Persistence and Sheaves : from Theory to Applications

Nicolas Berkouk

INSTITUT POLYTECHNIQUE DE PARIS

NNT : 2020IPPAX032

Persistence and Sheaves: from Theory to Applications

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'Ecole polytechnique

École doctorale n°626 École Doctorale de l'Institut Polytechnique de Paris (ED IPP)
Spécialité de doctorat : Informatique, Données et Intelligence Artificielle

Thèse présentée et soutenue à Palaiseau, le 24/09/2020, par

M. BERKOUK NICOLAS

Composition du Jury :

Pierre Schapira Professeur Émérite, Sorbonne Université (Institut de Mathématiques de Jussieu)	Président
Ezra Miller Professeur, Duke University (Mathematics Department)	Rapporteur
Claude Viterbo Professeur, École Normale Supérieure (Département de Mathématiques et Applications)	Rapporteur
Ulrich Bauer Professeur Assistant, Technical University of Munich (Applied & Computation Topology group)	Examinateur
Justin Curry Professeur Assistant, State University of New-York (Department of Mathematics and Statistics)	Examinateur
Steve Oudot Chargé de Recherche, INRIA (Datashape)	Directeur de thèse
Ulrike Tillmann Professeur, Université d'Oxford (Mathematical Institute)	Examinateur

Thèse de doctorat

À Hannah, Maxime, et Sarah...

Remerciements

Je tiens tout d'abord à remercier Steve Oudot, mon directeur de thèse, pour toute la confiance et le soutien qu'il m'a accordé durant ces années. Je suis extrêmement honoré que Claude Viterbo ait accepté de rapporter mon travail, et tiens à lui exprimer ici ma gratitude. Les travaux fondateurs d'Ezra Miller en multi-persistance ont eu une grande influence sur ma production mathématique, et j'aimerais le remercier d'avoir accepté d'être rapporteur de cette thèse. Tout au long de ces années, j'ai eu la chance de toujours trouver la porte du bureau de Pierre Schapira ouverte. En plus d'avoir accepté d'être membre de mon jury, je tiens à le remercier pour sa disponibilité, sa pédagogie et sa patience face à mes idées parfois saugrenues : son influence sur ce manuscript est immense. J'aimerais également témoigner ma gratitude aux membres de mon jury qui ont accepté d'examiner mon travail en cette période si particulière : Ulrich Bauer, Justin Curry et Ulrike Tillmann.

J'ai pu découvrir aux côtés de Grégory Ginot les joies de la collaboration mathématique (et du café de compétition!). Merci pour ta bienveillance et ton optimisme, j'ai énormément appris en travaillant avec toi. François Petit m'a invité à le visiter il y a deux ans, et nos échanges qui ont débuté en terre luxembourgeoise n'ont depuis cessé de nous amener en des contrées mathématiques à chaque fois différentes. Merci pour ton soutien permanent et ta générosité, j'ai hâte de la suite de nos aventures !

Ce manuscript doit également beaucoup à Magnus Botnan. Aux détours des conférences où nous nous sommes croisés, ses remarques et sa connaissance impressionnante de la zoologie de la persistance m'ont été d'une aide très précieuse. Merci aussi pour la musique! J'aimerais également remercier Francesco Vaccarino pour m'avoir aussi bien accueilli à Turin, ainsi que Vincent Humilière, pour avoir pris le temps d'essayer de me rendre un peu moins ignorant en topologie symplectique. Je suis très heureux de rejoindre cette année l'équipe de Kathryn Hess, et je tiens à lui exprimer toute ma reconnaissance pour m'accueillir.

Ces années de recherche au sein de l'équipe Datashape me laisseront de beaux souvenirs, notamment grâce à l'ambiance chaleureuse qui y règne. Cela ne serait pas possible sans la bienveillance de Fred, et la présence de Marc, toujours prêt à aider un thésard en détresse, quelque soit le sujet. Merci à Claire, pour les bons moments partagés dans notre bureau, ainsi qu'à Matthieu pour les pauses cafés plus que nécessaires en début de thèse. Travailler en parallèle de Théo ont rendues ces années particulièrement agréables, merci pour ton regard si juste et pour ton aide dans l'organisation périlleuse de ma soutenance! Bien que je ne t'aies pas suivi jusque dans les structures persistantes multiplicatives, tu es quand même venu jusqu'à Chantelouve et on s'est bien marré, Raphaël. La relève de

l'équipe a été admirablement assurée par Louis, Etienne et Alex. Enfin, Vadim, merci pour le temps que tu as passé sur ce manuscript, et bon voyage en terres faisceautiques! J'aimerais également exprimer toute ma grattitude aux assistantes de l'équipe, Christine, Stéphanie et Bahar, pour leur aide si précieuse.

Ma césure au sein du programme Entrepreneurs d'Intérêt Général fût une expérience extrêmement riche. Merci à Mathilde et Soizic d'avoir créé cet ilôt fantastique, et d'y imprimer votre énergie! Merci à Delphine de m'avoir donné la chance de travailler avec toi, et merci à Simon, PE, Clément et Lucas, d'avoir eu la patience d'écouter mes longs discours sur MAPPER! Merci aussi à Bastien d'avoir réussi le pari impossible de me mettre à la course.

Cette thèse est le résultat d'un parcours scolaire d'une vingtaine d'années, qui a été parfois sinueux et parfois plein d'évidences. À chacun de ces moments charnières, j'ai eu la chance d'y trouver des maîtres dont l'influence a été déterminante. Le premier en fut une. Ma soeur Hannah m'a inculqué la leçon qui, je pense, a eu la plus grande influence sur mon parcours intellectuel. Grâce à elle, j'ai appris à ne jamais avoir peur de l'apprentissage. En m'accompagnant, elle m'a montré qu'aucun savoir n'était inaccessible, pour peu qu'on y travaille suffisamment. Je ne serais pas là sans toi. Mon passage par le primaire et le collège, quelque peu tumultueux, a été accompagné par Mmes Simeoni et Gasmi, que j'aimerais ici remercier. Le lycée a été la découverte de mes premières amours pour les sciences physiques, grâce aux enseignements de Mmes Villepelet et de Reguardati. L'arrivée en classe préparatoire a constitué pour moi une véritable épiphanie, et mon professeur de mathématique de première année Cyril Germain y est pour beaucoup. Il se passe rarement une semaine où je fais des mathématiques sans ressentir l'influence de son enseignement. J'ai eu la chance de suivre la classe de physique de Frédéric Massias, qui a fondamentalement transformé mon rapport aux Sciences, et m'a donné une formidable introduction aux principes de la recherche. Aussi, bien que n'ayant pas suivi ses cours, l'aura bienveillante de Nicolas Tosel n'a jamais été bien loin. Enfin, mon passage par le Centre de Sociologie des Organisations de Sciences Po. a marqué un changement profond dans mon parcours mathématique et personnel. Je ne pourrais remercier assez Pierre François pour la richesse et la profondeur de ses enseignements, ainsi que pour ses conseils toujours précieux.

J'aimerais aussi rendre hommage ici à mes amis. A celles de la première heure, Alice, Olivia et Lou, que le temps n'éloignera jamais. Pauline, Valérian, Clara, Émilie, Benjamin, Julien, Anton aka *la série friends mais à Paris* (et un peu à New-York), votre présence à mes côtés compte plus

que les mots que je pourrais écrire ici. Alexandre, mon alter ego. Olivia sans qui la vie parisienne serait bien plus morne, merci d'avoir cru en moi. Bien qu'une frontière nous sépare désormais Jim, tu n'as jamais été bien loin, et ta musique a bercé l'écriture de ce manuscript. L'équipe du Monticule qui est en fait devenue beaucoup plus que ça: Maxime, Darius, Claire, Thomas, Chris. La création du très select *S boulettes club Squizza* avec Charles. L'aventure complètement folle du Chantelouve avec JB et Arthur. Les jeudis merveilleux à refaire le monde avec Thomas et Tania. L'aventure Cambo-Laotienne avec Valentin. Les "après-midis" chez Lolo, Gatien et Adé qui finissent le matin. Merci également à mes colocs Thomas, Hugo, Arnaud et Vincent, qui chacun à leur façon ont subit les affres ce travail doctoral, et m'ont soutenu dans cette aventure.
Vous donnez un sens à ma vie.

La rédaction de ce manuscript s'est faite en une période bien étrange, et je remercie mon père qui a réuni les meilleures conditions pour que je travaille sereinement. Merci également du fond du coeur à ma mère, d'avoir toujours veillé à ce que je puisse réaliser mes ambitions. Enfin Hannah, Maxime et Sarah, cette thèse est pour vous.

Contents

CHAPTER 1

Introduction

A short and imprecise history of persistence and its applications

The foundations

There is no need to recall how the field of machine learning has intensely developed over the last fifty years. One could probably explain this progression by three main factors: the theoretical advances in statistical methods, the technological progress in micro-electronics (increasing the computational power of computers), and the almost systematic harvest of data from multiple sources (allowing to train statistical models over ever bigger datasets). In this context, Topological Data Analysis (TDA) arose in the early 2000's, with the aim to provide computational means to estimate topological quantities (such as (co-)homology groups) of a dataset.

Although the philosophical roots of TDA probably lie before (see *eg.* [Bar94]), the concept of persistence was first introduced in its actual formulation by Afra Zomorodian in his Ph.D. thesis [Zom01]. Given a finite sequence of nested simplicial complexes $\emptyset = X_0 \subset X_2 \subset ... \subset X_n = X$, he defines an algorithm to determine when a i-cycle is born and dies in the i-th simplicial homology (with coefficients in a field $\mathbf{k}$) of this filtration. Four years later, A. Zomorodian and G. Carlsson define in [CZ05] the i-th persistence module of this filtration. It is the sequence of $\mathbf{k}$-vector spaces and linear maps obtained by applying the i-th simplicial homology functor to the filtration of simplicial complexes:

$$0 = \mathrm{H}_i(\emptyset) \longrightarrow \mathrm{H}_i(X_1) \longrightarrow \mathrm{H}_i(X_2) \longrightarrow ... \longrightarrow \mathrm{H}_i(X_n) = \mathrm{H}_i(X) \, .$$

In this situation where the simplicial complex X is finite, standard results in commutative algebra imply that one can find a decomposition of this persistence module as a direct sum of persistence modules which are worth $\mathbf{k}$ between the values b and d in the filtration, and with the linear maps between any pair of index degree which lies between b and d being identities. The collection of pairs (b, d) appearing in the decomposition is called the i-th *barcode* of the filtration. It totally determines the i-th persistence module of this filtration up to isomorphism. Therefore, the barcode is a discrete invariant encoding topological features of the filtration. Although at this point, it is hard to use barcodes for machine learning since we do not yet know how to compare how similar two barcodes are. This is precisely the reason why G. Carlsson et al. introduced the *bottleneck distance* between barcodes in [CZCG05]. Roughly speaking, two barcodes are at bottleneck distance less than δ, if there exists a pairing between all bars of length greater than 2δ of each barcode, such that two paired bars have endpoints at distance at most δ. This distance can be easily computed in polynomial time with respect to the total number of bars in the barcodes. It has had surprisingly deep interpretations and consequences, which are the cornerstone of modern methods of TDA.

In [CSEH05], Cohen-Steiner et al. proved a fundamental property of the bottleneck distance. Consider $f : X \to \mathbb{R}$ a function on the topological space X, one can define the i-th *sub-level sets persistence module* of f as the collection of $\mathbf{k}$-vector spaces $(\mathrm{H}_i^{\mathrm{sing}}(f^{-1}(] - \infty, s[)))_{s\in\mathbb{R}}$ equipped with the linear maps $(\varphi_{s,t})_{s<t\in\mathbb{R}}$ defined as the linear maps induced by inclusion:

$$\varphi_{s,t} : \mathrm{H}_i^{\mathrm{sing}}(f^{-1}(])\infty, s[)) \longrightarrow \mathrm{H}_i^{\mathrm{sing}}(f^{-1}(])\infty, t[))$$

Assume that $f, g : X \to \mathbb{R}$ are two continuous functions such that the homology of the preimage by f and g of intervals $(-\infty, s]$ changes only when crossing a finite number of values of the parameter s (*eg.* f and g are Morse functions and X is a manifold). Then one can define the barcodes of their respective i-th persistence modules, written $\mathcal{B}^i(f)$ and $\mathcal{B}^i(g)$. The authors of [CSEH05] then proved that one can control the bottleneck distance $d_B(\mathcal{B}^i(f), \mathcal{B}^i(g))$ by the L_∞ distance between f and g. More precisely:

$$d_B(\mathcal{B}^i(f), \mathcal{B}^i(g)) \leq \sup_{x\in X} \|f(x) - g(x)\| = d_{L_\infty}(f, g). \tag{1.1}$$

This result indicates that the bottleneck distance is a consistent choice for measuring differences between barcodes to perform machine learning tasks. Indeed, real-world datasets always come with noise. Therefore, one never observes the reality, but rather a noisy version of it. Thanks to this stability result, one can be assured that the barcode of a noisy version $\tilde{f}$ of f is never further apart from the real barcode of f, with respect to the bottleneck distance, than $\tilde{f}$ is of f for the L_∞ distance.

The algebraic viewpoint

The next key turn in the theory of persistence is a purely algebraic interpretation of (1.1) by Chazal et al. in [CSG+09]. The authors defined persistence modules as functors from the category associated to the partially ordered set $(\mathbb{R}, \leq)$ to the category of **k**-vector spaces. They introduced the notion of ε-*interleavings* between two persistence modules, which is thought of as an approximate isomorphism, and defined the *interleaving distance* between two persistence modules as the infimum of the values of ε such that they are ε-interleaved. From its definition, it is straightforward to prove that the interleaving distance between the i-th persistence modules of two functions $f, g : X \to \mathbb{R}$ is bounded by the L_∞ distance between f and g.

Crawley-Boevey proved in [CB12] that under mild assumptions, persistence modules (still considered as functors from the poset $(\mathbb{R}, \leq)$ to the category of **k**-vector spaces) admit a barcode. The algebraic miracle is then that under the assumptions of Crawley-Boevey's theorem, the interleaving distance between two persistence modules M and N is equal to the bottleneck distance between their barcode:

$$d_I(M, N) = d_B(\mathcal{B}(M), \mathcal{B}(N)). \tag{1.2}$$

The purely algebraic viewpoint of interleavings has instigated the categorification of many concepts of TDA, and allowed to lay down the theoretical foundations in order to further develop persistence. One of the beautiful manifestations of this approach is the induced matching theorem of Bauer and Lesnick [BL15]. It states that epimorphisms of persistence modules $f : M \to N$ induce functorially a matching between the barcodes of M and N, whose cost is determined by the length of the longest bar appearing in the barcode of $\ker(f)$ and $\operatorname{coker}(f)$.

Many fruitful applications

Since its introduction, the techniques of TDA have spread widely in very diverse areas of Science and of Mathematics. Let us mention a few of them, with no intention to be exhaustive.

Material sciences. In [NHH$^+$15], persistent homology allows to characterize the microscopic structure of different medium-range order in anomorphous silica.

Cellular Biology. In [RCK$^+$17], the authors use TDA to efficiently predict cellular differenciation within RNA sequence analysis.

Time series analysis. In [DUC19], the authors use persistent homology as a feature extraction technique on electro-cardiograms to detect arrhytmia through neural networks.

Comprehension of Neural Networks. In [NZL20], the authors describe by the mean of persistent homology, the transformations operated by the successive layers of a neural network performing binary classification of point cloud data. This allows them to draw important insights on some fundamental behaviours of deep learning.

Symplectic Topology. The Floer homology, a central construction in symplectic topology, is naturally equipped with a filtration by the real-valued *action* function. This induces a structure of persistence module on the Floer homology. Although the barcode vocabulary did not exist at the time, Viterbo introduced in [Vit92] the *spectral invariants*, which roughly speaking correspond to the set of values $s \in \mathbb{R}$ such that the interval $]s, +\infty[$ appears in the barcode of the Floer homology persistence module. The barcode formalism has been introduced in this context by Polterovich and Shelukhin in [PS15]. In [BHS18], the authors have proved a stability theorem for the bottleneck distance between barcodes originating from filtered Floer homology.

Towards richer topological invariants

Beyond sub-level sets persistence

Although the theory of sub-level sets persistence is particularly well suited for computer science, it is not difficult to exhibit certain of its limitations. The first one is that by nature, sub-level sets persistence forget many informations contained in the fibers of a function. For instance (see example 2.1.36), one can easily construct two functions $f, g : X \to \mathbb{R}$ with the same barcode, at arbitrarily large L_∞ distance one of another. One way to partially overcome this issue is by studying a richer type of perststistence modules obtained from f. Instead of looking only at the pre-images by f of intervals of type $] - \infty, s[$, one can study the i-th homology groups of the pre-images by f of open bounded intervals $] - x, y[$. This object is either called the i-th *zig-zag/interlevel-sets/level-sets* persistence module of f. In [Bot15], Botnan proved that one can define an interleaving distance for such persistence modules, which enjoy the same stability property as the usual sub-level sets persistence. The algebraic structure underlying this construction is given by functors from the poset $(\mathbb{R}^2, \leq)$ to the category of $\mathbf{k}$-vector spaces, satisfying a certain property called *middle-exactness*. Cochoy and Oudot have proved in [CO17] that such persistence modules enjoy a decomposition theorem that allows to define a notion of barcode for level sets persistence modules of real valued functions. Moreover, Bjerkevik [Bje16] proved that one can compute the interleaving distance between two middle-exact persistence modules as the bottleneck distance between their barcodes.

Therefore, level-sets persistence appears to be a good candidate as a first step to enrich the computational methods of persistence. Indeed, it has the same formal and computational properties as sublevel-sets persistence, but carries strictly more information.

Multi-parameter persistence

Another way to enrich the persistence module arising from a function $f : X \to \mathbb{R}$, is by adding extra information coming from another real valued function $g : X \to \mathbb{R}$ in it. One way to achieve so is to study the filtration of X obtained by studying the fibers of f and g simultaneously, that is, understanding the persistence module with two parameters, that associates to a couple of real numbers (s, t), the homology of the topological space

$$f^{-1}(]-\infty, s[) \cap g^{-1}(]-\infty, t[).$$

The underlying algebraic structure of this construction is that of a functor from the partially ordered set $(\mathbb{R}^2, \leq)$ to the category of $\mathbf{k}$-vector spaces. Unlike level-sets persistence modules, such functors do not satisfy in general a tameness property such as middle-exactness. This motivates the study of multi-parameter persistence modules, that is, functors from the partially ordered set $(\mathbb{R}^n, \leq)$ to the category of $\mathbf{k}$-vector spaces, with $n \geq 2$, in full generality.

In [CZ09], Carlsson and Zomorodian proved that the category of persistence modules with n discrete parameters is equivalent to a certain category of graded modules over a polynomial ring with n indeterminates. This result shows the impossibility of a barcode-like decomposition for persistence modules with more than one parameter, both in the discrete and continuous setting. Indeed, modules over a polynomial ring with more than one indeterminate have the representation type of a wild quiver – those quivers for which the problem of expliciting a “simple”list of indecomposable representations is impossible [Mik90]. Therefore, one direction of research in multi-persistence is to seek for algebraic invariants of multi-parameter persistence modules, rather than studying general decomposition results. To be useful in the context of machine learning, these invariants have to be *computable*, and to satisfy a certain form of stability with respect to the noise in the input datasets. The first invariant that was proposed by the authors is the *rank invariant*, which records the rank of every internal map of a given persistence module. However, this invariant has been shown not to be discriminative in many important situations.

In his Ph. D. thesis [Les12], M. Lesnick studied in great details the interleaving distance for multi-parameter persistence modules. In particular, he showed that it is universal (in a precise sense) amongst all distances on persistence modules which are stable. This justifies the choice of this metric for the study of multi-parameter persistence modules. However, Bjerkevik, Botnan and Kerber proved in [BBK18] that the interleaving distance between two persistence modules with n parameters is NP-hard to compute whenever $n \geq 2$.

By restricting a persistence modules with n parameters to a line of positive slope in $\mathbb{R}^n$, one can obtain a barcode. The collection for all lines of these barcodes form an invariant of multi-parameter persistence modules which is equivalent to the rank invariant. In the case where $n = 2$, those can be efficiently computed thanks to the software RIVET [LW15]. From this collection of barcodes, one can derive a distance between persistence modules with n parameters, by taking the supremum over the lines of positive slopes in $\mathbb{R}^n$ of the (weighted) bottleneck distance between the barcodes of each persistence modules restricted to this line. This distance is bounded by the interleaving distance, and therefore satisfy the same stability property. Some recent advances by Vipond [Vip20] shows that it is even locally equivalent to the interleaving distance on the class of finitely presented persistence modules.

One of the main challenges of the research in multi-parameter persistence today is to find invariants of multi-parameter persistence modules which are expressive, computable and stable. In view of the results we have recalled here, studying the category of persistence modules in general seems very ambitious.

Sheaves and persistence

The initial work of J. Curry

One key observation is that the persistence modules arising from filtrations of continuous functions carry extra-structure. One way to express this is by invoking the formalism of sheaves. Sheaves were invented by Jean Leray during his detention in Germany during World War two, as a way to express local to global properties of certain construction of algebraic topology. It has then been spectacularly extended by Alexander Grothendieck during the second half of the 20th century. Sheaf theory is a very powerful language in which a great part of the modern results in algebraic topology and geometry are expressed.

In his Ph.D. thesis [Cur14], Justin Curry expressed sheaf theoretical constructions in a computational framework, in particular by using the notion of cellular (co-)sheaf. He also made the first links between persistence modules and sheaves, and introduced a metric on sheaves inspired by the interleaving distance.

The derived approach of Kashiwara-Schapira

However, sheaf theory takes its full strength when working in the derived category, which was only partially done in the work of Curry. In particular, its interleaving distance for sheaves is defined at an abelian level. We will show in this manuscript that it is not sufficient to fully understand —for instance— the behaviour of sheaves that are at finite distance to sheaves constant on open subsets.

In their seminal work [KS18a], Kashiwara and Schapira introduced the convolution distance between objects of the derived category of sheaves on a finite dimensional real vector space, as a proposition for an interleaving-like distance on sheaves. The convolution functor —the equivalent of the shift functor for persistence modules in this setting— is expressed in terms of Grothendieck operations, which is appropriate to perform cohomological computations. The authors prove that this distance is stable with respect to the derived pushforward, mimicking the stability property of the interleaving distance. They introduce the notion of *piecewise linear sheaves* —sheaves that should be storable in a computer, and prove that any constructible sheaf can be approximated within any precision in the convolution distance, by a piecewise linear sheaf. Moreover, the authors suggest that a topology that they have studied in detail in [KS90] —the *γ-topology*— should be of interest to transfer ideas of persistence such as multi-parameter persistence modules into sheaf theory. They finally develop a theory of *γ-piecewise linear sheaves* which they further develop in [KS18b].

The challenge of applied sheaf theory

The mathematical content of this thesis could be named *applied sheaf theory*. We aim in this work at employing the powerful theoretical framework of Kashiwara-Schapira to develop topological invariants of a new type, to be useful in machine learning and potentially other fields of mathematics such as symplectic topology. The price to pay to use this theory is a high level of abstraction, which is not well suited at first sight for a computational approach. In this manuscript, we make efforts to develop results that link the usual theory of persistence —that can be easily implemented in a computer, but has many algebraic limitations— with the formalism of sheaves in the derived setting —for which there exists a very rich literature, but which lacks an easy combinatorial description.

Structure and contributions of the thesis

This manuscript is divided into six chapters (including this introduction), a bibliography, an appendix presenting a brief introduction to sheaf theory, an index of definitions and an index of notations. While chapter 5 can be read independently of the other chapters, chapter 4 uses results of chapter 3. We have tried to make this work as self contained as possible, in order to allow a wide range of mathematicians to read it. In particular, we have given in appendix a very short introduction to abelian sheaves in the derived setting, alhtough this material is standard in the community of algebraic topologists. Our main contributions are the following ones:

Stable free resolutions of persistence modules. The last section of chapter 2 is devoted to show in a very simple setting the importance of "deriving" constructions, in order to get homological invariants of persistence modules which are stable with respect to the interleaving distance. More precisely, we introduce an interleaving distance on the homotopy and derived categories of persistence modules such that resolution and localization functors are isometric. This construction explains why a certain invariant of a persistence module —the graded Betti numbers— is not stable in a naive way with respect to the interleaving distance, and gives an algebraic framework in which seeking for stable homological invariants. This content is submitted for publication and is available as a preprint [Ber19].

Derived isometry theorem for sheaves. In chapter 3, we start from the observation that constructible (derived) sheaves on the real line admit a barcode, which is naturally graded over the integers. We prove an isometry theorem in this setting: one can compute the convolution distance between two sheaves in a purely combinatorial way from their barcodes. This indicates that this sheaf barcode, together with the convolution distance, can easily be handled by computers. As a byproduct, we are able to study some properties of the convolution distance. We prove that it is *closed* (thus answering an open question of Kashiwara-Schapira in dimension one) and that the space of constructible sheaves over $\mathbb{R}$ is locally path-connected when equipped with this distance. This content is a collaboration with Grégory Ginot. It is submitted for publication and is available as a preprint [BG18].

Level-sets persistence and sheaves. In chapter 4, we make precise the intuition that the collection of the level-sets persistence modules associated to a continuous function $f : X \to \mathbb{R}$ carries the same information than $\mathrm{R}f_*\mathbf{k}_X$, the derived pushforward of the constant sheaf on X by f. To compare these objects, we construct two functors exchanging level-sets persistence of functions and derived pushforward of sheaves. This is possible only by observing that the collection of level-sets persistence of f carry extra structure, which we call *Mayer-Vietoris systems*. We classify all Mayer-Vietoris systems with an appropriate finiteness property that we call strongly finite dimensionality, and prove that our functors are isometric with respect to the interleaving distance and the convolution distance. Ultimately, we establish a bijection between the barcodes of level-sets persistence and the sheaf graded barcodes. We prove that we can compute the convolution distance in a rather simple way from the knowledge of the level-sets persistence barcode, which can be computed by several software suites. This content is a collaboration with Grégory Ginot and Steve Oudot. It is submitted for publication and is available as a preprint [BGO19].

Ephemeral persistence modules and distance comparison. In chapter 5, we follow Kashiwara-Schapira's program and study precisely the links between multi-parameter persistence modules, sheaves in the Alexandrov topology, and sheaves in the γ-topology —where all of these categories are equipped with interleaving-like distances. We first generalize the definition of *ephemeral persistence modules* that was introduced in the one-parameter setting by Chazal et al. in [CCBS16]. This enables us to construct the *observable category of persistence modules* in the multi-parameter case: the quotient category of persistence modules by the ephemeral persistence modules. We build an equivalence from the observable category of persistence modules to the category of γ-sheaves that is distance preserving in their respective interleaving distances. We then strengthen this result by showing that, with some finiteness assumptions always satisfied by persistence modules arising from data point cloud, the interleaving distance between γ-sheaves equals the convolution distance of Kashiwara-Schapira. This shows that the observable information contained in multi-parameter persistence can be cast isometrically into the (derived) category of γ-sheaves. This category was shown in [KS18a] to be equivalent to a subcategory of the derived category of sheaves in the usual euclidean topology. Therefore, our results allow to use usual modern sheaves techniques to tackle the multi-parameter persistence setting. This content is a collaboration with François Petit and is to appear in *Algebraic and Geometric Topology* [BPar].

CHAPTER 2

Background & first results

Abstract

This chapter is devoted to introducing the theoretical framework of the manuscript. We start by introducing the general algebraic context of persistence. We then present the theoretical framework introduced by Kashiwara-Schapira in [KS18a] which transfers ideas of persistence in the context of sheaves on a real vector space. Finally, as a first warming up result, we equip the homotopy and derived categories of persistence modules with interleaving distances so that the operation of taking free resolutions becomes stable in a precise sense.

2.1 Persistence theory

This section aims at introducing the algebraic concepts behind the theory of persistence, and the main theoretical results obtained in the last decade in the special cases of one-parameter persistence and level-sets persistence. In all the manuscript, we let $\mathbf{k}$ be a field.

2.1.1 The category $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$ and the interleaving distance

To any partially ordered set (poset for short) $(\mathcal{P}, \leq)$, one can associate a category, also noted $(\mathcal{P}, \leq)$, defined by:

$$\begin{cases} \mathrm{Obj}((\mathcal{P}, \leq)) = \mathcal{P}, \\ \mathrm{Hom}_{(\mathcal{P},\leq)}(x, y) = \{*\} \text{ if } x \leq y, \ \emptyset \text{ otherwise.} \end{cases}$$

Definition 2.1.1
The *category of $\boldsymbol{k}$-persistence modules over* $(\mathcal{P}, \leq)$ (or persistence modules over $\mathcal{P}$, when there is no risk of confusion) is the category of functors from $(\mathcal{P}, \leq)$ to the category of $\mathbf{k}$-vector spaces $\mathrm{Mod}(\mathbf{k})$. We shall denote it by $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$.

It is a classical result that functors categories inherit the properties of their target category. Therefore, $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$ is a Grothendieck category: it is abelian, complete (small limits exist), co-complete (small colimits exist), has enough injectives and projectives, small filtered colimits are exacts, and has a generator since $\mathrm{Mod}(\mathbf{k})$ has all these properties.

An object M of an additive category $\mathscr{C}$ is *indecomposable* if $M \neq 0$, and for any isomorphim $M \simeq M_1 \oplus M_2$, either M_1 or M_2 is equal to 0.

A subset $\mathcal{I}$ of $\mathcal{P}$ is an *interval*, if it satisfies: (i) for any $x, z \in \mathcal{I}$, and $y \in \mathcal{P}$ satisfying $x \leq y \leq z$, then $y \in \mathcal{I}$; (ii) for $x, y \in \mathcal{I}$ there exists a finite sequence $(r_i)_{i=0..n}$ in $\mathcal{P}$ such that $r_0 = x$, $r_n = y$ and for any $i < n$, $r_i \leq r_{i+1}$ or $r_{i+1} \leq r_i$.

Definition 2.1.2
Given $\mathcal{I}$ an interval of $(\mathcal{P}, \leq)$, one can define $\mathbf{k}^{\mathcal{I}}$ the *interval persistence module* by:

$$\begin{cases} \mathbf{k}^{\mathcal{I}}(x) = \mathbf{k} \text{ if } x \in \mathcal{I}, \ 0 \text{ otherwise,} \\ \mathbf{k}^{\mathcal{I}}(x \leq y) = \mathrm{id}_{\mathbf{k}} \text{ if } x \text{ and } y \in \mathcal{I}, \ 0 \text{ otherwise.} \end{cases}$$

Observing that $\mathrm{Hom}_{\mathcal{P}}(\mathbf{k}^{\mathcal{I}}, \mathbf{k}^{\mathcal{I}}) \simeq \mathbf{k}$, one can prove that $\mathbf{k}^{\mathcal{I}}$ is an indecomposable persistence module. Note that $\mathbf{k}^{\mathcal{I}}$ is the persistent counterpart of $\mathbf{k}_Z$, the constant sheaf on a locally closed subset Z of a topological space X (see example A.2.12). We write $\mathcal{I}$ as a superscript to avoid confusions between the two constructions.

Definition 2.1.3
A category $\mathscr{C}$ is a *Krull-Schmidt category* if it satisfies the following axioms.

(KS-1) $\mathscr{C}$ is an additive category.

(KS-2) For any object X of $\mathscr{C}$, there exists a family of indecomposable objects $\mathbb{B}(X)$ of $\mathscr{C}$ such that $X \simeq \bigoplus_{I \in \mathbb{B}(X)} I$ which is essentially unique. That is, for any other family of indecomposable objects $\mathbb{B}'(X)$ with the same property, there exists a bijection $\sigma : \mathbb{B}(X) \to \mathbb{B}'(X)$ such that $I \simeq \sigma(I)$, for all I in $\mathbb{B}(X)$.

(KS-3) For any object X of $\mathscr{C}$ such that $X \simeq \bigoplus_{I \in \mathbb{B}(X)} I$ with $\mathbb{B}(X)$ a collection of indecomposable objects of $\mathscr{C}$, then $\prod_{I \in \mathbb{B}(X)} I$ exists in $\mathscr{C}$ and the canonical morphism :

$$\bigoplus_{I \in \mathbb{B}(X)} I \longrightarrow \prod_{I \in \mathbb{B}(X)} I$$

is an isomorphism.

Definition 2.1.4
Let M be a persistence module over a poset $(\mathcal{P}, \leq)$. M is *pointwise finite dimensional* (pfd for short), if $M(x)$ is a finite dimensional $\mathbf{k}$-vector space, for all $x \in \mathcal{P}$. We denote by $\mathrm{Pers}_f(\mathbf{k}^{\mathcal{P}})$ the full subcategory of $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$ consisting of pfd persistence modules.

Remark 2.1.5
Note that the usual definition of a Krull-Schmidt category asks, with notations of definition 2.1.3, that $\mathbb{B}(X)$ is finite (see *eg.* [Kra14]). This will not be sufficient for our study of pfd persistence modules since they are potentially infinite direct sum of interval modules (theorem 2.1.21). However, one important behaviour of these direct sums is that they satisfy axiom (KS-3), as expressed in the following proposition.

Proposition 2.1.6
Let J be an indexing set, and $(M_j)_{j \in J}$ be a family of persistence modules over $(\mathcal{P}, \leq)$, such that $\oplus_j M_j$ is pfd. Then the canonical morphism :

$$\bigoplus_{j \in J} M_j \longrightarrow \prod_{j \in J} M_j,$$

is an isomorphism.

Proof

Since $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$ is the functor category from $(\mathcal{P}, \leq)$ to $\mathrm{Mod}(\mathbf{k})$, the latter being a complete and co-complete category, small products and co-products can be computed pointwise in $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$. Therefore, given $s \in \mathcal{P}$, we have the following isomorphisms:

$$\left(\bigoplus_{j \in J} M_j\right)(s) \simeq \bigoplus_{j \in J} M_j(s) \tag{2.1}$$

$$\simeq \prod_{j \in J} M_j(s) \quad \text{(finite dimensionality)} \tag{2.2}$$

$$\simeq \left(\prod_{j \in J} M_j\right)(s), \tag{2.3}$$

which concludes the proof.

$\square$

Theorem 2.1.7 ([BCB18] - Thm. 1.1)

Let $(\mathcal{P}, \leq)$ be a poset and $M \in \mathrm{Obj}(\mathrm{Pers}_f(\mathbf{k}^{\mathcal{P}}))$. Then M is isomorphic to a direct sum of indecomposable persistence modules with local endomorphism ring.

Combining proposition 2.1.6 and theorem 2.1.7, we deduce the following result:

Theorem 2.1.8

Let $(\mathcal{P}, \leq)$ be a poset, the category $\mathrm{Pers}_f(\mathbf{k}^{\mathcal{P}})$ is a Krull-Schmidt category.

We will now introduce the formalism of interleaving distance associated to a flow, as introduced in [dMS18]. Since, we do not need the full generality of the theory, we will give simpler definitions that are specific to our context.

We denote by $([0, +\infty), \leq)$ the poset of positive real numbers endowed with the usual order. Also, let $\mathrm{End}((\mathcal{P}, \leq))$ denote the category of endofunctors of $(\mathcal{P}, \leq)$.

Definition 2.1.9

A *flow* on $(\mathcal{P}, \leq)$ is a functor $T : ([0, +\infty), \leq) \to \mathrm{End}((\mathcal{P}, \leq))$ satisfying:

1. $T(0) = \mathrm{id}_{\mathcal{P}}$,
2. for any $\varepsilon, \varepsilon' \in [0, +\infty)$, $T(\varepsilon) \circ T(\varepsilon') = T(\varepsilon + \varepsilon')$.

Definition 2.1.10
Given T a flow on $(\mathcal{P}, \leq)$, M a persistence module over $(\mathcal{P}, \leq)$ and $\varepsilon \geq 0$, one defines a new persistence module $M[\varepsilon]_T$, the *ε-shift* of M along T, by:

$$M[\varepsilon]_T = M \circ T(\varepsilon). \tag{2.4}$$

For $0 \leq \varepsilon' \leq \varepsilon$, the morphism $T(\varepsilon' \leq \varepsilon)$ induces a morphism

$$\tau^M_{\varepsilon',\varepsilon}(M) : M[\varepsilon']_T \to M[\varepsilon]_T. \tag{2.5}$$

Since $T(0) = \mathrm{id}_{\mathcal{P}}$, the morphism $T(0 \leq \varepsilon)$ induces a morphism $\tau^M_\varepsilon : M \to M[\varepsilon]_T$ which we call the *ε-smoothing morphism*[1] of M along T.

Definition 2.1.11
Let T be a flow on the poset $(\mathcal{P}, \leq)$. Let M, N be two persistence modules over $\mathcal{P}$, and $\varepsilon \geq 0$. A *(ε, T)-interleaving* (or ε-interleaving when there is no risk of confusion) is the data of two morphisms $f : M \to M[\varepsilon]_T$ and $g : N \to N[\varepsilon]_T$ fitting in a commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr] \arrow[rr, bend left, "\tau^M_{2\varepsilon}"] & N[\varepsilon]_T \arrow[r, "g[\varepsilon]"] \arrow[dr] & M[2\varepsilon]_T \\
N \arrow[r, "g"] \arrow[ur] \arrow[rr, bend right, "\tau^N_{2\varepsilon}"'] & M[\varepsilon]_T \arrow[r, "f[\varepsilon]"] \arrow[ur] & N[2\varepsilon]_T
\end{tikzcd}$$

In this situation, we will say that M and N are (ε, T)-interleaved and write $M \sim^T_\varepsilon N$. We will omit T when there is no risk of confusion.

Observe that a 0-interleaving is an isomorphism. Hence, one shall understand ε-interleaving as a weaker form of isomorphism. However, one must pay attention that "being ε-interleaved" is not an equivalence relation since it is not transitive. Indeed, if we have $M \sim^T_\varepsilon N \sim^T_\varepsilon L$ one can only deduce $M \sim^T_{2\varepsilon} L$.

Definition 2.1.12
Let M and N be two persistence modules over $(\mathcal{P}, \leq)$, and T be a flow on $(\mathcal{P}, \leq)$. The *interleaving distance* with respect to T between M and N is the possibly infinite following quantity:

$$d^T_I(M, N) = \inf\{\varepsilon \geq 0 \mid M \sim^T_\varepsilon N\}.$$

1. The name comes from the fact that the image of τ^M_ε corresponds to elements of M that lives longer than ε in the filtration.

Proposition 2.1.13 ([dMS18]- Thm. 2.5)
The interleaving distance with respect to a flow T of $(\mathcal{P}, \leq)$ is an extended pseudo-distance on $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$. That is, it satisfies for $L, M, N \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\mathcal{P}}))$:

(M1) $d_I^T(M,N) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$,

(M2) $d_I^T(M,N) = d_I^T(N,M)$,

(M3) $d_I^T(M,N) \leq d_I^T(M,L) + d_I^T(L,N)$.

Observe also that d_I^T satisfies (M4): if $M \simeq N$, then $d_I^T(M,N) = 0$. Note that the converse is not true in general.

Definition 2.1.14
The interleaving distance with respect to the flow T is said to be *closed* if it satisfies, for M, N any persistence modules and all $\varepsilon \geq 0$:

$$d_I^T(M,N) \leq \varepsilon \Longrightarrow M \sim_\varepsilon^T N.$$

In particular, if d_I^T is closed, then the converse to $(M4)$ holds.

Remark 2.1.15
Note that not all interleaving distances associated with a flow are closed. An important example that will be presented in the next sections, is the interleaving distance over $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^n})$ associated with the flow $T(\varepsilon) : s \mapsto s + (\varepsilon, ..., \varepsilon)$. In this context, their exists some *ephemeral* persistence modules, which are non zero persistence modules which are at distance 0 from the zero persistence module (*eg.* the persistence module $\mathbf{k}^{\{0\}}$). In chapter 5, we will give a precise treatment of these persistence modules.

Let $\mathscr{C}$ be a Krull-Schmidt category, equipped with a map $d : \mathrm{Obj}(\mathscr{C}) \times \mathrm{Obj}(\mathscr{C}) \to \mathbb{R}_{\geq 0}$ satisfying $(M1) - (M4)$. For any object X of $\mathscr{C}$, one denotes by $\mathbb{B}(X)$ a collection of indecomposable objects of $\mathscr{C}$ such that $X \simeq \oplus_{I \in \mathbb{B}(X)} I$.

Definition 2.1.16
Let $X, Y \in \mathrm{Obj}(\mathscr{C})$ and $\varepsilon \geq 0$. An ε-*matching* between $\mathbb{B}(X)$ and $\mathbb{B}(Y)$ is the following data.

1. Two subcollections $\mathcal{X} \subset \mathbb{B}(X)$ and $\mathcal{Y} \subset \mathbb{B}(Y)$, and a bijection $\sigma : \mathcal{X} \to \mathcal{Y}$ satisiying $d(I, \sigma(I)) \leq \varepsilon$ for all I in $\mathcal{X}$.
2. Moreover: $d(I, 0) \leq \varepsilon$, for any I in $\mathbb{B}(X) \backslash \mathcal{X}$ or in $\mathbb{B}(Y) \backslash \mathcal{Y}$.

Since d satisfies $(M4)$, the existence of a ε-matching does not depend on the choice of representatives in $\mathbb{B}(X)$ and $\mathbb{B}(Y)$.

Definition 2.1.17
Let X and Y be two objects of $\mathscr{C}$. One defines the *bottleneck distance associated to d* between X and Y as the possibly infinite following quantity:

$$d_B(X,Y) = \inf\{\varepsilon \geq 0 \mid \text{there exists an } \varepsilon\text{-matching between } \mathbb{B}(X) \text{ and } \mathbb{B}(Y)\}.$$

Remark 2.1.18
At this point, d_B shall not satisfy any of the axioms $(M1) - (M3)$. **The isometry theorem problem** associated to the Krull-Schmidt category $\mathscr{C}$ equipped with the map d satisfying (M1)-(M4) is to determine whether $d = d_B$. Note that the isometry theorem is far from true in general. It has been proved (see theorem 2.1.22) that it holds for $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$ equipped with the usual interleaving distance (see section 2.1.2), and we will prove it in chapter 3 for the derived category of constructible sheaves over $\mathbb{R}$ equipped with the convolution distance (see 3.4.18).

However in [Bje16, Example 5.2], the author provide a counter-example to the isometry theorem for $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^2})$. In particular, the author constructs two persistence modules over $\mathbb{R}^2$, M and N such that:

$$d_I(M,N) = 1 \quad \text{and} \quad d_B(M,N).$$

A more down to earth way to understand this counterexample, is that an ε-interleaving between $M = \oplus_i M_i$ and $N = \oplus_i N_i$ does not necessarily imply the existence of a ε-matching between $\{M_i\}_i$ and $\{N_i\}_i$.

2.1.2 One parameter persistence

One-parameter persistence is the persistence theory associated to the poset $(\mathcal{P}, \leq) = (\mathbb{R}, \leq)$. Hence, a *one-parameter persistence module* is an object of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$.

The main example of one-parameter persistence module is: given a function $u : X \to \mathbb{R}$ (not necessarily continous), one defines its *sub-level sets filtration* as the functor $\mathcal{S}(u)$ from the poset category $(\mathbb{R}, \leq)$ to the category of topological spaces **Top**, defined by $\mathcal{S}(u)(s) = u^{-1}(-\infty, s)$, and $\mathcal{S}(u)(s \leq t)$ is the inclusion of $u^{-1}(-\infty, s)$ into $u^{-1}(-\infty, t)$. For $n \geq 0$, the *n-th sublevel set persistence modules* associated to u is the functor

$$\mathcal{S}_n(u) := \mathrm{H}_n^{\mathrm{sing}} \circ \mathcal{S}(u),$$

where $\mathrm{H}_n^{\mathrm{sing}}$ denotes the n-th singular homology functor with coefficients in $\mathbf{k}$.

It is easy to verify that the functor $T : ([0, +\infty), \leq) \to \mathrm{End}((\mathbb{R}, \leq))$ defined by $T(\varepsilon) : s \mapsto s + \varepsilon$ is a flow on $(\mathbb{R}, \leq)$. This flow induces an interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$ which allows us to express the stability properties of the mapping $u \mapsto \mathcal{S}_n(u)$.

Theorem 2.1.19 ([CdSGO16])
Let $u, v : X \to \mathbb{R}$ be two maps from a topological space X to $\mathbb{R}$, and $n \geq 0$. Then:

$$d_I^T(\mathcal{S}_n(u), \mathcal{S}_n(v)) \leq \sup_{x \in X} |u(x) - v(x)|.$$

Moreover, d_I^T is universal amongst all other metrics satisfying this stability property.

Theorem 2.1.20 ([Les12] - Thm. 5.5)
Let $\mathbf{k}$ be a prime field, *ie.* $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ with p a prime number or $\mathbf{k} = \mathbb{Q}$. Let $d : \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})) \times \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})) \to \mathbb{R}$ which satisfies axioms $(M1) - (M4)$ (see prop. 2.1.13). Assume also that for any $u, v : X \to \mathbb{R}$ and $n \geq 0$, d satisfies:

$$d(\mathcal{S}_n(u), \mathcal{S}_n(v)) \leq \sup_{x \in X} |u(x) - v(x)|.$$

Then $d \leq d_I^T$.

In addition to satisfying this stability property, pfd one-parameter persistence modules (def. 2.1.4) enjoy a combinatorial description:

Theorem 2.1.21 ([CB12] - Thm. 1.1)
For any pointwise finite dimensional one-parameter persistence module M, there exists a unique multi-set of intervals of $\mathbb{R}$ (that is, an interval can appear several times in the list) noted $\mathbb{B}(M)$, such that :

$$M \simeq \bigoplus_{\mathcal{I} \in \mathbb{B}(M)} \mathbf{k}^{\mathcal{I}}.$$

$\mathbb{B}(M)$ is the *barcode* of M.

Therefore, the barcode of M, which consists of a list (with repetition) of intervals of $\mathbb{R}$, completely determines the isomorphism class of M. If $u : X \to \mathbb{R}$ is such that $\mathcal{S}_n(u)$ is pfd, one defines its *n-th sublevel sets barcode* as the barcode of $\mathcal{S}_n(u)$. There is no reason *a priori* that one can compute the interleaving distance between two persistence modules in a combinatorial way, from the knowledge of their barcodes. Nevertheless, the following result asserts that it is possible in the very specific context of one-parameter persistence. Usually, it is referred to as the **isometry theorem**.

Theorem 2.1.22 ([Les15] - Thm. 3.4)
Let d_B^T be the bottleneck distance associated to d_I^T. Then for M and N any pointwise finite dimensional persistence modules over $\mathbb{R}$:

$$d_I^T(M,N) = d_B^T(M,N).$$

This theorem is at the core of all the applications of persistence, both in machine learning and in theoretical areas of mathematics such as symplectic geometry.

2.1.3 Level-sets persistence

Level-sets persistence is the persistence theory associated to a certain class of persistence modules over the poset $\Delta^+ = \{(x,y) \in \mathbb{R}^2 \mid x+y > 0\}$, endowed with the following order:

$$(x,y) \leq (x',y') \iff x \leq x' \text{ and } y \leq y'.$$

Level-sets persistence take its name from the level-set filtration associated to a real-valued function. Consider $u : X \to \mathbb{R}$ a map from a topological space X to $\mathbb{R}$. Then, one defines the *level-sets filtration* of u as the functor $\mathcal{L}(u)$ from the poset category $(\Delta^+, \leq)$ to the category of topological spaces **Top**, defined by $\mathcal{L}(u)((x,y)) = u^{-1}(]-x,y[)$, and $\mathcal{L}(u)((x,y) \leq (x',y'))$ is the inclusion of $u^{-1}(]-x,y[)$ into $u^{-1}(]-x',y'[)$. For $n \geq 0$, the *n-th level-sets persistence module* associated to u is the functor

$$\mathcal{L}_n(u) := \mathrm{H}_n^{\mathrm{sing}} \circ \mathcal{L}(u),$$

where $\mathrm{H}_n^{\mathrm{sing}}$ denotes the n-th singular homology functor with coefficients in $\mathbf{k}$.

It is easy to verify that the functor $T : ([0,+\infty), \leq) \to \mathrm{End}((\Delta^+, \leq))$ defined by $T(\varepsilon) : s \mapsto s + (\varepsilon, \varepsilon)$ is a flow on $(\Delta^+, \leq)$. This flow induces an interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ which allows us to express the stability properties of the mapping $u \mapsto \mathcal{L}_n(u)$.

Theorem 2.1.23 ([BL17])
Let $u, v : X \to \mathbb{R}$ be two maps from a topological space X to $\mathbb{R}$, and $n \geq 0$. Then:

$$d_I^T(\mathcal{L}_n(u), \mathcal{L}_n(v)) \leq \sup_{x \in X} |u(x) - v(x)|.$$

We shall extend the definition of the shift functor to vectors of $(\mathbb{R}_{\geq 0})^2$. Let $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$ and $s = (s_1, s_1) \in (\mathbb{R}_{\geq 0})^2$. We shall denote in the sequel $s_x = (s_1, 0)$ and $s_y = (0, s_2)$. The persistence module $M[s]$ is defined by: $M[s](v) = M(v+s)$ for any $v \in \Delta^+$, and $M[s](v \leq w) = M(v+s \leq w+s)$.

It is immediate to check that there is a canonical s-smoothing morphism:

$$\tau_s^M : M \longrightarrow M[s].$$

When the context is clear, we shall not make explicit the smoothing morphisms in our diagrams. To a persistence module M over Δ^+ and any $s \in (\mathbb{R}_{\geq 0})^2$, one can associate a commutative diagram in $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$:

$$\begin{tikzcd} M[s_y] \arrow[r] & M[s] \\ M \arrow[u] \arrow[r] & M[s_x] \arrow[u] \end{tikzcd}$$

This induces the short complex:

$$M\{s\} := M \longrightarrow M[s_x] \oplus M[s_y] \longrightarrow M[s] \tag{2.6}$$

where the first map is $\begin{pmatrix} \tau_{s_x}^M \\ -\tau_{s_y}^M \end{pmatrix}$ and the second one is $(\tau_{s_y}^{M[s_x]}, \tau_{s_x}^{M[s_y]})$ in matrix notations.

Definition 2.1.24
An object $M \in \mathrm{Pers}(\mathbf{k}^{\Delta^+})$ is a *middle-exact persistence module* if the complexes $M\{s\}$ are exact for every $s \in \mathbb{R}^2_{>0}$.

Remark 2.1.25
We think of middle-exact persistence modules as being the analogue for the poset Δ^+ of half the terms of the Mayer-Vietoris long exact sequence relating the various homology groups of two open subsets of a space, their reunion and intersection. What is missing to have a long exact sequence are precisely the connecting homomorphisms relating homology groups of different degrees. In Section 4.2.1, we will precisely introduce an additional data on a (graded) middle-exact object of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ to obtain such long exact sequences.

Middle-exact persistence modules have a barcode decomposition similar to persistence modules over $\mathbb{R}$ that we now describe. First we specify the various geometric types, called blocks, of the barcode.

Notation. Given $a < b$ in $\mathbb{R} \cup \{\pm\infty\}$, we shall denote by $\langle a, b\rangle$ any of the four intervals $]a, b[$, $[a, b]$, $]a, b]$, $[a, b[$.

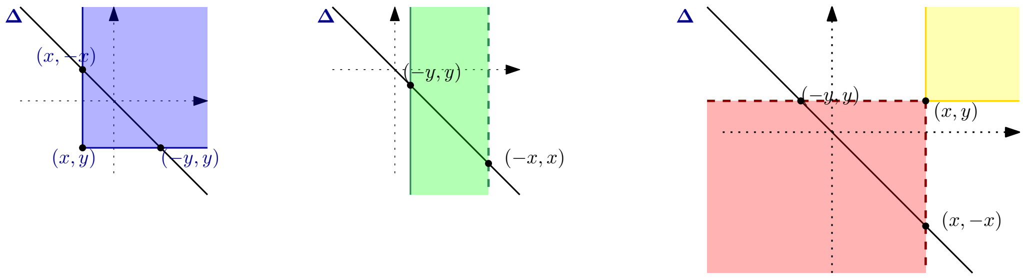


Figure 2.1 – On the left a block of type $\mathbf{bb}^-$ pictured in blue. On the middle a block of type **vb** pictured in green. On the right a block of type **db** in red and its dual block of type $\mathbf{bb}^+$ in yellow. The various coordinates refers to the intersection points of the boundaries of the blocks with the anti-diagonal Δ as well as the extremum of the birth or death blocks. The dashed boundary lines means that the boundary line is not part of the block.

Definition 2.1.26

A *block* B is a subset of $\mathbb{R}^2$ of the following type:

1. A *birthblock* (**bb** for short) if there exists $(a,b) \in \mathbb{R}^2$ such that $B = \langle a, \infty\rangle \times \langle b, \infty\rangle$, where a and b can both equal $-\infty$ simultaneously. Moreover, we will write that B is of type $\mathbf{bb}^+$ if $a+b > 0$, and of type $\mathbf{bb}^-$ if $a+b \leq 0$.
2. A *deathblock* (**db** for short) if there exists $(a,b) \in \mathbb{R}^2$ such that $B = \langle -\infty, a\rangle \times \langle -\infty, b\rangle$. Moreover, we will write that B is of type $\mathbf{db}^+$ if $a+b > 0$ and of type $\mathbf{db}^-$ if not.
3. A *horizontalblock* (**hb** for short) if there exists $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that $B = \mathbb{R} \times \langle a, b\rangle$.
4. A *verticalblock* (**vb** for short) if there exists $a \in \mathbb{R} \cup \{+\infty\}$ and $b \in \mathbb{R}$ such that $B = \langle a, b\rangle \times \mathbb{R}$.

Remark 2.1.27

Blocks are defined over the whole plane $\mathbb{R}^2$ and not just Δ^+.

Remark 2.1.28

Note that a deathblock B is characterized by its supremum [2], that is $\sup\{s \in B\}$ together with the data of whether its two boundary lines are in the block or not (note that the supremum is inside B if and only if both boundaries lines are). Similarly a birth block B' is characterized by its infimum $\inf\{s \in$

2. which is easily seen to be, if $B = \langle -\infty, x\rangle \times \langle -\infty, y\rangle$, the point $(x,y) \in \mathbb{R}^2$

$B'\}$ and the data of whether its boundary lines are in B or not. Note also that the vertical and horizontal blocks never have finite extrema.

Definition 2.1.29 (Duality between death and birth blocks)
The dual of a deathblock B is the birthblock $B^\dagger$ whose infimum is the supremum of B and whose vertical (resp. horizontal) boundary line is in $B^\dagger$ if and only if the vertical (resp. horizontal) boundary line of B is not.

Dually we define the dual $C^\dagger$ of a birthblock C as the death block whose supremum is the infimum of C and whose vertical (resp. horizontal) boundary line is in $C^\dagger$ if and only if the the vertical (resp. horizontal) boundary line of C are not.

Remark 2.1.30
The rule $B \mapsto B^\dagger$ is involutive: $(B^\dagger)^\dagger = B$, thereby it exhibits a perfect duality between death and birth blocks. Furthermore, note that the dual of a deathblock is of type $\mathbf{bb}^+$ if and only if the deathblock has a non-trivial intersection with Δ^+ i.e. is in $\mathbf{db}^+$.

One easily observe that for a block B, the set $B \cap \Delta^+$ is an interval of the poset Δ^+. Therefore the persistence module $\mathbf{k}^{B\cap\Delta^+}$ is well defined according to definition 2.1.2. For simplicity, we will omit Δ^+ and write $\mathbf{k}^B$ for $\mathbf{k}^{B\cap\Delta^+}$. This leads to the following remark:

Remark 2.1.31
If B is in $\mathbf{db}^-$, then $\mathbf{k}^B = 0$. Therefore we will usually not consider the block modules associated to such negative deathblocks. In what follows, the reader can safely assume that when we speak about a deathblock we mean an element of $\mathbf{db}^+$, unless otherwise stated.

Let us denote, for $s \in \Delta^+$, $B - s = \{t - s,\, t \in B\}$; this is a block of the same type as B (up to sign $\pm$).

Lemma 2.1.32
Let B be a block and $s \in \Delta^+$. There is a canonical isomorphism

$$\mathbf{k}^B[s] \cong \mathbf{k}^{B-s}.$$

Proof
By definition 2.1.2, we have that

$$\mathbf{k}^B[s](t) = \mathbf{k}^B(t + (s_1, s_2)) = \begin{cases} \mathbf{k} & \text{if t } \in B - s \\ 0 & \text{else.} \end{cases}$$

Therefore we have that

$$\mathbf{k}^B[s] \cong \mathbf{k}^{B-s}.$$

Theorem 2.1.33 ([CO17], [BCB18])
Let $M \in \mathrm{Pers}(\mathbf{k}^{\Delta^+})$ be middle exact and pointwise finite dimensional (pfd). Then there exists a unique multiset of blocks $\mathbb{B}(M)$ such that:

$$M \simeq \bigoplus_{B \in \mathbb{B}(M)} \mathbf{k}^B.$$

$\mathbb{B}(M)$ is the *barcode* of M.

Proposition 2.1.34
Let $u : X \to \mathbb{R}$ be a continuous map of topological spaces. Then $\mathcal{L}_n(u)$ is a middle-exact persistence module for all $n \geq 0$.

Proof
Let $s = (s_1, s_2) \in (\mathbb{R}_{>0})^2$ and $n \geq 0$. The complex $\mathcal{L}_n(u)\{s\}$ is exact if and only if it is exact when evaluated on all points $(x, y) \in \Delta^+$. The sequence

$$\mathcal{L}_n(u)(x, y) \longrightarrow \mathcal{L}_n(u)[s_x](x, y) \oplus \mathcal{L}_n(u)[s_y](x, y) \longrightarrow \mathcal{L}_n(u)[s](x, y)$$

is exact, since it is the middle part of the Mayer-Vietoris sequence associated to the open covering:

$$u^{-1}(]-x-s_1, y+s_2[) = u^{-1}(]-x-s_1, y[) \cup u^{-1}(]-x, y+s_2[),$$

$$u^{-1}(]-x-s_1, y[) \cap u^{-1}(]-x, y+s_2[) = u^{-1}(]-x, y[).$$

Therefore, if u is such that $\mathcal{L}_n(u)$ is pfd, it admits a decomposition as a direct sum of block modules. Therefore, we can define the *n-th level-sets barcode* of u as the multiset of blocks $\mathbb{B}(\mathcal{L}_n(u))$. Similarly to the case of one-parameter persistence, one can ask whether it is possible to find a combinatorial expression for the interleaving distance between two middle-exact persistence modules. More precisely, if the interleaving distance is equal to its associated bottleneck distance, noted d_B^T. Bjerkevik gave a positive answer to this question.

Theorem 2.1.35 ([Bje16] - Thm 3.3)
Let M and N be two pfd middle-exact persistence modules over Δ^+, then:

$$d_I^T(M, N) = d_B^T(M, N).$$

Note that this result is not true without the middle-exacteness assumption. In [Bje16, Ex. 5.2], the author gives an example of two persistence modules over Δ^+, M and N, which are finite direct sums of interval modules and verify:

$$d_I^T(M, N) = 1, \quad d_B^T(M, N) = 3.$$

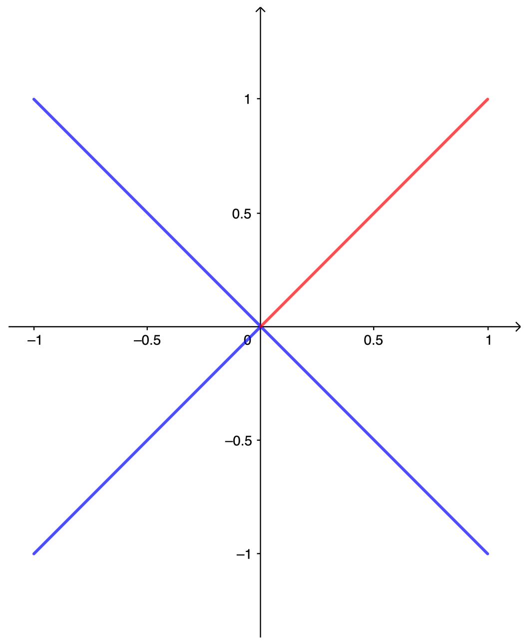


Figure 2.2 – The topological space X in $\mathbb{R}^2$ and its subset X' in red.

A natural question to ask is to compare the difference in topological information given by the sublevel-sets and the level-sets persistence modules construction. As one could expect, the level-sets persistence modules of a function contains strictly more information than its sub-levelsets module. This statement will be made precise in chapter 5. For now, we only present an example illustrating this phenomenon.

Example 2.1.36
We define the topological space

$$X = \{(x, x) \mid x \in [-1, 1]\} \cup \{(x, -x) \mid x \in [-1, 1]\}$$

endowed with the euclidean topology. We set $X' = \{(x, x) \mid x \in [0, 1]\} \subset X$.

We denote by $p : \mathbb{R}^2 \to \mathbb{R}$ the second coordinate projection. We define the maps from $u, v : X \to \mathbb{R}$ by:

$$u(x) = p(x), \quad \text{and} \quad v(x) = \begin{cases} p(x) \text{ if } x \in X \setminus X', \\ 0 \text{ if } x \in X'. \end{cases}$$

We can compute easily the homology of the pre-images by u and v of open intervals by the following, for $x, y \in \mathbb{R} \cup \{\pm\infty\}$:

$$u^{-1}(]x,y[) \simeq_h \begin{cases} \{*\} \text{ if } 0 \in]x,y[\\ \{*,\#\} \text{ if } 0 \notin]x,y[\text{ and }]x,y[\cap[-1,1] \neq \emptyset \\ \emptyset \text{ otherwise} \end{cases}$$

$$v^{-1}(]x,y[) \simeq_h \begin{cases} \{*\} \text{ if } [0,1]\cap]x,y[\neq \emptyset \\ \{*,\#\} \text{ if } y \in]-1,0] \\ \emptyset \text{ otherwise} \end{cases}$$

Where the notation $\simeq_h$ means "is homotopically equivalent to" and $\{*\}$ (resp. $\{*,\#\}$) is the set with one element (resp. two elements) equipped with the discrete topology.

From this, we deduce that the only degree where u and v have non-trivial homology of groups for pre-image of intervals is 0. Moreover, we have the following barcode decomposition:

$$\mathcal{S}_0(u) \simeq \mathbf{k}^{]-1,+\infty[} \oplus \mathbf{k}^{]-1,0]} \qquad \mathcal{S}_0(v) \simeq \mathbf{k}^{]-1,+\infty[} \oplus \mathbf{k}^{]-1,0]}$$

$$\mathcal{L}_0(u) \simeq \mathbf{k}^{B_b^{[-1,1]}} \oplus \mathbf{k}^{B_h^{[-1,0[}} \oplus \mathbf{k}^{B_v^{]0,1]}} \qquad \mathcal{L}_0(v) \simeq \mathbf{k}^{B_b^{[-1,1]}} \oplus \mathbf{k}^{B_h^{[-1,0[}}$$

where the notation for the block modules appearing in level-sets persistence is defined in lemma 4.3.9 (it is not necessary to understand the notation to get the intuition from the example). Then, we observe that $\mathcal{S}_0(u)$ is isomorphic to $\mathcal{S}_0(v)$. Nevertheless, the non-trivial summand $\mathbf{k}^{B_v^{]0,1]}}$ appears in the decomposition of $\mathcal{L}_0(u)$, yet not in the decomposition of $\mathcal{L}_0(v)$. This indicates that the level-sets persistence construction detects more information than the sublevel sets persistence. One can actually obtain the following computation, where the interleaving distances considered are the ones we have introduced in sections 2.1.2 and 2.1.3 :

$$d_I(\mathcal{S}_0(u), \mathcal{S}_0(v)) = 0 \text{ and } d_I(\mathcal{L}_0(u), \mathcal{L}_0(v)) = 1.$$

2.1.4 Persistence modules as modules, or the curse of multi-parameter persistence

A natural generalization of one-parameter is motivated by the study of the homology of pre-images of functions valued in a higher-dimensional real vector space. Given $d \geq 0$ one defines the *product order* over $\mathbb{R}^d$ by:

$$(x_1, ..., x_d) \leq (y_1, ..., y_d) \iff x_i \leq y_i \text{ for all } i.$$

Note that this order generalizes the one we introduced on $\mathbb{R}^2$ in the previous section. We define the category of *persistence modules with d parameters* as the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$. We will refer to multi-parameter persistence modules as objects of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ for a certain $d > 1$.

Let X be a topological space, and $u : X \to \mathbb{R}^d$ be a map. One defines its sublevel-set filtration as the functor $\mathcal{S}(u)$ from $(\mathbb{R}^d, \leq)$ to **Top** such that for $x, y \in \mathbb{R}^d$:

$$\mathcal{S}(u)(x) = u^{-1}(\{z \in \mathbb{R}^d \mid z \leq x\}),$$
$$\mathcal{S}(u)(x \leq y) = u^{-1}(\{z \in \mathbb{R}^d \mid y \leq x\}) \subset u^{-1}(\{z \in \mathbb{R}^d \mid z \leq y\}).$$

The *n-th persistence module associated to u* is the persistence module over $\mathbb{R}^d$ defined by:

$$\mathcal{S}_n(u) := \mathrm{H}_n^{\mathrm{sing}} \circ \mathcal{S}(u).$$

Similarly to the cases of one-parameter persistence and level-sets persistence, the functor $T : ([0, +\infty), \leq) \to \mathrm{End}((\mathbb{R}^d, \leq))$ defined by $T(\varepsilon) : s \mapsto s + (\varepsilon, ..., \varepsilon)$ is a flow on $(\mathbb{R}^d, \leq)$. It induces an interleaving satisfying the stability theorem for sublevel-sets filtration of functions to $\mathbb{R}^d$.

Theorem 2.1.37 ([Les12])
Let $u, v : X \to \mathbb{R}^d$ be two maps from a topological space X to $\mathbb{R}^d$, and $n \geq 0$. Then:

$$d_I^T(\mathcal{S}_n(u), \mathcal{S}_n(v)) \leq \sup_{x \in X} |u(x) - v(x)|.$$

Furthermore d_I^T satisfies the same universality property with respect to sublevel sets filtrations as in theorem 2.1.20.

One way to understand the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$, first explained by Carlsson and Zormorodian in [CZ09] in the specific case of persistence modules over $\mathbb{N}^d$, and then generalized to the continuous case of $\mathbb{R}^d$ in [Les12, Section 2.1.3], is to see persistence modules over $\mathbb{R}^d$ as $\mathbb{R}^d$-graded modules over the $\mathbb{R}^d$-graded algebra of generalized polynomials $\mathbf{k}\{x_1, ..., x_d\}$ (see definition below). This equivalence of categories explains the impossibility to give a combinatorial classification of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ when $d \geq 2$.

Definition 2.1.38
A *$\mathbb{R}^d$-graded $\mathbf{k}$-algebra* is a $\mathbf{k}$-algebra A together with a decomposition of $\mathbf{k}$-vector spaces $A = \bigoplus_{i \in \mathbb{R}^d} A_i$ such that $A_i \cdot A_j \subset A_{i+j}$ for all $i, j \in \mathbb{R}^d$.

Given A a $\mathbb{R}^d$-graded $\mathbf{k}$-algebra, a *$\mathbb{R}^d$-graded module* over A is a module M over the $\mathbf{k}$-algebra A together with a decomposition of $\mathbf{k}$-vector spaces $M = \bigoplus_{i \in \mathbb{R}^d} M_i$ such that $A_i \cdot M_j \subset M_{i+j}$ for all $i, j \in \mathbb{R}^d$.

In both cases, $\mathbb{R}^d$-graded morphisms are usual morphisms that respect the decomposition. We denote by A-$\mathbb{R}^d$-grad-mod the abelian category of $\mathbb{R}^d$-graded modules over A.

Let $\mathbf{k}\{x_1, ..., x_d\}$ be the algebra of generalized polynomials with coefficients in $\mathbf{k}$ and real positive exponents, that is polynomials that are expressed as a finite sum

$$P = \sum_i \alpha_i x_1^{e_i^1} ... x_d^{e_i^d},$$

with $\alpha_i \in \mathbf{k}$ and $e_i^j \in \mathbb{R}_{\geq 0}$. Given $e = (e_1, ..., e_n) \in (\mathbb{R}_{\geq 0})^d$, one denotes the monomial $x_1^{e_1} ... x_d^{e_d}$ by x^e.

The polynomial $\mathbf{k}$-algebra $\mathbf{k}\{x_1, ..., x_d\}$ is naturally endowed with a $\mathbb{R}^d$-grading, given by the decomposition of $\mathbf{k}$-vector spaces:

$$\mathbf{k}\{x_1, ..., x_d\} = \bigoplus_{e \in \mathbb{R}^d_{\geq 0}} \mathbf{k} \cdot x^e.$$

Let M be a persistence $\mathbf{k}$-module over $\mathbb{R}^d$. One defines $\alpha(M)$ to be the following $\mathbf{k}\{x_1, ..., x_d\}$-$\mathbb{R}^d$-graded module :

$\mathbb{R}^d$-grading : $\alpha(M) = \bigoplus_{i \in \mathbb{R}^d} M(i)$

Action of $\mathbf{k}\{x_1, ..., x_d\}$: for $e \in \mathbb{R}^d_{\geq 0}$, define the action of x^e component wise on $\alpha(M)$, that is for $i \in \mathbb{R}^d$ and $e \in \mathbb{R}^d_{\geq 0}$ let $\cdot\, x^e : \alpha(M)_i = M(i) \to \alpha(M)_{i+e} = M(i+e)$ be the morphism $M(i \leq i+e)$

Conversely, for $V = \bigoplus_{s \in \mathbb{R}^d} V_s$ a $\mathbf{k}\{x_1, ..., x_d\}$-$\mathbb{R}^d$-graded module, define $\gamma(V)$ the persistence module over $\mathbb{R}^d$ by, for $s \leq t$ in $\mathbb{R}^d$:

- $\gamma(V)(s) = V_s$
- $\gamma(V)(s \leq t)$ is the restriction of the action of x^{t-s} to the component V_s

The following was proved in [CZ09] (theorem 1) in the discrete case of persistence modules over $\mathbb{Z}^d$. Its proof generalizes readily to the continuous setting of persistence modules over $\mathbb{R}^d$.

Theorem 2.1.39
The mappings α and γ induce functors between $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ and $\mathbf{k}\{x_1, ..., x_d\}$-$\mathbb{R}^d$-grad-mod. These functors are additive exact isomorphisms of categories, inverse to each other.

This description of persistence modules as graded-modules over polynomial rings shows that there is no hope for the existence of a combinatorial invariant that entirely classifies the isomorphism classes of the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$, when $d > 1$. The module-theoretic approach to persistence for more general posets has been extended by Ezra Miller in [Mil19]. It has led to important results, such as the proof [Mil19, Corollary 8.25 and 8.26] of a conjecture of Kashiwara-Schapira [KS18b, Conjecture 3.17 and 3.20].

2.2 Persistence and sheaves

In the manuscript, we will use the classical notations of [KS90]. We have recalled in the appendix A the main classical results about sheaves that we will use. In particular given a topological space X, $\mathrm{Mod}(\mathbf{k}_X)$ is the category of sheaves of $\mathbf{k}$-vector spaces on X, $\mathrm{D}(\mathbf{k}_X)$ (resp. $\mathrm{D}^b(\mathbf{k}_X)$) its derived category (resp. bounded derived category).

2.2.1 Persistence modules as sheaves

Let $(\mathcal{P}, \leq)$ be a poset. We say that U is a *lower set of* $(\mathcal{P}, \leq)$ if for all $x \in U$ and $y \in \mathcal{P}$ with $y \leq x$, we have $y \in U$.

Proposition 2.2.1
The collection of lower sets of $(\mathcal{P}, \leq)$ is a topology on $\mathcal{P}$, called the *Alexandrov topology.*

We denote by $\mathcal{P}_{\mathfrak{a}}$ the topological space $\mathcal{P}$ endowed with the Alexandrov topology. For $x \in \mathcal{P}$, one defines $\mathscr{D}(x) = \{y \in \mathcal{P} \mid y \leq x\}$. It is the smallest open subset of $\mathcal{P}$ containing x. The family $(\mathscr{D}(x))_{x\in\mathcal{P}}$ is a basis for the Alexandrov topology of $\mathcal{P}$, since every lower set in $\mathcal{P}$ contains $\mathscr{D}(x)$ for each x therein.

Proposition 2.2.2
Let $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \leq')$ be two posets. A map $f : \mathcal{P} \to \mathcal{Q}$ is continuous for the Alexandrov topologies of $\mathcal{P}$ and $\mathcal{Q}$ if and only it is order preserving.

Let M be a persistence module over $(\mathcal{P}, \leq)$. One denotes by $(\mathcal{P}^{\mathrm{op}}, \leq^{\mathrm{op}})$ the poset whose underlying set is $\mathcal{P}$, and whose order is defined by $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$. A basis for the Alexandrov topology of $(\mathcal{P}^{\mathrm{op}}, \leq^{\mathrm{op}})$ is given by the family of upper sets $\mathscr{U}(x) = \{y \in \mathcal{P} \mid x \leq y\}$. Moreover, this family satisfies:

$$x \leq y \iff \mathscr{U}(y) \subset \mathscr{U}(x).$$

One defines $\theta^{-1}M$ as the object of $\mathrm{Mod}(\mathbf{k}_{\mathcal{P}^{\mathrm{op}}_{\mathfrak{a}}})$ defined on the basis $(\mathscr{U}(x))_{x\in\mathcal{P}}$ by:

$$\theta^{-1}M(\mathscr{U}(x)) = M(x),$$

$$\theta^{-1}M(\mathscr{U}(x)) \to \theta^{-1}M(\mathscr{U}(y)) = M(x \leq y),$$

for any $x \leq y$ in $\mathcal{P}$. We can extend θ^{-1} as a well defined functor from $\mathrm{Pers}(\mathbf{k}^{\mathcal{P}})$ to $\mathrm{Mod}(\mathbf{k}_{\mathcal{P}^{\mathrm{op}}_{\mathfrak{a}}})$.

Proposition 2.2.3 ([KS18a] - prop. 1.15)
The functor $\theta^{-1} : \mathrm{Pers}(\mathbf{k}^{\mathcal{P}}) \to \mathrm{Mod}(\mathbf{k}_{\mathcal{P}_a^{op}})$ is an equivalence of categories.

Therefore, one can treat persistence modules as sheaves on the Alexandrov topology. This point of view will be studied in details in chapter 5.

2.2.2 The convolution distance

In this section, we make a short review of the concepts introduced in [KS18a]. The framework is the study of sheaves on a real vector space $\mathbb{V}$ of finite dimension n equipped with a norm $\|\cdot\|$. For two such sheaves, one can define their convolution, which, as the name suggests, will be at the core of the definition 2.2.8 of the convolution distance.

The construction of the convolution of sheaves is as follows. Consider the following maps (addition and the canonical projections):

$$s : \mathbb{V} \times \mathbb{V} \to \mathbb{V}, \quad s(x,y) = x + y$$
$$q_i : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \;\; (i = 1,2) \quad q_1(x,y) = x, \;\; q_2(x,y) = y$$

Definition 2.2.4
For $F, G \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_{\mathbb{V}}))$, define the *convolution of F and G* by the formula:

$$F \star G = \mathrm{R}s_!(F \boxtimes G).$$

Where $F \boxtimes G := q_1^{-1}F \otimes_{\mathbf{k}_{\mathbb{V}\times\mathbb{V}}} q_2^{-1}G$ is the *external tensor product* of F and G. This defines a bi-functor $-\star- : \mathrm{D}^b(\mathbf{k}_{\mathbb{V}}) \times \mathrm{D}^b(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^b(\mathbf{k}_{\mathbb{V}})$. In the following, we will be interested in a more specific case: the convolution will be considered with one of the sheaves being the constant sheaf supported on a ball centered at 0. For $r > 0$, we denote $B_r := \{x \in \mathbb{V} \mid \|x\| \leq r\}$ the closed ball of radius r centered at 0, and $\overset{\circ}{B}_r$ its interior, that is, the open ball of radius r centered at 0. For $\varepsilon \in \mathbb{R}$ we define:

$$K_\varepsilon := \begin{cases} \mathbf{k}_{B_\varepsilon} \text{ if } \varepsilon \geq 0 \\ \mathbf{k}_{\overset{\circ}{B}_{-\varepsilon}}[n] \text{ if } \varepsilon < 0, \end{cases} \tag{2.7}$$

with $\mathbf{k}_{\overset{\circ}{B}_{-\varepsilon}}[n]$ seen as a complex concentrated in degree $-n$ (recall that n is the dimension of $\mathbb{V}$). We have the following properties:

Proposition 2.2.5
Let $\varepsilon, \varepsilon' \in \mathbb{R}$ and $F \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_{\mathbb{V}}))$.

1. There are functorial isomorphisms $(F \star K_\varepsilon) \star K_{\varepsilon'} \simeq F \star K_{\varepsilon+\varepsilon'}$ and $F \star K_0 \simeq F$.

2. If $\varepsilon \geq \varepsilon'$, there is a canonical morphism $K_\varepsilon \to K_{\varepsilon'}$ in $\mathrm{D}^b(\mathbf{k}_\mathbb{V})$ inducing a natural transformation $F \star K_\varepsilon \to F \star K_{\varepsilon'}$. In the special case where $\varepsilon' = 0$, we shall write $\phi_{F,\varepsilon}$ for this natural transformation.
3. The canonical morphism $F \star K_\varepsilon \to F \star K_{\varepsilon'}$ induces an isomorphism $\mathrm{R}\Gamma(\mathbb{V}; F \star K_\varepsilon) \tilde{\to} \mathrm{R}\Gamma(\mathbb{V}; F \star K'_\varepsilon)$.

Proof

We prove the third point, whose proof is omitted in [KS18a]. By 1., it is sufficient to prove the result for $\varepsilon' = 0$.

Let $a_1 : \mathbb{V} \to \mathrm{pt}$ and $a_2 : \mathbb{V} \times \mathbb{V} \to \mathrm{pt}$. As s is proper on $\mathrm{supp}(F) \times B_\varepsilon$, one has $\mathrm{R}s_!(F \boxtimes K_\varepsilon) \simeq \mathrm{R}s_*(F \boxtimes K_\varepsilon)$. Moreover, since $a_1 \circ s = a_2$, we have the isomorphisms :

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{V}; F \star K_\varepsilon) &\simeq \mathrm{R}a_{1*}\mathrm{R}s_*(F \boxtimes K_\varepsilon) \\ &\simeq \mathrm{R}a_{2*}(F \boxtimes K_\varepsilon) \end{aligned}$$

Hence, we are only left to prove the isomorphism $\mathrm{R}\Gamma(\mathbb{V} \times \mathbb{V}; F \boxtimes K_\varepsilon) \simeq \mathrm{R}\Gamma(\mathbb{V}; F)$.

First, observe that :

$$\mathrm{R}\Gamma(\mathbb{V} \times \mathbb{V}; F \boxtimes K_\varepsilon) \simeq \mathrm{R}\Gamma(\mathbb{V} \times \overline{B_\varepsilon}; F \boxtimes K_\varepsilon).$$

Let $f : \mathbb{V} \times \overline{B_\varepsilon} \to \mathbb{V}$ be the first coordinate projection. Since f is continuous, proper, and has contractible fibers, Cor 2.77 (iv) from [KS90] applies, and we have that the map $F \to \mathrm{R}f_* f^{-1}(F) \simeq \mathrm{R}f_*((F \boxtimes K_\varepsilon)|_{\mathbb{V} \times \overline{B_\varepsilon}})$ is an isomorphism. We conclude by taking global sections.

□

In particular, Proposition 2.2.5.(1) implies that any map $f : F \star K_\varepsilon \to G$ induces canonical maps

$$f \star K_\tau : F \star K_{\varepsilon+\tau} \simeq F \star K_\varepsilon \star K_\tau \to G \star K_\tau. \tag{2.8}$$

The following definition is central.

Definition 2.2.6

For $F, G \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_\mathbb{V}))$ and $\varepsilon \geq 0$, one says that F and G are ε-*interleaved* if there exists two morphisms in $\mathrm{D}^b(\mathbf{k}_\mathbb{V})$, $f : F \star K_\varepsilon \to G$ and $g : G \star K_\varepsilon \to F$ such that the compositions $F \star K_{2\varepsilon} \xrightarrow{f \star K_\varepsilon} G \star K_\varepsilon \xrightarrow{g} F$ and $G \star K_{2\varepsilon} \xrightarrow{g \star K_\varepsilon} F \star K_\varepsilon \xrightarrow{f} G$ are the natural morphisms $F \star K_{2\varepsilon} \xrightarrow{\phi_{F,2\varepsilon}} F$ and $G \star K_{2\varepsilon} \xrightarrow{\phi_{G,2\varepsilon}} G$, that is, we have a commutative diagram in $\mathrm{D}^b(\mathbf{k}_\mathbb{V})$:

$$\begin{tikzcd}
F \star K_{2\varepsilon} \arrow[r, "f \star K_\varepsilon"] \arrow[dr] \arrow[rr, bend left, "\phi_{F,2\varepsilon}"] & G \star K_\varepsilon \arrow[r, "g"] \arrow[dr] & F \\
G \star K_{2\varepsilon} \arrow[r, "g \star K_\varepsilon"] \arrow[ur] \arrow[rr, bend right, "\phi_{G,2\varepsilon}"'] & F \star K_\varepsilon \arrow[r, "f"] \arrow[ur] & G
\end{tikzcd}$$

In this case, we write $F \sim_\varepsilon G$.

Observe that F and G are 0-interleaved if and only if $F \simeq G$.

Remark 2.2.7
One must be aware that in [KS18a], the authors call this data an ε-isomorphism. Here, we choose to follow the usual terminology of persistence theory.

Since 0-interleavings are isomorphisms, the existence of an ε-interleaving between two sheaves expresses a notion of closeness. This leads the authors of [KS18a] to define the convolution distance as follows:

Definition 2.2.8
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_\mathbb{V}))$. Their *convolution distance* is the possibly infinite real number:

$$d_C(F,G) := \inf\left(\{a \in \mathbb{R}_{\geq 0} \mid F \text{ and } G \text{ are } a\text{-interleaved}\}\right)$$

Proposition 2.2.9
The convolution distance satisfies for $F, G, H \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_\mathbb{V}))$:

1. $d_C(F,G) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$,
2. $d_C(F,G) = d_C(G,F)$,
3. $d_C(F,G) \leq d_C(F,H) + d_C(H,G)$.

That is, it is an extended pseudo-distance on $\mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_\mathbb{V}))$.

Remark 2.2.10
In section 3.5.2, we will present an example of two sheaves $F, G \in \mathrm{D}^b(\mathbf{k}_\mathbb{R})$ such that $d_C(F,G) = 0$ but $F \not\simeq G$. We will also prove that if we assume F and G in $\mathrm{D}^b(\mathbf{k}_\mathbb{R})$ to be constructible, then $d_C(F,G) = 0$ implies $F \simeq G$.

The following proposition expresses that the functors $\mathrm{R}\Gamma(\mathbb{V}, -)$ and $\mathrm{R}\Gamma_c(\mathbb{V}, -)$ define some necessary conditions for two sheaves to be at finite distance.

Proposition 2.2.11 (Remark 2.5 - [KS18a])
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_{\mathbb{V}}))$.

1. If $d_C(F,G) < +\infty$ then $\mathrm{R}\Gamma(\mathbb{V}, G) \simeq \mathrm{R}\Gamma(\mathbb{V}; F)$ and $\mathrm{R}\Gamma_c(\mathbb{V}; G) \simeq \mathrm{R}\Gamma_c(\mathbb{V}; F)$.
2. If $\mathrm{supp}(F), \mathrm{supp}(G) \subset B_a$ then $d_C(F,G) \leq 2a$ if and only if $\mathrm{R}\Gamma(\mathbb{V}; G) \simeq \mathrm{R}\Gamma(\mathbb{V}; F)$.

There is a fundamental example that plays a central role in our work. Given X a topological space and $u : X \to \mathbb{V}$ a continuous map, one can consider the sheaves $\mathrm{R}u_*\mathbf{k}_X$ and $\mathrm{R}u_!\mathbf{k}_X$. Roughly speaking, and under some smoothness assumptions on X and f, they contain the information on how the cohomologies of the fibers of u evolve when moving on $\mathbb{V}$. For this information to be meaningful in practice, it has to be stable when we perturb u, that is, $\mathrm{R}u_*\mathbf{k}_X$ must stay in a neighborhood in the sense of the convolution distance, controlled by the size of the perturbation of u. This is expressed by the following theorem, which is the analogue of the stability theorem from persistence theory.

Theorem 2.2.12 ([KS18a] - Thm. 2.7)
Let X be a locally compact topological space, and $u, v : X \to \mathbb{V}$ two continuous functions. Then for any $F \in \mathrm{Obj}(\mathrm{D}^b(\mathbf{k}_{\mathbb{V}}))$ one has:

$$d_C(\mathrm{R}u_*F, \mathrm{R}v_*F) \leq \|u - v\| \quad \text{and} \quad d_C(\mathrm{R}u_!F, \mathrm{R}v_!F) \leq \|u - v\|$$

where $\|u - v\| = \sup_{x \in X} \|u(x) - v(x)\|$.

2.3 Stable resolutions of persistence modules

This section presents the content of [Ber19]. We shall introduce the notion of graded-Betti numbers for a multi-parameter persistence modules, a candidate as an invariant for multi-parameter persistence. While informative about the algebraic structure of a given persistence modules, we will explain why this information is not stable with respect to the interleaving distance in a suitable way. Therefore, it cannot be easily used to extract topological information from persistence modules arising from noisy datasets. We aim at finding the algebraic level at which the information contained in the graded-Betti numbers satisfies a form of stability with respect to the interleaving distance. Graded-Betti numbers are obtained as the graded rank of free minimal resolutions. We will prove here that it is possible to equip the homotopy category of persistence modules with an interleaving distance so that resolution functors (see def. 2.3.1) are always isometric.

From theorem 2.1.39, the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ is equivalent to a category of modules. As a consequence, it has enough projectives. Given $\mathscr{C}$ an additive category, we denote by $K(\mathscr{C})$ the homotopy category of $\mathscr{C}$, and $K^-(\mathscr{C})$ its full subcategory of complexes bounded from above (see definition A.1.17). For any object M in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ there exists a complex of projective modules $P(M)$ in $K^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and a quasi-isomorphism $P(M) \to M$, where M is seen as a complex concentrated in degree 0. Let $\mathscr{P}$ be the full subcategory of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ consisting of projective objects. The mapping $M \mapsto P(M)$ defines a fully-faithful functor $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}) \to K^-(\mathscr{P})$, which we call a projective resolution functor. More precisely:

Definition 2.3.1

A *projective resolution functor* of an abelian category $\mathscr{C}$ is the following data:

1. for all objects X in $K^-(\mathscr{C})$, a projective resolution $j(X)$,
2. for all objects X in $K^-(\mathscr{C})$, a quasi-isomorphism $i_X : j(X) \to X$.

Lemma 13.23.3 in [Pro] states that the data of a resolution functor induces a unique functor $j : K^-(\mathscr{C}) \to K^-(\mathscr{P})$, where $\mathscr{P}$ is the full subcategory of projective objects of X, together with a unique 2-isomorphism, such that the following diagram is 2-commutative :

$$\begin{tikzcd} & \mathrm{D}^-(\mathscr{C}) & \\ K^-(\mathscr{C}) \arrow[ur] \arrow[rr, "j"'] \arrow[rr, phantom, shift left=4, "\Uparrow\simeq"] & & K^-(\mathscr{P}) \arrow[ul] \end{tikzcd}$$

Therefore in our particular situation, we have the following diagram of categories, which commutes up to isomorphism of functors:

$$\begin{tikzcd} & \mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}) \arrow[dl, "P"'] \arrow[dr, "\iota"] & \\ K^-(\mathscr{P}) \arrow[rr, "\sim", "Q"'] & & D^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})) \end{tikzcd}$$

where P is a projective resolution functor restricted to the full subcategory of $K^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ consisting of complexes concentrated in degree 0 (which we identify with $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$), Q is the localization functor (def.

A.1.28) and ι is the functor which sends a persistence modules to the associated complex concentrated in degree 0. Following section 2.1.4, the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ can be endowed with an interleaving distance which is universal with respect to sublevel-sets filtration of functions. In this section, our aim will be to define interleaving distances on the categories $K^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and $D^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$, so that the functors P, Q and ι become isometries.

2.3.1 Graded-Betti numbers of persistence modules

In this section, we start by giving an abstract definition of the graded-Betti numbers associated to a persistence module over $\mathbb{R}^d$. In the case where a persistence module over $\mathbb{R}^d$ admits a minimal finite free resolution, we interpret the graded-Betti numbers in terms of the graded rank function at each step of this resolution. The conditions for the existence of minimal free resolutions of persistence over $\mathbb{R}^d$ is still an open question, as is the right notion of minimality in a continuous setting. It seems reasonable to think that finitely presented persistence modules over $\mathbb{R}^d$ admit minimal free resolutions in the sense of definition 2.3.2. Although, we do not provide a proof of this statement as it is very technical, and our focus here is not about the computation, but the stability, of invariants of persistence modules.

Definition 2.3.2
Let $I = \langle x^e \mid e \in \mathbb{R}^d_{>0} \rangle$ be the maximal graded ideal of $\mathbf{k}\{x_1, ..., x_d\}$, $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$, and (P, ∂) be a projective resolution of M. Then (P, ∂) is said to be a *minimal resolution* if for any $j \in \mathbb{Z}_{<0}$, $\mathrm{im}\, \alpha(\partial_j) \subset I \cdot \alpha(P_{j+1})$. With α as in theorem 2.1.39.

Definition 2.3.3
Let M be a persistence module over $\mathbb{R}^d$. Given $n \in \mathbb{Z}_{\geq 0}$, define its *i-th graded-Betti number* to be the function $\beta_n(M) : \mathbb{R}^d \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that for $s \in \mathbb{R}^d$:

$$\beta_n(M)(s) = \dim_{\mathbf{k}} \left(\mathrm{Tor}_n \left(\alpha(M), \mathbf{k}\{x_1, ..., x_d\}/I \cdot \mathbf{k}\{x_1, ..., x_d\}\right)_s\right).$$

Given $a \in \mathbb{R}^d$, we will denote by F_a the persistence module over $\mathbb{R}^d$ defined by, for $s \leq t$ in $\mathbb{R}^d$:

$$F_a(s) = \begin{cases} \mathbf{k} & \text{if } \ a \leq s \\ 0 & \text{else} \end{cases} \qquad F_a(s \leq t) = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } \ a \leq s \\ 0 & \text{else} \end{cases}$$

Definition 2.3.4
A *finite free persistence module* over $\mathbb{R}^d$ is a persistence module M over $\mathbb{R}^d$ such that there exists a function with finite support $\xi(M) : \mathbb{R}^d \to \mathbb{Z}_{\geq 0}$ such that:

$$M \simeq \bigoplus_{i \in \mathbb{R}^d} F_i^{\oplus \xi(M)(i)},$$

where $F_i^{\oplus \xi(M)(i)}$ corresponds to the direct sum of $\xi(M)(i)$ copies of F_i.

Remark 2.3.5

1. The name "finite free" module is coherent with the classical notion of free modules through the functor α (see section 2.1.4): indeed, M is free in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ if and only if $\alpha(M)$ is a finitely generated free graded-module. In particular, they are projective objects of the category $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$.
2. By the Krull-Schmidt theorem for pfd persistence modules (thm. 2.1.7), the function $\xi(M)$ is well-defined for each M, since $F_a \simeq F_b$ if and only if $a = b$.

Proposition 2.3.6
Let M be a persistence module over $\mathbb{R}^d$. Assume that M admits a minimal projective resolution such (P, ∂) such that for all $j \in \mathbb{Z}_{<0}$, P_j is a finite free persistence module. Then with the notations of definition 2.3.4, we have for all $n \in \mathbb{Z}_{\geq 0}$:

$$\beta_n(M) = \xi(P_{-n}).$$

Proof
We shall use the notation $B_d = \mathbf{k}\{x_1, ..., x_d\}$. By definition:

$$\mathrm{Tor}_n^{B_d}(\alpha(M), B_d/I \cdot B_d) \simeq \mathrm{H}_n(\alpha(P) \otimes_{B_d} B_d/I \cdot B_d).$$

By minimality of P, the complex $\alpha(P) \otimes_{B_d} B_d/I \cdot B_d$ has differentials which are worth 0. This ends the proof.

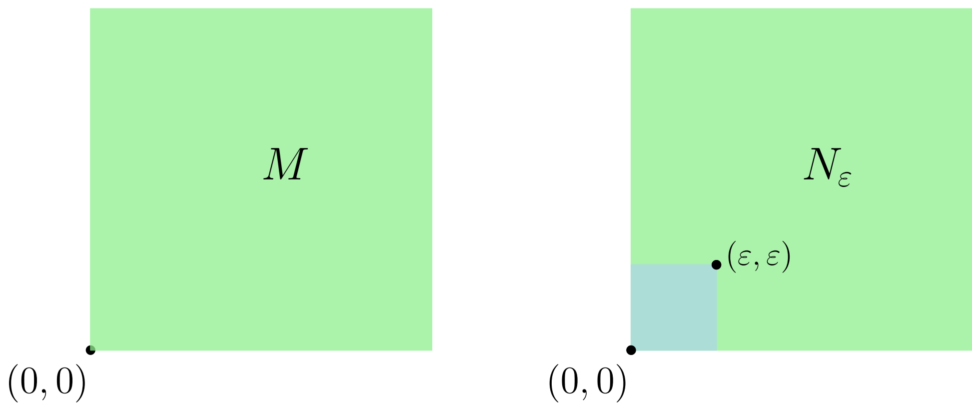
$\square$

2.3.2 A counter-example to naive stability of graded-Betti numbers

In the following, we propose to show that, although informative about the algebraic structure of a persistence module, graded-Betti numbers do not satisfy a naive form of stability with respect to the interleaving distance. Recall that we have defined an interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ in section 2.1.4. We can find a persistence module M and a family of persistence modules N_ε such that the interleaving distance between M and N_ε goes to 0 as ε goes to 0, and the graded-Betti numbers of N_ε

Consider $\varepsilon \geq 0$, M and N_ε the persistence modules over $\mathbb{R}^2$ defined by:

$$M = \mathbf{k}^{\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}} \quad \text{and} \quad N_\varepsilon = M \oplus \mathbf{k}^{[0,\varepsilon[^2}.$$



Observe that the interleaving distance between M and N_ε is $\frac{\varepsilon}{2}$. Since M is a free module (see definition 2.3.4), it is its own minimal free resolution. Hence, $\beta_0(M)(x) = 1$ for $x = (0,0)$ and $\beta_0(M)(x) = 0$ otherwise.

Also, N_ε has one more generator at $(0,0)$, thus $\beta_0(N_\varepsilon)(x) = 2$ for $x = (0,0)$ and $\beta_0(N)(x) = 0$ otherwise.

Therefore, for any $\varepsilon > 0$:

$$d_I(M, N_\varepsilon) = \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} |\beta_0(M)(x) - \beta_0(N_\varepsilon)(x)| \geq 1.$$

This very simple example shows that graded-Betti numbers are extremely sensitive to noise. An arbitrary small perturbation of the input persistence modules can lead to an arbitrary change in the graded-Betti numbers. In the rest of this section, our aim will be to show that it is possible to equip the homotopy (or derived) category of persistence modules over $\mathbb{R}^d$ with an interleaving distance. Taking homotopy into accounts, we will show that all projective resolution functors are isometric, in particular the minimal free resolution functor from which the graded-Betti numbers are computed. This result (thm. 2.3.15) opens the door to obtaining new stable homological invariants for multi-parameter persistence.

2.3.3 Homotopy and derived interleavings

For the usual flow T on $\mathbb{R}^d$ (section 2.1.4), the ε-shift functor along T (see equation 2.1.10) extends readily for negative values of ε. Recall that we denote $M \circ T(\varepsilon)$ by $M[\varepsilon]_T$, and shall omit T to simplify notations. We have for any $\varepsilon \in \mathbb{R}$ and persistence module M:

$$M[\varepsilon](s) = M\big(s + (\varepsilon, ..., \varepsilon)\big),$$

$$M[\varepsilon](s \leq t) = M\big(s + (\varepsilon, ..., \varepsilon) \leq t + (\varepsilon, ..., \varepsilon)\big),$$

for $s \leq t$ in $\mathbb{R}^d$.

From the obvious computation:

$$\cdot[\varepsilon] \circ \cdot[-\varepsilon] = \cdot[-\varepsilon] \circ \cdot[\varepsilon] = \mathrm{id}_{\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})},$$

one deduces the following.

Proposition 2.3.7
Let $\varepsilon \in \mathbb{R}$.

1. The functor $\cdot[\varepsilon]$ is exact.
2. The functor $\cdot[\varepsilon]$ sends projective objects to projective object.

Therefore, $\cdot[\varepsilon]$ preserves quasi-isomorphisms and induces a well-defined endofunctor of $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$, by applying $\cdot[\varepsilon]$ degree-wise to a complex. One shall still denote $\cdot[\varepsilon]$ the induced functor on $C(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ (the category of chain complexes of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ - see definition A.1.13), $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$. Let X be an object of $C(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and $0 \leq \varepsilon' \leq \varepsilon$. It is immediate to verify that the collection $(\tau^{X^i}_{\varepsilon',\varepsilon})_{i\in\mathbb{Z}}$ defines a homomorphism

$$\tau^X_{\varepsilon',\varepsilon} \in \mathrm{Hom}_{C(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))}(X[\varepsilon'], X[\varepsilon]).$$

We shall denote by $[\tau^X_\varepsilon]$ (resp. $\{\tau^X_\varepsilon\}$) the image of τ^X_ε in $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ (resp. $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$).

Proposition 2.3.8
Let X be an object of $C(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and $0 \leq \varepsilon'' \leq \varepsilon' \leq \varepsilon$. Then:

$$[\tau^X_{\varepsilon',\varepsilon}] \circ [\tau^X_{\varepsilon'',\varepsilon'}] = [\tau^X_{\varepsilon'',\varepsilon}] \quad \text{and} \quad \{\tau^X_{\varepsilon',\varepsilon}\} \circ \{\tau^X_{\varepsilon'',\varepsilon'}\} = \{\tau^X_{\varepsilon'',\varepsilon}\}.$$

Proof
The first equality is a consequence of the computation:

$$\tau^X_{\varepsilon',\varepsilon} \circ \tau^X_{\varepsilon'',\varepsilon'} = \tau^X_{\varepsilon'',\varepsilon}.$$

We deduce the second equality from the first one, by the exactness of the functor $\cdot[\varepsilon]$.

□

Definition 2.3.9
Let X, and Y be two objects of $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$. An *homotopy ε-interleaving* between X and Y is the data of two morphisms of $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$, $f : X \to Y[\varepsilon]$ and $g : X \to Y[\varepsilon]$ such that the following diagram commutes in $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] \arrow[rr, bend left, "{[\tau^X_{2\varepsilon}]}"] & Y[\varepsilon] \arrow[r, "{g[\varepsilon]}"] \arrow[dr] & X[2\varepsilon] \\
Y \arrow[r, "g"] \arrow[ur] \arrow[rr, bend right, "{[\tau^Y_{2\varepsilon}]}"'] & X[\varepsilon] \arrow[r, "{f[\varepsilon]}"] \arrow[ur] & Y[2\varepsilon]
\end{tikzcd}$$

If such a diagram exists, we say that X and Y are *homotopically ε-interleaved* and write $X \sim^K_\varepsilon Y$. If the context is clear, we will often drop “homotopically”.

Definition 2.3.10
Let $X, Y \in K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$. One defines their *homotopy interleaving distance* (interleaving distance when no confusion is possible) to be the possibly infinite number:

$$d^K_I(X,Y) = \inf\{\varepsilon \in \mathbb{Z}_{\geq 0} \mid X \sim^K_\varepsilon Y\}$$

Proposition 2.3.11
The interleaving distance d^K_I is an extended pseudo-distance on $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$.

Proof
For X an object of $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$, we have $d^K_I(X,X) = 0$ since isomorphisms are 0-interleavings. The triangle inequality is an easy consequence of proposition 2.3.8.

□

We proceed in the same way to define an interleaving distance on $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$.

Definition 2.3.12
Let $X, Y \in \mathrm{Obj}(\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})))$. A *derived ε-interleaving* between X and Y is the data of two derived morphisms $f : X \to Y[\varepsilon]$ and $g : X \to Y[\varepsilon]$ such that the following diagram commutes in $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] \arrow[rr, bend left, "\{\tau_{2\varepsilon}^X\}"] & Y[\varepsilon] \arrow[r, "g[\varepsilon]"] \arrow[dr] & X[2\varepsilon] \\
Y \arrow[r, "g"] \arrow[ur] \arrow[rr, bend right, "\{\tau_{2\varepsilon}^Y\}"'] & X[\varepsilon] \arrow[r, "f[\varepsilon]"] \arrow[ur] & Y[2\varepsilon]
\end{tikzcd}$$

If such a diagram exists, we say that X and Y are *derived ε-interleaved* and write $X \sim_{\varepsilon}^{D} Y$. We will drop derived"if the context is clear.

Definition 2.3.13
Let X and Y be two objects of $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$. Define their *derived interleaving distance* (or interleaving distance when no confusion is possible) to be the possibly infinite number:

$$d_I^D(X, Y) = \inf\{\varepsilon \in \mathbb{Z}_{\geq 0} \mid X \sim_{\varepsilon}^{D} Y\}$$

Proposition 2.3.14
The interleaving distance d_I^D is an extended pseudo-distance on $\mathrm{D}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$.

2.3.4 Distance comparisons

We can now state the distance comparison theorem, that we will prove in this subsection.

Theorem 2.3.15 ([Ber19])
Let P be a projective resolution functor on $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$, Q be the localization functor and ι the fully-faithful embedding of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ in its derived category. The following diagram of categories is commutative, and all functors are isometries.

$$\begin{tikzcd}
 & (\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}), d_I) \arrow[dl, "P"'] \arrow[dr, "\iota"] & \\
(K^-(\mathscr{P}), d_I^K) \arrow[rr, "\sim", "Q"'] & & (D^-(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})), d_I^D)
\end{tikzcd}$$

We recall the following important proposition about projective resolution functors (definition 2.3.1).

Proposition 2.3.16 ([Pro] - Lem. 13.23.4)
Assume $\mathscr{C}$ is an abelian category with enough projectives. Then there exists a projective resolution functor of $\mathscr{C}$. Moreover, for any two such functors P, P' there exists a unique isomorphism of functors $P \simeq P'$.

The functor $P(- \cdot [\varepsilon])[-\varepsilon]$ is a resolution functor. By proposition 2.3.16 for any $\varepsilon, \varepsilon' \geq 0$, there exists a unique isomorphism of functors

$$\chi_{\varepsilon,\varepsilon'} : P(- \cdot [\varepsilon])[-\varepsilon] \overset{\sim}{\Rightarrow} P(- \cdot [\varepsilon'])[-\varepsilon'].$$

By the uniqueness of $\chi_{\varepsilon,\varepsilon'}$, we also deduce the identity:

$$\chi_{\varepsilon',\varepsilon''} \circ \chi_{\varepsilon,\varepsilon'} = \chi_{\varepsilon,\varepsilon''}. \tag{2.9}$$

Lemma 2.3.17
Let $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$ and $\varepsilon \geq 0$. Then:

$$P(\tau_\varepsilon^M) = \chi_{\varepsilon,0}(M)[\varepsilon] \circ \left[\tau_\varepsilon^{P(M)}\right].$$

Consequently:

$$\mathrm{H}^0\left(\left[\tau_\varepsilon^{P(M)}\right]\right) = \tau_\varepsilon^M.$$

Proof
Consider a chain morphism $\varphi : P(M)[\varepsilon] \to P(M[\varepsilon])$ such that $[\varphi] = \chi_{\varepsilon,0}(M)[\varepsilon]$. One has the following commutative diagram in $C(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$:

$$\begin{tikzcd}
\dots \arrow[r] & P^i(M) \arrow[r] \arrow[d, "\tau_\varepsilon^{P^i(M)}"] & \dots \arrow[r] & P^1(M) \arrow[r] \arrow[d, "\tau_\varepsilon^{P^1(M)}"] & P^0(M) \arrow[r] \arrow[d, "\tau_\varepsilon^{P^0(M)}"] & M \arrow[r] \arrow[d, "\tau_\varepsilon^M"] & 0 \\
\dots \arrow[r] & P^i(M)[\varepsilon] \arrow[r] \arrow[d, "\varphi^i"] & \dots \arrow[r] & P^1(M)[\varepsilon] \arrow[r] \arrow[d, "\varphi^1"] & P^0(M)[\varepsilon] \arrow[r] \arrow[d, "\varphi^0"] & M[\varepsilon] \arrow[r] \arrow[d, "\mathrm{id}_{M[\varepsilon]}"] & 0 \\
\dots \arrow[r] & P^i(M[\varepsilon]) \arrow[r] & \dots \arrow[r] & P^1(M[\varepsilon]) \arrow[r] & P^0(M[\varepsilon]) \arrow[r] & M[\varepsilon] \arrow[r] & 0
\end{tikzcd}$$

Therefore, $\varphi \circ \tau_\varepsilon^{P(M)}$ is one lift of τ_ε^M, which by characterization of lifts of morphism to projective resolutions proves that

$$P(\tau_\varepsilon^M) = \left[\varphi \circ \tau_\varepsilon^{P(M)}\right] = [\varphi] \circ \left[\tau_\varepsilon^{P(M)}\right] = \chi_{\varepsilon,0}(M)[\varepsilon] \circ \left[\tau_\varepsilon^{P(M)}\right].$$

Lemma 2.3.18 (Homotopy comparison)
Let M and N be two persistence modules in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$ and $\varepsilon \geq 0$.

1. If M and N are ε-interleaved with respect to $f : M \to N[\varepsilon]$ and $g : N \to M[\varepsilon]$ in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$, then $P(M)$ and $P(N)$ are ε-interleaved with respect to $\chi_{\varepsilon,0}(N)[\varepsilon] \circ P(f)$ and $\chi_{\varepsilon,0}(M)[\varepsilon] \circ P(g)$ in $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$.
2. Conversely, if $P(M)$ and $P(N)$ are ε-interleaved with respect to $\alpha : P(M) \to P(N)[\varepsilon]$ and $\beta : P(N) \to P(M)[\varepsilon]$ in $K(\mathrm{Pers}(\mathbb{R}^d))$, then M and N are ε-interleaved with respect to $\mathrm{H}^0(\alpha)$ and $\mathrm{H}^0(\beta)$ in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$.

Proof

1. Applying the functor P to the interleaving diagram gives:

$$\begin{tikzcd} P(M) \arrow[r, "P(f)"] \arrow[rr, bend left, "P(\tau_{2\varepsilon}^M)"] & P(N[\varepsilon]) \arrow[r, "P(g[\varepsilon])"] & P(M[2\varepsilon]) \end{tikzcd}$$

$$\begin{tikzcd} P(N) \arrow[r, "P(g)"] \arrow[rr, bend right, "P(\tau_{2\varepsilon}^N)"'] & P(M[\varepsilon]) \arrow[r, "P(f[\varepsilon])"] & P(N[2\varepsilon]) \end{tikzcd}$$

Therefore we have a commutative diagram in $K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}))$:

$$\begin{tikzcd} P(M) \arrow[r, "P(f)"] \arrow[rr, bend left, "P(\tau_{2\varepsilon}^M)"] & P(N[\varepsilon]) \arrow[r, "P(g[\varepsilon])"] \arrow[d, "\chi_{\varepsilon,0}(N)[\varepsilon]"', "\simeq"] & P(M[2\varepsilon]) \arrow[d, "\simeq"', "\chi_{2\varepsilon,0}(M)[2\varepsilon]"] \\ & P(N)[\varepsilon] \arrow[r, "\chi_{2\varepsilon,0}(M)[2\varepsilon] \circ P(g[\varepsilon]) \circ (\chi_{\varepsilon,0}(N)[\varepsilon])^{-1}"'] & P(M)[2\varepsilon] \end{tikzcd}$$

We introduce the morphisms

$$\tilde{f} = \chi_{\varepsilon,0}(N)[\varepsilon] \circ P(f) : P(M) \to P(N)[\varepsilon],$$

$$\tilde{g} = \chi_{\varepsilon,0}(M)[\varepsilon] \circ P(g) : P(N) \to P(M)[\varepsilon].$$

One has the following computations:

$$\tilde{g}[\varepsilon] = \chi_{2\varepsilon,0}(M)[2\varepsilon] \circ P(g[\varepsilon]) \circ (\chi_{\varepsilon,0}(N)[\varepsilon])^{-1}, \tag{2.10}$$

$$\mathrm{H}^0\left(\tilde{g}[\varepsilon]\circ\tilde{f}\right)=\tau_{2\varepsilon}^M. \tag{2.11}$$

From equation 2.11 and lemma 2.3.17, we deduce:

$$\tilde{g}[\varepsilon]\circ\tilde{f}=[\tau_{2\varepsilon}^{P(M)}]. \tag{2.12}$$

Equations 2.10,2.11,2.12 also hold when intertwining f and g, M and N, $\tilde{f}$ and $\tilde{g}$. Therefore, $\tilde{f}$ and $\tilde{g}$ define a homotopy ε-interleaving between $P(N)$ and $P(M)$.

2. The converse of the theorem is obtained applying H^0 to the interleaving diagram, and lemma 2.3.17 according to which $\mathrm{H}^0([\tau_\varepsilon^{P(M)}])=\tau_\varepsilon^M$.

Corollary 2.3.19
The functor $P:(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d}),d_I)\to(K(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})),d_I^K)$ is distance preserving.

Proof (of theorem 2.3.15)
There only remains to prove that ι and Q are distance preserving.

Since by definition, $\iota(\tau_\varepsilon^M)=\{\tau_\varepsilon^M\}$, ι sends ε-interleavings to ε-interleavings. Conversely, assume that $\iota(M)$ and $\iota(N)$ are ε-interleaved. Then since ι is fully-faithful, applying H^0 to the ε-interleaving in the derived category leads to a ε-interleaving in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^d})$.

By definition, for X an object of $K^-(\mathscr{P})$, $Q([\tau_\varepsilon^X])=\{\tau_\varepsilon^X\}$, which shows that Q preserves interleavings. Now assume that $Q(X)$ and $Q(Y)$ are ε-interleaved. Since Q is fully-faithful and $Q([\tau_\varepsilon^X])=\{\tau_\varepsilon^X\}$, one deduces the existence of a ε-interleaving between X and Y in $K^-(\mathscr{P})$.

2.3.5 Computations and discussions

In this subsection, we develop the motivating example introduced in section 2.3.2 with the persistence modules over $\mathbb{R}^2$, M and N_ε. Using notations of section 2.3.1, the minimal finite free resolutions of M and N_ε are given, up to isomorphism, by the following:

$$\pi(M)\simeq 0\longrightarrow 0\longrightarrow 0\longrightarrow F_{(0,0)}\longrightarrow 0$$

$$\pi(N_\varepsilon)\simeq 0\longrightarrow F_{(\varepsilon,\varepsilon)}\xrightarrow{\begin{pmatrix}1\\-1\end{pmatrix}}F_{(\varepsilon,0)}\oplus F_{(0,\varepsilon)}\xrightarrow{\begin{pmatrix}1&-1\\0&0\end{pmatrix}}F_{(0,0)}^2\longrightarrow 0$$

Now, observe that for any $\eta>\frac{\varepsilon}{2}$, M and N_ε are η-interleaved with respect to the following morphisms:

$$M \xrightarrow{\begin{pmatrix} 0 \\ \tau_\eta^{F_{(0,0)}} \end{pmatrix}} N_\varepsilon[\eta]$$

$$N_\varepsilon \xrightarrow{\begin{pmatrix} 0 & \tau_\eta^{F_{(0,0)}} \end{pmatrix}} M[\eta]$$

However for every $i \in \{0, -1 -2\}$, $\pi^i(M)$ is not η-interleaved with $\pi^i(N_\varepsilon)$ in $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^2})$. Let us now construct a homotopy η-interleaving between $\pi(M)$ and $\pi(N_\varepsilon)$.

Define the following Koszul complex:

$$C_\varepsilon = 0 \longrightarrow F_{(\varepsilon,\varepsilon)} \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} F_{(\varepsilon,0)} \oplus F_{(0,\varepsilon)} \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} F_{(0,0)} \longrightarrow 0$$

And observe that since $\pi(N_\varepsilon) = \pi(M) \oplus C_\varepsilon$, it is sufficient to prove that C_ε is η-interleaved with 0 in $K(\mathrm{Pers}(\mathbb{R}^n))$. This is equivalent to proving that $[\tau_{2\eta}^{C_\varepsilon}] = 0$, that is, the chain map $\tau_{2\eta}^{C_\varepsilon} : C_\varepsilon \to C_\varepsilon[2\eta]$ is homotopic to 0.

Now define:

$$h^{-2} : C_\varepsilon^{-2} = F_{(\varepsilon,\varepsilon)} \xrightarrow{\quad 0 \quad} 0 = C_\varepsilon^{-3}[2\eta]$$

$$h^{-1} : C_\varepsilon^{-1} = F_{(\varepsilon,0)} \oplus F_{(0,\varepsilon)} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} F_{(\varepsilon-2\eta,\varepsilon-2\eta)} = C_\varepsilon^{-2}[2\eta]$$

$$h^{0} : C_\varepsilon^{0} = F_{(0,0)} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} F_{(\varepsilon-2\eta,-2\eta)} \oplus F_{(-2\eta,-2\eta)} = C_\varepsilon^{-1}[2\eta]$$

Note that h^{-1} and h^0 are well-defined only when $\eta \geq \frac{\varepsilon}{2}$. We have that $h = (h^i)$ defines an homotopy from $\tau_{2\eta}^{C_\varepsilon}$ to 0. That is, we have (with $h^i = 0$ for i different from $0, -1$):

$$\tau_{2\eta}^{C_\varepsilon} = d' \circ h + h \circ d,$$

where d stands for the differential of the complex C_ε and d' the differential of $C_\varepsilon[2\eta]$.

This example shows that, even in such a simple case, one cannot avoid taking into account homotopies in the problem of lifting interleavings to resolutions of persistence modules. Thus, to obtain homological invariants that are stable for persistence with multiple parameters, our work shows that a good algebraic framework is the homotopy category of persistence modules, equipped with the homotopy interleaving distance.

In [BL17], the authors prove that in the case of free persistence modules over $\mathbb{R}^2$, the interleaving distance can be computed exactly as a matching distance. This result has since been extended in [Bje16], where Bjerkevik proves an inequality bounding the bottleneck distance between two free persistence modules over $\mathbb{R}^n$ a multiple of their interleaving distance. One could ask whether it is possible to define a bottleneck distance between minimal free resolutions of two persistence modules (that would allow to match free indecomposables across degrees), and to bound this distance by a multiple of the homotopy interleaving distance. This would lead to a computable lower bound to the interleaving distance (which has been shown to be NP-hard to compute for persistence modules with more than one parameter in [BBK18]), not relying on any kind of decomposition theorems.

CHAPTER 3

The derived isometry theorem

Abstract

In this chapter, we prove an isometry theorem for derived constructible sheaves on the real line. That is, we express the convolution distance of sheaves as a matching distance between combinatorial objects associated to them that we call graded barcodes. This allows to consider sheaf-theoretical constructions as combinatorial, stable topological descriptors of data, and generalizes the situation of persistence with one parameter. As a byproduct of our isometry theorem, we prove that the convolution distance is closed between constructible sheaves on $\mathbb{R}$ and provide a counter-example without constructibility assumption, thus answering an open question of Kashiwara-Schapira in dimension one. We conjecture that this result extends to higher dimensional real vector-spaces. We also give a precise description of connected components of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. Furthermore, we provide some explicit examples of computation of the convolution distance. This chapter follows from a collaboration with Grégory Ginot [BG18].

3.1 Introduction and preliminaries

Persistence theory appeared in the early 2000's as an attempt to make some constructions inspired by Morse theory computable in practice. For instance, in the context of studying the underlying topology of a data set. It has since been widely developed and applied in many ways. We refer the reader to [Oud15, EH10] for extended expositions of the theory and of its applications. One promising expansion of the theory, initiated by Curry in his Ph.D. thesis [Cur14], is to combine the powerful theory of sheaves with computer-friendly ideas coming from persistence. However, sheaf theory takes its full strength in the derived setting and Kashiwara and Schapira developped persistent homology in this new framework in [KS18a]. It follows from theorems 3.1.2 and 3.1.4 that objects in the derived category of constructible sheaves on $\mathbb{R}$ admit a natural notion of barcode: a multi-set of intervals of $\mathbb{R}$ that entirely describes their isomorphism class. However, this barcode naturally comes with a grading (each cohomology object of a complex admits a barcode), leading to the notion of graded-barcodes. The aim of this chapter is to define a bottleneck distance between graded-barcodes and to prove an isometry theorem: the convolution distance between two complexes of sheaves is equal to the bottleneck distance between their graded-barcodes. This relates to classical one-dimensional persistence isometry theorem (theorem 2.1.22), with the particularity that our bottleneck distance heavily relies on the possibility of matching intervals across different degrees. The difficulty lies in the fact that although constructible sheaves on $\mathbb{R}$ have a graded description, there exists morphisms in the derived category between complexes concentrated in different degree.

The chapter is structured as follows:

Section 2 aims at introducing the mathematical context of the chapter.

Section 3 is dedicated to the complete description of the morphisms in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, the derived category of constructible sheaves on $\mathbb{R}$, and to compute the action of the convolution functor $- \star \mathrm{K}_\varepsilon$. To do so, we need to overcome the fact that the category of constructible sheaves on $\mathbb{R}$ does not have enough projective/injective. Hence to compute derived morphisms from F to G, we need to take a different type of resolutions for F and G. The tables of propositions 3.1 and 3.7, describing these homomorphisms, are the main output of this section, and may be of independent interest.

Section 4 describes the conditions for two indecomposable sheaves to be ε-close, and introduce the central-left-right decomposition for any sheaf $F \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$, which is adapted to the convolution distance in the following sense: two sheaves are ε-close with respect to

d_C if and only if their central (resp. left, resp. right) parts are.

Section 5 introduces rigorously the notion of graded-barcodes associated to an object of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, defines the appropriate notion of ε-matching between graded-barcodes, and then expresses the associated bottleneck distance d_B. We prove that this distance bounds the convolution distance: "$d_C \leq d_B$". We then prove that given an ε-interleaving between the central (resp. left, resp. right) parts of two sheaves, it induces an ε-matching between the graded-barcodes of their central (resp. left, resp. right) parts. We reduce the proof for left and right parts to the well-known case of one-parameter persistence modules by introducing fully faithful functors from sheaves supported on half-open intervals to persistence modules. The construction of the ε-matching between the central parts is far less direct. We adapt the proof by Bjerkevik [Bje16] to our setting, introducing a similar pre-order $\leq_\alpha$ on central parts, enabling us to "trigonalize" the interleaving morphisms. By a rank argument, this allows us to apply Hall's marriage theorem and to deduce the existence of an ε-matching. Note that our definition of $\leq_\alpha$ differs in nature from Bjerkevik's, for it enables us to compare elements of the graded-barcodes in *different degrees*. We conclude the section by proving the isometry theorem, which states that "$d_C = d_B$".

Section 6 provides some applications of the isometry theorem. We start by an example brought to our knowledge by Justin Curry and that motivated our work. Then, we prove that the convolution distance is closed (two constructible sheaves are ε-close if and only if they are ε-interleaved). We also provide a counter-example of two non-constructible sheaves which are non-isomorphic, and at distance 0 for the convolution distance - this answers in dimension one an open question of Kashiwara and Schapira in [KS18a]. Finally, we introduce the category of graded-barcodes, equipped with the graded-bottleneck distance, and prove that it is a locally path-connected metric space.

Preliminaries

Which metric for sheaves?

In [Cur14], Curry defined an interleaving-like distance on $\mathrm{Mod}(\mathbf{k}_X)$ for a metric space (X, d). It is based on what he calls the smoothing of opens. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, define $F^\varepsilon \in \mathrm{Mod}(\mathbf{k}_X)$ to be the the sheafification of the presheaf $U \mapsto F(U^\varepsilon)$, with

$$U^\varepsilon = \{x \in X \mid \exists u \in U, d(x,u) \leq \varepsilon\}.$$

This yields a functor $\cdot^\varepsilon : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$ together with a natural transformation $\cdot^\varepsilon \Rightarrow \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)}$. Although this seems to mimic the construction of interleaving distance for persistence modules, one must pay attention to the fact that $\cdot^\varepsilon$ is only left-exact. Since topological informations are obtained from sheaves by considering sheaf-cohomology, one needs to derive the functor $\cdot^\varepsilon$ in order to keep track of the cohomological informations while smoothing a sheaf.

This is precisely the sense of the convolution distance (definition 2.2.8) introduced by Kashiwara and Schapira in [KS18b] using convolution of sheaves, which has the advantage to have a nice expression in term of Grothendieck operations (that precisely allows appropriate operations for sheaf cohomology). Moreover, it satisfies a general stability result with respect to derived direct pushforward of continuous functions (theorem 2.2.12), which is of crucial importance for applications.

Constructible sheaves over $\mathbb{R}$

We let $\mathbf{k}$ be a field. We will write $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_M)$ for the abelian category of $\mathbb{R}$-constructible sheaves on M (definition A.2.14), and $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^b(\mathbf{k}_M)$ consisting of complexes of sheaves whose cohomology objects lies in $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_M)$. Note that Theorem 8.4.5 in [KS90] asserts that the natural functor $\mathrm{D}^b(\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_M)) \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_M)$ is an equivalence of triangulated categories. Theorem 3.1.2 below is proved in [KS18a] and generalizes Crawley-Boeyvey's theorem [CB12] to the context of constructible sheaves on the real line. Together with Theorem 3.1.4, they will be the cornerstone to define the graded-barcode of an object of $\mathrm{D}^b(\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ later on in Section 5.

Definition 3.1.1
Let $\mathcal{I} = \{I_\alpha\}_{\alpha \in A}$ be a multi-set of intervals of $\mathbb{R}$, that is, a list of interval where one interval can appear several times. Then $\mathcal{I}$ is said to be *locally finite* if and only if for every compact set $K \subset \mathbb{R}$, the set $A_K = \{\alpha \in A \mid K \cap I_\alpha \neq \emptyset\}$ is finite.

Theorem 3.1.2 ([KS18a] - Thm 1.17)
Let $F \in \mathrm{Obj}(\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$, then there exists a locally finite family of intervals $\{I_\alpha\}_{\alpha \in A}$ such that $F \simeq \bigoplus_{\alpha \in A} \mathbf{k}_{I_\alpha}$. Moreover, this decomposition is unique up to isomorphism.

Corollary 3.1.3
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$, and $j \geq 2$, then: $\mathrm{Ext}^j(F, G) = 0$.

A classical consequence of such a statement is the following:

Theorem 3.1.4 (Structure)
Let $F \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$. Then there exists an isomorphism in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$:

$$F \simeq \bigoplus_{j \in \mathbb{Z}} \mathrm{H}^j(F)[-j],$$

where $\mathrm{H}^j(F)[-j]$ is seen as a complex concentrated in degree j.

The isometry theorem problem

From the decomposition and structure theorems (theorems 3.1.2 and 3.1.4), a complex of sheaves in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ is entirely determined up to isomorphism by recording the intervals appearing in the decomposition of each of its cohomology objects. Hence, this *graded-barcode* (see Definition 3.4.1 below) is a complete and discrete invariant of the isomorphism classes of objects of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. As a consequence, $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ is a Krull-Schmidt category (definition 2.1.3).

Following notations of section 2.1.1, proposition 2.2.9 asserts that the convolution distance d_C satisfies axioms (M1)-(M4). Therefore, we can consider the bottleneck distance associated to d_C (definition 2.1.17), which we shall denote by d_B. The isometry theorem problem, which we address in this chapter, is to determine whether d_C is equal to d_B. We shall demonstrate that it is the case when considering the distance between two constructible sheaves (theorem 3.4.18). We will also provide an counter-example to an isometry theorem for non-constructible sheaves in section 3.5.2.

3.2 Homomorphisms in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$

This section aims at making explicit all the computations of morphisms in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Combining theorems 3.1.4 and 3.1.2, we see that any object of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ is isomorphic to a direct sum of sheaves constant on an interval seen as a complex concentrated in one degree. Hence, to give a full description of the morphisms, it is enough to compute $\mathrm{RHom}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I, \mathbf{k}_J)$ for I, J two intervals. To do so, we start by computing the morphisms in $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. To derive the functor $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}$, we introduce the subcategories $\mathscr{O}$ and $\mathscr{K}$ such that the pair $(\mathscr{O}^{\mathrm{op}}, \mathscr{K})$ is $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}$-injective. This gives us a systematic way to compute $\mathrm{Hom}_{\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I, \mathbf{k}_J[i])$: consider a left resolution of $\mathbf{k}_I$ by sheaves which are constant on open intervals $O^\bullet(\mathbf{k}_I)$ and a right resolution of $\mathbf{k}_J[i]$ by sheaves which are constant on closed intervals $K^\bullet(\mathbf{k}_J[i])$, and compute the first cohomology object of the totalization of the double complex $\mathrm{Hom}(O^\bullet(\mathbf{k}_I), K^\bullet(\mathbf{k}_J[i]))$. In addition, we also compute $\mathbf{k}_I \star K_\varepsilon$ in every cases. These computations are at the core of our proof of the isometry theorem. However, they might be useful by themselves outside of this context.

3.2.1 Homomorphisms in $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$

In this section, we give a description of all the morphisms in $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ that will enable us to make explicit computations in the derived setting.

Throughout the chapter, recall that we will write Hom instead of $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}$, except if stated otherwise.

Proposition 3.2.1
Let $a \leq b$ and $c \leq d$ be four elements of $\mathbb{R} \cup \{\pm\infty\}$. We have the following homomorphism groups, where the lines define the left-side object (i.e. the source) in $\mathrm{Hom}(-,-)$ and the columns the right-side one:

Hom$(-,-)$	(c,d)	$[c,d]$	$[c,d)$	$(c,d]$
(a,b)	$\begin{cases}\mathbf{k} \text{ if } (a,b)\subset(c,d)\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } (a,b)\cap[c,d]\neq\emptyset\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } c<b\leq d\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } c\leq a<d\\ 0 \text{ else}\end{cases}$
$[a,b]$	0	$\begin{cases}\mathbf{k} \text{ if } [c,d]\subset[a,b]\\ 0 \text{ else}\end{cases}$	0	0
$[a,b)$	0	$\begin{cases}\mathbf{k} \text{ if } a\leq c<b\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } a\leq c<b\leq d\\ 0 \text{ else}\end{cases}$	0
$(a,b]$	0	$\begin{cases}\mathbf{k} \text{ if } a<d\leq b\\ 0 \text{ else}\end{cases}$	0	$\begin{cases}\mathbf{k} \text{ if } c\leq a<d\leq b\\ 0 \text{ else}\end{cases}$

Where $a, b, c, d \in \mathbb{R} \cup \{-\infty, +\infty\}$, and we extend the order on $\mathbb{R}$ to the values $-\infty$ and $+\infty$ in the obvious manner.

Remark 3.2.2
Observe that some intervals with an infinite bound can be written with different type. For instance if $a \in \mathbb{R}$, $(a, +\infty)$ can be considered as open or half-open. In the above table, intervals with an infinite bound are seen as being open at the neighborhood of their bound if they are at the source and closed if they are the target of the morphism. Namely $(a, +\infty)$ is seen as open in the source and as half-open in the target.

Proof
We set the following notations for the proof:

$$U = (a, b),\ V = (c, d),\ S = [a, b],\ T = [c, d].$$

Open to open, closed to closed : We start from the general fact (corollary A.2.15) that for any non-empty open set $U \subset V$ and $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$, we have a canonical morphism $F_U \to F_V$ and dually for any closed sets $T \subset S$, $F_S \to F_T$. In the case were $F = \mathbf{k}_{\mathbb{R}}$ those morphisms are not zero. Indeed, for $x \in U$ (resp. $x \in T$), they induce the isomorphisms between the non zero vector spaces $(F_U)_x \xrightarrow{\sim} (F_V)_x$ (resp. $(F_S)_x \xrightarrow{\sim} (F_T)_x$).
Now observe that since the intervals of $\mathbb{R}$ are all contractible, $\mathrm{Hom}(\mathbf{k}_U, \mathbf{k}_V)$ and $\mathrm{Hom}(\mathbf{k}_S, \mathbf{k}_T)$ are at most of dimension 1.
It remains to prove that if $T \not\subset S$, then $\mathrm{Hom}(\mathbf{k}_S, \mathbf{k}_T) \simeq 0$. The dual case with open sets can be proved similarly.
Let us now suppose $S \cap T \neq \emptyset$ and $T \not\subset S$ and consider a natural morphism $\Psi : \mathbf{k}_S \to \mathbf{k}_T$. With $S = [a, b]$ and $T = [c, d]$ we will treat the case where we have $a < c < b < d$. Thus we can construct $U \subset V$ such that $U = (x, y)$, $b < x < y < d$ and $V = (z, w)$ with $z < b < y < w < d$. Therefore we get a commutative diagram:

$$\begin{tikzcd}
\mathbf{k}_S(V) \simeq \mathbf{k} \arrow[r] \arrow[d] & \mathbf{k}_S(U) \simeq 0 \arrow[d] \\
\mathbf{k}_T(V) \simeq k \arrow[r] & \mathbf{k}_T(U) \simeq \mathbf{k}
\end{tikzcd}$$

Hence $\Psi_V : \mathbf{k}_S(V) \to \mathbf{k}_T(V)$ is the zero map. As both S and T are contractible, Ψ has to be zero.
The remaining cases works the same.

Open to close : Just consider the composition $\mathbf{k}_U \to \mathbf{k}_{\mathbb{R}} \to \mathbf{k}_T$. It is not zero if $U \cap T \neq \emptyset$ since for $x \in U \cap T$ the composition $(\mathbf{k}_U)_x \to (\mathbf{k}_{\mathbb{R}})_x \to (\mathbf{k}_T)_x$ is an isomorphism.

Close to open: Suppose there exists a non zero morphism $\mathbf{k}_S \to \mathbf{k}_V$. Then, applying the functor $(-)_S$ we get a non zero morphism $\mathbf{k}_S \xrightarrow{s} \mathbf{k}_{V\cap S}$. Apply again this exact functor to the exact sequence:

$$0 \to \mathbf{k}_V \to \mathbf{k}_{\mathbb{R}} \to \mathbf{k}_{\mathbb{R}\setminus V} \to 0$$

we obtain the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{k}_{V\cap S} \arrow[r] & \mathbf{k}_S \arrow[r] \arrow[l, bend right, "s"'] & \mathbf{k}_{(\mathbb{R}\setminus V)\cap S} \arrow[r] & 0 \end{tikzcd}$$

And s is a section for this sequence. Hence $\mathbf{k}_S$ decomposes as a direct sum, which is a contradiction.

Half-open : To prove the non-existence of non-zero morphisms involving constant sheaves on half-open intervals, we can always, as before, consider the restrictions to some subsets so that we are left with some morphism between constant sheaves on either open or closed interval.

Hence, we are only left to construct the morphisms in the cases we are claiming that they exist.

Consider the case where the source is $\mathbf{k}_{[a,b)}$ and the target $\mathbf{k}_{[c,d)}$ with the condition $a \leq c < b \leq d$. Then consider the two morphisms $\mathbf{k}_{(a-1,b)} \to \mathbf{k}_{(a-1,d)}$ and $\mathbf{k}_{[a,d+1]} \to \mathbf{k}_{[c,d+a]}$. Then taking the tensor product of those two morphisms lead to a morphism

$$\mathbf{k}_{(a-1,b)} \otimes \mathbf{k}_{(a-1,d)} \simeq \mathbf{k}_{[a,b)} \longrightarrow \mathbf{k}_{(a-1,d)} \otimes \mathbf{k}_{[c,d+a]} \simeq \mathbf{k}_{[c,d)}$$

that is not zero.

□

Lemma 3.2.3

Let $F \in \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then there exists a locally finite set of open bounded intervals $\{(a_\alpha, b_\alpha)\}_{\alpha\in A}$ such that there exists an epimorphism:

$$\bigoplus_{\alpha\in A} \mathbf{k}_{(a_\alpha,b_\alpha)} \twoheadrightarrow F$$

Proof

It is sufficient to prove it for any $F \simeq \mathbf{k}_I$ with I any interval and thus to consider the following cases.

1. $I = (a, b)$ with $a, b \in \mathbb{R}$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha\in A} = \{(a, b)\}$
2. $I = [a, b]$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha\in A} = \{(a-1, b+1)\}$

3. $I = [a, b)$ with $a, b \in \mathbb{R}$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(a-1, b)\}$
4. $I = (a, b]$ with $a, b \in \mathbb{R}$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(a, b+1)\}$
5. $I = \mathbb{R}$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(n, n+2)\}_{n \in \mathbb{Z}}$
6. $I = (-\infty, b)$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(b-(n+2), b-n)\}_{n \in \mathbb{N}}$
7. $I = (-\infty, b]$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(b-(n+2), b-n)\}_{n \in \mathbb{N} \cup \{-1\}}$
8. $I = (a, \infty)$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(a+n, a+(n+2))\}_{n \in \mathbb{N}}$
9. $I = [a, \infty)$ then take $\{(a_\alpha, b_\alpha)\}_{\alpha \in A} = \{(a+n, a+(n+2))\}_{n \in \mathbb{N} \cup \{-1\}}$.

$\square$

The open bounded and compact intervals play dual role in $\mathrm{Mod}_{\mathbb{R}_c}(\mathbf{k}_\mathbb{R})$, hence we have the following :

Lemma 3.2.4
Let $F \in \mathrm{Mod}_{\mathbb{R}_c}(\mathbf{k}_\mathbb{R})$. Then there exists a locally finite set of compact intervals $\{[a_\alpha, b_\alpha]\}_{\alpha \in A}$ such that there exists a monomorphism:

$$F \rightarrowtail \bigoplus_{\alpha \in A} \mathbf{k}_{[a_\alpha, b_\alpha]}$$

The proof is similar to the one of Lemma 3.2.3.

3.2.2 Derived (bi-)functors and application to $- \star K_\epsilon$ and Hom

We now explain how to compute derived morphisms and the convolution of a sheaf with K_ε.

Computation of $- \star \mathrm{K}_\varepsilon$

To compute convolution (Definition 2.2.4) with K_ε (see Equation 2.7 of section 2.2.2), we start by computing it for sheaves constant over some closed intervals. We then deduce the other computations by sitting the other cases in distinguished triangles.

Proposition 3.2.5
Let $\varepsilon \geq 0$, and $a \leq b$ in $\mathbb{R} \cup \{\pm\infty\}$. Then:

- $\mathbf{k}_\mathbb{R} \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_\mathbb{R}$,
- $\mathbf{k}_{[a,b]} \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{[a-\varepsilon, b+\varepsilon]}$,
- $\mathbf{k}_{(a,b)} \star \mathrm{K}_\varepsilon \simeq \begin{cases} \mathbf{k}_{(a+\varepsilon, b-\varepsilon)} & \text{if } \varepsilon < \frac{|b-a|}{2}, \\ \mathbf{k}_{[\frac{b-a}{2}-\varepsilon, \frac{b-a}{2}+\varepsilon]}[-1] & \text{if } \varepsilon \geq \frac{|b-a|}{2}, \end{cases}$

- $\mathbf{k}_{(a,b]} \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{(a+\varepsilon,b+\varepsilon]}$,
- $\mathbf{k}_{[a,b)} \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{[a-\varepsilon,b-\varepsilon)}$.

The two first statements are particular cases of the more general following lemma.

Lemma 3.2.6
Let $A, B \subset \mathbb{V}$ two closed subsets of the finite dimensional real vector space $\mathbb{V}$ (endowed with the topology inherited from any norm) satisfying:

1. the map $s_{|A\times B} : A \times B \to \mathbb{V}$ is proper,
2. for any $x \in \mathbb{V}$, $s^{-1}_{|A\times B}(x)$ is contractible.

Then $\mathbf{k}_A \star \mathbf{k}_B \simeq \mathbf{k}_{A+B}$ with $A + B = \{a + b \mid a \in A, b \in B\}$.

Proof (of the lemma)
We start by proving the existence of a non zero morphism $\mathbf{k}_{A+B} \to \mathbf{k}_A \star \mathbf{k}_B$. In all the proof, Hom sets are implicitly understood as the ones in the corresponding derived category of sheaves.

Observe that since $A \times B = (A \times \mathbb{V}) \cap (\mathbb{V} \times B)$, we have:

$$\begin{aligned}
\mathbf{k}_{A\times B} &\simeq \Big((\mathbf{k}_{\mathbb{V}} \boxtimes \mathbf{k}_{\mathbb{V}})_{|A\times\mathbb{V}}\Big)_{|\mathbb{V}\times B} \\
&\simeq \big((q_1^{-1}\mathbf{k}_{\mathbb{V}})_{|A\times\mathbb{V}} \otimes (q_2^{-1}\mathbf{k}_{\mathbb{V}})_{|A\times\mathbb{V}}\big)_{|\mathbb{V}\times B} \\
&\simeq (q_1^{-1}\mathbf{k}_{\mathbb{V}})_{|A\times\mathbb{V}} \otimes (q_2^{-1}\mathbf{k}_{\mathbb{V}})_{|\mathbb{V}\times B} \\
&\simeq q_1^{-1}\mathbf{k}_A \otimes q_2^{-1}\mathbf{k}_B \\
&\simeq \mathbf{k}_A \boxtimes \mathbf{k}_B.
\end{aligned}$$

Now since s is proper on $A\times B$, we have an isomorphism $\mathrm{R}s_!(\mathbf{k}_A \boxtimes \mathbf{k}_B) \simeq \mathrm{R}s_*(\mathbf{k}_{A\times B})$. Therefore:

$$\mathrm{Hom}(\mathbf{k}_{A+B}, \mathbf{k}_A \star \mathbf{k}_B) \simeq \mathrm{Hom}(s^{-1}\mathbf{k}_{A+B}, \mathbf{k}_{A\times B}).$$

Since $A + B$ is closed, the inclusion $A + B \longrightarrow \mathbb{V}$ is proper and by base change we have $s^{-1}\mathbf{k}_{A+B} \simeq \mathbf{k}_{s^{-1}(A+B)}$. Let $i : A \times B \longrightarrow \mathbb{V} \times \mathbb{V}$ the inclusion, it is also proper since $A \times B$ is closed and we have by base change $i^{-1}\mathbf{k}_{s^{-1}(A+B)} \simeq \mathbf{k}_{i^{-1}(s^{-1}(A+B))} = \mathbf{k}_{A\times B}$. Using this computation in the former, we get :

$$\begin{aligned}
\mathrm{Hom}(\mathbf{k}_{A+B}, \mathbf{k}_A \star \mathbf{k}_B) &\simeq \mathrm{Hom}(\mathbf{k}_{s^{-1}(A+B)}, i_* i^{-1}\mathbf{k}_{A\times B}) \\
&\simeq \mathrm{Hom}(\mathbf{k}_{s^{-1}(A+B)}, i_* i^{-1}\mathbf{k}_{A\times B}) \\
&\simeq \mathrm{Hom}(i^{-1}\mathbf{k}_{s^{-1}(A+B)}, i^{-1}\mathbf{k}_{A\times B}) \\
&\simeq \mathrm{Hom}(\mathbf{k}_{A\times B}, \mathbf{k}_{A\times B}).
\end{aligned}$$

Now consider the image of $\mathrm{id}_{\mathbf{k}_{A\times B}}$ in $\mathrm{Hom}(\mathbf{k}_{A+B}, \mathbf{k}_A \star \mathbf{k}_B)$, written φ. One can prove that it induces the following isomorphisms on stalks at any $x \in \mathbb{V}$:

$$\begin{aligned}(\mathbf{k}_A \star \mathbf{k}_B)_x &\simeq \mathrm{R}\Gamma_c(s^{-1}(x), \mathbf{k}_{A\times B|s^{-1}(x)}) \\ &\simeq \mathrm{R}\Gamma(s^{-1}(x), \mathbf{k}_{A\times B|s^{-1}(x)}) \\ &\simeq (\mathbf{k}_{A+B})_x\end{aligned}$$

where the second isomorphism holds by properness of $s_{|A\times B}$ and the third one by contractibility of $s^{-1}(x)$.

This proves that φ is an isomorphism.

□

Proof (of the proposition)

We can obtain the computation for $\mathbf{k}_{(a,b)}$ by using the distinguished triangle $\mathbf{k}_{(a,b)} \longrightarrow \mathbf{k}_{\mathbb{R}} \longrightarrow \mathbf{k}_{\mathbb{R}\setminus(a,b)} \xrightarrow{+1}$, as $\mathbf{k}_{\mathbb{R}\setminus(a,b)}$ is the direct sum of one or two sheaves constant over closed intervals. Similarly for half-open intervals $\mathbf{k}_{[a,b)}$, we can use the distinguished triangles $\mathbf{k}_{[a,b)} \longrightarrow \mathbf{k}_{[a,b]} \longrightarrow \mathbf{k}_{\{b\}} \xrightarrow{+1}$.

□

Definition 3.2.7

We define $\mathscr{K}$ (resp. $\mathscr{O}$) to be the full sub-category of $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ whose objects are locally finite direct sums of sheaves of the type $\mathbf{k}_I$, with I a compact interval (resp. a relatively compact open interval).

3.2.3 Computation of derived homomorphisms

Proposition 3.2.8

The pair $(\mathscr{O}^{op}, \mathscr{K})$ is Hom-injective (see Definitions A.1.38 and A.1.43).

Proof

By lemma 3.2.3, we are left to prove that for any open interval (a, b) and closed interval $[c, d]$, $\mathscr{K}$ is $\mathrm{Hom}(\mathbf{k}_{(a,b)}, -)$-injective and $\mathscr{O}^{op}$ is $\mathrm{Hom}(-, \mathbf{k}_{[c,d]})$-injective. We will give the proof of the first part, as the second statement works similarly.

According to lemma 3.2.3, $\mathscr{K}$ satisfies the first axiom to be $\mathrm{Hom}(\mathbf{k}_{(a,b)}, -)$-injective. Let $X, X' \in \mathrm{Obj}(\mathscr{K})$ and $X'' \in \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ be objects such that there is an exact sequence in $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$:

$$0 \to X' \to X \to X'' \to 0.$$

Now decompose $X'' \simeq \oplus_{J\in B''}\mathbf{k}_J$ and $X \simeq \oplus_{I\in B}\mathbf{k}_I$. As $X \to X''$ is an epimorphism, for every $J \in B''$ there exists $I \in B$ such that the following composition is not zero:

$$\mathbf{k}_I \longrightarrow X \longrightarrow X'' \longrightarrow \mathbf{k}_J.$$

As X is an object of $\mathscr{K}$, I is a closed interval. From the computation of morphisms (Proposition 3.2.1) in $\mathrm{Mod}(\mathbf{k}_X)$, J must be a closed interval. Hence $X'' \in \mathrm{Obj}(\mathscr{K})$. This proves the second axiom.

Now let $0 \to X'' \to X \to X' \to 0$ be a short exact sequence of objects of $\mathscr{K}$. Applying $\mathrm{Hom}(\mathbf{k}_{(a,b)}, -)$ we get a long exact sequence in $\mathrm{Mod}(\mathbf{k})$:

$$0 \to \mathrm{Hom}(\mathbf{k}_{(a,b)}, X') \to \mathrm{Hom}(\mathbf{k}_{(a,b)}, X) \to \mathrm{Hom}(\mathbf{k}_{(a,b)}, X'') \to \mathrm{Ext}^1(\mathbf{k}_{(a,b)}, X') \to ...$$

Now recall that $\mathbf{k}_{(a,b)}$ represents the functor $\Gamma((a,b);-)$, that is we have a natural isomorphism $\mathrm{Hom}(\mathbf{k}_{(a,b)},-) \simeq \Gamma((a,b);-)$ and consequently $\mathrm{Ext}^1(\mathbf{k}_{(a,b)};X') \simeq \mathrm{R}^1\Gamma((a,b);X')$.

Finally, observe that $\mathrm{R}\Gamma^1((a,b);\mathbf{k}_S) \simeq 0$ for any closed interval S.

□

The following proposition is standard for derived categories of abelian categories with enough injectives.

Proposition 3.2.9

For any F, $G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$, we have an isomorphism of $\mathbf{k}$-vector spaces:

$$\mathrm{H}^0(R\mathrm{Hom}(F,G)) \simeq \mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(F,G).$$

Remark 3.2.10

As a consequence, we obtain the classical result for I, J two intervals of $\mathbb{R}$, then $\mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I,\mathbf{k}_J) \simeq \mathrm{Hom}(\mathbf{k}_I,\mathbf{k}_J)$ and $\mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I,\mathbf{k}_J[-1]) \simeq 0$. Where $\mathbf{k}_J[-1]$ is seen as a complex concentrated in degree $+1$.

It remains to see the case of homomorphisms where the target is shifted in cohomological degree 1.

Proposition 3.2.11

Let $a \le b$ and $c \le d$ be four elements of $\mathbb{R} \cup \{\pm\infty\}$. Recall that for J an interval, $\mathbf{k}_J[1]$ is the complex of sheaves concentrated in degree -1. We have the following homomorphism groups, where the lines define the left-side object (i.e. the source) in $\mathrm{Hom}(-,-[1])$ and the columns the right-side one:

$\mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(-,-[1])$	(c,d)	$[c,d]$	$(c,d)\cap[c,d)$	$(c,d]$
(a,b)	$\begin{cases}\mathbf{k} \text{ if } [c,d]\subset(a,b)\\ 0 \text{ else}\end{cases}$	0	0	0
$[a,b]$	$\begin{cases}\mathbf{k} \text{ if } [a,b]\cap(c,d)\neq\emptyset\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } (a,b)\subset[c,d]\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } c<a\\ 0 \text{ else}\end{cases}$	$\begin{cases}\mathbf{k} \text{ if } b<d\\ 0 \text{ else}\end{cases}$
$[a,b)$	$\begin{cases}\mathbf{k} \text{ if } a\leq d<b\\ 0 \text{ else}\end{cases}$	0	$\begin{cases}\mathbf{k} \text{ if } c<a\leq d<b\\ 0 \text{ else}\end{cases}$	0
$(a,b]$	$\begin{cases}\mathbf{k} \text{ if } a<c\leq b\\ 0 \text{ else}\end{cases}$	0	0	$\begin{cases}\mathbf{k} \text{ if } a<c\leq b<d\\ 0 \text{ else}\end{cases}$

Where $a, b, c, d \in \mathbb{R} \cup \{-\infty, +\infty\}$, and we extend the order on $\mathbb{R}$ to the values $-\infty$ and $+\infty$ in the obvious manner.

Remark 3.2.12

1. Some intervals can be written with different type. For instance if $a \in \mathbb{R}$, $(a, +\infty)$ can be considered as open or half-open. We follow the same convention as in Proposition 3.2.1 with respect to these choices.
2. Since $\mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_{[a,b]}, \mathbf{k}_{(c,d)}[1])$ can be non zero, observe that theorem 14.2.3 in [Cur14], which states that $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ is equivalent to the $\mathbb{Z}$-graded category of $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ cannot hold. In other words every object is isomorphic to its graded cohomology but morphisms are not the same in the two categories.

Proof

The strategy for the computations will always be the same: for I, J two intervals, consider $O^\bullet(\mathbf{k}_I)$ a left resolution of $\mathbf{k}_I$ by objects of O, and $K^\bullet(\mathbf{k}_J)$ a right resolution of $\mathbf{k}_J$ by objects of K. For simplicity, we will write $\mathrm{Hom}^\bullet$ instead of $\mathrm{Hom}^\bullet(O^\bullet(\mathbf{k}_I), K^\bullet(\mathbf{k}_J))$, and similarly $\mathrm{Hom}^{-1}, \mathrm{Hom}^0, \mathrm{Hom}^1$ will stand for $\mathrm{Hom}^{-1}(O^\bullet(\mathbf{k}_I), K^\bullet(\mathbf{k}_J))$ etc.. We note Z^i for the cycles of order i. Also in order to have lighter notations, we shall write only intervals, to stand for the constant sheaf supported on this interval.

— $I = (a, b)$ and $J = (c, d)$, then choose:

$$\begin{aligned} O^\bullet(\mathbf{k}_I) &= 0 \to 0 \to (a,b) \to 0, \\ K^\bullet(\mathbf{k}_J)[1] &= 0 \to [c,d] \to \cdot_c \oplus \cdot_d \to 0. \end{aligned}$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((a,b),[c,d])$$

$$\mathrm{Hom}^0 = 0 \times \mathrm{Hom}((a,b), \cdot_c \oplus \cdot_d)$$

$$\mathrm{Hom}^1 = 0 \times 0$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathrm{Hom}((a,b), \cdot_c \oplus \cdot_d)$ thus $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $\mathrm{Hom}((a,b), \cdot_c \oplus \cdot_d) \simeq \mathbf{k}^2$, that is $[c,d] \subset (a,b)$. In this case, $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k} \simeq \mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I, \mathbf{k}_J[1])$.

— $I = (a, b)$ and $J = [c, d]$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to 0 \to (a,b) \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] \ = \ 0 \to [c,d] \to 0 \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((a,b),[c,d]),$$

$$\mathrm{Hom}^0 = 0 \times 0,$$

$$\mathrm{Hom}^1 = 0 \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq 0$ thus $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq 0 \simeq \mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})}(\mathbf{k}_I, \mathbf{k}_J[1])$.

— $I = [a,b]$ and $J = (c,d)$, then choose :

$$O^\bullet(\mathbf{k}_I) \ = \ 0 \to (-\infty,a) \oplus (b,\infty) \to \mathbb{R} \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] \ = \ 0 \to [c,d] \to \cdot_c \oplus \cdot_d \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}(\mathbb{R},[c,d]) \simeq 0 \times \mathbf{k},$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty,a) \oplus (b,\infty),[c,d]) \times \mathrm{Hom}(\mathbb{R}, \cdot_c \oplus \cdot_d),$$

$$\mathrm{Hom}^1 = \mathrm{Hom}((-\infty,a) \oplus (b,\infty), \cdot_c \oplus \cdot_d) \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$ if and only if $[a,b] \cap (c,d) \neq \emptyset$ and since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective when not 0, we obtain $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $[a,b] \subset (c,d)$. In this case, $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k} \simeq \mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})}(\mathbf{k}_I, \mathbf{k}_J[1])$.

— $I = [a,b]$ and $J = [c,d]$, then choose:

$$O^\bullet(\mathbf{k}_I) \ = \ 0 \to (-\infty,a) \oplus (b,\infty) \to \mathbb{R} \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] \ = \ 0 \to [c,d] \to 0 \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}(\mathbb{R},[c,d]) \simeq 0 \times \mathbf{k},$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty,a) \oplus (b,\infty),[c,d]) \times 0,$$

$$\mathrm{Hom}^1 = 0 \times 0.$$

Hence $Z^0(\mathrm{Hom}^\bullet) \simeq \mathrm{Hom}((-\infty,a)\oplus(b,\infty),[c,d])$ and since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective, we obtain $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $\mathrm{Hom}((-\infty,a)\oplus(b,\infty),[c,d]) \simeq \mathbf{k}^2$ which is equivalent to $[a,b] \subset (c,d)$.

— $I = (a,b)$ and $J = [c,d]$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to 0 \to (a,b) \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] = 0 \to [c,d] \to \cdot_d \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((a,b),[c,d]),$$

$$\mathrm{Hom}^0 = 0 \times \mathrm{Hom}((a,b),\cdot_d),$$

$$\mathrm{Hom}^1 = 0 \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathrm{Hom}((a,b),\cdot_d)$ thus $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq 0$ since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective when $\mathrm{Hom}^{-1} \not\simeq 0$.

— $I = [a,b]$ and $J = [c,d)$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to (-\infty,a)\oplus(b,\infty) \to \mathbb{R} \to 0$$

$$K^\bullet(\mathbf{k}_J)[1] = 0 \to [c,d] \to \cdot_d \to 0$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}(\mathbb{R},[c,d]),$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty,a)\oplus(b,\infty),[c,d]) \times \mathrm{Hom}((a,b),\cdot_d),$$

$$\mathrm{Hom}^1 = \mathrm{Hom}((-\infty,a)\oplus(b,\infty),\cdot_d) \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \begin{cases} \mathbf{k}^2 \text{ if } c < a \\ \mathbf{k} \text{ else} \end{cases}$. Since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective when not zero, $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$, that is $c < a$. In this case, $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k} \simeq \mathrm{Hom}_{D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})}(\mathbf{k}_I, \mathbf{k}_J[1])$.

— $I = [a, b)$ and $J = (c, d)$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to (-\infty, a) \to (-\infty, b) \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] = 0 \to [c, d] \to \cdot_c \oplus \cdot_d \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((-\infty, b), [c, d]),$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty, a), [c, d]) \times \mathrm{Hom}((-\infty, b), \cdot_c \oplus \cdot_d),$$

$$\mathrm{Hom}^1 = \mathrm{Hom}((-\infty, a), \cdot_c \oplus \cdot_d) \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$ if and only if $a \leq d < b$ and since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective, we obtain $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$ which is equivalent to $a \leq d < b$.

— $I = [a, b)$ and $J = [c, d]$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to (-\infty, a) \to (-\infty, b) \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] = 0 \to [c, d] \to 0 \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((-\infty, b), [c, d]),$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty, a), [c, d]) \times 0,$$

$$\mathrm{Hom}^1 = 0 \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathrm{Hom}((-\infty, a), [c, d])$ and since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective when not zero, we obtain $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq 0$.

— $I = [a, b)$ and $J = [c, d)$, then choose:

$$O^\bullet(\mathbf{k}_I) = 0 \to (-\infty, a) \to (-\infty, b) \to 0,$$

$$K^\bullet(\mathbf{k}_J)[1] = 0 \to [c, d] \to \cdot_d \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((-\infty, b), [c, d]),$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty, a), [c, d]) \times \mathrm{Hom}((-\infty, b), \cdot_d),$$

$$\mathrm{Hom}^1 = \mathrm{Hom}((-\infty, a), \cdot_d) \times 0.$$

Hence we get that $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$ if and only if $c < a \leq d < b$ and since the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is injective, we obtain $\mathrm{H}^0(\mathrm{Hom}^\bullet) \not\simeq 0$ if and only if $Z^0(\mathrm{Hom}^\bullet) \simeq \mathbf{k}^2$ which is equivalent to $c < a \leq d < b$.

— $I = [a, b)$ and $J = (c, d]$, then choose:

$$O^\bullet(\mathbf{k}_I) \;=\; 0 \to (-\infty, a) \to (-\infty, b) \to 0,$$
$$K^\bullet(\mathbf{k}_J)[1] \;=\; 0 \to [c, d] \to \cdot_c \to 0.$$

Then we have:

$$\mathrm{Hom}^{-1} = 0 \times \mathrm{Hom}((-\infty, b), [c, d]),$$

$$\mathrm{Hom}^0 = \mathrm{Hom}((-\infty, a), [c, d]) \times \mathrm{Hom}((-\infty, b), \cdot_c),$$

$$\mathrm{Hom}^1 = \mathrm{Hom}((-\infty, a), \cdot_c) \times 0.$$

Hence we get that the differential $\mathrm{Hom}^{-1} \to \mathrm{Hom}^0$ is always a surjective onto $Z^0(\mathrm{Hom}^\bullet)$. Hence $\mathrm{H}^0(\mathrm{Hom}^\bullet) \simeq 0$.

□

3.3 Structure of ε-interleavings

The aim of this section is to start to study the convolution distance, now that we know effective computations of morphisms and how to compute the convolution in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. We start by giving some explicit conditions on the support of two sheaves, which are constant on an interval, for them to be ε-close with respect to d_C. This will lead us, in a second time, to prove that d_C has a specific behavior when considering two sheaves F and G. To do so we introduce what we call the CLR decomposition, that is decomposition into central part (made of sheaves whose summands have support which are either bounded open or compact intervals) and left and right parts (made of the two possible types of half-open intervals). Then we prove that the distance between F and G is nothing but the maximum of the distance between F_C and G_C, F_R and G_R, F_L and G_L, that is between the respective three parts.

3.3.1 Characterization of ε-interleavings between indecomposables sheaves

For any interval I and real number $\varepsilon \geq 0$, we will write $I^\varepsilon = \bigcup_{x \in I} \mathrm{B}(x, \varepsilon)$ where $\mathrm{B}(x, \varepsilon)$ is the euclidean closed ball centered at x with radius ε. Moreover if $I = (a, b)$ with $a, b \in \mathbb{R}$, and $\varepsilon < \frac{b-a}{2} = \frac{\mathrm{diam}(I)}{2}$, define $I^{-\varepsilon} = (a + \varepsilon, b - \varepsilon)$. Further, if I is bounded, we write $\mathrm{cent}(I)$ for its center, that is $(b + a)/2$ where a, b are the boundary points of I.

The following proposition describes the condition for sheaves constant on open/closed intervals to be ε-close (Definition 2.2.6).

Proposition 3.3.1 (closed/open)
Let S, T be two closed interval, U, V be two open intervals and $\varepsilon > 0$. Then:

1. $\mathbf{k}_S \sim_\varepsilon \mathbf{k}_T \iff S \subset T^\varepsilon$ and $T \subset S^\varepsilon$
2. $\mathbf{k}_U \sim_\varepsilon \mathbf{k}_V \iff U \subset V^\varepsilon$ and $V \subset U^\varepsilon$
3. $\mathbf{k}_S \sim_\varepsilon \mathbf{k}_U[-1] \iff \varepsilon \geq \frac{\mathrm{diam}(U)}{2}$ and $S \subset [\mathrm{cent}(U) - (\varepsilon - \frac{\mathrm{diam}(U)}{2}), \mathrm{cent}(U) + (\varepsilon - \frac{\mathrm{diam}(U)}{2})]$

Proof

1. Consider $f : \mathbf{k}_S \star \mathrm{K}_\varepsilon \to \mathbf{k}_T$ and $g : \mathbf{k}_T \star \mathrm{K}_\varepsilon \to \mathbf{k}_S$ the data of an ε-interleaving. Then f and g are in particular not zero as
$$\mathrm{R}\Gamma(\mathbb{R}, \phi_{\mathbf{k}_S, 2\varepsilon}) = \mathrm{R}\Gamma(\mathbb{R}, g) \circ \mathrm{R}\Gamma(\mathbb{R}, f \star \mathrm{K}_\varepsilon)$$
is an isomorphism between $\mathrm{R}\Gamma(\mathbb{R}, \mathbf{k}_S \star K_{2\varepsilon})$ and $\mathrm{R}\Gamma(\mathbb{R}, \mathbf{k}_T)$ that are not zero. Remark that $\mathbf{k}_S \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{S^\varepsilon}$ and $\mathbf{k}_T \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{T^\varepsilon}$ by Proposition 3.2.5. From our computations of morphisms (Proposition 3.2.1), we have necessarily $S \subset T^\varepsilon$ and $T \subset S^\varepsilon$. Conversely, if $S \subset T^\varepsilon$ and $T \subset S^\varepsilon$, it is easy to build an ε-interleaving.
2. Consider $f : \mathbf{k}_U \star \mathrm{K}_\varepsilon \to \mathbf{k}_V$ and $g : \mathbf{k}_V \star \mathrm{K}_\varepsilon \to \mathbf{k}_U$ the data of an ε-interleaving. For the same reason as above, f and g are not zero. Hence, $f \star K_{-\varepsilon} : \mathbf{k}_U \to \mathbf{k}_V \star K_{-\varepsilon}$ is not zero. As $\mathbf{k}_V \star K_{-\varepsilon} \simeq \mathbf{k}_{V^\varepsilon}$, we get again with our computations of morphisms (Proposition 3.2.1) that $U \subset V^\varepsilon$. Similarly we have $V \subset U^\varepsilon$.

   Conversely if we assume $U \subset V^\varepsilon$ and $U \subset V^\varepsilon$, it is easy to construct an ε-interleaving.
3. $f : \mathbf{k}_S \star \mathrm{K}_\varepsilon \to \mathbf{k}_U[-1]$ and $g : \mathbf{k}_U[-1] \star \mathrm{K}_\varepsilon \to \mathbf{k}_S$ the data of an ε-interleaving. For the same reason as above, f and g are not zero. Suppose $\varepsilon < \frac{\mathrm{diam}(U)}{2}$, then Proposition 3.2.5 implies that $\mathbf{k}_U[-1] \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{U^{-\varepsilon}}[-1]$, hence the fact that g is not zero is absurd.

Therefore we necessarily have $\varepsilon \geq \frac{\text{diam}(U)}{2}$, and

$$\mathbf{k}_U[-1] \star \mathrm{K}_\varepsilon \simeq \mathbf{k}_{[\text{cent}(U)-(\varepsilon-\frac{\text{diam}(U)}{2}),\text{cent}(U)+(\varepsilon-\frac{\text{diam}(U)}{2})]}.$$

Hence the existence of g implies that $S \subset [\text{cent}(U)-(\varepsilon-\frac{\text{diam}(U)}{2}), \text{cent}(U)+(\varepsilon-\frac{\text{diam}(U)}{2})]$, also the existence of f implies that $S^\varepsilon \cap U \neq \emptyset$, but this condition is weaker than the previous one.
Conversely, if $\varepsilon \geq \frac{\text{diam}(U)}{2}$ and $S \subset [\text{cent}(U) - (\varepsilon - \frac{\text{diam}(U)}{2}), \text{cent}(U) + (\varepsilon - \frac{\text{diam}(U)}{2})]$, we can construct the desired morphisms (for instance using Proposition 3.2.11) and have to check that their composition (after applying ε convolution to one of the two) is not zero. This can be obtained by taking stalks at any $x \in J$.

Proposition 3.3.2 (half-open)
Let $I = [a, b)$ and $J = [c, d)$ with $a, c \in \mathbb{R}$ and $b, d \in \mathbb{R} \cup \{+\infty\}$, and $\varepsilon \geq 0$, then:

$$\mathbf{k}_I \sim_\varepsilon \mathbf{k}_J \iff |\, a - c \,| \leq \varepsilon \text{ and } |\, b - d \,| \leq \varepsilon.$$

Similarly for $I = (a, b]$ and $J = (c, d]$,

$$\mathbf{k}_I \sim_\varepsilon \mathbf{k}_J \iff |\, a - c \,| \leq \varepsilon \text{ and } |\, b - d \,| \leq \varepsilon.$$

Proof
The proof works exactly the same as the open/closed case, that is Proposition 3.3.1.

□

3.3.2 CLR Decomposition

In order to define a matching between graded barcodes, we will have to distinguish between the topological nature of their support interval as the existence of shifted morphisms between them depends on this nature according to Proposition 3.2.11.

Definition 3.3.3
Let I be an interval.

1. I is said to be an *interval of type C* iff there exists $(a, b) \in \mathbb{R}^2$ such that $I = [a, b]$ or $I = (a, b)$.
2. I is said to be an *interval of type R* iff there exists $(a, b) \in \mathbb{R}^2$ such that either $I = [a, b)$, $I = (-\infty, b)$, $I = [a, +\infty)$ or $I = \mathbb{R}$.

3. I is said to be an *interval of type* L iff there exists $(a,b) \in \mathbb{R}^2$ such that $I = (a,b]$, $I = (-\infty,b]$ or $I = (a,+\infty)$.

Proposition 3.3.4
Let $F \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ then there exists a unique decomposition up to isomorphism $F \simeq F_C \oplus F_R \oplus F_L$ such that :

1. The cohomology objects of F_C are direct sums of constant sheaves over intervals of type C,
2. The cohomology objects of F_R are direct sums of constant sheaves over intervals of type R,
3. The cohomology objects of F_L are direct sums of constant sheaves over intervals of type L.

We will call F_C (resp. F_R, F_L) the *central* (resp. *right*, *left*) part of F, and name it the *CLR decomposition* of F.

Proof
Observe that the types C,L,R do form a partition of the set of intervals of $\mathbb{R}$, and apply the decomposition and structure theorems from section 2.

□

Definition 3.3.5
Let $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. F is said to be a *central sheaf* if $F \simeq F_C$. Similarly, F is a *left* (resp. *right*) sheaf if $F \simeq F_L$ (resp. $F \simeq F_R$).

The CLR decomposition is compatible with the convolution distance. More precisely, we have the following result.

Theorem 3.3.6 (Theorem 4.5 - [BG18])
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ and $\varepsilon \geq 0$, then the following holds:

$$d_C(F,G) \leq \varepsilon \iff \begin{cases} d_C(F_C, G_C) \leq \varepsilon \\ d_C(F_R, G_R) \leq \varepsilon \\ d_C(F_L, G_L) \leq \varepsilon \end{cases}$$

Proof
We will in fact prove the stronger statement: for any $\varepsilon \geq 0$, F and G are ε-interleaved if and only if F_C and G_C, F_R and G_R, F_L and G_L are. The right to left implication is an immediate consequence of the additivity of the convolution functor. Now let us consider the data of an ε-interleaving between F and G, that is, two morphisms $F \star \mathrm{K}_\varepsilon \xrightarrow{f} G$ and $G \star \mathrm{K}_\varepsilon \xrightarrow{g} F$ such that $f \star \mathrm{K}_\varepsilon \circ g : G \star K_{2\varepsilon} \longrightarrow F$ is the canonical arrow and similarly

for $g \star \mathrm{K}_\varepsilon \circ f : F \star K_{2\varepsilon} \longrightarrow G$. During the proof, we will use the letters i to denote a canonical injection of a summand into a sheaf, and p for the canonical projection.

1. Let I be a closed or open interval and $j \in \mathbb{Z}$ be such that $\mathbf{k}_I[-j]$ appears in the decomposition of the cohomology objects of F_C. Consider the composition:

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^F} F \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G \star \mathrm{K}_\varepsilon \xrightarrow{p_L^G} G_L \star \mathrm{K}_\varepsilon \xrightarrow{g} F \xrightarrow{p_I^F} \mathbf{k}_I[-j]$$

   From our previous computations (Proposition 3.3.1, this composition must be zero whether I is open or closed. The same results hold for G_R instead of G_L using Proposition 3.3.2. Hence, the composition:

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^F} F \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G \star \mathrm{K}_\varepsilon \xrightarrow{g} F \xrightarrow{p_I^F} \mathbf{k}_I[-j]$$

   is equal to the composition:

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^F} F \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G \xrightarrow{p_C^G} G_C \star \mathrm{K}_\varepsilon \xrightarrow{g} F \xrightarrow{p_I^F} \mathbf{k}_I[-j].$$

   As this is true for any direct summands of F_C, we get that the composition

$$F_C \star K_{2\varepsilon} \xrightarrow{i_C^F} F \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G \star \mathrm{K}_\varepsilon \xrightarrow{g} F \xrightarrow{p_C^F} F_C$$

   is equal to the composition:

$$F_C \star K_{2\varepsilon} \xrightarrow{i_C} F \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G \star \mathrm{K}_\varepsilon \xrightarrow{p_C^G} G_C \star \mathrm{K}_\varepsilon \xrightarrow{g} F \xrightarrow{p_C^F} F_C.$$

   In other words, it is just the canonical arrow $F_C \star \mathrm{K}_\varepsilon \longrightarrow F_C$. Intertwining F and G proves that the composition:

$$G_C \star K_{2\varepsilon} \xrightarrow{i_C} G \star K_{2\varepsilon} \xrightarrow{g \star \mathrm{K}_\varepsilon} F \star \mathrm{K}_\varepsilon \xrightarrow{p_C^G} F_C \star \mathrm{K}_\varepsilon \xrightarrow{f} G \xrightarrow{p_C^G} G_C$$

   is the canonical arrow. Hence $p_C^G \circ f \circ i_C^F$ and $p_C^F \circ g \circ i_C^G$ defines an ε-interleaving between F_C and G_C.

2. We proceed exactly similarly to prove that $p_L^G \circ f \circ i_L^F$ and $p_L^F \circ g \circ i_L^G$ (resp. $p_R^G \circ f \circ i_R^F$ and $p_R^F \circ g \circ i_R^G$) define ε-interleavings between F_L and G_L (resp. F_R and G_R).

3.4 Isometry theorem and graded barcodes

This section presents the proof of the isometry theorem after defining precisely the graded barcodes associated to a complexe of constructible sheaves in the derived category. To do so, we use the results of previous sections to first define graded barcodes, the combinatorial object that entirely encodes the isomorphism class of a sheaf. The properties of the CLR decomposition invite us to split this barcode into three parts: central, left, right, and to define ε-matchings between each of these parts. This leads us to a definition of a bottleneck distance between graded barcodes, which, as we prove, bounds the convolution distance.

In order to prove the reverse inequality, we prove that an ε-interleaving between two sheaves induces a ε-matching between their graded-barcodes. To do so, we construct the matching according to the CLR decomposition. We reduce the construction of the matching between the left and right parts to the well-known case of persistence modules with one parameter. To this end, we first prove that interleavings between right (resp. left) parts of two sheaves happen degree-wise at the level of their cohomology objects. This enables us to define functors Ψ_R^j, that send the j-th cohomology of the right part of a sheaf to a one parameter persistence module. We prove that the Ψ_R^j's are barcode preserving, and send interleavings of sheaves to interleavings of persistence modules.

To build the matching between central parts, we adapt Bjerkevik's approach [Bje16]: we introduce a similar pre-order to his, $\leq_\alpha$, between graded-intervals. This pre-order allows us to "trigonalize" the interleaving morphisms, and by a rank argument to deduce that the hypothesis of Hall's marriage theorem are satisfied.

3.4.1 Graded-barcodes, bottleneck distance and stability

We start by introducing the abstract notion of a graded barcode, and then define the graded barcode of a complex of sheaves.

Definition 3.4.1
A *graded-barcode* $\mathbb{B}$ is the data of three $\mathbb{Z}$-indexed sequence of multi-set of intervals $((\mathbb{B}_C^j)_{j\in\mathbb{Z}}, (\mathbb{B}_R^j)_{j\in\mathbb{Z}}, (\mathbb{B}_L^j)_{j\in\mathbb{Z}})$ such that for every $j \in \mathbb{Z}$:

1. $\mathbb{B}_C^j$ is a locally finite multiset of closed or open bounded intervals.
2. $\mathbb{B}_R^j$ is a locally finite multiset of half-open intervals of the form $[a, b)$ with a, $b \in \mathbb{R} \cup \{\pm\infty\}$.

3. $\mathbb{B}_L^j$ is a locally finite multiset of half-open intervals of the form $(a, b]$ with a, $b \in \mathbb{R} \cup \{\pm\infty\}$ and $(a, b] \neq \mathbb{R}$.

We can now define the graded barcode of a constructible sheaf.

Definition 3.4.2
Let $F \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$, that decomposes uniquely up to isomorphism as $F \simeq F_C \oplus F_R \oplus F_L$. We define the *graded-barcode of* F to be $\mathbb{B}(F) = (\mathbb{B}_C(F), \mathbb{B}_R(F), \mathbb{B}_L(F))$ where:

1. $\mathbb{B}_C(F) = (\mathbb{B}_C^j(F))_{j\in\mathbb{Z}}$, $\mathbb{B}_R(F) = (\mathbb{B}_R^j(F))_{j\in\mathbb{Z}}$, $\mathbb{B}_L(F) = (\mathbb{B}_L^j(F))_{j\in\mathbb{Z}}$.
2. For $j \in \mathbb{Z}$, and $\alpha \in \{L, R\}$, $\mathbb{B}_\alpha^j(F)$ is a complete enumeration of the intervals appearing in the decomposition of $\mathrm{H}^j(F_\alpha)$, that is, we have an isomorphism in $\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$:
$$\mathrm{H}^j(F_\alpha) \simeq \bigoplus_{I\in\mathbb{B}_\alpha^j(F)} \mathbf{k}_I.$$
3. For $j \in \mathbb{Z}$, $\mathbb{B}_C^j(F)$ is a complete enumeration of the open intervals appearing in the decomposition of $\mathrm{H}^j(F_C)$, and of the closed intervals appearing in the decomposition of $\mathrm{H}^{j+1}(F_C)$.

Remark 3.4.3
Since $F \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$, there exists $N \geq 0$ such that for $|j| \geq N$, $\mathbb{B}_C^j(F) = \mathbb{B}_R^j(F) = \mathbb{B}_L^j(F) = \emptyset$.

From the decomposition theorem 3.1.2 and structure theorem 3.1.4 of section 2, the isomorphism classes of objects in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ are completely determined by their graded-barcodes. And conversely to a graded-barcode corresponds a unique isomorphism class of object in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$.

Now, we will define a bottleneck distance between the graded barcodes. We start by defining ε-matchings between graded barcodes.

Definition 3.4.4
For S and T two sets, a *partial matching* between S and T is the data of two subsets $S' \subset S$ and $T' \subset T$ and a bijection $\sigma : S' \longrightarrow T'$. We define $T' = \mathrm{im}\, \sigma$ as the *image* of the partial matching, $S' = \mathrm{coim}\, \sigma$ its *co-image*.

We will refer to the partial matching just as σ and denote it $\sigma : S \nrightarrow T$.

Definition 3.4.5
Let $\mathbb{B}$ and $\mathbb{D}$ be two graded-barcodes and $\varepsilon \geq 0$. An *ε-matching* between $\mathbb{B}$ and $\mathbb{D}$ is the data $\sigma = ((\sigma_C^j)_{j\in\mathbb{Z}}, (\sigma_R^j)_{j\in\mathbb{Z}}, (\sigma_L^j)_{j\in\mathbb{Z}})$ where, for $j \in \mathbb{Z}$:

1. $\sigma_C^j : \mathbb{B}_C^j \longrightarrow \mathbb{D}_C^j$ is a bijection satisfying, for any $I \in \mathbb{B}_C^j$:$\mathbf{k}_I \sim_\varepsilon \mathbf{k}_{\sigma_C^j(I)}[-\delta]$, where $\delta = 0$ if I and $\sigma_C^j(I)$ are both open or both closed, $\delta = 1$ if I is open and $\sigma_C^j(I)$ is closed and $\delta = -1$ if I is closed and $\sigma_C^j(I)$ is open.
2. $\sigma_R^j : \mathbb{B}_R^j \not\rightarrow \mathbb{D}_R^j$ is a partial matching satisfying:
   (i) for any $I \in \text{im } \sigma_C^j \sqcup \text{coim } \sigma_C^j$, one has $\mathbf{k}_I[-j] \sim_\varepsilon \mathbf{k}_{\sigma_C^j(I)}[-j]$;
   (ii) for $I \in (\mathbb{B}_R^j \backslash \text{im } \sigma_C^j) \sqcup (\mathbb{D}_R^j \backslash \text{coim } \sigma_C^j)$, one has $\mathbf{k}_I[-j] \sim_\varepsilon 0$.
3. $\sigma_L^j : \mathbb{B}_L^j \not\rightarrow \mathbb{D}_L^j$ is a partial matching satisfying :
   (i) for any $I \in \text{im } \sigma_C^j \sqcup \text{coim } \sigma_C^j$, one has $\mathbf{k}_I[-j] \sim_\varepsilon \mathbf{k}_{\sigma_C^j(I)}[-j]$;
   (ii) for $I \in (\mathbb{B}_L^j \backslash \text{im } \sigma_C^j) \sqcup (\mathbb{D}_L^j \backslash \text{coim } \sigma_C^j)$, one has $\mathbf{k}_I[-j] \sim_\varepsilon 0$.

As one could expect, we can now define a bottleneck distance from this notion of matching in a standard way:

Definition 3.4.6
Let $\mathbb{B}$ and $\mathbb{D}$ be two graded-barcodes, then one defines their *graded bottleneck distance* to be the possibly infinite positive value:

$$d_B(\mathbb{B}, \mathbb{D}) = \inf\{\varepsilon \geq 0 \mid \text{there exists a } \varepsilon\text{-matching between } \mathbb{B} \text{ and } \mathbb{D}\}$$

Proposition 3.4.7
The distance d_B is indeed the bottleneck distance associated to d_C on the Krull-Schmidt category $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ according to definition 2.1.17.

Proof
This is a direct consequence of the fact that interleavings respect the CLR decomposition (theorem 3.3.6).

Remark 3.4.8
Note that our definition of the bottleneck distance allows to compare intervals of a barcodes defines in *different* degrees unlike in the traditional persistence module case. This is fundamental in order to take into account the derived nature of sheaves; a basic example demonstrating this is given in Section 3.5.1.

Lemma 3.4.9
Let F, G two objects of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, then:

$$d_C(F, G) \leq d_B(\mathbb{B}(F), \mathbb{B}(G)).$$

Proof
By definition, an ε-matching between the graded-barcodes of F and G implies the existence of an ε-interleaving between F and G. Hence we have the following inclusion:

$$\{\varepsilon \geq 0 \mid \text{there exists a } \varepsilon\text{-matching between } \mathbb{B}(F) \text{ and } \mathbb{B}(G)\}$$

$$\subset \{\varepsilon \geq 0 \mid \text{there exists a } \varepsilon\text{-isomorphism between } F \text{ and } G\}$$

which proves the lemma.

□

3.4.2 The cases $F_R \leftrightarrow G_R$ and $F_L \leftrightarrow G_L$

In this section, we give a description of the ε-interleavings between the right part of two complexes of sheaves. The proofs and statements for the left part are exactly the same.

Construction of Ψ_R^j

Proposition 3.4.10
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ and $\varepsilon \geq 0$ with right parts F_R and G_R. The following holds:

$$d_C(F_R, G_R) \leq \varepsilon \iff \forall j \in \mathbb{Z},\ d_C(\mathrm{H}^j(F_R), \mathrm{H}^j(G_R)) \leq \varepsilon.$$

Here in the last inequality, $\mathrm{H}^j(F_R)$ and $\mathrm{H}^j(G_R)$ are seen as complexes concentrated in degree j.

Proof
As in Theorem 3.3.6, we will prove the stronger statement (which is, in fact, equivalent by 3.5.3) that F_R and G_R are ε-interleaved if and only if each of their cohomologies are pairwise ε-interleaved. The right to left implication by additivity of the convolution functor and the structure theorem 3.1.4, so let us consider an ε-interleaving given by $F_R \star \mathrm{K}_\varepsilon \xrightarrow{f} G_R$ and $G_R \star \mathrm{K}_\varepsilon \xrightarrow{g} F_R$. Let $j \in \mathbb{Z}$ and pick $\mathbf{k}_I$ a direct summand of $\mathrm{H}^j(F_R)$ (I is a half-open interval of the type $[a, b)$). We consider again the composition:

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^{F_R}} F_R \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G_R \star \mathrm{K}_\varepsilon \xrightarrow{g} F_R \xrightarrow{p_I^{F_R}} \mathbf{k}_I[-j]$$

From our computations of derived morphisms (Proposition 3.2.11), this is equal to the composition :

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^{F_R}} F_R \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} \mathrm{H}^j(G_R \star \mathrm{K}_\varepsilon)[-j] \oplus \mathrm{H}^{j+1}(G_R \star \mathrm{K}_\varepsilon)[-j-1] \xrightarrow{g} F_R \xrightarrow{p_I^{F_R}} \mathbf{k}_I[-j].$$

We obtain using our computations of convolution (Proposition 3.2.5) and morphisms show that since G_R has only half-open intervals in the decomposition of its cohomologie objects, $\mathrm{H}^j(G_R \star \mathrm{K}_\varepsilon)[-j] \simeq \mathrm{H}^j(G_R)[-j] \star \mathrm{K}_\varepsilon$ and $\mathrm{H}^{j+1}(G_R \star \mathrm{K}_\varepsilon)[-j-1] \simeq \mathrm{H}^{j+1}(G_R)[-j-1] \star \mathrm{K}_\varepsilon$.

It follows again from our computations in Proposition 3.2.11 that any morphism of $\mathbf{k}_I[-j] \star \mathrm{K}_\varepsilon \to \mathbf{k}_I[-j]$ that factors through a complex concentrated in degree $j+1$ must be zero.

Finally, the first composition is thus equal to

$$\mathbf{k}_I[-j] \star K_{2\varepsilon} \xrightarrow{i_I^{F_R}} F_R \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} \mathrm{H}^j(G_R)[-j] \star \mathrm{K}_\varepsilon \xrightarrow{g} F_R \xrightarrow{p_I^{F_R}} \mathbf{k}_I[-j].$$

As this is true for any summands of $\mathrm{H}^j(F_R)$ we get that the composition:

$$\mathrm{H}^j(F_R)[-j] \star K_{2\varepsilon} \xrightarrow{i_I^{F_R}} F_R \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} G_R \star \mathrm{K}_\varepsilon \xrightarrow{g} F_R \xrightarrow{p_I^{F_R}} \mathrm{H}^j(F_R)[-j]$$

is equal to the composition

$$\mathrm{H}^j(F_R)[-j] \star K_{2\varepsilon} \xrightarrow{i_I^{F_R}} F_R \star K_{2\varepsilon} \xrightarrow{f \star \mathrm{K}_\varepsilon} \mathrm{H}^j(G_R)[-j] \star \mathrm{K}_\varepsilon \xrightarrow{g} F_R \xrightarrow{p_I^{F_R}} \mathrm{H}^j(F_R)[-j].$$

This gives the first part of the ε-interleaving. We get the second one by intertwining the roles of F_R and G_R.

□

The above result shows that when one wants to understand an ε-interleaving between the right parts of two sheaves, it is sufficient to understand it at the level of each of their cohomology objects, degree wise. In the sequel, we will show that the behavior of ε-interleavings between sheaves with cohomologies concentrated in degree $j \in \mathbb{Z}$ decomposing into direct summands of type R, is essentially the same as looking at ε-interleavings in the opposite category of one-parameter persistence modules, which is well understood. According to section 2.1.1, we denote by $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$ the category of persistence modules over the poset $(\mathbb{R}, \leq)$, we write $\mathrm{Pers}_f(\mathbf{k}^{\mathbb{R}})$ for its full subcategory of pointwise finite dimensional persistence modules. We also denote d_I the usual interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$.

Proposition 3.4.11
Let D^j_R be the full sub-category of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ with objects complexes of sheaves F such that $F \simeq F_R$ and $\mathrm{H}^i(F) = 0$ for $i \neq j$. Then there exists a well-defined functor $\Psi^j_R : \mathrm{D}^j_R \longrightarrow \mathrm{Pers}_f(\mathbf{k}^\mathbb{R})^{op}$ such that:

1. For $F \in \mathrm{Obj}(\mathrm{D}^j_R)$ such that $\mathrm{H}^j(F) \simeq \oplus_{I\in B}\mathbf{k}_I$, we have $\Psi^j_R(F) = \oplus_{I\in B}\mathbf{k}^I$.
2. Ψ^j_R is fully faithful.
3. For $\varepsilon \geq 0$ and $F \in \mathrm{Obj}(\mathrm{D}^j_R)$, $\Psi^j_R(F \star \mathrm{K}_\varepsilon) = \Psi^j_R(F)[\varepsilon]$ and $\Psi^j_R(\phi_{F,\varepsilon}) = s_\varepsilon^{\Psi^j_R(F)}$.
4. Ψ^j_R is isometric with respect to $d_C(\cdot,\cdot)$ and $d_I(\cdot,\cdot)$.

Proof
This is a combination of the computations of morphisms and convolution (Propositions 3.2.9, 3.2.5, 3.3.4), together with the observation that for I, J two intervals of type R and $j \in \mathbb{Z}$, then we have the functorial isomorphisms:

$$\mathrm{Hom}_{\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})}(\mathbf{k}_I[-j], \mathbf{k}_J[-j]) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_\mathbb{R})}(\mathbf{k}_I, \mathbf{k}_J) \simeq \mathrm{Hom}_{\mathrm{Pers}_f(\mathbf{k}^\mathbb{R})}(\mathbf{k}^J, \mathbf{k}^I).$$

Induced matching

Theorem 3.4.12 (Theorem 5.11 - [BG18])
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ which are ε-interleaved through the maps $F \star \mathrm{K}_\varepsilon \xrightarrow{f} G$ and $G \star \mathrm{K}_\varepsilon \xrightarrow{g} F$. For any $j \in \mathbb{Z}$, there exists an ε-matching $\sigma^j_R : \mathbb{B}^j_R(F) \not\longrightarrow \mathbb{B}^j_R(G)$.

Proof
Observe that $\Psi^j_R(F_R)$ (resp. $\Psi^j_R(G_R)$) is a persistence module with the same barcode than $\mathrm{H}^j(F_R)$ (resp. $\mathrm{H}^j(G_R)$). Also, from proposition 5.3, $\Psi^j_R(F_R)$ and $\Psi^j_R(G_R)$ are ε-interleaved as persistence modules. Hence, we can apply the isometry theorem for pointwise finite dimensional persistence modules [CdSGO16, Theorem 4.11] to $\Psi^j_R(F_R)$ and $\Psi^j_R(G_R)$ and deduce the existence of a ε-matching of barcodes of persistence modules between $\mathbb{B}^j_R(F)$ and $\mathbb{B}^j_R(F)$. Notice now that this matching is exactly what we ask for σ^j_R, by proposition 3.3.2.

□

3.4.3 The case $F_C \leftrightarrow G_C$

In this section, we construct the ε-matching between the central parts of two sheaves, assuming they are ε-interleaved. Using ideas of Bjerkevik [Bje16, Section 4], we introduce a pre-order $\leq_\alpha$ on the set of graded-intervals of type C whose purpose is to prove the existence of the ε-matching using Hall's marriage theorem. To apply this theorem, we must prove that given a finite list of interval in the barcode of one of the two sheaves, there exists, at least, the same number of intervals in the barcode of the second sheaf which are at distance less than ε from an interval in the first list.

We will show that ordering the graded-barcodes of the sheaves according to $\leq_\alpha$ will actually lead to a very nice expression of the interleaving morphisms, allowing us, by a rank argument, to deduce that this condition is satisfied.

Ordering graded-intervals of type C

We define a graded interval to be an interval I together with an integer $j \in \mathbb{Z}$. It will be written I^j henceforth. For I of type C such that either $I = [a, b]$ or $I = (a, b)$ with $a, b \in \mathbb{R}$, define $\mathrm{diam}(I) = b - a$ to be its diameter.

Definition 3.4.13

The relation $\leq_\alpha$ on the set of graded intervals of type C is defined by:

1. For R^i, T^j two closed intervals in degree i and j: $R^i \leq_\alpha T^j \iff i = j$ and $\mathrm{diam}(T) \leq \mathrm{diam}(R)$,
2. for U^i, V^j two open intervals in degree i and j: $U^i \leq_\alpha V^j \iff i = j$ and $\mathrm{diam}(U) \leq \mathrm{diam}(V)$,
3. for R^i a closed interval in degree i, and V^j an open interval in degree j: $R^i \leq_\alpha V^j \iff i = j + 1$.

Proposition 3.4.14

The relation $\leq_\alpha$ is a partial pre-order over the set of graded intervals, that is, it is reflexive and transitive. Moreover, it is total if restricted to subsets of graded intervals containing only, for a given $i \in \mathbb{Z}$, open intervals in degree i and closed intervals in degree $i + 1$.

The following is the analogous result in our setting to [Bje16, Lemma 4.6].

Proposition 3.4.15

Let I^i, J^i, S^l be three graded intervals of type C and $\varepsilon \geq 0$ such that $I^i \leq_\alpha J^j$ and there exists two non-zero morphisms $\chi : \mathbf{k}_S[-l] \star \mathrm{K}_\varepsilon \longrightarrow$

$\mathbf{k}_I[-i]$ and $\xi : \mathbf{k}_J[-j] \star \mathrm{K}_\varepsilon \longrightarrow \mathbf{k}_S[-l]$. Then either $\mathbf{k}_S[-l] \sim_\varepsilon \mathbf{k}_I[-i]$ or $\mathbf{k}_S[-l] \sim_\varepsilon \mathbf{k}_J[-j]$.

Proof

By definition of the pre-order $\leq_\alpha$, we only have to investigate the three cases of the above definition 3.4.13 :

1. Let $i \in \mathbb{Z}$ and R, T be two open intervals such that $R^i \leq_\alpha T^i$, that is, $\mathrm{diam}(T) \leq \mathrm{diam}(R)$. Let S^l be a graded interval such that there exists some non-zero χ and ξ. Then S must be a closed interval, and $l = i$. As a consequence, $R \subset S^\varepsilon$ and $S \subset T^\varepsilon$.

   Assume that $\mathbf{k}_R[-i] \not\sim_\varepsilon \mathbf{k}_S[-i]$. Then, as $R \subset S^\varepsilon$, $S \not\subset R^\varepsilon$. So either $\min(S) < \min(R) - \varepsilon$, or $\max(S) > \max(R) + \varepsilon$. Assume the latter.

   As $S \subset R^\varepsilon$, $\min(S) - \varepsilon < \min(R)$, we get subtracting the first inequality to this one:$\mathrm{diam}(S) + \varepsilon > \mathrm{diam}(R) + \varepsilon$. Hence $S <_\alpha R$. We get the same thing assuming $\min(S) < \min(R) - \varepsilon$.

   Moreover, one can prove this way that $\mathbf{k}_T[-i] \not\sim_\varepsilon \mathbf{k}_S[-i]$ implies $S <_\alpha R$.

   As we assumed $R^i \leq_\alpha T^i$, one has $\mathbf{k}_S[-l] \sim_\varepsilon \mathbf{k}_I[-i]$ or $\mathbf{k}_S[-l] \sim_\varepsilon \mathbf{k}_J[-j]$.

2. The proof for U^i, V^i where U and V are open intervals is similar.

3. Let R^i a closed interval, V^j an open interval, with $i = j + 1$. Let S^l be a graded interval and ε such that there exists χ and ξ such as in the proposition. Then S must be an open interval and $l = j$. By the existence of χ, we have that $\varepsilon \geq \frac{\mathrm{diam}(U)}{2}$ and $R \subset [\mathrm{cent}(U) - (\varepsilon - \frac{\mathrm{diam}(U)}{2}), \mathrm{cent}(U) + (\varepsilon - \frac{\mathrm{diam}(U)}{2})]$, which, according to our characterization of ε-interleaving between indecomposable sheaves (proposition 3.3.1), is equivalent to $\mathbf{k}_R[-j-1] \sim_\varepsilon \mathbf{k}_S[-j]$.

Induced matching

We now have the ingredients to prove the theorem.

Theorem 3.4.16 (Theorem 5.15 - [BG18])

Let $F = F_C$ and $G = G_C$ be two objects of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, and $\varepsilon \geq 0$ be such that F_C and G_C are ε-interleaved through maps $F_C \star \mathrm{K}_\varepsilon \xrightarrow{f} G_C$ and $G_C \star \mathrm{K}_\varepsilon \xrightarrow{g} F_C$. Then, for every $j \in \mathbb{Z}$, there exists a bijection

$$\sigma^j_C : \mathbb{B}^j_C(F_C) \longrightarrow \mathbb{B}^j_C(G_C)$$

such that, for $I \in \mathcal{B}^j_C$, we have $\mathbf{k}_I \sim_\varepsilon \mathbf{k}_{\sigma^j_C(I)}[-\delta]$, where $\delta = 0$ if I and $\sigma^j_C(I)$ are both open or both closed, and $\delta = 1$ if I is open and $\sigma^j_C(I)$ is closed, $\delta = -1$ if I is closed and $\sigma^j_C(I)$ is open.

Our proof will use a generalization of Hall's marriage theorem to the case of countable sets. It was first proved in 1976 by Podewski and Steffens [PS76]:

Theorem 3.4.17 (Hall)
Let X and Y be two countable sets, let $\mathcal{P}(Y)$ be the set of subsets of Y and $M : X \to \mathcal{P}(Y)$. Then the following are equivalent:

1. there exists an injective map $m : X \to Y$ satisfying $m(x) \in M(x)$ for every $x \in X$;
2. for every finite subset $A \subset X$, $|A| \leq |\cup_{x\in A} M(x)|$. Where $|A|$ is the cardinality of A.

We introduce some notations that will be used, developed and refined later on. For $F_C \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ a central sheaf, recall first that $F_C \simeq \oplus_{j\in\mathbb{Z}} \mathrm{H}^j(F_C)[-j]$, and for any $j \in \mathbb{Z}$, there exists a unique multi-set $\mathbb{B}^j_C(F)$ of bounded open or compact intervals of $\mathbb{R}$ (an interval can appear several times in the list) such that

$$\mathrm{H}^j(F) = \mathrm{H}^j(F_C) \simeq \bigoplus_{I\in\mathbb{B}^j_C(F)} \mathbf{k}_I,$$

where the last isomorphism is in the category $\mathrm{Mod}(\mathbf{k}_\mathbb{R})$.

In the following, we let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ for which we set two isomorphisms:

$$F \simeq \bigoplus_{j\in\mathbb{Z}} \bigoplus_{I^j\in\mathcal{B}^j(F)} \mathbf{k}_I[-j] \quad \text{and} \quad G \simeq \bigoplus_{j\in\mathbb{Z}} \bigoplus_{I^j\in\mathcal{B}^j(G)} \mathbf{k}_I[-j].$$

For any morphism $f : F \to G$, given $I^i \in \mathbb{B}^i(F)$ and $J^j \in \mathbb{B}^j(G)$, we will write:

$$f_{I^i,J^j} = \mathbf{k}_I[-i] \longrightarrow F \xrightarrow{f} G \longrightarrow \mathbf{k}_J[-j].$$

Similarly for $A \subset \mathbb{B}(F)$, let $f_{|A}$ be the composition:

$$\bigoplus_{I^i\in A} \mathbf{k}_I[-i] \longrightarrow F \xrightarrow{f} G.$$

Let F and G in $\mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$ be two central sheaves and $\varepsilon \geq 0$ such that F and G are ε-interleaved with respect to f and g. For $I^i \in \mathbb{B}^i_C(F)$ and $J^j \in \mathbb{B}^j_C(G)$ our computations of propositions 3.2.1, 3.2.11 and 3.3.1:

$$(f \star \mathrm{K}_\varepsilon)_{I^i,J^j} \circ g_{J^j,I^i} \neq 0 \text{ implies either: } \begin{cases} I, J \text{ are closed and } i = j, \\ I \text{ is open}, J \text{ is closed and } j = i+1, \\ J \text{ is open}, I \text{ is closed and } i = j+1. \end{cases}$$

Proof (matching of central parts)
Our strategy is to adapt Bjerkevik's proof of [Bje16, Theorem 4.2] to our setting. The pre-order $\leq_\alpha$ we have defined has exactly the same properties as the one defined in his proof.

Let $n \in \mathbb{Z}$, to define σ^n_C, we will apply Hall's theorem. Since graded-barcodes of F and G are locally finite, they are countable. We here consider multi-sets as sets, to make the proof easier to understand. Nevertheless, it would not be difficult to write the proof properly using multi-sets. Let $M : \mathbb{B}^n_C(F_C) \to \mathcal{P}(\mathbb{B}^n_C(G_C))$ defined by:

$$M(I) = \{J \in \mathbb{B}^n_C(G_C) \mid \mathbf{k}_I \sim_\varepsilon \mathbf{k}_J[-\delta],$$

where $\delta = 0$ if I and $\sigma^j_C(I)$ are both open or both closed, $\delta = 1$ if I is open

and $\sigma^j_C(I)$ is closed, $\delta = -1$ if I is closed and $\sigma^j_C(I)$ is open. $\}$,

for $I \in \mathbb{B}^n_C(F_C)$.

Thus, let A be a finite subset of $\mathbb{B}^n_C(F_C)$ and $M(A) = \cup_{I \in A} M(I)$. To apply Hall's theorem and deduce the existence of σ^n_C, we need to prove that $|A| \leq |M(A)|$. If $M(A)$ is infinite, the result is automatically true. Let us assume now that $M(A)$ is finite.

By proposition 3.4.14, $\leq_\alpha$ is a total pre-order on A. Hence, with $r = |A|$, there exists an enumeration $A = \{I_1^{i_1}, ..., I_r^{i_r}\}$, where $i_l = n$ if I_l is an open interval and $i_l = n+1$ if I_l is a closed interval, such that for $1 \leq l \leq m \leq r$ we have $I_l^{i_l} \leq_\alpha I_m^{i_m}$.

We have by assumption $g \circ (f \star \mathrm{K}_\varepsilon) = \phi_{F,2\varepsilon}$ (see definition 2.2.6), also, the additivity of the convolution functor implies the following equality for $I_l^{i_l} \in A$:

$$\phi_{\mathbf{k}_I[-i_l],2\varepsilon} = \mathbf{k}_I[-i_l] \star \mathrm{K}_\varepsilon \longrightarrow F \star \mathrm{K}_\varepsilon \xrightarrow{\phi_{F,2\varepsilon}} F \to \mathbf{k}_I[-i_l]$$

Therefore:

$$\begin{aligned}
\phi_{\mathbf{k}_I[-i_l],2\varepsilon} &= \sum_{J^j \in \mathbb{B}(G)} g_{J^j,I_l^{i_l}} \circ (f \star \mathrm{K}_\varepsilon)_{I_l^{i_l},J^j} \\
&= \sum_{J^j \in \mathbb{B}^l(G)} g_{J^j,I_l^{i_l}} \circ \left[f_{I_l^{i_l},J^j} \star \mathrm{K}_\varepsilon \right].
\end{aligned}$$

Now observe that if $g_{J^j,I_l^{i_l}} \circ \left[f_{I_l^{i_l},J^j} \star \mathrm{K}_\varepsilon \right] \neq 0$ then $\mathbf{k}_{I_l}[-i_l] \sim_\varepsilon \mathbf{k}_J[-j]$, hence:

$$\phi_{\mathbf{k}_I[-i_l],2\varepsilon} = \sum_{J^j \in M(A)} g_{J^j,I_l^{i_l}} \circ (f \star \mathrm{K}_\varepsilon)_{I_l^{i_l},J^j}$$

Similarly for $I_m \neq I_{m'}$ in A,

$$0 = \sum_{J^j \in \mathbb{B}^l(G)} g_{J^j,I_m^{i_m}} \circ \left[f_{I_{m'}^{i_{m'}},J^j} \star \mathrm{K}_\varepsilon \right].$$

Hence if $m < m'$ and $g_{J^j,I_m^{i_m}} \circ \left[f_{I_{m'}^{i_{m'}},J^j} \star \mathrm{K}_\varepsilon \right] \neq 0$, then $\mathbf{k}_J[-j]$ is ε-interleaved with either $\mathbf{k}_{I_m}[-m]$ or $\mathbf{k}_{I_{m'}}[-m']$. Therefore:

$$0 = \sum_{J^j \in M(A)} g_{J^j,I_m^{i_m}} \circ \left[f_{I_{m'}^{i_{m'}},J^j} \star \mathrm{K}_\varepsilon \right].$$

For $m > m'$, we can't say anything about the value of

$$\sum_{J^j \in M(A)} g_{J^j,I_m^{i_m}} \circ \left[f_{I_{m'}^{i_{m'}},J^j} \star \mathrm{K}_\varepsilon \right].$$

Writing those equalities in matrix form, we get :

$$\begin{pmatrix} g_{J^1,I_1^{i_1}} & \cdots & g_{J^1,I_r^{i_r}} \\ \vdots & \ddots & \vdots \\ g_{J^s,I_1^{i_1}} & \cdots & g_{J^s,I_r^{i_r}} \end{pmatrix} \begin{pmatrix} f_{I_1^{i_1},J^1} \star \mathrm{K}_\varepsilon & \cdots & f_{I_r^{i_r},J^1} \star \mathrm{K}_\varepsilon \\ \vdots & \ddots & \vdots \\ f_{I_1^{i_1},J^s} \star \mathrm{K}_\varepsilon & \cdots & f_{I_r^{i_r},J^s} \star \mathrm{K}_\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} \phi_{I_1^{i_1},2\varepsilon} & ? & ? & ? \\ 0 & \phi_{I_2^{i_2},2\varepsilon} & ? & ? \\ \vdots & \vdots & \ddots & ? \\ 0 & 0 & \cdots & \phi_{I_r^{i_r},2\varepsilon} \end{pmatrix}$$

Now recall that $\mathrm{R}\Gamma(\mathbb{R},-)$ is an additive functor. Hence, applying $\mathrm{R}\Gamma(\mathbb{R},-)$ to the above equality, we get:

$$\begin{pmatrix} \mathrm{R}\Gamma(\mathbb{R}, g_{J^1,I_1^{i_1}}) & \dots & \mathrm{R}\Gamma(\mathbb{R}, g_{J^1,I_r^{i_r}}) \\ \vdots & \ddots & \vdots \\ \mathrm{R}\Gamma(\mathbb{R}, g_{J^s,I_1^{i_1}}) & \dots & \mathrm{R}\Gamma(\mathbb{R}, g_{J^s,I_r^{i_r}}) \end{pmatrix} \begin{pmatrix} \mathrm{R}\Gamma(\mathbb{R}, f_{I_1^{i_1},J^1} \star \mathrm{K}_\varepsilon) & \dots & \mathrm{R}\Gamma(\mathbb{R}, f_{I_r^{i_r},J^1} \star \mathrm{K}_\varepsilon) \\ \vdots & \ddots & \vdots \\ \mathrm{R}\Gamma(\mathbb{R}, f_{I_1^{i_1},J^s} \star \mathrm{K}_\varepsilon) & \dots & \mathrm{R}\Gamma(\mathbb{R}, f_{I_r^{i_r},J^s} \star \mathrm{K}_\varepsilon) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{R}\Gamma(\mathbb{R}, \phi_{I_1^{i_1},2\varepsilon}) & ? & ? & ? \\ 0 & \mathrm{R}\Gamma(\mathbb{R}, \phi_{I_2^{i_2},2\varepsilon}) & ? & ? \\ \vdots & \vdots & \ddots & ? \\ 0 & 0 & \dots & \mathrm{R}\Gamma(\mathbb{R}, \phi_{I_r^{i_r},2\varepsilon}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ \vdots & \vdots & \ddots & ? \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Each entry in those matrices is uniquely characterized by one scalar. Hence, we can consider their rank. The left hand side has rank at most equal to the minimum of r and s, in particular it is less or equal to $|M(A)|$. The right-hand side has rank $r = |A|$. Therefore we get the inequality we wanted.

□

3.4.4 Isometry theorem

In this section, we put together the results proved before to prove that the convolution distance between two sheaves is exactly the same as the bottleneck distance between their graded-barcodes.

Theorem 3.4.18 (Theorem 5.17 - [BG18])
Let F, G be two objects of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, then:

$$d_C(F,G) = d_B(\mathbb{B}(F), \mathbb{B}(G)).$$

Proof
By lemma 3.4.9, there only remains to prove that $d_C(F,G) \geq d_B(\mathbb{B}(F), \mathbb{B}(G))$, or equivalently, that any ε-interleaving between F and G induces an ε-matching between $\mathbb{B}(F)$ and $\mathbb{B}(G)$.

According to sections 5.3 and 5.4, this interleaving induces a ε-matching between the central, left and right parts of F and G, which proves the theorem.

The formulation of the convolution distance as a matching distance we obtained here turns the computation of an algebraic problem into minimizing the cost of a matching, which is of combinatorial nature. This is in fact a variant of a very classical problem of linear programming, for which there exists an abundant literature and that can be solved in polynomial time in the total numbers of bars in the barcodes, using the *Hungarian algorithm* [Kuh09]. Hence, distances in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ *can be implemented* in a computer and *computed*. Although at this point, we can compute distances between sheaves, we do not yet know how to compute their barcodes. This will be the purpose of the next chapter. This will lead to an equivalent formulation to the matching problem associated with the graded bottleneck distance that is even simpler to implement (see corollary 4.5.5).

3.5 Applications

In this section, we expose some corollaries of the isometry theorem. We start with some explicit computations on an example, showing the fundamentally derived nature ouf our graded-bottleneck distance. Then, we prove that d_C is closed between constructible sheaves on $\mathbb{R}$, that is, two constructible sheaves are ε-close if and only if they are ε-interleaved, which in particular implies that d_C induces a metric on isomorphism classes of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. This allows us to consider the set of isomorphism classes of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ as a topological metric space. We prove that it is locally path-connected and give a characterization of its connected components. We also provide a counter-example of two non constructible sheaves which are at convolution distance 0 and which are not isomorphic.

3.5.1 Example: projection from the circle

We aim here to explain and compute an explicit example that was pointed to us by Justin Curry, that is two simple projections from the euclidean circle to the real line. Understanding this example has been at the origin of our reflexions. It is simple yet general enough to exhibit the phenomenons and issues that can happen with the matchings of graded barcodes.

Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle seen as a submanifold in $\mathbb{R}^2$. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be the first coordinate projection and $g : \mathbb{S}^1 \to \mathbb{R}$ be the constant map with value zero. Let $F = \mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}$ and $G = \mathrm{R}g_*\mathbf{k}_{\mathbb{S}^1}$. Since $\|f - g\| = 1$, the stability theorem by Kashiwara and

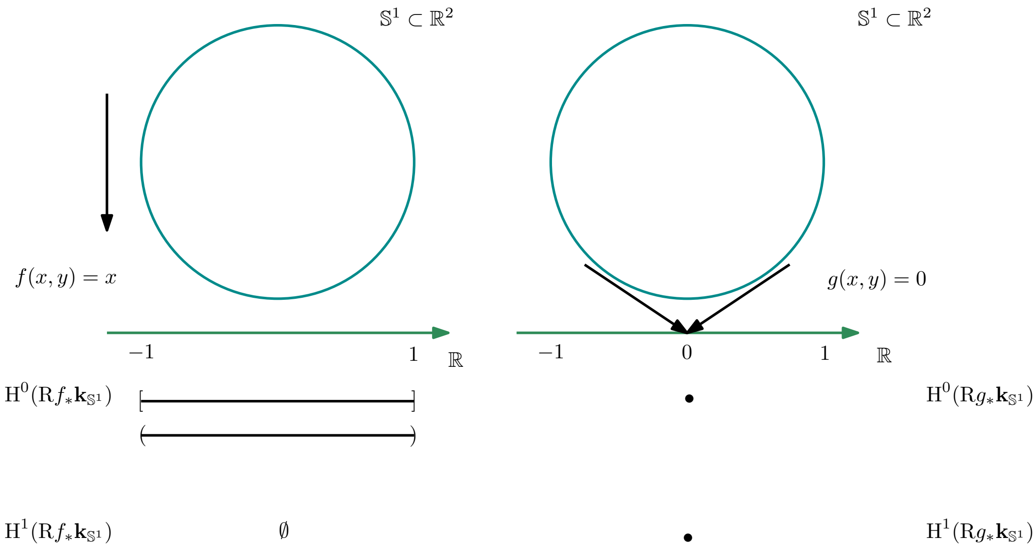


Figure 3.1 – The maps f and g and the graded-barcodes of $\mathrm{R}f_*\mathbf{k}_X$ and $\mathrm{R}g_*\mathbf{k}_X$.

Schapira [KS18a, theorem 2.7] implies:

$$d_C(F,G) \leq 1.$$

The CLR decomposition (Definition 3.3.3) of this two complexes of sheaves is easy to compute (and depicted in the figure below).

Proposition 3.5.1
The complexes F and G have non-zero cohomology spaces at most in degree 0 and 1. Moreover:

1. $\mathrm{H}^0(F) \simeq \mathbf{k}_{(-1,1)} \oplus \mathbf{k}_{[-1,1]}$ and $\mathrm{H}^1(F) \simeq 0$
2. $\mathrm{H}^0(G) \simeq \mathbf{k}_{\{0\}}$ and $\mathrm{H}^1(G) \simeq \mathbf{k}_{\{0\}}$

Hence, F and G are central sheaves and $\mathbb{B}^0_C(F) = \{[-1,1],(-1,1)\}$, $\mathbb{B}^1_C(F) = \emptyset$, $\mathbb{B}^0_C(G) = \{\{0\}\}$, $\mathbb{B}^1_C(G) = \{\{0\}\}$. Even in this simple example, there could be *no* ε-matching between the graded-barcodes if one was working in the ordinary graded category. Indeed, $d_C(\mathbf{k}_{\{0\}}[-1],0) = +\infty$. Hence this will not agree both with Kashiwara-Schapira (naturally derived) convolution distance and the intuition from persistence.

However, using our derived notion of interleavings and matching distance we get the expected answer and in fact prove that in this case the bound given by the L_∞-norm between the function is optimal.

Indeed, let $(\sigma_C^j)_{j\in\mathbb{Z}}$ be defined by :

$$\sigma_C^0([-1,1]^0) = \{0\}^0 \quad \text{and} \quad \sigma_C^0((-1,1)^0) = \{0\}^1$$

Then we claim that σ_C is a 1-matching between $\mathbb{B}_C(F)$ and $\mathbb{B}_C(G)$. As $F = F_C$ and $G = G_C$, it extends trivially to a 1-matching between $\mathbb{B}(F)$ and $\mathbb{B}(G)$. Moreover, since the convolution distances between any pair of graded intervals is at least 1, there can not exist an ε-matching between $\mathbb{B}(F)$ and $\mathbb{B}(G)$ for $0 \leq \varepsilon < 1$. Hence we have $F \sim_1 G$ and further

Proposition 3.5.2
The convolution distance of $F = \mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}$ and $G = \mathrm{R}g_*\mathbf{k}_{\mathbb{S}^1}$ is

$$d_C(\mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}, \mathrm{R}g_*\mathbf{k}_{\mathbb{S}^1}) = 1.$$

3.5.2 About the closedness of d_C

In this section we apply our isometry Theorem 3.4.18 to answer an open question on the closedness of the convolution distance (see Remark 2.3 of [KS18a]). More precisely, we show that the convolution distance is closed between constructible sheaves over $\mathbb{R}$. We also provide a counter-example to this statement without constructibility assumption.

Theorem 3.5.3 (Theorem 6.3 - [BG18])
The convolution distance is closed on $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. That is, for $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$ and $\varepsilon \geq 0$:

$$d_C(F,G) \leq \varepsilon \text{ if and only if } F \sim_\varepsilon G.$$

Corollary 3.5.4
d_C induces a metric on the isomorphism classes of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$.

We start with the following easy lemma, whose proof is a direct consequence of section 3.3.1.

Lemma 3.5.5
Let I^i, J^j two graded intervals (possibly empty, we set $\mathbf{k}_\emptyset = 0$) and $\varepsilon \geq 0$. Then:

$$d_C(\mathbf{k}_I[-i], \mathbf{k}_J[-j]) \leq \varepsilon \iff \mathbf{k}_I[-i] \sim_\varepsilon \mathbf{k}_J[-j].$$

Proof (of the theorem)
Suppose $d_C(F,G) \leq \varepsilon$. Then by definition there exists a decreasing sequence (ε_n) such that $\varepsilon_n \to \varepsilon$ when n goes to infinity and for every $n \in \mathbb{N}$, $F \sim_{\varepsilon_n} G$. For simplicity of the proof, we will assume the graded-barcodes of F and G to be finite, but the proof generalizes to the locally finite case.

Then by applying the isometry theorem, for $n \geq 0$, there exists a ε_n matching $\sigma_n : \mathbb{B}(F) \to \mathbb{B}(G)$.

Now by finiteness of the graded-barcodes, the set of matchings between $\mathbb{B}(F)$ and $\mathbb{B}(G)$ is finite. Hence, we can extract from (σ_n) a constant sequence, say $(\sigma_{\varphi(n)})$. Applying lemma 4.1 and making n going to infinity, we see that $\sigma := \sigma_{\varphi(0)}$ is an ε-matching between $\mathbb{B}(F)$ and $\mathbb{B}(G)$.

Remark 3.5.6

One must pay attention to the fact that in the case of persistence modules, the interleaving distance is *not* closed. There exists some ephemeral modules at distance 0 from 0: consider the one parameter persistence module $\mathbf{k}^{\{0\}}$ (keeping notations of section 5.2). To avoid this issue, Chazal, Crawley-Boevey and de Silva introduced the observable category of persistence modules $\mathrm{Obs}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}}))$ in [CCBS16]. It is defined as the quotient category of $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$ by the full sub-category of ephemeral persistent modules, which has objects $M \in \mathrm{Pers}(\mathbb{R})$ such that $M(s < t) = 0$ for every $s < t \in \mathbb{R}$. By construction, the interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}})$ induces a closed metric on $\mathrm{Obs}(\mathrm{Pers}(\mathbf{k}^{\mathbb{R}}))$. Note that we will generalize this construction in chapter 5.

Corollary 3.5.7

The functors $\Psi_R^j : \mathrm{D}_R^j \to \mathrm{Pers}(\mathbb{R})^{\mathrm{op}}$ introduced in 5.2 induces an isometric equivalence of category between D_R^j and $\mathrm{Obs}(\mathrm{Pers}(\mathbb{R}))^{op}$.

Note that, this result is the manifestation in dimension one of the more general proposition 5.3.13, since one can easily observe, that $\mathrm{D}_R^j \simeq \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}_\gamma})$, where $\gamma = [0, +\infty[$ and $\mathbb{R}_\gamma$ is the topological space $\mathbb{R}$ endowed with the γ-topology (section 5.2.1).

We now construct a counter-example to the closedness of d_C without constructibility assumptions. More precisely, we will construct two sheaves $F, G \in \mathrm{D}^b(\mathbf{k}_{\mathbb{R}})$ such that $d_C(F,G) = 0$ but $F \not\simeq G$. We consider the sets $X = \mathbb{Q} \cap [0,1]$ and $Y = \sqrt{2}\mathbb{Q} \cap [0,1] = \{\sqrt{2}q \mid q \in \mathbb{Q}\} \cap [0,1]$.

Proposition 3.5.8

There exists a sequence of functions $(r_n)_{n \in \mathbb{Z}_{>0}}$ from X to Y satisfying:

1. for any $n \in \mathbb{Z}_{>0}$, $r_n : X \to Y$ is bijective,
2. $\sup_{x \in X} |r_n(x) - x| \xrightarrow[n \to +\infty]{} 0$.

Proof

Let $n \in \mathbb{Z}_{n>0}$. We define r_n piecewise on $[0, 1 - \frac{1}{n}] \cap \mathbb{Q}$ and $]1 - \frac{1}{n}, 1] \cap \mathbb{Q}$. For $q \in [0, 1 - \frac{1}{n}] \cap \mathbb{Q}$, we set $r_n(q) = \frac{\sqrt{2}}{\lceil \sqrt{2} \rceil_{10}(n)} q$, with $\lceil \sqrt{2} \rceil_{10}(n) = \frac{\lceil 10^n \sqrt{2} \rceil}{\lceil 10^n \rceil}$

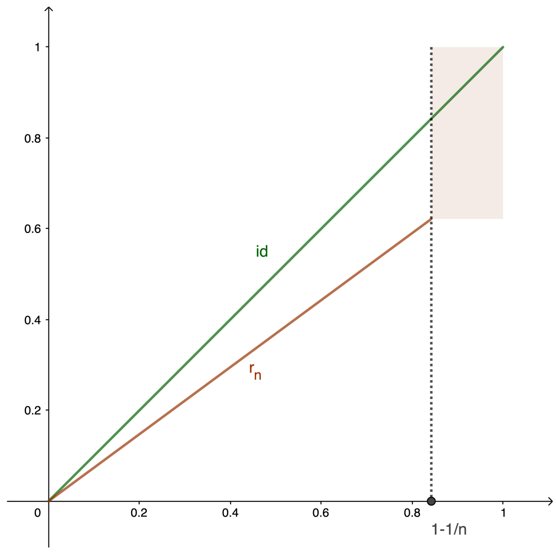


Figure 3.2 – Graphical representation of r_n

the n-th ceil decimal approximation of $\sqrt{2}$. Then $r_{n|[0,1-\frac{1}{n}]\cap\mathbb{Q}}$ is injective, and

$$\begin{aligned} r_n\left([0,1-\frac{1}{n}]\cap\mathbb{Q}\right) &= \left\{\sqrt{2}q \mid q\in\left[0,\frac{1-1/n}{\lceil\sqrt{2}\rceil_{10}(n)}\right]\cap\mathbb{Q}\right\} \\ &= Y\cap\left[0,\sqrt{2}\frac{1-1/n}{\lceil\sqrt{2}\rceil_{10}(n)}\right] \\ &\subsetneq Y. \end{aligned}$$

Now, since $]1-\frac{1}{n},1]\cap\mathbb{Q}$ and $Y\backslash r_n\left([0,1-\frac{1}{n}]\cap\mathbb{Q}\right)$ are both infinite subsets of $\mathbb{Q}$, there exists a bijection

$$\varphi_n : \left]1-\frac{1}{n},1\right]\cap\mathbb{Q}\xrightarrow{\sim} Y\backslash r_n\left([0,1-\frac{1}{n}]\cap\mathbb{Q}\right).$$

We define $r_{n|]1-\frac{1}{n},1]\cap\mathbb{Q}}=\varphi_n$. Then $r_{n|]1-\frac{1}{n},1]\cap\mathbb{Q}}$ is injective and

$$r_n(]1-\frac{1}{n},1]\cap\mathbb{Q}) = Y\backslash r_n\left([0,1-\frac{1}{n}]\cap\mathbb{Q}\right).$$

Finally, r_n is indeed a bijective function from X to $Y=\sqrt{2}\mathbb{Q}\cap[0,1]$. Observe that:

$$\sup_{x\in X}|r_n(x)-x| = \max\left(\sup_{x\in X\cap[0,1-\frac{1}{n}]}|r_n(x)-x|, \sup_{x\in X\cap[1-\frac{1}{n},1]}|r_n(x)-x|\right).$$

The first term of the maximum is worth $\left(\frac{\sqrt{2}}{\lceil\sqrt{2}\rceil_{10}(n)}-1\right)(1-1/n)$, and the second term is bounded from above by the diameter of the interval $[\sqrt{2}\frac{1-1/n}{\lceil\sqrt{2}\rceil_{10}(n)},1]$ which is worth $1-\sqrt{2}\frac{1-1/n}{\lceil\sqrt{2}\rceil_{10}(n)}$. Since both of these terms go to 0 as n goes to infinity, we deduce the desired property:

$$\sup_{x\in X}|r_n(x)-x| \xrightarrow[n\to+\infty]{} 0.$$

Proposition 3.5.9
Let $(F_i)_{i\in I}$ and $(G_j)_{j\in J}$ be two families of objects of $\mathrm{D}^b(\mathbf{k}_\mathbb{V})$. Assume that there exists a bijective function $\sigma : I \to J$, and $\varepsilon \geq 0$ such that for all $i \in I$, $d_C(F_i, G_{\sigma(i)}) \leq \varepsilon$. Then:

$$d_C\left(\bigoplus_{i\in I} F_i, \bigoplus_{j\in J} G_j\right) \leq \varepsilon.$$

Proof
Let $\varepsilon' > \varepsilon$ and $i \in I$. Then by assumptions, there exists some ε'-interleaving morphisms between F_i and $G_{\sigma(i)}$, $\varphi_i : F_i \star K_{\varepsilon'} \to G_{\sigma(i)}$ and $\psi_i : G_{\sigma(i)} \star K_{\varepsilon'} \to F_i$. Since $-\star K_{\varepsilon'}$ is a left-adjoint functor, it commutes with arbitrary colimits. Therefore, by taking direct sums of the previous ε'-interleaving morphisms, we get ε'-interleaving morphisms between $\bigoplus_{i\in I} F_i$ and $\bigoplus_{i\in I} G_{\sigma(i)} \simeq \bigoplus_{j\in J} G_j$, which proves the result.

$\square$

Let $F = \bigoplus_{x\in X} \mathbf{k}_{\{x\}}$ and $G = \bigoplus_{y\in Y} \mathbf{k}_{\{y\}}$.

Proposition 3.5.10
F is not isomorphic to G and $d_C(F,G) = 0$.

Proof
F and G cannot be isomorphic since $F_1 \simeq \mathbf{k}$ and $G_1 \simeq 0$.

Let $r_n : X \to Y$ as in proposition 3.5.8. Using proposition 3.5.9, and the fact that for $x, y \in \mathbb{R}$, $d_C(\mathbf{k}_{\{x\}}, \mathbf{k}_{\{y\}}) = |x-y|$, we obtain that for any $n \in \mathbb{Z}_{>0}$:

$$d_C(F,G) \leq \sup_{x\in X} |r_n(x) - x|.$$

Taking the limit as n goes to infinity, we deduce that $d_C(F,G) = 0$.

□

We conjecture that this results extend to higher dimensional real vector spaces. More precisely, consider $\mathbb{V}$ a finite-dimensional real vector space, endowed with a norm $\|\cdot\|$ and consider the convolution distance associated to this norm.

Conjecture 3.5.11
Let F and G in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{V}})$, and $\varepsilon \geq 0$. Then:

$$F \sim_\varepsilon G \iff d_C(F,G) \leq \varepsilon.$$

3.5.3 Description of the connected components of $\mathbf{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$

In this section, we introduce the small category **Barcode** as a combinatorial description of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. The category **Barcode** is shown to be skeletal (any two isomorphic objects are equal), and is equipped with the graded bottleneck distance. We thus obtain an extended metric space, which is locally path-connected. To do so, we prove an interpolation lemma in the same fashion as Chazal et al. [CdSGO16, Theorem 3.5], which stands that if two sheaves are ε-interleaved, there exists a 1-lipschitz path in $D^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ between them.

Lemma 3.5.12 (Interpolation)
Let $F, G \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}))$ be such that $F \sim_\varepsilon G$ for some $\varepsilon \geq 0$. Then there exists a family of sheaves $(U_t)_{t\in[0,\varepsilon]}$ in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ such that :

1. $U_0 = F$ and $U_\varepsilon = G$.
2. For $t \in [0,\varepsilon]$, $d_C(F,U_t) \leq t$ and $d_C(G,U_t) \leq \varepsilon - t$.
3. For $(t,t') \in [0,\varepsilon]^2$, $d_C(U_t,U_{t'}) \leq |t-t'|$.

Proof
Let $F \star K_\varepsilon \xrightarrow{\varphi} G$ and $G \star K_\varepsilon \xrightarrow{\psi} F$ be the interleaving morphisms between F and G.

We start by constructing U_t for $t \in [0, \frac{\varepsilon}{2}]$. The interleaving morphism and the canonical maps in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ give (by Proposition 2.2.5 and (2.8)) the following diagram $\mathbb{D}_t$:

$$\begin{tikzcd}
G \star K_{t-\varepsilon} \arrow[d, "\phi_{G,2t}\star K_{t-\varepsilon}"'] \arrow[dr, "\psi\star K_{t-\varepsilon}" near start] & F \star K_{-t} \arrow[d, "\phi_{F,2\varepsilon-2t}\star K_{-t}"] \arrow[dl, "\varphi\star K_{-t}"' near start] \\
G \star K_{-t-\varepsilon} & F \star K_{t-2\varepsilon}
\end{tikzcd} .$$

Taking resolutions in $\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$, one can assume this diagram is actually given by a diagram still denoted $\mathbb{D}_t$ in $C(\mathrm{Mod}(\mathbf{k}_{\mathbb{R}}))$ which we assume from now on. One can note that this diagram defines the two maps $\theta_t, \tilde{\phi}_t$: $(G \star K_{t-\varepsilon}) \oplus (F \star K_{-t}) \longrightarrow (G \star K_{-t-\varepsilon}) \oplus (F \star K_{t-2\varepsilon})$ given by $(x, y) \overset{\theta_t}{\mapsto} (\varphi \star K_{-t}(y), \psi \star K_{t-\varepsilon}(x))$ and $(x, y) \overset{\tilde{\phi}_t}{\mapsto} (\phi_{G,2t} \star K_{t-\varepsilon}(x), \phi_{F,2\varepsilon-2t} \star K_{-t}(y))$. The limit $\varprojlim \mathbb{D}_t$ of the diagram is precisely (isomorphic to) the equalizer of the two maps and thus to the kernel $\ker(\theta_t - \tilde{\phi}_t)$ of their difference.

Since we are dealing with a diagram in $C(\mathrm{Mod}(\mathbf{k}_{\mathbb{R}}))$ that we wish to see in the derived category we essentially only need to replace the limit by its homotopy limit. Namely, we define $\tilde{U}_t := \mathrm{ho}\varprojlim \mathbb{D}_t$ to be the homotopy limit in (the model category of sheaves [Cra95]) $C(\mathrm{Mod}(\mathbf{k}_{\mathbb{R}}))$ of the diagram $\mathbb{D}_t$. For the reader who wish to avoid the use of the technique of homotopy limits in model category (see [Dug]), in view of the above identification, an explicit model for this homotopy limit is given by the cocone of the map $(G \star K_{t-\varepsilon}) \oplus (F \star K_{-t}) \overset{\theta_t - \tilde{\phi}_t}{\longrightarrow} (G \star K_{-t-\varepsilon}) \oplus (F \star K_{t-2\varepsilon})$ and hence passes to $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$.

We need to prove that $\tilde{U}_t$ is t-interleaved with F. Note that by definition of an homotopy limit we have a canonical map $\tilde{U}_t \to (G \star K_{t-\varepsilon}) \oplus (F \star K_{-t})$, simply given by the obvious map on the cocone above. Hence we have a morphism to either factor, in particular we have $\tilde{U}_t \overset{f}{\to} F \star K_{-t}$ and hence (by proposition 2.2.5) a map

$$\tilde{U}_t \star K_t \overset{f}{\to} F. \tag{3.1}$$

We now need to define a map $g : F \star K_t \to \tilde{U}_t$. By definition of an homotopy limit, we have a canonical factorization $\varprojlim \mathbb{D}_t \to \tilde{U}_t \to (G \star K_{t-\varepsilon}) \oplus (F \star K_{-t})$ of the canonical map defined by the limit in $C(\mathrm{Mod}(\mathbf{k}_{\mathbb{R}}))$. At the explicit cocone level this is just given by the canonical map of the limit on its first summand $(G \star K_{t-\varepsilon}) \oplus (F \star K_{-t})$

The interleaving map $\varphi : F \star K_\varepsilon \to G$ induces the map

$$F \star K_t \overset{(\varphi \star K_{t-\epsilon}, \phi_{F,2t} \star K_t)}{\longrightarrow} (G \star K_{t-\varepsilon}) \oplus (F \star K_{-t}) \tag{3.2}$$

which makes the following diagram

$$\begin{tikzcd}
 & F \star K_t \arrow[dl, "\varphi\star K_{t-\epsilon}"'] \arrow[dr, "\phi_{F,2t}\star K_t"] & \\
G \star K_{t-\varepsilon} \arrow[d, "\phi_{G,2t}\star K_{t-\varepsilon}"'] \arrow[drr, "\psi\star K_{t-\varepsilon}" near start] & & F \star K_{-t} \arrow[d, "\phi_{F,2\varepsilon-2t}\star K_{-t}"] \arrow[dll, "\varphi\star K_{-t}"' near start] \\
G \star K_{-t-\varepsilon} & & F \star K_{t-2\varepsilon}
\end{tikzcd}$$

commutative since φ, ψ defines a ε-interleaving. This implies that the map (3.2) factors through $\varprojlim \mathbb{D}_t$ and hence we get a map $g : F \star K_t \to \varprojlim \mathbb{D}_t \to \tilde{U}_t$ in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$.

For $t \in]\frac{\varepsilon}{2}, \varepsilon]$, we construct U_t in a similar fashion by intertwining the roles of F and G in the diagram $C(\mathrm{Mod}(\mathbf{k}_\mathbb{R}))$.

Let $\Delta_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y - x \leq \varepsilon\}$ be equipped with the standard product order of $\mathbb{R}^2$: $(x, y) \leq (x', y') \iff x \leq x'$ and $y \leq y'$. Observe that the mapping :

$$\Delta_\varepsilon \ni (x, y) \rightsquigarrow U_{y-x} \star K_{-x-y}$$

induces a well defined functor $(\Delta_\varepsilon, \leq) \longrightarrow \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ whose restriction to the poset $\{(x, y) \in \mathbb{R}^2 \mid y - x = t\}$ is the functor : $(x, y) \longrightarrow U_t \star K_{-x-y}$ with internal maps given by the natural morphisms $(\phi_{U_t,\varepsilon})$. Hence, for $\varepsilon \geq t, t' \geq 0$, U_t and $U_{t'}$ are $|t - t'|$ interleaved.

□

We write $\mathrm{Int}(\mathbb{R})$ the set of intervals of $\mathbb{R}$. Let p_1 and p_2 be the two first coordinate projections of $\mathrm{Int}(\mathbb{R}) \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Let $\mathbb{B}$ be a subset of $\mathrm{Int}(\mathbb{R}) \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Then $\mathbb{B}$ is said to be *locally finite* if $p_1(\mathbb{B}) \cap K$ is finite for all compact subsets of $\mathbb{R}$. It is said to be *bounded* is $p_2(\mathbb{B}) \subset \mathbb{Z}$ is bounded. Moreover, $\mathbb{B}$ is *well-defined* if the fibers of the projection (p_1, p_2) have cardinality at most 1.

We define the category **Barcode** as follows :

$$\mathrm{Obj}(\mathbf{Barcode}) = \{\mathbb{B} \subset \mathrm{Int}(\mathbb{R}) \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid$$

$$\mathbb{B} \text{ is bounded, locally finite and well-defined}\},$$

for any $\mathbb{B}, \mathbb{B}' \in \mathrm{Obj}(\mathbf{Barcode})$,

$$\mathrm{Hom}_{\mathbf{Barcode}}(\mathbb{B}, \mathbb{B}') = \prod_{\substack{(I,j,n)\in\mathbb{B} \\ (I',j',n')\in\mathbb{B}'}} \mathrm{Hom}_{\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})}\left(\mathbf{k}_I^n[-j], \mathbf{k}_{I'}^{n'}[-j']\right).$$

We define the composition in **Barcode** so that the mapping :

$$\iota : \mathrm{Obj}(\mathbf{Barcode}) \ni \mathbb{B} \mapsto \bigoplus_{(I,j,n)\in\mathbb{B}} \mathbf{k}_I^n[-j] \in \mathrm{Obj}(\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}))$$

becomes a fully faithful functor :

$$\iota : \mathbf{Barcode} \longrightarrow \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}).$$

Note that this is possible only because the objects of **Barcode** are locally finite. Theorems 3.1.4 and 3.1.2 assert that ι is essentially surjective, therefore is an equivalence. We also deduce from these theorems that **Barcode** is a skeletal category: it satisfies for any $\mathbb{B}, \mathbb{B}' \in \mathrm{Obj}(\mathbf{Barcode})$,

$$\mathbb{B} \simeq \mathbb{B}' \text{ if and only if } \mathbb{B} = \mathbb{B}'.$$

The notion of equality is well-defined here since $\mathrm{Obj}(\mathbf{Barcode})$ is a set. Therefore ι identifies its image as a skeleton of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, a full-subcategory which is dense and skeletal.

Moreover, from these theorems, we can equip the set $\mathrm{Obj}(\mathbf{Barcode})$ with the graded-bottleneck distance (definition 3.4.6), and we deduce from the derived isometry theorem 3.4.18 that for any $\mathbb{B}, \mathbb{B}' \in \mathrm{Obj}(\mathbf{Barcode})$:

$$d_C(\iota(\mathbb{B}), \iota(\mathbb{B}')) = d_B(\mathbb{B}, \mathbb{B}').$$

Theorem 3.5.13
The following assertions hold :

1. $(\mathrm{Obj}(\mathbf{Barcode}), d_B)$ is an extended metric space,
2. $(\mathrm{Obj}(\mathbf{Barcode}), d_B)$ is locally path-connected.

Proof

1. The fact that d_B is a pseudo-extended metric is inherited from the properties of d_C (proposition 2.2.9) by the derived isometry theorem. Moreover, if $d_B(\mathbb{B}, \mathbb{B}') = 0$ then $\mathbb{B} = \mathbb{B}'$ by theorem 3.5.3.
2. We will prove that open balls are path-connected, that is, any two barcodes at finite distance can be connected by a continuous path. Let $\mathbb{B}_0$ and $\mathbb{B}_\varepsilon$ in $\mathrm{Obj}(\mathbf{Barcode})$ such that $d_B(\mathbb{B}_0, \mathbb{B}_\varepsilon) = \varepsilon$. According

to the interpolation lemma 3.5.12, there exists a family of objects $(F_t)_{t\in[0,\varepsilon]}$ of $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ such that $F_0 = \iota(\mathbb{B}_0)$, $F_\varepsilon = \iota(\mathbb{B}_\varepsilon)$, and for any $t, t' \in [0,\varepsilon]$, $d_C(F_t, F_{t'}) \leq |t - t'|$. Given $t \in [0,\varepsilon]$, define $\mathbb{B}_t$ to be the graded-barcode of F_t. Then, thanks to the derived isometry theorem, $(t \mapsto \mathbb{B}_t)$ defines a 1-lipschitz path between $\mathbb{B}_0$ and $\mathbb{B}_\varepsilon$.

Remark 3.5.14

One shall be aware that it is not sufficient for being in the same connected component to have graded-barcodes of the same "type", as one can see in the following counter-example. Define:

$$F = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathbf{k}_{\left(\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right)} \quad \text{and} \quad G = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathbf{k}_{\left(\frac{(n+1)^2}{2}-1, \frac{(n+1)^2}{2}+1\right)}$$

Then $d_C(F, G) = +\infty$, hence F and G do not belong to the same connected component, whereas one might think that the obvious matching between their barcodes leads to a continuous path between the two sheaves.

CHAPTER 4

Level-sets persistence and sheaves

Abstract

In this chapter we provide an explicit connection between persistence theory on the one hand, and sheaf theory on the other hand. To do so, we observe that the collection of 2-parameter persistence modules arising in level-sets persistence of real-valued functions carry extra-structure, that we call Mayer-Vietoris systems. We establish a functorial correspondence between Mayer-Vietoris systems and derived sheaves on $\mathbb{R}$, which exchanges the level-sets persistence of a real-valued function $f : X \to \mathbb{R}$ with the derived pushforward by this function of the constant sheaf on X. This correspondence yields what we call an isometric pseudo-equivalence of categories between level-sets persistence modules with the interleaving distance, and derived constructible sheaves with the convolution distance of Kashiwara and Schapira. Ultimately, it allows to relate the barcodes and bottleneck distances of each of these theories. This content was developed in collaboration with Grégory Ginot and Steve Oudot in [BGO19].

4.1 Introduction

Level-sets persistence studies the homology groups of preimages $\mathrm{H}_i^{\mathrm{sing}}(f^{-1}(]s,t[)$, where $]s,t[$ is the French notation [1] for the open interval (s,t). The collection of these groups for $s < t \in \mathbb{R}$, together with the collection of morphisms induced by inclusions of smaller intervals into larger intervals, form a two-parameter persistence module indexed over the upper half-plane $\Delta^+ = \{(x,y) \mid x + y > 0\}$ via the identification of each interval $]s,t[$ with the point $(-s,t)$. This module is called the *i-th level-sets persistence module* of f and denoted by $\mathcal{L}_i(f)$ (see 4.2.14). Note that the plane $\mathbb{R}^2$ is equipped with the partial product order, noted $\leq$. Henceforth we will write $\mathrm{Pers}(\mathbf{k}^{\mathbb{R}^2})$ for the category of persistence modules indexed over $(\mathbb{R}^2, \leq)$, and $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ for its counterpart over $(\Delta^+, \leq)$. Unfortunately, though very natural, the theory of general 2-parameter persistence modules is significantly more complicated than 1-parameter persistence. For instance, there is no analogue of barcodes, due to the poset $(\mathbb{R}^2, \leq)$ being a wild-type quiver with arbitrarily complicated indecomposables. Nevertheless, one can still define an interleaving distance in this context, satisfying the same stability and universality properties as in 1-parameter persistence as explained in section 2.1.4.

Another very promising direction of investigation is given by merging sheaf theory with persistence and computer-friendly techniques. It was pioneered by the work of Curry [Cur14], and a general framework was developed by Kashiwara and Schapira [KS18a, KS18b]. In order to benefit fully from the cohomology of sheaves, it is necessary to work with the derived category. Kashiwara-Schapira have equipped the derived category of sheaves on a real vector space with a distance, called the *convolution distance*, which is a derived analogue of the interleaving distance, see definition 2.2.8. Furthermore, there is a natural notion of barcode and a decomposition theorem for *constructible* sheaves over $\mathbb{R}$. To a function $f : X \to \mathbb{R}$, one can associate a canonical sheaf over $\mathbb{R}$, namely the derived pushforward $\mathrm{R}f_*\mathbf{k}_X$, which is a sheaf analogue of the level-sets persistence homology introduced earlier (here $\mathbf{k}$ is our ground field). We have already studied in depth in chapter 3 the persistence theory for sheaves over $\mathbb{R}$, following Kashiwara and Schapira's program. In particular, a *derived bottleneck distance* for constructible sheaves was developed and proven to be *isometric to the convolution distance.*

---

1. We adopt this notation for the sake of clarity, to avoid potential confusions with the point $(s,t) \in \mathbb{R}^2$.

The main motivation of this chapter is to relate *precisely* the above developments, namely: persistence modules over $\Delta^+ := \{(x,y),\, x+y>0\}$ on the one hand; sheaves over $\mathbb{R}$ on the other hand. Specifically, given a function $f : X \to \mathbb{R}$, we are interested in connecting the collection of level-sets persistence module $(\mathcal{L}_i(f))_{i\in\mathbb{Z}}$ with the derived pushforward $\mathrm{R}f_*\mathbf{k}_X$. In order to do so, we will construct a functor $\overline{(-)}$ from 2-parameter persistence modules to sheaves over $\mathbb{R}$ (see section 4.3.1). Note that this functor is not an equivalence of categories, and that it is not isometric nor reasonably Lipschitz either. Indeed, there can be no equivalences or almost equivalences between these two categories, since the general category of 2-parameter persistence modules is wild representation type as we have mentioned already, and since its objects do not, in general, satisfy any of the local-to-global properties of sheaves.

As mentioned above, of particular interest to us are the level-sets persistence modules $(\mathcal{L}_i(f))_{i\in\mathbb{Z}}$ arising from continuous functions $f : X \to \mathbb{R}$, which actually have *more structure* than general persistence modules over Δ^+.

Our idea is thus to consider a variant of the category of 2-parameter persistence modules taking into account the extra structure and properties carried by level-sets persistence modules.

This follows the fundamental credo of algebraic topology that extra structure on homology gives refined homotopy and geometric information. A general idea here is that to get a better-behaved category of 2-parameter persistence modules, it is key to consider and restrict to those objects having the extra structure and properties coming from data arising in practical applications.

Let us now explain where this extra structure comes from: the various homology groups of a topological space obtained as the union of two open subsets are connected through the well-known Mayer-Vietoris long exact sequence. This sequence involves the homology groups of the union, the sum of the homology groups of the two open subsets, and the homology of their intersection. We axiomatize this data to define a structure we call *Mayer-Vietoris (MV) persistence systems over* Δ^+, whose category is denoted by $\mathrm{MV}(\mathbb{R})$. A MV-system is a graded persistence module $(S_i)_{i\in\mathbb{Z}}$ over Δ^+, together with connecting morphisms $\delta_i^s : S_{i+1}[s] \to S_i$ for all vectors $s \in (\mathbb{R}_{>0})^2$ and grades $i \in \mathbb{Z}$, giving rise to the following *exact* sequences (see Definition 4.2.1):

$$S_{i+1}[s] \xrightarrow{\delta_{i+1}^s} S_i \longrightarrow S_i[s_x] \oplus S_i[s_y] \longrightarrow S_i[s] \xrightarrow{\delta_i^s} S_{i-1}$$

and satisfying some appropriate compatibility conditions. These sequences encode the interactions between the various homology groups at various points of Δ^+, and they carry both a derived and local-to-global information—in some sense that will be made precise in the chapter.

A key property that we leverage in our analysis, is that the category of Mayer-Vietoris systems is rather well behaved. In particular, we prove a *structure theorem* for Mayer-Vietoris persistence systems under standard pointwise finite dimensionality assumptions (see Theorem 4.2.6). According to this result, there are *four different types* of *indecomposable* Mayer-Vietoris systems, which all have pointwise dimension at most 1 and are therefore characterized by their supports. The supports can be either vertical or horizontal bands, or else birth or death blocks (see Definition 4.2.3 and Lemma 4.3.9). Degree-wise, these indecomposables behave like the so-called *block modules* from level-sets persistence and middle-exact bipersistence theories [BCB18, BL17, CdSKM19, CdSM09, CO17]. For this reason, in the following we abuse terms and also call our indecomposables *block MV-systems*. Our structure theorem (Theorem 4.2.6) takes the following form:

Theorem 4.1.1
A, bounded below, pointwise finite-dimensional (pfd) Mayer-Vietoris system has a unique decomposition as a direct sum of block MV-systems.

This result follows non-trivially from the decomposition theorem for middle-exact bipersistence modules [BCB18, CO17]. It provides a *barcode for Mayer-Vietoris systems*, made of the blocks involved in their decomposition. Furthermore, we have a canonical *interleaving distance for Mayer-Vietoris systems* inherited from the classical one of persistence modules.

The aforementioned functor $\overline{(-)}$ from 2-parameter persistence modules to sheaves lifts as a (contravariant) functor $\overline{(-)}^{\mathrm{MV}}$ from Mayer-Vietoris systems to the derived category $\mathrm{D}(\mathbf{k}_{\mathbb{R}})$ of sheaves on $\mathbb{R}$, which is essentially the sheafification of the duality functor. We *construct a pointwise section* of this functor, i.e. a functor Ψ from sheaves to Mayer-Vietoris systems such that the composition with $\overline{(-)}^{\mathrm{MV}}$ gives the identity pointwise on every sheaf $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ (see Corollary 4.4.18):

$$\left(\overline{(\cdot)}^{\mathrm{MV}} \circ \Psi\right)(F) \simeq F.$$

Roughly speaking, this functor Ψ is defined as the dual of the derived global sections of sheaves (see Definition 4.4.7). Both functors restrict to the subcategories of pointwise (resp. strongly pointwise see Definition 4.4.1) finite-dimensional Mayer-Vietoris systems on one side, and of constructible sheaves on $\mathbb{R}$ on the other side. Under standard pointwise finiteness conditions, we are able to prove that these two functors establish an isometric pseudo-equivalence between these categories. More precisely, our second main theorem (see Theorem 4.4.21 and Corollary 4.4.20) states as follows, where $\mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ denotes the category of strongly pointwise finite dimensional MV-systems (definition 4.4.1):

Theorem 4.1.2
The functors $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})^{\mathrm{op}}$ and $\Psi : \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})^{\mathrm{op}} \to \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ form an isometric pseudo-equivalence of categories, meaning:

- for all strongly pointwise finite-dimensional Mayer-Vietoris systems M, N, one has equality
$$d_I^{\mathrm{MV}}(M,N) = d_C(\overline{M}^{\mathrm{MV}}, \overline{N}^{\mathrm{MV}}) = d_B(\mathbb{B}(\overline{M}^{\mathrm{MV}}), \mathbb{B}(\overline{N}^{\mathrm{MV}}))$$
between the interleaving, convolution and derived bottleneck distances;
- for all constructible sheaves $F, G \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, one has $d_B(\mathbb{B}(F), \mathbb{B}(G)) = d_C(F,G) = d_I(\Psi(F), \Psi(G))$;
- $\overline{M}^{\mathrm{MV}} = \overline{N}^{\mathrm{MV}}$ if and only if $d_I(M,N) = 0$.

In particular, the derived distances can be computed using the 2-parameters interleaving distance for MV systems. We also strengthen this result by proving that the MV interleaving distance can be computed as the maximum of the degree-wise usual interleaving distances between the persistence modules which constitute the MV systems (theorem 4.5.4).

To prove theorem 4.1.2, we explicitly compute in Sections 4.4.1 and 4.4.2 the action of the sheafification of Mayer-Vietoris systems functor $\overline{(-)}^{\mathrm{MV}}$ and of its section Ψ on shifts and convolution for the building block modules of each theories. These are computations of independent interest.

Finally, we prove that the functors $\mathrm{R}f_*\mathbf{k}_X$ and $\mathcal{L}_*(f)$ are equivalent to each other under these transformations, i.e. $\overline{\mathcal{L}_*(f)}^{\mathrm{MV}} \cong (\mathrm{R}f_*\mathbf{k}_X)$ (Proposition 4.4.6). Combined with theorem 4.1.2, this establishes the sought-for correspondence between level-sets persistence and derived pushforward for continuous real-valued functions.

Theorem 4.1.3 below summarizes our main results connecting level-sets persistence, Mayer-Vietoris systems, and derived sheaves. The notation $\mathrm{Top}_{|\mathbb{R}}$ stands for the category of topological spaces over $\mathbb{R}$, whose objects are spaces X together with a continuous map $f : X \to \mathbb{R}$, and whose morphisms are commutative triangles $\begin{tikzcd} X \arrow[r, "\phi"'] \arrow[rr, bend left, "f"] & Y \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}$. We let $\mathrm{Top}^c_{|\mathbb{R}}$

denote the subcategory of those functions $f : X \to \mathbb{R}$ such that $\mathrm{R}f_*\mathbf{k}_X$ is constructible, $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ denote the bounded derived category of constructible sheaves on $\mathbb{R}$, and we denote $\mathrm{R}(-)_*\mathbf{k}_{(-)}$ the functor $f \mapsto \bigoplus \mathrm{R}^i f_*(\mathbf{k}_X)[-i]$.

Theorem 4.1.3

The following diagram of categories and functors commutes up to isomorphism of functors:

$$\begin{tikzcd}
\mathrm{Top}_{|\mathbb{R}} \arrow[r, "\mathcal{L}_*(-)"'] \arrow[rr, bend left, "\mathrm{R}(-)_*\mathbf{k}_{(-)}"] & \mathrm{MV}(\mathbb{R}) \arrow[r, "\overline{(-)}^{\mathrm{MV}}"'] & \mathrm{D}(\mathbf{k}_\mathbb{R})^{\mathrm{op}} \\
\mathrm{Top}^c_{|\mathbb{R}} \arrow[u, hook] \arrow[r, "\mathcal{L}_*(-)"] \arrow[rr, bend right, "\mathrm{R}(-)_*\mathbf{k}_{(-)}"'] & \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \arrow[u, hook] \arrow[r, "\overline{(-)}^{\mathrm{MV}}"] & \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})^{\mathrm{op}}. \arrow[u, hook]
\end{tikzcd}$$

Furthermore, $\overline{(-)}^{\mathrm{MV}}$ restricted to $\mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ and the vertical functors are isometries, while $\mathcal{L}_*(-)$ is 1-Lipschitz.

Proof

The existence and commutativity of the diagram is the content of Lemma 4.4.3, Propositions 4.2.15 and 4.4.6, and Theorem 4.4.21. The rest of the statement is given by Theorem 4.4.21 and the stability theorems 4.2.17, 2.2.12.

□

Theorem 4.1.3 relates precisely, and in fact essentially identifies, level-sets persistence and constructible sheaves. Moreover, it does so in a functorial way. In the final section of the chapter (Section 4.5.1) we give an example with full detail that illustrates this result.

4.1.1 Notations

Here we detail our notations and conventions, for the reader's convenience:

- We fix a ground (commutative) field denoted $\mathbf{k}$.
- For $s = (s_1, s_2) \in \mathbb{R}^2_{>0}$, we will use the notations $s_x := (s_1, 0)$ and $s_y := (0, s_2)$.
- Given a category $\mathscr{C}$, we denote $\mathscr{C}^{\mathrm{op}}$ its opposite category.

Notations for intervals

— We will use the *french notation* $]a, b[$ for *open intervals* (a, b) in $\mathbb{R}$. The reason is to avoid confusion with points $(a, b) \in \mathbb{R}^2$ which will both be possible values of persistent or sheaf objects (and usually appear with similar letters).

— For real numbers $a \leq b$, the notation $\langle a, b\rangle$ will mean an interval whose boundary points are a and b. We use this notation $\langle\, ,\, \rangle$ when we do not want to precise if the interval is open, compact, or half-open; in other words as a variable.

Notations and conventions for shifts of graded and persistent objects

Standard and convenient notations for shifting the degree of a (possibly differential) graded object or for shifted (or translated) persistent object are both given by $[-]$ in the literature. We will have to use objects which are both differential graded and persistent, and we now explain how to avoid confusion about this notation in the chapter. Note that we also have to deal with objects which are naturally homologically graded (for instance persistence modules) and cohomologically graded (sheaves).

For any (differential) cohomologically *graded* object C, we will use the notation $C[i]$ for the (differential) graded object $C[i]^n := C^{i+n}$ where $i \in \mathbb{Z}$. The letter i can be replaced by j, k, Q, m or n in the chapter, and the notation with one of these letters always means such a grading shift. These letters can also show up in subscripts.

Similarly, for a (differential) *homologically graded* object M_* we will use the notation $M_*[i]$ for the graded object $(M_*[i])_n = M_{i+n}$, following for instance the conventions of [Sta19, Section 12.13]. We warn the reader that there is also an opposite convention in the literature (which is the topological convention for suspension). The main advantage of this choice of convention in this chapter is that the duality functor commutes with the shift in grading (instead of changing to its opposite):

$$\mathrm{Hom}_{\mathbf{k}}(C^*[i], \mathbf{k}) \;\cong\; \mathrm{Hom}_{\mathbf{k}}(C^*, \mathbf{k})[i]$$

where following the usual convention for dual of (differential) graded objects we define

$$\mathrm{Hom}_{\mathbf{k}}(C^*, \mathbf{k})_n := \mathrm{Hom}_{\mathbf{k}}(C^n, \mathbf{k}), \quad \mathrm{Hom}_{\mathbf{k}}(D_*, \mathbf{k})^n := \mathrm{Hom}_{\mathbf{k}}(D_n, \mathbf{k})$$

for any integer n.

For a *persistence* module P (over Δ^+ or $\mathbb{R}^2$, see 2.1.1) we will also use the standard notation $P[s]$ (where s is in $\mathbb{R}^2$) for its shifted by the vector s, which is also a persistence module (Definition 2.1.10). Note that the shift is by a vector, i.e. a point in $\mathbb{R}^2$ not an integer. We will also use letters such as t, x or $\vec{\varepsilon}$, s_x, s_y for these operations. This should cause no confusion since the sets of letters used in the two types of shifts are disjoint.

For instance for a graded persistent object P, the notation $(P[i])[s] = (P[s])[i]$, where $i \in \mathbb{Z}$ and $s \in \mathbb{R}^2_{>0}$, stands for the persistent object defined by $(P[i])[s](x)^n := P^{n+i}(x+s)$.

4.2 The category $\mathrm{MV}(\mathbb{R})$

In this section we study the notion of Mayer-Vietoris system which are sequence of persistence modules over $\{(x,y) \mid x+y>0\}$ with additional structure.

4.2.1 Mayer-Vietoris systems over $\mathbb{R}$ and their classification

Definition 4.2.1

We define the category $\mathrm{MV}(\mathbb{R})$ of *Mayer-Vietoris persistent systems over* $\mathbb{R}$ as follows:

- Objects: collections $S = (S_i, \delta_i^s)_{i\in\mathbb{Z}, s\in\mathbb{R}^2_{>0}}$ where S_i is in $\mathrm{Pers}(\Delta^+)$ and $\delta_i^s \in \mathrm{Hom}_{\Delta^+}(S_i[s], S_{i-1})$, such that for all $i \in \mathbb{Z}$ and all $s \in \mathbb{R}^2_{>0}$, the following sequence

$$S_{i+1}[s] \xrightarrow{\delta_{i+1}^s} S_i \longrightarrow S_i[s_x] \oplus S_i[s_y] \longrightarrow S_i[s] \xrightarrow{\delta_i^s} S_{i-1} \tag{4.1}$$

  is exact and furthermore the following diagram is commutative, for $s' \geq s$:

$$\begin{tikzcd} S_i[s] \arrow[r, "\delta_i^s"] \arrow[d] & S_{i-1} \arrow[d, "\mathrm{id}_{S_{i-1}}"] \\ S_i[s'] \arrow[r, "\delta_i^{s'}"] & S_{i-1}. \end{tikzcd} \tag{4.2}$$

- Morphisms: for (S_i, δ_i^s) and $(T_i, \tilde{\delta}_i^s)$ two Mayer-Vietoris systems over $\mathbb{R}$, a morphism from (S_i, δ_i^s) to $(T_i, \tilde{\delta}_i^s)$ is a collection of morphisms $(\varphi_i)_{i\in\mathbb{Z}}$ where $\varphi_i \in \mathrm{Hom}_{\mathrm{Pers}(\mathbf{k}^{\Delta^+})}(S_i, T_i)$ such that the following diagram

$$\begin{tikzcd} S_i[s] \arrow[r, "\delta_i^s"] \arrow[d, "\varphi_i[s]"'] & S_{i-1} \arrow[d, "\varphi_{i-1}"] \\ T_i[s] \arrow[r, "\tilde{\delta}_i^s"] & T_{i-1} \end{tikzcd} \tag{4.3}$$

commutes for all $i \in \mathbb{Z}$ and $s \in \mathbb{R}^2_{>0}$.

For a Mayer-Vietoris system S and $i \in \mathbb{Z}$, we will write S_i for the associated object of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ of S which lies in degree i.

A natural class of examples of such MV systems is provided by homology of level-sets of a continuous function on a topological space X. See, example 4.2.14 below. Furthermore, we will see that any complex of sheaves F on $\mathbb{R}$ gives rise to a MV-system $\Psi(F)$ (see Proposition 4.4.11).

Remark 4.2.2

- Observe that if (S_i, δ_i^s) is a Mayer-Vietoris system, S_i is in particular a middle exact modules, for $i \in \mathbb{Z}$.
- The category MV($\mathbb{R}$) is indeed a category. It is easy from the definition to observe that it is additive. However, as we shall see later on, it is not abelian.

Our remaining goal in this section is to classify Mayer-Vietoris system in a way similar to Theorem 2.1.33. For this, we introduce building blocks for those.

Definition 4.2.3

Let B be a block (Definition 2.1.26) and $j \in \mathbb{Z}$. We define the Mayer-Vietoris system of degree j associated to B, denoted $\mathrm{S}_j^{\mathrm{B}}$, by:

- If B is of type $\mathbf{bb}^-$, $\mathbf{hb}$ or $\mathbf{vb}$ then $\mathrm{S}_j^{\mathrm{B}} = (M_i, 0)_{i,s}$ with $M_i = 0$ for all $i \neq j$ and $M_j = \mathbf{k}^B$
- If B is of type $\mathbf{db}^+$, then $\mathrm{S}_j^{\mathrm{B}} = (M_i, \delta_i^s)$ with $M_i = 0$ for all $i \notin \{j+1, j\}$, $\delta_i^s = 0$ for all $s \in \mathbb{R}^2$ and for $i \neq j+1$, we define $M_{j+1} = \mathbf{k}_{B^\dagger}$, $M_j = \mathbf{k}_B$, and $\delta_{j+1}^s : \mathbf{k}_{B^\dagger}[s] \to \mathbf{k}^B$ by pointwise identities on $B^\dagger[s] \cap B \cap \Delta^+$.
- Dually, if B is of type $\mathbf{bb}^+$, then define S_j^{B} as $S_{j-1}^{B^\dagger}$.
- If B is of type $\mathbf{db}^-$, then we set $S^B = 0$.

Of course the case of $\mathbf{db}^-$ matches remark 2.1.31.

Remark 4.2.4

One can easily see that the Mayer-Vietoris systems S_j^B are indecomposable objects of the additive category MV($\mathbb{R}$). We will refer to these Mayer-Vietoris systems to *block MV-systems* for short.

Also note that for a block B of type $\mathbf{db}^+$ or $\mathbf{bb}^+$ and $j \in \mathbb{Z}$, the graded persistence module $(M_i, 0)_{i,s}$ with $M_i = 0$ for all $i \neq j$ and $M_j = \mathbf{k}^B$ is *not* a Mayer-Vietoris system.

Lemma 4.2.5
The graded persistence modules S_j^B associated to blocks B in Definition 4.2.3 are Mayer-Vietoris systems for any j and block B.

Proof
We advise the reader to draw the different cases in a way similar to figure 4.1. Since $S_j^B = S_{j-1}^{B^\dagger}$, the case of $\mathbf{db}^+$ and $\mathbf{bb}^+$ are equivalent.

Note that every block which is not of type $\mathbf{db}$ is stable by upward vertical and/or left-to-right horizontal translations. It follows that $\mathbf{k}^B \to \mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y]$ is injective. Thus for blocks of type $\mathbf{vb}$, $\mathbf{hb}$ or $\mathbf{bb}^-$, $S_j^B \to S_j^B[s_x] \oplus S_j^B[s_y]$ is one to one as well in every degree, a well as is the map $S_j^B \to S_j^B[s_x] \oplus S_j^B[s_y]$ in degree $i \neq j$ for B of type $\mathbf{db}$ (and therefore also for S_{j-1}^B if B is of type $\mathbf{bb}^+$ by definition 4.2.3).

Note now that for a block B, if $z \in \mathbb{R}^2$ and $s \in \mathbb{R}^2_{>0}$ satisfies that $z+s \in B$, then either $z+s_x$ or $z+s_y$ is in B as well if B is of type different from $\mathbf{bb}$. Furthermore, for a block of type $\mathbf{bb}$, the latter property only fails if $x \in B \cap B^\dagger$ where $B^\dagger$ is its dual (death)block. When B is of type $\mathbf{bb}^-$, those points are not in Δ^+. Therefore, the maps $\mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y] \to \mathbf{k}^B[s]$ are surjective for all blocks of type different from $\mathbf{bb}^+$.

Let us now prove that the subsequences $\mathbf{k}^B \longrightarrow \mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y] \longrightarrow \mathbf{k}^B[s]$ are exact for any B; we have already seen that the composition is zero. Now, assume $(\alpha_x, \alpha_y) \in \mathbf{k}^B[s_x](v) \oplus \mathbf{k}^B[s_y](v)$ is a *nonzero* element in the kernel of $\tau_{s_x} \oplus \tau_{s_y}$. Then, if $v+s \in B$ then so are $v+s_x$ and $v+s_b$ and therefore τ_{s_x} and τ_y are the identity map $\mathbf{k} \to \mathbf{k}$. In particular $\alpha_x = \alpha_y =: \alpha$. But since $\mathbf{k}^B[s](v) = \mathbf{k}^B(s+v) = \mathbf{k}$ as well, then $\mathbf{k}^B(v) \to \mathbf{k}^B[s_x](v) \oplus \mathbf{k}^B[s_y](v)$ is the map $(\mathrm{id}, \mathrm{id})$ and hence (α, α) is in its image. If $v+s \notin B$, then B is not a birthblock and at least one element among $v+s_x$ and $v+s_y$ is not in B. If none are, then there is nothing to prove and if not then B is either a vertical or horizontal block. In the first case, $v+s_y \in B$ and therefore $\mathbf{k}^B \to \mathbf{k}^B[s_y]$ is the identity map so that we have a preimage for α_y. The other case is dual. This conclude the proof of the lemma for all blocks which are not of type $\mathbf{db}^+$.

To prove the result for blocks of type $\mathbf{db}^+$, since $S_i^B = S^B[-i]$ and by the injectivity result we have obtained at the beginning of that proof, it is enough to prove that the sequences

$$\mathbf{k}^{B^\dagger}[s](v) \to \mathbf{k}^B(v) \to \mathbf{k}^B[s_x](v) \oplus \mathbf{k}^B[s_y](v)$$

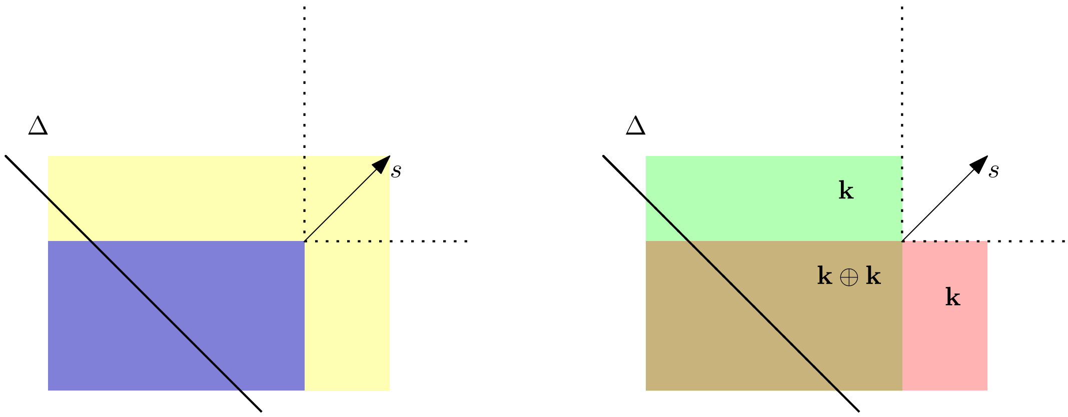


Figure 4.1 – *On the left:* A deathblock B in yellow (and blue) and the translated deathblock $B - s = B[s]$ in blue where s is the vector drawn. The dotted lines are the boundary of the dual birth block $B^\dagger[s]$. *On the right:* The value of $\mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y]$ in every region where the green region is the translated death block $B - s_x$, the red is the translated death block $B - s_y$. The value is 0 on the white region and below the antidiagonal.

are exact for any $s \in \mathbb{R}^2_{>0}$, $v \in \Delta^+$. If $v \notin B$, there is nothing to prove. Thus we assume $v \in B$. First, if both $\mathbf{k}^B[s_x]$ and $\mathbf{k}^B[s_y]$ are null, then $x \in B \cap B^\dagger[s] = B \cap (B - s)^\dagger$. Therefore, $\mathbf{k}^{B^\dagger}[s](v) \to \mathbf{k}^B(v)$ is the identity and the sequence is exact. If both $\mathbf{k}^B[s_x]$ and $\mathbf{k}^B[s_y]$ are non-null, then $\mathbf{k}^B(v) \to \mathbf{k}^B[s_x](v) \oplus \mathbf{k}^B[s_y](v)$ identifies with the necessarily injective diagonal inclusion and $v \notin B[s]^\dagger$ so that $\mathbf{k}^{B^\dagger}[s](v) = 0$ and the sequence is thus exact. Finally if only one $\mathbf{k}^B[s_x]$ or $\mathbf{k}^B[s_y]$ is non-null, one of the map $\mathbf{k}^B \to \mathbf{k}^B[s_x]$ or $\mathbf{k}^B \to \mathbf{k}^B[s_y]$ is the identity-hence injective-and we still have $v \notin B[s]^\dagger$. Thus $\mathbf{k}^{B^\dagger}[s](v) = 0$. The sequence is again exact and the lemma is proved.

□

Denote by $\mathrm{MV}^+(\mathbb{R})$ the full *sub-category of* Mayer-Vietoris systems over $\mathbb{R}$ whose objects are the MV systems $S = (S_j, \delta_j^s)$ such that there exists $N \in \mathbb{Z}$ with $S_j = 0$ for all $j < N$. In other words, $\mathrm{MV}^+(\mathbb{R})$ is the subcategory of lower-bounded Mayer-Vietoris systems. Moreover, S will be said to be point-wise finitely dimensional if all the S_j are.

Theorem 4.2.6 (Theorem 2.19 - [BGO19])
Let S be an object of $\mathrm{MV}^+(\mathbb{R})$ which is pointwise finite dimensional. Then there exists a unique collection of multisets of blocks $\mathbb{B}(S) = (\mathbb{B}_j(S))_{j\in\mathbb{Z}}$ of type $\mathbf{bb}^-$, $\mathbf{hb}$, $\mathbf{vb}$, and $\mathbf{db}^+$, such that we have an isomorphism in $\mathrm{MV}(\mathbb{R})$:

$$S \simeq \bigoplus_{j\in\mathbb{Z}} \bigoplus_{B\in\mathbb{B}_j(S)} S_j^B$$

We call $\mathbb{B}(S)$ the **barcode** of S. It completely determines S up to isomorphism of Mayer-Vietoris systems.

Remark 4.2.7
By Definition 2.1.26, birth blocks of type $\mathbf{bb}^+$ generate the same MV systems as their dual death blocks, therefore they come in pairs in the decomposition given by Theorem 4.2.6, which explains why the blocks of type $\mathbf{bb}^+$ are ignored in the barcode.

To prove the theorem 4.2.6, we will use the following technical lemmas :

Lemma 4.2.8
Let $S = (S_j, \delta_j^s)$ be a pfd MV-system over $\mathbb{R}$. If $\mathbb{B}(S_j)$ contains only blocks of type $\mathbf{db}^+$, then $S = 0$.

Proof
Given $s \in \mathbb{R}^2_{>0}$, the universal property of cokernels and the exactness of equation 4.1 imply that δ_j^s factorizes through

$$\mathrm{coker}\left(S_j[s_x] \oplus S_j[s_y] \longrightarrow S_j[s]\right).$$

Now, this cokernel is trivial since by assumption S_j is isomorphic to a direct sum of blocks of type $\mathbf{db}^+$. Therefore, $\delta_j^s = 0$.

Consequently, for every $B \in \mathbb{B}(S_j)$ and every $s \in \mathbb{R}^2_{>0}$, the exact sequence of persistence modules

$$0 \longrightarrow \mathbf{k}^B \longrightarrow \mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y]$$

yields $B = \emptyset$ since B is assumed to be of type $\mathbf{db}^+$.

$\square$

Lemma 4.2.9
Let S be a pfd MV-system over $\mathbb{R}$, such that there exists a block $B = \langle a, \infty\rangle \times \langle b, \infty\rangle$ of type $\mathbf{bb}^+$ such that $B \in \mathbb{B}(S_j)$ for some $j \in \mathbb{Z}$. Then there exists a pfd MV system Σ such that:

$$S \simeq S_j^B \oplus \Sigma = S_{j-1}^{B^\dagger} \oplus \Sigma$$

Proof
Let $s \geq \sqrt{2}(a+b, a+b)$, since $\operatorname{coker}\left(\mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y] \longrightarrow \mathbf{k}^B[s]\right) \simeq \mathbf{k}_{B^\dagger}$ we have the following commutative diagram, where the rows are exact sequences and where φ exists (and is injective) by the universal property of cokernels:

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathbf{k}^B \arrow[r] \arrow[d, hook] & \mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y] \arrow[r] \arrow[d, hook] & \mathbf{k}^B[s] \arrow[r] \arrow[d, hook] & \mathbf{k}_{B^\dagger} \arrow[r] \arrow[d, dotted, "\varphi"] & 0 \\
\dots \arrow[rr] & & S_j \arrow[r] & S_j[s_x] \oplus S_j[s_y] \arrow[r] & S_j[s] \arrow[r, "\delta_j^s"] & S_{j-1} \arrow[r] & \dots
\end{tikzcd} \tag{4.4}$$

Since $B^\dagger$ is a directed ideal of Δ^+, $\mathbf{k}_{B^\dagger}$ is an injective object of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ by lemma 2.1 of [BCB18]. Therefore, φ splits and $\mathrm{im}\varphi \simeq \mathbf{k}_{B^\dagger}$ is a summand of S_{j+1}. The commutativity of (4.4) then implies the existence of a complement X_j of $\mathrm{im}(\mathbf{k}^B \hookrightarrow S_j)$ in S_j, such that S decomposes locally as follows:

$$\begin{tikzcd}
X_j \oplus \mathbf{k}^B \arrow[r] \arrow[d, "\simeq"] & X_j[s_x] \oplus X_j[s_y] \oplus \mathbf{k}_b[s_x] \oplus \mathbf{k}_b[s_y] \arrow[r] \arrow[d, "\simeq"] & X_j[s] \oplus \mathbf{k}^B[s] \arrow[r] \arrow[d, "\simeq"] & X_{j-1} \oplus \mathbf{k}_{B^\dagger} \arrow[d, "\simeq"] \\
S_j \arrow[r, "\sigma"] & S_j[s_x] \oplus S_j[s_y] \arrow[r] & S_j[s] \arrow[r, "\delta_j^s"] & S_{j-1}
\end{tikzcd}$$

Note that we may assume without loss of generality that $X_j \supseteq \ker \sigma$. Then, by exactness of S, we have $\mathrm{im}\delta_{j+1}^s = \ker \sigma \subseteq X_j$, therefore our local decomposition extends to a full decomposition of S, which means that the upper row complex in (4.4) is a summand of S.

□

Proof (of theorem 4.2.6)
Let $S = (S_j, \delta_j^s) \in \mathrm{MV}^+(\mathbb{R})$, and assume without loss of generality that the lower bound N is equal to 1. Then, all the S_j's are middle-exact pfd persistence modules over Δ^+, therefore they decompose uniquely (up to isomorphism) as direct sums of block modules, by theorem 2.1.33. Note that for $j \leq 0$ the decomposition is trivial.

The finite barcode case:

We first show the result in the case where $\mathbb{B}(S_j)$ is finite for every $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, fix an isomorphism $\varphi_j : S_j \xrightarrow{\sim} \bigoplus_{B \in \mathbb{B}(S_j)} \mathbf{k}^B$. Thus, the family $(\varphi_j)_j$ induces an isomorphism of MV systems from S to

$$S' := \left(\bigoplus_{B \in \mathbb{B}(S_j)} \mathbf{k}^B,\ \varphi_{j-1} \circ \delta_j^s \circ \varphi_j^{-1}[s] \right)_{j \in \mathbb{Z}, s \in \mathbb{R}^2_{>0}}$$

Let $B \in \mathbb{B}(S_j)$ of type either $\mathbf{bb}^-$, $\mathbf{hb}$ or $\mathbf{vb}$, then for $s \in \mathbb{R}^2_{>0}$, the map:

$$\mathbf{k}^B[s_x] \oplus \mathbf{k}^B[s_y] \longrightarrow \mathbf{k}^B[s]$$

is surjective. Thus, $\varphi_{j-1} \circ \delta_j^s \circ \varphi_j^{-1}[s]$ is zero on $\mathbf{k}^B[s]$. This proves that S_j^B is a summand of S'. Finally, noting $\mathbb{B}^-(S_j)$ the multi-set of intervals of $\mathbb{B}(S_j)$ of type either $\mathbf{bb}^-$, $\mathbf{hb}$ or $\mathbf{vb}$, we have:

$$S' = \left(\bigoplus_{j\in\mathbb{Z}} \bigoplus_{B\in\mathbb{B}^-(S_j)} S_j^B \right) \oplus \left(\bigoplus_{j\in\mathbb{Z}} \bigoplus_{B\in\mathbb{B}(S_j)\backslash\mathbb{B}^-(S_j)} \mathbf{k}^B,\ \varphi_{j-1} \circ \delta_j^s \circ \varphi_j^{-1}[s] \right).$$

There remains to prove that the right-hand side of the direct sum, noted S'', decomposes in $\mathrm{MV}(\mathbb{R})$. For $j \in \mathbb{Z}$, the barcode $\mathbb{B}(S_j)\backslash\mathbb{B}^-(S_j)$ contains only blocks of type either $\mathbf{bb}^+$ or $\mathbf{db}^+$. Denote by $\mathbb{B}(S_j)^+$ the multiset of blocks of type $\mathbf{bb}^+$ involved in $\mathbb{B}(S_j)$. Let us prove by induction that, for any $j_0 \geq 1 (= N)$, there exists a MV system Σ^{j_0} such that

$$S'' \simeq \left(\bigoplus_{1\leq j<j_0} \bigoplus_{B\in\mathbb{B}(S_j)^+} S_j^B \right) \oplus \Sigma^{j_0}.$$

For $j_0 = 1$ the property holds with $\Sigma^{j_0} = S''$. Let us now assume the property holds up to some $j_0 \geq 1$. Since $\mathbb{B}(S_{j_0})^+$ has finite cardinality, Lemma 4.2.9 (applied repeatedly) decomposes Σ^{j_0} as

$$\Sigma^{j_0} \simeq \left(\bigoplus_{B\in\mathbb{B}(S_{j_0})^+} S_{j_0}^B \right) \oplus \Sigma^{j_0+1},$$

which yields the induction step.

Now, given $j_0 \geq 1$, for any $j < j_0$ the barcode of $\Sigma_j^{j_0}$ can only contain deathblocks by construction. Therefore, by Lemma 4.2.8, we have $\Sigma_j^{j_0} = 0$. It follows that

$$S'' \simeq \bigoplus_{j\in\mathbb{Z}} \bigoplus_{B\in\mathbb{B}(S_j)^+} S_j^B,$$

thus concluding the decomposition in the finite barcode case.

The infinite barcode case:

We now generalize to the case where the barcodes $\mathbb{B}(S_j)$ can be infinite. For the same reason as in the finite case, each block of type $\mathbf{bb}^-$, $\mathbf{hb}$ or $\mathbf{vb}$ involved in some barcode $\mathbb{B}(S_j)$ splits as a summand S_j^B of S. Hence, we are reduced to proving the existence of the decomposition in the case where S

is a pfd MV system, and $\mathbb{B}(S_j)$ contains only blocks of type $\mathbf{bb}^+$ or $\mathbf{db}^+$ for all $j \in \mathbb{Z}$. Given $n \in \mathbb{Z}_{>0}$, define $\Delta_n^+ := \Delta^+ \cap \{(x,y) \in \mathbb{R}^2 \mid x \leq n, y \leq n\}$. Define also

$$\mathbb{B}(S_j)_n := \{B \in \mathbb{B}(S_j) \mid B \text{ is of type } \mathbf{bb}^+ \text{ and } B \cap \Delta_n^+ \neq \emptyset$$
$$\text{or } B \text{ is of type } \mathbf{db}^+ \text{ and } B \subset \Delta_n^+\}.$$

Then we have that $\mathbb{B}(S_j) = \bigcup_n \mathbb{B}(S_j)_n$, and since S is pointwise finite dimensional, $\mathbb{B}(S_j)_n$ contains finitely many blocks of type $\mathbf{bb}^+$, for all $n \geq 0$. We now identify each S_j with its block decomposition via some fixed isomorphism, and for $n \geq 0$ we define ${}_n\tilde{S}$ as follows:

$${}_n\tilde{S} = \left(\bigoplus_{B \in \mathbb{B}(S_j)_n} \mathbf{k}^B, (\delta_j^s)_{|\bigoplus_{B\in\mathbb{B}(S_j)_n} \mathbf{k}_B} \right)$$

Let us prove that $\tilde{S}$ is a sub-MV system of S. To do so, it is sufficient to prove that for all $j \in \mathbb{Z}$, the image of $(\delta_j^s)_{|\bigoplus_{B\in\mathbb{B}(S_j)_n}}$ is contained in $\bigoplus_{B\in\mathbb{B}(S_{j-1})_n} \mathbf{k}_B$. Fix $j \in \mathbb{Z}$ and $s \in \mathbb{R}^2_{>0}$. Then $(\delta_j^s)_{|\bigoplus_{B\in\mathbb{B}(S_j)_n}}$ factorizes uniquely through:

$$\begin{aligned} &\operatorname{coker}\left(\bigoplus_{B\in\mathbb{B}(S_j)_n} \mathbf{k}_B[s_x] \oplus \mathbf{k}_B[s_y] \longrightarrow \bigoplus_{B\in\mathbb{B}(S_j)_n} \mathbf{k}_B[s] \right) \\ &\simeq \bigoplus_{B\in\mathbb{B}(S_j)_n} \operatorname{coker}\left(\mathbf{k}_B[s_x] \oplus \mathbf{k}_B[s_y] \longrightarrow \mathbf{k}_B[s]\right) \\ &= \bigoplus_{\substack{B\in\mathbb{B}(S_j)_n \\ B \text{ is of type } \mathbf{bb}^+}} \operatorname{coker}\left(\mathbf{k}_B[s_x] \oplus \mathbf{k}_B[s_y] \longrightarrow \mathbf{k}_B[s]\right) \end{aligned}$$

As previously, for every $B \in \mathbb{B}(S_j)_n$ of type $\mathbf{bb}^+$, we can find $s \in \mathbb{R}^2_{>0}$ such that the canonical map:

$$\operatorname{coker}\big(\mathbf{k}_B[s_x] \oplus \mathbf{k}_B[s_y] \longrightarrow \mathbf{k}_B[s]\big) \longrightarrow \bigoplus_{B\in\mathbb{B}(S_{j-1})} \mathbf{k}_B$$

is a monomorphism. And as seen in the proof of Lemma 4.2.9, $\operatorname{coker}\big(\mathbf{k}_B[s_x] \oplus \mathbf{k}_B[s_y] \longrightarrow \mathbf{k}_B[s]\big)$ is isomorphic to $\mathbf{k}_{B^\dagger}$, hence an injective object of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$, so its image splits off as a summand of $\bigoplus_{B\in\mathbb{B}(S_{j-1})} \mathbf{k}_B$ and is therefore included in

$$\mathbf{k}_{B^\dagger}^m \subset \bigoplus_{B\in\mathbb{B}(S_{j-1})} \mathbf{k}^B$$

where m is the multiplicity of $B^\dagger$ in $\mathbb{B}(S_{j-1})$. Since $B^\dagger \in \mathbb{B}(S_{j-1})_n$, we conclude that $\mathrm{im}((\delta_j^s)_{|\bigoplus_{B\in\mathbb{B}(S_j)_n}}) \subset \bigoplus_{B\in\mathbb{B}(S_{j-1})_n} \mathbf{k}_B$. This proves that ${}_n\tilde{S}$ is a sub-MV system of S.

Then, we can apply our decomposition result in the finite barcode case to ${}_n\tilde{S}$. And since we have the filtration

$$S = \bigcup_{n\geq 0} {}_n\tilde{S}$$

which stabilizes pointwise, we get a decomposition for S.

□

Remark 4.2.10

Let us finish by a remark on the "derived" meaning of Mayer-Vietoris systems. The axioms and structure we put on $\mathrm{MV}(\mathbb{R})$ are actually encoding a natural homotopy property. To state it, we have to consider the (derived) category of *2-parameter persistence chains complexes*, that is the (associated derived) category of functors $\Delta^+ \to \text{dg-Mod}(\mathbf{k})$. Taking the direct sum of homology groups of 2-parameter persistence chain complex gives a graded 2-parameter persistence module. Such graded 2-parameter persistence modules $(\mathrm{H}_i(C_\bullet)_{i\in\mathbb{Z}}$ that can be lifted to a Mayer-Vietoris system are precisely those such that the underlying 2-parameter persistence chain complex $C_\bullet$ satisfies the following property:

For any $s \in \Delta^+$, the canonical map $C_\bullet[s_x] \oplus C_\bullet[s_y] \to C_\bullet[s]$ exhibits $C_\bullet[s]$ as the *homotopy* quotient $\mathbf{hocoker}\Big(C_\bullet \to C_\bullet[s_x] \oplus C_\bullet[s_y]\Big)$ of the persistence chain complex morphisms $C_\bullet \to C_\bullet[s_x] \oplus C_\bullet[s_y]$.

A down to earth way of expressing this homotopy quotient property is to say $C_\bullet[s]$ is quasi-isomorphic to the cone of $C_\bullet \to C_\bullet[s_x] \oplus C_\bullet[s_y]$ as a persistent chain complex over Δ^+. In other words, the structure of Mayer-Vietoris systems is essentially encoding the data of a homotopy property carried by their underlying chain complexes; property expressing that the chain complex at a $(x, y) + s$ is determined by those of the chain complexes at the point (x, y), $(x, y) + s_x$, $(x, y) + s_y$ for any $s \in \Delta^+$ which exhibits a local to global coherence of the values of those special 2-parameter persistence chain complexes.

4.2.2 Interleaving distance for MV systems

Given $t \in (\mathbb{R}_{\geq 0})^2$, and $M = (M_i, \delta_i^s)$ a Mayer-Vietoris system over $\mathbb{R}$ (as in Definition 4.2.1), the collection of shifted modules $(M_i[t], \delta_i^s[t])$ is indeed a Mayer-Vietoris system, that we call the *t-shift* of M. The mapping $M \mapsto M[t]$ induces an endofunctor of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$.

Observe that the collection $\tau_t^M := (\tau_t^{M_i})_{i\in\mathbb{Z}}$ is a morphism of Mayer-Vietoris systems

$$\tau_t^M : M \longrightarrow M[t].$$

We shall denote the diagonal embedding $\varepsilon \mapsto \vec{\varepsilon}$ where $\vec{\varepsilon} = (\varepsilon, \varepsilon)$. We also $\overrightarrow{(-)} : (\mathbb{R}_{>0}, \leq) \to (\Delta^+, \leq)$ the induced functor.

Definition 4.2.11

Let M and N two Mayer-Vietoris systems over $\mathbb{R}$. An **ε-interleaving** between M and N is the data of two morphisms of MV systems $f = (f_i) : (M_i, \delta_i^s) \longrightarrow (N_i[\vec{\varepsilon}], \tilde{\delta}_i^s[\vec{\varepsilon}])$ and $g = (g_i) : (N_i, \tilde{\delta}_i^s) \longrightarrow (M_i[\vec{\varepsilon}], \delta_i^s[\vec{\varepsilon}])$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr] \arrow[rr, bend left, "\tau^M_{2\vec{\varepsilon}}"] & N[\vec{\varepsilon}] \arrow[r, "g[\vec{\varepsilon}]"] \arrow[dr] & M[2\vec{\varepsilon}] \\
N \arrow[r, "g"] \arrow[ur] \arrow[rr, bend right, "\tau^N_{2\vec{\varepsilon}}"'] & M[\vec{\varepsilon}] \arrow[r, "f[\vec{\varepsilon}]"] \arrow[ur] & N[2\vec{\varepsilon}]
\end{tikzcd} \tag{4.5}$$

If M and N are ε-interleaved, we shall write $M \sim_\varepsilon^{\mathrm{MV}} N$.

Remark 4.2.12

In particular for $i \in \mathbb{Z}$, f_i and g_i define an ε-interleaving of persistence modules between M_i and N_i. Therefore, interleavings of Mayer-Vietoris systems are just a derived [2] extension of the usual interleavings for persistence modules over Δ^+.

Definition 4.2.13

Define the interleaving distance between two Mayer-Vietoris systems M and N to be the non-negative or possibly infinite number:

$$d_I^{\mathrm{MV}}(M, N) := \inf\{\varepsilon \geq 0 \mid M \sim_\varepsilon^{\mathrm{MV}} N\}.$$

We say that a Mayer-Vietoris system $M = (M_i, \delta_i)_{i\in\mathbb{Z}}$ is *bounded* if there is only finitely many M_i which are non-zero.

We now turn to a main source of examples of Mayer-Vietoris systems.

2. because we precisely requires the morphisms to commute with the maps δ_s^i connecting homology groups of different degrees. This claim will be even more supported by the isometry theorem 4.4.21

Example 4.2.14 (MV system associated to continuous functions)
Let $u : X \to \mathbb{R}$ be a continuous function on a topological space X. For any $z = (z_1, z_2) \in \Delta^+$, recall that we set

$$\mathcal{L}_i(u)(z) := \mathrm{H}_i(u^{-1}(]-z_1, z_2[).$$

If $z' = (z_1', z_2') \geq x$, then we have the inclusion $]-z_1, z_2[\subset]-z_1', z_2'[$ inducing, for all i's, homomorphisms $\mathcal{L}_i(u)(z) = \mathrm{H}_i(u^{-1}(]-z_1, z_2[) \to \mathrm{H}_i(u^{-1}(]-z_1', z_2'[) = \mathcal{L}_i(u)(z')$ in homology. By Lemma 4.3.1, this makes $\mathcal{L}_i(u)(-)$ a persistence module over Δ^+, which is called the *i-th level sets persistence module associated to $h : X \to \mathbb{R}$.*

Now, let $s = (s_1, s_2) \in \mathbb{R}^2_{>0}$. For any $z = (z_1, z_2) \in \Delta^+$, we have that the open interval $]-z_1 - s_1, z_2, +s_2[$ has a cover given by the two open sub-intervals $]-z_1 - s_1, z_2[$ and $]-z_1, z_2, +s_2[$ whose intersection is $]-z_1, z_2[$. Therefore the Mayer-Vietoris sequence associated to this cover gives us linear maps $(\delta_i^{s,z})_{i\in\mathbb{N}}$ and exact sequences

$$\mathcal{L}_{i+1}(u)[s](z) \xrightarrow{\delta_{i+1}^{s,z}} \mathcal{L}_i(u)(z) \longrightarrow \mathcal{L}_i(u)[s_x](z) \oplus \mathcal{L}_i(u)[s_y](z) \longrightarrow \mathcal{L}_i(u)[s](z)$$
$$\xrightarrow{\qquad\delta_i^{s,z}\qquad} \mathcal{L}_{i-1}(u)(z). \tag{4.6}$$

We write $\delta_i^s : \mathcal{L}_i(u)[s] \to \mathcal{L}_{i-1}(u)$ the maps given at every point z by $\delta_i^{s,z}$ and for $i \leq 0$ we set $\delta^i = 0$.

Proposition 4.2.15
The δ_i^s's are persistence modules morphisms and makes the collection $\mathcal{L}_*(u) := (\mathcal{L}_i(u), \delta_i^s)_{i\in\mathbb{Z}, s\in\mathbb{R}^2_{>0}}$ a Mayer-Vietoris persistence system over $\mathbb{R}$. Furthermore, the assignment $u \mapsto \mathcal{L}_*(u)$ is a functor

$$\mathcal{L}_* : \mathrm{Top}_{|\mathbb{R}} \longrightarrow \mathrm{MV}(\mathbb{R}).$$

Proof
The fact that the δ_i^s are persistence modules maps as well as the commutativity of diagram (4.2) follow from the naturality of the Mayer-Vietoris sequence. The exactness of (4.6) implies the condition (4.1).

Recall from the introduction that $\mathrm{Top}_{|\mathbb{R}}$ is the category of topological spaces over $\mathbb{R}$ which by definition has objects given by continuous functions $u : X \to \mathbb{R}$ where X is a topological space. The set of morphisms from $u : X \to \mathbb{R}$ to $v : Y \to \mathbb{R}$ is the set of all continuous maps $\phi : X \to Y$ such that the diagram $\begin{tikzcd} X \arrow[r, "\phi"'] \arrow[rr, bend left, "u"] & Y \arrow[r, "v"'] & \mathbb{R} \end{tikzcd}$ is commutative. Since $u^{-1}(]-$

$x,y[) = \phi^{-1}\big(v^{-1}(]-x,y[)\big)$, we have that ϕ restricts to a continuous map $\phi : u^{-1}(]-x,y[) \hookrightarrow v^{-1}(]-x,y[)$. Therefore we have induced maps

$$\phi_*(x,y) : \mathcal{L}_i(u)(x,y) = \mathrm{H}_i(u^{-1}(]-x,y[) \to \mathrm{H}_i(v^{-1}(]-x,y[) = \mathcal{L}_i(v)(x,y)$$

after taking homology for all $(x,y) \in \Delta^+$. The functoriality of the homology functor and Mayer-Vietoris sequence prove that this ϕ_* is a morphism of Mayer-Vietoris system and furthermore that the assignment $u \mapsto \mathcal{L}_*(u)$, $\phi \mapsto (s \mapsto \phi_*(s))$ is a functor.

□

Example 4.2.16
Assume X is a smooth or topological manifold and $u : X \to \mathbb{R}$ is continuous. Then the Mayer Vietoris system $\mathcal{L}_*(u)$ (given by example 4.2.14) is bounded since an open subset of a manifold is a manifold and hence has no homology in degrees higher than its dimension.

The MV interleaving distance enjoy a similar stability property than the usual interleaving distance for persistence modules:

Proposition 4.2.17
Let $u, v : X \to \mathbb{R}$ two continuous functions defined on the topological space X, then:

$$d_I^{\mathrm{MV}}(\mathcal{L}_*(u), \mathcal{L}_*(v)) \leq \sup_{x\in X} |u(x) - v(x)|$$

Proof
If the distance is ∞, there is nothing to prove. Otherwise, let $\varepsilon = \sup_{x\in X} |u(x) - v(x)|$. Then for any $(x,y) \in \Delta^+$, we have level-sets inclusions $u^{-1}(]-x,y[) \subset v^{-1}(]-x-\varepsilon, y+\varepsilon[)$ and $v^{-1}(]-x,y[) \subset u^{-1}(]-x-\varepsilon, y+\varepsilon[)$ which induce persistence modules over Δ^+ morphisms

$$f : \big(\mathcal{L}_i(u)(x,y) = \mathrm{H}_i(u^{-1}(]-x,y[) \to \mathrm{H}_i(v^{-1}(]-x-\varepsilon, y+\varepsilon[) = \mathcal{L}_i(v)[\vec{\varepsilon}](x,y)\big)_{(x,y)\in\Delta^+},$$

$$g : \big(\mathcal{L}_i(v)(x,y) = \mathrm{H}_i(v^{-1}(]-x,y[) \to \mathrm{H}_i(u^{-1}(]-x-\varepsilon, y+\varepsilon[) = \mathcal{L}_i(u)[\vec{\varepsilon}](x,y)\big)_{(x,y)\in\Delta^+}$$

since taking homology groups is a functor and by lemma 4.3.1.

The fact that these maps are Mayer-Vietoris systems morphisms follows again as in proposition 4.2.15 by the naturality of the Mayer-Vietoris sequence associated to open covers of the intervals $]-x-\varepsilon, y+\varepsilon[$ by $]-x-\varepsilon, y[$ and $]-x, y+\varepsilon[$.

□

4.3 Stable sheaf theoretic interpretation of persistence

4.3.1 Extending level-sets persistence modules as pre-sheaves over $\mathbb{R}$

In this section we interpret level-sets persistence modules as (pre)sheaves on the line $\mathbb{R}$. In particular, we construct a functor:

$$\overline{(-)} : \mathrm{Pers}(\mathbf{k}^{\Delta^+}) \longrightarrow \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}},$$

which is, roughly speaking, the right Kan extension of the dual of a persistence module along the inclusion of bounded intervals in the category of open subsets of $\mathbb{R}$.

Let $(\mathfrak{Op}(\mathbb{R}), \subset)$ be the poset of open subsets of $\mathbb{R}$ ordered by the inclusion. We denote in the same way the associated category.

Lemma 4.3.1
Set $\iota : (\Delta^+, \leq) \to (\mathfrak{Op}(\mathbb{R}), \subset)$ to be given on objects by $\iota : s = (s_1, s_2) \mapsto]-s_1, s_2[$.Then ι is a well defined fully faithfull functor. The esential image of ι is precisely the full subcategory of bounded open intervals of $\mathbb{R}$.

In particular, restricting to objects of those categories, ι is a bijection from Δ^+ to bounded open intervals of $\mathbb{R}$.

Proof
By definition

$$s = (s_1, s_2) \in \Delta^+ \iff -s_1 < s_2$$

hence ι is well defined, injective on objects with image the bounded open intervals. Furthermore, if $(s_1, s_2) \leq (s_1', s_2')$ then $-s_1' \leq -s_1 < s_2 \leq s_2'$ which proves that ι is order presearving (and necessarily fully faithfull since the morphisms are empty or a singleton).

□

Given $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$ we can consider its pointwise dual $t \mapsto \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}(M(t); \mathbf{k})$ which has a canonical structure of a *persistence co-module*, that is of an object of

$$\mathrm{Fun}((\Delta^+)^{\mathrm{op}}; \mathrm{Mod}(\mathbf{k})) \cong \mathrm{Fun}(\Delta^+; \mathrm{Mod}(\mathbf{k})^{\mathrm{op}})^{\mathrm{op}}.$$

We denote by M^* this dual of M. More precisely, M^* is the composition of functors

$$M^* := \Delta^{+\mathrm{op}} \xrightarrow{M^{\mathrm{op}}} \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}(-;\mathbf{k})} \mathrm{Mod}(\mathbf{k}).$$

Since M^* is a persistence comodule, for any open $U \subset \mathbb{R}$, one sets

$$\tilde{M}(U) := \varprojlim_{]-x,y[\subset U} M^*((x,y)). \tag{4.7}$$

Lemma 4.3.2
There is a functor $(\tilde{-}) : \mathrm{Pers}(\mathbf{k}^{\Delta^+}) \to \mathrm{PSh}(\mathbb{R})^{\mathrm{op}}$ extending the formula (4.7) into a canonical presheaf on $\mathbb{R}$, that is such that for $U \in \mathrm{Obj}(\mathfrak{Op}(\mathbb{R}))$, one has

$$\tilde{M}(U) := \varprojlim_{]-x,y[\subset U} M^*((x,y)).$$

Proof
One notice that the formula exhibits $\tilde{M}$ as a Kan extension which makes it into a presheaf canonically. Indeed, consider $\iota^{\mathrm{op}} : (\Delta^+)^{\mathrm{op}} \longrightarrow \mathfrak{Op}(\mathbb{R})^{\mathrm{op}}$ the (opposite of the) functor defined previously (see 4.3.1) and let $\mathrm{Ran}_{\iota^{\mathrm{op}}} M^*$ be the right Kan extension along ι^{op} of M^*, which is therefore by definition an object of $\mathrm{PSh}(X)$:

$$\begin{tikzcd} \Delta^{+\mathrm{op}} \arrow[r, hook, "\iota^{\mathrm{op}}"] \arrow[d, "M^*"'] & \mathfrak{Op}(\mathbb{R})^{\mathrm{op}} \arrow[dl, dotted, "\mathrm{Ran}_{\iota^{\mathrm{op}}} M^* =: \tilde{M}"] \\ \mathrm{Mod}(\mathbf{k}) & \end{tikzcd}$$

As $\mathrm{Mod}(\mathbf{k})$ is complete, the pointwise formula (4.7) is an immediate consequence.

□

Composing $(\tilde{-})$ with the (opposite of the) sheafification functor $\mathrm{PSh}(\mathbb{R}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$ gives the functor from persistence modules on Δ^+ to sheaves on $\mathbb{R}$.

Definition 4.3.3
Let M be a persistence module over Δ^+. We set $\overline{M}$ to be the sheafification of the presheaf $\tilde{M}$ and we write

$$\overline{(-)} : \mathrm{Pers}(\mathbf{k}^{\Delta^+}) \longrightarrow \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}}$$

for the induced functor $M \mapsto \overline{M}$. We call $\overline{(-)}$ the *level-sets persistence to sheaves functor*.

Similarly there is a functor going in the other direction defined as follows. Given a sheaf on $\mathbb{R}$, by restriction to open intervals and using the identification of lemma 4.3.1, we get a persistence comodule. Since pointwise duality transforms a persistence comodule into a persistence module we obtain the functor

$$\pi : \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \longrightarrow \mathrm{Pers}(\mathbf{k}^{\Delta^+}), \quad F^{\mathrm{op}} \mapsto \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k}) \circ F^{\mathrm{op}} \circ \iota. \tag{4.8}$$

Given $\mathbb{V}$ a $\mathbf{k}$-vector space, one denotes for short

$$\mathbb{V}^* := \mathrm{Hom}_{\mathbf{k}}(\mathbb{V}, \mathbf{k}).$$

Let F be a sheaf in $\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$. We define the *pointwise bidual of F* as the sheaffification of the presheaf:

$$U \mapsto F(U)^{**}.$$

This defines a functor $\mathrm{bidual}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})} : \mathrm{Mod}(\mathbf{k}_{\mathbb{R}}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})$. There is a canonical natural transformation

$$\mathrm{id}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})} \longrightarrow \mathrm{bidual}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})} \tag{4.9}$$

given by the pointwise canonical morphism.

Proposition 4.3.4
The level-sets persistence to sheaves functor $\overline{(-)} : \mathrm{Pers}(\mathbf{k}^{\Delta^+}) \longrightarrow \mathrm{Mod}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}}$ from Definition 4.3.3 satisfies the following properties.

1. Its composition with the restriction-to-intervals functors is the canonical biduality functor: $\overline{(-)} \circ \pi = \mathrm{bidual}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})}$.
   In particular the restriction of this composition of functors to the subcategory of pointwise finite dimensional[3] objects is naturally isomorphic to $\mathrm{id}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})^{pfd}}$
2. If M is a pointwise finite dimensional persistence module, and $M \simeq \oplus_i M_i$, then $\overline{M} \simeq \oplus_i \overline{M_i}$
3. Assume that $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$ is pointwise finite dimensional. Then, for all $\alpha \in \mathbb{R}$, we have natural isomorphisms

$$\varprojlim_{]-x;y[\ni\alpha} M((x,y)) \simeq \tilde{M}_\alpha \simeq \overline{M}_\alpha$$

   provided that the left hand side is finite dimensional.
4. One can identify $\overline{M}$ with the image of pre-sheaves morphism:$\tilde{M} \longrightarrow \prod_{\alpha\in\mathbb{R}} \tilde{M}_\alpha$.

3. where we mean the sheaves whose stalk at each point are finite dimensional

Proof

1. First, once the formula $\overline{(-)} \circ \pi = \overline{(-)} \circ \pi = \mathrm{bidual}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})}$ is proved, to check that the asserted restriction of the composite $\overline{(-)} \circ \pi$ is canonically isomorphic to the identity, it is sufficient to prove that the canonical transformation $\mathrm{id}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})} \to \mathrm{bidual}_{\mathrm{Mod}(\mathbf{k}_{\mathbb{R}})}$ is an isomorphism on all stalks when restricted to a pointwise finite dimensional sheaf. This reduces the statement to the standard case of finite dimensional vector spaces. To prove the formula note that the section of a sheaf F on an open U is uniquely determined by its value on any open cover; furthermore, for any open interval I one has that

$$\varprojlim_{]-x,y[\subset I} ((F(I))^*)^* \cong (F(I)^*)^*.$$

In particular, on can restrict to cover by open intervals (I_j) of U and compute the value of the sheaf at U as a limit. Therefore, noticing that the intersection of two intervvals is an interval, we get that for a sheaf F, one has

$$\begin{aligned} \overline{\pi(F)}(U) &= \varprojlim \left(\overline{\pi(F)}(\textstyle\coprod U_k) \rightrightarrows \overline{\pi(F)}(\textstyle\coprod (U_i \cap U_j)) \right) \\ &\cong \varprojlim \left(\textstyle\prod (F(U_k)^*)^* \rightrightarrows \textstyle\prod (F(U_i \cap U_j)^*)^* \right) \\ &\cong (F(U)^*)^* \end{aligned}$$

2. The functors $\mathrm{Hom}(-, \mathbf{k})$, sheafification (which is a right adjoint) as well as right Kan extensions commute with finite direct sums. This gives the finite sums case. But the assumption ensures it is enough to estblish the resutls on the stalks and therefore the canonical map $\bigoplus \overline{M_i} \to \overline{\bigoplus M_i}$ is an isomorphism.
3. Write $\mathrm{Int}(\alpha)$ for the (full) subcategory of $\mathfrak{Op}(\mathbb{R})$ consisting of intervals containing α. Let us fix $G : (]0, \infty[, \leq) \longrightarrow \mathrm{Int}(\alpha)$ defined by $G(\varepsilon) =]\alpha - \varepsilon, \alpha + \varepsilon[$. Then G is a functor and is initial among functors $(]0, \infty[, \leq) \longrightarrow \mathrm{Int}(\alpha)$. Therefore:

$$\varprojlim_{]-x;y[\ni\alpha} M((x,y)) \simeq \varprojlim M \circ G = \varprojlim_{\varepsilon>0} M((\varepsilon - \alpha, \alpha + \varepsilon))$$

Since $(]0, \infty[, \leq)$ is a totally ordered set, we can apply the theorem of decomposition of pfd modules over totally ordered sets to $M \circ G$, thus there exists a multiset $\mathbb{B}(M \circ G)$ of intervals of $\mathbb{R}$ such that:

$$M \circ G \simeq \bigoplus_{I \in \mathbb{B}(M \circ G)} \mathbf{k}_I \tag{4.10}$$

It follows that

$$\varprojlim_{\varepsilon>0} M((\varepsilon-\alpha,\alpha+\varepsilon)) \simeq \prod_{\substack{I\in\mathbb{B}(M\circ G)\\ 0\in \mathrm{closure}(I)}} \mathbf{k} \tag{4.11}$$

Now if $\varprojlim_{]-x;y[\ni\alpha} M((x,y))$ is finite dimensional then the above product in the right hand side of (4.11) is a finite product and thus a direct sum: $\prod_{\substack{I\in\mathbb{B}(M\circ G)\\ 0\in \mathrm{closure}(I)}} \mathbf{k} \simeq \bigoplus_{\substack{I\in\mathbb{B}(M\circ G)\\ 0\in \mathrm{closure}(I)}} \mathbf{k}$. Therefore we have

$$\begin{aligned}\varprojlim_{]-x;y[\ni\alpha} M((x,y)) &\simeq \bigoplus_{\substack{I\in\mathbb{B}(M\circ G)\\ 0\in \mathrm{closure}(I)}} \mathbf{k}\\ &\simeq \varprojlim_{]-x;y[\ni\alpha} \mathrm{Hom}\left(M((x,y)),\mathbf{k}\right) \quad \text{(by (4.10) and finite dimensionality)}\\ &\simeq \tilde{M}_\alpha \quad \text{(by (4.7))}\\ &\simeq \overline{M}_\alpha.\end{aligned}$$

4. This is a direct consequence of the definition of the sheaffification functor for sheaves on a T_1-topological space, that is for sheaves on a space for which all points are closed.

□

Remark 4.3.5

We have sticked to the traditional point of view of looking at level-sets as being given by homology functors and thus as persistent objects; point of view for which computational models are well developed. This is the reason why some (bi)duality shows up in the picture. It is possible (and actually slightly easier) to construct an analogue of $\overline{(-)} : M \mapsto \overline{(M)}$ going from persistence co-modules to sheaves.

Let $\Delta = \{(-x,x) \mid x\in\mathbb{R}\}$, and $p:\Delta \longrightarrow \mathbb{R}$ be the projection $(x_1,x_2)\mapsto x_2$ onto the second coordinate. Recall that for any block B (Definition 2.1.2) we have defined a persistence module $\mathbf{k}^B \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$.

Proposition 4.3.6

Let B be a block. Let $a,b\in\mathbb{R}$ be such that $<a,b>= p(B\cap\Delta)$, with the convention that $a=1$ and $b=-1$ when $p(B\cap\Delta)=\emptyset$.

1. If B is of type **db**, then $\overline{\mathbf{k}^B} \simeq \mathbf{k}_{]a,b[}$.

2. If B is of type **bb**, then $\overline{\mathbf{k}^B} \simeq \mathbf{k}_{[a,b]}$.
3. If B is of type **vb**, then $\overline{\mathbf{k}^B} \simeq \mathbf{k}_{]a,b]}$.
4. If B is of type **hb**, then $\overline{\mathbf{k}^B} \simeq \mathbf{k}_{[a,b[}$.

Proof

Let B be of type **db**. If B is included in $\mathbb{R}^2 \setminus \Delta^+$, then $\mathbf{k}^B$ is identically null as well as $\mathbf{k}_{]a,b[}$ and there is nothing to prove. If not, B has a non-trivial intersection with Δ and $(-a, b) = \sup_B(s)$ are the coordinates of the supremum of B for the order relation of Δ^+. Then, for $s = (s_1, s_2) \in \mathbb{R}^2_{>0}$, one has

$$\mathbf{k}^B(s) = \begin{cases} \mathbf{k} & \text{if } (s_1, s_2) < (-a, b) \\ 0 & \text{if } (s_1, s_2) \nleq (-a, b) \end{cases}$$

Hence $\mathbf{k}^B(s)$ is non-zero if $a \leq -s_1 < s_2 \leq b$ and always null if either $-s_1 < a$ or $s_2 > b$. It follows that that for $\alpha \in \mathbb{R}$, then $\mathbf{k}^B(-\alpha-\varepsilon, \alpha+\varepsilon) = 0$ if $\alpha \notin]a, b[$ and for all $\alpha \in]a, b[$, there exists $\eta > 0$ such that $\mathbf{k}(-\alpha-\eta, \alpha+\eta) = \mathbf{k}$. We conclude that

$$\varinjlim_{]-x,y[\ni\alpha} \mathbf{k}^B(x, y) = \begin{cases} \mathbf{k} & \text{if } \alpha \in]a, b[\\ 0 & \text{else.} \end{cases} \tag{4.12}$$

By claim 5 of Proposition 4.3.4, we deduce that $\overline{\mathbf{k}^B} \cong \mathbf{k}_{]a,b[}$. Similarly, if B_v is a vertical block, delimited by the lines $x = -b$, $x = -a$ with $a < b$, then we have

$$\mathbf{k}^{B_v}(s_1, s_2) = \begin{cases} \mathbf{k} & \text{if } -b < s_1 < -a \\ 0 & \text{if } s_1 > -a \text{ or } s_1 < -b \end{cases}$$

independently of whether the boundary lines are part of B_v or not. In particular, for any $\alpha \in]a, b]$, there exists $\eta > 0$ such that $\mathbf{k}(-\alpha-\eta, \alpha+\eta) = \mathbf{k}$ while there exists $\varepsilon > 0$ such that $\mathbf{k}^B(-\alpha-\varepsilon, \alpha+\varepsilon) = 0$ if $\alpha \leq a$ or $\alpha > b$. As in the **db** case (4.12), we thus find that

$$\varinjlim_{]-x,y[\ni\alpha} \mathbf{k}^{B_v}(x, y) = \begin{cases} \mathbf{k} & \text{if } \alpha \in]a, b] \\ 0 & \text{else.} \end{cases}$$

The last two other cases are obtained using a similar analysis.

□

Remark 4.3.7

In particular, $\overline{\mathbf{k}^B}$ does not depend on whether B contains its boundary or not. If B is of type **bb**$^+$, then $\overline{\mathbf{k}^B} = 0$ (since $B \cap \Delta = \emptyset$).

Remark 4.3.8 (characterizations of (a, b))
If B is of type $\mathbf{bb}^-$, then the numbers a and b are characterized by the fact that the point $(-b, a)$ is the infimum of the points in B, see figure (2.1.3).

Similarly, if B is of type $\mathbf{db}$, then the numbers a and b satisfies that the point $(-a, b)$ is the supremum of the points in B.

Finally, for B of type $\mathbf{vb}$, a and b satisfies that B has boundary given by the lines of equation $x = -b$ and $x = -a$, while if it is of type $\mathbf{hb}$, a and b satisfies that the boundary of B are the horizontal lines of equations $y = a$ and $y = b$.

Blocks of type $\mathbf{db}^+$, $\mathbf{hb}$, $\mathbf{vb}$ and $\mathbf{bb}^-$ are actually uniquely determined by their intersection with the anti-diagonal, that is the interval $\langle a, b\rangle = p(\Delta \cap B)$ (as in Proposition 4.3.6). Precisely we have:

Lemma 4.3.9
Let $a < b$ be real numbers. Let $\{<, >\} \in \{[,]\}^2$. There are unique blocks $B_b^{\langle a,b\rangle}$, $B_h^{\langle a,b\rangle}$, $B_v^{\langle a,b\rangle}$ and $B_d^{\langle a,b\rangle}$ respectively of type $\mathbf{bb}^-$, $\mathbf{hb}$, $\mathbf{vb}$ and $\mathbf{db}^+$ such that $p(\Delta \cap B_\lambda^{\langle a,b\rangle}) = \langle a, b\rangle$, for $\lambda \in \{b, h, v, d\}$.

Proof
By definition 2.1.26, all the blocks except the birth blocks lying entirely in $\Delta^+_{>0}$, that is those of type $\mathbf{bb}^+$, are uniquely determined by their intersection with the anti-diagonal (also see figure 2.1.3). In fact the points a, b determine the block of any of these types as in remark 4.3.8 and more precisely it determines the boundary lines of the block. To determine if the lines are included in the block or not, we look to whether a or b are inside the interval $\langle a, b\rangle$. For instance, for $B_v^{[a,b]}$ we take the vertical block delimited by the vertical lines $x = -b$ and $x = -a$ and containing them, while $B_v^{]a,b[}$ is the block vertical delimited by the same lines but not containing any of them.

$\square$

Corollary 4.3.10
If $M \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$ is middle-exact and pointwise finite dimensional, then $\overline{M}$ is weakly constructible. Furthermore, if M is strongly pointwise finite dimensional (definition 4.4.1) and midddle-exact, then $\overline{M}$ is constructible.

In particular, the restriction of the sheafification functor $\overline{(-)} = \overline{(-)}$: $\mathrm{Pers}(\mathbf{k}^{\Delta^+}) \to \mathrm{Mod}(\mathbf{k}_\mathbb{R})$ to the full subcategory of strongly pointwise finite dimensional middle exact modules takes values in the subcategory $\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$ of constructible sheaves.

Proof

By the decomposition Theorem 2.1.33, the pfd module M is isomorphic to a direct sum of blocks $M \cong \bigoplus_{B\in\mathbb{B}(M)} \mathbf{k}^B$. Since $\overline{(-)}$ commutes with direct sum for pfd modules (Proposition 4.3.4), Proposition 4.3.6 yields that $\overline{M} \cong \bigoplus_{B\in\mathbb{B}(M)} \overline{\mathbf{k}^B}$ is a (pointwise finite when M is strongly pfd) direct sum of sheaves of the form $\mathbf{k}_I$ where I is an interval.

□

The level-sets persistence to sheaves functor $\overline{(-)}$ does not preserve interleavings in general. However, the trouble is only related to the death or $\mathbf{bb}^+$ quadrant. More precisely we have the following two lemmas.

Lemma 4.3.11

Let $M, N \in \mathrm{Obj}(\mathrm{Pers}(\mathbf{k}^{\Delta^+}))$ be middle exact pointwise finite dimensional and such that their barcodes contains only blocks of type $\mathbf{bb}^-$, $\mathbf{vb}$ and $\mathbf{hb}$. Then

$$\overline{M[\vec{\varepsilon}]} \cong \overline{M} \star K_\varepsilon.$$

Furthermore, if $M \sim_\varepsilon^{\Delta^+} N$, then

$$\overline{M} \sim_\varepsilon \overline{N}.$$

Proof

By Theorem 2.1.33, we have isomorphisms $M \cong \bigoplus\limits_{B\in\mathbb{B}(M)} \mathbf{k}^B$, $N \cong \bigoplus\limits_{B\in\mathbb{B}(N)} \mathbf{k}^B$ of persistence modules, such that the blocks B are of types $\mathbf{bb}^-$, $\mathbf{vb}$ and $\mathbf{hb}$. Lemma 2.1.32 implies that

$$\overline{M[\vec{\varepsilon}]} \cong \bigoplus_{B\in\mathbb{B}(M)} \overline{\mathbf{k}^{B-\vec{\varepsilon}}}$$

$$\overline{N[\vec{\varepsilon}]} \cong \bigoplus_{B'\in\mathbb{B}(N)} \overline{\mathbf{k}^{B'-\vec{\varepsilon}}}$$

where each $\overline{\mathbf{k}^{B-\vec{\varepsilon}}}$ is of the form $\mathbf{k}_{I(B,\varepsilon)}$ where $I(B,\varepsilon)$ is an interval $< a, b >= p((B-\vec{\varepsilon})\cap\Delta)$ which is

— a closed non-empty interval if B is of type $\mathbf{bb}^-$;

— a semi-open interval closed on the left (resp. closed on the right) if B is of type $\mathbf{hb}$ (resp. $\mathbf{vb}$).

Therefore we have:

$$\begin{aligned}
\text{if } B \text{ is of type } \mathbf{bb}^-\text{, then } \overline{\mathbf{k}^{B-\vec{\varepsilon}}} &\cong \mathbf{k}_{[a-\varepsilon,b+\varepsilon]},\\
\text{if } B \text{ is of type } \mathbf{vb}\text{, then } \overline{\mathbf{k}^{B-\vec{\varepsilon}}} &\cong \mathbf{k}_{]a+\varepsilon,b+\varepsilon]},\\
\text{if } B \text{ is of type } \mathbf{hb}\text{, then } \overline{\mathbf{k}^{B-\vec{\varepsilon}}} &\cong \mathbf{k}_{[a-\varepsilon,b-\varepsilon[}.
\end{aligned}$$

Using Proposition 3.2.5, we thus get that in all cases,

$$\overline{\mathbf{k}^{B-\vec{\varepsilon}}} \;\cong\; \overline{\mathbf{k}^B} \star K_\varepsilon$$

and by additivity of the convolution functor we obtain $\overline{M[\vec{\varepsilon}]} \cong \overline{M} \star K_\varepsilon$ as claimed.

The same results holds for the blocks $B' \in \mathbb{B}(N)$ so that

$$\overline{M[\vec{\varepsilon}]} \cong \overline{M} \star K_\varepsilon, \quad \overline{N[\vec{\varepsilon}]} \cong \overline{N} \star K_\varepsilon.$$

Note further that, for a $\mathbf{bb}^-$ block B, the canonical map $\overline{\mathbf{k}^B} \star K_\varepsilon \to \overline{\mathbf{k}^B}$ (of proposition 2.2.5) is identified with the canonical sheaf map $\mathbf{k}_{[a-\varepsilon,b+\varepsilon]} \to \mathbf{k}_{[a,b]}$ as follows from the proof of 3.2.6. Since the sheaf map is induced by restriction we obtain from the above equivalences, that the diagram

$$\begin{tikzcd}
\overline{\mathbf{k}^B} \star K_\varepsilon \arrow[r, "\cong"] \arrow[d] & \mathbf{k}_{[a-\varepsilon,b+\varepsilon]} \arrow[r, "\cong"] \arrow[d] & \overline{\mathbf{k}^B[\varepsilon]} \arrow[d, "\tau_\varepsilon^{\overline{\mathbf{k}^B}}"] \\
\overline{\mathbf{k}^B} \arrow[r, "\cong"] & \mathbf{k}_{[a,b]} \arrow[r, "\cong"] & \overline{\mathbf{k}^B}
\end{tikzcd}$$

is commutative. Using proposition 3.2.5, the above identification extends to the $\mathbf{vb}$ and $\mathbf{hb}$ blocks case as well: that is we have, for any block B of type $\mathbf{bb}^-$, $\mathbf{vb}$ and $\mathbf{hb}$ a commutative diagram

$$\begin{tikzcd}
\overline{\mathbf{k}^B} \star K_\varepsilon \arrow[r, "\cong"] \arrow[d] & \overline{\mathbf{k}^B[\varepsilon]} \arrow[d, "\tau_\varepsilon^{\overline{\mathbf{k}^B}}"] \\
\overline{\mathbf{k}^B} \arrow[r, "\cong"] & \overline{\mathbf{k}^B}
\end{tikzcd} \tag{4.13}$$

Now let $f : M \to N[\vec{\varepsilon}]$ and $g : N \to M[\vec{\varepsilon}]$ be an ε-interleaving between M and N, then applying the functor $\overline{(-)}$ to the latter isomorphisms, we obtain an ε-interleaving in sheaves given by $\overline{M} \xrightarrow{\overline{f}} \overline{N[\vec{\varepsilon}]} \cong \overline{N} \star K_\varepsilon$ and $\overline{N} \xrightarrow{\overline{g}} \overline{M[\vec{\varepsilon}]} \cong \overline{M} \star K_\varepsilon$.

□

For *deathblocks* or *birthblocks* of type $\mathbf{bb}^+$, the sheafification functor does not intertwine shifts with convolution in a naive way. However we have the following precise result. To state it, we first recall that to a block B, we can associate the two real numbers $a, b \in \mathbb{R}$ such that $< a, b >= p(B \cap \Delta)$; the convention being that $a = 1$ and $b = -1$ when $p(B \cap \Delta) = \emptyset$.

Lemma 4.3.12
Let B be a block of type **db** or **bb**$^+$. If B is a **bb**$^+$ block, its dual death block $B^\dagger$ intersects Δ and we denote $< a^\dagger, b^\dagger >= p(B^\dagger \cap \Delta)$.

Furthermore, for any $\varepsilon \geq 0$, we have that,

$$\text{if } B \text{ is of type } \mathbf{db}\text{, then, } \overline{\mathbf{k}^B[\vec{\varepsilon}]} \cong \begin{cases} \overline{\mathbf{k}^B} \star K_\varepsilon & \text{if } \varepsilon < \frac{b-a}{2} \\ 0 & \text{if } \varepsilon \geqslant \frac{b-a}{2}, \end{cases} \tag{4.14}$$

$$\text{if } B \text{ is of type } \mathbf{bb}^+\text{, then, } \overline{\mathbf{k}^B[\vec{\varepsilon}]} \cong \begin{cases} 0 & \text{if } \varepsilon < \frac{b^\dagger - a^\dagger}{2} \\ \mathbf{k}_{[\frac{a^\dagger+b^\dagger}{2}, \frac{a^\dagger+b^\dagger}{2}]} \star K_{\varepsilon - \frac{b^\dagger - a^\dagger}{2}} & \text{if } \varepsilon \geqslant \frac{b^\dagger - a^\dagger}{2}. \end{cases} \tag{4.15}$$

Proof
The proof will be similar to the one of lemma 4.3.11. First note that, if B is of birthtype **bb**$^+$, the supremum of $B^\dagger$ is the infimum of B by definition of the dual block. Therefore, by remark 4.3.8, we have that the infimum of the elements of B is the point $(-a^\dagger, b^\dagger) \in \Delta^+$. It follows that $B - \vec{\varepsilon}$ remains of type **bb**$^+$ as long as $\varepsilon < \frac{b^\dagger - a^\dagger}{2}$ and it becomes of type **bb**$^-$ when $\varepsilon \geqslant \frac{b^\dagger - a^\dagger}{2}$. Furthermore, in that latter case, we have that

$$p((B - \vec{\varepsilon}) \cap \Delta) = \langle b^\dagger - \varepsilon, a^\dagger + \varepsilon \rangle.$$

By lemma 2.1.32 and proposition 4.3.6, we thus have that if B is of type **bb**$^+$, then

$$\overline{\mathbf{k}^{B-\vec{\varepsilon}}} \cong \begin{cases} 0 & \text{if } \varepsilon < \frac{b^\dagger - a^\dagger}{2} \\ \mathbf{k}_{[b^\dagger - \varepsilon, a^\dagger + \varepsilon]} & \text{if } \varepsilon \geqslant \frac{b^\dagger - a^\dagger}{2}. \end{cases}$$

Using Proposition 3.2.5, we see that for $\varepsilon \geqslant \frac{b^\dagger - a^\dagger}{2}$, one has

$$\mathbf{k}_{[b^\dagger - \varepsilon, a^\dagger + \varepsilon]} \cong \mathbf{k}_{[\frac{a^\dagger+b^\dagger}{2}, \frac{a^\dagger+b^\dagger}{2}]} \star K_{\varepsilon - \frac{b^\dagger - a^\dagger}{2}}$$

which shows the formula (4.15).

Now, note that if B is of type **db**, then $B - \vec{\varepsilon}$ has a non-empty intersection with Δ as long as $\varepsilon < \dfrac{b-a}{2}$. And similarly we find, using proposition 4.3.6 that

$$\overline{\mathbf{k}^{B-\vec{\varepsilon}}} \cong \begin{cases} 0 & \text{if } \varepsilon \geq \frac{b^\dagger - a^\dagger}{2} \\ \mathbf{k}_{]a+\varepsilon, b-\varepsilon[} & \text{if } \varepsilon < \frac{b^\dagger - a^\dagger}{2}. \end{cases}$$

To prove formula (4.14), we are left to apply Proposition 3.2.5 a last time.

□

Remark 4.3.13
The proof of lemma 4.3.12 and proposition 4.3.6 also shows that the last equivalence in lemma 4.3.12 also reads, for $\varepsilon \geq \frac{b^\dagger - a^\dagger}{2}$, as

$$\mathbf{k}_{[\frac{a^\dagger+b^\dagger}{2}, \frac{a^\dagger+b^\dagger}{2}]} \star K_{\varepsilon - \frac{b^\dagger - a^\dagger}{2}} \cong \overline{\mathbf{k}^{B - \overrightarrow{\frac{b^\dagger - a^\dagger}{2}}}} \star K_{\varepsilon - \frac{b^\dagger - a^\dagger}{2}}. \tag{4.16}$$

4.4 Isometric pseudo-equivalence between $\mathbf{MV}(\mathbb{R})^{\mathbf{sf}}$ and $\mathbf{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$

In this section, we explain why the interleaving distance between level set persistence is essentially the same as the derived bottleneck distance between the associated sheaves (in the constructible case).

In order to express this we will relate constructible sheaves by an isometry to a specific type of graded persistence modules, that is those satisfying the following definition.

Definition 4.4.1
A middle-exact persistence module $M \in \mathrm{Pers}(\mathbf{k}^{\Delta^+})$, is a *strongly pointwise finite dimensional persistence module*, if it is pointwise finite dimensional and satisfies the following additional condition:

$$\text{For every } \alpha \in \Delta, \ \varprojlim_{]-x;y[\ni\alpha} M((x,y)) \text{ is finite dimensional}$$

A Mayer-Vietoris system $S = (S_i, \delta_i^s)$ is said to be strongly pointwise finite dimensional if each S_i is strongly pointwise finite dimensional and only finitely many S_i's are non-zero.

The full subcategory of $\mathrm{MV}(\mathbb{R})$ whose objects are strongly pointwise finite dimensional MV-systems is denoted by $\mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$.

Our goal now is to build two distance-preserving functors:

$$\overline{(-)}^{\mathrm{MV}} : \qquad \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \longrightarrow \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})^{\mathrm{op}}$$

$$\Psi : \qquad \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})^{\mathrm{op}} \longrightarrow \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$$

Satisfying for every $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, $\overline{(\cdot)}^{\mathrm{MV}} \circ \Psi(F) \simeq F$, in other words Ψ is a pointwise section, and that, for every $M \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$,

$$d_I^{\mathrm{MV}}\left(M, \Psi(\overline{M}^{\mathrm{MV}})\right) = 0.$$

This goal will be achieved by Corollary 4.4.18 and Corollary 4.4.20.

4.4.1 Sheafification of MV-systems: the functor $\overline{(\cdot)}^{\mathbf{MV}}$

We will now apply section 4.3.1 to compare the Mayer-Vietoris persistence systems and constructible sheaves. To do so, we first consider the direct sum of the level set persistence to sheaves functor:

Let $Q : \mathrm{K}(\mathrm{Mod}(\mathbf{k}_R)) \to \mathrm{D}(\mathbf{k}_{\mathbb{R}})$ be *the localization functor* sending the category of complexes of sheaves over $\mathbb{R}$ to its derived category $\mathrm{D}(\mathbf{k}_{\mathbb{R}})$ (definition A.1.28).

Definition 4.4.2
The *sheafification of MV-systems functor*: $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R}) \to \mathrm{D}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}}$ is the functor given, on objects $S = (S_i, \delta_i^S)_{i\in\mathbb{Z}, s\in\mathbb{R}^2_{>0}} \in \mathrm{Obj}(\mathrm{MV}(\mathbb{R}))$, by

$$\overline{S}^{\mathrm{MV}} := Q\Big(\bigoplus_{i\in\mathbb{Z}} \overline{S_i}[-i]\Big)$$

and, on morphisms $(S_i \xrightarrow{\varphi_i} T_i)_{i\in\mathbb{Z}}$, by

$$\overline{(\varphi_i)_{i\in\mathbb{Z}}} := Q\left(\bigoplus_{i\in\mathbb{Z}} \overline{\varphi_i}\right).$$

The notation $\overline{S_i}[-i]$ stands for the complex concentrated in degree i with value $\overline{S_i}$. The functoriality is a direct consequence of section 4.3.1.

Lemma 4.4.3
If S is a strongly pointwise finite dimensional Mayer-Vietoris system, then $\overline{S}^{\mathrm{MV}}$ is a constructible sheaf. In particular, we have a commutative diagram of functors:

$$\begin{tikzcd}
\mathrm{MV}(\mathbb{R}) \arrow[r, "\overline{(-)}^{\mathrm{MV}}"] & \mathrm{D}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \\
\mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \arrow[u, hook] \arrow[r, "\overline{(-)}^{\mathrm{MV}}"] & \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \arrow[u, hook]
\end{tikzcd}$$

Proof
We can apply Theorem 4.2.6 together with proposition 4.3.6 in a way similar to the proof of corollary 4.3.10.

□

The same argument shows that if S is pfd (but not necessarily strongly), then $\overline{S}^{\mathrm{MV}}$ is weakly constructible.

Proposition 4.4.4
The sheafification of MV-systems functor $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ satisfies the following properties:

1. it commutes with degree shifting operator:for all Mayer-Vietoris system M, one has $\overline{M[n]}^{\mathrm{MV}} \cong \overline{M}^{\mathrm{MV}}[n]$.
2. For a block B of type $\mathbf{bb}^-$, $\mathbf{hb}$, $\mathbf{vb}$, $\mathbf{db}^+$, $j \in \mathbb{Z}$ and $\varepsilon \geq 0$, we have:
$$(\overline{S_j^B[\varepsilon]})^{\mathrm{MV}} \simeq \mathbf{k}_{I(B)}[-j] \star K_\varepsilon$$
where, still denoting $\langle a, b\rangle = p(B \cap \Delta^+)$, $I(B)$ is the interval given by
$$I(B) = [a, b] \text{ if } B \text{ is of type } \mathbf{bb}^-, \qquad I(B) = [a, b[\text{ if } B \text{ is of type } \mathbf{hb},$$
$$I(B) =]a, b] \text{ if } B \text{ is of type } \mathbf{vb}, \qquad I(B) =]a, b[\text{ if } B \text{ is of type } \mathbf{db}^+.$$
3. If $M \sim_\varepsilon^{\Delta^+} N$, then $\overline{M}^{\mathrm{MV}} \sim_\varepsilon \overline{N}^{\mathrm{MV}}$.
4. If $\overline{M}^{\mathrm{MV}}$ is isomorphic to $\overline{N}^{\mathrm{MV}}$ (in the derived category), then $d_I^{\mathrm{MV}}(M, N) = 0$.

Proof
Note that assertion 1 is immediate from the definition since we put each $\overline{S_i}$ precisely in degree i.

2 and 3. First assume B is of type $\mathbf{bb}^-$, $\mathbf{hb}$ or $\mathbf{vb}$. Then definition 4.2.3 implies that $S_j^B \cong \mathbf{k}^B[-j]$. Since $\overline{(\cdot)}^{\mathrm{MV}}$ commutes with direct sum and shifts, Lemma 4.3.11 implies $(\overline{S_j^B[\varepsilon]})^{\mathrm{MV}} \simeq \mathbf{k}^B[-j] \star K_\varepsilon$ and further, for $\varepsilon \leq \varepsilon'$, this isomorphism sends the canonical structure maps $S_j^B[\varepsilon'] \to S_j^B[\varepsilon]$ onto the canonical map $\mathbf{k}^B[-j] \star K_\varepsilon \to \mathbf{k}^B[-j] \star K_{\varepsilon'}$ (see diagram (4.13)).

It remains to prove the same result in the case of a block of type $\mathbf{db}$. Then definition 4.2.3 says that as a graded persistence module, one has
$$S_j^B \cong \mathbf{k}^B[-j] \oplus \mathbf{k}_{B^\dagger}[-j-1]$$
and therefore
$$\overline{S_j^B[\varepsilon]}^{\mathrm{MV}} \cong \overline{\mathbf{k}^B[\varepsilon]}[-j] \oplus \overline{\mathbf{k}_{B^\dagger}[\varepsilon]}[-j-1].$$
Denote $\langle a, b\rangle = p(B \cap \Delta)$ as before Proposition 4.3.6. Following the notation of Lemma 4.3.12 we thus have that for the dual block $B^\dagger$ of type $\mathbf{bb}^+$, one has that $a^\dagger = a$, $b^\dagger = b$ by definition. Then, Lemma 4.3.12, the commutation of convolution wih shifts and Proposition 4.3.6 imply that
$$\overline{\mathbf{k}^B[\varepsilon]}[-j] \oplus \overline{\mathbf{k}^{B^\dagger}[\varepsilon]}[-j-1] \cong \begin{cases} \mathbf{k}_{]a,b[} \star K_\varepsilon[-j] & \text{if } \varepsilon < \frac{b-a}{2} \\ \mathbf{k}_{[\frac{a+b}{2}, \frac{a+b}{2}]} \star K_{\varepsilon - \frac{b-a}{2}}[-j-1] & \text{if } \varepsilon \geqslant \frac{b-a}{2}. \end{cases} \tag{4.17}$$

This formula (4.17) is precisely the formula for $\mathbf{k}_{]a,b[} \star K_\varepsilon$ according to Proposition 3.2.5. We obtain a commutative diagram similar to (4.13) in the same way as in Lemma 4.3.11. This concludes the proof of claim 2. Assertion 3 follows immediately of assertion 2 and the fact that the canonical translation maps of persistence modules are sent to the canonical maps $\mathbf{k}_B[-j] \star K_\varepsilon \to \mathbf{k}_B[-j] \star K_{\varepsilon'}$.

4. Assume $\overline{M}^{\mathrm{MV}} \cong \overline{N}^{\mathrm{MV}}$. By Theorem 4.2.6, we can decompose

$$M \cong \bigoplus_{j\in\mathbb{Z}} \left(\bigoplus_{B_M \in \mathbb{B}_j(M)} S_j^{B_M}[-j] \right) \text{ and } N \cong \bigoplus_{j\in\mathbb{Z}} \left(\bigoplus_{B_N \in \mathbb{B}_j(N)} S_j^{B_N}[-j] \right)$$

into Mayer-Vietoris blocks. Since $\overline{(-)}^{\mathrm{MV}}$ commutes with direct sum and shifts (by property 1), we have isomorphisms

$$\begin{aligned} \overline{\bigoplus_{j\in\mathbb{Z}} \bigoplus_{B_M \in \mathbb{B}_j(M)} S_j^{B_M}[-j]}^{\mathrm{MV}} &\cong \overline{\bigoplus_{j\in\mathbb{Z}} \bigoplus_{B_N \in \mathbb{B}_j(N)} S_j^{B_N}[-j]}^{\mathrm{MV}} \\ \bigoplus_{j\in\mathbb{Z}} \bigoplus_{B_M \in \mathbb{B}_j(M)} \overline{S_j^{B_M}}^{\mathrm{MV}}[-j] &\cong \bigoplus_{j\in\mathbb{Z}} \bigoplus_{B_N \in \mathbb{B}_j(N)} \overline{S_j^{B_N}}^{\mathrm{MV}}[-j]. \end{aligned}$$

For any vertical, horizontal or $\mathbf{bb}^-$ type block B, Proposition 4.3.6 tells us that $\overline{S_j^B}^{\mathrm{MV}} \cong \mathbf{k}_{I(B)}$ where $I(B)$ is a non-empty interval (uniquely determined by $p(B\cap\Delta)$). If B is of type $\mathbf{db}^+$, then

$$\overline{S_j^B} \cong \mathbf{k}_{I(B^\dagger)}[j-1]$$

according to definition 4.2.3 and 4.4.2. Therefore, we have an isomomorphism

$$\begin{aligned} &\bigoplus_{j\in\mathbb{Z}} \left(\Big(\bigoplus_{B_M \in \mathbb{B}_j(M)\setminus \mathbb{B}_j^{\mathbf{bb}^+}(M)} \mathbf{k}_{I(B_M)}[-j] \Big) \oplus \bigoplus_{B_M \in \mathbb{B}_j^{\mathbf{bb}^+}(M)} \mathbf{k}_{I(B_M^\dagger)}[-j-1] \right) \\ \cong &\bigoplus_{j\in\mathbb{Z}} \left(\Big(\bigoplus_{B_N \in \mathbb{B}_j(N)\setminus \mathbb{B}_j^{\mathbf{bb}^+}(N)} \mathbf{k}_{I(B_N)}[-j] \Big) \oplus \bigoplus_{B_N \in \mathbb{B}_j^{\mathbf{bb}^+}(N)} \mathbf{k}_{I(B_N^\dagger)}[-j-1] \right) \end{aligned} \tag{4.18}$$

of constructible sheaves. Here $\mathbb{B}^{\mathbf{bb}}{}_j^+(M)$, $\mathbb{B}^{\mathbf{bb}}{}_j^+(N)$ are the subsets of those bars that are of type $\mathbf{bb}^+$ in the respective decompositions of M and N.

By unicity of the decomposition in Theorem 3.1.2, we obtain degreewise bijections between the set of associated graded barcodes $\{I(B_M),\ B_M \in \mathbb{B}_j(M)\}$ and $\{I(B_N),\ B_N \in \mathbb{B}_j(N)\}$ and therefore bijections $\sigma_j : \mathbb{B}_j(M) \cong \mathbb{B}_j(N)$ with the property that for any $B_M \in \mathbb{B}_j(M)$, $\sigma_j(B_M)$ is a block of the same type as B_M and which is equal to B_M except maybe on the boundary.

Lemma 4.4.5
Let $\mathbb{B}, \mathbb{B}'$ be sets of MV blocks of types **db**, **vb**, **db** and $\mathbf{bb}^-$. If there is a bijection $\sigma : \mathbb{B} \to \mathbb{B}'$ such that for any $B \in \mathbb{B}$, $\sigma(B)$ is equal to B except maybe on the boundary, then

$$d_I^{\mathrm{MV}}\left(\bigoplus_{B\in\mathbb{B}} S_j^B, \bigoplus_{B'\in\mathbb{B}'} S_j^{B'}\right) = 0.$$

It is enough to check that, if B and B' are two blocks of the same type which differs only on their boundary, then B and B' are ε-interleaved for any $\varepsilon > 0$. This property follows from Lemma 2.1.32 and an immediate application of the definition of the blocks of each type. Then the direct sum of those interleavings relating each $\mathbb{B}$ to $\sigma(\mathbb{B})$ gives a ε-interleaving in between $\bigoplus_{B\in\mathbb{B}} S_{j_B}^B$ and $\bigoplus_{B'\in\mathbb{B}'} S_{j_{B'}}^{B'}$ for every $\varepsilon > 0$; the lemma follows. The claimed property 3 follows from the lemma since we have proved just above that we can find such a permutation relating $\mathbb{B}_j(M), \mathbb{B}_j(N)$ for each degree j.

□

Let $u : X \to \mathbb{R}$ be a continuous map. Then we have the derived functors of the direct image: $\mathrm{R}^i u_* \mathbf{k}_X \in \mathrm{Mod}(\mathbf{k}_\mathbb{R})$ which are the cohomology groups of the derived functor $\mathrm{R}u_*\mathbf{k}_X \in \mathrm{D}^+(\mathbf{k}_\mathbb{R})$. Note that this is just a special case of derived direct image, defined for any continuous map $\phi : X \to Y$, which is a functor $\mathrm{R}\phi_* : \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y)$. In particular, the

$$\textit{assignment } u \mapsto \bigoplus_{i\in\mathbb{N}} \mathrm{R}^i u_*(\mathbf{k}_X)[-i] \textit{ defines a functor}$$

$$\mathrm{R}(-)\mathbf{k}_{(-)} : \mathrm{Top}_{|\mathbb{R}} \to \mathrm{D}(\mathbf{k})^{\mathrm{op}}. \tag{4.19}$$

A morphism $\phi : (X, f) \to (Y, g)$ is mapped by this functor to the linear map $\bigoplus \mathrm{R}^i\phi_* : \mathrm{R}^i g_*\mathbf{k}_Y \to \mathrm{R}^i f_*\mathbf{k}_X$ and the fact that this defines a functor is an immediate consequence of the composition formula $\mathrm{R}(\psi \circ \kappa)_* \cong \mathrm{R}\psi_* \circ \mathrm{R}\kappa_*$ (corollary A.2.21).

Proposition 4.4.6

Assume X is locally contractible and paracompact. Then there is an natural isomorphism

$$\overline{\mathcal{L}_*(u)}^{\mathrm{MV}} \cong \bigoplus_{i\in\mathbb{N}} \mathrm{R}^i u_* \mathbf{k}_X[-i].$$

Proof

By example 4.2.14 and definition 4.4.2, we have

$$\overline{\mathcal{L}_*(u)}^{\mathrm{MV}} \cong \bigoplus_{i\in\mathbb{N}} Q(\overline{\mathrm{H}_i^{\mathrm{sing}}(u^{-1}(-))})[-i].$$

Now, from definition 4.3.3, we have that $\overline{\mathrm{H}_i^{\mathrm{sing}}(u^{-1}(-))}$ is the sheafification of the presheaf

$$\mathfrak{Op}(\mathbb{R}) \ni U \mapsto \varprojlim_{]-x,y[\subset U} \mathrm{Hom}_{\mathbf{k}}\left(\mathrm{H}_i^{\mathrm{sing}}(u^{-1}(]-x,y[)),\mathbf{k}\right) \cong \mathrm{H}^i_{\mathrm{sing}}(u^{-1}(U))$$

since $\mathbf{k}$ is a field (and therefore the cohomology of the dual of a chain complex is the dual of the homology) and every open in $\mathbb{R}$ is a disjoint union of intervals. It is well-known that for $u : X \to Y$ and any sheaf F, $\mathrm{R}^i u_*(F)$ is the sheaf associated to the presheaf $\mathfrak{Op}(Y) \ni U \mapsto \mathrm{H}^i(u^{-1}(U), F)$ (see [Ive86, Proposition 5.11] for instance). Furthermore, when X is locally contractible and paracompact, one has an isomorphism of presheaves (theorem A.2.23) :

$$V \mapsto \mathrm{H}^i(V;\mathbf{k}_X) \cong \mathrm{H}^i(V;\mathbf{k}_V) \cong \mathrm{H}^i_{\mathrm{sing}}(V),$$

where the first isomorphism is for the sheaf cohomology with value in a constant sheaf and its restriction $\mathbf{k}_{X|_V} \cong \mathbf{k}_V$ to an open subset, and the last isomorphism is the usual identification of sheaf cohomology with value on a constant sheaf with singular cohomology for locally contractible spaces.

□

4.4.2 The functor Ψ

We now turn to the construction of a section of the sheafification of MV-systems. We have the following intrinsic definition.

Definition 4.4.7

Given $F \in \mathrm{D}(\mathbf{k}_{\mathbb{R}})$ and $i \in \mathbb{Z}$, we define $\Psi(F)_i$ to be the object of $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ given, for $(x,y) \in \Delta^+$ by :

$$\Psi(F)_i(x,y) = \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}\left(\mathrm{R}^i\Gamma\left(]-x,y[;F\right),\mathbf{k}\right). \tag{4.20}$$

It follows from lemma 4.3.1 that $\Psi(F)_i$ is a persistence module. For constructible sheaves, one can give a simpler formula using their decomposition as direct sums of constant sheaves.

Lemma 4.4.8
Let F be in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then one has an isomorphism of persistence modules over Δ^+, given by the pointwise formula :

$$\Psi(F)_i(x,y) \cong \bigoplus_{k+l=i} \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}\left(\mathrm{R}^k\Gamma\left(]-x,y[,\mathrm{H}^l(F)\right),\mathbf{k}\right), \tag{4.21}$$

for any $i \in \mathbb{Z}$, and $(x,y) \in \Delta^+$.

Note that since F is assumed to be constructible, *there are only finitely many* pairs (k,l) such that the right-hand-side vector space is non zero.

Proof
The reader who knows the spectral sequences associated to hypercohomology can immediately deduce the result of the lemma by noticing that the assumption on F implies its degeneracy at the E_2-page which is exactly the right hand side of (4.21).

Alternatively, let F be any complex of sheaves on a space X. Then according to definition 4.4.7 $\Psi(F)_i$ is the persistent object given by the composite functor

$$\Delta^+ \xrightarrow{\iota} \mathfrak{Op}(\mathbb{R}) \xrightarrow{\mathrm{R}^*\Gamma(-,F)} \mathrm{D}(\mathrm{Mod}(\mathbf{k}))^{\mathrm{op}} \xrightarrow{\mathrm{H}^i(-)} \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbf{k}}(-,\mathbf{k})} \mathrm{Mod}(\mathbf{k}). \tag{4.22}$$

Now we asume $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. By Theorem 3.1.4, we have an isomorphism of complexes of sheaves $F \cong \bigoplus_{j\in\mathbb{Z}} \mathrm{H}^j(F)[-j]$. Therefore we can replace F by its homology in (4.22). Then, we can take I to be the direct sum of injective resolutions of each $\mathrm{H}^j(F)[-j]$. The lemma follows thanks to the fact that only finitely many k and l in (4.21) gives non-zero terms as noted above and therefore the functors in (4.22) commutes with the (finite) direct sum.

$\square$

Remark 4.4.9
The functor Ψ is not faithful. Indeed for any $a<b<c$, one has a non-split exact sequence of sheaves

$$0 \to \mathbf{k}_{[a,b[} \to \mathbf{k}_{[a,c[} \to \mathbf{k}_{[b,c[} \to 0$$

which gives a non zero homomorphism $\mathbf{k}_{[b,c[} \to \mathbf{k}_{[a,b[}[1]$ in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. However there are no non-zero Mayer-Vietoris systems homomorphism in between $\Psi(\mathbf{k}_{[b,c[})$ and $\Psi(\mathbf{k}_{[a,b[}[1]) = \Psi(\mathbf{k}_{[a,b[})[1]$ as follows from Proposition 4.4.14 (there are no non-zero homomorphims in between MV systems associated to horizontal blocks in *different* degrees).

Note also that the isomorphism of lemma 4.4.8 is *not* natural in F for similar reasons. For instance, the right hand side of (4.21) maps the non zero morphism $\mathbf{k}_{[b,c[} \to \mathbf{k}_{]a,b[}[1]$ (induced by the short exact sequence $0 \to \mathbf{k}_{]a,b[} \to \mathbf{k}_{]a,c[} \to \mathbf{k}_{[b,c[} \to 0$) to 0 but Ψ does not.

Remark 4.4.10

The functor Ψ is thus essentially defined as the dual of the derived section of F and not just as the dual of the homology sheaf of F which could have been a more naive approach. The main reason is that the latter will not carry a Mayer-Vietoris structure; in other words, it will forget too much of the structure of the constructible sheaf. However, the derived construction carries such a structure in a natural way as we will now see.

Proposition 4.4.11

The family $(\Psi(F)_i)_{i\in\mathbb{Z}}$ carries a natural structure of a Mayer-Vietoris system. In addition, if $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$, then it is strongly pointwise finite dimensional (Definition 4.4.1).

Proof

We have already seen that $\Psi(F)_i$ is a persistence module over Δ^+ as an immediate consequence of lemma 4.3.1. For $s = (s_1, s_2) \in \mathbb{R}^2_{>0}$ and $i \in \mathbb{Z}$, we have to build the connection morphism δ^2_i. Let $I \in C^b(\mathbf{k}_{\mathbb{R}})$ an injective resolution of F. Consider $(x, y) \in \Delta^+$, then we have the Mayer-Vietoris sequence associated to the cover $]-x-s_1, y[\cup]-x, y+s_2[$ of $]-x-s_1, y+s_2[$ which is the short exact sequence of complexes of sheaves

$$0 \longrightarrow \Gamma(]-x-s_1, y+s_2[, I) \longrightarrow \Gamma(]-x-s_1, y[, I) \oplus \Gamma(]-x, y+s_2[, I) \longrightarrow$$
$$\Gamma(]-x, y[, I) \longrightarrow 0.$$

For short, let us write $\mathrm{H}^i(U, I)$ for the i-th cohomology groups $\mathrm{R}^i\Gamma(U, I)$. Passing to cohomology, we thus obtain a long exact sequence

$$\begin{aligned} \ldots \longrightarrow \mathrm{H}^i(]-x-s_1, y+s_2[, I) \longrightarrow \mathrm{H}^i(]-x-s_1, y[, I) \oplus \mathrm{H}^i(]-x, y+s_2[, I) \\ \longrightarrow \mathrm{H}^i(]-x, y[, I) \xrightarrow{\delta} \mathrm{H}^{i+1}(]-x-s_1, y+s_2[, I) \longrightarrow \ldots . \end{aligned} \tag{4.23}$$

Since by definition of sheaf cohomology, one has, $\mathrm{H}^i(]{-x}, y[, I) \cong \mathrm{R}^i\Gamma\left((]x, y[, F\right)$, the linear dual of the maps δ given by the exact sequence (4.23) yields linear maps $\delta_i^s : \Psi(F)_i(x, y) \to \Psi(F)_i[s](x, y)$ for all $(x, y) \in \Delta^+$. The exactness of (4.23) and Lemma 4.3.1 also implies that the collection $(\Psi(F)_i, \delta_i^s)_{i,s}$ is a Mayer-Vietoris system over $\mathbb{R}$.

When F is constructible, it satisfies for all $(x, y) \in \Delta^+$ that the $\mathbf{k}$-vector spaces $\mathrm{R}^i\Gamma(] - x, y[; F)$ are finite dimensional. Therefore $\Psi(F)$ is pointwise finite dimensional. Now the proof that $\Psi(F)$ is strongly finite dimensional is an argument similar to the proof of property 4 in Proposition 4.3.4. Alternatively, one can simply use the structure theorem 3.1.4 and proposition 4.4.14 below to conlude directly since strongly pointwise finite dimensional modules are stable under locally finite direct sums.

Proposition 4.4.12

The rule $F \mapsto \Psi(F) := (\Psi(F)_i, \delta_i^s)_{i,s}$ defines functors $\Psi : \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \to \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$, $\Psi : \mathrm{D}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \to \mathrm{MV}(\mathbb{R})$ fitting in a commutative diagram:

$$\begin{tikzcd} \mathrm{D}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \arrow[r, "\Psi"] & \mathrm{MV}(\mathbb{R}) \\ \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})^{\mathrm{op}} \arrow[r, "\Psi"] \arrow[u, hook] & \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \arrow[u, hook] \end{tikzcd}$$

Furthermore, these functors are additive and commutes with shifts associated to the canonical triangulated structure of the derived category.

Proof

Since the definition of Ψ_i is functorial and the connecting morphism in Mayer-Vietoris long exact sequences is also functorial, we obtain that Ψ is indeed a functor. Proposition 4.4.11 gives the fact that Ψ sends the subcategory of constructible sheaves to the one of strongly pointwise finite dimensional systems. The last assertion follows from the fact that hyperchohomology commutes with direct sums and shifts.

□

Example 4.4.13

Let $F = \bigoplus_I \mathbf{k}_I[n_I]$ be constructible (derived) sheaf over $\mathbb{R}$. Then by Proposition 4.4.4 and Lemma 4.4.8 we obtain, for any $(x, y) \in \Delta^+$, the simple formula

$$\Psi(F)_i(x, y) \cong \bigoplus_I \bigoplus_k \mathrm{R}^k\Gamma\left(]{-}, x, y[, \mathbf{k}_I\right)[n_i + k]$$

for $\Psi(F)$ (one can also note that the only values of k for which we have a non zero term are 0 and 1 from Proposition 4.3.6).

Recall definition 4.2.3 of the canonical MV-systems S_i^B associated to blocks as well as Lemma 4.3.9. The following is the analogue of proposition 4.3.6, that is, it describes the action of Ψ on the building blocks of a constructible sheaf. Together with example 4.4.13, it allows to compute the value of Ψ explicitly.

Proposition 4.4.14
Let $I = \langle a, b\rangle$ be an interval in $\mathbb{R}$.

1. If I is open, then $\Psi(\mathbf{k}_{]a,b[}[-i]) \cong S_i^{B_d^{[a,b]}}$.
2. If $I =]a, b]$, then $\Psi(\mathbf{k}_{]a,b]}[-i]) \cong S_i^{B_v^{[a,b[}}$.
3. If $I = [a, b[$, then $\Psi(\mathbf{k}_{[a,b[}[-i]) \cong S_i^{B_h^{]a,b]}}$.
4. If I is compact, then $\Psi(\mathbf{k}_{[a,b]}[-i]) \cong S_i^{B_b^{]a,b[}}$.

Here all the isomorphisms are isomorphisms of Mayer-Vietoris systems and the blocks are given by lemma 4.3.9.

Remark 4.4.15
Note that when applying Ψ on an interval, the closed boundary becomes an open boundary lines in the associated block of the image and the open ones become closed.

As will be made clear by the proof, the claim 1 relies heavily on the fact that we have taken a derived functor approach for the definiton of Ψ.

Proof
Let us first prove the open interval case. In view of the proof of proposition 4.4.11, using compatibility with shifts and direct sums, we only need to compute the cohomology groups of $\mathrm{R}^k\Gamma\left(]-x, y[; \mathbf{k}_I\right)$ which by definition (see [KS90]) is isomorphic to $\mathrm{Ext}^k_{\mathbf{k}_{\mathbb{R}}}\left(\mathbf{k}_{]-x,y[}, \mathbf{k}_I\right)$. By Propositions 3.2.9 and 3.2.11, we have that it is always 0 for $k > 1$. Furthermore, the only case for which it is non-zero for $k = 1$ is when I is an open whose closure is included in $]-x, y[$. In that latter case (which means, if $I =]a, b[$, that $[a, b] \subset]-x, y[$ i.e. $x > -a$ and $y > b$) we then have $\mathrm{Ext}^1_{\mathbf{k}_{\mathbb{R}}}\left(\mathbf{k}_{]-x,y[}, \mathbf{k}_I\right) \cong \mathbf{k}$. Therefore, by functoriality of the $\mathrm{Ext}^1_{\mathbf{k}_{\mathbb{R}}}(-, \mathbf{k}_I)$ functor in its left variable, it follows that the persistence module associated to $\mathrm{Ext}^1_{\mathbf{k}_{\mathbb{R}}}(-, \mathbf{k}_I)$ in $\Psi(\mathbf{k}_I)$ is either 0 if I is not open or, if I is open, is precisely the block module in degree 1 which is supported on the type $\mathbf{bb}^+$ block whose infimum is $(-a, b)$ and contains none of its boundary lines. Here, by block module we refer to Definition 2.1.26. Therefore, by definition of duality 2.1.29, for an open $I =]a, b[$, the contribution of $\mathrm{Ext}^1_{\mathbf{k}_{\mathbb{R}}}(-, \mathbf{k}_I)$ in $\Psi(\mathbf{k}_I)$ is precisely $\mathbf{k}^{(B_d^{[a,b]})^\dagger}[-1]$ in degree 1 supported on the type $\mathbf{bb}^+$ block dual to the deathblock $B_d^{[a,b]}$.

It remains to compute the image of the $\mathrm{Ext}^0_{\mathbf{k}_\mathbb{R}}\left(\mathbf{k}_{]-x,y[},\mathbf{k}_I\right)$. By Propositions 3.2.1 and 3.2.11, we find that if I is open,

$$\mathrm{Ext}^0_{\mathbf{k}_\mathbb{R}}\left(\mathbf{k}_{]-x,y[},\mathbf{k}_I\right) \cong \begin{cases} \mathbf{k} & \text{if }]-x,y[\subset I \\ 0 & \text{else.} \end{cases}$$

For $I=]a,b[$, the condition $]-x,y[\subset]a,b[$ can be rewritten as $x \leq -a$ and $y \leq b$. Using functoriality of Ext again, we thus find that, when I is open, the persistence module associated to $\mathbf{k}_I$ is the block module $\mathbf{k}^{B_d^{[a,b]}}$ concentrated in degree 0 and supported on the type **db** block B_d^I. Combining the degree 0 and 1 part, the functoriality of the the Mayer-Vietoris long exact sequence (4.23) then shows that $\Psi(\mathbf{k}_I)$ is precisely the MV-block module $S_0^{B_d^{[a,b]}}$ as in Definition 2.1.26.

Now for the three other types of intervals, the computation is easier since we only have to consider $\mathrm{Ext}^0_{\mathbf{k}_\mathbb{R}}\left(\mathbf{k}_{]-x,y[},\mathbf{k}_I\right)$ in the computation of $\Psi(\mathbf{k}_I)$ (all other degrees are 0 by the Ext computations of chapter 3). Arguing as for the open interval case, using Propositions 3.2.1 and 3.2.11, we obtain that the persistence modules $\mathrm{Ext}^0_{\mathbf{k}_\mathbb{R}}(\mathbf{k}_-,\mathbf{k}_I)$ are respectively the block modules $S_0^{B_v^{[a,b[}}$, $S_0^{B_\mathrm{H}^{]a,b]}}$ and $S_0^{B_b^{]a,b[}}$ when I is of the type $]a,b]$, $[a,b[$ or $[a,b]$.

□

Proposition 4.4.16

Let $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. For any $\varepsilon \geq 0$, there is an isomorphism of MV-systems $\Psi(F \star K_\varepsilon) \cong \Psi(F)[\vec{\varepsilon}]$.

Proof

Using theorem 3.1.2, we have that $F \cong \bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}[-n_\alpha]$. By compatibility of convolution with direct sums and shifts, it is thus enough to prove the result for $\mathbf{k}_I$ for an interval I.

Let us start with the case where $I=[a,b]$ is compact. Then by Proposition 3.2.5, we obtain

$$\Psi(\mathbf{k}_{[a,b]}[-i]\star K_\varepsilon) \cong \Psi(\mathbf{k}_{[a-\varepsilon,b+\varepsilon]}[-i]) \cong S_i^{B_b^{\langle a-\varepsilon,b+\varepsilon\rangle}} \tag{4.24}$$

where the last isomorphism is given by Proposition 4.4.14. Note that, by definition, the block B_b is of type $\mathbf{bb}^-$ (see Lemma 4.3.9 and definition 4.2.3). Therefore as a persistence module over Δ^+, we have $S_i^{B_b^{\langle a-\varepsilon,b+\varepsilon\rangle}} \cong \mathbf{k}^{B_b^{\langle a-\varepsilon,b+\varepsilon\rangle}}[-i]$. By lemma 2.1.32 and remark 4.3.8 we find that

$$\mathbf{k}^{B_b^{\langle a-\varepsilon,b+\varepsilon\rangle}} \cong \mathbf{k}^{B_b^{\langle a,b\rangle}}[\vec{\varepsilon}].$$

Combining the last two isomorphisms with (4.24), we find that

$$\Psi(\mathbf{k}_{[a,b]}[-i]\star K_\varepsilon) \;\cong\; \left(\mathbf{k}^{B_b^{\langle a,b\rangle}}[-i]\right)[\vec{\varepsilon}] \;\cong\; \Psi\big(\mathbf{k}_{[a,b]}[-i]\big)[\vec{\varepsilon}] \tag{4.25}$$

using again Proposition 4.4.14 for the last isomorphism. Similarly, in the case where I is half-open, we obtain, for any $\varepsilon \geq 0$ and $i \in \mathbb{Z}$,

$$\Psi(\mathbf{k}_{[a,b[}[-i]\star K_\varepsilon) \;\cong\; \left(\mathbf{k}^{B_h^{\langle a,b\rangle}}[-i]\right)[\vec{\varepsilon}] \;\cong\; \Psi\big(\mathbf{k}_{[a,b[}[-i]\big)[\vec{\varepsilon}]. \tag{4.26}$$

$$\Psi(\mathbf{k}_{]a,b]}[-i]\star K_\varepsilon) \;\cong\; \left(\mathbf{k}^{B_v^{\langle a,b\rangle}}[-i]\right)[\vec{\varepsilon}] \;\cong\; \Psi\big(\mathbf{k}_{]a,b]}[-i]\big)[\vec{\varepsilon}]. \tag{4.27}$$

It remains to cover the case of an open interval $]a,b[$. Again by Proposition 3.2.5, we have

$$\Psi(\mathbf{k}_{]a,b[}[-i]\star K_\varepsilon) \;\cong\; \begin{cases} \Psi(\mathbf{k}_{]a+\varepsilon,b-\varepsilon[}[-i]) & \text{if } \varepsilon < \frac{b-a}{2} \\ \Psi(\mathbf{k}_{[b-\varepsilon,a+\varepsilon]}[-i]) & \text{if } \varepsilon \leqslant \frac{b-a}{2} \end{cases} \tag{4.28}$$

$$\cong \begin{cases} S_i^{B_d^{\langle a+\varepsilon,b-\varepsilon\rangle}} & \text{if } \varepsilon < \frac{b-a}{2} \\ S_i^{B_b^{\langle b-\varepsilon,a+\varepsilon\rangle}} & \text{if } \varepsilon \leqslant \frac{b-a}{2} \end{cases} \tag{4.29}$$

where the last isomorphism is given by proposition 4.4.14. Note that by definition the block B_d is of type $\mathbf{db}^+$ (see Lemma 4.3.9 and definition 4.2.3). Therefore as a persistence module over Δ_+, we have, for $\varepsilon < \dfrac{b-a}{2}$, that

$$S_i^{B_d^{\langle a+\varepsilon,b-\varepsilon\rangle}} \;\cong\; \mathbf{k}^{B_d^{\langle a-\varepsilon,b+\varepsilon\rangle}}[-i]\oplus \mathbf{k}^{(B_d^{\langle a-\varepsilon,b+\varepsilon\rangle})^\dagger}[-i-1]$$

where the dual block $(B_d^{\langle a-\varepsilon,b+\varepsilon\rangle})^\dagger$ is of type $\mathbf{bb}^+$. By Lemma 2.1.32 and Remark 4.3.8 we find that

$$\mathbf{k}^{B_d^{\langle a-\varepsilon,b+\varepsilon\rangle}}[-i]\oplus\mathbf{k}^{(B_d^{\langle a-\varepsilon,b+\varepsilon\rangle})^\dagger}[-i-1] \;\cong\; (\mathbf{k}^{B_d^{\langle a,b\rangle}}[-i])[\vec{\varepsilon}]\oplus(\mathbf{k}^{(B_d^{\langle a,b\rangle})^\dagger}[-i-1])[\vec{\varepsilon}].$$

Combining these last two isomorphisms with (4.28), we find that, for $\varepsilon < \dfrac{b-a}{2}$,

$$\Psi(\mathbf{k}_{]a,b[}[-i]\star K_\varepsilon) \;\cong\; S_i^{B_d^{\langle a,b\rangle}}[\vec{\varepsilon}] \;\cong\; \Psi(\mathbf{k}_{]a,b[}[-i])[\vec{\varepsilon}] \tag{4.30}$$

as claimed.

It remains to consider the case $\varepsilon \geq \dfrac{b-a}{2}$. We have still $\Psi(\mathbf{k}_{]a,b[}[-i])[\vec{\varepsilon}] \cong S_i^{B_d^{\langle a,b\rangle}}[\vec{\varepsilon}]$. As a graded persistence module over Δ_+, by Lemma 2.1.32, we have that

$$S_i^{B_d^{\langle a,b\rangle}}[\vec{\varepsilon}] \;\cong\; \mathbf{k}^{\left(B_d^{\langle a,b\rangle}-\vec{\varepsilon}\right)}[-i]\oplus\mathbf{k}^{\left((B_d^{\langle a,b\rangle})^\dagger-\vec{\varepsilon}\right)}[-i-1].$$

But since $\varepsilon \geq \dfrac{b-a}{2}$, we have that the death block $\left(B_d^{\langle a,b\rangle} - \vec{\varepsilon}\right)$ is concentrated below the anti-diagonal Δ, that is in $\mathbb{R}^2 \setminus \Delta^+_{>0}$ and therefore $\mathbf{k}^{\left(B_d^{\langle a,b\rangle} - \vec{\varepsilon}\right)} \cong 0$. Similarly, the birth block module $\left((B_d^{\langle a,b\rangle})^\dagger - \vec{\varepsilon}\right)$ is of type $\mathbf{bb}^-$ precisely for $\varepsilon \geq \dfrac{b-a}{2}$. The infimum of the points included in this birth block has coordinates $(\varepsilon - b, a + \varepsilon)$. Therefore,

$$\mathbf{k}^{\left((B_d^{\langle a,b\rangle})^\dagger - \vec{\varepsilon}\right)} \cong \mathbf{k}^{B_b^{\langle b-\varepsilon, a+\varepsilon\rangle}}.$$

Taking the (shifted) direct sum of this last two isomorphisms thus obtain, that, for $\varepsilon \geq \dfrac{b-a}{2}$, we have

$$S_i^{B_d^{\langle a,b\rangle}}[\vec{\varepsilon}] \cong 0 \oplus S_i^{B_b^{\langle b-\varepsilon, a+\varepsilon\rangle}}$$

and therefore the claim follows from the last case of (4.28).

$\square$

Remark 4.4.17
Note that we will generalize this result in section 5.5, without needing a decomposition theorem.

4.4.3 The isometry theorem between the interleaving distance on Δ^+ and the graded bottleneck distance for sheaves

In this section, we will state and prove our main isometry theorem. Before that, we derive a few corollaries of the results we have obtained in sections 4.4.2 and 4.4.1.

Corollary 4.4.18
Consider (the restrictions) $\Psi : \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R}) \to \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ and $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$. For any $F \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_\mathbb{R})$, one has an isomorphism

$$\overline{(\cdot)}^{\mathrm{MV}} \circ \Psi(F) \cong F.$$

In other words, Ψ is a natural "object-wise"section of the functor $\overline{(-)}^{\mathrm{MV}}$ on strongly pointwise finite dimensional modules.

Proof

Since both functors $\overline{(\cdot)}^{\mathrm{MV}}$ and Ψ commutes with shifts and direct sums (propositions 4.4.4 and 4.4.12), in view of the structure theorem 3.1.2, it is enough to construct the isomorphism for sheaves of the form $\mathbf{k}_I$. Now Proposition 4.4.4.2 (for $\varepsilon = 0$) and Proposition 4.4.14 we have

$$\overline{(\Psi(\mathbf{k}_I))}^{\mathrm{MV}} \ \cong \ \mathbf{k}_I \tag{4.31}$$

which is precisely giving such a claimed isomorphism for an interval.

□

Let $\mathcal{H}^*(-) : \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}}) \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ be the endofunctor given by the cohomology sheaf, that is, for any complex of sheaves F, by

$$\mathcal{H}^*(F) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(F)[-i].$$

Note that although $\mathcal{H}^*$ is essentially surjective, it is by no means an auto-equivalence. We claim that there is a natural isomorphism of functors $\overline{(\cdot)}^{\mathrm{MV}} \circ \Psi \ \simeq \ \mathcal{H}^*(-)$. Since the construction of the natural isomorphism is rather technical, and we do not need this result for the following, we omit the proof.

Corollary 4.4.19

The functor $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ is essentially surjective.

Proof

For any constructible sheaf F, $\Psi(F)$ is a strongly pointwise finite dimensional Mayer-Vietoris system and Corollary 4.4.18 gives a natural isomorphism $F \cong \overline{(\Psi(F))}^{\mathrm{MV}}$. Therefore, F is in the essential image of $\overline{(-)}^{\mathrm{MV}}$.

□

Corollary 4.4.20

Let $M \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ be a strongly point-wise finite dimensional Mayer-Vietoris system. Then

$$d_I^{\mathrm{MV}}(M, \Psi(\overline{M}^{\mathrm{MV}})) = 0.$$

In other words, though $\Psi \circ \overline{(-)}^{\mathrm{MV}}$ is not an equivalence, it maps an object to an object which is at distance 0 fom itself.

Proof

By statement 3 of Proposition 4.4.4, it is sufficient to prove that $\overline{M}^{\mathrm{MV}}$ and $\overline{\Psi(\overline{M}^{\mathrm{MV}})}^{\mathrm{MV}}$ are isomorphic in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. But Corollary 4.4.18 implies $\overline{\Psi(\overline{M}^{\mathrm{MV}})}^{\mathrm{MV}} \cong \overline{M}^{\mathrm{MV}}$ and the result follows.

□

We can now state our isometry theorem

Theorem 4.4.21 (Theorem 4.21 - [BGO19])

The functors $\overline{(-)}^{\mathrm{MV}}$ and Ψ are isometries between the interleaving distance and the convolution and bottleneck distances for sheaves. That is, for all $M, N \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$, one has equalities

$$d_I(M,N) = d_C(\overline{M}^{\mathrm{MV}}, \overline{N}^{\mathrm{MV}}) = d_B(\mathbb{B}(\overline{M}^{\mathrm{MV}}), \mathbb{B}(\overline{N}^{\mathrm{MV}})).$$

And for all constructible sheaves $F, G \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$, one has

$$d_B(\mathbb{B}(F), \mathbb{B}(G)) = d_C(F,G) = d_I(\Psi(F), \Psi(G)).$$

Proof

Theorem 3.4.18 implies already the equality between bottleneck and convolution distances.

By Proposition 4.4.4, the Mayer-Vietoris sheafification functor $\overline{(-)}^{\mathrm{MV}} : \mathrm{MV}(\mathbb{R})^{\mathbf{sf}} \to \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ maps the shift functor $[\vec{\varepsilon}]$ onto the convolution functor $(-) \star K_\varepsilon$ and therefore if $M, N \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ are ε-interleaved, then $\overline{M}^{\mathrm{MV}} \sim_\varepsilon \overline{N}^{\mathrm{MV}}$ in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Thus, for all $M, N \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$, one has

$$d_C(\overline{M}^{\mathrm{MV}}, \overline{N}^{\mathrm{MV}}) \leqslant d_I(M,N). \tag{4.32}$$

Similarly, Proposition 4.4.16 implies that Ψ sends the convolution $(-) \star K_\varepsilon$ functor to the shift functor and thus, we also have that, for all $F, G \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$, one has

$$d_I(\Psi(F), \Psi(G)) \leqslant d_C(F,G). \tag{4.33}$$

From (4.32) and (4.33) we get, for all $M, N \in \mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$, that

$$d_I(\Psi(\overline{M}^{\mathrm{MV}}), \Psi(\overline{N}^{\mathrm{MV}})) \leqslant d_C(\overline{M}^{\mathrm{MV}}, \overline{N}^{\mathrm{MV}}) \leqslant d_I(M,N). \tag{4.34}$$

The triangular inequality and Corollary 4.4.20 implies

$$\begin{aligned} d_I(M,N) \leqslant d_I(\Psi(M, \Psi(\overline{M}^{\mathrm{MV}})) + d_I(\Psi(\overline{M}^{\mathrm{MV}}), \Psi(\overline{N}^{\mathrm{MV}})) + d_I(\Psi(\overline{N}^{\mathrm{MV}}), N) \\ = d_I(\Psi(\overline{M}^{\mathrm{MV}}), \Psi(\overline{N}^{\mathrm{MV}})) \end{aligned} \tag{4.35}$$

Combining inequality (4.35) with (4.34), we obtain that all inequalities in (4.34) are equalities, which gives the first claim

$$d_I(M,N) = d_C(\overline{M}^{\mathrm{MV}}, \overline{N}^{\mathrm{MV}}).$$

To prove the remaining one, we use Corollary 4.4.18. This gives us, for any F, $G \in \mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ isomorphisms $F \cong \overline{\Psi(F)}^{\mathrm{MV}}$ and $G \cong \overline{\Psi(G)}^{\mathrm{MV}}$ and therefore we have

$$d_C(F,G) = d_C\big(\overline{\Psi(F)}^{\mathrm{MV}}, \overline{\Psi(G)}^{\mathrm{MV}}\big) = d_I(\Psi(F), \Psi(G)) \tag{4.36}$$

since we just proved that $\overline{(-)}^{\mathrm{MV}}$ is an isometry. The equality (4.36) concludes the proof of the theorem.

□

In particular the theorem allows to compute the bottleneck or convolution distance for sheaves using interleaving distance for persistence modules and vice-versa. Furthermore, we recover as a corollary the following result of [KS18a].

Corollary 4.4.22
If X is a locally contractible compact topological manifold, and $u, v : X \to \mathbb{R}$ are continuous constructible functions, one has:

$$d_C(\mathrm{R}u_*\mathbf{k}_X, \mathrm{R}v_*\mathbf{k}_X) \leq \sup_{x\in X} \|u(x) - v(x)\|$$

Proof
Theorem 4.4.21 and proposition 4.4.6 imply that

$$d_C(\mathrm{R}u_*\mathbf{k}_X, \mathrm{R}v_*\mathbf{k}_X) = d_I(\mathcal{L}_*(u), \mathcal{L}_*(v)) \leqslant \sup_{x\in X} \|u(x) - v(x)\|$$

by Proposition 4.2.17.

□

4.5 Applications

In this section, we provide concrete applications of theorem 4.4.21. We first go back to our example of section 3.5.1. We then prove that the MV-interleaving distance can actually be computed in a graded-fashion. This result allow a much simpler expression of both the MV-interleaving distance and the convolution distance in terms of bottleneck distances matching blocks *degree-wise*.

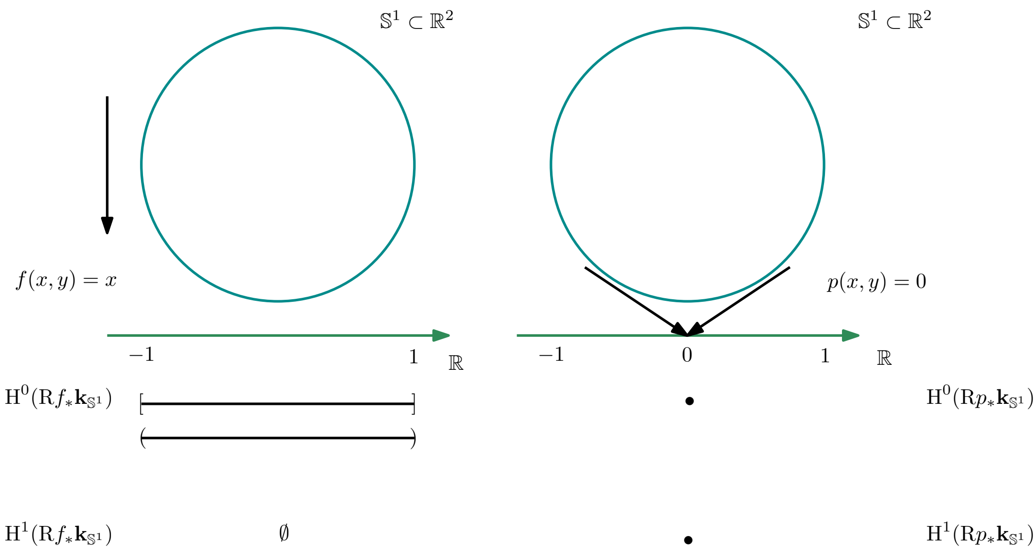


Figure 4.2 – The map f and p together with the intervals on which are supported the degree 0 and 1 part of the associated sheaves, see (4.37) and proposition 4.5.1.

4.5.1 Back to the circle

We illustrate the use of the theorems of this chapter with the example of the projection from the circle that we encountered in section 3.5.1.

Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle embedded in $\mathbb{R}^2$. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be the first coordinate projection and $p : \mathbb{S}^1 \to \mathbb{R}$ be the constant map with value zero.

From example 4.2.14 we obtain two Mayer-Vietoris systems $\mathcal{L}_*(f)$ and $\mathcal{L}_*(p)$, which are given, for any $(x, y) \in \Delta^+$, by $\mathcal{L}_*(f)(x, y) = \mathrm{H}_*(f^{-1}(]-x, y[))$ and $\mathcal{L}_*(p)(x, y) = \mathrm{H}_*(p^{-1}(]-x, y[))$.

Using the same notation as in Lemma 4.3.9 we have.

Proposition 4.5.1

One has

$$\mathcal{L}_*(f) \cong S_0^{B_b^{]-1,1[}} \oplus S_0^{B_d^{[-1,1]}}, \qquad \mathcal{L}_*(p) \cong S_0^{B_b^{]0,0[}} \oplus S_1^{B_b^{]0,0[}}.$$

In particular, $\overline{\mathcal{L}_*(f)}^{\mathrm{MV}} \simeq \mathbf{k}_{(-1,1)} \oplus \mathbf{k}_{[-1,1]}$ and $\overline{\mathcal{L}_*(p)}^{\mathrm{MV}} \simeq \mathbf{k}_{\{0\}} \oplus \mathbf{k}_{\{0\}}[-1]$. Furthermore, $\Psi(\mathbf{k}_{(-1,1)} \oplus \mathbf{k}_{[-1,1]}) \cong \mathcal{L}_*(f)$ and $\Psi(\mathbf{k}_{\{0\}} \oplus \mathbf{k}_{\{0\}}[-1]) \cong \mathcal{L}_*(p)$.

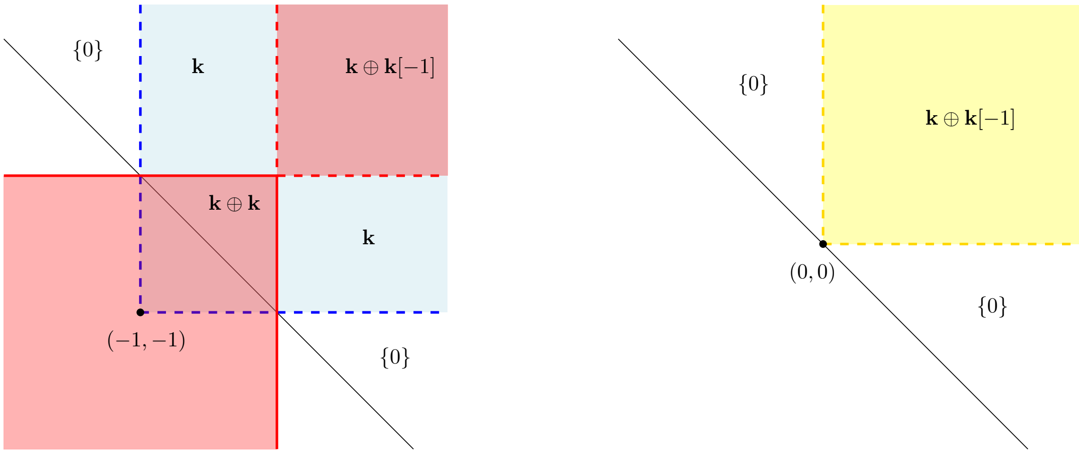


Figure 4.3 – On the left, the value of the MV system $\mathcal{L}_*(f)$ where the blue part is a birth block and the red part are a death block and its dual. On the right, the value of the MV system $\mathcal{L}_*(p)$ where the yellow part refers to the (reunion of) two birthblocks. The dashed lines pictures boundary which are not inside the blocks.

Proof

The preimage of f satisfies

$$f^{-1}(]-x,y[) = \begin{cases} \emptyset & \text{if } -x \geqslant 1 \text{ or } y \leqslant -1, \\ \mathbb{S}^1 & \text{if } -x < -1 \text{ and } y > 1, \\ \text{two intervals} & \text{if } -1 \leqslant -x < y \leqslant 1, \\ \text{one interval} & \text{if } -x < -1 < y \leqslant 1 \text{ or } -1 \leqslant -x < 1 < y. \end{cases}$$

This gives that $\mathcal{L}_*(f)$ has the module decomposition given in figure (4.5.1) which is exactly the decomposition of $\mathcal{L}_*(f)$ into a $\mathbf{bb}^-$ module with infimum $(-1,-1)$ in degree 0 and a module associated to the deathblock with supremum $(1,1)$. The image of Ψ is given by adidtivity and Proposition 4.4.14:

$$\Psi(\mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}) \cong \Psi(\mathbf{k}_{(-1,1)} \oplus \mathbf{k}_{[-1,1]}) \cong \Psi(\mathbf{k}_{(-1,1)}) \oplus \Psi(\mathbf{k}_{[-1,1]}) \cong S_0^{B_b^{]0,0[}} \oplus S_1^{B_b^{]0,0[}} \cong \mathcal{L}_*(f).$$

Applying Corollary 4.4.18 (or using Proposition 4.4.4 directly) yields $\overline{\mathcal{L}_*(f)}^{\mathrm{MV}} = \mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}$. Similarly, we have that

$$p^{-1}(]-x,y[) = \begin{cases} \mathbb{S}^1 & \text{if } -x < 0 < y \\ \emptyset & \text{else.} \end{cases}$$

and thus $\mathcal{L}_*(p) \cong S_0^{B_b^{]0,0[}} \oplus S_1^{B_b^{]0,0[}}$ as can be seen in figure (4.5.1) as well. We apply again Proposition 4.4.14 and Corollary 4.4.18 to conclude.

□

In particular we recover the computation of chapter 3 for the derived images sheaves $\mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1}$ and $\mathrm{R}p_*\mathbf{k}_{\mathbb{S}^1}$:

$$\mathrm{R}f_*\mathbf{k}_{\mathbb{S}^1} \cong \mathbf{k}_{]-1,1[} \oplus \mathbf{k}_{[-1,1]}, \qquad \mathrm{R}p_*\mathbf{k}_{\mathbb{S}^1} \cong \mathbf{k}_{\{0\}} \oplus \mathbf{k}_{\{0\}}[-1]. \tag{4.37}$$

Furthermore, we can find 1-interleaving between $S_0^{B_d^{[-1,1]}}$ and $S_1^{B_b^{]0,0[}}$ as well as 1-interleaving between $S_0^{B_b^{]-1,1[}}$ and $S_0^{B_b^{]0,0[}}$. Therefore, the decomposition of the proposition gives a 1-interleaving for $\mathcal{L}_*(f)$ and $\mathcal{L}_*(p)$.

4.5.2 Degree-wise description of $d_I^{\mathbf{MV}}$

The content of this chapter has been to prove that the category of Mayer-Vietoris systems contain most of the derived behaviour of derived constructible sheaves, although it has a much simpler algebraic structure to work with. This will allow us to give a new equivalent formulation of the MV-interleaving distance.

We start by setting some notations. We denote by d_I the usual interleaving distance on $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$. If M and N in $\mathrm{Pers}(\mathbf{k}^{\Delta^+})$ are ε-interleaved, we shall write $M \sim_\varepsilon N$. If M is pfd and middle-exact, we denote $\mathbb{B}(M)$ its barcode.

Let $\mathrm{MV}(\mathbb{R})^{\mathbf{sf}}$ be the category of strongly pointwise finite dimensional Mayer-Vietoris systems over $\mathbb{R}$. We denote by d_I^{MV} the interleaving distance in $MV(\mathbb{R})$. If S and NT in $MV(\mathbb{R})$ are ε-MV-interleaved, we shall write $S \sim_\varepsilon^{\mathrm{MV}} T$. If S is spfd, we denote $\mathbb{B}^{\mathrm{MV}}(S)$ its barcode. Recall that for MV systems, blocks of type $\mathbf{bb}^+$ are paired with their dual block of type $\mathbf{db}$, and that we only record the birth block of type $\mathbf{bb}^+$ in $\mathbb{B}^{\mathrm{MV}}(S)$.

Let d_B be the bottleneck distance associated to d_I between middle-exact pointwise finite dimensional persistence modules over Δ^+ (see definition 2.1.17). The following result was proved by Bjerkevik, and will be of use to prove our theorem.

Theorem 4.5.2 ([Bje16] - Theorem 3.3)
Let M and N be pointwise finite dimension middle exact persistence modules over Δ^+. If M and N are ε-interleaved, then there exists an ε-matching between $\mathbb{B}(M)$ and $\mathbb{B}(N)$.

In particular:

$$d_I(M,N) = d_B(M,N).$$

We have the following lemma:

Lemma 4.5.3
Let B, B' be blocks of type $\mathbf{bb}^{\pm}$, $\mathbf{hb}$ or $\mathbf{vb}$. (That is, blocks that are used to parametrize the barcodes of MV-systems). Then for all $i \in \mathbb{Z}$ and $\varepsilon \geq 0$,

$$S_i^B \sim_\varepsilon^{\mathrm{MV}} S_i^{B'} \iff \mathbf{k}^B \sim_\varepsilon \mathbf{k}^{B'}.$$

Consequently:

$$d_I^{\mathrm{MV}}(S_i^B, S_i^{B'}) = d_I(\mathbf{k}^B, \mathbf{k}^{B'}).$$

Proof
The only difficult case is for B and B' of type $\mathbf{bb}^{+}$, where there exists a derived"behaviour. We assume without loss of generality that $i = 1$. In this situation, for $j \in \mathbb{Z}$:

$$S_j^B = \begin{cases} \mathbf{k}^B \text{ if } j = 1, \\ \mathbf{k}^{B^\dagger} \text{ if } j = 0, \\ 0 \text{ otherwise,} \end{cases} \quad \text{and} \quad S_j^{B'} = \begin{cases} \mathbf{k}^{B'} \text{ if } j = 1, \\ \mathbf{k}^{B'^\dagger} \text{ if } j = 0, \\ 0 \text{ otherwise.} \end{cases}$$

The key observation to conclude is then the following, for any $\varepsilon \geq 0$:

$$\mathbf{k}^B \sim_\varepsilon \mathbf{k}^{B'} \Longrightarrow \mathbf{k}^{B^\dagger} \sim_\varepsilon \mathbf{k}^{B'^\dagger}.$$

We can now state our degree-wise description of d_I^{MV}.

Theorem 4.5.4
Let $S = (S_i, \delta_i^s)$ and $T = (T_i, \mu_i^s)$ be two strongly pointwise finite dimensional Mayer-Vietoris systems over $\mathbb{R}$. Then, for all $\varepsilon \geq 0$:

$$S \sim_\varepsilon^{\mathrm{MV}} T \iff \forall i \in \mathbb{Z},\ S_i \sim_\varepsilon T_i.$$

Consequently:

$$d_I^{\mathrm{MV}}(S, T) = \max_i d_I(S_i, T_i).$$

Proof
By definition, an ε-MV-interleaving between S and T induces some ε-interleaving between S_i and T_i for all i.

Conversely, let ε such that $S_i \sim_\varepsilon T_i$ for all i. By Bjerkevik's above result (theorem 4.5.2), there exists a ε-matching σ_i between the barcode of persistence modules of S_i and T_i. A matching between block preserves the types of the blocks, except maybe for blocks of type $\mathbf{db}$ that can be matched to 0.

Since MV-barcodes contain only blocks of type **bb**, **hb** or **vb**, the collection (σ_i) induces a degree-wise bijection between the blocks in $\mathbb{B}^{\mathrm{MV}}(S)$ and $\mathbb{B}^{\mathrm{MV}}(T)$. This bijection induces a ε-MV-interleaving according to lemma 4.5.3.

Applying our main isometry theorem 4.4.21, we obtain:

Corollary 4.5.5
Let F and G in $\mathrm{D}^b_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then:

$$d_C(F,G) = \max_i \; d_I(\Psi(F)_i, \Psi(G)_i).$$

Therefore, to compute the convolution distance between F and G, two constructible sheaves on $\mathbb{R}$, it is sufficient to compute the degree-wise usual interleaving distance between the persistence modules $\Psi(F)_i$ and $\Psi(G)_i$, and take the maximum over i. Moreover by Bjerkevik's result (theorem 4.5.2), those distances can be computed as matching problems, hence can be easily implemented.

We now consider the particular case where F and G are the pushforwards of the constant sheaf over a paracompact and locally contractible topological space X, through two continuous functions $u, v : X \to \mathbb{R}$, that is, $F = \mathrm{R}u_*\mathbf{k}_X$ and $G = \mathrm{R}v_*\mathbf{k}_X$. Then by proposition 4.4.6 and theorem 4.4.21 we deduce

$$d_C(\mathrm{R}u_*\mathbf{k}_X, \mathrm{R}v_*\mathbf{k}_X) = d_I^{\mathrm{MV}}(\mathcal{L}_*(u), \mathcal{L}_*(v)) = \max_i \; d_I(\mathcal{L}_i(u), \mathcal{L}_i(v)).$$

Combining all these results, we have proved that the convolution distance between $\mathrm{R}u_*\mathbf{k}_X$ and $\mathrm{R}v_*\mathbf{k}_X$ is the maximum over the degree of homology i of the bottleneck distances between the usual barcodes of the i-th level-sets persistence homology modules of u and v.

4.5.3 Discussion

Our analysis has shown that graded barcodes of sheaves can be computed from the barcodes of level-sets persistence modules. Thanks to our description of the algebraic structure of Mayer-Vietoris systems, we are also able to compare and prove the equality between the convolution distance of two sheaves and the maximum of the interleaving distances between their associated level sets persistence modules. This last result seems important to us in the view of applications. Indeed, it indicates that up to reformulation in the language of Mayer-Vietoris systems, the problem of matching sheaf graded barcodes (which has a fundamentally derived behaviour) is equivalent to a degree-wise matching problem. Moreover, some software solutions already exists in order to compute level-sets persistence barcodes [Mor]. Ultimately, our work proves that depending on the research question, one can enjoy the best properties of both constructions : the computational simplicity of level-sets persistence on the one hand, the theoretical power of the Grothendieck's six operations formalism of sheaf theory on the other hand. We think that the two following research questions could be of interest in the future :

Data analysis : to the best of our knowledge, the collection of level-sets barcode have not been used yet as a descriptor of datasets. Our sheaf interpretation of the collection of level-sets persistence barcodes shows that it carries information that should be valuable for machine learning tasks such as classification.

Symplectic topology : the filtered Floer Hamiltonian homology naturally admits a sublevel sets filtration by the action function, giving it a structure of a one-parameter persistence module [BHS18]. Although it is not obvious how to equip this homology theory with a filtration turning it into a level sets persistence module, this could lead to new interesting invariants of Hamiltonian diffeomorphisms.

CHAPTER 5

Ephemeral persistence modules and distance comparison

Abstract

We provide a definition of ephemeral multi-persistence module and further prove that the quotient of persistence modules by the ephemeral ones is equivalent to the category of γ-sheaves. In the case of one dimensional persistence, our definition agree with the usual one showing that the observable category and the category of γ-sheaves are equivalent. We also also establish isometry theorems between the category of persistence modules and γ-sheaves both endowed with the interleaving distances. Finally, we compare the interleaving and convolution distances.

This chapter presents the results obtained in collaboration with François Petit in [BPar] and is to appear in *Algebra and Geometric Topology*.

5.1 Introduction

One of the initial motivation of persistent homology was to provide a mean to estimate the topology of space from a finite sample of itself. Persistent homology and more generally the concept of persistence have since been developed and have spread among many areas of mathematics.

Though persistence theory is well understood in the one-parameter case (see for instance [Oud15] for an extensive exposition of the theory and its applications), its generalization to the multi-parameter case remains less understood, yet is important for applications [LW15]. The first approach to study the category of multi-parameter persistence modules was with an eye coming from algebraic geometry and representation theory [CZ09]. Roughly speaking, the idea was to consider persistence modules as graded-modules over a polynomial ring. This allowed to link the theory of persistence with more classical areas of mathematics and allowed to show that a complete classification of persistence modules with more than one parameter is impossible. Nevertheless, one thing not to be forgotten is that the category of persistence modules is naturally endowed with the interleaving distance. Having applications in mind, one is more interested in computing the distance between two persistence modules, than to explicate the structural difference between those.

In his thesis, J. Curry [Cur14] had the idea to take a sheaf-theorical point of view on persistence. More recently, M. Kashiwara and P. Schapira in [KS18a, KS18b] introduced derived sheaf-theoretic methods in persistent homology. Persistent homology studies filtration of topological spaces indexed by partially ordered groups. We will study here the case where the filtrations are indexed by the elements of an ordered vector space $\mathbb{V}$, defined by the choice of a closed convex proper cone $\gamma \subset \mathbb{V}$. Hence, the idea underlying both approaches is to endow $\mathbb{V}$ with a topology depending on this cone. Whereas J. Curry's approach relies on Alexandrov's topology, M. Kashiwara and P. Schapira's approach relies on the γ-topology which was introduced by the same authors in [KS90]. The goal of this chapter is to compare these two approaches. A key feature of persistence theory is that the various versions of the space of persistence modules can be endowed with pseudo-distances. We focus our attention on two main types of pseudo-distances: the interleaving distances studied by several authors among which [CdSGO16, dMS18, Les12, Les15] and the convolution distance introduced in [KS18a] and studied in detail in the one-dimensional case in the two previous chapters. Besides comparing the various categories of sheaves used in persistence theory (and especially multi-parameter persistence), we establish isometry theorems between these categories endowed with their respective distances.

To compare Alexandrov sheaves and γ-sheaves, we first study morphism of sites between the Alexandrov and the γ-topology. We make precise the results of [KS18a, Section 1.4] by introducing two morphisms of sites $\alpha\colon \mathbb{V}_\gamma \to \mathbb{V}_\mathfrak{a}$ and $\beta\colon \mathbb{V}_\mathfrak{a} \to \mathbb{V}_\gamma$ where $\mathbb{V}_\mathfrak{a}$ denotes the vector space $\mathbb{V}$ endowed with the Alexandrov topology while $\mathbb{V}_\gamma$ designates $\mathbb{V}$ endowed with the γ-topology. This provides us with three distinct functors $\alpha_*, \beta^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $\beta_* = \alpha^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ where $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ (resp. $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$) is the category of sheaves of $\mathbf{k}$-modules on $\mathbb{V}_\gamma$ (resp. $\mathbb{V}_\mathfrak{a}$). The properties of these functors allows us to define a well-behaved notion of ephemeral modules in arbitrary dimensions (Definition 5.3.4). They correspond to Alexandrov sheaves which vanishes when evaluated on open subset of the γ-topology. In dimension one, our notion of ephemeral module coincides with the one introduced in [CdSGO16] and further studied in [CCBdS16]. Then, we show that the quotient of the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ (which is equivalent to the category of persistence modules over $\mathbb{V}$ equipped with the order associated to the cone γ - see Theorem 5.2.7) by its subcategory of ephemeral modules is equivalent to the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ (Proposition 5.3.6). Specializing again our results to the situation where $\dim \mathbb{V} = 1$, we obtain a canonical equivalence of categories between the observable category $\mathbf{Ob}$ of [CCBdS16] and the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ (Corollary 5.3.9). This provides a natural description of the category of observable persistence modules and highlights the significance of the theory of γ-sheaves for studying persistent homology. We extend all these results to the derived setting.

We establish an isometry theorem between the category of Alexandrov sheaves and γ-sheaves on $\mathbb{V}$ endowed with their respective interleaving distances (Theorem 5.4.21). Note that our approach does not rely on a structure-theorem for persistence modules (as they are not available in arbitrary dimension) but on the properties of the morphisms of sites α and β. We also study the properties of ephemeral modules with respects to the notion of interleaving and show that they correspond to modules which are interleaved with zero in all the directions allowed by the Alexandrov topology. This shows that the notion of ephemeral modules is more delicate in higher dimension than in dimension one. This being essentially due to the fact that in dimension one the boundary of the cone associated with the usual order $\leq$ on $\mathbb{R}$ is of dimension zero.

Finally, we study the relation between the interleaving and the convolution distances on the category of γ-sheaves. The convolution distance depends on the choice of a norm on $\mathbb{V}$. Given an interleaving distance with respects to a vector v in the interior of the cone γ, we introduce a preferred norm (see formula (5.13)) and show that under a mild assumption on the persistence modules considered the convolution distance associated with this norm and interleaving distance associated with v are equal (Corollary 5.5.5).

5.2 Sheaves on γ and Alexandrov topology

5.2.1 γ and Alexandrov topology

γ-topology

Following [KS18a], we briefly review the notion of γ-topology. For more details, we refer the reader to [KS90].

Let $\mathbb{V}$ be a finite dimensional real vector space. We write s for the sum map $s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, $(x, y) \mapsto x + y$ and $a\colon x \mapsto -x$ for the antipodal map. If A is a subset of $\mathbb{V}$, we write A^a for the antipodal of A, that is the subset $\{x \in \mathbb{V} | -x \in A\}$.

A subset C of the vector space $\mathbb{V}$ is a cone if

(i) $0 \in C$,

(ii) $\mathbb{R}_{>0} C \subset C$.

We say that a convex cone C is proper if $C^a \cap C = \{0\}$.

Given a cone $C \subset \mathbb{V}$, we define its polar cone C° as the cone of $\mathbb{V}^*$

$$C^\circ = \{\xi \in \mathbb{V}^* \,|\, \forall v \in C, \langle \xi, v\rangle \geq 0\}.$$

From now on, γ denotes a

$$\textit{closed proper convex cone with non-empty interior.} \tag{5.1}$$

We still write $\mathbb{V}$ for the vector space $\mathbb{V}$ endowed with the Euclidian topology.

We say that a subset A of $\mathbb{V}$ is *γ-invariant* if $A = A + \gamma$. The set of γ-invariant open subset of $\mathbb{V}$ is a topology on $\mathbb{V}$ called the *γ-topology*. We denote by $\mathbb{V}_\gamma$ the vector space $\mathbb{V}$ endowed with the γ-topology. We write $\phi_\gamma\colon \mathbb{V} \to \mathbb{V}_\gamma$ for the continuous map whose underlying function is the identity.

If A is a subset of $\mathbb{V}$, we write $\mathrm{Int}(A)$ for the interior of A in the usual topology of $\mathbb{V}$.

Lemma 5.2.1
Let U be a γ-open. Then $U = \bigcup_{x \in U} x + \mathrm{Int}(\gamma)$

Proof

The inclusion $\bigcup_{x\in U} x + \mathrm{Int}(\gamma) \subset U$ is clear. Let us prove the reverse inclusion. Let $y \in U$. Since $0 \in \gamma = \overline{\mathrm{Int}(\gamma)}$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$, of points of $\mathrm{Int}(\gamma)$, such that $\lim_{n\to\infty} u_n = 0$. Hence, for n sufficiently large $y - u_n$ is in U as it is open and $u_n \in \mathrm{Int}(\gamma)$. As $y = y - u_n + u_n$, this implies that $y \in \bigcup_{x\in U} x + \mathrm{Int}(\gamma)$.

□

γ-sheaves

In this section, following [KS90], we recall the notion of γ-sheaves and results borrowed to [KS18a] and [GS14].

We denote by $\mathbf{k}_{\mathbb{V}_\gamma}$ the constant sheaf on $\mathbb{V}_\gamma$ and write $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ for the abelian category of $\mathbf{k}_{\mathbb{V}_\gamma}$-modules, $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ for its derived and $\mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$ for its bounded derived category.

We now state a result due to M. Kashiwara and P. Schapira that says that the derived category of γ-sheaves is equivalent to a subcategory of the usual derived category of sheaves $\mathrm{D}^b(\mathbf{k}_{\mathbb{V}})$. This subcategory can be characterized by a microsupport condition. Given F in $\mathrm{D}^b(\mathbf{k}_{\mathbb{V}})$, one denotes by $\mathrm{SS}(F)$ its microsupport. We refer the reader to [KS90, Chapter V] for the definition and properties of the microsupport.

Following [KS18a], we set

$$\begin{aligned}\mathrm{D}^b_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}}) &= \{F \in \mathrm{D}^b(\mathbf{k}_{\mathbb{V}}); \mathrm{SS}(F) \subset \gamma^{\circ,a}\}\\ \mathrm{Mod}_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}}) &= \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^b_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}})\end{aligned}$$

Theorem 5.2.2 ([KS18a, Theorem 1.5])

Let γ be a proper closed convex cone in $\mathbb{V}$. The functor $\mathrm{R}\phi_{\gamma_*}\colon \mathrm{D}^b_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of triangulated categories with quasi-inverse ϕ_γ^{-1}.

Corollary 5.2.3

The functor $\phi_{\gamma_*}\colon \mathrm{Mod}_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of categories with quasi-inverse ϕ_γ^{-1}.

Consider the following maps:

$$s : \mathbb{V}\times\mathbb{V} \to \mathbb{V}, \quad s(x,y) = x+y$$

$$q_i : \mathbb{V}\times\mathbb{V} \to \mathbb{V}\ (i=1,2) \quad q_1(x,y) = x,\ q_2(x,y) = y$$

Let F and G in $\mathrm{D}^b(\mathbf{k}_{\mathbb{V}})$, we set

$$F \underset{np}{*} G = \mathrm{R}s_*(q_1^{-1}F \otimes q_2^{-1}G) = \mathrm{R}s_*(F \boxtimes G).$$

The reader shall beware that $\underset{np}{*}$ differs from the covolution $\star$ that we have introduced in Definition 2.2.4, by the fact that we consider here the ordinary pushforward by s, and not the pushforward with compact support.

We denote by $\mathbf{k}_{\gamma^a}$ the sheaf associated to the closed subset γ^a. The canonical map $\mathbf{k}_{\gamma^a} \to \mathbf{k}_{\{0\}}$ induces a morphism

$$F \underset{np}{*} \mathbf{k}_{\gamma^a} \to F. \tag{5.2}$$

Proposition 5.2.4 ([GS14, Proposition 3.9])
Let $F \in \mathrm{D}^b(\mathbf{k}_{\mathbb{V}})$. Then $F \in \mathrm{D}^b_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}})$ if and only if the morphism (5.2) is an isomorphism.

We finally recall the following notion extracted from [KS18a].

Definition 5.2.5
Let A be a subset of $\mathbb{V}$. We say that A is *γ-proper* if the map s is proper on $\gamma \times A$.

Alexandrov sheaves

The datum of a closed proper convex cone γ of $\mathbb{V}$ endows $\mathbb{V}$ with the order

$$x \leq_\gamma y \text{ if and only if } x + \gamma \subset y + \gamma.$$

Let γ be a a closed proper convex cone of $\mathbb{V}$ we write it $\mathbb{V}_{\mathfrak{a}(\gamma)}$ for the topological space $\mathbb{V}$ endowed with the Alexandrov topology (proposition 2.2.1) associated to the pre-order $\leq_\gamma$. When the context is clear, we will write $\mathbb{V}_{\mathfrak{a}}$. An Alexandrov sheaf is an object of the abelian category $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$. We denote by $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ its derived category.

We denote by $\mathbb{V}_{\leq_\gamma}$ the category whose objects are the elements of $\mathbb{V}$ and given x and y in $\mathbb{V}$, there is exactly one morphism from x to y if an only if $x \leq_\gamma y$. If there is no risk of confusion, we simply write $\mathbb{V}_{\leq}$ and set

$$\mathrm{Mod}(\mathbb{V}_{\leq}) := \mathrm{Fun}((\mathbb{V}_{\leq})^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) = \mathrm{Pers}(\mathbf{k}^{(\mathbb{V}, \leq^{\mathrm{op}})}).$$

In this chapter only, for simplicity, we take the convention of calling the objects of Mod($\mathbb{V}_{\leq}$) "persistence modules over $\mathbb{V}_{\leq}$".

In order to compare γ-sheaves and Alexandrov sheaves we use morphisms of sites. These are morphisms between Grothendieck topologies and in particular usual topologies considered as Grothendieck topologies. It is important to keep in mind that some morphisms of sites between usual topological spaces are not induced by continuous maps. This is why we use this notion. Operations on sheaves can also be defined for morphisms of sites. These operations on sheaves generalize the ones induced by continuous maps between topological spaces. We refer the reader to [KS06] for a detailed presentation. Here, we provide, for the convenience of the reader, the definition of a morphism of sites in the special cases of topological spaces as it is sufficient for our needs.

Definition 5.2.6
Let X and Y be two topological spaces. A *morphism of sites* $f\colon X \to Y$ is a functor $f^t\colon \mathfrak{Op}(Y) \to \mathfrak{Op}(X)$ such that

(i) for any $U, V \in \mathfrak{Op}(Y)$, $f^t(U \cap V) = f^t(U) \cap f^t(V)$,

(ii) for any $V \in \mathfrak{Op}(Y)$ and any covering $\mathcal{V} = \{V_i\}_{i \in I}$ of V, $f^t(\mathcal{V}) = \{f^t(V_i)\}_{i \in I}$ is a covering of $f^t(V)$.

We write $\mathbb{V}_{\leq}^{\mathrm{top}}$ for $\mathbb{V}_{\leq}$ endowed with the trivial Grothendieck topology (that is the one for which all the sieve are representable). Note that on $\mathbb{V}_{\leq}^{\mathrm{top}}$ all presheaves are sheaves. Hence, the forgetful functor $for\colon \mathbb{V}_{\leq}^{\mathrm{top}} \to \mathbb{V}_{\leq}$ induces an equivalence

$$\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\leq}^{\mathrm{top}}}) \xrightarrow{\sim} \mathrm{Mod}(\mathbb{V}_{\leq})$$

For this reason, we will not distinguished between $\mathbb{V}_{\leq}^{\mathrm{top}}$ and $\mathbb{V}_{\leq}$. There is a morphism of sites $\theta\colon \mathbb{V}_{\mathfrak{a}} \to \mathbb{V}_{\leq}$ defined by

$$\theta^t\colon \mathbb{V}_{\leq} \to \mathfrak{Op}(\mathbb{V}_{\mathfrak{a}}), \quad x \mapsto x + \gamma.$$

The following statement is due to J. Curry. We refer to [KS18a] for a proof.

Theorem 5.2.7
The functor

$$\theta_*\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}) \to \mathrm{Mod}(\mathbb{V}_{\leq})$$

is an equivalence of categories.

5.2.2 Relation between γ-sheaves and Alexandrov sheaves

Let $\mathbb{V}$ be a finite dimensional real vector space and γ a cone of $\mathbb{V}$ satisfying hypothesis (5.1). Recall that we have defined a preorder $\leq$ on $\mathbb{V}$ as follow:

$$x \leq_\gamma y \Leftrightarrow x + \gamma \subset y + \gamma.$$

We denote by $\mathbb{V}_{\mathfrak{a}(\gamma)}$ the Alexandrov topology on $\mathbb{V}$ associated to the preorder $\leq_\gamma$. If there is no risk of confusion, we write $\mathbb{V}_{\mathfrak{a}}$ instead of $\mathbb{V}_{\mathfrak{a}(\gamma)}$. By definition the open sets $(x+\gamma)_{x\in\mathbb{V}} = (\mathscr{D}(x))_{x\in\mathbb{V}}$ form a basis of the topology $\mathbb{V}_{\mathfrak{a}}$.

We define the functor $\alpha^t \colon \mathfrak{Op}(\mathbb{V}_{\mathfrak{a}}) \to \mathfrak{Op}(\mathbb{V}_\gamma)$ by

$$\alpha^t \colon \mathfrak{Op}(\mathbb{V}_{\mathfrak{a}}) \to \mathfrak{Op}(\mathbb{V}_\gamma), \quad U = \bigcup_{x\in U} x + \gamma \mapsto \bigcup_{x \in U} x + \mathrm{Int}(\gamma).$$

Lemma 5.2.8
The functor α^t is a morphism of sites $\alpha \colon \mathbb{V}_\gamma \to \mathbb{V}_{\mathfrak{a}}$.

Proof
The functor α^t preserves covering. Let us check that it preserves finite limits. For that purpose it is sufficient to check that it preserves the final object (clear) and fibered products which reduces in this particular setting to show that

$$\alpha^t(U \cap V) = \alpha^t(U) \cap \alpha^t(V).$$

On one hand

$$\alpha^t(U\cap V) = \bigcup_{x\in U\cap V} x + \mathrm{Int}(\gamma).$$

On the other hand

$$\alpha^t(U) \cap \alpha^t(V) = \bigcup_{z \in \alpha^t(U)\cap\alpha^t(V)} z + \mathrm{Int}(\gamma).$$

Hence, $\alpha^t(U)\cap\alpha^t(V) \subset \alpha^t(U\cap V)$. As $U \cap V$ is included in U and V it follows by functoriality that $\alpha^t(U\cap V)$ is included in $\alpha^t(U)$ and $\alpha^t(V)$. This proves the reverse inclusion

□

We also have the following morphism of sites

$$\beta \colon \mathbb{V}_{\mathfrak{a}} \to \mathbb{V}_\gamma, \quad \beta^t(x + \mathrm{Int}(\gamma)) = x + \mathrm{Int}(\gamma)$$

The composition of β and α satisfies $\beta \circ \alpha = \mathrm{id}$.

The morphism of sites α and β provides the following adjunctions

$$\alpha^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}) \rightleftarrows \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\gamma}})\colon \alpha_*$$

$$\beta^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\gamma}}) \rightleftarrows \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})\colon \beta_*.$$

We define the functor

$$\alpha^{\dagger}\colon \mathrm{Fun}(\mathfrak{Op}(\mathbb{V}_{\mathfrak{a}})^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Fun}(\mathfrak{Op}(\mathbb{V}_{\gamma})^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})), \quad F \mapsto \alpha^{\dagger}F$$

where

$$\text{for every } U \in \mathfrak{Op}(\mathbb{V}_{\gamma}),\ \alpha^{\dagger}F(U) = \operatorname*{colim}_{U \subset \alpha^{t}(V)} F(V).$$

Recall that by definition $\alpha^{-1}F$ is the sheafification of $\alpha^{\dagger}F$.

Proposition 5.2.9

(i) The functors $\alpha^{-1} \simeq \alpha^{\dagger} \simeq \beta_*$ are isomorphic,

(ii) the functor α_* is fully faithful,

(iii) the functor β^{-1} is fully faithful.

Proof

(i) Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$. Then,

$$\alpha^{\dagger}F(U) = \operatorname*{colim}_{U \subset \alpha^{t}(V)} F(V) = F(U) = \beta_* F(U).$$

Hence, $\alpha^{\dagger} \simeq \beta_*$. Since $\alpha^{\dagger}F$ is a sheaf, it follows that $\alpha^{-1} \simeq \alpha^{\dagger}$.

(ii) Let $F,\ G \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\gamma}})$. The isomorphism of functors $\beta_*\alpha_* \simeq \mathrm{id}$ implies that the morphism $\mathrm{Hom}_{\mathbf{k}_{\mathbb{V}_{\gamma}}}(F, G) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}}(\alpha_*F, \alpha_*G)$ is injective. Let $\phi \in \mathrm{Hom}_{\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}}(\alpha_*F, \alpha_*G)$. Set $\psi_{x+\mathrm{Int}(\gamma)} := \phi_{x+\gamma}$. Since $\{x + \mathrm{Int}(\gamma)\}_{x \in \mathbb{V}_{\gamma}}$ is a basis of $\mathbb{V}_{\gamma}$, the family $(\psi_{x+\mathrm{Int}(\gamma)})_{x \in \mathbb{V}_{\gamma}}$ defines a morphism of sheaves $\psi\colon F \to G$ and $\alpha_*\psi = \phi$. This proves that α_* is fully faithful.

(iii) This follows from [KS06, Exercise 1.14].

□

We have the following sequence of adjunctions $\beta^{-1} \dashv \beta_* \simeq \alpha^{-1} \dashv \alpha_*$.

The functor α_* and β^{-1} are different as the following example shows. We set $\mathbb{V} = \mathbb{R}$ and $\gamma =]-\infty, 0]$. We consider the γ-closed set $[t, +\infty[$ with $t \in \mathbb{R}$ and the sheaf $\mathbf{k}_{[t,+\infty[}$ associated with it. Consider the sheaves

$$\beta^{-1}\mathbf{k}_{[t,+\infty[} \quad \text{and} \quad \alpha_*\mathbf{k}_{[t,+\infty[}.$$

We compute the stalk at t of these two sheaves. For the first one, observe that the continuous map $\beta\colon \mathbb{V}_{\mathfrak{a}} \longrightarrow \mathbb{V}_\gamma$ is the identity on the elements of $\mathbb{V}$. Therefore, we have $(\beta^{-1}\mathbf{k}_{[t,+\infty[})_t \simeq (\mathbf{k}_{[t,+\infty[})_t \simeq \mathbf{k}$. For the second one,

$$\begin{aligned}(\alpha_*\mathbf{k}_{[t,+\infty[})_t &\simeq \alpha_*\mathbf{k}_{[t,+\infty[}(t+\gamma)\\ &\simeq \mathbf{k}_{[t,+\infty[}(]-\infty,t[)\\ &\simeq 0.\end{aligned}$$

5.2.3 Compatibilities of operations

In this subsection, we study the compatibility between operations for sheaves in γ and Alexandrov topologies.

Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional real vector spaces endowed with cones γ and λ satisfying the hypothesis (5.1).

Lemma 5.2.10

Let $f\colon \mathbb{V} \to \mathbb{W}$ be a linear map. The following statements are equivalent.

(i) $f(\gamma) \subset \lambda$,

(ii) $f\colon \mathbb{V}_\gamma \to \mathbb{W}_\lambda$ is continuous,

(iii) $f\colon \mathbb{V}_{\mathfrak{a}(\gamma)} \to \mathbb{W}_{\mathfrak{a}(\lambda)}$ is continuous.

Proof

(i)⇒(ii) Let $y \in \mathbb{W}$. Let us show that $f^{-1}(y + \mathrm{Int}(\lambda))$ is a γ-open. As $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional, f is continuous for the usual topology. Hence $f^{-1}(y+\mathrm{Int}(\lambda))$ is open. The inclusion $f^{-1}(y+\mathrm{Int}(\lambda)) \subset f^{-1}(y+\mathrm{Int}(\lambda))+\gamma$ is clear. Let us show the reverse inclusion. Let $x \in f^{-1}(y + \mathrm{Int}(\lambda)) + \gamma$. There exists $u \in f^{-1}(y + \mathrm{Int}(\lambda))$ and $v \in \gamma$ such that $x = u + v$. Then $f(x) = y+l+f(v)$ with $l \in \mathrm{Int}(\lambda)$ and $f(v) \in \lambda$. Since $\mathrm{Int}(\lambda) = \mathrm{Int}(\lambda)+\lambda$ it follows that $f(x) \in y+\mathrm{Int}(\lambda)$. Hence $f^{-1}(y+\mathrm{Int}(\lambda))+\gamma = f^{-1}(y+\mathrm{Int}(\lambda))$. This proves that $f^{-1}(y + \mathrm{Int}(\lambda))$ is a γ-open.
(ii)⇒(i) Since $f(0) = 0$ and f is continuous, for every $\varepsilon > 0$ there exists $\eta > 0$ such that $f(B(0;\eta)+\gamma) \subset B(0,\varepsilon)+\lambda$. Hence, if $v \in \gamma$, $f(v) \in \overline{\lambda} = \lambda$.
(i)⇒(iii) The statement (i) implies that $f\colon (\mathbb{V}, \leq_\gamma) \to (\mathbb{W}, \leq_\lambda)$ is order preserving. Hence, $f\colon \mathbb{V}_{\mathfrak{a}(\gamma)} \to \mathbb{W}_{\mathfrak{a}(\lambda)}$ is continuous.
(iii)⇒(i) λ is an open subset of $\mathbb{W}_{\mathfrak{a}(\lambda)}$. As $f^{-1}(\lambda)$ is an open subset of $\mathbb{V}_{\mathfrak{a}(\gamma)}$ such that $0 \in f^{-1}(\lambda)$ it follows that $\gamma \subset f^{-1}(\lambda)$. Hence $f(\gamma) \subset \lambda$.

□

Let $f\colon \mathbb{V} \to \mathbb{W}$ be a linear map. Assume that $f(\gamma) \subset \lambda$. We denote by $\widetilde{f}\colon \mathbb{V}_{\mathfrak{a}(\gamma)} \to \mathbb{W}_{\mathfrak{a}(\lambda)}$ the continuous map between $\mathbb{V}$ and $\mathbb{W}$ endowed with the Alexandrov topologies respectively associated to the cones γ and λ and whose underlying linear map is f.

Proposition 5.2.11

(i) Assume that $f(\gamma) \subset \lambda$. Then the following diagram of morphisms of sites is commutative.

$$\begin{tikzcd} \mathbb{V}_{\mathfrak{a}} \arrow[r, "\beta"] \arrow[d, "\widetilde{f}"'] & \mathbb{V}_{\gamma} \arrow[d, "f"] \\ \mathbb{W}_{\mathfrak{a}} \arrow[r, "\beta"] & \mathbb{W}_{\lambda} \end{tikzcd}$$

(ii) Assume that $f(\mathrm{Int}(\gamma)) \subset \mathrm{Int}(\lambda)$. Then the following diagram of morphisms of sites is commutative.

$$\begin{tikzcd} \mathbb{V}_{\gamma} \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathbb{V}_{\mathfrak{a}} \arrow[d, "\widetilde{f}"] \\ \mathbb{W}_{\lambda} \arrow[r, "\alpha"] & \mathbb{W}_{\mathfrak{a}} \end{tikzcd}$$

Proof

(i) is clear.

(ii) Let $y \in \mathbb{W}$. On one hand, we have

$$\begin{aligned} \alpha^t \circ \widetilde{f}^t(y+\lambda) &= \alpha^t\Big(\bigcup_{\{x \in \mathbb{V} | f(x) \in y+\lambda\}} x+\gamma\Big) \\ &= \bigcup_{\{x \in \mathbb{V} | f(x) \in y+\lambda\}} x+\mathrm{Int}(\gamma). \end{aligned}$$

On the other hand,

$$\begin{aligned} f^t \circ \alpha^t(y+\lambda) &= f^t(y+\mathrm{Int}(\lambda)) \\ &= \bigcup_{\{x \in \mathbb{V} | f(x) \in y+\mathrm{Int}(\lambda)\}} x+\mathrm{Int}(\gamma). \end{aligned}$$

The inclusion

$$\bigcup_{\{x \in \mathbb{V} | f(x) \in y+\mathrm{Int}(\lambda)\}} x+\mathrm{Int}(\gamma) \subset \bigcup_{\{x \in \mathbb{V} | f(x) \in y+\lambda\}} x+\mathrm{Int}(\gamma)$$

is clear. Let us prove the reverse inclusion. Let $z \in \bigcup_{\{x \in \mathbb{V} | f(x) \in y+\lambda\}} x + \mathrm{Int}(\gamma)$. Then $z = x+g$ with $g \in \mathrm{Int}(\gamma)$ and $f(z) = y+l+f(g)$ with $l \in \lambda$. As $f(g) \in \mathrm{Int}(\lambda)$ then $l+f(g) \in \mathrm{Int}(\lambda)$. It follows that $f(z) \in y+\mathrm{Int}(\lambda)$.

□

In (ii) the hypothesis $f(\mathrm{Int}(\gamma)) \subset \mathrm{Int}(\lambda)$ is necessary as shown in the following example.

On $\mathbb{R}$, consider the cone $\gamma = \{x \in \mathbb{R} | \, x \leq 0\}$ and on $\mathbb{R}^2$ consider the cone $\lambda = \{(x,y) \in \mathbb{R}^2 | \, x \leq 0 \text{ and } y \leq 0\}$. Let $f\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x,0)$. Then, computing both $f^t\alpha^t(\lambda)$ and $\alpha^t \widetilde{f}^t(\lambda)$, we get

$$\begin{aligned} f^t\alpha^t(\lambda) &= f^t(\mathrm{Int}(\lambda)) \\ &= \emptyset \end{aligned} \qquad\qquad \begin{aligned} \alpha^t \widetilde{f}^t(\lambda) &= \alpha^t(\gamma) \\ &= \mathrm{Int}(\gamma) \end{aligned}$$

Note that the condition $f(\mathrm{Int}(\gamma)) \subset \mathrm{Int}(\lambda)$ is automatically satisfied when f is surjective.

5.3 Ephemeral persistence modules

5.3.1 The category of ephemeral modules

In this section, we propose a notion of ephemeral persistence module in arbitrary dimension, generalizing the one of [CdSGO16]. For the convenience of the reader, we start by recalling the definition of a Serre subcategory and of the quotient of an abelian category by a Serre subcategory that we subsequently use. We refer the reader to [Gab62] and [Sta19, Tag 02MN].

Definition 5.3.1
Let $\mathcal{A}$ be an abelian category. A Serre subcategory $\mathcal{C}$ is a full subcategory of $\mathcal{A}$, which contains 0, such that given an exact sequence

$$X \to A \to Y$$

with X and Y in $\mathcal{C}$ and $A \in \mathcal{A}$ then $A \in \mathcal{C}$.

If $\mathcal{C}$ is closed under isomorphism, we say that it is a strict Serre subcategory of $\mathcal{A}$.

Lemma 5.3.2
Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. There exists an abelian category denoted $\mathcal{A}/\mathcal{C}$ and an exact functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ whose kernel is $\mathcal{C}$ satisfying the following universal property: For any exact functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset \mathrm{Ker}(G)$ there exists a factorization $F = G \circ Q$ for a unique exact functor $G\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proposition 5.3.3 ([Gab62, Ch.2 §2 Proposition 5])
Let $L\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Assume that L has a fully faithful right adjoint R. Then $\mathrm{Ker}L$ is a Serre subcategory of $\mathcal{A}$ and L induces an equivalence between $\mathcal{A}/\mathrm{Ker}(L)$ and $\mathcal{B}$.

We now introduce our notion of ephemeral module.

Definition 5.3.4
An object $G \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is ephemeral if and only if $\beta_* G \simeq 0$. We denote by $\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ the full subcategory of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ spanned by ephemeral modules.

In other words, an object $G \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is ephemeral if and only if for every open subset of the usual topology of $\mathbb{V}$, $G(U + \gamma) = 0$.

Lemma 5.3.5
The full subcategory $\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is a Serre subcategory, stable by limits and colimits.

Proof
Since $\beta_* \simeq \alpha^{-1}$, $\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \simeq \ker(\alpha^{-1})$. Since α^{-1} is exact, Eph is a Serre subcategory of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. Since β_* commutes with limits (it is a right adjoint) and α^{-1} commutes with colimits (it is a left adjoint), $\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ has limits and colimits.

□

Proposition 5.3.6
The functor $\alpha^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ induces an equivalence of categories between $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})/\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$.

Proof
This is a direct consequences of Proposition 5.2.9 and 5.3.3.

□

5.3.2 Ephemeral module on $(\mathbb{R}, \leq)$

Ephemeral modules on $(\mathbb{R}, \leq)$ were introduced in [CdSGO16] as a way to express the intuition of persistence modules that cannot be measured with respect to the interleaving distance. The category of observable modules on $\mathbb{R}$ was then introduced and studied in [CCBdS16], as the quotient category of persistence modules by the subcategory of ephemeral ones. We show that our notion of ephemeral module generalize to arbitrary dimension the one of [CdSGO16] and [CCBdS16].

The choice of the order $\leq$ on $\mathbb{R}$ is equivalent to the choice of the proper closed convex cone $\gamma = [0, +\infty[$. We write d_I for the standard interleaving distance between one parameter persistence modules (see section 2.1.2).

Lemma 5.3.7
Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}})$. The following are equivalent :

(i) $F \in \mathrm{Eph}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}})$,

(ii) the restriction morphism $\rho_{t,s} \colon F(s+\gamma) \to F(t+\gamma)$ is null whenever $s < t$,

(iii) $d_I(F, 0) = 0$.

Proof
(i)$\Rightarrow$(ii). There exists $u \in \mathbb{R}$ such that $s < u < t$ and by hypothesis $F(u + \mathrm{Int}(\gamma)) \simeq (0)$. Hence, we have the following commutative diagram

$$\begin{tikzcd} F(s+\gamma) \arrow[rr, "\rho_{t,s}"] \arrow[dr, "\rho_{u+\mathrm{Int}(\gamma),s}"'] & & F(t+\gamma) . \\ & (0) \arrow[ur, "\rho_{t,u+\mathrm{Int}(\gamma)}"'] & \end{tikzcd}$$

This implies that $\rho_{t,s} = 0$.

(ii)$\Rightarrow$(i). As the family $(x + \mathrm{Int}(\gamma))_{x \in \mathbb{R}}$ is a basis of the γ-topology on $\mathbb{R}$, it is sufficient to show that for every $x \in \mathbb{R}$, $F(x + \mathrm{Int}(\gamma)) = (0)$. Let $x \in \mathbb{R}$. Since F is a sheaf for the Alexandrov topology, we have the following isomorphism

$$\lim_{u+\gamma \subset x+\mathrm{Int}(\gamma)} \rho_{x+\mathrm{Int}(\gamma),u} \colon F(x + \mathrm{Int}(\gamma)) \xrightarrow{\sim} \lim_{u+\gamma \subset x+\mathrm{Int}(\gamma)} F(u+\gamma). \tag{5.3}$$

Since $u + \gamma \subset x + \mathrm{Int}(\gamma)$, $x < u$. Then, there exists $t \in \mathbb{R}$ such that $x < t < u$. Hence, $\rho_{x+\mathrm{Int}(\gamma),u} = \rho_{t,u} \circ \rho_{x+\mathrm{Int}(\gamma),t} = 0$. It follows that the isomorphism (5.3) is the zero map. This implies that $F(x + \mathrm{Int}(\gamma)) \simeq 0$.
(iii) $\iff$ (ii) is an easy consequence of the definition of the interleaving distance for one-parameter persistence modules.

□

We refer the reader to [CCBdS16, Definition 2.3] for the definition of the observable category denoted **Ob** and recall the following result by the same authors

Theorem 5.3.8 ([CCBdS16, Corollary 2.13])
There is the following equivalence of categories $\mathbf{Ob} \simeq \mathrm{Mod}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}})/\mathrm{Eph}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}})$.

Corollary 5.3.9
The observable category **Ob** is equivalent to the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{R}_\gamma})$.

Proof
Using Proposition 5.3.6 and Theorem 5.3.8, we obtain the following sequence of equivalence

$$\mathbf{Ob} \simeq \mathrm{Mod}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}})/\mathrm{Eph}(\mathbf{k}_{\mathbb{R}_{\mathfrak{a}(\gamma)}}) \simeq \mathrm{Mod}(\mathbf{k}_{\mathbb{R}_\gamma}).$$

Note that we already obtained this result in corollary [?], but our approach was relying on a decomposition theorem, which is not the case here. This is precisely why it allows to generalize the notion of ephemeral persistence modules to higher dimensional vector spaces.

5.3.3 Ephemeral modules in the derived category

We write $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ for the derived category of Alexandrov sheaves and $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ for the one of γ-sheaves.

It follows from the preceding subsections that we have the following adjunctions

$$\beta_* = \alpha^{-1} \colon \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \rightleftarrows \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma}) \colon \beta^{-1},\ \mathrm{R}\alpha_*$$

Proposition 5.3.10

(i) the functor β^{-1} is fully faithful,

(ii) the functor $\mathrm{R}\alpha_*$ is fully faithful.

Proof
(i) follows from Proposition 5.2.9 as β_* and β^{-1} are exact.
(ii) This follows from [KS06, Exercices 1.14].

□

We write $\mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ for the full subcategory of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ consisting of objects $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ such that for every $i \in \mathbb{Z}$, $\mathrm{H}^i(F) \in \mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. Since $\mathrm{Eph}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is a thick abelian subcategory of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$, $\mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is a triangulated subcategory of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. We consider the full subcategory of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$

$$\mathrm{Ker}\alpha^{-1} = \{F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \mid \alpha^{-1}F \simeq 0\}.$$

Recall that a subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ is thick if it is triangulated and it contains all direct summands of its objects. It is clear that $\mathrm{Ker}\alpha^{-1}$ is thick.

Lemma 5.3.11
The triangulated category $\mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ is equivalent to the triangulated category $\mathrm{Ker}\alpha^{-1}$.

Proof
This follows immediately form the exactness of α^{-1}.

□

We now briefly review the notion of Verdier localization of triangulated categories. References are made to [KS06] and [Kra2].

Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ be a triangulated full subcategory of $\mathcal{T}$. We write $W(\mathcal{N})$ for the set of maps $f : X \to Y$ of $\mathcal{T}$ which sit into a triangle of the form

$$X \xrightarrow{f} Y \to Z \xrightarrow{+1}$$

where $Z \in \mathcal{N}$. By definition the Verdier quotient of $\mathcal{T}$ by $\mathcal{N}$ is the localization of $\mathcal{T}$ with respects to the set of maps $W(\mathcal{N})$. That is

$$\mathcal{T}/\mathcal{N} := \mathcal{T}[W(\mathcal{N})^{-1}]$$

together with the localization functor

$$Q\colon \ : \mathcal{T} \to \mathcal{T}/\mathcal{N}.$$

The following proposition is well-known

Proposition 5.3.12 ([Kra2] - Prop. 2.3.1)
Let $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ be an adjunction. Assume that the right adjoint R is fully faithful. Then $L\colon \mathcal{C} \to \mathcal{D}$ is the localization of $\mathcal{C}$ with respect to the set of morphisms

$$W = \{f : X \to Y \in \mathrm{Mor}(\mathcal{C}) \mid L(f) \text{ is an isomorphism}\}.$$

Proposition 5.3.13
The category $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ is the Verdier quotient of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ by $\mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ via the localization functor $\alpha^{-1}\colon \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \to \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$. In particular, $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})/\mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \simeq \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$.

Proof
Let $W = \{f \in \mathrm{Mor}(\mathcal{C}) \mid \alpha^{-1}(f) \text{ is an isomorphism}\}$. Let $f\colon F \to G$ be a morphism of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. By the axiom of triangulated categories, f sits in a distinguished triangle

$$F \xrightarrow{f} G \to H \xrightarrow{+1}.$$

Hence $\alpha^{-1}(f)$ is an isomorphism if and only if $\alpha^{-1}H \simeq 0$. That is if $H \in \mathrm{D}_{\mathrm{Eph}}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. This proves the claim.

□

5.4 Distances on categories of sheaves

5.4.1 Preliminary facts

Let $\mathbb{V}$ be a finite dimensional vector space, $\gamma \subset \mathbb{V}$ be a convex, proper cone with non-empty interior and $v \in \mathbb{V}$. The map

$$\tau_v \colon \mathbb{V} \to \mathbb{V},\ x \mapsto x - v$$

is continuous for the Euclidean, Alexandrov and the γ topologies on $\mathbb{V}$. Let $v,\ w \in \mathbb{V}$ and assume that $w \leq_\gamma v$.

Alexandrov & γ-topology

Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. Since $w + \gamma \subset v + \gamma$, it follows that for every $U \in \mathfrak{Op}(\mathbb{V}_\mathfrak{a})$, $U + w \subset U + v$. hence, replacing F by an homotopically injective resolution $\mathcal{I}$, and using the restriction morphisms, we obtain a morphism of sheaves

$$\tau_{v*}\mathcal{I} \to \tau_{w*}\mathcal{I}$$

This provides a morphism

$$\chi^\mathfrak{a}_{v,w}(F) \colon \tau_{v*}F \to \tau_{w*}F.$$

It follows that there is a morphism of functors from $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ to $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$

$$\chi^\mathfrak{a}_{v,w} \colon \tau_{v*} \to \tau_{w*}. \tag{5.4}$$

In a similar way, we obtain a morphism of functors from $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ to $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$

$$\chi^\gamma_{v,w} \colon \tau_{v*} \to \tau_{w*}. \tag{5.5}$$

One immediately verify that for every $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$

$$\beta_*\chi^\mathfrak{a}_{v,w}(F) \simeq \chi^\gamma_{v,w}\,(\beta_*F). \tag{5.6}$$

$$\mathrm{R}\alpha_*\chi^\gamma_{v,w}(G) \simeq \chi^\mathfrak{a}_{v,w}\,(\mathrm{R}\alpha_*G). \tag{5.7}$$

Lemma 5.4.1
For every $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, there is the following canonical isomorphism

$$\beta^{-1}\chi^\gamma_{v,w}(F) \simeq \chi^\mathfrak{a}_{v,w}\,(\beta^{-1}F).$$

Proof
Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ and consider the canonical morphism.

$$\chi^{\mathfrak{a}}_{v,w}(\beta^{-1}F) \colon \tau_{v*}\beta^{-1}F \to \tau_{w*}\beta^{-1}F.$$

Since β^{-1} is fully faithful and commutes with τ_{v*} and τ_{w*}, there exists a unique morphism $f \colon \tau_{v*}F \to \tau_{w*}F$ such that the following diagram commutes

$$\begin{tikzcd}
\tau_{v*}\beta^{-1}F \arrow[r, "\chi^{\mathfrak{a}}_{v,w}(\beta^{-1}F)"] \arrow[d, "\wr"'] & \tau_{w*}\beta^{-1}F \arrow[d, "\wr"] \\
\beta^{-1}\tau_{v*}F \arrow[r, "\beta^{-1}f"] & \beta^{-1}\tau_{w*}F.
\end{tikzcd}$$

Hence, $\beta_*\chi^{\mathfrak{a}}_{v,w}(\beta^{-1}F) \simeq \beta_*\beta^{-1}f$. It follows from the fully faithfulness of β^{-1} and from Formula (5.6) that

$$f \simeq \chi^{\gamma}_{v,w}(F).$$

Applying β^{-1} to the above isomorphism, we get that $\beta^{-1}\chi^{\gamma}_{v,w}(F) \simeq \chi^{\mathfrak{a}}_{v,w}(\beta^{-1}F)$.

Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$. If $w = 0$ and $v \in \gamma^a$, the morphisms (5.4) and (5.5) provide respectively the canonical morphisms

$$\chi^{\mathfrak{a}}_{v,0} \colon \tau_{v*}F \to F,$$

$$\chi^{\gamma}_{v,0} \colon \tau_{v*}G \to G.$$

Remark 5.4.2
In the abelian cases i.e. for the categories $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ similar morphisms exist. They can be constructed directly or induced from the derived cases by using the following facts. If $\mathcal{A}$ is an abelian category and $\mathrm{D}(\mathcal{A})$ is its derived category, then the canonical functor

$$\iota \colon \mathcal{A} \to \mathrm{D}(\mathcal{A})$$

which send an object of $\mathcal{A}$ to the corresponding complex concentrated in degree zero is fully faithful. Moreover, $\mathrm{H}^0 \circ \iota \simeq \mathrm{id}$ and for every $v \in V$, τ_{v*} commutes with H^0. Hence, we will focus on the derived situations as it implies, here, the abelian case.

The microlocal setting

We now construct similar morphisms for sheaves in $\mathrm{D}^b_{\gamma^{\circ,a}}(\mathbf{k}_{\mathbb{V}})$. This construction is classical (see for instance [GS14]). We provide it for the convenience of the reader.

Lemma 5.4.3
Let $F \in \mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_{\mathbb{V}})$ and $u \in \mathbb{V}$. Then there is a functorial isomorphism

$$\tau_{u*}F \simeq \mathbf{k}_{-u+\gamma^a} \underset{np}{*} F.$$

Proof
It follows from Proposition 5.2.4 that the canonical morphism $\mathbf{k}_{\gamma^a} \underset{np}{*} F \to F$ is an isomorphism and $\tau_u \circ s = s \circ (\tau_u \times \mathrm{id})$. Hence

$$\begin{aligned}\tau_{u*}F &\simeq \tau_{u*}(\mathbf{k}_{\gamma^a} \underset{np}{*} F)\\ &\simeq s_*(\tau_u \times \mathrm{id})_*(\mathbf{k}_{\gamma^a} \boxtimes F)\\ &\simeq \mathbf{k}_{-u+\gamma^a} \underset{np}{*} F.\end{aligned}$$

For $w \leq_\gamma v$, the canonical map

$$\mathbf{k}_{-v+\gamma^a} \to \mathbf{k}_{-w+\gamma^a}$$

induces a morphism of functors

$$\mathbf{k}_{-v+\gamma^a} \underset{np}{*} (\cdot) \to \mathbf{k}_{-w+\gamma^a} \underset{np}{*} (\cdot). \tag{5.8}$$

Using the Lemma 5.4.3, we obtain a morphism of functors from $\mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_{\mathbb{V}})$ to $\mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_{\mathbb{V}})$

$$\chi^\mu_{v,w} \colon \tau_{v*} \to \tau_{w*}. \tag{5.9}$$

Lemma 5.4.4
Let $F \in \mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_{\mathbb{V}})$ and $G \in \mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$. There are the following canonical isomorphisms

$$\mathrm{R}\phi_{\gamma*}\chi^\mu_{v,w}(F) \simeq \chi^\gamma_{v,w}(\mathrm{R}\phi_{\gamma*}F), \tag{5.10}$$

$$\phi_\gamma^{-1}\chi^\gamma_{v,w}(G) \simeq \chi^\mu_{v,w}\phi_\gamma^{-1}(G). \tag{5.11}$$

Proof

Let $F \in \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and U a γ-open set. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{RHom}_{\mathbf{k}_{\mathbb{V}_\gamma}}(\mathbf{k}_U, \mathrm{R}\phi_{\gamma_*}\tau_{v*}F) \arrow[r, "\mathrm{RHom}_{\mathbf{k}_{\mathbb{V}_\gamma}}(\mathbf{k}_U, \mathrm{R}\phi_{\gamma *}\chi^\mu_{v,w})"] \arrow[d] & \mathrm{RHom}_{\mathbf{k}_{\mathbb{V}_\gamma}}(\mathbf{k}_U, \mathrm{R}\phi_{\gamma_*}\tau_{w*}F) \arrow[d] \\
\mathrm{RHom}_{\mathbf{k}_{\mathbb{V}}}(\tau_v^{-1}\mathbf{k}_U, F) \arrow[d] & \mathrm{RHom}_{\mathbf{k}_{\mathbb{V}}}(\tau_w^{-1}\mathbf{k}_U, F) \arrow[d] \\
\mathrm{RHom}_{\mathbf{k}_{\mathbb{V}_\gamma}}(\mathbf{k}_{U+v}, \mathrm{R}\phi_{\gamma_*}F) \arrow[r] & \mathrm{RHom}_{\mathbf{k}_{\mathbb{V}}}(\mathbf{k}_{U+w}, \mathrm{R}\phi_{\gamma_*}F)
\end{tikzcd}$$

As $\mathrm{R}\phi_{\gamma_*} \colon \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of categories, it follows from the enriched Yoneda lemma that the bottom arrow on the diagram is induced by the canonical map $\mathbf{k}_{U+w} \to \mathbf{k}_{U+v}$ and hence is $\mathrm{RHom}_{\mathbf{k}_{\mathbb{V}_\gamma}}(\mathbf{k}_{U+v}, \chi^\gamma_{v,w}\mathrm{R}\phi_{\gamma*})$. which proves formula (5.10). The formula (5.11) follows by pre- and post-composing the preceding one by $\chi^\gamma_{v,w}\mathrm{R}\phi_{\gamma*}$ and using that $\mathrm{R}\phi_{\gamma_*}\phi_\gamma^{-1} \simeq \mathrm{id}$ and $\phi_\gamma^{-1}\mathrm{R}\phi_{\gamma_*} \simeq \mathrm{id}$.

□

Let $F \in \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. Again, if $w = 0$ and $v \in \gamma^a$, the morphism (5.9) provides the canonical morphism

$$\chi^\mu_{v,0} \colon \tau_{v*}F \to F.$$

Remark 5.4.5

Here, again, using Remark 5.4.2, we obtain, for every $F \in \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$ and every $w \leq_\gamma v$, a canonical morphism $\tau_{v*}F \to \tau_{w*}F$ by setting $\chi^\mu_{v,w}(F) := \mathrm{H}^0(\chi^\mu_{v,w}(\iota(F))$.

5.4.2 Interleavings and distances

Let $\mathcal{C}$ be any of the following category $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$. In the following, we will extend the notion of interleavings with respect to a vector.

Definition 5.4.6

Let F, $G \in \mathcal{C}$, and $v \in \gamma^a$. We say that F and G are v-interleaved if there exists $f \in \mathrm{Hom}_{\mathcal{C}}(\tau_{v*}F, G)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(\tau_{v*}G, F)$ such that the following diagram commutes.

$$
\begin{tikzcd}
\tau_{2v*}F \arrow[r, "\tau_{v*}f"] \arrow[rr, bend left, "\chi_{2v,0}(F)"] \arrow[dr] & \tau_{v*}G \arrow[r, "g"] \arrow[dr] & F \\
\tau_{2v*}G \arrow[r, "\tau_{v*}g"] \arrow[rr, bend right, "\chi_{2v,0}(G)"'] \arrow[ur] & \tau_{v*}F \arrow[r, "f"] \arrow[ur] & G
\end{tikzcd}
$$

Definition 5.4.7
With the same notations, define the interleaving distance between F and G with respect to $v \in \gamma^a$ to be the possibly infinite value:

$$d_I^v(F,G) := \inf(\{c \geq 0 \mid F \text{ and } G \text{ are } c \cdot v - \text{interleaved}\} \cup \{\infty\}).$$

Proposition 5.4.8
The interleaving distance d_I^v is a pseudo-extended metric on the objects of $\mathcal{C}$, that is it satisfies for F, G, H objects of $\mathcal{C}$:

1. $d_I^v(F,G) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$,
2. $d_I^v(F,G) = d_I^v(G,F)$,
3. $d_I^v(F,H) \leq d_I^v(F,G) + d_I^v(G,H)$.

We write $d^v_{I_{\mathfrak{a}}}$ for the interleaving distance on $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, $d^v_{I_\gamma}$ for the interleaving distance on $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, $d^v_{I_\mu}$ for the interleaving distance on $\mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$. We write $d^v_{I^{\mathrm{ab}}_{\mathfrak{a}}}$ for the interleaving distance on $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ and use similar notation in the cases of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ and $\mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$.

Remark 5.4.9
Again, here we focus on the derived case as the abelian one can be deduced from the derived one by using Remark 5.4.2.

Interleavings and ephemeral modules

This subsection is dedicated to the study of the relations between the notions of interleavings and ephemeral modules. We characterize ephemeral modules in terms of interleavings, and prove that they are exactly those modules which are at distance 0 from 0 for any direction of interleaving. Once again, we concentrate our attention on the derived setting as the abelian case can be deduced from the derived one by using Remark 5.4.2.

Proposition 5.4.10
Let F and G in $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$. The set

$$\mathrm{Inter}(F,G) = \{v \in \mathrm{Int}(\gamma^a) \mid F \text{ and } G \text{ are } v-\text{interleaved}\}$$

is Alexandrov-closed.

Proof
It is sufficient to show that $\mathrm{Inter}(F,G) + \gamma^a = \mathrm{Inter}(F,G)$. The inclusion $\mathrm{Inter}(F,G) \subset \mathrm{Inter}(F,G) + \gamma^a$ is clear. We prove the reverse inclusion. Let $w \in \gamma^a$ and $v \in \mathrm{Inter}(F,G)$. Let

$$f\colon \tau_{v*}F \to G \qquad\qquad g\colon \tau_{v*}G \to F$$

be a v-interleaving between F and G. The maps

$$\tau_{v+w*}F \xrightarrow{\tau_{w*}f} \tau_{w*}G \xrightarrow{\chi^{\mathfrak{a}}_{w,0}} G \qquad\qquad \tau_{v+w*}G \xrightarrow{\tau_{w*}g} \tau_{w*}F \xrightarrow{\chi^{\mathfrak{a}}_{w,0}} F$$

provides a $v+w$ interleaving between F and G as the following diagram

$$\begin{tikzcd}
\tau_{2(v+w)*}F \arrow[r, "\tau_{v+2w*}f"] \arrow[d, equal] & \tau_{v+2w*}G \arrow[r, "\chi^{\mathfrak{a}}_{v+2w,v+w}"] \arrow[d, equal] & \tau_{v+w*}G \arrow[r, "\tau_{*}g"] & \tau_{w*}F \arrow[r, "\chi_{w,0}"] & F \arrow[d, equal] \\
\tau_{2(v+w)*}F \arrow[r, "\tau_{2w*}\tau_{v*}f"] \arrow[rrrr, bend right, "\chi^{\mathfrak{a}}_{2(v+w),0}"'] & \tau_{v+2w*}G \arrow[r, "\tau_{2w*}g"] \arrow[ur, "\chi^{\mathfrak{a}}_{v+2w,v+w}"'] & \tau_{2w*}F \arrow[rr, "\chi_{2w,0}"] \arrow[ur, "\chi^{\mathfrak{a}}_{2w,w}"'] & & F
\end{tikzcd}$$

and its analogue with F and G interchanged are commutative.

□

Corollary 5.4.11
Let $w \geq_{\gamma^a} v$. Then,

$$d^v_{I_{\mathfrak{a}}} \geq d^w_{I_{\mathfrak{a}}}.$$

Remark 5.4.12
The proof of Proposition 5.4.10 proves also that for $F,\ G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$,

$$\mathrm{Inter}(F,G) + \gamma^a = \mathrm{Inter}(F,G).$$

Hence, if $w \geq_{\gamma^a} v$. Then, $d^v_{I_\gamma} \geq d^w_{I_\gamma}$.

Lemma 5.4.13
Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ and $v \in \mathrm{Int}(\gamma^a)$. Then F is v-interleaved with 0 if and only if the canonical morphism $\chi^{\mathfrak{a}}_{v,0}(F) \colon \tau_{v*}F \to F$ is null.

Proof
If $\chi^{\mathfrak{a}}_{v,0}(F)$ is zero then F is v-interleaved with zero.

Let us prove the converse. Suppose F is ephemeral and let $v \in \mathrm{Int}(\gamma^a)$. Since γ is closed, proper and convex, $\mathrm{Int}(\gamma^a)+\gamma^a = \mathrm{Int}(\gamma^a)$ and $\frac{v}{2} \in \mathrm{Int}(\gamma^a)$. Hence, we have the following sequence of inclusion

$$\gamma \subset \mathrm{Int}(\gamma) + \frac{v}{2} \subset \gamma + v$$

inducing for any $U \in \mathrm{Op}(\mathbb{V}_\mathfrak{a})$, the sequence of inclusions:

$$U \subset U + \mathrm{Int}(\gamma) + \frac{v}{2} \subset U + v.$$

Replacing F by a homotopically injective resolution $\mathcal{I}$, we obtain the following commutative diagram

$$\begin{tikzcd}
\Gamma(U;\mathcal{I}) \arrow[rr, "\chi^{\mathfrak{a}}_{v,0}"] \arrow[dr] & & \Gamma(U+v;\mathcal{I}) \\
& \Gamma(U+\mathrm{Int}(\gamma)+\frac{v}{2};\mathcal{I}). \arrow[ur] &
\end{tikzcd}$$

In $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$, $\Gamma(U+\mathrm{Int}(\gamma)+\frac{v}{2};\mathcal{I}) \simeq \mathrm{R}\Gamma(U+\mathrm{Int}(\gamma)+\frac{v}{2};F) \simeq 0$ as $U+\mathrm{Int}(\gamma)+\frac{v}{2} \in \mathfrak{Op}(\mathbb{V})$ and F is ephemeral. This proves that the canonical map $\chi^{\mathfrak{a}}_{v,0}$ is 0. Hence F is v-interleaved with 0.

□

The following propositions and corollaries prove that our notion of ephemeral modules captures the idea of "algebraic features of persistence modules that do not persist", and give a purely metric characterization of ephemeral Alexandrov sheaves.

Proposition 5.4.14
Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$, then F is ephemeral if and only if

$$\mathrm{Inter}(F,0) = \mathrm{Int}(\gamma^a).$$

Proof
(i) Assume F is ephemeral. Let $v \in \mathrm{Int}(\gamma^a)$ and U be an object of $\mathfrak{Op}(\mathbb{V}_\mathfrak{a})$. We have the following sequence of inclusion

$$U \subset U + \mathrm{Int}(\gamma) + \frac{v}{2} \subset U + v$$

and $U + \mathrm{Int}(\gamma) + \frac{v}{2} \in \mathfrak{Op}(\mathbb{V})$. Hence $\Gamma(U + \mathrm{Int}(\gamma) + \frac{v}{2}; F) \simeq 0$. It follows that $\chi^{\mathfrak{a}}_{v,0}(F) \colon \tau_{v*}F \to F$ factors through zero. This implies that $v \in \mathrm{Inter}(F, 0)$.
(ii) The proof is similar to the one of Lemma 5.3.7. Assume that $\mathrm{Inter}(F, 0) = \mathrm{Int}(\gamma^a)$. Let us show that $\beta_* F \simeq 0$. It is sufficient to show that for every $x \in \mathbb{V}$, $F(x + \mathrm{Int}(\gamma)) \simeq 0$. Let $x \in \mathbb{V}$. Then,

$$\lim_{u+\gamma \subset x+\mathrm{Int}(\gamma)} \rho_{x+\mathrm{Int}(\gamma),u} \colon F(x + \mathrm{Int}(\gamma)) \xrightarrow{\sim} \lim_{u+\gamma \subset x+\mathrm{Int}(\gamma)} F(u + \gamma). \tag{5.12}$$

Let $u \in x+\mathrm{Int}(\gamma)$, there exists $v \in \mathrm{Int}(\gamma^a)$ such $x = u+v$ and by assumption

$$\chi^{\mathfrak{a}}_{v,0}(F) \colon \tau_{v*}F \to F$$

factor through zero. Hence, the restriction map $\rho_{x+\mathrm{Int}(\gamma),u}$ is zero. This implies that the isomorphism (5.12) is null. It follows that $F(x+\mathrm{Int}(\gamma)) \simeq 0$ which proves the claim.

□

Remark 5.4.15
The previous proposition can be expressed in the language of graded modules over real polyhedral group, following the work of Ezra Miller [Mil20]. It precisely characterizes ephemeral persistence modules as those whose upper boundary [Mil20, Definition 3.11] atop the interior of the cone vanishes.

Corollary 5.4.16
Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, then F is ephemeral if and only if

$$\mathrm{Inter}(F, 0) = \mathrm{Int}(\gamma^a).$$

Proof
(i) Assume F is ephemeral. Then the proof is similar to the proof of Lemma 5.4.13.
(ii) Assume that $\mathrm{Inter}(F, 0) = \mathrm{Int}(\gamma^a)$. Then for every $i \in \mathbb{Z}$, $\mathrm{Inter}(\mathrm{H}^i(F), 0) = \mathrm{Int}(\gamma^a)$. Then the results follow from Proposition 5.4.14.

□

Corollary 5.4.17
Let $v \in \mathrm{Int}(\gamma^a)$ and $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$. Then, F is ephemeral if and only if $d^v_{I_{\mathfrak{a}}}(F, 0) = 0$.

Proof
The left to right implication is a direct consequence of corollary 5.4.16.

Let us prove the converse. Since $d^v_{I_{\mathfrak{a}}}(F, 0) = 0$, for all $\varepsilon > 0$, $\varepsilon \cdot v \in \mathrm{Inter}(F, 0)$. Let us prove that $\mathrm{Inter}(F, 0) = \mathrm{Int}(\gamma^a)$. Let $u \in \mathrm{Int}(\gamma^a)$. Since $\mathrm{Int}(\gamma^a)$ is open for the euclidean topology, there exists $\eta > 0$ such that $u - \eta \cdot v \in \mathrm{Int}(\gamma^a)$. Therefore $u = \eta \cdot v + (u - \eta \cdot v)$. The first element of the sum belongs to $\mathrm{Inter}(F, 0)$, and the second to $\mathrm{Int}(\gamma^a)$. By proposition 5.4.10, $\mathrm{Inter}(F, 0)$ is Alexandrov-closed, hence stable under addition by elements of $\mathrm{Int}(\gamma^a)$. This ends the proof.

□

Isometry theorems

We prove that there is an isometry between the category of Alexandrov sheaves and the category of γ-sheaves both of them endowed with their respective version of the interleaving distance.

Proposition 5.4.18
Let $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, then

(i) $\mathrm{Inter}(F, \beta^{-1}\alpha^{-1}F) = \mathrm{Int}(\gamma^a)$,

(ii) $\mathrm{Inter}(F, \mathrm{R}\alpha_*\beta_*F) = \mathrm{Int}(\gamma^a)$.

Proof
(i) We first prove that $\mathrm{Inter}(F, \beta^{-1}\alpha^{-1}F) = \mathrm{Int}(\gamma^a)$. Let $v \in \mathrm{Int}(\gamma^a)$, we first assume that $F \in (\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ the category of chain complexes of $\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}$-modules and remark that

$$\tau_{v*}\beta^{-1}\alpha^{-1}F \simeq (\alpha \circ \beta \circ \tau_{-v})^{-1}F.$$

Let U and V be open subsets of $\mathbb{V}_{\mathfrak{a}}$. As $\alpha\beta\tau_{-v}(V) = V + \mathrm{Int}(\gamma) + v$, if $U \subset \alpha\beta\tau_{-v}(V)$ then, $U \subset \alpha\beta\tau_{-v}(V) \subset V$. Hence, the restriction morphism $F(V) \to F(U)$ provides a map

$$(\alpha \circ \beta \circ \tau_{-v})^{\dagger}F(U) \simeq \operatorname*{colim}_{U \subset \alpha\beta\tau_{-v}(V)} F(V) \to F(U).$$

Sheafifying, we get a map

$$f\colon \tau_{v*}\beta^{-1}\alpha^{-1}F \to F.$$

Let U be an open subset of $\mathbb{V}_{\mathfrak{a}}$ and let $v \in \mathrm{Int}(\gamma^a)$. Then $U \subset U + \mathrm{Int}(\gamma) + v$. Thus, by definition of the following colimit, there is a morphism

$$F(U + v) \to \operatorname*{colim}_{U \subset V + \mathrm{Int}(\gamma)} F(V).$$

This induces a morphism of sheaves

$$g\colon \tau_{v*}F \to \beta^{-1}\alpha^{-1}F.$$

A straightforward computation shows that

$$\tau_{2v*}\beta^{-1}\alpha^{-1}F \xrightarrow{\tau_{v*}f} F \xrightarrow{g} \tau_{v*}\beta^{-1}\alpha^{-1}F \qquad \text{and} \quad \tau_{2v*}F \xrightarrow{\tau_{v*}f} \beta^{-1}\alpha^{-1}F \xrightarrow{g} \tau_{v*}F$$

are respectively equals to the morphisms $\chi^{\mathfrak{a}}_{2v,0}(\beta^{-1}\alpha^{-1}F)$ and $\chi^{\mathfrak{a}}_{2v,0}(F)$.

If $F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$, the preceding construction also provide an interleaving between F and $\beta^{-1}\alpha^{-1}F$, as the functors τ_{v*}, τ_{2v*}, β^{-1}, α^{-1} are exact.

(ii) Let $v \in \mathrm{Int}(\gamma^a)$ and $\mathcal{I}$ be an homotopically injective resolution of F. For every $U \in \mathfrak{Op}(\mathbb{V}_\mathfrak{a})$,

$$U \subset \alpha^t(U) + v \subset U + v.$$

Hence, we get the morphisms of sheaves

$$f\colon \tau_{v*}\alpha_*\beta_*\mathcal{I} \to \mathcal{I} \qquad\qquad g\colon \tau_{v*}\mathcal{I} \to \alpha_*\beta_*\mathcal{I}.$$

The morphisms f and g defines a v-interleaving between $\mathcal{I}$ and $\alpha_*\beta_*\mathcal{I}$. Hence, between F and $\mathrm{R}\alpha_*\beta_*F$.

□

Corollary 5.4.19

Let $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$, then

(i) $\mathrm{Inter}(F, \beta^{-1}\alpha^{-1}F) = \mathrm{Int}(\gamma^a)$,

(ii) $\mathrm{Inter}(F, \alpha_*\beta_*F) = \mathrm{Int}(\gamma^a)$.

Lemma 5.4.20

Let $v \in \mathrm{Int}(\gamma^a)$ and denote by $d^v_{I_\mathfrak{a}}$ (resp. $d^v_{I_\gamma}$) the interleaving distance on $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$). Then:

(i) The functor $\mathrm{R}\alpha_*$, β^{-1} (resp β_*) sends v-interleavings of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$) to v-interleavings of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$),

(ii) Let F, G in $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ then, $d^v_{I_\gamma}(F,G) = d^v_{I_\mathfrak{a}}(\beta^{-1}F, \beta^{-1}G) = d^v_{I_\mathfrak{a}}(\mathrm{R}\alpha_*F, \mathrm{R}\alpha_*G)$,

(iii) Let F, G in $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ then, $d^v_{I_\mathfrak{a}}(F,G) = d^v_{I_\mathfrak{a}}(\beta^{-1}\alpha^{-1}F, \beta^{-1}\alpha^{-1}G)$,

(iv) Let F, G in $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ then, $d^v_{I_\mathfrak{a}}(F,G) = d^v_{I_\mathfrak{a}}(\mathrm{R}\alpha_*\beta_*F, \mathrm{R}\alpha_*\beta_*G)$.

Proof

(i) This is a consequence of the fact that both morphisms of sites α and β commute with τ_v, combined with the isomorphisms (5.6), (5.7) and Lemma 5.4.1.

(ii) This follows from the fully faithfulness of $\mathrm{R}\alpha_*$ and $beta^{-1}$ and that they commute with τ_{v*}..

(iii) Using the triangular inequalities, we obtain

$$\begin{aligned} d^v_{I_a}(F,G) &\leq d^v_{I_a}(F,\beta^{-1}\alpha^{-1}F) + d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G) + d^v_{I_a}(\beta^{-1}\alpha^{-1}G,G) \\ &\leq d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G) \end{aligned}$$

as $d^v_{I_a}(F,\beta^{-1}\alpha^{-1}F) = d^v_{I_a}(\beta^{-1}\alpha^{-1}G,G) = 0$ by Proposition 5.4.18. Moreover, $\beta^{-1}\alpha^{-1}$ preserves interleaving. Hence,

$$d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G) \leq d^v_{I_a}(F,G)$$

It follows that $d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G) = d^v_{I_a}(F,G)$.

□

Theorem 5.4.21 (Theorem 4.21 - [BP])
Let $v \in \mathrm{Int}(\gamma^a)$, F, $G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_a})$ and denote by $d^v_{I_a}$ (resp. $d^v_{I_\gamma}$) the interleaving distance on $\mathrm{D}(\mathbf{k}_{\mathbb{V}_a})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$). Then :

$$d^v_{I_a}(F,G) = d^v_{I_\gamma}(\beta_*F,\beta_*G).$$

Proof
By Lemma 5.4.20 (i), β_* preserves v-interleavings. Hence, we obtain the inequality

$$d^v_{I_\gamma}(\beta_*F,\beta_*G) \leq d^v_{I_a}(F,G).$$

By Lemma 5.4.20 (ii), $d^v_{I_a}(F,G) = d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G)$ and β^{-1} preserves interleavings. Then,

$$d^v_{I_a}(\beta^{-1}\alpha^{-1}F,\beta^{-1}\alpha^{-1}G) \leq d^v_{I_\gamma}(\alpha^{-1}F,\alpha^{-1}G)$$

Finally, as $\beta_* = \alpha^{-1}$,

$$d^v_{I_a}(F,G) \leq d^v_{I_\gamma}(\beta_*F,\beta_*G).$$

Hence, $d^v_{I_a}(F,G) = d^v_{I_\gamma}(\beta_*F,\beta_*G)$.

□

Let $v \in \mathrm{Int}(\gamma^a)$, We write $d^v_{I_\mu}$ for the interleaving distance associated with v on $\mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$.

Proposition 5.4.22
The functor $\mathrm{R}\phi_{\gamma_*}\colon \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$ and its quasi inverse ϕ^{-1}_γ are isometries i.e.

(i) for every $F,\ G \in \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$, $d^v_{I_\mu}(F,G) = d^v_{I^\gamma}(\mathrm{R}\phi_{\gamma_*}F, \mathrm{R}\phi_{\gamma_*}G)$,

(ii) for every $F,\ G \in \mathrm{D}^b(\mathbf{k}_{\mathbb{V}_\gamma})$, $d^v_{I_\gamma}(F,G) = d^v_{I_\mu}(\phi_\gamma^{-1}F, \phi_\gamma^{-1}G)$.

Proof
First remark that the application ϕ_γ commutes with τ_v and that $\tau_{v*} \simeq \tau_{-v}^{-1}$. Finally, the result follows from Lemma 5.4.4.

□

The following lemma is a generalization of theorem 2.3.15.

Lemma 5.4.23
Let $v \in \mathrm{Int}(\gamma^a)$, $\iota_{\mathfrak{a}} \colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}) \to \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ the functor which sends an object of $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ to the corresponding complex in degree zero, $d^v_{I^{\mathrm{ab}}_{\mathfrak{a}}}$ the interleaving distance on $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ and $d^v_{I_{\mathfrak{a}}}$ the interleaving distance on $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$. Then, for every $F,\ G \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$,

$$d^v_{I_{\mathfrak{a}}}(\iota(F), \iota(G)) = d^v_{I^{\mathrm{ab}}_{\mathfrak{a}}}(F,G).$$

Proof
Clear in view of Remark 5.4.2.

□

Remark 5.4.24
Similar results hold when replacing

1. $\iota_{\mathfrak{a}} \colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}}) \to \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ by $\iota_\gamma \colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma}) \to \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$
(resp. $\iota_\mu \colon \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma}) \to \mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$),
2. $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ by $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ (resp. $\mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$),
3. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ by $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ (resp. $\mathrm{D}^b_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\gamma})$),
4. $d^v_{I^{\mathrm{ab}}_{\mathfrak{a}}}$ by $d^v_{I^{\mathrm{ab}}_\gamma}$ (resp. $d^v_{I^{\mathrm{ab}}_\mu}$),
5. $d^v_{I_{\mathfrak{a}}}$ by $d^v_{I_\gamma}$ (resp. $d^v_{I_\mu}$).

Corollary 5.4.25
Let $v \in \mathrm{Int}(\gamma^a)$, $F,\ G \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ and denote by $d^v_{I_{\mathfrak{a}}}$ (resp. $d^v_{I_\gamma}$) the interleaving distance on $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ (resp. $Mod(\mathbf{k}_{\mathbb{V}_\gamma})$). Then :

$$d^v_{I_{\mathfrak{a}}}(F,G) = d^v_{I_\gamma}(\beta_* F, \beta_* G).$$

5.5 Comparison of the convolution and the interleaving distance

We first review the notion of gauge (also called Minkowski functional) associated to a convex. We refer the reader to [Roc70, Ch. 15] for more details.

In all this subsection $\mathbb{V}$ is a finite dimensional real vector space endowed with a norm $\|\cdot\|$.

Definition 5.5.1
Let K be a non-empty convex of $\mathbb{V}$ such that $0 \in \operatorname{Int} K$. The gauge of K is the function

$$g_K \colon \mathbb{V} \to \mathbb{R},\ x \mapsto \inf\{\lambda > 0 \mid x \in \lambda K\}.$$

The following proposition is classic. We refer the reader to [Roc70, Theorem 15.2] for a proof.

Proposition 5.5.2
Let K be a symmetric closed bounded convex subset of $(\mathbb{V}, \|\cdot\|)$ such that $0 \in \operatorname{Int} K$. Then g_K is a norm on $\mathbb{V}$.

Assume now that $\mathbb{V}$ is endowed with a closed proper convex cone γ with non-empty interior. Let $v \in \operatorname{Int}(\gamma^a)$ and consider the set

$$B_v := (v + \gamma) \cap (-v + \gamma^a).$$

Lemma 5.5.3
The set B_v is a symmetric closed bounded convex subset of $\mathbb{V}$ such that $0 \in \operatorname{Int} B_v$.

Proof
The set B_v is symmetric by construction and is closed and convex as it is the intersection of two closed convex sets. Since $v \in \operatorname{Int}(\gamma)$, there exists $\varepsilon > 0$ such that $B(v, \varepsilon) \subset \gamma$. Hence $B(0, \varepsilon)$ is a subset of $(v + \gamma)$ and $(-v + \gamma^a)$. This implies that $0 \in \operatorname{Int} B_v$.

Assume that B_v is unbounded. Hence, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of B_v such that $\|x_n\| \underset{n\infty}{\longrightarrow} \infty$. The sequence $(x_n/\|x_n\|)_{n\in\mathbb{N}}$ is valued in the the compact $\partial B(0, 1)$. Thus, there is a subsequence $(\nu_n y_n)_{n\in\mathbb{N}}$ of $(x_n/\|x_n\|)_{n\in\mathbb{N}}$ with $|\nu_n| \underset{n\infty}{\longrightarrow} 0$ and such that for every $n \in \mathbb{N}$, $y_n \in B_v$ and y_n converges to a limit y. By [Roc70, Theorem 8.2],

$$y \in \{z \in \mathbb{V} \mid \forall x \in B_v, \forall \lambda \geq 0,\ x + \lambda z \in B_v\}.$$

Since $0 \in B_v$ the half line $\mathbb{R}_{\geq 0}\, y$ is contained in B_v. As B_v is symmetric it follows that $-y \in B_v$, this implies that $\mathbb{R}\, y \subset B_v \subset -v + \gamma$. This is absurd as γ is a proper cone. Hence B_v is bounded.

□

It follows from the previous lemma that the gauge

$$g_{B_v}(x) = \inf\{\lambda > 0 \,|x \in \lambda B_v\}. \tag{5.13}$$

is a norm, the unit ball of which is B_v. From now on, we consider $\mathbb{V}$ equipped with this norm. **In the rest of this section the balls are taken with respects to this norm**.

Moreover, to distinguish between the different notions of interleavings for sheaves, we will say in this section that $F, G \in \mathrm{D}^b(\mathbf{k}_\mathbb{V})$ are *c-isomorphic*, if they are c-interleaved in the sense of definition 2.2.6. Recall that γ-properness is introduced in definition 5.2.5.

Proposition 5.5.4
Let $v \in \mathrm{Int}(\gamma^a)$, $c \in \mathbb{R}_{\geq 0}$ and $F, G \in \mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_\mathbb{V})$. Assume that $\mathrm{supp}(F)$ and $\mathrm{supp}(G)$ are γ-proper subsets of $\mathbb{V}$. Then F and G are $c \cdot v$-interleaved if and only if they are c-isomorphic.

Proof
Let $F, G \in \mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_\mathbb{V})$. Assume that $\mathrm{supp}(F)$ and $\mathrm{supp}(G)$ are γ-proper subsets of $\mathbb{V}$ and that they are $c \cdot v$-interleaved. We set $w = c \cdot v$. Hence, we have the maps

$$\alpha\colon \tau_{w*}F \to G \qquad\qquad \beta\colon \tau_{w*}G \to F$$

such that the diagrams commute

$$\begin{tikzcd} \tau_{2w*}F \arrow[r, "\tau_{w*}\alpha"] \arrow[rr, bend right, "\chi^{\mu}_{0,2a}(F)"'] & \tau_{w*}G \arrow[r, "\tau_{w*}\beta"] & F \end{tikzcd} \qquad \begin{tikzcd} \tau_{2w*}G \arrow[r, "\tau_{w*}\beta"] \arrow[rr, bend right, "\chi^{\mu}_{0,2a}(G)"'] & \tau_{w*}F \arrow[r, "\tau_{w*}\alpha"] & G. \end{tikzcd}$$

Using Lemmas 5.4.3 and 5.4.4, we obtain

$$\begin{tikzcd} \mathbf{k}_{2w+\gamma^a} \underset{np}{*} F \arrow[r, "\mathbf{k}_{2w+\gamma^a} \underset{np}{*} f"] \arrow[rr, bend right, "\chi_{2w,0} \underset{np}{*} F"'] & \mathbf{k}_{w+\gamma^a} \underset{np}{*} G \arrow[r, "\mathbf{k}_{w+\gamma^a} \underset{np}{*} g"] & F. \end{tikzcd}$$

Hence, using the γ-properness of the supports of F and G and that for every $c \geq 0$,

$$\mathbf{k}_{c\cdot v+\gamma^a} \simeq \mathbf{k}_{c\cdot B_v+\gamma^a} \simeq K_c \star \mathbf{k}_{\gamma^a},$$

as well as Proposition 5.2.4, we get

$$\begin{tikzcd} K_{2c} \star F \arrow[r, "K_c \star f"] \arrow[rr, bend right, "\chi_{2c,0} \star F"'] & \mathbf{k}_{B_c} \star G \arrow[r, "g"] & F\,. \end{tikzcd}$$

Similarly we obtain the following commutative diagram

$$\begin{tikzcd} K_{2c} \star G \arrow[r, "K_c \star g"] \arrow[rr, bend right, "\chi_{2c,0} \star G"'] & \mathbf{k}_{B_c} \star F \arrow[r, "f"] & G\,. \end{tikzcd}$$

Hence, F and G are c-isomorphic.

A similar argument proves that if F and G are c-isomorphic then they are $c \cdot v$-interleaved.

□

Corollary 5.5.5
Let $v \in \operatorname{Int}\gamma^a$, $F, G \in \mathrm{D}^b_{\gamma^\circ a}(\mathbf{k}_{\mathbb{V}})$. Assume that $\operatorname{supp}(F)$ and $\operatorname{supp}(G)$ are γ-proper subsets of $\mathbb{V}$. Then

$$d_C(F,G) = d^v_{I^\mu}(F,G)$$

where d_C is the convolution distance associated with the norm g_{B_v} (definition 2.2.8).

CHAPTER 6

Conclusion

In this thesis, we have made precise the analogies that existed at different levels between persistence and sheaf theory. We started in the one-parameter case, where both theories enjoy a decomposition theorem. We first proved that the convolution distance between constructible sheaves over $\mathbb{R}$ is equal to its associated bottleneck distance, which is reminiscent of the isometry theorem for pointwise finite dimensional one-parameter persistence modules. We then constructed two functors between the categories of constructible sheaves on $\mathbb{R}$ and Mayer-Vietoris systems which, given a reasonably tame function $f : X \to \mathbb{R}$, exchange the derived direct image of the constant sheaf on X by f with the collection of level-sets persistence modules associated to f. We proved that these functors induce an isometric correspondence between the sheaf barcodes and the level-sets persistence barcodes. One particular important consequence of this result is that barcodes of sheaves can actually be computed using software solutions already implemented by the TDA community.

A second feature of our work has been to develop connections between multi-parameter persistence modules seen as sheaves for the Alexandrov topology and sheaves for the γ-topology. More precisely, we identified the category of γ-sheaves as the reflexive localization of the category of persistence modules by the full sub-category of ephemeral persistence modules, whose objects are persistence modules whose interleaving distance with the null persistence module is worth 0. We proved that the localization functor preserves the interleaving distance, and then compared the interleaving and convolution distances between γ-sheaves, by proving that under γ-properness assumptions, they coincide.

The guiding principle of our work has been to allow persistence theory and sheaf theory to enjoy best of both worlds' properties : computer friendliness on the one hand, theoretical deepness on the other. We hope that our work on the links between level-sets persistence and sheaves on $\mathbb{R}$ will reignite the interest for machine learning applications of level-sets persistence, by giving it a new theoretical framework. Moreover, it could be interesting to study the implications of our classification of Mayer-Vietoris systems to symplectic topology. Lastly, our identification of the observable category of multi-parameter persistence modules equipped with the interleaving distance, as the full sub-category of the category of sheaves for the usual topology equipped with the convolution distance, whose objects are sheaves with a certain micro-support condition, allow for the use of operations that do not exist in the formalism of persistence modules, such as derived direct image of arbitrary continuous functions. We believe that this observation should be carefully studied in the future, in view to obtain dimension reduction techniques for multi-parameter persistence.

APPENDIX A

Brief introduction to abelian sheaves

Abstract

This appendix aims at introducing the formalism of sheaves of vector-spaces in the derived setting. To read it, we assume basic knowledge in homological algebra, category theory and algebraic topology that can be found in many textbooks, such as [Wei94]. Our exposition will closely follow the one of Kashiwara-Schapira in [KS90]. Although we will expose the main definitions and results of the theory, a complete and detailed presentation is out of the scope of this section. We refer the reader who is interested in learning more about the details of the proofs to chapters 1 and 2 of [KS90].

A.1 Abelian categories and their derived category

A.1.1 Abelian categories and functors

Let $\mathscr{C}$ be a category. We denote by $\mathrm{Obj}(\mathscr{C})$ its class of objects. Given $X, Y \in \mathrm{Obj}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ denotes the collection of morphisms $X \to Y$ in $\mathscr{C}$. Given another category $\mathscr{C}'$, we write $\mathrm{Fun}(\mathscr{C}, \mathscr{C}')$ the category of functors from $\mathscr{C}$ to $\mathscr{C}'$ and natural transformations. Let $\mathfrak{Set}$ be the category of sets.

Definition A.1.1
A *representable functor* is a functor F from a category $\mathscr{C}$ to $\mathfrak{Set}$ such that there exists $X \in \mathscr{C}$ such that F is isomorphic to the functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$.

In this case, X is unique up to isomorphism.

Definition A.1.2
An *additive category* $\mathscr{C}$ is a category $\mathscr{C}$ such that:

1. for any pair (X, Y) of objects of $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ has the structure of an abelian group, and the composition law is bilinear,
2. there exists an object 0 such that $\mathrm{Hom}_{\mathscr{C}}(0, 0) = 0$,
3. for any pair (X, Y) of objects of $\mathscr{C}$ the functor from $\mathscr{C}$ to $\mathfrak{Set}$ defined by
$$W \mapsto \mathrm{Hom}_{\mathscr{C}}(X, W) \times \mathrm{Hom}_{\mathscr{C}}(Y, W) \text{ is representable,}$$
4. for any pair (X, Y) of objects of $\mathscr{C}$ the functor from $\mathscr{C}$ to $\mathfrak{Set}$ defined by
$$W \mapsto \mathrm{Hom}_{\mathscr{C}}(W, X) \times \mathrm{Hom}_{\mathscr{C}}(W, Y) \text{ is representable.}$$

It is a classical result that the representatives in 3. and 4. are isomorphic. We shall denote it by $X \oplus Y$ and call it the *direct sum* of X and Y.

Definition A.1.3
An *additive functor* is a functor $F : \mathscr{C} \to \mathscr{C}'$ between additive categories such that, for any pair (X, Y) of objects of $\mathscr{C}$, the induced map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is a group homomorphism.

From now on, $\mathscr{C}$ is an additive category.

Definition A.1.4
Let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

1. If the functor:
$$Z \mapsto \{u \in \mathrm{Hom}_{\mathscr{C}}(Z, X) \mid f \circ u = 0\}$$
is representable, its representative is called the *kernel of* f, and noted $\mathrm{Ker} f$.
2. Similarly if the functor:
$$Z \mapsto \{u \in \mathrm{Hom}_{\mathscr{C}}(Y, Z) \mid u \circ f = 0\}$$
is representable, its representative is called the *cokernel of* f, and noted $\mathrm{Coker} f$.
3. Assume that f has a kernel. Then there exists a natural morphism $\alpha : \mathrm{Ker} f \to X$. If α has a cokernel, we shall call it the *coimage of* f, noted $\mathrm{Coim} f$.

4. Similarly if f has a cokernel, there exists a natural morphism $\gamma : Y \to \mathrm{Coker} f$. If γ has a kernel, we shall call it the *image of* f, noted $\mathrm{Im} f$.

It follows from the universal properties of kernel and cokernel that if $\mathrm{Coim} f$ and $\mathrm{Im} f$ exist, there exists a natural morphism $\mathrm{Coim} f \to \mathrm{Im} f$.

Definition A.1.5
An additive category $\mathscr{C}$ is an *abelian category* if it satisfies:

1. Every morphism $f : X \to Y$ admits a kernel and a cokernel.
2. The canonical morphism $\mathrm{Coim} f \to \mathrm{Im} f$ is an isomorphism.

Example A.1.6
The main example of an abelian category one shall have in mind is, given a commutative ring R with unit, the category of left R-modules and R-linear maps noted $\mathrm{Mod}(R)$. A result of Freyd and Mitchell [Mit65] asserts that any abelian category can be embedded as a full sub-category of $\mathrm{Mod}(R)$, for a certain ring R.

We now assume $\mathscr{C}$ to be an abelian category.

Definition A.1.7
Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms in $\mathscr{C}$. It is said to be an *exact sequence* if :

1. $g \circ f = 0$,
2. the natural morphism $\mathrm{Im} f \to \mathrm{Ker} g$ is an isomorphism.

Definition A.1.8
A *left-exact functor* (resp. *right-exact functor*) is a functor $F : \mathscr{C} \to \mathscr{C}'$ between abelian categories such that for any exact sequence in $\mathscr{C}$ of the form:

$$0 \to X \to X' \to X''$$

(resp. $X \to X' \to X'' \to 0$), the sequence

$$0 \to F(X) \to F(X') \to F(X'')$$

(resp. $F(X) \to F(X') \to F(X'') \to 0$) is exact.

If F is both left and right exact, it is said to be an *exact functor*.

Example A.1.9
For any $X \in \mathscr{C}$, the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ is left-exact, but not exact in general.

Definition A.1.10
Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. Then $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are *adjoint functors* if there exists a natural isomorphism of functors:

$$\mathrm{Hom}_{\mathscr{B}}(F-, -) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(-, G-).$$

This is equivalent, for any $X \in \mathscr{A}$ and $Y \in \mathscr{B}$, to the data of a bijection $\Phi_{X,Y} : \mathrm{Hom}_{\mathscr{B}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(X, G(Y))$ which satisfies for any other X', Y' objects of $\mathscr{A}$ and $\mathscr{B}$ and morphisms $f : Y \to Y'$ and $g : X' \to X$ that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(F(X), Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\mathrm{Hom}(F(g),f)"'] & \mathrm{Hom}_{\mathscr{A}}(X, G(Y)) \arrow[d, "\mathrm{Hom}(g,G(f))"] \\
\mathrm{Hom}_{\mathscr{B}}(F(X'), Y') \arrow[r, "\Phi_{X',Y'}"'] & \mathrm{Hom}_{\mathscr{A}}(X', G(Y)')
\end{tikzcd} .$$

In this situation, we will say that F is left-adjoint to G (resp. G is right-adjoint to F), and we will denote:

$$F : \mathscr{A} \rightleftarrows \mathscr{B} : G .$$

Example A.1.11
For R a commutative ring with unit, and M a left R-module, we have the following adjunction

$$- \otimes_R M : \mathrm{Mod}(R) \rightleftarrows \mathrm{Mod}(R) : \mathrm{Hom}_{\mathrm{Mod}(R)}(M, -) .$$

Proposition A.1.12
Let $\mathscr{C}$ and $\mathscr{C}'$ be two abelian categories, and a pair of adjoint functors:

$$F : \mathscr{C} \rightleftarrows \mathscr{C}' : G .$$

Then F is right-exact, and G is left-exact.

A.1.2 Categories of complexes

Let $\mathscr{C}$ be an additive category.

Definition A.1.13
A *chain complex* X in $\mathscr{C}$ consists of the data $(X^n, d_X^n)_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$:

$$X^n \in \mathscr{C},\ d_X^n \in \mathrm{Hom}_{\mathscr{C}}(X^n, X^{n+1}),\ \text{and } d_X^{n+1} \circ d_X^n = 0.$$

The collection $d_X = (d_X^n)_{n\in\mathbb{Z}}$ is called the *differential of the complex* X. A morphism of complexes $f : X \to Y$ is a collection of morphisms $(f^n : X^n \to Y^n)_{n\in\mathbb{Z}}$ such that for any n:$d_Y^n \circ f = f^{n+1} \circ d_X^n$.

We denote by $C(\mathscr{C})$ the category thus obtained of complexes of $\mathscr{C}$. It is an abelian category whenever $\mathscr{C}$ is abelian.

Definition A.1.14
A complex $X \in C(\mathscr{C})$ is said to be a *bounded complex* (resp. *bounded below complex*, resp. *bounded above complex*) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$).

The full subcategory of $C(\mathscr{C})$ consisting of bounded complexes (resp. bounded below complexes resp. bounded above complexes), is noted $C^b(\mathscr{C})$ (resp. $C^+(\mathscr{C})$, resp. $C^-(\mathscr{C})$).

Definition A.1.15
Let $k \in \mathbb{Z}$ and $X \in C(\mathscr{C})$. One defines a new complex $X[k]$ by setting :

$$\begin{cases} (X[k])^n = X^{n+k} \\ d^n_{X[k]} = (-1)^k d_X^{n+k} \end{cases}$$

Given a morphism $f : X \to Y$ in $C(\mathscr{C})$, one defines $f[k] : X[k] \to Y[k]$ by setting

$$f[k]^n = f^{n+k}.$$

The functor $\cdot[k]$ from $C(\mathscr{C})$ to $C(\mathscr{C})$ is called the *shift functor* of degree k.

Definition A.1.16
A morphism $f : X \to$ in $C(\mathcal{C})$ is said to be *homotopic to zero* if there exists a sequence of morphisms $(s^n : X^n \to Y^{n-1})_{n\in\mathbb{Z}}$ in $\mathscr{C}$ such that

$$f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n.$$

Given an other morphism $g : X \to Y$, one says that f is **homotopic** to g if $f - g$ is homotopic to zero.

We denote by $\mathrm{Ht}(X, Y)$ the subgroup of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ consisting of morphisms homotopic to zero. Then the composition in $\mathscr{C}$ send $\mathrm{Ht}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z)$ and $\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Ht}(Y, Z)$ into $\mathrm{Ht}(X, Z)$. This ensures that the following is well-defined :

Definition A.1.17
The *homotopy category* of $\mathscr{C}$, noted $K(\mathscr{C})$, is defined by:

$$\begin{cases} \mathrm{Obj}(K(\mathscr{C})) = \mathrm{Obj}(C(\mathscr{C})), \\ \mathrm{Hom}_{K(\mathscr{C})}(X, Y) = \mathrm{Hom}_{C(\mathscr{C})}(X, Y)/\mathrm{Ht}(X, Y). \end{cases}$$

One similarly defines the categories $K^b(\mathscr{C}), K^+(\mathscr{C}), K^-(\mathscr{C})$, which are full subcategories of $K(\mathscr{C})$. Note that $K(\mathscr{C})$ is not abelian in the general case, even when $\mathscr{C}$ is. All of these categories are additive, although not abelian in the general case.

Until the end of the background section, we assume that $\mathscr{C}$ is abelian.

Definition A.1.18
For $X \in C(\mathscr{C})$, one sets :

$$Z^k(X) = \mathrm{Ker} d_X^k, B^k(X) = \mathrm{Im} d_X^{k-1},$$

$$\mathrm{H}^k(X) = \mathrm{Coker}(B^k(X) \to Z^k(X)) = \text{"}Z^k(X)/B^k(X)\text{"}.$$

One calls $\mathrm{H}^k(X)$ the *k-th cohomology of X*.

Note that the mapping $X \mapsto \mathrm{H}^k(X)$ is functorial, and defines an additive functor from $C(\mathscr{C})$ to $\mathscr{C}$. We also have the relation $\mathrm{H}^k = \mathrm{H}^0 \circ \cdot[k]$. If $f : X \to Y$ is homotopic to zero, then $\mathrm{H}^k(f) = 0$ for all k. This shows that H^k induces a well defined additive functor from $K(\mathscr{C})$ to $\mathscr{C}$.

Proposition A.1.19 (Long exact sequence in cohomology)
Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $C(\mathscr{C})$. Then there exists a canonical long exact sequence in $\mathscr{C}$:

$$\cdots \longrightarrow \mathrm{H}^n(X) \longrightarrow \mathrm{H}^n(Y) \longrightarrow \mathrm{H}^n(Z) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(X) \longrightarrow \cdots$$

and for any other exact sequence $0 \to X' \to Y' \to Z' \to 0$ that fits in a commutative diagram in $C(\mathscr{C})$:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}$$

all the diagrams:

$$\begin{tikzcd}
\mathrm{H}^n(Z) \arrow[r, "\delta^n"] \arrow[d] & \mathrm{H}^{n+1}(X) \arrow[d] \\
\mathrm{H}^n(Z') \arrow[r, "\delta'^n"] & \mathrm{H}^{n+1}(X')
\end{tikzcd}$$

commute.

Definition A.1.20
Let $f : X \to Y$. The *mapping cone* of f, noted $M(f)$, is the object of $C(\mathscr{C})$ defined by:

$$\begin{cases} M(f)^n = X^{n+1} \oplus Y^n, \\ d^n_{M(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}. \end{cases}$$

Recall that $d^n_{X[1]} = -d^{n+1}_X$.

We introduce the morphisms $\alpha(f) : Y \to M(f)$ and $\beta(f) : M(f) \to X[1]$ by:

$$\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix},$$

$$\beta(f)^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}.$$

Proposition A.1.21
The sequence:

$$0 \longrightarrow Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \longrightarrow 0$$

is exact in $C(\mathscr{C})$.

Lemma A.1.22
For any $f : X \to Y$ in $C(\mathscr{C})$, there exists $\phi : X[1] \to M(\alpha(f))$ in $C(\mathscr{C})$ such that:

(i) ϕ is an isomorphism in $K(\mathscr{C})$,

(ii) The following diagram commutes in $K(\mathscr{C})$:

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{id}_Y"] & M(f) \arrow[r, "\beta(f)"] \arrow[d, "\mathrm{id}_{M(f)}"] & X[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"] & Y[1] \arrow[d, "\mathrm{id}_{Y[1]}"] \\
Y \arrow[r, "\alpha(f)"'] & M(f) \arrow[r, "\alpha(\alpha(f))"'] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"'] & Y[1]
\end{tikzcd}$$

A **triangle** in $K(\mathscr{C})$ is a sequence of morphisms $X \to Y \to Z \to X[1]$, and a morphism of triangle is a commutative diagram in $K(\mathscr{C})$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\phi"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "\phi[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

Definition A.1.23
A *distinguished triangle* in $K(\mathscr{C})$ is a triangle isomorphic to a triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \ ,$$

for some f in $C(\mathscr{C})$.

For short, we will sometimes denote a distinguished triangle by:

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} \ .$$

Proposition A.1.24 ($K(\mathscr{C})$ is a triangulated category)
The collection of distinguished triangles in $K(\mathscr{C})$ satisfies the following properties :

(TR 0) It is closed under isomorphism in $K(\mathscr{C})$.

(TR 1) For any $X \in K(\mathscr{C})$, $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$ is a distinguished triangle.

(TR 2) Any $f : X \to Y$ in $K(\mathscr{C})$ can be embedded in a distinguished triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$.

(TR 3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if
$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(TR 4) Given two distinguished triangles

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] \text{ and } X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow X'[1],$$

any commutative diagram in $K(\mathscr{C})$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

can be embedded in a morphism of triangle (not necessarily unique).

(TR 5) (octahedral axiom) Given two distinguished triangles

$$X \longrightarrow Y \longrightarrow Z' \xrightarrow{+1} \quad \text{and} \quad Y \longrightarrow Z \longrightarrow X' \xrightarrow{+1} \ ,$$

and a morphism $X \to Z$, there exists a unique Y' (up to isomorphism in $K(\mathscr{C})$) together with the dotted morphisms such that the following diagram commutes in $K(\mathscr{C})$, and any sequence of three aligned morphisms is a distinguished triangle :

$$\begin{tikzcd}
X \arrow[r] \arrow[drr] & Y \arrow[rr] \arrow[dr] & & Z' \arrow[r, "+1"] \arrow[dd, dotted] & {} \\
& & Z \arrow[dr, dotted] \arrow[ddr] & & \\
& & & Y' \arrow[r, dotted, "+1"] \arrow[dr, dotted, "+1"] \arrow[d, dotted] & {} \\
& & & X' \arrow[d, dotted, "+1"] \arrow[dr, "+1"] & {} \\
& & & {} & {}
\end{tikzcd}$$

DEFINITION A.1.25
A *triangulated category* is an additive category $\mathscr{D}$ endowed with an additive endofunctor $\cdot[1] : \mathscr{D} \to \mathscr{D}$ and a class of triangles Δ which satisfies axioms (TR 0)-(TR 5). The triangles in Δ are called the distinguished triangles of $\mathscr{D}$.

A.1.3 Derived category and derived functors

Let $\mathscr{C}$ be a category, and let S be a family of morphisms in $\mathscr{C}$.

DEFINITION A.1.26
One says that S is a *multiplicative system* if it satisfies the following :

(S1) For any $X \in \mathscr{C}$, $\mathrm{id}_X \in S$.

(S2) For any pair (f, g) of S such that the composition $g \circ f$ exists, $g \circ f \in S$.

(S3) Any diagram:

$$\begin{tikzcd}
& Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

with $g \in S$, can be completed to a commutative diagam:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "h"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

with $h \in S$. Same thing with all the arrows reversed.

Definition A.1.27
The category $\mathscr{C}_S$, called the *localization* of $\mathscr{C}$ by S is defined by:

$$\mathrm{Obj}(\mathscr{C}_S) = \mathrm{Obj}(\mathscr{C}),$$

for any pair (X, Y) of $\mathrm{Obj}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}_S}(X, Y) = \{(X', s, f) \mid X' \in \mathrm{Obj}(\mathscr{C}), s : X' \to X, f : X' \to Y, s \in S\}/\mathscr{R}$$

where $\mathscr{R}$ is the equivalence relation:

$$(X', s, f)\mathscr{R}(X'', t, g)$$

if and only if there exists a commutative diagram

$$\begin{tikzcd}
 & X & \\
X' \arrow[ur, "s"] \arrow[dr, "f"'] & X''' \arrow[l] \arrow[r] \arrow[u, "u"] & X'' \arrow[ul, "t"'] \arrow[dl, "g"] \\
 & Y &
\end{tikzcd}$$

with $u \in S$. The composition of $(X', s, f) \in \mathrm{Hom}_{\mathscr{C}_S}(X, Y)$ and $(Y', t, g) \in \mathrm{Hom}_{\mathscr{C}_S}(Y, Z)$ is defined as follows, using (S3) to find a commutative diagram:

$$\begin{tikzcd}
 & & X'' \arrow[dl, "t'"'] \arrow[dr, "h"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

with $t' \in S$, and we set:

$$(Y', t, g) \circ (X', s, f) = (X'', s \circ t', g \circ h).$$

Using the axioms (S1)-(S4), one can prove that $\mathscr{C}_S$ is indeed a category.

Definition A.1.28
We define the *localization functor* with respect to S, noted by Q_S, as the functor from $\mathscr{C}$ to $\mathscr{C}_S$ given by :

$$\begin{cases} Q_S(X) = X, \text{ for } X \in \mathscr{C}, \\ Q_S(f) = (X, \mathrm{id}_X, f) \text{ for } f \in \mathrm{Hom}_{\mathscr{C}_S}(X, Y). \end{cases}$$

Proposition A.1.29

1. For $s \in S$, $Q_S(s)$ is an isomorphism in $\mathscr{C}_S$.
2. Let $\mathscr{C}'$ be another category and $F : \mathscr{C} \to \mathscr{C}'$ a functor that sends morphisms of S to isomorphisms in $\mathscr{C}'$. Then F factors uniquely through Q_S.

We let $\mathscr{C}$ be an abelian category. We shall apply the preceeding construction to the triangulated category $K(\mathscr{C})$. A morphism $f : X \to Y$ in $C(\mathscr{C})$ is said to be a *quasi-isomorphism* (qiso for short) if $\mathrm{H}^n(f)$ is an isomorphism for all n. The definition generalizes readily for morphisms in $K(\mathcal{C})$. One can prove easily that:

$$S = \{f \in \mathrm{Hom}_{K(\mathscr{C})}(X, Y) \mid f \text{ is a qiso}\}$$

is a multiplicative system in $K(\mathscr{C})$.

Definition A.1.30
The *derived category* of $\mathscr{C}$ is the category $K(\mathscr{C})_S$.

By replacing $K(\mathscr{C})$ with $K^b(\mathscr{C})$, (resp. $K^+(\mathscr{C})$, resp. $K^-(\mathscr{C})$), we define the derived categories $\mathrm{D}^b(\mathscr{C})$ (resp. $\mathrm{D}^+(\mathscr{C})$, resp. $\mathrm{D}^-(\mathscr{C})$). Observe that by proposition A.1.29, the functors $\mathrm{H}^n : K(\mathscr{C}) \to \mathscr{C}$ factors uniquely through $\mathrm{D}(\mathscr{C})$.

Proposition A.1.31

1. The category $\mathrm{D}^b(\mathscr{C})$ (resp. $\mathrm{D}^+(\mathscr{C})$, resp. $\mathrm{D}^-(\mathscr{C})$)) is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X such that $\mathrm{H}^n(X) = 0$ for $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$).
2. The composition of functors $\mathscr{C} \to K(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ is fully faithfull. Therefore, $\mathscr{C}$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects such that $\mathrm{H}^n(X) = 0$ for $n \neq 0$.

Proposition A.1.32
Let X be an object of $\mathrm{D}(\mathscr{C})$. Then $X \simeq 0$ if and only if $\mathrm{H}^n(X) \simeq 0$ for all n. Similarly, for $f : X \to Y \in C(\mathscr{C})$. The image of f in $\mathrm{D}(\mathscr{C})$ is 0 iff there exists a qiso g such that $f \circ g$ is homotopic to 0.

Proposition A.1.33
Let $\mathscr{I}$ be a full additive subcategory of $\mathscr{C}$ such that:for any object X of $\mathscr{C}$, there exists an object X' of $\mathscr{I}$ and an exact sequence $0 \to X \to X'$.

Then:

1. for any $X \in K^+(\mathscr{C})$, there exists $X' \in K^+(\mathscr{I})$ and a quasi-isomorphism $X \to X'$.

2. Let S' be the family of quasi-isomorphisms $K^+(\mathscr{I})$. Then the canonical functor:
$$K^+(\mathscr{I})_{S'} \longrightarrow \mathrm{D}^+(\mathscr{C})$$
is an equivalence.

3. Assume in addition that there exists $d \geq 0$ such that, for any exact sequence in $\mathscr{C}$, $X^0 \to X^1 \to .. \to X^d \to 0$ with X^j objects of $\mathscr{I}$ for $j < d$, we have X^d in $\mathscr{I}$. Then for any object X of $K^b(\mathscr{C})$ there exists an object X' in $K^b(\mathscr{I})$ and a quasi-isomorphism $X \to X'$.

Definition A.1.34

Let I and P be objects of the abelian category $\mathscr{C}$. We say that I is *injective* (resp. P is *projective*) if the functor $\mathrm{Hom}_{\mathscr{C}}(-, I)$ (resp. $\mathrm{Hom}_{\mathscr{C}}(P, -)$) is exact.

We say that $\mathscr{C}$ has *enough injectives* (resp. *enough projectives*) if for any object X of $\mathscr{C}$, there exists an injective object I (resp. a projective object P) and a monomorphism $X \to I$ (resp. an epimorphism $P \to X$).

Proposition A.1.35

Assume $\mathscr{C}$ has enough injectives. Let $\mathscr{I}$ be the full subcategory of $\mathscr{C}$ consisting of injective objects and let S' be the family of qiso of $\mathscr{I}$. Then the following composition:
$$K^+(\mathscr{I}) \longrightarrow K^+(\mathscr{I})_{S'} \longrightarrow \mathrm{D}^+(\mathscr{C})$$
is an equivalence of categories.

Let $F : \mathscr{C} \to \mathscr{C}'$ be an additive functor between abelian categories. We shall denote by Q (resp. Q') the natural localization functor $K^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C})$ (resp. $K^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}')$). We denote by $K^+(F)$ the functor induced by F from $K^+(\mathscr{C})$ to $K^+(\mathscr{C}')$.

Definition A.1.36

The *right-derived functor* of F, if it exists, is the functor noted $\mathrm{R}F : \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ defined as the left Kan extension of the functor $Q' \circ K^+(F)$ along Q.

Equivalently, it is a triangulated functor $T : \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ together with a natural transformation of functors $s : Q' \circ K^+(F) \Rightarrow T \circ Q$, which satisfies that for any other functor of triangulated categories $G : \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$, the morphism:
$$\mathrm{Hom}(T, G) \xrightarrow{s} \mathrm{Hom}(Q \circ K^+(F), G \circ Q)$$
is an isomorphism.

$$\begin{tikzcd} & \mathrm{D}^+(\mathscr{C}) \arrow[dr, dotted, "T=RF"] \arrow[d, Rightarrow, "s"] & \\ K^+(\mathscr{C}) \arrow[ur, "Q"] \arrow[rr, "Q'\circ K^+(F)"'] & {} & \mathrm{D}^+(\mathscr{C}') \end{tikzcd}$$

The functor $\mathrm{H}^n \circ \mathrm{R}F$ will be noted $\mathrm{R}^n F$ and is called the *n-th derived functor of F*.

Example A.1.37

If $F : \mathscr{C} \to \mathscr{C}'$ is an exact functor, then $Q' \circ K^+(F)$ sends quasi-isomorphisms to isomorphisms in $\mathrm{D}^+(\mathscr{C}')$. Therefore, if one denotes by S the family of quasi-isomorphisms of $K^+(\mathscr{C})$, $Q' \circ K^+(F)$ factorizes uniquely through $K^+(\mathscr{C})_S \simeq \mathrm{D}^+(\mathscr{C})$. One can prove that this factorization defines the right-derived functor of F, and can be computed by applying F degree-wise to an object X of $\mathrm{D}^+(\mathscr{C})$.

In the following, we will give a method to compute explicitly a derived functor for a left-exact functor $F : \mathscr{C} \to \mathscr{C}'$.

Definition A.1.38

A full subcategory $\mathscr{I}$ of $\mathscr{C}$ is called *F-injective* if :

1. for any object X of $\mathscr{C}$, there exists an object X' of $\mathscr{I}$ and an exact sequence $0 \to X \to X'$,
2. if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathscr{C}$, and if X' and X'' are objects of $\mathscr{I}$, then so is X,
3. if $0 \to X' \to X \to X'' \to 0$ is an exact sequence of objects of $\mathscr{I}$, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

We assume that F admits a F-injective subcategory $\mathscr{I}$. We still denote by S' the family of quasi-isomorphisms of $\mathscr{I}$. Observe that $K^+(F)$ sends elements of S' to quasi-isomorphisms. Hence, it factors through $K^+(\mathscr{I})_{S'}$ which is equivalent to $\mathrm{D}^+(\mathscr{C})$ by proposition A.1.35.

$$\begin{tikzcd} K^+(\mathscr{I}) \arrow[rr, "K^+(F)"] \arrow[dr, "Q"'] & & K^+(\mathscr{C}') \arrow[r, "Q'"] & \mathrm{D}^+(\mathscr{C}') \\ & K^+(\mathscr{I})_{S'} \simeq \mathrm{D}^+(\mathscr{C}) \arrow[ur, dotted] \arrow[urr, "\mathrm{R}F"'] & & \end{tikzcd}$$

Proposition A.1.39

The functor $\mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ constructed above is the right-derived functor of F.

Remark A.1.40
Therefore, given a left-exact functor $F : \mathscr{C} \to \mathscr{C}'$ which admits a F-injective category $\mathscr{I}$, we can compute $\mathrm{R}F(X)$ for X an object of $\mathrm{D}^+(\mathscr{C})$ as follows. First, find an object X' of $K^+(\mathscr{I})$ such that there exists a quasi-isomorphism $X \to X'$. Then, apply F degree-wise on X', that is, compute $K^+(F)(X)$. Finally, $\mathrm{R}F(X) \simeq K^+F(X)$ in $\mathrm{D}^+(\mathscr{C}')$.

Example A.1.41
The fundamental example of derived functor we will encounter is the situation of a left exact functor $F : \mathscr{C} \to \mathscr{C}'$ where $\mathscr{C}$ has enough injectives. In this case, it is easy to see that the full subcategory of $\mathscr{C}$ consisting of injective objects is F-injective (since any short exact sequence of injective objects splits). Therefore, every left-exact functor from $\mathscr{C}$ to an abelian category admits a right-derived functor.

Proposition A.1.42
Let $F : \mathscr{C} \to \mathscr{C}'$ and $F' : \mathscr{C}' \to \mathscr{C}''$ be two left-exact functors between abelian categories. Assume that there exist an F-injective category $\mathscr{I}$ and an F'-injective category $\mathscr{I}'$ satisfying $F(\mathrm{Obj}(\mathscr{I})) \subset \mathrm{Obj}(\mathscr{I}')$. Then $F' \circ F$ admits a right-derived functor, and:

$$\mathrm{R}(F' \circ F) \simeq \mathrm{R}F' \circ \mathrm{R}F.$$

We now briefly explain how to derive left-exact bi-functors.

Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be three abelian categories. Let $G : \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be a functor. G will be refered to as a bi-functor. Given two complexes X in $C^+(\mathscr{C})$ and X' in $C^+(\mathscr{C}')$, $G(X, X')$ has the structure of a bi-complex. There is a classical construction to associate functorially a simple complex to $G(X, X')$, noted by $s(G(X, X'))$. In particular, one has

$$s(G(X, X'))^k = \bigoplus_{k=n+m} G(X^n, X'^m).$$

Observe that the direct sum is finite since X and X' are bounded from below.

One can similarly define the notion of right-derived bi-functor as the right Kan extension (if it exists) of

$$Q'' \circ K^+(s(G)) = K^+(\mathscr{C}) \times K^+(\mathscr{C}') \longrightarrow K^+(\mathscr{C}''') \longrightarrow D^+(\mathscr{C}'')$$

along the product of the localization functors $Q \times Q' : K^+(\mathscr{C}) \times K^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}) \times \mathrm{D}^+(\mathscr{C}')$.

G will be said to be left-exact if it is with respect to each of its variables.

Definition A.1.43
For $\mathscr{I}$ (resp. $\mathscr{I}'$) a full additive subcategory of $\mathscr{C}$ (resp. $\mathscr{C}'$) we say that $(\mathscr{I}, \mathscr{I}')$ is G**-injective** if for any object $X \in \mathrm{Obj}(\mathscr{I})$ and any $X' \in \mathrm{Obj}(\mathscr{I}')$, $\mathscr{I}$ is $G(-, X')$-injective and $\mathscr{I}'$ is $G(X, -)$-injective.

Theorem A.1.44
Suppose $G : \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ is a left exact bifunctor such that there exists two full subcategories $\mathscr{I} \subset \mathscr{C}, \mathscr{I}' \subset \mathscr{C}'$) such that $(\mathscr{I}, \mathscr{I}')$ is G-injective. Then G admits a right derived functor. Moreover, it can be computed as follows. For $X \in \mathrm{Obj}(D^+(\mathscr{C}))$ and $X' \in \mathrm{Obj}(\mathrm{D}^+(\mathscr{C}'))$, there exists $Y \in \mathrm{Obj}(K^+(\mathscr{I}))$ (resp. $Y' \in \mathrm{Obj}(K^+(\mathscr{I}'))$) and a quasi-isomorphism in $K^+(\mathscr{C})$ (resp. $K^+(\mathscr{C}')$) $X \to Y$ (resp. $X' \to Y'$). Then, there is an isomorphism in $\mathrm{D}^+(\mathscr{C}''')$:

$$\mathrm{R}G(X, X') \simeq s(G(Y, Y')).$$

A.2 Sheaves of vector-spaces

A.2.1 Sheaves and operations

Let $\mathbf{k}$ be a field, X be a topological space and $\mathrm{OP}(X)$ be the collection of its open subsets. The *category of open sets* of X, noted by $\mathfrak{OP}(X)$ is given by:

$$\begin{cases} \mathrm{Obj}(\mathfrak{OP}(X)) = \mathrm{OP}(X) \\ \mathrm{Hom}_{\mathfrak{OP}(X)}(V, U) = \{*\} \text{ if } \mathrm{V} \subset U, \ \emptyset \text{ otherwise.} \end{cases}$$

Definition A.2.1
A *presheaf* of $\mathbf{k}$-vector spaces on X is a functor from $\mathfrak{OP}(X)^{\mathrm{op}}$, the opposite category of open sets of X, to $\mathrm{Mod}(\mathbf{k})$, the category of vector spaces over $\mathbf{k}$. Morphisms of presheaves are natural transformations of functors. The category obtained is written $\mathfrak{PSH}(X)$.

It is easy to observe that $\mathfrak{PSH}(X)$ is an abelian category. Given U an open set of X, an element $s \in F(U)$ is called a *section of F on U*. If one considers another open set $V \subset U$, one defines the *restriction of s to V*, defined by $s_{|V} := F(U \to V)(s)$. For $x \in X$, one sets:

$$F_x = \varinjlim_{W \ni x} F(W)$$

where W ranges over the open sets of X containing x. The vector space F_x is called the *stalk of F at x*. For $x \in U$, the image of $s \in F(U)$ in F_x is called the *germ of s at x*, written s_x. The *restriction of F* to the topological space U whose open sets are the open sets of X contained in U, is noted $F|_U$.

Definition A.2.2
A pre-sheaf F of $\mathbf{k}$-vector spaces on X is a *sheaf* if it satisfies the following conditions:

(S1) for any open set $U \subset X$, any open covering $U = \cup_{i\in I} U_i$, any section $s \in F(U)$, $s|_{U_i} = 0$ for all i implies $s = 0$

(S2) for any open set $U \subset X$, any open covering $U = \cup_{i\in I} U_i$, any family $s_i \in F(U_i)$, if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all pairs (i, j), there exists $s \in F(U)$ such that $s|_{U_i} = s_i$ for all i.

Note that (S1)-(S2) are equivalent to saying that for any open covering $U = \cup_{i\in I} U_i$ stable by finite intersection, the morphism $F(U) \to \varprojlim_i F(U_i)$ is an isomorphism. In particular if F is a sheaf, $F|_U$ is a sheaf on U.

The *support of a sheaf* F, noted $\mathrm{supp}(F)$ is defined as the set complement of the union of the open sets $U \subset X$ such that $F|_U = 0$. One also defines the *support of a section* $s \in F(U)$ by $\mathrm{supp}(s) = \{x \in X \mid s_x \neq 0\}$.

We denote by $\mathrm{Mod}(\mathbf{k}_X)$ the full subcategory of $\mathfrak{PSH}(X)$ whose objects consist of sheaves. One defines the functor $\Gamma(U; -) : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k})$ given by $\Gamma(U; F) = F(U)$.

Proposition A.2.3
Let $\phi : F \to G$ be a morphism in $\mathrm{Mod}(\mathbf{k}_X)$. Then ϕ is an isomorphism if and only if $\phi_x : F_x \to G_x$ is an isomorphism for all $x \in X$.

Proposition A.2.4
The inclusion functor $\iota : \mathrm{Mod}(\mathbf{k}_X) \to \mathfrak{PSH}(X)$ admits a right adjoint functor noted $(-)^+ : \mathfrak{PSH}(X) \to \mathrm{Mod}(\mathbf{k}_X)$, $F \mapsto F^+$. F^+ is called the *sheaf associated to* F, or the *sheaffification* of F.

For $M \in \mathrm{Mod}(\mathbf{k})$, one defines the *constant sheaf on X with stalk M*, noted by M_X as the sheafification of the presheaf $U \mapsto M$. One can prove that M_X is the sheaf of locally constant functions on X with value in M, that is, $M_X(U)$ is isomorphic to the $\mathbf{k}$-vector space of locally constant functions on U with value in M.

More generally, a sheaf F on X is *locally constant* if for every $x \in X$ there exists an open neighborhood U of x in X such that $F|_U$ is constant.

Proposition A.2.5
The sheafification functor has the following properties.

1. For $F \in \mathfrak{PSH}(X)$ and $x \in X$, $F_x \simeq (F^+)_x$.

2. Let $\phi : F \to G$ be a morphism in $\mathrm{Mod}(\mathbf{k}_X)$. Then, ϕ has a kernel and a cokernel in $\mathrm{Mod}(\mathbf{k}_X)$. More precisely, the presheaf $U \mapsto \mathrm{Ker}(\phi_U : F(U) \to G(U))$ is a sheaf, and it is the kernel of ϕ. The sheafification of the presheaf $U \mapsto \mathrm{Coker}(\phi_U : F(U) \to G(U))$ is the cokernel of ϕ.

One should beware that the cokernel of a morphism ϕ in $\mathrm{Mod}(\mathbf{k}_X)$ should not equal to the cokernel computed in $\mathfrak{PSH}(X)$ (ie the cokernel of $\iota \circ \phi$).

Corollary A.2.6
The category $\mathrm{Mod}(\mathbf{k}_X)$ is abelian.

Definition A.2.7
Let F and G be two sheaves of $\mathbf{k}$-vector spaces on X. The *sheaf of solutions of F in G*, noted $\mathscr{H}om_{\mathbf{k}_X}(F, G)$ is defined by $U \mapsto \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_U)}(F|_U, G|_U)$ (one can easily verify that it is indeed a sheaf).

The *tensor product of F and G*, noted $F \otimes_{\mathbf{k}_X} G$, is defined as the sheafification of the presheaf $U \mapsto F(U) \otimes_{\mathbf{k}} G(U)$.

Proposition A.2.8
Let F, G, H be sheaves of $\mathbf{k}$-vector spaces on X. There is an isomorphism (functorial in F, G, H):

$$\mathscr{H}om_{\mathbf{k}_X}(H \otimes_{\mathbf{k}_X} F, G) \xrightarrow{\sim} \mathscr{H}om_{\mathbf{k}_X}(F, \mathscr{H}om_{\mathbf{k}_X}(H, G))$$

This result proves that the functors $\mathscr{H}om_{\mathbf{k}_X}(F, -)$ and $\mathscr{H}om_{\mathbf{k}_X}(-, F)$ are left-exact. Also, the functors $F \otimes_{\mathbf{k}_X} - \simeq - \otimes_{\mathbf{k}_X} F$ are right-exact. Since we are considering sheaves of vector spaces, the tensor product of sheaves is in fact exact. Note that this would not be true if $\mathbf{k}$ were only a ring and not a field.

Let $u : Y \to X$ be a continuous map between topological spaces.

Definition A.2.9

1. Let G be a sheaf on Y. The *direct image* of G by u, noted u_*G, is the sheaf on X defined by:

$$U \mapsto u_*G(U) := G(u^{-1}(U)).$$

2. Let F be a sheaf on X. The *inverse image* of F by u, noted $u^{-1}F$, is the sheaf associated to the presheaf:

$$V \mapsto \varinjlim_{U \supset u(V)} F(U),$$

where U ranges over the open neighborhoods of $u(V)$ in X.

It is clear that u_* and u^{-1} induce functors:

$$u_* : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y),$$

$$u^{-1} : \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{Mod}(\mathbf{k}_X).$$

One fundamental example of direct image functor is given by $\Gamma(X; -)$. Indeed, if one denotes by $a_X : X \to \{*\}$ the constant map from X to the set with one element, then there is an isomorphism of functors $a_{X*} \simeq \Gamma(X; -)$.

Proposition A.2.10
Let F be a sheaf on X, G be a sheaf on Y, and $x \in X$. Then:

$$(u^{-1}G)_x \simeq G_{u(x)}.$$

Therefore, u^{-1} is an exact functor.

Proposition A.2.11
There is an adjunction :

$$u^{-1} : \mathrm{Mod}(\mathbf{k}_Y) \rightleftarrows \mathrm{Mod}(\mathbf{k}_X) : u_* \, .$$

For Z a closed subset of X, equipped with the induced topology, we denote by $j : Z \to X$ the inclusion. We define for $F \in \mathrm{Mod}(\mathbf{k}_X)$:

$$F|_Z = j^{-1}F \in \mathrm{Mod}(\mathbf{k}_Z),$$

$$\Gamma(Z; F) = \Gamma(Z; j^{-1}F),$$

$$F_Z = j_* j^{-1}F \in \mathrm{Mod}(\mathbf{k}_X).$$

Thanks to the adjunction $j^{-1} \longleftrightarrow j_*$, there is a natural morphism $F \to F_Z$. Moreover:

$$\begin{cases} F_Z|_Z = F|_Z, \\ F_Z|_{X\setminus Z} = 0. \end{cases}$$

In particular, $(F_Z)_x = F_x$ for $x \in Z$ and $(F_Z)_x = 0$ if $x \notin Z$.

If $U \subset X$ is open, one defines

$$F_U := \mathrm{Ker}(F \to F_{X\setminus U}).$$

We can generalize this construction to locally closed sets, that is, sets of the form $Z = U \cap A$ with U an open set of X, and A a closed set. Then, we define $F_Z = (F_U)_A$. One can show that this definition does not depend on the decomposition $Z = U \cap A$.

Example A.2.12
Let $M \in \mathrm{Mod}(\mathbf{k})$ and $Z \subset X$ be a locally closed subset. The *constant sheaf on Z with stalk M* is defined by $M_Z := (M_X)_Z$.

Proposition A.2.13
Let F be a sheaf on X. Let V be an open subset of X and T be a closed subset of X. Then, for any open subset U of X, one has:

$$F_V(U) = \begin{cases} F(U) \text{ if } U \subset V, \\ 0 \text{ otherwise.} \end{cases}$$

Moreover, F_T is the sheaffification of the presheaf:

$$U \mapsto \begin{cases} F(U) \text{ if } U \cap T \neq \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

Definition A.2.14
For M a real analytic manifold, and $F \in \mathrm{Obj}(\mathrm{Mod}(\mathbf{k}_M))$, F is said to be **weakly $\mathbb{R}$-constructible** if there exists a locally finite sub-analytic stratification of $M = \sqcup_\alpha M_\alpha$, such that for each stratum M_α, the restriction $F_{|M_\alpha}$ is locally constant. If, in addition, the stalks F_x are of finite dimension for every $x \in M$, we say that F is $\mathbb{R}$-*constructible*. We will often say constructible instead of $\mathbb{R}$-constructible, since it is the only notion of constructibility we use.

Proposition A.2.15
Let Z be a locally closed subset of X and F be a sheaf on X. Then:

1. the functor $(-)_Z$ is exact,
2. let Z' be another locally closed subset of X, then:$(F_Z)_{Z'} = F_{Z \cap Z'}$,
3. let Z' be a closed subset of X, then there is an exact sequence:
$$0 \to F_{Z \setminus Z'} \to F_Z \to F_{Z'} \to 0,$$
4. let Z_1, Z_2 be two closed subsets of X, then there is an exact sequence:
$$0 \longrightarrow F_{Z_1 \cup Z_2} \xrightarrow{\alpha} F_{Z_1} \oplus F_{Z_2} \xrightarrow{\beta} F_{Z_1 \cap Z_2} \longrightarrow 0,$$
where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, -\beta_2)$, with α_i and β_i being the natural morphisms $F_{Z_1 \cup Z_2} \to F_{Z_i}$ and $F_{Z_i} \to F_{Z_1 \cap Z_2}$,
5. let U_1, U_2 be two open subsets of X, then there is an exact sequence:
$$0 \longrightarrow F_{U_1 \cap U_2} \xrightarrow{\alpha} F_{U_1} \oplus F_{U_2} \xrightarrow{\beta} F_{U_1 \cup U_2} \longrightarrow 0,$$
where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, -\beta_2)$, with α_i and β_i being the natural morphisms $F_{U_1 \cap U_2} \to F_{U_i}$ and $F_{U_i} \to F_{U_1 \cup U_2}$.

Corollary A.2.16
Let Z be a closed subset of the topological space X, and F be a sheaf on X. There is an exact sequence:

$$0 \longrightarrow F_{X\setminus Z} \longrightarrow F \longrightarrow F_Z \longrightarrow 0$$

Assume X is locally compact. A continuous map $u : Y \to X$ (Y being not necessarily locally compact) is *proper* if it is closed (it sends closed sets to closed sets), its fibers are relatively Hausdorff (two points in a fiber have disjoints open neighborhoods in Y) and compact.

Definition A.2.17
Let G be a sheaf on Y. We define the *direct image with proper supports* of G by u, noted $u_!G$, by:

$$\Gamma(U; u_!G) = \{s \in \Gamma(U; G) \mid u : \mathrm{supp}(s) \to X \text{ is proper}\}.$$

One can prove that $u_!G$ is indeed a sheaf, and in fact, a subsheaf of u_*G. Moreover, the functor $u_!$ is left-exact. A natural question to ask is whether $u_!$ is an adjoint functor. We will see in the sequel that we need to go at the level of the derived category to construct a right-adjoint to $\mathrm{R}u_!$.

One defines the *global sections with compact support* of the sheaf F on X by:

$$\Gamma_c(X; F) = a_{X!}F = \{s \in \Gamma(U; F) \mid \mathrm{supp}(s) \text{ is compact and Hausdorff}\}.$$

One of the crucial properties of proper direct image is the following :

Proposition A.2.18
Let X and Y be locally compact spaces (in particular, they are Hausdorff), $u : Y \to X$ a continuous map, and G a sheaf on Y. Then for $x \in X$, the canonical morphism:

$$(u_!G)_x \to \Gamma_c(u^{-1}(x), G|_{u^{-1}(x)})$$

is an isomorphism.

A.2.2 Sheaves in the derived setting

Let $u : Y \to X$ be a continuous map between locally compact topological spaces, and U be an open subset of Y. So far, we have constructed the bifunctors $\mathscr{H}om_{\mathbf{k}_X}(-,-)$ and $-\otimes_{\mathbf{k}_X}-$ which are respectively left-exact and exact. We also have constructed the left exact functors $\Gamma(U; -)$, $\Gamma_c(U; -)$, u_* and $u_!$, and the exact functor u^{-1}.

Proposition A.2.19
The category $\mathrm{Mod}(\mathbf{k}_X)$ has enough injectives.

Therefore, all the aforementioned functors admit a right-derived functor. For the exact ones, this can be computed by example A.1.37. For the left-exact ones, one can use the procedure explained in example A.1.41. For short, we will denote by $\mathrm{D}(\mathbf{k}_X)$ the derived category of $\mathrm{Mod}(\mathbf{k}_X)$, and $\mathrm{D}^+(\mathbf{k}_X)$, $\mathrm{D}^-(\mathbf{k}_X)$, $\mathrm{D}^b(\mathbf{k}_X)$ its bounded counterparts. We will still call the objects of these categories by "sheaf".

Proposition A.2.20
Let $u : Y \to X$ be a morphism of topologcal spaces. The functor u_* sends injective sheaves of $\mathrm{Mod}(\mathbf{k}_Y)$ to injective sheaves of $\mathrm{Mod}(\mathbf{k}_X)$.

Combined with proposition A.1.42, this gives:

Corollary A.2.21
Let $u : Y \to X$ and $v : X \to Z$ be two morphisms of topological spaces. Then there is an isomorphism of functors:

$$\mathrm{R}(v \circ u)_* \simeq \mathrm{R}v_* \circ \mathrm{R}u_*.$$

Let $u, v : Y \to X$ be two continous maps between topological spaces, and F be an object of $\mathrm{D}^+(\mathbf{k}_X)$.

The spaces $\mathrm{R}^n\Gamma(X; F)$ will be called the *sheaf cohomology* groups of F. These groups generalize and unify the construction of many cohomology theories in algebraic topology. Let us give an illustration of this motto in the following.

The adjunction $u^{-1} \leftrightarrow u_*$ gives rise to a natural transformation $\mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)} \Rightarrow u_* \circ u^{-1}$ which induces a natural morphism:

$$u^\# : \mathrm{R}\Gamma(X; F) \longrightarrow \mathrm{R}\Gamma(Y; u^{-1}F) \quad \big(\simeq \mathrm{R}\Gamma(X; u_*u^{-1}F)\big)$$

Theorem A.2.22 (Homotopy invariance of sheaf cohomology)
Assume that u and v are homotopic. Assume also that F is locally constant. Then there exists an isomorphism $\theta : \mathrm{R}\Gamma(Y; u^{-1}F) \xrightarrow{\sim} \mathrm{R}\Gamma(Y; v^{-1}F)$ such that $\theta \circ u^\# = v$.

$$\begin{tikzcd} & \mathrm{R}\Gamma(X;F) \arrow[dl, "u^\#"'] \arrow[dr, "v^\#"] & \\ \mathrm{R}\Gamma(Y;u^{-1}F) \arrow[rr, "\sim", "\theta"'] & & \mathrm{R}\Gamma(Y;v^{-1}F) \end{tikzcd}$$

We denote by $\mathrm{H}^n_{\mathrm{sing}}(X;\mathbf{k})$ the n-th singular cohomology group of X with coefficients in $\mathbf{k}$. The proof of the following result can be found in [Mus].

Theorem A.2.23 (singular vs. sheaf cohomology)
Assume X is locally contractible and paracompact. Then there are natural isomorphisms:

$$\mathrm{R}^n\Gamma(X;\mathbf{k}_X) \simeq \mathrm{H}^n_{\mathrm{sing}}(X;\mathbf{k}),$$

for all $n \geq 0$.

The formalism of the derived category also allows us to construct a right-adjoint to the functor $\mathrm{R}u_!$. We say that the functor $u_!$ has *finite cohomological dimension* if there exists $r \geq 0$ such that for all $F \in \mathrm{Mod}(\mathbf{k}_Y)$ and $j > r$, $\mathrm{R}^j u_! F = 0$.

Theorem A.2.24 (Poincaré-Verdier duality)
Let $u : Y \to X$ be a continuous map of locally compact spaces, such that $u_!$ has finite cohomological dimension. Then there exists a triangulated functor $u^! : \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y)$ which is right-adjoint to $\mathrm{R}u_!$.

$$\mathrm{R}u_! : \mathrm{D}^+(\mathbf{k}_Y) \rightleftarrows \mathrm{D}^+(\mathbf{k}_X) : u^! \,.$$

Index of definitions

Index of notations

Bibliography

[Bar94] S. A. Barannikov. The framed Morse complex and its invariants. Singularities and bifurcations, *volume 21 of* Adv. Soviet Math., 1994. *Cited on page 7*

[BBK18] Håvard Bakke Bjerkevik, Magnus Bakke Botnan, and Michael Kerber. Computing the interleaving distance is np-hard. *Foundations of Computational Mathematics*, 2018. *Cited (2) on pages 12 and 50*

[BCB18] Magnus Bakke Botnan and William Crawley-Boevey. Decomposition of persistence modules. available at `https://arxiv.org/abs/1811.08946`, 2018. *Cited (4) on pages 20, 29, 100, and 109*

[Ber19] Nicolas Berkouk. Stable resolutions of multi-parameter persistence modules. preprint available at : `https://arxiv.org/abs/1901.09824`, 2019. *Cited (3) on pages 14, 38, and 45*

[BG18] Nicolas Berkouk and Grégory Ginot. A derived isometry theorem for constructible sheaves on $\mathbb{R}$. available at `https://arxiv.org/abs/1805.09694`, 2018. *Cited (7) on pages 15, 51, 72, 79, 81, 85, and 88*

[BGO19] Nicolas Berkouk, Gregory Ginot, and Steve Oudot. Level-sets persistence and sheaf theory. Preprint available at `https://arxiv.org/abs/1907.09759`, 2019. *Cited (4) on pages 15, 97, 108, and 140*

[BHS18] Lev Buhovsky, Vincent Humilière, and Sobhan Seyfaddini. The action spectrum and c^0 symplectic topology. Preprint available at `https://arxiv.org/abs/1808.09790`, 2018. *Cited (2) on pages 10 and 147*

[Bje16] Håvard Bakke Bjerkevik. Stability of higher-dimensional interval decomposable persistence modules. 2016. *Cited (9) on pages 11, 23, 29, 50, 53, 74, 80, 83, and 144*

[BL15] Ulrich Bauer and Michael Lesnick. Induced matchings and the algebraic stability of persistence barcodes. *Journal of Computational Geometry*, 2015. *Cited on page 9*

[BL17] Magnus Bakke Botnan and Michael Lesnick. Algebraic stability of zigzag persistence modules. arXiv preprint arXiv:1604.00655, 2017. *Cited (3) on pages 25, 50, and 100*

[Bot15] Magnus Botnan. *Applications and Generalizations of the Algebraic Stability Theorem.* PhD thesis, Norwegian University of Science and Technology, Trondheim, 2015. *Cited on page 10*

[BP] Nicolas Berkouk and François Petit. Ephemeral persistence modules and distance comparison. Preprint available at `https://arxiv.org/abs/1902.09933`. *Cited on page 175*

[BPar] Nicolas Berkouk and François Petit. Ephemeral persistence modules and distance comparison. *Algebraic and Geometric Topology*, To appear. Preprint available at `arXiv:1902.09933`. *Cited (2) on pages 16 and 149*

[CB12] William Crawley-Boevey. Decomposition of pointwise finite-dimensional persistence modules. 2012. *Cited (3) on pages 9, 24, and 54*

[CCBdS16] Frédéric Chazal, William Crawley-Boevey, and Vin de Silva. The observable structure of persistence modules. *Homology, Homotopy and Applications*, 18(2):247–265, 2016. *Cited (3) on pages 151, 161, and 162*

[CCBS16] Fredéric Chazal, William Crawley-Boevey, and Vin De Silva. The observable structure of persistence modules. available at `https://arxiv.org/pdf/1405.5644.pdf`, 2016. *Cited (2) on pages 15 and 89*

[CdSGO16] Frederic Chazal, Vin de Silva, Marc Glisse, and Steve Oudot. *The Structure and Stability of Persistence Modules*. Springer, 2016. *Cited (7) on pages 24, 79, 92, 150, 151, 160, and 161*

[CdSKM19] Gunnar Carlsson, Vin de Silva, Sara Kališnik, and Dmitriy Morozov. Parametrized homology via zigzag persistence. *Algebraic & Geometric Topology*, 19(2):657–700, 2019. *Cited on page 100*

[CdSM09] Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov. Zigzag persistent homology and real-valued functions. In *Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry*, SCG '09, pages 247–256, New York, NY, USA, 2009. ACM. *Cited on page 100*

[CO17] Jérémy Cochoy and Steve Oudot. Decomposition of exact pfd persistence bimodules. available at `arXiv:1605.09726`, 2017. *Cited (3) on pages 10, 29, and 100*

[Cra95] Sjoerd Crans. Quillen closed model structure for sheaves. *Journal of Pure and Applied Algebra*, 101, 1995. *Cited on page 93*

[CSEH05] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. *Discrete and Computational Geometry*, 2005. *Cited on page 8*

[CSG+09] F. Chazal, D. C. Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot. Proximity of persistence modules and their diagrams. In *Proceedings of the 25th Annual Symposium on Computational Geometry,*, 2009. *Cited on page 8*

[Cur14] Justin Curry. *Sheaves, Cosheaves and Applications*. PhD thesis, 2014. *Cited (6) on pages 13, 52, 53, 65, 98, and 150*

[CZ05] Gunnar Carlsson and Afra Zomorodian. Computing persistent homology. *Discrete and Computational Geometry*, 2005. *Cited on page 7*

[CZ09] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete and Computational Geometry*, 2009. *Cited (4) on pages 11, 32, 33, and 150*

[CZCG05] Gunnar Carlsson, Afra Zomorodian, Anne Collins, and Leonidas Guibas. Persistence barcodes for shapes. *Eurographics Symposium on Geometry Processing*, 2005. *Cited on page 8*

[dMS18] Vin de Silva, Elizabeth Munch, and Anastasios Stefanou. Theory of interleavings on categories with a flow. *Theory and Applications of Categories*, 33(21):583–607, 2018. *Cited (3) on pages 20, 22, and 150*

[DUC19] Meryll Dindin, Yuhei Umeda, and Frederic Chazal. Topological data analysis for arrhythmia detection through modular neural networks. Preprint available at `https://arxiv.org/abs/1906.05795`, 2019. *Cited on page 9*

[Dug] Daniel Dugger. A primer on homotopy colimit. Preprint available at `https://pages.uoregon.edu/ddugger/`. *Cited on page 93*

[EH10] Herbert Edelsbrunner and John L. Harer. *Computational Topology: An Introduction*. American Mathematical Society, 2010. *Cited on page 52*

[Gab62] Pierre Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, 79:323–448, 1962. *Cited (2) on pages 160 and 161*

[GS14] Stéphane Guillermou and Pierre Schapira. *Microlocal Theory of Sheaves and Tamarkin's Non Displaceability Theorem*, pages 43–85. Springer International Publishing, 2014. *Cited (3) on pages 153, 154, and 167*

[Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986. *Cited on page 131*

[Kra2] Henning Krause. Localization theory for triangulated categories. In Thorsten Holm, Peter Jorgensen, and Raphael Rouquier, editors, *Triangulated Categories*, pages 161–235. Cambridge University Press, 2. *Cited on page 164*

[Kra14] Henning Krause. Krull-schmidt categories and projective covers. available at `https://arxiv.org/pdf/1410.2822.pdf`, 2014. *Cited on page 19*

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. *Cited (9) on pages 13, 34, 36, 54, 135, 150, 152, 153, and 183*

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and Sheaves*, volume 332 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, 2006. *Cited (4) on pages 155, 157, 163, and 164*

[KS18a] Masaki Kashiwara and Pierre Schapira. Persistent homology and microlocal sheaf theory. *Journal of Applied and Computational Topology*, 2018. *Cited (20) on pages 13, 15, 17, 35, 36, 37, 38, 52, 53, 54, 87, 88, 98, 141, 150, 151,*

[KS18b] Masaki Kashiwara and Pierre Schapira. Piecewise linear sheaves. arXiv preprint at `https://arxiv.org/abs/1805.00349`, 2018. *Cited (5) on pages 13, 33, 54, 98, and 150*

[Kuh09] Harold W. Kuhn. The hungarian method for the assignment problem. *50 Years of Integer Programming 1958-2008*, 2009. *Cited on page 86*

[Les12] Michael Lesnick. *Multidimensional Interleavings and Applications to Topological Inference*. PhD thesis, Stanford University, 2012. *Cited (4) on pages 12, 24, 32, and 150*

[Les15] Michael Lesnick. The theory of the interleaving distance on multidimensional persistence modules. *Foundations of Computational Mathematics*, 2015. *Cited (2) on pages 25 and 150*

[LW15] Michael Lesnick and Matthew Wright. Interactive visualization of 2-d persistence modules. available at arXiv:1512.00180, 2015. *Cited (2) on pages 12 and 150*

[Mik90] Prest Mike. Wild representation type and undecidability. *Communications in Algebra*, 1990. *Cited on page 11*

[Mil19] Ezra Miller. Modules over posets : commutative and homological algebra. *arXiv preprint arXiv:1908.09750*, 2019. *Cited on page 33*

[Mil20] Ezra Miller. Essential graded algebra overpolynomial rings with real exponents. Preprint available at `https://arxiv.org/pdf/2008.03819.pdf`, 2020. *Cited on page 172*

[Mit65] B. Mitchell. *Theory of Categories.* Pure and Applied Mathematics - Academic Press. Academic Press, 1965. *Cited on page 185*

[Mor] Dmitriy Morozov. Dionysus 2 : a software to compute persistent homology. available at `https://www.mrzv.org/software/dionysus2/`. *Cited on page 147*

[Mus] Mircea Mustaţă. Singular cohomology as sheaf cohomology with constant coefficients. Preprint available at `http://www-personal.umich.edu/~mmustata/SingSheafcoho.pdf`. *Cited on page 204*

[NHH+15] T. Nakamura, Y. Hiraoka, A. Hirata, E.G. Escolar, and Y. Nishiura. Persistent homology and many-body atomic structure for medium-range order in the glass. *Nanotechnology*, 2015. *Cited on page 9*

[NZL20] Gregory Naitzat, Andrey Zhitnikov, and Lek-Heng Lim. Topology of deep neural networks. Preprint available at `https://arxiv.org/abs/2004.06093`, 2020. *Cited on page 10*

[Oud15] Steve Y. Oudot. *Persistence Theory: From Quiver Representations to Data Analysis.* American Mathematical Society, 2015. *Cited (2) on pages 52 and 150*

[Pro] Stacks Project. Resolution functors. `https://stacks.math.columbia.edu/tag/013U`. *Cited (2) on pages 39 and 46*

[PS76] Klaus-Peter Podewski and Karsten Steffens. lnjective choice functions for countable families. *Journal of Combinatorial Theory*, 21:40–46, 1976. *Cited on page 82*

[PS15] Leonid Polterovich and Egor Shelukhin. Autonomous hamiltonian flows, hofer's geometry and persistence modules. *Selecta Mathematica*, 2015. *Cited on page 10*

[RCK+17] Abbas H. Rizvi, Pablo G. Camara, Elena K. Kandror, Thomas J. Roberts, Ira Schieren, Tom Maniatis, and Raul Rabadan. Single-cell topological rna-seq analysis reveals insights into cellular differentiation and development. *Nature Biotechnonology*, 2017. *Cited on page 9*

[Roc70] Ralph Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970. *Cited on page 177*

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019. *Cited (2) on pages 103 and 160*

[Vip20] Oliver Vipond. Local equivalence of metrics for multiparameter persistence modules. available at `https://arxiv.org/pdf/2004.11926.pdf`, 2020. *Cited on page 12*

[Vit92] Claude Viterbo. Symplectic topology as the geometry of generating functions. *Mathematische Annalen*, 1992. *Cited on page 10*

[Wei94] Charles A. Weibel. *An Introduction To Homological Algebra*. Cambridge University Press, 1994. *Cited on page 183*

[Zom01] Afra Joze Zomorodian. *Computing and Comprehending Topology: Persistence and Hierarchical Morse Complexes*. PhD thesis, 2001. *Cited on page 7*

ECOLE DOCTORALE

Titre : Persistance et Faisceaux: de la Théorie aux Applications.

Mots clés : Persistance, Faisceau, Distances d'Entrelacements, Analyse de Données Topologique

Résumé : L'analyse de données topologique est un domaine de recherche récent qui vise à employer les techniques de la topologie algébrique pour concevoir des descripteurs de jeux de données. Pour être utiles en pratique, ces descripteurs doivent être calculables, et posséder une notion de métrique, afin de pouvoir exprimer leur stabilité vis à vis du bruit inhérent à toutes données réelles. La théorie de la persistance a été élaborée au début des années 2000 comme un premier cadre théorique permettant de définir de tels descripteurs - les désormais bien connus code-barres. Bien que très bien adaptée à un contexte informatique, la théorie de la persistance possède certaines limitations théoriques. Dans ce manuscript, nous établissons des liens explicites entre la théorie dérivée des faisceaux munie de la distance de convolution (d'après Kashiwara-Schapira) et la théorie de la persistance.
Nous commençons par montrer un théorème d'isométrie dérivée pour les faisceaux constructibles sur $\mathbb{R}$, c'est à dire, nous exprimons la distance de convolution comme une distance d'appariement entre les code-barres gradués de ces faisceaux. Cela nous permet de conclure dans ce cadre que la distance de convolution est fermée, ainsi que la classe des faisceaux constructibles sur $\mathbb{R}$ munie de la distance de convolution forme un espace topologique localement connexe par arcs.
Nous observons ensuite que la collection des modules de persistance *zig-zag* associée à une fonction à valeurs réelle possède une structure supplémentaire, que nous appelons *systèmes de Mayer-Vietoris*. Sous des hypothèses de finitude, nous classifions tous les systèmes de Mayer-Vietoris. Cela nous permet d'établir une correspondence fonctorielle et isométrique entre la catégorie dérivée des faisceaux constructibles sur $\mathbb{R}$ équipée de la distance de convolution, et la catégorie des systèmes de Mayer-Vietoris fortement finis munie de la distance d'entrelacement. Nous en déduisons une méthode de calcul des code-barres gradués faisceautiques à partir de programmes informatiques déjà implémentés par la communauté de la persistance.
Nous terminons par donner une définition purement faisceautique de la notion de module de persistance éphémère. Nous établissons que la catégorie observable des modules de persistance (le quotient de la catégorie des modules de persistance par la sous-catégorie des modules de persistance éphémères) est équivalente à la catégorie bien connue des γ-faisceaux.

Title : Persistence and Sheaves: from Theory to Applications

Keywords : Persistence, Sheaf Theory, Interleaving Distances, Topological Data Analysis

Abstract : Topological data analysis is a recent field of research aiming at using techniques coming from algebraic topology to define descriptors of datasets. To be useful in practice, these descriptors must be computable, and coming with a notion of metric, in order to express their stability properties with respect to the noise that always comes with real world data. Persistence theory was elaborated in the early 2000's as a first theoretical setting to define such descriptors - the now famous so-called barcodes. However very well suited to be implemented in a computer, persistence theory has certain limitations. In this manuscript, we establish explicit links between the theory of derived sheaves equipped with the convolution distance (after Kashiwara-Schapira) and persistence theory.
We start by showing a derived isometry theorem for constructible sheaves over $\mathbb{R}$, that is, we express the convolution distance between two sheaves as a matching distance between their graded barcodes. This enables us to conclude in this setting that the convolution distance is closed, and that the collection of constructible sheaves over $\mathbb{R}$ equipped with the convolution distance is locally path-connected.
Then, we observe that the collection of zig-zag/level sets persistence modules associated to a real valued function carry extra structure, which we call *Mayer-Vietoris systems*. We classify all Mayer-Vietoris systems under finiteness assumptions. This allows us to establish a functorial isometric correspondence between the derived category of constructible sheaves over $\mathbb{R}$ equipped with the convolution distance, and the category of strongly pfd Mayer-Vietoris systems endowed with the interleaving distance. We deduce from this result a way to compute barcodes of sheaves from already existing software.
Finally, we give a purely sheaf theoretic definition of the notion of ephemeral persistence module. We prove that the observable category of persistence modules (the quotient category of persistence modules by the sub-category of ephemeral ones) is equivalent to the well-known category of γ-sheaves.

Institut Polytechnique de Paris
91120 Palaiseau, France