoid of Nicomedes. Nicomedes, like many geometers of the third century B.C., tried to solve the problems of doubling the cube and trisecting the angle, whereby he created the conchoid. If one allows the extension of geometric constructions not through an extension of the constructive postulates but through the introduction of new curves, the conchoid allows solving the duplication of the cube and the trisection of the angle without the introduction of the neusis.

Hence, Euclid's limits can be overcome with other tools. However, when can the new methods be considered acceptable in geometry? In general, what can be considered a legitimate geometric construction? These complex questions are now said to be about "geometrical exactness." With a chronological jump, the need of a new canon for constructions became even more relevant when algebraic tools provided a different way of problem solving, an analytical method requiring a legitimation in a geometric paradigm.

8. In Greek "neusis" means inclination, tendency, or verging.

2.2 Geometric exactness in the early modern period

In this section, I introduce the early modern concern for "exactness"[9] of geometrical constructions, evincing the pivotal role of Descartes's *La Géométrie*.

2.2.1 Bos's perspective

Amongst historians of mathematics, Henk Bos's works on Cartesian analytical geometry and Leibnizian calculus have provided the starting point for all those who have tried to understand the conceptual developments of mathematics in the crucial period between the Renaissance and the Enlightenment.[10] I can especially use Bos's interpretation of early modern mathematics as the starting point of this thesis.

The peculiarity of Bos [2001] is the perspective. Often problems with respect to the changes in concepts of geometric constructions, typical of the early modern period, are looked from the later perspective of mature analytical geometry. Contrarily, Bos focuses on a set of problems that were of high significance for mathematicians in that period of mathematical revolutions. These mathematicians were troubled by the following questions: Which constructions can be considered legitimate? Which ones are simpler? Regarding mathematical entities, when can they be considered totally achieved? What does it mean for a problem to be solved and its solutions to be found?

In this context, the concepts of exactness, certitude, and precision were frequently used and discussed because of the conceptual and methodological problems owing to the introduction of algebra as an analytical tool for the solution of geometric problems: How can algebraic solutions be considered legitimate inside of a universally accepted geometric paradigm? The geometric interpretation of the new algebraic techniques posed enormous problems, as evident in the effort that, even if in different ways, mainly Viète and Descartes spent on the problem. All these topics totally disappeared in the 18th century because of the general affirmation of symbolic procedures as autonomous from geometry. Even if historically forgotten, the reflection on such topics is central to the understanding of the evolution of mathematics in that revolutionary period.

In Bos's book, the main role is played by Descartes, that is analyzed with in mind these early modern questions about constructions. From this perspective, in contrast to the future interpretation of reducing the study of curves to their defining equations, Bos supports that for Descartes the equation is just a part of the definition of a curve (analysis), because, in order to practice geometry, one had to produce the geometrical construction (synthesis). Thus, even introducing the power of algebra into geometry, it refers only to the analytical part, while the synthetic counterpart is still necessary. From this perspective, Descartes did not depart from the ancient view of considering a solution known only if constructed out of geometrical elements.

My aim is to adhere to the ancient paradigm of geometric constructions, and to extend it to differential objects along a direction different from the one

9. "Exactness" and not "rigor" because the latter is commonly used in connection with proofs rather than with constructions.
10. Cf. Guicciardini [2002].

that historically became dominant (the introduction of infinitesimal entities in analysis).

2.2.2 Defining the exactness problem

In this subsection, I take some useful concepts from the "General Introduction" of [Bos, 2001, pp. 3–22]. The "exactness" of mathematics is an evolving idea. It was especially really fluid in the period between the Renaissance and the Enlightenment. As observed at the beginning of the chapter, the problem of exactness in geometric constructions was present at least since the classical era, specially to face problems not easily solvable just with Euclid's lines and circles.

Though soon after its publication, Cartesian geometry became a widely accepted canon, prior to Descartes's criterion, there had been many other attempts of exactness in the early modern period. The *leitmotiv* of such attempts was the need to answer the foundational problems given by the introduction of algebra as an analytical tool for the solution of geometric problems. In fact, a solution was acceptable if it could be justified by a suitable geometric interpretation. It was thus necessary to define the acceptability of solutions and constructions in the geometric paradigm. From the 16th to the 18th centuries, many mathematicians asked themselves, in the new context, what it meant for a problem to be "solved" or for a mathematical object to be "known." According to classical Greek geometry, these questions were both answered by the acceptance of a canon for geometric constructions, usually by straight lines and circles constructions. However, in the early modern period, algebra strength in problem solving suggested overcoming the classical means of construction to interpret its solutions geometrically: There was a need for a new canon of acceptable procedures.

Since the classical times, Euclid's geometry was extended by families of curves out of lines and circles [11]. The introduction of algebra as a tool for geometric problem solving caused the growth of the constructible curves [12], so the question of when a curve was sufficiently known, or how it could acceptably be constructed, acquired a new urgency. Bos calls "representation of curves" the descriptions of curves that were considered to be sufficiently informative to make the curves known. For representing curves, mathematicians resorted to the means which geometry offered for making objects known—the conceptual apparatus of "construction."

Therefore, the exactness problem became the problem of choosing acceptable means of construction. Even if this choice was justified by a meta-mathematical argument (similar to the choice of axioms in a theory), the reasons for or against accepting procedures of construction were very important in the development of mathematical practice: they determined directions in mathematical research, and they reflected the mental images that mathematicians had of the objects they studied. To justify which procedures were acceptable, mathematicians had to explain, to themselves or to others, what requirements would make mathematical procedures exact in the above sense. Thus, they had to interpret what it means to proceed exactly in mathematics. Bos calls this activity

11. Mainly as loci or as intersections of solids.

12. Previously, the curves were just conic sections, the conchoid of Nicomedes, the cissoid of Diocles, the Archimedean spiral, and the quadratrix of Dinostratus.

the "interpretation of exactness," and suggests the following cases of basic attitudes: appeal to authority and tradition; idealization of practical methods; philosophical analysis of the geometrical intuition; appreciation of the resulting mathematics; refusal/rejection of any rules, or non-interest. The most influential cases were probably the "philosophical analysis of geometrical intuition" (Descartes's approach), which required a cognitive attention on how geometric intuition can be transformed in acceptable procedures, and the lack of interest (Leibniz' approach), in which, as in the dominant modern mathematical standpoint, procedures are justified by their utility in problem solving.

The structure of the story of construction and representation in early modern mathematics is basically simple. It comprises two slightly overlapping periods, c. 1590–c. 1650, c. 1635–c. 1750, and one central figure, Descartes. During the first period, questions about construction arose primarily in connection with geometrical problems that required a point or a line segment to be constructed and admitted one or at the most a finite number of solutions (e.g. dividing an angle in two equal parts, finding two mean proportionals between two given line segments). If translated into algebra, problems of this type led to equations in one unknown. Around these problems a considerable field of mathematical activity developed, which may be considered as the early modern tradition of geometrical problem solving. Indeed, the adoption of algebraic methods of analysis provided the principal dynamics of the developments in the field.

In Bos's opinion Descartes's *La Géométrie* of 1637 derived its structure and program from this field of geometrical problem solving. The two main themes of Descartes's book were the use of algebra in geometry and the choice of appropriate means of construction. The approach to geometrical construction that he formulated soon eclipsed all other answers to the question of how to construct in geometry. Thus, Descartes closed the first episode of the early modern story of construction by canonizing one special approach to the interpretation of exactness concerning geometrical constructions.

Nevertheless, *La Géométrie* may also be seen as the opening of a second period lasting until around 1750. In this period, the problems that gave rise to questions about construction and representation were primarily quadratures and inverse tangent problems. These belong to a class of problems in which it is required to find or construct a curve. If translated in terms of algebra, these problems lead to equations in two unknowns, either ordinary (finite) equations or differential equations. It was from this field that, in the period 1650–1750, infinitesimal analysis gradually emancipated itself as a separate mathematical discipline, independent of the geometrical imagery of coordinates, curves, quadratures, and tangents, and with its own subject matter, namely, analytical expressions and, later, functions. This process of emancipation, which might be called the "de-geometrization of analysis," constituted the principal dynamics within the area of mathematical activities around the investigation of curves by means of finite and infinitesimal analysis. It was strongly interrelated with the changing ideas on the interpretation of exactness with respect to construction and representation.

Although the interpretation of exactness with respect to geometrical construction and representation was discussed with some intensity during the early modern period, no ultimately convincing canon of geometric constructions was found to face the problems of the second period. By 1750 most mathematicians had lost interest in issues of geometrical exactness and construction; they found themselves working in the expanding field of infinitesimal analysis, which had by then outgrown its dependence of geometrical imagery and legitimation.

These changes were brought about by such processes as the habituation to new mathematical concepts and material, and the progressive shift of methodological restrictions. By habituation, a mathematical entity that was earlier seen as problematic (such as some transcendental curve) could later serve as solution of a problem, even though the mathematical knowledge about it had not changed essentially. Methodological restrictions were mitigated or lifted as the result of conflicts around the legitimacy of procedures and because of the appeal of new mathematical material.

2.2.3 Analysis and synthesis in *La Géométrie*

Early modern exactness problem dealt with the definition of appropriate norms for deciding if some objects, procedures or arguments can or cannot be considered geometrical. All the various attempts in this direction had a minimal common basis, given by Euclid's plane geometry, suitably extended. I previously said that Descartes provided a widely accepted canon of geometrical construction: His strength was given by his perspective of a "philosophical analysis of geometric intuition." In fact, I have to remind that, quoting Bos:

> "The *Geometry*[13] served as an illustrative essay accompanying the *Discourse on the method.* Descartes did not explicitly discuss the links between the method of the *Geometry* and the general rules of methodical thinking expounded in the *Discourse.* Yet, for instance, the second and third of the four rules expounded in Part 2 of the *Discourse*[14] might easily be seen as exemplified by the procedures of analysis and synthesis, respectively, as detailed in the *Geometry.*
>
> Indeed the method of the *Geometry* consisted of:
>
> - An analytic part, using algebra to reduce any problem to an appropriate equation;
> - A synthetic part, finding the appropriate construction of the problem on the basis of the equation."[15]

The construction of a curve had to be obtained as the simplest possible: This simplicity should be achieved reducing the geometrical problem to an algebraic equation (in one unknown) of lowest possible degree, later transformed in a certain standard form. This algebraic part, even if essential, was just one of the two parts of the method:

13. Bos calls Descartes's *La Géométrie* (appendix of Descartes [1637]) simply *Geometry.*
14. The translation in [Descartes, 1985, p. 120]: "The second, to divide each of the difficulties I examined into as many parts as possible and as may be required in order to resolve them better. The third, to direct my thoughts in an orderly manner, by beginning with the simplest and most easily known objects in order to ascend little by little, step by step, to knowledge of the most complex, and by supposing some order even among the objects that have no natural order of precedence."
15. Cf. [Bos, 2001, p. 287].

> "[t]he fact that algebra does not provide geometrical constructions merits emphasis because too often Descartes's contribution to geometry is presented as the brilliant removal of cumbersome geometrical procedures by simply applying algebra. In fact, algebra could only do half of the business, it could provide the analysis and reduce problems to equations. The other half of the job, the synthesis, the geometrical construction of the roots of the equations, remained to be done.
>
> The synthetic part of Descartes's program presented the most profound questions. They concerned the conception of geometrical construction itself, in other words the interpretation of constructional exactness. That interpretation required a demarcation of the class of curves acceptable for use in constructions and a criterion to judge the simplicity of these curves [...]: acceptable curves were traced by acceptable motions; they were precisely those that had algebraic equations; they were simpler in as much as their degree was lower." [16]

With regard to Descartes's main purpose in geometry, I propose two interpretations: the first one is that his purpose was to provide a general method for geometrical problem solving (Bos [2001]), while the second one (discussed in detail in the next subsection) is that the origin was to get a "conservative extension" of Euclid's geometry (Panza [2011]). These visions are not mutually exclusive, and for my standpoint it is not so important to consider one of them most basilar than the other. Quoting Bos:

> "By 1635 [...] the first generation of mathematicians active in the early modern tradition of geometrical problem solving had passed away. In their time the major innovation in the field was Viète's use of his new algebra [17]. Some mathematicians, Clavius , for instance, paid no attention to this innovation; Kepler even rejected the use of algebra in geometry. But it seems that by 1635 the practice of geometrical problem solving without algebra [...] had vanished from the scene of active mathematical investigation." [18]

After that, regarding Descartes's *La Géométrie*:

> "The core of its influence consisted in the spread of Descartes's insights and techniques about the relation between curves and their equations or, more generally, about the interplay between figures and formulas. [...] My analysis of the *Geometry* in the preceding chapters has shown, however, that Descartes's main motivation in writing

16. Cf. [Bos, 2001, p. 288].
17. Two kinds of analysis were distinguished in early modern geometry—the classical and the algebraic. The former method was known from examples in classical mathematical texts in which the constructions of problems were preceded by an argument referred to as "analysis;" in those cases the constructions were called "synthesis." Particularly for plane problems, the method of analysis by means of the concept "given" was codified in Euclid's *Data*, of which a Latin translation was available in print since 1505. The latter method, based on the use of algebra, consisted in reducing the problem to an equation, that would have later been explored by algebraic manipulations. This use of algebra in geometry had been pioneered by some Renaissance mathematicians before 1590, but it was Viète's conscious identification of this method with analysis that brought it into the center of attention.
18. Cf. [Bos, 2001, p. 415].

> the book was not to expose the equivalence of curve and equation. Rather, it was to provide an exact, complete method for solving "all the problems of geometry." [...] Thus, [...] the main influence of the book did not concur with its program. Indeed the *Geometry* exerted its main influence despite its primary motivation." [19]

2.2.4 Cartesian canon of constructions

Even if Panza [2011] agrees with the importance of problem solving in Descartes's program, he goes further and proposes that "Descartes's primary purpose in geometry appears to be a foundational one, and his addressing the exactness concern appears as a crucial ingredient of this purpose" [20], namely that of obtaining a "conservative extension" of Euclid's plane geometry. Panza bases his reconstruction on an analysis of the ontology of Euclid's geometry. In contrast to modern mathematical theories, Euclid's geometrical ontology "is composed of objects available within this system, rather than objects that are required or purported to exist by force of the assumptions that this system is based on and of the results proved within it" [21]. These objects to be available have to be constructed. Euclid's constructions require that appropriate diagrams be drawn, and these constructions are just "procedures for drawing diagrams in a licensed way, to the effect that an EPG [22] problem is solved when appropriate diagrams, representing some objects falling under the concepts this problem is concerned with, are so drawn, or imagined to have been drawn" [23]. To trace curves beyond straight lines and circles, it is fundamental to define the role that instruments have in "diagrammatic constructions." More precisely, these instruments can be constructively used on a plane in two ways:

> "either in the tracing way, i.e., by making them trace a curve; or in the pointing way, i.e., by making them indicate some points (which are then taken to be obtained) under the condition that some of their elements coincide with some given geometrical objects, or meet some other conditions relative to given objects. If an instrument is used in the former way, once a curve is traced, it can be put away, and this curve taken as constructed. If it is used in the latter way, the sought-after points can only be indicated by appropriate elements of it. [...] This suggests two different sorts of constructive clauses, licensing respectively obtaining curves by tracing them through instruments, and obtaining points by using instruments in the pointing way." [24]

The latter use implies that one can move (parts of) the instruments "until they reach a position that satisfies a coincidence condition relative to other diagrams representing some given geometrical objects" [25]. This is a use of diagrams essentially different from Euclid's, where coincidences are not acknowledged by inspecting moving diagrams but imposed on fixed diagrams by drawing them.

19. Cf. [Bos, 2001, p. 416].
20. Cf. [Panza, 2011, p. 44].
21. Cf. [Panza, 2011, p. 43].
22. Euclid's plane geometry.
23. Cf. [Panza, 2011, p. 51].
24. Cf. [Panza, 2011, p. 62].
25. Cf. [Panza, 2011, p. 65].

Descartes's geometry is today usually considered as the beginning of modern mathematics, because of the revolutionary possibility of describing and analyzing classes of geometrical curves through equations. However, here I want to focus on the genetic relation that Cartesian geometry has with classical one. In particular:

> "EPG is often described as dealing with ideal and immutable self-standing objects or forms, which we can only inaccurately depict. If EPG were so understood, the use of instruments in geometry (both in the pointing and in the tracing way), and more generally the appeal to motion, should be considered as entirely extraneous to its spirit, unless they were merely seen as tricks for achieving convenient depictions of ideal forms. The situation is different if it is granted that EPG objects are obtained through diagrammatic constructions. It then becomes natural to consider the admission of new procedures for drawing diagrams, also by using instruments, as a proper way of conservatively extending EPG.
>
> In classical geometry, the use of instruments to obtain geometrical objects did not go together with fixing precise conditions that such a use of an instrument had to submit to. As a matter of fact, this made the exactness norms of geometric objects inaccurate and contributed highly to the fluidity of classical geometry." [26]

This gives the motivation of Descartes's foundational program. In contrast to the other attempts before him, Descartes's geometry is a closed system with an ontology composed of objects available within it through precisely defined diagrammatic constructions: These well-framed boundaries can be seen as a conservative extension of Euclid's geometry. However, I still have to be precise about the admissible instruments and to justify their acceptance.

In the *La Géométrie* Descartes criticized the "ancients" for having termed "mechanical" any curves other than circles and conics, because also circles and straight lines "cannot be described on a paper without the use of a compass and a ruler, which may also be termed instruments" [27]. According to the previously introduced terminology, Descartes excluded from geometry the use of instruments in the "pointing way," according to an interpretation of diagrammatic construction coherent with Euclid's one. So the search for exactness norms is reduced to the identification of an appropriate class of instruments (later denoted "geometrical linkages" [28]) that, when used in the tracing way, trace curves that are admitted in geometry just because they can be so traced (Descartes named these curves "geometrical").

Regarding these instruments, Descartes did not precisely define geometrical linkages, but, in a more or less explicit and general way, he put some requirements that such machines have to satisfy. Even though I will not enter in the problems of suitably defining acceptable geometrical linkages, I have to cite that, according to [Panza, 2011, section 3.2], it is possible to characterize "geometrical" curves as objects obtained by ruler, compass and reiteration. Strengthening

26. Cf. [Panza, 2011, p. 74].
27. Cf. [Descartes, 1954, p. 43].
28. In few words, we can consider "geometrical linkages" as articulated devices basically working as joint systems, allowing a certain degree of freedom in movements between the two links they connect.

the connection between Descartes's and Euclid's canons, this perspective focuses on the way in which the first one is an extension of the second.

2.3 Beyond Cartesian tools

With respect to the consequences of Cartesian geometry, Bos asserted that:

> "fairly soon after Descartes's *Geometry* mathematicians were so far habituated to algebraic curves that the equation of such a curve no longer presented a problem (how to construct the curve with that equation); rather it represented an object (the curve with that equation).
>
> The habituation to non-algebraic curves took more time. This was partly because the representation of such curves was far from trivial; there were (at least until c. 1700) very few notational means available to express their equations. In the absence of analytical means of representation, a non-algebraic curve could only be imagined and talked or written about in terms of a geometrical procedure to construct or trace it. In the case of non-algebraic curves, these procedures involved combinations of motions, or pointwise constructions, which Descartes had expressly banned from genuine geometry because, in the case of non-algebraic curves, they did not provide proper knowledge of the objects.
>
> A number of mathematicians felt that a reinterpretation of geometrical exactness was needed, overcoming the obstacle of the restrictive Cartesian orthodoxy. Thus, in the second half of the seventeenth century, Descartes's ideas about genuine geometrical knowledge induced a new debate on the interpretation of exactness in connection with the proper representation of non-algebraic curves." [29]

Even though with some exceptions (specially in Great Britain), soon after the geometrical revolution of Descartes, it was suddenly accepted the analytical part of its program (a well-framed introduction of algebra in geometry), while the interest in geometric constructions remained alive just to justify transcendental curves (not treatable with polynomial algebra). Especially, even if non-algebraic curves were well known by Descartes (examples of mechanical curves included the quadratrix, the Archimedean spiral, the cycloid), it was the "inverse tangent problem" that generated a wide class of curves for which Descartes's tools were not powerful enough. Therefore, it was time to overcome Cartesian canons through an extension of the allowed "tracing machines" to perpetuate the paradigm of geometrical constructions (there was the acceptance of a kind of motion considered non-geometric by Cartesian canon, the "tractional" one). However, if Descartes's reasoning was oriented to a closed class of constructible objects, the new attempt was much more oriented to mathematical freedom. In this vision it is important the role of Leibniz, that hardly opposed to Cartesian restrictions [30]. Behind the collapse of the geometric paradigm in

29. Cf. [Bos, 2001, p. 424].

30. Another interesting vision is Newton's one. For him a curve, to be represented, needs a geometrical description (similarly, he also criticized the acceptance of the algebraic degree,

front of the power of the analytical counterpart there was the passage from "finite" to "infinitary" entities [31], unreachable with finite instruments of diagrammatic constructions. In this thesis I suggest to exhume the paradigm of geometric constructions in order to avoid the use of entities and procedures more or less implicitly recalling the infinity, so to finitely extend the balance between machines, geometry and algebra beyond Descartes.

2.3.1 A brief history of Tractional motion

The problem of extending geometry beyond Cartesian limits was dominant between 1650 and 1750, and in this subsection, I will shortly deal with it.

If direct tangent problems [32] are present since the classical period, it was only in the second half of the 17th century that the inverse ones [33] appeared. The main difference between direct and inverse tangent problems is the role of the curve: in the direct case it is given *a priori*, while in the second the curve is the solution. Even though beyond Cartesian geometry, to legitimate solutions of inverse tangent problems there was the introduction of certain machines, intended as both theoretical and practical instruments, able to trace such curves. The first documented curves constructed under tangent conditions were physically realized by the traction of a string tied to a load, which is why the study of these machines was named "tractional motion" [34]. During this period, mathematicians like Huygens began to consider instruments that, like the handlebars of a bike, could guide the tangent of a curve (in analytical mechanics terms, they introduced "non-holonomic" constraints), thus signaling the rise of tractional motion. Tractional motion suggested the possibility of constructing curves by imposing tangential conditions, generalizing (in a non-Cartesian way) the idea of geometrical objects, and constructing with new tools not only algebraic curves, but also some transcendental ones (seen as solutions of differential equations). During this period, the development of geometrical ideas often corresponded to the practical construction (or at least conception) of mechanical machines able to embody the theoretical properties, and thus able to trace the curves.

While questions about exactness in geometric constructions were so important in the early modern period, they disappeared in the 18th century because of the general affirmation of symbolic procedures, later considered autonomous from geometry. Hence, in contrast to what happened for algebraic curves, tractional motion did not reach a widely affirmed canon of constructions. Moreover, due to the change in paradigm, the geometric-mechanical ideas behind tractional machines remained forgotten for centuries, even for practical purposes, and were independently re-invented in the late 19th century, when they were used to build some grapho-mechanical instruments of integration (integraphs) to analogically

rather than a more geometrical criterion, as the measure of simplicity). Nevertheless, Newton's vision was not related to the use of machines (which is my point of view), which is why I will not treat him.

31. I have already specified what I mean for "infinitary" in note 4, pag. 3.

32. Given a curve and a point on it, the solution of a direct tangent problem is the line tangent to the given curve at the given point.

33. The solution of an inverse tangent problem is a curve, so its tangent has to satisfy some given properties.

34. Cf. Bos [1988, 1989].

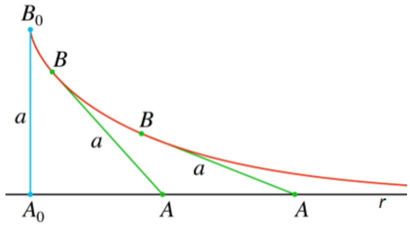

Figure 2.2: The heavy body is B, with initial position B_0, the string is a, and the other end of the string is A, with initial position A_0. Moving A along r, B describes the tractrix (obviously, the movement is not reversible because of the non-rigidity of the string). Note how a is tangent to the curve at every point.

compute symbolically non-solvable problems. Nevertheless, let me start with the first curve described in a tractional way, the "tractrix."

On a horizontal plane, consider a small heavy body (subjected to the friction on the plane) tied with an ideally weightless non-elastic string, and imagine (slowly) pulling the other end of the string along a straight line drawn on the plane. Because of the friction on the plane, the body offers resistance to the pulling of the string: if the motion is slow enough to neglect inertia, the curve described by the body is called a "tractrix." The first documented description of the tractrix is associated with Claude Perrault [35]. Examining Fig. 2.2, we can see how the curve is traced thanks to the property that the string is constantly tangent to the curve.

Christiaan Huygens [36] enhanced the theory of tractional motion, and moved toward a mechanical description to physically build some precise instruments for tracing. In fact, the original description of the tractrix is related to at least two physical problems: The tracing plane has to be perfectly horizontal, and the heavy body, when moved, acquires inertial velocity. Huygens suggested that, abstracting the problem from its physical complexity and considering it solely in terms of tangent properties, tractional motion can be seen as a "pure geometrical movement," independent from the motion speed. This is exactly the same as the circular motion of the compass, the straight motion of the ruler, and, in general, the continuous movement considered by Descartes as the basis of his geometry (even if with the strong difference that Huygens allows the presence of friction). In addition, Huygens introduced a technological change in the way straight components were considered: While a string only works in the case of traction, a physical rigid bar satisfies the tangent constraint (avoiding lateral motion) not only in traction, but also even in compression, making the curve realization reversible.

The foundations of tractional motion were laid, and, up to the first half of the 18th century, there was an improvement in related works, both in terms of practical machines (mechanical devices studied and realized to solve particular differential equations) and of theoretical studies. Concerning practical machines, I recall those introduced in Perks [1706, 1714] (see Fig. 2.3), which, for the first time, included a "rolling wheel" to guide the tangent (the same

35. In Leibniz [1693].
36. Cf. Huygens [1693].

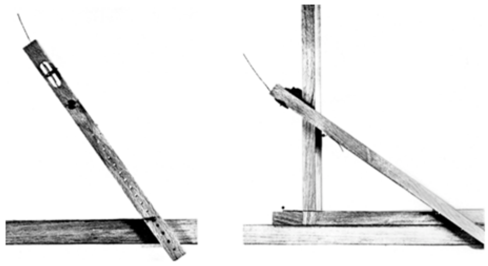

Figure 2.3: Reconstruction of Perks' instruments for the tractrix (left) and for the logarithmic curve (right) at the Institute for History of Science, Aarhus University. Regarding the tractrix, one can see the wheel taking the place of the load: in this case, the extreme point of the fixed-length bar can freely move along a straight line. In the machine for the logarithmic curve, a horizontal fixed-length plank moves along another horizontal bar to opportunely incline the slope.

solution was adopted in the 19th century for integraphs). Concerning theoretical evolutions, I have to recollect Leibniz's "universal tractional machine" [37]. According to him, tractional motion was the concrete realization of his vision of curves as "infinitangular polygons." Leibniz's approach was inextricably mixing the analytic representation with the physical execution, each one validating the other: from a certain point of view, kinematics forms the basis that mathematics without well-defined infinitesimal entities requires. Due to its complexity, the project, so important to a single theory able to realize the quadrature of any general curve with a continuous movement, never became a real device.

2.3.2 Leibniz's criticism of Descartes

With regard to geometry, Leibniz developed a concept very distinct from that of Descartes [38]. Even if Leibniz agreed to the Cartesian view that exactness is a geometrical matter, he criticized Descartes for his limits: While Descartes's geometry implied a static and inextensible class for acceptable geometrical objects, Leibniz adhered to a vision of mathematical objects as something fluid and dynamic. In particular, Leibniz main critic was based on the utility of transcendental curves, considered not exact in Cartesian geometry: Why objects such as spirals or the logarithmic curve have to be included or excluded from geometry? For him, if a construction is easy and useful, it has the right to become part of the mathematical practice. In fact, for Leibniz the core of mathematics is the solvability of problems. That is, both the exactness of geometric constructions and the analytic representation have not to be *a priori* delimited, but have to be suitably constructed.

With regard to the acceptable instruments for geometrical curves, Leibniz refused the restriction due to Cartesian "philosophical analysis of geometrical intuition," and proposed that acceptable curves be the ones that may be somehow physically defined in a simple way. The acceptance or not of strings is explicative of the differences between the idea of constructions of Descartes and Leibniz: Descartes argued that one should not accept lines in geometry which resemble strings

37. Cf. Leibniz [1693].
38. This subsection is based on Knobloch [2006].

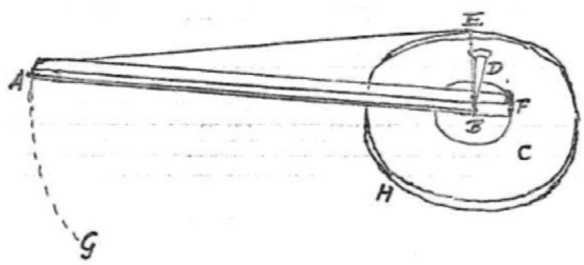

Figure 2.4: Huygens's instrument for tracing spirals from a manuscript of 1650 (described in [Huygens, 1888–1950, vol. 11, p. 216]).

> "that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and, I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact." [39]

On the contrary, the Leibnizian approach was much less restricted: A curve is acceptable if obtained by an exact, continuous motion, and this exactness may include a certain adaptation of a straight line to a curve so that the rectilinear motion is adapted to the circular. An example of curve acceptable to Leibniz but not to Descartes is provided by the spiral constructible with the instrument of Fig. 2.4.

Finally, through many different reformulations in which Leibniz enlarged more and more the class of geometrical objects, in 1693 he arrived at a bipartite geometry corresponding to a bipartite analysis. If the Cartesian canon corresponds to the "determinative geometry" (*geometria determinatrix*), the extension with tangent conditions [40] is the "metric geometry" (*geometria dimensoria*). These different kinds of geometry need different analytical tools. If the first one is translatable in the language of polynomial algebra, for the latter Leibniz introduced infinite entities (infinitesimals and infinite series to extend finite polynomials).

Therefore, Leibniz' introduction of infinitary entities in the analytic part was justified by its utility. This extension of polynomial algebra was adequate for transcendental objects that were simply constructible even if with tools beyond Cartesian geometrical linkages. In addition, the new exactness in geometrical constructions did not have to satisfy any particular idea of geometric intuition, but just had to be a suitable simple abstraction from the physical reality or just from imaginable machines.

In my opinion, the freedom so much desired by Leibniz (typical of modern mathematics) has to be mitigated by a reflection on what the acceptance of infinitary entities makes us lose from an intuitive and ontological perspective. Infinite is a powerful tool in analysis, but, given the related conceptual and practical problems (at least since Zeno), it is reasonable to ask ourselves when infinite is unavoidable and when it is just a possible way to approach a certain kind of problem. The main purpose of this thesis is to reconsider a class of transcendental (differential) problems from a point of view similar to

39. Cf. [Descartes, 1954, pp. 91–92].
40. Cf. Leibniz [1693].

Descartes', i.e. from a synthetic perspective with a clear and closed definition of the geometrical objects, and, from the corresponding analytical counterpart, with a finite extension of the polynomial algebra (through the reinterpretation of what a variable is). Specifically, my suggestion for a conservative extension of Descartes's machines is through a reinterpretation of "tractional motion" [41]. For this purpose, in the next subsection, I introduce what probably was the last original work on tractional constructions.

2.3.3 Vincenzo Riccati's theory of geometric integration

A unified theory for differential equations was actually developed by Vincenzo Riccati [42], the only complete theoretical work ever dedicated to the use of tractional motion in geometry [43]. The Italian mathematician (forth son of Jacopo Riccati, more famous than Vincenzo because of the differential equation named after him) found geometrical proofs corresponding to those that mathematicians such as Euler derived using series, arriving at the result that "every" curve defined in modern terminology by a differential equation $y' = f(x, y)$, can be drawn with tractional motion [44]. This result regarding transcendental curves overtook Descartes's announcement in relation to algebraic curves, and developed the theory of geometric construction with simple continuous movements. One characteristic of this work is the deep interaction among algebra, geometry, mechanics, and technology to develop an abstract unified theory of differential equations based on the conception of material instruments physically drawing the integral curves. His instruments plot the integral curve of a differential equation using tractional motion:

> "On a horizontal plane, one pulls one end of a tense string, or a rigid rod, along a given curve, and the other end of the string, the free end, describes during the motion a new curve that remains constantly tangent to the string. At this free end, one places a pen surmounted by a weight making pressure, or a sharp edged wheel cutting the paper, so that any lateral motion is neutralized. By suitably choosing the base curve along which the end of the string is dragged, and by suitably varying the length of the string according to a given law, one can integrate various types of differential equations. In this way of solving an inverse tangent problem, one actually materializes the tangent by a tense string and moves the string so that the given property of the tangents is verified at every moment. The length of the tangent is controlled at every moment by a mechanical system (a pulley or a slide channel) and by a second curve which is called the directrix of the motion." [45]

41. I consider "tractional motion" as the basis for the synthetic part and, even though I am not considering it in this chapter, "differential algebra" for the analytical one.
42. Cf. Riccati [1752].
43. This part is based on Tournès [2009].
44. Riccati showed that, adopting modern terminology, it is possible to integrate any differential equation $y' = f(x, y)$. However, he did not explicitly specify anything about the set of admissible functions f. According to the conceptions of the time, it is reasonable to assume that the function has to be obtained using only a finite number of algebraic operations and quadratures.
45. Cf. [Tournès, 2004, p. 2738].

Since the construction of the tractrix, it was already considered the idea of a tense string with an end (the tractor point) moving along a "base" curve and with on the other end (the point tracing the new curve) something avoiding lateral motion. The originality of Riccati is the introduction of a second curve, the "directrix."

To impose a certain constant length of the tangent string, one can consider the tracing point belonging to a moving circle centered in the point that generates the motion (the tractor point). This condition can thereby naturally be extended by considering not only constant radius circles, but also any general curve we call "directrix." Particularly, denoting the curves traced by tractional motion as "tractorias," Riccati's work begins treating tractorias with a constant tangent (described with a constant-length string dragged along a base curve), and then, as seen in Fig. 2.5, generalizes the constructions by allowing the integration of an increasing number of extended classes of differential equations. The final aim is to control the length of the tangent string by a variable directrix, whose form varies according to the position of the tractor point. The preliminary steps are tractorias with a constant directrix (the directrix, constant but not necessarily a circle, translates according to the motion of the tractor point) and tractorias with a variable tangent (the length of the string varies according to the position of the tractor point).

Using such tractorias, Riccati showed that tractional motion allows the integration of any differential equation having two independent variables x and y in which the coefficients of the infinitesimal elements dx and dy are obtained using only a finite number of algebraic operations and quadratures (cf. note 44). Under these conditions, all the auxiliary curves used by Riccati (base curves and directrix curves) are constructible by Cartesian means. Therefore, tractional motion is an additional means of construction that allows us to obtain new curves from previously known ones.

Furthermore, the integration of any specific differential equation is possible in an infinity of different constructions. It is always possible to integrate it using a tractoria with rectilinear base and variable directrix, but also with an arbitrary curvilinear base, so the problem consists in choosing the base so that the directrix is the simplest one.

Regarding the realization of practical instruments, the ones tracing tractorias with constant directrix are easily obtainable, whereas it is difficult to imagine the realization of a material curve that can continuously change its shape during the motion (as required for tractorias with variable directrix). To avoid these difficulties, when in next chapters I propose a theoretical model of some machines related to tractional motion, I do not use previously constructed curves as bases for new constructions, but I focus directly in the mechanical constraints that these machines have to respect.

Historically, even though Riccati's work overtook the ancient current of geometrical problem solving by the construction of curves, and proposed a very general theoretical model to explain in a unified way the operation of a great number of tractional instruments, it was neither celebrated nor influential. The book probably arrived too late, at the end of the period of curve construction. At this time, geometry was giving way to algebra, and series were becoming the principal tool to represent solutions to differential equations, making Riccati's

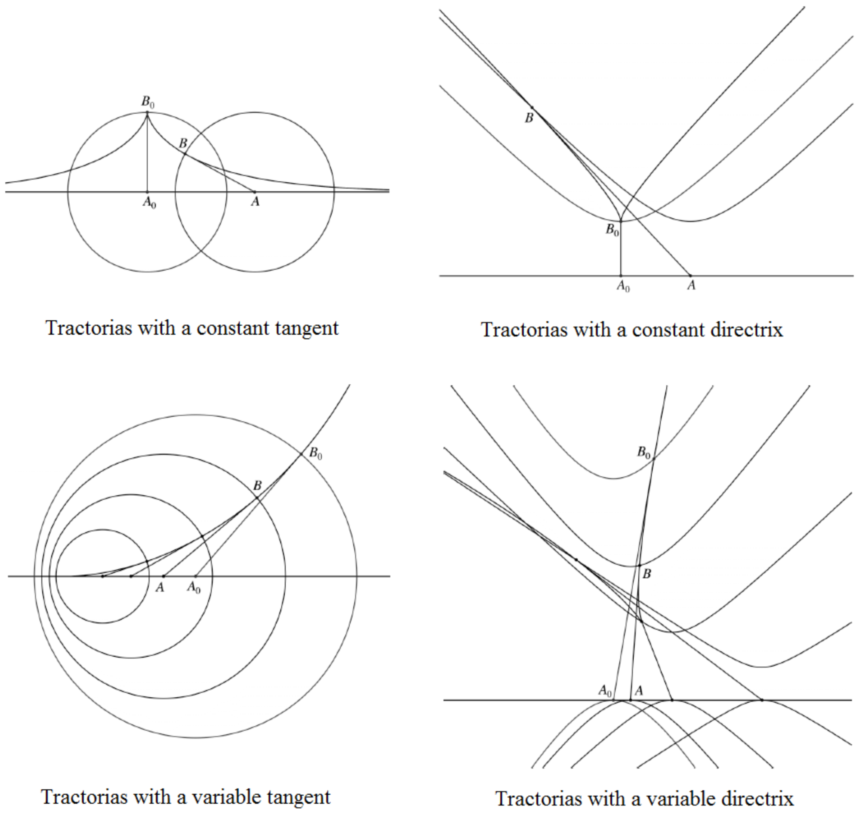

Figure 2.5: The four types of tractorias introduced by Riccati. The tractor point A with initial position A_0 moves on the base curve (in these cases rectilinear, even if in general can be curvilinear), and the motion of the point B (with initial position B_0) traces the tractoria. In particular, B is dragged by A according to the condition that B belongs to the directrix (i.e. a curve moving according to A). So in the first type, "tractorias with constant tangent," the distance AB is constant (i.e. the directrix is a circle of fixed radius); in the second type, "tractorias with constant directrix," the directrix is a general curve that translates according to the motion of A; in the third type, "tractorias with variable tangent," the distance AB varies in function of the position of A (i.e. the directrix is a circle with a changing radius); finally in the forth type, "tractorias with variable directrix," the directrix no longer just translates, but can also change its shape in function of the position of A.

work almost immediately outdated [46].

2.3.4 Changes of paradigm: Geometry, algebra, use of infinity

If the first effect of Cartesian geometry was the general habituation to the algebraic representation of curves, geometry for a period maintained its foundational role because it was still necessary to give a suitable representation of transcendental objects. It was by the introduction of the "infinite analysis" that things changed. The new introduced entities were no longer based on the two columns of classic and Cartesian mathematics, i.e. geometric constructions and finite analytical tools. While the rejection of the limit of finite analysis was obvious introducing infinitary entities, the decline of geometric constructions was given by reasons of efficiency. In fact, once accepted infinitary entities, formulas furnished a good representation for both the exact approach (symbolic manipulation) and the applicative one (numerical approximation).

That caused a shift to a new paradigm of mathematics, no longer based on circles and segments but based on numbers and functions, something where the infinite has an essential role, even if sometimes it generates paradoxical behaviors. These paradoxes required new standards of rigor, achieved only by the "arithmetization of analysis" of the 19th century.

Thus, the role of infinite is today considered indispensable, sometimes considered as a pride of freedom of the mathematical thought [47], sometimes as something necessary to obtain useful results. However, are infinitary tools really necessary, at least for part of differential calculus? I argue that they do not, and in the next chapters, I will propose a geometrical model, based on tractional motion, that would constitute an extension of Cartesian geometrical linkages. According to Cartesian interpretation, I propose a "method" (for a differential extension of algebraic geometry) made up by a synthetic and an analytical part: The synthetic part will be given by suitable geometric constructions, the analytic will consist of "differential algebra," an extension of polynomial algebra in which the indeterminates are not numbers but continuous functions.

An objection to such an approach could be that I am not really avoiding the infinite in the analytic part, because to define continuous functions I need limits or similar tools. With regard to this objection, I claim that, even if one considers continuity expressible only through infinitary tools, the allowed operations in differential algebra remain in the field of a finitist symbolic manipulation (in fact differential algebra is nowadays considered a field of computer algebra). The constructive role of infinite in differential algebra is avoidable as it is in the analysis of polynomial algebra. In classical algebra, indeterminates assume values on the field of the real numbers, the definition of which requires infinite, but algebra remains finite because it does not deal with general real numbers, one simply makes manipulations over them. Similarly, differential algebra does not deal with the definition of continuous functions. All we need to do is manipulate symbols that represent such functions, without formally defining what kind of objects we are dealing with.

46. Cf. Tournès [2004].
47. It is explicative Hermann Weyl's famous dictum: "Mathematics is the science of infinity."

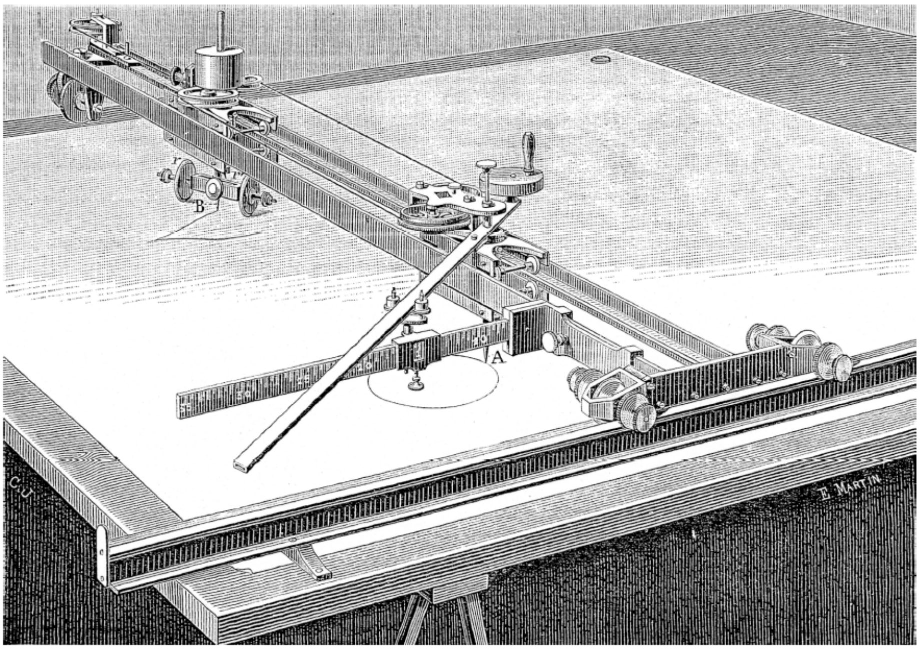

Figure 2.6: Example of an integraph taken from [Abdank-Abakanowicz, 1886, p. 43].

2.3.5 A note on computation

To conclude this chapter I want to have a look at the strange pathway (of oblivion and *ex novo* blooming) of the technical part of tractional motion, focusing on the relation with the more general field of analog computing and hypothesizing a possible application of the exactness concept on computation.

I have just shown that in the 18th century geometry began losing its importance in general and of tractional construction in particular. However, the machines used for tractional motion can be considered not only as theoretical tools, but also as computational ones, that can be practically used even if the dominant paradigm is not a geometrical one.

From this technical point of view, after about 150 years of interruption, during which there was no trace of tractional motion, engineers in the late 19th and early 20th centuries independently rediscovered the theoretical principles and technical solutions of the 18th century (see for example Fig. 2.6). Called "integraphs," these machines arrived to involve even more cutting wheels to integrate differential equations beyond the first order.

The biggest difference between the first appearance of tractional instruments and the second one is the aim. In the former appearance, these tools were mainly theoretical ones. Being involved in "pure geometry," they had to be an idealization of manufacturable objects, so concrete problems and precision had an almost marginal relevance [48]. In contrast, in the latter recurrence, engineers were much more practical, focused on the construction of really efficient artifacts to help in computation. No longer belonging to theoretical "exactness

48. An exception to this mainstream thought can be found in Poleni, who realized the first truly operational tractional instruments in the first half of the 18th century.

problems," it became more and more influent the practical problem of precision, and diagrammatic constructions became operative "analog computations." These tools belonged to the class of grapho-mechanical instruments to solve differential equations (for further reading, see Tournès [2003]).

It was generally necessary to construct a machine for any specific differential problem. The machine that overcame this limit was the *Differential Analyzer*, realized by Vannevar Bush[49] implementing the visionary program of Lord Kelvin of a mechanical machine able to solve general differential equations. Shannon would later go on to improve the machine by passing from mechanical components to electrical circuits. Therefore, being able to realize the same integration in different ways, it was possible to abstract its functioning in an abstract model called *General Purpose Analog Computer*[50], that even today is an important model of theoretical analog computation.

Nevertheless, if the *Differential Analyzer* was the apotheosis of analog computing, the downhill was close. Thanks to the theoretical foundation of Turing and Von Neumann, in the second half of the 20th century the technological thirst for equation-solving machines was quenched by digital computers, which improved the efficiency of analog computers with a very accurate error-control. Therefore, because of the technological digital revolution, there was another break of the paradigm even in computation, both from a technical and a theoretical perspective. I do not want to enter into the debate of the middle 20th century on analog and digital computation (and its influences on the theory of mind). All I want to point up is that nowadays the absolutely dominant computational paradigm is the digital one.

From the theoretical perspective, an important step was that, in the first half of the 20th century, several different independent attempts to formalize the notion of (digital) computability (recursion, the λ-calculus, and the Turing machine) were shown to be equivalent (defined the same class of functions). This led mathematicians and computer scientists to believe that the concept of computability is accurately characterized by these equivalent approaches, opening the way to the "Church-Turing thesis" that hypothesizes, in simple terms, that if some method (algorithm) exists to carry out a calculation, then the same calculation can also be carried out by a Turing machine.

An approach to break the Church-Turing thesis is to check if some results beyond Turing computational limits may be reached somehow (the "hypercomputation" problem[51]). With regard to this question, I think it could be interesting to set this problem from a purely mathematical point of view. Instead of considering the physical limits of analog computing, one could have an "exact" approach to analog computation through geometry. From this point of view, considering diagrammatic constructions and symbolic manipulations respectively as analog and digital computations, the evolution of mathematical foundational paradigms from the geometric/arithmetic perspectives (with their relative intercourses and extensions) can be considered an evolution of computational limits.

Considering the computational power of mathematical approaches, Pythagorean ratios (arithmetic perspective) were not sufficient to express the so-called "in-

49. Cf. Bush [1931].
50. Cf. Shannon [1941].
51. See, for example, Copeland [2002].

commensurable values" that have been generated by the arithmetic reinterpretation of the Euclidean geometric constructions. On the contrary, later polynomial algebra introduced values not geometrically constructible by ruler and compass (the exactness problem in the early modern period). However, the unbalance between the powers of the different paradigms is not a constant. Descartes balanced their powers in analytical geometry, and this powerful paradigm became the hard core over which calculus evolved, generating a rich symbolism inspired by ideas derived from geometry and mechanics. Something new happened with regard to calculus: If the geometrical paradigm had already been abandoned in other periods, there was the acceptance of entities generated by infinite processes[52]. This acceptance of infinite processes made it difficult to interpret the obtained entities suitably from an exact (finite) geometrical construction, hence the claim of this thesis: I want to reach part of infinitesimal calculus with suitable geometrical constructions (synthesis) and symbolic tools provided by a finite algebra (analysis).

Even if the theoretical model that will be soon introduced has no claim of constructing something beyond Turing limits, it is another case in which analog and digital constructive powers are balanced (as in Descartes's geometry). As differential calculus evolved on Cartesian geometry, I think that in future it could be interesting to investigate whether the new balance proposed in this thesis could become a step for new computational paradigms beyond the limits of today computation. I will neither hypothesize any answer about it, nor reflect on it in this thesis.

52. Infinite procedures were also adopted by Archimedes, but only as an investigative tool to be later interpreted from a synthetic perspective.

Chapter 3

Machines from Euclid to Descartes

In this chapter, I will introduce some classes of ideal geometric machines and the relative mathematical models. They constitute a necessary background for the definition of a suitable class of machines for "tractional motion." [1] Nevertheless, this chapter has another aim: Starting with an instrumental reinterpretation of the compass-and-ruler constructions, I plan to show that this interpretation can be naturally extended (still instrumentally) up to algebraic geometry. This unitary view of Euclid's and Cartesian geometry was evinced in Panza [2011]. However, the author, following in Descartes's footsteps, focused on curves, thus arriving to define the constructions allowed by Descartes as a recursive extension of the ones obtainable with ruler-and-compass. On the contrary, my point of view is purely instrumental, avoiding any constructive role of curves, which will allow providing a definition of algebraic spaces without recursive constructions, but just as the "configuration space" of machines assembled according to some rules.

In particular, I consider "classical machines," i.e. machines able to construct exactly the same constructible points available with Euclid's tools, and "algebraic machines," i.e. machines having real semi-algebraic sets as configuration space. I will see why algebraic machines can be considered a natural extension of classical ones.

Furthermore, for both these classes of machines, a marginal note will be introduced about the use of sliding objects (carts). The act of avoiding them is related to constructions by compass alone (for classical machines) and to configuration spaces of Kempe's mechanical linkages (for algebraic machines).

3.1 Mathematical modeling: A behavioral approach

Prior to studying the specific machines, I have to introduce a bit of notation to analyze them. I will use some basic tools and notations of the mathematical

1. In Chapter 4, I will introduce "differential machines," a class of machines for the tractional motion. These machines will be an extension of the machines introduced in this chapter.

theory of system and control theory. In particular, I adopt the "behavioral approach" of mathematical models. The present section will be a recall of [Polderman and Willems, 1998, pp. 1–8].

3.1.1 The universum and the behavior

Modeling is a cognitive activity in which we think about and make models to describe how devices or objects of interest behave. There are many ways in which devices and behaviors can be described. My interest lies in using the language of mathematics to make models of ideal machines working on a plane, but before doing that, I will need some basic concepts to able model any generic phenomenon.

First of all, we can view a mathematical model as an exclusion law. A mathematical model posits that some things can happen and are possible, while others cannot and are impossible. Thus, Kepler claims that planetary orbits that do not satisfy his three famous laws are impossible.

We can formalize these ideas by stating that a mathematical model selects a certain subset from a universum of possibilities. This subset consists of occurrences that the model allows, that it declares possible. We can refer to the subset in question as the "behavior" of the mathematical model [2].

We have been trained to think of mathematical models in terms of equations because an equation can be viewed as a law excluding the occurrence of certain outcomes, namely those combinations of variables for which the equations are not satisfied. Thus, equations define a behavior. I, therefore, speak of behavioral equations when mathematical equations are intended to model a phenomenon. It is important to emphasize already at this point that "behavioral equations" provide an effective, but at the same time non-unique, way of specifying a behavior. Different equations can define the same mathematical model. Hence, one should not exaggerate the intrinsic significance of a specific set of behavioral equations.

Now I can introduce in this language a distinction between the kinds of variables. I think of the variables that I try to model as "manifest" (or "external") variables. They are the attributes on which the modeler in principle focuses attention. However, in order to come up with a mathematical model for a phenomenon, one often has to consider other auxiliary variables. I refer to them as "latent" (or "internal") variables. These may be introduced for no other reason than to describe in a convenient way the laws governing a model. [3] The essential structure of this modeling language is given by three components—behavior, behavioral equations and variables (both manifest and latent).

2. The main difference between the behavioral approach and the input/output one is that in the first one we consider all the variables without the need of distinguishing them *a priori* between input and output. The advantage of missing this distinction comes from the fact that considering interconnection between components (the so-called "feedback"), it is generally impossible to easily understand which variables are inputs and which ones are outputs. In my initial work on machines for tractional motion, the approach was the input/output one (cf. Milici [2012a]), while in this thesis I will use the behavioral approach to analytically study the machines with differential algebra instead of classical infinitesimal calculus.

3. For example, when expressing the first and second laws of thermodynamics, it has been proven convenient to introduce the internal energy and entropy as latent variables. In my setting for machines, I will utilize the concept of manifest and latent variables: Even though all my variables will be physically manifest, I can be interested in focusing on just some of them, so the others will be considered latent.

When I want to model a phenomenon, I start by casting the situation in the language of mathematics by assuming that the phenomenon produces outcomes in a set $\mathbb{U}$, which I call "universum." Often $\mathbb{U}$ consists of a product space, for example a finite dimensional vector space. Now, a (deterministic) mathematical model for the phenomenon claims that certain outcomes are possible while others are not. Hence, a model recognizes a certain subset $\mathfrak{B}$ of $\mathbb{U}$. This subset is the behavior of the model. Formally:

Definition 1. *A mathematical model is a pair* $(\mathbb{U}, \mathfrak{B})$ *with* $\mathbb{U}$ *a set, called* universum *(its elements are called* outcomes*), and* $\mathfrak{B}$ *a subset of* $\mathbb{U}$*, called* behavior*.*

3.1.2 Behavioral equations

In applications, models are often described by equations. Thus, the behavior consists of those elements in the universum that satisfy certain equations.

Definition 2. *Let* $\mathbb{U}$ *be a universum,* $\mathbb{E}$ *a set, and* $f_1, f_2 : \mathbb{U} \to \mathbb{E}$*. The mathematical model* $(\mathbb{U}, \mathfrak{B})$ *with* $\mathfrak{B} = \{u \in \mathbb{U} | f_1(u) = f_2(u)\}$ *is said to be described by* behavioral equations *and is denoted by* $(\mathbb{U}, \mathbb{E}, f_1, f_2)$*. The set* $\mathbb{E}$ *is called the* equating space*. I also call* $(\mathbb{U}, \mathbb{E}, f_1, f_2)$ *a* behavioral equation representation *of* $(\mathbb{U}, \mathfrak{B})$*.*

Often, an appropriate way of looking at $f_1(u) = f_2(u)$ is as "equilibrium conditions": The behavior $\mathfrak{B}$ consists of those outcomes for which two (sets of) quantities are in equilibrium.

Consider, for example, an electrical resistor. We may view this as posing a relation between the voltage V across the resistor and the current I through it. Ohm recognized that (for metal wires) the voltage is proportional to the current: $V = RI$, with the proportionality factor R called resistance. This yields a mathematical model with universum $\mathbb{U} = \mathbb{R}^2$ and behavior $\mathfrak{B}$, induced by the behavioral equation: $V = RI$. Here $\mathbb{E} = \mathbb{R}^1, f_1 : (V, I) \to V$ and $f_2 : (V, I) \to RI$. Thus, $\mathfrak{B} = \{(V, I) \in \mathbb{R}^2 | V = RI\}$.

In many applications, models are described by behavioral inequalities. It is easy to accommodate this situation—simply assume $\mathbb{E}$ in the above definition as an ordered space and consider the behavioral inequality $f_1(u) \leq f_2(u)$ or $f_1(u) < f_2(u)$.

Note further that whereas behavioral equations specify the behavior uniquely, the converse is obviously not true. Clearly, if $f_1(u) = f_2(u)$ is a set of behavioral equations for a certain phenomenon and if $f : \mathbb{E} \to \mathbb{E}$ is any bijection, then the set of behavioral equations $(f \circ f_1)(u) = (f \circ f_2)(u)$ form another set of behavioral equations yielding the same mathematical model [4]. Since we tend to think of mathematical models in terms of behavioral equations, most models are being presented in this form. It is important to emphasize that the essential result of a modeling procedure is the behavior—the solution set of the behavioral equations, not the behavioral equations themselves.

4. The notation "$f \circ g$" stands for the composition of the functions f and g: $(f \circ g)(x) = f(g(x))$.

3.1.3 Manifest and latent variables

Often we need to introduce other variables in addition to the attributes in $\mathbb{U}$ that we try to model. As already said, I terms these other auxiliary variables "latent variables." Let me begin with a concrete example.

An economist is trying to figure out how much of a package of n economic goods will be produced. As a firm believer in equilibrium theory, the economist assumes that the production volumes consist of those points where, product for product, the supply equals the demand. This equilibrium set is a subset of $\mathbb{R}_+^n$. It is the behavior that we are looking for. In order to specify this set, we can proceed as follows. Introduce as latent variables the price, the supply, and the demand of each of the n products. Next determine, using economic theory or experimentation, the supply and demand functions $S_i : \mathbb{R}_+^n \to \mathbb{R}_+$ and $D_i : \mathbb{R}_+^n \to \mathbb{R}_+$. Thus, $S_i(p_1, p_2, \ldots, p_n)$ and $D_i(p_1, p_2, \ldots, p_n)$ are equal to the amount of product i that is bought and produced when the going market prices are $p_1, p_2, \ldots, p_n$. This yields the behavioral equations

$$\begin{aligned} s_i &= S_i(p_1, p_2, \ldots, p_n), \\ d_i &= D_i(p_1, p_2, \ldots, p_n), \\ s_i &= d_i = P_i \qquad i = 1, 2, \ldots, n. \end{aligned}$$

These behavioral equations describe the relation between the prices p_i, the supplies s_i, the demands d_i, and the production volumes P_i. The P_is, for which these equations are solvable, yield the desired behavior. Clearly, this behavior is most conveniently specified in terms of the above equations, that is, in terms of the behavior of the variables p_i, s_i, d_i, and $P_i (i = 1, 2, \ldots, n)$ jointly. The manifest behavioral equations would consist of an equation involving $P_1, P_2, \ldots, P_n$ only. This example illustrates the following definition.

Definition 3. A mathematical model with latent variables *is defined as a triple* $(\mathbb{U}, \mathbb{U}_l, \mathfrak{B}_f)$ *with* $\mathbb{U}$ *the* universum *of manifest variables,* $\mathbb{U}_l$ *the universum of* latent variables*, and* $\mathfrak{B}_f \subseteq \mathbb{U} \times \mathbb{U}_l$ *the* full behavior*. It defines the* manifest mathematical model $(\mathbb{U}, \mathfrak{B})$ *with* $\mathfrak{B} := \{u \in \mathbb{U} | \exists l \in \mathbb{U}_l$ *such that* $(u, l) \in \mathfrak{B}_f\}$*;* $\mathfrak{B}$ *is called the* manifest behavior *(or the* external behavior*)or simply the* behavior*. I call* $(\mathbb{U}, \mathbb{U}_l, \mathfrak{B}_f)$ a latent variable representation*of* $(\mathbb{U}, \mathfrak{B})$*.*

Of course, equations can also be used to express the full behavior $\mathfrak{B}_f$ of a latent variable model. I then speak of "full behavioral equations."

3.2 Classical machines

With respect to the behavioral approach to mathematical models, for all the rest of the thesis, my "phenomenon" will be the inspection of ideal machines working on a plane.

In this section, I want to give an instrumental foundation to Euclid's plane geometry, a foundation that will be extended to Cartesian geometry and beyond. Even though Euclid's foundation was based on circles and lines, the starting point will be its setting with tracing instruments (ruler and compass). Thus, to choose my tools, I will analyze the operations that characterize the practical use of straightedge and compass. I will then propose a slightly different solution that allows obtaining the same points constructible by classical geometry. I call the machines of my instrumental foundation "classical machines."

Particularly in this section, I want to compare Euclid's constructions with those of classical machines, focusing on the different rules and primitive objects, and evincing the equivalence about obtainable points.

3.2.1 Primitive objects of Euclid's geometry

In Euclid's geometry, the primitive objects are *points*, *straight lines* (or, better, segments), and *circles*. A construction starts from a certain outset made up of a finite number of points [5], and from them I can construct: [6]

E1. the line through two distinct points;

E2. the circle through one point with centre another point;

E3. the point which is the intersection of two previously constructed non-parallel lines;

E4. the one or two points in the intersection of a line and a circle (if they intersect);

E5. the one or two points in the intersection of two circles (if they intersect and do not coincide).

So lines are introduced based on two distinct points (through which the line passes). Circles are obtainable by a center and another passage point. Regarding points, I have to distinguish between generic and specific ones. If I consider a generic point on a line or on a circle, it is not distinguishable from another generic point on the same object, so I consider it "not denotable." For me, the denotable points are just the ones constructible as intersection of previously constructed objects (or as points given in the outset). I term these points "specific." Thus, in Euclid's setting the specific points are just the constructible ones.

Millennia after Euclid's *Elements*, in the 20th century, new axiomatizations of elementary geometry have been proposed and well formalized to overcome some deductive flaws of Euclid's formulation. The most influential modern formulation was Hilbert's.

Hilbert [7] adopted as primitive terms just *points* and *lines* (he also adopted *planes*, but I am not going beyond plane geometry), and as primitive relations *betweenness* (a ternary relation linking points), *containment* (a binary relation linking points and straight lines) and *congruence* (two binary relations, one

5. It is not generally true, an outset can be composed of geometric elements as polygons or figures, but they can be constructed by allowed tools starting from a finite number of points on the plane. It is different if we consider as given some objects that cannot be even piecewise obtained by straight lines and circles (e.g. other curves, as conics): I disregard the latter introduction of not constructible objects in the outset.

6. The label "E" in the numbered list stands for *Euclid*.

7. Cf. Hilbert [1913].

linking line segments and one linking angles) [8]. Thus, line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. Intersections with circles, even though circles were not introduced, may be defined using the congruence of segments (radii of circles) [9].

3.2.2 Components of classical machines

After recalling Euclid's rules, in order to introduce classical machines, I have to introduce the allowed primitive objects. To select these objects, I need to analyze the instrumental operations characterizing the classical allowed use of straightedge and compass. In particular, similarly to what is done in Hilbert's axioms, I will avoid circles and use only straight tools (cf. note 9). Furthermore, to instrumentally realize any diagram, I need some components and an idealized pencil, the motion of which traces a curve. Regarding the use of the straightedge, given two different points on a plane, I can trace any finite prolongation of the line passing through these points, and to instrumentally trace this line, I need to: [10]

S1. make a point of the straightedge coincide with the first point of the plane;

S2. make another point of the straightedge coincide with the second given point;

S3. move the pencil along the straightedge [11].

With respect to the use of the compass, to trace an arc of circumference (with a given center and passing through another point), I need to: [12]

8. Hilbert's *Grundlagen* purpose was to provide an axiomatic formal system for Euclid's geometry, to avoid any need of diagrams and geometric intuition in the verification of proofs. I also have to cite Tarski [1959], an axiom set for the so-called "elementary" fragment of geometry, i.e. the part that is formulable in first-order logic with identity and requires no set theory. Tarski's axioms comprise two primitive relations on *points* (these being the only primitive objects and Tarski's system being a first-order theory, it is not even possible to define lines as sets of points): *betweenness* (with the same meaning of Hilbert's one) and *congruence* (a tetradic relation: Applied on the points w, x, y, z can be interpreted as that the length of the line segment wx is equal to the length of the line segment yz). In particular, *betweenness* captures the affine aspect of Euclid's geometry, while *congruence* its metric aspect.

Moreover, with regard to geometric axiomatizations, I can also refer to Birkhoff [1932], but Birkhoff's postulates being built upon the real numbers (it has the possibility of measuring segments' lengths and angles through the use of scale and protractor), this axiomatization is out of my "purely geometric" interest.

9. So, paraphrasing Euclid's five construction rules and using Hilbert's objects, I can construct:

H1. the line through two distinct points (same as E1);

H2. the point which is the intersection of two previously constructed non-parallel lines (same as E3);

H3. the one or two points of a line at a given distance from a given point;

H4. the one or two points having a certain distance from a first given point and another distance from a second given point.

In this way, I *de facto* avoid circles (and in particular E2).

10. The label "S" in the numbered list stands for *straightedge*.

11. Note that, even though the final purpose of using the straightedge is the drawing of the line, this latter operation is not sufficient without the preceding operations. In fact, before tracing the line, I have to put the straightedge in the right position.

12. The label "C" in the numbered list stands for *compass*.

C1. make a point of the straightedge coincide with the center point; [13]

C2. mark a point on the straightedge in such a way that, in the initial configuration, it coincides with the second given point;

C3. move the previously marked point maintaining coincident both the first point of the straightedge with the center and the pencil with the marked point.

In addition to the possibility of tracing lines and circles, I have to face the possibility of identifying their intersections.

In order to perform these operations on an infinitely extensible plane, I can consider the following components:

- I adopt infinitely extensible **rods**, and assume that these have perfect straightness and negligible width. They are idealized straightedges [14], different from the Euclidean straight lines because they are not statically traced objects but planar rigid bodies (physical entities with three degrees of freedom—two characterizing the position of a specific point and the third identifying the slope with respect to a fixed line).
- To allow the motion along a rod (as the pencil does in S3), it is possible to put some **carts** on a rod, each one using the rod as a rail: A cart has one degree of freedom once placed on a rod (the cart can only move up and down the rod).

In contrast to the instrumental approach with straightedge and compass, I am not introducing components for the pencil. In fact, I am not considering classical machines as tools that trace diagrams, but as assembled mechanisms that move on the plane.

Without tracing curves, the points, defined as intersections of lines and circles, can be viewed as the points where the position of two different pencils (tracing different lines or circles) coincides. To avoid the introduction of pencils, I can introduce something mechanically constraining different points to assume the same position on the plane. That can be accomplished considering the cart not only as an additional component to be put on a rod (i.e. introducing a new point moving on the rod), but also as something able to constrain on the rod a previously constructed point (i.e. a cart can constrain a specific point to lie on a rod) [15].

In particular, as in Euclid's setting, the outset of a construction will be composed of some points on the plane, and I will recursively construct more and more of these points. I call such points *fixed on the plane*, and their construction will recursively extend and be extended by the construction of points *fixed on a rod* [16]. As I will specify in more detail, a point fixed on a rod will be constructed

13. Note that this condition is a repetition of S1.

14. I decided not to call them "straightedges" because it seemed that straightedges introduce the idea of a bar with a significant width.

15. The different uses of carts will be detailed and clarified in the next subsection.

16. Any rod will lie on the plane, so at first glance, the distinction between points fixed *on the plane* and *on a rod* may appear obscure. The idea is that these points have to remain *fixed* respectively to the plane or to the rod. As I will explain below, the orbit of a point fixed respect to a rod defines a circle if the rod rotates. When not generating confusion, in "points fixed on the plane" or "points fixed on a rod," I will sometimes omit the adjective "fixed."

marking on the rod a point that is coincident with a point fixed on the plane when the rod has a certain slope (as seen in C2).

3.2.3 Construction rules for classical machines

I introduced objects as rods, carts, points on the plane, and points on the rods. Now I have to specify how they can be used to satisfy the instrumental requirements of straightedge and compass.

With respect to the requirements S1,C1, a straightedge is considered only if constrained to be joined to a point fixed on the plane, so I can introduce my first rule: [17]

R1. A rod r is introduced after being constrained to rotate around a point on the plane P (through which r has to pass). I say that the rod r is joined in P.

I have to be precise with respect to the points that can be considered on a given rod. According to C2, I can mark a point on a rotating straightedge [18] if, for some slope, this rotating point coincides with another point of the plane [19]:

R2. On a rod r joined in P, I can consider a point A (maintaining invariant the distance PA) so that, for some slopes of the rod, A coincides with a point Q on the plane. I say that the point A on r is superimposable on Q.

The rotation of r forces A to move along a circle (as required by C3). Given a rod r joined in P and another point on the plane Q, the point A on r superimposable on Q is not uniquely defined. In fact, if Q is different from P, there are two points on r (symmetric with respect to P) that rotating can coincide with Q.

As introduced, a cart constrains a point B to lie on a rod. First of all, the point B constrained by the cart is of a different type with respect to the previously observed *points on the plane* and *points on a rod*, because B is not fixed neither with respect to the plane nor to the rod. I am not interested in using these free points in general: I want to specify how I can construct points fixed on the plane [20]. To restrict the use of free points, I need some definitions.

I say that a rod r joined in P is a *fixed rod* if, given a point on the plane Q distinct from P, I impose the point Q to lie on r, using a cart constraint (i.e.

17. The label "R" in the numbered list stands for *rule*.
18. Considering a rod r joined in P, the point on the rod A will satisfy the property that the distance PA will be fixed.
19. The property of marking just the points reachable during a rotation is the same constraint of the "collapsing" compass, i.e. a compass that collapses when lifted off the drawing surface, hence not usable to transfer distances. But, according to Euclid's *Elements* (I cite the translation in Heath et al. [1956]), Book 1, Prop. 2, it is possible "to place at a given point (as an extremity) a straight line equal to a given straight line." Thus, it is equivalent to consider "distance-transferring" compasses instead of collapsing ones, meaning that the collapsing compass generates (together with the straightedge) the same objects generable by the non-collapsing one.
20. With classical machines, I am not dealing with general motions of free points. For example, considering a rod r joined in a point, the motion of a cart on r can be a spiral (the cart moves while r rotates), which is outside my area of interests. I can note how the only reference to motion along a line in the instrumental approach is in S3.

the rod has to pass through a fixed point on the plane)[21]. I also call *rotating rod* any rod that is not fixed.

I have already mentioned that I have to distinguish between general points on rods (e.g. constrained by a cart) and constructible ones (the ones having definable position, i.e. points fixed on rods or on the plane). All I need yet to do is to define the rules to obtain new constructible points. Specifically, according to R2, I can obtain new points on rods if I construct new points on the plane. Thus, I can define some rules to construct new points on the plane:

R3. given two non-parallel fixed rods r, s, call A a point constrained with a cart to lie on r. If A is also constrained with a cart to lie on s, then A is a fixed point on the plane[22];

R4. given a point A fixed on a rotating rod r, and given a fixed rod s, if with a cart, I constrain A to lie on s, then A is a fixed point on the plane;

R5. given two rotating rods, r joined in P and s joined in Q (distinct from P), if the points A fixed on r and B fixed on s are constrained to assume the same position[23], then A is a point fixed on the plane[24].

Similarly to the case of classical constructions when there is no intersection, in some configurations it is not possible to instrumentally impose some constraints[25].

3.2.4 Characterizing ruler and compass constructions

In this section I want to show that the points on the plane obtainable with classical machines are exactly the points constructible with straightedge and compass (classical constructions). To prove the equivalence I will consider as outset a finite set of points[26] that will be considered both as given points (in

21. Compare with S2.

22. Note that here I used the cart in two different ways. First, I used it to introduce a new specific point I called A. Later, I constrained the particular point A to lie on s.

23. To constrain A in order to assume the same position of B, with a cart I can constrain B to lie on r and with another cart A to lie on s. That will instrumentally impose that A and B will assume the same position.

24. Obviously, being A and B coinciding, also B will be a point fixed on the plane.

25. For example, if in R5 I consider a configuration so that the distance PQ is greater than the sum of the distances $AP + BQ$, it will not be possible to impose that A assumes the same position of B.

26. Not every instance of Euclid's problems can be converted in this vision, for example no inquiry is possible for curves beyond straight-lines and circles (as said in note 5). But I also need another remark: Consider the problem to find the center of a given circle. Circles are objects of Euclid's plane geometry, but this problem cannot be converted only into a finite outset of points without introducing suitable restrictions on constructions. Given the points P and Q, if I translate the problem of finding the center of a circle (of center P and passing through Q) considering as given the points P and Q, the problem will be trivially solved by returning P. This problem could be translated as "find the center of the circle of center P and passing through Q without considering any rod passing through P," but I do not delve into this kind of problems "with restrictions" because it is out of my interests. Furthermore, to solve it I would require to have at least another point out of Q where I can put a rod, otherwise I cannot construct any new points. This last clarification introduces another difference from Euclid's vision: I cannot consider a general point of the plane as I can use only the points given at the outset (my approach is similar to the one adopted to define constructible numbers that one can find for example in Courant and Robbins [1996], Carrega [1981]). From this perspective, general points on lines or circles do not play an essential role in constructions, but are only useful for the visual representations.

Euclid's geometry) and as points fixed on the plane (classical machines).

Proof. Precisely I have to verify that points constructible in Euclid's geometry with the five rules of the subsection 3.2.1 are exactly the points fixed on the plane obtainable by classical machines using the five rules of the subsection 3.2.3. Even though the rules about the construction of points are the last three in both cases, the first two rules are important to introduce objects that allow the construction of points. About Euclid's setting I have that constructible points can be recursively obtained through the introduction of constructible lines (defined through the passage of two different constructible points) and constructible circles (defined with a given center and passing through another point, both these points being previously constructed). Also with classical machines the construction of new points fixed on the plane is obtained through the introduction of objects constructible in function of previously obtained points on the plane, in particular fixed rods and rotating rods.

Thus, considering the recursive nature of objects, I propose a proof by induction on the number of the constructed points.

Basis: As assumed, the points given at the outset can be considered both as constructible points (without any construction needed because they are given) and as points fixed on a plane. I also assume that these given points are at least two (otherwise no construction is available).

Inductive Step: Consider the points $P_1, \ldots, P_{n-1}$ that are both constructible points and points fixed on the plane. I have to show that, starting from these points:

$$P_n \text{ is a constructible point} \iff P_n \text{ is a fixed point on the plane.}$$

Before restricting my attention to points, it is useful to find a correspondence between the other objects of Euclid's and my instrumental approach.

At this level, all the constructible lines are the ones passing through two distinct points P_i, P_j $(i, j < n)$. Thus, any point [27] of a constructible line lies on a fixed rod [28] and vice versa [29].

Likewise, with respect to circles, any point of a constructible circle (centered in P_i and passing through P_j) is reachable by the motion of a point on a rod [30], and vice versa [31].

In light of these considerations, if now I restrict to the rules involving new constructible points or points fixed on the plane, I can easily note how E3, E4

27. Note that here "point" refers to any general point of the object, not just constructible points or points fixed on the plane.

28. P_i and P_j, for the induction hypothesis, are not only constructible points but also points fixed on the plane. Thus I can consider the rod r joined in P_i, and, putting a cart constraining P_j on r, I obtain that r is a fixed rod.

29. Any fixed rod obtainable at this level has to be constructed through two points previously obtained as fixed on the plane. But for the induction hypothesis these points are also constructible points, so I can consider the line passing through them, and any point of this line will also belong to the fixed rod.

30. Consider the rod r joined in P_i, then call A the point fixed on r superimposable on P_j. While r rotates, A moves on the previously considered circle.

31. Given a rotating rod and a point A fixed on it, there will be a slope of the rod so that A is superimposable on a point P_j. Thus, the motion of A defines uniquely a constructible circle.

and E5 are respectively equivalent to R3, R4 and R5 [32].

Thus, starting from $P_1, \ldots, P_{n-1}$, if P_n is a new constructed point, it will also be an obtainable point fixed on the plane, and vice-versa if P_n is a new obtained point fixed on the plane, it will also be a constructible one, concluding the proof. So, defining the same class of points, I will use "constructible point" also to indicate any point fixed on the plane.

□

3.2.5 Defining equivalence between classical machines

Using the behavioral approach of section 3.1, if I consider classical machines as the devices to be described and the points fixed on the plane that they define as the outcomes, I have that the universum over which they work is a subset of the points of the plane, so I can consider $\mathbb{U} = \mathbb{R}^2$. The outcome of a classical machine will be the finite set of the points fixed on the plane that have been constructed, so, considering their coordinates in a Cartesian plane, $\mathfrak{B}$ will be a finite subset of $\mathbb{R}^2$. [33] Two machines will be "equivalent" if their behaviors respect to the manifest variables are the same, i.e. if the points constructed with the first machine are the same as the points constructed with the second. These constructed points are in a finite number, hence I can restrict to the problem of checking whether a single constructed point is the same as the point constructed by another machine. Here I want to show that there is an algorithmic procedure to determine whether two machines are equivalent or not.

More precisely, given two classical machines $\mathcal{M}$ and $\mathcal{M}'$ (both working on the same outset of points $P_0, \ldots, P_n$), and considering the constructed points Q (obtained with $\mathcal{M}$) and Q' (obtained with $\mathcal{M}'$), I look for a general procedure to know if the point Q is equivalent to Q' [34].

In case of more constructed points, I say that two classical machines $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if all the constructible points of the first are the constructible ones of the other.

Constructible points correspond to points fixed on the plane, so I can use the ruler-and-compass construction terms. I begin considering an outset of just two given points P_0, P_1.

The first step is to introduce a coordinate system. As usual with constructible numbers, I identify any planar point with a couple of real numbers, and consider P_0 in the origin and P_1 in $(1, 0)$, thereby defining Cartesian coordinates. A real number is called *constructible* if it is a coordinate of a constructible point in a coordinate system.

As well known (and for example visible in [Courant and Robbins, 1996, pp. 127–133]), the set of constructible numbers can be completely characterized in

32. I just have to substitute the terms "constructible point," "point on a line" and "point on a circle" (in classical constructions) with "point fixed on the plane," "point constrained with a cart on a fixed rod," "point fixed on a rotating rod" (in classical machines), and to convert the concept of intersection using carts.

33. I have to admit that the introduction of universum and behavior is not useful for classical machines. Being the behavior given by a finite number of outcomes, it is not expressed by equations but simply by a list of points. However, I introduced here this model language to begin to become confident about its use for classes of idealized planar machines.

34. These points are equivalent if Q has the same position of Q' for every position of $P_0, \ldots, P_n$.

the language of field theory: in an elementary characterization, constructible numbers are the real ones which can be represented by a finite number of additions, subtractions, multiplications, divisions, and finite square root extractions of integers. It is easy, given the procedure of construction of a point, to find its algebraic form.

Thus, any constructible point has as coordinates real numbers written as combinations of the four field operations and square root extractions. So, coming back to my problem of testing the equivalence of two constructed points (obtained through different constructions), it may appear that in order to solve it, all I need to check is whether the abscissae and the ordinates of the constructed points are equal. This is not yet obvious, because I have to compare two real numbers that, even if equal, may be represented in different ways: For example, it may happen that I have to check whether $\sqrt{2} + \sqrt{3}$ is equal to $\sqrt{5 + 2\sqrt{6}}$. I need a general method to decide whether two different representations denote the same real value [35]. In so doing, I can look for a canonical form for any of such representations, obtaining that the symbolic representations are equal if and only if the real values are the same.

A "normal representation" can be found in [Bouhineau, 1996, sections 4,5]. The main idea is that, even if constructible numbers can be irrational, they can be identified in an exact way through a symbolic representation [36]. For what observed, a constructible number will belong to the quadratic extension k_n of $\mathbb{Q}$, obtained as it follows: [37]

- $k_0 = \mathbb{Q}$.
- $k_1 = k_0(\sqrt{\alpha_0})$ where $\alpha_0 \in k_0, \alpha_0 \geq 0, \alpha_0$ is the first square root introduced during the calculations.
- ...
- $k_n = k_{n-1}(\sqrt{\alpha_{n-1}})$ where $\alpha_{n-1} \in k_{n-1}, \alpha_{n-1} \geq 0, \alpha_{n-1}$ is the last square root introduced during the calculations.

Let $A \in k_n$, I can represent A as (a_1, a_2) where $a_1, a_2 \in k_{n-1}$ when $A = a_1 + a_2\sqrt{\alpha_{n-1}}$. For example, in $\mathbb{Q}(\sqrt{2})(\sqrt{1+\sqrt{2}})$, the real number represented as $((5, 2), (3, 1))$ is $5 + 2\sqrt{2} + (3 + 1\sqrt{2})\sqrt{1 + \sqrt{2}}$. So I can represent $\sqrt{2} + \sqrt{3}$ as $((0, 1), (1, 0))$ in $\mathbb{Q}(\sqrt{2})(\sqrt{3})$, and $\sqrt{5 + 2\sqrt{6}}$ as $((0, 0), (1, 0))$ in

35. The problem of deciding whether two different representations denote the same object is generally called the "equality test." The solvability of this problem depends on the setting: For example, if I consider not only constructible numbers (i.e. the ones obtainable with ruler and compass constructions), but also the computable ones (i.e. the ones approximable with any error by a Turing Machine), I have that the equality test is not computable.

36. In particular the proposed representation is adequate to recursively obtain sum, subtraction, multiplication, division and square root extraction of such symbolic representations.

37. A field F is called an extension of another field K if F contains K and the operations on F extend those on K (in other words, the sum or product in F of two elements of K are the same as the sum or product in K). Given a subset S of F, the smallest subfield of F which contains K and S is denoted by $K(S)$ (i.e. $K(S)$ is the field generated by adjoining the elements of S to K). If S consists of only one element s, $K(s)$ is a shorthand for $K(\{s\})$. For example, the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is an extension field of $\mathbb{Q}$, and the set $\mathbb{Q}(\sqrt{2})(\sqrt[3]{2}) = \mathbb{Q}(\{\sqrt{2}, \sqrt[3]{2}\})$ is an extension of both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}$.

The simplest case of field extension is the quadratic one. Supposing F contains an element a so that $a^2 \in K$ but $a \notin K$, and every element of F can be written as $x + ya$ with $x, y \in K$, then F is called a quadratic extension of K.

$\mathbb{Q}(\sqrt{6})(\sqrt{5+2\sqrt{6}})$. According to these representations, these values may appear to be different. The problem of this method is that the representation is not unique. However, this method can be improved, and there is an algorithm that allows us to arrive to a unique "normal representation" [38]. With this algorithm the representation of $\sqrt{5+2\sqrt{6}}$ is $((0,1),(1,0))$ in $\mathbb{Q}[\sqrt{2}][\sqrt{3}]$ [39], thus $\sqrt{2}+\sqrt{3}=\sqrt{5+2\sqrt{6}}$.

Presently, I considered only constructions starting from two given points P_0, P_1. If I also have the given points $P_2, \ldots, P_n$, I can introduce the real parameters $x_2, y_2, \ldots, x_n, y_n$ (the coordinates of these points in the coordinate system introduced by P_0, P_1), so any constructed point will have coordinates that can be represented by a finite number of additions, subtractions, multiplications, divisions, and finite square root extractions of integers and of the parameters $x_2, y_2, \ldots, x_n, y_n$. Thus, considering these parameters, I can easily extend what previously obtained to the case of many given points. [40]

3.2.6 The role of the cart in classical machines

I conclude this section with a marginal observation about the role of the cart in classical machines. This remark is not essential in the development of this work. However, I decided not to put it just in a note because the same rejection of carts will be analyzed not only for classical machines, but also for their extensions (algebraic and differential machines). More precisely, all the various models of machines introduced in this thesis allow the introduction of carts. On the other hand, if I restrict these models denying the introduction of carts, restricted models will involve different famous mathematical theorems.

About classical machines, without carts I can no longer pose any of R3, R4, R5, but if I let it possible to constrain a point on a rod to assume the same position of another one (like if I pin them together [41]), I obtain something interesting. Without carts I cannot find the intersection of fixed rods, and from the three rules to obtain points fixed on the plane only the rule R5 (about points fixed on rods) can still be used. If I consider the counterpart in classical constructions, it is equivalent to have only the compass (and not the straightedge) as means of construction. The problem of defining the class of

38. Here I am not interested in the specific definition of this algorithm, in general the idea is that a constructible number is a combination of sum, subtraction, multiplication, division and square root extraction of integers. So, starting from integers, I can represent a constructible number applying step by step the operations required by its representation: e.g. for $\sqrt{5+2\sqrt{6}}$ I have to calculate step by step the normal representation of $x_1 = 6, x_2 = \sqrt{x_1}, x_3 = 2 \cdot x_2, x_4 = 5 + x_3, x_5 = \sqrt{x_4}$. The procedures to calculate the various operations according to the normal representation for addition, subtraction, multiplication and division the algorithms are at [Bouhineau, 1996, pp. 279–280], while for the square root extraction the algorithm is at [Bouhineau, 1996, pp. 284–285]. Furthermore, about the sequence of quadratic extensions $k_0, \ldots, k_n$ (with $k_0 = \mathbb{Q}$), a new quadratic extension is introduced only when strictly necessary (i.e. when I set $k_n = k_{n-1}(\sqrt{\alpha_{n-1}})$, besides $\alpha_{n-1} \in k_{n-1}$ I also require that $\sqrt{\alpha_{n-1}} \notin k_{n-1}$).

39. Cf. [Bouhineau, 1996, p. 283].

40. Note that, as said in note 34, I am interested in the equivalence of two constructions for any value of the $n+1$ given points, so I do not have to evaluate $x_2, y_2, \ldots, x_n, y_n$ as real numbers but I just have to treat them as symbolic parameters. So the different constructions will be equivalent if the relative normal representations of the constructible numbers are symbolically equal (even about the use of the symbols $x_2, y_2, \ldots, x_n, y_n$).

41. Note that, in classical machines, this was provided by carts, as visible in note 23.

the points constructible just with compass was solved by the so-called Mohr-Mascheroni theorem. The solution is that any point constructed with compass and straightedge can be obtained using the compass alone. The history of this result is just as interesting. In fact, it was originally published in Mohr [1672], but this proof languished in obscurity until 1928. The well-known version of the theorem was the one published in Mascheroni [1797], proof independently obtained more than a century later.[42]

Referring back to my machines, even if any point on the plane obtainable with the introduction of the cart is also obtainable without it, I decided to let carts introduced to be closer to ruler-and-compass constructions.

In next section, extending classical machines to algebraic ones, I will see that the acceptance or rejection of carts will introduce some differences for the relative constructions (algebraic curves and in general varieties). In fact, without carts I can obtain just finite parts of the varieties constructible allowing also carts.

3.3 Algebraic machines

In the previous section I have seen that the constructive power of classical machines is equivalent to the one of Euclid's constructions. In particular I focused on the role of the points fixed on the plane, that, being equivalent to the ones constructible with straightedge-and-compass, I still call "constructible points." With Euclid's tools, as well known, it is impossible to solve the famous three classical geometry problems[43]. However, even though with more powerful tools, at least since Archimedes mathematicians conceived geometric constructions solving some of these problems. This quest for suitable extensions (the problem of "geometrical exactness") bring me in this section to introduce "algebraic machines," a new class of machines extending the classical ones. In particular, considering Descartes's canon about the acceptance of *neusis* constructions but not of "mechanical curves" (like the quadratrix), I begin reviewing the necessary extension of classical machines to introduce neusis constructions. Informally, I thought at the introduction of such machines as a canon for the machines used in Descartes's geometry (such as his proportional compass).

Once extended the constructive postulates beyond classical machines, I will explore the potential of the new machines focusing on their algebraic coun-

42. My machines are considered to work on a plane, but in general they can be considered to work on any two-dimensional space, so it is natural to ask myself something about the extendibility of the Mohr-Mascheroni theorem to non-Euclidean geometries. The starting point of the theorem is that, given a circle of center P passing through Q, called A an intersection with the circle centered in Q and passing through P, the angle $\angle APQ$ is $1/3$ of a flat angle. What happens if, instead of a plane, I am in a non-Euclidean setting, so the angle $\angle APQ$ is different? Which are the conditions on this angle or in general on a non-Euclidean two-dimensional space to satisfy the Mohr-Mascheroni theorem? Being this question very marginal respect to the aim of the thesis, I have not further developed it.

43. These problems were:

1. Trisecting an angle (given an arbitrary angle, divide it three equal angles).
2. Doubling a cube (given a cube, construct a new cube whose volume is double that of the first cube).
3. Squaring a circle (given a circle, construct a square of the same area).

Such impossibility (with straightedge and compass) was proved thanks to "abstract algebra" (it developed in the 19th century).

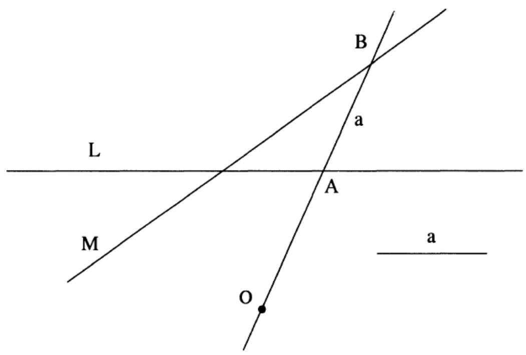

Figure 3.1: Neusis postulate.

terpart. In particular, instead of considering constructible points, I will be interested in the positions that the movable points of my machines can reach. So the role of postulates will be different. When it comes to classical machines, I am not interested in the position of every specific point, but just in the ones of points fixed on the plane (so postulates help us to known when a specific point is fixed on the plane). In algebraic machines, on the contrary, I am interested in the positions of all kinds of points.

Analytically, I will characterize the behavior of algebraic machines: considering as variables some coordinates of these points, I will see that the behavior will correspond to an n dimensional real semi-algebraic set.

3.3.1 Extending classical machines: Neusis constructions

In the historical part [44] I observed that neusis constructions extend classical ones. Here I propose how to extend classical machines in order to include a revision of the neusis postulate.

> **Neusis postulate.** As visible in Fig. 3.1, given two straight lines L and M, a point O (referred to as the "pole" of the neusis) and a segment a; It is possible to find a line through O, intersecting L and M in A and B, respectively, such that $AB = a$. [45]

To translate this postulate in my instrumental setting I have to start from two given fixed rods r, s (respectively for the lines L, M in Fig. 3.1) and three points O, P, Q fixed on the plane (O for the pole and the others to define the distance $a = PQ$). Then I have to consider a rod t joined in O, the carts A on r and B on s, and I have to constrain A and B to lie on t (with additional carts). The problem is that, to implement the neusis postulate, I do not have to impose the distances OA and OB, but AB [46]. So, if I construct a rod joined in A passing through O, I can intuitively report a length PQ on this rod, and I can constrain B to be such point fixed on the rod. However, rods, according to R1, can be considered only if joined in a point fixed on the plane, and *a priori* A is not

44. In the subsection 2.1.2, p. 8.
45. Cf. [Bos, 2001, p. 31].
46. Remind that, for R2, on a rod joined in O I can consider only the fixed points P given the distance OP.

fixed on the plane. Thus, to extend my constructions, I can modify R1 requiring the junction not only in a point fixed on the plane but in any specific point [47]. Thus I replace R1 with

R1′. A rod r is introduced after being constrained to rotate around a *specific* point P (through which r has to pass). I say that the rod r is joined in P.

I can so distinguish between *non-floating rods* (joined on points fixed on the plane, i.e. the fixed and rotating rods allowed in classical machines) and *floating rods* (joined on any other kind of specific points). I have to keep in mind that, to define constructible points (postulates R3, R4, R5), I have to restrict to non-floating rods, but it was already specified because in these postulates I explicitly considered just rotating and fixed rods.

I need more clarifications about R2. If I consider this postulate just on non-floating rods, then I am not yet able to face neusis constructions because I am not able to impose $AB = PQ$ on the rod joined in A. Thus I have to generalize it to floating rods.

Considering a floating rod r joined in a point O not fixed on the plane, given a point P on r that is superimposable to a point Q fixed on the plane, I have that P is not fixed on the rod, because r is joined in a moving point O, so the distance OQ will not be fixed even though Q is fixed on the plane. The instrumental possibility of realizing a device imposing $OP = OQ$ is then not trivial. In other words, it is not so simple to extend the idea of collapsing compass to the case of floating rods. On the contrary, it is quite natural to instrumentally implement the idea of distance transferring. Consider the following modification:

R2′. Given two points P, Q fixed on the plane and rod r joined in R, it is possible to consider a point S fixed on r so that the segment RS has the same length as that of PQ. [48]

However, I have to note that it is possible to transfer only distances between points fixed on the plane (and not between two general specific points). This is justified because of the following practical reasoning. To transfer the distance between two points A, B fixed on the plane on a rod r joined in O, I mark on r the distance AB from O. This mark indicates a point P fixed on the rod; on the contrary, if AB were a variable length (what happens in general if at least one between A and B is not fixed on the plane) then P would not have been fixed on r, and there would be no simple manner to practically construct it. That is why I restricted R2′ to the transferring of distances between points fixed on the plane.

As I am going to evince, the modification of these postulates allows me to pose the neusis condition. As already introduced, this is the problem of defining

47. Remind that in classical machines I introduced different kind of points: Points fixed on the plane, points fixed on a rod, points constrained by a cart to lie on a rod. Each point of these kinds is introduced with a well defined procedure, it is possible to refer exactly to it, so I call any of these points a specific one. Non-specific points are generic ones, i.e. points not defined by a procedure but considered as general part of a rod or simply on the plane. In my setting, generic points cannot be introduced in constructions.

48. Note that I am implementing Tarski's *congruence* tetradic relation, introduced in note 8.

the position of the points A, B (respectively on the fixed rods r, s) so that (given the points on the plane O, P, Q) O, A, B are aligned and $AB = PQ$. Owing to R1$'$, I can consider the rod u joined in A and passing through O. Owing to R2$'$ I can consider the point B' fixed on u so that $AB = PQ$. So, with carts, I can add the condition that the point B has to coincide with B', thus completing the definition of the construction of the wanted A, B.

Once used the neusis postulate to extend the constructive potentials of my old postulates [49], I do not want to add it to my set of postulates because I am not interested in extending the class of points fixed on the plane. More generally, I want to focus on the obtainable dynamic configurations of specific points.

More precisely I can define a new class of machines beyond classical ones, that I call "algebraic machines." The primitive objects will be the same of classical machines, with the differences that:

1. a rod can be joined in any specific point and not only in the ones fixed on the plane (R1$'$);
2. a fixed point on any rod can be introduced given the distance between any two fixed points (R2$'$).

The class of points fixed on the plane is not extended (they are still constructed according to R3, R4, R5), but there will be much more rods and points on a rod (both fixed and sliding). All the specific points (except the ones fixed on the plane) can move, and, as I will see, an algebraic machine will be defined by the configuration of its specific points.

3.3.2 Machine-based approach

Historically, the more widely accepted geometric canon extending Euclid's constructions was the Cartesian one. If Euclid based its setting on lines and circles, Descartes's objects were (algebraic) curves. In my mechanical setting of algebraic machines, the main objects will not be curves, but machines. The difference from the Cartesian canon is subtle: Machines had an important role in *La Géométrie*, but they were necessary only to trace curves (so, after the tracing, machines were no longer useful). On the contrary, in my approach the main objects will be machines, and curves (still defined as loci of moving points of machines) will not be used for any successive construction. This distinction is more visible when recursively constructing new objects. Based on the Cartesian method, one needs machines to trace curves, and then recursively uses the constructed curves to find new intersections (the locus of which will define new curves) [50], so one needs both machines at the beginning and then purely geometric curves. On the contrary from my purely instrumental perspective I do

49. Note that the neusis postulate allows constructing new points (not obtainable with ruler and compass), i.e. translated into my instrumental setting, to construct new points fixed on the plane. In particular, the postulate in an instrumental version may be translated as

> Given two fixed rods r, s and three points O, P, Q fixed on the plane (O for the pole and the others to define the distance PQ). If I consider a rod t joined in O, and on t the carts in A (also on r) and B (also on s) so that $AB = PQ$, I get that A and B are points fixed on the plane.

However, I am not interested in new static objects constructed by *neusis*, from now on I want to investigate dynamic configurations allowed by the new postulates R1$'$,R2$'$ (that have been introduced to face the neusis postulate).

50. Cf. [Panza, 2011, pp. 78–89].

no need to introduce two generating tools (machines and curves): e.g. to trace more and more complex curves, I will only consider more complicated machines having, as their parts, other simpler machines. The advantages of this instrumental perspective are that I can avoid any reference to a constructive role of curves (every object is defined by machines that have to satisfy specific construction rules), and also that, even if I adopt machines moving on a two-dimensional plane, I can handle n-dimensional varieties (for any positive integer value of n). On the contrary, in the curve-based approach one needs to refer to machines as primary source to trace curves, and has to restrict to one or two-dimensional objects (for being drawn in the plane).

Based on the assuming of my machine-based approach, I can observe the difference between classical and algebraic machines. In particular the objects constructed according to the postulates of classical machines are a finite set of constructible points (so the behavior, according to section 3.1, is a finite subset of $\mathbb{R}^2$), while objects of algebraic machines are not just points fixed on the plane, but in general they are specific points satisfying some constraints: The mutual positions of such points define the admissible configurations of the constructed machines. So algebraic machines define a (usually infinite) subset of $\mathbb{R}^n$ (in the curve-based approach the objects, i.e. the curves, are a generally infinite subset of $\mathbb{R}^2$).

3.3.3 Behavioral approach for algebraic machines

Consider an algebraic machine $\mathcal{M}$, i.e. an assembling of rods and carts given the outset of points $P_0, \ldots, P_n$ (fixed on the plane) according to R1′, R2′ [51]. To analytically characterize what is obtainable with $\mathcal{M}$, I introduce (as done in the subsection 3.2.5) a coordinate system in such a way that P_0 has Cartesian coordinates $(0, 0)$ and P_1 has coordinates $(1, 0)$. Differently from classical machines, that construct points fixed on the plane (i.e. static couple of numbers), $\mathcal{M}$ in general can constrain specific points to move along certain trajectories in relation with the position of other points. Therefore, to describe $\mathcal{M}$ analytically, it is not enough to give a finite vector of numbers. The objects defining $\mathcal{M}$ are the specific points and the rods: but, being a rod allowed if and only if joined on a specific point (by R1′), I can note that the configuration of $\mathcal{M}$ will depend only on specific points [52].

To express a configuration of a machine $\mathcal{M}$ in the full behavior $\mathfrak{B}_f$ (defining k specific points) I can use the vector $(a_1, \ldots, a_k)$ (with $a_i \in \mathbb{R}^2$) so that a_i is the couple of planar coordinates of the i-th specific point [53]. Making explicit

51. Note that in R2′ I recall the constructible points, so I somehow also need the postulates for classical machines.

52. A rod will be defined by the positions of its junction point and of another point on the rod: If there is no specific point on the rod in addition to the joint, it means that the rod is not useful in the machine (because the motion of the rod will not determine any change of any specific point).

53. Another widely spread nomenclature is the one taken from basic mechanics (and robotics). Considering my machine as "robot systems" (i.e. as constraints to the motion), I can use the definition: "The *configuration* of a robot system is a complete specification of the position of every point of that system. The *configuration space*, or *C-space*, of the robot system is the space of all possible configurations of the system" (cf. [Choset, 2005, p. 39]).

the single coordinates by $a_i = (x_{2i-1}, x_{2i})$, I have that a configuration is given by the vector $(x_1, \ldots, x_{2k})$, so $\mathfrak{B}_f$ is a subspace of $\mathbb{R}^{2k}$.

Now I can introduce the idea of (restricted) behavior [54]. From the $2k$-real variables, I can be interested only in some of them: Calling $J = \{1, \ldots, 2k\}$, for every set of "manifest components" $I \subset J$ I define $\mathfrak{B}_I$ (or, when not generating confusion, just $\mathfrak{B}$) the restriction of $\mathfrak{B}_f$ respect to the components with indexes in I. The definition of the behavior of a machine will be useful not only to analytically characterize a machine, but also to define the equality between machines.

Something strictly related with the behavior of a machine (even though different) is the concept of "reachable configuration." Given an initial configuration M_0, the set of the configurations (restricted to the components with indexes in I) reachable starting from M_0 is a subset of $\mathfrak{B}_I$, but it is not necessary the same. In fact, to reach a configuration physically, there must be a path connecting the initial configuration to any final reachable configuration. In particular, the set of reachable configurations (or just "reachable space") will be the connected part of $\mathfrak{B}_I$ containing M_0.

About the connectedness, I have to keep in mind that for me a variable indicates a coordinate of a specific point. That means that, if I consider the change of the position of the point respect to the time, variables have to vary in a continuous way (it is not physically possible that a point changes its position in a discontinuous way). But the setting of algebraic machines disregards the reference to the time: This continuity of the variables implies that, given any two configurations $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ in the space of the reachable configurations, there must be a path connecting the first to the latter configuration [55], thus the reachable space will be connected. Furthermore, for topological properties, the connectedness is inherited from a topological space to its projection, so even restricting only to some variables, their reachable space has to be connected. I can note how this interpretation of reachable space fits with Descartes's conception that a curve [56] has to be considered as a single connected branch, while in general the behavior fits with the concept of algebraic set (that can be made up by more unconnected branches).

In terms of the curve-based approach, one only considers the orbit reachable by a specific point of the machine (i.e. the restriction of my components with respect to abscissa and ordinate of the specific point), while, in my general setting, I can consider the relative positions of a component with respect to many others, moving from planar curve to any finite-dimensional variety. In particular in this section I will show some analytical characterizations: Any full behavior $\mathfrak{B}_f$ will be a real algebraic set, so, considering the restriction to the components in I, $\mathfrak{B}_I$ will be a real semi-algebraic set (I will define them later).

54. According to what introduced in section 3.1, I use the terms *manifest* and *latent* to respectively denote the variables I want or not to consider in the behavior. Just note that there is a little abuse of notation, in fact, differently from the name, all my variables can be considered as manifest; I restrict to some of them just because of my choice, not because their role on the machine is different from the role of the latent ones.

55. Restricting on the i-th variable and considering it in function of a generic time, I have that $f(t_0) = v_i$ and $f(t_1) = w_i$, where f is a continuous function, thus in $[t_0, t_1]$ f will cover (at least) any value between v_i and w_i.

56. I can consider a planar curve as the behavior of a single specific point with only one degree of freedom.

Also the converse holds, i.e. for every real semi-algebraic S I can construct a machine so that its behavior coincides with S [57]. Furthermore, the space reachable from an initial configuration M_0 is a connected part of a real algebraic set, i.e. a connected real semi-algebraic set. Conversely, any connected real semi-algebraic set can be seen as a reachable space.

3.3.4 Arithmetic operations with algebraic machines

In classical machines, the analytical counterpart was made up only of constructible numbers, and it was possible to perform arithmetic operations between static segment lengths [58]. Now I have to extend this static setting, allowing operations also between "dynamically changing" lengths. More precisely, to perform such operations with algebraic machines, I have to use *variables*. A variable (given a Cartesian coordinate system) is the abscissa or the ordinate of a specific point of the machine. In this subsection, I will see how I can perform the four field operations on such variables. Considering two variables (the coordinates of a single point or two components of two different points), I constrain a specific point to have one of its coordinates equivalent the result of an arithmetic operation between the two variables.

As already observed, I can fix a Cartesian coordinate system given two points O, X fixed on the plane. Being constructible points fixed on the plane equivalent with constructible points, I can consider the point Y (fixed on the plane) so that OY is obtained by an anticlockwise right angle rotation of OX around O. Then I can introduce two rods x and y (both joined in O). Putting a cart constraining X to lie on x and another to constrain Y to lie on y, I construct my set of Cartesian axes with two fixed rods considering OX as unit length. So, I am ready to see some problems and relative constructions in order to perform arithmetic operations on variables.

Problem 1. *Given a rod r and a specific point P, construct a rod perpendicular to r passing through P.* [59]

57. The possibility of *realizing* any real algebraic set is usually called "universality property." It was specially deepened respect to the so-called "mechanical linkages," that I will briefly introduce in the subsection 3.4.3.

58. I am considering binary operations from couples of segment lengths to a segment length. Arithmetic with geometric objects was introduced at least since Euclid, but multiplication was given by the construction of a bi-dimensional rectangle. For an *internal* multiplication, I need the introduction of an arbitrary *unit length.* These internal geometric constructions of arithmetic operations with segment lengths are visible at the beginning of Descartes's *Géométrie.*

59. In Euclid's *Elements* there is the construction of a line passing through a point and perpendicular to another line. That means that, using classical machines, such construction is also available substituting "lines" with "fixed rods." More formally, it is possible, with classical machines, to solve the problem: "Given a *fixed* rod r and a point *fixed on the plane* P, construct a rod perpendicular to r passing through P." However, I had to introduce a new construction because, in order to treat dynamically changing entities, I have to relax the condition that the rod r is a fixed one and that the point P is fixed on the plane (in general, both r and P can be moving objects). Classical machines (and Euclid's tools) are not able to deal with such dynamic objects. This reasoning has to be generalized to all the following problems and constructions in order to understand the need of their introduction.

Furthermore, I have to note that the proposed problems are not formally proved in a specific language. The proposed solutions are just sketches of the relative constructions, the soundness of which is left to elementary intuition (requiring just some very basic knowledge of geometry).

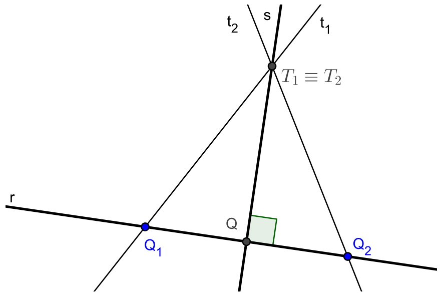

Figure 3.2: Construction of the rod t perpendicular to the rod r in Q (it was obtained imposing $QQ_1 = QQ_2$ and $Q_1T_1 = Q_2T_2 = Q_1Q_2$).

To begin with, I have to consider a rod r' sliding over r: Consider a point Q constrained by a cart to lie on r. For R1′, I can introduce a rod r' joined in Q, and for R2′, I can consider a point Q_1 so that $QQ_1 = OX$ (where O, X are two given points fixed on the plane). Thereafter, I can constrain with a cart Q_1 to lie on r. Hence, r' slides over r.

Furthermore, I can consider Q_2 on r' defined as the point (different from Q_1) satisfying $QQ_2 = OX$. I can also introduce the rods t_1, t_2 joined respectively in Q_1 and Q_2, and on these rods I can respectively consider the points T_1 and T_2 so that $Q_1T_1 = Q_2T_2 = 2OX$ (with an abuse of notation $2OX$ indicate a length double respect to OX, i.e. the distance Q_1Q_2). If with two carts I impose that T_2 belongs to t_1 and that T_1 belongs to t_2, hence T_1 and T_2 have to coincide (see Fig. 3.2): Call this point only T.[60]

Finally I can consider a rod s joined in Q, and, with a cart, I pose that T belongs to s. At the moment s is a rod perpendicular to r passing through Q (that is free to move). So, if I put another cart to constraint P to lie on s, s is the rod required by the problem. □

Thus, to construct the perpendicular projection of a point P with respect to a rod r, introduce a point Q constrained by a cart to lie on r, and then, considering the rod s perpendicular to r passing through Q, add the constraint that also P has to lie on s. The point Q satisfying all such constraints is the perpendicular projection of P on r. The difference with respect to classical machines is that the perpendicular projection is obtained even though P and r are not fixed. Thus, as in all the following problems, algebraic machines provide dynamic constructions.

Problem 2. *Given a rod r and a specific point P, construct a rod parallel to r passing through P.*

60. I can note how the construction of the triangle Q_1Q_2T is the step-by-step translation of Euclid's construction of an equilateral triangle (*Elements*, Book 1, Prop. 1).

According to Problem 1 I can construct a rod s perpendicular to r passing through P, and similarly I can construct a rod t perpendicular to s passing through P, so t is a wanted rod. □

In the following problems, to simplify the notation, I adopt Cartesian coordinates to shorten the notation. Furthermore, I consider the rods x and y, both joined in $(0,0)$ and passing respectively through $(1,0)$ and $(0,1)$ (i.e. they represent the axes of abscissae and ordinates).

Problem 3. *Given a specific point having Cartesian coordinates $(0,t)$, construct a point of coordinates $(t,0)$ and vice-versa.*

I will consider just the transposition of $(0,t)$, the vice-versa is totally similar.

Consider the constructible points $(1,0)$ and $(0,1)$. It is possible to put a rod r joint in $(1,0)$ with $(0,1)$ on it. According to Problem 2, I can construct the rod s joined in $(0,t)$ and parallel to r. Assuming that the ordinate axis is given by the rod y, the point $(t,0)$ will be defined (using carts) as the intersection of y and s. □

The possibility of projecting a point (on abscissae and ordinates) and of transposing a length from the ordinates to the abscissae, is important because it allows me to always consider the variables simply as points on the abscissae [61]. Now I want to show how I can do the internal binary operations of sum and multiplication with points on the abscissae [62].

Problem 4. *Given two specific points of Cartesian coordinates $(t_1,0)$ and $(t_2,0)$, construct a point of coordinates $(t_1+t_2,0)$.*

First of all I have to note that the obvious construction in Euclid's geometry (to open a compass of distance t_2 and to add it at t_1) is not translatable in my setting [63].

According to Problem 3, consider the point of coordinates $(0,t_1)$. Thanks to Problem 2 I can consider the rod r parallel to x passing through $(0,t_1)$ and the rod s parallel to y passing through $(t_2,0)$. By carts I can identify the point (t_2,t_1) (the one lying on both r and s). Finally I can consider the rod t joined in (t_2,t_1) parallel to the rod passing through $(t_1,0)$ and $(0,t_1)$: The point in the intersection of t and x will be in $(t_1+t_2,0)$ (cf. Fig. 3.3), so to solve the problem I just have to apply another time Problem 3. □

61. Given a point of coordinates (x_0,y_0), I can construct the points of coordinates $(x_0,0)$ and $(y_0,0)$. Conversely, given the points of coordinates $(x_0,0)$ and $(y_0,0)$, I can construct the point (x_0,y_0). Thus, to represent real variables in my setting, I can interpret them simply as points moving on the abscissae.

62. In my machines there is no difference between input and output points. So, being the subtraction and the division respectively the inverse of addition and multiplication, I do not need to introduce them. To perform $a-b=c$, I will simply impose $a=c+b$, and similarly for the division, $a/b=c$ will be posed as $a=bc$. About division, as in arithmetic, when $b=0$ the instrumental implementation will not be an operation: It will, however, imply $a=0$ and no constraints on c.

63. In my instrumental setting I do not have compasses, but a natural way to translate Euclid's construction could be to consider a rod joined in $(t_1,0)$, and to report on it the distance t_2. This is not allowed because, according to R2′, I can transfer just distances between points fixed on the plane, while in general $(t_1,0)$ and $(t_2,0)$ can be movable.

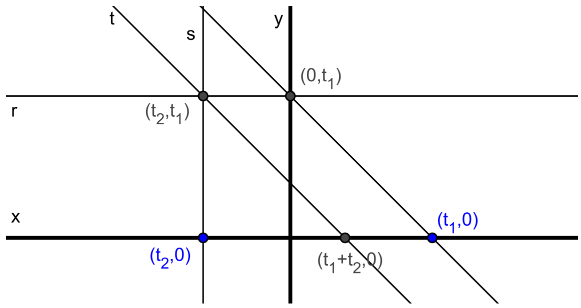

Figure 3.3: Construction of the point $(t_1 + t_2, 0)$ given the points $(t_1, 0)$ and $(t_2, 0)$.

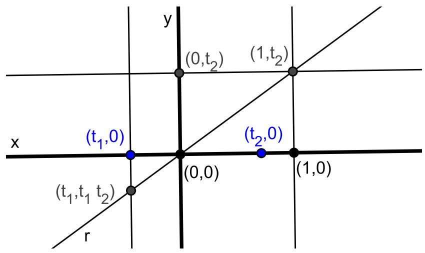

Figure 3.4: Construction of the point $(t_1 \cdot t_2, 0)$ given the points $(t_1, 0)$ and $(t_2, 0)$. To obtain $(t_1 \cdot t_2, 0)$ I have to project $(t_1, t_1 \cdot t_2)$ on y and to transpose $(0, t_1 \cdot t_2)$ on x. (I did not represent these final steps in the diagram to avoid too many lines.)

Problem 5. *Given two specific points of Cartesian coordinates* $(t_1, 0)$ *and* $(t_2, 0)$*, construct a point of coordinates* $(t_1 \cdot t_2, 0)$*.*

> As it happens in Descartes's interpretation of multiplication [64], I need to use the unit length. According to Problem 3, construct the point in $(0, t_2)$. Considering the intersection of the rod parallel to x through $(0, t_2)$ and the rod parallel to y through $(1, 0)$, I obtain the point of coordinates $(1, t_2)$. I can introduce the rod r joined in $(1, t_2)$ passing through the origin $(0, 0)$. As visible in Fig. 3.4, the intersection of r with the rod parallel to y passing through $(t_1, 0)$ will determine the point $(t_1, t_1 \cdot t_2)$. If I project it on y I have $(0, t_1 \cdot t_2)$, that for Problem 3 gives us the wanted $(t_1 \cdot t_2, 0)$. □

Summarizing, given two real variables a, b (that can be thought as the points $(a, 0)$ and $(b, 0)$ in a coordinate system) I can construct with my machines $a + b$ and $a \cdot b$ (i.e. the points $(a + b, 0)$ and $(a \cdot b, 0)$). Furthermore, note that it is possible to constrain a variable a to be null simply constraining the point $(a, 0)$ to coincide with the origin $(0, 0)$ (I just have to constrain with a cart the point $(a, 0)$ to lie on the ordinates). [65]

64. The definition of the length $c = a \cdot b$ is given by the proportion $a : c = 1 : b$.
65. This setting follows modern algebra concept of "field," that is erected on the operations of addition and multiplication. However, in his *La Géométrie*, Descartes also introduced the square root operation. I will discuss it just for the parametrization of the curves constructed as ruler-and-compass loci, as I will shortly show in the subsection 3.4.2.

3.3.5 Real algebraic geometry background

Algebraic geometry on a field $\mathbb{K}$ studies algebraic sets in $\mathbb{K}^n$ i.e. the sets of the form $\{x \in \mathbb{K}^n | P_1(x) = \ldots = P_k(x) = 0\}$, where P_i are polynomials with coefficients in $\mathbb{K}$. For my machines, I use real variables, which is why I restrict my attention to subsets of $\mathbb{R}^n$.[66] One of the difficulties when studying real algebraic sets is that the field $\mathbb{R}$ is not algebraically closed, e.g. the number of zeros (counted with multiplicity) of a real polynomial can be less than its degree. Besides, though the class of real algebraic sets is closed under taking finite unions and intersections, it is not closed under taking complement. Moreover, in general, images of algebraic sets by polynomial functions and their connected components are not algebraic sets. For example, the equation $xy-1=0$ defines a hyperbola in $\mathbb{R}^2$ consisting of the connected components: $\{(x,y) \in \mathbb{R}^2 | xy - 1 = 0, x > 0\}$ and $\{(x,y) \in \mathbb{R}^2 | xy - 1 = 0, x < 0\}$, and its image under the projection on the x coordinate is given by the union of the two disjoint intervals of negative and positive values (only the null value does not belong to the projection). The projected components are given by equations and inequalities, and in general they cannot be given by equations only. In particular, sets defined by a Boolean combination of equalities and inequalities of real polynomials are called "semi-algebraic sets," and this class of set is stable under projection (*Tarski-Seidenberg's Theorem*). Moreover, a semi-algebraic set has only finitely many connected components, and each of the components is also semi-algebraic (*Łojasiewicz's Theorem*)[67].

More precisely, the class of semi-algebraic sets in $\mathbb{R}^n$ is the smallest class of subsets of $\mathbb{R}^n$ satisfying the following properties:

1. it contains all the sets of the form $\{x \in \mathbb{R}^n | P(x) > 0\}, P \in \mathbb{R}[x_1, \ldots, x_n]$[68].
2. it is stable under taking finite unions, finite intersections and complements.

A consequence is that a subset of $\mathbb{R}^n$ is semi-algebraic if and only if it can be represented as a finite union of sets of the form:

$$\{x \in \mathbb{R}^n | f(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}, \qquad f, g_i \in \mathbb{R}[x_1, \ldots, x_n].$$

The *Tarski-Seidenberg Theorem* asserts that the image of a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^k$ under the natural projection $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a semi-algebraic set[69].

The *Łojasiewicz Theorem* asserts that the number of the connected components of a semi-algebraic set is finite, and each of the components is also semi-algebraic[70].

Another approach to semi-algebraic set is by logic: By a "Tarski sentence," I mean a sentence, possibly containing free variables, which can be formulated in

66. The following definitions and results are usually formulated in a more general way, instead of using "$\mathbb{R}^n$" the broader "R^n," where R is any real closed field (cf. Bochnak et al. [1998], Basu et al. [2006]), but I am not interested in this generalization.
67. This summary was taken from the introductory Ta [2011].
68. I adopt the notation that, if D is a ring, $D[x_1, \ldots, x_k]$ is the polynomials in k variables $x_1, \ldots, x_k$ with coefficients in D.
69. This theorem is named after Alfred Tarski and Abraham Seidenberg because of the works Tarski [1951] and Seidenberg [1954].
70. Published in Łojasiewicz [1964].

the following decidable quantified language studied by Tarski. In this language, variables designate real numbers and are quantified over the set of all real numbers. The operators allowed in the language are $+, -, *$, and $/$, designating the usual real arithmetic operators. The allowed comparators are $=, \neq, >, <, \geq, \leq$, all of which have their standard meanings. In addition quantifiers and Boolean connectives are allowed.

A semi-algebraic set S can so be considered as the subset of $\mathbb{R}^n$ satisfying a Tarski sentence $Q(x_1, \dots, x_n)$ containing exactly n free variables. Q is called the "defining formula" of S. By a result given in Tarski [1951], every semi-algebraic set has a quantifier-free defining formula, so Tarski sentences are decidable [71].

A useful constructive tool to prove both *Tarski-Seidenberg* and *Łojasiewicz* theorems is the "cylindrical algebraic decomposition" (commonly abbreviated CAD) introduced in Collins [1975] with a relative algorithm. Given a set S of polynomials in $\mathbb{R}^n$, a CAD is a decomposition of $\mathbb{R}^n$ into connected semialgebraic sets called "cells," on which each polynomial has constant sign, either $+, -$ or 0. For being "cylindrical," this decomposition must satisfy the following condition: If $1 \leq k < n$ and π is the projection from $\mathbb{R}^n$ onto $\mathbb{R}^{n-k}$ consisting in removing the k last coordinates, then for every cell c and d, one has either $\pi(c) = \pi(d)$ or $\pi(c) \cap \pi(d) = \emptyset$. This implies that the images by π of the cells define a cylindrical decomposition of $\mathbb{R}^{n-k}$.

With CAD, it is algorithmically possible to construct the connected components of a semi-algebraic set that will still be semi-algebraic sets.

3.3.6 The full behavior is a real algebraic set

The behavior of an algebraic machine is defined by the coordinates of its specific points. In particular these points can be:

1. given points fixed on the plane (shortly: "given points");
2. constructible points fixed on the plane (shortly: "constructible points");
3. points fixed on a rod (by R2′, using constructible points);
4. points free on a rod (introduced with a cart).

If a machine involves n specific points, the full behavior will be a real algebraic set on $\mathbb{R}^{2n}$. In order to show that, I have to see that conditions on every kind of point are translatable in polynomial equations.

The coordinates of both the "given points" and the "constructible points" are fixed real numbers (about the first kind they are given *a priori*, while for the second they are obtainable as seen in the subsection 3.2.5).

It is more interesting to consider non-static points. Consider a rod r joined in (x_i, y_i). For a point fixed on r, consider its coordinates (x, y). Its constraint is algebraically translatable in $(x_i - x)^2 + (y_i - y)^2 = d^2$, where d is the distance between two constructible points, so it is a well-known real number.

About carts, a point (x, y) is constrained on r if and only if it is constrained to be aligned with all the other points lying on r. So, if (x_j, y_j) and (x_k, y_k) are

71. This summary of the logic approach was essentially based on [Schwartz and Sharir, 1983, pp. 302–303].

on r, I can consider the equation $(x_j - x)(y_k - y) = (x_k - x)(y_j - y)$. Considering these equations for all the couples (x_j, y_j) and (x_k, y_k) of specific points on r, I set the constraint that (x, y) lies on the rod r [72].

Thus all the constraints of algebraic machines on specific points are translatable in algebraic equations. If I consider a machine with n specific points, the $2n$-variables have to satisfy an algebraic system of real polynomials [73], so the full behavior $\mathfrak{B}_f$ of an algebraic machine $\mathcal{M}$ has to be a real algebraic set. Furthermore, restricting my attention just to the variables in I (a subset of $\{1, 2, \ldots, 2n\}$), the external behavior [74] $\mathfrak{B}_I$ will be the projection of a real algebraic set, i.e. a real semi-algebraic set.

3.3.7 Any real semi-algebraic set is an external behavior

Let S be an algebraic set in $\mathbb{R}^n$. I want to construct a machine having as external behavior on n variables exactly S.

Being S a real algebraic set, it is the zero set of a polynomial $P(x_1, \ldots, x_n)$ [75]. I will show how to consider a machine having some variables satisfying $P = 0$.

To perform that, consider the points O, X defining a coordinate system for the plane. For every coefficient a_j appearing in P, consider as given also the point $(a_j, 0)$. Consider on the abscissa n carts and call their coordinates $(t_i, 0)$ $(i = 1, \ldots, n)$. For the constructions introduced in the subsection 3.3.4, I can construct the point $(P(t_1, \ldots, t_n), 0)$. With carts I can also constrain this point to lie both on the abscissae and on the ordinates, so constraining it to be in $(0, 0)$. This way I imposed $P(t_1, \ldots, t_n) = 0$, and if I consider as manifest variable of the full behavior just the abscissae of the points $(t_i, 0)$, I constructed the set $\{(t_1, \ldots, t_n) | P(t_1, \ldots, t_n) = 0\}$, i.e. S.

Note that the result can be extended to any real semi-algebraic set. According to the definition of subsection 3.3.5, every semi-algebraic set can be viewed as the projection of a real algebraic set [76]. Therefore, if I consider fewer

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72. At a first view it may appear redundant to consider all the possible combinations of points on the rod, and I can think to consider just the property of being aligned respect to two points (x_j, y_j) and (x_k, y_k) on r. This can be not enough: If (x_j, y_j) is a cart, it is possible that in a configuration it becomes superimposed to (x_k, y_k). In this case the equation $(x_j - x)(y_k - y) = (x_k - x)(y_j - y)$ simply becomes $0 = 0$. To avoid that, I require to pose the alignment conditions with all the possible couples of specific points on r.

73. In particular the ring of coefficients will be the field of the rational numbers extended with the coordinates of the given points. More precisely, let $\mathcal{P}$ be the set of points fixed on the plane given at the outset. I can consider the set $S \subset \mathbb{R}$ such that $S = \{\frac{AB}{CD} | A, B, C, D \in \mathcal{P}, A \neq B, C \neq D\}$ (PQ indicates the distance between P and Q. I consider all the possible ratios to avoid any dependence on the unit length of the coordinate system). So the ring of coefficients will be $\mathbb{Q}(S) = \{a + bs | a, b \in \mathbb{Q}, s \in S\}$. In particular, if I consider as given just two points, $S = \{1\}$ and so the polynomial in the constructed real algebraic sets will have integer coefficients (because integer and rational coefficients generate the same polynomials), and also their projections will have integer coefficients polynomials (because of the Tarski-Seidenberg theorem).

74. I introduced manifest (or external) variables and behaviors in the subsection 3.1.3.

75. A real algebraic set in $\mathbb{R}^n$ is the zero set of a system of polynomial equations $P_1(x_1, \ldots, x_n) = \ldots = P_m(x_1, \ldots, x_n) = 0$. But, being real polynomials, the zeros of such system will coincide with the zeros of the single polynomial $P = P_1^2 + \ldots + P_m^2$.

76. A semi-algebraic set is the union of sets satisfying polynomial equations and inequalities. But $f(X) \geq 0$ can be rephrased as the projection on X of the solution of the polynomial $f(X) - t_1^2$, and $f(X) \neq 0$ can be seen as the projection of the zeros of $t_2 \cdot f(X) - 1$.

variables from the configuration space generating my real algebraic set, I obtain any wanted semi-algebraic set.

3.3.8 Equality between algebraic machines

About classical machines I asked myself how to know whether two differently obtained constructible points are equivalent. With regard to algebraic machines, I can no longer consider only the equality of one point; I have to consider different configurations. According to my behavioral approach, the equality is easily defined: Two machines are equivalent (or "externally equivalent") if their external behaviors are equivalent, i.e. they define the same set. As in classical machines, the equality for algebraic ones is algorithmically testable. In particular, being the behavior of an algebraic machine a (finite procedurally constructible) real semi-algebraic set, I just have to test the equality between real semi-algebraic sets.

Consider two real semi-algebraic sets $A_1, A_2 \subset \mathbb{R}^n$. In the subsection 3.3.5 I have seen that "Tarski sentences" are decidable, thus, if I can express "A_1 has the same elements of A_2" as such a sentence, I will get that the equality test is decidable. Being semi-algebraic sets, let A_1 be the set defined by the Tarski sentence Q_1 and A_2 by Q_2 (both Q_1 and Q_2 containing n free variables), so the behaviors will be equal if and only if it holds $\forall x_1, \ldots, x_n (Q_1(x_1, \ldots, x_n) \iff Q_2(x_1, \ldots, x_n))$. Thus it is always possible to test whether two algebraic machines are equivalent or not.

The just observed definition of equality between machines considers machines as a set of configurations satisfying constraints over some variables, but it does not deal with the problem of reachability. Now I will investigate how even the equality of reachable spaces is computable, even though the same problem extended to differential machines will remain an open problem.

For the constraints-based interpretation, a machine is defined by a set of assembling operations, but, not considering any specific initial value, the allowed configurations can be composed by unconnected parts: I am not interested in having an actual machine that covers all the configurations in the behavior (restricted to some components), but I know that every configuration in the behavior can be reached by a machine satisfying the given constraints. For the reaching-based interpretation, the machine is given as a set of assembling operations plus an initial value. Being variables introduced as coordinates of physical points, they cannot physically change in a discontinuous way (in function of the time), so, as observed in the subsection 3.3.3, the obtained space of reachable configuration has to be connected. Now I want to show that it is possible to test whether two reachable spaces are equal.

Given two machines $\mathcal{M}, \mathcal{N}$ and the relative initial configurations M_0, N_0, to test the equality between the relative reachable spaces it is not enough that $\mathcal{M}$ and $\mathcal{N}$ have the same behavior, but also that M_0 and N_0 lie in the same connected part of the behavior. Problems like this are typical of robotics, and they are called "motional planning problems:" Is a certain target configuration reachable starting from an initial configuration while respecting a set of constraints? An algorithm answering this question can be found in Schwartz and Sharir [1983]. At the end of subsection 3.3.5, I have introduced Collins's "cylindrical algebraic decomposition." Using these cells, it is possible to algorithmically construct the connected component of any real semi-algebraic set containing an initial value. So the problem now is to test whether the connected components are equal, but these components are semi-algebraic sets (finite union of cells). Hence, it means checking the equality between semi-algebraic sets. In the first part of this subsection I observed that this problem is solvable using the decidability of Tarski sentences.

3.4 Notes on other algebraic constructions

This final section of the chapter deals with some remarks on algebraic objects [77] constructed with tools different from algebraic machines. I will superficially study machines obtained allowing strings (even if just in a restricted way), curves constructed as ruler-and-compass loci, and machines assembled without carts. I have chosen the problems of (restricted) strings and of ruler-and-compass loci because in Descartes's *Géométrie* they were introduced as allowed constructions of "acceptable" curves [78]; I also examined algebraic machines without carts because of the important role they played in the 18th century.

3.4.1 Machines with strings

In Cartesian geometric linkages, strings are accepted "only to determine straight lines whose lengths are perfectly known" [79], thus strings are not accepted if, in the constructive procedures, they are somewhere curved [80]. That means that the behavior of strings allowed by Descartes is, as in the gardner's construction of the ellipse, that of combining a finite numbers of straight components. In this subsection I want to show how this string behavior can be simulated by algebraic machines, i.e. that the introduction of strings (used as

77. As "algebraic objects" I want to generically indicate a family of objects that can be analyzed with polynomial algebra.

78. For example cf. [Bos, 2001, pp. 335– 339].

79. About the curves described by means of a string that can be accepted in Descartes's geometry, I have to cite the following translation of *La Géométrie* (taken from [Descartes, 1954, pp. 91–92]):

> "Nor should we reject the method in which a string or loop of thread is used to determine the equality of or the difference of two or more straight lines drawn from each point of the required curve to certain other points, or making fixed angles with certain other lines. We have used this method in "La Dioptrique" in the discussion of the ellipse and the hyperbola.
>
> On the other hand, geometry should not include lines that are like strings, in that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and, I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact. Nevertheless, since strings can be used in these constructions only to determine lines whose lengths are known, they need not to be wholly excluded."

80. As observed in the subsection 2.3.2, Descartes's idea of acceptable curves is different from the Leibnizian one. For an example of curve acceptable to Leibniz but not to Descartes, see the spiral constructible with Huygens's instrument (seen in Fig. 2.4, pag. 21). In this case, strings change from curved to straight during the motion.

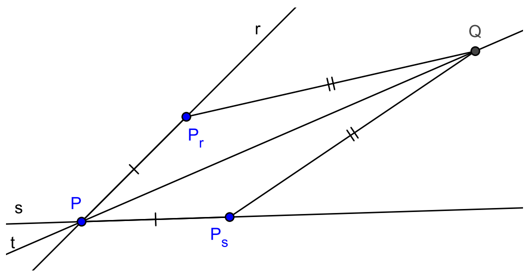

Figure 3.5: Given two rods r, s intersecting in P, construct the points P, P_r, P_s, Q so that $PP_r = PP_s$ and $P_rQ = P_sQ = 2PP_r$. These points are the vertices of a kite, so the rod t passing through P and Q bisects the rods r and s.

allowed by Descartes) does not extend my constructions.[81] In doing that, I need to solve some preliminary problems with my machines.

Problem 6. *Given two rods r, s intersecting in P, construct a rod t bisecting r and s.*

I can consider r, s joined in P (otherwise I have to consider the rod r' sliding on r and s' sliding on s). As visible in Fig. 3.5, consider the points P_r and P_s fixed respectively on r and s so that $PP_r = PP_s = OX$ (O, X can be any given points fixed on the plane). Then I can consider two more rods joined respectively in P_r and P_s, and on them two fixed points Q_r, Q_s so that $P_rQ_r = P_sQ_s = 2OX$ (in the figure, to avoid visual complications, I represented just a part of these rods). Imposing that Q_1 and Q_2 coincide in Q, the points P, P_r, P_s, Q define a kite. Thus the rod t passing through P and Q bisects the rods r and s. Note that I required that $P_rQ = P_sQ = 2OX$ to avoid degenerate or restricting cases.[82] □

Problem 7. *Let r, s be two rods intersecting in P. On each of them consider a cart (respectively P_r and P_s). Impose that it holds $PP_r = PP_s$.*

For Problem 6, construct a rod t joined in P that bisect r and s. For Problem 1, consider the rod u through P_r and perpendicular to t. If with a cart I impose that P_s lies on u, I am imposing the wanted condition. In fact, as visible in Fig. 3.6, it holds $PP_r = PP_s$ because, called Q the middle point between P_r and P_s, the right triangle P_r, Q, P is equivalent to the one of vertexes P_s, Q, P. So the triangle P, P_r, P_s is isosceles. □

As I am going to shortly evince, this construction makes it possible to simulate the behavior of a string folded in a finite number of straight parts. In

81. It will be interesting, even though not deepened for time constraints, to clearly explicate the relation between constructions with algebraic machines and the ones of Descartes.

82. Degenerate cases are present when, in some configuration, Q can coincide with P (thus leaving the slope of the rod t undefined). That happens if and only if the lengths $PP_r = PP_s$ are equal to $P_rQ = P_sQ$ and the angle between r and s is a flat one. Furthermore, if the distances $P_rQ = P_sQ$ are shorter than $PP_r = PP_s$, I am implicitly restricting the allowed angle between r and s. So, to avoid any problem, I need that $P_rQ = P_sQ$ have to be longer than $PP_r = PP_s$: The simplest implementation is to impose $P_rQ = P_sQ = 2PP_r = 2PP_s$.

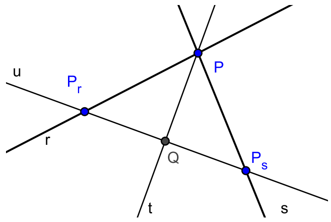

Figure 3.6: Given two rods r, s intersecting in P and with respectively the carts in P_r and P_s, impose $PP_r = PP_s$.

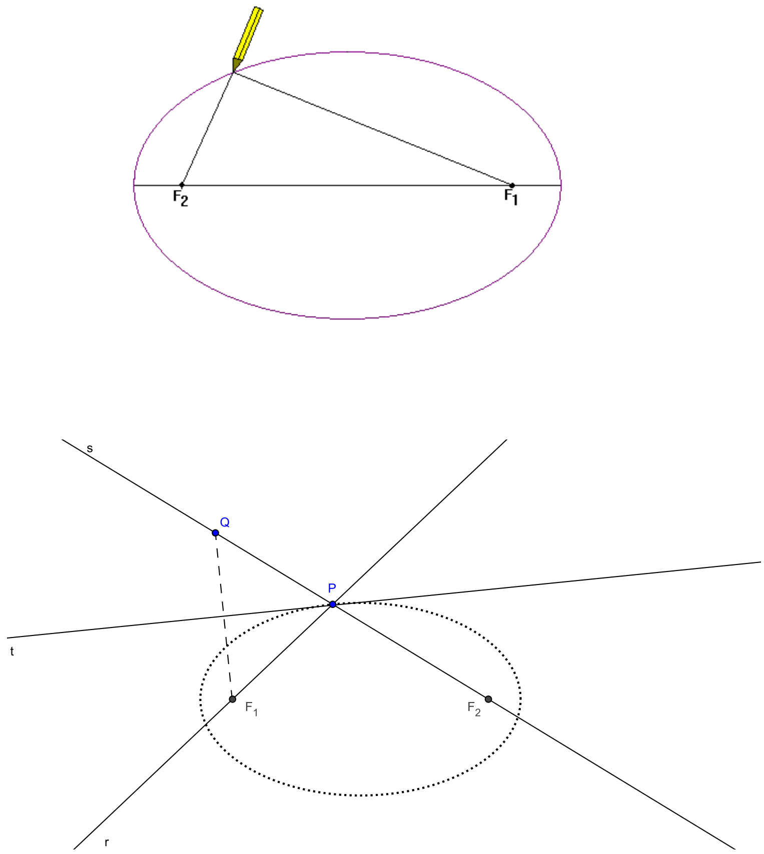

Figure 3.7: In the first diagram we can see the gardener's ellipse construction (by a string), while in the second the same construction obtained with an algebraic machine (through the construction seen in Problem 7). In the second diagram, we can observe the main idea of the simulation: The condition that the distance $PF_1 + PF_2$ has to be constant (because of the string) has been posed using the point Q (constructed on s by R2$'$ imposing the distance QF_2 to be equal to the length of the string), and finally posing that $PF_1 = PQ$ by Problem 7. In this case, I used the string folded in two straight parts, but the same method can be iterated for any number of folding.

fact, for such strings, the physical property useful for geometric constructions is the one of having a constant total length (i.e. the sum of all the straight parts have to be constant). Coming back to algebraic machines, I can transpose fixed lengths also on a rod by R2′. The difference is that the string may work on different straight parts, while the rod conditions have to be posed in the same straight part.

To simulate a string folded in a finite number of straight parts with algebraic machines, I can transpose (thanks to Problem 7) the length from one straight part of the string to the next straight part, recursively arriving to the transposition of the whole length of the string on a single straight part. This length on a single straight component can be finally posed thanks to R2′. An example of this simulation related to the construction of an ellipse is visible in Fig. 3.7 (in this case the string is folded in only two straight components).

3.4.2 Curves constructed as ruler-and-compass loci

Another method to introduce curves beyond lines and circles is to consider them as loci in which any general point is constructed with ruler and compass [83]. Similarly to procedures in "dynamic geometry software" (for example *GeoGebra* or *Cabri Géométre*), I start from a finite set P_0 of given points. I consider also given a straight line or a circle, that I simply call "the given curve," and introduce a point p free to move on the given curve. Now I can define the set $P_1 = P_0 \cup \{p\}$ [84]. I can now formalize the definition of curve constructed "as ruler-and-compass locus" (or "with ruler and compass").

> Given a set of points P_i, for every $a, b \in P_i$ I can consider the lines passing through a and b and the circles centered in a passing through b. Let $\mathcal{C}_i$ denote all these lines and circles. I can construct P_{i+1} as the set of all the points that are intersection of (distinct) curves $A, B \in \mathcal{C}_i$:
>
> $$p \in P_{i+1} \iff \exists A, B \in \mathcal{C}_i, A \neq B, p \in A \cap B.$$
>
> Note that, according to the construction, P_{i+1} includes P_i, and (for every i) contains a finite number of points because the possible couples A, B are finite and the possible intersections of (different) lines and/or circles are finite (at most two points).
>
> A "curve constructed as ruler-and-compass loci" for me is the locus of a point p^* belonging to P_k (where k is a finite positive integer) making p vary on the given curve [85].

In this subsection I will observe that these curves are just a strict subset of all the real algebraic curves in two variables. In particular, using modern terminology, I can consider the one degree of freedom of the free point p on

83. It is not important to construct the whole curve with a single locus, the curve has to be constructed with a finite number of ruler-and-compass loci. So I will consider that the locus defines just locally the curve.

84. Informally, P_i is a set of points over which I can construct new points, that will constitute P_{i+1}. The point p can be used for ruler and compass constructions, even though it moves on the given curve and has not a fixed position, which is why it is included in P_i since $i = 1$.

85. According to the construction, in general p^* can be constructed in function of the position of p.

the given curve (circle or line) as a "parameter." So I can easily arrive to a parametric characterization of the class of the curves (locally) constructible with ruler-and-compass.

In a Cartesian plane, the points constructible with ruler and compass once given the points of coordinates $(0, 0)$ and $(1, 0)$ are all and only the points with coordinates writable as combination of 1, the four field operations and the extraction of the square root. To extend constructions with the introduction of a parameter, I can consider the motion of a point on the abscissa axis, i.e. the point $(t, 0)$ (for every $t \in \mathbb{R}$)[86]. So, repeating all the passages necessary to prove that all the points constructible starting from $(0, 0)$ and $(1, 0)$ have certain coordinates, points constructible starting from even the "parametric point" $(t, 0)$ are defined by the combination of 1 and t with the four field operations and the square root. Thus curves constructible with ruler and compass are exactly the ones whose parametrization is a rational one extended with the square root (eventually nested). This kind of parametrization (at my knowledge) is neither well studied nor better characterized, but I can obtain some preliminary results.

First of all, these constructible curves are more than the rational ones (rational curves are exactly the curves of genus zero[87]), because every curve with equation of the form $y^2 = f(x)$ (where f is a polynomial of any degree) can be parametrized as $(t, \pm\sqrt{f(t)})$. That means that even elliptic and hyperelliptic curves are constructible with ruler and compass (thus there are constructible curves of every genus).

Furthermore, the "square root parametrization" can be considered as a restriction of the more famous "radical parametrization of algebraic curves" (where radicals do not have to be just square roots). It is well known that there are algebraic curves that are not parameterizable by radical because, quoting the introduction of Pirola and Schlesinger [2005]:

> "Zariski, solving a problem posed by Enriques at the Congress of Mathematicians held in Zurich in 1897, proves in Zariski [1926] that, given an algebraic equation $f(x, y) = 0$ of genus $p > 6$ with general moduli, it is not possible to introduce a parameter t, rational function of x and y, in such a way that x and y can be written by radicals as functions of t."

So, summarizing, curves constructible as ruler-and-compass loci are at least rational, elliptic and hyperelliptic ones (thus there are curves of every genus), but surely less than all the curves solution of a general algebraic equation $f(x, y) = 0$.

86. I could also assume the possibility of having a parameter varying along the coordinates of a circle, but it would be superfluous because it is possible to obtain (with the 5 operations and, for example, splitting the circle in two semi-circles) the two ordinates of a point on a circle in function of its projection on the abscissa, so in function of the variation of my "parametric point" $(t, 0)$.

87. The "genus" is a non-negative integer that constitutes an invariant of algebraic sets. In particular, considering a plane curve of degree d, the genus is at most $(d-1)(d-2)/2$. It is exactly $(d-1)(d-2)/2$ if and only if the curve is nonsingular (i.e. has no singular points, points in which the tangent space is not regularly defined). To exactly compute the genus you should previously know the multiplicity r of any singular point (for some clarifications about multiplicity see note 5 at pag. 64): A singularity of order r decreases the genus by $r(r-1)/2$.

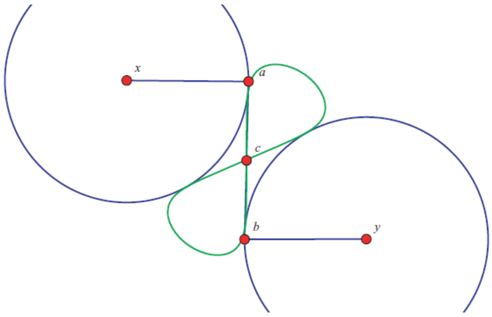

Figure 3.8: A Watt linkage: a is constrained to rotate around x and b around y. The distances xa and yb are equal, and the distance ab is fixed. Called c the midpoint of ab, it moves on a figure-8 algebraic curve called "lemniscate." Note that, close to the singular point of this curve, c moves approximately on a straight line. Diagram taken from [Demaine and O'Rourke, 2007, p. 29].

3.4.3 The role of the cart in algebraic machines

As observed for classical machines, also for algebraic ones the role of the cart is somehow not essential. In particular, if I avoid carts, all the possible constructions are made up by assembling machines where points fixed on rods can be joined with other points fixed on other rods. These machines are called "mechanical planar linkages" or simply "planar linkages." [88]

More precisely, I can recall the definition of linkages of [Demaine and O'Rourke, 2007, p. 9]:

> A "linkage" is a collection of fixed-length 1D segments joined at their endpoints to form a graph. A segment endpoint is also called a "vertex." The segments are often called "links" or "bars," and the shared endpoints are called "joints" or "vertices." [89]

I am interested in planar linkages, which have been thoroughly studied since the 18th century for practical engineering problems [90]. In particular, it was an interesting problem to design a linkage constraining a point to move along a straight line. It was a question of considerable practical importance, for example, to drive the piston rod of a steam engine. In 1784, James Watt invented a simple linkage "almost" achieving this, the so-called Watt's "parallel motion" linkage (see Fig. 3.8). After the discovery in the first half of the 19th century of several unsolvable geometric problems (like trisecting an angle with ruler and compass), for a while it was a common opinion that the problem of transforming linear to circular motion also has no solution, but in 1864, Charles-Nicolas Peaucellier, a captain in the French army, solved the problem of the exact straight-line motion (see Fig. 3.9).

88. It is nonetheless important to note that, as in the subsection 3.2.6, I can avoid carts if I allow to pin together two points. Furthermore, any rod can be considered of fixed length (there are no points sliding on it but just fixed ones).

89. Sometimes it is convenient to place an endpoint joint of one link in the interior of another rigid link. This structure can always be simulated by links that only share endpoint joints by adding extra links to ensure rigidity.

90. Today linkages are usually studied for robotics. About their use in math education (in laboratorial activities) see Bartolini Bussi and Maschietto [2006].

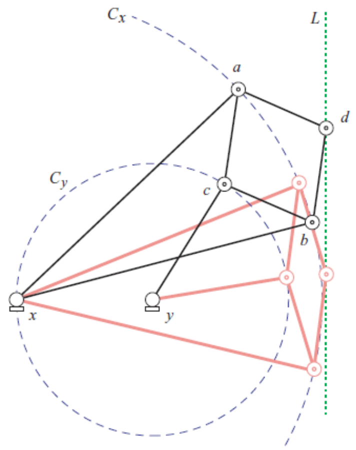

Figure 3.9: A Peaucellier linkage. The dark lines show it in one position, the light lines in another. Diagram taken from [Demaine and O'Rourke, 2007, p. 30].

Inspired by these works, Alfred Bray Kempe was the first to give a proof for a general theorem about linkages in Kempe [1876] [91]. He stated the so-called "universality theorem":

> **Kempe's Universality Theorem.** Let C be a bounded portion of an algebraic curve in the plane, that is, the intersection of zero-set of a real-coefficients polynomial $f(x, y) = 0$ with a closed disk. Then there exists a planar linkage such that the orbit of one joint is precisely C. [92]

Summarizing, now that I have seen the "universality theorem" of algebraic machines without carts, I can observe that carts are not allowing us to obtain new classes of algebraic sets. However, the orbit of a point of an algebraic machine with carts can entirely define any connected branch of a real algebraic set, not only its bounded restrictions.

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91. Kempe's proof was flawed, and his theorem first complete, detailed proof is generally acknowledged to be by Kapovich and Millson [2002]. See also Jordan and Steiner [1999].

92. This version of the statement of the theorem is taken from [Demaine and O'Rourke, 2007, p. 30].

Chapter 4

Differential machines

In the previous chapter, I introduced classical machines to set Euclid's constructions in a purely instrumental way, and saw how it was enough "natural" to extend them to algebraic machines, i.e. machines able to define real semi-algebraic sets. In this chapter, which is the core of the thesis, I extend algebraic machines to "differential machines," i.e. machines dealing with transcendental problems [1]. They will be a well-formalized class of machines intuitively able to convert the historical examples of "tractional motion." In particular, I will justify the introduction of a new geometric-mechanical tool: the wheel [2]. As it historically happened, this extension is introduced to solve "tangent problems."

4.1 Machines beyond algebraic ones

In this section, after the introduction of the direct tangent problem for algebraic curves, I will move on to the inverse problem using "slope fields." From a mechanical perspective, the "wheel" will be a tool naturally solving such inverse problems, and the extension of algebraic machines with wheels will allow us to define differential machines.

4.1.1 Tangent problems for algebraic curves

Let C be an algebraic curve on the plane defined as the zero set of a polynomial $p(x, y)$. At every point $(x_0, y_0) \in C$, I can consider the straight line tangent to C: this problem can be solved with the method of "implicit differentiation" [3]. In the polynomial equation $p(x, y) = 0$, I can consider y as a function of x (i.e. $y = y(x)$) so that using the techniques of differentiation, I arrive at a new equation in the form

$$a(x, y)y' + b(x, y) = 0 \tag{4.1}$$

where $y' = y'(x)$ is the derivative of y with respect to x. Note that $a(x, y)$ and $b(x, y)$ are respectively $\frac{\partial p}{\partial y}$ and $\frac{\partial p}{\partial x}$, so, being $p(x, y)$ a polynomial, they are still

1. I have introduced differential machines in some previous works, even if I called them "tractional motion machines:" cf. Milici [2012a,b, 2015].

2. As seen in the subsection 2.3.1, pag. 18.

3. With regard to implicit differentiation and slope field (which will be soon introduced), see for example [Hughes-Hallett et al., 1998, pp. 489–501].

polynomials. The tangent to C at (x_0, y_0) (on C) is the line of equation

$$a(x_0, y_0)(y - y_0) + b(x_0, y_0)(x - x_0) = 0. \quad (4.2)$$ [4]

The tangent line is not uniquely defined only when $a(x_0, y_0) = b(x_0, y_0) = 0$: in this case the point (x_0, y_0) of C is called "singular point"[5].

Up to now, I considered the tangent line to an algebraic curve C (defined as the zero set of the polynomial $p(x, y)$) at a point on C. Note that the equation (4.2) can also be considered for any point (x_0, y_0) even not on C[6]. In this case, for any point on the plane[7], I have a well-defined line (which generally is not tangent to C when (x_0, y_0) is not on C). Thus, at almost every point of the plane (x_0, y_0) I can associate the direction of the related line, i.e. I can define a "slope field."

Precisely, a slope field (also called "direction field") is a graphical representation useful to qualitatively visualize solutions, or to numerically approximate, first order differential equations (see Fig. 4.1). Given an ordinary differential

4. For example, given the circle of equation $p(x, y) = x^2 + y^2 - 1 = 0$, the tangent in $(1, 0)$ can be constructed as it follows. Using the implicit differentiation method I have to derive $x^2 + y(x)^2 - 1 = 0$ with respect to x, thereby obtaining $2x + 2y(x)y'(x) = 0$. So, according to the form $a(x, y)y' + b(x, y) = 0$, $a(x, y) = 2y$ and $b(x, y) = 2x$. The tangent at (x_0, y_0) satisfies the equation (4.2), so at $(x_0, y_0) = (1, 0)$ the tangent equation is $x - 1 = 0$. However, note that if $(x_0, y_0) \notin C$, the equation is no longer the one of the tangent to C at (x_0, y_0).

5. For a precise definition of "singular points" I can recall [Shafarevich, 2013, pp. 83–97]. Given an algebraic set $A \subset \mathbb{R}^n$ defined as zero of a system of polynomial equations $p_1 = \ldots = p_m = 0$, and considering any $X = (x_1, \ldots, x_n) \in A$, the *tangent space* to A in X is the set of all lines through X tangent to A. A line $L \subset \mathbb{R}^n$ passing through X is defined as $L = \{t \cdot \lambda + X | t \in \mathbb{R}\}$, where λ is the *direction* of L, i.e. any not-null fixed value of $\mathbb{R}^n$. To study whether L is tangent to A in X I have to consider $A \cap L$, that is given by the equations $p_1(t\lambda + X) = \ldots = p_m(t\lambda + X) = 0$. Since I am now dealing with polynomials in one variable t, their common roots are the roots of their highest common factor $p(t)$. In particular, $t = 0$ will be a root of the polynomial $p(t)$ because $X \in A$.

Thus, I can define the *intersection multiplicity* of a line L (of direction λ) with a variety A in X: it is the multiplicity of $t = 0$ as a root of $p(t)$ (where $p(t)$ is the highest common factor of $p_1(t\lambda + X), \ldots, p_m(t\lambda + X)$). Note that both $p(t)$ and the multiplicity of intersection are independent on the choice of the generators of A.

Formally, a line L is *tangent* to A at X if it has intersection multiplicity ≥ 2 with A at X.

The geometric locus of points on lines tangent to A at X is called the *tangent space* to A at X. It is possible to analytically define the tangent space: given the polynomial $p(T)$ (with $T = (t_1, \ldots, t_n)$) and a point $X = (x_1, \ldots, x_n)$, p has a Taylor series expansion $p(T) = p(X) + p^{(1)}(T) + \cdots + p^{(k)}(T)$, where $p^{(i)}$ are homogeneous polynomials of degree i in the variables $t_j - x_j$. The linear form $p^{(1)}$ is the *differential* of p at X, and is denoted $d_X p$. Therefore,

$$d_X p = \sum_{i=1}^{n} \frac{\partial p}{\partial t_i}(X)(t_i - x_i).$$

One can observe that the tangent space at A (defined by $p_1(X) = \ldots = p_m(X) = 0$) in X is made up by the points T, hence $d_X p_1 = \ldots = d_X p_m = 0$.

Approaching to dimension, I can distinguish between *nonsingular* and *singular* points of A. The dimension of the tangent space at a nonsingular point equals the dimension of the variety, while at singular ones the tangent has dimension greater than the one of the variety. Every point on A can be singular or nonsingular.

Analytically singular points are the ones satisfying all $\partial p / \partial t_i = 0$. Thus, the set of singular points is still a variety and of smaller dimension than A.

6. Even though $p(x_0, y_0)$ can be different from 0, also in this case the polynomials $a(x, y)$ and $b(x, y)$ are obtained from $p(x, y)$ with implicit differentiation.

7. Except the points (x_0, y_0) so that the partial derivatives of p with respect to x and y are both null (i.e. when $a(x_0, y_0) = b(x_0, y_0) = 0$).

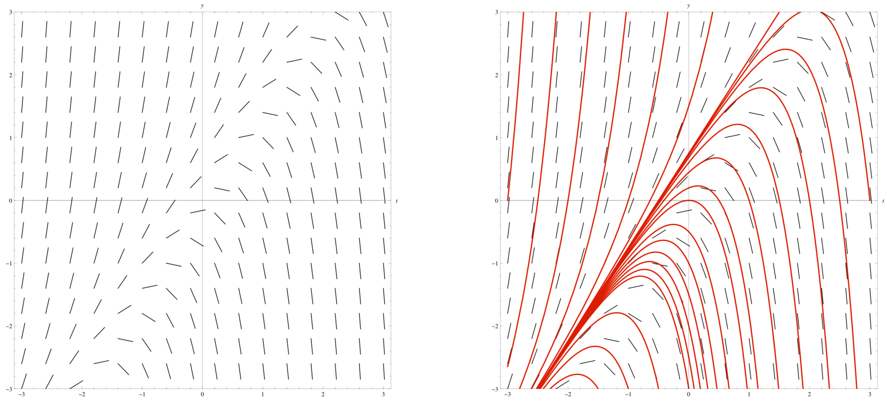


Figure 4.1: A representation of the slope field for $y' = 2y - 3x$ (left) along with several solution curves (right).

equation $y' = f(x, y)$, the slope field for that differential equation is the vector field that takes a point (x, y) to a unit vector with slope $f(x, y)$. Using the visualization of a slope field, it is easy to graphically trace out solution curves to initial value problems tracing a curve that locally satisfies the indicated direction condition [8]. In the present case, I am not introducing the derivative y' and in general differential equations. I am directly assigning a direction to any point of the plane (hence, in contrast to the analytical case of y', there will be no problem if the direction is vertical).

4.1.2 Dynamical slope field with algebraic machines

A slope field's graphical representation involves the simultaneous drawing of the directions at many points in the plane. This representation is a static one: With algebraic machines, I can extend this idea to "dynamical" slope fields.

I previously introduced the algebraic equation (4.2) of the line defining the slope in a point (x_0, y_0) given a polynomial $p(x, y)$ (and so $a = \frac{\partial p}{\partial y}$, $b = \frac{\partial p}{\partial x}$). Such a line can be instrumentally interpreted as a rod constrained by an algebraic machine to move in function of (x_0, y_0) [9]. That means that I can construct a machine that, according to the position of (x_0, y_0), constrains a rod r (joined in (x_0, y_0)) to satisfy (4.2). This setting is not a static representation of the direction (neither as an infinite theoretical vector field nor in a finite number of points as its diagrammatic representation). It is a dynamic instrumental construction of a rod that, when (x_0, y_0) satisfies $p(x_0, y_0) = 0$, corresponds to the tangent to the curve defined as the zero set of p. [10]

However, note that even though it is possible to define a dynamic slope field

8. Cf [Thomas and Finney, 1992, Slope Fields and Picard's Theorem, pp. 1088–1089 and 1101.].

9. For example, I can consider the rod through (x_0, y_0) and a point on the line at a given distance (this point is clearly expressible as a solution of a polynomial equation, so it is constructible with algebraic machines). Note that (4.2) defines a line except when $a(x_0, y_0) = b(x_0, y_0) = 0$.

10. That means that given an algebraic machine $\mathcal{M}$ with a point P tracing a certain curve C, I can construct a new algebraic machine $\mathcal{N}$ constraining a rod r joined in P to be tangent to C. That solves with algebraic machines the direct problem of tangent to a curve.

with r, I am not able to give any constraint to my machines to construct curves along the given directions starting from an initial value: I construct the dynamic vector field but I have no means to solve it. Then, considering this field as the one of the tangents, I can ask myself a tangent problem "inverse" with regard to the one previously solved: Which are the curves respecting such direction constraints? Analytically, this problem becomes the one of finding the solutions of (4.1).

Today the main approaches to solve inverse tangent problems are:

- numerical (or in general "approximate"): with finite numerical (rational numbers with the four field operations) or geometrical (planar constructions with ruler and compass constructions) operations one can construct an approximate solution for the integration of a ordinary differential equation. An example is Euler method, which is the simplest of the Runge-Kutta methods in numerical analysis.
- analytical: one can try to solve the formula rigorously manipulating it with tools conceptually involving infinite processes, such as limits or series or the geometrical idea of the tangent as the limit secant. This approach is the one of classical analysis, from both the more geometrical Newton's idea of "fluxions and fluents" and Leibniz's more algebraic idea of "infinitesimals."

I am interested in exploring a third way: an instrumental approach, the one suggested by the tractional constructions. It will permit from one way to introduce only finite tools, and from the other to obtain an exact solution (not an approximated one). My field will be the one of geometric constructions suitably extended, and the main idea is that I need something to find continuous solutions given a slope field. Thus, concretely I need something that could drive the curve as the steering of a bike in order to respect its direction constraints.

To sum up, even though I did not give yet a definition of "differential machines," they will be an extension of algebraic ones in order to solve the inverse tangent problem [11].

4.1.3 Tractional extension of machines

As previously done when extending classical machines with algebraic ones (with neusis constructions), even in this case I start from the problem of setting a specific problem, and from this starting point I proceed to a general extension. In particular, I am going to analyze the machine for the tractrix [12] introduced in the subsection 2.3.1 (Fig. 2.3 (left), pag. 20).

> *Instrumental definition of the tractrix.* Given a rod r fixed on the plane, consider a cart A moving on it. Consider a rod s joined in

11. I can note a deep difference between the extension from classical to algebraic machines and from algebraic to differential ones. In the former case I simply throw away some restrictions in the construction postulates, while in the latter the extension is introduced to somehow assure the "closure" of a certain class of inverse problems. However, even in the second case, the extension does not only remain logical and abstract; it also becomes well embodied in some new tools, as we will observe in the next subsections.

12. It is well known that the tractrix is not an algebraic curve, so it cannot be obtained only with algebraic machines.

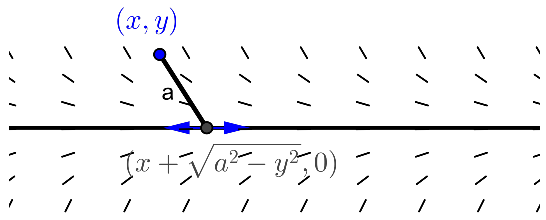


Figure 4.2: Construction of the slope field for the tractrix. A point (x, y) is constrained by a rod of length a to a point on the abscissa (I especially consider the case that the point on the abscissa is on the right in relation to (x, y)). If I consider the rod of length a as the tangent rod, I can consider the related slope field, and the slope in any generic point (x, y) will be $-\frac{y}{\sqrt{a^2-y^2}}$.

> A, and a point B fixed on s. If now I constrain B not to move perpendicularly to s, the trajectory of B will define a tractrix.

If I do not consider the (non-algebraic) constraint that avoids the perpendicular motion with respect to a rod, while I move the point B of coordinate (x, y), the rod s defines the slope field (denoting the tangent direction of B) visible in Fig. 4.2. The constraint that a point of a rod cannot move perpendicularly to the direction of the same rod can be considered the key to find a solution in the dynamic slope field. This constraint is not "geometric" from a Cartesian perspective, and, according to classical mechanics, I can better specify the differences.

In classical mechanics [13], "holonomic constraints" are relations between the coordinates $x_1, \ldots, x_n$ and t [14] which can be expressed in the form $f(x_1, \ldots, x_n, t) = 0$, where f is a function. A system is called holonomic if all its constraints are holonomic, i.e. the constraints on the coordinates are independent from the relative derivatives. If, as in my case, coordinates represent the position of particles, from the definition I can evince that a holonomic constraint imposes conditions only on the position of the particles, and not on their velocities: a constraint that cannot be expressed in the form shown above is a "nonholonomic" constraint. Velocity-dependent constraints such as $f(x_1, \ldots, x_n, x'_1, \ldots, x'_n, t) = 0$ are not usually holonomic [15].

From an analytical perspective, the difference between holonomic and nonholonomic systems lies in the introduction of derivatives of the coordinates. From a mechanical point of view, the difference is that the state of the system depends not only on the configuration of the coordinates, but also on the path that they go through. More precisely, holonomic constraints pose pointwise constraints, while nonholonomic ones pose path constraints.

Thus, referring back to my machines, algebraic ones imply holonomic constraints [16], while their extension for tractional motion will include nonholonomic

13. See for example Goldstein [1962].
14. "t" is the independent variable: it is usually considered the time of the dynamical system.
15. They can be holonomic if they can be reformulated as conditions not implying derivatives.
16. Every algebraic machine is defined by holonomic constraints, but the converse does not hold because in general the function f (such that, according to the definition, $f(x_1, \ldots, x_n, t) = 0$) can be transcendental (e.g. consider $x = \sin t$).

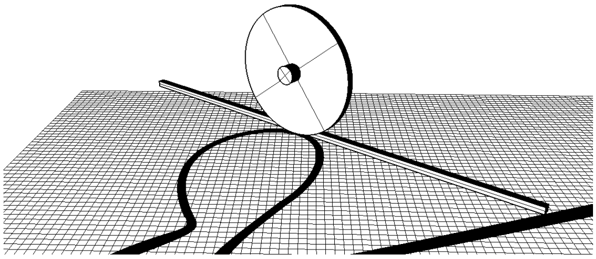


Figure 4.3: A wheel rolling while following any regular curve has the property that its own "direction" (represented in the picture by a bar) is always tangent to the curve.

constraints. As we will see in the subsection 4.2.1, this will imply the need for a different interpretation of universum and behavior of differential machines with respect to algebraic ones.

4.1.4 Defining differential machines

To extend algebraic machines, I introduce a new component posing nonholonomic constraints: the "wheel." A wheel on a point S of a rod r prevents S moving perpendicularly to r (considering the motion of S relative to the plane)[17]. The wheel poses a nonholonomic constraint because analytically—as we will see soon, though not in this subsection—its application implies the introduction of the derivative of the variables. Furthermore, from a mechanical view, the wheel constraint is nonholomic because it poses a condition on the path of the point on which I put the wheel. Hence, I can consider the following postulate:

> **Wheel postulate.** Given a rod r and a point S fixed on r, we can set a wheel at S that prevents S itself moving perpendicularly to r (considering the motion of S relative to the plane).

Technically, my wheel works as if I put a fixed caster (oriented like r) at S, with its wheel rotating without slipping on the plane. Thinking at the solution of slope fields, the avoidance of lateral motion with respect to the rod at a point is strongly related to the tangent. If I consider the caster wheel as a disk rolling perpendicularly to the base plane, the projection of the disk surface is always tangent to the curve described by the disk contact point[18] (see Fig. 4.3). Thus, the rod is tangent to the orbit of the wheeled point, having the same direction as the caster wheel.

As an example, I can consider a finite rod joined in the fixed point P and with other edge Q (so Q describes a circle), a rod r joined in Q, and a wheel on r in Q. For the tangent condition, r will always be tangent to the circle while

17. Even though I am mechanically introducing the wheel, the same constraint can be put using different practical solutions. For example, instead of a wheel rotating without slipping, I could have considered a blade, so avoiding the idea of rotation.

18. Given a wheel on a point S of r, the tangent at S (to the curve that the point traces) will be the direction r when S moves. If the rod rotates around S when S is not moving, r does not represent the tangent.

Q moves (so r will be perpendicular to PQ). Thus, if I consider any algebraic machines and, on any specific point S, put a rod r and pose a wheel on r in S, when S moves, r has to be tangent to the orbit of S. That means that the wheel somehow solves the "direct tangent problem."

However, the direct tangent problem was already solvable with algebraic machines: the wheel is particularly useful for the inverse problem. In fact, I can construct new curves given their tangent properties imposing conditions on the rod where the wheel is put. The concept of "dynamical slope field" can be used to evince the role of the wheel: Being the wheel put on a point S of a rod r, I can consider the slope of r in function of the position of the point S, defining a dynamical slope field. The construction of slope fields was already available with algebraic machines, but the wheel constraint allows to obtain the solution curves given an initial position of S. So the wheel can be considered a mechanical tool solving the dynamic slope field defined by the direction of r in function of the position of S.[19]

I can define a class of machines extending the algebraic ones: I call "differential machines" the machines constructed according to the postulates of algebraic machines extended with the "wheel postulate." Therefore, a differential machine is obtained adding to an algebraic machine any number of wheels (on points fixed on a rod). For a diagrammatic representation of the wheel in a differential machine, see Fig. 4.4.

Differential machines can be considered as a formalization of the machines of the tractional motion, and I will explore them in this chapter obtaining a precise characterization of constructible objects. If the orbit of a point of a differential machine describes a curve[20], the point can be considered to be obtained with tractional constructions. From this perspective, these machines can be used to solve the problem of curves traceable by means of tractional constructions.

However, there are some differences between the historical machines for tractional constructions and my differential machines. As seen for algebraic ones, even differential machines will not only construct curves (as it happens in tractional motion) but in general n-dimensional spaces. I will later deal with what I consider as universum for these machines.

To conclude, I can note some similarities between the wheel and the cart. First, both of them are posed given a rod and a point on it (even if for the wheel the point is fixed on the rod, while for the cart it is not). In addition, both of them constrain a point to move along the direction of the rod: The difference lies in the reference frame with regard to which the point moves along the direction. For the cart, the point constrained to lie on the rod can be considered as a point constrained to move along the direction of the rod in relation to the reference frame of the rod. On the contrary, the wheel constrains a point to move along the rod direction with regard to the reference frame of the plane. Thus, I could

19. If I consider a machine with more than one wheel, the direction of a wheel can determine and be determined by the direction of other wheels. Analytically, this is translated into the fact that differential machines solve ordinary differential equations (shortly: ODEs) not only of first order, but also of any order (see, for example, the machine of Fig. 4.10). Therefore, even though every wheel is solving a dynamical slope field, it is not always graphically representable as a static one. Moreover, the wheel can be considered as a tool solving Euler's method in a continuous setting.

20. In general it could define a 2D subspace of the plane.

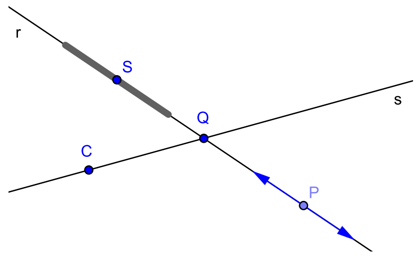


Figure 4.4: Schematic representation of the components: There are two rods (r and s) joined at Q. On r, there is also a cart P (the arrows stand for the possible motions the cart can have) and a wheel S (the gray thick line ideally represents the projection of a wheel).

introduce a generic "direction constraint" so that given a point S (that I want to constrain), a point P (that will give the direction of the motion of S), and a reference frame (i.e. two distinct points), S will move along the direction PS in the given reference frame. This new direction constraint will work as a cart if the reference frame coincides with a rod passing through P and S (i.e. the reference frame is given by two points fixed on such rod) but also as a wheel if the reference frame coincides with the plane (i.e. the reference frame is given by two points fixed on the plane).

In future it can be interesting to study the class of machines defined using not carts and wheels, but the more general "direction constraint." Another idea to be explored is whether it is possible to set these machines using only the relations (over quadruplets of points) "congruence"[21] and "direction."

4.2 Setting differential machines

Once defined differential machines, I have to set them in a behavioral approach. After evincing some major differences with respect to the algebraic case about the interpretation of variables and of the universum, I will come to define the full behavior of such machines. Note that to define the external behavior, I will need the introduction of more analytical tools (in the section 4.3).

4.2.1 Definition of the universum

The universum of algebraic machines is a subset of $\mathbb{R}^n$. For differential machines, having nonholonomic constraints, families of curves, not just families of points, will make up the behavior. According to constructions seen in the historical part, this distinction may appear useless, because tractional machines define single curves: the problem can arise if the set of the reachable points is of dimension greater than one. To evince it, in this subsection I propose the example of a differential machine: the set of its reachable points is a real semi-algebraic set, so it can be obtained with an algebraic machine, but we will intuitively see why its behavior is substantially different from the one of the algebraic one[22].

21. Introduced in note 8, pag. 34.
22. Example taken from Milici [2015].

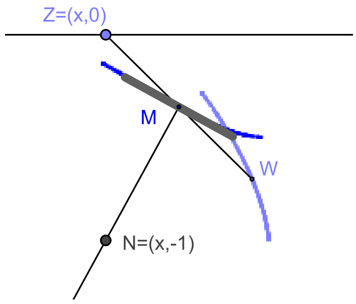


Figure 4.5: A simple differential machine (the point Z moves along a line, W rotates around).

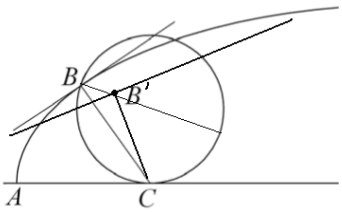


Figure 4.6: Property of the tangent to the curve traced by a point B' fixed on a rolling disk. Considering the contact point C, the tangent at B' will be perpendicular to CB'. If the point is on the circumference of the disk, then the traced curve will be a cycloid.

Given the unitary length and an oriented rod to be used as abscissa (so that I can consider the related Cartesian plane and coordinates), a cart $Z = (x, 0)$ on the rod, and the point $N = (x, -1)$, consider the rod ZW of unitary length (W is free to turn around Z), and let M be the middle point of ZW. I place a rod passing through M and N, and another one perpendicular to MN passing through M[23]: on the latter rod I place a wheel corresponding to M, so that the tangent to the curve traced by M will always be perpendicular to MN (see Fig. 4.5).

While I move Z along the abscissa, if the absolute value of the W ordinate is strictly less than 1, W has to describe a cycloid, because of the geometrical property shown in Fig. 4.6.[24] On the contrary, when W assumes coordinates $(x, \pm 1)$, the tangent to M must be horizontal, and so the motion of W can be both a cycloid and purely horizontal, losing the uniqueness.

It means that given any initial position (x_0, y_0) of W in the strip $]-\infty, +\infty[\times [-1, 1]$, any other value (x_1, y_1) in the strip can be reached by W: call $(x_0^*, 1)$ and $(x_1^*, 1)$ the first apex (going from left to right) of the cycloid starting respectively in (x_0, y_0) and (x_1, y_1). As can be seen in Fig. 4.7, with my differential machine, I can reach (x_1, y_1) from (x_0, y_0) decomposing the motion in three parts: First, I reach $(x_0^*, 1)$ (it is possible because they are on the same branch of cycloid); second, I reach $(x_1^*, 1)$ (it is possible because I am going horizontally on the line

23. According to Problem 1, pag. 48.
24. In particular, I imposed the wheel on M and not on W, because, while the rod ZW rotates around Z, W can become coincident with N, leaving the rod WN undetermined.

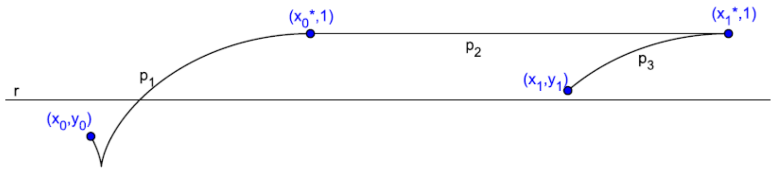


Figure 4.7: For any two points (x_0, y_0) and (x_1, y_1) in the strip $]-\infty, +\infty[\times[-1, 1]$ there is a path (the combination of the paths p_1, p_2, p_3) satisfying the constraints of the machine seen in Fig.4.5.

$y = 1$); third, I reach (x_1, y_1) (because they are on the same branch of cycloid).

So, if I consider as manifest variables the coordinates (x, y) of W, the space of the reachable configurations is exactly the strip $]-\infty, +\infty[\ \times [-1, 1]$. This strip is a real semi-algebraic set, so it can be considered as the behavior of an algebraic machine. For my behavioral approach, two systems/machines are equivalent if they have the same behavior. If I consider the set of the reachable configurations as behavior of my differential machines, my original machine is equivalent to an algebraic machine. However, for any algebraic machine, any path internal to the space of the reachable configuration is a path that can be walked by the machine. On the contrary, for my differential machine, I can walk only certain trajectories. Thus, I cannot consider a subset of $\mathbb{R}^n$ as universum for differential machines; I need something else. In particular, considering this example, I get that the universum of a differential machine has to be made up by a (generally infinite) set of curves satisfying both the configuration conditions of the holonomic constraints and the path conditions imposed by non-holonomic ones. Let me define it more precisely.

Starting in antiquity, many concrete curves have been investigated using the synthetic approach. Differential geometry takes another direction: Curves are represented in a parametrized form as a class of equivalence on vector-valued functions [25]. Coming back to my machines, I can continue the interpretation of variables as coordinates of specific points of machines as done in algebraic ones. But, unlike before, it is no longer enough to consider variable as real

25. Let n be a natural number, r a natural number or ∞, I a non-empty interval of real numbers and $t \in I$. A vector-valued function $\gamma : I \to \mathbb{R}^n$ of class C^r (i.e. γ is r times continuously differentiable) is called a "parametric curve" of class C^r, t is called the parameter of γ and $\gamma(I)$ is called the image of the curve. It is important to distinguish between a parametric curve γ and the image of a curve $\gamma(I)$ because a given image can be described by several different C^r parametric curves. One may consider the parameter t as representing time and $\gamma(t)$ as the trajectory of a moving particle in space.

Given the image of a curve one can define several different parameterizations of the curve. Differential geometry aims to describe properties of curves invariant under certain "reparametrizations." So we have to define a suitable equivalence relation on the set of all parametric curves. The differential geometric properties of a curve are invariant under reparametrization and therefore they are properties of the equivalence class. The equivalence classes are called C^r curves and are central objects studied in the differential geometry of curves.

Two parametric curves of class C^r $\boldsymbol{\gamma_1} : I_1 \to \mathbb{R}^n$ and $\boldsymbol{\gamma_2} : I_2 \to \mathbb{R}^n$ are said to be equivalent if there exists a bijective C^r map $\phi : I_1 \to I_2$, hence $\phi'(t) \neq 0 \quad (\forall t \in I_1)$ and $\boldsymbol{\gamma_2}(\phi(t)) = \boldsymbol{\gamma_1}(t) \quad (\forall t \in I_1)$. γ_2 is said to be a "reparametrization" of γ_1. This reparametrization of γ_1 defines the equivalence relation on the set of all parametric C^r curves. The equivalence class is called a C^r curve, and equivalent C^r curves have the same image. For a detailed discussion, see, for example, Do Carmo [1976].

numbers, but, to introduce path constraints, I can consider these variables as real functions ($\mathbb{R} \to \mathbb{R}$), where the parameter represents the time [26]. Being an idealization of physical machines, I can consider these functions to be C^∞ (sometimes I will refer to functions of the class C^∞ as "smooth" functions).

With reference to the example of the machine in Fig. 4.5, I need to consider as universum something like manifolds of curves. However, for what has just been observed, curves can be defined as classes of equivalence over vector-valued functions. So, to mathematically simplify the definition, I will consider a "manifold of C^∞ functions" as universum for differential machines. In particular, considering n variables, these functions have to be $\mathbb{R} \to \mathbb{R}^n$.

Algebraic machines are particular differential ones, so I have to observe how the interpretation of the universum/behavior as real semi-algebraic set is reformulated as manifold of functions. From the point of view of paths, algebraic machines allow any path moving inside the defined semi-algebraic set $S \subset \mathbb{R}^n$, so the manifold of functions will be made up by all the functions of class C^∞ having their image inside S.

4.2.2 Full behavior as solution of differential polynomial systems

I have just defined a manifold of smooth functions as universum of a differential machine. Variables are coordinates of specific points, and are considered as functions. Thus, given a machine with k specific points, its full behavior will be the manifold of all the smooth functions $\mathbb{R} \to \mathbb{R}^{2k}$, satisfying the constraints given by the specific machine. In particular, the used constraints will be the ones of algebraic machines (analytically convertible in polynomials) and wheels, so the first thing to do is to figure out how I can analytically translate wheel conditions.

Given a point P fixed on the rod r, consider the wheel on P. Consider also another point Q fixed on r (Q different from P [27]). As typical in physics, consider $P = (x(t), y(t))$, i.e. consider the Cartesian coordinates of the point in function of the time. The wheel poses the condition that P will not move perpendicularly to r: so, considering $P' = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$, P' has to be parallel to $Q - P$. Thus, considering $Q = (x^*, y^*)$ (also these coordinates are in function of the time t, but I will omit the dependence on t in the notation) and introducing the notation $\Delta x = x^* - x, \Delta y = y^* - y$, if I consider $P' = (x', y')$ and $Q - P = (\Delta x, \Delta y)$ as vectors, by proportions their parallelism will be given by the

26. It is hard to consider geometrical the introduction of the time. More precisely, being curves classes of equivalence over the parametrization, they are not depending on the parameter: similarly also for differential machines, the relation with the parameter (the time) is not important, but it is comfortable to use it to analytically characterize the universum.

27. It is always possible to construct (if not already present) such point Q fixed on the rod. Q will represent the direction toward which P can move, and I introduced it to avoid the introduction of rods coordinates. I posed that Q is a point fixed on r and not a cart to avoid the following case: consider a rod r joined in P, and consider on r just another specific point R that is a cart. Thus, in general R may go in the same position of P. In this case of coincidence, r may rotate without causing any motion of R. So I imposed to consider Q as a fixed point on r (a point different from P) to always satisfy the condition of expressing the direction of r with these two points.

condition

$$x'\Delta y = y'\Delta x. \tag{4.3}$$

If I consider as variables not only x, y, x^*, y^* but also x', y', the wheel constraint is translatable in a polynomial in x, y, x', y', x^*, y^*. As I will explain in section 4.3, the polynomials in variables and derivative of variables are called "differential polynomials." In summary, the wheel constraint is translatable in a differential polynomial condition on the variables (the coordinates of the points).

Coming back to the full behavior of a differential machine, both wheel constraints and algebraic conditions are translatable in differential polynomials [28]. Considering the system Σ of all the real differential polynomial equations obtained as counterpart of the constraints of the machine, the full behavior will be the manifold of all the smooth functions $\mathbb{R} \to \mathbb{R}^{2k}$ satisfying the system Σ.

Note that I gave an analytical form only to the full behavior. For an external behavior, I generally need to "eliminate" the unwanted variables of the full behavior (as has been done for algebraic machines when introducing semi-algebraic sets). As opposed to the real algebraic case, in my knowledge there is not a precise counterpart to semi-algebraic sets in the differential case [29]. However, in section 4.3, I will introduce some basic tools of "differential algebra" (specifically "differential elimination") that will allow me to answer some questions about these machines.

4.2.3 Differential systems are solved by differential machines

Given a system Σ of differential polynomial equations, I am going to show that I can construct a machine having as external behavior the manifold of the solutions of Σ (note that in this case I am referring to the external behavior, and not to the full one). First of all, working on real values, I can convert the system Σ in a single polynomial [30]. Being differential machines the extension of algebraic one, they are able to perform sum and multiplication, and it is possible to put the condition that a certain variable is equal to 0. So, all I need to show, is that I can construct the derivative of the variables $x_1, \ldots, x_n$ [31].

As Fig. 4.8 illustrates, consider the point $(t, 0)$ with a cart on the abscissa (t can assume any real value) [32], and, keeping in mind the constructions available

28. Trivially, any polynomial can be considered as a differential polynomial not involving any derivative.

29. In the non-differential case, the projection of any real algebraic set (i.e. defined as zero of some polynomial equations) is well expressed with a finite union of systems of polynomial equations and inequalities (semi-algebraic sets), which are also closed in respect of the projections. On the contrary, given a differential system of polynomial differential equations with real coefficients and considering the variables as real functions, it is an open problem whether the external behavior can be expressed as a finite union of systems defined by polynomial equations, inequations, and inequalities (see Mareels and Willems [1999]).

30. The system of real differential polynomials $p_1 = \ldots = p_l = 0$ is equivalent to $(p_1)^2 + \ldots + (p_l)^2 = 0$.

31. The first work when I expressed this construction was in Milici [2012a], even if there I expressed the possibility for such machines of solving polynomial Cauchy problems.

32. Note that t is arbitrary, the important thing is that all the various x_i are considered in correspondence of the same t. Thus, t can be viewed as the "independent variable" in function

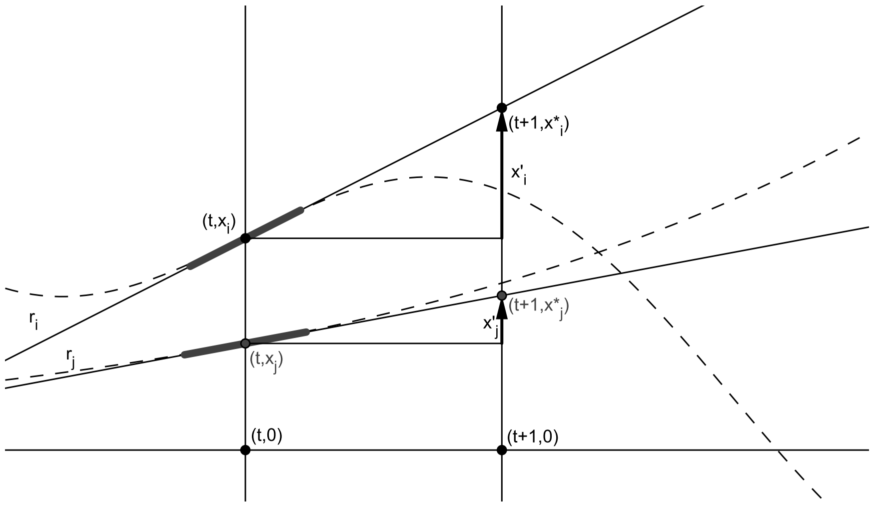


Figure 4.8: Construction of the derivative of the variables x_i, x_j.

for algebraic machines, consider the points $(t, x_1), \ldots, (t, x_n)$. On these points, I can put n rods: call r_i the rod joined in (t, x_i). Put also a wheel on every r_i in correspondence of (t, x_i). I can construct the rod of equation $x = t + 1$: call x_i^* the ordinate of the point in the intersection of $x = t + 1$ and r_i.

For what has been observed about the role of the wheel, r_i will be tangent to the graph of (t, x_i), hence x_i^* will be $x_i + x_i'$ [33]. It means that I can construct the point $(x_i', 0)$ that can be used as a new variable, and so the differential polynomial can be considered as a polynomial in the old variables $(x_1, \ldots, x_n)$ and in the new ones (their derivatives, not only first ones, but iteratively of any finite order). So, the possibility of solving polynomials with algebraic machines assures us that, for every system of differential polynomial equations Σ, I can consider a differential machine having as external behavior (restricted to $x_1, \ldots, x_n$) the solution of Σ.

4.2.4 First example and note on "independentization"

As a first example of passage from differential equation to differential machine, I can consider the problem $y' = y$. To construct a machine solving it I have to start considering a cart $(t, 0)$ on a fixed rod (that I will consider as abscissa), a rod perpendicular to the abscissa and translating according to the value of t, and on this rod the point (t, y). As seen in note 33, instead of the rod of equation $x = t+1$, I can consider any other form $x = t+a$. In particular, it is simpler if I adopt $a = -1$. Thus, y^* will be $y - y' = 0$ (for the problem is $y' = y$). Therefore, I have to introduce the rod r passing through (t, y) and $(t - 1, 0)$, and to put a wheel on it in correspondence of (t, y), obtaining the

of which the various functions (dependent variables) are computed.

33. Obviously, it was not strictly necessary to construct the rod of equation $x = t + 1$. In the case of a rod of equation $x = t + a$ (for any constant $a \neq 0$), the intersection of r_i with the new rod is $(t + a, x_i + ax_i')$ (in other words, $x_i^* = x_i + ax_i'$). However, having assumed the introduction of the unitary length, generally it is analytically simpler if I consider $a = 1$.

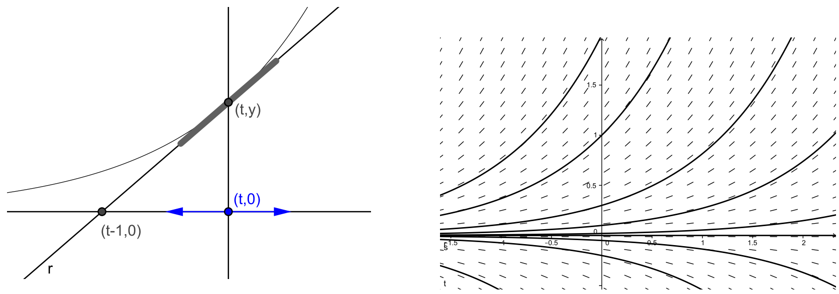


Figure 4.9: A machine solving the differential equation $y' = y$ (left) and the relative slope field (right). Note that, according to different initial conditions, the traced curve will be an exponential one or a line in the case of an initial condition $y(t_0) = 0$. This machine is the symmetric of the one in the right of Fig. 2.3, pag. 20.

machine of Fig. 4.9 (it is the symmetric of the machine seen in the right of Fig. 2.3, pag. 20). Conceptually, this machine is constructively using the property of the exponential curve of having a fixed-length subtangent (i.e. the segment connecting $(t-1,0)$ and $(t,0)$).

Until now, I passed from differential equation to differential machine: Conversely now I convert the machine in differential polynomials. Given the machine of Fig. 4.9, what is the differential polynomial system defining its behavior (according to the method of subsection 4.2.2)?

Because of the wheel, (t', y') has to be parallel to r. So, using the formula (4.3), I get $t'y - y' = 0$. Note that the obtained differential equation is different from the original one ($y' - y = 0$). The difference is given by the implicit assumption that $t' = 1$: When I solve a differential equation with the method seen in the subsection 4.2.3, I implicitly assume that t is the independent variable, so everything is obtained in function of its value [34].

In summary, given a system of differential polynomials Σ, with the method of subsection 4.2.3 I can construct a machine solving it, but the system Σ^* obtained analyzing this machine is slightly different from the original Σ. If I want to obtain Σ from Σ^* I have to add the condition $x_i' = 1$ for the variable x_i that represents the abscissa of the independent point $(t, 0)$. I may call such additional condition the "independentization of a variable" (because from $x_i' = 1$ it follows $x_i = t + k$, i.e. x_i is exactly the independent variable eventually translated of a constant k).

4.2.5 Note on initial conditions

The (full) behavior of differential machines can be analytically defined by a system of differential polynomials. However, when a machine is considered to work on a plane, the initial position of its components can be considered

34. The introduction of a new variable (with constant derivative 1) for the independent one is a standard method to pass from a differential polynomial involving also the independent variable to an equivalent polynomial not depending directly on the independent variable. The latter kind of polynomials is called "autonomous."

implementing the initial conditions of a polynomial Cauchy problem. In this subsection, I want to focus on how to apply these initial conditions.

As evident from the many examples in chapter 6, physical realizations of my planar machines are devices that can be lifted and downed on the plane. While the device is not yet downed on the plane, there are fewer working constraints (because of the lack of wheel friction), so I can move some points that will lose some degrees of freedom when wheels touch the plane. Therefore, if I consider my machines as physical devices, their assembly and use can be distinguished in two different steps:

1. composition: the various parts of the machine are assembled in order to construct the machines;

2. friction on the plane: the machine is “put on the plane,” so wheels avoid lateral motions.

The difference between these two steps is the role of the wheel. In the first case the machine is constructed but, considering it lifted from the plane, the wheel constraints do not work, so on the machine only the holonomic constraints are active (the ones of algebraic machines). When I ideally put the constructed machine on the plane, wheels begin to have friction on the plane, and consequently the related nonholonomic constraints begin to work.

While the composed machine is already defining differential polynomial equations, the activation of the friction is related to the posing of initial conditions. In fact, in the instant when the constructed machine touches the plane (and the wheel friction begins), all the points have a certain position: The values of the variables relative to these positions can be viewed analytically as the initial conditions of a Cauchy problem. Therefore, to pose an initial condition to some variables, I have to appropriately move the points (the position of which is related to the wanted variables) when the device is lifted. The downing of the device will assure that the variables will solve the Cauchy problem.

To clarify these ideas, as an example, I propose a machine solving the differential equation $-f''(t) = f(t)$. According to different initial conditions, the same machine can generate the sine (posing $f(0) = 0, f'(0) = 1$) and the cosine function (with initial conditions $f(0) = 1, f'(0) = 0$).

As seen in Fig. 4.10, once introduced the point $(t, f(t))$, I can construct the point $(t+1, f(t)+f'(t))$. Reporting the length $-f'(t)$ as represented in the figure (the dotted lines represent the translation of lengths, without visualizing all the step behind), I can construct $(t, -f'(t))$. Then, constructed $(t+1, -f'(t) - f''(t))$, it is possible to impose $f(t) = -f''(t)$ reporting the length $-f''(t)$.

Now it is time to impose initial conditions. If I want $f(t)$ be the sine function, I have to impose $f(0) = 0, f'(0) = 1$. Physically, this condition has to be posed after the construction of the machine, and before the “activation” of the friction of the wheels. First, I move the cart in $(t, 0)$ until it reaches the position $(0, 0)$, then I move the carts $(t, f(t))$ and $(t, -f'(t))$ until they (respectively) reach the positions $(0, 0)$ and $(0, -1)$. Once posed these conditions avoiding the nonholonomic constraints (ideally: when the machine is not yet put on the plane), the nonholonomic constraints of wheels can be activated (the machine can be finally put on the plane, allowing the friction of the wheels on the plane).

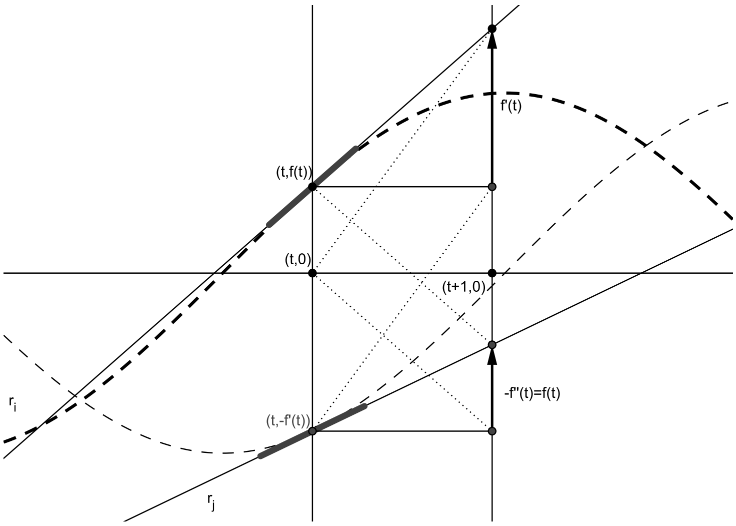


Figure 4.10: A machine for $f(t) = -f''(t)$.

In this way the machine will generate exactly the sine function.

In contrast to the case of the exponential, the sine function is constructed using a second order differential equation, so it is not possible to consider the wheel solving a static graphical slope field. In fact, the slope of the rod with a wheel is dynamically defined in function of the position of the other wheel. However, as in general, also in this case the machine can be considered as a continuous mechanical reinterpretation of Euler's method.

4.2.6 Role of the cart in differential machines

I can briefly discuss the potential infinite length of the rods and the non-minimality of the proposed components of differential machines [35].

The rods I adopted do not have to be considered "actually" infinite, but "potentially" infinite, so that they can be indefinitely extended. The following proposition shows that the model obtained by extending the mechanical linkage theory of jointed finite rods without carts (as seen in the subsection 3.4.3, pag. 61) with wheels can generate any limited curve traced by a differential machine, and these components are minimal (I cannot eliminate any other component while still tracing such curves). Even if the model of differential machines with carts is not minimal, this model was chosen to simplify constructions and to have a more direct correlation with the analytical counterpart (without the need of restriction to finite parts).

Proposition 1. *Any limited curve traced by a differential machine can be traced using finite rods, joints and wheels (without carts), and they make up a minimal set of components.*

35. This part is taken from [Milici, 2012a, pp. 224–225].

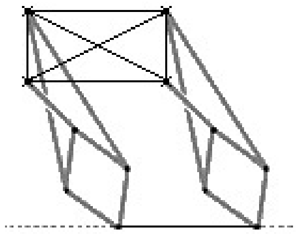


Figure 4.11: How to assembly two Peaucellier's devices in order to constrain a rod to move in a straight direction.

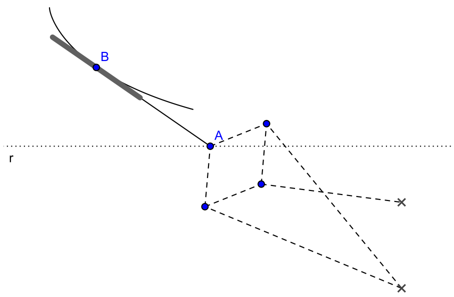


Figure 4.12: We can easily construct a machine tracing a piece of tractrix without carts: Consider the point A moving on a straight line r (thanks to a Peaucellier's inverter), consider the finite rod AB and put on B a wheel. B will trace a piece of tractrix.

Proof. Assuming the use of "potentially" infinite rods, I can construct any limited curve only by using finite rods. With finite rods, carts are superfluous because:

1. I can constrain a point to move on a straight line perpendicular to a given rod (that can be done using Peaucellier's inverter, as apparent in Fig. 3.9, pag. 62);

2. I can constrain a rod to move in a straight direction [36] (that can be done using two Peacellier's inverters nailed to parallel rods, as can be seen in Fig. 4.11).

With regard to the minimality of the components, I cannot avoid finite rods and joints. With regard to wheels, in Fig. 4.12 we can observe the construction of a machine for a finite piece of tractrix, which is not algebraic. It implies that with wheels I obtain some curves not previously traceable (Kempe's linkages construct only pieces of algebraic curves). □

36. This property is necessary to keep the possibility of putting a wheel on a rod passing through a point (passage granted by the use of a cart).

4.3 Analytical tools

The main strength of Cartesian geometry is the mathematical merging of algebra and geometry using suitable machines. In particular, polynomial algebra is used as finite tool for analysis, while geometric constructions compose the synthetic part. This merging allows the possibility of characterizing curves with algebraic tools.

In the proposed differential extension, I want to substitute polynomials with differential polynomials, machines for algebraic constructions (algebraic machines) with differential machines, and algebraic curves with manifolds of zeros of differential polynomials. In this section, I want to delve deeper into the analytical counterpart of differential machines, the "differential algebra," specifically "differential elimination." The peculiarity of this approach is that it is algorithmically implementable (it is part of computer algebra): Its finite symbolic manipulation does not need any reference to the concept of infinitary objects (as it happens in infinitesimal calculus). With these algebraic tools, I will be able to answer some questions about differential machines in section 4.4.

4.3.1 Brief history of differential algebra

As differential machines are an extension of algebraic ones, even differential algebra is an extension of polynomial algebra:

> "[i]t is common knowledge that algebra, including algebraic geometry, historically grew out of the study of algebraic equations with numerical coefficients. In much the same way, differential algebra sprang from the classical study of algebraic differential equations with coefficients that are meromorphic functions in a region of some complex space $\mathbb{C}^m$. As a consequence, differential algebra bears a considerable resemblance to the elementary parts of algebraic geometry. Indeed, since an algebraic equation can be considered a differential equation in which the derivatives do not occur, it is possible to consider algebraic geometry as a special case of differential geometry." [37]

Even if the pioneering work Janet [1929] gave a clear link between the theory of partial differential equations and the one of algebraic ideals, the first to introduce "differential algebra" as a new field of mathematics was Joseph Fels Ritt [1893–1951] [38]:

> "[a] complex analyst, who competed strenuously with Julia and Fatou for the prize offered by the French Academy for work on the iteration of rational functions [. . .], Ritt had a life-long interest in the properties of complex functions. He immersed himself in the literature of the 18th and 19th centuries that was concerned with the "transcendents" defined by algebraic differential equations with rational function coefficients. This concentration on what would be called "rationality questions" in algebraic geometry, which drew him to the work of Lagrange, Laplace, Liouville, Picard, Painlevé, and

37. Cf. [Kolchin, 1973, p. xi].
38. For a short biography see Smith [1956].

> Drach, led Ritt to the "new algebra" of Noether and van der Waerden. He patterned his approach to differential equations theory, which Kolchin named "differential algebra," on algebraic geometry, eschewing the transcendental methods of Lie."[39]

The history of differential algebra[40] can be said to have started with Ritt [1932], where Ritt introduced suitable algebraic tools for differential equations. These results have been later reformulated in Ritt [1950] with a great effort to meet the algebraist half ways, but for a completely algebraic approach we have to wait for Kolchin [1973]:

> "[a]lthough Ritt [...] devoted the greater part of his life to algebraicizing differential equations theory, his life blood was classical analysis. [...] A beginning of the complete algebraization of differential algebra was made by Ritt's students Raudenbush and Levi. However, it was his student Ellis Kolchin, deeply influenced by the "modern exploratory spirit" of Weil and Chevalley, who completed the task. Without deviating from the central philosophy of his teacher, whom he called differential algebra's "principal prophet and practitioner"[41], Kolchin deepened and modernized differential algebra, and developed differential algebraic geometry and differential algebraic groups."[42]

In particular, Kolchin opened the possibility of using differential algebraic geometry in diophantine geometry[43] and differential Galois theory[44].

Under both Ritt and Kolchin, basic differential algebra was developed from a constructive view point and the foundation they built has been advanced and extended to become applicable in symbolic computation, mainly thanks to the passage from old constructive methods (Ritt-Seidenberg algorithm of Seidenberg [1956]) to more recent computational complexity optimizations with Gröbner bases-like approach (firstly introduced in Carrà Ferro [1989])[45]. That brought, for example, to the development of a relative package in the commercial computer algebra system *Maple*[46].

39. Cf. [Buium and Cassidy, 1999, p. 568].

40. Some authors (as in [Grabmeier et al., 2003, p. 104]) refer to differential algebra as "differential ideal theory."

41. Cf. Kolchin [1973].

42. Cf. [Buium and Cassidy, 1999, p. 569].

43. Diophantine geometry is one approach to the theory of Diophantine equations, formulating questions about such equations in terms of algebraic geometry over a ground field K that is not algebraically closed (see for example Lang [1997]). For the application of differential algebraic groups to Diophantine questions over functions fields, see Buium [1992].

44. Whereas algebraic Galois theory studies extensions of algebraic fields, differential Galois theory studies extensions of differential fields, i.e. fields that are equipped with a derivation. Much of the theory of differential Galois theory is parallel to algebraic Galois theory. The problem of finding which integrals of elementary functions can be expressed with other elementary functions is analogous to the problem of solutions of polynomial equations by radicals in algebraic Galois theory, and is solved by Picard-Vessiot theory. For the topic, see, for example Magid [1994]. For Kolchin's contribution to this field, see Borel [1999], Singer [1999].

45. For a brief but complete introduction to these computational problems and the relative historical evolution see [Boulier, 2007, pp. 110–111].

46. The description of the package *DifferentialAlgebra* is available on-line at the address `http://www.maplesoft.com/support/help/maple/view.aspx?path=DifferentialAlgebra`. Regarding the methods adopted in the implementation, see Boulier et al. [1995], Boulier et al. [2009] and Hubert [1999].

Currently there is much interest in differential algebra also for practical reasons. In fact, there is a growing effort to use it in order to solve problems in control theory, dynamical systems and robotics [47]. Another important application of differential algebra is the "mechanic theorem proving" in differential geometries [48].

4.3.2 Differential algebra

The aim of differential algebra is to furnish an algebraic theory for differential equations both ordinary or with partial derivatives. In particular, its tools and notations are an extension of commutative algebra. To give a short introduction to differential algebra, I will recall [Boulier, 2007, pp. 112–116] because of the clarity, the brevity, and the adherence with my aims [49]. To begin I have to give some definitions beyond algebraic tools.

A *differential ring* (respectively *field*) is a ring (resp. field) R endowed with a derivation. As *derivation* I consider any unitary mapping $D : R \to R$ so that derivation must be distributive over addition [50] and must obey the product rule (also called "Leibniz rule"):

$$D(ab) = D(a)b + aD(b).$$

As I have already said, I am interested only in the case of a single derivation, but the theory is more general [51].

Note that any standard ring (resp. field) is a differential one with the trivial derivative $D(a) = 0$ (for every $a \in R$), and in this case all the elements of the ring (resp. field) can be considered as constants. For my purposes, it will be enough to consider the (differential) field of rational numbers. A non-trivial example of differential field is the field of the meromorphic functions $f(z)$ on a given region of the complex plane.

Similarly to classic algebraic geometry, we can consider the *differential polynomial ring* $K\{U\}$ where K is the differential field of coefficients and U is a finite set of *differential indeterminates*. The elements of $K\{U\}$, the *differential polynomials*, are polynomials in the usual sense built over the infinite set, denoted ΘU, of all the derivatives of the differential indeterminates. According to my aims, differential indeterminates can be considered simply as functions depending on the single independent variable t, which we may think as the time. Thus, I will also refer to differential indeterminates as *dependent variables*.

For example, $\mathbb{Q}\{x_1, x_2\}$ is composed by all the polynomials with rational coefficients in $x_1, x_1', x_1'', \ldots, x_2, x_2', x_2'', \ldots$, as $\frac{1}{3}x_2'^2 - 5x_1^3 x_1''^2 x_2 + \frac{2}{13}x_1''^2 x_1''' x_2 x_2'^4$.

47. See for example Mishra [2000] (for a general view) and Fliess and Glad [1993] (for applications to non-linear control theory).

48. See Wu [1991], Chou and Gao [1993] and Li [1995].

49. According to the kind of constraints obtained through my differential machines, I am only interested in ordinary differential equations. For an introduction to the partial derivatives case, see the tutorial article Hubert [2003], which I also used for some definitions and theorems.

50. I.e. for every $a, b \in R$ it holds $D(a + b) = D(a) + D(b)$.

51. In the general case, all the derivations have to satisfy distribution over addition and Leibniz rule.

In algebraic geometry it is well known that the set of polynomials which vanish over the solutions of a given polynomial system form an ideal and even a radical ideal [52]. Looking for something similar about differential equation, I can introduce the differential extension of these concepts.

In a differential ring R, an ideal I is a *differential ideal* if it is stable under derivation, which is $a' \in I$, for all $a \in I$. Besides, a differential ideal I is *radical* if $a^p \in I$ implies $a \in I$ for any integer $p > 0$.

The study of the radical of the differential ideal generated by a finite system of differential polynomials is intricately related to the study of the analytical solution of this system, as the following example illustrates.

Consider in $\mathbb{Q}\{x\}$ the differential polynomial $x'^2 - 4x$. The ordinary differential equation obtained posing this differential polynomial equal to 0 has as analytical solutions the zero function $x(t) = 0$ and the family of parabolas $x(t) = (t + c)^2$ where c is an arbitrary constant. These solutions are also solutions of all the derivatives of the differential equation:

$$2x'(x'' - 2) = 0, \qquad 2x'x'' + 2x''(x'' - 2) = 0, \qquad \ldots$$

More generally, they are solution of every differential polynomial, a power of which is a finite linear combination of the derivatives of $x'^2 - 4x$ with arbitrary differential polynomials as coefficients, i.e. every element of the radical of the differential ideal generated by $x'^2 - 4x$. With other words, the set of all the "differential and algebraic consequences" of the differential polynomials in a system Σ is the radical differential ideal generated by Σ, which I denote by $\sqrt{[\Sigma]}$. In general, given a differential system Σ (i.e. a system of differential polynomials), instead of studying directly the solutions of $\Sigma = 0$ I will go on to inspect the radical differential ideal generated by Σ, i.e. the intersection of all the radical differential ideals containing Σ.

The analogue of the Hilbert basis theorem for polynomial rings is given by the basis or Ritt-Raudenbush theorem [53]:

Theorem 1 (Ritt-Raudenbush)**.** *If J is any radical differential ideal in $K\{U\}$ there exists a finite subset Σ of $K\{U\}$ so that $J = \sqrt{[\Sigma]}$.*

We have to note that the result holds for radical differential ideals, in general it does not hold for differential ideals.

Furthermore, I remember that an ideal I is *prime* if whenever a product ab belongs to I at least one of the factors, a or b, belongs to I. In particular, a prime differential ideal is a prime ideal, which is also a differential ideal. Thus, the following theorem holds:

Theorem 2. *Any radical differential ideal J in $K\{U\}$ is the intersection of a finite number of prime differential ideals.*

52. An ideal I is a subset of a ring R that forms an additive group and has the property that, $\forall x \in R, \forall y \in I$, the product $xy \in I$. An ideal I is said to be *radical* if $a \in I$ whenever there exists some non-negative integer p so that $a^p \in I$. The radical of an ideal I is the set of all the ring elements whose power belongs to I.

53. The proof for ordinary differential rings is given in Ritt [1950], Kaplansky [1957]. For the general proof with partial derivatives, see Kolchin [1973].

Therefore, there exists a decomposition of J. Decomposition is minimal if none of the components contains another one. Having said that, the decomposition of J in prime differential ideals can be proved to be unique if it is minimal, and these components are called the *essential prime components* of J.

However, I am not interested only in existence proofs. I am looking for algorithms to be applied to the analysis of differential systems (and to differential machines). So, once associated a differential system Σ to a differential machine, my goal is to give an adequate representation of its radical differential ideal $\sqrt{[\Sigma]}$

4.3.3 Differential elimination

Differential elimination is an important application of differential algebra, and its methods are considered as computer algebra. It is a process which takes as input a system of differential equations (ordinary or with partial derivatives) and a ranking. Then it rewrites the input system into an equivalent system (or an equivalent family of systems when case splitting is necessary). The ranking permits to control the elimination process, indicating what should be eliminated. To present the main algorithm of differential elimination, called *Rosenfeld-Gröbner*, I previously have to define the concept of *ranking* and Ritt's reduction. Even if in practice the worst case complexity of the algorithms makes problems untreatable, in principle differential elimination is always possible [54].

Differential ranking

If U is a finite set of dependent variables, a *ranking* over U is a total ordering over the set ΘU of all the derivatives of the elements of U which satisfies, for all $a, b \in \Theta U$:

$$a' > a \qquad \text{and} \qquad a > b \Rightarrow a' > b'.$$

When $U = \{a\}$ (there is a unique dependent variable), there exists only one ranking: $\cdots > a'' > a' > a$. The choice of the ranking is non-trivial when I have more dependent variables. For my purposes, I will only introduce the most commonly used ones [55].

A ranking is said to be *orderly* if, for every $a, b \in U$ and for every positive integer value of i and j, $i > j \Rightarrow a^{(i)} > b^{(j)}$. This means that given $U = \{a, b\}$, the two possible orderly rankings will be

$$\cdots > b'' > a'' > b' > a' > b > a \qquad \text{and} \qquad \cdots > a'' > b'' > a' > b' > a > b.$$

If U and V are two finite sets of differential variables, one denotes $U \gg V$ every ranking so that any derivative of any element of U is greater than any derivative of any element of V. Such rankings are said to *eliminate* U with reference to V. Considering $U = \{a\}$ and $V = \{b\}$, the order eliminating a will be

$$\cdots > a'' > a' > a > \cdots > b'' > b' > b.$$

54. Also this subsection, as the previous one, is based on [Boulier, 2007, pp. 112–116].
55. Study and classification of general rankings are examined in Carrà Ferro and Sit [1993], Rust and Reid [1997].

Let f be a differential polynomial in $K\{U\}$ that is not in K (i.e. f has to really depend on some differential indeterminates or their derivatives). Given a ranking, the *leader* is the highest ranking derivative appearing in f. Thus, given the differential polynomial $\frac{1}{3}x_2'^2 - 5x_1^3x_1''^2x_2 + \frac{2}{13}x_1''^2x_1'''x_2x_2'^4$, with any orderly ranking the leader will be x_1''' (there are no x_2 with derivative more than 1). We have the same leader with the ranking eliminating x_1. On the contrary, with the ranking eliminating x_2 the leader will be x_2'.

Ritt's reduction algorithm

Once introduced the ranking, I can describe the "Ritt's reduction algorithm." In brief, it is a generalization of the Euclidean division. It is well known that if f and g are two polynomials (not differential ones) in one variable v with coefficients in a field, the Euclidean division of f by g is possible for every non-zero g. It yields two unique polynomials q and r so that $f = gq + r$ and $\deg r < \deg g$. If f and g have coefficients in a ring, the Euclidean division is no more possible in general for the leading coefficient of g may not be invertible. The closest available algorithm is the *pseudodivision* which consists in multiplying f by the leading coefficient c of g, raised at the power $p = \deg f - \deg g + 1$ before performing the Euclidean division. It yields a unique couple of polynomials q and r so that $c^p f = gq + r$ and $\deg r < \deg g$. The polynomial r is called the *pseudoremainder* of f by g and is denoted $prem(f, g)$ or $prem(f, g, v)$ when the variable v is not clear from the context (case of polynomials depending on many different variables). The pseudodivision generalizes to the differential setting, providing Ritt's reduction algorithm (note that I am only interested in the remainder).

Let f be a differential polynomial, to be reduced by a finite set $\Sigma = \{g_1, \dots, g_n\}$ of differential polynomials. Denote v_i the leader of g_i for $1 \leq i \leq n$ (assuming that none of the g_i lies in the base field). Ritt's reduction builds a sequence $f_0, \dots, f_r$ of differential polynomials starting at $f_0 = f$. The result is the polynomial

$$f_r = Ritt\_reduction(f, \Sigma).$$

To compute f_{l+1} from f_l I have to distinguish between three cases:

1. if, for each $1 \leq i \leq n$, the differential polynomial f_l does not depend on any proper derivative $v_i^{(k)}$ $(k \geq 1)$ of v_i and $\deg(f_l, v_i) < \deg(g_i, v_i)$ then the computation stops and $f_l = f_r$ is returned[56];
2. if there exists some index $1 \leq i \leq n$ such that $\deg(f_l, v_i) \geq \deg(g_i, v_i)$ then $f_{l+1} = prem(f_l, g_i, v_i)$;
3. if there exists some index $1 \leq i \leq n$ such that f_l depends on some proper derivative $v_i^{(k)}$ (with $k \geq 1$) of v_i then $f_{l+1} = prem(f_l, g_i^{(k)}, v_i^{(k)})$.

The sequence $f_0, \dots, f_r$ described above is not uniquely defined. One could define a precise algorithm by specifying that the sequence of the reduced derivatives $v_i^{(k)}$ must be decreasing. This is the usual strategy but any other strategy

56. The notation "$\deg(p, v)$" represents the highest power of the variable v appearing in the polynomial p. In case of differential polynomials the variable v has to be considered as a derivative (not necessary proper).

could be applied. Lastly, observe that whenever $k \geq 1$, the differential polynomial $g_i^{(k)}$ has degree one in $v_i^{(k)}$ and admits the *separant* $s_i = \frac{\partial g_i}{\partial v_i}$ for leading coefficient. In this case, writing $g_i^{(k)} = s_i v_i^{(k)} + t_{i,k}$, one sees that the pseudodivision of f_l by $g_i^{(k)}$ amounts to the following: first perform the following substitution in f_l

$$v_i^{(k)} \longrightarrow -\frac{t_{i,k}}{s_i}$$

then clear the denominator of the obtained rational fraction. The resulting polynomial is free of $v_i^{(k)}$.

An example will clarify the passages. Let us apply Ritt's reduction to $f_0 = u'' - vu'$ and $\Sigma = \{u'^2 + v\}$ with the ranking $u \gg v$ (eliminating u). The leader of $g = u'^2 + v$ is u': I am in the third case (not in the second because the leader u' in g has degree greater than u' in f), in fact, the polynomial f_0 depends on the first derivative of u'. Thus, I have to compute $f_1 = prem(f_0, g', u'')$. Deriving, I obtain $g' = 2u'u'' + v'$. Therefore, to obtain the pseudoreduction, I first have to substitute $u'' \longrightarrow -v'/(2u')$ over f_0, giving the rational fraction

$$-\frac{v'}{2u'} - vu'.$$

Second, I have to clear the denominator, obtaining $f_1 = -v' - 2vu'^2$. This polynomial f_1 is not pseudoreduced with respect to g. Now I am in the second case (the leader of g, u', is present power two both in g and f_1). Hence, $f_2 = prem(f_1, g, u')$: one has to substitute $u'^2 \longrightarrow -v$ over f_1, giving the differential polynomial f_2 (there is no denominator to clear)

$$f_2 = -v' + 2v^2.$$

Ritt's reduction stops at this step and $f_r = f_2$ is returned.

I have to observe that in general, the set of all the differential polynomials that are reduced to zero by Ritt's reduction has no clear structure. It does not even need to be an ideal. Observe also that the returned polynomial f_r is not equivalent to f modulo the differential ideal generated by Σ because of the denominator clearing step. In fact, with reference to the previous example, $f_1 = -v' - 2vu'^2$ is not equal to $f_1^* = -\frac{v'}{2u'} - vu'$ because, considering the denominator, $f_1^* = 0$ also requires $u' \neq 0$. A more careful version was designed in Boulier and Lemaire [2000] that returns a rational fraction instead of a polynomial.

Rosenfeld-Gröbner algorithm

Rosenfeld-Gröbner algorithm is useful to decide membership in a radical differential ideal [57]. It gathers as input a finite system Σ of differential polynomials and a ranking. It returns a finite family (possibly empty) $\Xi_1, \ldots, \Xi_r$ of finite

57. In polynomial algebra the test for ideal membership is achieved by Gröbner reduction, as evident in Buchberger [1985]. On the other side, Rosenfeld's lemma (appeared in Rosenfeld [1959]) was a link between differential algebra and polynomial algebra, and therefore the key to effective algorithms in differential algebra. Rosenfeld-Gröbner algorithm, introduced in Boulier [1994] and Boulier et al. [1995], computationally combines Rosenfeld's lemma and Gröbner bases.

subsets of $K\{U\} \setminus K$ (i.e. of polynomials really depending on some indeterminate or relative derivatives).[58] Each system Ξ_i defines a differential ideal C_i in the sense that, for any $f \in K\{U\}$, we have

$$f \in C_i \qquad \text{iff} \qquad \textit{Ritt\_reduction}(f, \Xi_i) = 0.$$

The relation with the radical I of the differential ideal generated by Σ (i.e. $I = \sqrt{[\Sigma]}$) is the following:

$$I = C_1 \cap \cdots \cap C_r.$$

When $r = 0$ we have $I = K\{U\}$. Combining both relations, one gets an algorithm to decide membership in I. Indeed, given any $f \in K\{U\}$ we have:

$$f \in I \qquad \text{iff} \qquad \textit{Ritt\_reduction}(f, \Xi_i) = 0, \qquad 1 \leq i \leq r.$$

The systems Ξ_i are often called *characteristic sets* or *differential regular chains*[59]. The differential ideals C_i do not need to be prime. They are however necessarily radical, thanks to Lazard's lemma. Observe that it is possible to refine further the intersection in order to get prime differential ideals. It is sufficient for this to apply a usual primary decomposition algorithm[60]. However, no algorithm is known to decide inclusion between differential ideals presented by characteristic sets, even when they are prime, thus the computed representation can by no means be guaranteed to be minimal though this latter theoretically exists. Inclusion problem is usually called "Ritt's problem"[61]:

> "A great unsolved problem is that of testing inclusion. Given two prime differential ideals I_1, I_2 defined by characteristic sets, can we decide whether $I_1 \subset I_2$? This is equivalent to finding and effective version of the Ritt-Raudenbush theorem, i.e. knowing a characteristic set of a prime differential ideal I to find a finite set Σ such that $I = \sqrt{[\Sigma]}$. See Péladan-Germa [1995], Hubert [1996], Hubert [1999] for more details on the subject. This problem is related to that of testing equalities in differential rings defined by differential algebraic systems and *initial conditions*."[62]

58. Considering Ritt's reduction returning rational fractions instead of polynomials, any Ξ_i will be composed of two subset of polynomials, so $\Xi_i = (A, H)$. I am interested in the differential ideal of all the consequences of the differential polynomial equations $A = 0$ (numerators of the fractions) and inequations $H \neq 0$ (denominators). To consider this ideal, inequations can be considered as polynomials that are invertible in the ideal. Indeed, if h is an inequation and some polynomial hq lies in the ideal then q lies in the ideal. The ideal theoretic corresponding operation is the *saturation*.

More precisely, let H be a subset of a differential ring R. I denote by H^∞ the minimal subset of R that contains 1 and H and is stable by multiplication and division i.e. $a, b \in H^\infty \iff ab \in H^\infty$. For a differential ideal I I define the saturation of I by a subset H of R as $I : H^\infty = \{q \in R | \exists h \in H^\infty \text{ s.t. } hq \in I\}$. $I : H^\infty$ is a differential ideal.

Thus, considering $[A]$ the differential ideal generated by A, the smallest ideal satisfying $A = 0, H \neq 0$ is the saturation of $[A]$ by H, i.e. $[A] : H^\infty$.

59. To be precise, regular differential chains, introduced in Lemaire [2002], slightly generalize Ritt's characteristic sets. An equivalent notion was introduced in Hubert [2000].

60. See for example Decker et al. [1999].

61. For a list of equivalent formulations of Ritt's problem, see [Golubitsky et al., 2009, section 3.1, pp. 517–519].

62. Cf. [Grabmeier et al., 2003, p. 105]: The part about differential algebra (in the handbook called "Differential Ideal Theory"), including the quoted paragraph, was written by F. Ollivier.

As an example, consider in $U = \{x\}$ (so a unique ranking is possible) $\Sigma = \{x'^2 - 4x\}$ [63] and denote I the radical differential ideal generated by Σ. If one applies the *Rosenfeld-Gröbner* algorithm to Σ, one gets an intersection $I = C_1 \cap C_2$ with C_1, C_2 generated respectively by

$$\Xi_1 = \{x'' - 2\} \qquad \text{and} \qquad \Xi_2 = \{x\}.$$

The differential polynomial x is reduced to zero by Ξ_2, not by Ξ_1. Hence, $x \notin I$. The differential polynomial $x'' - 2$ is reduced to zero by Ξ_1, not by Ξ_2. Thus, $x'' - 2 \notin I$. The product $x'(x'' - 2)$ is reduced to zero by Ξ_1 and Ξ_2. Thus, it lies in I (it is $D(x'^2 - 4x)/2$). This proves that the ideal I is not prime. The ideal C_1 corresponds to the family of parabolas $x(t) = (t + c)^2$. The ideal C_2 corresponds to the solution $x(t) = 0$.

4.3.4 Solved and unsolved problems

Differential algebra is an extension of polynomial algebra aimed at the analysis of systems of ordinary or partial differential equations that are polynomially nonlinear. To a differential equation I can associate a differential polynomial [64] and to a differential system I associate a radical differential ideal. Questions about the solution set of the differential system are best expressed in terms of that radical differential ideal. There are differential analogues to the Nullstellensatz, the Hilbert basis theorem, and the decomposition into prime ideals. The latter point gives light to the old problem of singular solution [65] of a single differential equation. The radical differential ideal of a single differential polynomial may split into several prime differential ideals. One of those describes the general solution and the others the singular solutions.

For a system of differential equations $\Sigma = 0$ and an appropriate choice of ranking, it is possible to solve with symbolic algorithms the following typical questions:

- Is a differential equation (not apparent in $\Sigma = 0$) satisfied by all the solutions of the system $\Sigma = 0$?
- What are the differential equations satisfied by the solutions of $\Sigma = 0$ in a subset of the dependent variables? If Σ is a differential system in the unknown functions $x_1, \ldots, x_n$, one might be interested in knowing the equations governing the behavior of the component x_1 independently of the others.

63. I have introduced this differential polynomial in the subsection 4.3.2.

64. In a differential equation, there may appear elementary functions like $\sin(t), e^t, \log(t)$. However, these functions are unique solutions to a specific initial value problem. Hence, it is possible to replace any such function with a new indeterminate function defined using differential polynomial equations (for example see [Pritchard and Sit, 2007, pp. 291–292]).

65. A singular solution is a solution of a differential equation that cannot be obtained from the general solution gotten by the usual method of solving the differential equation. When a differential equation is solved, a general solution consisting of a family of curves is obtained. Considering the already introduced example $x'^2 - 4x = 0$, it has the general solution $x = (t + c)^2$, which is a family of parabolas. The line $x(t) = 0$ is also a solution of the differential equation, but it is not a member of the family constituting the general solution. From another point of view, a singular solution of an ordinary differential equation is a solution for which the initial value problem (also called the "Cauchy problem") fails to have a unique solution at some point on the solution. Usually, singular solutions appear in differential equations when the usual methods of solving divide in a term that might be equal to zero.

- What are the lower order differential equations satisfied by the solutions of $\Sigma = 0$? In particular, one might inquire if the solutions of the system are constrained by purely algebraic equations i.e. differential equations of order zero.

All those questions require, a way or another, a membership test to the radical differential ideal generated by the set of differential polynomials Σ. For this purpose one can represent such radical differential ideals as the intersections of differential regular chains that can be obtained with the *Rosenfeld-Gröbner* algorithm [66]. In particular I will provide direct applications of differential algebra to differential machines in the following section.

Compared with the other languages for nonlinear differential equations [67],

> "[t]he language of differential algebra is better suited for expressing such properties (invariant properties of differential equations), and, puts at the disposal of the investigator the extensive apparatus of commutative algebra, differential algebra, and algebraic geometry. [...] The numerous "explicit formulas" for the solutions of the classical and newest differential equations have good interpretations in this language; the same may be said for conservation laws. However, the language of differential algebra which has been traditional since the work of Ritt does not contain the means for describing changes of the functions (dependent variables) and the variables t_i (independent variables), and for clarifying properties which are invariant under such changes." [68]

The development of "differential algebraic geometry" has begun since 1970s to overcome these limits of differential algebra. In particular it extended the classical language of Weil's algebraic geometry with the axiomatization of the notion of differential algebraic group [69]. However, with regard to initial value problems from a computational symbolic perspective, [70] a lot left to do. Even though Pritchard and Sit [2007] and the approach proposed by Markus Rosenkranz with regard to symbolic methods for (linear) boundary problems (e.g. Rosenkranz et al. [2012]), at my knowledge the symbolic solution of general initial value problems is far away from being solved.

4.4 Problem solving

With the introduction of differential machines, I overcame Cartesian geometry still relying on the idealization of suitable machines, and, thanks to differ-

66. The first part of this subsection was essentially taken from [Hubert, 2003, pp. 41–42].
67. Such as infinitesimal analysis or differential geometry.
68. Cf. Manin [1979].
69. See for example Cassidy [1972], Kolchin [1985] or, for a much modern approach with the theory of schemes, Kovacic [2002].
70. For example, regarding my machines, I am interested in the following problem: Given two differential machines with their relative initial configurations, are their behaviors equivalent? Analytically, the question arises: Given two systems of differential equations with the relative initial conditions, are the systems equivalent? I am looking for an algorithm to symbolically solve this problem. Differential algebra language does not permit even to express this problem because we need to explicitly state the relation between the dependent variables and the independent one (to pose the initial condition).

ential algebra, I also provided a well-defined language and set of algorithms for the analytical counterpart. In this section, I look at some applications of differential algebra for my machines. In particular, I will define external behavior and equality between machines.

Similar to the Cartesian geometric method, my steps will be the following:

1. start from a problem about differential machines,
2. convert it in differential equations,
3. solve the problem with differential algebra algorithms,
4. when requested, after the simplification, find the specific solution with diagrammatic construction of differential machines.

Regarding the third step, I will manipulate equations with the *DifferentialAlgebra* package of the computer algebra software *Maple*[71], of which I will include in notes the used commands.

4.4.1 The example of the cycloid

As a first example, I will prove in this subsection (only with mechanical reasoning) what is informally observed in the subsection 4.2.1 about the behavior of the machine of Fig. 4.5 (pag. 71).

Consider $Z = (x, 0)$, and $W = (x_w, y_w)$ moving around Z at unitary distance, so

$$(x_w - x)^2 + y_w^2 = 1. \tag{4.4}$$

Consider $N = (x, -1)$, M the middle point between Z and W, i.e. $M = \frac{Z+W}{2} = (\frac{x_w+x}{2}, \frac{y_w}{2})$, and its derivative $M' = (\frac{x'_w+x'}{2}, \frac{y'_w}{2})$. Considering the rod passing through M and perpendicular to MN, the wheel on it in M implies that M' has to be perpendicular to $M - N = (\frac{x_w-x}{2}, \frac{y_w+2}{2})$. So the scalar product $\langle M', M - N\rangle$ has to be null, i.e.

$$(x'_w + x')(x_w - x) + y'_w(y_w + 2) = 0. \tag{4.5}$$

Thus, I have two equations (the first purely algebraic and the second differential) in x, x_w, y_w. If I am interested in the curve traced by W, I can use differential elimination to eliminate the dependent variable x. I can proceed with the following steps:

1. consider the differential ring R having as dependent variables x, x_w, y_w, and adopt a ranking eliminating x;
2. consider the ideal I in R generated by my differential polynomials;
3. consider in I the *differential regular chains* reduced with respect to x.

71. I already mentioned it in note 46, pag. 81. The *DifferentialAlgebra* package is based on the software *BLAD* (standing for *Bibliothèques Lilloises d'Algèbre Différentielle*), developed in the *C* programming language by F. Boulier. The *BLAD* software is freely available online at `http://www.lifl.fr/~boulier/pmwiki/pmwiki.php?n=Main.BLAD`. Another free alternative is *ApCoCoA*, available at `www.apcocoa.org` (for my purposes, I have to cite the package *diffalg*), a software package based on *CoCoA*, `http://cocoa.dima.unige.it`.

I can translate these steps in commands for computer algebra software [72]. In particular I obtain that the *differential regular chains* (for the ideal generated by the two equations characterizing the differential machines) reduced with the ranking $x \gg x_w \gg y_w$ are:

$$C_1 = \{xy'_w + x'_w y_w - x'_w - x_w y'_w = 0, x'^2_w y_w - x'^2_w + y'^2_w y_w + y'^2_w = 0,$$
$$y'_w \neq 0, x'_w y_w - x'_w \neq 0, y_w - 1 \neq 0\};$$
$$C_2 = \{x_w^2 - 2x_w x + x^2 + y_w^2 - 1 = 0; x'_w = 0, y'_w = 0, x - x_w \neq 0\};$$
$$C_3 = \{x - x_w = 0, y_w^2 - 1 = 0, y_w \neq 0\}.$$

But, as said, I am not interested in the behavior of x, so, if I eliminate it [73], I obtain

$$C_1^* = \{x'^2_w y_w - x'^2_w + y'^2_w y_w + y'^2_w = 0, y'_w \neq 0, x'_w y_w - x'_w \neq 0, y_w - 1 \neq 0\} =$$
$$= \{x'^2_w (y_w - 1) + y'^2_w (y_w + 1) = 0, y'_w \neq 0, x'_w y_w - x'_w \neq 0, y_w - 1 \neq 0\};$$ [74]
$$C_2^* = \{x'_w = 0, y'_w = 0\};$$
$$C_3^* = \{y_w^2 - 1 = 0\}.$$

We can observe that C_2^* does not give us anything interesting: The case $x'_w = y'_w = 0$ only means that the point $W = (x_w, y_w)$ will not move.

On the contrary we can see how C_1^* contain as equation the general solution that, rewritten as an ODE, becomes

$$\left(\frac{dy_w}{dx_w}\right)^2 = \frac{1 - y_w}{1 + y_w}.$$

72. In Maple we can perform these operations with the following code lines (commented on the right):

`with(DifferentialAlgebra);`	load the package
`R := DifferentialRing(blocks= [x, x_w, y_w]), derivations= [t]);`	construct the differential ring with as independent variable t, and dependent ones x, x_w, y_w with the ranking $x \gg x_w \gg y_w$
`p := (x_w(t)-x(t))^2+y_w(t)^2 = 1;`	p is an algebraic equation
`q := ((D(x_w))(t) + (D(x))(t))*(x_w(t)-x(t)) + (D(y_w))(t)*(y_w(t)+2) = 0;`	q is a differential equation (`D(f)(t)` stands for the derivative df/dt)
`ideal := RosenfeldGroebner([p, q], R);`	`ideal` is the ideal generated by p and q
`Equations(ideal);`	returns the equations of `ideal`
`Inequations(ideal);`	returns the inequations of `ideal`

Note that the commands `Equations(ideal);` and `Inequations(ideal);` show the differential regular chains for the ideal in x, x_w, y_w.

73. Once obtained the differential regular chains reduced with respect to a certain ranking, the elimination of the greater depending variable only consists in taking all and only the equations and inequalities of the differential regular chains where the variable and its derivatives do not occur. Using Maple, it can be achieved with the command: `Equations(ideal, leader < x(t));`

74. Even though it is possible to do some simplifications (for example considering as inequations $y'_w \neq 0, x'_w \neq 0, y_w - 1 \neq 0$), I adopted the given form (that is exactly the one given by the Maple code) to evince the fact that any reasoning can be conducted in a purely formal way without considering the semantic meaning.

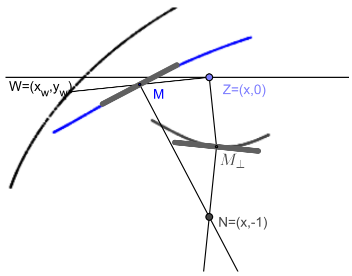


Figure 4.13: A machine for the cycloid. I constructed a new point $M_\perp$ through the rotation of a right angle M in relation to the point Z. Both in M and in $M_\perp$, I posed a wheel perpendicular to the rod passing through N.

Its solution is made up by arcs of cycloids and the lines $y_w = \pm 1$, lines that are the singular solutions described by C_3^*. However, none of the lines $y_w = \pm 1$ is a solution of C_1^*, because it includes the inequalities, which can be rewritten as

$$x'_w \neq 0, y'_w \neq 0, y_w \neq 1.$$

Therefore, the line $y_w - 1 = 0$ is explicitly avoided, but also $y_w + 1 = 0$ because any solution of C_1^* has to satisfy $y'_w \neq 0$.

I can also ask myself how it is possible to construct a cycloid exactly as external behavior. In particular, I will try to introduce more constraints. According to the property seen in Fig. 4.6 (pag. 71), if I consider the cycloid as traced by a circle rotating without slipping, the tangent at every point P on the circle has to be perpendicular to the PC (being C the contact point, as the cited figure shows). This property was used to construct the differential machine considered in this subsection. Nevertheless, I can slightly modify this machine posing new tangent conditions. For example, as evident in Fig. 4.13, I can put the tangent condition in the point $M_\perp$ obtained through the rotation of an anticlockwise right angle M in relation to Z[75]. The point $M_\perp$ has coordinate $(x,0) + \left(-\frac{y_w - y}{2}, \frac{x_w - x}{2}\right) = \left(\frac{2x - y_w}{2}, \frac{x_w - x}{2}\right)$, hence $M'_\perp = \left(\frac{2x' - y'_w}{2}, \frac{x'_w - x'}{2}\right)$. The new wheel condition means that $\langle M'_\perp, M_\perp - N\rangle = 0$. Thus, given that $M_\perp - N = \left(-\frac{y_w}{2}, \frac{x_w - x + 2}{2}\right)$, I obtain

$$-y_w(2x' - y'_w) + (x_w - x + 2)(x'_w - x') = 0. \tag{4.6}$$

If I consider the ideal generated by the three polynomial equations (4.4), (4.5) and (4.6), I can compute the relative differential regular chains eliminating x. I obtain that x_w and y_w have to satisfy the differential systems $C_1^{**}, C_2^{**}, C_3^{**}$: I find that $C_1^{**} = C_1^*, C_2^{**} = C_2^*$, but I obtain a new condition $x'_w = 0$ in C_3^{**}:

$$C_3^{**} = \{x'_w = 0, y_w^2 - 1 = 0\}.$$

This means that the old singular solutions $y_w = \pm 1$ are no longer available. In fact, having as new condition $x'_w = 0$, I get that C_3^{**} is satisfied on the plane only by the constant solutions $(k, 1)$ or $(k, -1)$ (for a real value of $k \in \mathbb{R}$), and no longer by the whole line[76].

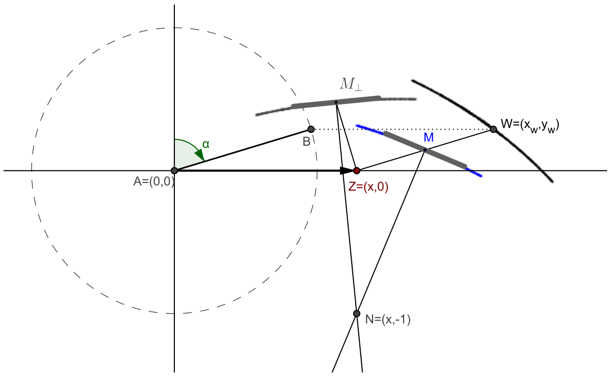


Figure 4.14: The machine for the cycloid used for the rectification of general angles. The angle α (determined between the vertical line and the segment AB) in radiant has the same value of the abscissa of Z.

As a curiosity, we can note that slightly modifying the machine of Fig. 4.13, we can assemble a machine for the rectification of general angles. As seen in Fig. 4.14, starting from the machine for the cycloid, I considered a point A of coordinate $(0, 0)$ in such a way that when Z is superimposed on A, W has coordinate $(0, 1)$. Considering B (joined by a rod of unitary length with the other edge joined in A) in such a manner that the vector $B - A$ is constrained to be equal to $W - Z$, I can introduce the angle α determined between the vertical line and the segment AB. In radiant the angle α has the same value of the abscissa of Z, i.e. the vector $Z - A$ is the rectification of α.

4.4.2 External behaviors and constructible functions

The procedure of the previous subsection is generalizable about any machine. Thus, given a differential machine $\mathcal{M}$ with manifest and latent variables, as seen in the subsection 4.2.2, the full behavior can be described by a system Σ of differential polynomial equations in both the manifest and latent variables. Then, considering any ranking eliminating latent variables, I get a representation of the external behavior given by a family of differential systems (each one given by equations and inequations of the regular chains rewritten using the given ranking, and taking only the ones where it does not appear any latent variable).

So the problem of characterizing the external behavior can be considered solved adopting the very basic tools of differential algebra. Indeed, given a radical ideal defined as the intersection of a finite number of differential systems (with equations and inequations), I can construct a machine solving exactly the wanted equations and inequations [77].

I can also define the nature of the functions that these machines characterize. The variables in differential algebra are functions. Now I want to give a

75. This machine was introduced in Milici [2012a].

76. Observing the machine in Fig. 4.13, when $y_w = \pm 1$ I have that $M_\perp$ is in position $(x \mp \frac{1}{2}, 0)$, and the tangent constraint in it is not satisfied if W moves along the horizontal line (in this hypothetical case the tangent in $M_\perp$ would have been horizontal).

77. To impose an inequation $p(X) \neq 0$, I can add a new variable y (i.e. a cart) so that $y \cdot p(X) - 1 = 0$.

classification of such constructible functions. Considering as manifest variables the coordinates (x, y) of a point of the machine, I find that the curves traced by that point are defined as all the value of x, y satisfying the external behavior, i.e. many systems of differential polynomial equations and inequations in x, y. Now I can be interested in interpreting a curve as the graph of a function (at least locally). So, I can consider the curve as a function $y = f(x)$ [78]. To achieve this aim, I can no longer consider x as a dependent variable, but as an independent one. Algebraically, this is translated (as seen in the subsection 4.2.4) by the "independentization condition" $x' = 1$. Indeed, if I add the new condition to the systems in x, y, I can again consider the elimination of x obtaining a family of differential regular chains only in y. Thus, I find that the curves $(t, y(t))$ are satisfied when y is (locally) solutions of differential polynomials in y: These functions are called "differentially algebraic" (shortly: D.A.) [79]. Conversely, every D.A. function is trivially constructible with my differential machines (being a differential polynomial in only one variable), so the constructible functions are all and only the differentially algebraic ones. That is important because it means that differential machines generate a new dualism beyond algebraic/transcendental (and this time about functions, not curves or varieties as done with algebraic machines). Note, however, that a machine can construct functions that are not D.A. globally, [80] but locally each of these functions has to be D.A.

All the elementary functions are D.A., and even most of the transcendental functions that we find in analysis handbooks. Historically, the first example of non-D.A. function was the Γ of Euler, as proven in Hölder [1886]. The history and development of the D.A. functions, together with the connection with analog computing and some correlations with the Cartesian dualism, will be explained in section 7.1. Note that Γ function is not D.A. not even locally, which is why it cannot be constructed with my tools.

As an example, I can continue with the cycloid. I will see some differences when I "independentize" different variables.

Adding the constraint $x'_w = 1$ to (4.4), (4.5) and (4.6), I consider y_w in function of x_w. This time, with a ranking eliminating x and x_w, I obtain only one regular chain:

$$C_{\{x'_w=1\}} = \{y'^2_w y_w + y'^2_w + y_w - 1 = 0; y'_w y_w + y'_w \neq 0; y_w + 1 \neq 0\}.$$

78. This parametrization is not possible at a point where the curve assumes a vertical tangent. In such an interval I can consider $x = g(y)$, so locally considering y as the independent variable. However, I will no longer consider this case because I only have to switch the role of x and y in the following reasoning.

79. A function y is differentially algebraic if it satisfies an algebraic differential equation (ADE), i.e. a differential equation in the form $P(t, y, y', \ldots, y^{(n)})$ where P is a nontrivial polynomial in $n+2$ variables (cf. Rubel [1989]). The nontriviality condition is essential because every function is solution of $0 = 0$. For example, if I consider the differential ideal generated only by (4.5) (without the other equations), I find that even with the ranking eliminating x, there is no equation in the differential regular chains where the dependent variable x or its derivatives do not appear. Hence, elimination does not produce any polynomial depending only on x_w, y_w. This happens because there are not enough conditions to eliminate x without arriving to a trivial polynomial.

80. This property has been visible since the first introduction of differential machines. Though called "tractional motion machines," their first appearance in Milici [2012a] concerned the construction of a machine tracing a curve that globally was not D.A. (the cycloid, considered as the graph of a function $y = f(x)$).

This representation is not useful to identify the traced curve as the usual parametrization of a cycloid. This identification is more visible if I "independentize" another variable. Consider the additional constraint $x' = 1$ instead of $x'_w = 1$. Even in this case, I obtain only one regular chain that, upon eliminating x, becomes:

$$C_{\{x'=1\}} = \{x'_w - y_w - 1 = 0; y'^2_w + y^2_w - 1 = 0; y'_w \neq 0\}$$

Now I can observe that this representation is the one of

$$\begin{cases} x_w = t + \cos t \\ y_w = -\sin t \end{cases}$$

Indeed, instead of the trigonometric functions I can convert the system in a purely differential polynomial one:

$$\begin{cases} x'_w = 1 + y_w \\ y'^2_w + y^2_w = 1 \\ y''_w = -y_w \end{cases}$$

Computationally, we can check that it has as regular chains exactly $C_{\{x'=1\}}$ [81]. Thus, I find that my machine exactly describes the cycloid.

Obviously I can have different machines constructing the same manifold of zeros. Remaining on the example of the cycloid, having a system of differential polynomials, I can construct a differential machine having a point of coordinate (x_w, y_w) satisfying a certain system with the standard method seen in the subsection 4.2.3. This way, I consider the variables x_w and y_w separately [82], impose all the conditions, and then I construct the point having as coordinate (x_w, y_w). This method is general, but of course, does not furnish the simplest machine [83].

4.4.3 Equivalence between differential machines

In the previous example, I showed that two radical ideals were equivalent because they had the same representation. However, the opposite in general does not hold.

Consider two differential machines $\mathcal{M}, \mathcal{N}$. As seen in the subsection 4.2.2, I can consider the full behavior of these machines as the solutions of two systems

81. In both cases the computed regular chain is $\{y_w - x'_w + 1 = 0; x''^2_w + x'^2_w - 2x'_w = 0; x''_w \neq 0\}$.

82. I can consider a cart $(t, 0)$ on the abscissae, and on the rod of equation $x = t$ I introduce the points (t, x_w) and (t, y_w). Then, I put the algebraic and differential conditions on both dependent variables.

83. Even if I have not introduced the notion of "simplicity" of a machine, I can consider a machine simpler than another in an intuitive way, i.e. if its construction recall less assembling instructions than the second. There are many possible metrics for the simplicity of the machine, some more concrete (for example the number of rods, wheel, carts), some others more analytical (e.g. about the system of differential polynomial equations describing the full behavior, or the system of differential equations and inequalities for the external behavior restrict to some variables). In every case, even in the more formalized metric about analytical counterpart, I have already observed that there are different possible rankings.

However, according to the trigonometrical definition of x_w and y_w, they can be constructed using the machine of Fig. 4.10 (with some minor modifications like the translation of t for x_w and the taking of the opposite for y_w).

of differential polynomial equations, so the external behavior is the restriction to the relative manifold of solutions on some variables. To begin the equality test I have to suppose that the variables of the external behavior in both $\mathcal{M}$ and $\mathcal{N}$ are in the same number. I call $x_1, \ldots, x_n$ the external variables for $\mathcal{M}$ and $y_1, \ldots, y_n$ the ones for $\mathcal{N}$.

If I have to check the equality between two full behaviors (i.e. between radical differential ideals given by a finite set of generators), I can fix a certain ranking and compute the regular differential chains using the Rosenfeld-Gröbner algorithm, and then I can test whether all the generators of the first ideal belong to the second and vice-versa [84].

The same procedure is not easily applicable to general external behaviors. Indeed, these behaviors are obtained by eliminating some variables. Thus, representations are given by the intersection of families of regular chains, and there is no known algorithm to pass from a representation of families of regular chains to a list of generators. Note that is it always theoretically possible, according to Ritt-Raudenbush theorem (see pag. 83), but no algorithm is known about it.

A different approach to check equality can be introduced using canonical representations. Fixed a ranking, there is an algorithm providing a "canonical prime decomposition" given a (generally not unique) prime decomposition of a radical differential ideal [85]. Thus, considering an ideal defined as an external behavior, this algorithm furnishes a method to compute a new prime decomposition that is independent from the initial representation of the ideal [86]. That means that given the machines $\mathcal{M}$ and $\mathcal{N}$, I can find a canonical representation of their external behaviors. So the two machines will be equivalent if their behaviors will define the same differential ideal, i.e. if they have the same canonical representation. Thus, even though there is no known algorithm to test inclusion between radical ideals (Ritt's open problem), it is possible to test their equality.

With regard to algebraic machines, I observed in the subsection 3.3.8 (pag.

84. Given the ideal A generated by the differential polynomials $p_1, \ldots, p_n$, and B generated by $q_1, \ldots, q_m$, I can test whether the ideals are equal using the Maple command `BelongsTo` of the *DifferentialAlgebra* package. Once given any ranking, and constructed with `RosenfeldGroebner` the ideals A and B, to check the equality I only have to test whether all the generators of A belongs to B and vice-versa. In Maple, the command `BelongsTo(`$[p_1, \ldots, p_n]$`,B)` produces as output a list of n true/false, the i-th of which indicates whether p_i belongs or not to B. Conversely `BelongsTo(`$[q_1, \ldots, q_n]$`,A)` can be used to check the belonging to A.

85. See [Golubitsky et al., 2009, section 3.2, pp. 519–520].

86. The algorithm of canonical decomposition can be applied given a prime decomposition of the differential ideal. I can easily find such prime decomposition for any ideal defined as the external behavior of a machine. Consider the manifest variables $x_1, \ldots, x_n$, the latent ones $z_1, \ldots, z_m$, and the full behavior defined by the differential system Σ in $x_1, \ldots, x_n, z_1, \ldots, z_m$. Fixed a ranking, I can compute a prime decomposition of $\sqrt{[\Sigma]}$ in canonical characteristic sets (as evident in Boulier and Lemaire [2000]). Calling I the ideal obtained eliminating the variables $z_1, \ldots, z_m$ from $\sqrt{[\Sigma]}$, I formally is the intersection of $\sqrt{[\Sigma]}$ with the ring of differential polynomials in $x_1, \ldots, x_n$ (that is a subring of the ring of the polynomials in $x_1, \ldots, x_n, z_1, \ldots, z_m$). From the definition of prime ideal, it holds that in commutative rings a prime ideal intersected with a subring is still a prime ideal. Thus, I can consider the prime decomposition of I obtained intersecting any component of the prime decomposition of $\sqrt{[\Sigma]}$ with the ring of differential polynomials in $x_1, \ldots, x_n$. Thus, having a prime decomposition of I, I can apply the algorithm in Golubitsky et al. [2009] to compute a "canonical" prime decomposition of I.

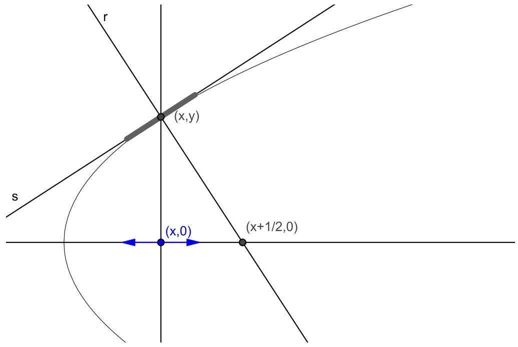


Figure 4.15: Machine with the tangent in (x, y) perpendicular to the line passing through $(x + 1/2, 0)$.

55) that one can consider different interpretations of the concept of equality between machines, according to the role of the machines. An algebraic machine can be viewed as a set of constraints or as the configurations reachable from a certain initial position—different interpretations lead to different behaviors. The possibility of posing initial conditions is also present in the differential case, as seen in the subsection 4.2.5. Also in this case there are different interpretations of equality.

I have so far treated differential machines without any reference to initial conditions. As far as my knowledge goes, the equality problem is still open if I introduce initial values. With regard to some positive results, I can consider Buchberger and Rosenkranz [2012], which furnishes an algorithm for the symbolic solution of linear boundary problems, passing from differential algebra to "integro-differential algebras" (Green's operators). For a Maple implementation of such integro-differential Green's operators for ordinary boundary problems, see the package *IntDiffOp*[87].

4.4.4 Differential machines equivalent to algebraic ones

Consider a differential machine defining the motion of a point P of coordinates (x, y) so that the tangent in P is perpendicular to the line passing through P and the point $(x + 1/2, 0)$ as the Fig. 4.15 shows. This machine is defined by the differential polynomial $x' - 2yy'$, which is the total derivative of $x - y^2$. Therefore, as can be seen in the figure, fixed any constant $c \in \mathbb{R}$, the solution will be the parabola satisfying $x = y^2 + c$. That means that I am able to trace any of the solutions of this differential machine with an algebraic one. Hence, the general question arises: Can I characterize the differential machines having solution constructible with algebraic machines (by eventually adding a finite number of real constants of integration)?

Given a differential system Σ or even its restriction on some variables, to find the algebraic constraints satisfied I can simply use the orderly ranking in the Rosenfeld-Gröbner algorithm. There will be algebraic constraints if and

87. Cf. Korporal et al. [2012].

only if in the obtained family of regular differential chains there are polynomial equations without any proper derivative (i.e. of order 0) [88].

It is more complicated if I am interested not only in algebraic constraints, but also on first integrals given by algebraic constraints. Given the system in the dependent variables $x_1, \ldots, x_n$ (depending on t)

$$\begin{cases} x_1' = p_1(x_1, \ldots, x_n) \\ x_2' = p_2(x_1, \ldots, x_n) \\ \cdots \\ x_n' = p_n(x_1, \ldots, x_n) \end{cases}$$

where all the p_i are polynomials, a function f is a "first integral" for this system if the total derivative with respect to t vanishes, i.e. if it satisfies $\frac{df}{dt} = 0$ under the constraints of the system. I am interested in such f that are polynomials in $x_1, \ldots, x_n$. There are known algorithms to solve this problem [89], but, at my knowledge, there are no known algorithms to find first integrals for general radical differential ideals (for example obtained as a manifest but not full behavior). So, at my knowledge, the general problem of defining when the solutions of a differential machine can be obtained with an algebraic one is still unsolved.

4.4.5 Conclusive notes

That is the core of the thesis, and in this chapter, I provided a definition of differential machines, and, based on differential algebra, explored the behavior of such machines. In particular, I have been successful for the characterization of the external behavior of my machines, which is given as the intersection of differential systems (each one composed by polynomial equations and inequations). I also observed that considering smooth functions as indeterminates, the constructible indeterminates are exactly the differentially algebraic functions: This can be considered as an "exact" extension of Cartesian dualism between algebraic and transcendental objects.

Moreover, as in the algebraic case, I furnished an algorithm to check the equality between two machines intended as set of constraints, i.e. constructible radical ideals. The problem is still open if I consider equality between the behaviors of differential machines with initial value conditions, i.e. intended as solutions of initial value problems. In section 7.2.3, I will claim that the equality test has a crucial role in the precise definition of exactness, so the possibility of testing equality between radical differential ideals will imply also philosophical consequences of the possibility of defining the "exactness" of differential machines. In contrast to other theories for infinitesimal analysis (for example "computable analysis"), it is furthermore important that it has not

88. In Maple, given the dependent variables $x_1, \ldots, x_n$ and the independent variable t, one can construct a differential ring with the orderly ranking by the command `R := DifferentialRing(blocks = [[`$x_1, \ldots, x_n$`]], derivations = [t]);` (the double square brackets `[[...]]` indicate the orderly ranking). After the usual construction of the ideal `ideal` with the Rosenfeld-Gröbner algorithm, the purely algebraic constraints are given by `Equations(ideal, order=0)`.

89. Cf. Schwarz [1985] or Sit [1989]. With regard to Maple implementations see `DEtools[firint]` or the testing version package *DifferentialAlgebra0* (available on-line at `http://www.lifl.fr/~boulier/BMI`) with the function `integrate` (also working for differential fractions, cf. Boulier et al. [2013]).

been proved the undecidability of the equality test between radical differential ideals with initial value conditions (note that differential algebra problems like the membership problem in an arbitrary differential ideal are well-known to be undecidable [90]).

Even if less important, the problem of detecting whether a differential machine can be reduced to an algebraic one remained open. However, the problem was algebraically translated to test whether a radical differential ideal has a family of generators where all the polynomial equations are algebraic or total derivatives of algebraic polynomials.

Regarding the definition of differential machines, I have to note that I still have to discuss the relationship with machines for tractional constructions. For example, in Riccati [1752] (see Fig. 2.5 at pag. 24), tractional constructions were allowed in cases of not clear instrumental realization (e.g. regarding "tractorias with variable directrix," where the directrix could change its shape). Thus, to consider that my formalization of differential machines is inherent to the historical tractional motion ones, I have to compare an instrumentally well-defined category of machines for tractional motion with mine. In particular, in section 6.1, I will observe how the behavior of integraphs (categorized in Pascal [1914]) can be obtained with differential machines.

To conclude, we can note how symbolic computation has an emphasis on "exactness" similar to the one of this thesis. Of course, symbolic computation stands from an analytic-symbolic perspective and not from the synthetic-diagrammatic one, but in both I find the same attention to operative procedures and finite general methods. That can be considered as a general vision of "computation," without the further division in digital and analog.

With respect to the similarity between symbolic computation and my geometrical approach, scholars of the first field consider the "approximated" or "approximating" methods of classical analysis (involving the use of non-finitary objects) as a class of methods to be overcome with new "exact" and finitary ones. Thus, in mathematics the role of algorithms (and of their effective computer implementations) can be more deeply analyzed, especially for infinitesimal analysis topics:

> "for many mathematicians, numerical mathematics is a compromise leading away from true mathematics by replacing the actual mathematical objects and domains by finitary approximations. In contrast, in our view, the *algorithmic treatment of mathematical problems* in the original, non-approximated, domains is the core of mathematical aspiration, which strives toward understanding a difficult problem so deeply that the infinitely many instances of the problem can be handled by a uniform "rule" (a theorem that has to be proved). However, how can problems in abstract mathematical structures, notably structures in analysis (in which we deal with uncountable sets of non-finitary objects like the field of real numbers or various function algebras) be turned into problems in algorithmic domains: domains consisting of countably many finitary (computer representable) objects with decidable membership and algorithmic

90. Cf. Gallo et al. [1991].

> functions and predicates on them?
>
> The clue is that, instead of solving problems in the actual mathematical domains (which are essentially non-algorithmic), one considers finitary representations of these domains — meaning finitary object representations for countable subsets of the domain carriers—and one develops a mathematical theory that maps the operations in the original domains to algorithmic ones on the finitary representations."[91]

So, in both symbolic manipulation and diagrammatic—or, in general, analog—constructions for infinitesimal calculus, the main aim is to somehow circumnavigate the need of a structural role of the concept of infinite. However, symbolic manipulation remains only an analytical tool, not quantitatively constructing new objects. Here we can find a big difference with classical calculus: even though introducing non-finitary objects, calculus allows evaluating quantitative results, arriving to avoid the need of diagrammatic/analog constructions.

In order to remain on finite representations, there is the need of synthetic analog constructions besides symbolic manipulation algorithms (as in Descartes's setting). In this perspective we can see the importance of the machines discussed in this thesis.

91. Cf. [Buchberger and Rosenkranz, 2012, p. 590].

Chapter 5

Machines for complex differential equations

I showed how differential machines can be used to solve real differential polynomial systems. Moreover, according to the Argand-Gauss plane, I can consider any point of the plane as a complex number. Hence, it is quite natural to ask whether it is possible to use these machines to solve complex differential equations given by differential polynomials. In the first section of this chapter, I will solve this problem. In the concrete solution of such problems with differential machines, it emerged the utility of the adoption of a particular point. With this point, called "pivot," I can simplify the construction of complex machines and I get some other preliminary results that are described in the second and last section.

5.1 Solving complex problems

Quoting [Needham, 1997, p. 194]:

> "In the ordinary real calculus we have a potent means of visualizing the derivative f' of a function f from $\mathbb{R}$ to $\mathbb{R}$, namely, as the slope of the graph $y = f(x)$. See Fig. 5.1[a]. Unfortunately, due to our lack of four-dimensional imagination, we can't draw the graph of a complex function, and hence we cannot generalize this particular conception of the derivative in any obvious way.
>
> As a first step towards a successful generalization, we simply split the axes apart, so that Fig. 5.1(a) becomes Fig. 5.1(b)."

According to Needham's purpose of visualizing the complex derivative as the two-dimensional case of the real one [1], the very first suggestion is to represent the motion of input and output of the function on two distinguished graphs. Thus, for the real case, we have two one-dimensional graphs, and for the complex case two two-dimensional graphs, one for the input and the other for the output.

In this chapter, I too start visualizing the complex derivative avoiding the introduction of a sensitively unimaginable four-dimensional space, but in a dif-

1. Respecting Cauchy-Riemann equations.

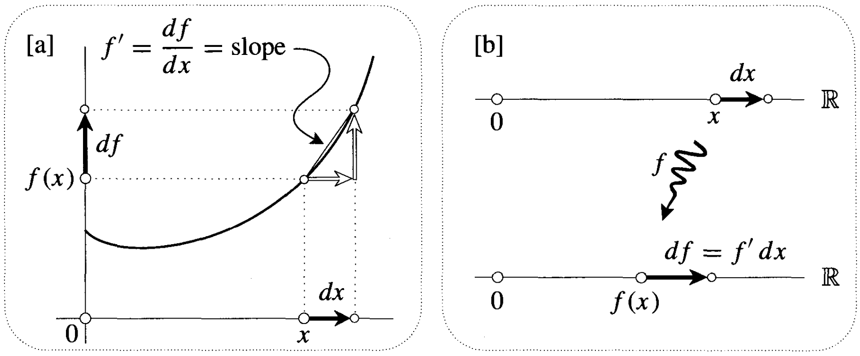


Figure 5.1: (a) The usual representation of the real derivative as the slope of the graph $y = f(x)$. (b) The representation of the same function f splitting the axes, evincing how the variation of the input dx implies a variation in the output df. Picture of (Needham [1997], p. 195).

ferent way with regard to Needham's one. In fact, instead of splitting the input and the output in different planes, I keep staying on a single two-dimensional plane so that, for example, I can generalize the visualization of the derivative as the slope of the real function. But staying on a two-dimensional plane instead of a four-dimensional space will cause that I will lose the bijective relation between the graph and the function. This means that I will no longer have the property that a point in the plane defines a unique input/output couple (so the function will no longer be defined by its planar graph), but in the meanwhile if I consider a certain point as input, I can dynamically find the position of the related output (i.e. I lose the static representation but I obtain a dynamic one). In particular, I try to reach the complex derivative extending in a "natural way" the real one.

5.1.1 Complex functions representation

Before introducing the solution of differential polynomials, I have to start giving a planar representation of complex functions. Although complex functions need a four-dimensional space to be statically represented, I can represent them through a planar transformation merging domain and range in the same two-dimensional plane, so that a function is given by a point-to-point correlation linking the motion of the input point with the one of the output point. Adopting the usual Argand-Gauss complex coordinate system, it is natural to assign a complex value to the position of any point of the plane. In order to represent the complex value of the function f, I can consider the complex value $w = z + f(z)$ as output point, so that (for every z) $f(z)$ can be seen as the difference vector between w and z.

I have to go deeper into the idea of representing $z + f(z)$ instead of $f(z)$ only. If, at first glance, it seems so different from the representation in the real case, the main condition behind both of them is that the motion of the output point has to be determined by the one of the input point, so it is necessary that the input "drags" the output. Mathematically, this is implemented by a vectors addition (input + output) both in the real and in the complex case. In the real case (as seen in the subsection 4.2.3 and in Fig. 4.8 at pag. 75), I consider the real independent variable t introducing a point of coordinates $(t, 0)$ and, called g the real function, the output point is $(t, g(t))$. Thus, the role of $(t, 0)$ is to somehow drag the point $(t, g(t))$. This propulsion allows wheel conditions to determine the values of g in function of t. If I consider this construction in complex coordinates instead of Cartesian ones, I find that the point for the independent variable will simply have coordinate t (it lies on the real axis), and the point for the dependent variable has coordinates $t + ig(t)$ (where i is the imaginary unit). To modify this setting for the complex case, I can naturally extend the real case considering point z (which is a complex one, not only a real one) as an independent variable, and $z + if(z)$ as the points for dependent variables (f being a complex function).

I can further pass from $z + if(z)$ to the representation $z + f(z)$. In fact, in contrast to the real case, multiplication by the imaginary unit in the complex one is useless for the following reasoning. While in the Cartesian plane, called the axes unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the graph is defined as $x\hat{\imath} + y\hat{\jmath}$ (so, being domain and range axes linearly independent, the graph of a real function "statically" represents all the information of the function), in the complex case domain and range have to be merged in the same planar coordinates, losing the property that any point of the plane identifies a single input/output couple. The introduction of i in the representation $t+ig(t)$ was useful to bijectively correlate real functions and their graphs, i.e. every point $x + iy$ belonging to the graph of a function g means $g(x) = y$. To obtain the same static representation for complex functions, I need a 4D-space, which is not possible. Hence, the multiplication for the imaginary unit becomes useless in the complex case (domain and range have to be merged). Thus, to avoid useless multiplications, I consider $z + f(z)$ as the output point.

Furthermore, we can observe how, constrained the complex input point to lie on a curve, different functions can map the same input in the same output image (intended as set of points, not input/output points correlation). In fact, if I consider the input point $z \in \mathbb{R}$, with my representation I have both $g(z) = iz$ and $h(z) = \frac{(i-1)}{2}z$ draw the same line, even if the functions are equal just for $z = 0$ (see Fig. 5.2).

5.1.2 From real to complex differential polynomials

At first, to solve a complex differential polynomial system [2], I have to understand what it changes in relation to the real case. Given the complex dependent variable z_j, with z'_j I denote $\frac{dz_j}{dz}$, where z is the complex independent variable. So, dependent variables $z_j = z_j(z)$ will be complex functions $\mathbb{C} \to \mathbb{C}$. Being the independent variable z a complex variable, it is natural to consider it as a free point on the plane (interpreted as an Argand-Gauss plane). However, if I consider the motion of the point z in relation to an arbitrary time, I can consider the function $z : \mathbb{R} \to \mathbb{C}$, so that $z(t)$ is the position of z in the time t. From another point of view, I can consider the arbitrary curve traced by z, so $z(t)$ will be a parametrization. Hence, I can assume $\frac{dz}{dt}$ to be always not null (eventually changing the parametrization). The introduction of the parametrization of z is useful because differential machines are not directly able to determine

2. Even in the complex case I consider only ordinary differential polynomials.

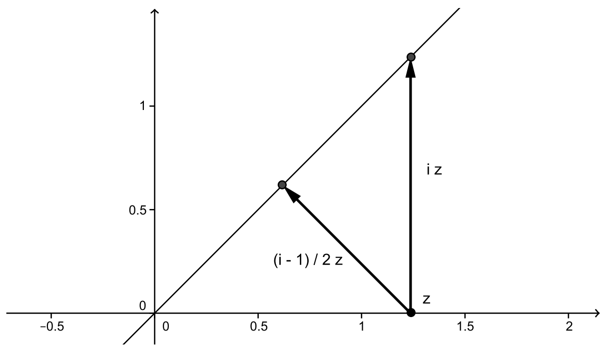


Figure 5.2: In the Cartesian representation, any point of the plane is uniquely determined as a couple of domain/range. The same does not hold in my complex representation, where, for example, for $z \in \mathbb{R}$, both $g(z) = iz$ and $h(z) = \frac{(i-1)}{2}z$ draw the same line.

the derivative with respect to a complex variable. It will be obtained using the property of the derivative of composed functions $\frac{dz_j}{dt} = \frac{dz_j}{dz}\frac{dz}{dt}$.

To solve differential polynomials, I first need to know how to do complex algebraic operations (sum and product) with my machines. Even though there are some works solving complex operations with specific machines [3], I can easily overcome this problem using algebra. In fact, I know that with algebraic machines (and as a fallout with differential ones) I can perform real operations and find the Cartesian coordinates of a point. I can thus consider the real and imaginary components of any complex number, and so I can construct a machine for $(a + ib) + (c + id) = (a + c) + i(b + d)$ and another for $(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$.

Posing tangent condition

Now the problem is to control the complex derivative z'_j. Considering $w = z + z_j(z)$ and $z(t)$, the tangent to the curve drawn by w has complex direction

$$\frac{dw}{dt} = \frac{dz}{dt}\left(1 + \frac{dz_j}{dz}\right) = \frac{dz}{dt}(1 + z'_j). \tag{5.1}$$

Thus, a first difference emerges with respect to the real case. I need to consider the derivative of the independent variable with respect to time (in the real case the independent variable is assimilable to the time). Being interested in the argument—and not in the modulus [4]—of the complex vector $\frac{dw}{dt}$, I get that $\arg\left(\frac{dw}{dt}\right) = \arg\left(\frac{dz}{dt}(1 + z'_j)\right) = \arg\left(\frac{dz}{dt}\right) + \arg(1 + z'_j)$. It is easy to set $1 + z'_j$ and

3. See for example Emch [1902], where the author showed how to perform any algebraic transformation of complex variables using only Kempe's planar linkages. More modern (and theoretical) machines for complex operations can be viewed in Kapovich and Millson [2002].

4. A complex number z may be represented as $z = x + iy = |z|(\cos\theta + i\sin\theta)$, where $|z|$ is a positive real number called the *complex modulus* of z, and θ is a real number in the range $[0, 2\pi[$ called the *argument*. This polar representation of a complex number is unique for every not null complex number. In contrast, 0 can be represented as $|z| = 0$ for each value of θ in $[0, 2\pi[$.

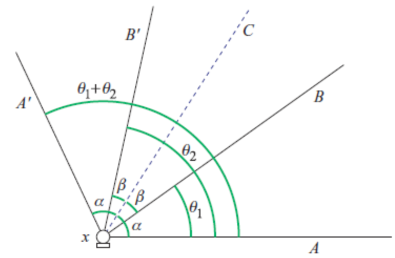


Figure 5.3: Addition of two angles using a machine that bisect/duplicate angles (seen in Fig. 3.5, pag. 57). Given the angle θ_1 defined by the lines A, B and θ_2 defined by the lines A, B', I consider the bisection of $\theta_2 - \theta_1$ defined by the lines A, C. Then I duplicate the angle defined by A, C, obtaining the line A' so that the angle defined by A, A' is $\theta_1 + \theta_2$. Figure taken from [Demaine and O'Rourke, 2007, p. 33].

find its argument: I need some clarifications about $\frac{dz}{dt}$. To begin with, $\frac{dz}{dt}$ can be supposed to always be not null (changing the parametrization of the curve traced by z), and its argument is trivially constructible after having posed a rod r tangent in z. Such rod r can be introduced as joined in z, and I put a wheel on r in z. These conditions mean that r has to be the tangent at z to the curve traced by the same point. Then I can construct a rod s joined in w and having as slope the argument of $\frac{dw}{dt}$. This rod is constructible adding the angles of $\frac{dz}{dt}$ and $1 + z_j'$ [5]. Adding a wheel on s in w will pose the tangent condition in w. However, I have to note that the direction of w is uniquely defined when $1 + z_j'$ is not null, i.e. if and only if $z_j' \neq -1$, otherwise no tangent conditions can be imposed on w.

Therefore, in the general case, it is easy to pose the tangent condition (5.1) but, in contrast to the real case, the posing of such condition is not enough to manage the behavior of z_j. Indeed, being the derivative given by a complex number, it has two dimensions, so the change of the function cannot be managed by a single tangent condition. In the real case, the input point $(t, 0)$ was imposing the abscissa of the output point. In the complex case, both the abscissa and the ordinate of the output point are not constrained to be the one of the input point z. Thus, at a first glance, for the complex derivative I need to impose not only a tangent condition (about the argument), but also a condition for the modulus. I will see that even modulus can be managed with tangent conditions on additional points, without the need of new mechanical tools (for example involving not only the direction but also the rotation speed of wheels [6]). It is time to introduce the "auxiliary output point" w_c. Given any complex constant

5. A simple machine to add angles was introduced in Kempe [1876]. The so-called "additor" is explained in modern notation in [Demaine and O'Rourke, 2007, pp. 32–33]. As visible in Fig. 5.3, given an angular bisector/duplication machine, I can easily construct the angle $\theta_1 + \theta_2$.

6. Using wheels, to set a complex derivative one can think to use a device such that, according to the direction and the rotation speed of the wheel in the input point z, imposes the direction and the rotation speed of the wheel in the output point w. The direction of the wheel in w is a tangent condition, so can be controlled by a differential machine. On the contrary, the speed of rotation of the wheel in w has to be the speed of rotation of the wheel in z times the modulus of the complex derivative: This latter constraint is not directly available with differential machine (it would require some mechanical components as gears). However, as I am going to introduce, there is no need of more components, it is possible to control complex derivatives (also modulus) with differential machines.

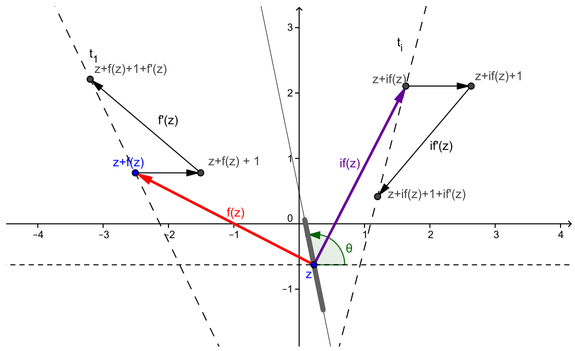


Figure 5.4: Call t_c the line (not defined if $1 + cf'(z) = 0$) passing through $w_c = z + cf(z)$ and $z + cf(z) + (1 + cf'(z))$. In this figure you can see an example when c assumes the values 1 and i. The tangent condition in w_c is given by the line t_c rotated of the angle $\theta = \arg\left(\frac{dz}{dt}\right)$ (this rotation is not represented in the figure).

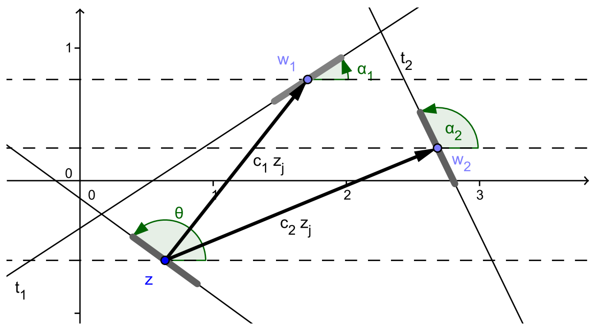


Figure 5.5: Managing the complex derivative with tangent conditions: Wheels are posed in z, w_1, w_2.

c, I can construct $w_c = z + cz_j$. The argument of $\frac{dw_c}{dt}$ is $\arg\left(\frac{dz}{dt}\right) + \arg(1 + cz'_j)$, so, as shown in Fig. 5.4, I can pose the tangent condition in w_c. The idea is that this way I can add a new condition without adding degrees of freedom. [7]

Managing complex derivative with more tangents

I have to precise the use of tangent conditions on auxiliary points to put constrains on z'_j. I will especially give some conditions under which the value of z'_j is uniquely determined given the tangents of z and of two output points.

Consider the auxiliary points $w_1 = z + c_1 z_j$ and $w_2 = z + c_2 z_j$ (where c_1, c_2 are complex constants). As seen in Fig. 5.5, call α_1, α_2 the angles of the directions of the wheels in w_1, w_2. Thus, taking the derivative with respect to t

7. It is implicitly intended that $c \neq 0$. In fact, looking for more tangent conditions, the case $w_0 = z$ is useless because it does not give any new tangent condition (I consider as given the direction of the independent point z).

I obtain

$$\begin{cases} \arg\left(\frac{dw_1}{dt}\right) = \alpha_1 \\ \arg\left(\frac{dw_2}{dt}\right) = \alpha_2 \end{cases}.$$

Calling $\theta = \arg\left(\frac{dz}{dt}\right)$, the system becomes

$$\begin{cases} \arg(1 + c_1 z'_j) = \alpha_1 - \theta \\ \arg(1 + c_2 z'_j) = \alpha_2 - \theta \end{cases}.$$

Considering that the argument φ of a complex number z has to satisfy the equation

$$\mathrm{Im}(z) \cdot \cos\varphi = \mathrm{Re}(z) \cdot \sin\varphi,$$

the system can be written

$$\begin{cases} \mathrm{Im}(1 + c_1 z'_j) \cdot \cos(\alpha_1 - \theta) = \mathrm{Re}(1 + c_1 z'_j) \cdot \sin(\alpha_1 - \theta) \\ \mathrm{Im}(1 + c_2 z'_j) \cdot \cos(\alpha_2 - \theta) = \mathrm{Re}(1 + c_2 z'_j) \cdot \sin(\alpha_2 - \theta) \end{cases}.$$

Thus, splitting the real and imaginary components of c_1, c_2, z'_j, I arrive to the real linear system

$$\begin{cases} \mathrm{Re}(z'_j)A_{11} + \mathrm{Im}(z'_j)A_{12} = b_1 \\ \mathrm{Re}(z'_j)A_{21} + \mathrm{Im}(z'_j)A_{22} = b_2 \end{cases} \tag{5.2}$$

where the matrix A is

$$\begin{bmatrix} \mathrm{Im}(c_1)\cos(\alpha_1 - \theta) - \mathrm{Re}(c_1)\sin(\alpha_1 - \theta) & \mathrm{Re}(c_1)\cos(\alpha_1 - \theta) + \mathrm{Im}(c_1)\sin(\alpha_1 - \theta) \\ \mathrm{Im}(c_2)\cos(\alpha_2 - \theta) - \mathrm{Re}(c_2)\sin(\alpha_2 - \theta) & \mathrm{Re}(c_2)\cos(\alpha_2 - \theta) + \mathrm{Im}(c_2)\sin(\alpha_2 - \theta) \end{bmatrix}$$

and

$$b = \begin{bmatrix} \sin(\alpha_1 - \theta) \\ \sin(\alpha_2 - \theta) \end{bmatrix}.$$

Using Rouché-Capelli theorem [8], my system will have a unique solution if and only if the determinant of A is not null [9]. However, my problem is not the lack of existence (when z'_j is a point at the infinity), I want to avoid when there is no unique definition of z'_j (infinite number of solutions). This happens when both the elements of b and the determinant of A are null. Being null the elements of b means that $\alpha_1 - \theta$ and $\alpha_2 - \theta$ are 0 (modulo π) [10], i.e. both $1 + c_1 z'_j$ and $1 + c_2 z'_j$ have to be pure real value. Thus, $c_1 z'_j$ and $c_2 z'_j$ have to be real. Hence, b is a null vector if and only if

$$(z'_j = 0) \vee (c_1 \parallel c_2 \parallel 1/z'_j).$$

8. In linear algebra Rouché-Capelli theorem allows computing the number of solutions in a system of linear equations given the ranks of its coefficient matrix A and the augmented matrix $[A|b]$ (i.e. the matrix obtained adding the column b to A). A system of linear equations with n real variables has a solution if and only if the rank of its coefficient matrix A is equal to the rank of its augmented matrix $[A|b]$. If there are solutions, they form an affine subspace of $\mathbb{R}^n$ of dimension $n - rank(A)$. If $n = rank(A)$, the solution is unique, otherwise the number of solutions is infinite. For a detailed discussion, see, for example, Lang [2010].

9. In case of 2×2 matrices, the determinant is $A_{11} \cdot A_{22} - A_{12} \cdot A_{21}$.

10. Two numbers a, b are said to be *congruent modulo n* (usually written $a \equiv b \mod n$) if their difference $a - b$ is an integer multiple of n. In my case, $\alpha_1 - \theta$ and $\alpha_2 - \theta$ have to be integer multiples of π.

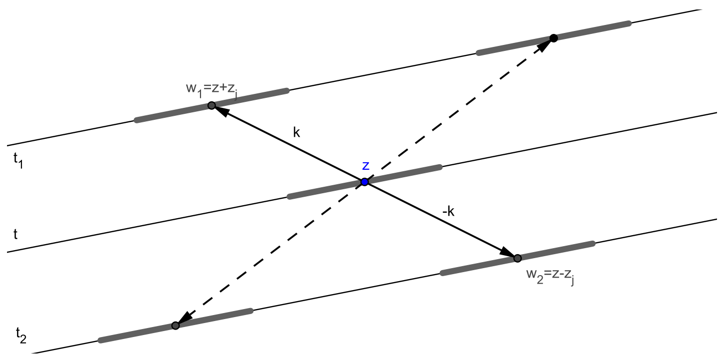


Figure 5.6: I consider $w_1 = z + c_1 z_j$ and $w_2 = z + c_2 z_j$ with $c_1 = 1, c_2 = -1$. This choice is not warranting the uniqueness of the z'_j because $c_1 \parallel c_2$. In fact, if the direction of $\frac{dz}{dt}$ is parallel to the directions of $\frac{dw_1}{dt}$ and $\frac{dw_2}{dt}$, the complex value z'_j is not uniquely defined.
In the figure, I consider the constant function $z_j(z) = k$ (where k is a complex constant). Only with the tangent conditions in w_1, w_2 I am not able to manage z_j and z'_j. In fact, if one avoids any motion of z, w_1 and w_2 are free to move respectively on t_1 and t_2 (the dashed vectors represent another possible position), always respecting the tangent conditions $\frac{dw_1}{dt} = \frac{dw_2}{dt} = \frac{dz}{dt}$ (i.e. keeping parallel t, t_1, t_2). So the tangent conditions do not define z_j in function of z (and so *a fortiori* z'_j is not determined).

In every case from $\alpha_1 - \theta = \alpha_2 - \theta = 0$ (modulo π) I get that the determinant of A becomes $\mathrm{Im}(c_1)\,\mathrm{Re}(c_2) - \mathrm{Re}(c_1)\,\mathrm{Im}(c_2)$. This determinant being null means that $c_1 \parallel c_2$. Summarizing, the non-uniqueness happens when both the elements of b and the determinant of A are null, i.e. when

$$(c_1 \parallel c_2 \parallel 1/z'_j) \vee ((c_1 \parallel c_2) \wedge z'_j = 0).$$

I want to consider c_1, c_2 as two fixed constants, so to avoid any problem of non-uniqueness for any value of $z, \frac{dz}{dt}, z_j, z'_j$ I can just take c_1, c_2 not parallel[11] (for an example see Fig. 5.6). It is also to be noted that when z'_j exists and is unique, it can be computed solving (5.2) with algebraic operations, and so can be constructed with my tools. This means that, as in the real case, I am able to manage derivatives also in the complex case with differential machines.

5.1.3 Some remarks

I have to note that, being the direction of tangents given by z'_j, there are problems when $1 + cz'_j = 0$. In this case $\frac{dw_c}{dt} = 0$, so there is no tangent defined at w_c. However, this problem can be solved if I consider in my machines not only two output points, but also another auxiliary output point that assures that at least two between $\frac{dw_1}{dt}$, $\frac{dw_2}{dt}$ and $\frac{dw_3}{dt}$ [12] are not null (if c_1, c_2, c_3 are all different and not null). Additionally, in order to assure that also in the case of $\frac{dw_i}{dt} = 0$ the tangent conditions defined by the other output points define uniquely z'_j, I take c_1, c_2, c_3 not parallel for every possible pair wise.

11. I am not yet considering the indetermination of the tangent in the case $1 + cz'_j = 0$.
12. Given the complex constants c_1, c_2, c_3, I am considering $w_i = z + c_i f(z)$ for $i = 1, 2, 3$.

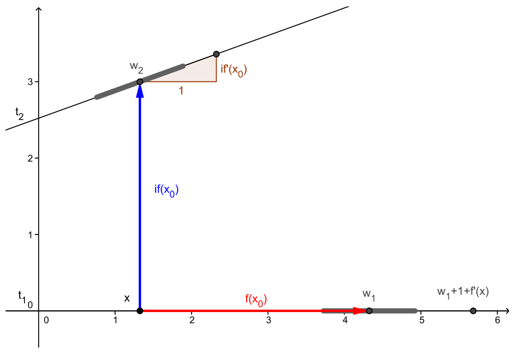


Figure 5.7: Considering the real case as a subcase of the complex one, I can introduce the points $w_1 = x + f(x)$ and $w_2 = x + if(x)$. I can impose that the vector $w_2 - x$ is vertical simply using the tangent conditions in my output points: The tangent in w_1 will always be horizontal, so w_1 will always lie on the abscissae. Hence, $w_2 - x$ will be parallel to ordinates because $w_1 - x$ will always be parallel to x, and $w_2 - x = i(w_1 - x)$. Continuing, the tangent condition in w_2 is exactly the tangent condition of the usual interpretation of the real derivative (derivative as slope of the tangent).

Finally, I conclude with a remark about the real case. The main difference with the complex one is that in the real case, adopting just a single tangent, one poses the perpendicularity between the input and output vectors. On the contrary, in the complex case, I lose the perpendicularity and to manage the derivative I need another tangent constraint [13]. Now I will observe how to manage real derivatives without the perpendicularity but with couples of tangents.

To set the tangent conditions, I compute the derivatives $\frac{dw_1}{dt} = 1 + \frac{df}{dx}$ and $\frac{dw_2}{dt} = 1 + i\frac{df}{dx}$. Thus, I find that the tangent in w_1 will always be an horizontal one (so w_1 will keep on lying on the abscissae) while the tangent in w_2 is the usual tangent to the real graph. That means that, with such c_1, c_2, $w_2 - x$ is constrained to be a vertical vector because $w_1 - x$ is always perpendicular to $w_2 - x$ and w_1 will always lie on the abscissae, as shown in Fig. 5.7.

5.1.4 A machine for the complex exponential

In this subsection [14], I will finally introduce a differential machine solving a complex differential equation. In particular, I will explain how to assemble a machine for the complex exponential function. Recall that, even in the complex case, the exponential function is the only solution to the Cauchy problem $f'(z) = f(z)$, $f(0) = 1$.

Given the general auxiliary output point $w_c = z + cf(z)$, I take as constant $c_1 = 1$ and $c_2 = i$. Being such coefficients not parallel, the tangent conditions in $w = z + f(z)$ and $w_\perp = z + if(z)$ will be enough to manage the behavior of the

13. As another difference, we can also note that in the real case for the tangent there is no additional rotation of $\frac{dz}{dt}$ with regard to the line passing through w_c and $w_c + 1 + cf'(z)$ because a real value x can move just in one direction.

14. Mainly taken from [Milici, 2015, pp. 14–16].

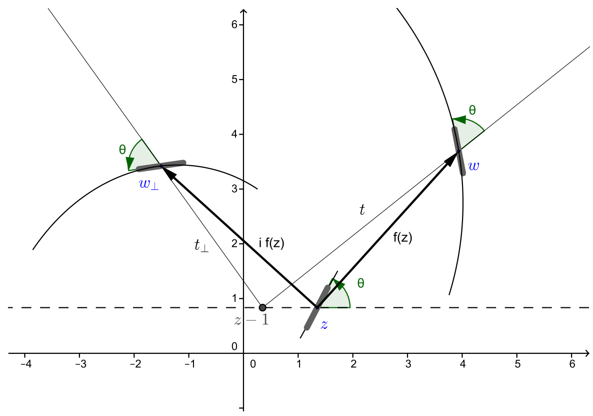


Figure 5.8: A machine for the complex exponential. Considering the points $w = z + f(z)$, $w_\perp = z + if(z)$, I want to impose the tangent conditions on them in such a way that $f'(z) = f(z)$ for every motion of z. Calling t and $t_\perp$ the lines passing through $z-1$ and respectively $w, w_\perp$, and θ the argument of $\frac{dz}{dt}$, the tangent condition in w (resp. $w_\perp$) is given by the line t (resp. $t_\perp$) rotated of the angle θ.

complex exponential machine when the tangent in both w and $w_\perp$ is defined [15]. Thus to construct the machine I have to pose the tangent conditions

$$\arg\left(\frac{dw_c}{dt}\right) = \arg\left(\frac{dz}{dt}\right) + \arg(1 + cf'(z)).$$

Neglecting for the moment the angular addition of $\arg\left(\frac{dz}{dt}\right)$, the line t_c passing through w_c with direction $1 + cf'(z)$ will be made up by all and only the points $t_c(\lambda) = w_c + \lambda(1 + cf'(z))$ (for every real value of λ). Being $f'(z) = f(z)$ and $w_c = z + cf(z)$, the point $t_c(-1)$ is $z - 1$. That means that the line t_c can be defined as the one passing through w_c and $z-1$ [16]. This line will not be defined if and only if w_c coincides with $z - 1$, i.e. $cf(z) = -1$, which is the problem of the tangent condition, as seen in note 15.

As shown in Fig. 5.8, call t and $t_\perp$ the lines passing through $z - 1$ and respectively $w, w_\perp$. Thus, considering the point z free to move on the plane, I can consider the rod tangent in it [17]. Calling θ the angle defined by the tangent rod with a horizontal one passing through z (thus $\theta = \arg\left(\frac{dz}{dt}\right)$), with an angle additor [18] I can impose the direction of the wheel in w (resp. $w_\perp$) to be t (resp. $t_\perp$) additionally rotated of the angle θ.

15. Even though I am not considering such cases in the construction of my machine, the direction of w_c is not defined when $\frac{dw_c}{dt} = 0$, i.e. when $1 + cf'(z) = 0$. In my case $f = f'$, so there are problems if $cf(z) = -1$. To overcome these problems, I would need to construct not only w and $w_\perp$, but also another output point w_c so that c is neither parallel to 1 nor to i (for example I can take $c = 1 + i$). This way, there would be at least two well-defined tangent conditions on the output points for every z.

16. Note that also in the real case the tangent condition was the passage through the point one unit at the left of the independent point (in Cartesian coordinates it was $(t-1, 0)$).

17. As already observed, I can introduce this tangent rod r as the one joined in z and with a wheel in z.

18. Cf. note 5, page 105.

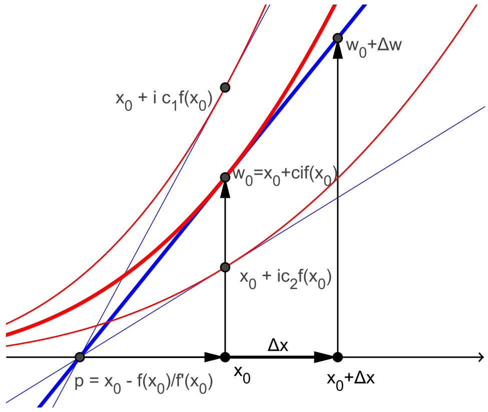


Figure 5.9: On the complex plane, representing the output point $w = x + icf(x)$, all the tangents when the input is x_0 (if $f'(x_0) \neq 0$) will pass through the pivot point $p = x_0 - \frac{f(x_0)}{f'(x_0)}$. To approximate $x + if(x)$ in $x_0 + \Delta x$ I can construct the triangle $p, x_0 + \Delta x, w_0 + \Delta w$ similar to the one with vertexes p, x_0, w_0 (if $f(x_0) \neq 0$ and $f'(x_0) \neq 0$).

Remember that such construction posed the complex condition $f'(z) = f(z)$. To obtain exactly the complex exponential function, I need to pose the initial condition $f(0) = 1$. It is also interesting to observe similarities to and differences from the machines for real exponential (Fig. 2.3, pag. 20, or Fig. 4.9, pag. 76).

5.2 Some properties of the pivot point

The introduction of machines for complex functions could be useful to visualize something not clearly visible otherwise. For example, in the complex exponential machine, we can observe that, to construct the tangent condition in an output point, I used the point $z - 1$. This point has the nice property that, when $\arg\left(\frac{dz}{dt}\right) = 0$, tangents to output points pass through it. The use of this point was useful in constructing a simple machine, which is why I generalized this property to any continuously differentiable complex function f, and found some possible application of such a point, which I call "pivot." In general, with the pivot point, I can reduce the complexity of graphical constructions and differential machines when dealing with the field of complex numbers.

5.2.1 Introduction of the pivot point

Given the complex input point z and the output point $w = z + f(z)$, I can introduce the "pivot point" $p = z - \frac{f(z)}{f'(z)}$ [19] (being at the denominator, the pivot is a finite point if and only if $f'(z) \neq 0$). I want to explore the possible uses and properties of such point.

19. It is the complex generalization of the Cartesian point $\left(t - \frac{f(t)}{f'(t)}, 0\right)$ in the case of real functions. This point is the intersection of the tangent to the graph with the abscissae, and it is used in Newton's numerical method to approximate the zeros of $f(t)$.

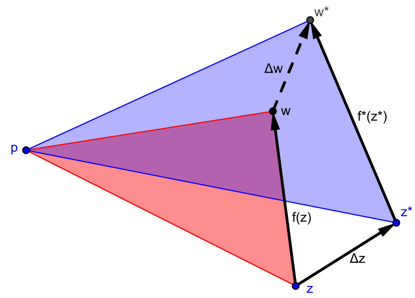


Figure 5.10: Given a complex function f, its linear approximation $f^*(z^*)$ can be obtained constructing the triangle of vertexes p, z^*, w^* similar to the one of vertex p, z, w.

I can begin by observing the role of pivot in the real case. Let f for the moment be a real function, and consider its graph in complex coordinates $x + if(x)$. As visible in Fig. 5.9, if I also introduce, $\forall c \in \mathbb{R}$, the functions $f_c(x) = c \cdot f(x)$,when $f'(x) \neq 0$ all the tangents t_c to the function $x + if_c(x)$ with input x_0 will meet in the point $p = x_0 - \frac{f(x_0)}{f'(x_0)}$ (the vector $-\frac{f(x_0)}{f'(x_0)}$ is usually denoted as "subtangent"). Called $w_0 = x_0 + icf(x_0)$, the linear approximation of $x+if(x)$ in $x_0+\Delta x$ is $w_0+\Delta w$, being Δw obtained constructing the triangle $(p, x_0+\Delta x, w_0+\Delta w)$ similar to the one with vertexes (p, x_0, w_0). As I am going to clarify, this similarity condition is at the origin of the name "pivot."

In order to extend the role of the pivot from the real to the complex case, I just have to remove the unique direction of input, output and Δx. Therefore, instead of $x + if(x)$ ($x \in \mathbb{R}$) I can represent $z + if(z)$ ($z \in \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$) with any direction of Δz. We will see that the pivot works even in the complex case. According to the possibility of considering the same pivot for every $z + icf(z)$, to simplify the notation I can directly adopt the representation $z + f(z)$ (so $c = -i$). However, to formally treat the complex case, I have to previously introduce some notations.

Given $a, b, c, a^*, b^*, c^* \in \mathbb{C}$ (that can be considered as points on the Argand-Gauss plane), by the notation $(a, b, c) \sim (a^*, b^*, c^*)$ I consider that the triangle of vertices a, b, c is similar (with the same orientation) to the one of vertices a^*, b^*, c^*. By proportions, this similitude in algebraic conditions becomes

$$(b - a)(c^* - b^*) = (b^* - a^*)(c - b). \tag{5.3}$$

Call $f^*(z)$ the linear approximation [20] of f in z, i.e. $f^*(z + \Delta z) = f(z) + f'(z)\Delta z$. Calling $z^* = z + \Delta z$, $w^* = z^* + f^*(z^*)$ and using (5.3) we can observe that, as visible in Fig. 5.10, it holds

$$(p, z, w) \sim (p, z^*, w^*) \tag{5.4}$$

20. I.e. the Taylor series of f truncated at the first order.

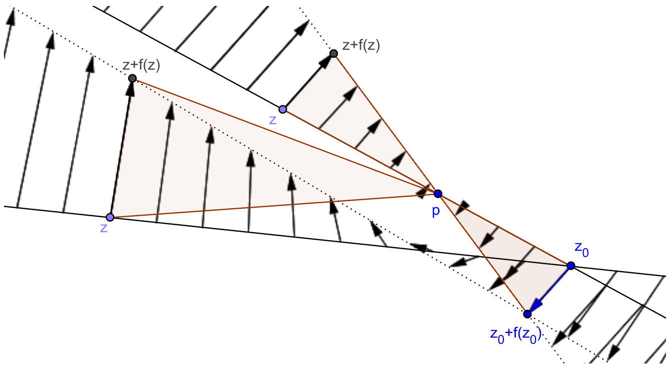


Figure 5.11: Visualization of the change of $f(z) = az + b$ along two different lines passing through z_0. For this function the pivot point does not change in function of z (p is constantly $-b/a$). You can note how if the line also passes through p than the behavior is similar to the real derivative (the triangle $p, z, z + f(z)$ does not rotate and the tangent to the curve drawn by the output passes through p), while on a generic line not passing through p the triangle $p, z, z + f(z)$ also rotates and the tangent to the output point is not passing through p (note that, considering z moving along a line and being $f(z)$ a linear function, the tangent coincides with the locus of $z + f(z)$).

that justifies why I called p the pivot point (it works like a pivot between the similar triangles (p, z, w) and (p, z^*, w^*)). To assure that this formula has a geometrical meaning, I have to impose that $f(z) \neq 0$ (else I have a similarity between a degenerated triangle collapsed on a point, which is similar to any other triangle). Thus, in the following, I am implicitly assuming that $f(z), f'(z) \neq 0$.

Before observing the pivot role in the geometric and kinematic graphical constructions with regard to complex functions, let me consider, as a basic example, $f(z) = az+b$ (with $a, b \in \mathbb{C}$). In this case, considering $a \neq 0$, the pivot p is $z - \frac{az+b}{a} = -\frac{b}{a}$, i.e. p is not varying in function of z but is always the same point on the plane. Particularly in this case $f(z) = f(z_0)+f'(z_0)(z-z_0)$ (and not only its linear approximation f^*), so the function $f(z)$ is always reconstructible through the similarity of the triangle $p, z_0, f(z_0)$, as shown in Fig. 5.11.

5.2.2 Tangents at output points in function of the pivot

The role of the pivot emerged from the complex exponential machine. In general, given any complex function defined by the first order equation $z_1' = P(z, z_1)$ (where z is the independent variable, $z_1(z)$ is a complex function and P is a complex polynomial), there is a simple general way to construct a differential machine solving it. In fact, for such z_1, the pivot is $p = z - \frac{z_1}{z_1'} = z - \frac{z_1}{P(z,z_1)}$, i.e. the position of p is well determined by an algebraic machine in function of z, z_1. Then I can generalize the machine of the complex exponential considering two

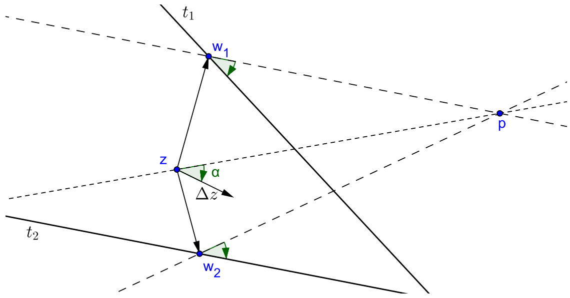


Figure 5.12: Construction of the tangents t_1, t_2 according to the displacement Δz given the pivot p.

output points (e.g. $w_1 = z + z_1, w_2 = z + iz_1$) [21] where the tangent direction is posed to be the one pointing to the pivot rotated of the angle arg $\frac{dz}{dt}$ (cf. Fig. 5.12). This result holds because the similitude condition (5.4) can be rewritten [22]

$$(p, z, z + \Delta z) \sim (p, w, w + \Delta w). \tag{5.5}$$

Out of the cases when this similitude does not work properly [23], in general, it is much simpler to construct tangents in output points given the pivot than given directly the function derivative. About the converse (the passage from tangents to information about the function), I observed that it is possible to obtain the value of the complex derivative given the tangents in two output points and the direction of z, but I obtained it using algebra and not too simple computations. I want to observe that the situation is much simpler if I want to obtain the position of the pivot point instead of directly the derivative, simple enough to furnish a synthetic construction to determine p. That will mean that pivot has a somehow more direct relation with tangents (both if I start from the pivot and want to construct tangents and vice-versa) than the complex derivative, somehow justifying its introduction. Before the synthetic construction I need a preliminary property.

Proposition 2. *As visible in Fig. 5.13, in the plane, given two non-parallel lines a, b passing respectively through the points A, B, denoted $C = a \cap b$, if I call a_α, b_α the lines a, b rotated respectively around A, B of an angle α, denoted $C_\alpha = a_\alpha \cap b_\alpha$, the locus described by C_α at the variation of the angle α is the circumscribed circle of the triangle ABC.*

Proof. That is a consequence of the "angle at the centre/angle at the circumference" theorem, Euclid's Elements, Book III, Proposition 20 (cf. Heath et al.

21. Even though I ignored in the construction of the complex exponential, to be precise I need even a third point $w_3 = z + (1 + i)z_1$ to have enough tangent conditions when $\frac{dw_n}{dt} = 0$ (for $n = 1$ or 2), i.e. when w_n coincides with p.

22. With Δw I consider $w^* - w$. Moreover, I can consider interchangeable Δa and $\frac{da}{dt}$. Their huge epistemological difference (the passage to the limit) is no longer present in graphical computation when the derivative is represented as a vector on the plane.

23. The tangent in w will not be defined if and only $\frac{dw}{dt} = 0$, and that happens if at least one of the triangles $(p, z, z + \frac{dz}{dt})$ and $(p, w, w + \frac{dw}{dt})$ collapses on a point. That happens in three cases: $\frac{dz}{dt} = 0$, $p = z$, $p = w$. The first case is excluded because we consider well defined the direction of z, the second is excluded by the assumption that $f, f' \neq 0$, but the third may happen.

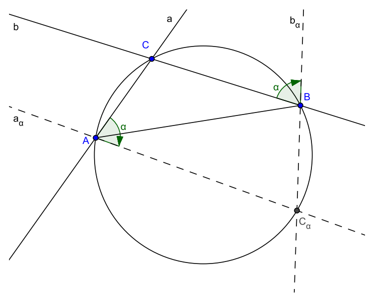


Figure 5.13: Given a triangle ABC, denoted a the line through AC and b the one through BC, if I rotate a, b respectively around A and B of the same angle, the rotated lines will intersect on the circumscribed circle of the triangle ABC.

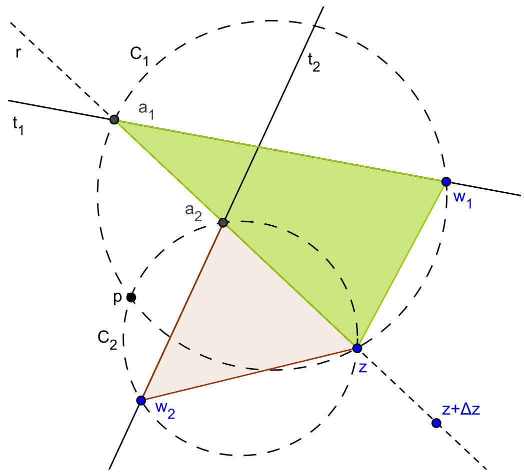


Figure 5.14: Given the all different points $z, z+\Delta z, w_1, w_2$ and the tangents t_1, t_2, to find the pivot I can construct the line r through $z, z+\Delta z$, consider $a_1 = r \cap t_1; a_2 = r \cap t_2$. Denoted C_1, C_2 the circumscribed circle respectively of the triangles (z, a_1, w_1) and (z, a_2, w_2), the pivot p will belong to the intersection $C_1 \cap C_2$.

[1956]). Moreover, note that if a, b are parallel, even a_α, b_α will always be parallel, so C_α can be considered as the infinity line in projective geometry. □

So I am ready for the following:

Theorem 3. *Given the points $z, z+\Delta z, w_1, w_2$ and the tangents t_1, t_2 respectively at w_1 and w_2, it is possible to geometrically find the pivot as it follows (see Fig. 5.14). Denote r the line through $z, z+\Delta z$, and consider $a_1 = r \cap t_1; a_2 = r \cap t_2$. Called C_1, C_2 the circumscribed circles respectively of the triangles (z, a_1, w_1) and (z, a_2, w_2), the intersection $C_1 \cap C_2$ will be made up by the points z and p. Therefore, I have a construction for p.*

Proof. Let t_i^γ denote the line obtained rotating t_i of an angle γ around w_i. According to (5.5), t_i^γ is the tangent along the direction $\arg(\Delta z)+\gamma$. In particular, if I denote r^γ the line passing through the input z and with direction $\arg(\Delta z)+\gamma$, to find the pivot I am looking for the angle $\overline{\gamma}$ so that $t_1^{\overline{\gamma}} \cap t_2^{\overline{\gamma}} \in r^{\overline{\gamma}}$ [24]. Assuming that $t_1^{\overline{\gamma}}, t_2^{\overline{\gamma}}$ and $r^{\overline{\gamma}}$ are pair wise distinct lines, I can rewrite $t_1^{\overline{\gamma}} \cap t_2^{\overline{\gamma}} \in r^{\overline{\gamma}}$ as $t_1^{\overline{\gamma}} \cap r^{\overline{\gamma}} = t_2^{\overline{\gamma}} \cap r^{\overline{\gamma}}$. Thus, denoted respectively C_1, C_2 the loci of $t_1^{\gamma} \cap r^{\gamma}$ and $t_2^{\gamma} \cap r^{\gamma}$ (at the variation of the angle γ), the pivot p belongs to $C_1 \cap C_2$. Particularly, denoted $a_1 = t_1 \cap r, a_2 = t_2 \cap r$, for Prop. 2, C_1, C_2 will be the circumscribed circles of respectively the triangles (z, w_1, a_1) and (z, w_2, a_2) (they can be constructed with ruler and compass by Euclid's Elements, Book IV, Proposition 5, e.g. see Heath et al. [1956]). Hence $z \in C_1 \cap C_2$, so $C_1 \cap C_2 \neq \emptyset$, which means that, if the intersection is of two points, the one different from z will be p. If the intersection point is unique, $p = z$. □

To be precise, the previous theorem has not been really proved, because I need some remarks about the conditions tacitly supposed:

1. the intersection $t_i \cap r$ always identifies a unique finite point;

2. $a_i \neq w_i, z$ (necessary to construct the circumscribed circle C_i);

3. the intersection $C_1 \cap C_2$ is finite ($C_1 \neq C_2$).

However, I am not really interested in the degenerated cases when at least one of these condition is not satisfied. So, conscious that I am leaving the proof at a sketch level, I can pass to another topic.

5.2.3 Planar kinematics

Given a planar kinematic problem, the main idea to graphically solve it is to represent the velocity of a point with a vector [25]. Thus, if I want to consider (5.4) from a kinematic perspective, I have to substitute, for every point a, Δa with the velocity $v_a = \frac{da}{dt}$.

Consider the point z moving in function of the time t according to the law $z : \mathbb{R} \to C$. The point w is moving in function of the position of z, precisely $w = z + f(z)$. Considering the relative velocity v_f of w with respect to an observer in z, it holds $v_f = \frac{df}{dt} = \frac{df}{dz} \cdot \frac{dz}{dt} = f'(z)v_z$. Knowing the pivot p I can graphically construct $f'(z)$ constructing the triangle $(p, z + v_z, w^*)$ similar to (p, z, w) (as done in Fig. 5.10 if I consider Δa instead of v_a). In fact, $w^* = z + v_z + f(z) + f'(z)v_z$, hence $w^* - w = v_w = v_z + v_f$.

However, as usual in graphical kinematics, I am interested in some particular points such as the instant center of rotation (shortly I.C.R.) [26]. But I am not dealing with rigid bodies: the triangle (p, z, w) is not rigid, it can expand and

24. Given the tangents t_1, t_2 and the line r passing through z with direction Δz, it holds $p \in t_1 \cap t_2 \cap r \iff p - z \parallel \Delta z$.

($\Rightarrow$) It is trivially true because $p \in r$.

($\Leftarrow$) If $p - z \parallel \Delta z$ then, for (5.5), $p - w_c$ will be parallel to t_c, so $p \in t_c$ ($c = 1, 2$).

25. See for example Mason [2001].

26. The instant centre of rotation can be considered the limiting case of the pole of a planar displacement. The planar displacement of a rigid body from position 1 to position 2 is defined by the combination of a planar rotation and planar translation. For any planar displacement there is a point in the moving body that is in the same place before and after the displacement. This point is the pole of the planar displacement, and the displacement can be viewed as a rotation around this pole. Taking I.C.R. will be such pole limit position while the change in the time tends to 0.

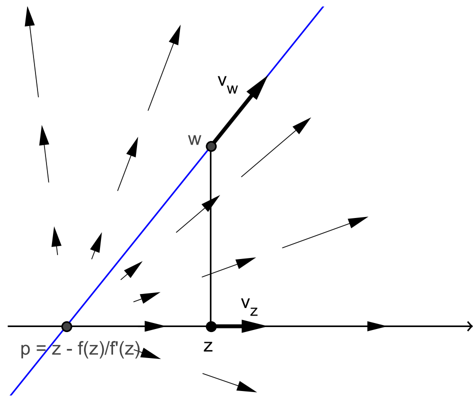


Figure 5.15: In the real case $z \in \mathbb{R}$, and $w = z + if(z)$ (where $f : \mathbb{R} \to \mathbb{R}$). I can consider z, w to stay fixed on a plane subject to a linear scaling with center p. This expansion/contraction in a generic point a obey to the proportion $p - a : p - z = v_a : v_z$ where p is the pivot $z - f(z)/f'(z)$, so in the real case p can be seen as a "Instant Center of Expansion" (I.C.E.). This property does not hold generally in the case of complex functions where, as we will see, I need to introduce also an "Instant Center of Rotation" to obtain the whole velocity of f (v_f is given by $v_w - v_z$).

contract in function of the time. So I have to introduce a new point that I call "instant center of expansion" (shortly I.C.E.) [27].

Real derivative as pure expansion

Let me restrict z to real values and f to a real function. Considering $w = z + if(z)$ as in the usual graph of real functions, the pivot $p = z - \frac{f(z)}{f'(z)}$ will be on the abscissae. So in every instant v_w will be parallel to $p - w$ (as obviously v_z is parallel to $p - z$). More specifically, the condition $\frac{v_w}{w-p} = \frac{v_z}{z-p}$ can be interpreted as if z, w are subject to a linear scaling with center in p [28]. As seen in Fig. 5.15, in the real case the pivot can be seen as an "instant center of expansion" (shortly I.C.E.), and the velocity of w is determined by v_z. Then v_f is simply given by $v_w - v_z$.

I have to note that this property will no longer generally hold in the case of complex functions, but I will soon observe how, even in the complex case, from the pivot and the direction of v_z I can easily obtain two points, an I.C.E. ($p_\parallel$) and an I.C.R. ($p_\perp$). These points can be used to obtain the components of the velocity v_f.

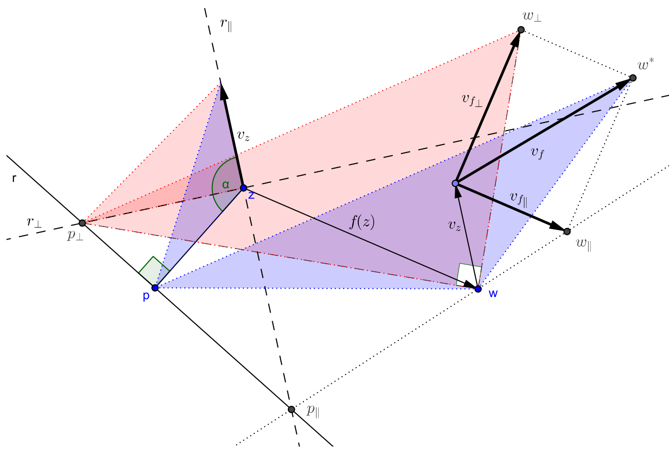


Figure 5.16: This is a kind of "graphical kinematics." Once given p, z, w, v_z, if I consider the line $r_\parallel$ passing through z with direction v_z, the line $r_\perp$ perpendicular to $r_\parallel$ through z and the line r perpendicular to $p-z$ through p, I can define the points $p_\parallel = r \cap r_\parallel$ and $p_\perp = r \cap r_\perp$. I can define the points $w^*, w_\parallel$ and $w_\perp$ in such a way that $(p, z, z+v_z) \sim (p, w, w^*)$, $(p_\parallel, z, z+v_z) \sim (p_\parallel, w, w_\parallel)$ and $(p_\perp, z, z+v_z) \sim (p_\perp, w, w_\perp)$. If I subtract $w+v_z$ respectively to the points w^*, $w_\parallel$ and $w_\perp$ I obtain the vectors v_f, $v_{f\parallel}$ and $v_{f\perp}$. The latter two velocities are respectively the parallel and perpendicular components of v_f with respect to $f(z) = w - z$.

Rotation and expansion centers for complex functions

I want to decompose v_f in two components respectively of "pure expansion" and "pure rotation" i.e. parallel and perpendicular components of v_f with respect to the direction of $f(z) = w - z$. To realize it, I will consider two auxiliary points that, similar to the pivot geometrical property $(p, z, z+v_z) \sim (p, w, w+v_w)$, will define the components of v_f.

As seen in Fig. 5.16, let me consider as given the pivot p, the input z with velocity v_z and the output $w = z + f(z)$. Thus, I can obtain v_f from the property that, with $w^* = w + v_w$, $v_f = w^* - (w + v_z)$. If I consider the line $r_\parallel$ passing through z with direction v_z, the line $r_\perp$ perpendicular to $r_\parallel$ through z and the line r perpendicular to $p - z$ through p, I can define the points $p_\parallel = r \cap r_\parallel$ and $p_\perp = r \cap r_\perp$. Now I can define the points $w_\parallel$ and $w_\perp$ so that $(p_\parallel, z, z+v_z) \sim (p_\parallel, w, w_\parallel)$ and $(p_\perp, z, z+v_z) \sim (p_\perp, w, w_\perp)$. For these similarities, $p_\parallel, w, w_\parallel$ will be aligned (as they are $p_\parallel, z, z+v_z$), and $w_\perp - w$ will be perpendicular to $p_\perp - w$ (as v_z is perpendicular to $p_\perp - z$).

Similar to the definition of $v_f = w^* - (w+v_z)$, I can denote $v_{f\parallel} = w_\parallel - (w+v_z)$

27. Note that, even though the name refers only to expansion, the scaling may be also a compression. At my knowledge this nomenclature is new, but I am not an expert of the field.

28. This situation is somehow similarly to what happens with the I.C.R. (the velocity of a point is proportional to the distance from the center). The difference is that, called q the I.C.R., in any point a the velocity vector v_a is perpendicular to $a - q$.

and $v_{f\perp} = w_\perp - (w + v_z)$. Formally, it holds

$$\begin{array}{rcl} v_f & = & \frac{v_z(z-w)}{p-z} \\ v_{f\|} & = & \frac{v_z(z-w)}{p_\|-z} \\ v_{f\perp} & = & \frac{v_z(z-w)}{p_\perp-z}. \end{array} \tag{5.6}$$

I want to show that the parallel and perpendicular components of v_f with respect to $w-z$ are respectively $v_{f\|}$ and $v_{f\perp}$. For the moment, I know that $v_{f\|}$ and $v_{f\perp}$ have the right direction (they are respectively parallel and perpendicular to $w-z$). Hence, I just have to prove that $v_f = v_{f\|} + v_{f\perp}$. Considering how I constructed $p_\|$ and $p_\perp$, $(p, z, p_\perp) \sim (p, p_\|, z)$, i.e., using (5.3),

$$p_\perp - z = \frac{(p_\| - z)(z-p)}{p - p_\|}.$$

Substituting it in (5.6), I get

$$v_{f\|} + v_{f\perp} = v_z(z-w)\frac{(z-p)+(p-p_\|)}{(p_\|-z)(z-p)} = v_z\frac{z-w}{p-z} = v_z f'(z) = v_f.$$

I can note that $v_{w\|} + v_{w\perp} = v_w + v_z$ and not only v_w, so $p_\perp$ and $p_\|$ are not the I.C.R. and I.C.E. with respect to the fixed reference frame. However, if I consider an observer in z, I get that the relative velocity of w (i.e. $v_f = v_w - v_z$) is exactly the sum of the one obtained by the pure rotation with I.C.R. $p_\perp$ and the pure expansion with I.C.E. $p_\|$. From this perspective, I can consider that the local behavior of a complex function is defined by an I.C.E. (like real functions) and an I.C.R., which are easily constructible given the pivot.

I can add some brief remarks. If p lies on $r_\|$ (resp. $r_\perp$), $p = p_\|$ (resp. $p = p_\perp$) while $p_\perp$ (resp. $p_\|$) is a point at infinity. In this case v_f will be a pure expansion (resp. rotation) velocity because $v_f = v_{f\|}$ (resp. $v_f = v_{f\perp}$).

In addition, using the complex polar form, I can write $f(z)$ as $\rho(\cos\theta + i\sin\theta)$ (with ρ and θ in function of z). I have seen that $\frac{df}{dt} = v_f = f'(z)v_z$. With $v_{f\|}$ and $v_{f\perp}$, I can easily express $\frac{d\rho}{dt}$ and $\frac{d\theta}{dt}$. In fact, considering the parallel and perpendicular components of v_f with respect to $f(z) = w - z$, it holds that, introducing the normalized vector $\hat{f}(z) = \frac{f(z)}{|f(z)|}$, it holds

$$v_{f\|} = \frac{d\rho}{dt}\hat{f}(z), \qquad v_{f\perp} = i\rho\frac{d\theta}{dt}\hat{f}(z)$$

(where $f(z), v_{f\|}, v_{f\perp}$ are complex values while ρ, θ and their derivatives are real). Using (5.6) and rewriting $\hat{f}(z) = \frac{w-z}{\rho}$, I obtain

$$\frac{d\rho}{dt} = -\rho\frac{v_z}{p_\| - z}, \qquad i\frac{d\theta}{dt} = -\frac{v_z}{p_\perp - z}.$$

Calling α the angle between v_z and $p-z$, I get that $\frac{v_z}{p_\|-z} = \cos\alpha\frac{|v_z|}{|p-z|}$ and $\frac{v_z}{p_\perp-z} = -i\sin\alpha\frac{|v_z|}{|p-z|}$. Thus,

$$\frac{d\rho}{dt} = -\rho\cos\alpha\frac{|v_z|}{|p-z|}; \qquad \frac{d\theta}{dt} = \sin\alpha\frac{|v_z|}{|p-z|}. \tag{5.7}$$

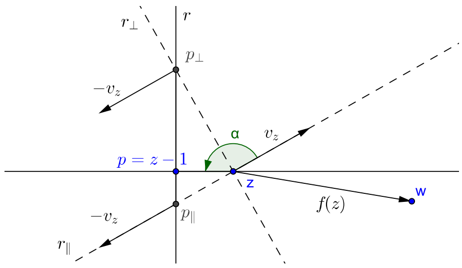


Figure 5.17: $f(z) = f'(z)$ in e^z, thus $p = z - 1$.

Note also that, being $|p - z| = |-\frac{f(z)}{f'(z)}| = \frac{\rho}{|f'(z)|}$, the equation for $\frac{d\rho}{dt}$ can be rewritten (this time using explicitly the modulus of the complex derivative):

$$\frac{d\rho}{dt} = -\cos\alpha \cdot |v_z| \cdot |f'(z)|.$$

As an elementary example I can use (5.7) for the differential equation of the complex exponential function. From the definition, it holds $p = z - 1$. Thus, considering the angle α between v_z and $p - z$ as seen in Fig. 5.17, I have $\frac{d\rho}{dt} = -\rho\cos\alpha|v_z|$ and $\frac{d\theta}{dt} = \sin\alpha|v_z|$. That means that, if v_z is a pure real value ($\alpha = 0 \mod \pi$), there is a pure expansion ($\frac{d\theta}{dt} = 0$), while if v_z is purely imaginary ($\alpha = \pi/2 \mod \pi$) then I have a pure rotation ($\frac{d\rho}{dt} = 0$). Again, if z moves with constant velocity ($z = kt + z_0$, with $k \in \mathbb{C}$), the rotation speed is constant ($\frac{d\theta}{dt} = \sin\alpha|k|$) and from $\frac{d\rho}{dt} = -\rho\cos\alpha|k|$ I get $\rho(t) = e^{\rho_0 - t|k|\cos\alpha}$.

Chapter 6

Differential machines as physical devices

I have already introduced differential machines as theoretical instruments. In this chapter, I will consider their concrete counterparts. In particular, I start interrelating differential machines with historical tractional devices. Specifically, instead of the 18th century theoretical approach of Riccati [1], I prefer the more practical classification of grapho-mechanical machines for integration of differential equation made in Pascal [1914] [2]. I will evince that all such machines are obtainable with my differential machines.

I then explore the diagrammatic constructions available thanks to a single differential machine extending and unifying ruler and compass, the "logarithmic compass," evincing some possible foundational fallouts. This machine, though not concretely realized, has been realistically designed.

I conclude this chapter by giving some didactic fallouts for my machines. Of course, the goal of this thesis is not to suggest a practical use of such machines to solve differential equations, but to face foundationally infinitesimal analysis problems with idealizations of concrete tools. This quest for such an instrumental and finitistic foundation in my perspective can be useful in didactics to make mathematics less abstract and more touchable. The use of such machines is suggested with the concrete manipulation of actual objects, but in the future could also be interesting to extend their constructive role in a piece of dynamic geometry software. Another future perspective is to explore the potentials of differential machines for a new educational pathway for calculus (with differential algebra).

1. In Riccati [1752], tractional constructions were allowed in cases of unclear instrumental realization, as it happens in the case of "tractorias with variable directrix," where the directrix can change its shape.

2. As shown in [Tournès, 2009, Chap. 9] and introduced briefly in the historical part (subsection 2.3.5 pag. 26), the methods of instrumental graphical integration of differential equations developed up to the mid-18th century were forgotten and later revived in the late 19th century with the same family of concepts and tools. The machines of the latter period were called "integraphs," and the work that better summarized such tools was Pascal [1914]. With a bit of localism, I can note how Italian were both Riccati and Pascal.

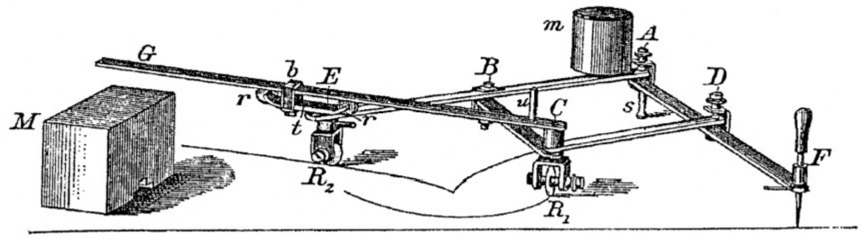


Figure 6.1: An example of integraph using more wheels is the "katenograph" [Schimmack, 1905, p. 344].

6.1 Differential machines and integraphs

To evince the relation between differential machines and tractional motion I will convert the machines of Pascal [1914] (probably the most complete classification of integraphs) in my differential machines. Out of wheels, integraphs use straight components and sliding parts (that correspond respectively to rods and carts). However, one can also find tools as curved bars or springs. I will show that their behaviors can be obtained by using only differential machines. From an analytical perspective, this attempt may appear useless because all the known integraphs solve algebraic differential equations (A.D.E.). However, I thought that it might be interesting because of the lacking of the proved closure of the class of the solutions constructible with integraphs. Theoretically, there might be some machines that solve something that is not an A.D.E. by applying methods of other integraphs as well as some slightly different parts. Hence, in this section I will see that all the practical ideas behind integraphs can be captured by differential machines.

All the machines in Pascal [1914] have only one wheel. Integraphs with a wheel can be thought to integrate tractionally differential equations of first order, in general to reach greater orders we need more wheels. An example of an integraph with more wheels is the "katenograph" introduced in Schimmack [1905] (seen in Fig. 6.1).

A general method to integrate differential equations of any order with graphomechanical instruments was suggested in Torres Quevedo [1901]. The Spanish engineer (1852–1936) considered the possibility of assembling together more "elementary machines," each one representing the values x, y, y' as points on three lines. The elementary machines impose with a wheel that $y' = \frac{dy}{dx}$ and, with a suitable mechanism, that the variables satisfy a relation $F(x, y, y') = 0$. Therefore, assembling many elementary machines, it was possible to mechanically integrate a system of n first order differential equations, or, equivalently, a differential equation of order n. I can note how this method is similar to the one seen in the subsection 4.2.3.

However, being not a problem to put any number of wheels with differential machines, I consider in this section only the one-wheel integraphs, so the ones of Pascal [1914]. Specifically, I am not interested in Ernesto Pascal's classification, but in observing that all his machines can be converted in differential machines. These integraphs have two fundamental components, the "differential cart" and "integral cart"[3]. The wheel is on the integral cart, and traces the integral curve, while the differential cart is rigidly coupled to a spike that the user has to move along the curve that has to be integrated. A general distinction of such machines resides in the coordinate system of the plane where we want to interpret the integral curve. If points are considered in Cartesian coordinates one constructs "Cartesian integraphs," while in polar coordinates "polar integraphs." The basic configuration for Cartesian integraphs is a rectangle sliding straightly, while for polar integraphs it is a rotating circular sector. On these basic configurations, there will be guides for both differential and integral carts. However, I will see them with more details in the following subsections.

6.1.1 Integraphs with only straight components

To begin with, I can observe the integraph for the differential equation $y' = f(x) - y$ in Fig. 6.2. This is a simple Cartesian integraph: I can note the rectangular frame that, owing to the two wheels united with an axis (top and bottom of the figure), can slide along the direction called x. On the right edge of the frame, there is a spike C that the user has to move along a curve to be integrated. This spike can move up and down the right edge due to the differential cart. On the left edge, there is the integral cart, which can move up and down but is constrained by a wheel D to go along the direction determined by the differential cart G[4]. Calling (x, y) the coordinates of the wheel D and $(x+1, f(x))$ the coordinates of the differential cart G[5], the wheel imposes the condition that $y' = f(x) - y$[6].

This integraph is composed only of straight components, and so it can trivially be considered as a differential machine. However, it is important to introduce this elementary case to understand how to extend Cartesian integraphs. Call "guide" the line along which the integral cart can move and "ruler" the line connecting differential and integral carts[7]. In the next subsections, I will explore the cases of non-straight guide and the ruler. Prior to that, I will consider the case of polar integraphs.

Pascal considered just one polar integraph, the one in Fig. 6.3. In this case, the main frame is a circular sector that can rotate around the centre. We can find differential and integral carts sliding on radial axes, as well as a straight rule to connect them. Even though the frame is a circular sector, the physical introduction of a curved element is not important at all[8], so even in this case the passage to the relative differential machine is trivial.

3. These are different from my carts because they do not have to slide only on straight rods.

4. Pascal considered the direction DG and not DC to make the integraph more user-friendly while moving the spike C along a curve.

5. I assume there is a curve $(x, f(x))$ (later transposed in $(x+1, f(x))$) traced on the plane.

6. As suggested by Pascal, we can also consider the direction of the wheel as the direction of DC rotated at a fixed angle. However, this is still implementable with the tools of differential machine (with the simple method described in note 8).

7. The direction of the wheel in the integral cart is determined by the direction (i.e. the tangent) of the "ruler" in correspondence of the wheel.

8. The external circular sector is only introduced to let the user define a constant angle between the radial axes of the differential and integral carts. However, given the two radial axes joined in a point fixed on the plane, I can simply constrain them to keep the same distance always using a chord instead of a circular sector.

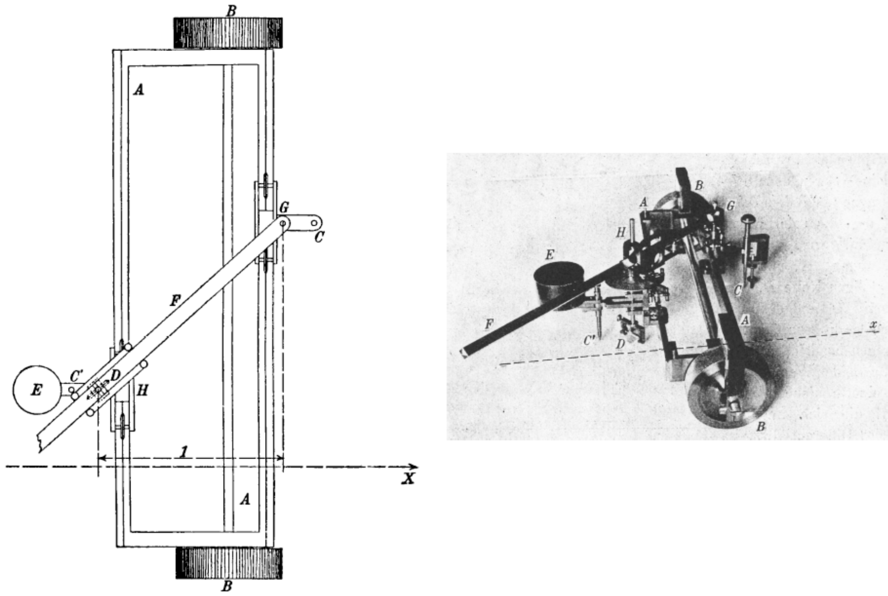


Figure 6.2: Ernesto Pascal's integraph for the first order differential equations $y' = f(x) - y$ [Willers, 1911, pp. 37–38].

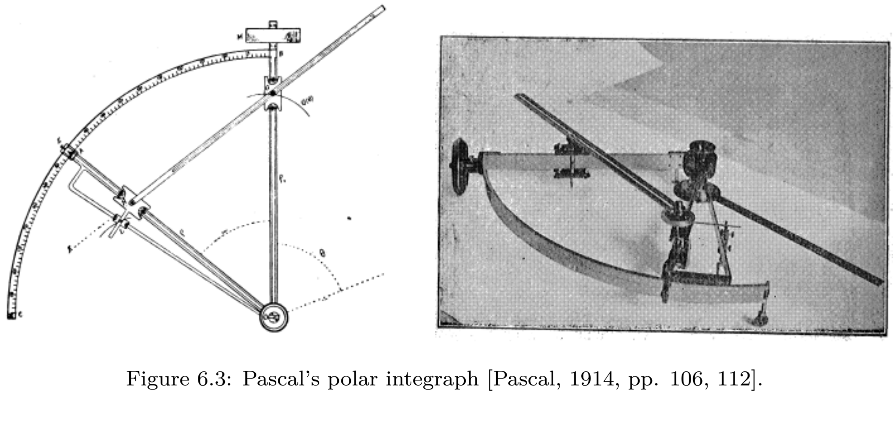


Figure 6.3: Pascal's polar integraph [Pascal, 1914, pp. 106, 112].

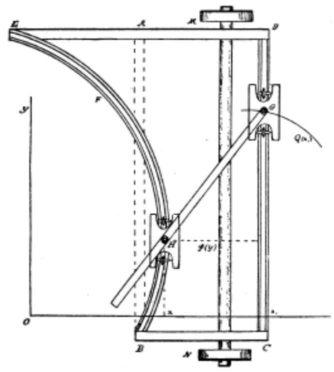


Figure 6.4: An integraph with curved guide [Pascal, 1914, p. 99].

6.1.2 Integraphs with curved guide

In the Cartesian integraph seen in Fig. 6.2, I can change the guide of the differential cart, using a curved guide instead of a straight one, as evident in Fig. 6.4. This time, the passage to a differential machine is no longer trivial, because the curve has an active role not directly substitutable by a straight rod.

First, I have to put some conditions on the curve defining the guide. If I allow any curve, I can consider the curve given by the graph of Euler's Γ function, that is not constructible with differential machines, and so a machine with this guide cannot be translated in a differential one. According to the curves introduced by Pascal, I can assume that the given curve γ is solution of an ADE [9], i.e., for what proved in the subsection 4.2.3, I can consider a differential machine tracing exactly the given curve. Call P the movable point of the differential machine tracing γ, and consider its coordinates (x_P, y_P). Introducing a point $(x, 0)$ [10] I can construct (for the construction of algebraic machines) the point Q of coordinates $(x + x_P, y_P)$. Thus, Q is constrained to lie on the given curve γ that slides united to $(x, 0)$. Hence, instead of introducing the physical curved guide for the integral cart, with differential machines I can impose the same condition making the integral cart coincide with Q.

Another integraph using a curve sliding along the abscissa is the one integrating the "odograph" equation, visible in Fig. 6.5 [11]. In this case, the machine is more complex: The integral cart does not lie on the curved guide but the wheel is constrained by a parallelogram to have the same direction of the rod KH (according to the letters used in the left diagram of the figure). However, regarding the conversion in differential machines, this case is analogous to the previous one: The position of K can be determined simulating the curved guide P with straight tools, and the parallelogram is naturally constructible with tools of algebraic machines.

9. Algebraic Differential Equation. Cf. note 79, pag. 94.
10. Back to integraphs, I can consider x the abscissa of any point fixed on the rectangular frame sliding on the abscissae.
11. This machine is useful in ballistic to compute the motion of a bullet subject to friction.

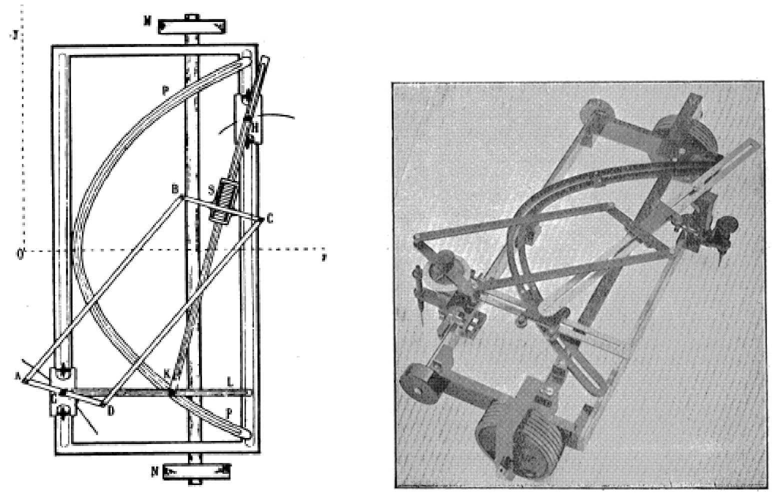


Figure 6.5: An integraph integrating the "odograph" equation [Pascal, 1914, pp. 66, 68].

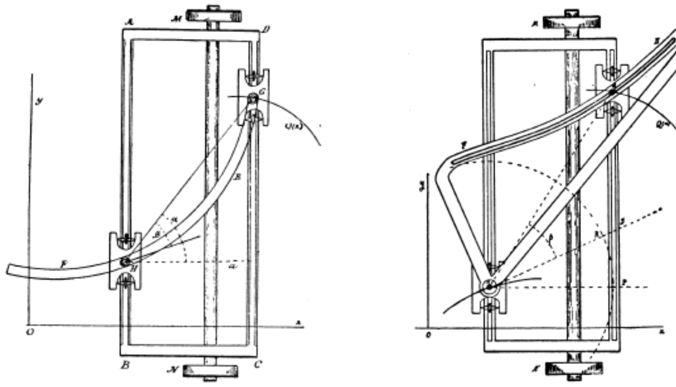


Figure 6.6: Integraphs with curved rulers: (left) jointed in the differential cart, (right) united to the integral cart [Pascal, 1914, pp. 10, 15].

6.1.3 Integraphs with curved ruler

Considering the possibility of using curved rulers connecting differential and integral carts, there are two possibilities: Either the curved ruler is jointed in the differential cart (with the possibility of rotating), or the ruler is united to the integral cart. These cases are evident in Fig. 6.6—the first case in the left and the second in the right.

Concerning both curved rulers, as seen in the previous subsection, consider the curve γ[12] traced by a point P (of coordinates (x_P, y_P)) of a differential machine. As distinct from before, the curve in this case has not only to translate, but also to rotate. With respect to the rotation, consider a point R of coordinates (x_R, y_R) constrained by a rod to lie in the unitary circumference centered in the origin O (i.e. $x_R^2 + y_R^2 = 1$). If I consider the angle α so that $x_R = \cos\alpha$ and $y_R = \sin\alpha$, to consider γ rotated of α I can introduce the

12. This curve has the shape of the curved ruler to be simulated.

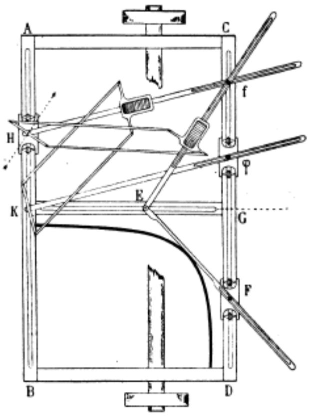


Figure 6.7: An integraph respecting tangent conditions: The rod EF has to be tangent to the branch of hyperbole (the thick line) [Pascal, 1914, p. 84]. In this integraph there is the introduction of a spring between E and G to impose the tangent condition.

point Q of coordinates $(x_R x_P - y_R y_P, x_R y_P + y_R x_P)$ that is constructible with algebraic machines [13]. Thus, Q is a point that lies on γ rotated of α around the origin (α is not *a priori* defined and can change according to the other conditions imposed by the machine). Furthermore, for every point (x, y), I can construct the point $Q^* = Q + (x, y)$.

In the case of curved ruler jointed in the differential cart, I can consider (x, y) to be the coordinates of the differential cart. So the point Q^* can assume any position available by the curve γ translated by the differential cart and rotating of any angle [14]. Constraining Q^* to lie also on the left edge of the sliding frame (as seen on the left of Fig. 6.6), I find the position of the integral cart. Finally, the direction of the wheel will be given by the tangent of the roto-translation of γ at Q^*, that is obtainable if the curve is an algebraic one (cf. subsection 4.1.1).

In the case of curved ruler united to the integral cart, I consider (x, y) to be the coordinates of the integral cart, and constrain Q^* to lie on the right edge of the frame (it coincides with the differential cart). This last constraint determines the angle of rotation of the curve γ, that determines the direction of the wheel (cf. the right of Fig. 6.6) [15].

6.1.4 Integraphs respecting tangent conditions

In the case of curved ruler jointed in the differential cart, I have used the tangent to a given curve to determine the direction of the wheel. In this subsection, I propose to go further. In the integraph in Fig. 6.7, the rod EF is posed

13. The formula of Q can be obtained thinking at the couple of coordinates as the real and imaginary part of complex numbers, and the rotation of α as the multiplication $OP \cdot OR$.

14. The curve γ has to pass through $(0, 0)$, and the point of the curve in the origin has to coincide with the point to be jointed in the differential cart.

15. Even in this case the curve γ has to pass through $(0, 0)$, and the point of the curve in the origin has to coincide with the point to be united to the integral cart. Called α the rotation of γ, the direction of the wheel in the integral cart has to be parallel to the tangent of γ at $(0, 0)$ rotated of α

to be tangent to the bent piece of metal modeled as a branch of hyperbole (in the figure the thick line) [16]. This condition is mechanically posed using a spring that moves E as far as possible (considering the other conditions) from G. This mechanism works because the branch of hyperbole is convex.

To convert this constraint in differential machines I miss the concept of "spring," but, as seen in the subsection 4.1.1, I can construct the tangent rod for any algebraic curve (at a non-singular point). Therefore, I consider an algebraic machine making a point move along a fixed hyperbole, and construct the rod tangent to the curve at this point. Furthermore, I can translate the machine and the tangent rod according to the position of the rectangular frame. Finally, using the trivial conversion for the straight components, I obtain a differential machine having the same behavior of the integraph.

6.2 The logarithmic compass

Probably the most simple non-algebraic polar integraph is the one tracing logarithmic spirals. Even though Pascal's polar integraph (Fig. 6.3) is able, out of many other uses, to trace such spirals, the approach of this section is original because I am interested in foundational questions.

Wantzel's characterization of constructible numbers [17] and Lindemann's proof of the transcendence of π [18] proved the impossibility, using a compass and an unmarked straightedge alone, of solving classical Greek geometric problems such as doubling a cube, trisecting an angle, squaring a circle, and constructing certain regular polygons. In this section, I introduce an instrument that unifies and extends the constructional powers of the compass and the straightedge, an instrument that I call the "logarithmic compass" (or "equiangular compass"). It can draw a logarithmic spiral about any given center, through any given point, with any given tangent at that point. [19]

6.2.1 Introducing the device

Physically, the logarithmic compass can be constructed as follows (see Fig. 6.8).

The wheel (A) rolls on the paper, constrained to follow a course at a fixed angle to the line through the center. Inconveniently, its point of contact with the paper is also the point whose locus I wish to mark; one can solve this problem by inking its rim. The wheel is mounted, perpendicular to the plane, in a fork (B) locked at a fixed angle with the rod (C). The rod is constrained by the

16. All the other components of the machine are straight, so for the translation in a differential machine I will focus just on this tangent condition.
17. Cf. Wantzel [1837].
18. Cf. Lindemann [1882].
19. All the results and images of this section appeared in Milici and Dawson [2012]. In particular, I am grateful to Robert M. Dawson for the wonderful rendering of the machine (in the first draft submitted to *The Mathematical Intelligencer*, the image was handmade) and the help in clarifying the general setting and the specific passages. The idea of a geometry based on the logarithmic compass and on a machine for the planar logarithmic curve was the spark that, in 2009, made me think of the possibility of a Ph.D. During that time, I did not know anything about tractional motion and integraphs.

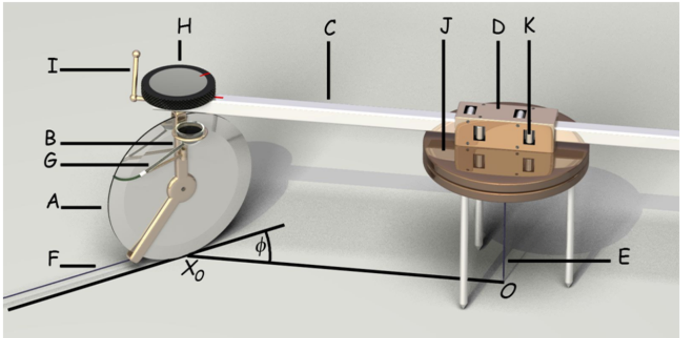


Figure 6.8: A logarithmic compass (image rendered by R. Dawson using the "POV-Ray" raytracer).

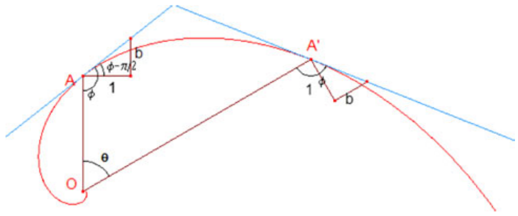


Figure 6.9: In logarithmic spirals, the angle between the tangent and the radial direction is constant.

rolling of the wheel, and by the pivot (D), which allows the rod to slide and which itself rotates over the chosen center point [20].

The compass is thus set by three parameters (two points and an angle [21]). I obtain the curve by rolling the wheel; its inked rim then traces the logarithmic spiral.

We can see that the compass forces the tangent to the curve to keep a constant angle ϕ with the radial direction. If we denote $a = OA$ and $b = \tan(\phi - \frac{\pi}{2})$, setting the origin of the polar reference system in O and the direction OA when $\theta = 0$, the compass will solve the Cauchy problem

$$\begin{cases} \rho(0) = a \\ \frac{\rho'(\theta)}{\rho(\theta)} = b \end{cases} .$$

The unique solution of this problem is $\rho(\theta) = a \cdot e^{b\theta}$. I call ϕ the *inclination* and b the *coefficient* of the spiral (see Fig. 6.9).

Note that for $\phi = 0$ I obtain a straight line (it is the only case in which the curve cannot be written in the form $\rho(\theta) = a \cdot e^{b\theta}$) and for $\phi = \frac{\pi}{2}$ I obtain a circle. I will show that both of these settings can be constructed, so I can emulate both straightedge and compass.

20. The figure also shows several features introduced for practicality. The pivot has a pointer (E) to align it accurately with the center of the spiral, and the wheel fork has a corresponding pointer (F) to let it remain aligned with an initial tangent line. A capillary feed (G) provides the wheel with ink. A knob (H) with a smooth concave top allows the wheel to be oriented accurately and then guided with a fingertip. There is a cam (I) to lock the fork in position. Finally, I must assume ball bearings between the discs (J) in the pivot and roller bearings (K) guiding the rod, as there must be no appreciable friction that might make the wheel slip.

21. I will consider angles in radians.

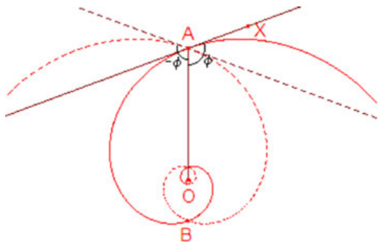


Figure 6.10: Constructing a line with the logarithmic compass.

To say exactly what we can do with this instrument, I will take the constructive power of the logarithmic compass to be represented by the following axiom:

> **Logarithmic compass.** Given 5 points O, A, B, C, D, where O, A are distinct and B, C, D are distinct, it is possible to construct a logarithmic spiral with center O, passing through A, and with inclination $\phi = \angle BCD$ (a signed angle). [22]

6.2.2 Logarithmic compass extend ruler and compass

As the first construction, given the points O, A, and a completely arbitrary point X anywhere in the plane, I will construct a point B collinear with O and A. This allows us to set the compass to draw a straight line.

Set the compass for the inclination $\phi = \angle OAX$ and draw an arc of a spiral (subtending at least π radians) with center at O and passing through A. Set it again for the inclination $-\phi = \angle XAO$ and draw another arc with center at O and passing through A. Any other point B of intersection between these spirals will be collinear with O and A (Fig. 6.10) [23]. If the compass is set with center at O, wheel at A, and tangent through B, it will draw the line AB.

A similar construction, with the second spiral centered at A and passing through O, yields two spiral arcs that intersect at points P, Q on the perpendicular bisector of OA (Fig. 6.11) [24]. Using the previous construction, I may use the compass to construct the lines PQ and OA. Taking R to be the intersection

22. This is slightly out of the Euclid's spirit of the "collapsing compass," i.e. the compass that can trace the circumference given the centre and a point, but not a centre and a radius. A "collapsing logarithmic compass" would allow only the more restrictive construction:

> Given 3 distinct points O, A, B, it is possible to construct a logarithmic spiral with center O, passing through A, and with inclination $\phi = \angle OAB$ (a signed angle).

I conjecture that the collapsing logarithmic compass on its own is strictly weaker than the logarithmic compass. However, if I have a straightedge and compass, as well as a classical logarithmic compass, I can copy any angle I like to the place where it is needed. Therefore, with regard to constructions, the logarithmic compass is equivalent to the collapsing logarithmic one extended with straightedge and compass.

23. This construction fails if $AX \parallel AO$ or $AX \perp AO$. However, in the first case, I am already able to construct a straight line. In the second, I have the possibility of tracing circles, and with a standard compass alone it is easy to construct a point B collinear to OA (the symmetric of A in relation to O).

24. The spirals intersect at other points as well, but P, Q are the only points so that the arcs AQ, AP subtend angles of less than π at O and vice-versa.

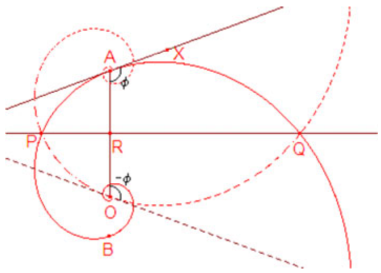


Figure 6.11: Constructing a perpendicular bisector with the logarithmic compass.

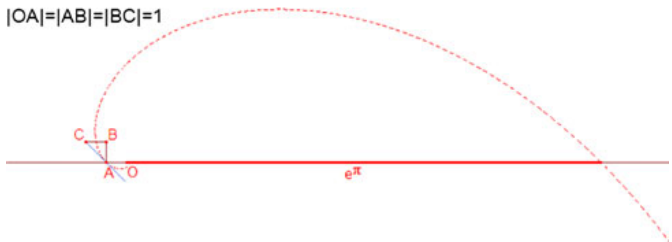


Figure 6.12: Constructing e^{π} with the logarithmic compass.

of these segments, $\angle ORP$ is a right angle, which may be used to set the compass to an inclination of a right angle, letting us draw a circle [25]. As explained previously, the radius is set in a separate operation.

I will now show that the logarithmic compass is strictly more powerful than classical straightedge and compass. I first construct three unit segments OA, AB, BC, forming a polygonal path with right angles as shown in Figure 6.12. I use this to set the compass with $OA = 1$ and $\phi = \angle OAC = 3\pi/4$. This gives the spiral $\rho(\theta) = e^{\theta}$, so the first intersection between the spiral and the line OA (rotating clockwise) will be distant from O by the length e^{π} (Gelfond's constant), which, being transcendental [26], is not constructible with Euclid's tools.

6.2.3 Applications to two classical problems

The traditional excuse for playing with new geometric construction devices is so that I can solve at least some of the classic "insoluble problems." In keeping with this tradition, I will show that the logarithmic compass allows both the trisection of an angle and the duplication of the cube. I will specifically show how to perform the multiplication of an angle by the ratio of two segments.

The Euclidean plane has, of course, no absolute unit of distance, and the product of two lengths is not a length. Thus, multiplication *per se* is always replaced in Euclidean constructions by the ternary operation of finding $x \cdot (y/z)$.

According to Fig. 6.13, we can consider the following construction given an angle α and two lengths k, l:

1. Given arbitrary P_1 and P_2 at distance R, construct a spiral S_1 with rota-

25. Even for the perpendicular bisector the construction fails if $AX \perp AO$ or $AX \parallel AO$. In the first case, I am already able to draw circles. In the second, there is no possible construction, so I have to consider X not collinear to OA.

26. See for example Baker [1990].

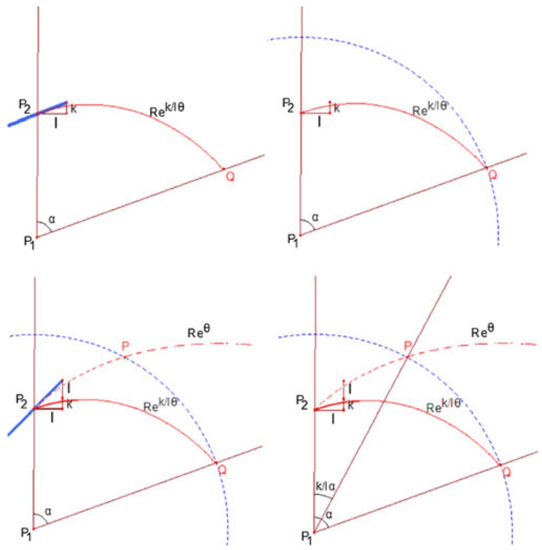


Figure 6.13: Multiplication of an angle by a ratio of segments with logarithmic compass.

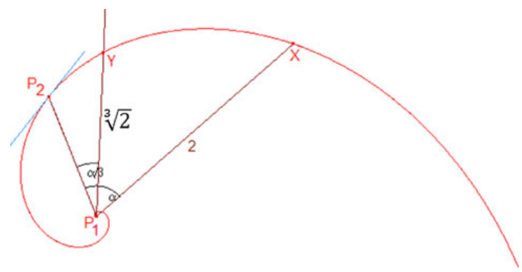


Figure 6.14: Construction of $\sqrt[3]{2}$ with the logarithmic compass.

tion center P_1, initial point P_2, and spiral coefficient k/l; call Q the point at $\theta = \alpha$.

2. Trace the circumference centered in P_1 and pointing at Q.

3. Construct a spiral S_2 with rotation center P_1, initial point P_2, and spiral coefficient 1; define P to be the intersection of the spiral with the circumference, so that $P_1P = P_1Q = R \cdot e^{\alpha k/l}$.

4. The angle $\angle P_2P_1P$ will be k/l times the angle α, because $P_1P = R \cdot e^{\alpha k/l}$ and the spiral S_2 has polar equation $\rho(\theta) = R \cdot e^{\theta}$.

Using this construction, the problem of the trisection (or n-section) of an angle becomes trivial. All I have to build is one segment three times the other's length and apply it to the multiplication of the desired angle. For general n, this is beyond the capability of cubic tools such as the marked straightedge [27].

This also gives us a construction for $\sqrt[3]{2}$ (Fig. 6.14). Given a spiral centered in P_1, starting in P_2 with coefficient $b \neq 0$, find the point X so that $P_1X = 2$.

27. See Gleason [1988].

Let the measure of the angle $\angle P_2P_1X$ be α; then I can construct $\alpha/3$. Let Y be the point of the spiral with angle $\alpha/3$; then $P_1Y = e^{b\alpha/3} = (e^{b\alpha})^{\frac{1}{3}} = \sqrt[3]{2}$.

Obviously, by the same method, I can (relative to a fixed length) construct any number of the form x^y where x, y are themselves constructible (and $x > 0$). Let K be the closure of $\mathbb{Q}$ under this process, along with addition, subtraction, multiplication, and division. [28]

Although quite large in some senses and comprising transcendental numbers such as $\sqrt{2}^{\sqrt{2}}$, K is still countable. So is the set L of all lengths constructible with the logarithmic compass. In each case, the argument is the same: Only a finite number of lengths can be constructed with $1, 2, 3, \ldots$ operations, and the nested union of finite sets must be countable.

6.2.4 Open questions

At this point, I frame some open questions. Some of these hinge on whether certain constructions are reversible [29].

- Are K and L the same set? (Probably not, but it seems difficult to prove.)
- Are e and π constructible with the logarithmic compass? Note that if I have the length π (as always, relative to a given unit length), I can always construct e. Let OA be a unit segment, and construct a spiral $\rho(\theta) = e^{\theta/\pi}$ with inclination $\frac{1}{\pi}$ through A. Then the ray with direction AO intersects the spiral again at a radius $e^{\pi/\pi} = e$. [30] Using a spiral of coefficient 1, we easily see that the constructions of the length e and a 1-radian angle are equivalent.
- I can, as shown previously, convert a ratio of lengths to a ratio of angles, constructing the angle $(l_1/l_2) \cdot \theta$. The inverse construction, of a length $(\theta_1/\theta_2) \cdot l$, is equivalent to constructing a spiral given its center O and two points A, B on it, or to constructing a length that is the logarithm of another length to a base e^α where α is a constructible or given angle.

  According to Fig. 6.15, let S be a spiral with coefficient 1, and P its intersection with the circle about O through B. Then the spiral about O with coefficient $\frac{\angle AOP}{\angle AOB}$ passes through B. This coefficient is the logarithm of $|OB|/|OA|$, to the base e^α where $\alpha = \angle AOB$. Conversely, given the spiral construction, let P, Q and B be on a circle about O, with $\angle QOP = \theta_1$, $\angle QOB = \theta_2$. If a spiral centered at O, through P, with coefficient 1 meets OQ at A, then the spiral centered at O through A and B has coefficient (θ_1/θ_2). I assume some construction for this spiral; the last step involves the use of the logarithmic compass to draw the desired curve. Let X (center), Y (wheel), and Z (pointer) be the points used to set the

28. If the classically constructible numbers are those that can be found with an (idealized) calculator with a square-root key, K is the set of numbers obtainable with a calculator that has an "x^y" key.

29. A typical example of a reversible construction in Euclid's planar geometry is the construction of the center of a given circle as the intersection of the perpendicular bisectors of two chords. This construction is fairly typical, in that the method is not a step-by-step reversal of the construction of the circle. Other constructions are not reversible; for example we can triple an angle but not trisect it.

30. I have no idea whether this is reversible—given e, can I construct π?

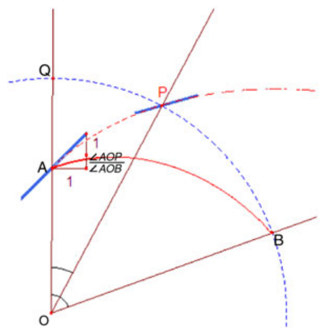


Figure 6.15: The logarithmic spiral with center O and passing through A, B has coefficient $\frac{\angle AOP}{\angle AOB}$.

inclination. Construct a right triangle with a side of length l perpendicular to XY at X, and the hypotenuse on YZ; its other side has length $(\theta_1/\theta_2)\cdot l$.

From a historical perspective [31], we can note that since Classical period the general angular section has been solved with the use of the quadratrix or the (Archimedean) spiral [32]. These curves also allowed solving the quadrature of the circle, while with the logarithmic compass (which is, however, an instrument, not a single curve) I still do not know whether it is possible to construct π. This problem becomes even more intriguing when I observe that my compass also solves the problem of the mean proportionals [33]. Therefore, if my compass constructed π, it could be considered a "universal device" to solve the classical problems. Otherwise, if the construction were impossible, it would have meant that area problems were "essentially different" with respect to angular sections and mean proportionals.

Even though I do not know whether with logarithmic compass alone I can solve the squaring of a circle, obviously I can do it if I adopt other differential machines. I have already seen a machine for the rectification of general angles (in Fig. 4.14, pag. 93), however it follows the description of a different machine for the same purpose.

A conceptually simple way to solve rectification of general angles is assembling two differential machines that I have already introduced—the one for the exponential curve and the logarithmic compass (with coefficient $b = 1$). Both these machines traces e^t, but the first one in Cartesian coordinates and the second one in polar ones. Consider a machine for the point (x, e^x) (so I can consider the point $(0, e^x)$ varying in function of x) and another for $\rho = e^\theta$. According to the construction 7 at pag. 57, I can constrain the point $(0, e^x)$ to have distance e^θ from the origin $(0, 0)$. As illustrated by Fig. 6.16 that means that, once set the initial conditions, I have the rectification of any angle (and vice-versa given

31. I have to thank Davide Crippa for helping me focus on such questions.

32. See Book IV, Prop. 35 of Pappus' *Mathematicae collectiones* (for example in the edition Pappus [1965]).

33. The problem of finding $x_1, x_2, \ldots, x_n$ *mean proportionals* between two values a and b means that $a : x_1 = x_1 : x_2 = \cdots = x_{n-1} : x_n = x_n : b$. This problem algebraically implies $x_1 = \sqrt[n+1]{a^n b}$, which is solvable with the logarithmic compass. For a survey on the tradition of geometrical problem solving regarding general angular section and mean proportionals (in the early modern period), see [Bos, 2001, pp. 70–79].

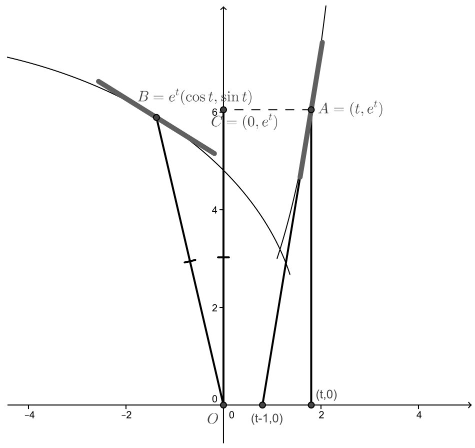


Figure 6.16: A machine using the logarithmic compass for the rectification of angles. A is constrained by a Cartesian exponential machines to have coordinates (x, e^x); B is constrained by a logarithmic compass to have coordinates $e^\theta(\cos\theta, \sin\theta)$. Considering C the projection of A on the ordinate, I can constrain $OB = OC$ with the construction 7 at pag. 57 (in the diagram it was not represented this part). That implies that we have a differential machine rectifying angles.

any segment, I can find the arc of unary circle with the same length).

6.3 Applications in math education

I think that one of the most important applications of my thesis could be to furnish the basis for a different approach to calculus. In fact, since the rigorous formalization of Cauchy, the main concept behind objects of calculus is the concept of limit. This makes the idea of infinite processes underlie classical calculus, with the related delicate epistemological problems in learning [34]. On the contrary, differential algebra allows manipulating differential polynomials in a deterministic way without any conceptual need of infinity. Differential algebra alone, however, misses the synthetic possibility of showing a solution of differential equations, and for its introduction at the moment it is necessary a preliminary knowledge of calculus (being its objects functions and their derivatives) [35]. In this perspective differential machines can introduce functions and derivatives without the need of calculus, and they can be considered as a finite synthetic method to solve problems of differential algebra with idealized machines. So, at a very first view, it appears to me that tractional motion could

34. Non-classical approaches instead of infinite processes require infinitesimals. I can cite Sullivan [1976] about teaching elementary calculus using nonstandard analysis. For a reflection on reforms in calculus, see Tall [1996].

35. Differential algebra manipulates differential polynomials, so it uses as variables smooth functions, that, to be precisely defined, need a preliminary knowledge of calculus. Furthermore, if one is interested in the evaluation of a function (e.g. defined as the solution of a Cauchy problem) given a certain input, differential algebra alone is not able to furnish an answer.

be useful to introduce calculus in math education [36]. Didactically, differential machines and differential algebra could help students in having a more concrete approach to these topics. Furthermore, the introduction of basic differential algebra could help in catching the continuity between algebraic and differential problems.

Concerning the didactical approaches to calculus and some possible reforms, of course the traditional approach with limits has to be introduced, but as an approach to such topics (essential for numerical methods and to overcome the limits of differential polynomial systems), not as the only possible approach, suggesting a rich multi-perspective. Obviously, it has to be empirically tested whether the approach with differential algebra and differential machines can be useful in math education.

6.3.1 Re-structuration of calculus

Even though for the moment without the support of any experimental result, I think it may be interesting to suggest a hypothetical re-structuration of the curricular introduction of calculus divided in the four following steps.

1. Preliminary introduction/revision of algebra with machines (both concrete and digital). These machines, intended as Kempe's linkages or algebraic machines, play a main role in diagrammatic constructions, and embody a lot of mathematical and technical knowledge. The use of machines similar to these in classroom activities has been dealt with in many works from around the world [37].

2. Introduction of some concrete differential machines to be manipulated and investigated by students in laboratorial activities. Even though students have not yet studied the mathematical counterpart, their considerations will be useful to pose the bases for the mathematical translation [38]. Some examples of possible pathways for such step will be furnished in next subsections.

3. Conversion of concrete machines in their digital counterparts. Even though less concrete [39], this step is important to give users the possibility not only of exploring but also of constructing differential machine (they would be too complicated to be physically realized). The problem of such step is that, at my knowledge, no didactical software for dynamic geometry is implementing "inverse tangent" conditions [40]. In this perspective, an optimal solution could be the development of a suitable package for the software

36. Historically Giovanni Poleni (1683–1761, university of Padua, Italy) and Ernesto Pascal (1865–1940, university of Naples, Italy) introduced tractional motion in math education, because they conceived tractional instruments for theirs students (they created mathematical laboratories in their universities). For further information see Tournès [2009].

37. For example Van Maanen [1992], Bartolini Bussi [2000], Isoda [2003], Sangaré [2003], Henderson and Taimina [2005].

38. As suggested in *Theory of the Semiotic Mediation* (cf. Bartolini Bussi and Mariotti [1999]), which focuses on the use of artifacts to transmit mathematical knowledge.

39. Computer simulations make it unreachable the physical mechanisms underlying the simulated behavior.

40. For my perspective, geometry is a tool not just for visualization but also for dynamic constructions. There are many software plotting solutions of differential equations, but not in a dynamic geometry perspective.

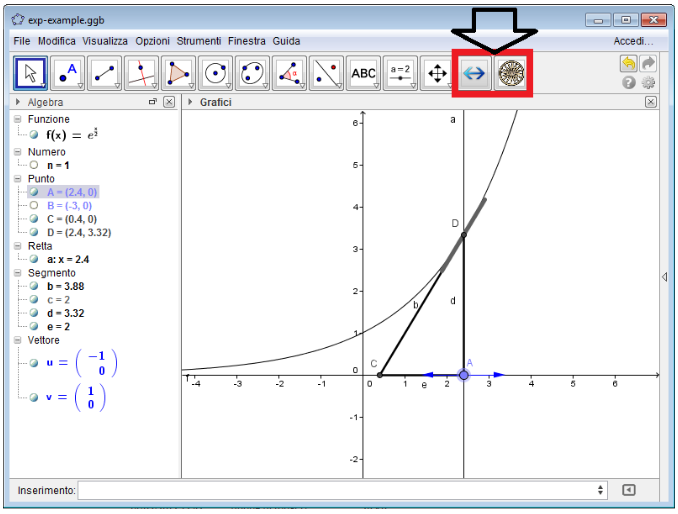


Figure 6.17: Possible interface for differential machines in dynamic geometry software. In the figure, the possible view of a windows of *GeoGebra* with the introduction, in the right-top, of the buttons for "cart" and "wheel" constraints. Out of the diagrammatic representation (blue double arrows for carts, thick grey segment for the wheel), the future project will be to implement dynamic objects so that geometric components move according to the differential constraints when the user moves some objects. Furthermore, being *GeoGebra* based on the multi-perspective (integration of geometry and algebra), it would also be interesting to implement in it differential algebra tools for the analytical part.

GeoGebra[41]. Indeed, the specific aim of *GeoGebra* is to integrate geometric constructions, algebra and calculus, so such package would introduce the possibility of managing inverse tangent problems both as wheel conditions (for tractional constructions) and as differential algebra equations. Then it will also be natural to pass to the numerical analysis perspective with the introduction of approximation methods. For a very preliminary example of the possible software interface see Fig. 6.17.

4. Theoretical unifying passage. Introduction to the mathematical formalism, first as symbolic manipulation (basic differential algebra) and later as classical/numerical analysis. That would realize the convergence of machines, algebra, and geometry beyond Cartesian boundaries. This part will require a strong effort considering the lack, at my knowledge, of any introduction of elementary differential algebra at school level, even though it may be organized as a more natural extension of polynomial algebra than classical analysis (specially thanks to the instrumental counterpart).

From this point of view, the aim of my thesis is to pave the way for future studies concerning a possible re-structuration of calculus. For the time being, let me suggest some laboratorial activities possibly related to the second step. In particular, after briefly proposing a suitable framework, I will look at a

41. See for example Hohenwarter and Preiner [2007]. The software can be downloaded from url `http://www.geogebra.org/`. It is free so there are no economic obstacles for its adoption at school. Furthermore, it is an open-source project based on Java platform, which means it will be easier to implement such package for all the operative systems. Of course, the package could be implemented in other proprietary software for dynamic geometry, as *Cabri Géomètre*.

first pathway dealing with the use of a very simple concrete artifact to shed light on the tangent concept. I will later go on to consider a more complex artifact, originally designed to permit a double use—a first explorative approach (from machine to mathematics), and a second constructive one (from formula to machine).

6.3.2 Artifacts in math education

I am interested in laboratorial activities based on the introduction of artifacts to develop mathematical meanings, processes and in general attitudes of arguing, conjecturing and demonstrating. Such artifacts can be considered as historical-cultural objects embodying mathematical knowledge [42], in my case instruments based on "tractional motion" are oriented to the devolution of the mathematical knowledge of direct and inverse tangent problems. Students very often learn just mechanically to "calculate" the tangent in analytical or geometric contexts, without having a unitary and conscious vision at a meta-cognitive level. Usually there is a strong break between the students' formal activity in the analytical register [43] and the conversion in the geometric one. My artifacts try to solve this cognitive gap.

Recent studies in laboratorial didactics are producing interesting inputs about students' participation in the construction of mathematical meanings through the adoption of problems, instruments, and teacher-student iterations. There are many researches on this topic, and they are set on different theoretical frameworks and inquiry methodologies. In particular, the didactical pathways proposed in next subsections are situated in the Vygotskian tradition and precisely in the theoretical construct of the "semiotic mediation" [44] as a complex process that belongs to a semantic structure including the content of mediation, the site in which they are set, the object mediated from the mediator and the "mediatee" [45]. In the pathways that I propose, the mediated object is the mathematical meaning of the tangent (for direct and inverse problems) with its analytical and kinematic properties, the means of semiotic mediation are the artifacts used in a laboratorial learning environment, the mediator is the teacher, and the receivers are the students. The aim is that students, starting from highly contextualizable signs strictly linked with the use of the artifact, reach the mathematical meaning "producing a particular chain of relations of

42. The concept of *artifact* can be understood very widely. From a historic-epistemological perspective, Wartofsky claims that "What constitutes a distinctively human form of action is the creation and use of artifacts, as tools, in the production of the means of existence and in the reproduction of the species. *Primary artifacts* are those directly used in this production; *secondary artifacts* are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artifacts are therefore representations of such modes of actions" [Wartofsky, 1979, p. 200]. There is also another class of artifacts (*tertiary artifacts*), "which can come to constitute a relatively autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or 'free' play or game activity. This is particularly true... when the relation to direct productive or communicative praxis is so weakened, that the formal structures of the representation are taken in their own right as primary, and are abstracted from their use in productive praxis" [Wartofsky, 1979, p. 208]. Mathematical theories are an example of tertiary artifacts. In fact, they organize mathematical models as secondary ones.

43. Semiotic representation registers, see Duval [1993].

44. Cf. Bartolini Bussi and Mariotti [1999], Bartolini Bussi and Mariotti [2008].

45. Cf. Hasan [2002].

signification, in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain" [46].

Concerning the role of teachers in such process, research discussions evince that it is not sufficient to propose an instrument suggesting its use in classes to mediate the underlying mathematical meanings. Instruments may in fact be used by the students without understanding the underlying mathematical knowledge. In these cases, the activity—even if motivating, intriguing, and pedagogically useful—can lose the devolution of the mathematical contents. Hence, a laboratorial activity generally needs the support of teachers [47]. Focusing on students' cognitive and meta-cognitive level, they have to pay attention at the didactic strategies centered on the use of an artifact and guide the evolution of signs and system of signs [48] toward what is recognizable as mathematics.

Vygotskij evinced the importance of a didactic use of artefacts [49] in semiotic mediation to introduce a new standpoint for a problem that the student probably would have otherwise solved with automatic reasoning. Thus, with a suitable pathway and an appropriate teacher orchestration, artefacts can mediate the knowledge embodied, fostering the internalization process. Specifically, the new situation obtained with artefacts will be as cognitively stimulating as it is deep and accurate the analysis of the tool and of its embodied knowledge. From this perspective I think tractional motion can constitute a rich treasure.

6.3.3 The tangentograph

The "tangentograph" [50] is a simple artifact designed to naturally determine the tangent to a curve. As a differential machine, it is simply a finite rod with a wheel, but the physical wheel has a handle that allows the user to keep the wheel contact point following a required trajectory on the plane. In particular, the direction of the wheel is given by the rod, and, while the wheel rotates, this direction is the tangent to the curve followed by the contact point of the wheel on the plane.

Tangentograph introduction is suggested exactly to make students focus on the relation between wheel direction and tangent. That would have some short-term consequences [51] and long-term ones about the systematic introduc-

46. Cf. [Walkerdine, 1988, p. 121].

47. Teachers have to "orchestrate the discussion." The term "orchestration" refers to the coordination of the different voices that are produced during classroom discussions, as explained in Bartolini Bussi [1998].

48. The term "sign" is used in a sense deeply inspired by Pierce, and consistent with the claims concerning the need of enlarging the notion of semiotic system (see Radford [2003], Arzarello [2006]) including different and more flexible kind of signs.

49. It is important to highlight the relationship between artifacts and knowledge at a cognitive level. Rabardel [1995] distinguishes between artifact (the material or symbolic object per se) and instrument (a mixed entity made up of both artifact-type components and *utilization schemes*). The mixed entity of instruments is born of both the subject and the object, and constitutes the instrument which has a functional value for the subject. Thus, in educational activities with artifacts, the aim is to make students achieve an "instrumental genesis." It is a complex cyclic process, and can be divided in *instrumentalization* (relative to the discovery of the different components of the artifact and the progressive recognition of its potential and limits) and *instrumentation* (relative to the begin and development of the use schemes which are progressively discovered/invented by the learner).

50. Introduced in Di Paola and Milici [2012].

51. The tangent concept, so important to introduce the geometric meaning of the derivative, is often misunderstood. A typical question evincing the difficult integration between the

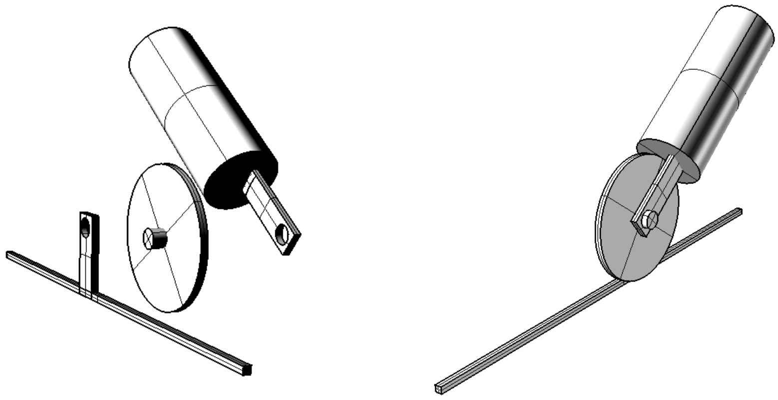


Figure 6.18: Left: the three components of the tangentograph; Right: the assembled artifact.

tion of tractional machines (concrete and simulated) in laboratorial activities and not only [52]. As evident in Fig. 6.18, the instrument is made up of three components—a handle, a wheel, and a rod. The wheel direction is given by the handle fork, and this direction will be clearly showed by the rod, that, for construction, will always lie on plane where the wheel rotates. The user has to grasp the handle and to move it in order to make the wheel contact point follow a curve (as we do with the pizza wheel cutter). The main idea is that it is "natural" to move a pizza wheel cutter along a path, suitably orienting the direction of the wheel according to the tangent to the path. So the simple introduction of the rod can help in evincing the idea of tangent line to a curve in a concrete and intuitive way.

I shortly describe the four steps for the hypothetical didactic activity. The pathway will proceed through a continuous cycle made up of stimulus questions, discussions, validations and institutionalizations of the knowledge. Students (indicatively 9–10th grades) are divided into small groups. For each step I suggest the goal and eventually some stimulus questions to direct the discussion.

1. *Exploration of the artifact.* Each group has to identify the components of the artifact, their possible movements, and to guess possible use schemes. Then the groups share their ideas, and the teacher tries to make connections between different notes.
2. *Artifact use.* Before the introduction of any mathematical content, the teacher suggests students to use the tangentograph, particularly in concrete settings, to find its potentials and limits. Possible stimulus questions

concept of tangent in classical geometry (the tangent line touches a curve only in one point) and modern one (tangent as limit of secant lines) is: What is the tangent to a straight line at a certain point? With regard to the different "concept images" about tangent, see Tall [1987].

52. Lakoff and Núñez [2000] suggests that "conceptual metaphors" play a deep role in developing mathematical ideas. Conceptual metaphors are seen as fundamental cognitive mechanisms which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference. Thus, linking the idea of the wheel to the one of tangent/derivative, could be a rich metaphor that students can use all over their studies and in their everyday life. In particular, this metaphor grounds the understanding of mathematical ideas in terms of everyday experience, so it is classified as a "grounding metaphor." It will be explored in subsection 7.2.1.

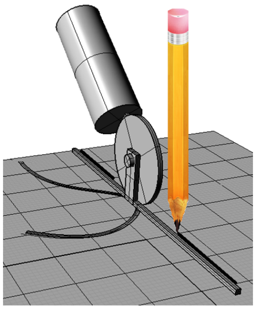


Figure 6.19: How to trace the tangent while the tangentograph walks along a curve.

can be: *Imagine you have to cut a square piece by a pizza*[53] *without lifting the artifact: Can you discuss what happens in the critical points of the square? What happens to the artifact rod in the square angles? In what other shapes can we find similar behaviors of the artifact?*

By these questions the teacher, without explicating it, will focus the attention to a naïf idea of singular points.

3. *Devolution of mathematical contents.* The teacher proposes problems of increasing difficulty and associated with suitable stimulus questions referred to tangents and singularity of curves. The topics of such problems can be:

   (a) Focus on the rod direction. Groups have to guess the artifact behavior while goes along some drawn curves (circles, straight lines, geometric shapes more or less familiar to students). The stimulus questions will be related to the rod direction and to the possibility of tracing it with a pencil (see Fig. 6.19).

   By these questions the teacher, without explicating it, focuses on the idea of the tangent to a curve in a point and on its variation when moving the tangency point.

   (b) Formal introduction of the tangent. The teacher adds some marked points to the drawn curves. Groups have to guess the tangent to curves in the marked points (without the artifact). Later it is suggested the use of the artifact to compare its rod direction with the hypothetical tangent. Finally students have to trace the tangent as in the previous point, concretely re-interpreting the previous conjecture. The teacher can also informally introduce the idea of tangent as limit secant.

   (c) Singular points. After a class discussion, learners recognize the difficulties in defining and tracing the tangent in some singular points. The teacher, through a continuous artifact-oriented laboratorial activity, evinces the parallelism between the concrete activity with the artifact on singular points and the related property of the tangent (specially focusing on the role of cusps, that, even though singular points, do not introduce problems having a single tangent).

53. This example has been chosen particularly for the typical *use scheme* of the artifact, so similar to a pizza cutter.

4. *Metacognition and generalization.* To shed light on the correlation between the structural element and the mathematical content, it would be useful to make pupils reflect on the correspondence between concrete components of the tangentograph and the idea of tangent. Some stimulus questions can be: *What happens if the wheel had not been designed to roll but to remain fixed to the handle (like a circular blade)? What if we had not only one but also two wheels on a line (like rollerblades)? And if the two wheels were parallel (like in a chariot)? Would it be different if the two wheels have a common axis of rotation or they can roll with different speeds? How can we differently build up a machine to find the tangent of a curve?*
   Regarding this last complex question, it will be interesting to compare students' solution with the ones given by the fathers of tractional motion.

The previous discussion of the tangentograph as a geometrical-mechanical artifact to manage the tangent concept suggest a more general reflection on the potentials of differential machines and their use in suitable didactic pathways. In fact, the steps I considered can be generalized beyond the specific device and the related mathematical meanings. Furthermore, this kind of artefacts can be useful not only to introduce and develop new mathematical concepts through a tractional motion approach, but also to break the "automatic reasoning" in students already knowing the related mathematical contents (e.g. in last years of high-school).

With regard to the tangentograph, it is just one of the most simple tractional machines. In particular, in the teaching experiment I focused just on the direct tangent problem (given a curve, find its tangent properties). Without changing the artifact, we could also introduce the inverse tangent problem asking students to apply it with a different use scheme. The stimulus question can be: *What does the wheel contact point describe if we move an extreme of the rod on a straight line? Which properties will this curve satisfy?*

Thus, students will construct a tractrix, opening new perspectives on more advanced mathematical knowledge, beyond algebraic boundaries. With regard to this, the teacher can introduce the historical origin of such curve [54] and the related foundational problems of tractional constructions.

6.3.4 A new concrete differential machine

What I am going to observe in this subsection is the possible use of a more complex artifact, one that was ideated and designed by me and realized in collaboration with Benedetto Di Paola [55]. The machine is illustrated in detail in Fig. 6.20. It is mainly made up of a wooden board over which a wooden frame can translate. In particular, we can attach a sheet of paper to the board with some tape. According to the assembling, the motion of the frame will determine the motion of the wheel for the same reasons of all tractional machines [56]. If one presses a marker on the external rubber tire of the wheel, the wheel will

54. Cf. subsection 2.3.1, page 18.
55. This artifact was presented in a workshop at the *64th Conference of the International Commission for Study and Improvement of Mathematics Education* (cf. Milici and Di Paola [2012]) and mathematically explored in Salvi and Milici [2013].
56. According to E. Pascal's classification, this machine can be considered as a simple Cartesian integraph with straight ruler and straight guide.

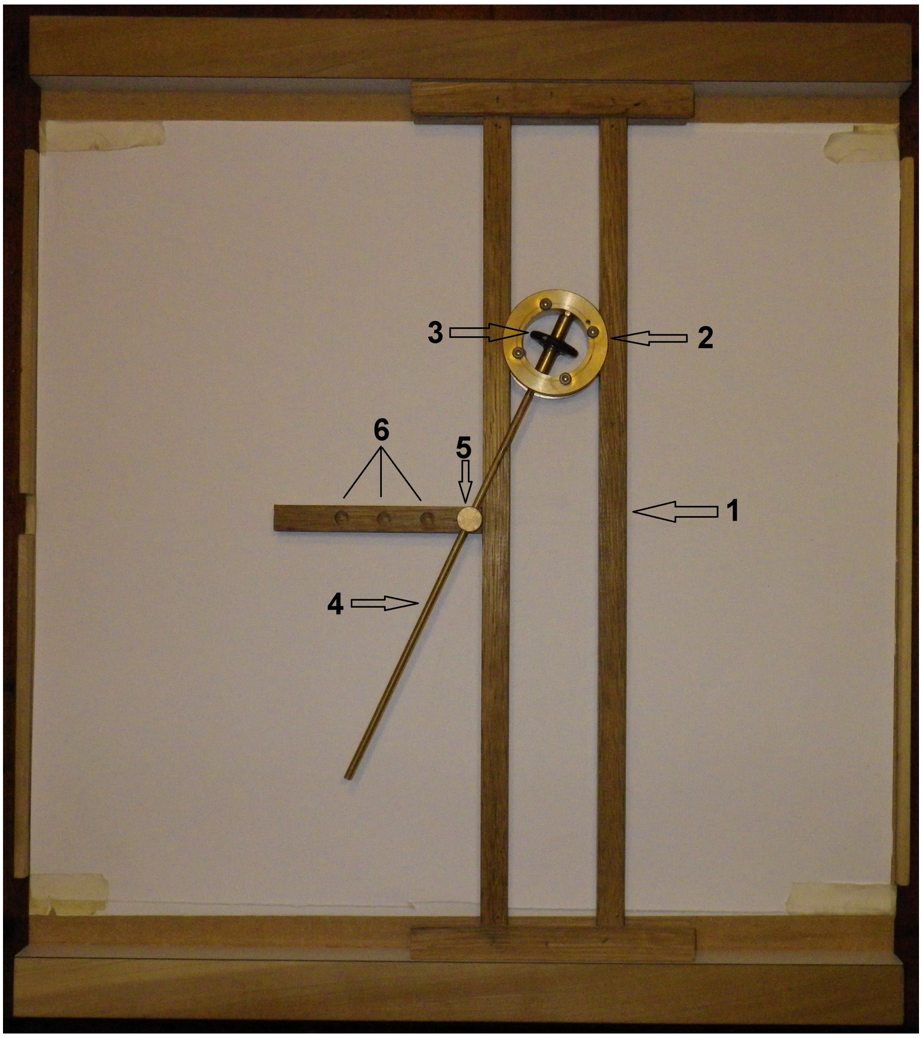


Figure 6.20: Legend: (1) wooden frame; (2) brass cylinder; (3) wheel (in brass with rubber tire); (4) brass rod; (5) brass peg (with a hole); (6) blind holes.
This tractional machine is made up of a wooden board (to be used in a horizontal position) covered with a paper sheet (attached with tape). Over this board the frame (1) can translate. According to the orientation of the figure, the left-right motion of (1) forces the motion of the inner brass cylinder (2), that however is free to move up-and-down. Inside (2) there is the wheel (3) that rotates over the paper on the board. Furthermore, on (2) there are two threaded holes: If the rod (4) is screwed in the first hole then the rod direction is the one of the wheel, if in the second the rod direction is perpendicular to the one of the wheel (as it appears in the picture). The direction of (4) is also given by the passage through the hole of the peg (5): This peg can be inserted in one of the four blind holes on the frame (1). In the picture, the peg (5) is set in the rightmost hole, the other blind holes being indicated by (6).

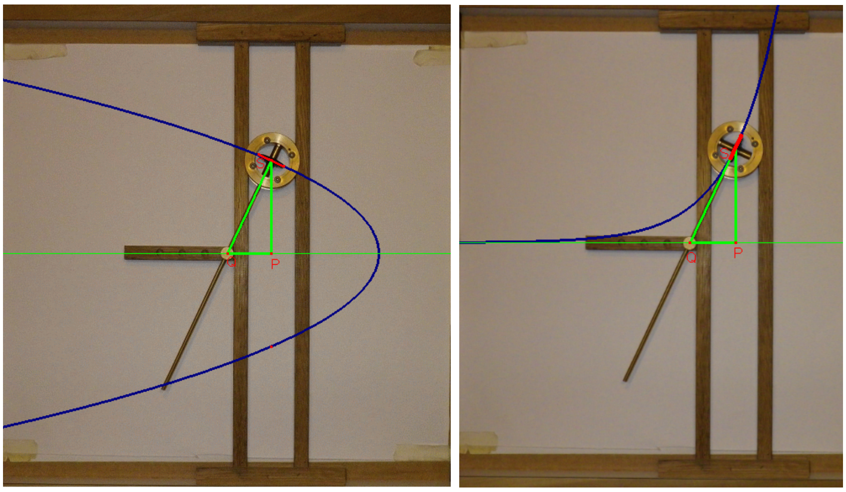


Figure 6.21: The artifact generating a parabola (left) and an exponential curve (right), according to the inclination of the wheel in relation to the brass rod. Note that I have not changed the position of the peg with a hole (indicated with (5) in Fig. 6.20) with respect to the other possible blind holes. However, the change of the hole for the peg does not modify the kind of traced curve.

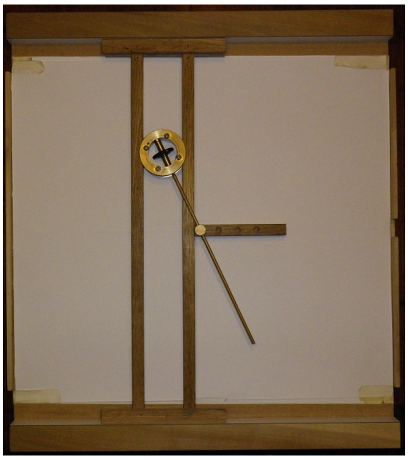
trace its trajectory on the sheet [57].

As shown in Fig. 6.21, based on the way we assemble its components, this artifact generates two different curves, one algebraic and one transcendental [58]. I realized digital simulations of these machines that are available on-line at the addresses `http://tube.geogebra.org/student/mzrMyFGdd` and `http://tube.geogebra.org/student/mLXugImiH`. These different uses of the artifact permit not only an explorative approach (from machine to mathematics), but also a concretely constructive one (from formula to machine). In particular, being interested in developing a pathway involving the field of infinitesimal calculus, instead of focusing on curves, I want to focus on the generated functions. Thus, I propose the exploration of a machine embodying the square root and the construction of a machine for the exponential curve (both machines can be obtained by assembling the same components in a different manner).

I chose these functions because, though very different from the usual didactic perspective, they can be markedly similar in their interpretation as differential

57. In a new upgrade of the machine (not represented in the pictures), I added a little sponge that can be easily attached and removed thanks to a magnet. When removed, this sponge can be filled of ink so that, when attached on the machine, it automatically dispenses ink over the rubber tire of the wheel (as in ballpoint pens). That simplify the operation of tracing the trajectory of the wheel.

58. In particular, the fact that two functions, one transcendent and the other algebraic, can be constructed through similar devices of equal complexity is an epistemological point, in contrast with the Cartesian dualism between the different legitimization of geometrical (algebraic) and mechanical (transcendental) curves. Concerning this, I may mention the letter that Poleni had written to Hermann in September 1728 (published in Poleni [1729]), in which the author wondered about the nature of tractional curves. With a simple modification to the exponential tractional machine (just changing an angle, which is essentially the same thing I did, as shown in Fig. 6.21), the author realized that tractional machines draw curves defined by differential equations in a uniform way, regardless of their algebraic or transcendental nature.



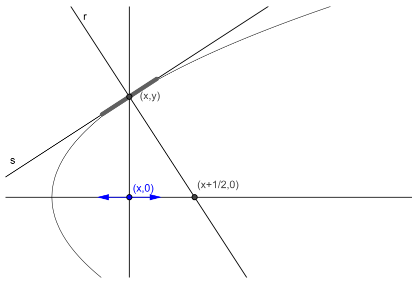


Figure 6.22: The actual artifact (compared to the left picture of Fig. 6.21, the machine has been rotated 180°) and the related differential machine for $f(x) = \sqrt{x}$ (we have already seen it in Fig. 4.15, page 97). To work, it must be shifted horizontally along the "basis cathetus" (the segment with extremes $(x, 0)$ and $(x + \frac{1}{2}, 0)$). The wheel at $(x, f(x))$ implements the condition that the tangent at $(x, f(x))$ must be perpendicular to the hypotenuse.

machines. The real exponential function has been discussed under many points of view because it is the solution of a very simple differential equation and can be constructed with simple tractional machines (e.g. remind Fig. 2.3 at page 20); on the other side the choice of the square root function has been made because of its nature, which, though simple, reveals many significant aspects that can be highlighted in the geometrical/mechanical interpretation.

Although the function $f(x) = \sqrt{x}$ is algebraic, I do not interpret it as the inverse of x^2. Specifically, the machine (Fig. 6.22) solves the differential equation $f'(x) = \frac{1}{2f(x)}$ with the initial condition $f(1) = 1$, whose single solution for positive values is the square root [59].

In particular, the following laboratorial activity [60] focuses on the mathematical concepts of tangent (geometric and analytical approach with the derivative), continuity, real function asymptotic behaviors, and differential equations. After students are gathered in small groups, the activity can be organized as follows:

1. *role of the wheel.* To begin with, students have to focus on the role of the wheel, on the avoidance of "lateral movement" in the contact point, and on the relation with the tangent (the main aim of the activity with the tangentograph).

2. *from concrete artefacts to formal language.* Students explore the artifact assembled in a predetermined shape (the machine for $f(x) = \sqrt{x}$) and build some related use schemes. In particular, they are oriented to focus on the following—identification of the mechanical components and constraints, transposition on geometrical constraints, exploration of the behavior (also asymptotic) without analytical tools, setting in analytical

59. This definition is solved by the square root only for the real values; it does not apply to the complex extension.
60. For students in last years of high-school or first ones of university.

geometry, and approach to the formula of the curve using calculus. The use of a dynamic geometry software can be added to help in highlighting the components and to give another perspective. Furthermore, the changes of the traced curve when changing the peg position [61] can be studied as a parametric problem.

3. *from formal language to concrete artefacts.* To facilitate a meta-reflection on the previous step, groups are proposed to theoretically design a machine to trace the exponential curve (the use of software can help students to test conjectures and to find useful properties). Then they are suggested to think at the changes that the artifact needs to trace the exponential curve, up to the concrete decomposition and reassembling of the machine.

In the previous point 2, the idea of "exploration of the behavior (also asymptotic) without analytical tools" was introduced. In contrast to the other steps, this one is not so standard. To get an idea, see the Table 6.1 (for the asymptotic observe the last two rows).

With regard to the "setting in analytical geometry," based on the Cartesian coordinate system seen in the right of Fig. 6.22, we can consider $x = f(y)$ and, from the geometric constraints, we get $f'(y) = 2y$. Thus, we can analytically solve it with an integral.

To sum up, I want to remind that the ideas of this section are just preliminary attempts. Before any real experimentation, the whole setting has to be explored in greater detail, refined, and more properly designed.

61. According to the notation of Fig. 6.20, the changing of the position of the peg with a hole (5) means that it is put in one of the other blind holes (6).

Table 6.1: Translation from the analytical to the geometrical/mechanical semiotic register for the exploration of the machine in Fig. 6.22 ($f(x) = \sqrt{x}$). These registers are currently not autonomous: I had to use some analytical properties in the geometrical/mechanical register (properties of continuous and monotonic functions).

Analytical Register	**Geometrical/Mechanical Register**
Domain: $\mathbb{R}^+$	Here there is a great difference in comparison to the analytical register. If not physically dragged on the plane, the artifact *does not allow us to evaluate the domain* (because the abscissa values are used in a dynamic way). On the other hand, it is possible to realize how the artifact becomes stuck when $f(x) = 0$ (the wheel becomes perpendicular to the left-right motion).
$f(x) \geq 0$	Considering that the artifact becomes stuck when $f(x) = 0$ and that $f(1) = 1$, the function is always non-negative as a result of its continuity.
$f'(x) > 0$	The tangent has to be perpendicular to the *hypotenuse*, so the derivative is positive when f is not negative (in the whole domain).
$\lim_{x\to+\infty} f(x) = +\infty$	Since it is increasing, f cannot oscillate. By *reductio ad absurdum*, suppose that f converges, so f' tends to 0. Mechanically, this implies that the *hypotenuse* tends to be parallel to the ordinates (even if this can never physically happen), and this occurs only if f tends to infinity. Hence, the absurdum (f had to converge).
$\lim_{x\to+\infty} f'(x) = 0$	Once the divergence has been observed, while f tends to infinity the *hypotenuse* tends to be parallel to the ordinates, so the tangent tends to be parallel to the abscissa.

Chapter 7

Conclusions and future perspectives

In this thesis, I discussed some of the most important approaches to geometry—synthetic (classical machines), analytic (algebraic machines), and differential (differential machines). For each of these approaches, I gave the class of allowed mechanical components, the counterpart of symbolic computation, and the definition of spaces as zero sets. As I will explore in section 7.1, the balance between machines, algebra, and geometry is central to a multi-perspectival view of the same object. From a historical perspective, this introduction of tractional motion can be considered as a "conservative extension" of the program of Descartes (finite analysis and synthesis with diagrammatic constructions). However, while algebraic objects statically limited Descartes's exact knowledge, if one looks for the balance between machines, algebra, and geometry, even the limit of differential algebraic objects appears temporary, waiting for further extensions.

Before concluding the chapter and the thesis with a summary of open problems and future perspectives, let me add some reflections about my setting and, more generally, about calculus. First, I will consider the point of view of the cognitive science of mathematics. From this perspective, mainly introduced by Lakoff and Núñez [2000], mathematical ideas are analyzed from the background of embodied cognition. Since Newton and Leibniz, the core concept of calculus is the constructive role of the methods involving the infinite. On the contrary, the proposed mechanical setting and the differential algebra counterpart suggest that it is possible to consider calculus (at least the part dealing with differential polynomials) without the need of infinity, but with the metaphor [1] "the wheel direction is the tangent." As I will show, this metaphor, being very concrete, can be considered a "grounding" one.

Machines can be considered not only as idealized instruments, but also as computing tools. The introduction of infinite approximations in construction is possible if we consider recursive methods. From a computational standpoint, recursion is the main tool of "digital" (symbolic) computing; differential machines

1. "Metaphor" has to be considered as the "conceptual metaphor" of Lakoff and Núñez [2000]. It will be explored in subsection 7.2.1.

can avoid the infinite because they are not based on recursion, being "analog" machines. The perspective of analog computation will be shortly explored.

However, the role of the infinity as a concept for calculus is something that is not lying inside mathematics but beyond. In fact, from the formalist perspective, mathematics deals just with the finite manipulation of finite formulas according to suitable deductive rules. From this perspective problems like Bos's exactness of constructions are clearly meta-mathematical: A mathematical theory is defined by arbitrary rules, so correctness of mathematics is simply given by suitable applications of deductive rules. However, I think that an algorithmic approach can be useful in characterizing the problem of exactness. In particular, I will propose the (still to be deepened) idea that exactness can be reconsidered within the framework of an algorithmic theory in relation to the solution of the "equality test." Therefore, the existence of a method to check equality between objects with different representations can be a tool to distinguish between exact and approximating settings.

7.1 Balance between machines, algebra and geometry

Machines, algebra, and geometry are the unavoidable components of my setting. I am going to shed light on their connection with Descartes's geometrical method, observing how my setting opens the possibility of a new dualism beyond algebraic/transcendental. However, the main difference from Descartes is that I do not consider differential machines as a static limit for geometric intuition, but as a step toward new future approaches.

7.1.1 A conservative extension of Descartes's canon

In subsection 2.2.3 (pag. 13) I introduced how Descartes's geometrical method was made up of two components—the analytical part (algebra) and the synthetic one (diagrammatic constructions). Furthermore, this theory, according to Panza [2011] and as observed in subsection 2.2.4 (pag. 15), can be considered as a conservative extension of Euclid's geometry. The richness of Cartesian setting depends on the correspondence between objects of the analytical and the synthetic part, i.e. equations and curves. From this perspective, the role of suitable ideal machines was central. Their role is somehow necessary to pass from the complex physical behavior to the simple one of geometry: If we consider the manipulation of physical artifacts as general analog computation, diagrammatic constructions with allowed tools can be seen as its restriction to an easily imaginable part. Informally, the will of specifying such machines guided me in the introduction of algebraic machines, and I tried to highlight their relation with Euclid's tools (viewed as classical machines).

The balance between machines, algebra, and geometry as suggested by Descartes was historically broken by the increase in importance of the analytical part with respect to geometric constructions. In particular, infinitesimal analysis also introduced infinitary tools in the analytical part such as series or infinitesimal elements. However, even though with some centuries of delay, I can consider the finite approach to calculus objects of differential algebra as a legitimate descendant of polynomial algebra. Contrarily, the synthetic part can be managed with the proposed differential machines, which, as a well-defined model for tractional constructions, can be considered as the extension of algebraic machines. The surprising result is that these heirs of Descartes's analytical and synthetic tools are still in balance, being the behavior of differential machines exactly the space of solutions definable with differential algebra.

Hence, I can consider differential machines, differential algebra, and differential manifolds as a conservative extension of Descartes's canon. This extension defines a closed class of objects based on a suitable interpretation of tangent problems. However, while Descartes's canon was justified for its limits by reasoning about the "geometric intuition," my setting is much more weak because I have no precise justification for the introduction of the wheel (necessary to extend algebraic machines). Therefore, like Leibniz, I am also adopting a utilitarian point of view, but with a mechanical extension instead of the introduction of infinite.

7.1.2 A new dualism beyond polynomial algebra

In this thesis, I have been able to define the behaviors of differential machines that have been introduced to formalize tractional constructions in a modern way. To my knowledge, it is the first clear definition of the limits of tractional motion. Such limits permit a distinction between objects that are constructible with differential machines and others that are not. To define the behavior of such machines, I used manifolds of functions: If Descartes's setting defined a dualism between algebraic and transcendental curves, my setting facilitates a new dualism between functions. As introduced in subsection 4.4.2, the obtainable functions are the "differential algebraic" ones (shortly: D.A.), i.e. solutions of algebraic differential equations [2]. As already mentioned, all elementary functions are D.A., and even most of the transcendental functions that we find in most of the analysis handbooks. Historically, the first example of non-D.A. function was the Γ of Euler, as proven in Hölder [1886]. As an example of function that is not D.A., it is interesting to look at this function more closely.

Γ function was introduced as an extension of the factorial, with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer, $\Gamma(n) = (n-1)!$. There are infinitely many continuous extensions of the factorial to non-integers, but Γ function is the most useful solution in practice, being analytical (except at the non-positive integers). In particular, this function is the unique satisfying the recurrence relation

$$\begin{aligned} f(1) &= 1 \\ f(x+1) &= xf(x), \end{aligned}$$

(for any $x \in \mathbb{R}, x > 0$) together with the assumption that f be logarithmically

2. An algebraic differential equations is a differential equation in the form $P(t, y, y', \dots, y^{(n)})$, where P is a nontrivial polynomial in $n+2$ variables, t is the independent variable, and y is the dependent one.

Table 7.1: Categorization of functions in one variable (taken from [Shannon, 1941, p. 501]).

Transcendental		**Algebraic**	
Trascendentally trascendental	*Algebraic-trascendental*	*Irrational algebraic*	*Rational*
Euler Γ, Riemann ζ	$e^x, \log(x)$; trigonometric, hyperbolic and inverses; Bessel, elliptic and probability functions	x^m (m a rational fraction); solutions of algebraic equations in terms of a parameter	polynomials, quotients of polynomials

convex [3], and its formula (defined for $x > 0$) is

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

So the class of functions not D.A. is non-empty: According to Moore [1896] they are named "transcendentally transcendental" functions (shortly: T.T.) [4]. Carmichael [1913] furnished an unlimited number of such functions, while for a more recent survey see Rubel [1989]. As discussed in the subsection 7.2.2, such dualism of functions is studied particularly because of its connection with analog computing.

About functions in one variable, I give a finer distinction beyond the algebraic and transcendental dualism. Of course algebraic functions are also D.A., so, calling "algebraic-transcendental" the functions that are D.A. but not algebraic, we can divide functions in the cases of Table 7.1 (with some examples).

To conclude the introduction of this new dualism, I have to consider that I have implicitly considered the functions satisfying some properties about the continuity of their derivatives, in particular I considered them locally smooth (i.e. of class C^{∞} in a certain domain). In general, to be a solution of an algebraic differential equation of order n, I have to assume that the function has to be derivable at least n times in a certain range. That means that, for example, Dirichlet function [5], nowhere being a continuous function, it is far from a solution of a differential equation. However, even with regard to continuous functions, it is possible that one cannot find any range in which such functions are smooth

3. By definition, a function f is logarithmically convex if and only if the composition of the logarithmic function with f is a convex function. About Γ function, *Bohr-Mollerup theorem* asserts that it is the unique solution of the recurrence relation for positive, real inputs. It can be extended by analytical continuation to all complex numbers except the non-positive integers (where the function has simple poles). To know further about this, see e.g. Artin [1964].

4. Other authors call D.A. functions "hypo-transcendental" and T.T. functions "hyper-transcendental."

5. Dirichlet function equals 1 if x is a rational number and 0 if x is not rational. It can be constructed as the double pointwise limit of a sequence of continuous functions:

$$f(x) = \lim_{k\to\infty} \left(\lim_{j\to\infty} (\cos(k!\pi x))^{2j} \right) \quad \text{for integer } j \text{ and } k.$$

enough. For example, I can consider Weierstrass function [6] that is continuous everywhere but differentiable nowhere. Thus, it is not always possible to restrict a continuous function in an interval over which it is smooth. That means that the basic condition of differential algebra is not always satisfied. Even though I will not explore them in detail, I have to cite that these cases are treated in Rubel [1983] in order to answer to the question "What is a solution of an algebraic differential equation?" For functions that are only C^n and not C^∞, most of the results (as Ritt-Raudenbush theorem, introduced at page 83) fail.

7.1.3 Beyond differential machines

How can I overcome the boundaries of differential machines still using finite tools in analysis and idealized machines in synthesis? Being unable to answer this question, I will focus on it in this subsection. From the analytical point of view, the answer is simple: I can still use differential algebra for differential polynomials, this time not only for ordinary but also with partial derivatives. However, it is not so easy to find a suitable class of idealized machines for such problems, even not requiring such machines being intuitively simple [7]. As I will show in the subsection 7.2.2, it is generally thought that machines working on continuous entities (analog computers) can construct only D.A. functions (as my differential machines).

In my search for an extension of differential machines, I think that a central role may be played by Euler Γ function. As I am going to examine, this function can play for D.A. functions the same role as the exponential curve played for algebraic curves.

Algebraic curves are defined as the zero set of polynomials, where a polynomial is an expression that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents. One can ask to relax the constraint of considering only non-negative integer exponents [8]: From this perspective, the exponential curve solves the problem of generic exponent. From the point of view of constructions, tractional motion justified the exponential curve with the introduction of loads subject to friction or with blades or wheels. Therefore, even though the extension from polynomials to formulas with any exponent is not enough to define analytically all the functions constructible with tractional motion, the construction of the exponential was important to focus on the role of the wheel for the expansion into the synthetic aspect.

D.A. functions are solutions of differential polynomials. Differential polynomials are polynomials in the variables and their derivatives, but these derivatives have to be of non-negative integer order. Negative integer-order derivatives can be considered integrals. However, what does it mean to consider derivatives of non-integer order? This question is older than three centuries and is at the core of "fractional calculus" [9]. Considering the Cauchy formula for repeated

6. Weierstrass function is defined as $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$ (cf. [Weierstrass, 1886, p. 97]).

7. An aim of differential machines was to furnish an intuitive geometric justification for the construction of solutions for differential equations.

8. For example, we may be interested in considering monomials in $x^{\frac{3}{2}}$, or in $x^{\sqrt{5}}$.

9. Quoting [Ross, 1977, p. 76]:

> "Fractional calculus has its origin in the question of the extension of meaning. A

integration [10]

$$D^{-n}f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt,$$

we can generalize $n!$ as arbitrary values since $n! = \Gamma(n+1)$, thus obtaining [11]

$$D^{-v}f(x) = \frac{1}{\Gamma(v)}\int_a^x (x-t)^{v-1} f(t)dt.$$

This formula links Γ function and fractional calculus. The construction of Γ with idealized machines can be important, because at the moment it is still missing a widely accepted geometric interpretation of fractional calculus [12]. Hence, from a historical/philosophical perspective, fractional calculus is now looking for a constructive-synthetic geometrical legitimation, as it happened in early modern period with transcendental curves. I hope that differential machines can constitute a solid step over which new extensions may come.

7.2 Foundational reflections on calculus

From a foundational perspective, the aim of differential machines is to evince that somehow infinites and infinitesimals are not strictly necessary to treat (part of) infinitesimal calculus. This aim has to be considered from different perspectives. First, from a cognitive perspective, I will suggest as underlying metaphor not the idea of unlimited processes but something about the role of wheel. Second, I will consider my machines as analog machines, evincing some differences with the digital counterpart. Third, I will propose a definition of "exactness," not as a metamathematical problem but as something determinable in an algorithmic perspective.

7.2.1 Cognitive approach

Since its appearance, calculus has posed foundational problems for its use of infinitesimal (from a geometric or algebraic standpoint). I am not interested in distinguishing between approaches based on infinites or infinitesimal [13], I just

well-known example is the extension of meaning of factorials of positive integers to factorials of complex numbers. The original question that led to the name fractional calculus was: Can the meaning of a derivative of integer order $d^n y/dx^n$ be extended to have meaning when n is a fraction? Later the question became: Can n be any number—fractional, irrational, or complex?"

The first appearance of such question was in a letter of Leibniz (cf. Leibniz [1849]), and later got many mathematicians interested in it: Euler (1730), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), H. Laurent (1884), Hadamard (1892), L. Schwartz (1945). For more precise historical references see Dugowson [1994],Ross [1975], Ross [1977].

Nowadays, fractional calculus finds use in many fields of science and engineering, including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, electromagnetic theory, and probability.

10. The Cauchy formula for repeated integration allows one to compress n antidifferentiations of a function into a single integral.

11. There is not a unique definition of fractional integral, but the following (usually called *Riemann-Liouville fractional integral*) is probably the most used version. I am giving it just to give a superficial idea. For clarifications and further reading, see Miller and Ross [1993].

12. For some attempts, see Adda [1997], Podlubny [2002], Tavassoli et al. [2013], Herrmann [2014].

13. Cf. Lolli [2012].

want to focus on the fact that, in one way or another, infinite was considered forming the basis of calculus. Nevertheless, while the role of infinity in modern mathematics is fundamental, historically (since Zeno), its use has implied paradoxes, up to foundational crisis of late 19th and early 20th centuries. To deal with it, Hilbert's formalism [14] suggested considering "signs" (finite and precise) as the only real mathematical objects. Thus, other mathematical objects (as infinite sets) are just ideal objects, but they can be accepted when they do not cause contradictions regarding finite real mathematical objects. Therefore, from a formalist perspective, the entire mathematical apparatus (so also calculus) trivially does not require infinite objects, because everything is already expressible with finite signs. On the contrary, the problem of defining calculus without the need of infinity is interesting if we want to analyze the intuitive ideas underlying mathematical concepts from a cognitive perspective, as suggested by the "cognitive science of mathematics" (introduced in Lakoff and Núñez [2000]).

Lakoff and Núñez assert that mathematics results from the human cognitive apparatus and must therefore be understood in cognitive terms. Looking for mathematical ideas in terms of the human experiences, metaphors, generalizations, and other cognitive mechanisms give rise to them. Starting from some human innate abilities, mathematics goes far beyond them mainly due to a large number of "metaphorical constructions." In fact, for the most part, human beings conceptualize abstract concepts in concrete terms, using precise inferential structures and modes of reasoning grounded in the sensory motor system. The cognitive mechanism by which the abstract is comprehended in terms of the concrete is called "conceptual metaphor" [15]. Mathematical thought also makes use of conceptual metaphor. In particular, Lakoff and Núñez distinguish between three important types of conceptual metaphors:

- *Grounding metaphors*, which ground our understanding of mathematical ideas in terms of everyday experience. In these cases, the target domain of the metaphor is mathematical, but the source domain lies outside of mathematics.
- *Redefinitional metaphors*, which are metaphors that impose a technical understanding replacing ordinary concepts (such as the conceptual metaphor used by Georg Cantor to reconceptualize the notions of "more than" and "as many as" for infinite sets).
- *Linking metaphors*, which are metaphors within mathematics itself that allow us to conceptualize one mathematical domain in terms of another mathematical domain. In these cases, both domains of the mapping are mathematical.

I will try to show how differential machines can be useful for all these three kind of metaphors.

According to the current interpretation, the main concept behind classical analysis is the idea of "limit." From a cognitive perspective, it may be seen as the idealization of an unlimited process of approaching without reaching. I am

14. Cf. Zach [2015].
15. Given a source domain (the conceptual domain from which we draw metaphorical expressions) and a target one (the conceptual domain that we try to understand), a metaphor is a systematic set of correspondences (i.e. a mapping) between constituent elements of the source and the target domain.

not interested in delving into such delicate topics [16], all I want to note is that differential machines can evince how it is possible an approach to differential objects without the infinity but just with the idealization of something like the wheel. More precisely, I can say that the main metaphor behind my approach to calculus is:

> *the direction of the wheel is the tangent to the curve traced by the wheel contact point.*

With the distinctions of metaphors just introduced, I can say that this wheel metaphor is a grounding one, so calculus may be based on something very close to our everyday experience. This improved concreteness, in my opinion, deserves to be explored not only for foundational reasons (what is the role of infinite in infinitesimal analysis?), but also for educational fallouts (as suggested in section 6.3).

With regard to the balance between machines, algebra, and geometry observed in section 7.1, I can interpret its cognitive richness as linking metaphors: Adding metaphors in different domains will enrich the vision of the one who is approaching, learning, or exploring that concept. Finally, regarding redefinitional metaphors, in subsection 7.2.3 I will propose an attempt to reinterpret the concept of exactness in an algorithmic setting.

7.2.2 Computational approach

Prior to 19th century efforts at "arithmetization of analysis" [17], calculus rested uneasily on two pillars: the discrete side on arithmetic, the continuous side on geometry. Considering the geometric foundations of calculus not solid enough for rigorous works, the arithmetization research program produced a foundation starting from the natural numbers. Of course, there was a price to pay in extending discrete tools to the continuous side—the role of infinity. If from a foundational standpoint the infinity became widely accepted, there is a field in which infinite tools are not accessible—computation. From an operative perspective, arithmetic is based on signs manipulations and geometry on diagrammatic constructions. Thus, their computational counterpart will be digital and analog computations respectively [18]. I will shortly introduce how the ob-

16. In particular, Lakoff and Núñez [2000] is largely developing the concept of a "Basic Metaphor of Infinity" unifying the various introduction of infinity in mathematics (infinite sets, points at infinity, limits of infinite series, infinite intersections and least upper bounds), that attracted so many critics (e.g. Gold [2001]).

17. Cf. [Boyer, 1968, Chapter XXV]. From a cognitive perspective, the arithmetization of analysis can be viewed as a foundational "discretization program" that shaped modern mathematics, as suggested in [Lakoff and Núñez, 2000, Part IV].

18. A computer is a machine working on inputs and giving outputs. Such machines can be considered dynamical systems, and distinguished according to the time and the state space (henceforth "space"). Both time and space can be considered as discrete or continuous. Turing machines, lambda calculus, and cellular automata are computational models with both discrete time (steps) and space (signs of an alphabet). Shannon's GPAC (which will be soon explored) and differential machines have both continuous time and space. The model of Blum et al. [1989] is continuous in space (works on real values) but discrete in time, while the one in Dee and Ghil [1984] is continuous in time and discrete in space. For a table with more examples see [Bournez and Campagnolo, 2008, p. 13, Fig. 3].

Today analog computing is no longer (mainly) studied for construction of actual machines. Analog computation being deeply related to differential equations, relations with the digital paradigm are searched to build a bridge between mathematical analysis (calculus) and the

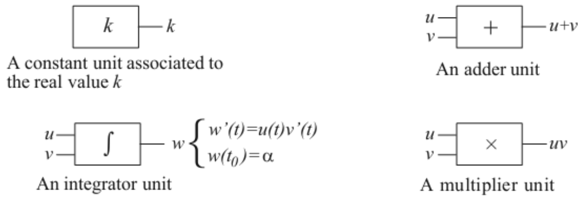


Figure 7.1: Different types of units used in a GPAC.

jects of calculus can be treated with such two computational paradigms [19].

Even thought analog computers are a large class of devices working on continuous data, the first analog device realized to be "programmable" was Vannevar Bush's "differential analyzer" [20]. It was designed to solve differential equations by integration [21], and its mathematical model, the General Purpose Analog Computer (shortly: GPAC) was given by Shannon [1941]. Even today, GPAC is widely accepted as the main model for analog computing.

A GPAC may be seen as a circuit built of interconnected black boxes [22], whose behavior is given by Figure 7.1, where inputs are functions of an independent variable. The fundamental result is that GPAC-generable real functions are the differentially algebraic ones [23]. In particular, differential machines can be thought as a model of analog computation: D.A. functions assure the link

classical theory of computability.

19. Even though I will not discuss it in detail, there are models where analog computation is extended with iterative methods (they are called "hybrid" models). For a comparison between different digital, analog, and hybrid models working on real numbers, see Hainry [2006].

20. Cf. Bush [1931]. The differential analyzer's input was the rotation of one or more drive shafts, and its output was the rotation of one or more output shafts. The main units were gearboxes and mechanical friction wheel integrators, the latter having been invented by Italian scientist Tito Gonella in 1825 (cf. Bowles [1996]). The origin of the idea of a machine using integrators to solve differential equations owes to Lord Kelvin, cf. Thomson [1875].

21. As opposed to the various integraphs, the differential analyzer was designed not to solve a specific differential equation but to be general-purpose. Since it is composed of mechanical components, it is possible to construct it with Meccano components: Scientists and researchers around the world until the end of the WWII built these Meccano models for serious works. Today building differential analyzers with Meccano parts has become a popular project among serious Meccano hobbyists. One of these models has been built at Marshall University, and is now used for educational purposes. By operating the machine, a student not only solves a differential equation, but also becomes the "calculator," and so develops a better understanding of what a differential equation is (cf. Brooks et al. [2008]).

22. Here we can note the main difference between GPAC and differential machines. My interest being to give a certain "geometric insight," my model deals with specific idealized components, not black boxes. However, the black boxes approach has the nice characteristic that it is not specific to any actual device (Shannon designed a device implementing the differential analyzer not with mechanical tools, but with electronic components, improving performances).

23. Cf. Shannon [1941],Pour-El [1974],Lipshitz and Rubel [1987]. Precisely it holds:

Proposition. *Let I and J be closed intervals of $\mathbb{R}$. If a function y is GPAC-generable on I then there is a closed subinterval $I' \subset I$ and a polynomial $p(t, y, y', \ldots, y^{(n)})$ such that $p = 0$ on I'. Vice versa if $y(t)$ is the unique solution of $p(t, y, y', \ldots, y^{(n)}) = 0$ satisfying a certain condition on J then there is a closed subinterval $J' \subset J$ on which $y(t)$ is GPAC-generable.*

between GPAC-generable and function generated by differential machines [24]. Thus, also my model of machines supports the usual idea that functions generable with analog machines are all and only the D.A. functions (so for example excluding Euler Γ, Weierstrass and Dirichlet functions) [25]. We will see some differences in the digital counterpart.

With regard to digital computation over the reals, I have to distinguish between two different approaches [26]. On the one hand, we have numerical analysis and scientific computation; on the other, we have the computation theory arising from logic and computer science. To perform computations on real numbers, numerical analysis approximates real values to rational ones. Thus, such algorithms (e.g. Newton's method) provide approximate solutions. The other approach, represented by "computable analysis," deals with "exact" computation, as I am going to explore. In the next subsection, I will distinguish "exact as symbolic" from "exact as not approximating," being more interested in the second interpretation for foundational reflections.

Computable analysis is a branch of computability theory studying those functions on the real numbers and related sets that can be computed by machines such as digital computers [27]. It avoids both approximations to rational numbers (as in numerical analysis) and the introduction of infinity (as in clas-

24. I need a note about functions generable by GPAC and differential machines. In Milici [2012a], I suggested that differential machines (in the paper called "Tractional Motion Machines") can generate more functions than GPAC. In fact, using the results of Graça [2004] and Graça and Costa [2003], GPAC generable functions can be characterized as solutions of *polynomial Cauchy problems* (shortly: *pCp*), i.e. Cauchy problems in the form

$$\begin{cases} \underline{y}' & = & \underline{p}(t, \underline{y}) \\ \underline{y}(t_0) & = & \underline{y}_0 \end{cases}$$

(with t the free variable, $\underline{y} = (y_1, \ldots, y_n)$ a vectorial function in t and $\underline{p} = (p_1, \ldots, p_n)$ a vectorial polynomial in $\underline{y}$ and t). The equivalence between functions that are GPAC-generable and solution of *pCp* implies that all the GPAC-generable functions have to be analytic.

Based on a suitable input/output interpretation of differential machines, I solved any generic *pCp* with my machines, and generated a real function having a cycloid as Cartesian graph, hence a non-analytic function. This means that, from an input/output interpretation, functions generated by differential machines are more than the ones generated by GPAC.

However, as I observed, every solution of a differential machines has to solve an algebraic differential equation, so solutions have to be locally D.A.: a general solution is made up gluing different local solutions (i.e. D.A. functions). I decided to avoid all such complications in this thesis setting differential machines in a behavioral approach and not in an input/output one. Analytically, the passage from behavioral (i.e. based on "relations") to input/output (i.e. based on "functions") approach is mathematically based on the *implicit function theorem*. This important theorem of multivariable calculus is a tool that allows relations to be converted to functions of several real variables (representing the relation as the graph of a function). There may not be a single function whose graph is the entire relation, but there exists such a function on a restriction of the domain of the relation.

25. For example, this idea that analog generable functions are all and only D.A. is visible in Rubel [1989]. A conceptual extension of GPAC solving partial differential equation is the "Extended Analog Computer" introduced by Rubel [1993]. However, the possibility of actually realizing such a model by physical devices has to be investigated.

26. Cf. Blum [2004].

27. Computable numbers (together with Turing machines) were introduced in Turing [1937], while computable real functions in Grzegorczyk [1955]. See Weihrauch [2000] for an up-to-date monograph of computable analysis from the computability point of view, or Ko [1991] for a presentation from a complexity point of view. Classical references are Aberth [1980], Pour-El and Richards [1988].

sical calculus) with the following definitions.

A real number a is *computable* if it can be approximated by a Turing-computable function $f : \mathbb{N} \to \mathbb{Z}$ such that, given any positive integer n, the function produces an integer $f(n)$ satisfying $f(n) - \frac{1}{n} \leq a \leq f(n) + \frac{1}{n}$. [28]. Representing computable numbers by the Gödel indices of the function f [29], we have an exact representation of these numbers [30]. The computable real numbers form a real closed field. The equality relation on computable real numbers is not computable, but for unequal computable real numbers, the order relation is computable.

To define computable real functions we can use the definitions of *sequentially computable* [31] and *effectively uniformly continuous* [32] functions. A real function is computable if it is both sequentially computable and effectively uniformly continuous. Computable real functions map computable real numbers to computable real numbers. The composition of computable real functions is again computable. Every computable real function has to be continuous.

It is now natural to compare GPAC-generable functions with computable ones. As has been seen, analog computation currently has a stronger connection with classical analysis (thanks to D.A. functions) than its digital counterpart does (to my knowledge, there is no analytical characterization of computable functions). Bournez et al. [2006] proved the computational equivalence between GPAC and Turing machines, but with some specifications. This computational equivalence is not about GPAC-generable functions, but about GPAC-computable functions [33]. However, my aim being to compare digital main tools (recursion) with analog ones (continuous solution of differential equations), I am not interested in the hybrid concept of GPAC-computability.

Referring back to GPAC-generable functions, each of these functions is Turing computable [34], but the converse does not hold. In fact, Euler Γ function, that is not D.A., is computable [35]. Thus at the moment analog computation (intended as D.A. functions, so generated by a GPAC or by a differential machine)

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28. Another definition is that the real number a is computable if there is a Turing-computable sequence of rational numbers q_i converging to a such that $|q_i - q_{i+1}| < 2^{-i}$ for each $i \in \mathbb{N}$.

29. The Gödel index (or Gödel number) of a Turing function f is an integer that encodes the rules of the Turing machine defining f. Any Turing function different from f will have a different Gödel index. The concept of Gödel numbering was used by Kurt Gödel for the proof of his incompleteness theorems in Gödel [1931].

30. While the set of real numbers is uncountable, the set of computable numbers is only countable. Therefore, almost all real numbers are not computable. However, all the real constants usually present in mathematics (such as algebraic numbers, e, π, Euler-Mascheroni constant γ) are computable. The first example of a real number that is definable but not computable is Chaitin's constant Ω, which is a type of real number that is Turing equivalent to the halting problem (cf. Chaitin [1975]).

31. A function $f : \mathbb{R} \to \mathbb{R}$ is *sequentially computable* if, for every computable sequence $\{x_i\}_{i=1}^{\infty}$ of real numbers, the sequence $\{f(x_i)\}_{i=1}^{\infty}$ is also computable.

32. A function $f : \mathbb{R} \to \mathbb{R}$ is *effectively uniformly continuous* if there exists a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that, if $|x - y| < \frac{1}{d(n)}$ then $|f(x) - f(y)| < \frac{1}{n}$.

33. GPAC-generable functions are functions computed with a GPAC in "real time," while GPAC-computable ones are obtained with a kind of "converging computation," as used in computable analysis. In other words, GPAC-computable functions are generated by a GPAC with a process of limit.

34. Every GPAC-generable function is GPAC-computable, and GPAC-computability is equivalent to Turing computability.

35. Cf. Pour-El and Richards [1988].

is weaker than digital one.

7.2.3 Toward a definition of exactness

According to section 2.2, the problem of geometric exactness is a metamathematical problem about the intuitive concept of geometry. What I claim is that, if we adopt a computational approach as the one of the previous subsection, the exactness problem can be suitably set inside mathematics. From this perspective, the arithmetization of analysis offers an important view: Passing from "exactness of constructions" to "rigor of proofs," this arithmetization allowed setting calculus totally inside formal mathematics. However, this rigorization was obtained by delving only into the discrete aspect of mathematics and releasing the continuous one, as the relation with geometric constructions. The first question is:

> Is it possible to bring the concept of "exactness" inside mathematics from a computational perspective, including both digital and analog paradigms?

I do not claim to answer this question, but I will give the sketch of an attempt to set the problem.

Heuristically, I looked for a property that could distinguish between a computational framework using infinitary approximation and another one purely finitistic [36]. An example of the first one is computable analysis: Even though this framework is computational (infinite is not directly introduced due to a suitable use of signs), objects as numbers and functions are defined only as arbitrary approximations. In such a setting, there are many algorithmically testable properties, but it is not generally possible to test whether two objects are equal or not. This is the case because checking equality implies infinite tests. Contrarily, in algebraic settings such as differential algebra, the equality can be tested [37], and differential algebra, even though dealing with objects of calculus, is not really involving infinite.

Thus, I propose the following definition of exactness for future research [38]:

> *a computational framework is exact if and only if the equality test is computable in it.*

Using this idea of exactness as solvability of equality test, I can justify in a new way the exactness of classical and algebraic machines (intuitively: of Euclid's and Descartes's geometries); not because of the adherence to a certain canon, but because they satisfy the equality test [39].

36. I am not dealing with approximated framework as numerical computation with finite precisions. I am focusing on purely symbolical computations.

37. Equality can be tested at least for the case of differential polynomial systems without initial value problems. However, it has not been proven that with IVP the equality relation is not computable.

38. In contrast to the historical setting of geometric constructions, this concept is available for both digital and analog computation.

39. Cf. subsection 3.2.5 at pag. 39, and subsection 3.3.8 at pag. 55.

Furthermore, I can answer about exactness of infinitesimal analysis: *Is it possible to have an exact approach to calculus?*[40] Owing to the analytical counterpart of differential algebra, differential machines exemplify how it is possible to find an exact approach to calculus. However, with regard to computable analysis, differential machines generate fewer functions (e.g. Γ function is computable but not D.A.), so now it is reasonable to ask ourselves: *Is it possible to have an exact approach to (a part of) transcendentally transcendental functions?* I expect this work to provide a basis to answer this and other questions about boundaries of exactness[41].

In summary, it can be said that this thesis was a quest for exactness of calculus. Nevertheless, I had to define what exactness is, and I suggested it on the basis of a computational approach. From a cognitive perspective (remind subsection 7.2.1), I can consider this definition a redefinitional metaphor for exactness.

7.3 Open problems and perspectives

In the present thesis, I proposed an approach for tractional constructions of differential equations with differential machines. I began exploring the historical example of tractional motion, giving to it a well-defined synthetic counterpart as machines, and suitable analytical tools.

While early modern exactness problem dealt with geometric constructions, I proposed a more general application of exactness to include any general compu-

40. From this perspective, I can reread the observations of Berkeley on infinitesimal as entities missing the equality test. *The Analyst* was a direct attack on the foundations of calculus, specifically on Newton's notion of fluxions and on Leibniz's notion of infinitesimal change. About my concept of exactness, I can see how Berkeley main argument (in [Berkeley, 1734, Section 16]) is about the ontological status of objects in calculus:

> "And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"

Algorithmically, that can be rephrased: it is not possible to check whether some objects (the infinitesimals) are equal or not to zero (still remaining in the field of real numbers, and not considering logic extensions as non-standard models).

41. Tait [1981] claims that finitist reasoning (rejecting all references to infinite totalities) is essentially primitive recursive reasoning. From this perspective, it can be interesting to consider primitive recursive real numbers (cf. Chen et al. [2007]) instead of computable ones: I have no idea about the decidability of the equality test on them.

With regard to the comparison between digital and analog computation, as recursive functions are constructed extending primitive recursive ones with the μ operator (i.e. the *minimization operator*, which searches for the least natural number with a given property), we can also consider an operator to extend the functions generable by GPAC. The first introduction of μ in analog computation is due to Moore [1996], who defined a class of recursive functions on the reals analogous to the classical recursive functions on the natural numbers. On a theoretical analog computer that operates in continuous time, μ was a zero-finding operator, and the class of functions obtainable with it turns out to be surprisingly large, including many functions which are uncomputable in the traditional sense. This conceptual computer is almost certainly unphysical (as probably the machine in Rubel [1993]), so, to address the degree of unphysicality, Moore stratified the class of functions according to the number of uses of the zero-finding operator μ (the lowest level of this hierarchy coincides with Shannon's GPAC). To evince the relationship with infinitary tools, Mycka [2003] replaced Moore's definition of μ-operator as zero finding by infinite limits.

tational framework (both analog and digital). The main purpose of my exactness definition is intuitively to distinguish between approaches that implicitly require infinite processes (approximating frameworks) and the ones that do not (exact ones). The idea is to interpret exactness as the algorithmic availability of the equality test between any two constructed objects.

Owing to the behavioral approach, differential machines construct solutions of systems of differential equations and inequations. Thanks to the symbolic manipulation methods of computer algebra, is it possible to test whether the solutions generated by different machines are equal or not (with the remarks of note 37), so differential machines can be considered an “exact framework” for calculus. With the proposed setting, a foundation for calculus can be achieved by avoiding infinitesimal, infinite, and approximations.

Furthermore, considering differential machines, differential algebra and D.A. functions as an extension of Cartesian setting (geometric linkages, polynomial algebra and algebraic curves), this approach provides a new balance between machines, algebra and geometry beyond Descartes's limits but still based on geometric constructions (in the synthetic part) and without the introduction of infinitary entities or procedures (in analysis). Nevertheless, in contrast to Descartes's view, my approach is not a closed one, which is why new extensions can be added in the future.

Out of the obtained results, there are many related open problems, and future perspective, that I am going to recall. The main topics to be deepened in the future are the following:

- Exactness as solution of equality test: Are there strong arguments for this idea? What is the role of finitism? (Especially with regard to calculus.)
- Is it possible to extend the balance between machines, algebra and geometry to construct Γ function, fractional calculus and differential polynomials with partial derivatives?
- Can algebraic machines be considered as a model of Descartes's machine? (Especially considering Descartes's canon a “conservative extension” of Euclid's geometry as algebraic machines are an extension of classical ones.)
- With regard to differential machines: Can they be considered as a “conservative extension” of algebraic ones? [42]
- Is equality between differential machine computable considering initial value problems? (cf. the final paragraph of the subsection 4.4.3 at pag. 95).

42. I introduced differential machines as an extension of algebraic machines to solve the inverse tangent problem: But why did I precisely choose the tangent problem instead of any other? Is there some logic/algorithmic justification for this?

The extension from classical to algebraic machines was just the relaxing of some postulates. Is it possible, using some setting different from the proposed one, to see differential machines as the ones obtained relaxing some postulates of algebraic ones, and not as an implementation solving an inverse problem? From this perspective, a possible idea is no longer to consider carts and wheels, but the more general “direction constraint” (cf. the final paragraph of the subsection 4.1.4 at pag. 68). Which class of machines is defined using not carts and wheels, but the direction constraint? Moreover, from a logical standpoint, is it possible to set these machines using only the tetradic relations “congruence” and “direction”?

I also met some other problems that, even though still open, are not important in the whole ambit of the thesis. However, at least for curiosity, I think it can be interesting to deepen them in future works. They are:

- Extension of Mohr-Mascheroni theorem to non Euclidean geometries: Considering constant curvature geometries, when can compass and straightedge constructions be performed by a compass alone? (cf. note 42 at pag. 42).
- Which class of the curves is constructible as ruler-and-compass loci? (cf. the subsection 3.4.2 at pag. 59).
- Is it possible to determine algorithmically when the solutions of a differential machine can be obtained with an algebraic one? (cf. subsection 4.4.4 at pag. 97).
- Which class of points is constructible using the logarithmic compass? (cf. section 6.2 at pag. 128).

Furthermore, as introduced in the subsection 6.3.1, to arrive at a more constructive, sensitive, and concrete approach to calculus in math education, future purposes are about the following:

- actual machines for laboratorial activities;
- the development of a suitable dynamic geometry software;
- a restructuration of calculus in the light of differential machines and differential algebra.

The possibility of a restructuration of calculus is interesting to be investigated from instrumental, visual, algebraic, cognitive, and foundational viewpoints.

Finally, I want to conclude this quest for exactness with a remark on the role that the exactness problem can assume in the future. In contrast to the rigor, exactness involves canons of constructions and not just of axiomatic settings. Hence, the role of exactness is strongly related to that of computation (from a theoretical perspective, not from a physical/engineering one). Thus, a reflection on exactness, on its nature, and on its boundaries, considering both geometric/analog and arithmetic/digital settings (specially about their mutual relations) could be useful for future definitions of what computation is. I hope that such an inquiry will open up a new way to go beyond the restrictions of today[43] in an evolutionary process.

43. As the Church-Turing thesis briefly discussed in the subsection 2.3.5.

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