Local mean field and energy transport in non-equilibrium systems

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Résumé (FR)

Les systèmes de chaînes d'oscillateurs permettent de modéliser microscopiquement un solide, dans le but d'étudier le transport d'énergie et de retrouver la loi de Fourier. Dans cette thèse, nous introduisons des nouveaux modèles de chaînes d'oscillateurs avec interaction mécanique de type champ moyen local et collisions stochastiques préservant l'énergie totale du système. Le premier modèle est un modèle avec échanges stochastiques de vitesses de type modèle de Kac. Le second est un modèle avec retournement de vitesses, où les vitesses sont changées en leurs opposées à des temps aléatoires.

Contrairement à la théorie classique des modèles de champ moyen, les particules du système ne sont pas indistinguables, et le caractère conservatif des échanges stochastiques pour le premier modèle représente une difficulté supplémentaire dans la preuve d'une limite de Vlasov. Nous prouvons dans un premier temps une limite quantitative de champ moyen, que nous utilisons ensuite pour prouver que l'énergie évolue diffusivement à une échelle de temps donnée pour le modèle avec échanges à longue portée pour une classe restreinte de potentiels anharmoniques. À cette même échelle de temps, nous prouvons également que l'énergie n'évolue pas pour le modèle avec retournement de vitesses.

Dans le cas d'interactions harmoniques, nous calculons ensuite la conductivité thermique via la formule de Green-Kubo pour ces deux modèles, afin de mettre en évidence que l'échelle de temps à laquelle l'énergie évolue pour le modèle avec retournements de vitesses est plus longue et donc que les mécanismes en jeu dans le transport d'énergie sont différents.

Abstract (EN)

Chains of oscillator systems enable to model microscopically a solid, in order to study energy transport and prove Fourier's law. In this thesis, we introduce two new models of chains of oscillators with local mean field mechanical interaction and stochastic collisions that preserve the system's total energy. The first model is a model with stochastic velocity exchanges of Kac type. The second one is a model with random flips of velocities, where the sign of the particles' velocities is changed at random times.

As we consider local mean field models, particles are not indistinguishable, and the conservative stochastic exchanges in our first model are an additional difficulty for the proof of a Vlasov limit. We first derive a quantitative mean field limit, that we then use to prove that energy evolves diffusively at a given timescale for the model with long-range exchanges and for a restricted class of anharmonic potentials. At the same timescale, we also prove that there is no evolution of energy for the model with flips of velocities.

For harmonic interactions, we then compute thermal conductivity via Green-Kubo formula for both models, to highlight that the timescale at which energy evolves for the model with velocity flips is longer and therefore that the mechanisms at play for energy transport are different.

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Chapitre 1

Introduction

Considérons un matériau dans un domaine $\Omega \subset \mathbb{R}^d$ et soit $e(t, r)$ la densité d'énergie en $r \in \Omega$ à l'instant t. Si le système est isolé, l'énergie totale est conservée et on peut définir la densité de flux thermique $j(t, r) \in \mathbb{R}^d$ par l'équation de continuité

$$\partial_t e + \nabla \cdot j = 0. \tag{1.1}$$

La loi de Fourier est une loi empirique qui exprime la densité de flux thermique j en fonction de la température T dans un domaine $\Omega \in \mathbb{R}^d$ par la relation

$$j = -\kappa(T)\nabla T, \tag{1.2}$$

où $\kappa > 0$ est la conductivité thermique du matériau, que l'on suppose isotrope. En notant $c = \partial e/\partial T$ la capacité thermique du matériau, et en combinant (1.1) et (1.2), on arrive [20] à l'équation de diffusion :

$$\partial_t e = \nabla \cdot \left(\frac{\kappa}{c}\nabla e\right) = \nabla \cdot (D\nabla e), \tag{1.3}$$

où $D = \kappa/c$ est la diffusivité thermique du matériau. S'il n'y a aucun doute sur la validité de la loi de Fourier, il n'existe à ce jour aucune preuve de sa dérivation à partir d'un modèle microscopique et cela reste à ce jour un des problèmes ouverts majeur en physique statistique, comme indiqué par Lebowitz *et al.* [13].

Une des difficultés essentielles réside dans le fait que les quantités thermodynamiques comme la température peuvent être définies dans des systèmes microscopiques à l'équilibre, tandis que la loi de Fourier décrit une situation hors équilibre. La loi de Fourier suppose donc l'existence d'un équilibre thermique local, c'est-à-dire un état pour lequel un petit volume macroscopique autour de $r \in \Omega$ contenant un nombre suffisamment grand de particules puisse être considéré à l'équilibre thermique à la température $T(t, r)$, mais que néanmoins ce paramètre puisse varier continûment à l'échelle macroscopique. En particulier, prouver un équilibre local pour un système de particules microscopiques revient à prouver que le système possède des propriétés ergodiques suffisantes pour atteindre localement l'équilibre en temps court, tandis que la température et les autres quantités conservées évoluent à des échelles de temps plus longues, dites macroscopiques.

Pour modéliser microscopiquement un solide cristallin, on utilise le modèle des chaînes d'oscillateurs. Dans cette thèse, nous introduisons et étudions un nouveau modèle de chaînes d'oscillateurs avec des interactions de type champ moyen local et des collisions stochastiques. Avant de présenter les contributions de la thèse, nous allons donc tout d'abord décrire le modèle classique des chaînes d'oscillateurs et les principaux résultats associés, puis introduire le modèle avec des interactions de type champ moyen local. Nous présenterons ensuite les limites de champ moyen pour divers systèmes de particules et analyserons les difficultés que nous avons dû résoudre dans notre contexte. Nous insisterons particulièrement sur des résultats récents concernant le modèle de Kac que nous avons adaptés dans la thèse.

1.1 Les chaînes d'oscillateurs

1.1.1 Modèle

Si dans un solide conducteur, l'énergie est transmise par les porteurs de charges, dans un solide cristallin isolant ce sont les vibrations des atomes ou des molécules sur le réseau qui transmettent l'énergie. Pour étudier ce phénomène, on modélise microscopiquement un solide d-dimensionnel par un système $(X^i, V^i)_{i\in\mathbb{Z}^d}$, où $X^i \in \mathbb{R}^d$ représente le déplacement de la particule i par rapport à sa position d'équilibre et $V^i \in \mathbb{R}^d$ le moment associé, les particules étant supposées de même masse unitaire. Habituellement, on suppose que seules les particules voisines dans le réseau interagissent entre elles et on associe à ce système l'hamiltonien

$$\mathcal{H} = \sum_{i\in\mathbb{Z}^d}\left(\frac{1}{2}|V^i|^2 + \frac{1}{2}\sum_{j\sim i} W(X^i - X^j) + U(X^i)\right). \tag{1.4}$$

Le potentiel W est un potentiel d'interaction, U est un potentiel extérieur d'accrochage, et on note $j \sim i$ si j et i sont voisins sur le réseau $\mathbb{Z}^d$. Les coordonnées des particules $(X^i_t, V^i_t)_{i\in\mathbb{Z}^d}$ évoluent donc selon les équations de Newton

$$\begin{cases} \dot{X}^i_t = V^i_t \\ \dot{V}^i_t = -\sum_{j\sim i}\nabla W(X^i_t - X^j_t) - \nabla U(X^i_t). \end{cases} \tag{1.5}$$

On peut alors définir l'énergie associée à la particule i à l'instant t par

$$e^i_t = \frac{1}{2}|V^i_t|^2 + \frac{1}{2}\sum_{j\sim i} W(X^i_t - X^j_t) + U(X^i_t). \tag{1.6}$$

A partir des équations de Newton, on peut alors exprimer le courant entre deux particules. Pour simplifier les notations, supposons que les particules sont désormais indexées par $\mathbb{Z}$ uniquement, tout en gardant des déplacements X^i dans $\mathbb{R}^d$. Le courant d'énergie $j^{i,i+1}$ entre les particules i et $i+1$ est alors défini par l'équivalent microscopique de (1.1) :

$$\frac{d}{dt}e^i_t + \nabla j^{i,i+1}_t = 0. \tag{1.7}$$

Dans cette équation, $\nabla f(i) = f(i+1) - f(i)$ désigne le gradient microscopique d'une fonction f définie sur $\mathbb{Z}$ et

$$j^{i,i+1}_t = \frac{1}{2}(V^i_t + V^{i+1}_t)\cdot\nabla W(X^i_t - X^{i+1}_t).$$

1.1.2 Lien avec la loi de Fourier

Il y a au moins trois cadres conceptuels différents pour lesquels on s'attend à ce que la loi de Fourier soit vérifiée pour le modèle microscopique. Nous allons présenter des arguments heuristiques déjà connus pour faire le lien entre les différents aspects qui seront étudiés dans cette thèse.

1. Le premier est purement et simplement la preuve d'une limite hydrodynamique, c'est-à-dire de la convergence de l'énergie microscopique vers la solution de (1.3) dans une limite d'échelle appropriée. Ce type de preuve requiert des propriétés ergodiques extrêmement fortes qui puissent permettre de prouver l'existence d'un équilibre local. Ces propriétés ergodiques ont été démontrées pour des systèmes hamiltoniens munis d'échanges stochastiques pour la première fois par Olla, Varadhan et Yau [57] afin de prouver une limite eulérienne (voir également [30] pour des modèles sur réseau). Il n'existe à ce jour aucune preuve d'une telle limite directe pour des modèles purement hamiltoniens.

En physique statistique, une autre approche en deux temps permet de justifier de telles limites, au moins heuristiquement. Il s'agit de prouver d'abord une limite cinétique, c'est-à-dire la convergence des statistiques du modèle microscopique vers celles des solutions d'une équation cinétique, comme l'équation de Boltzmann pour les gaz par exemple. Puis ensuite de dériver la limite hydrodynamique à partir de l'équation cinétique dans une certaine limite d'échelle, en fonction d'un paramètre cinétique. Dans le cadre des solides cristallins, une telle équation cinétique pour les phonons a été établie par Peierls. Elle est aujourd'hui notamment utilisée pour prédire la conductivité thermique d'un solide. Nous renvoyons à l'article de Spohn [63] pour la dérivation formelle d'une telle équation pour les phonons.

2. Le second cadre dans lequel on s'attend à observer la loi de Fourier est celui d'un forçage hors équilibre d'une chaîne de longueur N. On restreint les équations (1.5) aux particules $1 \leq i \leq N$ et on rajoute deux thermostats à températures T_L et T_R aux extrémités de la chaîne. La modélisation se fait généralement via des thermostats dits de Langevin, c'est-à-dire en modifiant les équations des particules 1 et N par l'ajout des termes stochastiques suivants :

$$\begin{cases} dX_t^1 = V_t^1 dt \\ dV_t^1 = -\left(\nabla W(X_t^1 - X_t^0) + \nabla W(X_t^1 - X_t^2) + -\nabla U(X_t^i)\right) dt - \lambda V_t^1 + \sqrt{2\lambda T_L}\, dW_t^1, \end{cases}$$

où λ est un paramètre de friction laissé libre, X^0 est une nouvelle variable permettant de fixer les conditions au bord (on prendra $X^0 = 0$) et W^1 est un processus de Wiener. On modifie de même l'équation au site N à l'aide d'un processus de Wiener W^N indépendant de W^1. Si $T_L = T_R = T$, alors la mesure de Gibbs à température T est l'unique mesure d'équilibre. Si les deux températures sont différentes, on s'attend alors à ce qu'un état stationnaire hors équilibre s'établisse. La preuve de l'existence et de l'unicité d'une mesure stationnaire régulière, ainsi que de la convergence vers cette mesure a été faite pour la première fois par Eckmann *et al.* [23, 24] pour des potentiels quadratiques à l'infini, puis pour des potentiels plus généraux par Eckmann et Hairer [22]. La convergence exponentielle vers cette mesure stationnaire a ensuite été prouvée par Rey-Bellet et Thomas [59]. Dans ces différents travaux, les thermostats ne sont pas des thermostats de Langevin, mais sont modélisés par des interactions hamiltoniennes. La convergence exponentielle vers l'unique mesure invariante pour des potentiels de Langevin a ensuite été prouvée par Carmona [15]. La loi de Fourier se traduit alors par la convergence de

$$\lim_{N\to\infty} N < j^{i,i+1} >_{T_L,T_R} =: f(T_L, T_R)$$

où $< \cdot >_{T_L,T_R}$ désigne l'espérance sous cette mesure stationnaire. En particulier, si les températures des thermostats sont proches $T_L = T + \delta T$ et $T_R = T - \delta T$, on s'attend alors à ce que le flux moyen soit proportionnel au gradient de température :

$$\lim_{N\to\infty} N < j^{i,i+1} >_{T_L,T_R} = -\kappa\delta T + O(\delta T), \tag{1.8}$$

où κ est la conductivité thermique. Aucune preuve d'une telle limite n'existe à ce jour.

3. Enfin, le dernier cadre se concentre sur la diffusion de l'énergie à l'équilibre. Définissons la mesure de Gibbs à température inverse β de la chaîne de longueur N par

$$\frac{1}{\mathcal{Z}_N} \exp\left(-\beta \mathcal{H}_N\right) \prod_{i=1}^{N} dX^i dV^i,$$

où $\mathcal{H}_N$ est donné par la formule (1.4) avec une somme restreinte à $1 \leq i \leq N$ et des termes de bord nuls $X^0 = X^{N+1} = 0$. En présence d'un potentiel d'accorchage, on peut définir la limite en volume infini de cette mesure de Gibbs, et nous noterons $< \cdot >_\beta$ l'espérance sous cette mesure. Notons alors

$$S(i,t) = < e_t^i e_0^0 >_\beta - < e_0^0 >_\beta^2$$

la corrélation spatio-temporelle de l'énergie (1.6). Par conservation de l'énergie totale de la chaîne, la quantité

$$\chi = \sum_{i \in \mathbb{Z}} S(i,t) \tag{1.9}$$

est constante en temps et appelée susceptibilité thermique. Si l'on pense que l'énergie e_t^i doit être proche d'une solution de l'équation de diffusion (1.3), $S(i,t)$ doit l'être aussi et on peut donc conjecturer [20, 62] que $S(i,t)$ devrait se comporter comme

$$\chi \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{i^2}{4Dt}\right),$$

où le préfacteur χ provient de (1.9). En particulier, on pourrait alors accéder à la diffusivité thermique D par les moments spatiaux d'ordre 2 en calculant la variance

$$\lim_{t\to\infty} \frac{1}{2\chi t} \sum_{i\in\mathbb{Z}} i^2 S(i,t).$$

En utilisant ensuite l'équation de conservation sur les courants (1.7) et l'invariance en temps et en espace par rapport à la mesure $< \cdot >_\beta$, on aboutit par un calcul formel à la formule de Green-Kubo

$$D(\beta) := \frac{1}{\chi} \int_0^\infty dt \sum_{i\in\mathbb{Z}} < j_t^{i,i+1} j_0^{0,1} >_\beta . \tag{1.10}$$

Ces calculs seront détaillés au Chapitre 3. La difficulté mathématique de la preuve de la convergence d'une telle formule réside dans le fait qu'il faut réussir à démontrer que les corrélations du courant décroissent suffisamment vite en temps et en espace.

Pour conclure, remarquons qu'il existe un lien formel entre ce cadre à l'équilibre et celui hors équilibre par la théorie de la réponse linéaire (c.f. le cours de Derrida [18] pour une introduction). En reprenant l'argument d'Olla [56], définissons la mesure de Gibbs inhomogène formelle

$$\mu_{\beta,\epsilon} = \prod_{i\in\mathbb{Z}} \frac{1}{\mathcal{Z}_{\beta+i\epsilon}} \exp\left(-(\beta+i\epsilon)e^i\right) dX^i dV^i,$$

présentant un petit gradient de température inverse ϵ. Par un développement limité en ϵ formel, on peut exprimer la moyenne du courant *hors équilibre* par

$$< j_t^{0,1} >_{\beta,\epsilon} = K_t(\beta)\epsilon + O(\epsilon), \tag{1.11}$$

où K_t est une quantité définie par les corrélations *à l'équilibre*

$$K_t(\beta) = -\sum_{i\in\mathbb{Z}} i < j_t^{0,1}(e_0^i - < e_0^i >_\beta) >_\beta, \tag{1.12}$$

et $< \cdot >_\beta = < \cdot >_{\beta,0}$ représente des moyennes à l'équilibre. La relation (1.11) est similaire à (1.8) au sens où elle exprime la moyenne du courant en fonction d'un gradient de température. En supposant que les coefficients de proportionnalité sont les mêmes dans les deux cas, on peut donc heuristiquement établir un lien entre les quantités D et χ définies à l'équilibre et la conductivité thermique κ définie hors équilibre par l'argument suivant. En utilisant successivement un argument de retournement du temps, puis le fait que $< j_0^{0,1} >_\beta = 0$ et $< j_0^{0,1} e_0^i >_\beta = 0$, on peut exprimer le coefficient (1.12) par

$$K_t(\beta) = \sum_{i\in\mathbb{Z}} i < j_0^{0,1}(e_t^i - < e_0^i >_\beta) >_\beta = \sum_{i\in\mathbb{Z}} i < j_0^{0,1}(e_t^i - e_0^i) >_\beta .$$

De plus, par l'équation de conservation (1.7), puis une sommation par parties, on obtient

$$K_t(\beta) = -\int_0^t \sum_{i\in\mathbb{Z}} i < j_0^{0,1} \nabla j_s^{i,i+1} >_\beta = \int_0^t \sum_{i\in\mathbb{Z}} < j_0^{0,1} j_s^{i,i+1} >_\beta .$$

Il s'agit toujours bien sûr d'un calcul formel, la convergence de ces sommes n'étant pas démontrée. En prenant la limite $t \to \infty$ dans cette dernière expression et en utilisant la formule de Green-Kubo (1.10), on obtient

$$\lim_{t\to\infty} K_t(\beta) = \chi D(\beta).$$

En remarquant qu'un petit gradient de température inverse ϵ autour de $\beta = 1/T$ doit être relié au gradient de température δT par

$$\epsilon \approx -\frac{\delta T}{T^2},$$

puis en comparant (1.11) et (1.8), on aboutit ainsi à la relation d'Einstein

$$\kappa = \frac{\chi}{T^2} D = cD,$$

en utilisant que la capacité thermique vérifie $c = \chi/T^2$ (*c.f.* [20]).

Pour les modèles étudiés dans cette thèse, nous nous intéresserons aux limites hydrodynamiques via une limite de champ moyen, ainsi qu'au calcul de la conductivité thermique via la formule de Green-Kubo, mais n'aborderont pas le cas d'une chaîne entre deux thermostats à températures différentes.

1.1.3 Quantités conservées et chaîne harmonique

Même si pour des choix de potentiels génériques U et W, les seules quantités conservées par le système (1.5) semblent être l'énergie et l'élongation $\sum_i (X^{i+1} - X^i)$, lorsque le potentiel d'accrochage est nul $U = 0$, le moment total $\sum_i V^i$ est également conservé. Dans ce cas, ces trois quantités interagissent de manière non triviale à diverses échelles de temps sous une forme non élucidée à ce jour. Il a été montré numériquement que le transport d'énergie peut être dans ce cas superdiffusif [50], c'est-à-dire que le courant dans (1.8) est en fait de l'ordre de $N^{-1+\alpha}$, sans qu'il n'y ait de consensus [20] sur la valeur de $\alpha > 0$ (*c.f.* [64] pour une prédiction).

Physiquement, le choix des potentiels U et W qui semblerait le plus naturel est celui de potentiels harmoniques, c'est-à-dire $W(x) = U(x) = |x|^2/2$, car ce sont les potentiels les plus simples modélisant une oscillation autour d'une position d'équilibre. De plus ces potentiels présentent l'avantage de fournir des équations d'évolution (1.5) linéaires. Néanmoins, on sait depuis les travaux de Rieder *et al.* [60] que le courant hors équilibre pour la chaîne entre deux thermostats ne vérifie pas (1.8), mais

$$\lim_{N\to\infty} < j^{i,i+1} >_{T_L,T_R} =: f(T_L, T_R).$$

On parle alors de transport ballistique. La non-ergodicité des chaînes harmoniques était un résultat connu bien avant les travaux de [60] car ces chaînes ont un grand nombre de quantités conservées. En effet, considérons la chaîne (1.5) sur un domaine périodique $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$. Les transformées de Fourier discrètes associées à $(X^i)_{i\in\mathbb{T}_N}$ et $(V^i)_{i\in\mathbb{T}_N}$ vérifient alors

$$\dot{\hat{X}}_t(\xi) = \hat{V}_t(\xi), \qquad \dot{\hat{V}}_t(\xi) = -\omega\left(\frac{\xi}{N}\right)^2 \hat{X}_t(\xi),$$

où

$$\omega(u) = \left(1 + \sin^2(\pi u)\right)^{1/2} \tag{1.13}$$

est la relation de dispersion de la chaîne et la transformée de Fourier discrète de f est définie par $\hat{f}(\xi) = \sum_{n\in\mathbb{T}_N} f_n \, \mathrm{e}^{i\frac{2\pi n \xi}{N}}$. On obtient donc N systèmes découplés (un pour chaque mode) et en particulier l'énergie associée au mode ξ

$$\hat{e}(\xi) = \frac{1}{2}|\hat{V}(\xi)|^2 + \frac{1}{2}\omega\left(\frac{\xi}{N}\right)^2 |\hat{X}(\xi)|^2$$

est une quantité conservée. On a longtemps pensé que l'ajout d'un terme anharmonique faible permettrait d'obtenir suffisamment de propriétés ergodiques pour obtenir une équipartition d'énergie entre ces différents modes, mais la fameuse expérience numérique de Fermi-Pasta-Ulam-Tsingou a prouvé le contraire, ouvrant la voie au développement de la théorie KAM pour expliquer le manque d'ergodicité dans des systèmes intégrables perturbés [32, 48].

1.1.4 Chaînes d'oscillateurs avec collisions stochastiques

Si l'étude du transport de l'énergie dans des chaînes anharmoniques reste à ce jour un problème non résolu, l'étude des chaînes avec des collisions stochastiques a été extrêmement fructueuse ces dernières années (*c.f.* [1] pour une synthèse récente sur cette approche). L'idée est de modéliser l'effet des anharmonicités ou des défauts du cristal par l'ajout de collisions aléatoires conservatives, donnant alors au système de bonnes propriétés ergodiques [9, 30].

Le premier modèle de la sorte a été étudié par Bernardin et Olla [7]. Entre chaque paire de particules voisines, une diffusion sur la surface formée par l'énergie cinétique des deux particules est rajoutée, permettant ainsi des échanges continus de vitesse. Cet échange stochastique conserve l'énergie totale du système, mais il ne conserve pas le moment. La relation (1.8), ainsi que le profil linéaire de l'énergie ont alors été prouvés dans ce cadre. De plus, la convergence des champs de fluctuations à l'équilibre des deux quantités conservées (énergie et élongation) ont été prouvées par Fritz *et al.* [31], et la limite hydrodynamique a été prouvée par Bernardin [6] dans le cas où les équations limites sont linéaires.

Par la suite, un modèle avec échanges stochastiques avec plusieurs quantités conservées a été considéré par Basile, Bernardin et Olla [2]. Il s'agit cette fois d'une diffusion sur l'intersection des surfaces formées par l'énergie cinétique et le moment de particules voisines. Ces échanges conservent donc le moment en plus de l'énergie. Le calcul de la conductivité thermique par la formule de Green-Kubo (1.10) pour cette dynamique a permis de mettre en évidence la diffusion anormale pour des systèmes avec plusieurs quantités conservées en dimension $d \leq 2$, ce qui est en accord avec les simulations numériques pour de telles dynamiques purement hamiltoniennes [50]. De plus, la diffusion des corrélations de l'énergie à l'équilibre a été prouvée par Basile et Olla [4] en dimension $d \geq 3$ dans le cas sans potentiel d'accrochage et pour toute dimension si un potentiel d'accrochage est présent.

Pour expliquer cette diffusion anormale, une réponse par une approche cinétique a été apportée par Basile *et al.* [5], où une équation de Boltzmann pour les phonons a été prouvée pour un bruit faible dans une échelle cinétique appropriée. A partir de cette équation cinétique, il a été ensuite prouvé que les solutions de cette équation convergeaient vers les solutions d'une équation de diffusion fractionnaire [3, 38].

Ces dernières années, une avancée a également été faite sur l'identification des conditions pour que la chaîne ait un comportement diffusif ou non [1]. S'il a longtemps été conjecturé que la conservation du moment impliquait un comportement diffusif, certaines théories ont mis en avant l'importance de la présence d'une vitesse du son non nul dans le solide [64]. Ainsi, Komorowski et Olla [45] ont prouvé que l'énergie se propageait via une équation de diffusion linéaire pour une chaîne où le moment est conservé, mais pour laquelle la vitesse du son est nulle.

Enfin, mentionnons un dernier type de collisions stochastiques introduit par Bernardin et Olla [8] que nous étudierons par la suite. Dans ce modèle, la vitesse V^i de la particule i est remplacée à des temps aléatoires par $-V^i$, indépendamment des autres particules. Les collisions portent donc individuellement sur chaque particule et n'induisent pas d'échange d'énergie. Elles conservent l'énergie totale, mais pas le moment. La preuve de la relation (1.8) a été prouvée dans un cas harmonique sans potentiel d'accrochage dans [8], et la limite hydrodynamique a été prouvée rigoureusement par Komorowski *et al.* [46].

1.1.5 Chaînes d'oscillateurs avec potentiels de Kac

Dans cette thèse, nous introduisons un nouveau modèle de chaînes d'oscillateurs. Dans ce modèle, chaque oscillateur peut interagir avec un grand nombre ℓ_N de particules voisines, où ℓ_N est un paramètre dépendant de la longueur de la chaîne N. ℓ_N vérifie $\lim_{N\to\infty} \ell_N = \infty$, mais la proportion de particules avec laquelle un certain oscillateur peut interagir est telle que $\lim_{N\to\infty} \ell_N/N = 0$. Nous supposerons que les N oscillateurs sont indexés par $\mathbb{T}_N$. L'hamiltonien est donné par

$$\mathcal{H} = \sum_{i\in\mathbb{T}_N} \left(\frac{1}{2}|V^i|^2 + \frac{1}{2} \sum_{k=-\ell_N}^{\ell_N} \phi_k W(X^i - X^{i+k}) + U(X^i) \right), \tag{1.14}$$

où ϕ_k est défini par

$$\phi_k = \frac{1}{\ell_N}\phi\left(\frac{k}{\ell_N}\right), \tag{1.15}$$

et ϕ est une fonction positive, symétrique, de classe C^∞ à support compact $[-1/2, 1/2]$ et telle que

$$\int_{-1/2}^{1/2} \phi(u)du = 1. \tag{1.16}$$

La fonction ϕ est décroissante sur $[0, 1/2]$ et module donc l'intensité des interactions : plus des particules ont un indice proche, plus elles interagissent. Etant donné qu'une particule interagit avec un nombre $\ell_N \gg 1$ de particules voisines, le système a localement une structure de champ moyen. On parle alors de potentiels de Kac (*c.f.* [58] pour une introduction aux potentiels de Kac et une synthèse de résultats concernant ces potentiels dans d'autres contextes). Les potentiels de Kac ont déjà été utilisés pour étudier le transport dans des dynamiques stochastiques dans [53, 54]. Pour la chaîne d'oscillateurs, les équations d'évolution du système $(X^i, V^i)_{i\in\mathbb{T}_N}$ s'écrivent alors

$$\begin{cases} \dot{X}_t^i = V_t^i \\ \dot{V}_t^i = -\sum_{k=-\ell_N}^{\ell_N} \phi_k \nabla W(X_t^i - X_t^{i+k}) - \nabla U(X_t^i). \end{cases} \tag{1.17}$$

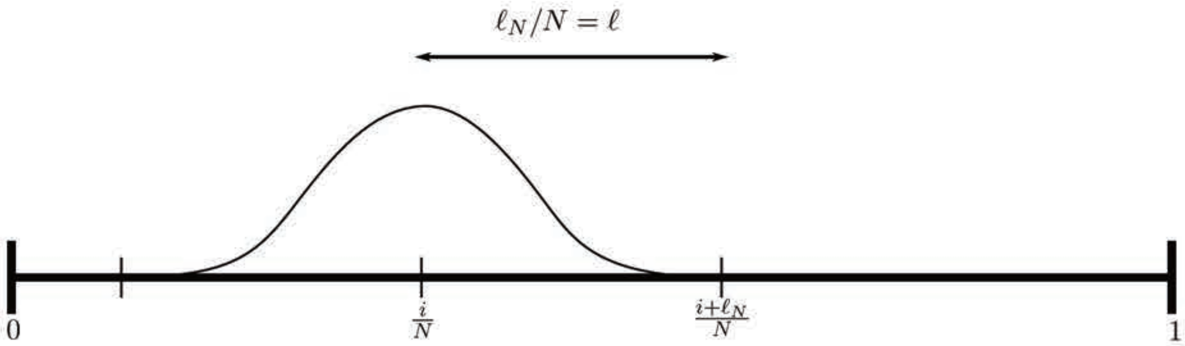

FIGURE 1.1 – Echelles du modèle et forme de la fonction ϕ

Les potentiels de Kac font apparaître une échelle mésoscopique intermédiaire et invitent naturellement à une approche en deux temps pour l'étude du transport d'énergie. En effet, posons $\ell_N = \ell N$, où $\ell \ll 1$ est un petit paramètre et

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{i}{N}, X_t^i, V_t^i} \tag{1.18}$$

la mesure empirique associée au système sur $\mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d$. Alors on peut faire apparaitre plus clairement la structure de champ moyen local en réécrivant (1.17) à l'aide de la mesure empirique

$$\begin{cases} \dot{X}_t^i = V_t^i \\ \dot{V}_t^i = -\int_{\mathbb{T}\times\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{\ell}\phi\left(\frac{i/N - r'}{\ell}\right)\nabla W(X_t^i - x')d\mu_t^N(r', x', v') - \nabla U(X_t^i). \end{cases} \tag{1.19}$$

On s'attend donc à ce que la mesure empirique μ_t^N converge vers la solution d'une équation de Vlasov :

$$\partial_t f_t + v\cdot\nabla_x f_t - \left(\int \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right)\nabla W(x-x')f_t(r',x',v')dr'dx'dz' + \nabla U(x)\right)\cdot\nabla_v f_t = 0, \tag{1.20}$$

comme nous le verrons plus en détail dans la section suivante. Notons toutefois que cette équation dépend du paramètre ℓ. Les équations de Vlasov ont la particularité d'être des équations de transport réversibles et de ne pas être dissipatives. En effet, elles présentent de nombreuses quantités conservées en plus de l'énergie. Nous verrons dans la section suivante qu'elles conservent notamment toutes les intégrales non linéaires de la densité, c'est-à-dire les intégrales de la forme

$$\int A(f_t(r,x,v))drdxdv, \tag{1.21}$$

où A est une fonction quelconque. En particulier, l'entropie

$$\int f_t(r,x,v)\log f_t(z,xv)drdxdv$$

est conservée. Pour pouvoir observer une diffusion, nous allons donc rajouter des collisions stochastiques dissipatives comme celles décrites précédemment. Nous considérerons principalement les deux échanges stochastiques suivants :

1. Echanges de vitesses à longue portée

Dans ce modèle, les collisions stochastiques ont lieu entre des particules à une distance du même ordre que les interactions mécaniques. Entre chaque paire de particules $(i, i+k)$ avec $|k| \le \ell_N$, on place un processus de Poisson $N^{i,i+k} = N^{i+k,i}$ d'intensité $\bar{\gamma}\gamma_k$, où

$$\gamma_k = \frac{1}{\ell_N}\gamma\left(\frac{k}{\ell_N}\right),$$

et γ est une fonction ayant les mêmes propriétés que ϕ. A chaque saut du processus $N^{i,i+k}$, les particules i et $i+k$ échangent leurs vitesses et $\bar{\gamma} > 0$ représente donc le taux d'échange individuel de chaque particule. Ainsi, plus des particules sont proches, plus elles subissent de collisions stochastiques. On peut interpréter ce modèle comme une variante du modèle de Kac pour des particules sur un réseau, où les collisions sont modulées en fonction de la position relative sur le réseau. Nous introduirons plus précisément le modèle de Kac dans la section 1.3 et reviendrons sur les liens entre ce modèle et le modèle présent. Les équations (1.17) deviennent alors

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\sum_{k=-\ell_N}^{\ell_N}\phi_k\nabla W(X_t^i - X_t^{i+k}) + \nabla U(X_t^i)\right)dt \\ \qquad\qquad + \sum_{k=-\ell_N}^{\ell_N}(V_{t_-}^{i+k} - V_{t_-}^i)dN_t^{i,i+k}. \end{cases} \tag{1.22}$$

Dans ce cas, l'équation de Vlasov associée est :

$$\partial_t f_t + v \cdot \nabla_x f_t - \left(\int \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \nabla W(x-x') f_t(r',x',v') dr'dx'dz' + \nabla U(x) \right) \cdot \nabla_v f_t$$
$$= \bar{\gamma} \int \frac{1}{\ell}\gamma\left(\frac{r-r'}{\ell}\right) \left(f_t(r',x',v) f_t(r,x,v') - f_t(r,x,v) f_t(r',x',v')\right) dr'dx'dv', \quad (1.23)$$

et possède donc un terme de collision rappelant l'équation de Boltzmann.

2. Retournement des vitesses
Le second modèle est un modèle avec collisions locales. Tout comme dans [8], on associe un processus de Poisson N^i d'intensité $\bar{\gamma}$ à chaque particule, de telle sorte à ce que V^i change sa vitesse en $-V^i$ à des temps aléatoires donnés par N^i. Les particules suivent alors les équations

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\sum_{k=-\ell_N}^{\ell_N} \phi_k \nabla W(X_t^i - X_t^{i+k}) + \nabla U(X_t^i)\right) dt - 2V_{t\_}^i dN_t^i, \end{cases} \quad (1.24)$$

et l'équation de Vlasov associée est

$$\partial_t f_t + v \cdot \nabla_x f_t - \left(\int \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \nabla W(x-x') f_t(r',x',v') dr'dx'dz' + \nabla U(x) \right) \cdot \nabla_v f_t$$
$$= \bar{\gamma}\left(f_t(r,x,-v) - f_t(r,x,v)\right). \quad (1.25)$$

Un des buts de cette thèse est de prouver la convergence de la mesure empirique (1.18) vers une solution de l'équation de Vlasov dans ces différents cadres (sans collision (1.17), avec échanges à longue portée (1.22) et avec retournement des vitesses (1.24)).

Nous n'étudierons pas de modèles de collisions impliquant uniquement les plus proches voisins. En effet, dans la limite de champ moyen, ces modèles ont une structure locale finalement similaire au flip de vitesses. Mais les équations qui les régissent ne rentrent pas directement dans un formalisme à champ moyen, ce qui rend la preuve de convergence de la mesure empirique plus délicate. Pour un modèle avec échanges de vitesses entre plus proches voisins, on peut néanmoins conjecturer que la mesure empirique converge dans ce cas vers une solution de l'équation de Vlasov suivante

$$\partial_t f_t + v \cdot \nabla_x f_t - \left(\int \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \nabla W(x-x') f_t(r',x',v') dr'dx'dz' + \nabla U(x) \right) \cdot \nabla_v f_t$$
$$= \bar{\gamma} \int_{\mathbb{R}^d} \left(f_t(r,x,v') f_t(r,x,v) - f_t(r,x,v') f_t(r,x,v)\right) dv'. \quad (1.26)$$

Enfin, mentionnons également qu'un modèle différent avec interactions à longue portée a été récemment étudié par Tamaki et Saito [68] et Suda [65]. Dans ces modèles, ϕ_k est une fonction à décroissance polynomiale en k : $\phi_k = |k|^{-\theta}$. Les collisions stochastiques sont des collisions aux plus proches voisins. La convergence de la formule de Green-Kubo en fonction du paramètre θ a été étudiée dans [68], et la propagation de l'énergie suivant une équation de diffusion fractionnaire pour des interactions harmoniques a été prouvée dans [65]. Ces modèles ne présentent pas de structure de champ moyen local comme ceux que nous étudions dans cette thèse, et les méthodes employées dans [65] sont en fait similaires à [5, 38, 39].

1.2 Limites de champ moyen

Comme expliqué précédemment, nous souhaitons prouver une limite de champ moyen, c'est-à-dire prouver que la mesure empirique (1.18) converge vers la solution d'une équation de Vlasov.

Les limites de champ moyen dans le cadre de dynamiques de gaz ont été étudiées rigoureusement à partir des années 70, notamment par Braun et Hepp [14], Dobrushin [21] et Neunzert et Wick [55]. Nous présentons ici la preuve classique d'une telle limite dans un cadre général en s'inspirant des références [33, 67, 72] et verrons quelles difficultés présente le modèle de chaînes d'oscillateurs avec potentiels de Kac.

1.2.1 Preuve dans un cadre classique

Commençons par un cadre sans terme stochastique : N particules de coordonnées $(Y_t^i)_{1\leq i\leq N}$ dans un espace euclidien $\mathcal{E}$, interagissent suivant

$$\dot{Y}_t^i = \frac{1}{N}\sum_{j=1}^N K(Y_t^i, Y_t^j) = \int_{\mathcal{E}} K(Y_t^i, y')d\mu_t^N(y'), \tag{1.27}$$

où $K : \mathcal{E}\times\mathcal{E}\to\mathcal{E}$ et

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}$$

est la mesure empirique associée au système $(Y_t^i)_{1\leq i\leq N}$. En particulier, en choisissant l'espace $\mathcal{E} = \mathbb{T}\times\mathbb{R}^d\times\mathbb{R}^d$, en prenant pour y les trois coordonnées $y = (r, x, v)$, en choisissant le noyau d'interaction

$$K(y, y') = \left(0, v, -\frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right)\nabla W(x-x') - \nabla U(x)\right), \tag{1.28}$$

et en prenant pour coordonnée initiale $r_0^i = i/N$, on retombe sur (1.17). On suppose que K est une fonction lipschitzienne en ses deux coordonnées. La preuve de la limite de champ moyen repose alors sur deux ingrédients.

1. Le premier est l'introduction de l'équation caractéristique non linéaire associée à la mesure initiale $\mu_0 \in \mathcal{M}^1(\mathcal{E})$, où $\mathcal{M}^1(\mathcal{E})$ est l'ensemble des mesures de probabilités sur $\mathcal{E}$. Cette équation nous fournit le comportement typique d'une particule dans la limite où $\mu_0^N \to \mu_0$ et est obtenue formellement en remplaçant la mesure empirique μ_t^N par sa limite μ_t dans (1.27). Plus rigoureusement, elle s'écrit :

$$\begin{cases} \dot{Y}_t = \int_{\mathcal{E}} K(Y_t, y')d\mu_t(y') \\ \mu_t = T_{t*}\mu_0 \\ Y_0 = y_0 \end{cases} \tag{1.29}$$

où $T_{t*}\mu_0$ est la mesure image de μ_0 par T_t et $T_t : \mathcal{E}\to\mathcal{E}$ est le flot associé à (1.29), c'est-à-dire l'application telle que $Y_t = T_t(y_0)$. Si la condition de moment $\int_{\mathcal{E}} |y| d\mu_0$ est satisfaite, alors il existe une unique solution à (1.29) (*c.f.* Théorème 1.3.2 dans [33]). En fait, cette équation ne fait pas qu'encoder le comportement limite d'une particule, mais elle encode également les équations du système de particules (1.27) ! En effet, si l'on prend comme mesure initiale μ_0^N dans (1.29), alors la mesure empirique à l'instant t vérifie $\mu_t^N = T_{t*}\mu_0^N$, et on retrouve alors (1.27).

2. Le second ingrédient est appelé estimation de Dobrushin [33]. Il s'agit d'un résultat de contraction entre les mesures images issues de (1.29). Soient $\mu_0, \widetilde{\mu}_0 \in \mathcal{M}^1(\mathcal{E})$ et notons T_t et $\widetilde{T}_t$ les flots des équations caractéristiques respectivement associées à μ_0 et $\widetilde{\mu}_0$. De même, notons $\mu_t = T_{t*}\mu_0$ et $\widetilde{\mu}_t = \widetilde{T}_{t*}\widetilde{\mu}_0$. Alors, en utilisant l'équation caractéristique (1.29), on peut borner pour tous $y_0, \widetilde{y}_0 \in \mathcal{E}$:

$$|T_t(y_0) - \widetilde{T}_t(\widetilde{y}_0)| \leq \int_0^t ds \left| \int_{\mathcal{E}} K(T_s(y_0), T_s(y'))d\mu_0(y') - \int_{\mathcal{E}} K(\widetilde{T}_s(\widetilde{y}_0), \widetilde{T}_s(\widetilde{y}'))d\widetilde{\mu}_0(\widetilde{y}')\right|, \tag{1.30}$$

où $|\cdot|$ désigne ici la norme euclidienne sur $\mathcal{E}$. Soit π_0 un couplage quelconque des données initiales μ_0 et $\widetilde{\mu}_0$. On peut alors introduire le terme croisé $\int K(\widetilde{T}_s(\widetilde{y}_0), T_s(y'))d\mu_0(y')$ dans le membre de droite, ainsi que le couplage π_0 et utiliser le fait que K soit Lipschitz pour obtenir

$$|T_t(y_0) - \widetilde{T}_t(\widetilde{y}_0)| \leq C \int_0^t ds \left| T_s(y_0) - \widetilde{T}_s(\widetilde{y}_0) \right| + C \int_0^t ds \int_{\mathcal{E}^2} |T_s(y') - \widetilde{T}_s(\widetilde{y}')| d\pi_0(y', \widetilde{y}'),$$

où C est une constante positive. Enfin, en intégrant à gauche et à droite le couple $(y_0, \widetilde{y}_0)$ par rapport à π_0, on obtient alors

$$\int_{\mathcal{E}^2} |T_t(y') - \widetilde{T}_t(\widetilde{y}')| d\pi_0(y', \widetilde{y}') \leq 2C \int_0^t ds \int_{\mathcal{E}^2} |T_s(y') - \widetilde{T}_s(\widetilde{y}')| d\pi_0(y', \widetilde{y}').$$

Par le lemme de Gronwall, on obtient donc

$$\int_{\mathcal{E}^2} |T_t(y') - \widetilde{T}_t(\widetilde{y}')| d\pi_0(y', \widetilde{y}') \leq \mathrm{e}^{2Ct} \int_{\mathcal{E}^2} |y' - \widetilde{y}'| d\pi_0(y', \widetilde{y}'), \tag{1.31}$$

pour tout couplage π_0 de μ_0 et de $\widetilde{\mu}_0$. Enfin, on introduit la distance de Wasserstein entre deux mesures $\mu, \nu \in \mathcal{M}^1(\mathcal{E})$ ayant un moment d'ordre un par

$$\mathcal{W}_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathcal{E}} |y - y'| d\pi(y, y'), \tag{1.32}$$

où $\Pi(\mu, \nu)$ désigne l'ensemble des couplages de μ et de ν. L'existence d'un couplage optimal, c'est-à-dire d'un couplage qui réalise l'infimum dans (2.9) est garantie par un argument simple d'analyse variationnelle (Théorème 4.1 dans [73]). On choisit donc π_0 dans (1.31) comme un couplage optimal de μ_0 et $\widetilde{\mu}_0$. La mesure image de π_0 par $T_t \otimes \widetilde{T}_t$ étant elle-même un couplage de μ_t et de $\widetilde{\mu}_t$, on peut borner inférieurement le membre de gauche de (1.31) par $\mathcal{W}_1(\mu_t, \widetilde{\mu}_t)$ pour arriver à l'estimation de Dobrushin :

$$\mathcal{W}_1(\mu_t, \widetilde{\mu}_t) \leq \mathcal{W}_1(\mu_0, \widetilde{\mu}_0) \, \mathrm{e}^{2Ct}. \tag{1.33}$$

On peut désormais conclure sur la limite de champ moyen. En appliquant l'estimation de Dobrushin (1.33) à μ_0^N et μ_0, on obtient

$$\mathcal{W}_1(\mu_t^N, \mu_t) \leq \mathcal{W}_1(\mu_0^N, \mu_0) \, \mathrm{e}^{2Ct}. \tag{1.34}$$

Par conséquent, si μ_0^N converge vers une mesure μ_0, alors, à un temps t fixé, μ_t^N converge vers μ_t. C'est par exemple le cas si les données initiales $(Y_0^i)_{1 \leq i \leq N}$ sont indépendantes et identiquement distribuées, et on peut même obtenir des taux de convergence précis de $\mathbb{E}[\mathcal{W}_1(\mu_0, \mu_0^N)]$ en $N^{-\kappa}$ par les résultats de Fournier et Guillin [26], où κ dépend de la dimension et d'hypothèses de moments sur μ_0.

1.2.2 Propriétés et première application

1. L'équation (1.29) est en fait l'équation caractéristique associée à l'équation de transport (de Vlasov) suivante

$$\partial_t \mu_t + \int_{\mathcal{E}} K(y, y') d\mu_t(y') \cdot \nabla_y \mu_t = 0. \tag{1.35}$$

On peut ainsi prouver l'unicité des solutions faibles pour cette équation [33]. De plus, si μ_0 a une densité f_0, et que T_t préserve la mesure de Lebesgue (ce qui est le cas pour un flot hamiltonien par exemple), on peut alors exprimer la densité f_t de μ_t par

$$f_t(y) = f_0(T_{-t}(y)),$$

où T_{-t} est l'inverse du flot T_t. On déduit alors directement de cette équation la conservation des intégrales non linéaires (1.21), comme annoncé précédemment.

2. Une des conséquences immédiates de cette preuve est qu'il est possible de déduire une limite de champ moyen pour la chaîne d'oscillateurs sans terme stochastique (1.17) dans le cas où la force ∇W est bornée. En effet, dans ce cas, l'application $(r, x) \mapsto 1/\ell\phi(r/\ell)\nabla W(x)$ est lipschitzienne de constante de Lipschitz $C\ell^{-2}$ pour un certain $C > 0$. Il en va donc de même pour K dans (1.28) et on obtient

$$\mathcal{W}_1(\mu_t^N, \mu_t) \leq \mathcal{W}_1(\mu_0^N, \mu_0)\, e^{2Ct\ell^{-2}} .$$

Néanmoins, cette preuve ne fonctionne pas si l'on suppose ∇W non borné, ce qui est précisément ce que nous voulons étudier puisque nous souhaitons prouver une limite de champ moyen pour des potentiels W proches d'un potentiel harmonique.

Plus généralement, prouver des limites de champ moyen pour des interactions non uniformément Lipschitz est un problème difficile. On peut citer [11] pour un exemple de preuve dans un cas seulement localement Lipschitz. En particulier le cas où le potentiel présente une singularité est un des challenges principaux du domaine (*c.f.* [37] pour un compte rendu récent sur les dernières avancées).

Dans notre cas, cette difficulté reflète une propriété essentielle qui différencie la chaîne d'oscillateurs avec potentiels de Kac des modèles à champ moyen pour des fluides. En effet, pour modéliser des situations hors équilibre dans la chaîne, les propriétés statistiques des particules doivent dépendre de leurs positions dans la chaîne : on ne peut pas supposer que les données initiales $(X_0^i, V_0^i)_{1\leq i\leq N}$ sont égales en loi, ni avoir par conséquent la propriété d'échangeabilité des variables $(X_t^i, V_t^i)_{1\leq i\leq N}$, qui est habituellement obtenue dans des modèles de fluides. Le comportement des coordonnées (X_t^i, V_t^i) est intrinsèquement lié à la position que la particule i occupe dans la chaîne et il faut alors adapter la preuve de l'estimation de Dobrushin pour en tenir compte.

1.2.3 Limite de champ moyen pour la chaîne avec potentiels de Kac sans collisions

Nous présentons ici l'idée de la preuve de la limite de champ moyen pour le modèle de la chaîne avec potentiels de Kac et sans collisions stochastiques (1.17), pour des forces ∇W non bornées. La preuve repose sur le rôle particulier joué par la variable r dont nous ne tenions pas compte dans les deux section précédentes.

Commençons par adapter l'estimation de Dobrushin (1.33) au cas de deux mesures de probabilité μ_0 et $\tilde{\mu}_0$ sur $\mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d$ dont la marginale sur $\mathbb{T}$ est la mesure de Lebesgue. Etant donné le rôle particulier joué par la variable r, les mesures transportées à l'instant t, μ_t et $\tilde{\mu}_t$, ont également pour marginale sur $\mathbb{T}$ la mesure de Lebesgue. Par désintégration de la mesure μ_t (resp. $\tilde{\mu}_t$), on introduit alors la mesure μ_t^r (resp. $\tilde{\mu}_t^r$) correspondant à la loi du couple (x, v) conditionnée à r. Pour simplifier, nous noterons dorénavant $E = \mathbb{R}^d \times \mathbb{R}^d$ et $z = (x, v)$. Nous allons prouver l'estimation de Dobrushin pour la nouvelle distance de Wasserstein suivante :

$$\mathcal{SW}_1(\mu_t, \tilde{\mu}_t) = \int_{\mathbb{T}} dr \mathcal{W}_1(\mu_t^r, \tilde{\mu}_t^r), \tag{1.36}$$

où $\mathcal{W}_1$ désigne ici une distance de Wasserstein sur E. Nous allons reprendre la borne (1.30) en prenant cette fois le noyau de la chaîne (1.28), et ce pour des données initiales $y_0 = (r, z_0)$ et

$\widetilde{y}_0 = (r, \widetilde{z}_0)$ ayant le même indice r. En utilisant le fait que ∇U soit Lipschitz, on a alors

$$\begin{aligned}\left|T_t(r,z_0) - \widetilde{T}_t(r,\widetilde{z}_0)\right| \leq & \int_0^t |T_s(v_0) - \widetilde{T}_s(\widetilde{v}_0)| ds + C\int_0^t |T_s(x_0) - \widetilde{T}_s(\widetilde{x}_0)| ds \\ & + \int_0^t ds \left| \int_{\mathbb{T}\times E} \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \nabla W\left(T_s(x_0) - T_s(x')\right) dr' \mu_0^{r'}(z') \right. \\ & \left. - \int_{\mathbb{T}\times E} \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \nabla W\left(\widetilde{T}_s(\widetilde{x}_0) - \widetilde{T}_s(\widetilde{x}')\right) dr' d\widetilde{\mu}_0^{r'}(\widetilde{z}')\right|,\end{aligned}$$

en utilisant l'abus de notation $T_t(x_0)$ pour la coordonnée en x de $T_t(y_0)$ et similairement pour $T_t(v_0)$. En regroupant les deux premiers termes, puis en regroupant les deux intégrales suivant r' dans le troisième terme, on obtient

$$\begin{aligned}\left|T_t(r,z_0) - \widetilde{T}_t(r,\widetilde{z}_0)\right| \leq & \, C'\int_0^t |T_s(y_0) - \widetilde{T}_s(\widetilde{y}_0)| ds \\ & + \int_0^t ds \int_{\mathbb{T}} dr' \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \left| \int_E \nabla W\left(T_s(x_0) - T_s(x')\right) \mu_0^{r'}(z') \right. \\ & \left. - \int_E \nabla W\left(\widetilde{T}_s(\widetilde{x}_0) - \widetilde{T}_s(\widetilde{x}')\right) d\widetilde{\mu}_0^{r'}(\widetilde{z}')\right|.\end{aligned}$$

Enfin, comme précédemment, on introduit le terme croisé $\int_E \nabla W\left(\widetilde{T}_s(\widetilde{x}_0) - T_s(x')\right) d\mu_0^{r'}(z')$, ainsi qu'un couplage $\pi_0^{r'}$ des données initiales $\mu_0^{r'}$ et $\widetilde{\mu}_0^{r'}$, pour aboutir à :

$$\begin{aligned}\left|T_t(r,z_0) - \widetilde{T}_t(r,\widetilde{z}_0)\right| \leq & \, C''\int_0^t |T_s(r,z_0) - \widetilde{T}_s(r,\widetilde{z}_0)| ds \\ & + \int_0^t ds \int_{\mathbb{T}} dr' \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) \int_{E^2} |T_s(r',z) - T_s(r',\widetilde{z}')|\, \pi_0^{r'}(z,z').\end{aligned}$$

La fin de l'estimation est ensuite similaire à ce que nous avions établi précédemment. On intègre alors à gauche et à droite le couple $(z_0, \widetilde{z}_0)$ par rapport à la mesure π_0^r, puis on intègre r par rapport à la mesure de Lebesgue et on utilise que ϕ vérifie (1.16) pour obtenir

$$\int_{\mathbb{T}} dr \int_{E^2} \left|T_t(r,z') - \widetilde{T}_t(r,\widetilde{z}')\right| d\pi_0^r(z',\widetilde{z}') \ \leq C\int_0^t ds \int_{\mathbb{T}} dr \int_{E^2} \left|T_s(r,z') - \widetilde{T}_s(r,\widetilde{z}')\right| d\pi_0^r(z',\widetilde{z}').$$

Puis, par le lemme de Gronwall, on obtient de même

$$\int_{\mathbb{T}} dr \int_{E^2} \left|T_t(r,z') - \widetilde{T}_t(r,\widetilde{z}')\right| d\pi_0^r(z',\widetilde{z}') \leq \mathrm{e}^{Ct} \int_{\mathbb{T}} dr \int_{E^2} |z' - \widetilde{z}'|\, d\pi_0^r(z',\widetilde{z}').$$

Enfin on choisit π_0^r comme couplage optimal de $(\mu_0^r, \widetilde{\mu}_0^r)$ et on minore le terme de gauche par $\mathcal{SW}_1(\mu_t, \tilde{\mu}_t)$ pour obtenir la nouvelle estimation à la Dobrushin :

$$\mathcal{SW}_1(\mu_t, \tilde{\mu}_t) \leq \mathcal{SW}_1(\mu_0, \tilde{\mu}_0)\, \mathrm{e}^{Ct}. \tag{1.37}$$

Le problème est que l'estimation que nous venons d'établir ne permet pas de conclure quant à la limite de champ moyen du système (1.17) ! En effet, la mesure empirique μ_t^N est atomique et on ne peut pas définir la distance $\mathcal{SW}_1(\mu_t^N, \mu_t)$ par la formule (1.36). L'idée que nous allons développer au chapitre 2 de cette thèse dans un cadre plus général est donc la suivante. Nous allons introduire des boîtes mésoscopiques de taille ϵ_N, où $1/N \ll \epsilon_N \ll 1$. On subdivise ainsi $\mathbb{T}$ en ϵ_N^{-1} boîtes $(B_j)_{1\leq j\leq \epsilon_N^{-1}}$ et on mène le calcul précédent en remplaçant la distance $\mathcal{SW}_1$ par

$$\frac{1}{\epsilon_N} \sum_{j=1}^{\epsilon_N^{-1}} \mathcal{W}_1(\mu_t^{N,j}, \mu_t^j),$$

où $\mu_t^{N,j}$ et μ_t^j sont respectivement les mesures μ_t^N et μ_t conditionnées à la boîte B_j.

1.2.4 Modèles de fluides avec interactions stochastiques locales

Revenons désormais à un cas de particules échangeables interagissant suivant l'équation (1.27), mais où chaque particule subit désormais en plus des forces stochastiques, indépendamment des autres particules. Par exemple, on peut citer le cas de diffusions de McKean-Vlasov

$$dY_t^i = \frac{1}{N}\sum_{j=1}^N K(Y_t^i, Y_t^j)dt + dW_t^i, \tag{1.38}$$

où les $(W^i)_{1\leq i\leq N}$ sont des processus de Wiener indépendants. On peut également considérer le cas de collisions aléatoires locales à des temps exponentiels. On modélise alors ce système par les équation différentielles stochastiques

$$dY_t^i = \frac{1}{N}\sum_{j=1}^N K(Y_t^i, Y_t^j)dt + F(Y_t^i)dN_t^i, \tag{1.39}$$

où $(N^i)_{1\leq i\leq N}$ sont des processus de Poisson indépendants de même intensité et F une fonction lipschitzienne. Le cas de la chaîne d'oscillateurs avec retournement des vitesses (1.24) pouvant être retrouvé à partir de ce modèle général en prenant $F(r,x,v) = -2v$, nous nous concentrerons sur le cas (1.39) par la suite.

Pour prouver la convergence de la mesure empirique μ_t^N, on peut alors procéder comme dans la section 1.2.1 et introduire tout d'abord une équation différentielle non linéaire similaire à (1.29) qui modélise le comportement limite attendu pour les particules du système. Cette équation est cette fois une équation différentielle stochastique, donnée par

$$dY_t = \int_{\mathcal{E}} K(Y_t, y')d\mu_t(y') + F(Y_t)dN_t, \tag{1.40}$$

où μ_t est cette fois la loi du processus Y_t et N a la même intensité que les $(N^i)_{1\leq i\leq N}$. Par un résultat de contraction pour une distance de Wasserstein appropriée, on prouve alors que cette équation est bien posée (c.f le cours de Sznitman [67] pour la preuve dans le cas des équations de McKean-Vlasov (1.38) et l'article de Graham [34] dans un cadre plus général avec des processus de sauts).

N'ayant pas à notre disposition les flots déterministes construits dans (1.29), on adapte l'argument de Dobrushin en introduisant un nouveau système $(\widetilde{Y}^i)_{1\leq i\leq N}$ de processus non linéaires solutions de (1.40), couplé au système $(Y^i)_{1\leq i\leq N}$ en choisissant le même processus de Poisson N^i dans (1.39) et dans (1.40) :

$$d\widetilde{Y}_t^i = \int_{\mathcal{E}} K(\widetilde{Y}_t^i, y')d\mu_t(y')dt + F(\widetilde{Y}_t^i)dN_t^i. \tag{1.41}$$

En particulier, par indépendance des mesures de Poisson $(N^i)_{1\leq i\leq N}$, si les conditions initiales $(\widetilde{Y}_0^i)_{1\leq i\leq N}$ sont indépendantes, alors les processus $(\widetilde{Y}^i)_{1\leq i\leq N}$ sont indépendants. Dans le cas de dynamiques de fluides ou plus généralement de particules identiques ayant les mêmes propriétés statistiques, il est naturel de supposer que les distributions initiales Y_0^i sont de même loi μ_0. On couple alors à l'instant 0 les systèmes $(Y^i)_{1\leq i\leq N}$ et $(\widetilde{Y}^i)_{1\leq i\leq N}$ en imposant les mêmes conditions initiales $Y_0^i = \widetilde{Y}_0^i$. Ainsi, toutes les variables $(\widetilde{Y}_t^i)_{1\leq i\leq N}$ ont la même loi μ_t. En notant

$$\widetilde{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{Y}_t^i}$$

la mesure empirique associée au système $(\widetilde{Y}_t^i)_{1\leq i\leq N}$, on peut alors borner

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \leq \mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \widetilde{\mu}_t^N)\right] + \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N, \mu_t)\right] \tag{1.42}$$

par inégalité triangulaire. En considérant le couplage des mesures atomiques μ_t^N et $\widetilde{\mu}_t^N$ qui associe Y_t^i à $\widetilde{Y}_t^i$ pour tout $1 \leq i \leq N$, on peut borner

$$\mathcal{W}_1(\mu_t^N, \widetilde{\mu}_t^N) \leq \frac{1}{N}\sum_{i=1}^N |Y_t^i - \widetilde{Y}_t^i|.$$

Les couples $(Y_t^i, \widetilde{Y}_t^i)_{1\leq i\leq N}$ ont tous la même loi, donc

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N |Y_t^i - \widetilde{Y}_t^i|\right] = \mathbb{E}\left[|Y_t^1 - \widetilde{Y}_t^1|\right], \tag{1.43}$$

et on peut donc finalement borner le membre de droite dans (1.42) et obtenir

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \leq \mathbb{E}\left[|Y_t^1 - \widetilde{Y}_t^1|\right] + \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N, \mu_t)\right]. \tag{1.44}$$

Les mêmes calculs que pour l'estimation de Dobrushin peuvent être maintenant répétés pour borner le premier terme dans (1.44) par

$$\mathbb{E}\left[|Y_t^1 - \widetilde{Y}_t^1|\right] \leq C\int_0^t \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_s^N, \mu_s)\right] \mathrm{e}^{C(t-s)}\, ds \leq \sup_{0\leq s\leq t} \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_s^N, \mu_s)\right] \mathrm{e}^{Ct}. \tag{1.45}$$

Pour conclure, il faut donc contrôler $\mathbb{E}[\mathcal{W}_1(\widetilde{\mu}_s^N, \mu_s)]$, c'est-à-dire prouver une loi des grands nombres quantitative, ce qui est possible par indépendance des $(\widetilde{Y}_s^i)_{1\leq i\leq N}$ et les résultats de [26] :

$$\mathbb{E}[\mathcal{W}_1(\widetilde{\mu}_s^N, \mu_s)] \leq C_s N^{-\kappa_s}, \tag{1.46}$$

où C_s et κ_s sont des termes dépendant du temps s. Si de plus, μ_t a des moments conservés, alors par [26] on peut trouver des constantes C_s et κ_s indépendantes du temps et conclure :

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \leq C\,\mathrm{e}^{Ct}. \tag{1.47}$$

1.2.5 Commentaires généraux et cas de la chaîne avec collisions

Il est à noter que les bornes (1.34) et (1.47) ne permettent de prouver la limite de champ moyen que pour des temps inférieurs à $\log N$. Comme remarqué dans [37], le cadre où l'on peut espérer obtenir des bornes uniformes en temps est celui où la solution de l'équation de type Vlasov μ_t converge exponentiellement vite vers un unique équilibre. Ainsi, ces bornes uniformes ont par exemple été prouvées pour des systèmes liés aux équations de milieu granulaire [51] ou aux équations de Vlasov-Fokker-Planck [12]. Ces systèmes ont en commun de posséder un terme de dissipation modélisant un équilibre thermique global, qui ramène donc rapidement tout le système à l'équilibre. Pour nos différents modèles de chaînes d'oscillateurs (1.17), (1.22) et (1.24), il n'y a donc pas de raison de penser que l'on puisse obtenir une limite de champ moyen avec une borne qui ne dépende pas du temps pour des conditions initiales hors équilibre. Dans l'optique d'une limite hydrodynamique à partir des équations de Vlasov avec collision, cela peut donc être potentiellement problématique si les échelles de temps permettant d'observer les équations hydrodynamiques sont trop longues.

Par ailleurs, on ne peut pas appliquer directement les résultats de la section 1.2.4 à la chaîne avec potentiels de Kac et retournement des vitesses (1.24), car la propriété d'échangeabilité des variables $(X_t^i, V_t^i)_{1\leq i\leq N}$ qui est habituellement obtenue pour des modèles de champ moyen pour un fluide n'est plus vraie. En particulier, l'égalité (1.43) ne peut plus être utilisée dans ce cadre. On ne peut plus choisir une particule représentative du système et contrôler l'écart $\mathbb{E}[|Y_t^1 - \widetilde{Y}_t^1|]$,

mais on doit désormais considérer le système $(Y^i)_{i\leq N} = (i/N, X^i, V^i)_{1\leq i\leq N}$ dans son ensemble et borner directement
$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|Y_t^i - \widetilde{Y}_t^i|\right].$$
De plus, une difficulté majeure apparait dans l'application de la loi des grands nombres à $\mathbb{E}[\mathcal{W}(\widetilde{\mu}_t^N, \mu_t)]$. En effet, les particules $(\widetilde{Y}^i)_{1\leq i\leq N}$ n'ont cette fois pas la même loi. On s'attend néanmoins à ce que des particules proches dans la chaîne aient des lois proches à l'instant t, et l'on pourrait alors appliquer localement des résultats de type loi des grands nombres.

Une difficulté supplémentaire se pose dans le cas du modèle avec échanges de vitesses (1.22), qui ne rentre pas dans le cadre que l'on vient de décrire. En effet, dans ce modèle les processus de Poisson induisent des échanges conservatifs et on retrouve ainsi plusieurs fois les mêmes processus dans les équations régissant le comportement de plusieurs particules distinctes. Comme nous le verrons, introduire pour ce modèle un système de processus non linéaires similaire à (1.41) induit que celui-ci sera forcément corrélé, et l'application de la loi des grands nombres sera donc impossible en l'état. Résoudre ce problème, également présent dans le modèle de Kac, représente le coeur de la thèse.

1.3 Modèle de Kac

Comme mentionné précédemment, le modèle avec échanges de vitesses à longue portée (1.22) présente une parenté forte avec le modèle de Kac, à la différence que les échanges de vitesses dans (1.22) se font désormais localement, avec une intensité modulée en fonction de la distance dans le cristal. Dans cette section, nous présentons une version simple du modèle de Kac. Les techniques de convergence d'un système de particules de Kac ont connu récemment des avancées importantes et nous allons voir comment adapter lesdites techniques pour prouver la limite de champ moyen dans la chaîne avec échanges à longue portée (1.22).

Le modèle de Kac est un modèle probabiliste introduit pour étudier l'équation de Boltzmann spatialement homogène
$$\partial_t f_t = \int_{\mathbb{R}^d\times[0,\pi]} \psi(|v-v^*|)b(\theta)\left(f_t(v')f_t(v'^*) - f_t(v)f_t(v^*)\right)dv^*d\theta, \tag{1.48}$$
où f_t est une densité de probabilité sur $\mathbb{R}^d$, (v', v'^*) sont des vitesses déterminées explicitement en fonction de (v, v^*, θ), θ l'angle de déviation entre $v - v^*$ et $v' - v'^*$, et $\psi(\cdot)b(\cdot)$ un noyau de collision dont la forme dépend des interactions physiques considérées. Cette équation modélise la distribution des vitesses d'un gaz raréfié spatialement homogène. La preuve de cette équation à partir d'une dynamique newtonienne modélisant les particules du gaz étant complexe, Kac [40] a introduit un système stochastique de N particules échangeables et sans positions spatiales, dont l'état est caractérisé uniquement par leurs vitesses $(V_t^i)_{1\leq i\leq N}$. A des temps exponentiels déterminés à partir de la fonction ψ, la vitesse de chaque paire de particules (V_t^i, V_t^j) est changée par la transformation $(V_t^i, V_t^j) \mapsto (V'^i_t, V'^j_t)$, où θ est un angle aléatoire dont la loi est obtenue à partir de b. Il existe toute une variété de modèles différents en fonction des comportements de ψ et b. Notre but étant de comparer le modèle (1.22) au modèle de Kac, nous prendrons par la suite $\psi = 1$ pour simplifier (molécules Maxweliennes) et n'aborderons pas la question de l'intégrabilité de b (modèles avec ou sans cutoff) en supposant dorénavant que $\int_0^\pi b(\theta)d\theta < \infty$.

Le modèle de Kac faisant également partie de la classe des modèles de champ moyen, il peut aussi être analysé en introduisant un processus de Markov non linéaire analogue aux solutions de (1.40). Plus précisément, on peut introduire l'équation différentielle stochastique
$$dV_t = \int_{\mathbb{R}^d} a(v', V_{t_-}, \theta)dN_t(\theta, v'), \tag{1.49}$$

où la fonction a est définie par $a(v, v', \theta) = v' - v$, N_t est une mesure de Poisson inhomogène d'intensité $b(\theta)d\theta f_t(v')dv'$ et f_t est la densité de la variable V_t. Cette fois, la non linéarité se retrouve dans l'intensité de la mesure de Poisson. Notons qu'originellement, Tanaka [69, 70] a plutôt introduit le processus de Markov V comme solution de l'équation différentielle stochastique

$$dV_t = \int_{[0,\pi]\times\Omega'} a(\bar{V}_t(\omega'), V_{t_-}, \theta) dN_t(\theta, \omega'),$$

où cette fois $\bar{V}$ est un processus défini sur l'espace de probabilité (Ω', P') ayant la même loi que le processus V sous P' et N_t est une mesure de Poisson homogène cette fois, d'intensité $b(\theta)d\theta dP'(\omega')$. Sznitman [66] a lui introduit le processus de Markov V par le biais d'un problème de martingales pour un modèle plus général. Dans le cas où $\int_0^\pi b(\theta)d\theta < \infty$, l'intensité des processus de Poisson susmentionnés est alors finie et on peut prouver par des résultats de représentation de martingales [25, 49] que ces trois points de vue sont en fait équivalents. Ce type de représentation de problème de martingales par des solutions d'équations différentielles stochastiques a également été étendu à des cas sans cutoff [19, 28, 67, 70, 71].

Comme pour les modèles de la section 1.2.4, l'idée est désormais de coupler le système de Kac $(V^i)_{1\le i\le N}$ à un système de processus non linéaires $(\widetilde{V}^i)_{1\le i\le N}$ solutions de (1.49). La difficulté réside dans le fait que si deux particules V^i et V^j subissent une collision dans le système original, le couplage naturel consiste à effectuer la même collision dans le système $(\widetilde{V}^i)_{1\le i\le N}$. Cela va donc induire des corrélations fortes entre les particules $(\widetilde{V}^i)_{1\le i\le N}$ et on ne peut donc pas appliquer de loi des grands nombres quantitative comme (1.46).

Un premier résultat de convergence d'un tel système a été prouvé par Graham et Méléard [36] par une analyse détaillée de l'historique des collisions (*c.f.* [19, 28] pour des extensions dans des cas sans cutoff et non Maxwelliens). Plus récemment, Cortez et Fontbona ont introduit une nouvelle approche pour prouver la convergence d'un système de Kac [16]. Cette approche se base sur un certain formalisme pour représenter les échanges dans le système et sur l'introduction d'un système de particules $(\bar{V}^i)_{1\le i\le N}$ supplémentaire, où les $\bar{V}^i$ sont également des solutions de (1.49), mais sont cette fois indépendants. Ce formalisme de représentation des échanges permet à la fois une construction fine des processus non linéaires corrélés $(\widetilde{V}^i)_{1\le i\le N}$ de telle sorte à ce que l'on puisse contrôler leur écart avec le système original $(V^i)_{1\le i\le N}$, mais aussi de construire les processus non linéaires indépendants $(\bar{V}^i)_{1\le i\le N}$ de telle sorte à ce qu'ils soient proches du système corrélé $(\widetilde{V}^i)_{1\le i\le N}$. En reprenant l'équation (1.42) et en rajoutant la mesure empirique $\bar{\mu}_t^N = 1/N \sum \delta_{\bar{V}_t^i}$ du système de processus non linéaires indépendants $(\bar{V}^i)_{1\le i\le N}$ à l'instant t, on obtient :

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \le \mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \widetilde{\mu}_t^N)\right] + \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N, \bar{\mu}_t)\right] + \mathbb{E}\left[\mathcal{W}_1(\bar{\mu}_t^N, \mu_t)\right]. \tag{1.50}$$

Le premier terme de (1.50) se contrôle comme précédemment dans (1.45), ce qui nous ramène finalement à prouver des estimations sur les deux termes suivants dans (1.50). Le second terme de (1.50) est borné par

$$\mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N, \bar{\mu}_t)\right] \le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N |\widetilde{V}_t^i - \bar{V}_t^i|\right] = \mathbb{E}\left[|\widetilde{V}_t^1 - \bar{V}_t^1|\right],$$

en utilisant l'échangeabilité de $(\widetilde{V}_t^1, \bar{V}_t^1)_{1\le i\le N}$. Ce terme est ainsi contrôlé par le couplage de [16]. Enfin, les particules du système $(\bar{V}^i)_{1\le i\le N}$ étant cette fois indépendantes et de même loi μ_t, on peut contrôler le troisième terme de (1.50) par une loi des grands nombres.

Enfin, si la convergence prouvée dans [16] pour le modèle de Kac est dépendante du temps, notons que Cortez et Fontbona ont par la suite développé leur méthode dans [17] pour prouver une convergence uniforme en temps. Par ailleurs, Fournier et Guillin ont également prouvé la convergence d'un système de particules avec échanges de vitesses continus dans [27] par l'introduction d'un nouveau système de particules indépendantes, l'équation limite étant alors l'équation de Landau.

Nous allons adapter l'approche de Cortez et Fontbona introduite dans [16] dans le chapitre 2 pour prouver la limite de champ moyen pour la chaîne avec échanges à longue portée (1.22). A la différence du modèle de Kac, les particules ne sont plus échangeables et la structure d'échanges est cette fois localisée : plus les particules sont proches dans la chaîne, et plus l'intensité des collisions stochastiques est forte. Nous verrons que le formalisme de [16] pour modéliser les échanges s'adapte néanmoins naturellement à notre cadre.

1.4 Contributions et perspectives

1.4.1 Chapitre 2

Dans le chapitre 2, nous prouvons le théorème 2.1 qui établit que la mesure empirique μ_t^N associée à la chaîne d'oscillateurs avec échanges à longue portée (1.22) converge vers une solution faible μ_t de l'équation de Vlasov (1.23) pour des forces ∇U et ∇W uniformément Lipschitz et non bornées. La mesure μ_t est définie à partir d'un problème de martingale dont nous prouvons également l'existence et l'unicité par la proposition 2.1. Le résultat de convergence de la mesure empirique μ_t^N est quantitatif : sous des hypothèses que nous préciserons au chapitre 2, on prouve le théorème suivant.

Théorème 1.1. *Il existe deux constantes positives K_1 et K_2 indépendantes des paramètres N, ℓ et γ, telles que, pour tout $1/N < \epsilon_N < \ell$,*

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \leq K_1 \left((N\epsilon_N)^{-\frac{1}{4(d+1)}} + \frac{\epsilon_N}{\ell} + \frac{\bar{\gamma}}{1+\bar{\gamma}} \frac{\epsilon_N^{1/2}}{\ell^{1/2}} \right) \mathrm{e}^{K_2(1+\bar{\gamma})t} .$$

Cette preuve s'appuie en partie sur la méthode de couplage de Cortez et Fontbona [16] précédemment évoquée pour traiter les problèmes induits par le caractère conservatif des échanges stochastiques. De plus, pour prendre en compte les problèmes liés à la non-échangeabilité du système de particules, nous introduirons des couplages par boîtes mésoscopiques, comme décrits en section 1.2.1. Cette preuve par boîtes mésoscopiques peut être adaptée pour étudier directement la convergence des modèles sans collisions (1.17) et avec retournement des vitesses (1.24) dans des cas où les forces ∇U et ∇W ne sont pas bornées.

A partir de cette limite de champ moyen, nous étudions ensuite la convergence de l'énergie dans le système (1.22). Même si les interactions hamiltoniennes et les collisions stochastiques semblent avoir la même portée dans ce cadre, on remarque que si le potentiel d'interaction W est harmonique ($W(x) = |x|^2/2$) et que les solutions μ_t de (1.23) vérifient $\int x d\mu_t = 0$, alors l'équation de Vlasov (1.23) se découple et devient

$$\begin{aligned} &\partial_t f_t + v \cdot \nabla_x f_t - (x + \nabla U(x)) \cdot \nabla_v f_t \\ &\qquad = \bar{\gamma} \int \frac{1}{\ell} \gamma\left(\frac{r-r'}{\ell}\right) \left(f_t(r',x',v) f_t(r,x,v') - f_t(r,x,v) f_t(r',x',v')\right) dr'dx'dv'. \end{aligned} \tag{1.51}$$

Cette équation décrit le comportement de particules confinées par un potentiel $|x|^2/2 + U(x)$, indexées par un paramètre r, qui n'interagissent pas mécaniquement mais dont les vitesses sont échangées à des temps exponentiels avec des particules de paramètre r' proche de r. Elle rappelle ainsi le modèle de chaîne d'oscillateurs de Kipnis *et al.* [43], dans lequel les échanges sont purement stochastiques également et pour lequel la loi de Fourier est prouvée. La différence essentielle est que la partie hamiltonienne est totalement enlevée dans [43] et le mécanisme d'échange est plus simple : la somme des énergies de deux particules voisines est uniformément redistribuée entre ces deux particules à des temps exponentiels.

On peut introduire l'énergie locale en r associée à une solution de (1.51) :

$$\mathcal{E}_t(r) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|v|^2 + \frac{1}{2}\int_{\mathbb{T}\times\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{\ell}\phi\left(\frac{r-r'}{\ell}\right) W(x-x')f_t(r',x',v')dr'dx'dv' + U(x)\right) f_t(r,x,v)dxdv. \tag{1.52}$$

En calculant la dérivée temporelle de l'énergie à partir de (1.51), à l'échelle $t\ell^{-2}$, on trouve formellement (en supposant que f est régulier en sa coordonnée r et par un développement de Taylor de f en r) :

$$\partial_t \mathcal{E}_{t\ell^{-2}}(r) = \bar{\gamma} c_\gamma \partial_{rr}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}|v|^2 f_{t\ell^{-2}}(r,x,v)dxdv\right) + O(\ell), \tag{1.53}$$

où

$$c_\gamma = \frac{1}{2}\int_{-1/2}^{1/2} u^2\gamma(u)du.$$

Si l'on arrive à remplacer l'énergie cinétique locale $\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}|v|^2 f_{t\ell^{-2}}(r,x,v)$ par l'énergie locale $\mathcal{E}_{t\ell^{-2}}(r)$, on peut alors fermer cette équation et prouver que l'énergie suit une équation de diffusion à l'échelle $t\ell^{-2}$. C'est cette approche que nous abordons en section 2.4 du chapitre 2, en prouvant un résultat d'équipartition d'énergie pour une classe de potentiels anharmoniques U. Par le résultat de convergence de la mesure empirique μ_t^N du théorème 1.2, on prouve ensuite que l'énergie microscopique converge vers l'énergie associée à une solution faible de (1.53). On prouve alors :

Proposition 1.1. *Si le potentiel d'interaction W est harmonique et le potentiel d'accrochage U vérifie $U(x) = |x|^2\psi(x/|x|)$, où $\psi \in C^2(\mathbb{S}^{d-1}, \mathbb{R}_+^*)$, alors l'énergie associée au système microscopique $\frac{1}{N}\sum_{i=1}^N e^i_{t\ell^{-2}}\delta_{i/N}$ converge vers la solution de l'équation de diffusion*

$$\begin{cases} \partial_t e_t = \bar{\gamma}\frac{c_\gamma}{2}\partial^2_{xx} e_t \\ e_0(r) = \int_E \left(\frac{1}{2}|v|^2 + U(x) + \frac{1}{2}|x|^2\right) d\mu_0(r,x,v), \end{cases}$$

au sens où, pour tout $g \in \mathcal{C}^4(\mathbb{T})$ et tout $T > 0$,

$$\lim_{\ell\to 0}\lim_{N\to\infty}\sup_{t\le T}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N e^i_{t\ell^{-2}} g\left(\frac{i}{N}\right) - \int_{\mathbb{T}} e_t(r)g(r)dr\right|\right] = 0.$$

En particulier, nous verrons que l'on peut choisir $\ell = \ell(N) = c(\log N)^{-1/2}$ et avoir pour tout $g \in \mathcal{C}^4(\mathbb{T})$ et tout $T > 0$,

$$\lim_{N\to\infty}\sup_{t\le T}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N e^i_{t\ell(N)^{-2}} g\left(\frac{i}{N}\right) - \int_{\mathbb{T}} e_t(r)g(r)dr\right|\right] = 0.$$

1.4.2 Chapitre 3

Les résultats du chapitre précédent nous ont permis d'identifier que le transport d'énergie avait lieu à l'échelle de temps $t\ell^{-2}$ pour le modèle avec échanges de vitesses à longue portée (1.22). En appliquant la même méthode pour le modèle avec retournement des vitesses (1.24), on constate néanmoins que l'énergie n'évolue pas à cette échelle de temps. En effet, l'énergie était uniquement transportée par les collisions stochastiques à longue portée dans le modèle (1.22) et ce mécanisme n'est plus présent pour le modèle avec retournement des vitesses. Pour déterminer l'échelle de temps caractéristique du modèle avec retournement des vitesses, nous avons calculé la limite d'échelle de la conductivité thermique associée à la chaîne (1.24) avec potentiels harmoniques par la formule de Green-Kubo, en adaptant les résultats prouvés par Basile, Bernardin et Olla [2] pour des modèles avec interactions aux plus proches voisins. Nous verrons au chapitre 3 que la conductivité thermique du modèle microscopique peut-être définie comme

$$\kappa_N(T) = \frac{1}{2T^2 t_N N}\mathbb{E}_T\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}J^{i,i+k}([0,t_N])\right)^2\right],$$

où $\mathbb{E}_T$ désigne une espérance par rapport à la mesure de Gibbs à température T, t_N désigne un temps dépendant de la taille N du système qui sera défini plus précisément au chapitre 3, tout comme $J^{i,i+k}([0,t_N])$, qui désigne l'intégrale temporelle du courant d'énergie entre les particules des sites i et $i+k$, entre les temps 0 et t_N. Nous prouvons alors le résultat suivant.

Théorème 1.2. *Supposons que les coefficients $(\phi_k)_{-\ell_N\le k\le \ell_N}$ sont constants en k et valent $\phi_k = 1/\ell_N$. Alors, pour toute portée d'interaction ℓ_N telle que $\ell_N \xrightarrow[N\to\infty]{} \infty$ et $\ell_N = o(N^{5/7})$, la conductivité thermique associée à la chaîne (1.24) avec retournement des vitesses et potentiels harmoniques vérifie*

$$\kappa_N(T) \underset{N\to\infty}{\sim} \frac{\bar\kappa}{\ell_N},$$

où $\bar\kappa$ est une constante explicite.

Nous confirmons par ailleurs les résultats du chapitre 2, en prouvant également le résultat suivant.

Théorème 1.3. *Supposons que les coefficients $(\phi_k)_{-\ell_N\le k\le \ell_N}$ sont constants en k et valent $\phi_k = 1/\ell_N$. Alors la conductivité thermique associée à la chaîne (1.22) avec échanges de vitesses à longue portée, avec potentiel d'accrochage non nul et potentiels harmoniques vérifie*

$$\kappa_N(T) \xrightarrow{N\to\infty} \frac{d}{2}\bar\gamma c_\gamma,$$

où

$$c_\gamma = \frac{1}{2}\int_{-1/2}^{1/2}\gamma(u)u^2 du.$$

Nous conjecturons que ces résultats sont valides pour des coefficients $(\phi_k)_{-\ell_N\le k\le \ell_N}$ définis par (1.15) pour une fonction ϕ de classe $\mathcal{C}^\infty$ à support compact. Nous conjecturons également que les résultats du théorème 1.3 sont valables pour des potentiels d'accrochage anharmoniques, uniformément convexes et donnons une ébauche de preuve pour sa résolution.

1.4.3 Perspectives

1. Equation de diffusion pour des potentiels anharmoniques plus généraux

Pour le modèle avec échanges de vitesses à longue portée (1.22), le résultat de la proposition 1.1 nous montre que l'énergie suit une équation de diffusion pour une classe restreinte de potentiels d'accrochage anharmoniques pour lesquels on peut prouver un résultat d'équipartition d'énergie. Le théorème 1.3 et la conjecture associée nous suggèrent que ce résultat reste vrai pour des potentiels anharmoniques plus généraux. Il serait donc intéressant d'étudier ce problème à partir de l'équation de Vlasov (1.51).

L'approche la plus naturelle consiste à comparer les solutions de (1.51) avec des mesures de Gibbs locales de la forme

$$\varphi_t(r,x,v) = \frac{1}{Z_{\beta_t(r)}}\exp\left(-\beta_t(r)\left(\frac{1}{2}|v|^2 + U(x) + \frac{1}{2}|x|^2\right)\right),$$

où $\beta_t(r)$ est la température inverse en r à l'instant t et $Z_{\beta_t(r)}$ est une constante de normalisation, puis de trouver l'équation que devrait alors satisfaire $\beta_t(r)$ à partir de (1.51). On peut prouver que les seules mesures de Gibbs locales solutions de (1.51) sont en fait des mesures de Gibbs à l'équilibre global, c'est-à-dire des mesures où la température $\beta_t(r)$ est constante en temps et en espace.

Puisque les mesures de Gibbs locales avec température variable ne peuvent être solution de (1.51), il faut donc prouver que les solutions f_t de (1.51) convergent vers des mesures de Gibbs φ_t locales lorsque $\ell \to 0$ en un certain sens, par exemple en prouvant que l'entropie relative de f_t par rapport à φ_t tend vers 0. Comme cela a été indiqué dans les articles [6, 57], la difficulté majeure pour appliquer la méthode d'entropie relative dans ce type de modèles réside dans le fait que les vitesses ne sont pas bornées.

2. Fluctuations autour de l'équation de Vlasov

Tandis qu'en appliquant les méthodes du chapitre 2, on ne voit pas d'évolution d'énergie pour le modèle avec retournement de vitesses (1.24) dans la limite de champ moyen à l'échelle temporelle $t\ell^{-2}$, le théorème 1.2 suggère que le transport d'énergie s'effectue à une échelle de temps plus longue. Il serait alors intéressant d'étudier les fluctuations autour de l'équation de Vlasov (1.25) hors équilibre, comme cela a pu être fait dans [47] dans un cadre plus classique sans terme stochastique, et de voir dans quelle mesure ces fluctuations contribuent au transport d'énergie.

3. Cas sans potentiel d'accrochage

Le moment est conservé par les échanges à longue portée. Si le potentiel d'accrochage est absent, la dynamique (1.22) préserve donc le moment en plus de l'énergie. Au vu des calculs présentés au chapitre 3 et des résultats de [2], on peut s'attendre à une conductivité thermique plus élevée si le potentiel d'accrochage est nul, ce qui représente une piste de recherche supplémentaire.

Chapter 2

Vlasov limit for a chain of oscillators with Kac potentials

Abstract
We consider a chain of anharmonic oscillators with local mean field interaction and long-range stochastic exchanges of velocity. We prove the convergence of the empirical measure associated with this chain to a solution of a Vlasov-type equation. We use this convergence to prove energy diffusion for a class of anharmonic potentials.

Contents

2.1 Model and results

2.1.1 Introduction

The study of chains of interacting oscillators has drawn a lot of attention over the past few years. Deriving Fourier's law from an anharmonic chain is a major open problem in statistical mechanics [13]. Mathematically, the model consists in a system of N oscillators, whose displacement and momentum are denoted by $X^i \in \mathbb{R}^d$ and $V^i \in \mathbb{R}^d$ respectively for $1 \leq i \leq N$. Particles interact via a Hamiltonian dynamics, with Hamiltonian given by

$$\mathcal{H} = \sum_{i=1}^{N} \left(\frac{1}{2}|V^i|^2 + \frac{1}{2}\sum_k \phi_k W(X^i - X^{i+k}) + U(X^i) \right),$$

where generally $(\phi_k)_{k\in\mathbb{Z}}$ is such that $\phi_k = 0$ when $|k| > K$ for some integer value K, *i.e.* the interaction is only between oscillators with close lattice index. W is a pair potential modelling the interaction between particles and U is a pinning potential. It is known since [60] that for nearest neighbor harmonic interaction ($i.e$ $K = 1$, $W(x) = x^2$, $U(x) = x^2$) the transport of energy is ballistic and therefore Fourier's law is not valid. The study of the anharmonic chain seems nevertheless out of reach for the moment. However, the model has drawn attention over the past few years with on the one hand the proof of existence of stationary measures for the anharmonic chain coupled to two heat baths with different temperatures (see *e.g.* [22] and [15]), and on the other hand the study of the harmonic chain with additional conservative stochastic collisions that enable to derive hydrodynamic limits (see [1] for a review). In fact, stochastic collisions give enough ergodicity to derive such limits, but for long time scales, calculations rely heavily on the harmonic structure of the interactions.

In this paper, we consider a chain with so-called Kac potentials (see [58] for a detailed introduction), *i.e.* we define the coefficients ϕ_k by

$$\phi_k = \frac{1}{\ell N}\phi\left(\frac{k}{\ell N}\right), \tag{2.1}$$

for $|k| \leq \ell N$, where ℓ is a small parameter and ϕ is a smooth even function, with support included in $[-1/2, 1/2]$ and normalized so that $\int_{-1/2}^{1/2}\phi(r)dr = 1$. Therefore, the model has a local mean field structure at macroscopic distance ℓ. In addition to the Hamiltonian dynamics, we also add stochastic exchanges of velocity between neighbors at distance of order ℓN. To do so, we introduce a smooth function γ with the same properties as ϕ, modulating the intensity of the stochastic exchanges and define for $|k| \leq \ell N$

$$\gamma_k = \frac{1}{\ell}\int_{\left[\frac{k-1/2}{N},\frac{k+1/2}{N}\right]}\gamma\left(\frac{r}{\ell}\right)dr. \tag{2.2}$$

We exchange the velocities of two neighbors at distance k at rate $\bar{\gamma}\gamma_k$, where $\bar{\gamma}$ is a positive parameter that gives the global rate at which a particle undergoes an exchange of velocity. The stochastic exchanges conserve the total energy of the system. We use the local mean field structure of the problem to prove the convergence of the empirical measure associated with the particle system to a Vlasov-type equation, and prove diffusion of the energy for a class of anharmonic pinning potentials.

We also mention that another model of chain of oscillators with long-range interaction has also been studied in [68] and [65]. In this model, the stochastic collisions are short-range and there is no local mean field structure in the mechanical interactions. The techniques used are then different from this paper and are similar to the short-range case [2, 39].

2.1.2 Model and notations

In our setting, particles are indexed by the discrete periodic lattice $\mathbb{Z}/N\mathbb{Z}$. For every $1 \leq i \leq N$, we set $r^i = i/N$. More generally, in what follows, the letter r will refer to a position in the periodic domain $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $z = (x, v)$ stands for the phase space coordinates of a single particle in the set $E := \mathbb{R}^d \times \mathbb{R}^d$. The dynamics followed by (X_t^i, V_t^i), for $1 \leq i \leq N$, is

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\int_{\mathbb{T}\times E}\Phi_\ell(r^i - r')\nabla W(X_t^i - x')d\mu_t^N(r', z') + \nabla U(X_t^i)\right)dt \\ \qquad\qquad + \int_{\mathbb{R}^d}(v' - V_{t-}^i)d\mathcal{N}^{\mu^N, r^i}(t, v'). \end{cases} \tag{2.3}$$

In the system (2.3), we wrote the Hamiltonian contribution by introducing the empirical measure μ_t^N associated with the system of particles:

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{r^i, X_t^i, V_t^i}. \tag{2.4}$$

$\mathcal{N}^{\mu^N, r^i}$ is a point process on $\mathbb{R}_+ \times \mathbb{R}^d$ that directly selects the new velocity of V^i at rate $\bar{\gamma}$ among its neighbors' velocity. More precisely, it is given by

$$\mathcal{N}^{\mu^N, r^i} = \sum_n \delta_{T^{i,n}, V^{i,n}},$$

where $(T^{i,n})_{n \geq 0}$ is the set of jump times of a Poisson process with intensity $\bar{\gamma}$ and, for any $n \geq 0$, $V^{i,n}$ is a random variable whose law given $T^{i,n}$ is

$$\mathbb{P}\left(V^{i,n} = V^{i+k}_{T^{i,n}_-} \mid T^{i,n}\right) = \gamma_k,$$

for all $-\ell N \leq k \leq \ell N$. In (2.3), we also used the notation

$$\Phi_\ell(u) = \frac{1}{\ell}\phi\left(\frac{u}{\ell}\right), \tag{2.5}$$

and we define similarly

$$\Gamma_\ell(u) = \frac{1}{\ell}\gamma\left(\frac{u}{\ell}\right).$$

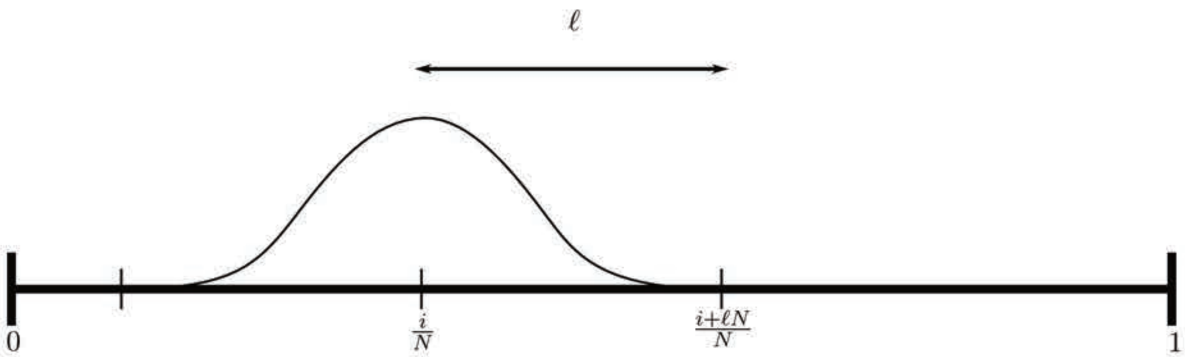

Figure 2.1 – Different scales of the model and shape of the functions Φ_ℓ and Γ_ℓ

To properly define the dynamics for the whole system, we should specify the correlations between the point processes $\mathcal{N}^{\mu^N, r^i}$, which is essentially that the jumps between two different pairs are independent. The detailed definition of those measures is left to Section 2.2. The equation (2.3) is thus a mean field type equation and we expect that the empirical measure μ_t^N converges to a weak solution of the following Vlasov-type equation

$$\begin{aligned}\partial_t f_t + v \cdot \nabla_x f_t - \left(\int_{\mathbb{T}\times E} \Phi_\ell(r-r')\nabla W(x-x') f_t(r',z')dr'dz' + \nabla U(x)\right) \cdot \nabla_v f_t \\ = \bar{\gamma}\int_{\mathbb{T}\times E} \Gamma_\ell(r-r')\left(f_t(r,x,v')f_t(r',x',v) - f_t(r,z)f_t(r',z')\right) dr'dz'.\end{aligned} \tag{2.6}$$

We will actually prove this statement by introducing the following nonlinear martingale problem. For any measure $\nu \in \mathcal{M}^1(\mathbb{T}\times E)$, the space of probability measures on $\mathbb{T}\times E$, define the operator $\mathcal{L}[\nu]$ by

$$\mathcal{L}[\nu]\psi = \mathcal{A}[\nu]\psi + \bar{\gamma}\mathcal{S}[\nu]\psi, \tag{2.7}$$

for all $\psi \in C_b^1(\mathbb{T}\times E)$, the space of bounded continuously differentiable real functions of $\mathbb{T}\times E$. $\mathcal{A}[\nu]$ is a drift operator given by

$$\mathcal{A}[\nu]\psi(r,x,v) = v\cdot\nabla_x\psi(r,x,v) - \left(\int_{\mathbb{T}\times E} \Phi_\ell(r-r')\nabla W(x-x')d\nu(r',z') + \nabla U(x)\right)\cdot\nabla_v\psi(r,x,v),$$

and

$$\mathcal{S}[\nu]\psi(r,x,v) = \int_{\mathbb{T}\times E} (\psi(r,x,v') - \psi(r,x,v))\Gamma_\ell(r-r')d\nu(r',z')$$

is a pure jump operator that exchanges velocities. Denote by $\mathcal{D} = D(\mathbb{R}_+, \mathbb{T}\times E)$ the set of right continuous functions with left limits on $\mathbb{R}_+$ with values in $\mathbb{T}\times E$, by $\mathcal{F}$ the product σ-field on $\mathcal{D}$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by the canonical process $Y = (r, Z) = (r, X, V)$ on $(\mathcal{D}, \mathcal{F}, \mu)$. Then the probability measure $\mu \in M^1(\mathcal{D})$ is said to solve the *nonlinear martingale problem* starting at $\nu_0 \in \mathcal{M}^1(\mathbb{T}\times E)$ if $\mu_0 = \nu_0$ and, for any $\psi \in C_b^1(\mathbb{T}\times E)$,

$$M_t^\psi = \psi(Y_t) - \psi(Y_0) - \int_0^t \mathcal{L}[\mu_s]\psi(Y_s)ds \tag{2.8}$$

is a martingale under μ, μ_s denoting the time marginal of μ at time s. In particular, $r_t = r_0$ for any $t > 0$, and we will indifferently use either the notation r_t, r_0 or r to refer to the spatial coordinate of $(Y_t)_{t\geq 0}$ in $\mathbb{T}$. Taking expectations in (2.8), it is then straightforward to check that the flow of time-marginals $(\mu_t)_{t\geq 0}$ associated with a solution μ to the martingale problem is a weak solution to the Vlasov-type equation (2.6) in the sense of distributions.

2.1.3 Results

We will always assume that the following hypotheses hold

(H1) $W, U \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, the set of twice differentiable functions of $\mathbb{R}^d$ taking values in $\mathbb{R}$. Moreover ∇W and ∇U are uniformly Lipschitz and satisfy $\nabla W(0) = \nabla U(0) = 0$. There exists a positive constant c such that for all $x \in \mathbb{R}^d$,

$$|x|^2 \leq cU(x)$$

$$|\nabla W(x)|^2 \leq cW(x).$$

(H2) $\phi, \gamma \in \mathcal{C}_c^\infty(\mathbb{R})$, the set of infinitely differentiable functions of $\mathbb{R}$ with compact support, and in particular, their support is included in $[-1/2, 1/2]$. Moreover ϕ and γ are non-negative, even functions that are non-increasing on $[0, 1/2]$ and such that $\int_{-1/2}^{1/2}\phi(r)dr = \int_{-1/2}^{1/2}\gamma(r)dr = 1$.

(H3) $\int_{\mathbb{T}\times E}\left(\frac{1}{2}|v|^2 + \frac{1}{2}\int_{\mathbb{T}\times E}\Phi_\ell(r-r')W(x-x')d\mu_0(r',z') + U(x)\right)d\mu_0(r,z) < \infty$.

(H4) $\mu_0 \in \mathcal{M}^1(\mathbb{T}\times E)$ has a density f_0 with respect to the Lebesgue measure and its r-marginal is the uniform measure on $\mathbb{T}$. In particular, for any $r \in \mathbb{T}$, $f_0(r, \cdot)$ is a probability density on E.
Moreover, there exist a probability density h on E with finite first moment $\int_E |z|h(z)dz < \infty$ and a constant $C > 0$ such that for any $r, r' \in \mathbb{T}$ and $z \in E$,

$$|f_0(r,z) - f_0(r',z)| \leq C|r-r'|h(z).$$

The Lipschitz assumption in **(H1)** is classical in mean field theory. The two inequalities on U and W are technical assumptions that hold for harmonic potentials. Hypothesis **(H3)** is a moment assumption which, by conservation of energy, is crucial. **(H4)** is the minimal regularity hypothesis on the r variable at time 0 that we will need to prove the mean field limit. It holds for local Gibbs measures, for which the temperature is a regular function of the r variable for instance. Under these hypotheses, we then have the following proposition.

Proposition 2.1. *There is a unique solution to the nonlinear martingale problem* (2.8) *starting at* μ_0.

In particular, Proposition 2.1 implies existence of weak solutions to (2.6). Denote by $\mathcal{W}_1$ the Wasserstein distance associated with the Euclidean norm $|\cdot|$ on $\mathbb{T}\times E$

$$\mathcal{W}_1(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{T}\times E} |y-y'| d\pi(y,y'), \tag{2.9}$$

where $\Pi(\mu,\nu)$ is the set of couplings of the probability measures $\mu,\nu \in \mathcal{M}^1(\mathbb{T}\times E)$, and we used the shortened notation $y := (r,z) = (r,x,v)$. We are now able to state the mean field convergence result.

Theorem 2.1. *Let μ be the solution of the nonlinear martingale problem (2.8) starting at μ_0. Assume that the initial coordinates $(Z_0^i)_{i\le N}$, with $Z_0^i = (X_0^i, V_0^i)$, are independent and with respective density distribution $f_0(i/N, z)$ for all $1 \le i \le N$. Then there exist two positive constants K_1 and K_2 such that for any $1/N < \epsilon_N < \ell$,*

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N,\mu_t)\right] \le K_1 \left((N\epsilon_N)^{-\frac{1}{4(d+1)}} + \frac{\epsilon_N}{\ell} + \frac{\bar\gamma}{1+\bar\gamma}\frac{\epsilon_N^{1/2}}{\ell^{1/2}} \right) \mathrm{e}^{K_2(1+\bar\gamma)t}.$$

The constants K_1 and K_2 in the theorem do not depend on the three parameters N, ℓ and $\bar\gamma$, but only on the potentials W and U, on the functions ϕ and γ, and the initial measure μ_0. ϵ_N is a coarse-graining parameter that naturally appears in the proof of Theorem 2.1 and is precisely defined in Section 2.2. In particular, for fixed $\bar\gamma$, choosing $\epsilon_N = \ell^{\frac{2d+2}{2d+3}} N^{-\frac{1}{2d+3}}$, we deduce that there exist positive constants K and K' such that for any $\ell > 1/N$,

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N,\mu_t)\right] \le K(\ell N)^{-\frac{1}{2(2d+3)}} \mathrm{e}^{K't}.$$

One of the main features of the particle system we consider is that, contrary to classical mean field theory for gases, the sequence $(Z_t^i)_{i\le N}$ is not exchangeable. The behavior of Z_t^i is intrinsically tied with the position r^i in the chain. In particular, one cannot prove the mean field limit by comparing the law of one typical oscillator Z_t^i at time t to μ_t, as usually done in the mean field theory (see [21, 33, 67]). Instead, the whole system $(r^i, Z_t^i)_{i\le N}$ has to be compared to μ_t in its entirety.

Another difficulty comes from the fact that, even if ∇W is uniformly Lipschitz over $\mathbb{R}^d$, the map $(r,x)\mapsto \Phi_\ell(r)\nabla W(x)$ is not uniformly Lipschitz over $\mathbb{T}\times\mathbb{R}^d$ in general. Consequently, classical mean field limit proofs do not readily work in this situation, and even the proof of Proposition 2.1 is not straightforward. We bypass this difficulty for Proposition 2.1 by proving a contraction estimate for a well-suited distance, the sliced Wasserstein distance (see Definition 2.1). Its proof is postponed to Section 2.3.

Theorem 2.1 is proved in Section 2.2. The proof is based on a coupling of the particle system $(Y^i)_{i\le N} := (r^i, Z^i)_{i\le N}$ to a new system $(\widetilde{Y}^i)_{i\le N}$, whose law is based on the solution μ to the nonlinear martingale problem (2.8), and which are driven by the same Poisson measures as the original particle system. We use an original coupling over mesoscopic boxes in $\mathbb{T}$, and control directly the averages $1/N\sum \mathbb{E}[|Y^i - \widetilde{Y}^i|]$ to circumvent the aforementioned difficulties arising in this mean field limit. Moreover, contrary to classical McKean-Vlasov theory (see [67] for instance), since the stochastic terms contribute via conservative exchanges, the new system $(\widetilde{r}^i, \widetilde{Z}^i)_{i\le N}$ is heavily correlated. We will therefore introduce a third system of independent processes, based on the techniques developed in [16] for the Kac model (see also [27] in the case of Brownian exchanges).

Finally, in Section 2.4, we study transport of energy in appropriate scales for the system by using this mean field limit. We will consider the following additional moment and symmetry hypothesis on the initial distribution f_0:

(H5) for any $(r,z)\in\mathbb{T}\times E$, $f_0(r,z) = f_0(r,-z)$.

(H6) $\int_{\mathbb{T}\times E}\left(|v|^{2+2b}+|x|^{2+2b}\right)d\mu_0(r,x,v)<\infty$ for some $b>0$.

We prove in Lemma 2.7 that the symmetry **(H5)** is preserved at any later time t for the solution of the nonlinear martingale problem. Therefore, for a harmonic interaction potential $W(x)=|x|^2/2$, μ is a weak solution to the simpler equation

$$\begin{aligned}\partial_t f_t + v\cdot\nabla_x f_t - \Big(x+\nabla U(x)\Big)\cdot\nabla_v f_t \\ = \bar\gamma\int_{\mathbb{T}\times E}\Gamma_\ell(r-r')\left(f_t(r,x,v')f_t(r',x',v)-f_t(r,z)f_t(r',z')\right)dr'dz', \end{aligned} \tag{2.10}$$

and the term coming from the interaction potential is therefore reduced to an additional pinning term (associated with a harmonic potential). Therefore, energy is only transmitted by the noise in the mean field limit. Define the energy of particle i at time t by

$$\mathcal{E}_t^i := \frac{1}{2}|V_t^i|^2 + U(X_t^i) + \frac{1}{4}\sum_{k=-\ell N}^{\ell N}\phi_k|X_t^i - X_t^{i+k}|^2, \tag{2.11}$$

we prove that $\mathcal{E}_t^i$ evolves diffusively for a class of anharmonic pining potentials U.

Proposition 2.2. *Suppose W is harmonic and $U(x)=|x|^2\psi(x/|x|)$, where $\psi\in C^2(\mathbb{S}^{d-1},\mathbb{R}_+^*)$. $\frac{1}{N}\sum_{i=1}^N\mathcal{E}^i_{t\ell^{-2}}\delta_{i/N}$ converges to the solution of*

$$\begin{cases}\partial_t e_t = \bar\gamma\frac{c_\gamma}{2}\partial^2_{xx}e_t\\ e_0(r)=\int_E\left(\frac{1}{2}|v|^2+U(x)+\frac{1}{2}|x|^2\right)d\mu_0(r,x,v),\end{cases}$$

where $c_\gamma=\frac{1}{2}\int_{-1/2}^{1/2}u^2\gamma(u)du$, in the sense that for any $g\in\mathcal{C}^4(\mathbb{T})$ and any $T>0$,

$$\lim_{\ell\to0}\lim_{N\to\infty}\sup_{t\le T}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N\mathcal{E}^i_{t\ell^{-2}}g\left(\frac{i}{N}\right)-\int_{\mathbb{T}}e_t(r)g(r)dr\right|\right]=0.$$

The potentials U in Proposition 2.2 are exactly C^2 homogeneous functions of degree 2, which satisfy **(H1)** and for which one can prove an equipartition theorem to close the diffusion equation for the energy. In particular, we will see that one can construct a function $\ell=\ell(N)=c(\log N)^{-1/2}$ for some constant $c>0$, such that

$$\lim_{N\to\infty}\sup_{t\le T}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N\mathcal{E}^i_{t\ell(N)^{-2}}g\left(\frac{i}{N}\right)-\int_{\mathbb{T}}e_t(r)g(r)dr\right|\right]=0. \tag{2.12}$$

2.2 Mean field limit

To define fully define the system (2.3) with the correlation structure of the different Poisson random measures, we use a construction similar to [16] for the Kac model. Instead of considering a collection of Poisson random measure on $\mathbb{R}_+\times\mathbb{R}^d$ that select velocities as in (2.3), we first define a global Poisson random measure $\mathcal{N}$ on $\mathbb{R}_+\times\mathbb{T}^2$ that selects the pairs of particles which exchange velocities. $\mathcal{N}$ selects points in the periodic square $\mathbb{T}^2$ with intensity

$$\bar\gamma N^2\sum_{i=1}^N\left(\gamma_0\mathbb{1}_{(r,r')\in\Lambda^i\times\Lambda^i}+\frac{1}{2}\sum_{\substack{k=-\ell N\\k\neq0}}^{\ell N}\gamma_k\mathbb{1}_{(r,r')\in\Lambda^i\times\Lambda^{i+k}}\right)dtdrdr', \tag{2.13}$$

where $\Lambda^i=[(i-1/2)/N,(i+1/2)/N]$ for $-N<i\le N$. Note that we identify Λ^i with Λ^{i-N} for any $1\le i\le N$ as we work in the torus. The velocity exchanges between two different particles

with indices i and $i+k$ only happen when the Poisson random measure hits either $\Lambda^i \times \Lambda^{i+k}$ or $\Lambda^{i+k} \times \Lambda^i$, which happens at rate $\bar{\gamma}\gamma_k$.

We associate to the particle indexed by i a Poisson random measure $\mathcal{N}^i$ on $\mathbb{R}_+ \times \mathbb{T}$ that only selects the velocity exchanges between this particle and one of its neighbours:

$$\mathcal{N}^i(dt, dr) = \mathcal{N}(dt, \Lambda^i, dr) + \mathcal{N}(dt, dr, \Lambda^i) - \mathcal{N}(dt, \Lambda^i, \Lambda^i). \tag{2.14}$$

$\mathcal{N}^i$ has thus intensity

$$\bar{\gamma} N \sum_{k=-\ell N}^{\ell N} \gamma_k \mathbb{1}_{r \in \Lambda^{i+k}} dt dr. \tag{2.15}$$

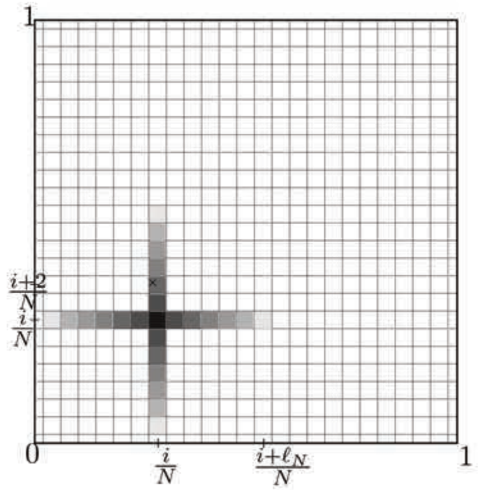

Figure 2.2 – The colors indicate the intensity of the Poisson random measure $\mathcal{N}$ over the line $\mathbb{T} \times \Lambda^i$ and the column $\Lambda^i \times \mathbb{T}$, from which we construct the Poisson random measure $\mathcal{N}^i$. In this figure, the random measure hits the square $\Lambda^i \times \Lambda^{i+2}$, which thus produces an exchange of velocity between particles i and $i+2$.

With these notations, the evolution of the coordinates (X_t^i, V_t^i) for any $1 \leq i \leq N$ is given by:

$$\left\{ \begin{array}{l} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\sum_{k=-\ell N}^{\ell N} \phi_k \nabla W(X_t^i - X_t^{i+k}) + \nabla U(X_t^i) \right) dt + \int_{\mathbb{T}} (V_t^{[Nr']} - V_{t-}^i) d\mathcal{N}^i(t, r'). \end{array} \right. \tag{2.16}$$

We denote by the brackets in $[Nr']$ the rounded value to the nearest integer of Nr', which might be bigger or smaller than Nr', and with any arbitrary convention for half-integers. The time intensity of $\mathcal{N}^i$ being bounded by $\bar{\gamma}$, this evolution is well defined for any N under hypothesis **(H1)** and **(H2)** on the potentials. Adapting the construction from [16], we now define a family of processes $(\widetilde{Y}^i)_{i \leq N} = (\widetilde{r}^i, \widetilde{X}^i, \widetilde{V}^i)_{i \leq N}$ coupled to the original system $(Y^i)_{i \leq N} = (r^i, X^i, V^i)_{i \leq N}$, by

$$\left\{ \begin{array}{l} d\widetilde{X}_t^i = \widetilde{V}_t^i dt \\ d\widetilde{V}_t^i = -\left(\int_{\mathbb{T} \times E} \Phi_\ell(\widetilde{r}_0^i - r') \nabla W(\widetilde{X}_t^i - x') d\mu_t(y') + \nabla U(\widetilde{X}_t^i) \right) dt \\ \qquad\qquad + \int_{\mathbb{T}} (\Pi_t^i(r') - \widetilde{V}_{t-}^i) d\mathcal{N}^i(t, r'), \end{array} \right. \tag{2.17}$$

where μ is the solution of the nonlinear martingale problem (2.8). $\widetilde{Y}^i$ and Y^i are thus driven by the same Poisson random measure $\mathcal{N}^i$ for any $i \leq N$. Π^i is a measurable mapping that will be precisely defined in Lemma 2.1, in such a way that when $\mathcal{N}^i$ selects a neighbor, the update of velocity for Y^i and $\widetilde{Y}^i$ is close enough. $\Pi_t^i(r')$ should be actually also written as a function

$\Pi_t^i(r', \tilde{r}_0^i, \mathbf{V}_{t_-}^i)$ of $\tilde{r}_0^i$ and of the vector $\mathbf{V}_{t_-}^i := (V_{t_-}^{i+k})_{-\ell N \leq k \leq \ell N}$, but we omit this dependence for notational convenience. Also, notice that the spatial parameters r^i and $\tilde{r}^i$ are constant and equal to their initial values r_0^i and $\tilde{r}_0^i$, and we will often write r^i to refer to r_0^i. We call $(\widetilde{Y}^i)_{i\leq N}$ the *nonlinear processes*.

It just remains to choose the initial distribution of $(\widetilde{Y}^i)_{i\leq N}$. Let us first comment on the proof strategy before defining $(\widetilde{Y}_0^i)_{i\leq N}$. By triangular inequality,

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \mu_t)\right] \leq \mathbb{E}\left[\mathcal{W}_1(\mu_t^N, \widetilde{\mu}_t^N)\right] + \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N, \mu_t)\right]. \tag{2.18}$$

where $\widetilde{\mu}_t^N = 1/N \sum \delta_{Y_t^i}$ is the empirical measure associated with the nonlinear processes. The first term on the right-hand side can be controlled by the coupling, while we need to prove an instance of law of large numbers for empirical measures to control the second term. To prove a law of large numbers result, we face two major issues. First, the system $(\widetilde{Y}^i)_{i\leq N}$ is strongly correlated since the Poisson random measures $(\mathcal{N}^i)_{i\leq N}$ share atoms. Following [16], we will define a new system of nonlinear independent processes $(\bar{Y}^i)_{i\leq N}$ from $(\widetilde{Y}^i)_{i\leq N}$ and compare both systems. In particular, we will take the initial conditions $\bar{Y}_0^i$ and $\widetilde{Y}_0^i$ to be equal almost surely for any $i \leq N$. This boils down to prove now that the empirical measure $\bar{\mu}_t^N$ associated with the system of independent nonlinear processes converges to μ_t.

Secondly, to prove a law of large numbers type result for $\bar{\mu}_t^N$, we need the variables $\bar{Y}_t^i$ to have a similar enough distribution. But, even if we have some regularity on the r-parameter for the initial distribution μ_0 by **(H3)**, we do not know any regularity property at time t. In particular, we dot not know to what extent two nonlinear processes $\bar{Y}^i$ and $\bar{Y}^j$ with close spatial parameters $\bar{r}^i$ and $\bar{r}^j$ have a similar distribution.

Therefore, it is easier to deal with truly identically distributed random variables. To do so, we first introduce mesoscopic boxes. Let ϵ_N be a parameter representing the macroscopic size of the aforementioned boxes such that $1/N < \epsilon_N < \ell$, and assume ϵ_N goes to 0 with N at a rate to be defined later on. Assume for simplicity that the parameter ϵ_N is such that ϵ_N^{-1} is an integer. We subdivide $\mathbb{T}$ in exactly ϵ_N^{-1} boxes $(B_j)_{1\leq j\leq \epsilon_N^{-1}}$, where $B_j =]j\epsilon_N, (j+1)\epsilon_N]$, for $1 \leq j \leq \epsilon_N^{-1}$. We add the extra technical assumption that the integer ϵ_N^{-1} divides N, so that each box contains exactly $N\epsilon_N$ particles of the original system. Denote by NB_j the set of indices of the particles in the box B_j:

$$NB_j = \{jN\epsilon_N + 1, jN\epsilon_N + 2, ..., (j+1)N\epsilon_N\}.$$

Denote by μ_t^j the distribution μ_t conditioned on the fact that the coordinate r belongs to B_j:

$$\mu_t^j(r, z) = \epsilon_N^{-1}\mathbb{1}_{r\in B_j}\mu_t(r, z). \tag{2.19}$$

For any $1 \leq j \leq \epsilon_N^{-1}$, the variables $(\widetilde{Y}_0^i)_{i\in NB_j}$ (and therefore $(\bar{Y}_0^i)_{i\in NB_j}$ as well) are then taken independent and with distribution μ_0^j. In particular, $\tilde{r}_0^i$ is a uniformly distributed over B_j for any $i \in NB_j$.

Notice that, in a given box B_j, the spatial parameters $(\widetilde{r}^i)_{i\in NB_j}$ are not ordered as their respective indices $i \in NB_j$ in general. Yet, with this construction, we can always localize $\widetilde{r}^i$ and get the uniform estimate

$$|i/N - \widetilde{r}^i| \leq \epsilon_N. \tag{2.20}$$

This estimate, together with energy bounds involving ∇W, will help compensating the fact we do not have the uniform Lipschitz property for $\Phi_\ell(r)\nabla W(x)$. The idea of doing couplings over mesoscopic boxes $(B_j)_{1\leq j\leq \epsilon_N^{-1}}$ will appear several times during the proof due to the local mean field structure of our problem.

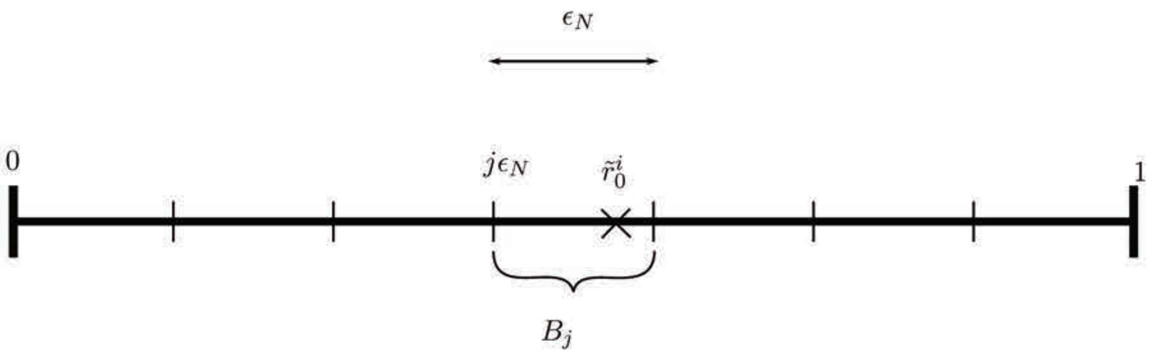

Figure 2.3 – Boxes B_j of size ϵ_N. For $i \in NB_j$, $\widetilde{r}_0^i$ is a uniform random variable in B_j

Before turning to the proof of Theorem 2.1, let us state a series of lemmas that we will use and whose proofs are postponed to the end of the section. In the whole proof and in the lemmas, K, K' are constants that change from one line to another that depend on U, W, ϕ, γ, but not on N, ℓ, $\bar{\gamma}$ since we want to track the dependence of our convergence rates on these terms. Define for any $1 \leq i \leq N$ the probability density on $\mathbb{T}$

$$\sigma^i(r) = N \sum_{k=-\ell N}^{\ell N} \gamma_k \mathbb{1}_{r \in \Lambda^{i+k}}, \tag{2.21}$$

from which $\mathcal{N}^i$ selects a neighbor of particle i to exchange velocities with, see (2.15). Define also the weighted distribution

$$w^{\mu_t, \widetilde{r}_0^i}(v') = \int_{\mathbb{T}\times\mathbb{R}^d} \Gamma_\ell(\widetilde{r}_0^i - r') d\mu_t(r', x', v'), \tag{2.22}$$

which is the velocity marginal of μ_t, modulated by the function Γ_ℓ around the axial parameter $\widetilde{r}_0^i$. This is the distribution we expect for the updates of velocity of the nonlinear processes with axial parameter $\widetilde{r}_0^i$. In the following lemma, adapted from [16], we construct the map Π^i introduced in (2.17).

Lemma 2.1. *Let $1 \leq j \leq \epsilon_N^{-1}$ and $i \in NB_j$. Then for any N, there exists a measurable mapping*

$$\Pi^i : \mathbb{R}_+ \times \mathbb{T} \times B_j \times (\mathbb{R}^d)^{2\ell N+1} \to \mathbb{R}^d,$$

such that, for any $\widetilde{r} \in B_j$ and any $\mathbf{v} \in (\mathbb{R}^d)^{2\ell N+1}$, if r is drawn from (2.21), then $\Pi_t^i(r, \widetilde{r}, \mathbf{v})$ has distribution $w^{\mu_t, \widetilde{r}}$ (see (2.22)). Moreover, for any coupling π_t of μ_t^N and μ_t, the following bounds hold

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{\mathbb{T}} |V_t^{[Nr]} - \Pi_t^i(r, \widetilde{r}_0^i, \mathbf{V}_t^i)|\sigma^i(r)dr\right] &\leq K\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} |v' - v''| d\pi_t(y', y'')\right] \\ &+ \frac{K}{\ell^2}\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} |v''| \mathbb{1}_{\{|\widetilde{r}_0^i - r''| < \ell/2\}} \left|r' - r''\right| d\pi_t(y', y'')\right] + \frac{K\epsilon_N}{\ell}, \end{aligned} \tag{2.23}$$

and also

$$\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}} \sum_{i \in NB_j} \mathbb{E}\left[\int_{B_j} \left|\Pi_t^i(r, \widetilde{r}_0^i, \mathbf{V}_t^i)\right| \sigma^i(r) dr\right] \leq K\frac{\epsilon_N^{1/2}}{\ell^{1/2}}. \tag{2.24}$$

From (2.23), we can control the difference of the velocities' updates between the original system and the nonlinear processes. (2.24) is a useful bound that we will need. The next lemma shows the marginal distribution at time t of the nonlinear process $\widetilde{Y}^i$ for $i \in NB_j$ defined in (2.17) is indeed μ_t^j, where μ_t^j is defined in (2.19).

Lemma 2.2. *Let $1 \leq j \leq \epsilon_N^{-1}$ and $i \in NB_j$. The process $(\widetilde{Y}^i_t)_{t\geq 0}$ satisfying (2.17) and such that $\widetilde{Y}^i_0$ has distribution μ^j_0 is well defined and has distribution μ^j_t.*

The next lemma shows that we can choose a good coupling of the initial conditions of the original system Y^i_0 and of the nonlinear processes $\widetilde{Y}^i_0$.

Lemma 2.3. *There exists a coupling of $(Y^i_0)_{i\leq N}$ and $(\widetilde{Y}^i_0)_{i\leq N}$ such that, for any $1 \leq i \leq N$,*

$$\mathbb{E}\left[|Y^i_0 - \widetilde{Y}^i_0|\right] \leq K\epsilon_N,$$

for some constant K.

This last lemma gives uniform in time moment bounds by conservation type arguments.

Lemma 2.4. *The solution μ of the nonlinear martingale problem (2.8) satisfies*

$$\sup_{t\geq 0} \int_{\mathbb{T}\times E} |z|^2 d\mu_t(r,z) < \infty.$$

For the particle system, the following bound holds

$$\sup_{N\geq 1}\sup_{t\geq 0} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|V^i_t|^2\right] < \infty.$$

2.2.1 Proof of Theorem 2.1

Proof. Recall that in (2.18), we got

$$\mathbb{E}\left[\mathcal{W}_1(\mu^N_t, \mu_t)\right] \leq \mathbb{E}\left[\mathcal{W}_1(\mu^N_t, \widetilde{\mu}^N_t)\right] + \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}^N_t, \mu_t)\right]$$

by triangular inequality. It suffices to prove the convergence of the two terms on the right-hand side to conclude. Following the terminology in [16], we will call the first step that consists in bounding $\mathbb{E}[\mathcal{W}_1(\mu^N_t, \widetilde{\mu}^N_t)]$ the *coupling* and the second step that consists in bounding $\mathbb{E}[\mathcal{W}_1(\widetilde{\mu}^N_t, \mu_t)]$ the *decoupling*.

Step 1: Coupling

First, as the two measures μ^N_t and $\widetilde{\mu}^N_t$ are atomic, it is easy to bound

$$\mathcal{W}_1(\mu^N_t, \widetilde{\mu}^N_t) \leq \frac{1}{N}\sum_{i=1}^N |Y^i_t - \widetilde{Y}^i_t|,$$

and we are led to bound the expectation of the term on the right-hand side. By the evolution equations (2.16) for $(Y^i)_{1\leq i\leq N}$ and (2.17) for $(\widetilde{Y}^i)_{1\leq i\leq N}$, we can bound:

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|Y^i_t - \widetilde{Y}^i_t|\right] \leq{}& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|Y^i_0 - \widetilde{Y}^i_0|\right] + \int_0^t \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|Y^i_s - \widetilde{Y}^i_s|\right] ds \\
&+ \int_0^t ds \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Bigg[\Bigg| \int_{\mathbb{T}\times E} \Phi_\ell\left(\frac{i}{N} - r'\right) \nabla W(X^i_s - x') d\mu^N_s(r',z') \\
&\qquad - \int_{\mathbb{T}\times E} \Phi_\ell(\widetilde{r}^i_0 - r'')\nabla W(\widetilde{X}^i_s - x'') d\mu_s(r'',z'')\Bigg|\Bigg] \\
&+ \int_0^t \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|\nabla U(X^i_s) - \nabla U(\widetilde{X}^i_s)|\right] ds \\
&+ \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_0^t \int_{\mathbb{T}} |V^{[Nr]} - \Pi^i_s(r) - \left(V^i_{s_-} - \widetilde{V}^i_{s_-}\right)| d\mathcal{N}^i(s,r)\right].
\end{aligned} \tag{2.25}$$

The difference of the force terms $|\nabla U(X_s^i) - \nabla U(\widetilde{X}_s^i)|$ is easy to bound by $K|X_s^i - \widetilde{X}_s^i|$ by the Lipschitz property for ∇U. As for the force term involving ∇W in (2.25), if we denote by π_s a coupling between μ_s^N and μ_s, we can bound it by:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \left|\Phi_\ell\left(\frac{i}{N}-r'\right)\nabla W(X_s^i - x') - \Phi_\ell(\widetilde{r}_0^i - r'')\nabla W(\widetilde{X}_s^i - x'')\right| d\pi_s(y',y'')\right] \\
&\quad\le \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{\mathbb{T}\times E} \Phi_\ell\left(\frac{i}{N}-r'\right)\left|\nabla W(X_s^i - x') - \nabla W(\widetilde{X}_s^i - x')\right| d\mu_s^N(y')\right] \\
&\quad\quad+ \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \Phi_\ell\left(\frac{i}{N}-r'\right)\left|\nabla W(\widetilde{X}_s^i - x') - \nabla W(\widetilde{X}_s^i - x'')\right| d\pi_s(y',y'')\right] \\
&\quad\quad+ \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \left|\Phi_\ell\left(\frac{i}{N}-r'\right) - \Phi_\ell\left(\frac{i}{N}-r''\right)\right|\left|\nabla W(\widetilde{X}_s^i - x'')\right| d\pi_s(y',y'')\right] \\
&\quad\quad+ \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \left|\Phi_\ell\left(\frac{i}{N}-r''\right) - \Phi_\ell(\widetilde{r}_0^i - r'')\right|\left|\nabla W(\widetilde{X}_s^i - x'')\right| d\mu_s(y'')\right] \\
&\quad\le A_1 + A_2 + A_3 + A_4,
\end{aligned}$$

by introducing the four cross terms in the second line and denoting by A_k the term numbered $1 \le k \le 4$ at the last line. By the Lipschitz property for ∇W, the first term is bounded by

$$A_1 \le K\frac{1}{N}\sum_{i=1}^N \sum_{k=-\ell N}^{\ell N} \Phi_\ell\left(\frac{i-k}{N}\right)\mathbb{E}\left[\left|X_s^i - \widetilde{X}_s^i\right|\right] \le K'\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i - \widetilde{X}_s^i\right|\right], \tag{2.26}$$

since $\sum_k \Phi_\ell(k/N)$ is bounded. Similarly, the second term is bounded by

$$A_2 \le K'\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} |x' - x''|\, d\pi_s(y',y'')\right]. \tag{2.27}$$

It remains to choose a good coupling to bound this term. The fourth term is also relatively easy to bound by the Lipschitz property for Φ_ℓ, which has Lipschitz constant equal to $Lip(\phi)\ell^{-2}$ by (2.5):

$$\begin{aligned}
A_4 &\le \frac{K}{\ell^2}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \left|\frac{i}{N} - \widetilde{r}_0^i\right| \mathbb{1}_{|\widetilde{r}_0^i - r''|<\ell/2}\left|\nabla W(\widetilde{X}_s^i - x'')\right| d\mu_s(y'')\right] \\
&\le \frac{K\epsilon_N}{\ell^2}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \mathbb{1}_{|\widetilde{r}_0^i - r''|<\ell/2}\left|\nabla W(\widetilde{X}_s^i - x'')\right| d\mu_s(y'')\right] \\
&\le \frac{K'\epsilon_N}{\ell}.
\end{aligned} \tag{2.28}$$

We used at the first line that Φ_ℓ has support included in $[-\ell/2, \ell/2]$ to introduce the indicator function, and we used at second line the uniform bound (2.20). For the last line, we used the following lemma.

Lemma 2.5. *Under hypothesis (H1) on W, the following uniform in time bound holds*

$$\sup_{0<\ell<1}\sup_{N\ge 1}\sup_{t\ge 0}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{\mathbb{T}\times E} \frac{1}{\ell}\mathbb{1}_{|\widetilde{r}_0^i - r''|<\ell/2}\left|\nabla W(\widetilde{X}_t^i - x'')\right| d\mu_t(y'')\right] < \infty.$$

The proof of Lemma 2.5 is postponed to the end of the section. Using the same arguments, the third term is bounded by

$$A_3 \le \frac{K}{\ell^2}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} |r' - r''|\, \mathbb{1}_{|\widetilde{r}_0^i - r''|<\ell/2}\left|\nabla W(\widetilde{X}_s^i - x'')\right| d\pi_s(y',y'')\right]. \tag{2.29}$$

It just remains to choose an appropriate coupling π_s of μ_s^N and μ_s to bound (2.27) and (2.29), keeping in mind that we will need to apply Gronwall's lemma at some point. A first option would be to take π_s to be the optimal coupling, which would give the best bound for (2.27), but this gives no information on the term (2.29) since ∇W is unbounded.

Contrary to the Kac model, the local mean field structure of the system we consider leads us to introduce again a coupling over mesoscopic boxes, which immediately give uniform bounds over the term $|r' - r''|$ in (2.29). More precisely, recall from (2.19) that for all $1 \leq j \leq \epsilon_N^{-1}$, we denoted by $\mu_s^j = \epsilon_N^{-1} \mathbb{1}_{r \in B_j} \mu_s$ the measure μ_s conditioned on the fact that $r \in B_j$. We similarly define the empirical probability measure on the box B_j:

$$\mu_s^{N,j} := \frac{1}{N\epsilon_N} \sum_{i \in NB_j} \delta_{Y_s^i}.$$

Then, denoting by π_s^j the optimal coupling between $\mu_s^{N,j}$ and μ_s^j for $\mathcal{W}_1$, it is easy to see that

$$\pi_s := \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \pi_s^j$$

is a coupling of μ_s^N and μ_s. In words, this coupling consists in choosing a box B_j uniformly at random over $(B_j)_{1 \leq j \leq \epsilon_N^{-1}}$ and then, given that the coordinates $r', r'' \in B_j$, select the optimal coupling π^j. With this approach, the term (2.27) can then be bounded by

$$A_2 \leq K' \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} |x' - x''| \, d\pi_s(y', y'')\right] \leq K' \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\mu_s^{N,j}, \mu_s^j)\right],$$

and the term (2.29) is bounded by

$$\begin{aligned}
A_3 &\leq K \frac{\epsilon_N}{\ell^2} \frac{1}{N} \sum_{i=1}^{N} \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \mathbb{1}_{|\widetilde{r}_0^i - r''| < \ell/2} \left|\nabla W(\widetilde{X}_s^i - x'')\right| d\mu_s^j(y'')\right] \\
&\leq K \frac{\epsilon_N}{\ell^2} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\int_{(\mathbb{T}\times E)^2} \mathbb{1}_{|\widetilde{r}_0^i - r''| < \ell/2} \left|\nabla W(\widetilde{X}_s^i - x'')\right| d\mu_s(y'')\right] \\
&\leq K' \frac{\epsilon_N}{\ell},
\end{aligned}$$

by Lemma 2.5. All in all, assembling the last two inequalities and the bounds (2.26) and (2.28), the force terms involving ∇W and ∇U in (2.25) are bounded by

$$K \int_0^t \left(\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\left|X_s^i - \widetilde{X}_s^i\right|\right] + \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\mu_s^{N,j}, \mu_s^j)\right] + \frac{\epsilon_N}{\ell} \right) ds. \tag{2.30}$$

Finally, it remains to control the terms coming from the exchanges of velocities. Recall that $\mathcal{N}^i$ has intensity $\bar{\gamma}\sigma^i(r) dr dt$, with σ^i defined in (2.21). Since $(t, r) \mapsto \Pi_t^i(r)$ is measurable, one can bound

$$\begin{aligned}
B &:= \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\int_0^t |V^{[Nr]} - \Pi_s^i(r) - \left(V_{s_-}^i - \widetilde{V}_{s_-}^i\right)| d\mathcal{N}^i(s, r)\right] \\
&\leq \bar{\gamma} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\int_0^t ds \int_{\mathbb{T}} \left|V^{[Nr]} - \Pi_s^i(r)\right| \sigma^i(r) dr\right] + \bar{\gamma} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\int_0^t ds \int_{\mathbb{T}} \left|V_s^i - \widetilde{V}_s^i\right| d\sigma^i(r) dr\right] \\
&\leq \bar{\gamma} \int_0^t \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\int_{\mathbb{T}} \left|V^{[Nr]} - \Pi_s^i(r)\right| \sigma^i(r) dr\right] ds + \bar{\gamma} \int_0^t \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\left|V_s^i - \widetilde{V}_s^i\right|\right] ds.
\end{aligned}$$

By (2.23) in Lemma 2.1, the first term is bounded by:

$$K\bar{\gamma}\int_0^t ds\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2}|v'-v''|d\pi_s(y',y'')\right] \\ +K\int_0^t\left(\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2}|v''|\frac{1}{\ell^2}\mathbb{1}_{\{|\widetilde{r}_0^i-r''|<\ell/2\}}\left|r'-r''\right|d\pi_s(y',y'')\right]+\frac{\epsilon_N}{\ell}\right)ds,$$

for any coupling π_s of μ_s^N and μ_s. Once again we choose the coupling $\pi_s=\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\pi_s^j$ inside the boxes $(B_j)_{1\le j\le\epsilon_N^{-1}}$ that enabled us to control (2.27) and (2.29), and therefore get the bound:

$$K\bar{\gamma}\epsilon_N\int_0^t\left(\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\mathcal{W}_1(\mu_s^{N,j},\mu_s^j)\right]+\frac{1}{\ell}\int_{\mathbb{T}\times E}|v''|d\mu_s+\frac{1}{\ell}\right)ds.$$

Using Lemma 2.4 to bound $\int_{\mathbb{T}\times E}|v''|d\mu_s$ by a constant, we finally can bound the terms coming from the exchanges of velocity in (2.25) by

$$B\le K\bar{\gamma}\int_0^t\left(\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\left|V_s^i-\widetilde{V}_s^i\right|\right]+\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\mathcal{W}_1(\mu_s^{N,j},\mu_s^j)\right]+\frac{\epsilon_N}{\ell}\right)ds. \tag{2.31}$$

Combining the bound on the force terms (2.30) and the bound on the terms coming from the exchanges of velocities (2.31), we obtain

$$\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[|Y_t^i-\widetilde{Y}_t^i|\right]\le\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[|Y_0^i-\widetilde{Y}_0^i|\right] \tag{2.32}$$
$$+K(1+\bar{\gamma})\int_0^t\left(\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\left|Y_s^i-\widetilde{Y}_s^i\right|\right]+\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\mathcal{W}_1(\mu_s^{N,j},\mu_s^j)\right]+\frac{\epsilon_N}{\ell}\right)ds.$$

To be able to combine this inequality with (2.18), we have to introduce the empirical measure associated with the nonlinear processes $(\widetilde{Y}_s^i)_{i\in NB_j}$ in $\mathcal{W}_1(\mu_s^{N,j},\mu_s^j)$. Define

$$\widetilde{\mu}_s^{N,j}=\frac{1}{N\epsilon_N}\sum_{i\in NB_j}\delta_{\widetilde{Y}_s^i},$$

the obvious analogue of $\mu_s^{N,j}$ for the system of nonlinear processes $(\widetilde{Y}_s^i)_{i\in NB_j}$. We can apply the triangular inequality and a straightforward choice of coupling between $\mu_s^{N,j}$ and $\widetilde{\mu}_s^{N,j}$ to bound

$$\mathcal{W}_1(\mu_s^{N,j},\mu_s^j)\le\mathcal{W}_1(\mu_s^{N,j},\widetilde{\mu}_s^{N,j})+\mathcal{W}_1(\widetilde{\mu}_s^{N,j},\mu_s^j)\le\frac{1}{N\epsilon_N}\sum_{i\in NB_j}|Y_s^i-\widetilde{Y}_s^i|+\mathcal{W}_1(\widetilde{\mu}_s^{N,j},\mu_s^j).$$

We see that summing these inequalities over j, we will obtain the mean value $1/N\sum_{i=1}^N|Y_s^i-\widetilde{Y}_s^i|$ for the first term on the right-hand side. Inserting it in (2.32), we get

$$\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[|Y_t^i-\widetilde{Y}_t^i|\right]\le\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[|Y_0^i-\widetilde{Y}_0^i|\right] \tag{2.33}$$
$$+K(1+\bar{\gamma})\int_0^t\left(\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\left|Y_s^i-\widetilde{Y}_s^i\right|\right]+\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_s^{N,j},\mu_s^j)\right]+\frac{\epsilon_N}{\ell}\right)ds.$$

Step 2: Decoupling

Observing (2.18) and (2.33), and since

$$\mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^N,\mu_t)\right] \le \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j},\mu_t^j)\right],$$

we can see that it suffices to control the right-hand side term

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}-1} \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j},\mu_t^j)\right]$$

to conclude the proof. As noted earlier, the sequence of variables $(\widetilde{Y}_t^i)_{i\le N}$ is correlated since particles with close indices i,i' are driven by two Poisson $\mathcal{N}^i$ and $\mathcal{N}^{i'}$ that share some atoms. In particular, when an atom (r,r') lies in $\Lambda^i\times\Lambda^{i'}$ or $\Lambda^{i'}\times\Lambda^i$, the updates of the two velocities are correlated by the definition of $(\Pi^i)_{i\le N}$. Following [16], we introduce a new system $(\bar{Y}^i)_{i\le N}$ of nonlinear processes by replacing those shared atoms. In particular, we take advantage of our box structure $(B_j)_{1\le j\le \epsilon_N^{-1}}$ to only require independence of $(\bar{Y}_t^i)_{i\in NB_j}$ on each block.

The construction of the nonlinear processes $(\bar{Y}_t^i)_{1\le i\le N}$ is the following. Recall that $\mathcal{N}^i$ was defined from a macroscopic Poisson random measure $\mathcal{N}$ by the formula

$$\mathcal{N}^i(dt,dr) = \mathcal{N}(dt,\Lambda^i,dr) + \mathcal{N}(dt,dr,\Lambda^i) - \mathcal{N}(dt,\Lambda^i,\Lambda^i).$$

We define a Poisson random measure $\mathcal{M}$ is on $\mathbb{R}_+\times\mathbb{T}^2$, independent of $\mathcal{N}$, with same intensity given by (2.13). For $1\le j\le \epsilon_N^{-1}$ and $i\in NB_j$, $\mathcal{M}^i$ is defined by

$$\mathcal{M}^i(dt,dr) = \mathcal{N}(dt,\Lambda^i,dr) + \mathcal{N}(dt,dr,\Lambda^i)\mathbb{1}_{r\notin B_j} + \mathcal{M}(dt,dr,\Lambda^i)\mathbb{1}_{r\in B_j\setminus\Lambda^i}.$$

By their definitions, the Poisson random measures $\mathcal{M}^i$ and $\mathcal{M}^{i'}$ do not share any atom if i,i' are in the same set NB_j for some j. The processes $(\bar{Y}_t^i)_{1\le i\le N}$ are now defined by the same equations (2.17) as $(\widetilde{Y}_t^i)_{1\le i\le N}$, but replacing $\mathcal{N}^i$ by $\mathcal{M}^i$, and with initial values $\bar{Y}_0^i = \widetilde{Y}_0^i$.

Lemma 2.6. *For any $1\le j\le\epsilon_N^{-1}$, the processes $(\bar{Y}^i)_{i\in NB_j}$, defined by*

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt \\ d\bar{V}_t^i = -\left(\int_{\mathbb{T}\times E}\Phi_\ell(\bar{r}_0^i - r')\nabla W(\widetilde{X}_t^i - x')d\mu_t(y') + \nabla U(\bar{X}_t^i)\right)dt \\ \qquad\qquad + \int_{\mathbb{T}}(\Pi_t^i(r') - \bar{V}_{t-}^i)d\mathcal{M}^i(t,r') \\ \bar{Y}_0^i = \widetilde{Y}_0^i, \end{cases} \tag{2.34}$$

are independent.

Denoting by $\bar{\mu}_t^{N,j}$ the empirical measure associated with $(\bar{Y}_t^i)_{i\in NB_j}$, the triangular inequality gives

$$\mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j},\mu_t^j)\right] \le \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j},\bar{\mu}_t^{N,j})\right] + \mathbb{E}\left[\mathcal{W}_1(\bar{\mu}_t^{N,j},\mu_t^j)\right], \tag{2.35}$$

and it suffices to control $\mathbb{E}[\mathcal{W}_1(\widetilde{\mu}_t^{N,j},\bar{\mu}_t^{N,j})]$ to conclude, the other term being controlled by the law of large numbers result in Appendix 2.A. The bound

$$\mathcal{W}_1(\widetilde{\mu}_t^j,\bar{\mu}_t^j) \le \frac{1}{N\epsilon_N}\sum_{i\in NB_j}|\widetilde{Y}_t^i - \bar{Y}_t^i| \tag{2.36}$$

is easily obtained for all $1 \le j \le \epsilon_N^{-1}$, and, as we are ultimately interested in founding a bound for the expectation and mean over j of $\mathcal{W}_1(\widetilde{\mu}_t^j, \bar{\mu}_t^j)$, we are finally led to bound

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big[|\widetilde{Y}_t^i - \bar{Y}_t^i|\Big] &\le \int_0^t ds \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\widetilde{V}_s^i - \bar{V}_s^i|\right] \\
&+ \int_0^t ds \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\int_{\mathbb{T}\times E}\Phi_\ell\left(\widetilde{r}_0^i - r'\right)\left|\nabla W(\widetilde{X}_s^i - x') - \nabla W(\bar{X}_s^i - x')\right| d\mu_s(r', z')\right] \\
&+ \int_0^t ds \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|\nabla U(\widetilde{X}_s^i) - \nabla U(\bar{X}_s^i)\right|\right] \\
&+ \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{T}}\left(\Pi_s^i(r) - \widetilde{V}_{s_-}^i\right) d\mathcal{N}^i(s,r) - \int_0^t\int_{\mathbb{T}}\left(\Pi_s^i(r) - \bar{V}_{s_-}^i\right) d\mathcal{M}^i(s,r)\right|\right].
\end{aligned} \tag{2.37}$$

Notice that we used that $\widetilde{r}_0^i = \bar{r}_0^i$ for the term at the second line. A bound

$$K\int_0^t ds \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\widetilde{Y}_s^i - \bar{Y}_s^i|\right] \tag{2.38}$$

is easily obtained for the first three terms. The last term in (2.37) can be bounded by splitting it in the three following terms:

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\Big[\int_0^t\int_{\mathbb{T}}\left|\widetilde{V}_{s_-}^i - \bar{V}_{s_-}^i\right|\left(d\mathcal{N}(s,\Lambda^i,r) + d\mathcal{N}(s,r,\Lambda^i)\mathbb{1}_{r\notin B_j}\right)\Big] \\
&\qquad + \frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\left[\int_0^t\int_{\mathbb{T}}\left|\Pi_s^i(r) - \widetilde{V}_{s_-}^i\right| d\mathcal{N}(s,r,\Lambda^i)\mathbb{1}_{r\in B_j\setminus\Lambda^i}\right] \\
&\qquad + \frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\left[\int_0^t\int_{\mathbb{T}}\left|\Pi_s^i(r) - \bar{V}_{s_-}^i\right| d\mathcal{M}(s,r,\Lambda^i)\mathbb{1}_{r\in B_j\setminus\Lambda^i}\right] \\
&= C_1 + C_2 + C_3,
\end{aligned}$$

where the first term C_1 corresponds to the simultaneous jumps of $\widetilde{V}^i$ and $\bar{V}^i$, *i.e.* the shared atoms of $\mathcal{N}^i$ and $\mathcal{M}^i$ for $1 \le i \le N$, and the two other terms C_2 and C_3 respectively correspond to the jumps of $\widetilde{Y}^i$ alone and to the jumps of $\bar{Y}^i$ alone. The term coming from the simultaneous jumps is equal to

$$C_1 = \bar{\gamma}\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\int_0^t ds\mathbb{E}\left[\left|\widetilde{V}_s^i - \bar{V}_s^i\right|\right]\sum_{k\,:\,i+k\notin NB_j}\gamma_k \le \bar{\gamma}\int_0^t ds\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|\widetilde{V}_s^i - \bar{V}_s^i\right|\right]. \tag{2.39}$$

As for the term C_2 coming from the jumps of $\mathcal{N}^i$ alone, one immediately gets:

$$\begin{aligned}
C_2 &\le \bar{\gamma}\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\left[\int_0^t ds\int_{B_j}\left(\left|\Pi_s^i(r)\right| + \left|\widetilde{V}_s^i\right|\right)\sigma^i(r)dr\right] \\
&\le \bar{\gamma}\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\left[\int_0^t ds\int_{B_j}\left|\Pi_s^i(r)\right|\sigma^i(r)dr\right] + \bar{\gamma}\sum_{k=-N\epsilon_N}^{N\epsilon_N}\gamma_k\int_0^t\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|\widetilde{V}_s^i\right|\right]ds \\
&\le \bar{\gamma}\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{i\in NB_j}\mathbb{E}\left[\int_0^t ds\int_{B_j}\left|\Pi_s^i(r)\right|\sigma^i(r)dr\right] + K\bar{\gamma}\frac{\epsilon_N}{\ell}\int_0^t\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left|\widetilde{V}_s^i\right|\right]ds,
\end{aligned} \tag{2.40}$$

where we used the symmetry and monotonic properties **(H2)** of γ to bound the integral over B_j by the sum $\sum_{k=-N\epsilon_N}^{N\epsilon_N} \gamma_k$ at the second line, and then bounded this sum by $K\epsilon_N/\ell$ by the definition (2.2) of γ_k. The sum appearing in the second term of (2.40) is equal to

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[\left|\widetilde{V}_s^i\right|\right] = \sum_{j=1}^{\epsilon_N^{-1}} \int_{B_j\times E} |v| d\mu_s = \int_{\mathbb{T}\times E} |v| d\mu_s < \infty,$$

which is finite by Lemma 2.4. By the bound (2.24) of Lemma 2.1, we can bound the first term in (2.40) and finally get the following bound for C_2:

$$C_2 \leq Kt\bar{\gamma}\left(\frac{\epsilon_N^{1/2}}{\ell^{1/2}} + \frac{\epsilon_N}{\ell}\right) \leq K' t\bar{\gamma}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}, \tag{2.41}$$

for N large enough. The same bound can be obtained for the term C_3 coming from the jumps of $\mathcal{M}^i$ alone. Combining the bound on the force terms (2.38), the bound on the simultaneous jumps (2.39) and the bounds on the asynchronous jumps (2.41) in (2.37), we get

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\Big[|\widetilde{Y}_t^i - \bar{Y}_t^i|\Big] \leq K\int_0^t \left((1+\bar{\gamma})\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|\widetilde{Y}_s^i - \bar{Y}_s^i|\right] + \bar{\gamma}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\right) ds.$$

By Gronwall's lemma and (2.36), we finally get

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j}, \bar{\mu}_t^{N,j})\right] \leq \frac{\bar{\gamma}}{1+\bar{\gamma}}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\left(\mathrm{e}^{K(1+\bar{\gamma})t} - 1\right). \tag{2.42}$$

Since by Lemma 2.4, the measure μ_t admits a moment of order 2 for all $t \geq 0$, we can apply the results of Appendix 2.A and get

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\bar{\mu}_t^{N,j}, \mu_t^j)\right] \leq K'(N\epsilon_N)^{-\frac{1}{4(d+1)}} + K'\epsilon_N.$$

Combining this result with (2.42) in (2.35) finally gives

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\widetilde{\mu}_t^{N,j}, \mu_t^j)\right] \leq \frac{\bar{\gamma}}{1+\bar{\gamma}}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\,\mathrm{e}^{K(1+\bar{\gamma})t} + K'(N\epsilon_N)^{-\frac{1}{4(d+1)}} + K'\epsilon_N. \tag{2.43}$$

Conclusion of the proof of Theorem 2.1

Combining (2.33) and (2.43), and applying Gronwall's lemma, we get the bound:

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|Y_t^i - \widetilde{Y}_t^i|\right] \leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|Y_0^i - \widetilde{Y}_0^i|\right]\mathrm{e}^{K(1+\bar{\gamma})t}$$
$$+ K'\left(\frac{\epsilon_N}{\ell} + (N\epsilon_N)^{-\frac{1}{4(d+1)}} + \epsilon_N + \frac{\bar{\gamma}}{1+\bar{\gamma}}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\right)\mathrm{e}^{K(1+\bar{\gamma})t}.$$

By Lemma 2.3, we can bound $\mathbb{E}[|Y_0^i - \widetilde{Y}_0^i|]$ by ϵ_N for all i. Since ϵ_N is smaller than ϵ_N/ℓ, we finally get

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|Y_t^i - \widetilde{Y}_t^i|\right] \leq K'\left(\frac{\epsilon_N}{\ell} + (N\epsilon_N)^{-\frac{1}{4(d+1)}} + \frac{\bar{\gamma}}{1+\bar{\gamma}}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\right)\mathrm{e}^{K(1+\bar{\gamma})t}.$$

Finally, combining this bound together with (2.43) in (2.18), gives

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N,\mu_t)\right] \le K'\left(\frac{\epsilon_N}{\ell} + (N\epsilon_N)^{-\frac{1}{4(d+1)}} + \frac{\bar{\gamma}}{1+\bar{\gamma}}\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\right)\mathrm{e}^{K(1+\bar{\gamma})t}.$$

□

2.2.2 Proof of the intermediate lemmas

Proof of Lemma 2.1. *1. Construction of* Π^i

The construction of Π^i is based on Lemma 3 in [16]. Let $1 \le j \le \epsilon_N^{-1}$ and $i \in NB_j$. Fix $\widetilde{r} \in B_j$ and $\mathbf{v}^i = (v^{i+k})_{-\ell N \le k \le \ell N} \in (\mathbb{R}^d)^{2\ell N+1}$. Recall from (2.22) the notation for the weighted measure around $\widetilde{r}$, from which we select a new velocity for particle $\widetilde{Y}^i$:

$$w^{\mu_t,\widetilde{r}}(v) = \int_{\mathbb{T}\times\mathbb{R}^d} \Gamma_\ell(\widetilde{r}-r)d\mu_t(r,x,v),$$

where the integration is only on the r and x variables. Denote also $w^{\mathbf{v}^i}$ the weighted measure on $\mathbb{R}^d$ corresponding to the update of velocity of Y^i in the original particle system:

$$w^{\mathbf{v}^i} = \sum_{k=-\ell N}^{\ell N} \gamma_k \delta_{v^{i+k}}.$$

From a random variable r with distribution σ^i (see (2.21)), we want to construct a couple of variables $(v^{[Nr]}, \Pi_t^i(r,\tilde{r},\mathbf{v}^i))$ with respective distributions $w^{\mathbf{v}^i}$ and $w^{\mu_t,\widetilde{r}}$ for all $(t,\tilde{r},\mathbf{v}^i)$. In addition, we want the distance between those variables to be small in the sense that

$$\int_{\mathbb{T}} |v^{[Nr]} - \Pi_t^i(r,\tilde{r},\mathbf{v}^i)|\sigma^i(r)dr = \mathcal{W}_1(w^{\mathbf{v}^i}, w^{\mu_t,\widetilde{r}}). \tag{2.44}$$

It is straightforward to see that $v^{[Nr]}$ has indeed distribution $w^{\mathbf{v}^i}$ in that case. The construction of a function $\Pi^i(r)$ with distribution $w^{\mu_t,\widetilde{r}}$ such that (2.44) holds principally amounts to the construction of a random variable with given law from a uniform random variable.

In fact, let $p^i_{t,\tilde{r},\mathbf{v}^i}(v,\tilde{v})$ be the optimal coupling of $w^{\mathbf{v}^i}$ and $w^{\mu_t,\widetilde{r}}$. For a couple of random variables $(V,\widetilde{V})$ with distribution $p^i_{t,\tilde{r},\mathbf{v}^i}$, define $P^{i,k}_{t,\tilde{r},\mathbf{v}^i}$ to be the conditional law of $\widetilde{V}$, given that $V = V^{i+k}$, *i.e.*:

$$P^{i,k}_{t,\tilde{r},\mathbf{v}^i}(B) = p^i_{t,\tilde{r},\mathbf{v}^i}\left(\{v^{i+k}\}\times B \middle| \{v^{i+k}\}\times\mathbb{R}^d\right) = \frac{1}{\gamma_k}p^i_{t,\tilde{r},\mathbf{v}^i}(\{v^{i+k}\}\times B), \tag{2.45}$$

for any Borel subset B of $\mathbb{R}^d$. The second equality comes from the fact that $p^i_{t,\tilde{r},\mathbf{v}^i}(\{v^{i+k}\}\times\mathbb{R}^d) = w^{\mathbf{v}^i}(\{v^{i+k}\}) = \gamma_k$. Then there exists a function

$$q^{i,k}_{t,\tilde{r},\mathbf{v}^i} : \Lambda^{i+k} \to \mathbb{R}^d, \tag{2.46}$$

such that, if r is uniformly distributed on Λ^{i+k}, $q^{i,k}_{t,\tilde{r},\mathbf{v}^i}(r)$ has distribution $P^{i,k}_{t,\tilde{r},\mathbf{v}^i}$. Defining Π^i by

$$\Pi_t^i(r,\tilde{r},\mathbf{v}^i) = \sum_{k=-\ell N}^{\ell N} q^{i,k}_{t,\tilde{r},\mathbf{v}^i}(r)\mathbb{1}_{r\in\Lambda^{i+k}}, \tag{2.47}$$

the pair $(v^{[Nr]}, \Pi_t^i(r,\tilde{r},\mathbf{v}^i))$ has indeed distribution $p^i_{t,\tilde{r},\mathbf{v}^i}$ if r has distribution $\sigma^i(r)dr$. Indeed,

$$\begin{aligned}\mathbb{P}\left(v^{[Nr]}=V_t^{i+k},\Pi_t^i\in B\right)&=\mathbb{P}\left(r\in\Lambda^{i+k},\Pi_t^i\in B\right)\\&=\mathbb{P}\left(r\in\Lambda^{i+k}\right)\mathbb{P}\left(\Pi_t^i\in B|r\in\Lambda^{i+k}\right)\\&=\gamma_k\mathbb{P}\left(q^{i,k}_{t,\tilde{r},\mathbf{v}^i}(r)\in B\right)\\&=p^i_{t,\tilde{r},\mathbf{v}^i}(\{V_t^{i+k}\}\times B).\end{aligned}$$

(2.44) is now obvious from the fact $(V_t^{[Nr]},\Pi_t(r))$ has distribution p_t^i when r is distributed by $\sigma^i(r)dr$. To ensure that the construction of Π^i detailed above can be done with the desired measurability properties, we follow [16]. Since the mapping $(t,\tilde{r},\mathbf{v}^i)\mapsto(w^{\mathbf{v}^i},w^{\mu_t,\widetilde{r}})$ is measurable, then we know from Corollary 5.22 in [73] that there exists a measurable mapping $(t,\tilde{r},\mathbf{v}^i)\mapsto p^i_{t,\tilde{r},\mathbf{v}^i}$ such that $p^i_{t,\tilde{r},\mathbf{v}^i}(v,\widetilde{v})$ is an optimal coupling between those two measures. $P^{i,k}_{t,\tilde{r},\mathbf{v}^i}$ defined in (2.45) is a kernel from $\mathbb{R}_+\times B_j\times(\mathbb{R}^d)^{2\ell N+1}$ to $\mathbb{R}^d$, so by Lemma 2.22 in [41], $q^{i,k}_{t,\tilde{r},\mathbf{v}^i}$ in (2.46) can be defined as a mesurable function of $(t,r,\tilde{r},\mathbf{v}^i)$. Finally the definition (2.47) ensures that Π^i is a mesurable function of $(t,r,\tilde{r},\mathbf{v}^i)$.

2. *Proof of* (2.23)

By (2.44), we can directly bound $\mathcal{W}_1(w^{\mathbf{V}_t^i},w^{\mu_t,\widetilde{r}_0^i})$ to obtain a bound on (2.23). By Kantorovich-Rubinstein duality formula (see [73]), this term can be rewritten:

$$\mathcal{W}_1(w^{\mathbf{V}_t^i},w^{\mu_t,\widetilde{r}_0^i})=\sup_{\substack{Lip(\varphi)\le 1\\ \varphi(0)=0}}\left|\sum_{k=-\ell N}^{\ell N}\gamma_k\varphi(V_t^{i+k})-\int_{\mathbb{T}\times E}\Gamma_\ell(\widetilde{r}_0^i-r'')\varphi(v'')d\mu_t(r'',z'')\right|. \tag{2.48}$$

By the definition (2.2) of γ_k, we have

$$\left|\gamma_k-\frac{1}{N}\Gamma_\ell\left(\frac{k}{N}\right)\right|=\left|\int_{\Lambda^k}\left(\Gamma_\ell(r)-\Gamma_\ell\left(\frac{k}{N}\right)\right)dr\right|\le\frac{K}{\ell^2}\int_{\Lambda^0}|r|dr\le\frac{K'}{N^2\ell^2},$$

using the Lipschitz property for Γ_ℓ for the first inequality. As a consequence, we can introduce the empirical measure μ_t^N in (2.48) by replacing γ_k and get the bound:

$$\begin{aligned}&\mathcal{W}_1(w^{\mathbf{V}_t^i},w^{\mu_t,\widetilde{r}_0^i})\\&\quad\le\sup_{\substack{Lip(\varphi)\le 1\\ \varphi(0)=0}}\left|\int_{\mathbb{T}\times E}\varphi(v')\Gamma_\ell\left(\frac{i}{N}-r'\right)d\mu_t^N(r',z')-\int_{\mathbb{T}\times E}\varphi(v'')\Gamma_\ell(\widetilde{r}_0^i-r'')d\mu_t(r'',z'')\right|\\&\qquad+\frac{K}{N\ell^2}\sup_{\substack{Lip(\varphi)\le 1\\ \varphi(0)=0}}\int_{\mathbb{T}\times E}|\varphi(v')|\mathbb{1}_{\{|i/N-r'|<\ell/2\}}d\mu_t^N(r',z').\end{aligned}$$

Therefore, bounding $\varphi(v')\le|v'|$ in the last term on the one hand, and taking expectations and the mean over $1\le i\le N$ on the other hand, we get:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\mathcal{W}_1(w^{\mathbf{V}_t^i},w^{\mu_t,\widetilde{r}_0^i})\right]\le\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Bigg[&\sup_{\substack{Lip(\varphi)\le 1\\ \varphi(0)=0}}\Bigg|\int_{\mathbb{T}\times E}\varphi(v')\Gamma_\ell\left(\frac{i}{N}-r'\right)d\mu_t^N(r',z')\\&-\int_{\mathbb{T}\times E}\varphi(v'')\Gamma_\ell(\widetilde{r}_0^i-r'')d\mu_t(r'',z'')\Bigg|\Bigg]\\&+\frac{K}{N\ell^2}\frac{1}{N}\sum_{i=1}^N\frac{1}{N}\sum_{k=-\ell N}^{\ell N}\mathbb{E}\left[|V_t^{i+k}|\right].\end{aligned} \tag{2.49}$$

The sum $1/N\sum_i \mathbb{E}[|V_t^i|]$ is bounded by Lemma 2.4, therefore the last term in (2.49) is bounded by

$$\frac{1}{N\ell^2}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{k=-\ell N}^{\ell N}\mathbb{E}\left[|V_t^{i+k}|\right] \leq \frac{K}{N\ell}. \tag{2.50}$$

We can now focus on the first term of (2.49). Let π_t be a coupling between μ_t^N and μ_t. Introducing cross terms, we bound the supremum term in the expectation:

$$\begin{aligned}
&\sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}}\left|\int_{\mathbb{T}\times E}\varphi(v')\Gamma_\ell\left(\frac{i}{N}-r'\right)d\mu_t^N(r',z') - \int_{\mathbb{T}\times E}\varphi(v'')\Gamma_\ell(\tilde{r}_0^i-r'')d\mu_t(r'',z'')\right| \\
&\leq \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}}\int_{(\mathbb{T}\times E)^2}|\varphi(v')-\varphi(v'')|\Gamma_\ell\left(\frac{i}{N}-r'\right)d\pi_t(y',y'') \\
&\qquad + \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}}\int_{(\mathbb{T}\times E)^2}|\varphi(v'')|\left|\Gamma_\ell\left(\frac{i}{N}-r'\right)-\Gamma_\ell\left(\frac{i}{N}-r''\right)\right|d\pi_t(y',y'') \\
&\qquad + \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}}\int_{(\mathbb{T}\times E)^2}|\varphi(v'')|\left|\Gamma_\ell\left(\frac{i}{N}-r''\right)-\Gamma_\ell(\tilde{r}_0^i-r'')\right|d\mu_t(r'',z'') \\
&\leq \int_{(\mathbb{T}\times E)^2}|v'-v''|\Gamma_\ell\left(\frac{i}{N}-r'\right)d\pi_t(y',y'') + \frac{K}{\ell^2}\int_{(\mathbb{T}\times E)^2}|v''|\mathbb{1}_{\{|\tilde{r}_0^i-r''|<\ell/2\}}\left|r'-r''\right|d\pi_t(y',y'') \\
&\qquad + \frac{K\epsilon_N}{\ell^2}\int_{(\mathbb{T}\times E)^2}\mathbb{1}_{\{|\tilde{r}_0^i-r''|<\ell/2\}}|v''|d\mu_t(r'',z''),
\end{aligned} \tag{2.51}$$

where we used that $|\phi(v'')|\leq|v''|$ and the Lipschitz property for Γ_ℓ at the last line, as well as the fact that $i/N, \tilde{r}_0^i \in B_j$ for some j and therefore their difference is smaller than ϵ_N for the last term (see (2.20)). Using Lemma 2.4, we can bound $\int_{\mathbb{T}\times E}|v''|d\mu_t$ by a constant. Combining (2.50) and (2.51) in (2.49), and taking into account the expectation and the mean over i, we finally get the bound

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\mathcal{W}_1(w^{\mathbf{V}_t^i}, w^{\mu_t,\tilde{r}_0^i})\right] &\leq K\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2}|v'-v''|d\pi_t(y',y'')\right] \\
&\quad + \frac{K}{\ell^2}\mathbb{E}\left[\int_{(\mathbb{T}\times E)^2}|v''|\mathbb{1}_{\{|\tilde{r}_0^i-r''|<\ell/2\}}\left|r'-r''\right|d\pi_t(y',y'')\right] + \frac{K\epsilon_N}{\ell} + \frac{K}{N\ell},
\end{aligned}$$

for all coupling π_t of μ_t^N and μ_t. As $\epsilon_N > 1/N$, (2.23) follows.

3. Proof of (2.24)

Using the definition (2.47) of Π^i, and then the definition of $q^{i,k}$ we get

$$\begin{aligned}
\int_{B_j}\left|\Pi_t^i(r,\tilde{r}_0^i,\mathbf{V}_t^i)\right|\sigma^i(r)dr &= \sum_{\substack{k=-\ell N,\\ i+k\in NB_j}}^{\ell N}\gamma_k\int_{\Lambda^{i+k}}\left|q^{i,k}_{t,\tilde{r}_0^i,\mathbf{V}_t^i}(r)\right|dr \\
&= \sum_{\substack{k=-\ell N,\\ i+k\in NB_j}}^{\ell N}\int_{\mathbb{R}^d\times\mathbb{R}^d}|v''|\mathbb{1}_{\{v'=V_t^{i+k}\}}dp^i_{t,\tilde{r}_0^i,\mathbf{V}_t^i}(v',v'') \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d}|v''|\mathbb{1}_{\{v'\in\{V_t^k,\,k\in NB_j\}\}}dp^i_{t,\tilde{r}_0^i,\mathbf{V}_t^i}(v',v''),
\end{aligned}$$

where the summation on the first two lines was on indices k satisfying both $-\ell N \leq k \leq \ell N$ and $i+k \in NB_j$. By Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{B_j} \left|\Pi^i_t(r, \tilde{r}^i_0, \mathbf{V}^i_t)\right| \sigma^i(r)dr &\leq \left(\int_{\mathbb{R}^d} \mathbb{1}_{\{v' \in \{V^k, k\in NB_j\}\}} dw^{\mathbf{V}^i_t}(v')\right)^{1/2} \left(\int_{\mathbb{R}^d} |v''|^2 dw^{\mu_t, \tilde{r}^i_0}(v'')\right)^{1/2} \\ &\leq \left(\sum_{\substack{k=-\ell N,\\ i+k\in NB_j}}^{\ell N} \gamma_k\right)^{1/2} \left(\int_{\mathbb{T}\times E} \Gamma_\ell\left(\tilde{r}^i_0 - r''\right) |v''|^2 d\mu_t(r'', z'')\right)^{1/2} \\ &\leq K\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\left(1 + \frac{1}{2}\int_{\mathbb{T}\times E} \Gamma_\ell\left(\tilde{r}^i_0 - r\right)|v|^2 d\mu_t(r'', z'')\right),\end{aligned}$$

using at the last line that the coefficients γ_k are bounded by K/ℓ and using the standard inequality $(1+x)^{1/2} \leq 1 + x/2$ to bound the square root. Summing this last inequality over i, j, taking expectations and using that $\tilde{r}^i_0$ is uniform random variable on B_j we get:

$$\frac{1}{N}\sum_{j=1}^{\epsilon_N^{-1}} \sum_{i\in NB_j} \mathbb{E}\left[\int_{B_j} \left|\Pi^i_t(r)\right| \sigma^i(r)dr\right] \leq K\frac{\epsilon_N^{1/2}}{\ell^{1/2}}\left(1 + \frac{1}{2}\int_{\mathbb{T}\times E} |v''|^2 d\mu_t(r'', z'')\right).$$

Applying Lemma 2.4 gives finally (2.24).

□

Proof of Lemma 2.2. Fix $1 \leq j \leq \epsilon_N^{-1}$ and $i \in NB_j$. First, by the construction of Π^i in Lemma 2.1, $(t, \omega, r) \mapsto \Pi^i_t(r, \tilde{r}^i_0(\omega), \mathbf{V}^i_{t\_}(\omega))$ is measurable with respect to the product of the predictable sigma field in (t, ω) and the Borel sigma field of $\mathbb{T}$. Therefore the integral with respect to the Poisson random measure in (2.17) is well-defined. Using the terminology of the next section, we are first going to check that the law of the process $\tilde{Y}^i$ solves the *linear* martingale problem (2.57) associated with μ and starting at μ^j_0 and finish with a uniqueness argument to prove that its law is in fact μ^j.

For any $\psi \in C^1_b(\mathbb{T} \times E)$, by a direct computation from (2.17), we have

$$\begin{aligned}\psi(\tilde{Y}^i_t) = \psi(\tilde{Y}^i_0) &+ \int_0^t \tilde{V}^i_s \cdot \nabla_x \psi(\tilde{Y}^i_s) ds \\ &- \int_0^t ds\left(\int_{\mathbb{T}\times E} \Phi_\ell(\tilde{r}^i_s - r')\nabla W(\tilde{X}^i_s - x') d\mu_s(r', z') + \nabla U(\tilde{X}^i_s)\right) \cdot \nabla_v \psi(\tilde{Y}^i_s) \\ &+ \bar{\gamma}\int_0^t ds \int_{\mathbb{T}} \left(\psi(\tilde{r}^i_s, \tilde{X}^i_s, \Pi^i_s(r')) - \psi(\tilde{r}^i_s, \tilde{X}^i_s, \tilde{V}^i_s)\right)\sigma^i(r')dr' + M_t, \end{aligned} \tag{2.52}$$

where M_t is the compensated martingale associated with the Poisson integral. By the definition of Π^i in Lemma 2.1, the last integral can be written

$$\begin{aligned}&\int_{\mathbb{T}} \left(\psi(\tilde{r}^i_s, \tilde{X}^i_s, \Pi^i_s(r')) - \psi(\tilde{r}^i_s, \tilde{X}^i_s, \tilde{V}^i_s)\right)\sigma^i(r')dr' \\ &\qquad = \int_{\mathbb{R}^d}\left(\psi(\tilde{r}^i_s, \tilde{X}^i_s, v') - \psi(\tilde{Y}^i_s)\right) dw^{\mu_s, \tilde{r}^i_0}(v') \\ &\qquad = \int_{\mathbb{T}\times E} \Gamma_\ell(\tilde{r}^i_0 - r')\left(\psi(\tilde{r}^i_s, \tilde{X}^i_s, v') - \psi(\tilde{Y}^i_s)\right) d\mu_s(r', x', v'),\end{aligned}$$

and we can therefore rewrite (2.52) as

$$\psi(\tilde{Y}^i_t) = \psi(\tilde{Y}^i_0) + \int_0^t \mathcal{L}[\mu_s]\psi(\tilde{Y}_s)ds + M_t.$$

Hence the law of $\widetilde{Y}^i$ solves the linear martingale problem associated with μ and starting at μ_0^j. We are now left with the task of proving that this property is sufficient to characterize the law $\mu_t^j(r,z) = \epsilon_N^{-1}\mathbb{1}_{r\in B_j}\mu_t(r,z)$.

To do so, we construct ϵ_N^{-1} processes $(\hat{Y}^j)_{1\le j\le \epsilon_N^{-1}}$, whose respective laws solve the linear martingale problem associated with μ and starting at μ_0^j. Let U be a uniform $\mathcal{F}_0$-measurable random variable on $\mathbb{T}$, independent of any $\hat{Y}^j$. Now define the process Y on $\mathbb{T}\times E$ by

$$Y = \sum_{j=1}^{\epsilon_N^{-1}} \hat{Y}^j \mathbb{1}_{U\in B_j}.$$

We can check that the law of Y solves the linear martingale problem associated with μ and starting at μ_0. This suffices to deduce that Y has law μ by uniqueness of the solutions to the *linear* martingale problem. In fact, it is easy to see that Y_0 has distribution μ_0 and, for any $\psi \in C_b^1(\mathbb{T}\times E)$, one has

$$\begin{aligned}\psi(Y_t) = \sum_{j=1}^{\epsilon_N^{-1}} \psi(\hat{Y}_t^j)\mathbb{1}_{U\in B_j} &= \sum_{j=1}^{\epsilon_N^{-1}} \left(\psi(\hat{Y}_0^j) + \int_0^t \mathcal{L}[\mu_s]\psi(\hat{Y}_s^j)ds + M_t^{\psi,j}\right)\mathbb{1}_{U\in B_j} \\ &= \psi(Y_0) + \int_0^t \mathcal{L}[\mu_s]\psi(Y_s)ds + M_t^{\psi},\end{aligned}$$

where $M^{\psi,j}$ are martingales and $M^{\psi} = \sum M^{\psi,j}\mathbb{1}_{U\in B_j}$ is therefore a martingale.

It now remains to check that $\hat{Y}^j$ has distribution μ^j. For any measurable function φ and for any $1\le j\le \epsilon_N^{-1}$, by independence of U and $\hat{Y}^j$,

$$\mathbb{E}\left[\varphi(Y_t)\mathbb{1}_{r_t\in B_j}\right] = \mathbb{E}\left[\varphi(\hat{Y}_t^j)\mathbb{1}_{U\in B_j}\right] = \epsilon_N\mathbb{E}\left[\varphi(\hat{Y}_t^j)\right].$$

Therefore $\hat{Y}^j$ has distribution μ^j as expected, since

$$\mathbb{E}\left[\varphi(\hat{Y}_t^j)\right] = \epsilon_N^{-1}\mathbb{E}\left[\varphi(Y_t)\mathbb{1}_{r_t\in B_j}\right] = \int_{\mathbb{T}\times E}\varphi d\mu_t^j,$$

and this concludes the proof.

□

Proof of Lemma 2.3. Recall that r_0^i is uniformly distributed over B_j, where j is such that $i\in NB_j$. We therefore already have the uniform bound

$$\left|\frac{i}{N} - \widetilde{r}_0^i\right| \le \epsilon_N.$$

Recall that Z_0^i has distribution $f_0(i/N,z)dz$ and $\widetilde{Z}_0^i$ has distribution $\epsilon_N^{-1}\int_{B_j} f(r,z)drdz$, where the integral is only over r, *i.e.* the distribution of $\widetilde{Z}_0^i$ is the marignal of μ_0^j on the z coordinates. Therefore, we can choose $(Z_0^i, \widetilde{Z}_0^i)$ to be distributed as the optimal coupling between those two measures and get, by Kantorovich-Rubinstein duality:

$$\begin{aligned}
\mathbb{E}\left[|Z_0^i - \widetilde{Z}_0^i|\right] &= \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_E \varphi(z) f_0\left(\frac{i}{N}, z\right) dz - \epsilon_N^{-1} \int_{B_j\times E} \varphi(z) f_0(r,z) drdz \\
&= \epsilon_N^{-1} \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{B_j\times E} \varphi(z) \left(f_0\left(\frac{i}{N}, z\right) - f_0(r,z)\right) drdz \\
&\leq \epsilon_N^{-1} \sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{B_j\times E} \varphi(z) \left|\frac{i}{N} - r\right| h(z) drdz \\
&\leq \epsilon_N^{-1} \int_{B_j} \left|\frac{i}{N} - r\right| dr \int_E |z| h(z) dz \\
&\leq K\epsilon_N
\end{aligned}$$

using hypothesis **(H4)** at the third line. Hence

$$\mathbb{E}\left[|Y_0^i - \widetilde{Y}_0^i|\right] \leq K\epsilon_N.$$

□

Proof of Lemma 2.4. Let Y_t be the canonical process associated with the solution to the non-linear martingale problem (2.8) starting at μ_0. Defining the energy

$$\mathcal{E}[\mu_t](Y_t) := \frac{1}{2}|V_t|^2 + \frac{1}{2}\int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r') W(X_t - x') d\mu_t(y') + U(X_t), \tag{2.53}$$

we can see that its expectation is constant, *i.e.*

$$\mathbb{E}[\mathcal{E}[\mu_t](Y_t)] = \int_{\mathbb{T}\times E} \mathcal{E}[\mu_t](y) d\mu_t(y) = \int_{\mathbb{T}\times E} \mathcal{E}[\mu_0](y) d\mu_0(y),$$

which is finite by assumption **(H3)**. In particular, by assumption **(H1)** on U, we get the moment bound we expected.

A similar argument holds for the particle system defined by (2.16), which clearly conserves energy in the sense that:

$$\begin{aligned}
\int_{\mathbb{T}\times E} \mathcal{E}[\mu_t^N](y) d\mu_t^N(y) &= \sum_{i=1}^N \left(\frac{1}{2}|V_t^i|^2 + \frac{1}{2}\sum_{k=-\ell N}^{\ell N} \phi_k W(X_t^i - X_t^{i+k}) + U(X_t^i)\right) \\
&= \int_{\mathbb{T}\times E} \mathcal{E}[\mu_0^N](y) d\mu_0^N(y)
\end{aligned}$$

almost surely for all $t > 0$. □

Proof of Lemma 2.5. The proof of this lemma is similar to that of Lemma 2.4 and is again based on conservation of energy. Since the law of $\widetilde{Y}_t^i$ is given by μ_t^j if $i \in NB_j$, one has:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_{\mathbb{T}\times E} \frac{1}{\ell} \mathbb{1}_{|\widetilde{r}_0^i - r''|<\ell/2} |\nabla W(\widetilde{X}_t^i - x'')| d\mu_t(r'', z'')\right] \\
&\qquad = \epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \int_{(\mathbb{T}\times E)^2} \frac{1}{\ell} \mathbb{1}_{|r'-r''|<\ell/2} |\nabla W(x' - x'')| d\mu_t^j(r', z') d\mu_t(r'', z'') \\
&\qquad = \int_{(\mathbb{T}\times E)^2} \frac{1}{\ell} \mathbb{1}_{|r'-r''|<\ell/2} |\nabla W(x' - x'')| d\mu_t(r', z') d\mu_t(r'', z'').
\end{aligned}$$

By Jensen's inequality and assumption **(H1)** on W, we get

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big[\int_{\mathbb{T}\times E}\frac{1}{\ell}\mathbb{1}_{|\widetilde{r}_0^i-r''|<\ell/2}|\nabla W(\widetilde{X}_t^i-x'')|d\mu_t(r'',z'')\Big]$$
$$\leq\left(c\int_{(\mathbb{T}\times E)^2}\frac{1}{\ell}\mathbb{1}_{|r'-r''|<\ell/2}W(x'-x'')d\mu_t(r',z')d\mu_t(r'',z'')\right)^{1/2}.$$

It just remains to show that this last term is bounded by a constant that does not depend on ℓ. This fact is true if we replace $\ell^{-1}\mathbb{1}_{|r'-r''|<\ell/2}$ by $\Phi_\ell(r'-r'')$, by conservation of energy, following the proof of Lemma 2.4. However, $\Phi_\ell(r)$ is not necessarily bigger than $\ell^{-1}\mathbb{1}_{|r|<\ell/2}$, in particular for r close to $\ell/2$ or $-\ell/2$. Instead, we are going to take advantage of the integration over r' and r'' to properly achieve this bound.

Let a be the only real number in $[0,1/2]$ such that $\phi(a)=\phi(-a)=\|\phi\|_\infty/2$. Adding several times the function ϕ shifted, we can upper bound the indicator function. More precisely, there exists a finite integer m such that

$$\frac{\|\phi\|_\infty}{2}\mathbb{1}_{|r|\leq 1/2}\leq\sum_{k=-m}^{m}\phi(r+ka).$$

From this, we deduce that

$$\frac{1}{\ell}\mathbb{1}_{|r|\leq\ell/2}\leq\frac{2}{\|\phi\|_\infty}\sum_{k=-m}^{m}\Phi_\ell(r+ka).$$

Inserting this in the integral term, it now suffices to bound it by

$$\int_{(\mathbb{T}\times E)^2}\frac{1}{\ell}\mathbb{1}_{|r'-r''|<\ell/2}W(x'-x'')d\mu_t(r',z')d\mu_t(r'',z'')$$
$$\leq\frac{2(2m+1)}{\|\phi\|_\infty}\int_{(\mathbb{T}\times E)^2}\Phi_\ell(r'-r'')W(x'-x'')d\mu_t(r',z')d\mu_t(r'',z''),$$

by translation invariance of the Lebesgue measure on $\mathbb{T}$.

□

Proof of Lemma 2.6. Denote by $\bar{\mathcal{M}}^i$ the point process on $\mathbb{T}\times\mathbb{R}^d$ with atoms given by $(t,\Pi_t(r))$ for any atom (t,r) of $\mathcal{M}^i$. Since the initial positions $(\bar{Y}_0^i)_{i\in NB_j}$ are independent, by the equation (2.34), independence of the processes $(\bar{Y}^i)_{i\in NB_j}$ will follow from the independence of $(\bar{\mathcal{M}}^i)_{i\in NB_j}$.

The independence of the point processes $(\mathcal{M}^i)_{i\in NB_j}$ is straightforward from the fact they do not share atoms, but by the definition of Π^i in Lemma 2.1, this argument alone does not guarantee the independence of $(\bar{\mathcal{M}}^i)_{i\in NB_j}$. Let $g_1,...,g_n$ be n nonnegative compactly supported functions defined on $\mathbb{R}^d$ and $i_1,...,i_n$ be n indices in NB_j. Let

$$G_t:=\sum_{k=1}^{n}\int_0^t ds\int_{\mathbb{R}^d}g_k(v)d\bar{\mathcal{M}}^{i_k}(s,v)=\sum_{k=1}^{n}\int_0^t ds\int_{\mathbb{T}}g_k(\Pi_s^{i_k}(r))d\mathcal{M}^{i_k}(s,r),$$

by definition of $\bar{\mathcal{M}}^{i_k}$. Since $(\mathcal{M}^{i_k})_{1\leq k\leq n}$ do not share atoms almost surely, then with probability one,

$$\mathrm{e}^{-G_t}=1+\sum_{k=1}^{n}\int_0^t ds\int_{\mathbb{T}}\left(\exp\left(-G_{s_-}-g_k(\Pi_s^{i_k}(r))\right)-\exp\left(-G_{s_-}\right)\right)d\mathcal{M}^{i_k}(s,r).$$

Taking expectations, we get

$$\mathbb{E}\left[\mathrm{e}^{-G_t}\right]=1+\int_0^t ds\mathbb{E}\left[\mathrm{e}^{-G_s}\right]\left(\sum_{k=1}^{n}\int_{\mathbb{R}^d}\left(\mathrm{e}^{-g_k(v')}-1\right)dw^{\mu_s,\bar{r}_0^{i_k}}(v')\right),$$

which gives

$$\mathbb{E}\left[\mathrm{e}^{-G_t}\right] = \prod_{k=1}^{n} \exp\left(\int_0^t ds \int_{\mathbb{R}^d} \left(\mathrm{e}^{-g_k(v')} - 1\right) dw^{\mu_s, \tilde{r}_0^{i_k}}(v')\right),$$

and this concludes the independence of $(\bar{\mathcal{M}}^i)_{i \in NB_j}$.

□

2.3 The nonlinear martingale problem

This section is devoted to the proof of Proposition 2.1. Recall the formalism of (2.3) with a random point process that selects directly the neighbours' velocities. We expect that, as N goes to infinity, the typical trajectories of the particles in the system (2.16) should be close to the solution of the nonlinear stochastic differential equation

$$\begin{cases} dX_t = V_t dt \\ dV_t = -\left(\int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r')\nabla W(X_t - x') d\mu_t(r', z') + \nabla U(X_t)\right) dt \\ \qquad\qquad + \int_{\mathbb{R}^d} (v' - V_{t^-}) d\mathcal{N}^{\mu, r_0}(t, v'), \end{cases} \tag{2.54}$$

where $\mathcal{N}^{\mu, r_0}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, independent of (X_0, V_0) given r_0, with intensity

$$\bar{\gamma} dt dw^{\mu_t, r_0}(v'), \tag{2.55}$$

given r_0. w^{μ_t, r_0} is the velocity marginal of μ_t, weighted by the function Γ_ℓ around r_0, as defined in (2.22) (replacing obviously $\tilde{r}_0^i$ by r_0 in this formula). Moreover, we set $r_t = r_0$ for all time $t \geq 0$. Formally, (2.54) is given by taking the limit μ of μ^N in (2.3).

The nonlinear martingale problem (2.8) and the equation (2.54) are nonlinear in the sense that they are both defined using the marginals $(\mu_t)_{t\geq 0}$, which are themselves only defined as a byproduct of the solutions to these problems. Several existence and uniqueness results exist for the solutions of SDEs and nonlinear martingale problems with similar Poisson exchanges in the case of the Kac model (see for instance [19, 28], and also [35] for a model with additional mean field interaction). Since $(r, x) \mapsto \Phi_\ell(r)\nabla W(x)$ is generally not Lipschitz continuous, those results cannot be applied straightforwardly in our setting and we have to study it from scratch.

To deal with the nonlinear problems, we first study the well-posedness of the associated *linear* problems. Let $Q \in \mathcal{M}^1(\mathcal{D})$ be a fixed probability measure. The linear mean field SDE associated with Q is

$$\begin{cases} dX_t = V_t dt \\ dV_t = -\left(\int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r')\nabla W(X_t - x') dQ_t(r', z') + \nabla U(X_t)\right) dt \\ \qquad\qquad + \int_{\mathbb{R}^d} (v' - V_{t^-}) d\mathcal{N}^{Q, r_0}(t, v') \end{cases} \tag{2.56}$$

and $\mathcal{N}^{Q, r_0}$ is associated with the same measure (2.55) (with Q instead of μ). Similarly we say that μ is a solution to the *linear martingale problem* associated with Q and starting at $\nu_0 \in \mathcal{M}^1(\mathbb{T} \times E)$ if $\mu_0 = \nu_0$ and

$$M_t^\psi = \psi(Y_t) - \psi(Y_0) - \int_0^t \mathcal{L}[Q_s]\psi(Y_s) ds \tag{2.57}$$

is a martingale under μ for any $\psi \in C_b^1(\mathbb{T} \times E)$, where $\mathcal{L}$ is the operator defined by (2.7).

Proposition 2.3. *Let $\nu_0 \in \mathcal{M}^1(\mathbb{T} \times E)$ satisfy **(H4)**, let Y_0 be random variable with law ν_0 and let $Q \in \mathcal{D}$ such that for all $T \geq 0$*

$$\sup_{t \leq T} \int_{\mathbb{T}\times E} (|x| + |v|)\, dQ_t(r, z) < \infty. \tag{2.58}$$

There is a unique solution to the stochastic differential equation (2.56), both pathwise and in law. Its law μ is the unique solution to the linear martingale problem (2.57) associated with Q starting at ν_0. Moreover, for any $T > 0$, μ satisfies

$$\sup_{t\leq T} \int (|x| + |v|)\, d\mu_t(r,z) < \infty, \tag{2.59}$$

and for any $t > 0$, and any measurable $\varphi : \mathbb{T} \to \mathbb{R}$

$$\int_{\mathbb{T}\times E} \varphi(r) d\mu_t(r,z) = \int_{\mathbb{T}\times E} \varphi(r) d\nu_0(r,z). \tag{2.60}$$

The proof of this proposition relies on classical techniques and there is no particular difficulty in deriving it. We therefore postpone it to the Appendix 2.B. With this result, we are now able to prove Proposition 2.1. The proof is inspired by [35] and relies on a contraction estimate in the space $\mathcal{M}^1(\mathcal{D}_T)$ for some $T > 0$, where $\mathcal{D}_T := D([0,T], \mathbb{T} \times E)$ is the set of right-continuous functions with left-limits defined on the time interval $[0,T]$ with values in $\mathbb{T} \times E$. To take into account the particular role of the parameter r, we will use a new appropriate Wasserstein-type distance. Let us first introduce the more classical distance that we will use throughout the proof. From the uniform distance

$$\|y^1 - y^2\|_{\infty,T} := \sup_{t\leq T} |y^1_t - y^2_t| \tag{2.61}$$

between $y^1, y^2 \in \mathcal{D}_T$ and from the Skorokhod distance $\|\cdot\|_{0,T}$ that makes the space $\mathcal{D}_T$ a complete space [10], we can consider the associated Wasserstein distances between $Q^1, Q^2 \in \mathcal{M}^1(\mathcal{D}_T)$:

$$\mathcal{W}_{\infty,T}(Q^1,Q^2) = \inf_{\pi\in\Pi(Q^1,Q^2)} \int \|y - y'\|_{\infty,T} d\pi(y,y'),$$

$$\mathcal{W}_{0,T}(Q^1,Q^2) = \inf_{\pi\in\Pi(Q^1,Q^2)} \int \|y - y'\|_{0,T} d\pi(y,y').$$

In particular

$$\mathcal{W}_{0,T}(Q^1,Q^2) \leq \mathcal{W}_{\infty,T}(Q^1,Q^2), \tag{2.62}$$

holds for any measures Q^1 and Q^2 by comparison of the two distances $\|\cdot\|_{\infty,T}$ and $\|\cdot\|_{0,T}$. From the completeness property for $(\mathcal{D}_T, \|\cdot\|_{0,T})$, we immediately get that $\mathcal{W}_{0,T}$ makes $\mathcal{M}^1(\mathcal{D}_T)$ a complete metric space (see [73]).

We will also use Wasserstein distances on the spaces of probability measures on $\mathbb{T} \times E$, E and $\mathbb{R}^d$. We will always use the notation $\mathcal{W}_1$ for each of them, and the space which it refers to will be clear from the context. We will also always denote by $\mathcal{W}_{\infty,T}$ the Wasserstein distances between probability measures on the spaces of trajectories on $\mathbb{T} \times E$, E and $\mathbb{R}^d$.

We will only deal with measures $Q^1, Q^2 \in \mathcal{M}^1(\mathcal{D})$ satisfying the following properties

(h1) $\sup_{t\leq T} \int (|x| + |v|)\, dQ^i_t(r,x,v) < \infty$ for $i = 1,2$ and for all $T > 0$.

(h2) Q^1 and Q^2 only charge trajectories $(r_t, x_t, v_t)_{t\geq 0}$ with constant $r_t = r_0$. Moreover, the r-marginal of Q^1_0 and Q^2_0 is the Lebesgue measure on $\mathbb{T}$.

Under **(h2)**, define the probability measures $Q^{1,r}$ and $Q^{2,r}$ on $D(\mathbb{R}_+, E)$ to be the conditional distributions of Q^1 and Q^2 on the z-trajectories given r.

Definition 2.1 (Sliced Wasserstein distance)**.** *The sliced Wasserstein distance between two measures $Q^1, Q^2 \in \mathcal{M}^1(\mathcal{D}_T)$ satisfying **(h2)** is the distance*

$$\mathcal{SW}_{\infty,T}(Q^1,Q^2) := \int_{\mathbb{T}} dr \mathcal{W}_{\infty,T}(Q^{1,r},Q^{2,r}). \tag{2.63}$$

Such a terminology has been used in a different context (see Chapter 5.5.4 in [61]), although the ideas behind both definitions are close. Therefore we will stick to this terminology. We will actually prove the contraction estimate for sliced Wasserstein distance by constructing an explicit coupling of the solutions of the linear martingale problem associated with Q^1 and Q^2. This distance enables us to deal with the particular structure we are considering and in particular to control the difference of the terms coming from the interaction potential and from the exchange of velocity. Moreover, the bound

$$\mathcal{W}_{\infty,T}(Q^1,Q^2) \leq \mathcal{SW}_{\infty,T}(Q^1,Q^2).$$

follows directly by considering the coupling $(r,Z,r,Z') \in (\mathcal{D}_T)^2$ of Q^1 and Q^2, where r is a uniform random variable on $\mathbb{T}$ and (Z,Z') has distribution given by the optimal coupling of $Q^{1,r}$ and $Q^{2,r}$. Combined with (2.62), we get

$$\mathcal{W}_{0,T}(Q^1,Q^2) \leq \mathcal{SW}_{\infty,T}(Q^1,Q^2). \tag{2.64}$$

Proof of Proposition 2.1. In the whole proof, K and K' are positive constants that change from one line to another and depend on the parameters of the problem W, U, ϕ γ but also ℓ and $\bar{\gamma}$, since we do not need to keep track of these elements anymore.

1. Contraction estimate on a short time interval

Let Q^1 and Q^2 be two elements of $\mathcal{M}^1(\mathcal{D})$ satisfying **(h1)** and **(h2)**. We are going to construct two coupled solutions $Y^1 = (r^1,Z^1)$ and $Y^2 = (r^2,Z^2)$ of the linear SDE (2.56) associated with Q^1 and Q^2 respectively and with initial distribution ν_0. By this coupling, our goal is to control the distance $\mathcal{SW}_{\infty,T}(\mu^1,\mu^2)$ by $\mathcal{SW}_{\infty,T}(Q^1,Q^2)$, where μ^1 and μ^2 are the law of Y^1 and Y^2 respectively.

We first couple their initial positions by setting $r_0^1 = r_0^2 = r_0$, where r_0 is a uniform random variable on $\mathbb{T}$, and we set $Z_0^1 = Z_0^2$, with Z_0^1 distributed with the probability measure $\nu_0^{r_0}$. In particular, the conditional law of Z^i given r_0 is μ^{i,r_0} for $i=1,2$. We straightforwardly get

$$\mathcal{SW}_{\infty,T}(\mu^1,\mu^2) \leq \mathbb{E}\left[\|Z^1-Z^2\|_{\infty,T}\right]. \tag{2.65}$$

We then construct the point processes $\mathcal{N}^{Q^1,r_0}$, $\mathcal{N}^{Q^1,r_0}$ that drive the two processes. By **(h2)**, the time intensities of the processes $\mathcal{N}^{Q^1,r_0}$ and $\mathcal{N}^{Q^1,r_0}$ are equal (to $\bar{\gamma}$), thus we can take the same jump times $(T_n)_{n\geq 0}$ for both. We will take $V^1_{T_n}$ and $V^2_{T_n}$, which are the new values of V^1 and V^2 after the jump, to be distributed as the optimal coupling of $w^{Q^1_{T_n},r_0}$ and $w^{Q^2_{T_n},r_0}$, that is to say:

$$\mathbb{E}\left[|V^1_{T_n}-V^2_{T_n}||r_0,T_n\right] = \mathcal{W}_1(w^{Q^1_{T_n},r_0},w^{Q^2_{T_n},r_0}). \tag{2.66}$$

We have a good control on (2.66) with the sliced distance by the following argument. Let $\pi^r \in \Pi(Q^{1,r},Q^{2,r})$ be the optimal coupling between $Q^{1,r}$ and $Q^{2,r}$ for $\mathcal{W}_{\infty,T}$, for $r\in\mathbb{T}$. Using Kantorovich-Rubinstein duality formula and hypothesis **(h2)**, we can bound $\mathcal{W}_1(w^{Q^1_t,r_0},w^{Q^2_t,r_0})$

for all $t \geq 0$:

$$\begin{aligned}
&\mathcal{W}_1(w^{Q^1_t,r_0}, w^{Q^2_t,r_0}) \\
&\qquad = \sup_{Lip(\varphi)\leq 1} \left| \int_{\mathbb{T}\times E} \varphi(v')\Gamma_\ell(r_0 - r')dQ^1_t(r',z') - \int_{\mathbb{T}\times E} \varphi(v'')\Gamma_\ell(r_0 - r'')dQ^2_t(r'',z'') \right| \\
&\qquad = \sup_{Lip(\varphi)\leq 1} \left| \int_{\mathbb{T}} dr\Gamma_\ell(r_0 - r) \int_{E^2} (\varphi(v') - \varphi(v''))d\pi^r_t(z',z'') \right| \\
&\qquad \leq \int_{\mathbb{T}} dr\Gamma_\ell(r_0 - r) \int_{E^2} |v' - v''| d\pi^r_t(z',z'') \\
&\qquad \leq \int_{\mathbb{T}} dr\Gamma_\ell(r_0 - r)\mathcal{W}_{\infty,T}(Q^{1,r}, Q^{2,r}) \\
&\qquad \leq K\mathcal{SW}_{\infty,T}(Q^1, Q^2),
\end{aligned} \tag{2.67}$$

where π^r_t is the time marginal of π^r at time t. A similar control by $\mathcal{W}_{\infty,T}$ instead of $\mathcal{SW}_{\infty,T}$ is much more difficult to get. As a consequence we control what happens at jump times. For the rest of the proof, we will adapt the proof techniques of [35]. We will only be concerned by the jump times before T, so set $S_n = T \wedge T_n$. The quantity $d_n := \|Z^1_t - Z^2_t\|_{\infty,S_n}$ is clearly bounded by

$$d_n \leq \sup_{t<S_n} |Z^1_t - Z^2_t| + |Z^1_{S_n} - Z^2_{S_n}|.$$

The last term can be bounded in expectation as follows: by continuity of X and continuity of V when there is no jump,

$$\begin{aligned}
\mathbb{E}\left[|Z^1_{S_n} - Z^2_{S_n}|\right] &\leq \mathbb{E}\left[|X^1_{S_n-} - X^2_{S_n-}| + |V^1_{S_n} - V^2_{S_n}|\right] \\
&\leq K\mathbb{E}\left[|Z^1_{S_n-} - Z^2_{S_n-}|\right] + \mathbb{E}\left[\mathcal{W}(w^{Q^1_{S_n},r_0}, w^{Q^2_{S_n},r_0})\right] \\
&\leq K\mathbb{E}\left[\sup_{t<S_n} |Z^1_t - Z^2_t|\right] + K\mathcal{SW}_{\infty,T}(Q^1, Q^2),
\end{aligned}$$

using (2.67) at the last line. Combining this with the previous inequality gives:

$$\mathbb{E}[d_n] \leq K'\mathbb{E}[d_{n-1}] + K'\mathbb{E}\left[\sup_{S_{n-1}<t<S_n} |Z^1_t - Z^2_t|\right] + K'\mathcal{SW}_{\infty,T}(Q^1, Q^2). \tag{2.68}$$

It just remains to control properly the second term of (2.68) to get a recursive inequality on $\mathbb{E}[d_n]$. This second term corresponds to a deterministic evolution between two jump times, so we need in particular to control the difference between the force terms at time t. By the Lipschitz property for ∇W, we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r')\nabla W(X^1_t - x')dQ^1_t(r',z') - \int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r'')\nabla W(X^2_t - x'')dQ^2_t(r'',z'') \right| \\
&\qquad\qquad \leq \int_{\mathbb{T}} dr\Phi_\ell(r_0 - r) \int_{E^2} \left|\nabla W(X^1_t - x') - \nabla W(X^2_t - x'')\right| d\pi^r_t(z',z'') \\
&\qquad\qquad \leq K|X^1_t - X^2_t| + K\mathcal{SW}_{\infty,T}(Q^1, Q^2),
\end{aligned}$$

introducing cross terms at the last line. This is the second place where we needed the sliced distance $\mathcal{SW}_{\infty,T}$ to bypass the difficulties coming from the r-parameter. The contribution of the difference between the pinning terms ∇U gives an extra $K|X^1_t - X^2_t|$ term. Hence, we get for $S_{n-1} < t < S_n$

$$\begin{aligned}
|Z^1_t - Z^2_t| &\leq |Z^1_{S_{n-1}} - Z^2_{S_{n-1}}| + K\int_{S_{n-1}}^t |Z^1_s - Z^2_s|ds + KT\mathcal{SW}_{\infty,T}(Q^1, Q^2) \\
&\leq d_{n-1} + KT \sup_{S_{n-1}<s<S_n} |Z^1_s - Z^2_s| + KT\mathcal{SW}_{\infty,T}(Q^1, Q^2).
\end{aligned}$$

Taking the supremum for $S_{n-1} < t < S_n$ on the left-hand side and choosing T sufficiently small so that $KT < 1$, we get the following bound:

$$\sup_{S_{n-1}<t<S_n} |Z_t^1 - Z_t^2| \le \frac{1}{1-KT} d_{n-1} + \frac{KT}{1-KT} \mathcal{SW}_{\infty,T}(Q^1, Q^2). \tag{2.69}$$

We can now combine this bound (2.69) for the deterministic dynamic evolution to (2.68) to get that:

$$\begin{aligned}\mathbb{E}[d_n] &\le K\frac{2-KT}{1-KT}\mathbb{E}[d_{n-1}] + \frac{K}{1-KT}\mathcal{SW}_{\infty,T}(Q^1, Q^2)\\ &\le a_T \mathbb{E}[d_{n-1}] + b_T \mathcal{SW}_{\infty,T}(Q^1, Q^2),\end{aligned}$$

where we denoted for simplicity

$$a_T = K\frac{2-KT}{1-KT}, \qquad b_T = \frac{K}{1-KT}.$$

By recursion and using the initial condition $d_0 = 0$, we get

$$\mathbb{E}[d_n] \le \frac{a_T^n - 1}{a_T - 1} b_T \mathcal{SW}_{\infty,T}(Q^1, Q^2). \tag{2.70}$$

Now that we control the trajectories until time S_n, we can extend this control up to time T. Let N_T be the number of jumps during the interval of time $[0, T]$. Then

$$\mathbb{E}\left[\|Z^1 - Z^2\|_{\infty,T}\right] \le \mathbb{E}[d_{N_T}] + \mathbb{E}\left[\sup_{T_{N_T}<t\le T} |Z_t^1 - Z_t^2|\right]. \tag{2.71}$$

As there is no jump between times T_{N_T} and T, we can apply the same estimates that lead to (2.69) to the second term on the right-hand side of (2.71):

$$\begin{aligned}\mathbb{E}\left[\sup_{T_{N_T}<t\le T} |Z_t^1 - Z_t^2|\right] &\le \frac{1}{1-KT}\mathbb{E}[d_{N_T}] + \frac{KT}{1-KT}\mathcal{SW}_{\infty,T}(Q^1, Q^2)\\ &\le \left(\frac{a_T}{K} - 1\right)\mathbb{E}[d_{N_T}] + \left(\frac{b_T}{K} - 1\right)\mathcal{SW}_{\infty,T}(Q^1, Q^2),\end{aligned}$$

in terms of a_T and b_T. All in all, combining this last inequality with (2.70) and (2.71) leads to

$$\begin{aligned}\mathbb{E}\left[\|Z_t^1 - Z_t^2\|_{\infty,T}\right] &\le \frac{a_T}{K}\mathbb{E}[d_{N_T}] + \left(\frac{b_T}{K} - 1\right)\mathcal{SW}_{\infty,T}(Q^1, Q^2)\\ &\le \left(\frac{a_T b_T}{K(a_T - 1)}\left(\mathbb{E}[a_T^{N_T}] - 1\right) + \frac{b_T}{K} - 1\right)\mathcal{SW}_{\infty,T}(Q^1, Q^2). \end{aligned} \tag{2.72}$$

As N_T has a Poisson distribution with parameter $\bar{\gamma}T$, the coefficient multiplying $\mathcal{SW}_{\infty,T}(Q^1, Q^2)$ can be computed explicitly:

$$\begin{aligned}c_T &:= \frac{a_T b_T}{K(a_T - 1)}\left(\mathbb{E}[a_T^{N_T}] - 1\right) + \frac{b_T}{K} - 1\\ &= \frac{a_T b_T}{K(a_T - 1)}\left(\exp\left(\bar{\gamma}T(a_T - 1)\right) - 1\right) + \frac{b_T}{K} - 1.\end{aligned}$$

Since $b_0 = K$, it is easy to see that $c_0 = 0$. As c_T is a continuous function of T, we can choose T small enough so that $c_T < 1$. Combining (2.72) with (2.65) finally gives the following contraction estimate for a short time $T > 0$:

$$\mathcal{SW}_{\infty,T}(\mu^1, \mu^2) \le c_T \mathcal{SW}_{\infty,T}(Q^1, Q^2). \tag{2.73}$$

2. Existence and uniqueness in $\mathcal{M}^1(\mathcal{D}_T)$

The existence part follows from a classical iteration method. Start from a measure $\mu^0 \in \mathcal{M}^1(\mathcal{D})$ such that $\mu^0_t = \nu_0$ for all $t \geq 0$. Then define iteratively $(\mu^n)_{n\geq 1}$ as solution to the linear martingale problem (2.57) associated with μ^{n-1} starting at ν_0. This is possible since, by (2.59) in Proposition 2.3, the solution μ^n to the linear martingale problem associated with μ^{n-1} satisfies

$$\sup_{t\leq T} \int_{\mathbb{T}\times E} (|x| + |v|)\, d\mu^n_t(r,z) < \infty,$$

which enables us to define such an iteration by Proposition 2.3. Moreover, by (2.59) and (2.60) in Proposition 2.3, we know that for any $n \geq 0$, μ^n and μ^{n+1} satisfy the hypotheses **(h1)** and **(h2)**. We can thus apply the recursion estimate (2.73) recursively and get

$$\mathcal{SW}_{\infty,T}(\mu^{n+1},\mu^n) \leq (c_T)^n \mathcal{SW}_{\infty,T}(\mu^1,\mu^0)$$

and the sequence is thus Cauchy for $\mathcal{SW}_{\infty,T}$ and therefore for $\mathcal{W}_{0,T}$ by (2.64). By completeness of $\mathcal{M}^1(\mathcal{D}_T)$ equipped with the distance $\mathcal{W}_{0,T}$, the sequence converges in this space. Uniqueness is also immediate by the contraction estimate.

3. Existence and uniqueness in $\mathcal{M}^1(\mathcal{D})$

Let μ be the solution to the nonlinear martingale problem starting at ν_0 on $\mathcal{M}^1(\mathcal{D}_T)$ and let Q^1 and Q^2 be two measures on $\mathcal{M}^1(\mathcal{D})$ such that $Q^1_t = Q^2_t = \mu_t$ for $t \leq T$. Define μ^1 and μ^2 to be the solutions of the linear martingale problem associated with Q^1 and Q^2 on $\mathcal{M}^1(\mathcal{D}_{2T})$. Thus $\mu^1_T = \mu^2_T$ by uniqueness of the linear martingale problem on $\mathcal{M}^1(\mathcal{D}_T)$ and the same contraction estimate as (2.73) hold for the interval $[T, 2T]$ between μ^1, μ^2 and Q^1, Q^2. Thus existence and uniqueness extends to $\mathcal{M}^1(\mathcal{D}_{2T})$ and by immediate recurrence to $\mathcal{M}^1(\mathcal{D})$.

□

The same Picard iteration used in the previous proof enables to prove the following Proposition.

Proposition 2.4. *There is existence and uniqueness in law of solutions to the SDE* (2.54).

2.4 Energy transport

We finish by studying the transport of energy in our model for a class of pinning potentials U and proving Proposition 2.2. The proof is in two parts. In the first one, we prove a diffusion equation for the energy associated with the limit measure μ_t at the timescale $t\ell^{-2}$. In the second part, we prove that the energy of the particle system converges to the energy associated with μ_t in that timescale.

2.4.1 Diffusion equation for the limit measure

First, let us start by proving that the symmetry **(H5)** is preserved at any time $t > 0$ for symmetric potentials.

Lemma 2.7. *Under **(H5)**, if U and W are symmetric, then the symmetry $\mu_t(r,z) = \mu_t(r,-z)$ for any $(r,z) \in \mathbb{T} \times E$ is preserved at any later time $t > 0$.*

Proof. Let $(Y_t)_{t\geq 0} = (r_t, X_t, V_t)_{t\geq 0}$ be the canonical process associated with the solution μ to the nonlinear martingale problem (2.8). Define $\mu^*(r,z) = \mu(r,-z)$, $\psi^*(r,z) = \psi(r,-z)$ and $Y_t^* = (r_t, -Z_t)$ for simplicity. Then, we can rewrite

$$\begin{aligned}
&\mathcal{A}[\mu_s]\psi(Y_s)\\
&\quad = V_s \cdot \nabla_v \psi(Y_s) - \left(\nabla U(X_s) + \int_{\mathbb{T}\times E} \Phi_\ell(r_s - r')\nabla W(X_s - x')d\mu_s(r',z')\right)\cdot \nabla_x \psi(Y_s)\\
&\quad = -V_s \cdot \nabla_v \psi^*(Y_s^*) + \left(\nabla U(X_s) + \int_{\mathbb{T}\times E} \Phi_\ell(r_s - r')\nabla W(X_s + x')d\mu_s^*(r',z')\right)\cdot \nabla_x \psi^*(Y_s^*)\\
&\quad = \mathcal{A}[\mu_s^*]\psi^*(Y_s^*),
\end{aligned}$$

using antisymmetry of ∇U and ∇W at the last line. Similarly,

$$\mathcal{S}[\mu_s]\psi(Y_s) = \int_{\mathbb{T}\times E} \Gamma_\ell(r_s - r')\left(\psi^*(r_s, -X_s, v') - \psi^*(Y_s^*)\right)d\mu_s^*(r',z') = \mathcal{S}[\mu_s^*]\psi^*(Y_s^*).$$

As a consequence, for any $\psi \in C_b^1(\mathbb{T}\times E)$,

$$\begin{aligned}
\psi^*(Y_t^*) - \psi^*(Y_0^*) + \int_0^t \mathcal{L}[\mu_s^*]\psi^*(Y_s^*)ds &= \psi(Y_t) - \psi(Y_0) + \int_0^t \mathcal{L}[\mu_s]\psi(Y_s)ds\\
&= M_t^\psi,
\end{aligned}$$

where M_t^ψ is a martingale. Since Y^* has distribution μ^*, μ^* is also solution to the nonlinear martingale problem (2.8) starting at $\mu_0^* = \mu_0$. By uniqueness, $\mu^* = \mu$, which implies the property we were looking for. □

As a consequence of this symmetry property, the integral $\int_{\mathbb{R}^d} x' d\mu_t(r',z')$ is null. In particular, if the interaction potential is harmonic $W(x) = |x|^2/2$, then the Vlasov equation (2.6) is reduced to the simplified version (2.10). We will always consider harmonic interaction potentials from now on. Let Y be a solution to the nonlinear SDE (2.54) with spatial parameter $r\in\mathbb{T}$ and recall the notation from (2.53):

$$\begin{aligned}
\mathcal{E}[\mu_t](Y_t) &= \frac{1}{2}|V_t|^2 + \frac{1}{2}\int_{\mathbb{T}\times E}\Phi_\ell(r-r')W(X_t - x')d\mu_t(r',z') + U(X_t)\\
&= \frac{1}{2}|V_t|^2 + \frac{1}{4}|X_t|^2 + \frac{1}{4}\int_{\mathbb{T}\times E}\Phi_\ell(r-r')|x'|^2 d\mu_t(r',z') + U(X_t), \qquad (2.74)
\end{aligned}$$

since the term $\frac{1}{2}X_t\cdot\int_{\mathbb{T}\times E}\Phi_\ell(r-r')x'd\mu_t(r',z') = 0$ by Lemma 2.7. Define also the local energy associated with the distribution μ of Y, given $r_0 = r$, by

$$\begin{aligned}
\mathcal{E}_t(r) &= \mathbb{E}\Big[\mathcal{E}[\mu_t](Y_t)|r_0 = r\Big]\\
&= \int_E\left(\frac{1}{2}|v|^2 + \frac{1}{2}\int_{\mathbb{T}\times E}\Phi_\ell(r-r')W(x-x')d\mu_t(r',z') + U(x)\right)d\mu_t(r,x,v)\\
&= \int_E\left(\frac{1}{2}|v|^2 + \frac{1}{4}\int_{\mathbb{T}\times E}\Phi_\ell(r-r')|x'|^2 d\mu_t(r',z') + \frac{1}{4}|x|^2 + U(x)\right)d\mu_t(r,x,v). \qquad (2.75)
\end{aligned}$$

When W is harmonic, the contribution of the neighboring parts near the axial position r in (2.75) is only given by the integral $\int_{\mathbb{T}\times E}\Phi_\ell(r-r')|x'|^2 d\mu_t(r',z')$, which does not involve the variable x. Let us investigate in more detail the evolution of $\mathcal{E}_t(r)$. By a straightforward calculation from (2.54), we get that for any $g\in\mathcal{C}^2(\mathbb{T})$,

$$\begin{aligned}
&\int_{\mathbb{T}}\mathcal{E}_{t\ell^{-2}}(r)g(r)dr - \int_{\mathbb{T}}\mathcal{E}_0(r)g(r)dr\\
&\quad = \mathbb{E}\Big[\mathcal{E}[\mu_{t\ell^{-2}}](Y_{t\ell^{-2}})g(r_0)\Big] - \mathbb{E}\Big[\mathcal{E}[\mu_0](Y_0)g(r_0)\Big]\\
&\quad = -\int_0^{t\ell^{-2}} ds\int_{\mathbb{T}^2} g(r)\Phi_\ell(u)j_s^{r,r+u;a}drdu - \bar{\gamma}\int_0^{t\ell^{-2}} ds\int_{\mathbb{T}^2} g(r)\Gamma_\ell(u)j_s^{r,r+u;s}drdu, \qquad (2.76)
\end{aligned}$$

where

$$\begin{aligned} j_s^{r,r+u;a} &= \frac{1}{2}\int_{E^2}(v+v')\cdot\nabla W(x-x')d\mu_s(r+u,z')d\mu_s(r,z) \\ &= \frac{1}{2}\int_E v\cdot x d\mu_s(r,z) - \frac{1}{2}\int_E v'\cdot x' d\mu_s(r+u,z') \end{aligned}$$

is the mechanical contribution to the energy current and

$$j_s^{r,r+u;s} = \int_E \frac{1}{2}|v|^2 d\mu_s(r,z) - \int_E \frac{1}{2}|v|^2 d\mu_s(r+u,z').$$

is the stochastic contribution to the energy current. Rearranging terms by a summation by parts in (2.76), since $r\in\mathbb{T}$ and Φ_ℓ and Γ_ℓ are symmetric, we get

$$\begin{aligned} &\int_{\mathbb{T}}\mathcal{E}_{t\ell^{-2}}(r)g(r)dr - \int_{\mathbb{T}}\mathcal{E}_0(r)g(r)dr \\ &= \int_0^{t\ell^{-2}} ds\int_{\mathbb{T}\times E}\frac{1}{2}v\cdot x d\mu_s(r,z)\int_{\mathbb{T}}\Phi_\ell(u)\left(g(r+u)-g(r)\right)du \\ &\qquad\qquad + \bar\gamma\int_0^{t\ell^{-2}} ds\int_{\mathbb{T}\times E}\frac{1}{2}|v|^2 d\mu_s(r,z)\int_{\mathbb{T}}\Gamma_\ell(u)\left(g(r+u)-g(r)\right)du \\ &= c_\phi\int_0^t ds\int_{\mathbb{T}\times E}\frac{1}{2}v\cdot x g''(r)d\mu_{s\ell^{-2}}(r,z) \\ &\qquad\qquad + \bar\gamma c_\gamma\int_0^t ds\int_{\mathbb{T}\times E}\frac{1}{2}|v|^2 g''(r)d\mu_{s\ell^{-2}}(r,z) + O(t\ell), \end{aligned} \tag{2.77}$$

where the error term comes from a Taylor expansion of g at the last line and we usied the uniform in time moment bound of Lemma 2.4. The constant c_ϕ is equal to

$$c_\phi = \frac{1}{2}\int_{-1/2}^{1/2} u^2\phi(u)du,$$

and similarly for c_γ. Notice that, even if we have the symmetry property given by Lemma 2.7, this is not sufficient to prove that the Hamiltonian contribution $\int x\cdot v d\mu_t(r,z)$ in (2.77) vanishes. However, it is easy to prove from (2.54) that

$$\begin{aligned} &\int_{\mathbb{T}\times E}|x|^2 g''(r)d\mu_{t\ell^{-2}}(r,z) - \int_{\mathbb{T}\times E}|x|^2 g''(r)d\mu_0(r,z) \\ &\qquad\qquad = \ell^{-2}\int_0^t ds\int_{\mathbb{T}\times E}\frac{1}{2}x\cdot v g''(r)d\mu_{s\ell^{-2}}(r,z), \end{aligned} \tag{2.78}$$

and consequently, applying uniform in time moment bound of Lemma 2.4 to the left-hand side of (2.78), we get an estimate of the right-hand side of (2.78):

$$\int_0^t ds\int_{\mathbb{T}\times E}\frac{1}{2}x\cdot v g''(r)d\mu_{s\ell^{-2}}(r,z) = O(t\ell^2). \tag{2.79}$$

To prove that $\mathcal{E}_{t\ell^{-2}}(r)$ evolves diffusively, it remains to close the equation (2.77) by replacing the kinetic energy integral $\int_E \frac{1}{2}|v|^2 d\mu_{s\ell^{-2}}(r,z)$, by the energy $\mathcal{E}_{t\ell^{-2}}(r)$. To do so, we prove a result of equipartition of energy in Lemma 2.8. This result states that long time integrals of kinetic energy can be actually replaced by a fraction of time integrals of the total energy. We prove it only for specific pinning potentials for which the identity

$$x\cdot\nabla U(x) = 2U(x)$$

holds. For a positive continuously differentiable function U, it is well known that satisfying this identity is equivalent to be homogeneous of degree 2. In particular, U is such that $U(x) = |x|^2\psi(x/|x|)$, where $\psi\in C^2(\mathbb{S}^{d-1},\mathbb{R}_+^*)$. It is straightforward to check that U satisfies **(H1)** in this setting. We furthermore require ψ to be symmetric for Lemma 2.7 to hold.

Lemma 2.8. *Let $U(x) = |x|^2\psi(x/|x|)$, where $\psi \in C^2(\mathbb{S}^{d-1}, \mathbb{R}_+^*)$ is a symmetric function. For any $G \in \mathcal{C}^2(\mathbb{T})$, we have the following time equipartition of energy.*

$$\int_0^t ds \int_{\mathbb{T}\times E} \frac{1}{2}|v|^2 G(r) d\mu_{s\ell^{-2}}(r,z) = \frac{1}{2}\int_0^t ds \int_{\mathbb{T}} \mathcal{E}_{s\ell^{-2}}(r) G(r) dr + O((1+\bar{\gamma})t\ell^2). \tag{2.80}$$

Proof. Let $(Y_s)_{s\geq 0} = (r_0, X_s, V_s)_{s\geq 0}$ be a solution to the nonlinear SDE (2.54). We compute for any $T > 0$:

$$\begin{aligned} &X_T \cdot V_T - X_0 \cdot V_0 \\ &\quad= \int_0^T |V_s|^2 ds - \int_0^T X_s \cdot (\nabla U(X_s) + X_s)\, ds + \int_0^T \int_{\mathbb{R}^d} X_s \cdot (v' - V_{s\_})\, d\mathcal{N}^{\mu,r_0}(s,v'). \end{aligned}$$

Using that $x \cdot \nabla U(x) = 2U(x)$, we can introduce the potential U at the last line. Then, using (2.74), we can introduce $\mathcal{E}[\mu_s](Y_s)$ and get

$$\begin{aligned} &X_T \cdot V_T - X_0 \cdot V_0 \\ &= 4\int_0^T \frac{1}{2}|V_s|^2 ds - 2\int_0^T \mathcal{E}[\mu_s](Y_s) ds + \int_0^T ds \left(\frac{1}{2}\int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r')|x'|^2 d\mu_s(r',z') - \frac{1}{2}|X_s|^2\right) \\ &\qquad + \int_0^T \int_{\mathbb{R}^d} X_s \cdot (v' - V_{s\_})\, d\mathcal{N}^{\mu,r_0}(s,v'). \end{aligned}$$

Multiplying by $G(r_0)$ on both sides for $G \in \mathcal{C}^2(\mathbb{T})$ and taking expectations gives:

$$\begin{aligned} &\int_{\mathbb{T}\times E} x\cdot v G(r) d\mu_T(r,z) - \int_{\mathbb{T}\times E} x\cdot v G(r) d\mu_0(r,z) \\ &= 4\int_0^T ds \int_{\mathbb{T}\times E} \frac{1}{2}|v|^2 G(r) d\mu_s(r,z) - 2\int_0^T ds \int_{\mathbb{T}} \mathcal{E}_s(r) G(r) dr \\ &\quad + \ell^2 c_\phi \int_0^T ds \int_{\mathbb{T}\times E} \frac{1}{2}|x|^2 G''(r) d\mu_s(r,z) + \ell^2 \bar{\gamma} c_\gamma \int_0^T ds \int_{\mathbb{T}\times E} x\cdot v G''(r) d\mu_s(r,z) + o(T\ell^2), \end{aligned}$$

where we used a Taylor expansion for the two terms at the last line. Applying this result for $T = t\ell^{-2}$, and rearranging terms, we get:

$$\begin{aligned} &\int_0^t ds \int_{\mathbb{T}\times E} \frac{1}{2}|v|^2 G(r) d\mu_{s\ell^{-2}}(r,z) - \frac{1}{2}\int_0^t ds \int_{\mathbb{T}} \mathcal{E}_{s\ell^{-2}}(r) G(r) dr \\ &= \ell^2 \frac{1}{4}\left(\int_{\mathbb{T}\times E} x\cdot v G(r) d\mu_{t\ell^{-2}}(r,z) - \int_{\mathbb{T}\times E} x\cdot v G(r) d\mu_0(r,z)\right) \\ &\quad + \ell^2 \frac{c_\phi}{4}\int_0^t ds \int_{\mathbb{T}\times E} \frac{1}{2}|x|^2 G''(r) d\mu_{s\ell^{-2}}(r,z) + \ell^2 \bar{\gamma}\frac{c_\gamma}{4}\int_0^t ds \int_{\mathbb{T}\times E} x\cdot v G''(r) d\mu_{s\ell^{-2}}(r,z) + o(t\ell^2). \end{aligned}$$

Applying the uniform moment bound in Lemma 2.4, we deduce that the right-hand side is a $O((1+\bar{\gamma})\ell^2)$ and this concludes the proof. ☐

Combining the equipartition result (2.80) and the control on the Hamiltonian current (2.79) in (2.77), we finally get that $\mathcal{E}_{t\ell^{-2}}(r)$ evolves diffusively:

$$\int_{\mathbb{T}} \mathcal{E}_{t\ell^{-2}}(r) g(r) dr - \int_{\mathbb{T}} \mathcal{E}_0(r) g(r) dr = \bar{\gamma}\frac{c_\gamma}{2}\int_0^t ds \int_{\mathbb{T}} \mathcal{E}_{s\ell^{-2}}(r) g''(r) dr + O(t\ell + t\bar{\gamma}\ell^2), \tag{2.81}$$

for any $g \in \mathcal{C}^4(\mathbb{T})$.

2.4.2 Convergence of the particle system's energy

We deduce Proposition 2.2 from the convergence of the microscopic energy to $\mathcal{E}_t(r)$ proven in the next lemma. Let us fix $\bar{\gamma}$ and let $c(N,\ell,t)$ denote the constant appearing in Theorem 2.1, *i.e*

$$c(N,\ell,t) = K_1\left((N\epsilon_N)^{-\frac{1}{4(d+1)}} + \frac{\epsilon_N^{1/2}}{\ell^{1/2}}\right)\mathrm{e}^{K_2 t}.$$

Recall from (2.11) the definition of the microscopic energy

$$\begin{aligned}\mathcal{E}_t^i &= \mathcal{E}[\mu_t^N](Y_t^i)\\ &= \frac{1}{2}|V_t^i|^2 + \frac{1}{4}\int_{\mathbb{T}\times E}\Phi_\ell\left(\frac{i}{N}-r'\right)|X^i - x'|^2 d\mu_t^N(r',z') + U(X_t^i).\end{aligned}$$

Lemma 2.9. *Let $G \in \mathcal{C}^1(\mathbb{T})$. Under hypothesis **(H6)**, there exist positive constants K, K' such that for any time $T > 0$ and a constant $M > 0$ large enough*

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \mathcal{E}_T^i G\left(\frac{i}{N}\right) - \int_{\mathbb{T}}\mathcal{E}_T(r)G(r)dr\right|\right] \leq M^2 c(N,\ell,T) + \frac{K}{M^b}\mathrm{e}^{K'T} + O(\ell^2).$$

In this result, b is the constant appearing in the moment hypothesis **(H6)**. Taking $T = t\ell^{-2}$, $M = c(N,\ell,T)^{-1/3}$ and $\epsilon_N = \ell^{\frac{2d+2}{2d+3}}N^{-\frac{1}{2d+3}}$ gives the result stated in Proposition 2.2. In particular, one can find a constant $c > 0$ and choose $\ell = \ell(N) = c(\log N)^{-1/2}$ so that (2.12) holds.

Proof. First, let us simplify the problem. In the definition (2.75) of the energy $\mathcal{E}_T(r)$, we expect that the approximation

$$\frac{1}{4}\int_{\mathbb{T}}\Phi_\ell(r-r')|x'|^2 d\mu_T(r',z') \approx \frac{1}{4}|x|^2$$

is true when ℓ is small. We can actually make this approximation rigorous when we integrate $\mathcal{E}_T(r)G(r)$:

$$\begin{aligned}\int_{\mathbb{T}}\mathcal{E}_T(r)G(r)dr &= \int_{\mathbb{T}\times E}\left(\left(\frac{1}{2}|v|^2 + \frac{1}{4}|x|^2 + U(x)\right)G(r) + \frac{1}{4}|x|^2\int_{\mathbb{T}}\Phi_\ell(u)G(r+u)du\right)d\mu_T(r,z)\\ &= \int_{\mathbb{T}\times E}\left(\frac{1}{2}|v|^2 + \frac{1}{2}|x|^2 + U(x)\right)G(r)d\mu_T(r,z) + O(\ell^2),\end{aligned}$$

by a Taylor expansion of G. By the same procedure for the microscopic system, we have

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N \mathcal{E}_T^i G\left(\frac{i}{N}\right) = \int_{\mathbb{T}\times E}&\left(\frac{1}{2}|v|^2 + \frac{1}{2}|x|^2 + U(x)\right)G(r)d\mu_T^N(r,z)\\ &- \frac{1}{2}\int_{(\mathbb{T}\times E)^2} x\cdot x' G(r)d\mu_T^N(r,z)d\mu_T^N(r',z') + \delta_\ell,\end{aligned}$$

where δ_ℓ is such that $\mathbb{E}[|\delta_\ell|] = O(\ell^2)$ by conservation of energy. Notice that we have a non-vanishing extra term for the microscopic version. We introduce

$$H(x,v) = \frac{1}{2}|v|^2 + \frac{1}{2}|x|^2 + U(x)$$

for notational convenience. The proof of the lemma now boils down to bound the following two terms

$$\mathbb{E}\left[\left|\int_{\mathbb{T}\times E}H(x,v)G(r)d\mu_T^N(r,z) - \int_{\mathbb{T}\times E}H(x,v)G(r)d\mu_T(r,z)\right|\right] \tag{2.82}$$

and

$$\mathbb{E}\left[\left|\int_{(\mathbb{T}\times E)^2} x\cdot x' G(r) d\mu_T^N(r,z) d\mu_T^N(r',z') - \int_{(\mathbb{T}\times E)^2} x\cdot x' G(r) d\mu_T^N(r,z) d\mu_T(r',z')\right|\right], \tag{2.83}$$

since the second integral in (2.83) is null by Lemma 2.7.

For that, we are going to use Theorem 2.1. However, since the Wasserstein $\mathcal{W}_1$ distance only enables to control differences of integrals with respect to Lipschitz functions with Lipschitz constant less than 1, we have to cut the large values of H and $x\cdot x'$. To control (2.82), we therefore introduce a sequence of functions H^M depending on a parameter M such that H^M approximates H:

$$H^M(z) := H(z)\mathbb{1}_{H(z)<M} + M\left(2 - \exp\left(1 - \frac{H(z)}{M}\right)\right)\mathbb{1}_{H(z)\geq M}.$$

With this choice, it is easy to check that H^M/M^2 is a Lipschitz continuous function such that

$$\left\|\frac{1}{M^2}H^M\right\|_\infty \leq \frac{2}{M}, \qquad Lip\left(\frac{1}{M^2}H^M\right) \leq \frac{1}{M^2}.$$

In particular,

$$\begin{aligned} Lip\left(\frac{1}{M^2}H^M G\right) &\leq Lip\left(\frac{1}{M^2}H^M\right)\|G\|_\infty + \left\|\frac{1}{M^2}H^M\right\|_\infty Lip(G) \\ &\leq \frac{1}{M^2}\|G\|_\infty + \frac{2}{M}Lip(G), \end{aligned}$$

which is less than one for M large enough. Introducing cross terms in (2.82), we now bound it by

$$\begin{aligned} &\mathbb{E}\left[\left|\int_{\mathbb{T}\times E} H(z)G(r)d\mu_T^N(r,z) - \int_{\mathbb{T}\times E} H(z)G(r)d\mu_T(r,z)\right|\right] \\ &\quad\leq \|G\|_\infty \mathbb{E}\left[\int_{\mathbb{T}\times E}\left|H(z)-H^M(z)\right| d\mu_T^N(r,z)\right] \\ &\qquad + \|G\|_\infty \int_{\mathbb{T}\times E}\left|H(z)-H^M(z)\right| d\mu_T(r,z) \\ &\qquad + M^2\mathbb{E}\left[\left|\int_{\mathbb{T}\times E}\frac{1}{M^2}H^M(z)G(r)d\mu_T^N(r,z) - \int_{\mathbb{T}\times E}\frac{1}{M^2}H^M(z)G(r)d\mu_T(r,z)\right|\right]. \end{aligned} \tag{2.84}$$

We can now bound the last term by $M^2\mathbb{E}[\mathcal{W}(\mu_t^N,\mu_t)]$ and get

$$\mathbb{E}\left[\left|\int_{\mathbb{T}\times E} H^M(z)G(r)d\mu_T^N(r,z) - \int_{\mathbb{T}\times E} H^M(z)G(r)d\mu_T(r,z)\right|\right] \leq M^2 c(N,\ell,T). \tag{2.85}$$

It just remains to control the first two terms in (2.84) which correspond to the cut parts. The first term in (2.84) can be bounded by

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{T}\times E}\left|H(z)-H^M(z)\right| d\mu_T^N(r,z)\right] &\leq \mathbb{E}\left[\int_{\mathbb{T}\times E} H(z)\mathbb{1}_{H(z)\geq M} d\mu_T^N(r,z)\right] \\ &\leq \frac{1}{M^b}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N H(X_T^i,V_T^i)^{1+b}\right], \end{aligned} \tag{2.86}$$

by a Markov inequality, with the view of using the moment hypothesis **(H6)**. We now derive a moment type bound at time T. Using the dynamics (2.16), and the fact that $\nabla_v H = v$ and

$\nabla_x H = x + \nabla U(x)$, we compute the last term:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N H(X_T^i, V_T^i)^{1+b}\right] - \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N H(X_0^i, V_0^i)^{1+b}\right] \\
= (1+b)\int_0^T \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sum_{k=-\ell N}^{\ell N} \phi_k V^i \cdot X^{i+k} H(X_s^i, V_s^i)^b\right] \\
+ \bar{\gamma}\int_0^T \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sum_{k=-\ell N}^{\ell N} \gamma_k \left(H(X_s^i, V_s^{i+k})^{1+b} - H(X_s^i, V_s^i)^{1+b}\right)\right] ds,
\end{aligned} \tag{2.87}$$

the first term being the hamiltonian contribution and the second one the stochastic contribution, as usual. We then bound the product

$$\begin{aligned}
V^i \cdot X^{i+k} H(X_s^i, V_s^i)^b &\leq \left(\frac{1}{2}|V^i|^2 + \frac{1}{2}|X^{i+k}|^2\right) H(X_s^i, V_s^i)^b \\
&\leq H(X_s^i, V_s^i)^{1+b} + H(X^{i+k}, V^{i+k}) H(X_s^i, V_s^i)^b \\
&\leq K H(X_s^i, V_s^i)^{1+b} + K' H(X_s^{i+k}, V_s^{i+k})^{1+b},
\end{aligned}$$

where K and K' are two constants. We bounded directly the two terms by H at the second line, and then used Young's inequality at the third line. Summing this inequality over i and k, we obtain that the first term in the right-hand side of (2.87) is thus bounded by

$$\int_0^T \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sum_{k=-\ell N}^{\ell N} \phi_k V^i \cdot X^{i+k} H(X_s^i, V_s^i)^b\right] \leq K \int_0^T \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N H(X_s^i, V_s^i)^{1+b}\right], \tag{2.88}$$

for some constant K. The same bound can be obtained for the second term in the right-hand side of (2.87). All in all, By Gronwall's inequality and **(H6)**, we get:

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N H(X_T^i, V_T^i)^{1+b}\right] \leq K\, \mathrm{e}^{K'T}.$$

Eventually, the term (2.86) corresponding to the large energies is bounded by

$$\mathbb{E}\left[\int_{\mathbb{T}\times E} \left|H(z) - H^M(z)\right| d\mu_T^N(r,z)\right] \leq \frac{K}{M^b}\, \mathrm{e}^{K'T}.$$

The same bound can be obtained for the second term in (2.84). Combining both bounds together with (2.85) in (2.84) gives the bound we wanted for the convergence of H in (2.82):

$$\mathbb{E}\left[\left|\int_{\mathbb{T}\times E} H(z)G(r) d\mu_T^N(r,z) - \int_{\mathbb{T}\times E} H(z)G(r) d\mu_T(r,z)\right|\right] \leq M^2 c(N,\ell,T) + \frac{K}{M^b}\, \mathrm{e}^{K'T}.$$

To conclude the proof, it just remains to control (2.83) in the same spirit. Introducing an indicator function to cut large values of x, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{(\mathbb{T}\times E)^2} x \cdot x' G(r) d\mu_T^N(r,z) d\mu_T^N(r',z') - \int_{(\mathbb{T}\times E)^2} x \cdot x' G(r) d\mu_T^N(r,z) d\mu_T(r',z')\right|\right] \\
&\leq \|G\|_\infty \mathbb{E}\left[\int_{\mathbb{T}\times E} |x| \mathbb{1}_{|x|\leq M} d\mu_T^N(r,z) \left|\int_{\mathbb{T}\times E} x' d\mu_T^N(r',z') - \int_{\mathbb{T}\times E} x' d\mu_T(r',z')\right|\right] \\
&\qquad + \mathbb{E}\left[\left|\int_{(\mathbb{T}\times E)^2} x \cdot x' \mathbb{1}_{|x|>M} G(r) d\mu_T^N(r,z) d\mu_T^N(r',z')\right|\right] \\
&\leq M\|G\|_\infty \mathbb{E}\left[\mathcal{W}_1(\mu_T^N, \mu_T)\right] \\
&\qquad + \|G\|_\infty \mathbb{E}\left[\int_{\mathbb{T}\times E} |x|^2 \mathbb{1}_{|x|>M} d\mu_T^N(r,z)\right] \mathbb{E}\left[\int_{\mathbb{T}\times E} |x'|^2 d\mu_T^N(r',z')\right],
\end{aligned}$$

by Cauchy-Schwarz inequality at the last line. We then use the moment bounds of Lemma 2.4 and a Markov inequality to get

$$\begin{aligned}\mathbb{E}\left[\left|\int_{(\mathbb{T}\times E)^2} x\cdot x' G(r) d\mu_T^N(r,z) d\mu_T^N(r',z') - \int_{(\mathbb{T}\times E)^2} x\cdot x' G(r) d\mu_T^N(r,z) d\mu_T(r',z')\right|\right] \\ \leq KMc(N,\ell,T) + \frac{K}{M^{2b}} E\left[\int_{\mathbb{T}\times E} |x|^{2(1+b)} d\mu_T^N(r,z)\right],\end{aligned}$$

which gives the desired result since $|x|^2 \leq H(x,v)$. $\square$

2.A Convergence of some empirical measure for the Wasserstein distance

For any $1 \leq j \leq \epsilon_N^{-1}$, let $(\bar{Y}^i)_{i \in NB_j}$ be a family of independent random variables on $B_j \times \mathbb{R}^d \times \mathbb{R}^d$ with law $\mu^j(r,z) = \epsilon_N^{-1}\mu(r,z)\mathbb{1}_{r \in B_j}$, and $\bar{Y}^i = (\bar{r}^i, \bar{X}^i, \bar{V}^i)$. In this section, we prove that the empirical measure $\bar{\mu}^{N,j} = 1/(N\epsilon_N)\sum \delta_{\bar{Y}_i}$ satisfies a law of large numbers in some sense. More precisely, we prove that

Proposition 2.5. *If μ has a finite moment of order 2 in the sense that $\int_{\mathbb{T}\times E} |z|^2 d\mu(r,z) < \infty$. Then there exists some positive constant K such that*

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\bar{\mu}^{N,j}, \mu^j)\right] \leq K(N\epsilon_N)^{-\frac{1}{4(d+1)}} + K\epsilon_N.$$

Notice that more refined versions of law of large numbers for empirical measures in the Wasserstein distance exist (see [26] for instance). But the proposition we prove here has an easy proof and is well suited for our problem.

Proof. Let $M > 0$. For any $1 \leq j \leq \epsilon_N^{-1}$, using Kantorovich-Rubinstein duality formula, we can bound:

$$\mathbb{E}\left[\mathcal{W}_1(\bar{\mu}^{N,j}, \mu^j)\right] \leq \mathbb{E}\left[\sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{B_j\times E} \varphi(r,z)\mathbb{1}_{\{|z|\leq M\}} d(\bar{\mu}^{N,j} - \mu^j)\right] + \mathbb{E}\left[\sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{B_j\times E} \varphi(r,z)\mathbb{1}_{\{|z|> M\}} d(\bar{\mu}^{N,j} - \mu^j)\right]. \quad (2.89)$$

The second term in (2.89) can be bounded using Markov inequality type arguments. First, we bound it by

$$\begin{aligned}\mathbb{E}&\left[\sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{B_j\times E} \varphi(r,z)\mathbb{1}_{\{|z|> M\}} d(\bar{\mu}^{N,j} - \mu^j)\right] \\ &\leq \mathbb{E}\left[\int_{B_j\times E} (r + |z|)\, \mathbb{1}_{\{|z|> M\}} d(\bar{\mu}^{N,j} + \mu^j)\right] \\ &\leq 2\epsilon_N^{-1} \int_{B_j\times E} \mathbb{1}_{\{|z|> M\}} d\mu\,(r,z) + 2\epsilon_N^{-1}\int_{B_j\times E} |z|\mathbb{1}_{\{|z|> M\}} d\mu(r,z).\end{aligned}$$

Taking the mean over j and then applying Markov inequality in the last expression gives:

$$\begin{aligned}\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}&\left[\sup_{\substack{Lip(\varphi)\leq 1\\ \varphi(0)=0}} \int_{\mathbb{T}\times E} \varphi(r,z)\mathbb{1}_{\{|z|> M\}} d(\bar{\mu}^{N,j} - \mu^j)\right] \\ &\leq 2\int_{\mathbb{T}\times E} \mathbb{1}_{\{|z|> M\}} d\mu\,(r,z) + 2\int_{\mathbb{T}\times E} |z|\mathbb{1}_{\{|z|> M\}} d\mu(r,z) \\ &\leq \frac{K}{M}, \quad (2.90)\end{aligned}$$

for some constant K. To bound the first term in (2.89), let us subdivide $[-M, M]^{2d}$ in n^{2d} disjoint cubes $(C_k)_{1\leq k\leq n^{2d}}$, for some integer $n \geq 1$. More precisely, each cube C_k is of the form

$[i_1M/n,(i_1+1)M/n]\times\ldots\times[i_{2d}M/n,(i_{2d}+1)M/n]$ for $-n\le i_1,\ldots,i_{2d}\le n-1$. Let $z_k\in C_k$ be a point in the cube C_k, let us say the center of C_k to fix ideas. Then for any φ such that $Lip(\varphi)\le 1$ and $\varphi(0)=0$, one has

$$\begin{aligned}
&\left|\int_{B_j\times E}\varphi(r,z)\mathbb{1}_{\{|z|\le M\}}d(\bar\mu^{N,j}-\mu^j)\right|\\
&\quad=\left|\sum_{k=1}^{n^{2d}}\int_{B_j\times C_k}\varphi(r,z)d(\bar\mu^{N,j}-\mu^j)\right|\\
&\quad\le\sum_{k=1}^{n^{2d}}\left(\left|\int_{B_j\times C_k}\varphi(j\epsilon_N,z_k)d(\bar\mu^{N,j}-\mu^j)\right|+\int_{B_j\times C_k}|\varphi(r,z)-\varphi(j\epsilon_N,z_k)|\,d(\bar\mu^{N,j}+\mu^j)\right)\\
&\quad\le\sum_{k=1}^{n^{2d}}\left(\int_{B_j\times C_k}(1+|z_k|)\,d\left|\bar\mu^{N,j}-\mu^j\right|+\left(\epsilon_N+\frac{M}{n}\right)\left(\bar\mu^{N,j}(B_j\times C_k)+\mu^j(B_j\times C_k)\right)\right)\\
&\quad\le(1+M)\sum_{k=1}^{n^{2d}}\left|\bar\mu^{N,j}(B_j\times C_k)-\mu^j(B_j\times C_k)\right|+2\frac{M}{n}+2\epsilon_N,
\end{aligned}\tag{2.91}$$

where we used at the third line that $\varphi(j\epsilon_N,z_k)\le r+|z_k|$ for the first term, and the Lipschitz property for the second term. Then we used that $|z_k|\le M$ at the last line. For any $k\le n^{2d}$ and any $1\le j\le\epsilon_N^{-1}$, by independence of the $(\bar Y^i)_{i\in NB_j}$ we get

$$\begin{aligned}
\mathbb{E}\left[\left(\bar\mu^{N,j}(B_j\times C_k)-\mu^j(B_j\times C_k)\right)^2\right]&=\frac{1}{(N\epsilon_N)^2}\sum_{i,i'\in NB_j}\mathbb{E}\left[\mathbb{1}_{\bar Z^i\in C_k}\mathbb{1}_{\bar Z^{i'}\in C_k}\right]\\
&\quad-\frac{2}{N\epsilon_N}\sum_{i\in NB_j}\mathbb{P}(\bar Z^i\in C_k)\mu^j(B_j\times C_k)+\mu^j(B_j\times C_k)^2\\
&=\frac{1}{N\epsilon_N}\mu^j(B_j\times C_k)\left(1-\mu^j(B_j\times C_k)\right)\\
&\le\frac{1}{N\epsilon_N}\mu^j(B_j\times C_k).
\end{aligned}\tag{2.92}$$

Taking the mean over j in (2.91) and using (2.92) gives, by Cauchy-Schwarz inequality

$$\begin{aligned}
&\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\sup_{\substack{Lip(\varphi)\le1\\\varphi(0)=0}}\int_{B_j\times E}\varphi(r,z)\mathbb{1}_{\{|z|\le M\}}d(\bar\mu^{N,j}-\mu^j)\right]\\
&\qquad\le(1+M)\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\sum_{k=1}^{n^{2d}}\mathbb{E}\left[\left|\bar\mu^{N,j}(B_j\times C_k)-\mu^j(B_j\times C_k)\right|\right]+2\frac{M}{n}+2\epsilon_N\\
&\qquad\le\frac{2M\epsilon_N}{(N\epsilon_N)^{1/2}}\sum_{j=1}^{\epsilon_N^{-1}}\sum_{k=1}^{n^{2d}}\sqrt{\mu^j(B_j\times C_k)}+2\frac{M}{n}+2\epsilon_N,
\end{aligned}$$

for $M>1$. Applying Cauchy-Schwarz inequality to the two sums at the last line and using that for any $1\le j\le\epsilon_N^{-1}$, we have $\sum_{k=1}^{n^{2d}}\mu^j(B_j\times C_k)\le 1$, we get

$$\epsilon_N\sum_{j=1}^{\epsilon_N^{-1}}\mathbb{E}\left[\sup_{\substack{Lip(\varphi)\le1\\\varphi(0)=0}}\int_{B_j\times E}\varphi(r,z)\mathbb{1}_{\{|z|\le M\}}d(\bar\mu^{N,j}-\mu^j)\right]\le\frac{2Mn^d}{(N\epsilon_N)^{1/2}}+2\frac{M}{n}+2\epsilon_N.\tag{2.93}$$

Combining (2.90) and (2.93) into (2.89) finally gives

$$\epsilon_N \sum_{j=1}^{\epsilon_N^{-1}} \mathbb{E}\left[\mathcal{W}_1(\bar{\mu}^{N,j}, \mu^j)\right] \leq K\left(\frac{Mn^d}{(N\epsilon_N)^{1/2}} + \frac{M}{n} + \epsilon_N + \frac{1}{M}\right),$$

which choosing $M = (N\epsilon_N)^{\frac{1}{4(d+1)}}$ and $n = \lfloor M^2 \rfloor$ yields the desired result.

$\square$

2.B Linear Martingale problem and SDE

This appendix is devoted to the proof of Proposition 2.3, which is divided as follows. We first prove existence of a solution to the SDE (2.56) by an iteration procedure, then prove pathwise uniqueness. To conclude on the uniqueness of the martingale problem, we then use classical representation theorems. We finish by proving (2.59) and (2.60).

Proof of Proposition 2.3.

We are not interested in keeping track of the influence of the parameters ℓ and $\bar{\gamma}$ yet, so here and in the rest of the section, K and K' denote constants depending on W, U, ϕ, γ, $\bar{\gamma}$ and ℓ that may change from one calculation to another.

1. Existence of solutions to (2.56)

Let us fix some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $Y_0 = (r_0, X_0, V_0)$ be a random variable with law ν_0 and $\mathcal{N}^{Q,r_0}$ an $(\mathcal{F}_t)$-adapted Poisson point process, which is independent of (X_0, V_0) given r_0. We are going to prove existence of a solution of (2.56) by a fix point procedure. Define by iteration the processes $(Y^n)_{n\geq 0}$ in $\mathcal{D}$. Y^0 is set to be constant equal to Y_0 and for any $n \geq 1$, Y^{n+1} is the solution of

$$\begin{cases} dX_t^{n+1} = V_t^n dt \\ dV_t^{n+1} = -\left(\int_{\mathbb{T}\times E} \Phi_\ell(r_0^n - r')\nabla W(X_t^n - x')dQ_t(y') + \nabla U(X_t^n)\right) dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \int_{\mathbb{R}^d} (v' - V_{t^-}^n) d\mathcal{N}^{Q,r_0}(t, v') \\ Y_0^{n+1} = Y_0. \end{cases} \tag{2.94}$$

Notice that, for any $n \geq 0$, $r_0^n =: r_0$ is constant. The processes $(Y_n)_{n\geq 0}$ are well defined since the intensity of $\mathcal{N}^{Q,r_0}$ is finite for the velocity and locally finite in time. Then for any $t > 0$ and $n \geq 1$, since the updates of velocities are the same for every process Y_n we get

$$\begin{aligned} \left|Y_t^{n+1} - Y_t^n\right| &\leq \left|X_t^{n+1} - X_t^n\right| + \left|V_t^{n+1} - V_t^n\right| \\ &\leq \int_0^t \left|V_s^n - V_s^{n-1}\right| ds \\ &\qquad + \int_0^t ds \int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r') \left|\nabla W(X_s^n - x') - \nabla W(X_s^{n-1} - x')\right| dQ_s(r', z') \\ &\qquad + \int_0^t ds \int_{\mathbb{T}\times E} \left|\nabla U(X_s^n) - \nabla U(X_s^{n-1})\right| dQ_s(r', z') \\ &\qquad + \int_0^t \int_{\mathbb{R}^d} \left|V_{s^-}^n - V_{s^-}^{n-1}\right| d\mathcal{N}^{Q,r_0}(s, v') \\ &\leq K \int_0^t \left|Y_s^n - Y_s^{n-1}\right| ds + \int_0^t \int_{\mathbb{R}^d} \left|Y_{s^-}^n - Y_{s^-}^{n-1}\right| d\mathcal{N}^{Q,r_0}(s, v'), \end{aligned}$$

using the Lipschitz property for U and W. We deduce that for any $T > 0$

$$\mathbb{E}\left[\sup_{t\leq T}\left|Y_t^{n+1} - Y_t^n\right|\right] \leq K' \int_0^T \mathbb{E}\left[\sup_{s\leq t}\left|Y_s^n - Y_s^{n-1}\right|\right] dt, \tag{2.95}$$

using finiteness of $\mathcal{N}^{Q,r_0}(s, \mathbb{R}^d)$ at the last line. By immediate recursion, we get

$$\mathbb{E}\left[\|Y^{n+1} - Y^n\|_{\infty,T}\right] \leq \frac{(K'T)^n}{n!}\mathbb{E}\left[\|Y^1 - Y^0\|_{\infty,T}\right], \tag{2.96}$$

where $\|\cdot\|_{\infty,T}$ denotes the uniform norm on $\mathcal{D}_T$, see (2.61). Bounding

$$\mathbb{E}\left[\sup_{t\leq T}\left|Y_t^1 - Y_t^0\right|\right] \leq C_T, .$$

for some constant C_T is again straightforward using the hypotheses **(H1)** on the potentials, **(H4)** on ν_0 and (2.58) on Q. Combining this with (2.96) proves that

$$\sum_{n\geq 0} \|Y^{n+1} - Y^n\|_{\infty,T} < \infty,$$

almost surely for all $T > 0$. Hence, for any $T > 0$, $(Y_n)_{n\geq 0}$ is a Cauchy sequence in the space $\mathcal{D}_T$ for the uniform norm. Thus, it is also a Cauchy sequence for the Skorokhod distance, which makes $\mathcal{D}_T$ a complete space [10]. As a consequence, the sequence converges almost surely to a process $Y = (r, X, V) \in \mathcal{D}_T$, and it remains to show that Y solves (2.56). This is straightforward by first deducing that the r-coordinate r_t of Y is constant in time almost surely from the fact that $\|Y^n - Y\|_{\infty,T} \to 0$. Then it easy and left to the reader to prove that the time integrals in (2.94) converge to those where Y^n is replaced by Y. As T was arbitrary, we finally get existence of a solution in $\mathcal{D}$.

2. Pathwise uniqueness

Let Y^1 and Y^2 be two solutions of (2.56) on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the same initial condition $Y_0^1 = Y_0^2$, driven by the same Poisson random measure $\mathcal{N}^{Q,r_0}(s, v')$. Then, by the same arguments leading to (2.95), we get

$$E\left[\sup_{t\leq T}\left|Y_t^1 - Y_t^2\right|\right] \leq K' \int_0^T \mathbb{E}\left[\sup_{s\leq t}\left|Y_s^1 - Y_s^2\right|\right] dt.$$

By Gronwall's inequality and the initial conditions, we deduce that Y_1 and Y_2 are indistinguishable. Moreover the law of the solution of (2.56) constructed by Picard iteration does not depend on the probability space, but only on the marginals $(Q_t)_{t\geq 0}$, so by classical results [49] we also deduce uniqueness in law.

3. Uniqueness for the Martingale Problem

It is easy to see that if Y is the solution to the stochastic differential equation (2.56), then its law solves the linear martingale problem (2.57). To prove that it is the only one, we follow [19, 28] and adapt the classical representation arguments from [25], [49] or [70] by proving that the canonical process associated with a solution to the martingale problem can be represented as a solution to stochastic differential equation (2.56) on an enlarged probability space. For the sake of brevity, we will only give the principal steps in this proof and refer to those classical references for details.

Let $\mu \in \mathcal{M}^1(\mathcal{D})$ be a solution to the nonlinear martingale problem (2.57). Adapting Theorem 10 in [49] and Theorem 2 in [70], we have that for any measurable positive φ defined on $\mathbb{R}_+ \times E^2$ such that $\varphi(., y, .) \leq v$, with $y = (r, x, v)$,

$$\sum_{s\leq t} \varphi(s, Y_s, Y_{s\_})\mathbb{1}_{\Delta Y_s \neq 0} - \int_0^t ds \int_{\mathbb{R}^d} \varphi\left(s, (r_s, X_s, v'), Y_s\right) dw^{Q_s, r_0}(v') \tag{2.97}$$

is a local martingale under μ. In particular, we deduce that

$$Y_t = Y_0 + \int_0^t \mathcal{L}[Q_s] Y_s ds + M_t,$$

where M_t is the vectorial martingale compensated sum of jumps of Y (see Theorem 12 in [49]). By Itô's formula applied to $(Y_t \cdot \theta)^2$, where $\theta = (\theta_r, \theta_x, \theta_v) \in \mathbb{T} \times E$, and comparison to the martingale problem (2.57), we get that $\theta \cdot M_t$ has predictable quadratic variation

$$\bar{\gamma} \int_0^t ds \int_{\mathbb{R}^d} (\theta_v \cdot (v' - V_s))^2 dw^{Q_s, r_0}(v').$$

We therefore have a description of the compensator of ΔY, whose associated Levy measure is $dtdw^{Q_t,r_0}(v')$. By the representation theorem in [25] (see also part II in [49] or section 4 in [70]), on an enlarged probability space $\tilde{\Omega}$, there exists a Poisson random measure $\mathcal{N}^{Q,r_0}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity $dsdw^{Q_t,r_0}(v')$, such that

$$\begin{cases} dX_t = V_t dt \\ dV_t = -\left(\int_{\mathbb{T}\times E} \Phi_\ell(r_0 - r') \nabla W(X_t - x') d\mu_t(r', z') + \nabla U(X_t) \right) dt + \int_{\mathbb{R}^d} (v' - V_{t-}) d\mathcal{N}^{Q,r_0}(t, v') \end{cases}$$

By uniqueness in law of solutions to (2.56), we get uniqueness of solutions to the martingale problem.

4. Proof of (2.59) *and* (2.60)

If Y is the solution to the stochastic differential equation (2.56) on some filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then

$$\begin{aligned} \mathbb{E}\left[|X_t| + |V_t|\right] \leq \mathbb{E}\left[|X_0| + |V_0|\right] &+ \int_0^t \mathbb{E}[|V_s|] ds \\ &+ \int_0^t ds \int_{\mathbb{T}\times E} \mathbb{E}\left[\Phi_\ell(r_0 - r') |\nabla W(X_t - x')|\right] dQ_s(r', z') \\ &+ \int_0^t \mathbb{E}\left[|\nabla U(X_s)|\right] ds + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} |v' - V_{s-}| d\mathcal{N}^{Q,r_0}(s, v')\right] \\ \leq \mathbb{E}\left[|X_0| + |V_0|\right] + K \int_0^t \mathbb{E}\left[|X_s| + |V_s|\right] ds &+ K \int_0^t ds \int_{\mathbb{T}\times E} (|x'| + |v'|) \, dQ_s(r', z'), \end{aligned}$$

where K is some positive constant and we used the Lipschitz property for ∇W and ∇U. Applying Gronwall's inequality and using the hypotheses **(H4)** on ν_0 and (2.58) on Q yield (2.59). (2.60) comes directly from the structure of the linear SDE (2.56).

□

Chapter 3

Thermal Conductivity for a chain of oscillators with Kac potentials

Abstract
In this chapter, we compute the finite-size scaling for the thermal conductivity by using Green-Kubo formula for the chain with Kac potentials and two different stochastic collisions. This allows us to derive then precise asymptotics for the conductivity with respect to N and ℓ_N.

Contents

In this chapter, we compute the thermal conductivity for different chains of oscillators with Kac harmonic potentials, based on calculations initially introduced by Basile, Bernardin and Olla in [2]. In the previous chapter, we identified that the timescale at which the diffusion of energy occurs for the chain with Kac harmonic potentials and long-range exchanges of velocity is $t\ell^{-2}$, where $\ell = \ell_N/N$, N is the size of the chain and ℓ_N the range of the interaction ($i.e$ the number of particles directly interacting with a single particle). Even if the range of interaction of the mechanical part and the collisions was the same, the energy was only transported by the stochastic collisions. In fact, the averaging effects of the local mean field interaction destroyed the contribution of the mechanical part in the energy transport. Notice that, applying the method of Chapter 2 for the chain with random flips of velocity instead of stochastic long-range exchanges of velocity, we can conclude that there is no evolution of energy at this timescale. Consequently, a different timescale has to be considered.

Therefore, an exact computation of the thermal conductivity for a harmonic chain with Kac potentials and random flip of velocities by Green-Kubo formula will help us prove that energy is transported at a longer timescale for this model. This is the main goal of this chapter. We also compute the thermal conductivity for a harmonic pinned chain with long-range exchanges of velocity to confirm the results in the previous chapter.

Before defining the finite-size thermal conductivity, recall that we consider the dynamics

$$\begin{cases} \dot{X}_t^i = V_t^i \\ \dot{V}_t^i = -\sum_{k=-\ell_N}^{\ell_N} \phi_k \nabla W(X_t^i - X_t^{i+k}) - \nabla U(X_t^i), \end{cases} \tag{3.1}$$

for $i \in \mathbb{T}_N$, where $\mathbb{T}_N$ is the finite periodic domain $\mathbb{Z}/N\mathbb{Z}$. The coefficients ϕ_k are obtained from a function ϕ defined on $[-1/2, 1/2]$ by $\phi_k = 1/\ell_N \phi(k/\ell_N)$, and the coordinates X^i, V^i are in $\mathbb{R}^d \times \mathbb{R}^d$. The canonical Gibbs measure at temperature T

$$\frac{1}{Z_{N,T}} \prod_{i \in \mathbb{T}_N} \exp\left(-\frac{1}{T}\mathcal{E}^i\right) dX^i dV^i, \tag{3.2}$$

is an invariant measure for this dynamics. In (3.2), $\mathcal{E}^i$ stands for the energy associated with particle i:

$$\mathcal{E}^i = \frac{1}{2}|V^i|^2 + \frac{1}{2}\sum_{k=-\ell_N}^{\ell_N} \phi_k W(X^i - X^{i+k}) + U(X^i), \tag{3.3}$$

and $Z_{N,T}$ is a normalization constant. We denote by $\mathbb{E}_T[\cdot]$ the expectation with respect to the trajectories $((X_t^i, V_t^i)_{t \geq 0})_{i \in \mathbb{T}_N}$ with initial distribution given by the Gibbs measure at temperature T. Let $J^{i,i+k}([0,t])$ be the time-integral of the energy current between particles i and $i+k$, between time 0 and time t, which satisfies the conservation equation

$$\mathcal{E}_t^i - \mathcal{E}_0^i = -\sum_{k=1}^{\ell_N} \left(J^{i,i+k}([0,t]) - J^{i-k,i}([0,t])\right). \tag{3.4}$$

Following [2], we define a finite-size conductivity for the periodic chain of length N by the formula

$$\kappa_N(T) = \frac{1}{2T^2 t_N N} \mathbb{E}_T\left[\left(\sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \frac{k}{\ell_N} J^{i,i+k}([0,t_N])\right)^2\right], \tag{3.5}$$

where t_N is a time depending on the size of the chain that is equal to

$$t_N = \tau N/\ell_N, \tag{3.6}$$

where τ is a constant. We will give a justification of the formulas (3.5) and (3.6) in Section 3.1.

We first consider the harmonic chain with random flip of velocities as studied in [8]. Recall that this dynamics is obtained by adding to (3.1) independent Poisson processes N^i with intensity $\bar{\gamma}$ to every particle, so that the velocity V^i of particle i is changed into $-V^i$ at random times given by N^i. Therefore (3.1) becomes

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\sum_{k=-\ell_N}^{\ell_N} \phi_k \nabla W(X_t^i - X_t^{i+k}) + \nabla U(X_t^i)\right) dt - 2V_{t-}^i dN_t^i, \end{cases} \tag{3.7}$$

with harmonic interactions

$$U(x) = \nu\frac{|x|^2}{2}, \qquad W(x) = \alpha\frac{|x|^2}{2}, \tag{3.8}$$

where $\nu \geq 0$ and $\alpha > 0$. We adapt the calculations in [2] to compute the finite-size conductivity (3.5) (see also [68] for the current correlations in a long-range interaction model) and prove the following result.

Theorem 3.1. *Suppose the coefficients $(\phi_k)_{-\ell_N \leq k \leq \ell_N}$ are constant in k and equal to $\phi_k = 1/\ell_N$. Then, for every range of interaction ℓ_N such that $\ell_N \underset{N\to\infty}{\longrightarrow} \infty$ and $\ell_N = o(N^{5/7})$, and for both pinned and unpinned chains ($\nu \geq 0$),*

$$\kappa_N(T) \underset{N\to\infty}{\sim} \frac{\bar{\kappa}}{\ell_N}, \tag{3.9}$$

where

$$\bar{\kappa} = \frac{\alpha^2 d}{\bar{\gamma}} \int_0^\infty dv \frac{\left(\int_0^1 u \sin(2\pi uv) du\right)^2}{\nu + 4\alpha \int_0^1 \sin^2(\pi uv) du}. \tag{3.10}$$

In Section 3.3, we consider the harmonic chain (3.1) with long-range exchanges of velocity. Recall that this chain is obtained by associating independent Poisson processes $N^{i,i+k}$ to every pair $(i, i+k)$ with $i \in \mathbb{T}_N$ and $k \leq \ell_N$, with respective intensity $\bar{\gamma}\gamma_k$. The dynamics is given by

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = -\left(\sum_{k=-\ell_N}^{\ell_N} \phi_k \nabla W(X_t^i - X_t^{i+k}) + \nabla U(X_t^i)\right) dt + \sum_{k=-\ell_N}^{\ell_N} (V_t^{i+k} - V_{t-}^i) dN_t^{i,i+k}, \end{cases} \tag{3.11}$$

with the natural convention $N^{i,j} = N^{j,i}$. We prove in that case that the thermal conductivity as defined in (3.5) converges when the chain is pinned.

Theorem 3.2. *Suppose the coefficients $(\phi_k)_{-\ell_N \leq k \leq \ell_N}$ are constant in k and equal to $\phi_k = 1/\ell_N$. For the pinned harmonic chain ($\nu > 0$) with long-range stochastic velocity exchanges, the thermal conductivity as defined in (3.5) satisfies*

$$\kappa_N(T) \xrightarrow{N\to\infty} \frac{d}{2}\bar{\gamma} c_\gamma,$$

where

$$c_\gamma = \frac{1}{2} \int_{-1/2}^{1/2} \gamma(u) u^2 du. \tag{3.12}$$

We conjecture that this result also holds when the pinning potential is anharmonic and convex and will give some ideas on how to prove it. As we have seen in the previous chapter, only the stochastic collisions contribute to energy transport in that case.

The plan of this chapter is the following. In Section 3.1, we give a justification of the definition (3.5) for the finite-size thermal conductivity. In Section 3.2, we study the thermal conductivity for the chain with flips of velocity and we prove Theorem 3.1. In Section 3.3, we analyze the thermal conductivity for the chain with long-range exchanges. We prove Theorem 3.2 for the harmonic pinned chain and give ideas on how to prove it for anharmonic pinning. We then give a heuristic justification of the asymptotic behavior of the conductivity in Theorem 3.2 by analogy with random walks.

3.1 Definition of the finite-size conductivity

3.1.1 Formal derivation

To justify the definition (3.5) for the finite-size thermal conductivity, we first formally derive a Green-Kubo type formula, as introduced in 1.10 in the introduction. In 1.10, we wrote the Green-Kubo formula for nearest neighbor interaction, but we can actually derive it for arbitrary range of interaction.

Notice that this formula is only computed for infinite chains. But as the range of interaction ℓ_N depends on the length of the chain N in our finite volume model (3.1), it is not clear how to define the infinite chain associated with (3.1). As a consequence, in this section, we will fix the range of interaction ℓ_N independently of any value N but keep the notation ℓ_N for convenience.

For the pinned chain, we can define the infinite volume Gibbs measure at temperature T from the finite volume Gibbs measures (3.2) with a limiting procedure. In this section, $\mathbb{E}_T[\cdot]$ will refer to the expectation with respect to the infinite Gibbs measure at temperature T.

It is straightforward to adapt the techniques in the lecture notes of Bernardin and Olla [9] to prove that the infinite dynamics with finite range of interaction ℓ_N is well defined on an appropriate Banach space. The dynamics of $((X_t^i, V_t^i)_{t\geq 0})_{i\in\mathbb{Z}}$ is stationary with respect the infinite volume Gibbs measure. By time-stationarity of the infinite dynamics with initial distribution given by $\mathbb{E}_T[\cdot]$, this dynamics can be extended to negative times. Without risk of confusion, we also denote by $\mathbb{E}_T[\cdot]$ the expectation with respect to the trajectories $((X_t^i, V_t^i)_{t\in\mathbb{R}})_{i\in\mathbb{Z}}$.

We now follow the derivation of the Green-Kubo formula given in [62] (Part II, Section 2). As we expect the energy $\mathcal{E}_t^i$ to evolve diffusively, we expect also a diffusive behavior for the space-time energy-energy correlation function

$$S(i,t) = \mathbb{E}_T[\mathcal{E}_t^i \mathcal{E}_0^0] - \mathbb{E}_T[\mathcal{E}_0^0]^2.$$

Notice that, by time-reversal symmetry,

$$S(i,t) = \mathbb{E}_T[\mathcal{E}_{-t}^i \mathcal{E}_0^0] - \mathbb{E}_T[\mathcal{E}_0^0]^2 = S(i,-t). \tag{3.13}$$

Then, by the argument presented in Subsection 1.1.2 of the introduction, the thermal conductivity should be accessible by computing the time-limit of the variance of $S(i,t)$. Since the interactions are of order ℓ_N, we naturally divide the distances by ℓ_N in the definition of the thermal conductivity:

$$\kappa(T) = \lim_{t\to\infty} \frac{1}{2tT^2} \sum_{i\in\mathbb{Z}} \frac{i^2}{\ell_N^2} S(i,t). \tag{3.14}$$

As the energy is globally conserved by the dynamics, one can define the time-integral of the energy current $J^{i,i+k}([0,t])$ by (3.4), and $J^{i,i+k}([0,t])$ is given by

$$J^{i,i+k}([0,t]) = \phi_k \int_0^t j_s^{i,i+k} ds,$$

where

$$j^{i,i+k} = \frac{1}{2}\left(V^i + V^{i+k}\right) \cdot \nabla W \left(X^i - X^{i+k}\right).$$

Our goal is now to express formally the conductivity (3.14) with the space correlations of the current by introducing the conservation equation (3.4) in (3.14). Using (3.4) and then a discrete summation by parts, we can express the variance

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} i^2 S(i,t) &= \sum_{i\in\mathbb{Z}} i^2 S(i,0) - \sum_{i\in\mathbb{Z}} i^2 \sum_{k=1}^{\ell_N} \mathbb{E}_T\left[\left(J^{i,i+k}([0,t]) - J^{i-k,i}([0,t])\right)\mathcal{E}_0^0\right] \\
&= \sum_{i\in\mathbb{Z}} i^2 S(i,0) + \sum_{i\in\mathbb{Z}} \sum_{k=1}^{\ell_N} \left((i+k)^2 - i^2\right) \mathbb{E}_T\left[J^{i,i+k}([0,t])\mathcal{E}_0^0\right] \\
&= \sum_{i\in\mathbb{Z}} i^2 S(i,0) + \sum_{i\in\mathbb{Z}} \sum_{k=1}^{\ell_N} (2i+k)k\mathbb{E}_T\left[J^{i,i+k}([0,t])\mathcal{E}_0^0\right],
\end{aligned} \tag{3.15}$$

simplifying the squares at the last line. Similarly, we can compute the variance associated with $S(i,-t)$

$$\sum_{i\in\mathbb{Z}} i^2 S(i,-t) = \sum_{i\in\mathbb{Z}} i^2 S(i,0) - \sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}(2i+k)k\mathbb{E}_T\left[J^{i,i+k}([-t,0])\mathcal{E}_0^0\right],$$

and by time stationarity get

$$\sum_{i\in\mathbb{Z}} i^2 S(i,-t) = \sum_{i\in\mathbb{Z}} i^2 S(i,0) - \sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}(2i+k)k\mathbb{E}_T\left[J^{i,i+k}([0,t])\mathcal{E}_t^0\right]. \tag{3.16}$$

Summing the two expressions (3.15) and (3.16), and using the symmetry (3.13), we deduce

$$2\sum_{i\in\mathbb{Z}} i^2 S(i,t) = 2\sum_{i\in\mathbb{Z}} i^2 S(i,0) + \sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}(2i+k)k\mathbb{E}_T\left[J^{i,i+k}([0,t])(\mathcal{E}_0^0-\mathcal{E}_t^0)\right].$$

Inserting the conservation equation (3.4) to replace the energy difference in this expression, we get

$$\begin{aligned}2\sum_{i\in\mathbb{Z}} i^2 S(i,t) &= 2\sum_{i\in\mathbb{Z}} i^2 S(i,0)\\ &\quad+\sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}\sum_{k'=1}^{\ell_N}(2i+k)k\mathbb{E}_T\left[J^{i,i+k}([0,t])\left(J^{0,k'}([0,t])-J^{-k',0}([0,t])\right)\right].\end{aligned}$$

By translation invariance, this is also equal to

$$\begin{aligned}2\sum_{i\in\mathbb{Z}} i^2 S(i,t) &= 2\sum_{i\in\mathbb{Z}} i^2 S(i,0)\\ &\quad+\sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}\sum_{k'=1}^{\ell_N}(2i+k)k\mathbb{E}_T\left[\left(J^{i,i+k}([0,t])-J^{i+k',i+k+k'}([0,t])\right)J^{0,k'}([0,t])\right].\end{aligned}$$

And finally, by a discrete summation by parts on $i\in\mathbb{Z}$, we get

$$2\sum_{i\in\mathbb{Z}} i^2 S(i,t) = 2\sum_{i\in\mathbb{Z}} i^2 S(i,0) + 2\sum_{i\in\mathbb{Z}}\sum_{k=1}^{\ell_N}\sum_{k'=1}^{\ell_N}kk'\mathbb{E}_T\left[J^{i,i+k}([0,t])J^{0,k'}([0,t])\right].$$

Hence the thermal conductivity (3.14) can be formally written

$$\kappa(T) = \lim_{t\to\infty}\frac{1}{2tT^2}\sum_{i\in\mathbb{Z}}\mathbb{E}_T\left[\left(\sum_{k'=1}^{\ell_N}\frac{k'}{\ell_N}J^{0,k'}([0,t])\right)\left(\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}J^{i,i+k}([0,t])\right)\right]. \tag{3.17}$$

Once again, recall that in the formal calculations above, ℓ_N does not play a particular role since it is not related to the size of the system as we dealt with the infinite dynamics. Nevertheless, (3.17) gives us an appropriate way to define the finite size thermal conductivity $\kappa_N(T)$ for the finite chain in the periodic domaine $\mathbb{T}_N$ by the formula

$$\kappa_N(T) = \frac{1}{2t_NT^2}\sum_{i\in\mathbb{T}_N}\mathbb{E}_T\left[\left(\sum_{k'=1}^{\ell_N}\frac{k'}{\ell_N}J^{0,k'}([0,t_N])\right)\left(\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}J^{i,i+k}([0,t_N])\right)\right], \tag{3.18}$$

which by translation invariance can be recasted into (3.5). Following [2], we avoid the limit in time in (3.5) by introducing an appropriate time t_N depending on the size N of the chain that we define now.

3.1.2 Dispersion relation

We want to define t_N large enough in (3.18) to be consistent with (3.17), but not too much so that we do not see the effect of the periodicity of $\mathbb{T}_N$ on the current correlations. As in [2], we therefore choose t_N to be the typical time it takes for the fastest energy modes to travel across the system. To define it, we first have to define the dispersion relation of the chain. We consider the case of harmonic potentials $W(x) = \alpha|x|^2/2$ and $U(x) = \nu|x|^2/2$. The dynamics (3.1) is then given by

$$\begin{cases} \dot{X}_t^i = V_t^i \\ \dot{V}_t^i = \sum_{k=-\ell_N}^{\ell_N} \phi_k \Delta_k X_t^i - \nu X_t^i, \end{cases} \tag{3.19}$$

where $\Delta_k f(n) = f(n+k) + f(n-k) - 2f(n)$ is a discrete Laplacian at distance k. Transforming (3.19) into Fourier space, we get N independent harmonic oscillator equations:

$$\begin{cases} \dot{\hat{X}}_t(\xi) = \hat{V}_t(\xi) \\ \dot{\hat{V}}_t(\xi) = -\omega_N\left(\frac{\xi}{N}\right)^2 \hat{X}_t(\xi), \end{cases}$$

for $\xi \in \mathbb{T}_N$, where the discrete Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(\xi) = \sum_{n\in\mathbb{T}_N} f(n) \exp\left(\frac{2\mathbf{i}\pi\xi n}{N}\right). \tag{3.20}$$

The function ω_N is the dispersion relation of the chain and is given by

$$\omega_N(u) = \left(\nu + 4\alpha\sum_{k=1}^{\ell_N} \phi_k \sin^2(\pi k u)\right)^{1/2}. \tag{3.21}$$

Notice that, due to the fact the range of interaction ℓ_N depends on the length N of the chain, the dispersion relation also depends on N. Deriving the dispersion relation (3.21), we get

$$\omega_N'(u) = \ell_N \frac{2\alpha\pi\sum_{k=1}^{\ell_N} \frac{k}{\ell_N}\phi_k \sin(2\pi k u)}{\omega_N(u)},$$

which is the group velocity of u-mode waves. This group velocity is therefore at most of order ℓ_N (obtained for modes u of order $1/\ell_N$ if $\nu > 0$ and for modes of order smaller than $1/\ell_N$ if $\nu = 0$). Defining $t_N = N/(\sup_u \omega_N'(u))$, t_N represents the typical time it takes for the fastest modes to cross the system, and we are led to take

$$t_N = \tau N \ell_N^{-1},$$

for some constant τ.

3.2 Random flip of velocity

In this section, we study the thermal conductivity for the harmonic chain with random flips of velocity (3.7). We first prove the following explicit formula for the thermal conductivity.

Proposition 3.1. *The finite-size thermal conductivity for the harmonic chain* (3.7) *is given by*

$$\kappa_N(T) = \frac{\alpha^2 d}{2\bar{\gamma}}\left(1 + \frac{\mathrm{e}^{-2\bar{\gamma}t_N} - 1}{2\bar{\gamma}t_N}\right)\frac{1}{N}\sum_{\xi=1}^{N}\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu + 4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}. \tag{3.22}$$

This proposition is proven in Subsection 3.2.1, following the computations of [2]. At first glance at (3.22), having in mind that $\phi_k = 1/\ell_N \phi(k/\ell_N)$, one would expect that the nested sums converge as Riemann sums and that therefore $\kappa_N(T)$ is of order 1. Actually, we will see that the contribution of the large values of ξ vanish and we prove Theorem 3.1 in Subsection 3.2.2, *i.e.* that the thermal conductivity in (3.22) is of order $1/\ell_N$. This theorem is only proven for constant coefficients $\phi_k = 1/\ell_N$ and we conclude this section by showing in Subsection 3.2.3 numerical evidences of a convergence when $\phi_k = 1/\ell_N \phi(k/\ell_N)$ and ϕ is a smooth compactly supported function.

3.2.1 Computation of thermal conductivity: proof of Proposition 3.1

Energy current correlations

We consider the dynamics with flips of velocity (3.7) and harmonic interaction (3.8). This collision has the particularity to conserve energy, but not momentum. In particular, even in the case when the chain is unpinned ($U = 0$), then the whole dynamics only conserves energy. We denote by

$$\mathcal{L} = \mathcal{A} + \bar{\gamma}\mathcal{S} \tag{3.23}$$

the generator of this dynamics on $\Omega_N = (\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{T}_N}$. $\mathcal{A}$ is the generator of the Hamiltonian part and $\mathcal{S}$ is the generator of the flip. For any $\psi \in \mathcal{C}_b^1(\Omega_N)$,

$$\begin{aligned}
\mathcal{A}\psi(\eta) &= -\sum_{i\in\mathbb{T}_N}\left(v^i \cdot \nabla_{x^i}\psi(\eta) - \left(\sum_{k=-\ell_N}^{\ell_N}\phi_k \nabla W(x^i - x^{i+k}) + \nabla U(x^i)\right)\cdot \nabla_{v^i}\psi(\eta)\right),\\
\mathcal{S}\psi(\eta) &= \sum_{i\in\mathbb{T}_N}\left(\psi\left(\bar{\eta}^i\right) - \psi(\eta)\right),
\end{aligned}$$

where $\eta = (x^i, v^i)_{i\in\mathbb{T}_N}$ and $\bar{\eta}^i$ is equal to η except for the ith velocity, which is equal to $-v^i$. Recall the definition (3.3) of the energy $\mathcal{E}_t^i$ of particle i at time t. One has

$$\mathcal{E}_t^i - \mathcal{E}_0^i = -\sum_{k=1}^{\ell_N}\left(J^{i,i+k}([0,t]) - J^{i-k,i}([0,t])\right), \tag{3.24}$$

where

$$J^{i,i+k}([0,t]) = \phi_k \int_0^t j_s^{i,i+k;a} ds, \tag{3.25}$$

and the Hamiltonian current $j_{i,i+k}^a$ is the only contribution to the energy current and is given by

$$j_s^{i,i+k;a} := \frac{\alpha}{2}(V_s^i + V_s^{i+k}) \cdot (X_s^i - X_s^{i+k}). \tag{3.26}$$

Inserting it in the formula for the finite-size thermal conductivity (3.5), we get

$$\kappa_N(T) = \frac{1}{2T^2 t_N N}\mathbb{E}_T\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k \frac{k}{\ell_N}\int_0^{t_N} j_s^{i,i+k;a} ds\right)^2\right].$$

Using Fubini's theorem, the conductivity can be rewritten

$$\begin{aligned}
\kappa_N(T) &= \frac{1}{2T^2 t_N N}\int_0^{t_N} ds \int_0^{t_N} du \mathbb{E}_T\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k \frac{k}{\ell_N} j_s^{i,i+k;a}\right)\left(\sum_{i'\in\mathbb{T}_N}\sum_{k'=1}^{\ell_N}\phi_{k'}\frac{k'}{\ell_N} j_u^{i',i'+k';a}\right)\right]\\
&= \frac{1}{T^2 t_N N}\int_0^{t_N} ds \int_0^{s} du \mathbb{E}_T\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k \frac{k}{\ell_N} j_s^{i,i+k;a}\right)\left(\sum_{i'\in\mathbb{T}_N}\sum_{k'=1}^{\ell_N}\phi_{k'}\frac{k'}{\ell_N} j_u^{i',i'+k';a}\right)\right].
\end{aligned}$$

By a time-stationarity and a change of variables,

$$\kappa_N(T) = \frac{1}{T^2 t_N N} \int_0^{t_N} ds \int_0^s du \mathbb{E}_T \left[\left(\sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} j_u^{i,i+k;a} \right) \left(\sum_{i' \in \mathbb{T}_N} \sum_{k'=1}^{\ell_N} \phi_{k'} \frac{k'}{\ell_N} j_0^{i',i'+k';a} \right) \right].$$

Expanding the product and using translation invariance, we get

$$\kappa_N(T) = \frac{1}{T^2 t_N} \int_0^{t_N} ds \int_0^s du \sum_{i \in \mathbb{T}_N} \sum_{k,k'=1}^{\ell_N} \frac{kk'}{(\ell_N)^2} \phi_k \phi_{k'} \mathbb{E}_T \left[j_u^{i,i+k;a} j_0^{0,k';a} \right].$$

Finally, after integrating in time, $\kappa_N(T)$ can be recasted

$$\kappa_N(T) = \frac{1}{T^2} \int_0^\infty \left(1 - \frac{s}{t_N}\right)^+ C^N(s) ds, \tag{3.27}$$

with

$$C^N(s) = \sum_{i \in \mathbb{T}_N} \sum_{k,k'=1}^{\ell_N} \frac{kk'}{(\ell_N)^2} \phi_k \phi_{k'} \mathbb{E}_T \left[j_s^{i,i+k;a} j_0^{0,k';a} \right]. \tag{3.28}$$

Laplace transform of C^N

Following the method in [2], we compute the Laplace transform $\mathbf{C}^N$ of C^N:

$$\begin{aligned} \mathbf{C}^N(\lambda) &= \int_0^\infty C^N(s)\, \mathrm{e}^{-\lambda s}\, ds \\ &= \int_0^\infty ds\, \mathrm{e}^{-\lambda s}\, \mathbb{E}_T \left[\left(\sum_{k'=1}^{\ell_N} \frac{k'}{\ell_N} \phi_k j_0^{0,k';a} \right) \left(\sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \frac{k}{\ell_N} \phi_k j_s^{i,i+k;a} \right) \right] \\ &= \mathbb{E}_T \left[\left(\sum_{k'=1}^{\ell_N} \frac{k'}{\ell_N} \phi_k j^{0,k';a} \right) (\lambda - \mathcal{L})^{-1} \left(\sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \frac{k}{\ell_N} \phi_k j^{i,i+k;a} \right) \right], \end{aligned} \tag{3.29}$$

where $\mathcal{L}$ is the generator defined in (3.23) for the dynamics (3.7). We therefore have to find a solution $u_{\lambda,N}$ to the resolvent equation

$$\lambda u_{\lambda,N} - \mathcal{L} u_{\lambda,N} = \sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} j^{i,i+k;a}. \tag{3.30}$$

Lemma 3.1. *The solution of* (3.30) *is given by*

$$u_{\lambda,N} = \frac{\alpha}{2} \sum_{i,i' \in \mathbb{T}_N} g_{\lambda,N}(i - i') V^i \cdot X^{i'}, \tag{3.31}$$

where $g_{\lambda,N}$ is a function defined on $\mathbb{T}_N$, given by

$$g_{\lambda,N}(n) = \frac{1}{\lambda + 2\bar{\gamma}} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} (\delta_{n-k} - \delta_{n+k}) \tag{3.32}$$

δ_n stands here for the Kronecker symbol equal to 1 if $n = 0$ and equal to 0 else.

Proof. For notational convenience, we define $u_{\lambda,N}$ by the formula (3.31) in this proof and prove that $u_{\lambda,N}$ solves (3.30). Extending $g_{\lambda,N}$ to a periodic function on $\mathbb{Z}$, it is easy to see that $g_{\lambda,N}$

is antisymmetric. Thus, computing $\mathcal{A}u_{\lambda,N}$, we get

$$\begin{aligned}\mathcal{A}u_{\lambda,N} &= \alpha \sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')\left(\alpha\sum_{k=1}^{\ell_N}\phi_k\Delta_k X^i - \nu X^i\right)\cdot X^{i'} + \alpha\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')V^i\cdot V^{i'}\\ &= \alpha^2\sum_{k=1}^{\ell_N}\phi_k\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')\left(\Delta_k X^i\right)\cdot X^{i'},\end{aligned}$$

using antisymmetry of $g_{\lambda,N}$ to remove the sums $\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')V^i\cdot V^{i'}$ and $\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')X^i\cdot X^{i'}$. Then, by summation by parts,

$$\mathcal{A}u_{\lambda,N} = \alpha^2\sum_{k=1}^{\ell_N}\phi_k\sum_{i,i'\in\mathbb{T}_N}\Delta_k g_{\lambda,N}(i-i')X^i\cdot X^{i'} = 0,$$

by antisymmetry of $\Delta_k g_{\lambda,N}(i)$. It is straightforward to see that

$$\mathcal{S}u_{\lambda,N} = -2\mathcal{S}u_{\lambda,N}.$$

Thus,

$$\begin{aligned}\lambda u_{\lambda,N} - \mathcal{L}u_{\lambda,N} &= \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N}(\lambda+2\bar{\gamma})\, g_{\lambda,N}(i-i')V^i\cdot X^{i'}\\ &= \frac{\alpha}{2}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}V^i\cdot(X^{i-k}-X^{i+k}).\end{aligned}$$

Rewriting this expression as

$$\lambda u_{\lambda,N} - \mathcal{L}u_{\lambda,N} = \frac{\alpha}{2}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\left(V^i+V^{i+k}\right)\cdot(X^i-X^{i+k}),$$

and by the formula (3.26) for the energy current, we find (3.30) as expected.

□

Inserting (3.30) in the correlation (3.29), we can express the Laplace transform of C^N as

$$\mathbf{C}^N(\lambda) = \frac{\alpha^2}{4}\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')\sum_{k'=1}^{\ell_N}\phi_{k'}\frac{k'}{\ell_N}\mathbb{E}_T[(V^0+V^{k'})\cdot(X^0-X^{k'})V^i\cdot X^{i'}].$$

Since under the Gibbs measure, the velocities are independent gaussian variables, independent of the positions, we have

$$\mathbf{C}^N(\lambda) = \frac{\alpha^2 T}{4}\sum_{i'\in\mathbb{T}_N}\sum_{k'=1}^{\ell_N}\frac{k'}{\ell_N}\phi_{k'}\left(g_{\lambda,N}(-i')+g_{\lambda,N}(k'-i')\right)\mathbb{E}_T[(X^0-X^{k'})\cdot X^{i'}].$$

Then, rearranging terms and by antisymmetry of $g_{\lambda,N}$, we get

$$\begin{aligned}\mathbf{C}^N(\lambda) &= \frac{\alpha^2 T}{4}\sum_{i'\in\mathbb{T}_N} g_{\lambda,N}(i')\sum_{k'=1}^{\ell_N}\phi_{k'}\frac{k'}{\ell_N}\mathbb{E}_T[(X^{k'}-X^{-k'})\cdot X^{i'}]\\ &= \frac{\alpha^2T^2}{4}\sum_{i'\in\mathbb{T}_N} g_{\lambda,N}(i')\Gamma_N(i'),\end{aligned}\tag{3.33}$$

with

$$\Gamma_N(n) := \frac{1}{T}\sum_{k'=1}^{\ell_N}\phi_{k'}\frac{k'}{\ell_N}\mathbb{E}_T[(X^{k'}-X^{-k'})\cdot X^n].\tag{3.34}$$

Lemma 3.2. *Γ_N is the unique solution on $\mathbb{T}_N$ to the equation*

$$\begin{cases} \left(\nu - \alpha \sum_{k=1}^{\ell_N} \phi_k \Delta_k\right) \Gamma_N(n) = d \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} (\delta_{n-k} - \delta_{n+k}) \\ \sum_{n \in \mathbb{T}_N} \Gamma_N(n) = 0. \end{cases} \tag{3.35}$$

Proof. Since

$$\mathcal{A} V_n = -\left(\nu - \alpha \sum_{k=1}^{\ell_N} \phi_k \Delta_k\right) X_n,$$

we have

$$\left(\nu - \alpha \sum_{k=1}^{\ell_N} \phi_k \Delta_k\right) \Gamma_N(n) = -\frac{1}{T} \sum_{k'=1}^{\ell_N} \phi_{k'} \frac{k'}{\ell_N} \mathbb{E}_T \left[(X^{k'} - X^{-k'}) \cdot \mathcal{A} V^n\right].$$

By antisymmetry of $\mathcal{A}$ with respect to the Gibbs measure, this can be rewritten

$$\begin{aligned} \left(\nu - \alpha \sum_{k=1}^{\ell_N} \phi_k \Delta_k\right) \Gamma_N(n) &= \frac{1}{T} \sum_{k'=1}^{\ell_N} \phi_{k'} \frac{k'}{\ell_N} \mathbb{E}_T \left[(V^{k'} - V^{-k'}) \cdot V^n\right] \\ &= d \sum_{k'=1}^{\ell_N} \phi_{k'} \frac{k'}{\ell_N} (\delta_{n-k'} - \delta_{n+k'}), \end{aligned}$$

by a direct calculation at the last line. □

Let us now compute the discrete Fourier transforms (3.20) of $g_{\lambda,N}$ and Γ_N. From (3.32), we get

$$\hat{g}_{\lambda,N}(\xi) = \frac{2\mathbf{i}}{\lambda + 2\bar{\gamma}} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} \sin\left(\frac{2\pi k \xi}{N}\right), \tag{3.36}$$

and similarly from (3.35), we get

$$\hat{\Gamma}_N(\xi) = \frac{2d\mathbf{i} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} \sin\left(\frac{2\pi k \xi}{N}\right)}{\nu + 4\alpha \sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\frac{\pi k \xi}{N}\right)}. \tag{3.37}$$

Using Parseval's formula in (3.33), one thus gets

$$\begin{aligned} \mathbf{C}^N(\lambda) &= \frac{\alpha^2 T^2}{4N} \sum_{\xi \in \mathbb{T}_N} \hat{g}_{\lambda,N}(\xi) \left(\hat{\Gamma}_N(\xi)\right)^* \\ &= \frac{\alpha^2 T^2 d}{N} \sum_{\xi \in \mathbb{T}_N} \left[\frac{\left(\sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} \sin\left(\frac{2\pi k \xi}{N}\right)\right)^2}{\nu + 4\alpha \sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\frac{\pi k \xi}{N}\right)} \frac{1}{\lambda + 2\bar{\gamma}} \right], \end{aligned}$$

where the factor $1/N$ comes from the convention (3.20) used for the discrete Fourier transform. For clarity, let us write these expressions with the dispersion relation of the chain (3.21):

$$\mathbf{C}^N(\lambda) = \frac{T^2 d}{4\pi^2(\lambda + 2\bar{\gamma})} \frac{1}{N \ell_N^2} \sum_{\xi \in \mathbb{T}_N} \omega_N' \left(\frac{\xi}{N}\right)^2.$$

As we will see, only modes ξ of order less than $N\ell_N^{-1}$ contribute to the thermal conductivity in the limit $N \to \infty$, and it is clearer to introduce the rescaled dispersion relation

$$\bar{\omega}_N(u) = \omega_N(u/\ell_N), \tag{3.38}$$

so that finally

$$\mathbf{C}^N(\lambda) = \frac{T^2 d}{4\pi^2(\lambda+2\bar{\gamma})}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}_N'\left(\frac{\xi\ell_N}{N}\right)^2. \tag{3.39}$$

By injectivity of the Laplace transform, we find the following formula for C^N

$$C^N(s) = \frac{T^2 d\,\mathrm{e}^{-2\bar{\gamma}s}}{4\pi^2}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}_N'\left(\frac{\xi\ell_N}{N}\right)^2.$$

Inserting it in (3.27) gives finally after time integration

$$\begin{aligned}\kappa_N(T) &= \left(1+\frac{\mathrm{e}^{-2\bar{\gamma}t_N}-1}{2\bar{\gamma}t_N}\right)\frac{d}{8\pi^2\bar{\gamma}}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}_N'\left(\frac{\xi\ell_N}{N}\right)^2\\ &= \frac{\alpha^2 d}{2\bar{\gamma}}\left(1+\frac{\mathrm{e}^{-2\bar{\gamma}t_N}-1}{2\bar{\gamma}t_N}\right)\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right).\end{aligned} \tag{3.40}$$

And this concludes the proof of Proposition 3.1.

3.2.2 Asymptotic analysis of the thermal conductivity

Now that we have an exact expression (3.40) for the thermal conductivity $\kappa_N(T)$, we are interested in its convergence as N goes to infinity. As $t_N = \tau N\ell_N^{-1}$, then

$$1+\frac{\mathrm{e}^{-2\bar{\gamma}t_N}-1}{2\bar{\gamma}t_N}\xrightarrow[N\to\infty]{}1,$$

and one can focus on the convergence of

$$\frac{d}{8\pi^2\bar{\gamma}}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}_N'\left(\frac{\xi\ell_N}{N},\right)^2. \tag{3.41}$$

where we used the notation $\bar{\omega}_N$ defined in (3.38). We first mention the nearest neighbour interaction case for which $\ell_N = 1$ and then prove Theorem 3.1 for the long-range interaction case.

1. Nearest neighbour interaction: $\ell_N = 1$

A first special case is the nearest neighbour interaction ($\ell_N = 1$ and $\phi_1 = 1$), for which one has

$$\frac{\alpha^2 d}{2\bar{\gamma}}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\frac{\sin^2\left(\frac{2\pi\xi}{N}\right)}{\nu+4\alpha\sin^2\left(\frac{\pi\xi}{N}\right)}\xrightarrow[N\to\infty]{}\frac{\alpha^2 d}{2\bar{\gamma}}\int_0^1\frac{\sin^2(2\pi u)}{\nu+4\alpha\sin^2(\pi u)}du$$

by convergence of the Riemann sum, even in the unpinned case ($\nu \neq 0$). The diffusive behaviour of this chain has in fact already been proven in [8].

2. Long range interaction: proof of Theorem 3.1

We now prove that

$$\ell_N\kappa_N(T)\xrightarrow{N\to\infty}\bar{\kappa}$$

for a coupling constant of the form $\phi_k = 1/\ell_N$. We will see where we need ϕ to be constant, but give all the detailed proofs for general compactly supported smooth ϕ whenever we have been able to. Notice that, since for any $1 \le k \le \ell_N$ and $1 \le \xi \le N$,

$$\sin\left(2\pi\frac{k(N-\xi)}{N}\right) = -\sin\left(2\pi\frac{k\xi}{N}\right),\qquad \sin^2\left(\pi\frac{k(N-\xi)}{N}\right) = \sin^2\left(\pi\frac{k\xi}{N}\right).$$

We can rewrite the sum

$$\frac{\ell_N}{N}\sum_{\xi=1}^{N}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)=2\frac{\ell_N}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)+O\left(\frac{\ell_N}{N}\right) \tag{3.42}$$

the $O(\ell_N/N)$ term taking into account the error made when N is odd. The proof of the convergence (3.10) in Theorem 3.1 ultimately relies on two arguments. The first one is that the terms of order bigger than N/ℓ_N in (3.42) have a vanishing contribution as $N\to\infty$. The second argument is that the smaller order terms converge as two nested Riemann sums. To see that, let us fix a real number $b\le\ell_N/2$ such that $bN\ell_N^{-1}$ is an integer. We split the sum

$$\begin{aligned}&\frac{\ell_N}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)\\&=\frac{\ell_N}{N}\sum_{\xi=1}^{bN\ell_N^{-1}}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)+\frac{\ell_N}{N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right).\end{aligned}$$

We will prove the two following lemmas.

Lemma 3.3. *For $\phi_k=1/\ell_N$, there exists a constant $C>0$ such that*

$$\frac{\ell_N}{N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\left(\frac{\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)\le C\left(\frac{1}{b}+\frac{\ell_N^2}{b^5}+\frac{1}{N}\right). \tag{3.43}$$

Lemma 3.4. *Suppose $\phi\in\mathcal{C}^1([-1,1])$ is non-increasing over $[0,1]$. There exists a constant $C>0$ and a function ϵ satisfying $\lim_{t\to0}\epsilon(t)=0$, such that*

$$\begin{aligned}&\left|\frac{\ell_N}{N}\sum_{\xi=0}^{bN\ell_N^{-1}}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)-\int_0^b dv\frac{\left(\int_0^1\phi(u)u\sin(2\pi uv)du\right)^2}{\nu+4\alpha\int_0^1\phi(u)\sin^2(\pi uv)du}\right|\\&\qquad\le C\frac{b^2}{\ell_N}+C\frac{b}{N\ell_N^{-1}}+\epsilon\left(\frac{\ell_N}{N}\right).\end{aligned} \tag{3.44}$$

The proofs of Lemma 3.4 and Lemma 3.3 are postponed to the end of the section. Notice that Lemma 3.3 only holds for constant ϕ_k since it is based on explicit trigonometric computations. We conjecture however that the result holds for general interactions.

We are now able to prove Theorem 3.1. Choose $b=N^\beta$ and $\ell_N=N^\delta$, where $0<\beta,\delta<1$. b and ℓ_N are supposed to be integers, so we should actually define them as the integers parts of N^β and N^δ, but for sake of clarity we keep this definition and let the reader check the detailed proof. Combining the error terms of (3.43) and (3.44), β and δ must satisfy

$$2\delta<5\beta,\qquad 2\beta<\delta,\qquad \beta+\delta<1.$$

Therefore, the maximal value δ_{max} we can get for δ is obtained by achieving the equality in the first and last inequalities, which gives $\delta_{max}=5/7$. It suffices to choose then $\delta<5/7$ and an appropriate β to conclude. The value $\delta_{max}=5/7$ is not optimal, and might be improved by a more detailed analysis in Lemma 3.4.

We highlight that there is no minimal speed of convergence of ℓ_N to infinity for Theorem 3.1 to hold, since choosing b such that

$$\frac{\ell_N^2}{b^5}=\frac{b^2}{\ell_N},$$

i.e. $b=\ell_N^{3/7}$, all the error terms go to 0 whenever ℓ_N goes to ∞ slower than $N^{7/10}$.

3. Proof of lemmas 3.3 and 3.4

To prove those lemmas, we will need a lower bound on the denominator of the integral

$$\int_0^\infty dv \frac{\left(\int_0^1 \phi(u) u \sin(2\pi u v) du\right)^2}{\nu + 4\alpha \int_0^1 \phi(u) \sin^2(\pi u v) du}, \tag{3.45}$$

and of the sum in the finite-size thermal conductivity formula (3.22). These integrals can be trivially lower bounded if $\nu > 0$, *i.e.* in the pinned case. To cover also the convergence of the thermal conductivity in the unpinned case, we will need the following lemma.

Lemma 3.5. *For a general non-increasing function ϕ on $[0,1]$, there exists a constant $c > 0$ such that for any $v \geq 0$*

$$\int_0^1 \phi(u) \sin^2(\pi u v)\, du > c \min(1, v^2), \tag{3.46}$$

and for any $N \geq 1$ and $\xi \leq \lfloor N/2 \rfloor$

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi \frac{k\xi}{N}\right) > c \min\left(1, \left(\frac{\xi \ell_N}{N}\right)^2\right). \tag{3.47}$$

The proof of Lemma 3.5 is left to the appendix, as well as the proof of the convergence of the integral in (3.10). We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3. By Lemma 3.11 in the appendix, we have

$$\sum_{k=1}^{\ell_N} k \sin\left(2\pi \frac{k\xi}{N}\right) = -\frac{\ell_N \cos\left((2\ell_N + 1)\frac{\pi\xi}{N}\right)}{2 \sin\left(\frac{\pi\xi}{N}\right)} + \frac{F_N(2\pi\xi/N)}{\sin^2\left(\frac{\pi\xi}{N}\right)},$$

where F_N is a uniformly bounded function. As a consequence, we can bound for $\xi > bN\ell_N^{-1}$ the numerator in the left-hand side of (3.43) by:

$$\left(\frac{1}{\ell_N} \sum_{k=1}^{\ell_N} \frac{k}{\ell_N} \sin\left(2\pi \frac{k\xi}{N}\right)\right)^2 \leq \frac{C}{\ell_N^2 \sin^2\left(\pi \frac{\xi}{N}\right)} + \frac{C}{\ell_N^4 \sin^4\left(\pi \frac{\xi}{N}\right)} \tag{3.48}$$

for some constant C. Since we are dealing with modes $\xi > N\ell_N^{-1}$, we can use Lemma 3.5 to bound the denominator in the left-hand side of (3.43) by a constant, even if $\nu = 0$. Using (3.48) to bound the numerator, we get

$$\frac{\ell_N}{N} \sum_{\xi = bN\ell_N^{-1}+1}^{\lfloor N/2 \rfloor} \left(\frac{\left(\frac{1}{\ell_N} \sum_{k=1}^{\ell_N} \frac{k}{\ell_N} \sin\left(2\pi \frac{k\xi}{N}\right)\right)^2}{\nu + 4\alpha \frac{1}{\ell_N} \sum_{k=1}^{\ell_N} \sin^2\left(\pi \frac{k\xi}{N}\right)} \right)$$
$$\leq C' \frac{1}{N\ell_N} \sum_{\xi = bN\ell_N^{-1}+1}^{\lfloor N/2 \rfloor} \frac{1}{\sin^2\left(\pi \frac{\xi}{N}\right)} + C' \frac{1}{N\ell_N^3} \sum_{\xi = bN\ell_N^{-1}+1}^{\lfloor N/2 \rfloor} \frac{1}{\sin^4\left(\pi \frac{\xi}{N}\right)}.$$

For $\xi \leq N/2$, using the inequality

$$\sin\left(\pi \frac{\xi}{N}\right) \geq \pi \frac{\xi}{N} \left(1 - \pi^2 \frac{\xi^2}{6N^2}\right) \geq c \frac{\xi}{N},$$

for a positive constant c, we finally get

$$\frac{\ell_N}{N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\left(\frac{\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\sin^2\left(\pi\frac{k\xi}{N}\right)}\right)$$
$$\leq C''\frac{1}{N\ell_N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\frac{1}{\left(\frac{\xi}{N}\right)^2}+C''\frac{1}{N\ell_N^3}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\frac{1}{\left(\frac{\xi}{N}\right)^4}. \tag{3.49}$$

Finally, using that

$$\frac{1}{N}\frac{1}{\left(\frac{\xi}{N}\right)^2}\leq\int_{\frac{\xi-1}{N}}^{\frac{\xi}{N}}\frac{1}{u^2}du,$$

we can bound the sum appearing in the first term of (3.49) by an integral:

$$\frac{1}{N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\frac{1}{\left(\frac{\xi}{N}\right)^2}\leq\int_{b\ell_N^{-1}}^{1/2}\frac{1}{u^2}du+O\left(\frac{1}{N}\right),$$

where the term $O(1/N)$ takes into account the error made when N is odd. Therefore, the first term in the right-hand side of (3.49) is bounded by

$$\frac{1}{N\ell_N}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\frac{1}{\left(\frac{\xi}{N}\right)^2}\leq\frac{1}{b}+O\left(\frac{1}{N}\right). \tag{3.50}$$

Similarly, for the the second term in (3.49), we get the bound

$$\frac{1}{N\ell_N^3}\sum_{\xi=bN\ell_N^{-1}+1}^{\lfloor N/2\rfloor}\frac{1}{\left(\frac{\xi}{N}\right)^4}\leq\frac{1}{\ell_N^3}\int_{b\ell_N^{-1}}^{1/2}\frac{1}{u^4}du+O\left(\frac{1}{N}\right)\leq\frac{\ell_N^2}{b^5}+O\left(\frac{1}{N}\right). \tag{3.51}$$

Combining (3.50) and (3.51) in (3.49), we get the desired result. □

Proof of Lemma 3.4. Since the denominators in (3.44) are uniformly lower bounded in the pinned case ($\nu>0$), the proof is easier and left to the reader. We restrict ourselves to the proof in the unpinned case ($\nu=0$). We introduce for notational purpose the two functions

$$F(u,v)=\phi(u)u\sin(2\pi uv),\qquad G(u,v)=4\alpha\phi(u)\sin^2(\pi uv),$$

defined on $[0,1]\times\mathbb{R}_+$. Using that $\sin(x)\leq\min(1,x)$, it is straightforward to prove the following bounds on F,G and their partial derivatives: for any $(u,v)\in[0,1]\times\mathbb{R}_+$,

$$|F(u,v)|\leq C\min(1,v), \tag{3.52}$$

$$|\partial_u F(u,v)|\leq C\min(1,v)+C'v, \tag{3.53}$$

$$|\partial_u G(u,v)|\leq C\min(1,v^2)+C'\min(1,v)v, \tag{3.54}$$

$$|\partial_v F(u,v)|\leq C, \tag{3.55}$$

$$|\partial_v G(u,v)|\leq C\min(1,v), \tag{3.56}$$

where C, C' are some positive constants.

The proof is now in two steps: we first control the difference between the expression with double sums in (3.44) by a similar one in which the sum over k is replaced by integrals over u

$$A := \left| \frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)\right)^2}{\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} G\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)} \right) - \frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{\left(\int_0^1 F\left(u, \frac{\xi\ell_N}{N}\right) du\right)^2}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du} \right) \right|, \tag{3.57}$$

and then control the difference between this new term and the one with double integrals

$$B := \left| \frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{\left(\int_0^1 F\left(u, \frac{\xi\ell_N}{N}\right) du\right)^2}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du} \right) - \int_0^b dv \left(\frac{\left(\int_0^1 F(u,v)\, du\right)^2}{\int_0^1 G(u,v)\, du} \right) \right|. \tag{3.58}$$

1. We start by (3.57). First, for any $\xi \leq bN\ell_N^{-1}$, we can bound the difference

$$\begin{aligned} &\left| \frac{1}{\ell_N} \sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right) - \int_0^1 F\left(u, \frac{\xi\ell_N}{N}\right) du \right| \\ &\qquad \leq \sum_{k=1}^{\ell_N} \sup_{u' \in [(k-1)/\ell_N, k/\ell_N]} \left| \partial_u F\left(u', \frac{\xi\ell_N}{N}\right) \right| \int_{(k-1)/\ell_N}^{k/\ell_N} \left| \frac{k}{\ell_N} - u \right| du \\ &\qquad \leq \left(K \min\left(1, \frac{\xi\ell_N}{N}\right) + K' \frac{\xi\ell_N}{N} \right) \frac{1}{\ell_N}, \end{aligned} \tag{3.59}$$

and similarly for G:

$$\begin{aligned} &\left| \frac{1}{\ell_N} \sum_{k=1}^{\ell_N} G\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right) - \int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du \right| \\ &\qquad \leq \left(K \min\left(1, \frac{\xi\ell_N}{N}\right)^2 + K' \min\left(1, \frac{\xi\ell_N}{N}\right) \frac{\xi\ell_N}{N} \right) \frac{1}{\ell_N}. \end{aligned} \tag{3.60}$$

We can now introduce the cross term

$$\frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)\right)^2}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du} \right)$$

in (3.57) to bound

$$A \leq \frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{1}{\ell_N} \sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right) \right)^2 \left| \frac{1}{\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} G\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)} - \frac{1}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du} \right| \tag{3.61}$$

$$+ \frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \frac{\left| \left(\int_0^1 F\left(u, \frac{\xi\ell_N}{N}\right) du\right)^2 - \left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)\right)^2 \right|}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du}. \tag{3.62}$$

To control the term (3.61), we use the upper bound (3.52) on F, the upper bound (3.60) and the lower bounds of Lemma 3.5 to get:

$$\begin{aligned} &\frac{\ell_N}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(\frac{1}{\ell_N} \sum_{k=1}^{\ell_N} F\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right) \right)^2 \left| \frac{1}{\frac{1}{\ell_N}\sum_{k=1}^{\ell_N} G\left(\frac{k}{\ell_N}, \frac{\xi\ell_N}{N}\right)} - \frac{1}{\int_0^1 G\left(u, \frac{\xi\ell_N}{N}\right) du} \right| \\ &\qquad \leq \frac{1}{N} \sum_{\xi=1}^{bN\ell_N^{-1}} \left(K + K' \frac{\xi\ell_N}{N} \right). \end{aligned} \tag{3.63}$$

For the term (3.62), the difference in the numerator can be bounded using both (3.52) and (3.59), yielding

$$\left|\left(\int_0^1 F\left(u,\frac{\xi\ell_N}{N}\right)du\right)^2-\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}F\left(\frac{k}{\ell_N},\frac{\xi\ell_N}{N}\right)\right)^2\right| \\ \le \min\left(1,\frac{\xi\ell_N}{N}\right)\left(K\min\left(1,\frac{\xi\ell_N}{N}\right)+K'\frac{\xi\ell_N}{N}\right)\frac{1}{\ell_N}.$$

Combined with the lower bound of Lemma 3.5, this gives the control on (3.62):

$$\frac{\ell_N}{N}\sum_{\xi=1}^{bN\ell_N^{-1}}\frac{\left|\left(\int_0^1 F\left(u,\frac{\xi\ell_N}{N}\right)du\right)^2-\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}F\left(\frac{k}{\ell_N},\frac{\xi\ell_N}{N}\right)\right)^2\right|}{\int_0^1 G\left(u,\frac{\xi\ell_N}{N}\right)du}\le\frac{K}{N}\sum_{\xi=1}^{bN\ell_N^{-1}}\left(1+\max\left(1,\frac{\xi\ell_N}{N}\right)\right). \tag{3.64}$$

All in all, inserting (3.63) in (3.61), and (3.64) in (3.62), we get

$$\begin{aligned} A &\le \frac{1}{N}\sum_{\xi=1}^{bN\ell_N^{-1}}\left(K+K'\max\left(1,\frac{\xi\ell_N}{N}\right)\right)\\ &\le K''\frac{b^2}{\ell_N}, \end{aligned} \tag{3.65}$$

where we used that $b>1$ to get the bound at the last line.

2. It now remains to control (3.58). Denote by H the function

$$H(v)=\frac{\left(\int_0^1 F(u,v)\,du\right)^2}{\int_0^1 G(u,v)\,du}.$$

Deriving H, we get

$$H'(v)=\frac{2\left(\int_0^1 F(u,v)\,du\right)\left(\int_0^1\partial_v F(u,v)\,du\right)}{\int_0^1 G(u,v)\,du}-\frac{\left(\int_0^1 F(u,v)\,du\right)^2\left(\int_0^1\partial_u G(u,v)\,du\right)}{\left(\int_0^1 G(u,v)\,du\right)^2}.$$

Using the bounds (3.52) for F, (3.55) for its partial derivative and Lemma 3.5 of the first term, and then (3.52), (3.56) and Lemma 3.5 for the second term, we get

$$H'(v)\le C\frac{1}{\min(1,v)}. \tag{3.66}$$

Since this bound blows up for small values of v, we are going to control differently the sum for the first N/ℓ_N terms and the rest.

H is continuous on $\mathbb{R}_+$ and therefore uniformly continuous on $[0,1]$. Let ϵ be its modulus of continuity on $[0,1]$. Then

$$\left|\frac{\ell_N}{N}\sum_{\xi=1}^{N\ell_N^{-1}}H\left(\frac{\xi\ell_N}{N}\right)-\int_0^1 H(v)dv\right|\le\epsilon\left(\frac{\ell_N}{N}\right), \tag{3.67}$$

and the right-hand side term goes to 0 with N. For the higher order terms, we use that the derivative of H is bounded (3.66) to get

$$\left|\frac{\ell_N}{N}\sum_{\xi=N\ell_N^{-1}+1}^{bN\ell_N^{-1}}H\left(\frac{\xi\ell_N}{N}\right)-\int_1^b H(v)dv\right|\le K\sum_{\xi=N\ell_N^{-1}+1}^{bN\ell_N^{-1}}\int_0^{\ell_N/N}vdv\le K'\frac{b\ell_N}{N},$$

for some constant K'. Combined with (3.67) gives

$$B \leq \epsilon\left(\frac{\ell_N}{N}\right) + K'\frac{b\ell_N}{N}.$$

The conclusion of the proof of the lemma follows from combining this with (3.65). $\square$

3.2.3 Case of general $\phi \in \mathcal{C}_c^\infty$

In this part, we present some numerical evidences of the convergence of $\ell_N\kappa_N(T)$ for an arbitrary function $\phi \in \mathcal{C}_c^\infty$. We compute numerically

$$\tilde{\kappa}_N = \frac{\ell_N}{N}\sum_{\xi=1}^{N}\left(\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu + 4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}\right) \tag{3.68}$$

for ϕ_k given by

$$\phi_k = \frac{1}{\ell_N}\phi\left(\frac{k}{\ell_N}\right),$$

where ϕ is the bump function

$$\phi(x) = \exp\left(-\frac{1}{1-x^2}\right)\mathbb{1}_{|x|<1}.$$

We compute (3.68) for different values of the parameter $\ell_N = \lfloor N^\delta \rfloor$. We then plot (3.68) as a function of $0 \leq \delta \leq 1$ for $\nu = 0$ and $\alpha = 1/4$. As we expect $\tilde{\kappa}_N$ to converge towards

$$\tilde{\kappa}_N \xrightarrow{N\to\infty} 2\int_0^\infty\left(\frac{\left(\int_0^1\phi(u)u\sin(2\pi uv)\right)^2}{\int_0^1\phi(u)\sin^2(\pi uv)}\right) =: \tilde{\kappa} \approx 1,28,$$

we compare these results to the value of $\tilde{\kappa} \approx 1,28$, that we computed numerically. The results are displayed in Figure 3.2.3.

The constant parts for small δ are finite size effects due to the fact that $\lfloor N^\delta \rfloor$ is equal to 1 (and then 2, 3,...) for a large number of small values of δ, at least for the range of values N we consider.

3.3 Long range exchanges of velocities

In this section, we study the thermal conductivity for the chain with long-range stochastic exchanges of velocity and prove in particular Theorem 3.2. The section is organized as follows. We prove in the Subsection 3.3.1 the following two formulas for the conductivity that we will use throughout the whole section.

Proposition 3.2. *In both the harmonic and anharmonic cases, the conductivity is given by*

$$\kappa_N(T) = \frac{1}{2T^2t_NN}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j_s^{i,i+k;a}ds\right)^2\right] + \frac{\bar{\gamma}d}{2}c_\gamma^N, \tag{3.69}$$

where

$$j^{i,i+k;a} = \frac{1}{2}\left(V^i + V^{i+k}\right)\cdot\nabla W\left(X^i - X^{i+k}\right)$$

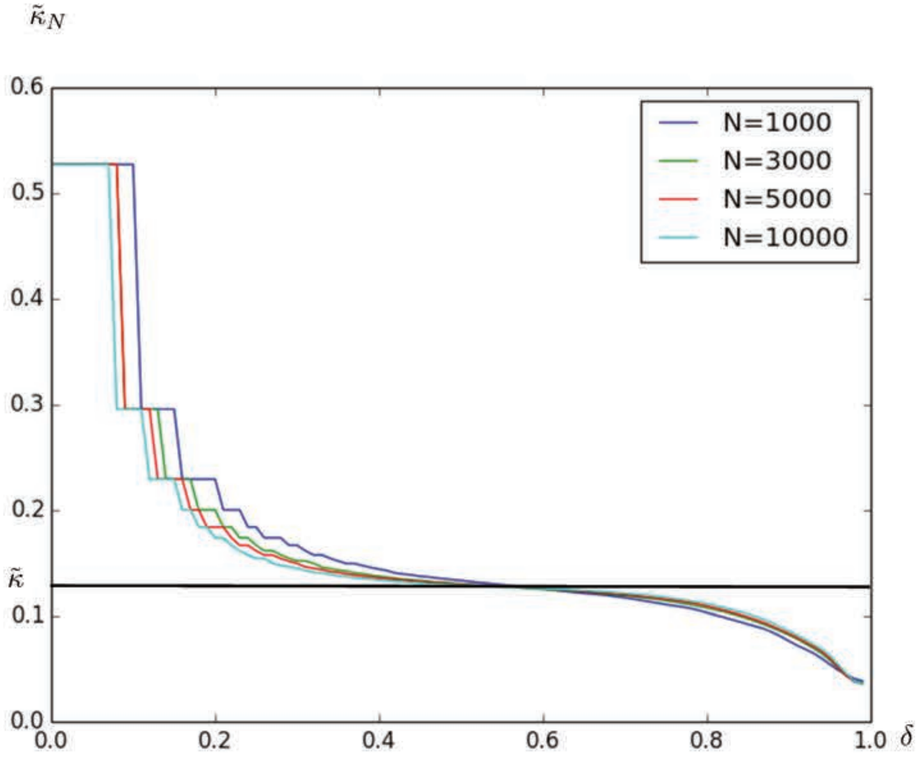

Figure 3.1 – $\tilde{\kappa}_N$ as a function of δ, where $\ell_N = N^\delta$, for different values of N. $\tilde{\kappa}_N$ seems to converge to $\tilde{\kappa}$ for a large range of values of ℓ_N.

is the mechanical contribution to the energy current and

$$c_\gamma^N = \sum_{k=1}^{\ell_N} \gamma_k \frac{k^2}{\ell_N^2}. \tag{3.70}$$

In the harmonic case, the correlations of the current can be explicitly computed and we get

$$\kappa_N(T) = \frac{d}{4\pi^2}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N} \bar{\omega}_N'\left(\frac{\xi\ell_N}{N}\right)^2 \sigma_N\left(\frac{\xi\ell_N}{N}\right) + \frac{\bar{\gamma}d}{2}c_\gamma^N, \tag{3.71}$$

where

$$\bar{\omega}_N'\left(\frac{\xi\ell_N}{N}\right)^2 = 4\alpha^2\pi^2\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\pi\frac{k\xi}{N}\right)}, \tag{3.72}$$

and

$$\sigma_N(u) = \frac{1}{\bar{\gamma}\psi_N(u)}\left(1+\frac{\mathrm{e}^{-\bar{\gamma}\psi_N(u)t_N}-1}{\bar{\gamma}\psi_N(u)t_N}\right) \tag{3.73}$$

is the scattering term, with

$$\psi_N(u) = 4\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\pi\frac{k}{\ell_N}u\right). \tag{3.74}$$

We then prove Theorem 3.2 in Subsection 3.3.2, *i.e.* that the thermal conductivity (3.71) converges for the pinned harmonic chain. In fact, the contribution of the hamiltonian currents vanish and energy is transported by the stochastic exchanges. We conjecture that this result is true it for an anharmonic uniformly convex pinning, and give a strategy to prove it in Subsection 3.3.3. We then analyse the result by analogy with a system of noninteracting random walks in Subsection 3.3.4.

3.3.1 Thermal conductivity computations: proof of Proposition 3.2

The calculations are similar to those of Section 3.2, and we only highlight the main differences.

1. Proof of (3.69) for any potential

The generator of the stochastic part of (3.11) is now

$$\bar{\gamma}\mathcal{S}f = \bar{\gamma}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\gamma_k(f(\eta^{i,i+k}) - f(\eta)),$$

with $\eta = (x^i, v^i)_{i\in\mathbb{T}_N}$, and $\eta^{i,i+k}$ is equal to η except that v^i and v^{i+k} are exchanged. We can still define the current by (3.24), but this time, the collisions contribute to the energy current. We get

$$J^{i,i+k}([0,t]) = \phi_k\int_0^t j_s^{i,i+k;a}ds + \bar{\gamma}\gamma_k\int_0^t j_s^{i,i+k;s}ds + M_t^{i,i+k}, \tag{3.75}$$

and

$$j_s^{i,i+k;s} = -\nabla_k\left(\frac{|V_s^i|^2}{2}\right) = \frac{|V_s^i|^2}{2} - \frac{|V_s^{i+k}|^2}{2}$$

is the contribution of the stochastic part to the energy current. Since $j_s^{i,i+k;s}$ is a spatial gradient, its contribution to (3.5) vanishes. $M_t^{i,i+k}$ is a martingale equal to

$$M_t^{i,i+k} = -\int_0^t\left(\frac{|V_s^{i+k}|^2}{2} - \frac{|V_s^i|^2}{2}\right)d\left(N_s^{i,i+k} - \bar{\gamma}\gamma_k s\right). \tag{3.76}$$

The thermal conductivity is thus equal to

$$\kappa_N(T) = \frac{1}{2T^2t_NN}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j_s^{i,i+k;a}ds + \sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}M_{t_N}^{i,i+k}\right)^2\right].$$

Expanding the square, the contribution of the product of the martingales with the energy current is null by the same time reversal argument as presented in [2], leading to

$$\kappa_N(T) = \frac{1}{2T^2t_NN}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j_s^{i,i+k;a}ds\right)^2 + \left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}M_{t_N}^{i,i+k}\right)^2\right]. \tag{3.77}$$

From (3.76), we compute the variance of the martingale $M_t^{i,i+k}$ using invariance of the Gibbs measure:

$$\begin{aligned}\mathbb{E}_T\left[\left(M_t^{i,i+k}\right)^2\right] &= \mathbb{E}_T\left[\frac{\bar{\gamma}\gamma_k}{4}\int_0^t\left(|V_s^i|^2 - |V_s^{i+k}|^2\right)^2ds\right]\\ &= \frac{t\bar{\gamma}\gamma_k}{2}\left(\mathbb{E}_T\left[|V^i|^4\right] - \mathbb{E}_T\left[|V^i|^2\right]^2\right).\end{aligned}$$

Since the coordinates of $V^i = (V^{i,j})_{1\le j\le d}$ are independent gaussians of variance T under the Gibbs measure, we have

$$\begin{aligned}\mathbb{E}_T\left[|V^i|^4\right] &= \mathbb{E}_T\left[\left(\sum_{j=1}^d|V^{i,j}|^2\right)^2\right]\\ &= \sum_{j=1}^d\mathbb{E}_T\left[|V^{i,j}|^4\right] + \sum_{1\le j\ne j'\le d}\mathbb{E}_T\left[|V^{i,j}|^2\right]\mathbb{E}_T\left[|V^{i,j'}|^2\right]\\ &= 3dT^2 + d(d-1)T^2,\end{aligned}$$

and

$$\mathbb{E}_T\left[|V^i|^2\right]^2 = d^2T^2,$$

which gives

$$\mathbb{E}_T\left[\left(M_t^{i,i+k}\right)^2\right] = t\bar{\gamma}\gamma_k dT^2.$$

By independence of the martingales we can finally compute the martingale contribution to (3.77):

$$\begin{aligned}\frac{1}{2T^2t_NN}\mathbb{E}_T\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}M_{t_N}^{i,i+k}\right)^2\right] &= \frac{1}{2T^2t_NN}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k^2}{\ell_N^2}\mathbb{E}_T\left[(M_{t_N}^{i,i+k})^2\right]\\ &= \frac{\bar{\gamma}d}{2}\sum_{k=1}^{\ell_N}\gamma_k\frac{k^2}{\ell_N^2}\\ &= \frac{\bar{\gamma}d}{2}c_\gamma^N, \end{aligned} \tag{3.78}$$

where c_γ^N is given by (3.70). Inserting (3.78) in (3.77) we conclude that

$$\kappa_N(T) = \frac{1}{2T^2t_NN}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j_s^{i,i+k;a}ds\right)^2\right] + \frac{\bar{\gamma}d}{2}c_\gamma^N,$$

which is (3.69) as expected.

2. Proof of (3.71) for harmonic potentials

For harmonic potentials, we can compute explicitly the contribution of the mechanical current correlations. The method is the same as in the first section. We start by solving explicitly the resolvent equation

$$\lambda u_{\lambda,N} - \mathcal{L}u_{\lambda,N} = \sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j^{i,i+k;a}. \tag{3.79}$$

This time, $u_{\lambda,N}$ is obtained via the following lemma.

Lemma 3.6. *$u_{\lambda,N}$ is given by*

$$u_{\lambda,N} = \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N} g_{\lambda,N}(i-i')V^i\cdot X^{i'}, \tag{3.80}$$

where this time $g_{\lambda,N}$ is the unique solution on $\mathbb{T}_N$ to the equation

$$\begin{cases}\lambda g_{\lambda,N}(n) - \bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\Delta_k g_{\lambda,N}(n) = \sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}(\delta_{n-k}-\delta_{n+k})\\ \sum_{n\in\mathbb{T}_N} g_{\lambda,N}(n) = 0.\end{cases} \tag{3.81}$$

Proof. We define $u_{\lambda,N}$ by (3.80) and check again that it solves (3.79). Extending $g_{\lambda,N}$ to a periodic function on $\mathbb{Z}$, $g_{\lambda,N}$ is again an antisymmetric function. Therefore, the same argument as in Lemma 3.1 applies to prove that $\mathcal{A}u_{\lambda,N} = 0$. We compute $\mathcal{S}u_{\lambda,N}$ by summation by parts:

$$\begin{aligned}\mathcal{S}u_{\lambda,N} &= \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\gamma_k g_{\lambda,N}(i-i')\Delta_k(V^i)\cdot X^{i'}\\ &= \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\gamma_k\Delta_k g_{\lambda,N}(i-i')V^i\cdot X^{i'}.\end{aligned}$$

Using (3.81), we get

$$\begin{aligned}
\lambda u_{\lambda,N} - \mathcal{L}u_{\lambda,N} &= \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N}\left(\lambda g_{\lambda,N}(i-i') - \bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\Delta_k g_{\lambda,N}(i-i')\right)V^i\cdot X^{i'}\\
&= \frac{\alpha}{2}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}V^i\cdot(X^{i-k}-X^{i+k})\\
&= \sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j^{i,i+k;a}
\end{aligned}$$

as expected. □

The rest of the proof follows straightforwardly. From (3.69), we can rewrite the conductivity as

$$\kappa_N(T) = \frac{1}{T^2}\int_0^\infty\left(1-\frac{s}{t_N}\right)^+ C^N(s)ds + \frac{\bar{\gamma}d}{2}c_\gamma^N, \tag{3.82}$$

where $C^N(s)$ is the correlation function

$$C^N(s) = \sum_{i\in\mathbb{T}_N}\sum_{k,k'=1}^{\ell_N}\frac{kk'}{(\ell_N)^2}\phi_k\phi_{k'}\mathbb{E}_T\left[j_s^{i,i+k;a}j_0^{0,k';a}\right]$$

As in (3.33), the Laplace transform of the correlation function can be written as

$$\mathbf{C}^N(\lambda) = \frac{\alpha^2T^2}{4}\sum_{n\in\mathbb{T}_N}g_{\lambda,N}(n)\Gamma_N(n),$$

Γ_N being still the solution to the same equation (3.35), that we recall

$$\begin{cases}\left(\nu-\alpha\sum_{k=1}^{\ell_N}\phi_k\Delta_k\right)\Gamma_N(n) = d\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}(\delta_{n-k}-\delta_{n+k})\\ \sum_{n\in\mathbb{T}_N}\Gamma_N(n)=0.\end{cases}$$

It suffices now to compute the discrete Fourier transforms of the new function $g_{\lambda,N}$ from the equation (3.81), the Fourier transform of Γ_N being still given by (3.37). We get

$$\hat{g}_{\lambda,N}(\xi) = \frac{2\mathbf{i}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)}{\lambda+4\bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right)}. \tag{3.83}$$

By Parseval's formula one thus gets

$$\mathbf{C}^N(\lambda) = \frac{\alpha^2T^2d}{N}\sum_{\xi\in\mathbb{T}_N}\left[\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\frac{\pi k\xi}{N}\right)}\frac{1}{\lambda+4\bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right)}\right].$$

For clarity, we rewrite these expressions by introducing the dispersion relation of the chain (3.38) and the scattering relation (3.74), so that

$$\mathbf{C}^N(\lambda) = \frac{T^2d}{4\pi^2}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\frac{\bar{\omega}'\left(\frac{\xi\ell_N}{N}\right)^2}{\lambda+\bar{\gamma}\psi_N\left(\frac{\xi\ell_N}{N}\right)}.$$

By injectivity of the Laplace transform, we find the following formula for C^N:

$$C^N(s) = \frac{T^2d}{4\pi^2N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}'\left(\frac{\xi\ell_N}{N}\right)^2\exp\left(-\bar{\gamma}\psi_N\left(\frac{\xi\ell_N}{N}\right)s\right).$$

Inserting it in (3.82) gives finally after time integration

$$\begin{aligned}\kappa_N(T) &= \frac{d}{4\pi^2}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\left[\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2\frac{1}{\bar{\gamma}\psi_N\left(\frac{\xi\ell_N}{N}\right)}\left(1+\frac{\mathrm{e}^{-\bar{\gamma}\psi_N\left(\frac{\xi\ell_N}{N}\right)t_N}-1}{\bar{\gamma}\psi_N\left(\frac{\xi\ell_N}{N}\right)t_N}\right)\right]+\frac{\bar{\gamma}d}{2}c_\gamma^N\\ &= \frac{d}{4\pi^2}\frac{1}{N}\sum_{\xi\in\mathbb{T}_N}\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2\sigma_N\left(\frac{\xi\ell_N}{N}\right)+\frac{\bar{\gamma}d}{2}c_\gamma^N,\end{aligned}$$

which is (3.71) as expected.

3.3.2 Convergence of the thermal conductivity for a pinned harmonic chain: proof of Theorem 3.2

From (3.70), it is easy to see that

$$c_\gamma^N \xrightarrow{N\to\infty} c_\gamma,$$

where c_γ is defined in (3.12). Moreover, the first term of (3.71) can be split as in (3.42) to get

$$\kappa_N(T) = 2\frac{d}{4\pi^2}\frac{1}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor}\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2\sigma_N\left(\frac{\xi\ell_N}{N}\right)+\frac{\bar{\gamma}d}{2}c_\gamma^N+O\left(\frac{\ell_N}{N}\right).$$

As a consequence, proving that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor}\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2\sigma_N\left(\frac{\xi\ell_N}{N}\right)=0 \tag{3.84}$$

suffices to conclude Theorem 3.2. To prove (3.84), we first give a bound on $\bar{\omega}'^2_N$ and then on σ_N.

1. Recall that $\bar{\omega}'^2_N$ is given by

$$\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2 = K\frac{\left(\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)\right)^2}{\nu+4\alpha\sum_{k=1}^{\ell_N}\phi_k\sin^2\left(\frac{\pi k\xi}{N}\right)},$$

for some constant K. Since the chain is unpinned ($\nu > 0$), the denominator is uniformly lower bounded. For the numerator, we can use the inequality $\sin(x)\le x$ for $\xi < N/\ell_N$. For $N/\ell_N \le \xi \le N/2$, by the same argument as in (3.48), we can use that ϕ_k is constant to obtain the bound

$$\begin{aligned}\left(\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}\sin\left(2\pi\frac{k\xi}{N}\right)\right)^2 &\le \frac{K}{\ell_N^2\sin^2\left(\pi\frac{\xi}{N}\right)}+\frac{K}{\ell_N^4\sin^4\left(\pi\frac{\xi}{N}\right)}\\ &\le \frac{K'}{\left(\frac{\xi\ell_N}{N}\right)^2}+\frac{K'}{\left(\frac{\xi\ell_N}{N}\right)^4},\end{aligned}$$

where the bound at the second line comes from the fact $\xi/N \le 1/2$. All in all, we obtain

$$\bar{\omega}'_N\left(\frac{\xi\ell_N}{N}\right)^2 \le \begin{cases} K\left(\frac{\xi\ell_N}{N}\right)^2 & \text{if } \ \xi<\frac{N}{\ell_N}\\ K\left(\frac{\xi\ell_N}{N}\right)^{-2}+K\left(\frac{\xi\ell_N}{N}\right)^{-4} & \text{if } \ \xi\ge\frac{N}{\ell_N}.\end{cases} \tag{3.85}$$

2. We now control the scattering part σ_N. Introduce the function

$$f(u) = 1 + \frac{e^{-u} - 1}{u}.$$

Then it is easy to see that f is bounded. Recall from the definition (3.73) of σ_N, that

$$\sigma_N\left(\frac{\xi\ell_N}{N}\right) = \frac{1}{\bar\gamma\psi_N\left(\frac{\xi\ell_N}{N}\right)} f\left(\bar\gamma t_N \psi_N\left(\frac{\xi\ell_N}{N}\right)\right),$$

yielding

$$\sigma_N\left(\frac{\xi\ell_N}{N}\right) \leq \frac{K}{\psi_N\left(\frac{\xi\ell_N}{N}\right)}.$$

Adapting the proof of Lemma 3.5, it is straightforward to see that $\psi_N(\xi\ell_N/N)$ is lower bounded by

$$\psi_N\left(\frac{\xi\ell_N}{N}\right) \geq c\min\left(1, \frac{\xi\ell_N}{N}\right)^2,$$

for $\xi < \lfloor N/2 \rfloor$. Therefore, we obtain

$$\sigma_N\left(\frac{\xi\ell_N}{N}\right) \leq \begin{cases} K\left(\frac{\xi\ell_N}{N}\right)^{-2} & \text{if} \quad \xi < \frac{N}{\ell_N} \\ K & \text{if} \quad \xi \geq \frac{N}{\ell_N}. \end{cases} \tag{3.86}$$

Combining (3.85) and (3.86) yields

$$\bar\omega'_N\left(\frac{\xi\ell_N}{N}\right)^2 \sigma_N\left(\frac{\xi\ell_N}{N}\right) \leq \begin{cases} K' & \text{if} \quad \xi < \frac{N}{\ell_N} \\ K'\left(\frac{\xi\ell_N}{N}\right)^{-2} + K'\left(\frac{\xi\ell_N}{N}\right)^{-4} & \text{if} \quad \xi \geq \frac{N}{\ell_N}. \end{cases}$$

Therefore, we can finally bound

$$\begin{aligned} \frac{1}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor} \bar\omega'_N\left(\frac{\xi\ell_N}{N}\right)^2 \sigma_N\left(\frac{\xi\ell_N}{N}\right) &\leq \frac{K'}{\ell_N} + \frac{K'}{N}\sum_{\xi=\lfloor N/\ell_N\rfloor}^{\lfloor N/2\rfloor}\left(\frac{\xi\ell_N}{N}\right)^{-2} + \frac{K'}{N}\sum_{\xi=\lfloor N/\ell_N\rfloor}^{\lfloor N/2\rfloor}\left(\frac{\xi\ell_N}{N}\right)^{-4} \\ &\leq \frac{K''}{\ell_N}, \end{aligned}$$

and this concludes the proof of (3.84) and thus of Theorem 3.2.

3.3.3 Conjecture and strategy for anharmonic pinning

In this subsection, we suppose that the interaction potential V is harmonic ($V(x) = \alpha|x|^2/2$) and the pinning potential U is anharmonic, uniformly convex and with Hessian lower bounded by ν. To prove the convergence of the thermal conductivity with an anharmonic pinning, we can follow the strategy of [2], in which bounds on the conductivity are obtained (in a nearest neighbor interaction model). Here, the contribution of the mechanical term is smaller than that of the stochastic collisions, so this type of bounds should be sufficient to conclude. As the range of interaction ℓ_N is a function of the size N of the system, we can not apply directly the strategy of [2] and we have to prove all the intermediate results. Unfortunately, one part of the argument is missing and we will just give a conjecture.

We first sketch the strategy and state the needed lemmas, and postpone the proof of these lemmas at the end of this subsection. By (3.69), the goal is to prove that

$$\frac{1}{2T^2 t_N N}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}j_s^{i,i+k;a}ds\right)^2\right] \tag{3.87}$$

goes to 0 with N. Denote

$$J_s^N = \sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N} \phi_k k j_s^{i,i+k;a},$$

and let $u_{\lambda,N}$ be the solution of the resolvent equation

$$(\lambda - \mathcal{L})\, u_{\lambda,N} = J^N. \tag{3.88}$$

By classical calculations for fluctuations of additive functionals of Markov processes (see [42] Appendix 1, Section 6 or [44] for a more complete reference), we can prove the following bound:

Lemma 3.7.

$$\frac{1}{2T^2 t_N N \ell_N^2}\mathbb{E}_T\left[\left(\int_0^{t_N}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k k j_s^{i,i+k;a} ds\right)^2\right] \leq \frac{15}{2T^2 N\ell_N^2}\mathbb{E}_T\left[J^N u_{t_N^{-1},N}\right]. \tag{3.89}$$

We can then compute the right-hand side of (3.89) thanks to the following lemma.

Lemma 3.8. *Denoting $u_N = u_{t_N^{-1},N}$ for simplicity, then u_N is given by*

$$u_N = \frac{\alpha}{2}\sum_{i,i'\in\mathbb{T}_N} g_N(i-i')V^i\cdot X^{i'}, \tag{3.90}$$

where g_N is the unique solution on $\mathbb{T}_N$ to the equation

$$\begin{cases} \frac{1}{t_N}g_N(n) - \bar\gamma\sum_{k=1}^{\ell_N}\gamma_k\Delta_k g_N(n) = \sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}(\delta_{n-k}-\delta_{n+k}) \\ \sum_{n\in\mathbb{T}_N} g_N(n) = 0. \end{cases} \tag{3.91}$$

Recall that the mechanical current is given by

$$j^{i,i+k;a} := \frac{\alpha}{2}(V^i + V^{i+k})\cdot(X^i - X^{i+k})$$

Using that velocities are independent gaussians under the Gibbs measure, we can compute the right-hand side term of (3.89) and get

$$\frac{5}{T^2 N\ell_N}\mathbb{E}_T\left[J_N u_N\right] = \frac{5\alpha^2}{4TN\ell_N}\sum_{i,i'\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k k\left(g_N(i-i') + g_N(i-i'+k)\right)\mathbb{E}_T\left[X^{i'}\cdot\left(X^i - X^{i+k}\right)\right].$$

By translation invariance, this can be rewritten

$$\begin{aligned}\frac{5}{T^2 N\ell_N^2}\mathbb{E}_T\left[J_N u_N\right] &= \frac{5\alpha^2}{4T\ell_N^2}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\phi_k k\left(g_N(i)+g_N(i+k)\right)\mathbb{E}_T\left[X^i\cdot\left(X^0 - X^k\right)\right] \\ &= \frac{5\alpha^2}{4T\ell_N}\sum_{i\in\mathbb{T}_N} g_N(i)\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\mathbb{E}_T\left[X^i\cdot\left(X^{-k}-X^k\right)\right] \\ &= \frac{5\alpha^2}{4T\ell_N}\sum_{i\in\mathbb{T}_N} g_N(i)\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\mathbb{E}_T\left[\left(X^{i+k}-X^{i-k}\right)\cdot X^0\right]. \end{aligned} \tag{3.92}$$

By the following lemma, we can bound g_N in (3.92).

Lemma 3.9. *Suppose the coefficients $(\phi_k)_{k\leq\ell_N}$ are constant and equal to $1/\ell_N$ for all $-\ell_N \leq k \leq \ell_N$. Then there exists a constant $C>0$ such that*

$$\|g_N\|_\infty < \frac{C\log\ell_N}{\ell_N}.$$

Finally, if the correlations $\mathbb{E}_T[X^i \cdot X^0]$ decay fast enough with i, we will be able to conclude. Actually, we can prove the following lemma for **harmonic pinning**.

Lemma 3.10. *There exists two constants $K, c' > 0$, independent of the length of the chain N, such that the correlations satisfy*

$$|\mathbb{E}_T [X_0 \cdot X_i]| \leq K\, \mathrm{e}^{-c' \frac{dist(i,0)}{\ell_N}}, \tag{3.93}$$

where $dist(i,0)$ is the distance between i and 0 in the discrete torus $\mathbb{T}_N$.

Combining Lemma 3.9 and (3.93) in (3.92) gives

$$\begin{aligned}
\frac{5}{T^2 N \ell_N^2} \mathbb{E}_T [J_N u_N] &\leq \frac{K \log(\ell_N)}{\ell_N^2} \sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \phi_k \mathbb{E}_T \left[\left(X^{i+k} - X^{i-k} \right) \cdot X^0 \right] \\
&\leq \frac{K' \log(\ell_N)}{\ell_N^2} \sum_{i=0}^{\lfloor N/2 \rfloor} \mathrm{e}^{-c' \frac{i}{\ell_N}} \sum_{k=1}^{\ell_N} \phi_k\, \mathrm{e}^{c' \frac{k}{\ell_N}} \\
&\leq K' \frac{\log(\ell_N)}{\ell_N}.
\end{aligned}$$

Together with Lemma 3.7, this concludes the proof of

$$\lim_{N \to \infty} \frac{1}{2T^2 t_N N} \mathbb{E}_T \left[\left(\int_0^{t_N} \sum_{i \in \mathbb{T}_N} \sum_{k=1}^{\ell_N} \phi_k \frac{k}{\ell_N} j_s^{i,i+k;a} ds \right)^2 \right] = 0,$$

for constant ϕ_k. The only part where we used that the pinning is harmonic is Lemma 3.10.

Conjecture 3.1. *We conjecture that Lemma 3.10 also holds for uniformly convex anharmonic pinning.*

Brascamp-Lieb type inequalities as proven in [52] should help us prove this conjecture, but we have not proven it yet. We finish this subsection by proving Lemma 3.7, Lemma 3.9 and Lemma 3.10. The proof of Lemma 3.8 being very similar to that of Lemma 3.1 and Lemma 3.6, we leave it for the reader.

Bound on the fluctuations of an additive functional

Proof of Lemma 3.7. $u_{\lambda,N}$ being a function of a configuration $(X^i, V^i)_{i \in \mathbb{T}_N}$, let us write $u_{\lambda,N}(s)$ for a function of the configuration $(X_s^i, V_s^i)_{i \in \mathbb{T}_N}$ at time s. Then

$$u_{\lambda,N}(t) = u_{\lambda,N}(0) + \int_0^t \mathcal{L} u_{\lambda,N}(s) ds + M_t,$$

where M_t is a martingale whose quadratic variation can be computed by stationarity (see [42], Appendix 1, Section 5)

$$\mathbb{E}_T \left[M_t^2 \right] = 2t \mathbb{E}_T [u_{\lambda,N}(-\mathcal{L}) u_{\lambda,N}].$$

From (3.88), we can thus write

$$\int_0^{t_N} J_s^N ds = \lambda \int_0^{t_N} u_{\lambda,N}(s) ds - u_{\lambda,N}(t_N) + u_{\lambda,N}(0) - M_{t_N}.$$

Computing the variance, we get the bound:

$$
\begin{aligned}
&\mathbb{E}_T\left[\left(\int_0^{t_N} J_s^N ds\right)^2\right] \\
&\quad \leq 3\lambda^2\mathbb{E}_T\left[\left(\int_0^{t_N} u_{\lambda,N}(s)ds\right)^2\right] + 3\mathbb{E}_T\left[(u_{\lambda,N}(t_N) - u_{\lambda,N}(0))^2\right] + 3\mathbb{E}_T\left[M_{t_N}^2\right] \\
&\quad \leq 3\left(\lambda^2 t_N^2 + 2\right)\mathbb{E}_T\left[u_{\lambda,N}^2\right] + 6t_N\mathbb{E}_T\left[u_{\lambda,N}(-\mathcal{L})u_{\lambda,N}\right], \qquad (3.94)
\end{aligned}
$$

where the last line is obtained by Cauchy-Schwarz inequality and using that the Gibbs measure is an invariant measure. From (3.88), multiplying by $u_{\lambda,N}$ and taking expectations, we deduce

$$
\lambda\mathbb{E}_T\left[u_{\lambda,N}^2\right] + \mathbb{E}_T\left[u_{\lambda,N}(-\mathcal{L})u_{\lambda,N}\right] = \mathbb{E}_T\left[u_{\lambda,N}J_N\right].
$$

Finally, using this last equality to bound the right-hand side terms in (3.94) gives

$$
\mathbb{E}_T\left[\left(\int_0^{t_N} J_s^N ds\right)^2\right] \leq \left(\frac{3\lambda^2 t_N^2 + 6}{\lambda} + 6t_N\right)\mathbb{E}_T\left[u_{\lambda,N}J_N\right].
$$

Choosing $\lambda = t_N^{-1}$ yields (3.89). □

Bound on g_N

Proof of Lemma 3.9. The proof is very similar to the computations in Subsection 3.3.2. By (3.91), the Fourier transform of g_N is given by

$$
\hat{g}_N(\xi) = \frac{2\mathbf{i}\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)}{\frac{1}{t_N} + 4\bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right)}.
$$

Therefore,

$$
\begin{aligned}
|g_N(n)| &\leq \frac{1}{N}\sum_{\xi=1}^{N}|\hat{g}_N(\xi)| \\
&\leq \frac{1}{2N}\sum_{\xi=1}^{N}\frac{\left|\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)\right|}{\bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right)} \\
&\leq \frac{1}{N}\sum_{\xi=1}^{\lfloor N/2\rfloor}\frac{\left|\sum_{k=1}^{\ell_N}\phi_k\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)\right|}{\bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right)} + O\left(\frac{1}{N}\right). \qquad (3.95)
\end{aligned}
$$

By the same arguments as in Subsection 3.3.2, if the coefficients $(\phi_k)_{-\ell_N\leq k\leq\ell_N}$ are constant, the numerator of the term in the right-hand side of (3.95) can be upper bounded by

$$
\left|\frac{1}{\ell_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}\sin\left(\frac{2\pi k\xi}{N}\right)\right| \leq \begin{cases} K\frac{\xi\ell_N}{N} & \text{if} \quad \xi < \frac{N}{\ell_N} \\ K\left(\frac{\xi\ell_N}{N}\right)^{-1} + K\left(\frac{\xi\ell_N}{N}\right)^{-2} & \text{if} \quad \xi \geq \frac{N}{\ell_N}. \end{cases} \qquad (3.96)
$$

The denominator of the term in the right-hand side of (3.95) can be lower bounded using Lemma 3.5:

$$
\sum_{k=1}^{\ell_N}\gamma_k\sin^2\left(\frac{\pi\xi k}{N}\right) \geq \begin{cases} K\left(\frac{\xi\ell_N}{N}\right)^2 & \text{if} \quad \xi < \frac{N}{\ell_N} \\ K & \text{if} \quad \xi \geq \frac{N}{\ell_N}. \end{cases} \qquad (3.97)
$$

Combining (3.96) and (3.97) in (3.95), we get

$$\begin{aligned}\|g\|_\infty &\leq \frac{K}{\ell_N} + \frac{K}{N}\sum_{\xi=\lfloor N/\ell_N\rfloor}^{\lfloor N/2\rfloor}\left(\frac{\xi\ell_N}{N}\right)^{-1} + \frac{K}{N}\sum_{\xi=\lfloor N/\ell_N\rfloor}^{\lfloor N/2\rfloor}\left(\frac{\xi\ell_N}{N}\right)^{-2} \\ &\leq \frac{K'}{\ell_N} + \frac{K'\log(\ell_N)}{\ell_N}.\end{aligned}$$

□

Equilibrium correlations in the positions

At equilibrium, it is well-known that in the Gaussian case with nearest neighbor interaction, the correlations in positions under the Gibbs measure decay exponentially if a pinning term is present (see [29], Chapter 8). Since the range of interactions ℓ_N in our model depends on the size of the system N, we adapt the calculations of [29] to our model.

Proof of Lemma 3.10. Following [29], Chapter 8, we define the discrete-time killed random walk $(Z_n)_{n\geq 0}$ on $\mathbb{T}_N \times \dagger$, where $\dagger$ is a graveyard point. Its probability transitions are given by

$$\begin{cases} P_N(i,i+k) = P_N(i+k,i) = \frac{1}{1+\nu}\phi_k \quad \text{if} \quad k \leq \ell_N \\ P_N(i,\dagger) = \frac{\nu}{1+\nu} \\ P_N(\dagger,\dagger) = 1 \\ P_N(i,j) = 0 \quad \text{otherwise} \end{cases}$$

We suppose $\alpha = 1$ for notational convenience, but the proof can be easily adapted for $\alpha \neq 1$ by defining a continuous time random walk and changing its rates of transition. We denote by P_j^N and E_j^N the probability and the expectation with respect to this random walk starting at $j \in \mathbb{T}_N$. Define for any $i \in \mathbb{T}_N \times \dagger$

$$\tau_i = \min\{n \geq 0 : Z_n = i\}.$$

Without modification, the proof of Theorem 8.26 in [29] can be adapted to obtain that, in the Gaussian case

$$\mathbb{E}_T\left[X_j \cdot X_i\right] = \frac{1}{1+\nu}E_j^N\left[\sum_{n=0}^{\tau_\dagger - 1}\mathbb{1}_{Z_n=i}\right]. \tag{3.98}$$

Then, applying the strong Markov property at the stopping time τ_i, we get

$$\begin{aligned}\mathbb{E}_T\left[X_0 \cdot X_i\right] &= \frac{1}{1+\nu}\sum_{n=0}^{\infty}P_0^N\left(Z_n = i, \tau_i < \tau_\dagger\right) \\ &= P_0^N\left(\tau_i < \tau_\dagger\right)\frac{1}{1+\nu}\sum_{n=0}^{\infty}P_i^N\left(Z_n = i, \tau_i < \tau_\dagger\right) \\ &= P_0^N\left(\tau_i < \tau_\dagger\right)\mathbb{E}_T\left[X_i^2\right] \\ &\leq CP_0^N\left(\tau_i < \tau_\dagger\right), \end{aligned} \tag{3.99}$$

where C is a constant independent of N. For any site i such that $dist(i,0) > \ell_N$, it takes to the random walk at least $\lceil dist(i,0)/\ell_N \rceil$ jumps to reach the site i before going to the graveyard $\dagger$. Therefore

$$P_0^N\left(\tau_i < \tau_\dagger\right) \leq \left(1 - \frac{\nu}{1+\nu}\right)^{dist(i,0)/\ell_N} = \exp\left(-c'\frac{dist(i,0)}{\ell_N}\right),$$

for some positive constant c'. This concludes the proof.

□

3.3.4 Analogy with noninteracting random walks

One can wonder if the definition of the finite-size thermal conductivity (3.5) is at the good scale and if it is normal to find a conductivity of order 1 of the model with long-range exchanges. It is actually not surprising to find a thermal conductivity of that order by analogy with non-interacting random walks on $\mathbb{T}_N$ in continuous time, with jumps at rate $\bar{\gamma}$, whose law is given by $(\gamma_k)_{-\ell_N \le k \le \ell_N}$. In fact, consider the Markov process $\eta_t = (\eta_t^i)_{i\in\mathbb{T}_n}$, where η_t^i is the number of particles at site i at time t. The conserved quantity in this model is the total of particles $\sum_i \eta^i$, and one can write

$$\eta_t^i = \sum_{k=-\ell_N}^{\ell_N} N_t^{i+k,i} - N_t^{i,i+k} = \sum_{k=1}^{\ell_N}\left(N_t^{i+k,i} - N_t^{i,i+k}\right) - \sum_{k=1}^{\ell_N}\left(N_t^{i,i-k} - N_t^{i-k,i}\right),$$

where $N_t^{i,i+k}$ is the number of times a particle jumped from i to $i+k$ before time t. It is easy to check that $N^{i,i+k}$ is an inhomogeneous Poisson process $N^{i,i+k}$ with rate $\bar{\gamma}\gamma_k\eta_t^i$. The difference $N_t^{i+k,i} - N_t^{i,i+k}$ is the analogue of the integrated current $J^{i,i+k}([0,t])$, and can be decomposed in a similar way as (3.75)

$$N_t^{i+k,i} - N_t^{i,i+k} = \bar{\gamma}\gamma_k \int_0^t (\eta_s^{i+k} - \eta_s^i)ds + M_t^{i,i+k},$$

where $M_t^{i,i+k}$ is a martingale. For the process η, also called the zero range process, the product Poisson distributions with density parameter ρ on $(\mathbb{T}_N)^{\mathbb{N}}$ form a family of invariant measures (see [42]). We write $\mathbb{E}_\rho$ for the expectation with respect to such a measure with parameter ρ. By the gradient argument we used for the chains of oscillators, we can compute the analogue quantity of (3.5)

$$\begin{aligned}
\frac{1}{2\rho^2 Nt}\mathbb{E}_\rho\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{j=1}^{\ell_N}\frac{k}{\ell_N}(N_t^{i+k,i} - N_t^{i,i+k})\right)^2\right] &= \frac{1}{\rho^2 Nt}\mathbb{E}_\rho\left[\left(\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k}{\ell_N}M_t^{i,i+k}\right)^2\right] \\
&= \frac{1}{2\rho^2 Nt}\sum_{i\in\mathbb{T}_N}\sum_{k=1}^{\ell_N}\frac{k^2}{\ell_N^2}\mathbb{E}_\rho\left[\left(M_t^{i,i+k}\right)^2\right] \\
&= \bar{\gamma}\sum_{k=1}^{\ell_N}\gamma_k\frac{k^2}{\ell_N^2} \\
&\xrightarrow{N\to\infty} 2\bar{\gamma}c_\gamma,
\end{aligned}$$

and we find a conductivity of order 1 as expected.

3.A Appendix: a trigonometric identity

In this appendix, we prove that

Lemma 3.11. *There exists a sequence of functions $(F_N)_{N\geq 1}$, bounded uniformly in N, such that, for $\theta \neq 0$,*

$$\sum_{k=1}^{\ell_N} k\sin(k\theta) = -\frac{\ell_N \cos\left((2\ell_N+1)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} + \frac{F_N(\theta)}{\sin^2\left(\frac{\theta}{2}\right)}.$$

Proof. We start by computing

$$-\sum_{k=1}^{\ell_N} \cos(k\theta) = -\mathrm{Re}\left(\sum_{k=1}^{\ell_N} \mathrm{e}^{ik\theta}\right) = -\mathrm{Re}\left(\mathrm{e}^{i\theta}\frac{1-\mathrm{e}^{i\ell_N\theta}}{1-\mathrm{e}^{i\theta}}\right) = -\frac{\cos\left((\ell_N+1)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

Differentiating with respect to θ, we get

$$\sum_{k=1}^{\ell_N} k\sin(k\theta) = \frac{(\ell_N+1)\sin\left((\ell_N+1)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{\ell_N\cos\left((\ell_N+1)\frac{\theta}{2}\right)\cos\left(\ell_N\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} + \frac{\cos\left((\ell_N+2)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}. \tag{3.100}$$

Then, using the identity

$$\sin\left((\ell_N+1)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right) - \cos\left((\ell_N+1)\frac{\theta}{2}\right)\cos\left(\ell_N\frac{\theta}{2}\right) = -\cos\left((2\ell_N+1)\frac{\theta}{2}\right)$$

to combine the two first terms in the right-hand side of (3.100), we get

$$\sum_{k=1}^{\ell_N} k\sin(k\theta) = -\frac{\ell_N\cos\left((2\ell_N+1)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} + \frac{\sin\left((\ell_N+1)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \cos\left((\ell_N+2)\frac{\theta}{2}\right)\sin\left(\ell_N\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}.$$

This completes the proof. ∎

3.B Appendix: proof of Lemma 3.5

This appendix is devoted to the proof of Lemma 3.5 and the convergence of the integral (3.45). We start by proving the convergence of the integral.

Lemma 3.12. *For a general function $\phi \in \mathcal{C}^1([-1,1])$, non-increasing on $[0,1]$, the integral*

$$\int_0^\infty dv \frac{\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2}{\nu + 4\alpha\int_0^1 \phi(u)\sin^2(\pi uv)du}$$

is convergent.

Proof of Lemma 3.12. For $v > 1$, the integral in the numerator can be split into the following terms

$$\begin{aligned}\int_0^1 &\phi(u)u\sin(2\pi uv)du\\ &= \sum_{k=0}^{\lfloor v\rfloor-1}\left(\int_{k/v}^{(k+1/2)/v}\phi(u)\,u\sin(2\pi uv)du + \int_{(k+1/2)/v}^{(k+1)/v}\phi(u)\,u\sin(2\pi uv)du\right)\\ &\qquad + \int_{\lfloor v\rfloor/v}^1 \phi(u)\,u\sin(2\pi uv)du\\ &= \sum_{k=0}^{\lfloor v\rfloor-1}\int_{k/v}^{(k+1/2)/v}\left(\phi(u)\,u - \phi\left(u+\frac{1}{2v}\right)\left(u+\frac{1}{2v}\right)\right)\sin(2\pi uv)du\\ &\qquad + \int_{\lfloor v\rfloor/v}^1 \phi(u)\,u\sin(2\pi uv)du,\end{aligned}$$

using that $\sin(2\pi uv + \pi) = -\sin(2\pi uv)$ at the last line. Hence, applying the mean value inequality to the function $u \mapsto \phi(u)u$ in the first integral at the last line, we deduce that there exists a constant C such that, for $v > 1$,

$$\begin{aligned}\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2 &\leq C\left(\frac{1}{v}\sum_{k=0}^{\lfloor v\rfloor-1}\int_{k/v}^{(k+1/2)/v}|\sin(2\pi uv)|du + \int_{\lfloor v\rfloor/v}^1|\sin(2\pi uv)|du\right)^2\\ &\leq C\left(\frac{1}{v}\int_0^1|\sin(2\pi uv)|du + \int_{\lfloor v\rfloor/v}^1|\sin(2\pi uv)|du\right)^2.\end{aligned}$$

Bounding the first integral in the last line by 1, and using that the integration interval of the second integral is smaller than $1/v$, we finally get that

$$\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2 \leq C\frac{1}{v^2}, \tag{3.101}$$

for $v > 1$. Moreover, for any $v > 0$, the numerator in (3.45) can be bounded by

$$\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2 \leq 4\pi^2v^2\left(\int_0^1 \phi(u)u^2du\right)^2. \tag{3.102}$$

Combining the bound (3.101) for $v > 1$ and (3.102), we finally get the following bound on the numerator in (3.45) for all $v > 0$:

$$\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2 \leq C\min(v^2, 1/v^2).$$

This already proves the convergence of (3.45) in the pinned case. For the unpinned case, we lower bound the denominator using (3.46) in Lemma 3.5, yielding

$$\frac{\left(\int_0^1 \phi(u)u\sin(2\pi uv)du\right)^2}{\nu + 4\alpha\int_0^1 \phi(u)\sin^2(\pi uv)du} \leq \frac{C\min(v^2, 1/v^2)}{c\min(1, v^2)} \leq \frac{C}{c}\min\left(1, \frac{1}{v^2}\right),$$

which concludes the proof. □

We finally prove Lemma 3.5.

Proof of Lemma 3.5. 1. We start by (3.46). As the integrand is positive, we can lower bound

$$\int_0^1 \phi(u)\sin^2(\pi uv)du \geq \int_0^{1/2} \phi(u)\sin^2(\pi uv)du, \tag{3.103}$$

and focus on the right-hand side term. We treat separately the case $v > 1$, for which we just have to find a uniform lower bound, and $v \leq 1$, for which we have to find a lower bound quadratic in v.

a. For $v > 1$, we use that ϕ is non-increasing and then an explicit integral calculation to get

$$\begin{aligned}\int_0^1 \phi(u)\sin^2(\pi uv)du &\geq \phi\left(\frac{1}{2}\right)\int_0^{1/2}\sin^2(\pi uv)du \\ &\geq \frac{1}{2}\phi\left(\frac{1}{2}\right)\left(\frac{1}{2} - \frac{\sin(\pi v)}{2\pi v}\right).\end{aligned} \tag{3.104}$$

It is then straightforward to check that there exists a constant $0 < c < 1$ such that for any $v > 1$,

$$\sin(\pi v) < c\pi v,$$

which inserted in (3.104) yields

$$\forall v > 1, \quad \int_0^1 \phi(u)\sin^2(\pi uv)du \geq \frac{1}{8}\phi\left(\frac{1}{4}\right)(1-c), \tag{3.105}$$

b. For $v \leq 1$, using the inequality $\sin(x) \geq x(1 - x^2/6)$, we have that for $0 \leq u \leq 1/2$,

$$\sin^2(\pi uv) \geq (\pi uv)^2\left(1 - \frac{\pi^2}{24}\right)^2 \geq c'u^2v^2,$$

where c' is a positive constant. Therefore, inserting in (3.103), we have

$$\forall v \leq 1, \quad \int_0^1 \phi(u)\sin^2(\pi uv)du \geq c'v^2\int_0^{1/2}\phi(u)u^2du,$$

Combining with (3.105), we therefore get (3.46).

2. For the discrete inequality (3.47), we use the same strategy, splitting into the case of small and large values of ξ. But for technical reasons we also have to consider an additional case of intermediate values of ξ separately. More precisely, we will first look at the large modes $\xi > \Delta N/\ell_N$, then the intermediate modes $\delta N/\ell_N < \xi \leq \Delta N/\ell_N$ and finally the small modes $\xi \leq \delta N/\ell_N$, where δ and Δ are two positive constants such that

$$1 < \delta^2 < \frac{24}{\pi^2}, \tag{3.106}$$

$$\Delta > \frac{2}{\pi(1 - \pi^2/24)}, \tag{3.107}$$

whose choice is motivated for technical reasons that will appear in the proof. We also start by lower bounding

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right), \tag{3.108}$$

and focus on the right-hand side term.

a. For the large modes $\xi > \Delta N/\ell_N$, we lower bound (3.108) using that ϕ is decreasing:

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \phi_{\lfloor \ell_N/2 \rfloor} \sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \sin^2\left(\pi k \frac{\xi}{N}\right).$$

We then compute the sum with the sines squared and get

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \frac{1}{2}\phi_{\lfloor \ell_N/2 \rfloor} \left(\lfloor \ell_N/2 \rfloor - \frac{\cos\left((\lfloor \ell_N/2 \rfloor + 1)\pi \frac{\xi}{N}\right) \sin\left(\lfloor \ell_N/2 \rfloor \pi \frac{\xi}{N}\right)}{\sin\left(\pi \frac{\xi}{N}\right)} \right).$$

Since $\xi/N \leq 2$, we have

$$\sin\left(\pi \frac{\xi}{N}\right) \geq \pi \left(1 - \frac{\pi^2}{24}\right) \frac{\xi}{N},$$

and we can therefore use this to lower bound

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \frac{1}{2}\phi_{\lfloor \ell_N/2 \rfloor} \lfloor \ell_N/2 \rfloor \left(1 - \frac{1}{\lfloor \ell_N/2 \rfloor \pi \left(1 - \frac{\pi^2}{24}\right) \frac{\xi}{N}}\right).$$

Finally, since $\xi > \Delta \ell_N/N$, we get

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \frac{1}{2}\phi_{\lfloor \ell_N/2 \rfloor} \lfloor \ell_N/2 \rfloor \left(1 - \frac{1}{\left(1 - \ell_N^{-1}\right) \pi \left(1 - \frac{\pi^2}{24}\right) \Delta}\right).$$

Since Δ is such that (3.106) holds, and $(\phi_{\lfloor \ell_N/2 \rfloor} \lfloor \ell_N/2 \rfloor)_{N \geq 1}$ is a positive sequence that converges to $\phi(1/2)$, this suffices to conclude that there exists a constant $c > 0$ such that

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) > c$$

for any $\xi \geq \delta N/\ell_N$.

b. For the intermediate modes $\delta N/\ell_N < \xi \leq \Delta N/\ell_N$, we can first lower bound

$$\sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \phi_{\lfloor \ell_N/2 \rfloor} \sum_{k=1}^{\lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor} \sin^2\left(\pi k \frac{\xi}{N}\right).$$

Since the function $u \mapsto \sin^2(\pi u)$ is increasing on $[0, 1/2]$ and since $k\frac{\xi}{N} \leq 1/2$ for $\xi \leq \Delta N/\ell_N$ and $k \leq \lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor$, we can lower bound

$$\phi_{\lfloor \ell_N/2 \rfloor} \sum_{k=1}^{\lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor} \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \phi_{\lfloor \ell_N/2 \rfloor} \ell_N \int_0^{\lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor / \ell_N} \sin^2\left(\pi u \frac{\xi}{N \ell_N^{-1}}\right) du.$$

Lower bounding the term in ξ in the integral, we finally get

$$\sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq \phi_{\lfloor \ell_N/2 \rfloor} \ell_N \int_0^{\lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor / \ell_N} \sin^2\left(\pi u \delta\right) du.$$

As $(\phi_{\lfloor \ell_N/2 \rfloor}\ell_N)_{N\geq 1}$ is a positive sequence converging to $\phi(1/2)$ and $\int_0^{\lfloor \lfloor \ell_N/2 \rfloor \Delta^{-1} \rfloor/\ell_N} \sin^2(\pi u \delta)\, du$ is also a positive converging sequence, we conclude that there exists a positive constant c such that for any $\delta N/\ell_N < \xi \leq \Delta N/\ell_N$,

$$\sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) > c.$$

c. For the small modes $\xi \leq \delta N/\ell_N$, we use that for $k \leq \lfloor \ell_N/2 \rfloor$ and $\xi < \delta N/\ell_N$,

$$\sin\left(\pi k \frac{\xi}{N}\right) \geq \pi k \frac{\xi}{N}\left(1 - \frac{\delta^2\pi^2}{24}\right) \geq c\frac{k}{\ell_N}\frac{\xi \ell_N}{N},$$

where c is a positive constant by (3.106). Therefore

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq c\frac{\xi^2\ell_N^2}{N^2}\sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k \frac{k^2}{\ell_N^2},$$

and since the positive terms $\sum_{k=1}^{\lfloor \ell_N/2 \rfloor} \phi_k k^2/\ell_N^2$ converges as a Riemann sum to $\int_0^{1/2} \phi(u)u^2 du$, there exists a constant $c' > 0$ such that for small modes $\xi < \delta N/\ell_N$,

$$\sum_{k=1}^{\ell_N} \phi_k \sin^2\left(\pi k \frac{\xi}{N}\right) \geq c'\frac{\xi^2\ell_N^2}{N^2}.$$

This concludes the proof.

$\square$

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Champ moyen local et transport de l'énergie dans des systèmes hors équilibre

Mots clés : Probabilités, Systèmes de particules en interaction, Champ moyen, Équation de Vlasov, Modèle de Kac

Résumé : Les systèmes de chaînes d'oscillateurs permettent de modéliser microscopiquement un solide, dans le but d'étudier le transport d'énergie et de retrouver la loi de Fourier. Dans cette thèse, nous introduisons des nouveaux modèles de chaînes d'oscillateurs avec interaction mécanique de type champ moyen local et collisions stochastiques préservant l'énergie totale du système. Le premier modèle est un modèle avec échanges stochastiques de vitesses de type modèle de Kac. Le second est un modèle avec retournement de vitesses, où les vitesses sont changées en leurs opposées à des temps aléatoires.
Contrairement à la théorie classique des modèles de champ moyen, les particules du système ne sont pas indistinguables, et le caractère conservatif des échanges stochastiques pour le premier modèle représente une difficulté supplémentaire dans la preuve d'une limite de Vlasov. Nous prouvons dans un premier temps une limite quantitative de champ moyen, que nous utilisons ensuite pour prouver que l'énergie évolue diffusivement à une échelle de temps donnée pour le modèle avec échanges à longue portée pour une classe restreinte de potentiels anharmoniques. À cette même échelle de temps, nous prouvons également que l'énergie n'évolue pas pour le modèle avec retournement de vitesses.
Dans le cas d'interactions harmoniques, nous calculons ensuite la conductivité thermique via la formule de Green-Kubo pour ces deux modèles, afin de mettre en évidence que l'échelle de temps à laquelle l'énergie évolue pour le modèle avec retournements de vitesses est plus longue et donc que les mécanismes en jeu dans le transport d'énergie sont différents.

Title : Local mean field and energy transport in non-equilibrium systems

Keywords : Probability, Interacting particle systems, Mean field, Vlasov equation, Kac model

Abstract : Chains of oscillator systems enable to model microscopically a solid, in order to study energy transport and prove Fourier's law. In this thesis, we introduce two new models of chains of oscillators with local mean field mechanical interaction and stochastic collisions that preserve the system's total energy. The first model is a model with stochastic velocity exchanges of Kac type. The second one is a model with random flips of velocities, where the sign of the particles' velocities is changed at random times.
As we consider local mean field models, particles are not indistinguishable, and the conservative stochastic exchanges in our first model are an additional difficulty for the proof of a Vlasov limit. We first derive a quantitative mean field limit, that we then use to prove that energy evolves diffusively at a given timescale for the model with long-range exchanges and for a restricted class of anharmonic potentials. At the same timescale, we also prove that there is no evolution of energy for the model with flips of velocities.
For harmonic interactions, we then compute thermal conductivity via Green-Kubo formula for both models, to highlight that the timescale at which energy evolves for the model with velocity flips is longer and therefore that the mechanisms at play for energy transport are different.

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