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Manh Tu NGUYEN

Higher Hida theory on unitary group GU(2,1)

Théorie de Hida supérieur pour le groupe unitaire GU(2,1)

Après l'avis des rapporteurs :

Fabrizio Andreatta	Professeur Université de Milan, Milan
Toby Gee	Professeur, Imperial College, Londres

Devant le jury composé de :

Fabrizio Andreatta	Professeur Université de Milan, Milan	Rapporteur
Andrian Iovita	Professeur, Concordia University, Montréal	Examinateur
Benoit Stroh	Professeur, UPMC Paris 6, Paris	Examinateur
Sandra Rozensztajn	Maître de conférences, ENS de Lyon, Lyon	Examinatrice
Vincent Pilloni	Chargé de recherche-HDR, ENS de Lyon, Lyon	Directeur de thèse

Résumé. Le travaux récent de Calegari et Geraghty[CG12] ont enlevé les restrictions de la méthode originale de Taylor-Wiles, cela nous permet d'attacquer les conjectures de modularité plus générales. Leur méthode se fonde sur deux autres conjectures, l'une est reliée au problème d'attacher les représentations galoisiennes aux classes de torsion dans le groupe de cohomologie de la varieté de Shimura sous entendue et l'autre à la dégrée de concentration de ces groupes de cohomologie localisés. La premiere conjecture a été adressée dans une grande généralité par Peter Scholze [Sch13] mais la seconde reste évasive. Récemment, pour la cohomologie cohérente, Vincent Pilloni a dévelopé une version de la théorie de Hida pour les groupes de cohomologie supérieurs qui construit une interpolation p-adique du complex de cohomologie en question. Comme une application importante, nous pouvons contourner la second conjecture au dessus et en effet dans un travail commun récent [BCGP18], ils ont montré que toutes les variétés abéliennes sur un corps totalement réel est potentiellement modulaire. Dans cette thèse, nous adaptons l'argument de Vincent Pilloni pour contruire un complex qui interpole les classes de cohomologie supérieurs de la varieté de Picard. Ces résultats servent comme le premier pas vers la modularité potentielle des variétés abéliennes de dimension 3 qui proviennent des Jacobiens de la courbe de Picard.

Abstract. In their breakthrough work [CG12], Calegari and Geraghty have shown how to bypass some serious restrictions of the original method by Taylor-Wiles, thus allowing us to attack more general modularity conjectures and related questions. Their method hinges on two conjectures, one is related to the problem of attaching Galois representations to torsion classes in the cohomology of Shimura varieties and the other to the requirement that these cohomology groups, localised at an appropriate ideal are non zero only in a certain range. The first conjecture is addressed in a great generality by Peter Scholze [Sch13], but the second remains elusive. Recently, for coherent cohomology, inspired by the classical Hida theory, Vincent Pilloni has proposed a method consisting of p-adically interpolating the entire complex of coherent sheaves of automorphic forms on the Siegel threefold. This serves as a way to get around the second conjecture above and plays a crucial role in a recent work [BCGP18], where they show that abelian surfaces over a totally real field are potentially modular. In this thesis, we adapt the argument in [Pil18] to construct a Hida complex interpolating classes in higher cohomology groups of the Picard modular surface. In a future work, we hope to use this to obtain some similar modularity results for abelian three-folds arising as Jacobians of some Picard curves in the spirit of [BCGP18].

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Contents

Part 1. General introduction

1. Notations

For any field K, let K^s denote its separable closure. When it makes sense, we denote by $\mathcal{O}_K$ the ring of integers of K and by $\mathfrak{m}_K$ the maximal ideal $\mathcal{O}_K$. Recall that we have the absolute Galois group G_K of K, i.e the group of all automorphisms of K^s that fix K. This group comes equipped with the profinite toplolgy, i.e the weakest topology making the projections $G_K \to Gal(K'/K)$ continuous, where $K \subset K' \subset K^s$ runs through finite extensions of K. In particular, G_K is compact, Hausdorff, totally disconnected with a basis of open subsets given by subgroups of form $Gal(K^s/K')$ with K' being some finite separable extension of K. Throughout, we will be only interested in the case when K is a number field or an extension of $\mathbb{Q}_p$ for some prime number p.

As it happens in practice, the most fruitful way to study these groups is to examine their representations.

Let R be any separated, commutative topological ring, then a Galois representation of K with coefficient in R of dimension n is a continuous group morphism: $\rho : G_K \to GL_n(R)$ (the topology of $GL_n(R)$ is induced by the product topology on $M_{n^2}(R) \times R$ via the embedding $GL_n(R) \to M_{n^2}(R) \times R$ $(M \mapsto M \times det(M))$). We often consider the cases where:

(1) $R = \mathbb{C}$ with its usual topology. Galois representations with complex coefficient are often called Artin representations, they have finite images. More over these are conjugated to representations with coefficients in $\bar{\mathbb{Q}}$.
(2) R is an extension of $\mathbb{Q}_p$ for some prime number p with p-adic topology, in this case we say p-adic Galois representation.
(3) R is a finite field of characteristic p.

That being said, we need to find a source of Galois representations.

1.1. Geometry and Galois representation.

There is a natural source of Galois representation from geometry. For example:

(1) Let $\mathbb{G}_m$ be the multiplicative group over $\mathbb{Z}$ (i.e $\mathbb{G}_m := Spec(\mathbb{Z}[T, T^{-1}])$), then we can define its l-adic Tate module to be $T_l(\mathbb{G}_m) := \varprojlim \mu_{l^n}(K^s)$ where $\mu_{l^n}(K^s)$ is the group of $l^n - th$ roots of unity in K^s. This is a free $\mathbb{Z}_l$-module of rank 1. The group G_K acts on each $\mu_{l^n}(K^s)$, thus on $T_l(\mathbb{G}_m)$. As a result we obtain a one dimensional representation $\rho : G_K \to GL_1(\mathbb{Z}_l)$ or equivalently

a character $\chi_K : G_K \to \mathbb{Z}_l^\times$. We call this the cyclotomic character. For any l-adic Galois representation ρ, we often denote by $\rho(n)$ the tensor product of ρ and χ^n for $n \in \mathbb{Z}$.

(2) Let A be an abelian variety of dimension g over K. Then we can also take its l-adic Tate module $T_l(A) := \varprojlim A[l^n](K^s)$ where $A[l^n](K^s)$ is the group of points of order l^n in K^s. It is well known that $A[l^n](K^s) \cong (\mathbb{Z}/l^n\mathbb{Z})^{2g}$. Similarly, the absolute Galois group G_K acts on $A[l^n](K^s)$, and thus on $T_l(A)$ and we obtain a Galois representation of dimension 2.

$$\rho_{A,l} : G_K \to GL_{2g}(\mathbb{Z}_l)$$

(3) More generally, if X is any proper smooth algebraic variety over K, then we can consider its étale cohomology groups $H^i(X_{K^s}, \mathbb{Q}_l)$. By constructions they carry an action of G_K and this forms a rich source of Galois representations.

Let K be a number field. Given a representation $\rho : G_K \to GL_n(\bar{\mathbb{Q}}_p)$. For a finite place v, the local Galois group G_{K_v} can be identified as a subgroup of G_K and we obtain a representation:

$$\rho_v : G_{K_v} \to G_K \to GL_n(\bar{\mathbb{Q}}_p)$$

We then say that $\rho : G_K \to GL_n(R)$ is unramified at v if the image of the inertia group $I_v \subset G_{K_v}$ is trivial. In other words, the representation ρ_v factors through $G_{K_v}/I_v \cong G_{\mathbb{F}_q}$ where $\mathbb{F}_q$ is the residue field at v. As a result, the unramified local Galois representation ρ_v is completely determined by the image $\rho_v(Frob_v)$ where $Frob_v$ is a preimage of the generator of $G_{\mathbb{F}_q}$.

Generally speaking, Galois representations coming from geometry enjoy the following property/definition below.

Definition 1.1.1. Let L be a finite extension of $\mathbb{Q}_p$ and $\rho : G_K \to GL_n(L)$ be a Galois representation. We say that:

(1) ρ is geometric if there exists a proper smooth variety X over K such that ρ is a subquotient to $H^i_{et}(X_{K^s}, \mathbb{Q}_p) \otimes L$ (up to a cyclotomic twist).

(2) ρ is weakly geometric if ρ is unramified at places outside a finite set of places and $\rho_v := \rho|_{G_{F_v}}$ is de Rham at all places v over p.

Remark 1.1.1. *By Cebotarev density theorem, such a geometric Galois representation is then determined (up to semi-simplification) by the images $\rho(Frob_v)$ where v runs through the set of all unramified places of ρ.*

In fact, it is well known that a geometric representation is unramified outside a finitely many places. This is a theorem due to Grothendieck [MJ73][L.I77]. More over geometric representation is de Rahm at all places over p (Fontaine, Messing,Faltings, Kato, Tsuji, de Jong, see e.g. [P.B] [L.I90]). Thus being geometric implies indeed being weakly geometric. The famous Fontaine-Mazur conjecture predicts that they are actually equivalent. This conjecture is recently confirmed in many cases for $n = 2$ and $K = \mathbb{Q}$ by Emerton and Kisin independently using p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$.

Example 1.1.1. *In the case of a rational elliptic curve E, the associated Galois representation $\rho_{E,p} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$ is unramified at all primes $l \neq p$ where E*

has good reduction. Further more the characteristic polynomial of $\bar{\rho}_{E,p}(Frob_l)$ reads $det(xI - \rho_{E,p}(Frob_l)) = x^2 - (l+1-N_l)x + l$ where N_l is the number of points of E with coordinates in $\mathbb{F}_l$.

We will not forget here another very important invariant : the L-function attached to an elliptic curve or an algebraic variety. It contains slightly less information about the elliptic curve than the Galois representation. Indeed, to an elliptic curve E over $\mathbb{Q}$ we associate the L-function $L(E,s) = \prod_p L_p(E,s)$ where p runs through all primes and for almost all prime p the local L-factor $L_p(E,s) = (1-(p+1-N_p)p^{-s}+p^{1-2s})^{-1}$.

1.2. **Automorphic forms and Galois representations.** Another source of Galois representation comes from the so-called automorphic forms or automorphic representations. We can define autormorphic form for any reductive groups but for introductory purpose, we can content ourselves with modular forms, the simplest example of automorphic form.

Let $\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 (\mathrm{mod} N)\}$ and $\Gamma_1(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a, d \equiv 1 (\mathrm{mod} N\}$. These two subgroups of $SL_2(\mathbb{Z})$ are of finite index and $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$.

Definition 1.2.1. A modular form of level N, weight k is a holomorphic function: $f : \mathcal{H} = \{z \in \mathbb{C} | im(z) > 0\} \to \mathbb{C}$ such that:

(1) $f|_k\gamma(z) := (cz+d)^{-k} f(\gamma z) = f(z)$ for all matrix $\gamma \in \Gamma_1(N)$.
(2) f is holomorphic at the cusps.

We also say that f is of character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ if $f|_k\gamma = \epsilon(\gamma) f$ for all $\gamma \in \Gamma_0(N)$.

Let us explain the second condition. First of all, because of condition (1), all modular forms are periodic, meaning that $f(z+L) = f(z)$ for some $L \in \mathbb{Z}$. As a result a modular form admits a Fourier transform $f(z) = \sum_{n \in \mathbb{Z}} a_n(f) q^n$ where $q = e^{2i\pi z/L}$. This is often refereed to as the q-expansion of f. The condition that f is holomorphic at the cusps then translates to the fact that the q-expansion of $f|_k\gamma$ has only non zero coefficients in non negative degrees in q for all $\gamma \in SL_2(\mathbb{Z})$. If furthermore, these q-expansion have only non zero coefficient in positive degree we say that f is a cusp form. When the leading coefficient of the q-expansion of a cusp form is 1, we say that it is normalized.

The set of all modular forms of level N, weight k forms a complex vector space and is denoted by $M(k,N,\mathbb{C})$ whereas its sub space of cusp forms is denoted by $S(k,N,\mathbb{C})$. For any $l \in \mathbb{N}$ prime to N we can define a Hecke operator $T_l \in End(M(k,N,\mathbb{C}))$ preserving the subspace of cusp forms $S(k,N,\mathbb{C})$. These operators commute and are simultaneously diagonalizable, meaning that $M(k,N,\mathbb{C})$ can be written as a direct sum of eigenspaces of $\{T_l\}_{l \wedge N = 1}$. We also denote the $\mathbb{Z}$-algebra generated by all T_l for l prime to N by $\mathbb{T}$, this algebra plays a central role in the theory of modular forms.

There is also an alternative algebraic approach to modular forms. We first point out that if $N \geq 3$, the group $\Gamma_1(N)$ acts on $\mathcal{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. This action is properly discontinuous and the quotient $Y_N(\mathbb{C}) := \Gamma_1(N)\backslash\mathcal{H}$ admits a structure of a non compact Riemanien surface. We can then add a finitely many points, called the cusps to $Y_N(\mathbb{C})$ to obtain the so called compactified modular curve $X_N(\mathbb{C})$. Moreover, $Y_N(\mathbb{C})$ and $X_N(\mathbb{C})$ are also the set of complex points of algebraic varieties Y_N and X_N respectively defined over $\mathbb{Z}[\frac{1}{N}]$ with the latter being proper.

We can even give Y_N (resp. X_N) a modular interpretation, i.e it parametrizes elliptic curve (resp. generalized elliptic curves) with additional level structure. In particular, there is a universal generalized elliptic curve E over X_N. We denote by ω the pull back of the differential sheaf of E along the identity section $e : X_N \to E$. This is an invertible sheaf on X_N and the Kodaira-Spencer isomorphism tells us that $\omega \otimes \omega(-D) \cong \Omega^1_{X_N}$ where $\Omega^1_{X_N}$ is the sheaf of differential forms of X_N and $D = X_N - Y_N$ is the divisor corresponding to the cusps.

Using this interpretation, the entire theory of modular forms can be recast algebraically (see e.g [Kat73]). More specifically, for all $\mathbb{Z}[\frac{1}{N}]$-algebra R we can define an R-valued modular form of level N, weight k as a section of the line bundle $\omega^{\otimes k}$ on X_N. The result is that we obtain a finite R-module $M(k, N, R) := H^0(X_N \times_{\mathbb{Z}[\frac{1}{N}]} R, \omega^{\otimes k})$ of R-valued modular form of level N, weight k which satisfies very nice functorial properties in R. Additionally its R-sub module $S(k, N, R)$ of cusp forms is identified with $H^0(X_N \times_{\mathbb{Z}[\frac{1}{N}]} R, \omega^{\otimes k}(-D))$, where D is the complementary of Y_N in X_N. In particular, when $R = \mathbb{C}$, we recover the classical spaces $M(k, N, \mathbb{C})$ and $S(k, N, \mathbb{C})$.

This approach also shows the existence of a $\mathbb{Z}[1/N]$-basis for $M(k, N, \mathbb{C})$ and $S(k, N, \mathbb{C})$. Under this angle, it is immediate that $\mathbb{T}$ is a finite $\mathbb{Z}[\frac{1}{N}]$-algebra. As a result if f is an eigenform, then all of its eigenvalues lie in a finite extension of $\mathbb{Q}$, we call it the coefficient field of f. By the same token, each normalized cuspidal eigenform f corresponds to a prime ideal of $\mathbb{T}$. Indeed, there is a character $\eta_f : \mathbb{T} \to \mathbb{C}$ given by $T_l \mapsto \lambda_l$ where $T_l(f) = \lambda_l f$. The image of η_f is clearly the coefficient field of f and the kernel of η_f is a prime ideal of $\mathbb{T}$.

The existence of Hecke operators $\{T_l\}_{l\in\mathbb{N}}$ on the space $M(k, N, \mathbb{C})$ make these seemingly analytic objects more relevant to the number theory. Indeed we have the following theorem due to Taniyama-Shimura in the case $k = 2$ [G.S71], Deligne in the case $k \geq 2$ [Del71] and later Deligne-Serre in the case $k = 1$ [DS74].

Theorem 1.2.1. *Let f be an cuspidal eigenform of level $\Gamma_1(N)$ of weight k and character χ with q-expansion $f(q) = \sum_{n\geq 1} a_n q^n$. Let F denote the coefficient field of f.*

(1) *If $k \geq 2$ then for any place λ above p of F, we have a representation:*

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_{F,\lambda})$$

This representation is unramified outside the primes dividing Np and for all prime $l \nmid pN$, we have

$$trace(\rho_{f,\lambda}(Frob_l)) = a_l \qquad and \qquad det(\rho_{f,\lambda}(Frob_l)) = \chi(l)l^{k-1}$$

(2) *If $k = 1$ then there exists an odd irreducible Galois representation:*

$$\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$$

such that ρ_f is unramified at $l \nmid N$ and $det(XI-\rho_f(Frob_l)) = X^2-a_lX+\chi(l)$.

Proof. [*Sketch*]

(1) We only outline the proof when $k = 2$ and we also suppose furthermore that f is a new form. Recall that each modular form of weight 2 and level $\Gamma_1(N)$ can be identified with a section $f \in H^0(X_N, \Omega^1_{X_N})$. Now by Hodge decomposition theorem, the Betti cohomology with complex coefficient can be written as : $H^1(X_N, \mathbb{C}) = H^0(X_N, \Omega^1_{X_N}) \oplus H^1(X_N, \mathcal{O}_{X_N})$ with $H^1(X_N, \mathcal{O}_{X_N}) = \overline{H^0(X_N, \Omega^1_{X_N})}$. The Hecke algebra $\mathbb{T}$ acts on these vector spaces and we can see them as $\mathbb{T}$-modules. Finally, by choosing an isomorphism $\iota : \bar{\mathbb{Q}}_p \cong \mathbb{C}$, we have the comparison theorem

$$H^1_{et}((X_N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes \mathbb{C} \cong H^1(X_N, \mathbb{C}) = H^1(X_N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \tag{1.1}$$

There is also natural action of $\mathbb{T}$ on $H^1_{et}((X_N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ and the above isomorphism is in fact an isomorphism of $\mathbb{T}$-modules.

Now giving an eigenform f we can compose $\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(H^1_{et}((X_N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p))$ with the projection to the f-eigenspace $\eta_f : \mathbb{T} \to \mathcal{O}_F$ where F is the coefficient field of f to obtain $\rho_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_{F,\lambda})$. It can be checked that $\rho_{f,\lambda}$ satisfies all the listed properties above (see appendix of [RS11]).

(2) For modular forms of weight 1, we proceed quite differently. Let $f = \sum_{n \geq 0} a_n q^n$ be a such eigenform and let K be its coefficient field. For any even k we have a normalized Eisenstein series $E_k = \sum_{n \geq 0} b_n q^n$ of weight k. It is easy to see that if $p-1|k$ then $b_n \equiv 0 \mod (p)$ for all $n \geq 1$ so that $fE_k \equiv f \mod \mathfrak{m}_{K_\lambda}$ where λ is a place above p.

Now fE_k might not be an eigenform but following Deligne-Serre's lifting theorem, there exists an eigenform $g = \sum_{n \geq 0} c_n q^n$ of weight $k+1$ with coefficient field K' such that $fE_k \mod \mathfrak{m}_{K_\lambda} = g \mod \mathfrak{m}_{K'_{\lambda'}}$ where λ' is a place above p. We can consider the representation $\rho_{g,\lambda} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_{K'_{\lambda'}})$. We let $\bar{\rho}_{f,p} = G_{\mathbb{Q}} \to GL_2(k)$ be the reduction $\rho_{g,\lambda'} \mod \mathfrak{m}_{K'_{\lambda'}}$, where k is some finite extension of $\mathbb{F}_p$. This representation (up to semi simplification) depends only on f.

More importantly, outside a finite set of prime, we can prove that the image of $\bar{\rho}_{f,p}$ is bounded uniformly. Therefore if p is big enough $\bar{\rho}_{f,p}$ can be lifted to a representation $\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(F)$ where F is the coefficient field of f

□

Remark 1.2.1. *The L-function as an invariant is not exclusive to algebraic varieties, we can attach an L-function to a modular form or more generally an automorphic representations. Take a cusp form f of weight $k \geq 2$ with Fourier expansion $f(z) = \sum_n a_n q^n$ we can put $L(f, s) := \sum_n a_n n^{-s}$. This carries all the numerical datum that we can read off from the Galois representation attached to f. Moreover, $L(f, s)$ converges uniformly to a holomorphic functions for $s \in \mathbb{C}$ such that $re(s) > 1 + k/2$ and admits an entire continuation.*

One might observe that $\rho_{f,\lambda}$ is weakly geometric and by construction, it comes from the étale cohomology of a modular curve, as predicted by the Fontaine-Mazur conjecture.

Definition 1.2.2. We say that a two dimensional p-adic Galois representation of $G_{\mathbb{Q}}$ is modular if it is isomorphic to a representation of the form $\rho_{f,\lambda}$.

This is just an instance of the much deeper phenomenon captured and studied by the Langlands programs. In fact there is a central object in the Langlands program called automorphic forms or automorphic representations that vastly generalize modular forms. We also expect to be able to attach to any such automorphic form a Galois representation. More precisely, we have the following conjecture/guiding philosophy:

Conjecture (Langlands-Clozel-Fontaine-Mazur): Let K be a number field, and a prime p, let us also fix an isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$. Then there is a unique bijection between the following sets:

(1) the set of L-algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_K)$
(2) the set of irreducible weakly geometric Galois representations $G_K \to GL_n(\bar{\mathbb{Q}}_p)$.

This bijection is supposed to satisfy a number of compatibility that we will not spell all out but notably, their L-functions are supposed to match.

The direction (1) $\to$ (2) consists of attaching Galois representations to automorphic forms, this is still wide open in general. When K is a totally real field or a CM field, we know how to do this in the case where automorphic forms show up in the Betti cohomology group of locally symmetric space (similar to how modular form appears in the cohomology of modular curve) (see [Sch15]). The requirement is that this automorphic representation be regular, a typical example would be modular forms of weight $k \geq 2$. The other case of non regular representations are often hopeless except for the so-called limit of discrete series representations, a simplest example of this would be modular form of weight 1. These automorphic representations still contribute to coherent cohomology of relevant shimura variety and following Scholze, Pilloni-Stroh [Sch13] [PS16b] we know how to treat these cases as well.

The direction (2) $\to$ (1) is known as the modularity conjecture and the only known method is due to Wiles, and Taylor-Wiles. They first introduced the method to prove that all semi stable elliptic curves over the rational are modular. Their method rests on a crucial "numerical coincidence" that makes it functional only in certain cases. Recently a breakthrough of Calegari-Geraghty gave a very satisfying explanation of this then mysterious coincidence and better, they showed how to fix it. However still, the new method requires us to show two additional conjectures that we are going to explain using the case of group GL_2 in the next section.

2. Modularity

As mentioned in the previous section, the modularity direction aims to prove that certain Galois representation comes from automorphic forms. (The case of 1-dimensional representations is handled by the class field theory). The case of two dimensional representation is already quite difficult.

In this section we review some of the main ingredients of Taylor-Wiles method by focusing on the case of elliptic curve over the rational, which is the simplest example, and in the next section we will study Artin representations, which requires a considerable modification of Taylor-Wiles method.

Definition 2.0.1. We say that an Elliptic curve E over $\mathbb{Q}$ is modular if for a prime p, the representation $\rho_{E,p}$ is modular, i.e there exist a modular form f with rational coefficient field such that $\rho_{f,p} \cong \rho_{E,p}$.

This definition does not depend on the chosen prime p. Indeed, suppose that E is modular, and let $f = \sum_n \alpha_n q^n$ be the associated form. Since the Galois representation $\rho_{E,p}$ and $\rho_{f,p}$ give the same representation, we have (see example 1.1.1 and theorem 1.2.1):

$$\alpha_l = l + 1 - N_l$$

for unramified $l \neq p, p'$. As a result, if p' is another prime, the representation $\rho_{E,p'}$ is also modular, because up to semi-simplification a geometric Galois representation is determined by the traces of the Frobenii at l for almost all l, and we have $Trace(\rho_{E,p}(Frob_l)) = Trace(\rho_{E,p'}(Frob_l)) = \alpha_l$ which depends only on f for almost all l. Thus $\rho_{E,p'}$ is isomorphic to $\rho_{f,p'}$.

The starting point is the observation due to Mazur (Conjecture (*) [MT90]) that in certain situations, modularity can be checked residually, in other words if there is a modular form f such that $\rho_{f,p} = \rho \pmod p$, then ρ is very likely to be modular. An instance of this observation was first proved by Wiles for the semi-stable rational elliptic curves which produce semi-stable representations.

Let us briefly explain the terminology before stating the theorem. An elliptic curve over $\mathbb{Q}$ is semi-stable if its reductions at bad primes are of multiplicative type and a Galois representation ρ is semi-simple if its reduction $\bar{\rho}$ is. Now a representation $\bar{\rho} : G_{\mathbb{Q}} \to \bar{\mathbb{F}}_p$ is semi-stable if

(1) for all $l \neq p$ the restriction $\bar{\rho}|_{I_l}$ is nilpotent.
(2) at p the restriction $\bar{\rho}|_{I_p}$ is either of form $\begin{pmatrix} \chi_p|_{I_p} & * \\ 0 & 1 \end{pmatrix}$ or $det(\bar{\rho}) = \chi_p$ and $\bar{\rho}$ can be obtained via base change from a finite flat group over $\mathbb{Z}_p$.

Theorem 2.0.1 (Modular lifting)**.** *[Wil95, TW95] Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_K)$ be a Galois representation where K is a finite extension of $\mathbb{Q}_p$ ($p > 2$). If ρ is semi stable and has cyclotomic determinant and the reduction $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_q)$ is irreducible and modular, then ρ itself is modular.*

Given this theorem, we can check the modularity by working with any primes and check the residual modularity instead. With this in mind, the attention is shifted to the Serre's conjecture.

Conjecture 1 (Serre's conjecture)**.** *Let k be a finite field and $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(k)$ be an odd, absolutely irreducible Galois representation, then $\bar{\rho}$ is modular.*

This conjecture is now a theorem due to Khare-Wintenberger [KW09a] [KW09b], but back then we had to resort to a special case of this conjecture.

Theorem 2.0.2 (Langlands-Tunell)**.** *[Lan80, Tun81] All odd irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_3)$ is (residually) modular meaning that there exists a modular form f such that the reduction of $\rho_{f,3}$ is $\bar{\rho}$.*

Now, given a semi-stable rational elliptic curve E, the associated representation $\rho_{E,3}$ can be checked to satisfy the condition of theorem (2.0.1) and thus by theorem 2.0.2, if $\bar{\rho}_{E,3}$ is irreducible then E is modular. Unfortunately, $\bar{\rho}_{E,3}$ can be very well reducible, such is the case when E has a rational subgroup of order 3. To fix this problem, there is a clever lemma known as the $3-5$ trick.

Lemma 2.0.1 (3 − 5 trick). *[Wil95] Let E be a semi-stable rational elliptic curve, then if $\bar{\rho}_{E,3}$ is reducible, then $\bar{\rho}_{E,5}$ is irreducible and there exists another rational elliptic curve E' such that $\bar{\rho}_{E,5} \cong \bar{\rho}_{E',5}$ and $\bar{\rho}_{E',3}$ is irreducible.*

The proof is based on our concrete understanding of the modular curve $Y(5)$ and $Y(15)$. In fact, if $\bar{\rho}_{E,3}$ is reducible, then $\bar{\rho}_{E,5}$ must be irreducible because otherwise, E has a rational subgroup of order 15 and it defines a rational point of the modular curve $Y(15)$. We happen to know all of the rational points of $Y(15)$, there are 4 of them, and each corresponds to an elliptic curve of conductor 50, therefore non of them are semi-stable (conductor of a semi-stable elliptic curve must be square-free).

In short, let E be a semi-stable rational elliptic curve, we can consider $\rho_{E,3}$. If $\bar{\rho}_{E,3}$ is irreducible, we are done by theorems 2.0.1 and 2.0.2. If $\bar{\rho}_{E,3}$ is not irreducible, we apply the $3-5$ trick and find another curve E'. Since $\bar{\rho}_{E',3}$ is irreducible, it is modular. As a result E is modular too because $\bar{\rho}_{E,5} \cong \bar{\rho}_{E',5}$.

3. Taylor-Wiles method

In this section, we review the proof of the theorem 2.0.1. We start with an odd, absolutely irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$. Suppose that $\bar{\rho}$ satisfies:

(1) $det(\bar{\rho})$ is the inverse of the cyclotomic character.
(2) the restriction $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is finite flat (i.e the restriction can be obtained via a group scheme).

The modularity lifting theorem proposes to show that if $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p)$ is a semi-stable Galois representation and the reduction mod p of ρ is $\bar{\rho}$. Then if $\bar{\rho}$ is modular, so is ρ.

First of all, when $\bar{\rho}$ is modular, and f is a modular form of weight 2 such that $\rho_{f,p} \equiv \bar{\rho} \pmod{p}$, we can consider the Hecke algebra $\mathbb{T}_{\mathfrak{m}}$ localized at the maximal ideal $\mathfrak{m}$ corresponding to f. This algebra actually parametrizes all modular forms whose associated Galois representations reduce mod p to $\bar{\rho}$.

The idea is to compare the $\mathbb{T}_{\mathfrak{m}}$ with the deformation ring $\mathcal{R}_{\bar{\rho}}$ where :

$\mathcal{R}_{\bar{\rho}}$ is the universal deformation ring of all representations $\rho : G_{\mathbb{Q}} \to GL_2(A)$ (up to strict equivalence) where A is any object in the category $Art_{\mathbb{F}_p}$ of Artinien local $\mathbb{Z}_p$-algebras of residue field k and

(1) the determinant $det(\rho) = \epsilon^{-1}$.
(2) the representation ρ is semi-stable.
(3) the reduction of ρ is $\bar{\rho}$.

One remarks in particular that our target representation ρ can be identified with a point in $Spec(\mathcal{R}_{\bar{\rho}})$ and if we are to show that $\mathcal{R}_{\bar{\rho}} \cong \mathbb{T}_{\mathfrak{m}}$ then the theorem 2.0.1 obviously follows.

The link between $\mathcal{R}_{\bar{\rho}}$ and $\mathbb{T}_{\mathfrak{m}}$ is subtle and it goes back to the proof of theorem 1.2.1. Recall that there is a representation $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{T}_{\mathfrak{m}})$ satisfying the condition $(1), (2)$ above. Consequently, we have a surjection $\iota : \mathcal{R}_{\bar{\rho}} \to \mathbb{T}_{\mathfrak{m}}$ so as to $M := H^0(X_N, \Omega^1_{X_N})_{\mathfrak{m}}$ becomes a $\mathcal{R}_{\bar{\rho}}$-module. To prove that ι is an isomorphism we can show a stronger result that M is actually a free $\mathcal{R}_{\bar{\rho}}$-module.

We take a look at some properties of $\mathcal{R}_{\bar{\rho}}$. By a result of Mazur [Maz89], $\mathcal{R}_{\bar{\rho}}$ is a complete noetherian local $\mathbb{Z}_p$-algebra, with residue field $\mathbb{F}_p$. There are two important invariants attached to $\bar{\rho}$ depending on $\mathcal{R}_{\bar{\rho}}$, an $\mathbb{F}_p$-linear subspace Sel of $H^1(G_{\mathbb{Q}}, ad^0\bar{\rho})$ called the Selmer group, and its dual $Sel^{\perp}$ which is an $\mathbb{F}_p$-linear subspace of $H^1(G_{\mathbb{Q}}, ad^0\bar{\rho}(1))$. These two groups control the dimension of various deformation rings that we care about. Concretely let $g = dim_{\mathbb{F}_p}Sel$ and $r = dim_{\mathbb{F}_p}Sel^{\perp}$ (in fact in our situation, $g = r$ but we pretend for the moment that they have nothing to do with each other). We know the dimension of the Zariski tangent space of $\mathcal{R}_{\bar{\rho}}$ is g, i.e there is a surjection $\mathbb{Z}_p[[X_1, ..., X_g]] \to \mathcal{R}_{\bar{\rho}}$. This surjection is very hard to understand, but through an ingenious machinery of Taylor-Wiles method, we can cook up a pair $(M_\infty, \mathcal{R}_\infty)$ where $\mathcal{R}_\infty$ is a $\mathbb{Z}_p[[S_1, ..., S_r]]$-algebra isomorphic to $\mathbb{Z}_p[[X_1, ..., X_g]]$ and M_∞ is a free $\mathbb{Z}_p[[S_1, ..., S_r]]$-module. The difference is that we have a very simple relationship of $(M_\infty, \mathcal{R}_\infty)$ and $(M, \mathcal{R}_{\bar{\rho}})$. In fact by construction $\mathcal{R}_\infty \otimes_{\mathbb{Z}_p[[S_1,...,S_r]]} \mathbb{Z}_p \cong \mathcal{R}_{\bar{\rho}}$ and $M_\infty \otimes_{\mathbb{Z}_p[[S_1,...,S_r]]} \mathbb{Z}_p \cong M$. In our case, it turns out that M_∞ is free as a $\mathcal{R}_\infty$-module, this implies immediately that M is free as $\mathcal{R}_{\bar{\rho}}$-module. We now go about the detailed construction of such a mysterious pair (M_∞, R_∞).

Definition 3.0.1. Given our residual representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_n(\mathbb{F}_p)$, a prime l is called a Taylor-Wiles prime if:

(1) $l \equiv 1 (\text{mod } p)$.
(2) $\bar{\rho}|_{G_{\mathbb{Q}_l}}$ is unramified and the eigenvalues of $\bar{\rho}(Frob_l)$ are distinct.

A set $\mathcal{Q}$ of Taylor-Wiles prime is called a Taylor-Wiles set if $\#\mathcal{Q} = r$ and furthermore the localization map $Sel^{\perp} \to \oplus_{p\in\mathcal{Q}} H^1(G_{\mathbb{Q}_p}, ad^0\bar{\rho}(1))$ is injective.

If $\mathcal{Q}$ is a Taylor-Wiles set, let $N_{\mathcal{Q}} := \prod_{p\in\mathcal{Q}} p$ and $\Lambda_{\mathcal{Q}} := \mathbb{Z}_p[\Delta_{\mathcal{Q}}]$ where $\Delta_{\mathcal{Q}}$ is the p-sylow of the finite group $\prod_{l\in\mathcal{Q}} \mathbb{F}_l^{\times}$ (notice the condition (1) above). Now for each Taylor-Wiles set $\mathcal{Q}$ we can define $\mathcal{R}_{\bar{\rho},\mathcal{Q}}$ as the deformation ring parametrizing all representations $\rho : G_{\mathbb{Q}} \to GL_2(A)$ for $A \in Art_{\mathbb{F}_p}$ that satisfy:

(1) $det\rho = \epsilon^{-1}$
(2) ρ is semi-stable.
(3) ρ is allowed to ramify at primes dividing $\mathcal{Q}$.

In other words, we now allow ramification at places in $\mathcal{Q}$. Because of the condition (2) of definition 3.0.1 above, $\mathcal{R}_{\bar{\rho},\mathcal{Q}}$ has a natural $\Lambda_{\mathcal{Q}}$-algebra structure. More over we have a surjection $\mathcal{R}_{\bar{\rho},\mathcal{Q}} \to \mathcal{R}_{\bar{\rho}}$. We can show that (since $\#\mathcal{Q} = r$) this surjection induces an insomorphism on tangent spaces, that means we have a surjection $\mathbb{Z}_p[[X_1, ..., X_g]] \to \mathcal{R}_{\bar{\rho},\mathcal{Q}}$.

Now, for each n we can choose a of Taylor Wiles set $\mathcal{Q}_n$ (following Wiles [Wil95], this is possible). Let $\Lambda_n := \mathbb{Z}_p[\Delta_{\mathcal{Q}_n}] \cong \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^r]$. Regarding $\mathcal{R}_n$, it is nothing but the deformation ring $\mathcal{R}_{\bar{\rho},\mathcal{Q}_n}$ introduced above. For M_n we can take the space of modular forms $M_n := H^0(X(N\mathcal{Q}_n), \omega^k)_{\mathfrak{m}}$ and let $\mathbb{T}_n$ be the Hecke algebra acting on M_n. We can also show that there is a surjection $\mathcal{R}_n \to \mathbb{T}_n$ which again comes from the existence of Galois representation attached to modular forms. All of these fit together and produce the following compatible system:

Theorem 3.0.1. *[Dia97]*

For each $n \in \mathbb{N}$ there is a triple $(\Lambda_n, \mathcal{R}_n, M_n)$ where

1. Λ_n *isomorphic to* $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^r]$.
2. M_n *is a free* Λ_n*-module such that* $M_n \otimes_{\Lambda_n} \mathbb{Z}_p \cong M$.
3. $\mathcal{R}_n$ *is a* Λ_n*-algebra and there is a surjection* $\mathcal{R}_n \to \mathcal{R}_{\bar{\rho}}$. *Moreover* $\mathcal{R}_n$ *acts on* M_n *such that we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{R}_n & \longrightarrow & End_{\Lambda_n}(M_n) \\ \downarrow & & \downarrow \otimes_{\Lambda_n}\mathbb{Z}_p \\ \mathcal{R}_{\bar{\rho}} & \longrightarrow & End_{\Lambda}(M) \end{array}$$

By construction, we have surjections $\mathbb{Z}_p[[X_1, ..., X_g]] \to \mathcal{R}_n$ for all n and one can imagine that as n varies, the triple $(\Lambda_n, \mathcal{R}_n, M_n)_n$ form a compatible system and we can produce a triple $(\Lambda_\infty, \mathcal{R}_\infty, M_\infty)$ where

1. $\Lambda_\infty \cong \mathbb{Z}_p[[S_1, ..., S_r]]$.
2. M_∞ is a free Λ_∞-module such that $M_\infty \otimes_{\Lambda_\infty} \mathbb{Z}_p \cong M$.
3. $\mathcal{R}_\infty$ is a Λ_∞-algebra and there is a surjection $\mathcal{R}_n \to \mathcal{R}_{\bar{\rho}}$. Moreover $\mathcal{R}_\infty$ acts on M_∞ such that we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}_\infty & \longrightarrow & End_{\Lambda_\infty}(M_\infty) \\ \downarrow & & \downarrow \otimes_{\Lambda_\infty}\mathbb{Z}_p \\ \mathcal{R}_{\bar{\rho}} & \longrightarrow & End_{\Lambda}(M) \end{array}$$

With these datum we can easily see that M is a free $\mathcal{R}_{\bar{\rho}}$-module. Indeed, recall the following Auslander-Buchsbaum formula:

$$depth_R M + projdim_R M = dim R$$

for any finitely generated module M over a regular ring R. This yields us the inequality:

$$dim\Lambda_\infty = depth_{\Lambda_\infty} M_\infty \leq depth_{\mathcal{R}_{\rho,\bar{\infty}}} M_\infty \leq dim\mathcal{R}_{\bar{\rho},\infty} \leq g$$

In our case, the famous "numerical incidence" $g = r$ happens so that $dim\Lambda_\infty = dim\mathcal{R}_{\bar{\rho},\infty}$ and M_∞ is a free $\mathcal{R}_{\bar{\rho},\infty}$-module. From the diagram above we conclude that M is a free $\mathcal{R}_{\bar{\rho}}$-module.

4. Beyond Taylor-Wiles

As we start to examine more general modularity statements, we will notice that the crucial equality $g = r$ need not hold at all, and the classical Taylor-Wiles method falls apart in such situations.

Luckily, Calegari-Geraghty showed how to fix this problems. To illustrate their work, we can put ourselves in the case of $g = r + 1$. We choose to review the case of modular form of weight 1 which is a typical example of the so-called limit of discrete series- a type of irregular automorphic forms.

Recall that to a cuspidal eigenform $f = \sum_n a_n q^n$ of level N, weight 1 we can attach an irreducible odd Galois representation $\rho_f : G_\mathbb{Q} \to GL_2(\mathbb{C})$ which has finite image and is unramified outside N. In fact by construction, if we pick a random prime p big enough, there exists a representation $\bar{\rho}_{f,p} : G_\mathbb{Q} \to GL_2(k)$, where k is some finite extension of $\mathbb{F}_p$, this can be lifted to a representation $\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_K)$ where K is some finite extension of $\mathbb{Q}_p$ and λ is a place diving p. For different p' we obtain another $\rho_{f,p'}$. Moreover $\rho_{f,p}$ and $\rho_{f,p'}$ are isomorphic as complex representation. As a result we can choose a big prime p and work with $\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_K)$.

Inspired by what happens to modular form of weight 2 one would like to in this case prove the following

Theorem 4.0.1. *Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_K)$ be an odd representation such that:*

(1) *ρ is unramified outside a finite set of primes.*
(2) *the reduction $\bar{\rho}$ mod $\mathfrak{m}_K$ of $\rho|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible.*
(3) *ρ is unramified at p and the eigenvalues of $\rho(Frob_p)$ are distinct mod $\mathfrak{m}_K$.*

Then ρ comes from modular form of weight 1.

Like before, we would like to point out the appropriate candidate for the pair of deformation rings $(\mathcal{R}_{\bar{\rho}}, \mathbb{T}_{\mathfrak{m}})$ as in the previous case.

Now we can look at the deformation ring $\mathcal{R}_{\bar{\rho}}$ parametrizing all representations $\rho : G_{\mathbb{Q}} \to GL_2(R)$ with $R \in Art_k$ satisfying:

(1) the determinant $det(\rho) = \epsilon^{-1}$.
(2) ρ is unramified at p.

Again, there is a selmer group $Sel \subset H^1(G_{\mathbb{Q}}, ad^0\rho)$ and its dual $Sel^{\perp} \subset H^1(G_{\mathbb{Q}}, ad^0\rho(1)$ and if we let $g = dim_{\mathbb{F}_p} Sel$ and $r = dim_{\mathbb{F}_p} Sel^{\perp}$ then this time $g = r + 1$ and the dimension of the Zariski tangent space of $\mathcal{R}_{\bar{\rho}}$ is r which means there is a surjection $\mathcal{O}[[X_1, ..., X_r]] \twoheadrightarrow \mathcal{R}_{\bar{\rho}}$.

Now if we try to mimic the Taylor-Wiles method to proceed, i.e by putting $M = H^0(X(N), \omega)_{\mathfrak{m}}$ we will not be able to conclude since $g \neq r$. The problem is that our deformation ring $\mathcal{R}_{\bar{\rho}}$ is of dimension "1-less", so there must be something going on on the automorphic side. In fact the peculiar thing that happens to modular forms of weight 1 is that they appear not only in $H^0(X(N), \omega)$ but also in $H^1(X(N), \omega)$. Let us be more clear on what this means. Consider the exact sequence:

$$0 \to \omega \xrightarrow{\times p} \omega \to \omega/(p) \to 0$$

and an induced long exact sequence :

$$H^0(X(N), \omega) \xrightarrow{red} H^0(X(N), \omega/(p)) \to H^1(X(N), \omega)$$

The fact that $H^1(X(N), \omega)$ does not vanish (unlike the case of modular forms of weight at least 2) implies there are modular form of weight 1 mod p that do not come from modular forms of characteristic 0. More surprisingly, these modular forms in positive characteristic do produce Galois representations, and they account for the "missing" representations here. Now according to Calegari-Geraghty method, we should instead look at the complex $\mathrm{R}\Gamma(X(N), \omega)$. Let $\mathbb{T}$ be the Hecke algebra acting on this complex, and let $\mathbb{T}^{coh}$ the Hecke algebra acting on $H^*(X(N), \omega)$. We have a surjection $\mathbb{T} \to \mathbb{T}^{coh}$ with nilpotent kernel. Again, $\mathbb{T}$ is a semi local ring, and for any maximal idea $\mathfrak{m}$ of $\mathbb{T}$, the image of $\mathfrak{m}$ in $\mathbb{T}^{coh}$ is included in the support of $H^*(X_N, \omega)$. In other words, $\mathbb{T}$ does parametrize all the modular forms of weight 1, be it in $H^0(X(N), \omega)$ or $H^1(X(N), \omega)$. The generalized Taylor-Wiles method will have as an input the complex $C := \mathrm{R}\Gamma(X(N), \omega)_{\mathfrak{m}}$ instead of $H^0(X(N), \omega)_{\mathfrak{m}}$.

Now, the slogan is that if we replace modules of modular forms by complexes of modular forms everywhere we can proceed as usual. Indeed, for each n, and a chosen Taylor-Wiles set $\mathcal{Q}_n$ (with $\#\mathcal{Q}_n = r$) we can build the following datum([CG12]):

(1) A perfect complex $C_n^\bullet$ of $\Lambda_n := \mathcal{O}[\Delta_{\mathcal{Q}_n}]$-module concentrated in degrees $[0,1]$.
(2) A deformation ring $\mathcal{R}_n$ which has a structure of Λ_n-algebra and there is a surjection $\mathcal{O}[[S_1,...,S_r]] \twoheadrightarrow \mathcal{R}_n$.
(3) A morphism of Λ_n-algebra $\mathcal{R}_n \to End_{\Lambda_n}(C_n^\bullet)$.
(4) An isomorphism $C_n^\bullet \otimes^L_{\Lambda_n} \mathcal{O} \cong C^\bullet$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{R}_n \arrow[r] \arrow[d] & End(C_n^\bullet) \arrow[d, "\otimes_{\Lambda_n}\mathcal{O}"] \\ \mathcal{R}_{\bar\rho} \arrow[r] & End(C^\bullet) \end{tikzcd}$$

Remark 4.0.1. *What is essential for the construction of these datum is the existence of Galois representation $\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}_{\mathfrak{m}})$ attached to each maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ that satisfies required conditions, and this is highly non trivial.*

A similar patching process yields us a system of :

(1) A perfect complex of Λ_∞-modules $C_\infty^\bullet$ concentrated in degrees $[0,1]$ where $\Lambda_\infty \cong \mathcal{O}[[X_1,...,X_g]]$.
(2) A complete local Λ_∞-algebra $\mathcal{R}_\infty$ with a surjection $W[[S_1,...,S_r]] \to \mathcal{R}_{\bar\rho,\infty}$ and a map of Λ_∞-algebras: $R_\infty \to End_{\Lambda_\infty}(C_\infty^\bullet)$.
(3) An isomorphism $C_\infty^\bullet \otimes^L_{\Lambda_\infty} \mathcal{O} \cong C^\bullet$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{R}_\infty \arrow[r] \arrow[d] & End(C_\infty^\bullet) \arrow[d, "\otimes^L_{\Lambda_\infty}\mathcal{O}"] \\ \mathcal{R}_{\bar\rho} \arrow[r] & End(C^\bullet) \end{tikzcd}$$

To conclude, we need the following lemma

Lemma 4.0.1 (Lemma 6.2[CG12])**.** *Let S be a regular local ring of dimension d and $C^\bullet$ be a perfect complex over S concentrated in degree $[0,l]$ with $0 \le l \le d$. Then $depth_S H^*(C^\bullet) \ge d-l$ where $H^*(C^\bullet) := \oplus_i H^i(C^\bullet)$ and if equality holds then :*

(1) *$C^\bullet$ is a projective resolution of $H^l(C^\bullet)$.*
(2) *$H^l(C^\bullet)$ as depth $d-l$ and has projective dimension l.*

We can now show that $\mathcal{R}_{\bar\rho} \to \mathbb{T}_{\mathfrak{m}}$ is an isomorphism. Indeed we have the following equality:

$$depth_{\Lambda_\infty} H^*(C_\infty^\bullet) \le depth_{\mathcal{R}_\infty} H^*(C_\infty^\bullet) \le dim \mathcal{R}_\infty \le r = g-1$$

then from the lemma above, we further have $dim_{\Lambda_\infty} H^*(C_\infty^\bullet) \ge g-1$.

As a result, $depth_{\mathcal{R}_\infty} H^*(C_\infty^\bullet) = g-1$ and the lemma above also tells us that $depth_{\Lambda_\infty} H^1(C_\infty^\bullet) = g-1$. Once again, we can use the Auslander-Buchsbaum formula to get

$$g - 1 = depth_{\Lambda_\infty} H^1(C^\bullet_\infty) \leq depth_{\mathcal{R}_\infty} H^1(C^\bullet_\infty) \leq dim \mathcal{R}_\infty = g - 1$$

This of course means that $H^1(C^\bullet_\infty)$ is free as $\mathcal{R}_\infty$-module. In other words, $\mathcal{R}_{\bar{\rho}} \to \mathbb{T}^{coh}$ is an isomorphism. Finally since we have a surjection $\mathbb{T}_\mathfrak{m} \to \mathbb{T}^{coh}$ with nilpotent kernel, the surjection $\mathcal{R}_{\bar{\rho}} \to \mathbb{T}_\mathfrak{m}$ is an isomorphism.

5. Hida theory and Taylor-Wiles method

P-adic modular forms has been a central topic in number theory. The idea is to put a p-adic topology on the space of modular forms (regardless of their weights) in a way that if two p-adic modular forms are close then their associated Galois representations look very similar too. One such theory was proposed by Hida. Concretely, let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. This is the Hida's weight space. For each k there is a projector $e_p \in End(M(k, pN, \mathbb{Z}_p))$ commuting with the action of U_p such that on $e_p M(k, pN, \mathbb{Z}_p)$ (resp. $(1 - e_p)M(k, pN, \mathbb{Z}_p)$) the operator U_p becomes an isomorphism (resp. nilpotent). Hida further proved that there is a projective Λ-module M such that for any $k \geq 2$, we have the identification $M \otimes_{\Lambda,k} \mathbb{Z}_p \cong e_p M(k, N, \mathbb{Z}_p)$. An element of M is called an ordinary p-adic modular form. There is also a natural Hecke algebra $\mathbb{T}$ acting on M and to each maximal idea $\mathfrak{m}$ of $\mathbb{T}$ we can construct a Galois representation $\rho : G_\mathbb{Q} \to GL_2(\mathbb{T}_\mathfrak{m})$ satisfying expected properties.

This theory of Hida has proved to be very fruitful in various situations. One of such illuminating example is the use of Hida theory in modularity. For this we can look at the pioneering work of Buzzard and Taylor. Well before Calegari-Geraghty's breakthrough, they could show that all odd irreducible Artin representations ρ (unramified at p such that the two eigenvalues of $\rho(Frob_p)$, say α and β, are distinct mod p) come from modular forms of weight 1[BT99].

Recall that the problem with modular forms of weight 1 is that they contribute to two consecutive cohomology degrees, and thus there are some modular forms of weight 1 in $H^0(X(N)_{\mathbb{F}_p}, \omega)$ that do not lift to the characteristic zero, and the Galois representations attached to these forms account for the missing representations not observed by the deformation ring $\mathcal{R}_{\bar{\rho}}$. The idea of Buzzard and Taylor was to work with p-adic modular forms instead because unlike classical modular forms, Hida's p-adic forms are sections of certain line bundle over $X^{ord}(N)$ where $X^{ord}(N)$ is an affine open subscheme of $\hat{X}(N)$, the formal completion of X_N along its special fiber. The difference it makes is that the higher cohomology groups $H^i(X^{ord}(N), \omega)$ vanish for all $i \geq 1$. In other words, congruence is not obstructed, or a modular form mod p now will always lift to a p-adic modular forms in characteristic 0. As a result classical Taylor-Wiles method carries over perfectly to the p-adic forms. The main drawback is that we can only show a kind of p-adic modularity. Indeed, they showed that there are two p-adic modular forms f_α and f_β of weight 1 attached to ρ with property that $U_p f_\alpha = \alpha f_\alpha$ and $U_p f_\beta = \beta f_\beta$. Finally using the assumption $\alpha \neq \beta$ mod p, they could show that both f_α and f_β are classical.

Remark 5.0.1. *We can remove the assumption $\alpha \neq \beta$ mod p following an idea of Taylor [PS16a], or we can use the modified Taylor-Wiles method of Calegari-Geraghty, though we are limited to the minimal case [CG12] .*

This method of Buzzard and Taylor inspired many generalizations. Notably, and maybe more relevant to us, in [Pil12a], Vincent Pilloni, using a version of Hida theory for GSp_4 [Pil12b], showed that Tate modules of a big class of abelian surfaces over $\mathbb{Q}$ come from p-adic Siegel forms. Unfortunately, he could only show that these p-adic forms are classical when certain assumptions on the weight are satisfied [Pil11]. We will come back to this in the next section.

6. Higher Hida theory and beyond Taylor-Wiles

The modified Taylor-Wiles method supposedly allows us to tackle much more general modularity statements. It comes however with a catch, we have to verify two highly nontrivial conjectures, especially the one predicting the expected degrees of concentration of complexes we use in the construction of Taylor-Wiles systems. This is very difficult and we do not know the answer most of the time. In this section, we illustrate how a generalized version of Hida theory can help us circumvent this difficulty.

As an example, let us look at the case of abelian surface A over the rational. The Tate module of A at p gives a Galois representation $\rho_{A,p} : G_{\mathbb{Q}} \to GL_4(\bar{\mathbb{Q}}_p)$ which factors through $GSp_4(\bar{\mathbb{Q}}_p) \subset GL_4(\bar{\mathbb{Q}}_p)$.

On the automorphic side, to each pair of integer $\kappa := (k_1, k_2)$ we can define Siegel modular forms of weight κ, a direct generalization of modular forms, which are sections of a certain vector bundle Ω^κ over Siegel modular variety $X/\mathbb{Z}_p$. This variety parametrizes abelian surfaces with additional structures. Let us denote the space of Siegel modular forms of weight κ by $M(\kappa)$. We have a natural action of a Hecke algebra $\mathbb{T}$ on $M(\kappa)$ and to each eigenforms f (which is the same as giving a maximal ideal $\mathfrak{m}_f$ of $\mathbb{T}$) we can construct a Galois representations $\rho_{f,p} : G_{\mathbb{Q}} \to GSp_4(\bar{\mathbb{Q}}_p)$ which satisfies very similar properties to those arising from abelian surfaces. In fact, it is conjectured that all rational abelian surfaces are modular in the sense that their Galois representations match with those coming from Hilbert modular forms of weight $(2,2)$. This conjecture looks a lot like the Taniyama-Shimura conjecture, but it is actually more similar to the Artin conjecture. From a technical standpoint, Siegel modular forms of weight $(2,2)$ also contribute to multiple cohomology degrees, hence the picture resembles the case of modular forms of weight 1. As a result, we should direct our attention to Calegari-Geraghty method, which can in theory be applied to all obstructed situation. This method requires us to among other things establish that the complex $\mathrm{R}\Gamma(X, \omega^\kappa)_{\mathfrak{m}_f}$ is concentrated in 2 degrees which we are clueless about.

Nonetheless, inspired by how p-adic modular forms helped us get around the problems in the case of modular forms of weight 1, Vincent Pilloni has proposed a generalization of Hida theory in which we p-adically interpolate the complex $\mathrm{R}\Gamma(X, \omega^\kappa)$ instead of only $H^0(X, \omega^\kappa)$ like in classical Hida theory. More concretely, he constructed a perfect complex $M^\bullet$ of $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-module concentrated in degrees $[0,1]$ and for each κ a projector $e_p \in End(\mathrm{R}\Gamma(X, \omega^\kappa))$ such that under some mild condition on κ, we have

$$M^\bullet \otimes^L_{\Lambda,\kappa} \mathbb{Z}_p \cong e_p \mathrm{R}\Gamma(X, \omega^\kappa)$$

More over there is a Hecke algebra $\mathbb{T}'$ acting on $M^\bullet$ and to any maximal ideal $\mathfrak{m}$ of $\mathbb{T}'$ we can construct a Galois representation $G_{\mathbb{Q}} \to GSp_4(\mathbb{T}_{\mathfrak{m}})$ which satisfy some desired conditions. Above all, the fact that this complex is concentrated in degrees $[0,1]$ makes it an immediate input for the Calegari-Geraghty method. This is indeed used in [BCGP18] and they obtain the following spectacular result:

Theorem 6.0.1 ([BCGP18])**.** *Let X be an abelian surface or a curve of genus 2 over a totally real field. Then X is potentially modular.*

Remark 6.0.1.
(1) *In the theorem above, we can only prove that X is potentially modular since right from the beginning, we only strive for a more general form of the modularity lifting theorem 2.0.1, this theorem assumes that the residual representation is modular. In the case of representations of dimension 2, we had the help from Langland-Tunell theorem, or Serre's conjecture, but in higher dimension, an analogue of any of these two is completely out of reach.*
(2) *Other thing we have not talked about is the problem of local-global compatibility which is another input of Calegari-Geraghty method as well as a crucial ingredient in the construction of Taylor-Wiles systems. Nevertheless, it seems easier to handle in general, despite being also quite delicate to establish.*

7. A case of abelian threefold

It is true that this new powerful Calegari-Geraghty method can in theory handle all obstructed situations, but on the flipped side, it also requires some highly non trivial inputs, and to get around one we could resort to higher Hida theory, but in any cases, we have taken for granted the fact that our targeted automorphic forms contribute to cohomology groups, be it coherent, or étale of some relevant Shimura varieties. Moreover, in order to develop higher Hida theory, we necessarily need these automorphic forms to sit inside the coherent cohomolgy groups, and this is flat out wrong in general, in particular for the case of abelian variety of dimension at least 3. There is a rather simple way to see this. In fact for an abelian variety A of dimension n over, say $\mathbb{Q}$, the Galois representation $\rho_{A,p}$ is conjectured to come from an automorphic form π for the split orthogonal group SO_{2n+1}. In particular, the Hodge-Tate weight of $\rho_{A,p}$ must coincide with the infinitesimal character of π. Up to some normalization, this means the infinitesimal character of π reads $(0,...,0,1,..,1)$ where the weight $0,1$ appear n times. However an automorphic form that appears in Betty or coherent cohomology must be regular or weakly regular, which means the Hodge-Tate weight can be allowed to repeat at most twice (see e.g section 9 [FP19] or introduction of [BCGP18]). In other words, for a generic abelian variety of dimension at least 3, the expected automorphic forms do not even contribute to any known cohomology theories. As a result, a modularity conjecture for such variety is still completely out of reach. However, there is a case of abelian threefold which is still amenable to the current method.

More concretely, it arises as the Jacobian of a Picard curve which is a non-singular projective curve (P) of genus 3 defined by (affine)equation:

$$(P) \qquad y^3 = f(x) = x^4 + ax^3 + bx^2 + cx + d$$

with coefficients $a, b, c, d \in \mathbb{Q}$. Let $V_l(P) := H^1_{et}(P_{\bar{\mathbb{Q}}}, \mathbb{Q}_l) \cong H^1(Jac(P))$. Remark that $K := \mathbb{Q}[\zeta_3]$ acts on P_K, and we have an induced action of K on $V_l(P)$. As a result, over $\bar{\mathbb{Q}}$ we have the following decomposition:

$$V_l(P) \otimes \bar{\mathbb{Q}} = V_{1,l} \oplus V_{2,l}$$

where $x \in K$ acts by x on $V_{1,l}$ an by $\bar{x}$ on $V_{2,l}$.

Since the complex conjugate P^σ of P is isomorphic to P, we conclude that as l-adic Galois representation $V_{1,l}^\vee = V_{1,l}^\sigma(1)$ where $V_{1,l}^\vee$ is the dual of $V_{1,l}$. We have the following conjecture.

Conjecture[Appendix [Til06]]: Given an l-adic irreducible geometric Galois representation $\rho : G_K \to V$ of dimension 3 satisfying $V^\vee \cong V^\sigma(n)$ then there exists a set Π(called an L-packet) of cuspidal automorphic representations π of $G(\mathbb{A}_\mathbb{Q})$ where G is the unitary group over $\mathbb{Q}$ of signature $(2,1)$ such that for almost all places v of K we have $L_v(\pi, s) = L_v(\rho, s)$.

Remark 7.0.1. (1) *Such packet is in fact unique if it exists.*
(2) *Using base change from G to $GL_3 \times GL_1$ we can state the precise compatibility of their L-functions at remaining places.*

Under this correspondence the representations (ρ, V) with regular Hodge-Tate weights are sent to discrete series and those with "weakly" regular Hodge-Tate weights (repeated weights of multiplicity at most 2) are sent to limit of discrete series.

Now in our case the l-adic Galois representation $V_{1,l}$ satisfies the condition of the conjecture and since $Jac(P)$ is an abelian of dimension 3, the Hodge-Tate weights of $V_{1,l}$ is $(0,0,1)$ and we expect a packet Π of automorphic representations π such that the infinity components π_∞ are holomorphic limit of discrete series. In fact the expected automorphic forms are the Picard modular forms (see definition below) of weight $(1,1,1)$ and we will see that these forms contribute to multiple cohomological degrees of certain Shimura variety.

Following the this discussion, we are in the framework of the Calegari-Geraghty method. In this situation, the main object will be the (compactified) Picard modular surface $\mathcal{S}$ which parametrizes abelian threefolds with action of a quadratic imaginary field and some additional structures. For each triple $\kappa := (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 \geq k_2$, there is a coherent sheaf Ω^κ over $\mathcal{S}$ so that Picard modular forms can be identified with $H^0(S, \Omega^\kappa)$. We would be interested in the complex $\mathrm{R}\Gamma(S, \Omega^\kappa)$. More precisely, there is a Hecke algebra $\mathbb{T}$ acting on this complex and to each maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ we can construct a Galois representation $\rho_\mathfrak{m} : G_K \to GL_3(\bar{\mathbb{Q}}_p)$. According to the numerical inputs of this new method, when $\mathfrak{m}$ corresponds to an absolutely irreducible $\rho_\mathfrak{m}$, we would like to prove that the localized complex $\mathrm{R}\Gamma(\mathcal{S}, \Omega^\kappa)_\mathfrak{m}$ is concentrated only in two degrees. This task is however not possible to establish with current techniques, but similarly to the case of abelian surfaces, we will use higher Hida theory to get around this. Indeed, in this thesis we construct a perfect p-adic complex over the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, concentrated in degrees $[0,1]$ and interpolate the ordinary part of $\mathrm{R}\Gamma(S, \Omega^{\kappa})$. This will be the first step to prove the conjecture above and we hope to be able to establish a suitable modularity lifting theorem in the future work.

8. Hasse-Weil conjecture

Another central topic in number theory is the L-function. Similar to Galois representation, L-function can be attached to an algebraic variety or an automorphic form and it carries a lot of arithmetic information about these objects. We have previously seen examples of L-function attached to elliptic curves and mordular forms (more generally to algebraic variety and automorphic forms). Since L-functions coming from automorphic forms enjoy a lot of properties that we wish to establish for L-functions coming from algebraic variety, it is thus natural to use the correspondence between these two classes of objects to answer questions about the latter.

Let X be a smooth proper algebraic variety over a number filed K and let S be the finite set of places of K outside of which X has a good reduction . This means X admits an integral model $\mathfrak{X}$ over $\mathcal{O}_{K,S}$ where $\mathcal{O}_{K,S}$ is the localization of $\mathcal{O}_K$ at places in S. The Hasse-Weil zeta function of X is defined as:

$$\zeta_X(s) = \prod_x \frac{1}{1 - N(x)^{-s}}$$

where x runs through the set of closed points of $\mathfrak{X}$ and $N(x)$ is the degree of extension of residue field at x. This definition does not depend on the choice of integral model $\mathfrak{X}$ up to a finite number of Euler factors. Notice that when $K = \mathbb{Q}$ and we take X to be just the point $Spec(\mathbb{Q})$ over $\mathbb{Q}$, then we can choose $\mathfrak{X} = Spec(\mathbb{Z})$ and recover the well known Riemann zeta function. In general it is known that $\zeta_X(s)$ is absolutely convergent for $s \in \mathbb{C}$ such that $re(s) \geq 1 + d$ where $d = dim(X)$. However just as the Riemann zeta function, many of its properties remain wide open, the following conjecture, known as the Hasse-Weil conjecture, (proposed by [Ser70]) is one of them:

Conjecture: The Hasse-Weil zeta function $\zeta_X(s)$ admits a meromorphic continuation to the whole complex plane and satisfy the functional equation similar to the one in the case of Riemann zeta function.

It is also well known that this conjecture would be a direct consequence of a suitable modularity theorem (or potential modularity theorem).

So far, very few case of Hasse-Weil conjecture is known. The most recent and strongest result so far is the verification of this conjecture for the curve of genus 2 or equivalently of abelian surface of dimension 2 over a totally real field. This is a consequence of the modularity of ablien surfaces of dimension 2 over totally real fields. The case of curve of genus 3 seems hopeless. However, as mentioned above there is a special case of Picard curve which is still amenable to the method used in [BCGP18] and we also wish to establish the first case Hasse-Weil conjecture for a curve of genus 3 as an application of our work.

Part 2. Higher Hida theory

As mentioned earlier in the general introduction, the case of interest for us will be abelian variety that comes as the jacobian of the Picard curve of genus 3, let's call it A. This abelian variety is defined over $\mathbb{Q}$ and has dimension 3 as well as an action of an imaginary quadratic field. Following conjecture 7 there should be a Picard modular form associated to A in the sense that their L-functions agree. Now, the Galois representation attached to A has Hodge-Tate weights $(1,1,1,0,0,0)$ and we expect that the Picard modular form (see definition below) showing up has weight $(1,1,1)$. In order to carry out the method of Calegari and Geraghty, we would like to know the range of concentration of higher cohomology groups to which our targeted Picard modular forms contribute. Somewhat more concretely, in this situation, the main object is the (compactified) Picard modular surface $\mathcal{S}$. For each triple $\kappa := (k_1,k_2,k_3) \in \mathbb{Z}^3$ with $k_1 \geq k_2$, we have a coherent sheaf Ω^κ over $\mathcal{S}$. As an input, the Calegari-Geraghty method requires us to prove that the localized complex $\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa)_\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal associated to the Picard modular form above) is concentrated only in two degrees, which we do not know how to prove directly. For this reason, we follow the idea of [Pil18] to construct p-adic Picard modular forms whose cohomology groups are non zero only in degree 0 and 1.

8.1. **Overview of main results and methods.** Now let us jump into the precise settings and the main results of this thesis. Let K be an imaginary quadratic field over $\mathbb{Q}$ with ring of integer $\mathcal{O}_K$ and p be a rational prime that splits in K. Let us also denote by π and $\bar{\pi}$ two primes of $\mathcal{O}_K$ above p. We consider the Picard variety $\mathcal{M}_U$ which is a Shimura variety of PEL type associated to unitary group $GU(2,1)$ over $Spec(\mathbb{Z}_p)$ (see below for a detailed definition). This variety classifies (up to isogenies) abelian varieties of dimension 3 with an action of $\mathcal{O}_K$ of signature $(2,1)$, a prime to p polarization and a level structure $\mathcal{U}$ that is hyperspecial at p. Let $\mathcal{S}$ be the toroidal compactification of $\mathcal{M}_\mathcal{U}$ with the boundary $D := \mathcal{S} - \mathcal{M}_\mathcal{U}$ (we should have denoted it by $\mathcal{S}_\mathcal{U}$ but we drop the subscript $\mathcal{U}$ to simplify notations). Let $\mathcal{A}$ be the universal semi-abelian scheme over $\mathcal{S}$, and $\omega_\mathcal{A}$ the relative differential sheaf of $\mathcal{A}$. The action of $\mathcal{O}_K$ induces a splitting $\omega_\mathcal{A} = \omega_{\mathcal{A},\pi} \oplus \omega_{\mathcal{A},\bar{\pi}}$. For any triple $\kappa = (k_1,k_2,k_3) \in \mathbb{Z}^3$ such that $k_1 \geq k_2$ we define the automorphic sheaf Ω^κ of weight κ over $\mathcal{M}_\mathcal{U}$ by:

$$\Omega^\kappa = Sym^{k_1-k_2}\omega_{\mathcal{A},\pi} \otimes det^{k_2}\omega_{\mathcal{A},\pi} \otimes \omega_{\mathcal{A},\bar{\pi}}^{k_3}$$

This coherent sheaf extends canonically to $\mathcal{S}$ and we still denote it by Ω^κ, as well as its cuspidal subsheaf by $\Omega^\kappa_D := \Omega^\kappa(-D)$. In this paper we develop a version of Hida theory that, for each fixed k_2, k_3, interpolates the complex $\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa_D)$ letting k_1 vary. More precisely there is a projector $t_p \in End(\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa_D))$ cutting out an ordinary part $t_p\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa_D)$ that will be interpolated.

We have our first theorem:

Theorem A: For any weight κ such that $k_1 - k_2 \geq 1$ and $k_1 + k_3 > 3$ the complex $t_p\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa_D)$ is perfect and concentrated in degree $[0,1]$.

For each fixed weight $\kappa^- := (k_2,k_3)$ we also show the existence of a perfect complex $\mathcal{V}^\bullet(\kappa^-)$ of Λ-module concentrated in degree $[0,1]$, where $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the Iwasawa algebra of dimension 1. We have our second main theorem:

Theorem B[Control theorem]: For any weight κ such that $k_1 - k_2 \geq 1$ and $k_1 + k_3 > 3$ we have a quasi-isomorphism:

$$\mathcal{V}^\bullet(\kappa^-) \otimes^L_{\Lambda,k_1} \mathbb{Z}_p \cong t_p\mathrm{R}\Gamma(\mathcal{S},\Omega^\kappa_D)$$

Remark 8.1.1. *Readers can observe that in the theorem (B), we have an (integral) control theorem for weight κ such that $k_1 - k_2 \geq 1$ and $k_1 + k_3 > 3$. This condition fails for the weight $\kappa = (1,1,1)$ which is our target. However, if we work rationally by inverting p, we can obtain the control theorem for such weight. This is achieved by working with overconvergent forms in the third part of this thesis.*

Now let us explain the strategy of the proofs.

For the first theorem, let $\mathcal{S}_1$ is the special fiber $\mathcal{S} \times_{\mathbb{Z}_p} Spec(\mathbb{F}_p)$, it is enough to prove that the complex $t_p \mathrm{R}\Gamma(\mathcal{S}_1, \Omega^{\kappa}_D)$, i.e the reduction mod p of $t_p \mathrm{R}\Gamma(\mathcal{S}, \Omega^{\kappa}_D)$ is perfect and concentrated in degree $[0,1]$. Now, on this special fiber, there is a convenient stratification by p-rank of the universal semi-group $\mathcal{A}$. In our situation the p-torsion $\mathcal{A}[p]$ is the product of $\mathcal{A}[\pi]$ and $\mathcal{A}[\bar{\pi}]$ and the Cartier dual $\mathcal{A}[\pi]^D$ of $\mathcal{A}[\pi]$ is $\mathcal{A}[\bar{\pi}]$. The factor $\mathcal{A}[\pi]$ is a truncated Barsotti-Tate group of level 1 of height 3, dimension 2, thus its multiplicative rank can only be $0, 1$ or 2. We can then define two Hasse invariants. The first one is the classical Hasse invariant $Ha \in H^0(\mathcal{S}_1, det^{p-1}\omega_{\mathcal{A}[\pi]})$ that vanishes on the locus where the multiplicative rank of $\mathcal{A}[\pi]$ is less than 2. Let us denote this vanishing locus by $\mathcal{S}_1^{\leq 1}$ and its complementary $\mathcal{S}_1^{=2} := \mathcal{S}_1 \backslash \mathcal{S}_1^{\leq 1}$. We can also construct (see section 10.1.2) a second Hasse invariant $Ha' \in H^0(\mathcal{S}^{\leq 1}, det^{p^2-1}\omega_{\mathcal{A},\pi})$. The section Ha' vanishes on the locus where the multiplicative rank of $\mathcal{A}[\pi]$ is 0, we denote this locus by $\mathcal{S}_1^{=0}$ and its complementary $\mathcal{S}^{\geq 1} := \mathcal{S}_1 \backslash \mathcal{S}_1^{=0}$. The reason for the introduction of this stratification is twofold. First of all, the projector t_p is obtained by iterating a Hecke operator $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1, \Omega^{\kappa}_D))$ and under some assumptions on the weight κ, the Hecke projector T_p interacts well with this stratification. By that we mean T_p commutes with two Hasse invariants and thus induces two operators $T_p^{\leq 1} \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{\leq 1}, \Omega_D^{k_1+p-1, k_2+p-1, k_3}))$ and $T_p^{=0} \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=0}, \Omega_D^{k_1+p+p^2-2, k_2+p+p^2-2, k_3})$. Second of all, under the assumption on the weight of the theorem, $T_p^{=0} = 0$, which implies $t_p \mathrm{R}\Gamma(\mathcal{S}_1^{=0}, \Omega^{\kappa}_D) = 0$. As a result, we can write down an explicit resolution for the sheaf Ω^{κ}_D, which has two terms that are supported on the loci $\mathcal{S}_1^{=2}$ and $\mathcal{S}_1^{=1}$. Finally the fact that these two loci are both affine in the minimal compactification helps us conclude.

For the next theorem, we need to understand the construction of the complex $\mathcal{V}^{\bullet}(\kappa)$. Recall that in the classical Hida theory, we construct the family of p-adic forms on the formal completion of the ordinary locus. The idea of higher Hida theory is similar, but we work with strictly bigger locus $\mathcal{S}_1^{\geq 1}$ instead. More precisely, let $\mathfrak{S}$ the completion of $\mathcal{S}$ along its special fiber, and let $\mathfrak{S}^{\geq 1}$ to be open formal subscheme of $\mathfrak{S}$ where the multiplicative rank of $\mathcal{A}[\pi]$ is at least 1, this locus strictly contains the ordinary locus $\mathfrak{S}^{=2}$ where Ha is invertible. Analogous to the classical case, we will build a pro-covering of $\mathfrak{S}^{\geq 1}$, called the Igusa tower with an action of $\mathbb{Z}_p^{\times}$. Explicitly, we have an affine étale map $\mathfrak{S}^{\geq 1}(p^n) \to \mathfrak{S}^{\geq 1}$ for each n, where $\mathfrak{S}^{\geq 1}(p^n)$ is the moduli space over $\mathfrak{S}^{\geq 1}$ parametrizing multiplicative subgroup $H_n \in \mathcal{A}[p^n]$ that is étale locally isomorphic to μ_{p^n}. We define $\mathfrak{S}^{\geq 1}(p^{\infty}) = \varprojlim_n \mathfrak{S}^{\geq 1}(p^n)$. Over each $\mathfrak{S}^{\geq 1}(p^n)$, we can define a $(\mathbb{Z}/p^n\mathbb{Z})^{\times}$-torsor by putting $Ig_n := \underline{Isom}_{\mathfrak{S}^{\geq 1}(p^n)}(\mu_{p^n}, H_n)$. By passing to the limit we obtain the Igusa tower $\pi : Ig_{\infty} \to \mathfrak{S}^{\geq 1}(p^{\infty}) \to \mathfrak{S}^{\geq 1}(p)$

which carries a natural action of $\mathbb{Z}_p^\times$. We define

$$\mathscr{P} := (\pi_* \mathcal{O}_{Ig_\infty} \hat{\otimes}_{\mathbb{Z}_p} \Lambda)^{\mathbb{Z}_p^\times}$$

where $\mathbb{Z}_p^\times$ acts on Λ by the universal character. This is the sheaf of p-adic modular forms and it lives over $\mathfrak{S}^{\geq 1}(p)$. Let us sketch its relationship with the classical sheaf Ω^κ. For any continuous character $w : \Lambda \to \mathbb{Z}_p^\times$, we denote $\mathscr{P}^w := \mathscr{P} \hat{\otimes}_{\Lambda, w} \mathbb{Z}_p$. In particular when w is induced from an algebraic character $k_1 - k_2 : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times (t \mapsto t^{k_1 - k_2})$ we can use Hodge-Tate map $HT : H_\infty^D \otimes \mathcal{O}_{\mathfrak{S}^{\geq 1}(p^\infty)} \to \omega_{H_\infty}$ for the universal multiplicative group $H_\infty \subset \mathcal{A}[\pi^\infty] \subset \mathcal{A}[p^\infty]$ to construct a surjection over $\mathfrak{S}^{\geq 1}(p^\infty)$:

$$\Omega_D^{k_1,k_2,k_3} \to \mathscr{P}^{k_1-k_2} \otimes \Omega_D^{k_2,k_2,k_3} \tag{8.1}$$

and thus relating classical forms and p-adic forms. Now, of course this is not satisfying because this surjection has nontrivial kernel, we get around this by defining a Hecke operator U_p acting on the cohomology groups of $\mathscr{P}^{k_1-k_2} \otimes \Omega_D^{k_2,k_2,k_3}$. One important feature of this operator is that it acts equivariantly with respect to the action of operator U_p on $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega_D^\kappa)$ via the surjection (8.1) and furthermore its action on the kernel of (8.1) is topologically nilpotent (i.e divisible by p). As a result, after iterating U_p to obtain a projector $u_p \in End(\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}^{k_1-k_2} \otimes \Omega_D^{k_2,k_2,k_3}))$, one can expect a good control theorem (see theorem 13.0.2):

$$u_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega_D^\kappa) \cong u_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}^{k_1-k_2} \otimes \Omega_D^{k_2,k_2,k_3}) \tag{8.2}$$

Finally, a detailed study of the relationship between the operator U_p and T_p shows that when the weight κ is regular enough, we have a quasi-isomorphism (see section (12.5)):

$$u_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega_D^\kappa) \cong t_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}, \Omega_D^\kappa) \tag{8.3}$$

This allows us to conclude the proof of the theorem B.

9. Picard variety

For any number field E, its ring of integer is denoted by $\mathcal{O}_E$ and we will denote by $\mathbb{A}_E = \mathbb{A}_{E,f} \times \mathbb{A}_{E,\infty}$ the ring of adeles of E, with ring of finite adeles $\mathbb{A}_{E,f}$ and ring of infinite adeles $\mathbb{A}_{E,\infty}$. For any set S of places of E, we denote by $\mathbb{A}_E^S$ the ring of adeles away from S. Lastly, when $E = \mathbb{Q}$ we drop the subscript E (i.e we only denote the ring of adeles of $\mathbb{Q}$ by $\mathbb{A} = \mathbb{A}_f \times \mathbb{A}_\infty$).

Let K be a quadratic imaginary extension of $\mathbb{Q}$, i.e there exists a square free $\mathbb{Z} \ni \delta < 0$ such that $K = \mathbb{Q}[\sqrt{\delta}]$. We fix a rational prime p that is prime to δ and throughout the rest, we suppose that p splits in K and let us denote by π and $\bar{\pi}$ two primes of $\mathcal{O}_K$ above p. Let $(\bar{.})$ be the unique nontrivial element in $Gal(K/\mathbb{Q})$ (i.e the complex conjugation). Let $(M, <,>, h)$ be a triple where:

(1) M is a $\mathcal{O}_K$-lattice of rank 3, i.e a finitely generated, free $\mathbb{Z}$-module of rank 6 equipped with an $\mathcal{O}_K$-action.
(2) $<,>$: $M \times M \to \mathbb{Z}$ is a non degenerate $\mathcal{O}_K$-linear Hermitian form on M, i.e, we have $< ax, y >=< x, \bar{a}y > \forall a \in \mathcal{O}_K$ and $x, y \in M$. This pairing induces an involution $*$ on $End_{\mathcal{O}_K}(M)$ by sending $f \in End_{\mathcal{O}_K}(M)$ to its $<,>$-adjoint.

(3) $h : \mathbb{C} \to End_{\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{R}}(M \otimes_{\mathbb{Z}} \mathbb{R})$ is a morphism of $\mathbb{R}$-algebras with involutions (involution of $\mathbb{C}$ is given by the usual conjugation), such that $(x, y) \mapsto < x, h(i)y >$ is symmetric and positive definite. Furthermore we require $< h(a)x, y >=< x, h(\bar{a})y >$.

Now the involution $h(i) \in End_{\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{R}}(M \otimes_{\mathbb{Z}} \mathbb{R})$ gives us a decomposition of $V := M\otimes_{\mathbb{Z}}\mathbb{C}$ into $V = V_1 \oplus V_2$ where $h(z)\otimes 1$ acts by z on V_1 and by $\bar{z}$ on V_2. Both V_1 and V_2 can be also seen as $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}$-modules and as a result we have a decomposition $V_i = V_{i,\pi} \oplus V_{i,\bar{\pi}}$ for $i \in \{1, 2\}$ compatible with the decomposition of $K\otimes_{\mathbb{Z}}\mathbb{C} = \mathbb{C}_\pi \oplus \mathbb{C}_{\bar{\pi}}$ (induced by two embedding of K to $\mathbb{C}$) in the sense that only the copy $\mathbb{C}_\pi \cong \mathbb{C}$ (resp. $\mathbb{C}_{\bar{\pi}}$) acts non trivially on $V_{1,\pi} \oplus V_{2,\pi}$ (resp.$V_{1,\bar{\pi}} \oplus V_{2,\bar{\pi}}$). We also make two additional assumptions:

(1) The signature of $(M, <, >, h)$ is $(2, 1)$, meaning that $2 = dim(V_{1,\pi}) = dim(V_{2,\bar{\pi}})$ and $1 = dim(V_{2,\pi}) = dim(V_{1,\bar{\pi}})$.
(2) The paring induced by $<, >$ on $M \otimes \mathbb{Z}_p$ is perfect (this condition is required for the group G defined right below to be reductive over $\mathbb{Z}_p$).

Associated with this triple $(M, <, >, h)$, we can define the following algebraic groups:

(1) The unitary similitude group $G := GU(2, 1)$ whose R-point for each $\mathbb{Z}$-algebra R can be described as:

$$G(R) := \{(g, \eta_g) \in End_{\mathcal{O}_K\otimes_{\mathbb{Z}}R}(M \otimes_{\mathbb{Z}} R) \times R^\times| < g(x), g(y) >= \eta_g < x, y >\}$$

where $\eta(g) : g \mapsto \eta_g$ is called the similitude character .
(2) The unitary group $G' := U(2, 1)$ which sends each $\mathbb{Z}$-algebra R to the set

$$G'(R) := \{(g, \eta) \in G(R)|\eta(g) = 1 \in R^\times\}$$

Remind ourself that the morphism h in the definition of the triple $(M, <, >, h)$ is an $\mathbb{R}$-algebra morphism, thus we have $h_{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \to End_{\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{C}}(M \otimes_{\mathbb{Z}} \mathbb{C})$ and from this we can define a map $\mu : \mathbb{C} \to M \otimes_{\mathbb{Z}} \mathbb{C}$ by $\mu(z) = h_{\mathbb{C}}(z, 1)$. The field of definition of a $GU(\mathbb{C})$-conjugacy class of μ is a finite extension of $\mathbb{Q}$ and we call it the reflex field associated triple $(M, <, >, h)$ above. In this case the reflex field is K itself because of the signature of $<, >$ is $(2, 1)$.

As p splits in K we have $G(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times \times GL_3(\mathbb{Z}_p)$. A level subgroup is an open compact subgroup $\mathcal{U} \subset G(\mathbb{A}_f)$ that we suppose of the form $\mathcal{U} = \mathcal{U}^p\mathcal{U}_p$ with $\mathcal{U}^p \subset G(\mathbb{A}^{p,\infty})$ and $\mathcal{U}_p \subset G(\mathbb{Z}_p)$. Throughout this paper, we will only be interested in the case where the level $\mathcal{U}_p$ is hyperspecial, i.e $\mathcal{U}_p = G(\mathbb{Z}_p)$ or parahoric. More precisely, if we take the canonical basis $\{e_1, e_2, e_3\}$ for $\mathbb{F}_p^3$, there are two parabolic subgroups P_1 and P_2 of $GL_3(\mathbb{Z}_p)$ which are the preimages via the reduction map $GL_3(\mathbb{Z}_p) \to GL_3(\mathbb{F}_p)$ of the subgroups of $GL_3(\mathbb{F}_p)$ that stabilize the line $< e_1 >$ and the plane $< e_1, e_2 >$ respectively. Via the isomorphism $G(\mathbb{Z}_p) \cong \mathbb{G}_m(\mathbb{Z}_p)\times GL_3(\mathbb{Z}_p)$, the parahoric level structure that we consider corresponds to the case where $\mathcal{U}_p = \mathbb{G}_m(\mathbb{Z}_p) \times P_2$ (see also the remark 10.6.2 for an explanation of this choice).

Fix a prime ideal, say π over p. For each level subgroup $\mathcal{U} = \mathcal{U}^p\mathcal{U}_p \subset G(\mathbb{A}_f)$, we define the following moduli problem that sends each connected scheme S over $Spec(\mathcal{O}_{K,\pi})$ to the set of tuples $(A, \lambda, \eta_{\mathcal{U}}, i)$ up to equivalence by isogenies, where

(1) $A \to S$ is an abelian scheme of relative dimension 3.
(2) $i : \mathcal{O}_K \to End(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is an injective morphism of $\mathbb{Z}_{(p)}$-algebra.

(3) A prime-to-p polarization $\lambda : A \to A^\vee$ inducing a Rossati involution which sends $i(k)$ to $i(\bar{k})$ for all $k \in K$.
(4) $\eta_{\mathcal{U}}$ is a $\mathcal{U}$-level structure.

We explain what the level structure $\eta_{\mathcal{U}}$ means. For any abelian variety A of dimension g over S satisfying the conditions $(1),(2),(3)$ above and a geometric point $x \hookrightarrow S$, we can look at the trivialization of the Tate module:

$$H_1(A_x, \mathbb{A}_f^p) \cong M \otimes \mathbb{A}_f^p \tag{9.1}$$

This is an isomorphism of symplectic modules with the symplectic structure on $H_1(A_x, \mathbb{A}_f^p)$ given by the Weil paring and on $M \otimes \mathbb{A}_f^p$ by the standard symplectic structure. The group $G(\mathbb{A}_f^p)$ acts on $M \otimes \mathbb{A}_f^p$ and so does $\mathcal{U}^p \subset G(\mathbb{A}_f^p)$. We are interested in two cases:

(1) If $\mathcal{U}_p = G(\mathbb{Z}_p)$ then $\eta_{\mathcal{U}}$ means a choice of an $\mathcal{U}^p$-orbit of the above isomorphism (9.1).
(2) If $\mathcal{U}_p = \mathbb{Z}_p^\times \times P_2$ then $\eta_{\mathcal{U}}$ means a choice of an $\mathcal{U}^p$-orbit of the isomorphism (9.1) plus a choice of a subgroup H of $A[\pi]$ of order p^2.

The following is well known and can be found in [Lan12].

Theorem 9.0.1. *If the level $\mathcal{U}^p$ is neat, the above moduli problem is represented by a smooth, quasi-projective scheme $\mathcal{M}_{\mathcal{U}}$ over Spec$(\mathcal{O}_{K,\pi})$.*

Above $\mathcal{M}_{\mathcal{U}}$, there is a universal abelian scheme $\mathcal{A}$ of dimension 3 with an $\mathcal{O}_K$ action. We also have the relative differential sheaf $\omega_{\mathcal{A}} := e^*\Omega^1_{\mathcal{A}/\mathcal{M}_{\mathcal{U}}}$ where $e : \mathcal{M}_{\mathcal{U}} \to \mathcal{A}$ is the identity section. This is a locally free sheaf of rank 3 which inherits the $\mathcal{O}_K$-action of $\mathcal{A}$. Since K is also the reflex field, we have a decomposition $\omega_{\mathcal{A}} = \omega_\pi \oplus \omega_{\bar{\pi}}$. In addition, by our assumption on the signature of the triple $(M, <,>, h)$, the type of $\omega_{\mathcal{A}}$ as an $\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}$-module is also $(2,1)$. Thus $rank_{\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}}(\omega_\pi) = 2$ and $rank_{\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}}(\omega_{\bar{\pi}}) = 1$ on all connected components of $\mathcal{M}_{\mathcal{U}}$.

For each triple $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$, with $k_1 \geq k_2$ we define the following coherent sheaf on $\mathcal{M}_{\mathcal{U}}$:

$$\Omega^\kappa := Sym(\omega_\pi)^{k_1-k_2} \otimes det(\omega_\pi)^{k_2} \otimes \omega_{\bar{\pi}}^{k_3}$$

This sheaf extends to the toroidal and minimal compactification whose construction we recall in the next subsection.

9.1. **Compactification.** Fix a level structure $\mathcal{U} = \mathcal{U}^p\mathcal{U}_p$, where the level $\mathcal{U}^p$ is neat and $\mathcal{U}_p \subset G(\mathbb{Z}_p)$. We assume that $\mathcal{U}_p$ is either hyperspecial, i.e $\mathcal{U}_p = G(\mathbb{Z}_p)$ or parahoric, i.e $\mathcal{U}_p \cong \mathbb{Z}_p^\times \times P_2$.

Let $\mathfrak{C}$ be the set of all totally isotropic factor $W \subset V = \mathcal{O}_K^3$ with respect to the hermitian form $<,>$ given by the matrix $diag(1,1,-1)$. For each $W \in \mathfrak{C}$ we denote by $C(V/W^\perp)$ the cone of symmetric hermitian semi-definite positive forms on $(V/W^\perp)_{\mathbb{R}}$ with rational radical. In particular if $W \neq 0$, it can only be of rank 1 and so we can see that $C(V/W^\perp) = \mathbb{R}_{>0}$. For any $W' \subset W$ we have an obvious inclusion $C(V/W^\perp) \subset C(V/W'^\perp)$, this generates an equivalence relation $\sim$ in the set $\coprod_{W \in \mathfrak{C}} C(V/W^\perp)$. Let $\mathcal{C} := \coprod_{W \in \mathfrak{C}} C(V/W^\perp)/\sim$. As there are only two classes of equivalence, 0 and $< e_1 + e_3 >$, we can see that $\mathcal{C} = \mathbb{R}_{>0} \coprod \{0\} = \mathbb{R}_{\geq 0}$. Usually, in order to define a toroidal compactification, we need to make a choice of a rational polyhedral cone decomposition, but here there is only a unique rational polyhedral cone for each $C(V/W^{\perp})$ which is $C(V/W^{\perp})$ itself. Now for each $W \in \mathfrak{C}$ we have a following tower :

$$\mathcal{M}_W \to \mathcal{B}_W \to Y_W$$

Where Y_W is the moduli of elliptic curves over $\mathbb{Z}_{(p)}$ with complex multiplication by $\mathcal{O}_K$, it also comes equipped with a (neat)level structure outside p, and a level at p (depending on the relative positions of the W and the lattice $p\mathbb{Z} \oplus p\mathbb{Z} \oplus \mathbb{Z}$). Let us denote by $\mathcal{E}_W$ the universal elliptic curve over Y_W. Above Y_W there is $\mathcal{B}_W := Ext^1_{\mathcal{O}_K}(\mathcal{E}, \mathbb{G}_m \times \mathcal{O}_K)$ which is also isogenous to the dual $\mathcal{E}_W^{\vee}$ of $\mathcal{E}_W$. Lastly $\mathcal{M}_W$ is a $\mathbb{G}_m$-torsor, relatively affine over $\mathcal{B}_W$ parametrizing 1-motives which carry the level structure $\mathcal{U}_p$ at p (recall that we allow hyperspecial or parahoric level structure). With the interpretation of 1-motifs as in proposition (10.2.14)[Del74], there is a coherent $\mathcal{O}_{\mathcal{B}_W}$-algebra $\mathcal{L}$ with an action of $\mathbb{G}_m$ such that $\mathcal{M}_W = Spec_{\mathcal{B}_W}(\mathcal{L})$. Additionally, there is toroidal embedding of $\mathcal{M}_W$ associated with an affine toroidal embedding of $\mathbb{G}_m$:

$$\mathcal{M}_W \to \mathcal{M}_W^{tor}$$

The action of $\mathbb{G}_m$ on $\mathcal{L}$ induces a decomposition $\mathcal{L} = \oplus_k \mathcal{L}(k)$ via the action of $\mathbb{G}_m$ on it. With these notations we have the description $\mathcal{M}_W^{tor} = Spec_{\mathcal{B}_W}(\oplus_{k\geq 0}\mathcal{L}(k))$. Inside $\mathcal{M}_W^{tor}$ there is a closed strata, which is the zero section of $\mathcal{M}_W^{tor}$ and can be identified with $\mathcal{B}_W$. Let $\widehat{\mathcal{M}_W^{tor}}$ be the completion of $\mathcal{M}_W^{tor}$ along its closed strata. We have the following theorem:

Theorem 9.1.1. *When the level $\mathcal{U}^p$ is neat, and $\mathcal{U}_p$ is either hyperspecial or parahoric, there exists a normal proper scheme, of complete intersection $\mathcal{S}_{\mathcal{U}}$ such that:*

1. *$\mathcal{M}_{\mathcal{U}}$ can be identified as its open subscheme, with the complementary $D = \mathcal{S}_{\mathcal{U}} - \mathcal{M}_{\mathcal{U}}$ is a normal crossing divisor(it is just a finite disjoint union of curves in our case).*
2. *There is a stratification of D indexed by the set $\mathfrak{C}/\mathcal{U}$, with each $W \in \mathfrak{C}$ we have a strata $\mathcal{Z}_W \subset D$.*
3. *The completion of $\mathcal{S}_{\mathcal{U}}$ along one strata $\mathcal{Z}_W$ is isomorphic to $\widehat{\mathcal{M}_W^{tor}}$.*

We will often drop the subscript $\mathcal{U}$ if the context is clear. The minimal compactification $\mathcal{S}^*_{\mathcal{U}}$ of $\mathcal{M}_{\mathcal{U}}$ is obtained by adding a finite number points corresponding to elliptic curves with CM structure by $\mathcal{O}_K$, one for each component of the boundary D. Let x be such a geometric point of $\mathcal{S}^*_{\mathcal{U}} - \mathcal{M}_{\mathcal{U}}$, it can be identified with a point $x \in Y_W$, and if we denote $\widehat{\mathcal{B}_W}, x$ the completion of $\mathcal{B}_W$ along its fibers over x via $\mathcal{B}_W \to Y_W$, then we have the following.

Theorem 9.1.2. *The completed local ring $\widehat{\mathcal{O}}_{\mathcal{S}^*_{\mathcal{U}},x}$ is isomorphic to $\prod_{k\geq 0} H^0(\widehat{\mathcal{B}}_{W,x}, \mathcal{L}(k))$.*

Now the coherent sheaf Ω^{κ} over $\mathcal{M}_{\mathcal{U}}$ that we have introduced in the previous section extends canonically to $\mathcal{S}_{\mathcal{U}}$, and we call $H^0(\mathcal{S}_{\mathcal{U}}, \Omega^{\kappa})$ the space of Picard modular forms of weight κ and level $\mathcal{U}$. We also have cuspidal subspace $H^0(\mathcal{S}_{\mathcal{U}}, \Omega^{\kappa}_D)$ (recall that for any coherent sheaf $\mathcal{F}$ on $\mathcal{S}_{\mathcal{U}}$ we denote by $\mathcal{F}_D := \mathcal{F}(-D)$).

9.2. **Complex Picard variety.** In this section, we briefly go over another description of the complex points $\mathcal{M}_{\mathcal{U}}(\mathbb{C})$ as a finite disjoint union of locally symmetric spaces associated to $G(\mathbb{R})$.

Recall that we have an $\mathcal{O}_K$-lattice M. If we denote $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, then $M_{\mathbb{R}}$ is equipped with an action of $\mathcal{O}_K$ and a complex structure by $h : \mathbb{C} \to End_{\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}} M_{\mathbb{R}}$ and a Hermitian form of signature $(2,1)$. In other word $M_{\mathbb{R}} \cong \mathbb{C}^3$ is a 3-dimensional Hermitian $\mathbb{C}$-vector space of signature $(2,1)$. Up to ismorphism, there is only one such Hermitian space, and thus without loss of generality, we suppose that the Hermitian form is given by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The group $G(\mathbb{R})$ now is just the group of linear automorphism of $M_{\mathbb{R}}$ preserving the form J. Let $\mathbb{X}$ be the set of negative definite lines in the projective space $\mathbb{P}(M_{\mathbb{R}}) = \mathbb{P}^2(\mathbb{C})$ with respect to J, concretely it can be identified with:

$$\mathbb{X} = \{(z_1, z_2) \in \mathbb{C}^2 | -2Im(z_1) + |z_2|^2 < 0\}$$

with this description the action of $G(\mathbb{R})$ on $\mathbb{X}$ is given by:

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} z = \frac{1}{cz+d}(Az+b)$$

where $A \in Mat_{2\times 2}(\mathbb{C})$. It is straightforward to show that $G(\mathbb{R})$ acts transitively on $\mathbb{X}$ with stabilizer $K_\infty := Stab(i,0) = \{A, z \in G(2)(\mathbb{R}) \times \mathbb{C}^\times | \eta(A) = |z|\}$ where $\eta(A)$ is the similitude factor of A, as a result $\mathbb{X} = G(\mathbb{R})/K_\infty$. In fact the following theorem is well known :

Theorem 9.2.1. *Suppose that $\mathcal{U}^p$ is neat. Then the complex point $\mathcal{M}_{\mathcal{U}}(\mathbb{C})$ can be identified with the set $G(\mathbb{Q})\backslash(\mathbb{X} \times G(\mathbb{A})/\mathcal{U}) = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_\infty\mathcal{U}$. In other words, it is a disjoint union of finitely many quotients of $\mathbb{X}$.*

Proof. See section 2.3.2 [CEF$^+$14]. □

Next, we recall the definition of (analytic)Picard modular form and compare it with the algebraic definition we have given earlier.

Let P be the parabolic $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ and let $\rho : P \to GL(W)$ be a finite dimensional complex representation of P. Now for any map $f : \mathbb{X} \to W$ and any $g = (\begin{pmatrix} A & b \\ c & d \end{pmatrix}, \eta) \in G_+(\mathbb{R})$ we define $f|_\rho g : \mathbb{X} \to W$ to be the map given by

$$f|_\rho g(x) := \rho(\eta^{\frac{-1}{2}}(\bar{b}^t x + \bar{A}), \eta^{\frac{-1}{2}}(cz+d))^{-1} f(gx)$$

Definition 9.2.1. For a congruence subgroup Γ of $G(\mathbb{Q})$ and a finite dimensional representation $\rho : P \to GL(W)$. A holomorphic automorphic form of weight ρ and level Γ is a holomorphic function $f : \mathbb{X} \to W$ that satisfies: $f|_\rho \gamma = f$ for any $\gamma \in \Gamma$. We denote the set of automorphic form of weight ρ level $\mathcal{U}$ by $S(\rho, \mathcal{U})$.

Consider the universal abelian variety $\mathcal{A}$ over $\mathcal{M}_{\mathcal{U}}$. Over $\mathcal{M}_{\mathcal{U}}$ there is a $GL_2 \times GL_1$-torsor $\mathcal{T} := \underline{Isom}(\mathcal{O}^2_{\mathcal{M}_{\mathcal{U}}}, \omega_\pi) \times \underline{Isom}(\mathcal{O}_{\mathcal{M}_{\mathcal{U}}}, \omega_{\bar{\pi}})$. For each scheme $S = Spec(R)$ over

$\mathbb{Z}_p$ a S-point of $\mathcal{T}$ corresponds to a pair of trivializations $R^2 \cong \omega_{\mathcal{A},\pi}$ and $R \cong \omega_{\mathcal{A},\bar{\pi}}$. The group $GL_2(R) \times GL_1(R)$ acts naturally on $R^2 \times R$ and thus on $\mathcal{T}(R)$.

Let R be a $\mathbb{Z}_p$-algebra and by abuse of notation, we will still denote by $\mathcal{M}_{\mathcal{U}}$ the base change of $\mathcal{M}_{\mathcal{U}}$ over $\mathbb{Z}_p$ to R

Definition 9.2.2. Let $\rho : GL_2 \times GL_1 \to GL_n$ be an algebraic representation (for some n). An algebraic automorphic form f defined over R for G of weight ρ, and level $\mathcal{U}$ is a law that for each R-algebra R' sends each pair $(A, \psi) \in \mathcal{M}_{\mathcal{U}}(R') \times \mathcal{T}(R')$ to $f(A, \psi) \in R'^n$ such that:

(1) $f(A, g\psi) = \rho({}^tg^{-1})f(A, \psi)$
(2) For any morphism of R-algebra $R' \to R''$ and any $(A, \psi) \in \mathcal{M}_{\mathcal{U}}(R') \times \mathcal{T}(R')$ we have $f(A''_R, \psi_{R''}) = f(A, \psi) \otimes_{R'} 1_{R''} \in R''^n$.

We denote by $S^{alg}(\rho, \mathcal{U}, R)$ the set of all algebraic automorphic forms defined over R of weight ρ, level $\mathcal{U}$. In particular when $R = \mathbb{C}$ we wish to compare $S(\rho, \mathcal{U})$ and $S^{alg}(\rho, \mathcal{U}, \mathbb{C})$.

Proposition 9.2.1. *We have a bijection between $S^{alg}(\rho, \mathcal{U}, \mathbb{C})$ and $S(\rho, \mathcal{U})$.*

Proof. See proposition 3.17[CEF+14]. □

There is yet another way to see $S^{alg}(\rho, \mathcal{U}, R)$ as a global section of some coherent sheaf $\mathcal{V}^\rho$ on $\mathcal{M}_{\mathcal{U}} \times R$ ($\mathcal{V}^\rho$ can be defined as $\mathcal{T} \times^\rho R^n$, where R^n is the underlying module of the representation ρ). In particular if we consider $\rho(\kappa)$ to be the highest weight (algebraic)representation of $GL_2 \times GL_1$ of weight κ over some $\mathbb{Z}_p$-algebra, using Frobenius reciprocity, we can see that the $\mathcal{V}^\rho$ is isomorphic to Ω^κ, and thus $S^{alg}(\rho(\kappa), \mathcal{U}, R)$ is in bijection with $H^0(\mathcal{M}_{\mathcal{U}} \times R, \Omega^\kappa)$.

9.3. **Structure of the p-torsion.** In this section we recall some facts regarding the p-torsion $\mathcal{A}[p]$ of the universal abelian schemes which plays a crucial role in later studies of local properties of Picard variety. Fortunately, in the case that we have specialized in, it is not difficult to give an explicit description of $\mathcal{A}[p]$.

Let A be any abelian variety of dimension 3 over $\mathbb{Z}_p$ with $\mathcal{O}_K$-action of type $(2, 1)$. For each $n \in \mathbb{N}$ the action by $\mathcal{O}_K$ induces an action by $\mathcal{O}_K/p^n$ on $A[p^n]$, the p^n-torsion of A. Notice that $\mathcal{O}_K/p^n \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$, as a result we have an action of $\mathcal{O}_{K,p} = \mathbb{Z}_p \times \mathbb{Z}_p$ on $A[p^\infty]$ by passing to the limit. Let e_1 and e_2 be the projectors on the first and second factor of $\mathcal{O}_{K,p}$, then $A[\pi^\infty] = e_1A[p^\infty]$ and $A[\bar{\pi}^\infty] = e_2A[p^\infty]$. Both of these factors are p-divisible group of height 3. By our assumption on the signature of the action of $\mathcal{O}_K$, we see that the dimension of $A[\pi^\infty]$ is 2 and $A[\bar{\pi}^\infty]$ is 1.

Let k be a perfect field of characteristic p and $W := W(k)$ its ring of Witt vector as well as $K := W[\frac{1}{p}]$. Let $\sigma : W \to W$ be Frobenius lift of k. Recall that a Dieudonné module over W is a free W-module equipped with a σ-linear endomorphism F and a σ^{-1}-linear endomorphism V. There is a contravariant functor from the category of p-divisible groups over k to the category of Dieudonné modules over W.

$$\begin{aligned} \{p\text{-divisible groups}/k\} &\longrightarrow \{\text{Dieudonné modules}\} \\ G &\mapsto D(G) = \varinjlim_n Hom_{k-group}(G[p^n], W_n(k)) \end{aligned} \tag{9.2}$$

This induces an (anti-)equivalence between the category of p-divisible group over k and the sub-category of Dieudonné modules of slopes in $[0,1]$. If G and G' are two isogenous p-divisible group over k then $D(G)\otimes_W K \cong D(G')\otimes_W K$. Furthermore, if we fix G then isomorphism classes of p-divisible groups isogenous to G are in bijection with (F,V)-stable W-lattices inside $D(G)\otimes_W K$.

When k is furthermore assumed to be algebraically closed, the category of Dieudonné modules up to isogeny is semisimple with simple objects completely determined by their slopes. For example, if $\lambda = \frac{s}{r} \in [0,1]\cap\mathbb{Q}$ with s,r coprime. The simple Dieudonné module over K of slope $\frac{s}{r}$ is given by E_λ with a K-basis $e, Fv, ..., F^{r-1}v$ and $F^r v = p^s v$ whereas $V = pF^{-1}$.

As a consequence, any given p-divisible group G over $k=\bar{k}$ is isogenous to a direct sum $D(G)\otimes_W K = \oplus_{i=1}^n E_{\lambda_i}^{m_i}$ with $\lambda_1 < \lambda_2 < ... < \lambda_n$ and $m_i \in \mathbb{N}$. We say that G is of slope $(\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_n, ..., \lambda_n)$ with λ_i is repeated m_i times. In particular for a p-divisible group of height 3, dimension 2 there are only 3 cases and more specially, isogeny classes become in fact isomorphism classes (see p92,93 [Dem72]) :

(1) If $A[\pi^\infty]$ is of slope $(0,1,1)$, its Cartier dual $A[\bar{\pi}^\infty]$ is of slope $(0,0,1)$ and their respective Dieudonné modules are given by $(D(A[\pi^\infty]), F_1, V_1)$ and $(D(A[\bar{\pi}^\infty]), F_2, V_2)$ where $D(A[\pi^\infty]) = D(A[\bar{\pi}^\infty]) = W(k)^3$ with

$$F_1 = V_2 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$V_1 = F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}$$

(2) If $A[\pi^\infty]$ is of slope $(\frac{1}{2},\frac{1}{2},1)$, its Cartier dual $A[\bar{\pi}^\infty]$ is of slope $(0,\frac{1}{2},\frac{1}{2})$ and their respective Dieudonné modules are given by $(D(A[\pi^\infty]), F_1, V_1)$ and $(D(A[\bar{\pi}^\infty]), F_2, V_2)$ where $D(A[\pi^\infty]) = D(A[\bar{\pi}^\infty]) = W(k)^3$ with

$$F_1 = V_2 = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & p & 0 \end{pmatrix}$$

and

$$V_1 = F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & p & 0 \end{pmatrix}$$

(3) If $A[\pi^\infty]$ is of slope $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$, its Cartier dual $A[\bar{\pi}^\infty]$ is of slope $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and their respective Dieudonné modules are given by $(D(A[\pi^\infty]), F_1, V_1)$ and $(D(A[\bar{\pi}^\infty]), F_2, V_2)$ where $D(A[\pi^\infty]) = D(A[\bar{\pi}^\infty]) = W(k)^3$ with

$$F_1 = V_2 = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & p \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$V_1 = F_2 = \begin{pmatrix} 0 & 0 & p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

10. Higher Hida theory for $U(2,1)$

The goal of this section is to construct the generalized Hasse invariant, together with the classical Hasse invariant, they are used in the study of the stratification of the special fiber $\mathcal{S}_1 = \mathcal{S} \times Spec(\mathbb{F}_p)$. Later in (10.2) we will construct a Hecke operator T_p that acts locally finitely on the complex $\mathrm{R}\Gamma(\mathcal{S}, \Omega_D^\kappa)$ and show various compatibilities of this operator with respect to the Hasse invariants. In the last subsection (11.2) we use the strategy outlined in the introduction to deduce the basic theorems regarding some preliminary properties of the ordinary part $t_p\mathrm{R}\Gamma(\mathcal{S}, \Omega_D^\kappa)$.

10.1. **Generalized Hasse invariant.** In this section S is always a reduced $\mathbb{F}_p$-scheme, we denote by $BT_n(S)$ the category of truncated Barsotti-Tate group of level n. For any $G \in BT_n(S)$, let $\Omega^1_{G/S}$ be its differential sheaf and $\omega_G := e^*\Omega^1_{G/S}$ with e the identity section of $G \to S$ and $\det_G := det(\omega_G)$ the determinant of ω_G.

We have a contravarient functor D from $BT_1(S)$ to the category of locally free $\mathcal{O}_S$-modules of finite rank (see [Ber79]). This functor sends each $G \in BT_1(S)$ to its Dieudonné crystal:

$$D(G) := Ext^1_{cris}(G, \mathcal{O}_S/\mathbb{F}_p)_S$$

It is a locally free $\mathcal{O}_S$-module and we have $D(G^{(p)}) = D(G)^{(p)}$. By functoriality, these modules come equipped with actions of Frobenius $F : D(G)^{(p)} \to D(G)$ and of Verschibung $V : D(G) \to D(G)^{(p)}$. Furthermore, there exists an exact sequence(Hodge filtration) :

$$0 \to \omega_G \to D(G) \to \omega^{-1}_{G^D} \to 0$$

10.1.1. *Classical Hasse invariant.* Let G be a finite locally free group over some $\mathbb{F}_p$-scheme S. The Verschibung $V : G^{(p)} \to G$ induces a map $det(V^*) : det_G \to \det_{G^{(p)}} = \det_G^p$, and gives us the so-called Hasse invariant which is a section $Ha(G) \in H^0(S, det_G^{p-1})$. The following example says why this invariant is useful.

Example 10.1.1. *let A be an abelian scheme of dimension g over S and $A[p]$ its p-torsion. The p-rank of A at each geometric point $x \in S$ is defined as the dimension of $\bar{\mathbb{F}}_p$-vector space $A_x[p](\bar{\mathbb{F}}_p)$ and we say A_x is ordinary if its p-rank is g. It is well known that the Hasse invariant $Ha(A_x[p])$ is invertible if and only if A_x is ordinary (cf.[Far11] 2.3). As a result, when viewed as a section $Ha(A[p]) \in H^0(S, det_A^{p-1})$ of the line bundle det_A^{p-1}, the non-zero locus of $Ha(A[p])$ is exactly the ordinary locus of S.*

10.1.2. *Generalized Hasse invariant.* We now begin the construction of the generalized Hasse invariant. Let S be a reduced $\mathbb{F}_p$-scheme and $G \in BT_1(S)$ of height 3 and dimension 2 such that the multiplicative rank of G over S is less than 2. The goal is to construct a section of $H^0(S, det_G^{p^2-1})$ that vanishes exactly at points where G is of multiplicative rank 0.

We first construct such Hasse invariant when G is of multiplicative rank 1, étale rank 0. This means that G is connected and there is a sub group $G_m \subset G$ that is étale locally isomorphic to μ_p. In this case, the quotient $H := G/G_m$ is in $BT_1(S)$ and it is of height 2 with dimension 1. The Hodge filtration for $D(H)$ and $D(H)^{(p)}$ gives us:

$$\begin{tikzcd} 0 \arrow[r] & \omega_H \arrow[r, "i_1"] \arrow[d, "D(V)_{|\omega_H}"] & D(H) \arrow[r, "q_1"] \arrow[d, "D(V)"] & \omega_{H^D}^{\vee} \arrow[r] \arrow[d, "D(V)"] & 0 \\ 0 \arrow[r] & \omega_H^{(p)} \arrow[r, "i_2"] & D(H)^{(p)} \arrow[r, "q_2"] & (\omega_{H^D}^{\vee})^{(p)} \arrow[r] & 0 \end{tikzcd}$$

As H is not ordinary, the restriction of $D(V)$ to ω_H is zero pointwise, hence zero as S is reduced, this means that the map $D(V) : D(H) \to D(H)^{(p)}$ factors through $D(V) : \omega_{H^D}^{\vee} \to D(H)^{(p)}$. The map $D(V) : \omega_{H^D}^{\vee} \to (\omega_{H^D}^{\vee})^{(p)}$ is always zero by construction, so that in the end we have an isomorphism $D(V) : \omega_{H^D}^{\vee} \to \omega_H^{(p)}$.

Raise both sides of this isomorphism to the $p-1$ power, we obtain (we just choose to write det_H instead of ω_H here):

$$det(D(V))^{p-1} : det_{H^D}^{1-p} \to det_H^{p(p-1)}$$

In other words we obtain a section that we denote by $Ha^1 \in H^0(S, \det_H^{p(p-1)} \otimes \det_{H^D}^{p-1})$. Finally we define:

$$Ha'(G) := Ha(G_m)^{p+1} \otimes Ha^1(H) \in H^0(S, det_G^{p(p-1)} \otimes det_{G_m}^{p-1} \otimes det_{H^D}^{p-1}) \tag{10.1}$$

In order to get a section in $H^0(S, det_G^{p^2-1})$ we use the following lemma.

Lemma 10.1.1. *Let $G \in BT_n(S)$. There is a canonical isomorphism:*

$$\theta : det_G^{p-1} \cong det_{G^D}^{p-1}$$

Proof. Proposition 2 [Far11]. □

In particular this lemma gives: $det_{H^D}^{p-1} \cong det_H^{p-1}$ and thus an isomorphism:

$$det_G^{p(p-1)} \otimes det_{G_m}^{p-1} \otimes det_{H^D}^{p-1} \cong det_G^{p^2-1}$$

As a result, we get a section that we still denote by $Ha'(G) \in H^0(S, det_G^{p^2-1})$.

Now, we explain how to extend this construction of generalized Hasse invariant to the situation where:

(1) $G \in BT_1(S)$ is of height 3, dimension 2, and multiplicative rank less than 2 whereas S is a normal reduced scheme.
(2) There is a dense subscheme $S' \hookrightarrow S$ such that $G|_{S'}$ is of multiplicative rank 1, étale rank 0. Furthermore, we suppose that S' is maximal with this property.

Lemma 10.1.2. *The hypothesis* (1) *and* (2) *above imply that the map $V^* : \omega_G \to \omega_G^{(p)}$ has constant rank* 1 *over S and the map $V^* : \omega_{G^D} \to \omega_{G^D}^{(p)}$ is zero.*

Proof. Since S is reduced, it is enough to check the rank of the maps point by point. Over any geometric point, using the fact that $\omega_G \cong V(D(G^D))/pD(G^D)$ and

$\omega_{G^D} \cong V(D(G))/pD(G)$ we can use the explicit description of $D(G)$ and $D(G^D)$ given in section (9.3) to verify the claims. $\square$

Under these assumptions, we will define a section $Ha'(G) \in H^0(S, det_G^{p^2-1})$ such that the restriction of $Ha'(G)$ to S' is the Hasse invariant we have previously defined, i.e $Ha'(G)|_{S'} = Ha'(G|_{S'})$. Furthermore, this section vanishes on the complement of S' in S at order 1.

We first observe that, since $G|_{S'}$ is of multiplicative rank 1, étale rank 0, we have a multiplicative subgroup $H \subset G$ of rank 1, but this group can be seen as the image of the map $V : G[F]^{(p)} \to G[F]$ as well. Indeed, let $K := ker(V : G[F]^{(p)} \to G[F])$. We show that K is locally isomorphic to α_p, the additive group of order p. As S is reduced, we can verify this at each geometric points. Over each geometric point of S', the crystal of G is given by $D(G) = W(\bar{\mathbb{F}}_p)^3$ with $F = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & p & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & p & 0 \end{pmatrix}$ with respect to the canonical basis $\{e_1, e_2, e_3\}$ and so $D(G[F])$ is given by $\bar{\mathbb{F}}_p\bar{e}_1 \oplus \bar{\mathbb{F}}_p\bar{e}_3$ with $F = 0$ and $V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. As a result $D(K) \cong \bar{\mathbb{F}}_p\bar{e}_3$ with $V = F = 0$, and we can conclude that K is locally isomorphic to α_p. Finally, since $G|_{S'}$ is of slope $(\frac{1}{2}, \frac{1}{2}, 1)$ the quotient $G[F]^{(p)}/K$, now seen as a subgroup of $G[F]$ via $V : G[F]^{(p)} \to G[F]$, is multiplicative over S'. We have the following lemma:

Lemma 10.1.3. *There exists a finite flat subgroup $H \subset G[F]$ such that $H|_{S'}$ is locally isomorphic to μ_p.*

Proof. Let $K := ker(V : G[F]^{(p)} \to G[F])$ and put $H := G[F]^{(p)}/K$. If we can show that H is finite flat, then by the previous discussion, the restriction of H to S' is locally isomorphic to μ_p.

We first show that K is finite flat. It is obviously finite, and for the flatness, we can examine the rank function which to each point $s \in S$ associates $rank_s(K)$ which is simply the rank of the kernel of V at this point. If this function is locally constant then K is finite flat.

For the points of slope $(\frac{1}{2}, \frac{1}{2}, 1)$ we have seen just above that K is of rank p. For the point of slope $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, we proceed similar. The crystal of G at this point is given by $D(G) = W(\bar{\mathbb{F}})_p^3$ with $F = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & p \\ 1 & 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with respect to the canonical basis $\{e_1, e_2, e_3\}$ and so $D(G[F])$ is given by $\bar{\mathbb{F}}_p\bar{e}_1 \oplus \bar{\mathbb{F}}_p\bar{e}_2$ with $F = 0$ and $V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus K is indeed of constant rank p.

Finally since K is flat, the quotient map $H = G[F]^{(p)}/K$ is also flat, even faithfully flat because we clearly have these Cartesian squares and flatness is a property local on the target in the fppf topology:

$$\begin{tikzcd} K \times_S G[F]^{(p)} \arrow[r, "\sim"] \arrow[d, "pr_2"'] & G[F]^{(p)} \times_H G[F]^{(p)} \arrow[r, "pr_2"] \arrow[d, "pr_2"'] & G[F]^{(p)} \arrow[d, "pr"] \\ G[F]^{(p)} \arrow[r, equal] & G[F]^{(p)} \arrow[r, "pr"] & H \end{tikzcd} \tag{10.2}$$

This shows that H is finite flat as we have a flat composition $G[F]^{(p)} \to H \to S$ with the first map being faithfully flat.

□

Now, the exact sequence $H \to G[F] \to G[F]/H$ induces the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \omega_{G[F]/H} \arrow[r] \arrow[d, "0"] & \omega_{G[F]} \arrow[r] \arrow[d, "V^*"] & \omega_H \arrow[r] \arrow[d, "V_H^*"] & 0 \\ 0 \arrow[r] & \omega^{(p)}_{G[F]/H} \arrow[r] & \omega^{(p)}_{G[F]} \arrow[r] & \omega^{(p)}_H \arrow[r] & 0 \end{tikzcd} \tag{10.3}$$

The left vertical map is 0 by the definition of H, so this gives us a map: $V_H^* : \omega_H \to \omega_H^{(p)}$ and $W : \omega^{(p)}_{G[F]/H} \to \omega^{(p)}_G / V^*\omega_G$. The most important thing here is that these two map vanish on $S \backslash S'$ and have the same order of vanishing. Indeed, we can check this point by point as S is reduced. Let $x \in S\backslash S'$ be a generic point, the localizations $\omega_{G,x}$ and $\omega^{(p)}_{G,x}$ are free $\mathcal{O}_{S,x}$-modules of rank 2, so we can choose a basis (e_1, e_2) for $\omega_{G,x}$ and (f_1, f_2) for $\omega^{(p)}_{G,x}$ so that $\omega_{G[F]/H,x} \cong \mathcal{O}_S e_1$ and $\omega^{(p)}_{G[F]/H,x} \cong \mathcal{O}_S f_1$. By lemma (10.1.2) $V : \omega_{G,x} \to \omega^{(p)}_{G,x}$ is of rank 1. So the matrix for V^* with respect to previously chosen basis is $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$. Plus V^* vanishes over x and has rank 1, so $a \in \mathcal{O}^*_{S,x}$ and $b \in \mathfrak{m}_x$, the maximal ideal of $\mathcal{O}_{S,x}$. As a result, both V_H^* and W vanish at x with the same vanishing order.

We have the following diagram:

$$\begin{tikzcd} \omega_G \arrow[r] \arrow[d] & D(G) \arrow[r] \arrow[d, "V^*"] & \omega^\vee_{G^D} \arrow[d, "0"] \\ \omega^{(p)}_G \arrow[r] & D(G)^{(p)} \arrow[r] & \omega^{\vee\,(p)}_{G^D} \end{tikzcd} \tag{10.4}$$

Since the rows are exact and the right vertical arrow is 0, we have a map $Z : \omega^\vee_{G^D} \to \omega^{(p)}_G / V^*\omega_G$ by the snake lemma. This map is in fact an isomorphism. Indeed, this can be checked on points as S is reduced. By definition of S, the multiplicative rank of G at any geometric points is less than 2, and so the rank of $V^* : D(G) \to D(G)^{(p)}$ is 2 (see (9.3)). Further more, by lemma (10.1.2), the rank of $V^* : \omega_G \to \omega^{(p)}_G$ is 1. As a result, the map $Z : \omega^\vee_{G^D} \to \omega^{(p)}_G / V^*\omega_G$ is not zero and thus an isomorphism since both sheaves are of rank 1.

Now, let $v = det(V_H^*)$; $w = det(W)$; $z = det(Z)$ and set $Ha'(G) := v^p \otimes (w^{-1} \circ z)^{p-1}$ which is a section in:

$$H^0(S, det_H^{p(p-1)}) \otimes det_{G[F]/H}^{p(p-1)} \otimes det_{G^D}^{p-1}) = H^0(S, det_G^{p(p-1)} \otimes det_{G^D}^{p-1})$$

Using the isomorphism $\theta : det_G^{p-1} \cong det_{G^D}^{p-1}$ we can think of $Ha'(G)$ as a section of $H^0(S, det_G^{p(p-1)})$. This is the generalized Hasse invariant that we want to construct.

Lemma 10.1.4. *The section $Ha'(G)$ vanishes at order 1 along the complementary $S\backslash S'$ of S' and it extends the section $Ha'(G|_{S'})$ constructed over S' earlier.*

Proof. Indeed, the map V_H^* and W vanish over $S\backslash S'$ at the same order and the map Z is an isomorphism. By definition $Ha'(G) = v^p \otimes (w^{-1} \circ z)^{p-1}$ has vanishing order 1 over $S\backslash S'$. Plus, since the restriction of H to S' is the multiplicative group $G_m = \mu_p$, the restriction of $Ha'(G)$ to S' agrees with $Ha'(G|_{S'})$. □

10.1.3. *Compatibility with base change.* Let S be a scheme of characteristic p. Suppose that we have an isogeny $\lambda : G \to G'$ where $G, G' \in BT_1(S)$, both of height 3, dimension 2. This map induces a map : $\lambda^* : det_G \to det_{G'}$ and we would like to ask if $\lambda^{*p-1} : det_{G'}^{p-1} \to det_G^{p-1}$ sends $Ha(G') \in H^0(S, det_{G'}^{p-1})$ to $Ha(G) \in H^0(S, det_G^{p-1})$. Similarly suppose that the second Hasse invariant exists in this case, does λ^{*p^2-1} : $det_{G'}^{p^2-1} \to det_G^{p^2-1}$ sends $Ha'(G')$ to $Ha'(G)$? The answer to both questions is the obvious yes if the isogeny is étale but in general, they are false. However we have the following propositions.

Proposition 10.1.1. *Suppose that G and G' are of multiplicative type, and $\lambda : G \to G'$ is an isogeny. Then we can define an isomorphism (depending on λ):*

$$\lambda_{norm}^* : det_{G'} \to det_G$$

such that $\lambda_{norm}^{*,p-1}$ sends $Ha(G')$ to $Ha(G)$.

Proof. This is lemma 6.2.4.1 in [Pil18]. □

Proposition 10.1.2. *Suppose that G and G' are of constant multiplicative rank over S and $\lambda : G \to G'$ is an isogeny with kernel L such that geometrically, L is a direct factor of $G[p]$ (i.e for every geometric point $x \to S$, there exists a subgroup $H_x \in G_x[p]$ such that $L_x \oplus H_x = G_x[p]$). Then there exist an isomorphisms λ_{norm}^* : $det_{G'} \to det_G$ such that $\lambda_{norm}^{*,p-1}(Ha(G')) = Ha(G)$ and $\lambda_{norm}^{*,p^2-1}(Ha'(G')) = Ha'(G)$.*

Proof. This is lemma 6.2.4.2 in [Pil18]

□

Remark 10.1.1. *The isomorphism λ_{norm}^* is a normalization of $det(\lambda^*) : det_{G'} \to det_G$ where λ^* is the differential $\lambda^* : \omega_{G'} \to \omega_G$ and this is the morphism that will be used to define the Hecke operator later.*

10.2. **Hecke operator T_p.** An important ingredient of the classical Hida theory is a Hecke operator T_p which we use to construct the so called ordinary automorphic forms. In this section we explain the construction of such operator.

So far, we have introduced the Picard varieties with hyperspecial and parahoric level structures that we will denote by $\mathcal{M}$ and $\mathcal{M}_{par}$ respectively. Recall that we have a universal abelian variety $\mathcal{A}$ over $\mathcal{M}$ and that $\mathcal{A}[p] = \mathcal{A}[\pi] \times \mathcal{A}[\bar{\pi}]$. The scheme $\mathcal{M}_{par}$ is simply the solution to the following moduli problem:

$$\mathcal{M}_{par} : \{\mathbb{Z}_p - algebra\} \to \{Sets\}$$
$$R \mapsto (A, H, i, \eta_{\mathcal{U}}, \lambda)/_\sim$$

with $(A, i, \eta_{\mathcal{U}}, \lambda)/R)/_\sim \in \mathcal{M}(R)$ and $H \subset A[\pi]$ is a totally isotropic subgroup of order p^2.

We have seen that $\mathcal{M}_{par}$ is quasi-projective schemes of local complete intersection but it might help us understand the geometry of $\mathcal{M}_{par}$ better by spelling out the structure of its local rings.

Proposition 10.2.1. *The completed local rings of $\mathcal{M}_{par}$ is either isomorphic to $\mathbb{Z}_p[[a,b]]$ or $\mathbb{Z}_p[[a,b,c]]/(ac-p)$.*

Proof. To handle this kind of question, we use the theory of local model (see [Bel02] section 4, 6). Let $M = M' = \mathbb{Z}_p^6$, both equipped with the standard symplectic structure. The action of $\mathcal{O}_K$ on these modules factors through the $\mathbb{Z}_p$-linear action of $\mathcal{O}_K \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Via this isomorphism, the action of $(1,0)$ is given by the matrix $diag(1,1,1,0,0,0)$ while the action of $(0,1)$ is given by $diag(0,0,0,1,1,1)$. There is a map(isogegy): $\phi : M' \to M$ given by the matrix $diag(p,p,1,1,1,1)$.

The local model for $\mathcal{M}_{par}$ will be given as the moduli space of totally isotropic direct factors $\omega \subset M$ and $\omega' \subset M'$ such that both have signature $(2,1)$ with respect to the action of $\mathcal{O}$ and $\phi(\omega') \subset \omega$. With the basis of M and M' we have chosen, the universal deformation can be written down according to which point $x \in \mathcal{M}$ we start with, and up to choosing a suitable basis, there are two types of points :

(1) If x corresponds to the pair $\omega =< e_2, e_3, e_6 >; \omega' =< e_2', e_3', e_6' >$ then the universal pair are:
$$\omega =< e_2 + ae_1, e_3 + be_1, e_6 + ae_5 + be_4 >$$
$$\omega' =< e_2' + ae_1', e_3' + be_1', e_6' + ae_5' + be_4' >$$
With the condition $\phi(\omega') \subset \omega$, we see that the deformation ring is isomorphic to $\mathbb{Z}_p[[a,b]]$.

(2) If x corresponds to the pair $\omega =< e_2, e_3, e_6 >; \omega' =< e_1', e_3', e_6' >$ then the universal pair are:
$$\omega =< e_2 + ae_1, e_3 - abe_1, e_6 - ae_5 + abe_4 >$$
$$\omega' =< e_1' + ce_2', e_3' + be_2', e_6' - ae_5' + abe_4' >$$
That means the deformation ring is isomorphic to $\mathbb{Z}_p[[a,b,c]]/(ac-p)$.

□

We have two natural maps from $\mathcal{M}_{par}$ to $\mathcal{M}$:

$$\begin{tikzcd} & \mathcal{M}_{par} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{M} & & \mathcal{M} \end{tikzcd}$$

The first map p_1 just forgets the subgroup H, sending $(A, H, i, \eta_{\mathcal{U}}, \lambda)$ to $(A, i, \eta_{\mathcal{U}}, \lambda)$. The second map p_2 sends $(A, H, i, \eta_{\mathcal{U}}, \lambda)$ to $(A' := \frac{A}{H+H^{\perp}}, \tilde{i}, \tilde{\eta}_{\mathcal{U}}, \tilde{\lambda})$, where :

(1) $\tilde{i}$ is the induced $\mathcal{O}_K$ action on A, this is well defined because the action of $\mathcal{O}_K$ on $A[\pi]$ factorizes through the action of $\mathbb{Z}_p$ on $A[\pi]$ and this obviously stabilizes $H \subset A[\pi]$, similarly for $H^{\perp} \in A[\bar{\pi}]$. As a result, $H + H^{\perp}$ is stable by the action of $\mathcal{O}_K$, and thus we have an action of $\mathcal{O}_K$ on the quotient A'.
(2) $\tilde{\eta}_{\mathcal{U}}$ is the induced level structure, via the isomorphism $H_1(A_x, \mathbb{A}_f^p) \cong H_1(A'_x, \mathbb{A}_f^p)$ for any geometric point $x \hookrightarrow S$.
(3) $\tilde{\lambda} : A' \to (A')^t$ is the descended polarization from $p\lambda : A \to A^t$ which is still principle because $H + H^{\perp}$ is maximal isotropic. In other words, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p\lambda"] & A' \arrow[d, "\tilde{\lambda}"] \\ A^t & A'^t \arrow[l, "f^t"] \end{tikzcd}$$

where $f : A \to A'$ is the isogeny with kernel $H + H^{\perp}$.

Now we also have a unique toroidal compactification $\mathcal{S}_{par}$ of $\mathcal{M}_{par}$, and the previous correspondence extends to:

$$\begin{tikzcd} & \mathcal{S}_{par} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{S} & & \mathcal{S} \end{tikzcd}$$

Lemma 10.2.1. *The morphisms p_1 and p_2 are finite flat.*

Proof. The scheme $\mathcal{S}$ is smooth and by proposition (10.2.1), the scheme $\mathcal{M}_{par}$ is of local complete intersection, in particular, it is Cohen-Macaulay. By the general theory of toroidal compactification, $\mathcal{S}_{par}$ is Cohen-Macaulay too (in general we can choose a toroidal compactification to be Cohen Macaulay if the interior is (see 14.2 [LAN16]), but in our case the compactification is unique, so this is harmless). The theorem of miracle flatness then tells us that it is enough to check that the projections p_1 and p_2 have finite fibers at each points. This is indeed true and can be checked as follow. For each geometric point $x \hookrightarrow \mathcal{S}$ outside the boundary with residue field $\bar{k}_x$ and represented by an abelian variety $\mathcal{A}_x$, the fibers $p_1^{-1}(\mathcal{A}_x)$ simply counts all subgroups $H \subset \mathcal{A}_x[\pi]$ of order p^2, and the fibers $p_2^{-1}(\mathcal{A}_x)$ counts the number of isogenies $\mathcal{A}'_x[\pi^{\infty}] \to \mathcal{A}_x[\pi^{\infty}]$ with kernel $H \subset \mathcal{A}'_x[\pi]$ of order p^2. Now, we can use the classification of Dieudonné modules given in (9.3) to count such set. For example for the map p_1, let (D_x, F, V) be the underlying Dieudonné module of $\mathcal{A}_x[\pi]$, the set of subgroup $H \subset \mathcal{A}_x[\pi]$ of order p^2 is then in bijection with the set of sub modules of D_x of rank 1 (i.e sub $\bar{k}_x$-vector space of dimension 1 of D_x stable by the action of F, V). We can work type by type, for instance if the point $\mathcal{A}_x[\pi]$ is of multiplicative rank 0, its Dieudonné module is given by $D(x) = \bar{k}_x^3$ with:

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This means that the only rank one stable module of D_x is $\bar{k}_x e_3$ (where $\{e_1, e_2, e_3\}$ is the canonical basis of $\bar{k}_x^3$). Hence $p_1^{-1}(\mathcal{A}_x)$ has only one point. We can similarly check that $p_1^{-1}(\mathcal{A}_x)$ and $p_2^{-1}(\mathcal{A}_x)$ are finite for x of any types.

On the boundary, we remark that in our case the rational polyhedral cone decomposition is unique and for a geometric point $x \hookrightarrow D$, the underlying semi-abelian scheme is ordinary, so there is always finite number of subgroups of fixed order. Thus the maps p_1 and p_2 are quasi-finite over the boundary too. $\square$

This finite flatness makes it easier for us to study the Hecke operator associated with the correspondence $\mathcal{S} \xleftarrow{p_2} \mathcal{S}_{par} \xrightarrow{p_1} \mathcal{S}$ or other similar correspondences later on.

10.3. **Correspondence.** Let X be a scheme. We denote by $D(\mathcal{O}_X)$ the derived category of coherent sheaf on X. There are full subcategory $D^+(\mathcal{O}_X)$ (resp. $D^-(\mathcal{O}_X)$, resp. $D^b(\mathcal{O}_X)$) of objects isomorphic to complexes that are bounded above (resp. below; resp. above and below). Let $f : X \to Y$ be a morphism of S-schemes. We suppose that f is finite flat. In this setting one can define the exceptional inverse image functor $f^! : Coh(Y) \to Coh(X)$ between the categories of coherent sheaves on Y and X. Explicitly to each $\mathcal{F} \in Coh(Y)$ we can define [1] $f^!\mathcal{F} = \underline{Hom}(f_*\mathcal{O}_X, \mathcal{F})$. This functor satisfies two following crucial properties:

Proposition 10.3.1. (1) *We have a natural transformation $Tr : f_*f^! \to Id$.*
(2) *For a flat coherent sheaf $\mathcal{F} \in Coh(Y)$ we have $f^!\mathcal{F} = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^!\mathcal{O}_Y$*

Proof. Since the map f is finite flat, we can suppose that $X = Spec(B)$ and $Y = Spec(A)$ are affine, with finite flat map $f^\# : A \to B$. The verification is now very direct:

(1) We want to prove that $f^!$ is the right adjoint of f_* as then the counit of this adjunction will give us the map $Tr : f_*f^! \to Id$. Indeed, this is equivalent to the fact that the co-restriction functor $Mod_A \to Mod_B (N \mapsto Hom_A(B, N))$ is the right adjoint of the scalar restriction from from B to A. Since $f^\#$ is finite flat, this is true by Shapiro's lemma.
(2) Suppose that $\mathcal{F}(Y) = M$ with M a flat A-module. We have a canonical map $Hom_A(B, A) \otimes_A M \xrightarrow{h_M} Hom_A(B, M)$. We want to prove that h is an isomorphism. Observe that since we work over a noetherien base we can find

[1]There is a bijection between the set of coherent sheaves over X and the set of coherent sheaves over Y with an action of $f_*\mathcal{O}_X$. As a result $Hom(f_*\mathcal{O}_X, \mathcal{F})$ with the obvious action of $f_*\mathcal{O}_X$ defines indeed a sheaf over Y.

a presentation $A^l \to A^m \to B \to 0$. This gives us the following diagram (the rows are exact sequence as M is flat):

$$\begin{tikzcd} 0 \arrow[r] & Hom_A(B,A)\otimes_A M \arrow[r] \arrow[d, "h_M"] & Hom_A(A^m,A)\otimes_A M \arrow[r] \arrow[d, "h_{A^m}"] & Hom_A(A^l,A)\otimes_A M \arrow[d, "h_{A^l}"] \\ 0 \arrow[r] & Hom_A(B,M) \arrow[r] & Hom_A(A^m,M) \arrow[r] & Hom_A(A^l,M) \end{tikzcd}$$

Since h_{A^m} and h_{A^l} are isomorphisms, h_M is an isomorphism too.

□

Remark 10.3.1. *In general the existence of a functor $f^! : D^+(\mathcal{O}_Y) \to D^+(\mathcal{O}_X)$ is not at all obvious and has only been constructed in some special cases, but when it does exist, it satisfies similar properties like above, i.e (cf. thm* (10.5) *and prop*(8.8) *[Har66]):*

(1) *There is a natural transformation $Tr : Rf_* f^! \to Id$ of endofunctors of $D^+(\mathcal{O}_X)$ and it induces via the adjoint formula:*
$$Hom_{D(\mathcal{O}_X)}(\mathcal{F}, f^!\mathcal{G}) \overset{Ad}{\cong} Hom_{D(\mathcal{O}_Y)}(Rf_*\mathcal{F}, \mathcal{G})$$
for any $\mathcal{G} \in D^+(\mathcal{O}_Y)$ and $\mathcal{F} \in D^-(\mathcal{O}_X)$

(2) *For any $\mathcal{G} \in D^b(\mathcal{O}_Y)$ is isomorphic to a bounded complex of flat $\mathcal{O}_Y$-sheaves then we have the following formula :*
$$f^!\mathcal{F} \otimes^L Rf^*\mathcal{G} = f^!(\mathcal{F} \otimes^L \mathcal{G})$$

We explain how to apply these in our situation to a flat coherent sheaf $\mathcal{F}$ on $\mathcal{S}$. Recall that we have seen: $\mathcal{S} \xleftarrow{p_2} \mathcal{S}_{par} \xrightarrow{p_1} \mathcal{S}$. Here, the map p_1 is finite flat, so that the functor $p_1^!$ is available. We have natural maps (the third equality results from the fact that p_1 is affine).

$$\mathrm{R}\Gamma(\mathcal{S}, \mathcal{F}) \to \mathrm{R}\Gamma(\mathcal{S}_{par}, p_2^*\mathcal{F}) = \mathrm{R}\Gamma(\mathcal{S}, p_{1*}p_2^*\mathcal{F})$$

Now, in our case of interests, we will be able to define a map $\lambda : p_2^*\mathcal{F} \to p_1^*\mathcal{F}$ and a map $\theta : p_1^*\mathcal{F} \to p_1^!\mathcal{F}$ which we call the fundamental class. Remark that by proposition(10.3.1), $p_1^!\mathcal{F} = p_1^*\mathcal{F} \otimes p_1^!\mathcal{O}_{\mathcal{S}}$, so to define the fundamental class, it is enough to exhibit a map $p_1^*\mathcal{O}_{\mathcal{S}} \to p_1^!\mathcal{O}_{\mathcal{S}}$. Since p_1 is finite flat, there is the trace map $Tr : p_{1*}\mathcal{O}_{\mathcal{S}_{par}} \to \mathcal{O}_{\mathcal{S}}$, and this induces a map $p_{1*}\mathcal{O}_{\mathcal{S}_{par}} \to p_{1*}p_1^!\mathcal{O}_{\mathcal{S}} = \underline{Hom}(p_{1*}\mathcal{O}_{\mathcal{S}_{par}}, \mathcal{O}_{\mathcal{S}})$.

Combining everything we obtain a map:

$$T : \mathrm{R}\Gamma(\mathcal{S}, \mathcal{F}) \to \mathrm{R}\Gamma(\mathcal{S}, p_{1*}p_2^*\mathcal{F}) \xrightarrow{\lambda} \mathrm{R}\Gamma(\mathcal{S}, p_{1*}p_1^*\mathcal{F}) \xrightarrow{\theta} \mathrm{R}\Gamma(\mathcal{S}, p_{1*}p_1^!\mathcal{F}) \xrightarrow{Tr} \mathrm{R}\Gamma(\mathcal{S}, \mathcal{F})$$

It should be noted that as it happens in practice, the map λ above is often suitably normalized to improve the integrality of T depending on specific situations.

This construction applies most frequently to the automorphic sheaf Ω^κ (or its cuspidal counterpart $\Omega^\kappa_D := \Omega^\kappa \otimes \mathcal{I}_D$) of weight κ. In this occasion, the universal isogeny $\lambda : \mathcal{A} \to \mathcal{A}'$ over $\mathcal{S}_{par}$ induces a map $\lambda^*_\pi : \omega_{\mathcal{A}',\pi} \to \omega_{\mathcal{A},\pi}$ and $\lambda^*_{\bar{\pi}} : \omega_{\mathcal{A}',\bar{\pi}} \to \omega_{\mathcal{A},\bar{\pi}}$ and hence we obtain a (rational)map, still denoted by $\lambda : p_2^*\Omega^\kappa \xrightarrow{\lambda} p_1^*\Omega^\kappa$ which is nothing but $sym^{k_1-k_2}\lambda_\pi^* \otimes det^{k_2}(\lambda_{\bar\pi}^*) \otimes det^{k_3}(\lambda_{\bar\pi}^*))$. Furthermore the maps p_1, p_2 are finite flat and $p_1^!\mathcal{O}_{\mathcal{S}}$ is just the sheaf $\underline{Hom}(p_{1*}\mathcal{O}_{\mathcal{S}_{par}}, \mathcal{O}_{\mathcal{S}})$, as a result the fundamental class $\theta : p_1^*\Omega^\kappa \to p_1^!\Omega^\kappa$ is induced by the trace map $Tr : p_{1*}\mathcal{O}_{par} \to \mathcal{O}_{\mathcal{S}}$.

We thus get the operator $\mathcal{T}_p \in End(\mathrm{R}\Gamma(\mathcal{S}, \Omega^\kappa))$ and call it the Hecke operator at p. Sometimes, we will write $\mathcal{T}_p$ as $\mathcal{T}_p : p_2^*\Omega^\kappa \to p_1^*\Omega^\kappa \to p_1^!\Omega^\kappa$ to stress which maps are involved in the construction of $\mathcal{T}_p$.

10.4. **Hecke correspondences on special fiber.** It is important for us that our Hecke correspondence gives rise to a correspondence mod p. The following proposition tells us that the first step is to normalize $\mathcal{T}_p : p_2^*\Omega^\kappa \to p_1^*\Omega^\kappa \to p_1^!\Omega^\kappa$.

Proposition 10.4.1. *Write $\kappa = (k_1, k_2; k_3)$. If $k_1 + k_3 \geq 1$ the map $\mathcal{T}_p$ factorizes through $p^{k_2+1}p_1^!\Omega^\kappa$.*

Proof. We can check this property over the ordinary locus outside the cusps as it is dense. Now over each point x of this locus corresponding to a pair $(H_x, \mathcal{A}_x)$ we have an isogeny $\lambda : \mathcal{A}_x \to \mathcal{A}'_x = \mathcal{A}_x/(H_x + H_x^\perp)$, this induces an isogeny of p-divisible groups :

$$\begin{array}{ccc} \mathcal{A}_x[p^\infty] & \xrightarrow{\quad\lambda\quad} & \mathcal{A}'_x[p^\infty] \\ \downarrow & & \downarrow \\ \mathcal{A}_x[\pi^\infty] \times \mathcal{A}_x[\bar\pi^\infty] & \xrightarrow{\lambda_\pi \times \lambda_{\bar\pi}} & \mathcal{A}'_x[\pi^\infty] \times \mathcal{A}'_x[\bar\pi^\infty] \end{array}$$

Where the isogeny $\lambda_\pi : \mathcal{A}_x[\pi^\infty] \to \mathcal{A}'_x[\pi^\infty]$ (resp.$\lambda_{\bar\pi}$) is the quotient by $H_x \subset \mathcal{A}_x[\pi^\infty]$ (resp. $H_x^\perp \subset \mathcal{A}_x[\bar\pi^\infty]$). These isogenies induce : $\lambda_\pi^* : \omega_{\mathcal{A}'_x,\pi} \to \omega_{\mathcal{A}_x,\pi}$ and $\lambda_{\bar\pi}^* : \omega_{\mathcal{A}'_x,\bar\pi} \to \omega_{\mathcal{A}_x,\bar\pi}$. Now over ordinary locus outside the boundary, there are two possibility for H_x:

(1) If H_x is of multiplicative rank 1 and étale rank 1 then $H_x^\perp$ is of étale rank 1 (indeed, we have $H_x^D = \mathcal{A}[\bar\pi]/H_x^\perp$), so the map λ_π^* factors through $p\omega_{\mathcal{A}_x,\pi}$ and $\lambda_{\bar\pi}^*$ is an isomorphism. As a consequence the map $p_2^*\Omega^\kappa \to p_1^*\Omega^\kappa$ factors through $p^{k_2}p_1^*\Omega^\kappa$.
(2) If H_x is of multiplicative rank 2, then $H_x^\perp$ is of multiplicative rank 1, so that λ_π^* factors through $p^2\omega_{\mathcal{A}_x,\pi}$ and $\lambda_{\bar\pi}^*$ through $p\omega_{\mathcal{A}_x,\bar\pi}$, and the map $p_2^*\Omega^\kappa \to p_1^*\Omega^\kappa$ factors through $p^{k_1+k_2+k_3}p_1^*\Omega^\kappa$.

For the trace map $p_1^*\Omega^\kappa \to p_1^!\Omega^\kappa$, we need an explicit description of this map locally at each point. We will use Serre-Tate theory which firstly says that deforming our polarized abelian variety $\mathcal{A}_x$ at point x is equivalent to deforming its p-divisible group $\mathcal{A}_x[p^\infty] = \mathcal{A}_x[\pi^\infty] \times \mathcal{A}[\bar\pi^\infty]$. But since $\mathcal{A}_x[\pi^\infty] \cong \mathcal{A}_x[\bar\pi^\infty]^D$, we just need to look at the deformation space for $\mathcal{A}_x[\pi^\infty]$. Secondly, this theory tells us that each such deformation corresponds to a map $q : T_\pi(\mathcal{A}_x) \times T_{\bar\pi}(\mathcal{A}_x) \to \hat{\mathbb{G}}_m$, called Serre-Tate coordinate. We have a similar Serre-Tate coordinate for each deformation of $\mathcal{A}'_x[p^\infty]$, or equivalently of $\mathcal{A}'_x[\pi^\infty]$. These two deformation spaces are linked by the isogeny $\lambda_\pi : T_\pi\mathcal{A}_x \to T_\pi\mathcal{A}'_x$. Now, given that :

$$T_\pi(\mathcal{A}_x) \cong T_\pi(\mathcal{A}'_x) \cong \mathbb{Z}_p \qquad T_{\bar\pi}(\mathcal{A}_x) \cong T_{\bar\pi}(\mathcal{A}'_x) \cong \mathbb{Z}_p^2 \tag{10.5}$$

we have the following diagrams:

$$\begin{tikzcd} \mathbb{Z}_p \times \mathbb{Z}_p^2 \arrow[r, "q"] \arrow[d, "M_x(\lambda_\pi)"'] & \hat{\mathbb{G}}_m \arrow[d, equal] \\ \mathbb{Z}_p \times \mathbb{Z}_p^2 \arrow[r, "q"] & \hat{\mathbb{G}}_m \end{tikzcd}$$

Here $M_x(\lambda_\pi)$ is the matrix given by the isogeny λ_π. The trivializations of Tate modules (10.5) give an isomorphism between the deformation space of $\mathcal{A}_x$ and $\mathbb{Z}_p[[X,Y]]$ via Serre-Tate coordinates. We again have two cases:

(1) If H_x is of multiplicative rank 1, étale rank 1, then $H_x^\perp$ is of étale rank 1 and the matrix $M_x = diag(p,p,1)$. Then the deformation space of $\mathcal{A}'_x$ is isomorphic to $\mathbb{Z}_p[[X,Y,X',Y']]/((1+X')^p - X - 1, Y' - Y))$ and this implies that the trace map $p_1^*\mathcal{O}_{\mathcal{S}_{par}} \to p_1^!\mathcal{O}_{\mathcal{S}_{par}}$ factorizes through $p.p_1^!\mathcal{O}_{\mathcal{S}_{par}}$.

(2) If H_x is of multiplicative rank 2 , we can repeat the whole thing and see that $p_1^*\mathcal{O}_{\mathcal{S}_{par}} \to p_1^!\mathcal{O}_{\mathcal{S}_{par}}$ is actually an isomorphism.

So we conclude that if H_x is of multiplicative rank 1, then $\mathcal{T}_p$ is divisible by p^{k_2+1}, and if H_x is of multiplicative rank 2, then $\mathcal{T}_p$ is divisible by $p^{k_1+k_2+k_3}$. So if $k_1+k_2+k_3 \geq k_2+1$, then $\mathcal{T}_p$ factors through $p^{k_2+1}p_1^!\Omega^\kappa$. □

This proposition leads us to set $T_p := \frac{1}{p^{k_2+1}}\mathcal{T}_p$. This normalized correspondence induces a non trivial one on the special fiber $\mathcal{S}_1 := \mathcal{S} \times_{\mathbb{Z}_p} \mathbb{F}_p$, ie we have $T_p|_{\mathcal{S}_{par,1}} : p_2^*\Omega^\kappa|\mathcal{S}_{par,1} \to p_1^!\Omega^\kappa|_{\mathcal{S}_{par,1}}$ which induces an endomorphism of complex $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1, \Omega^\kappa))$. As in the introduction of this section, we will study the interaction of this correspondence with various Hasse invariants that we have defined.

10.5. **Stratification of special fiber.** In this section we study the stratification by multiplicative rank of the special fiber $\mathcal{S}_1$ and its interaction with the Hecke operator T_p. Let us denote by $\mathcal{S}_1^{\geq i}$ (resp.$\mathcal{S}_1^{\leq i}$, resp.$\mathcal{S}_1^{=i}$) the locus where the multiplicative rank of universal p-torsion group $\mathcal{A}[\pi]$ is no less than (resp.no bigger than, resp. equal to) i. Recall that we have the decomposition $\mathcal{A}[p] = \mathcal{A}[\pi] \times \mathcal{A}[\bar{\pi}]$ and since $\mathcal{A}[\pi]$ is the Cartier dual of $\mathcal{A}[\bar{\pi}]$, the multiplicative rank of $\mathcal{A}[p]$ is dictated by that of $\mathcal{A}[\pi]$.

Now, the classical Hasse invariants $Ha := Ha(\mathcal{A}[\pi]) \in H^0(\mathcal{S}_1, det(\omega_{\mathcal{A},\pi})^{p-1})$ vanishes exactly over $\mathcal{S}_1^{\leq 1}$. We want to construct a second Hasse invariant of $\mathcal{A}[\pi]$ that vanishes exactly on the locus $\mathcal{S}_1^{=0}$. After what we have explained in section 10.1.2, it is enough to show that $\mathcal{S}_1^{\leq 1}$ is smooth.

Before we do so, we would like to simplify notations by denoting $det_{\mathcal{A},\pi} := det(\omega_{\mathcal{A},\pi})/\mathcal{S}^?)$, where (?) can be $(\leq i; \geq i; = i)$, hopefully this will not cause confusions. Finally, for any scheme X over $Spec(\mathbb{Z}_p)$ we denote by X_n the base change $X_n := X \times_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$.

Theorem 10.5.1. *The non-ordinary locus $\mathcal{S}_1^{\leq 1}$ is smooth.*

Proof. As $\mathcal{S}_1^{\leq 1}$ is cut out by the Hasse invariant Ha, we need to show that Ha defines a local regular parameter at each points. More precisely, let $x \in \mathcal{S}_1^{\leq 1}$ be any geometric point with residue field k_x, then if Ha induces a non trivial morphism on the tangent space of $\mathcal{S}_1^{\leq 1}$ at x, then the point x is smooth by the Jacobian criterion.

We treat this case by case.
If x corresponds to a point whose underlying p-divisible group $\mathcal{A}[\pi^\infty]$ is of slope $(\frac{1}{2}, \frac{1}{2}, 1)$, we have seen that the crystal corresponding $\mathcal{A}_x[\pi]$ is given by $D_x = W(k_x)^3$ with Frobenius given by the matrix :

$$F = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & p & 0 \end{pmatrix}$$

By the standard theory, the crystal D_x is also equipped with a Hodge filtration $\omega_{\mathcal{A}_x} = ker(F : D_x/pD_x \to D_x/pD_x)$, so if we fix a basis (e_1, e_2, e_3) for D_x and $(\overline{e_1}, \overline{e_2}, \overline{e_3})$ for D_x/pD_x this filtration is given by $\omega_{\mathcal{A}_x} =< \overline{e_1}, \overline{e_2} >$. The first order deformation ring at x is isomorphic to $R = k[X,Y]/(X,Y)^2$, this is obtained by formally deforming the filtration $\omega_{\mathcal{A}_x} \subset D_x$. this means that a deformation of a point x of slope $(1,0)$ correspond to a lift of $\omega_{\mathcal{A}}$ inside $M = D_x \otimes_{W(k_x)} R$ which is of the following form:

$$Fil^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ X & Y \end{pmatrix}$$

We then easily see the induced Verschibung $V : M/Fil^1 \to M/Fil^1$ is given by the multiplication by $-Y$, and so the Hasse invariant, which is the determinant of V is non zero on the tangent space at x.

For the point x of slope $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ (i.e corresponding to a supersingular point), we can do a completely analogous calculation, based on the explicit formula for the associated crystal given in [9.3], and see that the Hasse invariant also defines a regular local parameter at this point, and we conclude that $\mathcal{S}_1^{\leq 1}$ is smooth. □

As $\mathcal{S}_1^{\leq 1}$ is smooth and the locus $\mathcal{S}_1^{=0}$ has only finitely many closed points while $\mathcal{S}_1^{=1}$ has dimension 1 and thus is dense, we can apply the earlier result (10.1.2) to get the second Hasse invariant $Ha' := Ha'(\mathcal{A}[\pi]) \in H^0(\mathcal{S}_1^{\leq 1}, det^{p^2-1}(\omega_{\mathcal{A},\pi}))$, whose zero locus is $\mathcal{S}_1^{=0}$.

10.6. **Restrictions of Hecke operator and finiteness of coherent cohomology.** In this section, if $\kappa = (k_1, k_2; k_3)$ is a weight, and $n \in \mathbb{Z}$ we will denote by:

(1) $\kappa(n)$ for the weight $(k_1 + n(p-1), k_2 + n(p-1); k_3)$.
(2) $\kappa'(n)$ for the weight $(k_1 + n(p^2-1), k_2 + n(p^2-1); k_3)$
(3) $\kappa(n,m)$ for the weight $(k_1 + n(p-1) + m(p^2-1), k_2 + n(p-1) + m(p^2-1); k_3)$

We have defined above a map $T_p : p_2^*\Omega^\kappa \to p_1^!\Omega^\kappa$ over $S_{par,1}$ which induces an operator $T_p : \mathrm{R}\Gamma(S_1, \Omega^\kappa) \to \mathrm{R}\Gamma(S_1, \Omega^\kappa)$ when $k_1 + k_3 \geq 1$. A natural question to ask is how this operator interacts with the stratification by p-rank introduced above of $S_{par,1}$. We have our first proposition.

Proposition 10.6.1. *For all weight κ satisfying $k_1+k_3>1$. The following diagram over $S_{par,1}$ commutes:*

$$\begin{array}{ccc} p_2^*\Omega^\kappa & \xrightarrow{T_p} & p_1^!\Omega^\kappa \\ \times p_2^*Ha \downarrow & & \downarrow \times p_1^! Ha \\ p_2^*\Omega^{\kappa(1)} & \xrightarrow{T_p} & p_1^!\Omega^{\kappa(1)} \end{array} \tag{10.6}$$

where in the diagram above, we are using $Ha := Ha(\mathcal{A}[\pi])$.

Proof. It is enough to check this point by point on ordinary locus because it is dense and our schemes are Cohen-Macaulay. Over the ordinary locus of $\mathcal{S}_{par,1}$ we have the universal isogeny $\lambda : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ with kernel $H \in \mathcal{A}[\pi]$ of order p^2. As a consequence, we can decompose $\mathcal{S}_{par,1}$ into a disjoint union of two components depending on the type of λ (i.e depending on whether H is multiplicative rank 1 or 2).

Over a point x in the component where H is of multiplicative rank 2, the map $p_2^*\Omega^\kappa \to p_1^*\Omega^\kappa$ over a lift of x to $\mathcal{S}_{par}$ factors through $p^{k_1+k_2+k_3}$ and thus if $k_1+k_3>1$, our Hecke operator over this point vanishes due to the normalization (cf.prop(10.4)). As the result the commutativity is obvious.

Over the component where H is of multiplicative rank 1, we see that the fundamental class $p_1^*\Omega^\kappa \to p_1^!\Omega^\kappa$ is actually obtained by simply tensoring the fundamental class $p_1^*\mathcal{O}_{\mathcal{S}_1} \to p_1^!\mathcal{O}_{\mathcal{S}_1}$ with $p_1^*\Omega^\kappa$ (Since $p_1^!\Omega^\kappa = p_1^*\Omega^\kappa \otimes p_1^!\mathcal{O}_{\mathcal{S}_1}$) and by $p_1^!Ha$ we just mean the image of $p_1^*Ha \in H^0(\mathcal{S}_{par,1}, p_1^*det_{\mathcal{A},\pi}^{p-1})$ under $p_1^*det_{\mathcal{A},\pi}^{p-1} \to p_1^!det_{\mathcal{A},\pi}^{p-1} = p_1^*det_{\mathcal{A},\pi}^{p-1} \otimes p_1^!\mathcal{O}_{\mathcal{S}_1}$. As a result, the above diagram (10.6) can be rewritten as :

$$\begin{array}{ccccc} p_2^*\Omega^\kappa & \xrightarrow{\lambda^*_{norm}} & p_1^*\Omega^\kappa & \longrightarrow & p_1^*\Omega^\kappa \otimes p_1^!\mathcal{O}_{\mathcal{S}_1} \\ \times p_2^*Ha \downarrow & & \downarrow \times p_1^*Ha & & \downarrow p_1^*Ha\otimes 1 \\ p_2^*\Omega^{\kappa(1)} & \xrightarrow{\lambda^*_{norm}} & p_1^*\Omega^{\kappa(1)} & \longrightarrow & p_1^*\Omega^{\kappa(1)} \otimes p_1^!\mathcal{O}_{\mathcal{S}_1} \end{array} \tag{10.7}$$

with λ^*_{norm} is the normalized map induced by the isogeny λ.

The commutativity of the right square is tautological, so what is left is that on the left square. This amounts to checking if the map λ^*_{norm} sends $p_2^*Ha(\mathcal{A}'[\pi])$ to $p_1^*Ha(\mathcal{A}[\pi])$. Let $\bar{x} \in \mathcal{S}_{par,1}$ be a point with residue field k. Let $W(k)$ be the ring of Witt vector with residue field k and $x \in \mathcal{S}_{par}$ be a $W(k)$-point whose reduction mod p is $\bar{x}$ (this is possible since the ordinary locus is smooth). We have the universal isogeny $\lambda_x : \mathcal{A}_x[\pi^\infty] \to \mathcal{A}'_x[\pi^\infty]$ with kernel H_x multiplicative of rank 1, and this isogeny induces a map $\lambda_x^* : det_{\mathcal{A}'_{x},\pi} \to det_{\mathcal{A}_x,\pi}$. We have normalized this map to obtain $\lambda^*_{x,norm} := \frac{\lambda_x^*}{p}$. Now as H_x is of multiplicative rank 1, this is exactly the way we normalized to obtain the map λ^*_{norm} in proposition 10.1.1 from λ^*. By that proposition, λ^*_{norm} sends $Ha(\mathcal{A}'_x[\pi])$ to $Ha(\mathcal{A}_x[\pi])$ and we are done.

□

Because $\mathcal{S}_1^{\leq 1}$ is the vanishing locus of the classical Hasse invariant $p_2^*Ha(\mathcal{A}')$ (or $p_1^*Ha(\mathcal{A})$), an immediate consequence of this proposition is that we have a restricted operator:

$$T_p : \mathrm{R}\Gamma(\mathcal{S}_1^{\leq 1}, \Omega^{\kappa(1)}) \to \mathrm{R}\Gamma(\mathcal{S}_1^{\leq 1}, \Omega^{\kappa(1)})$$

Inside the locus $\mathcal{S}_1^{\leq 1}$ the interaction is more involved, we will need to take a detour. Let us begin with the following lemma :

Lemma 10.6.1. *Let G be a truncated Barsotti-Tate of level 1 over an $\mathbb{F}_p$-scheme S. If the multiplicative rank and étale rank of G are constant over S and if $H \subset G$ is a subgroup of order p, then S can be decomposed into a disjoint union of sub-schemes $S = S^{et} \coprod S^{oo} \coprod S^m$, where S^{et}(resp. S^m, resp. S^{oo}) is open and closed subscheme over which H is étale (resp. multiplicative, resp. biconnected).*

Proof. This is lemma 7.4.2.1 in [Pil18]. □

We use this to decompose our correspondence T_p into two parts as follows: over $\mathcal{S}_{par,1}$ we have the universal isogenies : $H \to \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ and $H^\perp \to \mathcal{A}[\bar{\pi}^\infty] \to \mathcal{A}'[\bar{\pi}^\infty]$ (the group $H^\perp$ is of order p). Because the multiplicative ranks of $\mathcal{A}$ and $\mathcal{A}'$ are equal, it makes sense to define $\mathcal{S}_{par,1}^{=1} \subset \mathcal{S}_{par,1}$ as the locus where multiplicative rank of $\mathcal{A}$ (hence of $\mathcal{A}'$) is 1. Further more, over this locus the p-ranks of $\mathcal{A}[\bar{\pi}]$ and $\mathcal{A}'[\bar{\pi}]$ are constant, we can thus apply the lemma to get the decomposition $\mathcal{S}_{par,1}^{=1} = \mathcal{S}_{par,1}^{et} \coprod \mathcal{S}_{par,1}^{oo}$ where the $\mathcal{S}_{par,1}^{et}$(resp.$\mathcal{S}_{par,1}^{oo}$) is the subscheme of $\mathcal{S}_{par,1}^{=1}$ where $H^\perp \subset \mathcal{A}[\bar{\pi}]$ is étale (resp.biconnected). Overall we obtain two correspondences over $\mathcal{S}_{par,1}^{=1}$:

$$T_p^{et} : p_2^*\Omega^\kappa|_{\mathcal{S}_{par,1}^{et}} \to p_1^!\Omega^\kappa|_{\mathcal{S}_{par,1}^{et}} \qquad \text{and} \qquad T_p^{oo} : p_2^*\Omega^\kappa|_{\mathcal{S}_{par,1}^{oo}} \to p_1^!\Omega^\kappa|_{\mathcal{S}_{par,1}^{oo}}$$

Remark 10.6.1. *The formation of fundamental class is the only thing we need to pay attention in this decomposition of correspondence. Indeed, the formation of fundamental class is stable by restriction to open or closed subscheme, so at the start we obtain a correspondence over $\mathcal{S}_{par,1}^{\leq 1}$ and then over $\mathcal{S}_{par,1}^{=1} \hookrightarrow \mathcal{S}_{par,1}^{\leq 1}$ by restriction and finally over open and closed subschemes $\mathcal{S}_{par,1}^{et}$ as well as $\mathcal{S}_{par,1}^{oo}$.*

We are now ready to prove that the Hecke correspondence commutes with the second Hasse invariant.

Proposition 10.6.2. *For any weight κ such that $k_1 + k_3 > p + 1$, the following diagram (defined over $\mathcal{S}_{par,1}^{\leq 1}$) commutes:*

$$\begin{tikzcd}
p_2^*\Omega^\kappa \arrow[r, "T_p"] \arrow[d, "\times p_2^* Ha'"'] & p_1^!\Omega^\kappa \arrow[d, "\times p_1^* Ha'"] \\
p_2^*\Omega^{\kappa'(1)} \arrow[r, "T_p"] & p_1^!\Omega^{\kappa'(1)}
\end{tikzcd}$$

where in the diagram above, we are using $Ha' := Ha'(\mathcal{A}[\pi])$.

Proof. Similarly to the proof of the proposition (10.6.1), we check this point by point over dense locus $\mathcal{S}_{par,1}^{=1}$ of $\mathcal{S}_{par,1}^{\leq 1}$. Now over this locus, we have a decomposition $T_p = T_p^{et} + T_p^{oo}$. We will actually prove that T_p^{et} commutes with Ha' and T_p^{oo} vanishes.

(1) For the commutativity of T_p^{et} with Ha' , the argument of proposition (10.6.1) carries over, and so it amounts in the end to proving that for any point $\bar{x} \in \mathcal{S}_{par,1}^{=1}$ with a lift $x \in \mathcal{S}_{par}^{=1}$ corresponding to a universal isogeny $\lambda_x : \mathcal{A}_x[\pi^\infty] \to \mathcal{A}'_x[\pi^\infty]$ with kernel H_x such that $H_x^\perp$ is étale, the induced normalized morphism : $\frac{1}{p^{p^2-1}}\lambda^* : det^{p^2-1}_{\mathcal{A}',\pi} \to det^{p^2-1}_{\mathcal{A},\pi}$ sends $Ha'(\mathcal{A}'_x[\pi])$ to $Ha'(\mathcal{A}_x[\pi])$. Now by our definition of T_p^{et}, the group $H_x^\perp$ is étale, implying that H_x is biconnected, so geometrically, H_x is a direct factor of $\mathcal{A}_x[\pi]$. This setting fits perfectly for the conditions of proposition (10.1.2), because the morphism $\frac{1}{p^{p^2-1}}\lambda_x^*$ is exactly the morphism $\lambda^*_{x,norm}$ there, thus we are done.

(2) Regarding the part $T_p^{oo}(\kappa)$, it actually vanishes over $\mathcal{S}_{par,1}^{=1}$. It is enough to prove this point by point . Let $\bar{x} : Spec(k) \to \mathcal{S}_1^{=1}$ be a geometric point with residue field k. By deformation theory, we can lift $\bar{x}$ to a point $x : Spec(\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{S}^{=1}$. Let $\bar{y} \in \mathcal{S}_{par,1}^{=1}$ mapping to $\bar{x}$ via p_2 and $y : Spec(\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{S}_{par}^{=1}$ a lift of $\bar{y}$ mapping to x. Now over y we have the isogeny $\lambda_y : \mathcal{A}_y[p^\infty] \to \mathcal{A}'_y[p^\infty]$ with kernel $H_y + H_y^\perp$ which can be seen as the product of two isogenies $\lambda_{y,\pi} : \mathcal{A}_y[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ and $\lambda_{y,\bar{\pi}} : \mathcal{A}_y[\bar{\pi}^\infty] \to \mathcal{A}'_y[\bar{\pi}^\infty]$.

We have $H_y = ker\lambda_{y,\pi} \cong \mu_p \times \mu^{oo}$ and $H_u^\perp \cong \mu^{oo}$ over $\mathcal{O}_{\mathbb{C}_p}$, where μ^{oo} is the biconnected group of order p. Since the degree of μ^{oo} is $\frac{1}{p+1}$ in the sense of [Far11] (See also A.2.2 [Pil11]). It then follows that the differential $\lambda^*_{y,\pi}$ of $\lambda_{y,\pi}$ factors through an ideal generated by (p, ϖ) with ϖ an element of valuation $\frac{1}{p+1}$ and the differential $\lambda^*_{y,\bar{\pi}}$ is divisible by $p^{\frac{1}{p+1}}$. The result is that the non normalized map $\mathcal{T}_p^{oo} : p_2^*\Omega_y^\kappa \to p_1^!\Omega^\kappa$ which is nothing but $sym^{k_1-k_2}\lambda^*_{y,\pi} \otimes det^{k_2}\lambda^*_{y,\pi} \otimes det^{k_3}\lambda^*_{y,\bar{\pi}}$ factors through $p^{k_2+\frac{k_1+k_3}{p+1}} p_1^!\Omega_y^\kappa$ (the map $p_1^*\Omega_y^\kappa \to p_1^!\Omega_y^\kappa$ is an isomorphism in this case). Finally, we conclude that if $k_1 + k_3 > p+1$, the normalized map T_p^{oo} vanishes.

□

By this preposition, we have a well defined operator $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=0}, \Omega^{\kappa'(1)})$ as Ha' vanishes exactly over $\mathcal{S}_1^{=0}$. We would like to show that when the weight is regular enough, this operator vanishes on this locus which is crucial for the higher Hida theory as we will see later on.

Proposition 10.6.3. *For all weight $\kappa = (k_1, k_2, k_3)$ such that $k_1 - k_2 \geq 1$ and $k_1 + k_3 > 3$, the Hecke operator $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1, \Omega^\kappa)$ acts trivially on $\mathrm{R}\Gamma(\mathcal{S}_1^{=0}, \Omega^{\kappa(1,1)})$.*

Proof. First of all, under the condition of the weight, $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1, \Omega^\kappa)$ gives rise to $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{\leq 1}, \Omega^{\kappa(1)})$. Let $\mathcal{I}$ be the sheaf of ideal defining the closed subscheme $\mathcal{S}_1^{=0}$ inside $\mathcal{S}_1^{\leq 1}$ (i.e it is generated by Ha'). The proposition can then be rephrased as under the condition on the weight, the correspondence $T_p : p_2^*\Omega^{\kappa(1,1)} \to p_1^!\Omega^{\kappa(1,1)}$ over $\mathcal{S}_{par,1}^{\leq 1}$ factors through:

$$T_p : p_2^*\Omega^{\kappa(1,1)} \to p_1^*\mathcal{I}p_1^!\Omega^{\kappa(1,1)}$$

We can write $\mathcal{S}^{\leq 1}_{par,1} = \mathcal{S}^{\leq 1,et}_{par,1} \bigcup \mathcal{S}^{\leq 1,oo}_{par,1}$ where $\mathcal{S}^{\leq 1,et}_{par,1}$(resp. $\mathcal{S}^{\leq 1,oo}_{par,1}$) is the closure of the subscheme $\mathcal{S}^{=1,et}_{par,1} \subset \mathcal{S}^{\leq 1}_{par,1}$ (resp.$\mathcal{S}^{=1,oo}_{par,1} \subset \mathcal{S}^{\leq 1}_{par,1}$)where the $\bar{\pi}$-isogeny $\mathcal{A}[\bar{\pi}^\infty] \to \mathcal{A}'[\bar{\pi}^\infty]$ has étale (resp. biconnected) kernel $H^\perp$. Due to the normalization, the restriction of T_p to $\mathcal{S}^{\leq 1,oo}_{par,1}$ is zero, we are finally left to prove that the restriction of T_p to $\mathcal{S}^{\leq 1,et}_{par,1}$ vanishes on $\mathcal{S}^{\leq 1,et}_{par,1} \bigcap V(p_1^*\mathcal{I})$.

Right off the bat, we have to comment on the scheme structure of $\mathcal{S}^{\leq 1,et}_{par,1} \bigcap V(p_1^*\mathcal{I})$ since a priori $\mathcal{S}^{\leq 1}_{par,1}$ and its decomposition above are merely topological. There is no canonical underlying subscheme structure of $\mathcal{S}^{\leq 1}_{par,1}$ inside $\mathcal{S}_{par,1}$, since two Hasse invariants $Ha(\mathcal{A}[\pi])$ and $Ha(\mathcal{A}'[\pi])$ would define two different scheme structures of $\mathcal{S}^{\leq 1}_{par,1}$. Fortunately, we have already ignore the locus $\mathcal{S}^{\leq 1,oo}_{par,1}$ and the scheme structure $\mathcal{S}^{\leq 1,et}_{par,1}$ can be naturally defined as :

$$\mathcal{S}^{\leq 1,et}_{par,1} = V(Ha(\mathcal{A}[\pi]) \cap \mathcal{S}^{\leq 1,et}_{par,1} = V(Ha(\mathcal{A}'[\pi]) \cap \mathcal{S}^{\leq 1,et}_{par,1}$$

It comes from the fact that we can define a canonical isomorphism λ_{norm} : $det(\omega_{\mathcal{A}'[\pi^\infty]}) \to det(\omega_{\mathcal{A}[\pi^\infty]})$ such that $\lambda_{norm}^{p-1} Ha(\mathcal{A}'[\pi]) \to Ha(\mathcal{A}[\pi])$ (we can use the deformation theory and prove this similarly as in the lemma 6.3.4.2 [Pil18]).

Once the scheme structure on $\mathcal{S}^{\leq 1,et}_{par,1}$ has been determined, we can proceed as follows . First, observe that the maps $p_1, p_2 : \mathcal{S}^{\leq 1,et}_{par,1} \to \mathcal{S}_1$ land in $\mathcal{S}^{\leq 1}_1$ and as $\mathcal{S}^{\leq,et}_{par,1}$ parameterizes étale subgroups $H^\perp \subset \mathcal{A}[\bar{\pi}]$ of order p, we can see that $p_2 : \mathcal{S}^{\leq 1,et}_{par,1} \cong \mathcal{S}^{\leq 1}_1$ is actually an isomorphism and $p_1 : \mathcal{S}^{\leq 1,et}_{par,1} \to \mathcal{S}^{\leq 1}_1$ is nothing but $Frob^2$, the square of the Frobenius. In particular $V(p_1^*(\mathcal{I})) = p^2 V(p_2^*(\mathcal{I}))$ as divisors of $\mathcal{S}^{\leq 1}_{par,1}$, which implies that it suffices to prove that T_p vanishes at order p^2 along $V(p_2^*(\mathcal{I}))$.

Let us denote by κ_- the weight $(k_1 - 1, k_2, k_3)$ so that T_p is the tensor product of two maps : $T_p(\kappa_-) : p_2^*\Omega^{\kappa_-(1,1)} \to p_1^!\Omega^{\kappa_-(1,1)}$(the correspondence of weight κ_-) and $\lambda_\pi^* : \omega_{\mathcal{A}'[\pi^\infty]} \to \omega_{\mathcal{A}[\pi^\infty]}$(the differential of the isogeny $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$). We claim that λ_π^* vanishes at $\mathcal{S}^{=0}_1$ and $T_p(\kappa_-)$ vanishes at order $p^2 - 1$ along $\mathcal{S}^{=0}_1$, seen as the subscheme of $\mathcal{S}^{\leq 1,et}_{par,1}$ via the isomorphism $p_2 : \mathcal{S}^{\leq 1,et}_{par,1} \cong \mathcal{S}^{\leq 1}_1$.

Indeed, over $\mathcal{S}^{\leq 1,et}_{par,1}$, the kernel H of $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ is biconnected, thus $V_H^{*2} = 0$, which implies that the morphism $D(A[\pi^\infty]) \hookrightarrow D(\mathcal{A}[\pi^\infty)] \xrightarrow{V^{*2}} D(\mathcal{A}[\pi^\infty])^{(p^2)}$ factors through $D(\mathcal{A}[\pi^\infty]/H)^{(p^2)}$ and induces an isomorphism $\omega_{\mathcal{A}[\pi^\infty]} \cong \omega^{(p^2)}_{\mathcal{A}[\pi^\infty]/H}$ and as a result we have the following diagram:

$$\begin{tikzcd}
\omega_{\mathcal{A}[\pi^\infty]/H} \arrow[r, "\lambda_\pi^*"] \arrow[d, equal] & \omega_{\mathcal{A}[\pi^\infty]} \arrow[d, "\sim"] \\
\omega_{\mathcal{A}[\pi^\infty]/H} \arrow[r, "V^2"] & \omega^{(p^2)}_{\mathcal{A}[\pi^\infty]/H}
\end{tikzcd} \tag{10.8}$$

In other words, we can identify λ_π^* with V^2, but as over $\mathcal{S}^{=0}_1$ we have $V^2 = F$ and as the Frobenius is infinitesimal, we conclude that λ_π^* is identically zero along $\mathcal{S}^{=0}_1$.

We will show that $T_p(\kappa_-)$ vanishes at order p^2-1 along $\mathcal{S}_1^{=0}$. This is a simple consequence of the fact that, under our assumption on the weight, we still have:

$$\begin{tikzcd}
p_2^*\Omega^{\kappa_-(1)} \arrow[r, "T_p"] \arrow[d, "\times p_2^* Ha'"'] & p_1^!\Omega^{\kappa_-(1)} \arrow[d, "\times p_1^* Ha'"] \\
p_2^*\Omega^{\kappa_-(1,1)} \arrow[r, "T_p^{\kappa_-}"] & p_1^!\Omega^{\kappa_-(1,1)}
\end{tikzcd}$$

Since the section p_2^*Ha' and p_1^* vanishes along $\mathcal{S}_1^{=0}$ at order 1 and p^2 respectively, this means that $T_p(\kappa_-) : p_2^*\Omega^{\kappa(1,1)} \to p_1^!\Omega^{\kappa(1,1)}$ vanishes at order p^2-1 along $\mathcal{S}_1^{=0}$. $\square$

Remark 10.6.2. (1) *The fact that this proposition holds is very important in establishing the finiteness of various complexes we will construct in later sections. This is the very reason that we have chosen to define the Hecke operator that way. In fact, there are two parahorique subgroups of $\mathcal{A}[p]$ of order p and p^2 respectively and we chose the one of order p^2. Another choice of parahorique subgroup $H \in \mathcal{A}[\pi] \subset \mathcal{A}[p]$ of order p would gives us another correspondence $\mathcal{S} \leftarrow \mathcal{S}_{par'} \to \mathcal{S}$. We could still prove that the normalized operator associated to this correspondence commutes with both Hasse invariants but we would not have the important property that T_p acts trivially on multiplicative rank 0 locus.*

11. Ordinary projector and perfect complexes

In this section we show how one can define an ordinary projector t_p as in the classical Hida theory, but in this context, it will cut out a perfect complex $t_p\mathrm{R}\Gamma(\mathcal{S}_1, \Omega_D^\kappa)$ inside $\mathrm{R}\Gamma(\mathcal{S}_1, \Omega_D^\kappa)$.

11.1. **Locally finite operator.** We borrow the presentation from section 2 [Pil18] to recall some basic facts about locally finite operators. Let R be a complete local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let $Mod_{\mathfrak{m}}(R)$ be the category of $\mathfrak{m}$-complete, $\mathfrak{m}$-separated R-module. For any $M \in Mod_{\mathfrak{m}}(R)$ we write M_n for $M \otimes_R R/\mathfrak{m}^n$.

Definition 11.1.1. Let $M \in Mod_{\mathfrak{m}}(R)$. We say that an operator $T \in End_R(M)$ is locally finite if for all n and any $x \in M_n$ the sub R_n-module of M_n generated by $\{T^i(x)\}_{i\in\mathbb{N}}$ is finite.

Lemma 11.1.1. *Let $M \in Mod_{\mathfrak{m}}(R)$ and $T \in End_R(M)$ a locally finite operator, then there exists a projector $e_T \in End_R(M)$ such that T is invertible on e_TM and topologically nilpotent on $(1-e_T)M$.*

Proof. First of all, if such decomposition exists, it is necessarily unique, so it suffices to exhibit one such decomposition. Now, as M is $\mathfrak{m}$-adically separated, it is enough to prove the lemma for M_n. Furthermore, by our hypothesis T is locally finite so we can assume that M_n is a finite R_n-module.

Let us first suppose that M_n is a free R_n-module. Then, it makes sense to take the characteristic polynomial $P \in R_n[X]$ of $T \in End_{R_n}(M_n)$. Let and $\bar{P}$ the image of P in $k[X]$. We can decompose $\bar{P}$ into $\bar{P} = \bar{P}_1\bar{P}_2$ where $\bar{P}_2$ is of form X^m for some $m \in \mathbb{N}$ and $\bar{P}_1$ has nonzero constant term. Since $\bar{P}_1$ is prime to $\bar{P}_2$, by Hensel lemma there is a decomposition of $P = P_1P_2$ that lifts the decomposition $\bar{P} = \bar{P}_1\bar{P}_2$, where P_1, P_2 monic and $deg(P_i) = deg(\bar{P}_i)$ with $i \in \{1,2\}$. We define $M_n^o := P_2(T)M_n$ and $M_n^{no} := P_1(T)M_n$. Now, we observe that as $P_2(T)M_n^{no} = 0$, and P_2 is of form $X^m + a$ with $a \in \mathfrak{m}R_n[X]$, the action of T on M_n^{no} is nilpotent. In addition, by Bézout's identity (P_1 is prime to P_2), it is clear that $M_n = M_n^o \oplus M_n^{no}$.

We just denote by e_T the projector $M_n \to M_n^o$ so that the decomposition above can be rewritten as $M_n = e_TM_n \oplus (1-e_T)M_n$.

If M_n is not free, there exists a surjective map of R_n-module $p : R_n^l \twoheadrightarrow M_n$ for some $l \in \mathbb{N}^*$. We can then lift the action of T on M_n to R_n^l so that p becomes T-equivariant. Now by earlier construction, there is a projector e_T such that $R_n^l = e_TR_n^l \oplus (1-e_T)R_n^l$. We define $M_n^o := p(e_TR_n^l)$ and $M_n^{no} = p(1-e_T)R_n^l$. It is immediate that $M_n = M_n^o + M_n^{no}$ and T is nilpotent on M_n^{no} and is invertible on M_n^o. If $x \in M_n^o \bigcap M_n^{no}$ then $T^k(x) = 0$ for some $k \in \mathbb{N}^*$ but as T is invertible on M_n^o, we deduce that $x = 0$ and then $M_n = M_n^o \oplus M_n^{no}$.

□

We recall some useful lemmas used in the next sections.

Lemma 11.1.2. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of modules in $Mod_{\mathfrak{m}}(R)$, and T be an R-linear operator that acts equivariantly on M_1, M_2, M_3.*

(1) *If T acts locally finitely on M_3 and M_1, its action on M_2 is also locally finite.*
(2) *If the action of T on M_2 is locally finite, then it acts locally finitely on M_3.*
(3) *If $\mathfrak{m}^nM_2 = 0$ for some $n \in \mathbb{N}$ and T acts locally finitely on M_2, then it acts locally finitely on M_1*

Proof. See lemma 2.1.1 [Pil18]. □

An immediate and useful corollary of this lemma is that we can check the local finiteness of an operator $T \in End_R(M)$ by looking at its action on the reduction $M/\mathfrak{m}$ by using the exact sequence $\mathfrak{m}^{n-1}/\mathfrak{m}^n \otimes_R M \to M/\mathfrak{m}^n \to M/\mathfrak{m}^{n-1} \to 0$ and induction on n.

Move on to the situation where we have complexes of R-modules instead. Let $\mathbf{D}(R)$ be the derived category of R-modules. Let $\mathbf{C}^{comp}(R)$ the category of bounded complexes of $\mathfrak{m}$-adically complete, separated R-modules and $\mathbf{C}^{flat}(R)$ its subcategory of complexes of $\mathfrak{m}$-adically complete, separated, flat R-modules. Let $\mathbf{D}^{comp}(R)$ (resp. $\mathbf{D}^{flat}(R)$) be the full sub-category of $\mathbf{D}(R)$ generated by objects in $\mathbf{C}^{comp}(R)$ (resp. $\mathbf{C}^{flat}(R)$). We recall that a complex $M^\bullet \in \mathbf{D}^{flat}(R)$ is called perfect if it is quasi-isomorphic to a bounded complex of finite projective modules . We have the following useful criteria:

Proposition 11.1.1. *Let $M^\bullet \in \boldsymbol{D}^{flat}(R)$, concentrated in degree $[a,b]$. If $M^\bullet \otimes_R R/\mathfrak{m}$ has finite cohomology groups, then $M^\bullet$ is perfect and concentrated in degree $[a,b]$*

Proof. See proposition 2.2.1[Pil18]. □

In our paper, we study complex of coherent sheaf over $\mathcal{S}$ or $\mathcal{S}_{par}$ and this proposition says in particular that we can study the restriction of these complexes on the special fibers to show that they are perfect and concentrated in certain degrees.

Definition 11.1.2. (1) Given a complex $M^\bullet \in \mathbf{C}^{flat}(R)$ and an endomorphism $T \in End_{\mathbf{C}^{flat}(R)}(M^\bullet)$. We say that T acts locally finitely on $M^\bullet$ if its action is locally finite on each M^i.

(2) Given a complex $M^\bullet \in \mathbf{D}^{flat}(R)$ and $T \in End_{\mathbf{D}^{flat}(R)}(M^\bullet)$. We say that the action of T is locally finite if there exists a representative $M_0^\bullet \in \mathbf{C}^{flat}(R)$ and a locally finite representative $T_0 \in End_{\mathbf{C}^{flat}(R)}(M_0^\bullet)$ of T, i.e we have the following diagram:

$$\begin{tikzcd} & M_0^\bullet \arrow[rr, "T_0"] \arrow[dl] \arrow[dr, "T'"] & & M_0^\bullet \arrow[dl, "T'"] \\ M^\bullet \arrow[rr, dotted, "T"] & & M^\bullet & \end{tikzcd} \tag{11.1}$$

We have another useful criteria for detecting locally finite operator:

Proposition 11.1.2. *Let $M^\bullet \in \boldsymbol{D}^{flat}(R)$. Let $T \in End_{\boldsymbol{D}^{flat}(R)}(M^\bullet)$, then T acts locally finitely on $M^\bullet$ if and only if there is a representative representative $M_0^\bullet \in \boldsymbol{C}^{flat}(R)$ along with a locally finite representative $T_0 \in End_{\boldsymbol{C}^{flat}(R)}(M_0^\bullet)$ of T and the induced action of T on $H^i(M^\bullet \otimes_R^L R/\mathfrak{m})$ is locally finite.*

Proof. See proposition 2.3.1 [Pil18]. □

We have seen that if T acts locally finitely on a module $M \in Mod_{\mathfrak{m}}(R)$ then there is a projector $e_T \in End_R(M)$ associated to T. Now if $M^\bullet \in \mathbf{C}^{flat}(R)$ and T acts locally finitely on $M^\bullet$ then we can apply the projector e_T to each terms to give a new complex $e_T M^\bullet$. For an object $M^\bullet \in \mathbf{D}^{flat}(R)$ and a locally finite operator T on $M^\bullet$, it is more subtle to define a projector $e_T \in End_{\mathbf{D}^{flat}(R)}(M^\bullet)$ as two representatives $(M_0^\bullet, T_0)$ and $(M_1^\bullet, T_1)$ might give rise to two different projectors of $M^\bullet$ (the images of e_{T_0} and e_{T_1} in $End_{\mathbf{D}^{flat}(R)}(M^\bullet)$ are not homotopic). Nevertheless, the subcomplexes $e_{T_0}M_0^\bullet$ and $e_{T_1}M_1^\bullet$ are quasi-isomorphic (cf. lemma 2.3.2 [Pil18]), thus the factor $e_T M^\bullet$ is well defined and can be represented by $e_{T_0}M_0^\bullet$ for any representative $(M_0^\bullet, T_0)$. Finally we have the following lemma:

Lemma 11.1.3. *If the action of T on $M^\bullet \in \boldsymbol{D}^{flat}(R)$ is locally finite. We have : $H^i(e_T M^\bullet) = e_T H^i(M^\bullet)$*

Proof. Let $(M_0^\bullet, T_0)$ to be a representative of (M, T) with $M_0^\bullet \in \mathbf{C}^{flat}(R)$. As we have seen, the direct factor $e_T M^\bullet$ does not depend on a representative and is quasi isomorphic to $e_{T_0}M_0^\bullet$. Thus we need to show that $H^i(e_{T_0}M^\bullet) = e_{T_0}H^i(M_0^\bullet)$. A standard induction procedure reduces us to the case where we have an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of modules in $\mathbf{C}^{flat}(R)$ and we need to prove that :

$$e_{T_0}M_0 \to e_{T_0}M_1 \to e_{T_0}M_2$$

$$(1 - e_{T_0})M_0 \to (1 - e_{T_0})M_1 \to (1 - e_{T_0})M_2$$

are both exact. We will show this for the first complex, the treatment for the second complex is similar. On the first complex, the exactness rests on the fact that T_0 acts invertibly on $e_{T_0}M_i$. Indeed, for the exactness on the left, if $u_0 \in e_{T_0}M_0$ is mapped to 0 then let $u_0' \in e_{T_0}M_0$ such that $T_0u_0' = u_0$, then $u_0' \in e_{T_0}M_0 \subset M_0$ is sent to zero in M_1, so $u_0' = 0$ and $u_0 = 0$.

For the right exactness, let $u_2 \in e_{T_0}M_2$ and write $u_2 = Tu_2'$ with $u_2' \in e_{T_0}M_2$. Let $u_1 \in M_1$ be some lift of u_2'. Now it is clear that $T_0e_{T_0}u_1$ is sent to u_2, and as $e_{T_0}u_1 \in e_{T_0}M_1$, we are done.

Finally for the exactness in the middle, let $u \in e_{T_0}M_1$ that is sent to 0 in $e_{T_0}M_2$. Again as T_0 acts invertibly on $e_{T_0}M_1$, we can write $T_0u' = u$ for some $u' \in e_{T_0}M_1$. Since $u_1 \in e_{T_0}M_1 \subset M_1$ is also sent to zero in M_2, it must be the image of some $u_0 \in M_0$, and now, obviously $Te_{T_0}u_1$ is sent to u. □

Remark 11.1.1. *All the complexes we consider in this paper satisfy the assumptions of the lemmas and propositions above, i.e they can be represented by flat objects in relevant category, as can be checked using Cĕch complex for some finite affine covering.*

11.2. **Perfect complex of classical forms.** Now we are ready to prove our main theorem for the complex of classical forms. Let us start with following easy lemma that is used repeatedly through out this paper:

Lemma 11.2.1. *Let X be a scheme, $\mathcal{L}$ a line bundle and $\mathcal{F}$ a coherent sheaf on X. Let $f \in H^0(X, \mathcal{L})$ be any section and $j : Z \hookrightarrow X$ be open subscheme of X which is the nonzero locus of f. We have :*

$$j_*j^*\mathcal{F} \cong \varinjlim_{\times f} \mathcal{F} \otimes \mathcal{L}^n$$

In particular $H^0(Z, \mathcal{F}) = \varinjlim_n H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$.

Proof. There is an isomorphism $j^* \varinjlim_{\times f} \mathcal{F} \otimes \mathcal{L}^n \to j^*\mathcal{F}$. Indeed, on each affine open subset $U = Spec(R)$ small enough of X we can write $\mathcal{L}|_U = R.e_L$ where $e_L \in \mathcal{L}(U)$. Let also $M := \mathcal{F}(U)$.

If we write $f|_U = se_L$ with $s \in R$ then the isomorphism $j^* \varinjlim_{\times f} \mathcal{F} \otimes \mathcal{L}^n \to j^*\mathcal{F}$ over U is nothing but the canonical isomorphism $\varinjlim_{\times s} M \cong M \otimes R_s$.

Finally by adjunction, we obtain the isomorphism

$$\varinjlim_{\times f} \mathcal{F} \otimes \mathcal{L}^n \cong j_*j^*\mathcal{F}$$

and this finishes the lemma. □

In order to show that the action of T_p on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1})$ is locally finite we begin with:

Proposition 11.2.1. *For any weight $\kappa = (k_1, k_2, k_3)$ such that $k_1 + k_3 > p + 1$, the action of T_p on $H^0(\mathcal{S}_1^{=1}, \Omega_D^\kappa)$ is locally finite.*

Proof. We use lemma (11.2.1) for the line bundle $det_{\mathcal{A}[\pi]}^{p^2-1}$ and the section $Ha' \in H^0(\mathcal{S}_1^{\leq 1}, det_{\mathcal{A}[\pi]}^{p^2-1})$ to see that :

$$H^0(\mathcal{S}_1^{=1}, \Omega_D^\kappa) = \varinjlim_{\times Ha'} H^0(\mathcal{S}_1^{\leq 1}, \Omega_D^{\kappa'(n)})$$

The locus $\mathcal{S}_1^{\leq 1}$ is closed in $\mathcal{S}_1$, hence proper, so that each $H^0(\mathcal{S}_1^{\leq 1}, \Omega_D^{\kappa'(n)})$ is a finite dimensional $\mathbb{F}_p$-vector space. Further more, following the proposition (10.6.2), when $k_1 + k_3 > p + 1$, we have $T_p.Ha' = Ha'.T_p$ and each $H^0(\mathcal{S}_1^{\leq 1}, \Omega^{\kappa'(n)})$ is thus stable under T_p. This means that $H^0(\mathcal{S}_1^{=1}, \Omega_D^{\kappa})$ can be written as a direct limit of T_p-stable finite dimensional spaces, as a result T_p is locally finite. □

Now over $\mathcal{S}_1^{\geq 1}$, we can cook up a resolution for Ω_D^{κ} in the following way. First, take a look at the exact sequence (for each n):

$$\Omega_D^{\kappa} \xrightarrow{\times Ha^n} \Omega_D^{\kappa(n)} \to \Omega_D^{\kappa(n)}/(Ha^n) \to 0 \tag{11.2}$$

This induces :

$$\Omega_D^{\kappa} \to \varinjlim_n \Omega_D^{\kappa(n)} \to \varinjlim_n \Omega_D^{\kappa(n)}/(Ha^n) \to 0 \tag{11.3}$$

We observe that the support of the sheaf $\varinjlim_n \Omega_D^{\kappa(n)}$ is the ordinary locus $\mathcal{S}_1^{=2}$ (See lemma 11.2.1), and the (reduced) support of the sheaf $\varinjlim_n \Omega_D^{\kappa(n)}/(Ha^n)$ is clearly the locus $\mathcal{S}_1^{=1}$. Both these loci $\mathcal{S}_1^{=2}, \mathcal{S}_1^{=1}$ are affine in the minimal compactification. As a result the sheaves $\varinjlim_n \Omega_D^{\kappa(n)}$ and $\varinjlim_n \Omega_D^{\kappa(n)}/(Ha^n)$ are acyclic and the above exact sequence is a resolution of Ω_D^{κ} over $\mathcal{S}_1^{\geq 1}$. In other words the complex $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa})$ is represented by :

$$H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa}) \to \varinjlim_n H^0(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa(n)}/(Ha^n)) \tag{11.4}$$

Proposition 11.2.2. *For all weight κ such that $k_1 + k_3 > 2$, the action of T_p on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa})$ is locally finite.*

Proof. For such a weight, we have seen that the Hecke operator T_p commutes with the classical Hasse invariant so that the above resolution (11.4) of Ω_D^{κ} over $\mathcal{S}_1^{\geq 1}$ is T_p-equivariant. By lemma (11.1.2), it is enough to show the local finiteness of T_p on both terms $H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa})$ and $\varinjlim_{\times Ha} H^0(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa(n)}/(Ha^n)$.

(1) For the first terms, we can write $H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa})$ as $\varinjlim_{\times Ha} H^0(\mathcal{S}_1, \Omega_D^{\kappa(n)})$ and the condition on the weight κ guarantees that $T_p.Ha = Ha.T_p$ (see 10.6.1)). As a consequence T_p on ordinary locus is locally finite since it acts finitely locally on each $H^0(\mathcal{S}_1, \Omega_D^{\kappa(n)})$.
(2) For the second term, we proceed by induction. If $n = 1$, notice that $\kappa(1)$ is the weight $(k_1 + p - 1, k_2 + p - 1, k_3)$ so the proposition(11.2.1) applies. For general n we use the exact sequence:

$$\begin{aligned} 0 \to H^0(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa(n)}/(Ha^n) &\to H^0(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa(n+1)})/(Ha)^{n+1}) \\ &\to H^0(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa(n)}/(Ha)) \to 0 \end{aligned} \tag{11.5}$$

Because T_p acts locally finitely on the third term by lemma (11.2.1), and on the first term by induction hypothesis, it acts locally finitely on the second term.

□

From now on, whenever we have a complex $M^\bullet$ on which T_p acts locally finitely, we will denote by t_p the projector associated to T_p (see section (11.1)). The following result is important:

Proposition 11.2.3. *For all weight κ such that $k_1 - k_2 \geq 1$ and $k_1 + k_3 > 3$, the complexes $t_p\mathrm{R}\Gamma(\mathcal{S}_1, \Omega^\kappa_D)$ and $t_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}, \Omega^\kappa_D)$ are quasi-isomorphic.*

Proof. We reconsider the short exact sequence (11.4) over $\mathcal{S}_1$ and $\mathcal{S}_1^{\geq 1}$, this gives us two long exact sequences:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(\mathcal{S}_1, \Omega^\kappa_D) & \longrightarrow & \varinjlim_{\times Ha} H^0(\mathcal{S}_1, \Omega^{\kappa(n)}_D) & \longrightarrow & \varinjlim_{\times Ha} H^0(\mathcal{S}_1, \Omega^{\kappa(n)}_D/(Ha^n)) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^0(\mathcal{S}_1^{\geq 1}, \Omega^\kappa_D) & \longrightarrow & \varinjlim_{\times Ha} H^0(\mathcal{S}_1^{\geq 1}, \Omega^{\kappa(n)}_D) & \longrightarrow & \varinjlim_{\times Ha} H^1(\mathcal{S}_1, \Omega^{\kappa(n)}/(Ha^n))
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & H^1(\mathcal{S}_1, \Omega^\kappa_D) & \longrightarrow & 0 & \longrightarrow & \varinjlim_{\times Ha} H^0(\mathcal{S}_1, \Omega^{\kappa(n)}_D/(Ha^n)) & \longrightarrow & H^2(\mathcal{S}_1, \Omega^\kappa_D) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & H^1(\mathcal{S}_1^{\geq 1}, \Omega^\kappa_D) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The goal is to show that each vertical arrows is an isomorphism after applying the projector t_p. The first one is an isomorphism because $\mathcal{S}_1$ is Cohen-Macaulay and $\mathcal{S}_1 \backslash \mathcal{S}_1^{\geq 1}$ has codimension 2. The second vertical arrow is also an isomorphism since both are isomorphic to $H^0(\mathcal{S}_1^{=2}, \Omega^\kappa)$. Proving the third one is an isomorphism is harder. Firstly, it is enough to prove that for each n, we have: $t_pH^0(\mathcal{S}_1, \Omega^{\kappa(n)}_D/(Ha^n)) \cong t_pH^0(\mathcal{S}_1^{\geq 1}, \Omega^{\kappa(n)}_D/(Ha^n))$. We proceed by induction.

(1) For $n = 1$, then we need to prove that

$$t_pH^0(\mathcal{S}_1, \Omega^{\kappa(1)}_D/(Ha)) = t_pH^0(\mathcal{S}_1^{\leq 1}, \Omega^{\kappa(1)}_D) = t_pH^0(\mathcal{S}_1^{=1}, \Omega^{\kappa(1)}_D) = t_pH^0(\mathcal{S}_1^{\geq 1}, \Omega^{\kappa(1)}_D/(Ha))$$

This is handled by lemma (11.2.2) down below which establishes the middle equality.

(2) For general n, we consider the exact sequence over $\mathcal{S}_1$ and $\mathcal{S}^{\geq 1}$:

$$0 \to \Omega^{\kappa(n)}_D/(Ha^n) \to \Omega^{\kappa(n+1)}_D/(Ha^{n+1}) \to \Omega^{\kappa(n+1)}_D/(Ha) \to 0 \tag{11.6}$$

which induces two long exact sequences:

$$\begin{array}{ccccc}
0 \longrightarrow & H^0(\mathcal{S}_1, \Omega^{\kappa(n)}_D/(Ha^n)) & \longrightarrow & H^0(\mathcal{S}_1, \Omega^{\kappa(n+1)}_D/(Ha^{n+1})) & \longrightarrow \dots \\
& \downarrow & & \downarrow & \\
0 \longrightarrow & H^0(\mathcal{S}_1^{\geq 1}, \Omega^{\kappa(n)}_D/(Ha^n)) & \longrightarrow & H^0(\mathcal{S}_1^{\geq 1}, \Omega^{\kappa(n+1)}_D/(Ha^{n+1})) & \longrightarrow \dots
\end{array}$$

$$\begin{array}{ccccc} H^0(\mathcal{S}_1,\Omega_D^{\kappa(n+1)}/(Ha)) & \longrightarrow & H^1(\mathcal{S}_1,\Omega_D^{\kappa(n)}/(Ha^n)) & \longrightarrow & ... \\ \downarrow & & \downarrow & & \\ H^0(\mathcal{S}_1^{\geq 1},\Omega_D^{\kappa(n+1)}/(Ha)) & \longrightarrow & H^1(\mathcal{S}_1^{\geq 1},\Omega_D^{\kappa(n)}/(Ha^n)) & \longrightarrow & ... \end{array}$$

Again, lemma (11.2.2), point (2) says that under our initial condition on the weight κ we have:

$$t_pH^1(\mathcal{S}_1,\Omega_D^{\kappa(n)}/(Ha^n)) = t_pH^1(\mathcal{S}_1^{\geq 1},\Omega_D^{\kappa(n)}/(Ha^n)) = 0$$

Applying the projector t_p on the last diagram, and use the induction hypothesis, we obtain the result.

Finally, $t_pH^1(\mathcal{S}_1,\Omega^{\kappa(n)}/(Ha^n)) = 0$ is handled by point (3) of lemma (11.2.2) below.

□

We have an immediate corollary of this proposition.

Corollary 11.2.1. *For all weight κ with $k_1-k_2 \geq 1$, and $k_1+k_3 > 3$. The complexes of $\mathbb{F}_p$-vector space $t_p\mathrm{R}\Gamma(\mathcal{S}_1,\Omega_D^\kappa)$ and $t_p\mathrm{R}\Gamma(\mathcal{S}^{\geq 1},\Omega_D^\kappa)$ are perfect of amplitude $[0,1]$.*

Proof. A perfect complex over a field is just a complex of finite dimensional vector spaces. By proposition (11.2.3), we need to prove that $t_p\mathrm{R}\Gamma(\mathcal{S}_1,\Omega_D^\kappa)$ is perfect. For this we need to check its cohomology groups are finite dimensional which is an immediate consequence of the fact that $\mathcal{S}_1$ is proper. □

We recollect some technical lemmas that we need for the proofs of propositions above.

Lemma 11.2.2.
(1) *If $k_1-k_2 \geq 1$ and $k_1+k_3 \geq p+2$, we have $t_pH^0(\mathcal{S}_1^{\leq 1},\Omega_D^\kappa) = t_pH^0(\mathcal{S}_1^{=1},\Omega_D^\kappa)$.*
(2) *If $k_1-k_2 \geq 1$ and $k_1+k_3 \geq p+2$, we have $t_pH^i(\mathcal{S}_1^{\leq 1},\Omega_D^\kappa) = t_pH^i(\mathcal{S}_1^{=1},\Omega_D^\kappa) = 0$ for $i \geq 1$.*
(3) *If $k_1-k_2 \geq 1$ and $k_1+k_3 \geq 3$, we have $t_pH^i(\mathcal{S}_1,\Omega_D^{\kappa(n-1)}/(Ha^n)) = 0$ for $i = 1,2$.*

Proof. (1) The short exact sequence:

$$0 \to \Omega_D^\kappa \to \Omega_D^{\kappa'(1)} \to \Omega_D^{\kappa'(1)}/(Ha') \to 0$$

gives a long exact sequence:

$$\to H^i(\mathcal{S}_1^{\leq 1},\Omega_D^\kappa) \to H^i(\mathcal{S}_1^{\leq 1},\Omega_D^{\kappa'(1)}) \to H^i(\mathcal{S}_1^{=0},\Omega_D^{\kappa'(1)}) \to H^{i+1}(\mathcal{S}_1^{\leq 1},\Omega_D^\kappa) \to$$

With the condition on the weight κ, this long exact sequence is T_p-equivariant by proposition (10.6.2). Further more when $k_1-k_2 \geq 1$ and $k_1+k_3 \geq p+2$, the action of T_p on $H^i(\mathcal{S}^{=0},\Omega_D^{\kappa'(1)})$ is trivial. We deduce that for any $f \in t_pH^i(\mathcal{S}_1^{\leq 1},\Omega_D^{\kappa(1)})$, we can find $g \in H^i(\mathcal{S}_1^{\leq 1},\Omega_D^{\kappa(1)})$ such that $t_p(g) = f$. In particular, for $i = 0$, combining with the fact that there is a restriction $H^0(\mathcal{S}_1^{\leq 1},\Omega_D^{\kappa(1)}) \hookrightarrow H^0(\mathcal{S}_1^{=1},\Omega_D^{\kappa(1)})$ we have:

$$t_pH^0(\mathcal{S}_1^{\leq 1},\Omega_D^\kappa) = t_pH^0(\mathcal{S}_1^{\leq 1},\Omega_D^{\kappa'(1)})$$

Passing to the limit and using the lemma (11.2.1) for the line bundle $det_{\mathcal{A},\pi}^{p^2-1}$ and the second Hasse invariant Ha', we obtain:

$$t_pH^0(\mathcal{S}_1^{\leq 1}, \Omega^\kappa) = t_pH^0(\mathcal{S}_1^{=1}, \Omega_D^\kappa)$$

(2) By argument in (1), for all i we have $t_pH^i(\mathcal{S}_1^{\leq 1}, \Omega^\kappa) = t_pH^i(\mathcal{S}_1^{=1}, \Omega_D^\kappa)$, but as $\mathcal{S}_1^{=1}$ has affine image in the minimal compactification, the cohomology group $H^i(\mathcal{S}_1^{=1}, \Omega_D^\kappa)$ is zero if $i \geq 1$.

(3) we prove this by induction: For $n = 1$, this follows from the point (2) as $t_pH^i(\mathcal{S}_1, \Omega_D^{\kappa(1)}/Ha) = t_pH^i(\mathcal{S}_1^{\leq 1}, \Omega_D^{\kappa(n)})$.

For general n, take the long exact sequence associated with the short exact sequence (11.6) and the induction hypothesis gives the answer:

$$H^0(\mathcal{S}_1, \Omega_D^{\kappa(n)}/(Ha^n)) \longrightarrow H^0(\mathcal{S}_1, \Omega_D^{\kappa(n+1)}/(Ha^{n+1})) \longrightarrow H^0(\mathcal{S}_1, \Omega_D^{\kappa(n+1)}/(Ha)) \longrightarrow ...$$

$$H^1(\mathcal{S}_1, \Omega^{\kappa+n(p-1)}/(Ha^n)) \longrightarrow H^1(\mathcal{S}_1, \Omega_D^{\kappa(n+1)}/(Ha^n)) \longrightarrow H^1(\mathcal{S}_1, \Omega_D^{\kappa(n+1)}/(Ha)) \longrightarrow ...$$

$\square$

12. p-adic modular forms

In this section, we define the Hida complex of p-adic modular forms for each pair of integer (k_1, k_2), denoted by $\mathcal{V}^\bullet(k_1, k_2)$. This will be the "ordinary" part of a huge complex over the weight space $Spec(\Lambda)$ where $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. To cut out this ordinary part, we will define a Hecke operator U_p on this complex, similar to the operator T_p in the previous sections. Finally, for each weight $\kappa = (k_1, k_2, k_3)$ satisfying certain regularity condition, there will be a map $t_p\mathrm{R}\Gamma(\mathcal{S}, \Omega_D^\kappa) \to \mathcal{V}^\bullet(k_2, k_3) \otimes^L_{\Lambda, k_1} \mathbb{Z}_p$ which induces injections on each cohomology groups, thus $\mathcal{V}^\bullet(k_2, k_3)$ interpolates the complex of classical forms.

12.1. **Igusa tower and p-adic modular forms.** Let $\mathfrak{S}$ be the completion along the special fiber $\mathcal{S}_1$ of $\mathcal{S}$, and $\mathfrak{S}^{\geq 1}$ be the open subscheme of $\mathfrak{S}$ where the multiplicative rank of abelian scheme is greater than or equal to 1. We define the formal moduli scheme $\mathfrak{S}^{\geq 1}(p^m)$ parametrizing the subgroup $H_m \subset \mathcal{A}[\pi^\infty]$ locally isomorphic in the étale topology to μ_{p^m}. For any $m \geq n$ we have a natural map $\mathfrak{S}^{\geq 1}(p^m) \to \mathfrak{S}^{\geq 1}(p^n)$ sending each point (H_m, A) to $(H_m[p^n], A)$.

Remark 12.1.1. *If we let $\mathfrak{S}_{par}$ be the completion of $\mathcal{S}_{par}$ along its special fiber then by definition the forgetful map $\mathfrak{S}^{\geq 1}(p^m) \to \mathfrak{S}^{\geq 1}$ factors through $\mathfrak{S}^{\geq 1}_{par} \xrightarrow{p_1} \mathfrak{S}^{\geq 1}$. So we should denote $\mathfrak{S}^{\geq 1}(p^m)$ by $\mathfrak{S}^{\geq 1}_{par}(p^m)$ instead, but we drop the subscript "par".*

Lemma 12.1.1. *The natural map $\mathfrak{S}^{\geq 1}(p^m) \to \mathfrak{S}^{\geq 1}(p^n)$ is affine, étale for any $m \geq n$.*

Proof. The proof uses the deformation theory and can be done exactly as in lemma (9.1.1.1) [Pil18]. $\square$

We can define a partial Igusa tower $Ig_m := \underline{Isom}_{\mathfrak{S}^{\geq 1}(p^m)}(\mu_{p^m}, H_m)$ with a natural action of $(\mathbb{Z}_p/p^m\mathbb{Z}_p)^\times$. More precisely, each point $x \in Ig_n$ corresponds to a trivialization $\psi_x : \mu_{p^m} \cong H_m$ and an element $\lambda \in (\mathbb{Z}_p/p^m\mathbb{Z}_p)^\times = End(\mu_{p^m})$ sends x to $\lambda(x)$ corresponding to the trivialization $\psi_x \circ \lambda : \mu_{p^m} \cong H_m$. Let $\mathfrak{S}^{\geq 1}(p^\infty)$ be the inverse limit of $\mathfrak{S}^{\geq 1}(p^m)$ and we call $Ig_\infty := \underline{Isom}_{\mathfrak{S}^{\geq 1}(p^\infty)}(\mu_{p^\infty}, H_\infty)$ the Igusa tower, this has an action of $\mathbb{Z}_p^\times$. Before really defining the sheaf of p-adic forms, we take a brief moment to recall some facts about the p-adic weight space in our case.

Let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the completed group algebra. Recall that we have an isomorphism $\mathbb{Z}_p[[1+p\mathbb{Z}_p]] \cong \mathbb{Z}_p[[T]]$, sending $[1+p]$ to $(1+T)$. As a result, $\Lambda \cong \mathbb{Z}_p[[T]][\mathbb{F}_p^\times]$ since $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p)$. We denote by $\chi : \mathbb{Z}_p^\times \to \Lambda^\times$ the canonical character sending x to $[x]$. This character is universal in the sense that, for any $\mathbb{Z}_p$-complete algebra R, any continuous character $w : \mathbb{Z}_p^\times \to R^\times$ factors through χ. In particular, any character $w : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ gives rise to a continuous function, still denoted by $w : \Lambda \to \mathbb{Z}_p$.

We now return to the problem of defining the p-adic forms. Let

$$\mathscr{P} := (\pi_*\mathcal{O}_{Ig_\infty}\hat{\otimes}_{\mathbb{Z}_p}\Lambda)^{\mathbb{Z}_p^\times}$$

where $\pi : \mathfrak{S}^{\geq 1}(p^\infty) \to \mathfrak{S}^{\geq 1}(p)$ and $\mathbb{Z}_p^\times$ acts on $\pi_*\mathcal{O}_{Ig_\infty}$ as described above, and it acts on Λ via the universal character χ. Lastly, for any complete $\mathbb{Z}_p$-algebra R and a continuous character $w : \mathbb{Z}_p^\times \to R$, let :

$$\mathscr{P}^w := \mathscr{P}\hat{\otimes}_{\Lambda,w}R$$

12.2. **Truncated Igusa tower and Comparison morphism.** Recall that we have denoted by $\mathcal{M}$ the interior of the scheme $\mathcal{S}$, where the semi-abelian scheme is an abelian scheme. For each n, we introduce the following schemes :

(1) $\mathcal{S}_n := \mathcal{S} \times Spec(\mathbb{Z}/p^n\mathbb{Z})$ and $\mathcal{M}_n$ the interior of $\mathcal{S}_n$ away from the boundary.
(2) $\mathcal{S}_n^{\geq 1}$ is a subscheme of $\mathcal{S}_n$ where the multiplicative rank of the universal semi-abelian scheme is at least 1 and $\mathcal{M}_n^{\geq 1}$ the interior away from the boundary.
(3) $\mathcal{S}_n^{\geq 1}(p^m)$ be the moduli problem parametrizing subgroups $H_m \subset \mathcal{A}[\pi^m] \subset \mathcal{A}[p^m]$ étale locally isomorphic to μ_{p^m} and again $\mathcal{M}_n^{\geq 1}(p^m)$ is its interior.

We can define a truncated weight space $\Lambda_n := \mathbb{Z}/p^n\mathbb{Z}[[(\mathbb{Z}/p^n\mathbb{Z})^\times]]$ and a truncated version of Igusa tower over $\mathcal{S}_n^{\geq 1}(p^m)$. Indeed when $m \geq n$ (so that $\omega_{\mu_{p^m}}$ is locally free), we can consider $Ig_{n,m} := \underline{Isom}_{\mathcal{S}_n^{\geq 1}(p^m)}(\mu_{p^m}, H_m)$ which carries an action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$, and let $\mathscr{P}_{n,m} := (\pi_*\mathcal{O}_{Ig_{n,m}}\hat{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}}\Lambda_n)^{(\mathbb{Z}/p^m\mathbb{Z})^\times}$ with $\pi : \mathcal{S}_n^{\geq 1}(p^m) \to \mathcal{S}_n^{\geq 1}(p)$. Here, $(\mathbb{Z}/p^m\mathbb{Z})^\times$ acts on Λ_n via the obvious map $(\mathbb{Z}/p^m\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times \to \Lambda_n$. It is clear that we have $\mathscr{P} = \varprojlim_n \varinjlim_m \mathscr{P}_{n,m}$.

Over $\mathcal{S}_n^{\geq 1}(p)$ we have the automorphic sheaf Ω^κ and one can ask how $\mathscr{P}_{n,m}$ and Ω^κ are related. To see this, we remark that over $\mathcal{S}_n^{\geq 1}(p^m)$, the universal multiplicative group $H_m \subset \mathcal{A}[p^m]$ induces a surjection $\omega_{\mathcal{A}[p^m]} \to \omega_{H_m}$. We can think about $\mathscr{P}_{n,m}$ as a sheaf on $\mathcal{S}_n^{\geq}(p)$ whose sections are "functions" of the trivialization $H_m \cong \mu_p^m$. We also have the Hodge Tate map: $HT : H_m^D \otimes \mathcal{O}_{\mathcal{S}_n^{\geq 1}(p^m)} \to \omega_{H_m}$. Given the definition of the Hodge Tate map, this is just a short way of saying that there is a map $c : \pi_*\mathcal{O}_{Ig_{n,m}} \to \omega_{H_m}$. It is also immediate that this comparison map induces an isomorphism $c : \mathscr{P}^{k_1}_{n,m} \to \omega^{k_1}_{H_m}$ over $\mathcal{S}^{\geq 1}_n(p^m)$ for any $k_1 \in \mathbb{Z}$ (we see k_1 as a character $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ sending t to t^{k_1}). Combined with the surjections, $\omega_{\mathcal{A}[p^m]} \to \omega_{H_m}$ we deduce a surjection:

$$\Omega^{k_1,0,0} := sym^{k_1}\omega_\pi \to \mathscr{P}^{k_1}_{n,m}$$

As a consequence for any weight κ ($k_1 \geq k_2$),we obtain a surjection $c(\kappa) : \Omega^\kappa \to \mathscr{P}^{k_1-k_2}_{n,m} \otimes \Omega^{k_2,k_2,k_3}$. Let us denote the kernel of this map by $K\Omega^\kappa$.

12.3. **Operator U_p.** The sheaf $\mathscr{P}$ we have just defined over $\mathcal{S}^{\geq 1}(p)$ is too big for any practical purpose, we now show how to construct a projector that cuts out a direct factor which interpolates classical forms similar to how one would do in the classical Hida theory.

Over $\mathcal{M}^{\geq 1}_n(p^m)$ we can define the correspondence:

$$\begin{tikzcd} & \mathcal{U}_n(p^m) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{M}^{\geq 1}_n(p^m) & & \mathcal{M}^{\geq 1}_n(p^m) \end{tikzcd} \tag{12.1}$$

Where $\mathcal{U}_n(p^m)$ parametrizes the triple $(L, H_m, \mathcal{A})$ with $L \subset \mathcal{A}[\pi]$ of order p^2 and $L \cap H_m = \{0\}$. The map p_1 then sends each triple $(L, H_m, \mathcal{A})$ to $(H_m, \mathcal{A})$ and p_2 sends each triple $(L, H_m, \mathcal{A})$ to $(\frac{H_m+L+L^\perp}{L+L^\perp}, \frac{\mathcal{A}}{L+L^\perp})$.

We can also construct a toroidal compactifications of $\mathcal{U}(p^m)$ and $\mathcal{M}^{\geq 1}_n(p^m)$ which we denote by $\mathfrak{U}_n(p^m)$ and $\mathcal{S}^{\geq 1}_n(p^m)$ respectively. The correspondence above extends to a correspondence:

$$\begin{tikzcd} & \mathfrak{U}_n(p^m) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathcal{S}^{\geq 1}_n(p^m) & & \mathcal{S}^{\geq 1}_n(p^m) \end{tikzcd} \tag{12.2}$$

Lemma 12.3.1. *The morphism p_1 over $\mathcal{M}^{\geq 1}_n(p^m)$ is finite, flat.*

Proof. Since the base $\mathcal{M}^{\geq 1}_n(p^m)$ is smooth and $\mathfrak{U}_n(p^m)$ is Cohen-Macaulay , we can use the miracle flatness theorem. It then remains to check that p_1 is quasi-finite which we can check using Dieudonné modules like in lemma (10.2.1).

□

For each sheaf $\mathcal{F}$ over $\mathcal{M}^{\geq 1}(p^m)$ we can form the correspondence : $p_2^*\mathcal{F} \to p_1^!\mathcal{F}$ with the fundamental class given by the trace map and define a non normalized operator $\mathcal{U}_p \in End(\mathrm{R}\Gamma(\mathcal{M}^{\geq 1}_1(p^m), \mathcal{F}))$ as before. We extend this correspondence to the compactifications as in previous section for the operator T_p. The result is that for $\mathcal{F} = \Omega^\kappa$, we obtain a *normalized* correspondence $U_p \in End(\mathrm{R}\Gamma(\mathcal{S}^{\geq 1}_1(p^m), \Omega^\kappa))$ with $U_p = \frac{1}{p^{k_2+1}}\mathcal{U}_p$.

Finally, we show how to construct a correspondence for formal schemes $\mathfrak{S}^{\geq 1}(p^m)$ by some limit procedure. The construction of $p_2^*\mathcal{F} \to p_1^*\mathcal{F}$ is obvious, the only subtle issue is the fundamental class $p_1^*\mathcal{F} \to p_1^!\mathcal{F}$. The point is that we can build such fundamental class over $\mathcal{S}^{\geq 1}_n(p^m)$, but we need to verify that the formation of fundamental class is compatible with the embeddings $\mathcal{S}_n^{\geq 1}(p^m) \to \mathcal{S}_l^{\geq 1}(p^m)$ for $n \leq l$. To see this, we remark that the sheaf of ideal that defines $\mathcal{S}_n^{\geq 1}(p^m)$ inside $\mathcal{S}_l^{\geq 1}(p^m)$ is of finite Tor-dimension. As the result the base change theorem(10.3.1) works, and it means that we get the fundamental class $p_1^*\mathcal{F} \to p_1^!\mathcal{F}$ over $\mathcal{S}_n^{\geq 1}(p^m)$ from the one on $\mathcal{S}_l^{\geq 1}(p^m)$ by tensoring with $p_1^*\mathcal{O}_{\mathcal{S}_n^{\geq 1}(p^m)}$.This shows the compatibility with respect to restriction and allows us to pass to the limit and obtain the fundamental class: $p_1^*\mathcal{F} \to p_1^!\mathcal{F}$ over $\mathfrak{S}^{\geq 1}(p^m)$. As a consequence we can produce an (normalized) operator $U_p \in End(\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p^m), \Omega^\kappa)$

12.4. **Compatibility with Hasse invariants.** As we have done with the complex of classical modular forms, we want to establish that under some conditions on the weight κ, the operator U_p acts locally finitely on $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p^m), \Omega^\kappa)$ and then the ordinary part $e_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p^m), \Omega^\kappa)$ (where e_p is the projector associated to U_p) is a perfect complex. We can deduce this from working on the special fiber. For this reason we will study the correspondence (12.2) on the special fiber and its interaction with Hasse invariants.

Proposition 12.4.1. *For all weight κ the operator U_p commutes with the classical Hasse invariant, i.e the following diagram commutes:*

$$\begin{tikzcd} p_2^*\Omega^\kappa \arrow[r, "U_p"] \arrow[d, "\times p_2^*Ha(\mathcal{A}')"'] & p_1^!\Omega^\kappa \arrow[d, "\times p_1^!Ha(\mathcal{A})"] \\ p_2^*\Omega^{\kappa(1)} \arrow[r, "U_p"'] & p_1^!\Omega^{\kappa(1)} \end{tikzcd} \tag{12.3}$$

Proof. Our schemes are still Cohen-Macaulay, and so, we can check the commutativity points by points on a locus whose complementary is of codimension at least 2, which in our case is the ordinary locus away from the boundary. After this, the proof can be repeated verbatim as in (10.6.1). □

Similarly for the generalized Hasse invariant, we also have:

Proposition 12.4.2. *For all weight κ, the following diagram defined over $\mathcal{S}_1^{\leq 1}(p)$ commutes :*

$$\begin{tikzcd} p_2^*\Omega^\kappa \arrow[r, "U_p"] \arrow[d, "\times p_2^*Ha'"'] & p_1^!\Omega^\kappa \arrow[d, "\times p_1^!Ha'"] \\ p_2^*\Omega^{\kappa'(1)} \arrow[r, "U_p"'] & p_1^!\Omega^{\kappa'(1)} \end{tikzcd}$$

Proof. Exactly as in proposition(10.6.2) □

Remark 12.4.1. *Reader can observe that there is no restriction on the weight κ compared to the case of the operator T_p in propositions (10.6.1,10.6.2)earlier.*

12.5. **Operator U_p and T_p.** We have just defined an action U_p on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^\kappa)$, this operator commutes with Hasse invariants and induces operators $U_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa))$ and $U_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=1}(p), \Omega_D^\kappa)$. Recall that we also have similar operators $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=2}, \Omega_D^\kappa))$ and $T_p \in End(\mathrm{R}\Gamma(\mathcal{S}_1^{=1}, \Omega_D^\kappa))$. We wish to study the relationship between the operators T_p and U_p on each stratum.

12.5.1. *Ordinary locus.* In this subsection, we show that with some hypothesis on κ, the action of U_p on $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$ is locally finite. For this, we will show that T_p also acts on $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$ and deduce the local finiteness of U_p based on that of T_p.

Away from the boundary, we have a very clear description of Hecke operator T_p as follows, let $f \in H^0(\mathcal{M}_1^{=2}, \Omega^{\kappa})$ regarded as a "function" on $\mathcal{M}_1^{=2}$. Inside the ordinary locus, the correspondence is finite flat and the fundamental class is given by the trace map as usual, so that if for any $x \in \mathcal{M}_1^{=2}$, we denote by A_x the underlying abelian scheme, the action of T_p on f is given by the formula:

$$T_p(f)(A_x) = \frac{1}{p^{k_2+1}} \sum_L f(\frac{A_x}{L + L^{\perp}})$$

Where L runs through the set of all subgroups of $A_x[\pi]$ of order p^2. Since we have the restriction $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}) \hookrightarrow H^0(\mathcal{M}_1^{=2}, \Omega)$, we can pretend that T_p is given by the formula above when analyzing action of T_p on $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$.

With this description, it is immediate to see that T_p can be written as a sum of two operators $T_p^{1,1}$ and $T_p^{2,0}$ where $T_p^{1,1}$ accounts for all the isogenies by subgroup L of multiplicative rank 1, étale rank 1, and $T_p^{2,0}$ for all isogenies by L of multiplicative rank 2, étale rank 0. Now for $k_1 + k_3 > 1$, due to the normalization (see the proof of prop(10.4.1)) tells us that $T_p^{2,0}$ acts trivially on $H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa})$. What is left is then the part $T_p^{1,1}$ and it factors through $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$, we have the following diagram:

(12.4)

$$\begin{tikzcd}
H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa}) \arrow[rr, "T_p^{1,1}"] \arrow[dr, "res"'] & & H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa}) \arrow[dr, "res"] & \\
& H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}) \arrow[ur, "T_p'"] \arrow[rr, "T_p"] & & H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})
\end{tikzcd}$$

Indeed, as A_x is ordinary, and L is of multiplicative rank 1, étale rank 1, the point $\frac{A_x}{L+L^{\perp}}$ actually lifts to a point of $\mathcal{S}_1^{=2}(p)$ corresponding to a pair $(\frac{A_x}{L+L^{\perp}}, H_1 = \frac{A_x[\pi]}{L})$. By composing with the restriction $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}) \to H^0(\mathcal{S}_1^{=2}, \Omega_D^{\kappa})$ we have an operator, still denoted by $T_p := res \circ T_p' \in End(H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}))$.

Proposition 12.5.1. *For all weight κ such that $k_1 > k_2$, we have $U_p \circ T_p = U_p \circ U_p$ as endomorphism of $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$.*

Proof. Remark that $T_p = U_p + I_p$ where I_p accounts for all igosenies with kernel L such that $L \cap H_1 \neq \{0\}$, so that we only need to prove that $U_p \circ I_p = 0$. Over $\mathcal{S}^{=2}(p)$ we can analyze $\tilde{U_p}$ and $\tilde{I_p}$, the non normalized versions of U_p and I_p. For each point $x \in \mathcal{S}^{=2}(p)$, we denote by (A_x, H_x) the corresponding universal pair of groups at x. For any $f \in H^0(\mathcal{S}^{=2}, \Omega_D^{\kappa})$, we have:

$$\tilde{U_p}(f)(x) = \sum_{L \text{ such that } L \cap H_x = \{0\}} f\Big(\frac{A_x}{L + L^{\perp}}, \frac{H_x + L + L^{\perp}}{L + L^{\perp}}\Big)$$

and

$$\tilde{I}_p(f)(x) = \sum_{L \text{ such that } L\cap H_x \neq \{0\}} f\Big(\frac{A_x}{L+L^\perp}, \frac{H_x+L+L^\perp}{L+L^\perp}\Big)$$

So we can write:

$$\tilde{U}_p \circ \tilde{I}_p(f)(x) = \sum_{\substack{L\cap H_x=\{0\} \\ L'\cap H_x\neq\{0\}}} f\Big(\frac{A_x}{L+L^\perp+L'+L'^\perp}, \frac{H_x+L+L^\perp+L'+L'^\perp}{L+L^\perp+L'+L'^\perp}\Big)$$

Now, as $A_x[\pi] \subset L + L'$ (since $L \cap H_x = \{0\}$ whereas $L' \cap H_x \neq \{0\}$) and $L^\perp + L'^\perp$ is étale, we can do exactly as in proposition 10.4 to see that the correspondence $\tilde{U}_p \circ \tilde{I}_p$ is divisible by $p^{k_1+k_2+2}$, with the factor $p^{k_1+k_2}$ coming from the differential of $A_x \to \frac{A_x}{L+L^\perp+L'+L'^\perp}$ and the factor p^2 coming from the fundamental classes of $\tilde{I}_p$ and $\tilde{U}_p$.

As a consequence $U_p \circ I_p = \tilde{U}_p \circ \tilde{I}_p/p^{2k_2+2} = 0(\text{mod } p^{k_1-k_2})$, so as long as $k_1 > k_2$ we have the claimed result.

□

Corollary 12.5.1. *For weight κ satisfying $k_1 > k_2$, the action of U_p on $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$ is locally finite.*

Proof. We need to prove that for each vector $v \in H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$, there exists a subvector space $V \subset H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$ of finite dimension and stable under the action of U_p.

Indeed, the action of T_p is locally finite on $H^0(\mathcal{S}_1^{=2}, \Omega_D^\kappa)$, hence via diagram (12.4) it also acts locally finitely on $H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$, this means that we can choose a finite dimensional subspace $W \in H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$ containing vector $T_p(v)$ and stable by T_p. Now let $V =< U_p(W), v, T_p(v), W >$, the sub vector space generated by $U_p(W), v, T_p(v), W$. This is a subspace of finite dimension, and by previous proposition 12.5.1 it is also stable by U_p.

□

Let u_p be the projector associated to U_p, we have:

Theorem 12.5.1. *Assume that $k_1 > k_2$ and $k_1 + k_3 > 1$, the following composition is a bijection:*

$$\theta : t_p H^0(\mathcal{S}_1^{=2}, \Omega_D^\kappa) \xrightarrow{res} t_p H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa) \xrightarrow{u_p} u_p H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$$

Proof. We take a look at the composition :

$$\Theta : t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa) \xrightarrow{res} t_p^{1,1} H^0(\mathfrak{S}^{=2}(p), \Omega_D^\kappa) \xrightarrow{u_p} u_p H^0(\mathfrak{S}^{=2}(p), \Omega_D^\kappa)$$

where $t_p^{1,1}$ is the projector associated to $T_p^{1,1}$. Recall that $T_p = T_p^{1,1} + T_p^{2,0}$ and $T_p^{2,0}$ acts trivially on the ordinary locus of the special fiber (when $k_1 + k_3 > 1$, due to our normalization). Thus the reduction mod p of Θ is the map θ of the theorem. As a result, if we can show that Θ is in fact an isomorphism, we are finished.

First of all, to show that Θ is surjective, it is enough by Nakayama's lemma to show that θ is surjective. Indeed, let $g \in u_p H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$. By definition of u_p we have $u_p g = g$. Let $g_1 \in H^0(\mathcal{S}_1^{=2}(p), \Omega_D^\kappa)$ such that $U_p g_1 = g$. From the proposition (12.5.1) we can write:

$$u_p g = u_p U_p g_1 = u_p T_p g_1 = u_p t_p T_p g_1$$

we can see that $T_p g_1 \in H^0(\mathcal{S}_1^{=2}, \Omega_D^\kappa)$ and so $t_p T_p g_1 \in t_p H^0(\mathcal{S}_1^{=2}, \Omega_D^\kappa)$.

For the injectivity of Θ, we claim that there is a section :

$$u_p H^0(\mathfrak{S}^{=2}(p), \Omega_D^\kappa) \to t_p H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$$

This section is indeed given by the trace map $Tr : H^0(\mathfrak{S}^{=2}(p), \Omega_D^\kappa) \to H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ since the projection $\pi : \mathfrak{S}^{=2}(p) \to \mathfrak{S}^{=2}$ is finite étale. We need an extra lemma:

Lemma 12.5.1. *We have $Tr \circ U_p^n \circ res = (T_p^{1,1})^n$ for all $n \geq 1$.*

Proof. We can check this point by point away from the boundary. Let $f \in H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ and let $x \in \mathfrak{S}^{=2}$ be a geometric point with underlying abelian variety (A_x), we have:

$$Tr \circ U_p^n \circ res(f)(x) = \frac{1}{p^{k_2+1}} \sum_{H_x} \sum_{L_n} f\Big(\frac{A_x}{L_n + L_n^\perp}, \frac{H_x + L_n + L_n^\perp}{L_n + L_n^\perp}\Big)$$

Where H_x runs through the set of all multiplicative subgroups of $A_x[\pi]$ locally isomorphic to μ_p and L_n runs through the set of all subgroup of $A_x[\pi^n]$ of order p^{2n} and $L_n \cap H_x = 0$. Now by definition, we also have :

$$(T_p^{1,1})^n(f)(x) = \frac{1}{p^{k_2+1}} \sum_{L_n} f\Big(\frac{A_x}{L_n + L_n^\perp}, \frac{H_x + L_n + L_n^\perp}{L_n + L_n^\perp}\Big)$$

where L_n runs through the set of all all subgroup of $A_x[\pi^n]$ of order p^{2n} which is locally isomorphic to $\mu_{p^n} \rtimes \mathbb{Z}/p^n\mathbb{Z}$. Now given such subgroup L_n there are exactly p multiplicative subgroups of $A_x[\pi]$ which intersect L_n trivially, this allows us to conclude. $\square$

Go back to our problem, let $f \in t_p H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ such that $\Theta(f) = 0$. Now, because $\Theta(f) = u_p \circ res(f) = 0$ and U_p is nilpotent on $(1 - u_p)H^0(\mathfrak{S}^{=2}(p), \Omega_D^\kappa)$, there exists $n \in \mathbb{N}^*$ such that $U_p^n \circ res(f) = 0$, implying $Tr \circ U_p^n \circ res(f) = 0$. By invoking the lemma above, we have $p(T_p^{1,1})^n(f) = 0$.

Now since $T_p^{1,1}$ is invertible on $t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ we deduce that $f \in p t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$. By the obvious induction, $f \in p^m t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ for all m too. Since $t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa)$ is a finite $\mathbb{Z}_p$-module, hence p-adically separated, we can finally conclude that $f \in \cap_m p^m t_p^{1,1} H^0(\mathfrak{S}^{=2}, \Omega_D^\kappa) = 0$.

$\square$

12.5.2. *Rank* 1 *locus.* By definition, $\mathcal{S}_1^{=1}$ is isomorphic to $\mathcal{S}_1^{=1}(p)$ as for a point $x \in \mathcal{S}_1^{=1}$ corresponding to A_x of multiplicative rank 1 then we have a canonical multiplicative subgroup $H \subset A_x[\pi]^o$, defined as the the image of $A_x[\pi][F]^{(p)}/ker(V) \hookrightarrow A_x[F]$ (See arguments right before lemma 10.1.3). On this locus we have $U_p = T_p$. Indeed, just as before we can decompose T_p into $T_p = T_p^{1,0} + T_p^{0,0}$ where $T_p^{1,0}$ (resp. $T_p^{0,0}$) accounts for all isogenies $\mathcal{A} \to \mathcal{A}'$ of kernel $L + L^\perp$ with $L^\perp$ of multiplicative rank 1 and étale rank 0 (resp. biconnected).

When $k_1 + k_3 > p + 1$, essentially by the renormalization of the Hecke correspondence, $T_p^{1,0}$ vanishes on $\mathcal{S}_1^{=1}$ (see the end of the proof the proposition (10.6.2),

$T_p^{1,0}$ is what we call T_p^{oo} there). As a result, only $T_p^{0,0}$ remains, but this operator accounts for all the isogenies $\mathcal{A} \to \mathcal{A}'$ of kernel $L + L^{\perp}$ with $L^{\perp}$ biconnected, hence L intersects trivially with the multiplicative subgroup $H_1 \subset \mathcal{A}[\pi]$. This means over $\mathcal{S}_1^{=1} \cong \mathcal{S}_1^{=1}(p)$ we have $T_p = T_p^{0,0} = U_p$.

13. Hida complex

The comparison between T_p and U_p on various strata of the special fiber allows us to easily carry over most of the properties of T_p to U_p. We begin by:

Proposition 13.0.1. (1) *For any κ such that $k_1 > k_2$ and $k_1 + k_3 > 2$, the action of U_p on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is locally finite.*

(2) *For $k_1 > k_2$ and $k_1+k_3 > 2$, the natural map $t_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa}) \to u_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is a quasi-isomorphism.*

(3) *For all κ with $k_1 > k_2$ and $k_1+k_3 > 3$, then $t_p\mathrm{R}\Gamma(\mathcal{S}_1, \Omega_D^{\kappa}) \to u_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is a quasi-isomorphism.*

(4) *For all κ with $k_1 > k_2$ and $k_1 + k_3 > 2$, the $\mathbb{F}_p$-complex $u_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is perfect of amplitude $[0, 1]$.*

Proof. (1) It suffices to show that the complex $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ admits a presentations by a complex of modules on each term of which $_p$ acts locally finitely. Now as in previous section, we can write down a resolution of Ω_D^{κ} over $\mathcal{S}_1^{\geq 1}(p)$:

$$0 \longrightarrow \Omega_D^{\kappa} \longrightarrow \varinjlim_n \Omega_D^{\kappa(n)} \longrightarrow \varinjlim_n \Omega_D^{\kappa(n)}/(Ha)^n \to 0 \tag{13.1}$$

Just as before, $\varinjlim_n \Omega_D^{\kappa(n)}$ is supported on the ordinary locus $\mathcal{S}_1^{=2}(p)$ and $\varinjlim_n \Omega^{\kappa(n)}/(Ha)^n$ is supported on the multiplicative rank 1 locus $\mathcal{S}_1^{=1}(p)$, both of these loci are affine in the minimal compactification, implying that both sheaves are acyclic and $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is represented by:

$$H^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}) \to \varinjlim_n H^0(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa(n)}/(Ha)^n) \tag{13.2}$$

By corollary 12.5.1, the action of U_p is finite on the first term. We can also show by induction that the action of U_p on $H^0(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa(n)}/(Ha)^n)$ is locally finite, exactly as in proposition(11.2.2), using the result of section 12.5.2 which states that $T_p = U_p$ over $\mathcal{S}_1^{=1} = \mathcal{S}_1^{=1}(p)$ to establish the base case $n = 1$ (it is here that we need the condition $k_1 + k_3 > 2$). We conclude that U_p acts locally finitely on $H^0(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$.

(2) As we have seen, the complex $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \Omega_D^{\kappa})$ is represented by the complex (17.2), so it is enough to establish that $e_pH^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa}) = u_pH^0(\mathcal{S}_1^{=2}(p), \Omega_D^{\kappa})$ and $e_pH^0(\mathcal{S}_1^{\geq 1}(p), \varinjlim_n \Omega^{\kappa_D(n)}/(Ha)^n) = u_pH^0(\mathcal{S}_1^{\geq 1}(p), \varinjlim_n \Omega^{\kappa(n)}/(Ha)^n)$. But these are settled by theorem 12.5.1.

(3) When κ satisfies such hypothesis, we know that T_p acts trivially on $\mathcal{S}_1^{=0}$, and we have a quasi isomorphism: $t_p\mathrm{R}\Gamma(\mathcal{S}_1, \Omega_D^{\kappa}) \cong t_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}, \Omega_D^{\kappa})$. As a consequence, (3) is then a direct corollary of (2).

(4) This is an immediate consequence of the above.

$\square$

Before we state similar statements but for formal schemes $\mathfrak{S}^{\geq 1}(p); \mathfrak{S}^{\geq 1}; \mathfrak{S}$, we recall the following lemmas about perfect complex of Λ-modules.

Lemma 13.0.1. *Let $M^\bullet \in D(\Lambda)$ be a complex representable by a bounded complex of flat, complete Λ-module. Assume that for some weight k_1, the cohomology groups $H^i(M^\bullet \otimes^L_{\Lambda,k_1} \mathbb{Z}_p)$ are finite as $\mathbb{Z}_p$-modules, then $M^\bullet$ is perfect.*

Proof. See Prop 2.2.1 [Pil18]. $\square$

Lemma 13.0.2. *Let $M^\bullet \in D(\Lambda)$ be a complex representable by a bounded complex of flat, complete Λ-module and $T \in End_{\boldsymbol{D}^{flat}(\Lambda)}(M^\bullet)$. If for some weight k_1, the action of T on $M^\bullet \otimes^L_{\Lambda,k_1} \mathbb{Z}_p$ is locally finite, then T is locally finite.*

Proof. See Prop 2.3.1 [Pil18]. $\square$

Theorem 13.0.1. (1) *If $k_1 > k_2$ and $k_1 + k_3 > 2$, the action of U_p (resp. T_p)on $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega^\kappa_D)$ (resp. $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}, \Omega^\kappa_D)$) is locally finite. We will denote by t_p and u_p the ordinary projector associated to T_p and U_p respectively.*

(2) *If $k_1 > k_2$ and $k_1 + k_3 > 2$, then we have a natural quasi-isomorphism :*

$$t_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}, \Omega^\kappa_D) \to u_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega^\kappa_D)$$

(3) *For all κ with $k_1 > k_2$ and $k_1 + k_3 > 3$, we have a quasi-isomorphism:*

$$t_p \mathrm{R}\Gamma(\mathfrak{S}, \Omega^\kappa_D) \to t_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}, \Omega^\kappa_D)$$

(4) *For all κ with $k_1 > k_2$ and $k_1 + k_3 > 3$,the natural map $t_p H^i(\mathfrak{S}, \Omega^\kappa_D) \to u_p H^i(\mathfrak{S}^{\geq 1}(p), \Omega^\kappa_D)$ is an isomorphism for $i = 0, 1$.*

(5) *For $k_1 > k_2$, and $k_1 + k_3 > 2$, the $\mathbb{Z}_p$-complex $u_p \mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega^\kappa_D)$ is perfect of amplitude $[0, 1]$.*

Proof. All of these statements have been proved for the special fiber. We just need to perform some limit procedure, except for the point (5) where we use the last lemma (13.0.1). We prove the point (1) as an example of how one can deduce the claims from the results we have shown earlier, the other points are completely analogous.

For this, we would like to write down a resolution of Ω^κ_D over $\mathfrak{S}^{\geq 1}$ using Hasse invariant. Recall that over $\mathcal{S}^{\geq 1}_n$, the sheaf $det_{\mathcal{A}}$ is ample in the minimal compactifications so let N be an integer(depending on n) such that the pull back of $det_{\mathcal{A}}^{\frac{N}{2}(p-1)}$ in the minimal compactification is ample, with this assumption, $Ha^N \in H^0(\mathcal{S}_1, det^{N(p-1)}_{\mathcal{A},\bar{\pi}})$ lifts to a section $\widetilde{Ha} \in H^0(\mathcal{S}, det^{N(p-1)}_{\mathcal{A},\bar{\pi}})$. Now consider the following exact sequence:

$$0 \to \Omega^\kappa \to \varinjlim_r \Omega^{\kappa(rN)} \to \varinjlim_n \Omega^{\kappa(rN)}/(\widetilde{Ha})^r \to 0$$

The middle term is supported on ordinary locus $\mathfrak{S}^{=2}$, and the third term is supported on multiplicative rank 1 locus $\mathfrak{S}^{=1}$. We need to show that the sheaves $\varinjlim_r \Omega^\kappa_D|_{\mathfrak{S}^{=2}}$

and $\varinjlim_n \Omega^{\kappa(rN)}|_{\mathfrak{S}^{=1}}$ are really acyclic. But the formal scheme $\mathfrak{S}^{=2}$ and $\mathfrak{S}^{=1}$ are completions of $\mathcal{S}^{=2}$ and $\mathcal{S}^{=1}$ along their respective special fibers, hence we have :

$$H^i(\mathfrak{S}^{=2}, \varinjlim_r \Omega_D^{\kappa(rN)}) = lim_n H^i(\mathcal{S}_n^{=2}, \Omega_D^{\kappa(rN)}) = 0$$

and similarly $H^i(\mathfrak{S}^{=1}, \varinjlim_r \Omega_D^{\kappa(rN)}) = 0$ for $i \geq 1$.

This implies that the exact sequence above is a resolution by acyclic sheaves of Ω^κ over $\mathfrak{S}^{\geq 1}$. Now we proceed completely similarly as in proposition (11.2.2), i.e we can prove by induction on r that T_p acts locally finitely on $\Omega_D^{\kappa(rN)})$ and on $\Omega^{\kappa(rN)}/(\widetilde{Ha})^r$. Passing to limit we see that the action of T_p on $\mathrm{R}\Gamma(\mathcal{S}_n^{\geq 1}(p), \Omega^\kappa)$ is locally finite. We can repeat the proof for the action on U_p on $\mathrm{R}\Gamma(\mathcal{S}_n^{\geq 1}(p), \Omega^\kappa)$.

□

Recall that we have constructed a complex of p-adic forms $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P} \otimes \Omega_D^{k_2,k_2,k_3})$. In what follows, we will show the main theorem stated in the introduction.

To lighten some notations, with any weight $\kappa = (k_1, k_2, k_3)$, we denote κ^+ for the weight $(k_1 - k_2, 0, 0)$ and κ^- for (k_2, k_2, k_3).

Lemma 13.0.3. *For any weight κ such that $k_1 + k_3 > 2$, the action of U_p on $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P} \otimes \Omega_D^{\kappa^-})$ is locally finite.*

Proof. Recall that we have an Igusa tower Ig and a projection $\pi : Ig \to \mathfrak{S}^{\geq 1}(p^\infty) \to \mathfrak{S}^{\geq 1}(p)$, and we have defined $\mathscr{P} := (\pi_* \mathcal{O}_{Ig_\infty} \hat{\otimes}_{\mathbb{Z}_p} \Lambda)^{\mathbb{Z}_p^\times}$. We also have a truncated version (when $m \geq n$) $\mathscr{P}_{m,n}$ of $\mathscr{P}$ over $\mathcal{S}_n^{\geq 1}(p)$. It is enough to show the lemma for the truncated version, and then by the remark (13.0.2) above, it is enough to show that for any $k_1 - k_2 \in \mathbb{Z}_+$, the action of U_p on $\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}_{m,n}^{k_1-k_2} \otimes \Omega_D^{\kappa^-})$ is locally finite.

For any n, and any coherent sheaf $\mathcal{G}$ of flat $\mathbb{Z}/p^n\mathbb{Z}$-module over $\mathcal{S}_n^{\geq 1}(p)$, we have an exact sequence:

$$0 \to \mathcal{G} \otimes \Omega_D^{\kappa^-} \to \varinjlim_r \mathcal{G} \otimes \Omega_D^{k^-(rN)} \to \varinjlim_r \mathcal{G} \otimes \Omega_D^{\kappa^-(rN)}/(\widetilde{Ha}^r) \to 0 \tag{13.3}$$

For any weight κ, we have introduced an exact sequence : $0 \to K\Omega_D^{\kappa^+} \to \Omega_D^{\kappa^+} \to \mathscr{P}_{m,n}^{\kappa^+} := \mathscr{P}_{m,n}^{k_1-k_2} \to 0$ Taking the resolution (13.3) above for each terms of this exact

sequence gives us a diagram:
(13.4)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0(\mathcal{S}_n^{=2}(p), K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-}) \arrow[r] \arrow[d] & \varinjlim_r H^0(\mathcal{S}_n^{\geq 1}(p), K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-(rN)}/(\widetilde{Ha}^r)) \arrow[d] \\
H^0(\mathcal{S}_n^{=2}(p), \Omega_D^{\kappa}) \arrow[r] \arrow[d] & \varinjlim_r H^0(\mathcal{S}_n^{\geq 1}(p), \Omega_D^{\kappa(rN)}/(\widetilde{Ha}^r)) \arrow[d] \\
H^0(\mathcal{S}_n^{=2}(p), \mathscr{P}_{m,n}^{\kappa^+}\otimes\Omega_D^{\kappa^-}) \arrow[r] \arrow[d] & \varinjlim_r H^0(\mathcal{S}_n^{\geq 1}(p), \mathscr{P}_{m,n}^{\kappa^+}\otimes\Omega_D^{\kappa^-(rN)}/(\widetilde{Ha}^r)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

By lemma(11.1.2) if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ over a local ring (R, m_R), where all arrows commutes with action of U_p then if the action of U_p is locally finite on M_2, it is so on M_3, and when M_1 is killed by some power of m_R, then U_p also acts locally finitely on M_1 too. As a result, we see that U_p acts locally finitely on all of the terms of the commutative diagram above. In particular, passing to the limit, one deduce that U_p acts locally finitely on $\varinjlim_m \mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \mathscr{P}_{m,1}^{\kappa^+}\otimes\Omega_D^{\kappa^-}) = \mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p), \mathscr{P}_{\infty,1}^{\kappa^+}\otimes\Omega_D^{\kappa^-})$, and similarly, U_p is locally finite on $\varinjlim_r H^0(\mathcal{S}_n^{\geq 1}(p), \mathscr{P}_{\infty,n}^{\kappa^+}\otimes\Omega_D^{\kappa^-(rN)}/(\widetilde{Ha}^r)$. Finally we use the proposition (11.1.2) above to conclude.

□

Let u_p be the ordinary projector associated with $U_p \in End(\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}\otimes\Omega_D^{\kappa^-})$.

Definition 13.0.1. For each weight $\kappa := (k_1, k_2, k_3)$ such that $k_1 + k_3 > 2$, we call

$$\mathcal{V}^\bullet(\kappa^-) := u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}\otimes\Omega_D^{\kappa^-})$$

the Hida complex of p-adic Picard modular form of base weight $\kappa^- := (k_2, k_2, k_3)$. We denote also by $\mathcal{V}^\bullet(\kappa)$ the specialization $u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}^{\kappa^+}\otimes\Omega_D^{\kappa^-})$ of $\mathcal{V}^\bullet(\kappa^-)$.

Remark 13.0.1. *The complex $\mathcal{V}^\bullet(\kappa^-)$ will interpolates only the part Ω^{κ^+} of the classical sheaf $\Omega^\kappa = \Omega^{\kappa^+}\otimes\Omega^{\kappa^-}$.*

Theorem 13.0.2. *The Λ-complex $\mathcal{V}^\bullet(\kappa^-)$ is perfect and concentrated in two degrees 0 and 1.*

Proof. By lemma(13.0.1), it suffices to prove that $\mathcal{V}^\bullet(\kappa) := u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}^{\kappa^+}\otimes\Omega_D^{\kappa^-}$ is perfect in $D(\mathbb{Z}_p)$. Then, the previous theorem 13.0.1, point (5) says that it suffices to show the quasi-isomorphism:

$$u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \Omega_D^{\kappa}) \cong u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}(p), \mathscr{P}^{\kappa^+}\otimes\Omega_D^{\kappa^-}) \tag{13.5}$$

By proposition 2.2.2 [Pil18], we can prove this by checking that the morphism on special fibers :

$$u_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p),\Omega_D^{\kappa}) \cong u_p\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p),\mathscr{P}_{\infty,1}^{\kappa^+}\otimes\Omega_D^{\kappa^-}) \tag{13.6}$$

is a quasi-isomorphism. To do so, we reconsider the diagram (13.4) but specialize in the case of $n=1$. Following the lemma (13.0.4) below, we deduce that U_p acts trivially on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}, K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-})$. As a consequence the action of U_p on $H^0(\mathcal{S}_1^{=2}, K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-})$ and $\varinjlim_r H^0(\mathcal{S}_1^{\geq 1}, K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-(rN)}/(\widetilde{Ha}^r))$ is topologically nilpotent, and so when we apply the projector u_p on the diagram (13.4), we obtain two isomorphisms:

$$u_pH^0(\mathcal{S}_1^{=2}(p),\mathscr{P}_{m,1}^{\kappa^+}\otimes\Omega_D^{\kappa^-}) \cong u_pH^0(\mathcal{S}_1^{=2}(p),\Omega_D^{\kappa}) \tag{13.7}$$

(13.8)

$$u_p\varinjlim_r H^0(\mathcal{S}_1^{=2}(p),\mathscr{P}_{m,1}^{\kappa^+}\otimes\Omega_D^{\kappa^-(rN)}/(\widetilde{Ha})^r) \cong u_p\varinjlim_r H^0(\mathcal{S}_1^{=2}(p),\Omega_D^{\kappa(rN)}/(\widetilde{Ha})^r)$$

The two $\mathbb{Z}_p$-modules on the right of the above isomorphisms are both finite by point (5), theorem 13.0.1. We need to check now that it is safe to pass to the limit (with respect to m) the left hand side of the above isomorphisms. In other words, we should verify that the iterated U_p^m acting on $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p),\mathscr{P}_{1,1}^{\kappa^+}\otimes\Omega^{\kappa^-})$ is compatible with restrictions $\mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p),\mathscr{P}_{1,1}^{\kappa^+}\otimes\Omega^{\kappa^-}) \xrightarrow{res} \mathrm{R}\Gamma(\mathcal{S}_1^{\geq 1}(p),\mathscr{P}_{m,1}^{\kappa^+}\otimes\Omega^{\kappa^-})$. Indeed the correspondence underlying the iterated U_p^m can be defined as follows. Let C_m be a scheme over $\mathcal{S}_1^{\geq 1}(p)$ parametrizing the triple (A,H,L_m) with $L_m\in A[\pi^m]$ of multiplicative rank 1, étale rank 0 and order p^{2m} (i.e L_m is locally isomorphic to $\mu_{p^m}\times\mathbb{Z}/p^m\mathbb{Z}$) such that $L_m\cap H=\{0\}$. We have two maps $p_1: C_m\to\mathcal{S}_1^{\geq 1}(p)$ that simply forgets the group L_m and $p_2: C_m\to\mathcal{S}_1^{\geq 1}(p)$ that sends (A,H,L_m) to $\left(\frac{A}{L_m+L_m^{\perp}},\frac{H+L_m+L_m^{\perp}}{L_m+L_m^{\perp}}\right)$. Obviously, the correspondence (C_m,p_1,p_2) gives rise to the operator U_p^m, and more over the map p_2 actually factors through $\mathcal{S}_1^{\geq 1}(p^m)\to\mathcal{S}_1^{\geq 1}(p)$, by sending (A,H,L_m) to $\left(\frac{A}{L_m+L_m^{\perp}},H_m\right)\in\mathcal{S}_1^{\geq 1}(p^m)$, where $H_m = Im\left(A[p^m]\to\frac{A}{L_m+L_m^{\perp}}\right)$. The upshot is that we get the compatibility we want.

Finally we can pass to the limit with respect to m the right hand sides of (13.7)(13.8), and "patch" two isomorphisms into the desired quasi-isomorphism (13.6).

□

Lemma 13.0.4. *The action of U_p on $\mathrm{R}\Gamma(\mathcal{S}_n^{\geq 1}, K\Omega^{\kappa^+}\otimes\Omega_D^{\kappa^-})$ is divisible by p.*

Proof. It is enough to show this property when $k^+=(1,0,0)$. Recall that by definition, we have an exact sequence : $K\Omega^{\kappa^+}\to\Omega^{\kappa^+}\to\mathscr{P}_{n,m}^{k_1-k_2}$, and an isomorphism $\mathscr{P}_{n,m}^{k_1-k_2}\cong\omega_{H_m}^{k_1-k_2}$ for all κ.

Now we prove that U_p acts naturally on the surjection: $\omega_{\mathcal{A}[\pi^m]}\to\omega_{H_m}$ and its action is divisible by p on the kernel K of this surjection. Indeed, we have the universal isogeny $\lambda(\pi):\mathcal{A}[\pi^m]\to\mathcal{A}'[\pi^m]$ with kernel $L\subset\mathcal{A}[\pi^m]$ of multiplicative rank 1, étale rank 0 which intersects trivially with H_m. On each affine open $Spec(R)\subset\mathcal{S}_n^{\geq 1}(p^m)$ such that we have a trivialization $R^2\cong\omega_{\mathcal{A}[\pi^m]}\cong\omega_{\mathcal{A}'[\pi^m]}$, and $R\cong\omega_{H_m}\cong\omega_{H_m'}$, where H_m' is the image of H_m in $\mathcal{A}'[\pi^m]$ via the isogeny $\lambda(\pi)$ above.

We can identify ω_{H_m} and $\omega_{H'_m}$ along with choosing a basis of R^2 in a way that the map $\omega_{\mathcal{A}'[\pi^m]} = p_2^*\omega_{\mathcal{A}[\pi^m]} \to p_1^*\omega_{\mathcal{A}[\pi^m]} = \omega_{\mathcal{A}[\pi^m]}$ appearing in the definition of U_p can be written locally over $Spec(R)$ as: $R^2 \to R^2$ with matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and the projections $\omega_{\mathcal{A}[\pi^m]} \to \omega_{H_m}$ as well as $\omega_{\mathcal{A}'[\pi^m]} \to \omega_{H'_m}$ are given by $R^2 \ni (x,y) \mapsto y \in R$. The result is that the map $p_2^*K \to p_1^*K$ over $Spec(R)$ factors through pR. As a consequence the action of U_p on K is also divisible by p. $\square$

Part 3. Higher Coleman theory

14. Introduction

In the first part, we have defined a complex of Λ-modules that interpolates the ordinary complex computing the ordinary cohomology of Picard modular forms, all integrally. In this part we formulate a version of Coleman theory for the complex of overconvergent Picard modular forms. First, let us recall the classical Coleman theory for p-adic overconvergent modular forms. Let $X_0(p)$ be the adic modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$ over $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$. In particular, there is a universal subgroup H of order p inside the universal elliptic curve E. Let $X_0(p)^m$ be the open subspace of $X_0(p)$ where H has multiplicative reduction, and let $X_0(p)^{m,\dagger}$ be the inductive limit of all strict neighborhoods of $X_0(p)^m$. We call the section of $H^0(X_0(p)^{m,\dagger}, \omega^k)$ the overconvergent modular forms of weight k. There exists a Hecke operator U_p that acts on $H^0(X_0(p)^{m,\dagger}, \omega^k)$ and we have the famous classicity theorem due to Coleman[Col97]:

Theorem 14.0.1. *There is an isomorphism:*

$$H^0(X_0(p), \omega^k)^{<k-1} \cong H^0(X_0(p)^{m,\dagger}, \omega^k)^{<k-1}$$

where the superscript " $< k-1$" means the subspace of slope $< k-1$ with respect to U_p.

What is amazing is that such a classicity theorem should exist for higher cohomology too. In the case of Siegel modular form of genus 2, see [Pil18]. In this section, we will develop a similar theory for higher cohomology of Picard modular forms.

Somewhat more concretely we will study the strict neighborhood system $\{\mathcal{S}_{par}(p^n, \epsilon)\}_\epsilon$ of $\mathcal{S}_{par}^{\geq 1}(p^n)$ and establish some precise classicity theorem for overconvergent classes in higher cohomology, in other words, when the restriction

$$\mathrm{R}\Gamma(\mathcal{S}_{par}(p^n), \Omega^\kappa) \to \mathrm{R}\Gamma(\mathcal{S}_{par}(p^n, \epsilon), \Omega^\kappa)$$

becomes a quasi-isomorphism.

As an application of this theory, we prove that rationally we can obtain a good control theorem. In other words, we have the following theorem:

Theorem 14.0.2. *For all algebraic weight $\kappa = (k_1, k_2, k_3)$ (with $k_1 \geq k_2$, $k_1 + k_3 > 1$), we have the quasi-isomorphism:*

$$u_p \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \Omega_D^\kappa) \otimes^L \mathbb{Q}_p \cong u_p \mathrm{R}\Gamma(\mathfrak{S}_{par}^{\geq 1}(p), \Omega_D^\kappa) \otimes^L \mathbb{Q}_p.$$

This is very technical, but roughly the idea is to remark that it is enough to establish this quasi-isomorphism over $\mathbb{C}_p$ and once over $\mathbb{C}_p$, we will be able prove that both complexes are quasi-isomorphic to a third complex whose terms compute the cohomology of certain overconvergent automorphic sheaves.

15. Analytic geometry

Let $\mathbb{C}_p$ be the p-adic completion of algebraic closure of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integers. We choose a valuation v_p on $\mathbb{C}_p$ such that $v_p(p) = 1$. Let $\mathbf{Adm}_{\mathcal{O}}$ be the category of admissible $\mathcal{O}$-algebra (i.e flat $\mathcal{O}$-algebra that is isomorphic to a quotient of covergent series $\mathcal{O} < x_1, ..., x_n > /I$ for some n by some finitely generated ideal I) and $\mathbf{Nadm}_{\mathcal{O}}$ the category of normal admissible $\mathcal{O}$-algebras. For us, a formal scheme is a locally topologically ringed space with a structure sheaf that is locally isomorphic to formal scheme $Spf(A)$ where A is some adic $\mathcal{O}$-algebra with an ideal of definition containing p. To each affine formal scheme $Spf(A)$ we can associate an adic space $Spa(A, A)$ over $\mathrm{Spa}(\mathcal{O}, \mathcal{O})$, this construction can be easily extended to general formal schemes. Let us denote by $\mathfrak{Form}_{\mathcal{O}}$ the category of formal schemes over $Spf(\mathcal{O})$. For any $\mathfrak{X} \in \mathfrak{Form}_{\mathcal{O}}$ we can define its generic fiber by taking an affine covering $\mathfrak{U}$ of $\mathfrak{X}$ and then for each affine formal subschemes $Spf(R) \in \mathfrak{U}$ we define its generic fiber to be $Spa(R, R) \times_{Spa(\mathcal{O},\mathcal{O})} \mathrm{Spa}(\mathbb{C}_p, \mathcal{O})$, this construction glues and gives us the generic fiber, denoted by $\mathfrak{X}_\eta$ of $\mathfrak{X}$.

Now let $\mathfrak{S}_{par} \in \mathfrak{Form}_{\mathcal{O}}$ be the p-adic formal completion of $\mathcal{S}_{par}$ and let $\mathscr{S}_{par}$ be its generic fiber, let $\mathscr{S}_{par}^{\geq 1}$ be the subspace of points whose associated universal subgroup is multiplicative. We will study an appropriate system $\{\mathscr{S}_{par}(\epsilon)\}_\epsilon$ of open subspaces of $\mathscr{S}_{par}$ parametrized by $\epsilon \in \mathbb{R}_{\geq 0}$, each $\mathscr{S}_{par}(\epsilon)$ strictly contains $\mathscr{S}_{par}^{\geq 1}$. There exists also a systems of sheaves of overconvergent forms $\mathscr{F}(\kappa, \epsilon)$ of some analytic weight κ on each $\mathscr{S}_{par}(\epsilon)$, and we are interested in the complex $\mathrm{R}\Gamma(\mathscr{S}_{par}(\epsilon), \mathscr{F}(\kappa, \epsilon))$. The goal of the first section is to prove that this complex is concentrated in degree 0 and 1 for any ϵ, κ.

15.1. **Hodge-Tate map.** We need to study the locus $\mathscr{S}_{par}^{\geq 1}$ and a compatible system of its strict neighborhoods. It is possible to use Hasse invariant and its generalization to define such system of neighborhoods like in the case of modular curves, but it is more convenient and conceptual to use Hodge-Tate maps as we are going to explain now.

Recall that we have defined the Picard scheme $\mathcal{M}$ and its toroidal compactification $\mathcal{S}$ over $\mathbb{Z}_p$ but from now on we (by abuse of notation) denote $\mathcal{S}$ the base change $\mathcal{S} \times_{Spec\mathbb{Z}_p} Spec(\mathcal{O})$. We now define some objects needed for the construction of overconvergent sheaves.

(1) Let $\mathfrak{S}$ be the p-adic completion of $\mathcal{S}$ and $\mathfrak{M}$ be the interior of $\mathfrak{S}$ away from the boundary (i.e p-adic completion of $\mathcal{M}$). They are all formal schemes topologically of finite type over $Spf(\mathcal{O})$.
(2) $\mathscr{S}$ be the adic generic fiber of $\mathfrak{S}$, it is an adic space over $Spa(\mathbb{C}_p, \mathcal{O})$. Similarly we denote by $\mathscr{M}$ the generic fiber of $\mathfrak{M}$. By abuse of notation, we still denote by $\mathcal{A}$ the universal semi-abelian scheme over $\mathscr{S}$ or the universal abelian scheme over $\mathscr{M}$.

(3) $\mathscr{M}(p^n)$ be the analytic space over $\mathscr{M}$ parametrizing isomorphisms

$$(\mathbb{Z}/p^n\mathbb{Z})^6 \to \mathcal{A}[p^n]$$

commuting with action of $\mathcal{O}_K$. As p splits this is also just the space over $\mathscr{M}$ parametrizing isomorphisms $(\mathbb{Z}/p^n)^3 \cong \mathcal{A}[\pi^n]$ or $(\mathbb{Z}/p^n)^3 \cong \mathcal{A}[\bar{\pi}^n]$. The projection $l_n : \mathscr{M}(p^n) \to \mathscr{M}$ is finite flat. Furthermore, as $\mathscr{M}(p^n)$ is smooth over $spa(\mathbb{C}_p, \mathcal{O})$, we can define its toroidal compactification denoted by $\mathscr{S}(p^n)$ and minimal compactification, denoted by $\mathscr{S}^*(p^n)$ (In our case, the toroidal compactification is unique).

(4) By proposition (1.2) in [PS16b] we can normalize $\mathfrak{S}$ inside $\mathscr{S}(p^n)$ to obtain a formal scheme topologically of finite type over $Spf(\mathcal{O})$ and we denote it by $\mathfrak{S}(p^n)$, similarly we have the normalization $\mathfrak{S}^*(p^n)$ of $\mathfrak{S}^*$ inside $\mathscr{S}^*(p^n)$. We can define also $\mathfrak{S}^*(p^n)$ as the Stein factorization of $\mathfrak{S}(p^n) \to \mathfrak{S}^*$, that is to say the unique formal scheme affine over $\mathfrak{S}^*$ that fits into the chain of maps $\mathfrak{S}(p^n) \xrightarrow{pr} \mathfrak{S}^*(p^n) \to \mathfrak{S}^*$ such that $pr_*\mathcal{O}_{\mathfrak{S}(p^n)} = \mathcal{O}_{\mathfrak{S}^*(p^n)}$. Let $\mathfrak{M}(p^n)$ be the interior of $\mathfrak{S}(p^n)$.

Recall that for any $\mathcal{O}$ scheme S and finite locally free group scheme G over S we have the Hodge-Tate map $HT : G \to \omega_{G^D}$ where G and ω_{G^D} are identified with their canonically associated $fppf$-sheaves over S. Concretely it is defined by sending $x \in G \cong Hom(G^D, \mathbb{G}_m)$ to $x^* \frac{dT}{T}$.

Now, over $\mathfrak{M}(p^n)$ we have the universal abelian scheme $\mathcal{A}$ and the Hodge Tate map $HT : \mathcal{A}[p^n] \to \omega_{\mathcal{A}}/p^n\omega_{\mathcal{A}}$. As p splits , this map is the product of two maps:

$$HT_\pi : \mathcal{A}[\pi^n] \to \omega_{\bar{\pi}}/p^n\omega_{\bar{\pi}} \quad \text{and} \quad HT_{\bar{\pi}} : \mathcal{A}[\bar{\pi}] \to \omega_\pi/p^n\omega_\pi$$

Theorem 15.1.1. *The map $HT_{\bar{\pi}}$ and HT_π extend to the toroidal compactification $\mathfrak{S}(p^n)$.*

Proof. Identical to the proof of proposition 1.5 [PS16b]

□

Furthermore, following prop 1.2 (loc. cit.) the map HT_π and the exterior product

$$\wedge^2 HT_{\bar{\pi}} : \wedge^2(\mathbb{Z}/p^n\mathbb{Z})^3 \to \wedge^2\omega_\pi =: det(\omega_\pi)$$

defined over $\mathfrak{S}(p^n)$ descends to the minimal compactification $\mathfrak{S}^*(p^n)$.

We have a rather useful characterization of the images of these Hodge-Tate maps.

Theorem 15.1.2 ([Far10],thm.7)**.** *For $p \geq 3$, and $S = Spec(R)$ be an affine scheme where R is a normal admissible $\mathcal{O}$-algebra. Let G be a Barsotti-Tate group over S, truncated of level n, and of height h. Suppose furthermore over S we have the trivialization $G(S) \cong (\mathbb{Z}/p^n\mathbb{Z})^h$, then the cokernel of Hodge-Tate map:*

$$HT : G(S) \times_{\mathbb{Z}} S \to \omega_{G^D}$$

is killed by the ideal generated by all elements of valuation greater than or equal to $\frac{1}{p-1}$.

Remark 15.1.1. *This theorem also holds for $p = 2$ but with $\frac{1}{p-1}$ replaced by 2. As a result it is possible to extend all the theorems below to the case $p = 2$ up to changing the numerical inputs. For this reason, from now on we will suppose that $p \geq 3$.*

In particular, the maps HT_π and $HT_{\bar\pi}$, as well as $\wedge^2 HT_{\bar\pi}$ are not surjective and not suitable for our purposes. However, if we let $\omega_\pi^{mod} \subset \omega_\pi$ be the subsheaf of ω_π generated by the inverse images of the Hodge-Tate map $HT_{\bar\pi}$ via the projection $\omega_\pi \to \omega_\pi/(p^n)$, then ω_π^{mod} does not depend on $n \geq 1$. Indeed, according to theorem 15.1.2, the cokernel of $HT_{\bar\pi}$ is killed by $p^{\frac{1}{p-1}}$, so a fortiori by p, the sheaves ω_π^{mod} (resp. $\omega_{\bar\pi}$) then sits in the sequence $p\omega_\pi \subset \omega_\pi^{mod} \subset \omega_\pi$. In other words, ω_π^{mod} is the unique subsheaf of ω_π containing $p\omega_\pi$ and whose reduction mod p is the image of $HT_{\bar\pi}$. The same goes for the sheaves $\omega_{\bar\pi}$ and $det(\omega_\pi)$ and we have the following corollary

Corollary 15.1.1. *There exist unique subsheaves ω_π^{mod} and $\omega_{\bar\pi}^{mod}$ (resp. $det(\omega_\pi)^{mod}$) be of ω_π and $\omega_{\bar\pi}$ (resp. $det(\omega_\pi)$) generated by the inverse images of the $HT_{\bar\pi}$ and HT_π (resp. $\wedge^2 HT_{\bar\pi}$) in ω_π and $\omega_{\bar\pi}$ (resp. $det(\omega_\pi)$). These modified sheaves do not depend on $n \geq 1$.*

Notice however that ω_π^{mod} and $\omega_{\bar\pi}^{mod}$ are only locally free over the generic fiber of $\mathfrak{S}(p^n)$ but not over the special fiber. We can remedy this by blowing up $\mathfrak{S}(p^n)$ successively along the sheaves of ideals I_1, I_2 followed by normalizations inside its generic fiber, where I_1(resp. I_2) is generated the lifts of coefficients(resp. determinants of minors) of the matrices of HT_π and $HT_{\bar\pi}$. The result is a formal scheme $\mathfrak{S}^{mod}(p^n)$ with a projective map $Bl : \mathfrak{S}^{mod}(p^n) \to \mathfrak{S}(p^n)$ such that $Bl^*\omega_\pi^{mod}$ and $Bl^*\omega_{\bar\pi}^{mod}$ are locally free over $\mathfrak{S}^{mod}$. To simplify notations, we still denote $Bl^*\omega_\pi^{mod}$ or $Bl^*\omega_{\bar\pi}^{mod}$ by ω_π^{mod} and $\omega_{\bar\pi}^{mod}$. A crucial thing to keep in mind is that since the divisors cut out by I_1, I_2 lie inside the special fiber, we are modifying $\mathfrak{S}(p^n)$ without messing up its generic fiber $\mathscr{S}(p^n)$.

By a similar process, we can obtain a formal scheme $\mathfrak{S}(p^n)^{*-mod}$ with a projective map $Bl' : \mathfrak{S}(p^n)^*$ such that the pull back $Bl'^*(det(\omega_\pi))$ is locally free.

Now let $\{e_i\}_{1\leq i\leq 3}$ and $\{e_i\}_{4\leq i\leq 6}$ be a basis of $(\mathbb{Z}/p^n\mathbb{Z})^3 \cong \mathcal{A}[\pi^n]$ and $(\mathbb{Z}/p^n\mathbb{Z})^3 \cong \mathcal{A}[\bar\pi^n]$ respectively such that e_1 is sent to e_6 via the Cartier duality and the orthogonal of e_1 is generated by e_4, e_5. Following the theorem(15.1.2) and the "mod" construction above, we have the surjections:

$$HT_{\bar\pi} : (\mathbb{Z}/p^n\mathbb{Z})^3 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{S}^{mod}(p^n)} \to \omega_\pi^{mod}/p^{n-\frac{1}{p-1}}\omega_\pi^{mod}$$

$$HT_\pi : (\mathbb{Z}/p^n\mathbb{Z})^3 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{S}^{mod}(p^n)} \to \omega_{\bar\pi}^{mod}/p^{n-\frac{1}{p-1}}\omega_{\bar\pi}^{mod}$$

15.2. **Overconvergent neighborhood and flag variety.** For each $\epsilon \in [0, n-\frac{1}{p-1}[$ we define $\mathfrak{S}(p^n,\epsilon) \subset \mathfrak{S}^{mod}(p^n)$ to be the formal scheme where $v_p(HT(e_1)) \geq \epsilon$. In particular if $n - \frac{1}{p-1} \geq \epsilon' \geq \epsilon \geq 0$ we have $\mathfrak{S}(p^n,\epsilon') \subset \mathfrak{S}(p^n,\epsilon)$.

Over $\mathfrak{S}(p^n,\epsilon)$, we still have a surjections:

$$HT_\pi : (\mathbb{Z}/p^n\mathbb{Z})^6 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{S}(p^n,\epsilon)} \to \omega_{\bar\pi}^{mod}/p^\epsilon\omega_{\bar\pi}^{mod}$$

$$HT_{\bar\pi} : (\mathbb{Z}/p^n\mathbb{Z})^6 \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{S}(p^n,\epsilon)} \to \omega_\pi^{mod}/p^\epsilon\omega_\pi^{mod}$$

Suppose that over some affine open $\mathrm{Spf} R \subset \mathfrak{S}^{mod}(p^n,\epsilon)$ with $R \in \mathbf{Adm}_{\mathcal{O}}$ the sheaf ω^{mod} becomes trivial, then we can write the matrices of $HT_{\bar\pi}$ and HT_π respectively as:

$$Mat(HT_{\bar{\pi}}) = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \qquad\qquad Mat(HT_{\pi}) = \begin{pmatrix} 0 & g & h \end{pmatrix}$$

Let Fil^{can} be the sub module of ω_{π}^{mod} generated by $HT_{\bar{\pi}}(e_4), HT_{\bar{\pi}}(e_5)$. Since the map HT_{π} is surjective, the coefficients g and h are co-linear (i.e there is some $\lambda \in R^{\times}$ such that $g = \lambda h$. By duality, e_2, e_3 are sent to e_4, e_5 respectively and thus $HT_{\bar{\pi}}(e_4)$ and $HT_{\bar{\pi}}(e_5)$ are also co-linear. Furthermore $< e_1 >^{\perp} = < e_4, e_5 >$ and the dual of e_1 is e_6, as a result ω_{π}/Fil^{can} is generated by the image of $HT_{\bar{\pi}}(e_6)$. Finally since $HT_{\bar{\pi}}$ is surjective, $HT_{\bar{\pi}}(e_6)$ must be non zero and independent of $HT_{\bar{\pi}}(e_4), HT_{\bar{\pi}}(e_5)$, we conclude that Fil^{can} is locally direct factor of rank 1 of ω_{π}^{mod} with quotient $\omega_{\pi}^{mod}/Fil^{can}$ generated by $HT_{\bar{\pi}}(e_6)$.

Let us denote by $\mathfrak{S}(p^n, \epsilon, e_4)$ (resp. $\mathfrak{S}(p^n, \epsilon, e_5)$) the open subscheme of $\mathfrak{S}(p^n, \epsilon)$ where Fil^{can} is generated by $HT_{\bar{\pi}}(e_4)$ (resp. $HT_{\bar{\pi}}(e_5)$).

We denote by $\mathfrak{FL}_n$ the flag variety over $\mathfrak{S}^{mod}(p^n)$ parametrizing direct factors of rank 1 of ω_{π}^{mod}, this means over $\mathfrak{FL}_n$ there is a universal filtration $Fil_n \subset \omega_{\pi}^{mod}$. Let also $\mathfrak{FL}_n(\epsilon)$ be the pullback of $\mathfrak{FL}_n$ to $\mathfrak{S}^{mod}(p^n, \epsilon)$. For each $w \in [0, \epsilon]$, we consider the subspace $\mathfrak{FL}_n(\epsilon, w)$ of $\mathfrak{FL}_n(\epsilon)$ where $Fil_n = Fil_{\epsilon}^{can} (\text{mod } p^w)$.

More precisely, on each open subscheme $Spf(R) \subset \mathfrak{S}^{mod}(p^n)$ with $R \in \mathbf{Adm}_{\mathcal{O}}$, such that $\omega_{\pi}^{mod} \cong R\widetilde{e}_{45} \oplus R\widetilde{e}_6$, where $\widetilde{e_6}$ lifts $HT(e_6) \in \omega_{\pi}^{mod}/p^{\epsilon}\omega_{\pi}^{mod}$ and $\widetilde{e}_{45}$ lifts a basis for $Fil_{\epsilon}^{can} \subset \omega_{\pi}^{mod}/p^{\epsilon}\omega_{\pi}^{mod}$. Now it is immediate that

$$\mathfrak{FL}_n(\epsilon, w)_{Spf(R)} \cong Spf(R\langle T \rangle) \tag{15.1}$$

with the universal locally free direct factor of rank 1 of ω_{π}^{mod} given by $(\widetilde{e}_{45} + p^w T\widetilde{e_6})R \subset R\widetilde{e}_{45} \oplus R\widetilde{e_6}$.

Over $\mathfrak{FL}_n(\epsilon, w)$, for each $0 \leq w' \leq w$ we can define yet another scheme $\mathfrak{FL}_n(\epsilon, w, w')$ parametrizing all trivializations $\phi : \mathcal{O}_{\mathfrak{FL}_n(\epsilon,w,w')} \cong \omega_{\pi}^{mod}/Fil_n$ such that $\phi(1) = HT_{\bar{\pi}}(e_6)$ (mod $p^{w'}$). Locally over each open affine $Spf(R)$ as above, we have:

$$\mathfrak{FL}_n(\epsilon, w, w')|_{SpfR} \cong Spf(R\langle T, T' \rangle) \tag{15.2}$$

with the projection $p_1 : \mathfrak{FL}_n(\epsilon, w, w') \to \mathfrak{FL}_n(\epsilon, w)$ over $Spf(R)$ given by $R\langle T \rangle \to R\langle T, T' \rangle$ sending T to T. The universal trivialization over Spf(R) is given by $\phi : R < T, T' > \to R^2/(\widetilde{e}_{45} + p^w T\widetilde{e_6})$ such that $\phi(1) = (1 + p^{w'}T')\widetilde{e}_6$. Thus, we obtain :

$$\mathfrak{FL}_n(\epsilon, w, w') \xrightarrow{p_1} \mathfrak{FL}_n(\epsilon, w) \xrightarrow{p_2} \mathfrak{S}(p^n, \epsilon)$$

We define $\mathfrak{P}(\epsilon, w) := p_{1*}\mathcal{O}_{\mathfrak{FL}_n(\epsilon,w,w')}$ and $\mathfrak{P}(\epsilon) := p_{2*}\mathfrak{P}(\epsilon, \omega)$. We denote by $\mathscr{P}(p^n, \epsilon)$ and $\mathscr{P}(\epsilon)$ the generic fiber of $\mathfrak{S}(p^n, \epsilon)$ and $\mathfrak{P}(\epsilon)$.

15.2.1. *Analytic weight.* Let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]$ be the Iwasawa algebra of dimension 1 and $\mathfrak{W} := Spf(\Lambda)$ be the associated formal weight space with its rigid analytic fiber $\mathcal{W} := \mathfrak{W} \times_{Spa(\mathbb{Z}_p,\mathbb{Z}_p)} Spa(\mathbb{Q}_p, \mathbb{Z}_p)$. To each affine adic space $T := Spa(R, R^+)$ the points $\mathcal{W}(T)$ of the weight space $\mathcal{W}$ is identified with $Hom_{cont}(\mathbb{Z}_p^{\times}, R^{\times})$. Thus, in particular $\mathcal{W}(Spa(\mathbb{Q}_p, \mathbb{Z}_p)) = Hom_{cont}(\mathbb{Z}_p^{\times}, \mathbb{Q}_p^{\times})$ contains all algebraic weights. For each $w \in [0, n - \frac{1}{p-1}]$ we have a formal subgroup $\mathfrak{W}_w$ defined by $\mathfrak{W}_w(R) =$

$\mathbb{Z}_p^\times(1+p^wR)$ for any R $\in$ $\mathbf{Adm}_\mathcal{O}$. Let also $\mathfrak{W}_w^o$ be the connected component of $\mathfrak{W}_w$ (i.e the sub formal torus that sends any $R \in \mathbf{Adm}_\mathcal{O}$ to $(1+p^w)R$.

For each Huber pair (A, A^+) and a character $\kappa_A : \Lambda^\times \to A^\times$ we say that A is w'-analytic if κ_A extends to a pairing $\mathfrak{W}_{w'} \times Spf(A) \to \mathbb{G}_m$.

Recall that we have a series of affine maps:

$$\mathfrak{FL}_n(\epsilon, w, w') \xrightarrow{p_1} \mathfrak{FL}_n(\epsilon, w) \xrightarrow{p_2} \mathfrak{S}(p^n, \epsilon)$$

There is a natural action of the formal group $\mathfrak{W}_{w'}^\circ$ on $p_{1*}\mathfrak{FL}_n(\epsilon, w, w')$. More concretely, let $\mathfrak{U} := \{\mathfrak{U}_i\}_{i\in I}$ be a cover by affine formal subschemes of $\mathfrak{S}(p^n,\epsilon)$ such that over each $\mathfrak{U}_i = Spf(R_i)$ we have a description of $\mathfrak{FL}_n(\epsilon, w, w')_{\mathfrak{U}_i}$ and $\mathfrak{FL}_n(w)_{\mathfrak{U}_i}$ as in (15.1, 15.2). An element $g \in \mathfrak{W}_{w'}^o(R_i)$ acts on $\mathfrak{FL}_n(\epsilon, w, w')_{\mathfrak{U}_i}$ by sending the trivialization $\phi : R_i \cong R_i^2/(\widetilde{e} + p^w\widetilde{e_6})$ to $g(\phi) : R_i \cong R_i^2/(\widetilde{e} + p^w\widetilde{e_6})$ such that $g(\phi)(1) = g(1 + p^{w'}T')$.

Given a w'-analytic character k_A we can consider the sheaf:

$$\mathfrak{D}^{k_A}(w) := (p_{1*}\mathfrak{FL}_n(\epsilon, w, w')\hat{\otimes}A)^{\mathfrak{W}_{w'}}$$

as well as the sheaf $\mathfrak{C}^{k_A}(w) := p_{2*}\mathfrak{D}^{k_A}(w)$ over $\mathfrak{S}(p^n,\epsilon)$. We denote by $\mathscr{D}^{\kappa_A}(w)$ and $\mathscr{C}^{\kappa_A}(w)$ their generic fibers.

Remark 15.2.1. *Notice that the sheaf $\mathfrak{D}^{k_A}(w)$ does not depend on w'.*

15.2.2. *Formal Picard scheme with parahoric level structure.* Let $\mathfrak{G}$ be the p-adic completion of $GL_3 \times GL_3$ over $\mathcal{O}$ and $\mathfrak{Par}$ be the parahoric subgroup of $\mathfrak{G}$ consisting of upper triangular matrix with blocks of size 1×1, 2×2 , 2×2 and 1×1 on the diagonal in that order. Over $\mathscr{S}(p^n)$ we have a trivialization $\rho : (\mathbb{Z}/p^n\mathbb{Z})^6 \cong \mathcal{A}[\pi^n] \times \mathcal{A}[\bar{\pi}^n]$. As a consequence, $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\mathscr{S}(p^n)$ by acting on ρ. This action is trivial on $\mathscr{S}$, thus $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ also acts on $\mathfrak{S}(p^n)$ since it is the normalization of $\mathfrak{S}$ inside $\mathscr{S}(p^n)$. Furthermore $\mathfrak{S}^{mod}(p^n)$ is obtained by blowing up along the divisors cut by ideals I_1, I_2 followed by normalization, both of these divisors are stable under the action of $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$. As a result $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\mathfrak{S}^{mod}(p^n)$, and we denote by $\mathfrak{S}^{mod}_{par}(p^n)$ as the quotient of $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ by $\mathfrak{S}^{mod}(p^n)$.

Now as $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\mathfrak{S}^{mod}(p^n)$ by acting on the trivialization ρ, it clearly fixes the vector e_1, hence there is an induced action of $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathfrak{S}^{mod}(p^n,\epsilon)$. We denote by $\mathfrak{S}^{mod}_{par}(p^n,\epsilon)$ its quotient and then by $\mathscr{S}_{par}(p^n,\epsilon)$ the generic fiber (recall that the "modification" does not change the generic fiber, so we do not have superscript "mod" when taking the generic fiber). Denote by $q : \mathscr{S}(p^n,\epsilon) \to \mathscr{S}_{par}(p^n,\epsilon)$ the natural projection.

15.3. **Analytic setting.** Over $\mathscr{S}(p^n)$ as well as $\mathscr{S}(p^n,\epsilon)$ we still have Hodge Tate map :

$$HT : (\mathbb{Z}/p^n\mathbb{Z})^6 \otimes_\mathbb{Z} \mathcal{O}_{\mathscr{S}_{par}(p^n)} \to \omega_\mathcal{A}$$

We also care about the integral structure as well, i.e we have a subsheaf $\mathcal{O}^+_{\mathscr{S}_{par}(p^n)} \subset \mathcal{O}_{\mathscr{S}_{par}(p^n)}$ and a sheaf of integral differential forms $\omega^+_\mathcal{A}$ but the "integral" Hodge-Tate map :

$$HT : (\mathbb{Z}/p^n\mathbb{Z})^6 \otimes_\mathbb{Z} \mathcal{O}^+_{\mathscr{S}_{par}(p^n)} \to \omega^+_\mathcal{A}/(p^{n-\frac{1}{p-1}})$$

is not surjective, just like the case of formal scheme earlier. We thus define $\omega_{\pi}^{+,mod}$ and $\omega_{\bar{\pi}}^{+,mod}$ as the sub sheaves of ω_{π}^{+} and $\omega_{\bar{\pi}}^{+}$ generated by the images of Hodge-Tate maps above. Be careful that these modified sheaves are really defined on the étale site since we use the Hodge-Tate maps to define them.

Now we explain why we call $\mathscr{S}_{par}(p^n,\epsilon)$ an overconvergent neighborhood of $\mathscr{S}_{par}^{\geq 1}(p^n)$. Firstly, the space $\mathscr{S}_{par}(p^n)$ parametrizes the pair $(H_n,\mathcal{A})$ with $H_n\subset\mathcal{A}[\pi^n]$ a subgroup étale locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, and so we have a diagram:

$$\begin{array}{ccc} H_n & \longrightarrow & \mathcal{A}[\pi^n] \\ \downarrow & & \downarrow \\ \omega_{H_n^D} & \longrightarrow & \omega_{\mathcal{A}[\bar{\pi}]} \end{array}$$

we can fix an isomorphism $\mathcal{A}[\pi^n]\cong(\mathbb{Z}/p^n\mathbb{Z})^3$ and a choice of basis $\{e_i\}_{1\leq i\leq 3}$ for $(\mathbb{Z}/p^n\mathbb{Z})^3$ such that $\mathcal{A}[\pi^n]\supset H_n\cong(\mathbb{Z}/p^n\mathbb{Z})e_1$. Then if the group H_n is multiplicative then $HT(e_1)=0 \mod p^{n-\frac{1}{p-1}}$. This means that $\mathscr{S}_{par}^{\geq 1}(p^n)$ corresponds to a subspace of $\mathscr{S}_{par}(p^n,n-\frac{1}{p-1})$ and as ϵ varies the schemes $\{\mathscr{S}_{par}(p^n,\epsilon)\}_\epsilon$ are strict neighborhoods of $\mathscr{S}_{par}^{\geq 1}(p^n)$.

Now, over $\mathscr{S}_{par}(p^n,\epsilon)$ we have a trivialization $\mathcal{A}[\bar{\pi}^n]\cong(\mathbb{Z}/p^n\mathbb{Z})^3$ and we choose a basis $\{e_i\}_{4\leq i\leq 6}$ for $\mathbb{Z}/p^n\mathbb{Z}$ such that e_1 is sent to e_6 by Cartier duality (recall that $\mathcal{A}[\pi^n]^D=\mathcal{A}[\bar{\pi}^n]$. Moreover, the orthogonal $H_n^{\perp}$ is isomorphic to $\mathcal{A}[\bar{\pi}^n]/H_n^D$, this means that in the trivialization above, $H_n^{\perp}$ corresponds to sub module of $(\mathbb{Z}/p^n\mathbb{Z})^3\cong\mathcal{A}[\bar{\pi}^n]$ generated by e_4,e_5. Similarly to earlier section, we can define (recall that for any $\mathcal{O}^+$-module M, and $\epsilon\in\mathbb{Q}_{>0}$ we set $M_\epsilon:=M/p^\epsilon M$) :

(1) a filtration $Fil_\epsilon^{can}\subset\omega_{\pi}^{+,mod}$ over $\mathscr{S}_{par}(p^n,\epsilon)$ defined as the images of e_4,e_5 via the integral Hodge-Tate map $HT_{\bar{\pi}}:(\mathbb{Z}/p^n\mathbb{Z})^3\otimes\mathcal{O}^+_{\mathscr{S}_{par}(p^n,\epsilon)}\to\omega_{\pi,\epsilon}^{+,mod}$.
(2) Gr_ϵ^{can} as the quotient $(\omega_{\pi,\epsilon}^{+,mod})/Fil_\epsilon^{can}$
(3) $\mathscr{F}\mathscr{L}$ is the flag variety over $\mathscr{S}_{par}(p^n)$ parametrizing locally direct factor (in zariski topology) of rank one $Fil^1\subset\omega_\pi$. Let also $\mathscr{F}\mathscr{L}_{par}(\epsilon):=\mathscr{F}\mathscr{L}\times\mathscr{S}_{par}(p^n,\epsilon)$.
(4) $\mathscr{F}\mathscr{L}_{par}(\epsilon,w)$ is an open subspace of $\mathscr{F}\mathscr{L}(\epsilon)$ such that étale locally we have $Fil^1\bigcap\omega_{\pi}^{+,mod}=Fil_\epsilon^{can}(\mathrm{mod}\ p^w)$.
(5) $\mathscr{F}\mathscr{L}_{par}(\epsilon,w,w')$ is a torsor over $\mathscr{F}\mathscr{L}_{par}(\epsilon,w)$ that parametrizes the trivializations $\phi:\mathcal{O}^+_{\mathscr{F}\mathscr{L}_{par}(\epsilon,w)}\cong\omega_{\pi}^{+,mod}/(Fil^1\bigcap\omega_{\pi}^{+,mod})$ such that $\phi=HT(1)(\mathrm{mod}\ p^{w'})$.

We have the following very important fact:

Lemma 15.3.1. (1) *The generic fiber $\mathscr{F}\mathscr{L}_n(\epsilon,w)$ of $\mathfrak{F}\mathfrak{L}_n(\epsilon,w)$ over $\mathscr{S}(p^n,\epsilon)$ is the pull back of $\mathscr{F}\mathscr{L}_{par}(\epsilon,w)$ via the projection: $\mathscr{S}(p^n,\epsilon)\to\mathscr{S}_{par}(p^n,\epsilon)$.*
(2) *Similarly,the generic fiber $\mathscr{F}\mathscr{L}_n(\epsilon,w,w')$ of $\mathfrak{F}\mathfrak{L}_{nn}(\epsilon,w,w')$ over $\mathscr{S}(p^n,\epsilon)$ is the pull back of $\mathscr{F}\mathscr{L}_{par}(\epsilon,w,w')$ via the projection: $\mathscr{S}(p^n,\epsilon)\to\mathscr{S}_{par}(p^n,\epsilon)$.*

Proof. The proof is an application of the following descent lemma (see lemma 4.2.4([Con06]))

Lemma: Let $f:X\to X'$ be a faithfully flat map of rigid analytic spaces. Let $W\subset X$ be an admissible open subspace. Suppose that:

(1) f admits a fpqc quasi-section (i.e there is a faithfully flat quasi-compact map $U \to Y$ and a Y-map $U \to X$).
(2) if $x \in X$ and $f(x) \in im(W)$ then $x \in W$.

then the image of W is an admissible open subspace of X'.

Indeed we can consider the flag variety $\mathscr{FL}$ over $\mathscr{S}_{par}(n, \epsilon, w)$ parametrizing locally free sheaf of rank 1 of ω_π. The map $\mathscr{X} := \mathscr{FL} \times_{\mathscr{S}_{par}(n,\epsilon,w,w')} \mathscr{S}(p^n, \epsilon, w) \to \mathscr{FL}$ is faithfully flat map (since the quotient $\mathscr{S}(p^n, \epsilon, w) \to \mathscr{S}_{par}(n, \epsilon, w, w')$ is). Now the space $\mathscr{FL}_n(\epsilon, w)$ can be identified as a subspace of $\mathscr{X}$. The fpqc quasi-section is given by the subgroup $H_n \subset A[\pi]$ using the modular interpretation of $\mathscr{S}_{par}(n, \epsilon, w)$. The second condition of the descent lemma above can also be readily verified on points. Thus $\mathscr{FL}_n(\epsilon, w)$ arises as a pull back of an open admissible subspace of $\mathscr{FL}$. This space is exactly $\mathscr{FL}_{par}(\epsilon, w)$.

The point (2) is proved similarly.

□

To sum it up, we have a sequence of maps:

$$\mathscr{FL}_{par,n}(\epsilon, w, w') \xrightarrow{p_1} \mathscr{FL}_{par,n}(\epsilon, w) \xrightarrow{p_2} \mathscr{S}_{par}(p^n, \epsilon)$$

For each w-analytic weight k_A, we define a sheaf :

$$\mathscr{P}^{k_A}_{n,\epsilon}(w) := (p_{1*}\mathcal{O}_{\mathscr{FL}_{par,n}(\epsilon,w,w')} \hat{\otimes} A)^{\mathcal{W}_{w',n}}$$

Recall that for each triple $\kappa := (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 \geq k_2$ we denote by $\kappa^+ := k_1 - k_2$ and $\kappa^- := (k_2, k_3)$. By ω^{κ^-} we mean the sheaf $det(\omega_\pi)^{k_2} \otimes \omega_{\bar{\pi}}^{k_2}$ and by $\mathscr{P}^{\kappa}_{n,\epsilon}$ the sheaf $\mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \omega^{\kappa^-}$ (an algebraic weight is always analytic). Lastly, if k_A is an A-valued w-analytic weight, by abuse of notation, let κ_A be the triple (k_A, k_2, k_3) with $\kappa_A^+ := k_A - k_2$ and $\kappa_A^- := (k_2, k_3)$ and we simply refer to κ_A as A-valued w-analytic weight. This time $\mathscr{P}^{\kappa_A}_{n,\epsilon}(w)$ denotes $\mathscr{P}^{k_A - k_2}_{n,\epsilon}(w) \otimes \omega^{\kappa_A^-}$.

Definition 15.3.1. For each A-valued w-analytic weight κ_A we call

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) := \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{k_A}_{n,\epsilon}(w) \otimes \omega^{\kappa^-})$$

the (n, ϵ)-overconvergent w-analytic complex of weight κ_A. We can also defined the cuspidal version:

$$\mathscr{V}^{cusp}_{n,\epsilon}(\kappa_A, w) := \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{k_A}_{n,\epsilon}(w) \otimes \omega^{\kappa^-}(-D))$$

Another thing worth pointing out is that we can use a different equivalent description of $\mathscr{V}_{n,\epsilon}(\kappa_A, w)$. This description uses the fact that $\mathscr{FL}_n(\epsilon, w)$ is locally affine over $\mathscr{S}(p^n, \epsilon)$ so that we can find a cover $\mathscr{U} = \{\mathscr{U}_i\}_{i \in I}$ by affinoids $\mathscr{U}_i = Spa(R, R^+)$ such that $p_2^{-1}\mathscr{U}_i$ are also affinoids. We can always refine this covering so that each affinoids $\mathscr{U}_i$ is stable under the action of $\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z})$, then using Čech complex we see that :

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) = H^0(\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z}), \mathrm{R}\Gamma(\mathscr{FL}_n(\epsilon, w), \mathscr{D}^{k_A}(w) \otimes \omega^{\kappa-}) \tag{15.3}$$

and we also have $R^i p_{1*}$ vanishes for $i \geq 1$ (indeed, the map p_1 is also locally affine), this implies :

$$\mathrm{R}\Gamma(\mathscr{FL}_n(\epsilon, w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa-}) \cong \mathrm{R}\Gamma(\mathscr{S}(p^n, \epsilon, w), \mathscr{C}^{k_A}(w) \otimes \omega^{\kappa-}) \tag{15.4}$$

15.3.1. $\mathscr{V}^{cusp}_{n,\epsilon}(\kappa_A, w)$ *is concentrated in two degrees.* In this section, we prove that $\mathscr{V}^{cusp}_{n,\epsilon}(\kappa_A, w)$ is concentrated in two degrees when n is big enough.

Recall that we have the minimal compactification $\mathfrak{S}^*(p^n)$ and over it there are Hodge-Tate maps:

$$\wedge^2 HT_\pi : \wedge^2(\mathbb{Z}/p^n\mathbb{Z})^3 \otimes \mathcal{O}_{\mathfrak{S}^*(p^n)} \to det(\omega_\pi)/(p^{n-\frac{1}{p-1}})$$

$$\text{and} \quad HT_{\bar\pi} \otimes 1 : (\mathbb{Z}/p^n\mathbb{Z})^3 \otimes \mathcal{O}_{\mathfrak{S}^*(p^n)} \to \omega_{\bar\pi}/(p^{n-\frac{1}{p-1}})$$

Let us denote by e_{ij} the exterior product $e_i \wedge e_j$ for all $0 \le i, j \le 6$. We have also shown the existence of a modified sheaf $det(\omega_\pi)^{mod}$ over $\mathfrak{S}^*(p^n)$ and a formal scheme $\mathfrak{S}(p^n)^{*-mod}$ with a map $\mathfrak{S}(p^n)^{*-mod} \to \mathfrak{S}^*(p^n)$ such that the pull back of $det(\omega_\pi)^{mod}$ along this map is locally free.

Now, above $\mathfrak{S}(p^n)^{*-mod}$ we can define the scheme $\mathfrak{S}^*(p^n, \epsilon)$ such that $\wedge^2 HT_\pi(e_{45}) = 0 \bmod p^\epsilon$ and $\wedge^2 HT_\pi(e_{46})$ and $\wedge^2 HT_\pi(e_{56})$ are not simultaneously $0 \bmod p^\epsilon$.

Lemma 15.3.2. *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathfrak{S}^*(p^n, \epsilon) & \longrightarrow & \mathfrak{S}(p^n)^{mod} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{S}^*(p^n, \epsilon) & \longrightarrow & \mathfrak{S}(p^n)^{*-mod} \end{array}$$

Proof. Since over $\mathfrak{S}(p^n, \epsilon)$ we have $HT_{\bar\pi}(e_1) = 0 \bmod p^\epsilon$, in particular this would imply that e_6 is non zero mod p^ϵ. Moreover as e_2 and e_3 are dual to e_4 and e_5 respectively, $\wedge^2 HT_\pi(e_{45}) = 0 \bmod p^\epsilon$ implies that $HT_{\bar\pi}(e_2)$ and $HT_{\bar\pi}(e_3) = 0$ simultaneously mod p^ϵ and this forces $HT_{\bar\pi}(e_1)$ to be non zero. $\square$

Let $\mathcal{L} := det(\omega_\pi)^{mod} \otimes \omega^{mod}_{\bar\pi}$. For $1 \le i \le 3$ and $4 \le j, k \le 6$ we let s_{ijk} the section of $\mathcal{L}$ locally defined by the Hodge-Tate map $\wedge^2 HT_{\bar\pi} \otimes HT_\pi(e_j \wedge e_k \otimes e_i)$. We would like to use these section to cook up some affine covering of $\mathfrak{S}^*(p^n, \epsilon)$ but unfortunately $\mathcal{L}$ is not ample. However we have the following theorem

Theorem 15.3.1. *There exists $N \in \mathbb{N}$ such that for any $n \ge N$, there exists a formal model $\mathfrak{S}(p^n)^{*-HT}$ (actually a Stein factorization) with a projective map $\mathfrak{S}(p^n)^{*-mod} \xrightarrow{ht} \mathfrak{S}(p^n)^{*-HT}$ inducing an isomorphism on generic fiber with following properties :*

1. *The line bundle $\mathcal{L}$ descends to an ample line bundle, denoted by $\mathcal{L}^{HT}$ over $\mathfrak{S}(p^n)^{*-HT}$*
2. *For each ϵ there is an integer $n(\epsilon)$ such that for all $n \ge n(\epsilon)$ there exist sections $s^{HT}_{ijk} \in H^0(\mathfrak{S}(p^n)^{*-HT}, \mathcal{L})$ such that their pull backs to $\mathfrak{S}(p^n)^{*-mod}$ agree with s_{ijk} (mod p^ϵ).*

Now we can define the scheme $\mathfrak{S}(p^n, \epsilon)^{HT} \to \mathfrak{S}(p^n)^{HT}$ by the condition $s^{HT}_{1jk} = 0$ mod (p^ϵ) for all j, k. We also denote by $\mathfrak{S}(p^n, \epsilon, e_k)^{HT}$ two open formal subschemes of $\mathfrak{S}(p^n, \epsilon)^{HT}$ where $s^{HT}_{1k_6}$ is non zero mod (p^ϵ) for $k \in \{4, 5\}$. Since $\mathcal{L}^{HT}$ is ample, the formal schemes $\mathfrak{S}(p^n, \epsilon, e_k)^{HT}$ are affine. Furthermore we can easily verify that the following diagram is commutative for $k \in \{4,5\}$.

$$\begin{array}{ccc} \mathfrak{S}(p^n,\epsilon,e_k)^{*-mod} & \longrightarrow & \mathfrak{S}(p^n,\epsilon,e_k)^{HT} \\ \downarrow & & \downarrow \\ \mathfrak{S}(p^n,\epsilon)^{*-mod} & \xrightarrow{ht} & \mathfrak{S}(p^n,\epsilon)^{HT} \end{array} \tag{15.5}$$

This diagram implies that the top horizontal map is projective. This suggests us to use $\mathfrak{S}(p^n,\epsilon)^{HT}$ to compute coherent cohomology $H^i(\mathfrak{S}(p^n,\epsilon)^{*-mod},\mathfrak{F})$ via $H^i(\mathfrak{S}(p^n,\epsilon)^{HT}, ht_*\mathfrak{F})$. Fortunately when the coherent sheaf $\mathfrak{F}$ is nice enough we can indeed compare the two.

Definition 15.3.2. Let $\mathfrak{X}$ be an admissible formal scheme, a formal Banach sheaf $\mathfrak{F}$ over $\mathfrak{X}$ is a family $\{\mathfrak{F}_i\}_i$ of quasi-coherent sheaves such that :

1. Each $\mathfrak{F}_i$ is a sheaf of $\mathcal{O}_{\mathfrak{X}}/p^i$-module.
2. Each $\mathfrak{F}_i$ is flat over $\mathcal{O}/p^i$
3. For all $0 \le m \le n$ we have the isomorphism $\mathfrak{F}_n \otimes_{\mathcal{O}} \mathcal{O}/p^m \cong \mathfrak{F}_m$.

We say furthermore that

1. $\mathfrak{F}$ is flat if each $\mathfrak{F}_i$ is a sheaf of flat $\mathcal{O}_{\mathfrak{X}}/p^i$-module.
2. $\mathfrak{F} = \varprojlim_i \mathfrak{F}_i$ is small if the sheaf $\mathfrak{F}_1$ can be written as $\varinjlim_j \mathfrak{F}_{1,j}$ of coherent sheaves $\mathfrak{F}_{1,j}$ and there exists a coherent sheaf $\mathcal{G}$ over $\mathfrak{S}$ such that successive quotients $\mathfrak{F}_{1,j}/\mathfrak{F}_{1,j+1}$ are direct sums of $\mathcal{G}$.

Lemma 15.3.3. *The coherent sheaf $f_*\mathfrak{C}^{\kappa_A}(w)\otimes\omega_D^{\kappa^-}$ is a small formal Banach sheaf over $\mathfrak{S}(p^n,\epsilon)^{*-mod}$.*

Proof. See proposition A.1.3.1 [AIP12]. □

We invoke the following theorem (see thm A.1.2.2 [AIP12])

Theorem 15.3.2. *Let $\mathfrak{X}$ be an admissible formal scheme. If $\mathfrak{X} \to \mathfrak{Y}$ is a projective map with $\mathfrak{Y}$ an admissible affine formal scheme which induces an isomorphism on their generic fiber. Let $\mathfrak{F}$ be a small Banach sheaf over $\mathfrak{X}$ and $\mathfrak{U}$ be an affine cover of $\mathfrak{X}$. Then the Čhech complex $\check{C}ech(\mathfrak{U},\mathfrak{F})[1/p]$ is acyclic in positive degree.*

Corollary 15.3.1. *Suppose that we have a coherent sheaf $\mathcal{G}$ over $\mathscr{S}^{mod}(p^n,\epsilon)$ and that $\mathcal{G}$ descends to a small formal sheaf $\mathfrak{G}$ over $\mathfrak{S}^{mod}(p^n,\epsilon)$. Then the complex $\mathrm{R}\Gamma(\mathscr{S}^{mod}(p^n,\epsilon),\mathcal{G})$ is concentrated in degrees 0 and 1.*

Proof. The space $\mathfrak{S}^{mod}(p^n,\epsilon)$ is covered by two open subspaces $\mathfrak{U}_4 := \mathfrak{S}^{mod}(p^n,\epsilon,e_4)$ and $\mathfrak{U}_5 := \mathfrak{S}^{mod}(p^n,\epsilon,e_5)$. Let $\mathfrak{U} = \{\mathfrak{U}_i\}_{(i=4,5)}$. We have a spectral sequence :

$$E_2^{p,q} = \check{H}^p(\mathfrak{U},\mathcal{H}^q) \Rightarrow H^{p+q}(\mathfrak{S}^{mod}(p^n,\epsilon),\mathfrak{G})$$

Where $\mathcal{H}^q$ is the sheaf associated to the presheaf over $\mathfrak{S}^{mod}(p^n,\epsilon)$ sending each open subset U to $H^q(U,\mathfrak{G})$.

Now, we have projective maps $\mathfrak{U}_i \to \mathfrak{S}(p^n,\epsilon,\tilde{s}_i)^{*-HT}$ for $i = 4,5$. By theorem(15.3.2) above, we conclude that the sheaf $\mathcal{H}^q$ restricted to $\mathfrak{U}_i$ doesn't have non trivial torsion-free global sections for $q \geq 1$. As a result we see that when p is inverted we have:

$$\check{H}^i(\mathfrak{U}, \mathfrak{G})[\frac{1}{p}] \Rightarrow H^i(\mathfrak{S}^{mod}(p^n, \epsilon), \mathfrak{G})[\frac{1}{p}] = H^i(\mathscr{S}^{mod}(p^n, \epsilon), \mathcal{G})$$

As $\check{H}^i(\mathfrak{U}, \mathfrak{G})[\frac{1}{p}]$ is clearly non zero in degree 0 and 1, we're done. □

We cite another vanishing theorem before concluding:

Theorem 15.3.3. *Let $\pi : \mathfrak{S}(p^n, \epsilon) \to \mathfrak{S}(p^n, \epsilon)^{*-mod}$, then $R\pi_* \mathcal{O}_{\mathfrak{S}(p^n,\epsilon)}(-D) = \pi_* \mathcal{O}_{\mathfrak{S}(p^n,\epsilon)(-D)}$.*

Proof. See proposition 12.9.2.1[Pil18] □

Theorem 15.3.4. *For any $\epsilon \in]0, n - \frac{1}{p-1}[$, then there exists $n(\epsilon)$ such that for all $n \geq n(\epsilon)$, the overconvergent complex $\mathscr{V}_{n,\epsilon}(\kappa_A, w)$ is concentrated in degree 0 and 1.*

Proof. We have seen that (15.3):

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) = H^0(\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z}), \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon, w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa-}))$$

So we can show that the complex $\mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon, w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa-})$ is concentrated in degree 0, 1 instead. But recall that the projection $\mathscr{F}\mathscr{L}_n(\epsilon, w) \to \mathscr{S}(p^n, \epsilon)$ is locally affine, so we have a quasi-isomorphism:

$$\mathrm{R}\Gamma(\mathscr{S}(p^n, \epsilon), \mathscr{C}^{\kappa_A}(w) \cong \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon, w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa-})$$

Finally, we apply the corollary (15.3.1) for the sheaf $\mathscr{C}^{\kappa_A}(w)$ which is the generic fiber of the small sheaf $\mathfrak{C}^{\kappa_A}(w)$. □

16. Comparison theorem

16.1. **Comparison map.** For each A-valued w-analytic triple $\kappa_A = (k_A^+, k_A^-)$ we have defined the (n, ϵ)-overconvergent w-analytic complex $\mathscr{V}_{n,\epsilon}(\kappa, w)$. When $\kappa := (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 \geq k_2$ we also have the (n, ϵ)-overconvergent complex of classical sheaf $\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \Omega_D^\kappa)$. In fact, there is a natural map:

$$\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa) \to \mathcal{V}_{n,\epsilon}(\kappa, w) := \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}_{n,\epsilon}^{\kappa^+} \otimes \omega^{\kappa^-}) \tag{16.1}$$

Indeed, it is enough to exhibit a map $\Omega^{\kappa^+,0,0} \to \mathscr{P}_{n,\epsilon}^{\kappa^+}$. Let $\mathscr{F}\mathscr{L}$ be the flag variety over $\mathscr{S}_{par}(p^n, \epsilon)$ parametrizing the filtration $Fil^1 \subset \omega_\pi$, and $p : \mathscr{F}\mathscr{L}^+ \to \mathscr{S}_{par}(p^n, \epsilon)$ parametrizing the trivialization of $\mathrm{Gr}^1 := \omega_\pi / Fil^1$. Since ω_π is of rank 2 and Fil^1 is of rank 1, the flag $\mathscr{F}\mathscr{L}^+$ is a $\mathbb{G}_m$-torsor over $\mathscr{S}_{par}(p^n, \epsilon)$ and the symmetric power $\Omega^{\kappa^+,0,0} = Sym^{\kappa^+}\omega_\pi$ is nothing but $p_* \mathcal{O}_{\mathscr{F}\mathscr{L}^+}[-\kappa^+]$ where $p_* \mathcal{O}_{\mathscr{F}\mathscr{L}^+}[-\kappa^+]$ is the part on which $\mathbb{G}_m$ acts by the character $t \to t^{-\kappa^+}$. Now the open immersion $\mathscr{F}\mathscr{L}_{par}(\epsilon, w, w') \hookrightarrow \mathscr{F}\mathscr{L}^+$ is equivariant with respect to the action of $\mathcal{W}_{w'}$ on the left and $\mathbb{G}_m$ on the right (under the natural map $\mathcal{W}_{w'} \to \mathbb{G}_m$). As a result, by taking $-\kappa^+$ invariant part, we obtain a map $\Omega^{\kappa^+,0,0} \to \mathscr{P}_{n,\epsilon}^{\kappa^+}$ defined on each affine $U = Spa(A, A^+) \subset \mathscr{S}_{par}(p^n, \epsilon)$ by

$$\Omega^{\kappa^+,0,0}|_U \cong p_* \mathcal{O}_{\mathscr{F}\mathscr{L}^+}[-\kappa^+] \to (p_* \mathscr{F}\mathscr{L}_{par}(\epsilon, w, w') \hat{\otimes}_{\kappa^+} A^+)^{\mathcal{W}_{w'}} = \mathscr{P}_{n,\epsilon}^{\kappa^+}|_U$$

Moreover, the map (16.1) respects the actions of a Hecke operator $U_p^\dagger$ constructed in the next section. Above all, this operator is compact and spectral theory allows us to make sense of cohomology classes of prescribed slopes with respect to $U_p^\dagger$. We will actually prove that (16.1) become a quasi-isomorphism when restricted to the finite slope part.

16.2. **Operator $U_p^\dagger$.** In this subsection, we define the Hecke operator $U_p^\dagger$ acting on overconvergent complexes.

Let C_n° be the correspondence defined over $\mathscr{M}_{par}(p^n)$ which parametrizes the triple (A, H_n, L) where (A, H_n) is a point of $\mathscr{M}_{par}(p^n)$ and L is a totally isotropic subgroup of $A[\pi]$ of order p^2 and $L \cap H_n = \{0\}$. We have two maps $p_1 : C_n^\circ \to \mathscr{M}(p^n)$ sending (A, H_n, L) to (A, H_n) and $p_2 : C_n^\circ \to \mathscr{M}_{par}(p^{n+1})$ sending (A, H_n, L) to $(\frac{A}{L+L^\perp}, \frac{p^{-1}H_n+L+L^\perp}{L+L^\perp})$. As before, this correspondence extends to C_n over the toroidal compactification $\mathscr{S}_{par}(p^n)$.

$$\begin{tikzcd} & C_n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathscr{S}_{par}(p^n) & & \mathscr{S}_{par}(p^{n+1}) \end{tikzcd}$$

Lemma 16.2.1. *The pull back of C_n to $\mathscr{S}_{par}(p^n, \epsilon)$ gives us the correspondence:*

$$\begin{tikzcd} & C_{n,\epsilon} := C_n \times_{\mathscr{S}_{par}(p^n), p_1} \mathscr{S}_{par}(p^n, \epsilon) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathscr{S}_{par}(p^n, \epsilon) & & \mathscr{S}_{par}(p^{n+1}, \epsilon + 1) \end{tikzcd}$$

Proof. Indeed, we have to check that the map $C_{n,\epsilon} \xrightarrow{p_2} \mathscr{S}_{par}(p^{n+1}, \epsilon)$ factors through $\mathscr{S}_{par}(p^{n+1}, \epsilon + 1)$ and it is enough to show this on rank 1 points.

Over $C_{n,\epsilon}$ we have the universal isogeny $\lambda : \mathcal{A} \to \mathcal{A}'$ with kernel $Ker(\lambda) = L \times L^\perp$ where $L \subset \mathcal{A}[\pi]$ is totally isotropic of order p^2 and $L^\perp \subset \mathcal{A}[\bar{\pi}]$ is the orthogonal of L. In other words we have two commutative diagrams:

$$\begin{tikzcd} \mathcal{A}[\bar{\pi}^n] \arrow[r, "\lambda_{\bar{\pi}}"] \arrow[d, "HT"] & \mathcal{A}'[\bar{\pi}^n] \arrow[d, "HT'"] \\ \omega^{+,mod}_{\mathcal{A},\pi} \arrow[r, "\lambda^*_\pi"] & \omega^{+,mod}_{\mathcal{A}',\pi} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}[\pi^n] \arrow[r, "\lambda_\pi"] \arrow[d, "HT"] & \mathcal{A}'[\pi^n] \arrow[d, "HT'"] \\ \omega^{+,mod}_{\mathcal{A},\bar{\pi}} \arrow[r, "\lambda^*_{\bar{\pi}}"] & \omega^{+,mod}_{\mathcal{A}',\bar{\pi}} \end{tikzcd} \tag{16.2}$$

Let $x : Spa(K, \mathcal{O}_K) \to C_\epsilon$ be a geometric point of rank 1. Then using the trivialization

$$\mathcal{A}_x[\pi^\infty] \times \mathcal{A}_x[\bar{\pi}^\infty] \cong \mathbb{Z}_p^3 \times \mathbb{Z}_p^3 \cong \mathcal{A}'_x[\pi^\infty] \times \mathcal{A}'_x[\bar{\pi}^\infty]$$

and $\omega^{+,mod}_{\mathcal{A},\pi}(x) \cong \mathcal{O}_K^2 \cong \omega^{+,mod}_{\mathcal{A}',\pi}(x)$ as well as $\omega^{+,mod}_{\mathcal{A},\pi}(x) \cong \mathcal{O}_K \cong \omega^{mod}_{\mathcal{A}',\bar{\pi}}(x)$. Further more as the map λ^*_π is the differential of the isogeny $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ and $\lambda^*_{\bar{\pi}}$ is the differential of $\lambda_{\bar\pi} : \mathcal{A}[\bar\pi^\infty] \to \mathcal{A}'[\bar\pi^\infty]$ of kernel $L^\perp$, we obtain two diagrams:

$$
\begin{tikzcd}
\mathbb{Z}_p^3 \arrow[r, "{(p,1,1)}"] \arrow[d, "HT"] & \mathbb{Z}_p^3 \arrow[d, "HT'"] \\
\mathcal{O}_K^2 \arrow[r, "{(p,p)}"] & \mathcal{O}_K^2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{Z}_p^3 \arrow[r, "{(1,p,p)}"] \arrow[d, "HT"] & \mathbb{Z}_p^3 \arrow[d, "HT'"] \\
\mathcal{O}_K \arrow[r, "(p)"] & \mathcal{O}_K
\end{tikzcd}
\tag{16.3}
$$

Now, by definition $v_p(HT(e_1)) \geq \epsilon$ since $x \in C_{n,\epsilon}$, and the diagram 16.3 on the right says that λ_π sends e_1 to e_1 so we must have $v_p(HT'(e_1)) = v_p(pHT(e_1)) \geq \epsilon+1$. This means that the projection p_2 lands in $\mathscr{S}_{par}(p^{n+1}, \epsilon+1)$.

□

Proposition 16.2.1. *Over $C_{n,\epsilon}$ we have a natural map :*

$$p_2^* \mathscr{P}^{\kappa_A}_{n+1,\epsilon+1}(w+1) \to p_1^* \mathscr{P}^{\kappa_A}_{n,\epsilon}(w)$$

Proof. We just need to unwind the definition of overconvergent sheaves $\mathscr{P}^{\kappa_A}_{n,\epsilon}(w)$. Firstly, observe that over $C_{n,\epsilon}$ we have a map $\lambda_\pi^* : p_1^*\mathscr{F}\mathscr{L} \to p_2^*\mathscr{F}\mathscr{L}$ where $\mathscr{F}\mathscr{L}$ is the flag variety parametrizing locally free factors of rank 1 of $\omega_{\mathcal{A},\pi}$. Now, for any such factor, say $Fil^1 \subset \omega_{\mathcal{A},\pi}$, we get a locally free factor $\lambda_\pi^* Fil^1$ of rank 1 of $\omega_{\mathcal{A}',\pi}$ by pull-back via the isogeny $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$.

we claim that λ_π^* induces a map

$$p_1^* \mathscr{F}\mathscr{L}_{par}(n,\epsilon,w) \to p_2 * \mathscr{F}\mathscr{L}_{par}(n+1,\epsilon+1,w+1) \tag{16.4}$$

By definition $\mathscr{F}\mathscr{L}_{par}(n,\epsilon,w) \subset \mathscr{F}\mathscr{L}$ is the open subspace where

$$Fil^1 \bigcap \omega^{+,mod}_{\mathcal{A},\pi} = Fil^{can} (\text{mod} \quad p^w)$$

More concretely, for each point $x \in \mathscr{S}_{par}(p^n,\epsilon)$ of rank 1, we can find an affine neighborhood of x such that the filtration is given explicitly by $Fil^1_x = HT(e_4) + \alpha(x)p^w HT(e_6)$ or $Fil^1_x = HT(e_5) + \alpha(x)p^w HT(e_6)$, depending where the point $p_1(x)$ lies in $\mathscr{S}_{par}(p^n, \epsilon = \mathscr{S}_{par}(p^n,\epsilon,e_4) \cup \mathscr{S}_{par}(p^n,\epsilon,e_5)$. We reuse the left diagram (16.3) above to see that Fil^1_x is sent via λ_π^* to $pHT(e_4) + \alpha(x)p^{w+1}HT(e_6)$. This means that $\lambda_\pi^* Fil^1_x$ lands in $\mathscr{F}\mathscr{L}^{\kappa^+}(w+1)$.

Similarly, we can show that

$$\lambda_\pi^* : p_1^* \mathscr{F}\mathscr{L}_{par}(n,\epsilon,w,w') \to p_1^* \mathscr{F}\mathscr{L}_{par}(n+1,\epsilon+1,w+1,w'+1) \tag{16.5}$$

Indeed, given a filtration $Fil^1_{\mathcal{A}',x} \subset \omega_{\mathcal{A}'_x,\pi}$ and a trivialization

$$\phi : \mathcal{O}_K \cong \omega^{+,mod}_{\mathcal{A}'_x,\pi} / Fil^1_{\mathcal{A}',x} \bigcap \omega^{+,mod}_{\mathcal{A}'_x,\pi}$$

By pull-back via λ_π^* we obtain a filtration $Fil^1_{\mathcal{A}',x} := \lambda_\pi^* Fil^1_{\mathcal{A},x} \subset \omega_{\mathcal{A}_x,\pi}$ and obviously a trivialization $\phi : \mathcal{O}_K \cong \omega^{+,mod}_{\mathcal{A}'_x,\pi}/Fil^1_{\mathcal{A}',x} \bigcap \omega^{+,mod}_{\mathcal{A}'_x,\pi} \cong \omega^{+,mod}_{\mathcal{A}_x,\pi}/Fil^1_{\mathcal{A},x} \bigcap \omega^{+,mod}_{\mathcal{A}_x,\pi}$.

Finally, the map $p_2^* \mathscr{P}^{\kappa_A}_{n+1,\epsilon+1}(w+1) \to p_1^* \mathscr{P}^{\kappa_A}_{n,\epsilon}(w)$ is defined as follows, for any affine open subset $U \subset C_{n,\epsilon}$, and a section $s \in H^0(U, p_2^* \mathscr{P}^{\kappa_A}_{n+1,\epsilon+1}(w+1))$ we can associate a section s' of $H^0(U, p_1^* \mathscr{P}^{\kappa_A}_{n,\epsilon}(w))$ by putting $s'(x) = s(\lambda_\pi^*(x))$. □

As a consequence can define a series of map:

(16.6)

$$\mathscr{V}_{n+1,\epsilon+1}(\kappa_A, w+1) \to \mathrm{R}\Gamma(C_{n,\epsilon}, p_2^*\mathscr{P}^{\kappa_A^+}_{n+1,\epsilon+1}(w+1) \otimes \omega^{\kappa_A^-}) \xrightarrow{1/p^{k_2}\lambda_\pi^*} \mathrm{R}\Gamma(C_{n,\epsilon}, p_1^*\mathscr{P}^{\kappa_A^+}_{n,\epsilon}(w) \otimes \omega^{\kappa_A^-})$$
$$\to \mathrm{R}\Gamma(C_{n,\epsilon}, p_{1*}p_1^*\mathscr{P}^{\kappa_A^+}_{n,\epsilon}(w) \otimes \omega^{\kappa_A^-}) \xrightarrow{\frac{1}{p}Tr} \mathscr{V}_{n,\epsilon}(\kappa_A, w)$$

where the map $\frac{1}{p^{k_2}}\lambda_\pi^*$ is normalized map of λ_π^*, and $\frac{1}{p}Tr$ is the normalized trace map with the normalization calculated as in proposition (10.4). We also have a natural restriction map

$$\mathscr{V}_{n,\epsilon}(\kappa, w) \xrightarrow{res} \mathscr{V}_{n,\epsilon+1}(\kappa, w+1)$$

combined with the above map we obtain the operator for overconvergent complex :

$$U_p^\dagger : \mathscr{V}_{n+1,\epsilon+1}(\kappa, w+1) \to \mathscr{V}_{n+1,\epsilon+1}(\kappa, w+1)$$

16.2.1. *$U_p^\dagger$ and integral structure of overconvergent sheaves.* Recall that if $\mathscr{F}$ is a sheaf on an adic space $\mathscr{X}$, we are often interested in the subsheaf $\mathscr{F}^+$ and $\mathscr{F}^{++}$ of power bounded elements and topologically nilpotent elements respectively. In this section we analyze the effect of $U_p^\dagger$ on the subcomplexes

$$\mathscr{V}_{n,\epsilon}(\kappa^+, w)^+ := \mathrm{R}\Gamma(\mathscr{S}(p^n, \epsilon), \mathscr{P}^\kappa_{n,\epsilon}(w)^+)$$
$$\mathscr{V}_{n,\epsilon}(\kappa^+, w)^{++} := \mathrm{R}\Gamma(\mathscr{S}(p^n, \epsilon), \mathscr{P}^\kappa_{n,\epsilon}(w)^{++})$$

Where $\mathscr{P}^\kappa_{n,\epsilon}(w)^+ := \mathscr{P}^{\kappa^+}_{n,\epsilon}(w)^+ \otimes \omega^{\kappa^-,+}$ and $\mathscr{P}^\kappa_{n,\epsilon}(w)^{++} := \mathscr{P}^{\kappa^+}_{n,\epsilon}(w)^{++} \otimes \omega^{\kappa^-,++}$
For example, a non normalized version of the chain of maps (16.6) would give us an operator that stabilizes the integral structure, but since we have normalized it by a factor $1/p^{k_2+1}$, our operator $U_p^\dagger$ does not stabilize these subcomplexes anymore.

To see exactly what $U_p^\dagger$ does with the integral structure of $\mathscr{V}_{n,\epsilon}(\kappa_A, w)$, we need to unwind yet one more time the definition of $U_p^\dagger$. Above $\mathscr{S}_{par}(p^n, \epsilon)$ we have the universal isogeny $\lambda : \mathcal{A}[p^\infty] \to \mathcal{A}'[p^\infty]$, which can be written as a product of $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}'[\pi^\infty]$ with kernel L and $\lambda_{\bar\pi} : \mathcal{A}[\bar\pi^\infty] \to \mathcal{A}'[\bar\pi^\infty]$ with kernel $L^\perp$. These induce maps:

$$\lambda_\pi^* : \omega_{\mathcal{A}',\pi} \to \omega_{\mathcal{A},\pi} \quad \text{and} \quad \lambda_{\bar\pi}^* : \omega_{\mathcal{A}',\bar\pi} \to \omega_{\mathcal{A},\bar\pi}$$

Now the map λ_π^* induces a map $\lambda_\pi^* : \omega^{++}_{\mathcal{A}',\pi} \to \omega^{++}_{\mathcal{A},\pi}$ and then $det(\lambda_\pi^*) : det(\omega_{\mathcal{A}',\pi}) \to det(\omega^{++}_{\mathcal{A},\pi})$ which factors through $p.det(\omega^{++}_{\mathcal{A}',\pi})$ due to the fact that $deg(L) \geq 1$. By definition :

$$\mathscr{F}^{++} := Sym^{\kappa^+}(\omega^{++}_{\mathcal{A},\pi}) \otimes det^{k_2}(\omega^{++}_{\mathcal{A},\pi}) \otimes det^{k_3}(\omega^{++}_{\mathcal{A},\bar\pi})$$

We deduce that $(\lambda^*)^\kappa : p_2^*\mathscr{P}^{\kappa_A,++}_{n,\epsilon} \to p_1^*\mathscr{P}^{\kappa_A,++}_{n,\epsilon}$ factors through $p^{k_2}p_1^*\mathscr{P}^{\kappa_A,++}_{n,\epsilon}$. This implies that after normalization $\frac{1}{p^{k_2}}det(\lambda_*)^\kappa$ still preserves the integral structure. For the Trace map $Tr : p_{1*}\mathcal{O}_{C_{n,\epsilon}} \to \mathcal{O}_{\mathscr{S}_{par}(p^n,\epsilon)}$, we observe that as p_1 is finite, we have the restricted trace map $Tr : p_{1*}\mathcal{O}^{++}_{C_{n,\epsilon}} \to \mathcal{O}_{\mathscr{S}_{par}(p^n,\epsilon)^{++}}$. But at the end we have also normalized the trace map by dividing p which is the very reason why U_p does not stabilize $\mathscr{F}^{++}$. As a consequence, we conclude that $pU_p^\dagger$ preserves the integral structure.

Proposition 16.2.2. *The operator $U_p^\dagger \in End(\mathrm{R}\Gamma(\mathscr{S}_{par}(n,\epsilon), \mathscr{P}^{\kappa_A}_{n,\epsilon}(w)))$ is compact, and we let $u_p^\dagger$ be the associated projector.*

Proof. A composition of a compact map and continuous map is still compact, so it suffices to prove that the restriction map:

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) \to \mathscr{V}_{n+1,\epsilon+1}(\kappa, w+1)$$

is compact. Intuitively, using Čhech complex to represent these overconvergent complex, we can see that locally this restriction map looks like an honest restriction of sections of sheaves, hence compact. More rigorously , using the local description of $\mathscr{V}_{n,\epsilon}(\kappa_A, w)$ as in (15.3), there exists a cover $\mathcal{U} = \{U\}_i$ by affinoids of $\mathscr{S}(p^{n+1}, \epsilon+1)$ such that we have :

$$\mathscr{V}_{n+1,\epsilon+1}(\kappa_A, w{+}1) = H^0(\mathfrak{Par}(\mathbb{Z}/p^{n+1}\mathbb{Z}), \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_{n+1}(\epsilon{+}1, w{+}1), \mathscr{D}^{k_A}(w{+}1) {\otimes} \omega^{\kappa-})$$

and

$$\mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_{n+1}(\epsilon, w), \mathscr{D}^{k_A}(w{+}1) {\otimes} \omega^{\kappa-}) \cong \mathrm{R}\Gamma(\mathscr{S}(p^{n+1}, \epsilon{+}1, w{+}1), \mathscr{C}^{k_A}(w{+}1) {\otimes} \omega^{\kappa-})$$

we observe as well that the closure of $\mathscr{F}\mathscr{L}_{n+1}(\epsilon+1, w+1)$ is contained in

$$\mathscr{S}(p^{n+1}, \epsilon+1) \times_{\mathscr{S}(p^n,\epsilon)} \mathscr{F}\mathscr{L}_n(\epsilon, w)$$

Moreover, we can choose a cover by affinoids $\mathcal{U} = \{U_i\}_i$ s of $\mathscr{F}\mathscr{L}_{n+1}(\epsilon+1, w+1)$ stable by the action of $\mathfrak{Par}(\mathbb{Z}/p^{n+1}\mathbb{Z})$ and such that there is also a cover by affinoids $\mathcal{V} = \{V_i\}_i$ of $\mathscr{S}(p^{n+1}, \epsilon+1) {\times_{\mathscr{S}(p^n,\epsilon)}} \mathscr{F}\mathscr{L}_n(\epsilon, w)$ such that $\bar{U}_i \subset V_i$. Now, we can always refine these covers so that:

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) = H^0(\mathfrak{Par}(\mathbb{Z}/p^{n+1}\mathbb{Z}), Cech(\mathcal{V}, \mathscr{C}^{\kappa_A}(w) \otimes \omega^{\kappa-}))$$

$$\mathscr{V}_{n+1,\epsilon+1}(\kappa, w+1) = H^0(\mathfrak{Par}(\mathbb{Z}/p^{n+1}\mathbb{Z}), Cech(\mathcal{U}, \mathscr{C}^{\kappa_A}(w+1) \otimes \omega^{\kappa-}))$$

But then the restriction map

$$Cech(\mathcal{V}, \mathscr{C}^{\kappa_A}(w) \otimes \omega^{\kappa-}) \xrightarrow{res} Cech(\mathcal{V}, \mathscr{C}^{\kappa_A}(w+1) \otimes \omega^{\kappa-})$$

is compact because each $\mathscr{C}^{\kappa_A}(w) \otimes \omega^{\kappa-}(V_i) \xrightarrow{res} \mathscr{C}^{\kappa_A}(w+1) \otimes \omega^{\kappa-}(U_i)$ is. □

We denote by $u_p^\dagger$ the projector associated to $U_p^\dagger$, then we have a map between the ordinary parts:

$$u_p \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \Omega^{(k_1,k_2,k_3)}) \to u_p^\dagger \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{k_1-k_2}_{n,\epsilon}(w) \otimes \omega^{(k_2,k_3)})$$

It is reasonable now to ask if we can characterizes the image of this map. This is achieved via a formulation of slope decomposition theory for overconvergent complex in the spirit of [Pil18], which is a generalization of the original theory due to Coleman [Col97]. Before giving a precise statement, we review some vocabulary of spectral theory.

17. Spectral theory

Let L be a complete extension of $\mathbb{Q}_p$ with valuation v_p on L is normalized as usual (i.e $v_p(p) = 1$). Let (A, A^+) be a Tate algebra over L, we can view A as a Banach L-algebra in the sense of Coleman [Col97]. A Banach module over A is an A-module equipped with a norm compatible with that on A. The set of all Banach modules over A forms a category that we denote by $\mathbf{Ban}(A)$.

Definition 17.0.1. A Banach module $M \in \mathbf{Ban}(A)$ is orthonormizable if there exists a subset of $\{e_i\}_{i\in\mathbb{N}}$ of elements of M with the following properties:

(1) $\{e_i\}_{i\in\mathbb{N}}$ is a basis of M, in the sense that $|e_i| = 1$ and every element $m \in M$ can be written as a sum $m = \Sigma_{i\in\mathbb{N}}\lambda_i e_i$ with $\underset{i\to\infty}{lim}\,\lambda_i = 0$.
(2) If we write $m = \Sigma_{i\in\mathbb{N}}\lambda_i e_i$ then $|m| = max_{i\in\mathbb{N}}|\lambda_i|$.

We also say that a Banach module over A is projective if it is a direct factor of an orthornomizable Banach A-module.

Let $M \in \mathbf{Ban}(A)$ and $U \in End_A(M)$ be a compact operator. By some basic properties of compact operator, we can represent U in terms of an infinite matrix $\{\lambda_{i,j}\}$ such that $\underset{j\to\infty}{lim}\, sup_i|\lambda_{i,j}| = 0$. This means that we can actually define a characteristic series of U as $P(U)_M := det(1 - XU) \in A\langle X\rangle$ by formally adopting the usual formula for characteristic polynomial of operators of finite dimensional vector spaces. Another advantages of working with compact operator is that we have a very nice spectral theory with many properties analogous to the classical spectral theory, some of which we are going to recall.

Definition 17.0.2. Let K be a complete extension of $\mathbb{Q}_p$, seen as a Banach $\mathbb{Q}_p$-algebra and $M \in \mathbf{Ban}(K)$ and $U \in End_K(M)$. For each $h \in \mathbb{Q}$, an h-slope decomposition of U on M is a decomposition $M = M^{\leq h} \oplus M^{>h}$ with :

(1) $M^{\leq h}$ and $M^{>h}$ are sub vector spaces stable by U .
(2) $M^{\leq h}$ is of finite dimension over K and all eigenvalues of U on $M^{\leq h}$ are of valuation $\leq h$
(3) Let $P \in K[X]$ a polynomial such that the valuations of its roots are strictly less than h, then $P^{-1}(U)$ (where P^{-1} is the reciprocal of P) exists as an operator of $End(M^{>h})$ and it is invertible.

It is not hard to prove that if M admits h-decomposition then this decomposition is unique, and we say that M has slope decomposition if it has h-decomposition for any $h \in \mathbb{Q}$.

Now let $M^\bullet$ a perfect complex in the derived category $D_{\mathbf{Ban}(A)}$ of Banach A-modules, that is to say, $M^\bullet$ is quasi-isomorphic to a bounded complex of projective Banach A-modules. Let U be compact operator of $M^\bullet$ (i.e there is a representative of $\tilde{M}^\bullet$ such that U acts compactly on each $\tilde{M}_i$). For each point $x : Spa(K, \mathcal{O}_K) \to Spa(A, A^+)$ of rank 1 coming from the rigid space $Sp(A)$, we see that $U_x \in End_{D_{\mathbf{Ban}(A)}}(M^\bullet_x)$ is still compact, thus by the proposition A4.2 [Col97], each cohomology group $H^i(M^\bullet_x)$ has slope decomposition with respect to U_x. This, together with the uniqueness of the slope decomposition also implies that for each $h \in \mathbb{Q}$, we can define $H^i(M^\bullet_x)^{=h}$ as a subspace of $H^i(M^\bullet_x)^{\leq h}$ of eigenvectors with valuation h. We put

$$\chi_x(M^\bullet) := \oplus_i (-1)^i dim(H^i(M_x^\bullet)^{=h})$$

then $\chi_{(.)}(M^\bullet)$ is seen as a function $\chi_{(.)}M^\bullet : Spa(A, A^+) \to \mathbb{Z}$, this function is locally constant by [Col97].

Suppose furthermore that $M^\bullet \in D_{\mathbf{Ban}(A)}$ is perfect and admits an h-slope decomposition with respect to U (i.e there is a representative of $M^\bullet$ such that each terms admits an h-slope decomposition). Since the full subcategory of perfect complex of $D_{\mathbf{Ban}(A)}$ is idempotent complete we can form its perfect subcomplex $M^{\bullet,=h}$ of slopes h.

17.0.1. *Slope decomposition of $U_p^\dagger$.* Let $K = \mathbb{C}_p$ and $M^\bullet \in D_{\mathbf{Ban}(\mathbb{C}_p)}$ and $U \in End_{D_{\mathbf{Ban}(\mathbb{C}_p)}}(M^\bullet)$ be a compact operator. Now let $M^{+,\bullet}$ be a sub complex of $M^\bullet$ whose i^{th}-terms $M_i^+ := \{v \in M||v| \leq 1\}$ and $M^{++,\bullet}$ a subcomplex of M^i whose i^{th}-term is $M_i^{++} := \{v \in M||v| < 1\}$. We have the following easy but useful lemma:

Lemma 17.0.1. *Let $M^\bullet \in D_{\mathbf{Ban}(\mathbb{C}_p)}$ and $U \in End(M^\bullet)$ is a compact operator such that U stabilizes the subcomplex $M^{++,\bullet}$, then for all i the cohomology group $H^i(M^\bullet)^{=h}$ is zero if $h < 0$.*

Proof. Let $pr : M^\bullet \to (M^\bullet)^{=h}$ be the projection to the h-slope part of $M^\bullet$ inducing a continuous map $H^i(M^\bullet) \to H^i(M^\bullet)^{=h}$. Under the projection pr the image of the subcomplex $M^{++,\bullet}$ in $(M^\bullet)^{=h}$ is open and bounded $\mathcal{O}$-module by definition, and hence so is the image $H^i(M^{++,\bullet})^{=h}$ of $H^i(M^{++,\bullet})$ inside $H^i(M^\bullet)^{=h}$. Now, by our hypothesis, $M^{++,\bullet}$ is stable under the action of U and as a result, so is $H^i(M^{++,\bullet})^{=h}$. Since U stabilizes the open bounded submodule $H^i(M^{++,\bullet})^{=h}$ the slope h must be positive. $\square$

Go back to our situation, we have define the complex $\mathscr{V}_{n,\epsilon}(\kappa, w)$. The index triple (n, ϵ, w) is obviously filtered and we can define

$$H^i(\dagger, \kappa, w) := \varinjlim_{n,\epsilon,w} H^i(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}_{n,\epsilon}^{\kappa+}(w) \otimes \omega^{\kappa-})$$

Similarly for the cuspidal overconvergent forms:

$$H^i(\dagger, \kappa, w)^{cusp} := \varinjlim_{n,\epsilon,w} H^i(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}_{n,\epsilon}^{\kappa+}(w) \otimes \omega_D^{\kappa-})$$

These groups inherits an action of $U_p^\dagger$ and we have following important theorem

Theorem 17.0.1. *The slope of $U_p^\dagger$ on $H^i(\dagger, \kappa, w)$ is greater than or equal to -1 (resp. 0) for all $i > 0$(resp. $i = 0$)*

Proof. Intuitively, if there exists a complex $M^\bullet$ that computes the cohomology groupes $H^i(\dagger, \kappa, \epsilon)$ and the compact operator $p.U_p^\dagger$ preserves the integral subcomplex $M^{++,\bullet}$, hence by lemma (17.0.1), we can conclude. So, we will point out how to get the complex $M^\bullet$ with such property. Recall that we can find a cover $\mathscr{U} = \{\mathscr{U}_i\}_{i\in I}$ such that $\mathscr{FL}_{par,n}(\epsilon, w, w')$ becomes an affinoid when restricted to $\mathscr{U}_i$ (see 15.3), then let $M^\bullet$ be the complex that computes the Cech cohomology of $\mathscr{P}^{\kappa_A}_{n,\epsilon} \otimes \omega^{\kappa^-}$ with respect to $\mathscr{U}$. We have a map

$$H^i(M^\bullet) \to H^i(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{\kappa_A}_{n,\epsilon} \otimes \omega^{\kappa^-})$$

Inside $H^i(M^\bullet)$

$$\mathscr{V}_{n,\epsilon}(\kappa_A, w) = H^0(\mathfrak{Par}(\mathbb{Z}/p^n\mathbb{Z}), \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon,w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa -})$$

Now we need to show that the complex $M^\bullet := \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon,w), \mathfrak{D}^{k_A}(w) \otimes \omega^{\kappa -})$ has an integral structure stable by $pU_p^\dagger$. For this it is enough to take the subcomplex:

$$M^{++,\bullet} := \mathrm{R}\Gamma(\mathscr{F}\mathscr{L}_n(\epsilon,w), \mathfrak{D}^{k_A}(w)^{++} \otimes \omega^{++,\kappa^-})$$

For $i = 0$ one can embed $H^0(\dagger, \kappa, w)$ into the space of p-adic modular forms of weight κ, this embedding is compatible with the action of $U_p^\dagger$ on $H^0(\dagger,\kappa,w)$ and U_p on the latter. Moreover, U_p stablizes the integral structure of p-adic forms, hence the possible slope is positive on $H^0(\dagger,\kappa,w)$ (compared with 13.3.3.1 [Pil18] or see [PS16a]). □

Recall that we have a map $\Omega^\kappa \to \mathscr{P}^{\kappa^+,w}_{n,\epsilon} \otimes \Omega^{\kappa^-}$. For all $k \in \mathbb{Z}$, we put $\mathscr{P}^{k}_{n,\epsilon} := colim \mathscr{P}^{k,w}_{n,\epsilon}$. There is an exact sequence (relative BGG resolution)

$$0 \to \Omega^\kappa \to \mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-} \xrightarrow{\partial} \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-} \to 0 \tag{17.1}$$

where $r : \mathbb{Z}^3_{dom} \to \mathbb{Z}^3_{dom}$ is a certain reflection of weight (see 2.4[AIP12]) and $\mathbb{Z}_{dom} := \{k_1, k_2, k_3 | k_1 \geq k_2\}$. The precise construction of this relative BGG resolution might not be important here, but the fact to bear in mind is that the differential ∂ does not commute with $U_p^\dagger$ but instead it induces the following commutative diagram (see section 6[AIP12]):
(17.2)

$$\begin{array}{ccc}
\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-}) & \xrightarrow{U_p^\dagger} & \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-}) \\
\downarrow \partial & & \downarrow \partial \\
\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-}) & \xrightarrow{p^{1-\kappa^+}U_p^\dagger} & \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-})
\end{array}$$

However, recall that $pU_p^\dagger$ stabilizes the integral structure of $\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-})$, as a result, $p^{2-\kappa^+}U_p^\dagger$ stabilizes the integral structure of $\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa^-)})$. We deduce that (following lemma 17.0.1, and theorem 17.0.1) the complex $\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n,\epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa^-)})^{<\kappa^+-2}$ is zero. We deduce the following classicity criteria.

Remark 17.0.1. *To see why the factor $p^{1-\kappa^+}$ appears in the diagram 17.2, it is enough to see this locally on $\mathscr{S}_{par}(p^n,\epsilon)$ and locally $\mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-}$ is isomorphic to the space of analytic representations of GL_2 (see section 2 [AIP12] for definition). Using this interpretation, one can deduce the factor $p^{1-\kappa^+}$. For more details and general computation, see section 6.2, 2.5 and proposition 7.2.3 of [AIP12].*

Theorem 17.0.2. *For all weight $\kappa = (k_1, k_2, k_3)$ with $k_1 \geq k_2$ we have:*

1. $H^i(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa)^{<(\kappa^+-2)} \cong H^i(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon}(w) \otimes \omega^{\kappa^-})^{<(\kappa^+-2)}$
2. $H^0(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa)^{<(\kappa^+-1)} \cong H^0(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon}(w) \otimes \omega^{\kappa^-})^{<(\kappa^+-1)}$
3. $H^1(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa)^{<(\kappa^+-1)} \to H^1(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon}(w) \otimes \omega^{\kappa^-})^{<(\kappa^+-1)}$ *is an injection.*

Proof. The short exact sequence 17.1 induces the long exact sequence:

$$\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{\kappa^+}_{n,\epsilon} \otimes \Omega^{\kappa^-}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-}) \xrightarrow{+1}$$

from what we have discussed above, the cohomology group

$$H^i(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-})^{<\kappa^+-2} = 0$$

for all i and for $i = 0$, we have

$$H^0(\mathscr{S}_{par}(p^n, \epsilon), \mathscr{P}^{r(\kappa)^+}_{n,\epsilon} \otimes \Omega^{r(\kappa)^-})^{<\kappa^+-1} = 0$$

The claims follows immediately. □

18. Classicity of small slope class

In the previous section we have studied the relationship of w-analytic (n, ϵ)-overconverent classes and the overconvergent cohomology of classical sheaf. In this section, we determine which overconvergent cohomology classes of classical is classical, i.e when does the restriction $\mathrm{R}\Gamma(\mathscr{S}_{par}(p^n), \Omega^\kappa) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p^n, \epsilon), \Omega^\kappa)$ becomes a quasi-isomorphism. To do this we will use a system of strict neighborhood $\{\mathscr{S}_{par}(p)_\epsilon\}_\epsilon$ of $\mathscr{S}^{\geq 1}_{par}(p^n)$.This parametrization uses the function deg as introduced in [Far10]. The goal is to determine a precise bound h such that the restriction $\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \Omega^\kappa)^{\leq h} \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \Omega^\kappa)^{\leq h}$ becomes a quasi-isomorphism, and the bound h does not depend on ϵ.

18.1. **Dynamic of operator $U_p^\dagger$.** In this subsection, we define the new strict neighborhood system $\{\mathscr{S}_{par}(p)_\epsilon\}_\epsilon$ of $\mathscr{S}_{par}(p)^{\geq 1}$, and study the dynamical properties of $U_p^\dagger$ with respect to this system. In order to do so, we first recall all the properties of this function that we need here for reader's convenience.

Let K be an extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ its ring of integer. We fix a normalized valuation v_p on K, so that $v_p(p) = 1$. Let G be a finite flat group over $\mathcal{O}_K$, and ω_G its sheaf of relative differential form. As G is finite flat , we can write $\omega_G = \oplus_i \mathcal{O}_K / a_i \mathcal{O}_K$ with $a_i \in \mathcal{O}_K$. Now can its degree $deg(G) := \sum_i v_p(a_i)$. We recap some the properties of this function that we need :

Proposition 18.1.1. *Keeping the notation as above.*

1. *If G is a Barsotti-Tate group truncated of level n and dimension d then $deg(G) = n.d$*
2. *we have $deg(G) + deg(G^D) = height(G)$.*
3. *If $0 \to G_1 \to G \to G_2 \to 0$ is an exact sequence of finite flat groups. Then we have $deg(G) = deg(G_1) + deg(G_2)$.*
4. *If $\phi : G_1 \to G_2$ is a morphism inducing an isomorphism on generic fiber , we have $deg(G_2) \geq deg(G_1)$. Furthermore ϕ is an isomorphism if and only if $deg(G_1) = deg(G_2)$.*

(5) *If L/K is an extension such that L admits a valuation extending that of K then we have $deg(G \otimes_{\mathcal{O}_K} \mathcal{O}_L) = deg(G)$.*
(6) *If H and L are two finite flat subgroup of G, and generically we have $G = H \oplus L$ then $deg(G) \leq deg(H) + deg(L)$*

Proof. For the point (1) to (5), see [Far10], the point (6) is an easy application of point $(4), (5)$. ☐

For any $\epsilon \in \mathbb{R}_+$ we can define $\mathscr{S}_{par}(p)_\epsilon$ as the open subspace of $\mathscr{S}_{par}(p)$ where $deg(H) \geq \epsilon$.

Lemma 18.1.1. *For all $\epsilon \in [0, n - \frac{1}{p-1}]$, the map $\mathscr{S}_{par}(p^n, \epsilon) \to \mathscr{S}_{par}(p)$ factorizes through $\mathscr{S}_{par}(p)_{1-\frac{2}{n}(\epsilon+\frac{1}{p-1})}$.*

Proof. We check this point by point and It is enough to check on rank 1 points. Let $x : Spa(K, \mathcal{O}_K) \to \mathscr{S}_{par}(p^n, \epsilon)$ be a rank 1 point corresponding to a pair (A, H_n). We choose a trivialization $\phi : (\mathbb{Z}/p^n\mathbb{Z})^3 \to A[\pi^n]$ and a basis $\{e_i\}_{1 \leq i \leq 3}$ such that via ϕ, the subgroup $H_n \subset A[\pi^n]$ is generated by e_1. If we look at the Hodge-Tate map $\mathbb{Z}/p^n\mathbb{Z} \to \omega_{H_n^D} \to \omega_{A^D[\pi^n]}$. By theorem 15.1.2, and the fact that $x \in \mathscr{S}_{par}(p^n, \epsilon)$, we see that $\omega_{H_n^D}$ is killed by $p^{\frac{1}{p-1}+\epsilon}$. As $\omega_{H_n^D}$ is generated by 2 elements, i.e $\omega_{H_n^D} = \mathcal{O}_K/(a) \oplus \mathcal{O}_K/(b))$, we deduce that $deg(H_n^D) \leq 2(\epsilon + 1/(p-1))$. Also by point 2 of property (18.1.1) above, one sees that $deg(H_n) \geq n - 2(\epsilon + 1/(p-1)$. Now the natural map $\mathscr{S}_{par}(p^n, \epsilon) \to \mathscr{S}_{par}(p)$ sends (A, H_n) to (A, H_1) with $H_1 = H_n[p^{n-1}]$. Plus, remark also that we have a map $H_n[p^k]/H_n[p^{k-1}] \to H_1$, which is an isomorphism on generic fiber (by comparing ranks). Varying k and using the points (2),(3) prop. (18.1.1)) we easily check that $deg(H_1) + deg(H_n[p^{k-1}] \geq deg(H_n[p^k])$. Add everything up, we see that $n.deg(H_1) \geq deg(H_n[p^n]) = deg(H_n) \geq n - 2(\epsilon + 1/(p-1))$. ☐

On the other hand, we have $\mathscr{S}_{par}(p)_\epsilon \subset \mathscr{S}_{par}(p)$ factors through $\mathscr{S}_{par}(p, 1 - \frac{1}{p-1}) \to \mathscr{S}_{par}(p)$ (see lemma 14.1.2 [Pil18]). In other words, we can switch back and forth between the two neighborhood systems.

We introduce some extra correspondences over $\mathscr{S}_{par}(p)$, these correspondences correspond to the iteration $U_p^{\dagger,n}$. Over $\mathscr{M}_{par}(p)$ we define a correspondence :

$$\begin{tikzcd} & C_n^o \arrow[dl, "p_{1,n}"'] \arrow[dr, "p_{2,n}"] & \\ \mathscr{M}_{par}(p) & & \mathscr{M}_{par}(p) \end{tikzcd}$$

where as usual $\mathscr{M}_{par}(p)$ parametrizes all pair (A, H_1) with $H_1 \in A[\pi]$ étale locally isomorphic to μ_p and C_n^o is the space above $\mathscr{M}_{par}$ parametrizing all triples (A, H_1, L_n) with $(A, H_1) \in \mathscr{M}_{par}(p)$ and $L_n \subset A[\pi^n]$ étale locally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^2$ such that $H_1 \cap L_n = \{0\}$. The map $p_{1,n}$ sends each triple (A, H_1, L_n) to (A, H_1) and the map $p_{2,n}$ sends (A, H_1, L_n) to $(\frac{A}{L_n + L_n^\perp}, \frac{H_1 + L_n}{L_n})$. We can compactify it to a correspondence $\mathscr{S}_{par}(p) \xleftarrow{p_{2,n}} C_n \xrightarrow{p_{1,n}} \mathscr{S}_{par}(p)$.

We have the first general theorem that summarizes the dynamic of operators corresponding to the correspondences C_n

Theorem 18.1.1. *Let* $[a,b] \subset]0,1[$, *then there is contraction radius* $r_{a,b}$ *such that for all* $\epsilon \in [a,b]$, *we have* $p_{2,1}(p_{1,1}^{-1}\mathscr{S}_{par}(p)_\epsilon) \subset \mathscr{S}_{par}(p)_{\epsilon+r_{a,b}}$.

Proof. This is based on two facts

(1) let $x \in \mathscr{S}_{par}(p)_\epsilon$ and $y \in p_{2,1}p_{1,1}^{-1}(x)$ then if H_x is not a Barsotti-Tate, then $deg(H_y) > deg(H_x)$ (see corrolary 2.2 [Pil11]).
(2) Consider the isogeny $A \to A/H$, this induces a morphism $det(\omega_{A/H}) \to det(\omega_A)$ and hence a section $\delta_H \in H^0(\mathscr{S}_{par}(p)_\epsilon, \mathcal{L})$, where $\mathcal{L} := det(\omega_{A/H}^{-1}) \otimes det(\omega_A)$ we have $deg(H_x) = val(\delta_x)$ for any $x \in \mathscr{S}_{par}(p)_\epsilon$. Now if we consider the section $p_{2,1}^*\delta_H(p_{1,1}^{\delta_H})^{-1} \in H^0(\mathscr{S}_{par}(p)_\epsilon, p_{2,1}^*\mathcal{L} \otimes p_{1,1}^*\mathcal{L}^{-1})$. This section attains its minimum valuation $r_{a,b}$ over the quasi-compact $U_{a,b} \times_{p_{1,1}} C_1$ where $U_{a,b} = deg^{-1}[a,b]$. Combined with the first point, we get the theorem.

□

Over C_n we have the universal isogeny $\lambda : \mathcal{A}[\pi^\infty] \to \mathcal{A}_n[\pi^\infty]$ with kernel $L_n \subset \mathcal{A}[\pi^n]$, this induces a morphism $\lambda^* : \omega^+_{\mathcal{A}_n,\pi} \to \omega^+_{\mathcal{A},\pi}$. Taking the determinant we obtain a section $s_\lambda \in H^0(\mathscr{S}_{par}(p), det(\omega^+_{\mathcal{A}_n,\pi})^{-1} \otimes det(\omega^+_{\mathcal{A}.\pi}))$, and we can define $deg(L_n) = v_p(s_\lambda)$.

Proposition 18.1.2. *Let* $x : Spa(K,K^+) \to C_1$ *be a rank* 1 *point and* (A, H_1, L_1) *be the underlying triple, we have the following:*

(1) $deg(L_1) - deg(L_1^\perp) = 1$.
(2) $deg(A[\pi]/L_1) \geq deg(H_1)$ *and we have equality if and only if* H_1 *is of étale or multiplicative type.*

Proof. As the complementary of the boundary is dense and deg is a continuous function, it is enough to prove this for all point of rank 1 where A is an honest abelian variety.

(1) As $L_1 \subset A[\pi]$ and $A[\bar\pi]^D = A[\bar\pi]$, we have $L_1^D = A[\bar\pi]/L_1^\perp$, the properties of the function deg recalled in (18.1.1) tells us that $deg(L_1^\perp) + deg(L_1^D) = deg(A[\bar\pi]) = 1$ and that $deg(L_1) + deg(L_1^D) = ht(L_1) = 2$. As a consequence $deg(L_1) - deg(L_1^\perp) = 1$.
(2) The morphism $H_1 \to A[\pi]/L_1$ is an isomorphism on generic fiber (by comparing ranks), so we have by the proposition (18.1.1) that $deg(A[\pi]/L_1) \geq deg(H_1)$. If more over we have an equality, then this means $H_1 \to A[\pi]/L_1$ is an isomorphism and $deg(H_1)+deg(L_1) = deg(A[\pi]) = 2$. We can conclude by the classification of Dieudonné module attached to $A[\pi]$ that there are only two possibilities for this. The first case is when $deg(H_1) = 0$ and $deg(L_1) = 2$ corresponding to the case where $A[\pi]$ is ordinary with H_1 is its étale quotient. The second case is when $deg(H_1) = 1, deg(L_1) = 1$ corresponding to the case where H_1 is multiplicative.

□

18.2. **Analytic continuation.** Previously, we defined $\{\mathscr{S}_{par}(p)_\epsilon\}_\epsilon$ and correspondences C_n and see how they contract the overconvergent neighborhoods. Now we apply this to show that ϵ'-overconvergent class in $H^i(\mathscr{S}_{par}(p)'_\epsilon), \Omega^\kappa)$ extends to the ϵ-overconvergent class in $H^i(\mathscr{S}_{par}(p)_\epsilon, \Omega^\kappa)$ (when $\epsilon' > \epsilon$).

Let $\mathscr{F}$ be either Ω^κ of Ω^κ_D over $\mathscr{S}_{par}(p)$, and we denote by $\mathscr{F}_\epsilon$ the restriction of $\mathscr{F}$ to $\mathscr{S}_{par}(p)_\epsilon$. Recall that over $\mathscr{S}_{par}(p)$ we have the correspondence $\mathscr{S}_{par}(p) \xleftarrow{p_{2,n}} C_n \xrightarrow{p_{1,n}} \mathscr{S}_{par}(p)$. We define the operator $U_p^{\dagger,n} \in \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$ starting with the map $p_{2,n}^*\mathscr{F} \to p_{1,n}^*\mathscr{F}$ as usual, we set:

$$
\begin{aligned}
U_p^{\dagger,n} :&\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p), p_{2,n}^*\mathscr{F}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p), p_{1,n}^*\mathscr{F}) \\
&\to \mathrm{R}\Gamma(\mathscr{S}_{par}(p), p_{1,n*}p_{1,n}^*\mathscr{F}) \xrightarrow{\frac{1}{p^{n(k_2+1)}}Tr} \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F})
\end{aligned} \tag{18.1}
$$

We can actually construct this correspondence over $\mathcal{S}_{par}(p)$ and then take formal completion and pass to the generic fibers to get the correspondence over adic spaces. The upshot is that we can obtain a natural map $p_{2,n}^*\mathscr{F}^{++} \to p_{1,n}^*\mathscr{F}^{++}$.

Remark 18.2.1. *The reason we define a new correspondence C_n is to make sense of the n^{th}-iteration of $U_p^\dagger$. Indeed, we can check for example that the following diagram commutes:*

$$
\begin{array}{ccccc}
H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) & \xrightarrow{U_p^{\dagger,n}} & H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F}) & \xrightarrow{U_p^{\dagger}} & H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F}) \\
\downarrow & & & & \downarrow \\
H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) & \xrightarrow{U_p^{\dagger}} & H^i(\mathscr{S}_{par}(p)_{\epsilon}, \mathscr{F}) & \xrightarrow{U_p^{\dagger,n}} & H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})
\end{array}
$$

It is easier to have a modular interpretation of $U_p^{\dagger,n}$ this way.

For each $\epsilon' \geq \epsilon$, there is the natural restriction map $res : \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})$. In what follows we will study when a class of cohomology of $\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})$ extends to that of $\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$

Theorem 18.2.1. *Let $\epsilon < 1$ and $f \in H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$. Suppose that there exists a monic polynomial $P \in \mathcal{O}[X]$ with nonzero constant term, such that $P(U_p^\dagger)(f) = 0$, then for all $1 > \epsilon' \geq \epsilon$ we can find $g \in H^i(\mathscr{S}_{par}(p)'_\epsilon, \mathscr{F})$ satisfying:*

1. $P(U_p^\dagger)(g) = 0$
2. *under the natural restriction: $res : \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$ we have $res(f) = g$*

Proof. We first show this for $P = X - \lambda \in \mathcal{O}[X]$. Choose an interval $[a, b] \subset]0, 1[$ such that $\epsilon, \epsilon' \in [a, b]$, and let $r_{a,b}$ the contraction radius as in the theorem **??**. Let n be an integer such that $nr_{a,b} + \epsilon' \geq \epsilon$. We set $g = \lambda^{-n}U_p^n f$. It is immediate to see that g is also of slope h (because we can pretend that $U_p^n = (U_p^\dagger)^n$). This element g is also unique because the oprator $U_p^n \in End(\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}))$ is actually a composition :

$$U_p^n : H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) \to H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F}) \xrightarrow{U_p^n} H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$$

For general polynomial $P = X^n + a_{n-1}X^{n-1} + ... + a_0$, we can set $Q = -a_0^{-1}(X^n + ... + a_1X)$, we have obviously $Q(f) = f$. We can now repeat the argument above for the polynomial $X - 1$.

$\square$

18.3. **Analytic continuation for torsion classes.** Under some conditions, torsion classes also extends to bigger overconvergent neighborhood. Before going into precise statement, let us start with a lemma:

Lemma 18.3.1. *Let $x : Spa(K, \mathcal{O}_K) \to C_n$ be a point of rank 1 with the underlying universal isogeny $\mathcal{A} \to \mathcal{A}_n$ of kernel $L_n + L_n^{\perp}$ with $deg(L^{\perp}) \geq 2n - \epsilon$). Let us denote $\alpha = \kappa^+(n-\epsilon) + 2k_2(n-\epsilon) + k_3(n-\epsilon) - n$, then the operator $U_p^{\dagger,n} : (p_1)_* p_2^* \mathscr{F} \to \mathscr{F}^+$ factors through $p^\alpha \mathscr{F}^+$.*

Proof. As usual, we write the universal isogny of p-divisible groupes $\lambda : \mathcal{A}[p^\infty] \to \mathcal{A}_n[p^\infty]$ as the product of $\lambda_\pi : \mathcal{A}[\pi^\infty] \to \mathcal{A}_n[\pi^\infty]$ with kernel L_n and $\lambda_{\bar{\pi}} : \mathcal{A}[\bar{\pi}^\infty] \to \mathcal{A}_n[\bar{\pi}^\infty]$ with kernel $L_n^{\perp}$. As a result we have induced morphisms:

$$\lambda_\pi^* : \omega_{\mathcal{A}_n,\pi} \to \omega_{\mathcal{A},\pi} \quad \text{and} \quad \lambda_{\bar{\pi}}^* : \omega^+_{\mathcal{A}_n,\bar{\pi}} \to \omega^+_{\mathcal{A},\bar{\pi}}$$

Now, we can choose isomorphisms $\omega_{\mathcal{A}_n,\pi} \cong \mathcal{O}_k^2 \cong \omega_{\mathcal{A},\pi}$ and $\omega^+_{\mathcal{A}_n,\bar{\pi}} \cong \mathcal{O}_K \cong \omega^+_{\mathcal{A},\bar{\pi}}$ such that the matrix for λ_π^* is $diag(a_1, a_2)$ and $\lambda_{\bar{\pi}}^*$ is a multiplication by $a_3 \in \mathcal{O}_K$. According to our hypothesis, $|a_1(x)a_2(x)| \leq |p^{2n-\epsilon}(x)|$ but as $L_n \in \mathcal{A}[\pi^n]$, it is killed by p^n and therefore $|a_i(x)| \geq |p^n(x)|$ for both i, so $|a_i(x)| \leq |p^{n-\epsilon}(x)|$ for $i \in \{1,2\}$. Plus, we have that $deg(L_n) = n + deg(L_n^{\perp})$. As a consequence the map

$$\lambda^*(\kappa) := sym^{\kappa^+}(\lambda_\pi^*) \otimes det^{k_2}(\lambda_\pi^*) \otimes det^{k_3}(\lambda_{\bar{\pi}}^*) : \mathscr{F}^+ \to \mathscr{F}^+$$

factorizes through $p^{\kappa^+(n-\epsilon)+k_2(2n-\epsilon)+k_3(n-\epsilon)}\mathscr{F}^+$. Finally, taking into account the trace map (see remark (16.2.1)), we have the claim of the lemma. $\square$

Lemma 18.3.2. *Let $a_\epsilon = \kappa^+(1-\epsilon) + 2k_2(1-\epsilon) + k_3(1-\epsilon) - 1$. The map*

$$U_p^{\dagger,n} : \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}^{++}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}/p^{na_\epsilon}\mathscr{F}^{++})$$

factors through

$$\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}^{++}) \xrightarrow{res} \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}^{++}) \xrightarrow{U_p} \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}/p^{na_\epsilon}\mathscr{F}^{++})$$

Proof. We want to define a map :

$$p_{1,n_*} p_{2,n}^* \mathscr{F}^{++} \to \mathscr{F}/p^{na_\epsilon}\mathscr{F}^{++}$$

It is enough to define this map over a dense open subspace of $\mathscr{S}_{par}(p)$. Now let $x : Spa(K, \mathcal{O}_K) \to \mathscr{S}_{par}(p)$ be a rank 1 point away from the boundary. As the map $p_{1,n}$ is finite away from the boundary, the cardinal of $p_{1,n}^{-1}(x)$ is finite and so we can choose $\epsilon" \leq \epsilon$ such that $deg(L_n)_y \neq 2n - \epsilon", \forall y \in p_{1,n}^{-1}(x)$, this means that there exist an open neighborhood U of x in $\mathscr{S}_{par}(p)$, such that $deg(L_n) \neq 2n - \epsilon$ for all $x \in U$. Let $V = p_{1,n}^{-1}(U)$ and let $V^>$ and $V^<$ be two disjoint open subsets of V where $deg(L_n) < 2n - \epsilon$ and $deg(L_n) > 2n - \epsilon$ respectively. We have an obvious map:

$$p_{2,n}^*\mathscr{F}^{++}(V^> \coprod V^<) = p_{2,n}^*\mathscr{F}^{++}(V^>) \oplus p_{2,n}^*\mathscr{F}^{++}(V^<) \to \mathscr{F}(U) \to \mathscr{F}/p^\alpha\mathscr{F}^{++}(U)$$

Now for $y \in V^>$, the lemma above tells us that $p_{2,n}^*\mathscr{F}^{++}(V^>)$ maps to zero in $\mathscr{F}/p^\alpha\mathscr{F}^{++}$ and for $y \in V^<$, by definition we have $deg(L_n) \leq 2n - \epsilon$ which implies that $deg(H) > \epsilon$ and so $p_{1,n}(y) \in \mathscr{S}_{par}(p)_\epsilon$. To recap , we obtain a map :

$$p_{1,n_*} p_{2,n}^*\mathscr{F}^{++}(U) \to p_{2,n}^*\mathscr{F}^{++}(V^<) \to \mathscr{F}/p^{na_\epsilon}\mathscr{F}^{++}(U)$$

$\square$

Theorem 18.3.1. *Let $f \in H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$ be an eigenclass with $U_p^\dagger f = \lambda f$. If $v_p(\lambda) < \kappa^+ + 2k_2 + k_3 - 1 = k_1 + k_2 + k_3 - 1$ then there is a projective system*

$$\{f_n\}_n \in \varprojlim_n H^i(\mathscr{S}_{par}(p), \mathscr{F}/p^n\mathscr{F}^{++})$$

such that $U_p^\dagger f_n = \lambda f_n$ for all n and the restriction of $f_n \in H^i(\mathscr{S}_{par}(p), \mathscr{F}/p^n\mathscr{F}^{++})$ to $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^{++})$ is also the image of f in $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^{++})$.

Proof. Firstly, as $\kappa^+ + 2k_2 + k_3 - 1 - v_p(\lambda) > 0$ by hypothesis, there exists ϵ' such that $\kappa^+(1-\epsilon') + 2k_2(1-\epsilon') + k_3(1-\epsilon') - 1 - v_p(\lambda) > 0$. If $\epsilon' > \epsilon$ we can consider the restriction of f to $H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})$, and if $\epsilon' \leq \epsilon$, by theorem 18.2.1 we can find $f' \in H^i(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})$ extending f and such that $U_p^\dagger f' = \lambda f'$. Thus, we can always suppose that :

$$\kappa^+(1-\epsilon) + 2k_2(1-\epsilon) + k_3(1-\epsilon) - 1 - v_p(\lambda) > 0$$

Now, observe that as the map $\mathscr{S}_{par}(p)_\epsilon \to \mathscr{S}_{par}(p)$ is locally affine, we have $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) = H^i(\mathscr{S}_{par}(p), \mathscr{F}_\epsilon)$. Plus, up to scaling, we can always assume that $f \in H^i(\mathscr{S}_{par}(p)), \mathscr{F}_\epsilon^{++})$ and f has maximal possible norm (i.e we have $U_p^\dagger f \in H^i(\mathscr{S}_{par}(p), p^{-1}\mathscr{F}_\epsilon^{++})$). For each n the lemma (18.3.2) gives us a map

$$U_n : \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}_\epsilon^{++}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F}/p^{n\alpha}\mathscr{F}^{++})$$

Let $f_n = \lambda^{-n} U_n f$. We can easily see that image of f_n in $H^i(\mathscr{S}_{par}(p), \mathscr{F}/p^{m\alpha}\mathscr{F}^{++})$ for any $n \geq m$ is indeed f_m. As a result, we have a projective systems $\{f_n\}_n$. We verify immediately all other properties. $\square$

18.4. **Classicity of overconvergent cohomology.** For any smooth and separated adic space $\mathscr{X}$ and a locally free sheaf of $\mathcal{O}_{\mathscr{X}}^+$-module $\mathscr{F}^+$ with $\mathscr{F} = \mathscr{F}^+ \otimes \mathcal{O}_{\mathscr{X}}$. We have a natural surjection (which is an isomorphism if $\mathscr{X}$ is proper) (see lemma 3.3.2 [Pil18]).

$$H^i(\mathscr{X}, \mathscr{F}) \twoheadrightarrow \varinjlim_n H^i(\mathscr{X}, \mathscr{F}/p^n\mathscr{F}^+)$$

We want to apply this fact to prove the following lemma

Lemma 18.4.1. *The natural map*

$$H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{\leq a} \to \varprojlim_n H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^+)$$

is injective for any rational finite slope a.

Proof. Let V be the image of $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}^+)$ in $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$. It is enough to show that $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{\leq a} \cap V$ is bounded.

Let $U = \{U_i\}_{i\in I}$ and $U = \{U'_i\}_{i\in I}$ be two finite affinoid coverings of $\mathscr{S}_{par}(p)$. We assume that the closure of each U'_i is contained in U_i for all $i \in I$. Let $U_\epsilon = \{U_{i,\epsilon}\}_i$ be the finite affinoid covering $U \cap \mathscr{S}_{par}(p)_\epsilon$. Let $\epsilon < \epsilon'$ be such that $U_p^\dagger(\mathscr{S}_{par}(p)_{\epsilon'}) \subset \mathscr{S}_{par}(p)_\epsilon$. Let $U_{\epsilon'} = \{U_{i,\epsilon'}\}$ be the covering $U' \cap \mathscr{S}_{par}(p)_{\epsilon'}$. For all $i \in I$, we have $U_{i,\epsilon'} \subset U_{i,\epsilon}$. The operator $U_p^\dagger$ is defined as the composite

$$\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}) \xrightarrow{res} \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$$

If we represent $\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$ by the Cech complex $M^\bullet = Cech(U_\epsilon, \mathscr{F})$ and $\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_{\epsilon'}, \mathscr{F})$ by $N^\bullet = Cech(U_{\epsilon'}, \mathscr{F})$. The operator $U_p^\dagger$ can then be represented by $M^\bullet \xrightarrow{res} N^\bullet \xrightarrow{U_{\epsilon,\epsilon'}} M^\bullet$. The subcomplex of finite dimensional vector spaces $M^{\bullet,\leq a}$ of $M^\bullet$ is its direct summand and $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{\leq a} = H^i((M^{\bullet,\leq a}))$. Now, since the natural map $H^i_{cech}(U_\epsilon, \mathscr{F}^+) \to H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F}^+)$ has cokernel of bounded torsion, we can replace V by the image V' of $H^i_{cech}(U_\epsilon, \mathscr{F})$ in $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})$. Let $Z_i((M^{\bullet,\leq a}) \subset M^i$ be the cocycles of slope less than h. This is a finite dimensional vector space. We denote by $M^{\bullet,+}$ the Cech complex $Cech(U_\epsilon, \mathscr{F}^+)$. Then $M^{i,+}$ is bounded in M^i. It follows that $M^{i,+} \cap Z_i((M^{\bullet,\leq a})$ is bounded and thus a lattice. As a result, its image in $H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{\leq a}$ (which is $H^i(\mathscr{S}_{par}(p)_\epsilon,)^{\leq a} \cap V'$) is bounded and the lemma is proved. □

The following theorem says that small slope overconvergent class is classical.

Theorem 18.4.1. *We still denote $h = \kappa^+ + 2k_2 + k_3 - 1 = k_1 + k_2 + k_3 - 1$. The restriction map :*

$$\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \mathscr{F})^{<h} \xrightarrow{res} \mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{<h}$$

is a quasi isomorphism.

Proof. We need to show that for each i, we have an isomorphism:

$$H^i(\mathscr{S}_{par}(p), \mathscr{F})^{<h} \xrightarrow{res} H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{<h}$$

For any $f \in H^i(\mathscr{S}_{par}(p)_\epsilon, \mathscr{F})^{<h}$, we obtain a projective system

$$\{f_n\}_n \in \varprojlim H^i(\mathscr{S}_{par}(p), \mathscr{F}/p^n\mathscr{F}^{++})$$

by theorem 18.3.1. Now as $\mathscr{S}_{par}(p)$ is proper, we can apply proposition (3.2.1) in [Pil18] which says that we have an isomorphism:

$$\varprojlim H^i(\mathscr{S}_{par}(p), \mathscr{F}/p^n\mathscr{F}^{++}) \cong H^i(\mathscr{S}_{par}(p), \mathscr{F})$$

□

As an application, we are now able to prove the rational control theorem :

Theorem 18.4.2. *For all $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$ such that $k_1 \geq k_2, k_1 + k_3 > 1$, we have a quasi-isomorphism:*

$$u_p\mathrm{R}\Gamma(\mathcal{S}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{Q}_p \cong u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{Q}_p$$

Proof. As already mentioned earlier, it is enough to show:

$$u_p\mathrm{R}\Gamma(\mathcal{S}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{C}_p \cong u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{C}_p$$

Over $\mathbb{C}_p$ we can see that :

(1) $u_p\mathrm{R}\Gamma(\mathcal{S}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{C}_p \cong u_p\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \Omega^\kappa_D)$
(2) $u_p\mathrm{R}\Gamma(\mathfrak{S}^{\geq 1}_{par}(p), \Omega^\kappa_D) \otimes^L \mathbb{C}_p \cong u_p\mathrm{R}\Gamma(\mathscr{S}^{\geq 1}_{par}(p), \Omega^\kappa_D)$

Since the ordinary part corresponds to 0-slope part, by classicity criteria (thm(18.4.1) above), there is a quasi-isomorphism.

$$u_p\mathrm{R}\Gamma(\mathscr{S}_{par}(p), \Omega^\kappa_D) \cong u_p\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \Omega^\kappa_D)$$

As a consequence it suffices to show that the restriction :

$$u_p\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) \to u_p\mathrm{R}\Gamma(\mathscr{S}_{par}(p)^{\geq 1}, \Omega_D^\kappa) \tag{18.2}$$

is a quasi-isomorphism. Let us denote by $d_i(\kappa)$ and $d_i^\dagger(\kappa)$ for the $\mathbb{C}_p$-dimension of $u_pH^i(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$ and $colim_\epsilon u_pH^i(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa))$ with $(i = 0, 1)$. We claim $d_0(\kappa) \geq d_0^\dagger(\kappa)$ and $d_1(\kappa) \leq d_1^\dagger(\kappa)$ for all κ such that $k_1 \geq k_2, k_1+k_3 > 1$ and these inequalities become equality when $k_1 \geq k_2+1, k_1+k_3 > 1$. For this we need the following lemma.

Lemma 18.4.2. *The induced morphism $u_pH^i(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) \to u_pH^i(\mathscr{S}_{par}^{\geq 1}(p), \Omega_D^\kappa)$ is injective for $i = 0$ and surjective for $i = 1$.*

Proof. First we notice that both complexes are perfect and concentrated in degrees $[0, 1]$. There is an injection $u_pH^0(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) \to u_pH^0(\mathscr{S}_{par}^{\geq 1}(p)_\epsilon, \Omega_D^\kappa)$ because $u_pH^0(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) = H^0(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa)^{=0} = colim_{n,\epsilon} u_pH^0(\mathscr{S}_{par})$

Recall that we have the minimal compactification $\mathfrak{S}_{par}^*(p)$ of and there is a projection $\pi : \mathfrak{S}_{par}(p) \to \mathfrak{S}_{par}^*(p)$ such that $R\pi_*\Omega_D^\kappa = \pi_*\Omega_D^\kappa$. The image of $\mathfrak{S}_{par}^{\geq 1}(p)$ in $\mathfrak{S}_{par}^*(p)$ is covered by two affines, indeed this image is nothing but $\mathfrak{S}_{par}^*(p) \times_{\mathfrak{S}^*} \mathfrak{S}^{*,\geq 1}$ but since the map $\mathfrak{S}_{par}^*(p) \to \mathfrak{S}$ is proper and quasi affine, hence affine, and $\mathfrak{S}^{*,\geq 1}$ can be covered by two affines $\mathfrak{U}_1, \mathfrak{U}_2$(corresponding to the invertible loci of some lifts of Hasse invariants Ha and Ha') with generic fibers $\mathscr{U}_1, \mathscr{U}_2$. As a result, $\mathrm{R}\Gamma(\mathscr{S}_{par}^{\geq 1}(p), \Omega_D^\kappa)$ is represented by

$$H^0(\mathscr{U}_1, \Omega_D^\kappa) \oplus H^0(\mathscr{U}_2, \Omega_D^\kappa) \to H^0(\mathscr{U}_1 \cap \mathscr{U}_2, \Omega_D^\kappa)$$

and $colim_{\epsilon,w}\mathrm{R}\Gamma(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa)$ is represented by

$$H^0(\mathscr{U}_1, \Omega_D^\kappa, \dagger) \oplus H^0(\mathscr{U}_2, \Omega_D^\kappa, \dagger) \to H^0(\mathscr{U}_1 \cap \mathscr{U}_2, \Omega_D^\kappa, \dagger)$$

Where the " $\dagger$ " means we are taking the overconvergent sections.

From this representation, we see that $colim_\epsilon H^1(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa)$ has a dense image in $H^1(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$. It follows immediately that $colim_\epsilon u_pH^1(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) \to u_pH^1(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$ is surjective as ordinary parts are finite dimensional.

Lastly $H^1(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa)$ does not depend on $\epsilon \in [0, 1]$ based on what we established above. We deduce that $u_pH^1(\mathscr{S}_{par}(p)_\epsilon, \Omega_D^\kappa) \to u_pH^1(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$ is surjective or put differently $d_1(\kappa) \geq d_1^\dagger(\kappa)$. ☐

From the above, $d_0(\kappa) \geq d_0^\dagger(\kappa)$ and $d_1^\dagger(\kappa) \geq d_1(\kappa)$. More over for $k_1 - k_2 \geq 1, k_1 + k_3 > 1$ we have an isomorphism $t_pH^0(\mathscr{S}, \Omega^\kappa) \to u_pH^0(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$ and an injection $t_pH^1(\mathscr{S}, \Omega_D^\kappa) \to u_pH^1(\mathscr{S}_{par}^{\geq 1}(p), \Omega_D^\kappa)$. In addition

$$t_pH^i(\mathscr{S}, \Omega_D^\kappa) \cong u_pH^i(\mathscr{S}_{par}(p), \Omega_D^\kappa)$$

This means that for such κ we have $u_pH^i(\mathscr{S}_{par}(p), \Omega_D^\kappa) \cong u_pH^i(\mathscr{S}_{par}^{\geq 1}, \Omega_D^\kappa)$ $(i = 0, 1)$. The result is that $d_1(\kappa) - d_0(\kappa) = d_1^\dagger(\kappa) - d_0^\dagger(\kappa)$ for all κ such that $k_1 \geq k_2+1, k_1+k_3 > 1$.

Finally, since the Euler characteristic $d_1(\kappa) - d_0(\kappa)$ and $d_1^\dagger(\kappa) - d_0^\dagger(\kappa)$ is locally constant, we conclude that $d_1(\kappa) - d_0(\kappa) = d_1^\dagger(\kappa) = d_0^\dagger(\kappa)$ and consequently $d_i(\kappa) = d_i^\dagger(\kappa)$ for all κ such that $k_1 \geq k_2, k_1 + k_3 > 1$. This finishes the proof of the theorem. ☐

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