On one point functions in the Supersymmetric Sine Gordon model

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Summary of the results

This PhD deals with the calculation of one point functions in integrable two dimensional Quantum Field Theories. These quantities are of first interest. Indeed, any general n point function can be reduced (in the Ultra-Violet limit) to a sum of products of one point functions and of coefficients of the Operator Product Expansion. The latter coefficients can be computed by a usual perturbative approach, whereas the one point functions carry important Infra-Red information that cannot be accessed by perturbation theory. Therefore, it is an essential issue to develop a systematic, alternative method for their calculation.

Such a method was elaborated in the last 15 years in the context of the sine-Gordon theory [1, 2, 3, 4, 5]. The authors of this series of papers used the integrability of the model to first build a basis of local operators in the context of the six vertex model (the lattice regularization of the sine-Gordon Quantum Field Theory). This basis, termed *fermionic*, is powerful because the vacuum expectation values of its operators on the lattice are easy to calculate : they are expressed in terms of determinants of only two functions. Another remarkable property is that the fermionic structure can be extended to the continuum limit of the six vertex model and be used to characterize local operators in the Conformal Field Theory. Then, besides the usual Virasoro description, one gets the fermionic basis to classify the fields. Furthermore, the fermionic basis is well adapted to the Conformal Perturbation Theory. From the correspondence between the Virasoro description and the fermionic basis, and the possibility to compute the one point functions of the latter in the massive theory, one access the one point functions of local fields in the sine-Gordon model. This was a major breakthrough in the general calculation of one point functions in integrable Quantum Field Theories.

In this thesis, we continue the work on this very promising approach, aiming to generalize the fermionic basis construction to a more complex system. The logical choice to make is to consider a "higher spin" theory, which naturally leads us to apply the previous ideas to the more complicated Supersymmetric sine-Gordon model. The main objective is to achieve the following goal : *calculate the one point functions in the Supersymmetric sine-Gordon model.* In order to do so, we are first concerned with the generalization of the fermionic basis at the lattice level (in this new case it is the 19 vertex model). The study of this new basis has already been initiated [6], and shows a much richer structure since it involves not only operators of fermionic nature but also Kac Moody currents. It will be referred to as the fermion-current structure. In the paper [8], we further confirmed the well foundedness of the fermion-current basis at the lattice level by using it to explicitly decompose several local, invariant operators. Moreover we computed the density matrix and the entanglement entropy for a small number of lattice sites. In the paper [7] we derived scaling equations governing the thermodynamics of the Supersymmetric sine-Gordon theory. In particular in the conformal limit we reproduced from them the Bazhanov-Lukyanov-Zamolodchikov generating function of local integrals of motion, and checked the scaling equations against known results from the ODE-CFT correspondence. Finally, in the work [9] we described the integrable structure of the Supersymmetric sine-Gordon model in terms of the fermion-current basis. We focused on the fermionic part of the latter and computed the one point functions of fermionic operators. In the conformal limit, these results were checked to be consistent with an alternative approach relying on the reflection relations.

Résumé des résultats

Cette thèse porte sur le calcul des fonctions à un point dans les théories des champs quantiques bidimensionnelles intégrables. Ces quantités sont de premier intérêt. En effet, toute fonction générale à n points peut être réduite (dans la limite Ultra-Violette) à une somme de produits de fonctions à un point et de coefficients provenant de l'Expansion du Produit d'Opérateur. Ces derniers coefficients peuvent être calculés à l'aide d'une approche perturbatrive habituelle, alors que les fonctions à un point contiennent d'importantes informations infrarouges qui ne sont pas accessibles par la théorie des perturbations. Il est donc essentiel de mettre au point une méthode de calcul systématique et alternative des fonctions à un point.

Une telle méthode a été élaborée au cours des 15 dernières années dans le contexte de la théorie de sine-Gordon [1, 2, 3, 4, 5]. Les auteurs de cette série d'articles ont utilisé l'intégrabilité du modèle pour construire d'abord une base d'opérateurs locaux dans le contexte du modèle à six sommets (qui est la régularisation sur réseau de la théorie des champs quantiques de sine-Gordon). Cette base, appelé *fermionique*, est puissante parce que les valeurs moyennes dans le vide de ses opérateurs sur le réseau sont faciles à calculer : ils sont exprimés en termes de déterminants de seulement deux fonctions. Une autre propriété remarquable est que la structure fermionique peut être étendue jusqu'à la limite continue du modèle à six sommets et être utilisée pour caractériser les opérateurs locaux dans le modèle de la théorie des champs conformes. Cela implique qu'additionnellement à la description habituelle de Virasoro, on peut utiliser la base fermionique pour classifier les champs. De plus, la base fermionique est bien adaptée à la théorie de la perturbation dans le cas d'une théorie initiale conforme. D'après la correspondance entre la description de Virasoro et la description de la base fermionique, et la possibilité de calculer les fonctions à un point de cette dernière dans la théorie massive, on accède aux fonctions à un point des champs locaux dans le modèle sine-Gordon. Il s'agit donc d'une percée majeure dans le calcul général des fonctions à un point dans les théories des champs quantiques intégrables.

Dans cette thèse, nous poursuivons le travail sur cette approche très prometteuse visant à généraliser la construction de la base fermionique à un système plus complexe. Le choix logique à faire est d'envisager une théorie de "spin plus élevé", ce qui nous conduit naturellement à appliquer les idées précédentes au modèle plus compliqué de sine-Gordon supersymétrique. L'objectif principal est donc le suivant : *calculer les fonctions à un point dans le modèle sine-Gordon supersymétrique*. Pour ce faire, nous devons d'abord généraliser la base fermionique au niveau du réseau (dans ce nouveau cas, il s'agit du modèle à 19 sommets). L'étude de cette nouvelle base a déjà été initiée [6], et montre une structure beaucoup plus riche puisqu'elle contient non seulement des opérateurs de nature fermionique mais aussi des courants de Kac Moody. On l'appellera donc la structure de type "fermion-courant". Dans l'article [8], nous avons confirmé le bien-fondé de la base de type fermion-courant au niveau du réseau et nous l'avons utilisée pour décomposer explicitement plusieurs opérateurs locaux. De plus, nous avons calculé la matrice de densité sur le réseau et l'entropie d'intrication, pour un petit nombre de sites. Dans l'article [7] nous avons dérivé des équations de scaling régissant la thermodynamique de la théorie de sine-Gordon supersymétrique. En particulier dans la limite conforme de ces équations nous avons reproduit la fonction génératrice des intégrales locales du mouvement de Bazhanov-Lukyanov-Zamolodchikov, et vérifié ces calculs par comparaison avec des résultats connus de la correspondance ODE-CFT. Enfin, dans l'article [9] nous avons décrit la structure intégrable du modèle de sine-Gordon supersymétrique grâce à la base de type fermion-courant. Nous nous sommes concentrés sur la partie fermionique de cette dernière et avons calculé les fonctions à un point des opérateurs fermioniques. Dans la limite conforme, la cohérence de ces résultats a été vérifié grâce à une approche alternative s'appuyant sur les relations de réflexion.

Publications

This PhD work resulted in the following three publications :

[7] C. Babenko, F. Smirnov.
Suzuki equations and integrals of motion for Supersymmetric CFT. Nuclear Physics B, **924**, (2017), 406-416.

[8] C. Babenko, F. Smirnov.
Fermion-current basis and correlation functions for the integrable spin 1 chain. Int. J. of Modern Phys. A, **34**, (2019), 1950075.

[9] C. Babenko, F. Smirnov.
One point functions of fermionic operators in the Super sine-Gordon model. Nuclear Physics B, **946**, (2019), 114698.

Structure of the manuscript

The text is divided in 4 chapters. The first two chapters are intended to recall some introductory material, the two last present the new results obtained during the PhD research. They are structured as follows : the chapter 1 motivates the subject of one point functions and recall the known results and methods on their computation for Integrable Quantum Field Theories. In the chapter 2, we review some basic facts about lattice integrable system and describe the integrable structure of Quantum Field Theories. In particular, the spin $\frac{1}{2}$ fermionic basis is introduced. In the chapter 3 we then move to the lattice fermion-current construction and present the results of [8]. Finally, in the chapter 4, we discuss the Super sine-Gordon model and explain the achievements of [7], [9].

List of abbreviations

- QFT : Quantum Field Theory
- CFT : Conformal Field Theory
- SCFT : Supersymmetric Conformal Field Theory
- IQFT : Integrable Quantum Field Theory
- OPE : Operator Product Expansion
- sG : sine-Gordon
- shG : sinh-Gordon
- ssG : Super sine-Gordon
- IR : Infra-Red
- UV : Ultra-Violet
- TBA : Thermodynamic Bethe Ansatz
- SUSY : Supersymmetry
- 6 V : 6-vertex (model)
- 19 V : 19-vertex (model)

List of figures

Notations

We shall attempt as much as possible to observe the following conventions : $V, V_i, V_j, V_k, ...$ will be used for generic fields in QFT, V_Δ will denote a primary field of conformal dimension Δ in CFT, V_a will stand for a vertex operator in (Super) Liouville CFT with weight a. Simple letters (such as b^*, c^*) will be used for the spin $\frac{1}{2}$ fermionic basis and bold letters (such as $\mathbf{b}^*, \mathbf{c}^*$) will serve for the spin 1 fermion-current basis.

Contents

Chapter 1

One point functions in Quantum Field Theories

This thesis aims at calculating the one point functions in a specific model of two dimensional *Quantum Field Theory* (QFT) : the *Supersymmetric sine-Gordon* model (ssG). Two questions that should be addressed in the first place are : why are the one point functions important quantities ? And what are the existing methods to calculate them ? This first chapter will provide answers to these two guiding questions. As a basis for our discussion, we present the ssG action :

$$\mathcal{A}_{\text{ssG}} = \int \Big[\Big(\frac{1}{4\pi} \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{1}{2\pi} \left(\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi} \right) - 2\mu \bar{\psi} \psi \cos\Big(\frac{\beta}{\sqrt{2}} \varphi \Big) \Big] d^2 z \,, \tag{1.1}$$

where φ is a bosonic field, ψ a Majorana fermion, β, μ are the coupling constants. Let us mention, that we will handle the ssG model as a perturbation of a specific *Conformal Field Theory* (CFT) : the *complex Supersymmetric Liouville CFT*. From a Lagrangian point of view this means that :

$$\mathcal{A}_{\text{ssG}} = \mathcal{A}_{\text{CFT}} + \mathcal{A}_{\text{pert}} \,, \tag{1.2}$$

where the terms $\mathcal{A}_{\text{CFT}}$ and $\mathcal{A}_{\text{pert}}$ will be explicited later. It implies that the Ultra-Violet (UV) behavior of the ssG model is described by a conformal theory. Hence, it is of crucial importance to know how to operate in the context of CFT and to be able to connect the ssG model with its UV limit. Technically, this will be done through the use of scaling equations, that will be described in great detail later. First we are going to motivate the importance of the computation of the one point functions, in particular in the context of conformal perturbation theory. Do to so, we will take the following steps :

1. Explain the relevance of the calculation of one point functions in the course of solving a QFT.
2. Recall some basic facts about Conformal Field Theory, explain how one deals with CFT on a cylinder and introduce the Liouville CFT. As we said, the UV limit of the ssG model is described by a particular CFT, and this is also true for all other theories where we know how to obtain the one point functions. Moreover, this part will be a preparation for the calculations carried out in the case of the Super Liouville model in the chapter 4.
3. Give a brief overview of the Conformal Perturbation Theory, since this is the framework in which we will study the ssG. We will explain how to compute perturbatively the coefficients of the Operator Product Expansion and hence further underline the particularity and the importance of one point functions.
4. Discuss the existence of conservation laws in QFT, especially the deformation of conformal ones and the notion of Integrable QFT.
5. Recall the principles of the Thermodynamical Bethe Ansatz. It is stated in terms of Non Linear Integral Equations, that share some similarities with the scaling *Suzuki* equations that we will use to get new results in the ssG model.
6. Finally, explain how we can use reflection relations to obtain the one point functions, and the limits of this method. This will be illustrated for the sine-Gordon model, for the primary and the simplest descendant fields.

1.1 One point functions : a fundamental object

Let us consider a general QFT in two Euclidean dimensions, described by the action $\mathcal{A}_{\text{QFT}}$. By "solving" a QFT we usually understand that we should compute its spectrum $\mathcal{S}$ (the complete field content), as well as all correlation functions among the elements of $\mathcal{S}$. To be more precise, suppose that we have a theory with a space of fields $\mathcal{S} = \{V_i\}_{i\in\mathcal{I}}$ for some set $\mathcal{I}$, and that the general correlation function can be written from a path integral perspective :

$$\langle V_{i_1}(x_1)...V_{i_n}(x_n)\rangle = \int \mathcal{D}\varphi\, V_{i_1}(x_1)...V_{i_n}(x_n)e^{-\mathcal{A}_{\text{QFT}}[\varphi]}\,. \tag{1.3}$$

At this point, we should remark that even if we are not going to use the path integral to carry out explicit calculations, it will reveal several times to be a useful tool to obtain an intuition about the result (for example this will be case when we will be studying Conformal Perturbation Theory). As is well known, the path integral is hard to manipulate, so one should take another way to compute $\langle V_{i_1}(x_1)...V_{i_n}(x_n)\rangle$. To do so, let us assume that our theory admits a short distance *Operator Product Expansion* (OPE), that is for any two elements of $\mathcal{S}$ we can write :

$$V_i(x)V_j(y) \simeq \sum_{k\in\mathcal{I}} C_{ij}^k(x,y)V_k(y)\,, \quad x\to y\,, \tag{1.4}$$

where $C_{ij}^k(x,y)$ are functions depending on the positions of the fields. The OPE assumption has been proposed in [10], [11] and reflects the fact that the field space $\mathcal{S}$ can be considered as an algebra, with a basis indexed by $\mathcal{I}$ and structure constants C_{ij}^k. Here we should make a warning about identities between fields : they will have to be always understood as holding inside correlation functions (X is any product of elements of $\mathcal{S}$) :

$$\langle V_i(x)V_j(y)X\rangle \simeq \sum_{k\in\mathcal{I}} C_{ij}^k(x,y)\,\langle V_k(y)X\rangle\,. \tag{1.5}$$

Consider now (1.5) with only $\langle V_i(x)V_j(y)\rangle$ in the left hand side. In this decomposition, two objects of very different nature are involved. The coefficients $C_{ij}^k(x,y)$ are defined in the limit $x\to y$, and are thus purely Ultra-Violet data, meaning that they depend on the short distance interaction of the theory. On the other hand, the one point functions $\langle V_k(y)\rangle$ are by definition the average of the fields V_k over all space-time, and have hence to depend on the overall long distance or Infra-Red (IR) environment of the theory. The above reduction (1.5) holds for any n point function in the UV limit. Therefore, we claim that the fundamental quantities that we should aim at in a study of a QFT are :

1. The spectrum $\mathcal{S}$.
2. The structure constants C_{ij}^k.
3. The one point functions $\langle V_k(y)\rangle$.

In the following sections, we will show how to compute the spectrum and the structure constants in the case of the theories that we will consider : CFTs and Perturbation of CFTs. The one point functions appear to be the most challenging quantities for which straightforward approaches do not exist for the moment.

One field that we will always assume contained in $\mathcal{S}$ is the stress energy tensor of the theory $T_{\mu\nu}$, that measures the response of the model under deformations of the space-time. To get an explicit formula, we first consider our theory in curved space, and take the variation of the action with respect to the metric :

$$T_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta\mathcal{A}_{\text{QFT}}}{\delta g^{\mu\nu}}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\,. \tag{1.6}$$

Since we work in Euclidean signature in two dimensions, it will be helpful to consider complex coordinates $z = x+iy$, for which we define the following notations :

$$T = T_{z,z}\,, \quad \bar{T} = T_{\bar{z},\bar{z}}\,, \quad \Theta = -T_{z,\bar{z}}\,. \tag{1.7}$$

Let us make a remark here. The assumption that the theories we consider are invariant under translations implies that the one point functions can be simply calculated at the origin :

$$\langle V_k(y)\rangle = \langle V_k(0)\rangle\,. \tag{1.8}$$

Notice that Θ is often referred to as the trace of the stress energy tensor, however the exact relation is $\Theta = -\frac{1}{4}T^a_a$. Since $T_{\mu\nu}$ is a Noether current, it is conserved, and the conservation laws in the complex coordinates are :

$$\partial_{\bar{z}}T(z,\bar{z}) = \partial_z\Theta(z,\bar{z})\,, \quad \partial_z\bar{T}(z,\bar{z}) = \partial_{\bar{z}}\Theta(z,\bar{z})\,, \tag{1.9}$$

with the convention

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)\,, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)\,. \tag{1.10}$$

The relations (1.9) are very important, and we will see that in the context of integrable QFTs they represent the first occurrence of an entire tower of conservation laws. Notice also that from them we can construct conserved charges :

$$\mathcal{P} = \oint_{c_0}\frac{dz}{2\pi i}\left(Tdz + \Theta d\bar{z}\right)\,, \quad \bar{\mathcal{P}} = \oint_{c_0}\frac{dz}{2\pi i}\left(\bar{T}d\bar{z} + \Theta dz\right)\,. \tag{1.11}$$

where c_0 is a small circular contour around the origin. These are simply integrated infinitesimal complex translations, and this means that one can act with $\mathcal{P}, \bar{\mathcal{P}}$ on a field of $\mathcal{S}$ to get :

$$\mathcal{P}V_i(w,\bar{w}) = \partial_w V_i(w,\bar{w})\,, \quad \bar{\mathcal{P}}V_i(w,\bar{w}) = \partial_{\bar{w}}V_i(w,\bar{w})\,. \tag{1.12}$$

At this moment in our discussion, we should start to look a specific theories. We will first see how the above questions (spectrum, coefficients of OPE, one point functions) can be fully answered in the case of Conformal Field Theories, and then what remain hidden if we consider Perturbed CFT.

1.2 Conformal Field Theory

The first type of theories to look at are the Conformal Field Theories (CFT) that have been very extensively studied in the last 40 years. They allowed, in particular, a Quantum Field Theoretic interpretation of critical phenomena and phase transitions. The fundamental paper where the modern description of CFT was given is [12]. It was followed by a plethora of articles on the subject. There are many excellent books and reviews, we used [13, 14, 15, 16].

In this section our modest objective is not to give an extensive introduction to CFT, but rather to briefly recall the most important concepts, introducing notations for further calculations and illustrating the very general notions that have been considered above. We would like to prepare the ground for the study of more complicate QFTs, which will be described by a CFT in their UV limits. As we will recall, the CFTs are theories that are completely solved in the sense given above, so we will heavily rely on them in our further calculations to extract information about the one point functions.

1.2.1 The Virasoro algebra $\mathfrak{V}$.

Consider such a theory on the plane. Conformal means that this theory is invariant under conformal transformations, in particular scale transformations. Therefore, the β-function is identically zero, and this implies the vanishing of the field $\Theta(z,\bar{z})$ [1]. The conservation laws (1.9) simplifies to holomorphicity (anti-holomorphicity) conditions :

$$\partial_{\bar{z}}T(z,\bar{z}) = 0\,, \quad \partial_z\bar{T}(z,\bar{z}) = 0\,. \tag{1.14}$$

The stress energy tensor is by definition the generator of conformal transformations. It is well known that in a general QFT, the infinitesimal effect of a symmetry on a field corresponds to the space integral of the commutator of the field with the Noether charge. In the two dimensional setting, using complex coordinates, we can rewrite the integrals of commutators as contour integrals. Focusing on

[1] Consider a general QFT given by the action $\mathcal{A}_{\mathrm{QFT}}$ and depending on some coupling constants $g = \{g^a\}_a$. Define the fields $\xi_a = \frac{\delta\mathcal{A}_{\mathrm{QFT}}}{\delta g^a}$. One can express the "trace" of the stress energy tensor Θ in terms of the beta-functions by the formula :

$$\Theta = \sum_a \beta^a(g)\xi_a\,. \tag{1.13}$$

T, the action of an infinitesimal conformal transformation $z \to \epsilon(z)$ on a field V is then given by (c_w is a small circular contour around w) :

$$\delta_\epsilon V(w,\bar{w}) = \oint_{c_w} \frac{dz}{2\pi i}\epsilon(z)T(z)V(w,\bar{w})\,. \tag{1.15}$$

From locality and holomorphicity assumptions, the stress energy tensor can be expanded in a Laurent series :

$$T(z)V(w,\overline{w}) = \sum_{n\in\mathbb{Z}}(z-w)^{-n-2}L_n V(w,\overline{w})\,. \tag{1.16}$$

One obtains the action of the modes on the fields by reversing the previous formula :

$$L_n V(w,\bar{w}) = \oint_{c_w} \frac{du}{2\pi i}(u-w)^{n+1}T(u)V(w,\overline{w})\,. \tag{1.17}$$

An important assumption to make is that the spectrum $\mathcal{S}$ includes the so called *primary* fields, that are denoted $V_\Delta(w,\bar{w})$ and satisfy the properties :

$$L_m V_\Delta(w,\bar{w}) = 0\,, \quad m>0\,, \quad L_0 V_\Delta(w,\bar{w}) = \Delta V_\Delta(w,\bar{w})\,. \tag{1.18}$$

The parameter Δ is called the (holomorphic) *conformal dimension* , and characterizes the field V_Δ. The relation (1.12) gives :

$$L_{-1}V_\Delta(w,\bar{w}) = \partial_w V_\Delta(w,\bar{w})\,. \tag{1.19}$$

We are mostly interested in the singular part of the OPEs. Applying the above properties of primary fields to the general expression (1.16), we get

$$T(z)V_\Delta(w,\bar{w}) = \frac{\Delta}{(z-w)^2}V_\Delta(w,\bar{w}) + \frac{1}{z-w}\partial_w V_\Delta(w,\bar{w}) + O(1)\,. \tag{1.20}$$

The expressions (1.15), (1.17) and (1.18),(1.20), as well as most of the following considerations have dual formulae that involve the field $\bar{T}(\bar{z})$ and its modes $\bar{L}_n$. In particular one can also define the anti-holomorphic conformal dimension $\overline{\Delta}$.

Plugging the OPE (1.20) in (1.15), one obtains the infinitesimal transformation law for the primary fields :

$$\delta_\epsilon V_\Delta(z,\bar{z}) = \epsilon(z)\partial_z V_\Delta(z,\bar{z}) + \Delta\epsilon'(z)V_\Delta(z,\bar{z})\,, \tag{1.21}$$

(and an analogous expression for the second chirality). From this we deduce the transformation law of a primary field under a generic conformal transformation $z \to w(z)$ (together with $\bar{z}\to\bar{w}(\bar{z})$) :

$$V_\Delta(z,\bar{z}) = V_\Delta(w,\bar{w})\left(\frac{dw}{dz}\right)^{\Delta}\left(\overline{\frac{dw}{dz}}\right)^{\overline{\Delta}}\,. \tag{1.22}$$

Let us now turn to the field T. The most general infinitesimal conformal transformation that we can write is [12] :

$$\delta_\epsilon T(z) = \epsilon(z)\partial_z T(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\epsilon'''(z)\,, \tag{1.23}$$

where in the last term we have introduced the parameter c, which will be referred to as the *central charge* of theory. This expression is equivalent the OPE between T and itself :

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + O(1)\,. \tag{1.24}$$

Notice, that from this OPE we read that T is not a primary field, since the series does not truncate at the term $\frac{1}{(z-w)^2}$, T will be therefore termed a *quasi-primary* field. Moreover it is clear that the conformal dimension of T is 2 : $L_0 T = 2T$. Finally (1.24) implies the following behavior of the stress energy tensor at infinity :

$$T(z) = O\left(\frac{1}{z^4}\right)\,, \quad z\to\infty\,. \tag{1.25}$$

Then, computing the action of $T(z)T(w)V(u,\bar{u})$ we can deduce the commutation relations of the modes of stress energy tensor. They satisfy the celebrated Virasoro algebra $\mathfrak{V}$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,. \tag{1.26}$$

Exactly the same analysis holds for the algebra of the modes $\bar{L}_n$. The full symmetry algebra of a CFT is then $\mathfrak{V} \times \overline{\mathfrak{V}}$, made of two copies of the Virasoro algebra, one for each chirality. Since for our needs the calculations for both chiralities will be identical, we will only speak about $\mathfrak{V}$ and drop the dependence of primary fields on the variables $\bar{z}$ most of the time. Another important consequence of (1.23) is the transformation law of T under global conformal transformations :

$$T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w, z\}\,, \tag{1.27}$$

where $\{w, z\}$ is the Schwarzian derivative given by :

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 . \tag{1.28}$$

We will apply this transformation rule to calculate the stress energy tensor on the cylinder.

Ward-Takahashi identities. Now that we have identified the structure of the primary fields in $\mathcal{S}$, one can look at the form of the correlation functions in a CFT. To understand the effects of conformal transformations on correlation functions we must consider $\langle T(z)V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle$. This should be a meromorphic function of z, with the poles located at the points w_i and with singularities given by (1.20). Hence we deduce the relation :

$$\langle T(z)V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle = \sum_{k=1}^{n}\left(\frac{\Delta_k}{(z-w_k)^2} + \frac{\partial_{w_k}}{z-w_k}\right)\langle V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle\ , \tag{1.29}$$

This formula is termed the *(local) Ward-Takahashi identity*, and from the knowledge of (1.24) it is possible to write similar expressions for any correlation function of the form

$$\langle T(z_1)...T(z_m)V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle\ . \tag{1.30}$$

They will be presented later in the case of the CFT on a cylinder. From the Ward-Takahashi identities, one can compute correlation functions of more complex fields. Using the inversion formula

$$\langle L_{-m}V_\Delta(z)...\rangle = \oint_{c_z}\frac{du}{2\pi i}(u-z)^{-m+1}\langle T(u)V_\Delta(z)...\rangle\ , \tag{1.31}$$

we get the following correlation function :

$$\begin{aligned}&\langle L_{-m}V_\Delta(z)V_{\Delta_1}(z_1)...V_{\Delta_n}(z_n)\rangle = \\ &= \sum_{k=1}^{n}\left(\frac{(m-1)\Delta_k}{(z_i-z)^m} - \frac{1}{(z_i-z)^{m-1}}\partial_{z_i}\right)\langle V_\Delta(z)V_{\Delta_1}(z_1)...V_{\Delta_n}(z_n)\rangle\ .\end{aligned} \tag{1.32}$$

Finally taking the limit $z \to \infty$ of (1.29) and using (1.25) we recover the three *global* Ward-Takahashi identities :

$$\sum_{i=1}^{n}\partial_{w_i}\langle V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle = 0\,, \tag{1.33}$$

$$\sum_{i=1}^{n}(w_i\partial_{w_i} + \Delta_i)\langle V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle = 0\,, \tag{1.34}$$

$$\sum_{i=1}^{n}(w_i^2\partial_{w_i} + 2\Delta_i w_i)\langle V_{\Delta_1}(w_1)...V_{\Delta_n}(w_n)\rangle = 0\,. \tag{1.35}$$

They are named "global" since they describe the invariance of the correlation functions of primary fields V_{Δ_k} under the global conformal transformations on the Riemann sphere. These transformations form the $SL(2,\mathbb{C})$ subgroup of the full conformal group. The global Ward-Takahashi identities can be seen as the invariance of the correlation functions under translations (1.33), rotations (1.34) and special conformal transformations (1.35).

The global Ward identities are important since they fix strong constraints on the correlation functions of primary fields. It follows immediately that the one point functions vanish on the plane, due to the translational symmetry (1.33) :

$$\langle V_\Delta(0)\rangle = 0\,. \tag{1.36}$$

Furthermore, they allow to determine the form of the two and three point functions. Set $z_{ij} = z_i - z_j$, the full result is :

$$\langle V_{\Delta_1}(z_1,\bar{z}_1)V_{\Delta_2}(z_2,\bar{z}_2)\rangle = \frac{D_{1,2}\ \delta_{\Delta_1,\Delta_2}\delta_{\bar{\Delta}_1,\bar{\Delta}_2}}{z_{12}^{2\Delta_1}\bar{z}_{12}^{2\bar{\Delta}_2}}\,, \tag{1.37}$$

$$\langle V_{\Delta_1}(z_1,\bar{z}_1)V_{\Delta_2}(z_2,\bar{z}_2)V_{\Delta_3}(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{\kappa_3}z_{13}^{\kappa_2}z_{23}^{\kappa_1}\bar{z}_{12}^{\bar{\kappa}_3}\bar{z}_{13}^{\bar{\kappa}_2}\bar{z}_{23}^{\bar{\kappa}_1}}\,, \tag{1.38}$$

where :

$$\kappa_i = \Delta - 2\Delta_i\,, \quad \Delta = \Delta_1 + \Delta_2 + \Delta_3\,, \tag{1.39}$$

and similarly for $\bar{\kappa}_i$. The constant $D_{1,2}$ depends only on the normalization of the fields, we will work with $D_{1,2} = 1$, calling this the *conformal normalization* of the two point function. On the other hand, the functions C_{123} are extremely important. From the normalization of the two point function, it follows that they are equal to the coefficients of the field V_{Δ_3} in the OPE between the fields V_{Δ_1} and V_{Δ_2}. From dimensional reasons the latter is written

$$V_{\Delta_1}(z,\bar{z})V_{\Delta_2}(0) = \sum_k C_{1,2}^k z^{\Delta_k-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_k-\bar{\Delta}_1-\bar{\Delta}_2}V_k(0)\,, \tag{1.40}$$

where the field V_k contains all the contributions of the primary field V_{Δ_k} and of its *descendants*, that will be defined in the next paragraph. The correspondence between the three-point function and OPE coefficients is given by :

$$C_{12}^3 = C_{123}\,. \tag{1.41}$$

Finally the evaluation of (1.37) and (1.38) at the points ∞ and 0 gives (instead of taking naively $z \to \infty$ we should first perform the conformal mapping $z \to \frac{1}{z}$ and then send $z \to 0$) :

$$\langle V_{\Delta_1}(\infty)V_{\Delta_2}(0)\rangle = \delta_{\Delta_1,\Delta_2}\delta_{\bar{\Delta}_1,\bar{\Delta}_2}\,, \tag{1.42}$$

$$\langle V_{\Delta_1}(\infty)V_{\Delta_2}(z,\bar{z})V_{\Delta_3}(0\rangle = C_{1,2,3}z^{\Delta_1-\Delta_2-\Delta_3}\bar{z}^{\bar{\Delta}_1-\bar{\Delta}_2-\bar{\Delta}_3}\,. \tag{1.43}$$

Therefore the knowledge of the three point functions answers the question about the structure constants of the operator algebra in the CFT context.

Representation Theory and CFT. Finally, we would like to give some representation theoretic interpretation of CFT. As we have seen, a CFT contains primary fields V_Δ that satisfy (1.18). But this condition is exactly the property that V_Δ is considered as a highest weight vector for $\mathfrak{V}$, with weight Δ. Keeping in mind this fact, we shall use the following obvious notation, representing the primary field V_Δ by a state [2] :

$$V_\Delta \equiv |\Delta\rangle\ . \tag{1.44}$$

From this highest weight vector, it is possible to generate a Verma module $\mathcal{V}_\Delta$ for $\mathfrak{V}$ by acting with modes with negative index :

$$\mathcal{V}_\Delta = \mathrm{Vect}\left(L_{-n_1}...L_{-n_p}\,|\Delta\rangle\ , n_k \geq 0\right)\ . \tag{1.45}$$

Elements in $\mathcal{V}_\Delta$ can be classified by their level l, which is defined for a generic element as

$$l = \sum_k n_k\,. \tag{1.46}$$

From the field theoretic point of view, this is the same as acting on a primary field with the modes by the formula (1.17) :

$$L_{-n_1}...L_{-n_p}\,|\Delta\rangle \equiv L_{-n_1}...L_{-n_p}V_\Delta\,, \quad n_k \geq 0\,. \tag{1.47}$$

We shall call these fields *descendants* of the primary field V_Δ, at *level* l. The vector space generated by all descendants of a primary field (that is the Verma module $\mathcal{V}_\Delta$) will be called a *conformal family* in the QFT language, and denoted by $[V_\Delta]$.
Let us introduce two more important quantities : the *spin* s and the *scaling* or *anomalous dimension*

[2] This is a manifestation of the *state-operator correspondence* [17, 18].

d : they are respectively the eigenvalues of the operators $L_0 - \bar{L}_0$ and $L_0 + \bar{L}_0$ for a given field. For a primary field V_Δ :

$$s = \Delta - \overline{\Delta}\,, \quad d = \Delta + \overline{\Delta}\,. \tag{1.48}$$

For generic values of c and of the conformal dimension Δ, the Verma module $\mathcal{V}_\Delta$ is an irreducible representation of $\mathfrak{V}$. The spectrum of any CFT is then classified by the representation of the Virasoro algebra, and can be written formally as a sum of the above mentioned Verma modules (or conformal families) [3]:

$$\mathcal{S} = \bigoplus_\Delta \mathcal{V}_\Delta\,. \tag{1.49}$$

As usual in representation theory, a good numerical tool for the study of the spectrum are the characters, or the partition function in the CFT language. From (1.49) the partition function of $\mathcal{S}$ is :

$$Z_\mathcal{S}(t) = \sum_\Delta \chi_\Delta(t)\,, \quad \chi_\Delta(t) = \mathrm{Tr}_{\mathcal{V}_\Delta}\left(t^{L_0 - \frac{c}{24}}\right) = \sum_{n=0}^{\infty} \dim(n+\Delta) t^{n+\Delta-\frac{c}{24}}\,, \tag{1.50}$$

where $\dim(n+\Delta)$ counts the dimension of the vector space of the descendants of V_Δ at level n.

1.2.2 CFT on the cylinder.

In the following we will use a lot the CFT defined on a cylinder. Usually we will consider the cylinder with the following boundary conditions : two primary fields with respective dimensions $\Delta_\pm$ will be located at the infinities. At the origin we insert a generic operator $\mathcal{O}$, this is represented on the picture 1.1.

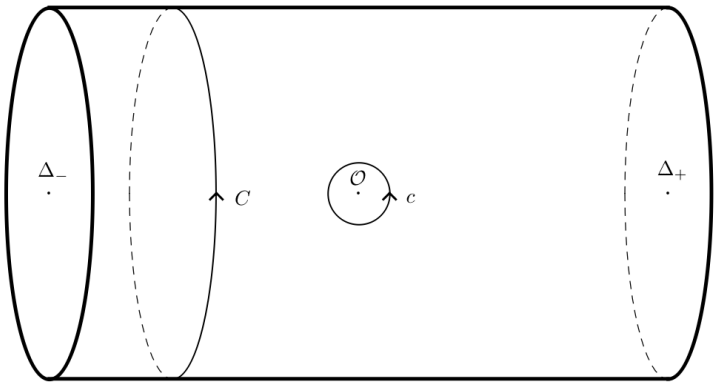

Figure 1.1: Insertion of a local operator $\mathcal{O}$ on a cylinder with boundary conditions $\Delta_\pm$.

Moreover we will distinguish two types of cycles on the cylinder : cycles c that are contractible and encircle the origin, and cycles C that are wrapping around the cylinder.

The passage from the plane with coordinate z to the cylinder with coordinate x is made by the transformation :

$$z = e^{ax}\,, \tag{1.51}$$

where a is a parameter characterizing the radius of the cylinder. First we need to define the stress energy tensor on the cylinder. We have $\{x, z\} = \frac{1}{2z^2}$. With the help of (1.27) we then obtain :

$$T(x) = \left(\frac{dz}{dx}\right)^2 \left(T(z) - \frac{c}{12}\{x, z\}\right) = a^2 \left(\sum_n e^{-anx} L_n - \frac{c}{24}\right)\,. \tag{1.52}$$

[3] We write here only the holomorphic part for simplicity, the complete Hilbert space of a CFT is

$$\mathcal{S} = \bigoplus_{\Delta,\bar{\Delta}} \mathcal{V}_\Delta \otimes \mathcal{V}_{\bar{\Delta}}$$

To make contact with [4], we take $a = -\frac{1}{R}$, and split the stress energy tensor $T(x)$:

$$T(x) = \frac{1}{R^2}\left(\sum_n e^{\frac{nx}{R}} L_n - \frac{c}{24}\right) = T_+(x) + T_-(x)\,, \tag{1.53}$$

with

$$T_+(x) = \frac{1}{R^2}\left(\sum_{n\geq 1} L_n e^{nx} + \frac{L_0}{2} - \frac{c}{48}\right)\,, \quad T_-(x) = \frac{1}{R^2}\left(\sum_{n\geq 1} L_{-n} e^{-nx} + \frac{L_0}{2} - \frac{c}{48}\right)\,. \tag{1.54}$$

From the expressions (1.54) the boundary conditions can be rewritten in terms of the stress energy tensor :

$$\lim_{x\to\pm\infty} T(x) = \frac{1}{R^2}\left(\Delta_\pm - \frac{c}{24}\right) \tag{1.55}$$

Define the local action of the stress energy tensor on a field V by :

$$(\mathbf{l}_n V)(y) = \oint_{c_y} \frac{dx}{2\pi i}(x-y)^{n+1}\mathcal{T}\left(T(x)V(y)\right)\,. \tag{1.56}$$

where $\mathcal{T}$ is the time ordering symbol defined as follows :

$$\mathcal{T}(V(x)W(y)) = \begin{cases} V(x)W(y), & \text{if } x < y\,, \\ W(y)V(x), & \text{if } y < x\,, \end{cases} \tag{1.57}$$

and x, y denote the coordinates along the non-compact direction of the cylinder.

To perform explicit calculations of one point functions on the cylinder (for example one point functions of descendant fields $\mathbf{l}_{-n}V_\Delta$), we need the commutation relations between the two parts of the stress energy tensor $T_\pm$ and the primary field V_Δ. They are obtained from the commutation relation of the modes L_n of the Virasoro algebra with those of the primary field [4]. First we define the following basic function (B_n are the Bernoulli numbers) :

$$\chi(z) = \frac{1}{2}\mathrm{cth}\left(\frac{z}{2R}\right) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}\left(\frac{z}{R}\right)^{2n-1}\,. \tag{1.60}$$

Then (with the notation $\chi^{(n)} = \partial_z^n \chi(z)$) the following relations hold :

$$[T_+(x), V_\Delta(y)] = \frac{\partial_y V_\Delta(y)}{R}\chi(x-y) - \frac{\Delta V_\Delta(y)}{R}\chi'(x-y)\,, \quad x < y\,. \tag{1.61}$$

$$[V_\Delta(y), T_-(x)] = \frac{\partial_y V_\Delta(y)}{R}\chi(x-y) - \frac{\Delta V_\Delta(y)}{R}\chi'(x-y)\,, \quad x > y\,. \tag{1.62}$$

In addition, we have :

$$[T_+(x), T(y)] = \frac{\partial_y T(y)}{R}\chi(x-y) - \frac{2T(y)}{R}\chi'(x-y) - \frac{c}{12R}\chi'''(x-y)\,, \quad x < y\,, \tag{1.63}$$

$$[T(y), T_-(x)] = \frac{\partial_y T(y)}{R}\chi(x-y) - \frac{2T(y)}{R}\chi'(x-y) - \frac{c}{12R}\chi'''(x-y)\,, \quad x > y\,. \tag{1.64}$$

With these relations in hand, we can explicitly calculate the action of the modes $\mathbf{l}_m$ applying (1.56). To simplify our calculations we will from now *compute the one point functions* in the setting $R = 1$. A generic value of R can be restored by taking an appropriate conformal transformation. For example we can easily obtain :

$$\mathbf{l}_0 V_\Delta = \Delta V_\Delta\,, \quad \mathbf{l}_{-1} V_\Delta(y) = \partial_y V_\Delta(y)\,. \tag{1.65}$$

[4] On the plane, expanding the primary field V_Δ in modes :

$$V_\Delta(z) = \sum_n z^{-n-\Delta} V_n\,, \tag{1.58}$$

the OPE (1.20) implies that :

$$[L_m, V_n] = (m(\Delta-1) - n)V_{m+n}\,. \tag{1.59}$$

They are special cases of the more general result (with $n \geq 1$) :

$$\langle (\mathbf{l}_{-n} V_\Delta)(y) \rangle = \frac{\delta_{n,2}}{2} \left(\Delta_+ + \Delta_- - \frac{c}{12} \right) - \delta_{n,E} \Delta \frac{B_n}{n(n-2)!} + \delta_{n,O} \left(\Delta_+ - \Delta_- \right) \frac{B_{n-1}}{(n-1)!} . \tag{1.66}$$

First, the symbol $\delta_{n,O}$ equals 1 if n is odd and 0 otherwise, a similar definitions holds for $\delta_{n,E}$ where E means even. Second, we use the following short-hand notation for one point functions :

$$\langle \mathbf{l}_{-n} \mathcal{O}(y) \rangle = \frac{\langle \mathbf{l}_{-n} \mathcal{O}(y) \rangle_{\Delta_+, \Delta_-}}{\langle \mathcal{O}(y) \rangle_{\Delta_+, \Delta_-}} , \tag{1.67}$$

meaning that we consider the one point functions in presence of the two primary fields $|\Delta_\pm\rangle$ at infinity. The proof is as follows : split the contour c_y on two small half-circular parts located on the left (c_y^-) and right (c_y^+) of y :

$$c_y = c_y^- + c_y^+ . \tag{1.68}$$

Then :

$$\begin{aligned}
(\mathbf{l}_{-n} V_\Delta)(y) &= \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \mathcal{T}(T(x) V_\Delta(y)) \\
&= \oint_{c_y^-} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} T(x) V_\Delta(y) + \oint_{c_y^+} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} V_\Delta(y) T(x) . \\
&= \oint_{c_y^-} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \Big[T_-(x) V_\Delta(y) + V_\Delta(y) T_+(x) + [T_+(x), V_\Delta(y)] \Big] \\
&+ \oint_{c_y^+} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \Big[V_\Delta(y) T_+(x) + T_-(x) V_\Delta(y) + [V_\Delta(y), T_-(x)] \Big] .
\end{aligned} \tag{1.69}$$

The application of (1.61),(1.62) gives :

$$\begin{aligned}
(\mathbf{l}_{-n} V_\Delta)(y) &= \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \left(T_-(x) V(y) + V(y) T_+(x) \right) \\
&+ \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \Big[\partial_y V_\Delta(y) \chi(x-y) - \Delta V_\Delta(y) \chi'(x-y) \Big] \\
&= \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \left(T_-(x) V_\Delta(y) + V_\Delta(y) T_+(x) \right) \\
&+ \partial_y V_\Delta(y) \mathrm{Res}_{z \to 0} \Big[\frac{1}{z^{n-1}} \chi(z) \Big] - \Delta V_\Delta(y) \mathrm{Res}_{z \to 0} \Big[\frac{1}{z^{n-1}} \chi'(z) \Big] \\
&= \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{n-1}} \left(T_-(x) V_\Delta(y) + V_\Delta(y) T_+(x) \right) \\
&+ \partial_y V_\Delta(y) \delta_{n,O} \frac{B_{n-1}}{(n-1)!} - \Delta V_\Delta(y) \delta_{n,E} \frac{B_n}{n(n-2)!} .
\end{aligned} \tag{1.70}$$

Now, let us define another deformation of the contours : we can split c_y into two large circular contours $C_\pm$ (see 1.1) located on the right and on the left of y respectively and running in opposite directions :

$$c_y = C_+ - C_- . \tag{1.71}$$

It implies that :

$$\begin{aligned}
\mathbf{l}_{-1} V_\Delta(y) &= \oint_{c_y} \frac{dx}{2\pi i} \mathcal{T}(T(x) V_\Delta(y)) = - \oint_{C_-} \frac{dx}{2\pi i} T(x) V_\Delta(y) + \oint_{C_+} \frac{dx}{2\pi i} V_\Delta(y) T(x) \\
&= \left(\Delta_+ - \Delta_- \right) V_\Delta(y) .
\end{aligned} \tag{1.72}$$

To get the last line we pushed the contours to infinity $C_\pm \to \pm\infty$. Applying (1.72) and the boundary conditions (1.55) to the previous calculations (1.70), one recovers (1.66).

The multiple action of the modes is calculated by iteration, we will see more examples in the section 4.1.2. Another illustration that will be useful is the one point function of the normal ordered square of T. By definition it is given by

$$\begin{aligned}
(TT)(y) = (\mathbf{l}_{-2} T)(y) &= \oint_{C_y} \frac{dx}{2\pi i} \frac{\mathcal{T}(T(x) T(y))}{x-y} , \\
&= T_-(y) T(y) + T(y) T_+(y) - \frac{1}{6} T(y) + \frac{c}{1440} .
\end{aligned} \tag{1.73}$$

With this formula one can first calculate the one point function of this field (there is here no normalization by a primary field like in (1.67)) :

$$\langle (TT)(y)\rangle = \left(\left(\Delta_+ - \frac{c}{24}\right)^2 + \frac{1}{6}\left(\Delta_+ - \frac{c}{24}\right) + \frac{c}{1440}\right)\delta_{\Delta_-,\Delta_+}\,, \tag{1.74}$$

as well as the following expression of the Virasoro modes (the notation I_3 will be clarified latter) :

$$I_3 = \int_C (TT)(y)\frac{dy}{2\pi i} = 2\sum_{n=1}^{\infty} L_{-n}L_n + \left(L_0 - \frac{c}{24}\right)^2 - \frac{1}{6}\left(L_0 - \frac{c}{24}\right) + \frac{c}{1440}\,. \tag{1.75}$$

The formulae like (1.66) will prove to be very useful when we will calculate the one point functions of descendants fields on the cylinder. However, in the case of an action of multiple modes on V_Δ, the above method is not very efficient to perform the calculations. There is a more adequate way to compute the one point functions by using the Ward-Takahashi identities in the case of the cylinder. First start with the OPE on the cylinder obtained from the commutation relations (1.61), (1.62), (1.63), (1.64) in [4] :

$$T(x)T(y) = \frac{\partial_y T(y)}{R}\chi(x-y) - \frac{2T(y)}{R}\chi'(x-y) - \frac{c}{12R}\chi'''(x-y) + O(1)\,, \tag{1.76}$$

$$T(x)V_\Delta(y) = \frac{\partial_y V_\Delta(y)}{R}\chi(x-y) - \frac{\Delta V_\Delta(y)}{R}\chi'(x-y) + O(1)\,. \tag{1.77}$$

Then the Ward-Takahashi identities on a cylinder are given by (in the second line the $\widehat{\ }$ means the omission of the j-th term) :

$$\begin{aligned}
&\langle T(x_k)...T(x_1)V_\Delta(y)\rangle = \\
&-\frac{c}{12R}\sum_{j=2}^{k}\chi'''(x_1-x_j)\,\langle T(x_k)...\widehat{\ }...T(x_2)\Delta(y)\rangle \\
&+\left[\sum_{j=2}^{k}\left(-\frac{2}{R}\chi'(x_1-x_j) + \frac{1}{R}(\chi(x_1-x_j)-\chi(x_1-y))\frac{\partial}{\partial x_j}\right) - \frac{\Delta}{R}\chi'(x_1-y)\right. \\
&\left.+(\Delta_+-\Delta_-)\frac{1}{R^2}\chi(x_1-y) + \frac{1}{2R^2}(\Delta_++\Delta_-) - \frac{c}{24R^2}\right]\langle T(x_k)...T(x_2)V_\Delta(y)\rangle\,.
\end{aligned} \tag{1.78}$$

Now one can apply this to calculate any one point function $\langle \mathbf{l}_{-n_k}...\mathbf{l}_{-n_1}V_\Delta\rangle$ by iterating the formula (1.78) and taking appropriate integrals :

$$\langle \mathbf{l}_{-n_k}...\mathbf{l}_{-n_1}V_\Delta(y)\rangle = \oint_{c_{x_k}}\frac{dx_k}{2\pi i(x_k-y)^{n_k-1}}...\oint_{c_{x_1}}\frac{dx_1}{2\pi i(x_1-y)^{n_1-1}}\,\langle T(x_k)...T(x_1)V_\Delta(y)\rangle\,,$$

We shall also use the notation :

$$\oint_{c_{x_j}} \boldsymbol{dx_j} = \oint_{c_{x_j}}\frac{dx_j}{2\pi i(x_j-y)^{n_j-1}}\,. \tag{1.79}$$

In the what follows, we will write similar equations in the context of Supersymmetric CFT.

Effective central charge. Finally, lets us give two more definitions. The *Hamiltonian operator* H in a CFT is defined by [5] :

$$H = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)\,. \tag{1.80}$$

and the *Momentum operator* P is :

$$P = \frac{2\pi}{iR}(L_0 - \bar{L}_0)\,. \tag{1.81}$$

[5] To match the notations of [19], we take $a = \frac{2\pi}{R}$ in (1.51). Then, on the cylinder we have $w = \tau + i\sigma\,,\quad \sigma + R = \sigma$. To get the Hamiltonian, we naturally integrate the energy density : $\frac{1}{2\pi}\int_0^R T_{\tau\tau}d\sigma = \frac{1}{2\pi}\int_0^R (T_{ww}+T_{\bar{w}\bar{w}})d\sigma = \frac{1}{2\pi}\left(\frac{2\pi}{R}\right)^2\oint\left(Tz^2+\bar{T}\bar{z}^2-\frac{c}{12}\right)\frac{R}{2\pi i}\frac{dz}{z} = \frac{2\pi}{R}\left(L_0+\bar{L}_0-\frac{c}{12}\right)$. For the momentum a similar calculation holds starting with $\frac{1}{2\pi}\int_0^R T_{\tau\sigma}d\sigma = \frac{2\pi}{iR}(L_0-\bar{L}_0)$.

The ground state energy E_0 is the minimal eigenvalue of H :

$$H = -\frac{\pi}{6R}\left(c - 12(L_0 + \bar{L}_0)\right) \implies E_0 = -\frac{\pi}{6R}(c - 24\Delta_{\min})\,. \tag{1.82}$$

We introduce the *effective central charge* : $\widetilde{c} = c - 24\Delta_{\min}$ and get :

$$E_0 = -\frac{\pi\widetilde{c}}{6R}\,. \tag{1.83}$$

This formula can be interpreted as a Casimir energy : the non zero energy of the ground state in a finite volume. One recovers $E_0 = 0$ as expected on the plane when we take $R \to \infty$. Moreover, in the course of the identification between statistical models at criticality and their continuum CFT limit, the ground state energy E_0 is a universal term obtained in the expansion of the free energy of the statistical model with respect to the size of the system. We will tell more about this later.

Now we are ready to present explicit examples of CFTs, that will be important for future investigations.

1.2.3 Liouville CFT

A particular theory that will be useful is the *Liouville CFT* given by the Lagrangian [16],[20] [6] :

$$\mathcal{A}_{\rm L} = \int \left(\frac{1}{4\pi}\partial_z\varphi(z,\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z}) + \mu e^{b\varphi(z,\bar{z})}\right)\frac{idz\wedge d\bar{z}}{2}\,. \tag{1.84}$$

The stress energy tensor

$$T = -\frac{1}{4}(\partial_z\varphi(z,\bar{z}))^2 + \frac{Q}{2}\partial_z^2\varphi(z,\bar{z})\,, \quad Q = b + \frac{1}{b}\,,$$

ensures the conformal invariance of this theory, with the central charge :

$$c = 1 + 6Q^2\,. \tag{1.85}$$

What is the spectrum of Liouville CFT ? The primary fields in this theory are given by the expressions $V_a(z,\bar{z}) = e^{a\varphi(z,\bar{z})}$ and are parametrized by a *weight* a. $V_a(z,\bar{z})$ has the conformal dimension [7] :

$$\Delta_a = a(Q-a)\,. \tag{1.86}$$

Notice that we will also use another parametrization of the conformal dimension. Setting

$$a = \frac{Q}{2} + iP\,, \tag{1.87}$$

then

$$\Delta_a = \frac{Q^2}{4} + P^2\,. \tag{1.88}$$

One very important result is that the 3 point function for this theory is explicitly known, and has been obtained in [21],[22]. It is the famous DOZZ formula, given by :

$$C(a_1,a_2,a_3) = \left(\pi\mu\gamma(b^2)^2b^{2-2b^2}\right)^{\frac{Q-\sum_i a_i}{b}}\Upsilon(2a_1)\times \\ \frac{\Upsilon_0\Upsilon(2a_2)\Upsilon(2a_3)}{\Upsilon(a_1+a_2+a_3-Q)\Upsilon(a_1+a_2-a_3)\Upsilon(a_2+a_3-a_1)\Upsilon(a_3+a_1-a_2)}\,, \tag{1.89}$$

with $\Upsilon_0 = \frac{d\Upsilon}{dx}\big|_{x=0}$. The function Υ is a solution of the functional equations :

$$\frac{\Upsilon(x+b)}{\Upsilon(x)} = \gamma(bx)b^{1-2bx}\,, \quad \frac{\Upsilon(x+1/b)}{\Upsilon(x)} = \gamma(x/b)b^{-1+2x/b}\,. \tag{1.90}$$

[6] Recall that in our conventions $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\frac{idz\wedge d\bar{z}}{2} = dx\wedge dy$. This implies that a vertex operator in the free boson theory has the "natural" conformal dimension : $\Delta(e^{h\varphi}) = h^2$.

[7] The conformal dimensions of vertex operators in the free boson theory have to be shifted in such a way that the exponential interaction $e^{b\varphi}$ has dimension $\Delta(e^{b\varphi}) = 1$. Hence the above formula, which can be demonstrated by algebraic methods when we make contact with the Heisenberg algebra.

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{1.91}$$

Υ is a meromorphic function, that admits an integral representation in the range $0 < \mathrm{Re}(x) < Q$:

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t}\left\{ \left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\mathrm{sh}^2\left((\frac{Q}{2} - x)\frac{t}{2}\right)}{\mathrm{sh}(\frac{tb}{2})\mathrm{sh}(\frac{t}{2b})} \right\}. \tag{1.92}$$

This explicit formula for Υ implies the symmetry property

$$\Upsilon(x) = \Upsilon(Q-x)\,. \tag{1.93}$$

In turn, it leads to the following remarkable property for the structure function [21] :

$$C(Q-a_1, a_2, a_3)R(a_1) = C(a_1, a_2, a_3)\,. \tag{1.94}$$

where R is called the Liouville reflection amplitude. To calculate R one can remark that the second line of (1.89) is invariant under the reflection $a_1 \to Q - a_1$. This implies

$$R(a) = \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2}\gamma(2ab - b^2)\gamma(2ab^{-1} - b^{-2} - 1)\,. \tag{1.95}$$

Indeed

$$\begin{aligned} R(a) &= \frac{C(a, a_2, a_3)}{C(Q-a, a_2, a_3)} = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2a}{b}} \frac{\Upsilon(2a)}{\Upsilon(2a-Q)} \\ &= \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2a}{b}} \frac{\Upsilon(2a)}{\Upsilon(2a-b)} \frac{\Upsilon(2a-b)}{\Upsilon((2a-b)-b^{-1})} \\ &= \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2}\gamma(2ab - b^2)\gamma(2ab^{-1} - b^{-2} - 1)\,. \end{aligned} \tag{1.96}$$

We will also use the expression of the reflection amplitude in terms of the quasi-momentum P. First define

$$S(ia - iQ/2) = R(a)\,. \tag{1.97}$$

Recalling the relation (1.87), it implies :

$$S(P) = -\left(\pi\mu\gamma(b^2)\right)^{-\frac{2iP}{b}} \frac{\Gamma(1 + \frac{2iP}{b})\Gamma(1 + 2iPb)}{\Gamma(1 - \frac{2iP}{b})\Gamma(1 - 2iPb)}\,. \tag{1.98}$$

The property (1.94) of the Liouville three point function is used to infer the following reflection relation among the fields (holding inside correlation functions) :

$$e^{a\varphi(z,\bar{z})} = R(a)e^{(Q-a)\varphi(z,\bar{z})}\,. \tag{1.99}$$

The Liouville CFT possesses a special position among the CFTs. Indeed, since its central charge takes continuous values, it is possible to restrict it to specific Conformal Field Theories. The values of the three point functions in these particular cases can be obtained by appropriately restricting the general DOZZ formula (1.89).

Minimal models Let us introduce some more general notions. A field ξ of a CFT is called a *singular vector*, if it is a descendant of some primary field V_Δ that satisfy :

$$L_n\xi = 0\,, \quad n > 0\,. \tag{1.100}$$

Basically this means that the representation $[V_\Delta]$ of $\mathfrak{V}$ supported by the highest weight vector V_Δ is reducible, and we need to factor out the descendants of the field ξ to recover an irreducible representation. But how to spoil singular vectors ? The following result answers this question.

Consider a general CFT with the parametrization of the central charge $c = 1 + 6Q^2$, $Q = b + b^{-1}$ and of the dimensions $\Delta_a = a(Q-a)$. If the parameter a is equal to :

$$a_{mn} = \frac{(1-m)b^{-1} + (1-n)b}{2}\,, \quad m, n \in \mathbb{N}\,, \tag{1.101}$$

then there is a singular vector at level $m \times n$, which is a descendant of the primary field with conformal dimension $\Delta_{a_{mn}}$. This special primary field is often called a *degenerate* field and we will denote it :

$$V_{mn} = V_{\Delta_{a_{mn}}} . \tag{1.102}$$

We can then try to construct theories containing only degenerate primary fields. This is very meaningful, since each module $[V_{mn}]$ becomes finite dimensional after the factorization by its submodule generated by the singular descendant. Moreover, the factorization condition $\xi = 0$ imposes constraints on the OPE of V_{mn} with other fields of the theory. The construction of a CFT out of a finite number of degenerate fields, with a closed operator algebra under the OPE procedure leads to the the so-called *Minimal Models*. They are denoted $M_{p\,q}$ where $p, q \in \mathbb{N}$ with the condition $p \wedge q = 1$. To fulfill the above restrictions (finite number of primary fields and closed operator algebra), the central charge and the conformal dimensions have to be equal to :

$$c = 1 - 6\frac{(p-q)^2}{pq} , \tag{1.103}$$

$$\Delta_{rs} = \frac{(pr - qs)^2 - (p-q)^2}{4pq} , \quad r \leq p , \quad s \leq q . \tag{1.104}$$

Among the minimal models we single out the *unitary* minimal models defined by $q = p + 1$ and denoted simply M_p. They can be endowed with a Hermitian structure, giving rise to a unitary theory. In such models the field $V_{1,3}$ will play a special role, its conformal dimension is :

$$\Delta_{1,3} = 1 - \epsilon < 1 , \quad \epsilon = \frac{2}{p+1} , \quad \forall p . \tag{1.105}$$

The field $V_{1,3}$ has the smallest (except the identity field $V_{1,1}$ with $\Delta_{1,1} = 0$) conformal dimension. Remark that we can formally pass from the Liouville CFT parametrized by b to the minimal model M_p by setting :

$$b^2 = -\frac{p}{p+1} . \tag{1.106}$$

The conformal dimension of the Liouville field $V_{-b} = e^{-b\varphi(z,\bar{z})}$ is then :

$$\Delta_{-b} = -b(Q+b) = -(1+2b^2) = 1 - \frac{2}{p+1} , \tag{1.107}$$

and hence, it can be interpreted as the $V_{1,3}$ field in the Liouville context. This will be of some importance latter.

Partial conclusion. We have seen that for two dimensional CFTs, all the information needed to describe the field theory is accessible :

1. The spectrum is classified by the representations of the Virasoro algebra (1.49).

2. The structure constants are given by the DOZZ formula (or its variations) (1.89).

3. On the plane the one point functions vanish. On the cylinder with the boundary conditions (1.55), the one point functions of primary fields are given (as a result of the conformal mapping (1.51)) by the plane 3-point functions of the DOZZ formula. For the one point functions of descendant fields we have the formulae from the Ward-Takahashi identities (1.78).

Therefore, we can now move to more complex theories that will involve a length scale and hence massive particles. Bearing in mind the example of the ssG model, it is natural to look at perturbations of CFT.

1.3 Perturbed CFT

The 2d Conformal Field Theories describe critical phenomena in 2d statistical systems. If we want to consider massive Quantum Field Theories, or equivalently theories with a length scale, we should therefore focus instead on the vicinity of such critical points. This approach to CFT, termed the *Perturbation of Conformal Field Theory* (PCFT) was initiated in the paper [23] by Al. Zamolodchikov. A very complete review on this (as well as other forthcoming) topic can be found in [24]. We are going to follow closely this two sources in this section.

The perturbation of CFT is significant for us since our main model of investigation, the Super sine-Gordon model, will be treated exactly in this set-up. Therefore, in this section we will briefly recall the ideas of conformal perturbation theory, concentrating on the two following objectives :

1. Explain the renormalization of the fields in PCFT. The renormalization in this context will actually allow to consider that the fields in the perturbed model are just deformations of CFT fields. Consequently, one will still have the notion of primary and descendant fields in the PCFT. This will answer the question of the spectrum of such theories.

2. Demonstrate how one can calculate perturbatively the OPE coefficients.

These two points indicate that the single object that is (theoretically) left unknown in dealing with perturbed CFT are precisely the one point functions.

Generalities. We would like here to construct a perturbation theory of some CFT given by the formal action $\mathcal{A}_{\text{CFT}}$. Let us consider a primary field φ in the CFT spectrum (take this field to be spinless $\Delta = \overline{\Delta}$), introduce a coupling constant λ, and define the resulting Quantum Field Theory by :

$$\mathcal{A}_{\text{QFT}} = \mathcal{A}_{\text{CFT}} + \lambda \int \varphi(x) d^2x \,. \tag{1.108}$$

This is not the most general perturbation. It is possible to perturb the CFT by several primary fields, or to make a perturbation by complex composite operators [25]. This triggered a huge scientific activity recently [26, 27]. However, for our needs, that is for dealing with the Supersymmetric sine-Gordon QFT we concentrate on this simplest case. We start by making two fundamental comments. First, as is pointed out in [23], we suppose that the local fields in the perturbed theory are in one to one correspondence with those of the short distance CFT, this will be demonstrated in this section and means that we can still continue to label our fields exactly as in the CFT. The second remark concerns the fact that in order to get exact results it is enough to execute the calculations up to the first order in perturbation theory, the outcomes are supposed to hold at all orders.

It is known from Renormalization Group analysis, that to ensure the production of a new theory from the perturbation, we need the perturbing field φ to be relevant. It means that its anomalous dimension should satisfy :

$$d = 2\Delta < 2 \,. \tag{1.109}$$

We assume that the introduction of a perturbation generates a mass M (see [28, 29, 30] for counterexamples), which is from dimensional reasons related to the coupling constant by[8] :

$$M = C\lambda^{\frac{1}{2-2\Delta}} \,, \tag{1.112}$$

where C is a numerical constant, that can be calculated thanks to the Thermodynamic Bethe Ansatz, this will be done in the section 1.6.

[8] The mass dimension of a field is the same as the anomalous dimension

$$[\varphi] = 2\Delta \,. \tag{1.110}$$

This can be seen for example from the two points function $\langle \varphi\varphi \rangle \sim r^{-4\Delta}$. It implies that the mass dimension of the coupling constant is

$$[\lambda] = 2(1-\Delta) \,, \tag{1.111}$$

since $[\lambda \int \varphi d^2x] = 0$ with $[d^2x] = -2$.

Renormalization. Let us consider the one point function $\langle V(0)\rangle_{\rm QFT}$ of a generic field in the PCFT. Since in the conformal theory we know, in principle, all correlation functions, we can use this information to compute the one point functions in the perturbed theory :

$$\begin{aligned}\langle V(0)\rangle_{\rm QFT} &= \int \mathcal{D}\phi\, V(0)\, e^{-\mathcal{A}_{\rm CFT}-\lambda\int\varphi(w)d^2w} \\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int d^2w_1...d^2w_n\,\langle\widetilde{V}(0)\varphi(w_1)...\varphi(w_n)\rangle_{\rm CFT}\ ,\end{aligned} \tag{1.113}$$

where we explicitly indicated in which theory the correlation functions are calculated. Moreover, we assume that the field V is a deformation of the CFT field $\widetilde{V}$, with conformal dimension Δ_V. If the field φ is such that its conformal dimension is $\Delta > \frac{1}{2}$, then the above integrals suffer from both UV and IR singularities. As we will see shortly, the type of these singularities is very different. The UV singularities can be dealt with by a renormalization of the fields. The IR singularities are more complicated. For the moment, to get finite correlation functions, we need to introduce two cutoffs (respectively UV and IR) : ϵ and R. A general correlation function is then at the first order :

$$\langle X\, V(0)\rangle_{\rm QFT} = \langle X\,\widetilde{V}(0)\rangle_{\rm CFT} - \lambda\int_{\epsilon<|w|<R} d^2w\,\langle X\,\widetilde{V}(0)\,\varphi(w)\rangle_{\rm CFT} + O(\lambda^2)\,. \tag{1.114}$$

The OPE at the CFT level is given by :

$$\varphi(z,\bar{z})V(0) = \sum_k C^k_{\varphi V}(z\bar{z})^{\Delta_k-\Delta_V-\Delta}A_k(0)\,, \tag{1.115}$$

where A_k are a set of fields with dimensions Δ_k. Recall a trivial formula that we will write explicitly :

$$\int_\epsilon^R (z\bar{z})^\alpha d^2z = \int_0^\pi\int_\epsilon^R r^{2\alpha}rdrd\theta = \frac{\pi}{\alpha+1}(R^{2(\alpha+1)}-\epsilon^{2(\alpha+1)})\,,\quad \text{with}\quad \alpha\neq 1\,. \tag{1.116}$$

Then the integral of order $O(\lambda)$ in (1.114) is divergent if the condition among the conformal dimensions

$$\Delta_k - \Delta_V - \Delta + 1 \leq 0\,, \tag{1.117}$$

is satisfied. In this case, by defining the renormalized field V by :

$$V = \widetilde{V} + \lambda\sum_k \pi C^k_{\varphi V}\frac{\epsilon^{2(\Delta_k-\Delta_V-\Delta+1)}}{\Delta_k-\Delta_V-\Delta+1}A_k + O(\lambda^2)\,, \tag{1.118}$$

we get UV-finite correlation function at the order $O(\lambda)$. This formula implies a mixing of the original field $\widetilde{V}$ with fields of lower dimensions.

OPE in the perturbed theory. Let us now turn to the analysis of the OPE in the PCFT. We start with the two point functions, which are written analogously to (1.114) :

$$\langle V(z)V(0)\rangle_{\rm QFT} = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int\langle\widetilde{V}(z)\widetilde{V}(0)\,\varphi(w_1)\,...\,\varphi(w_n)\rangle_{\rm CFT}\,d^2w_1\,...\,d^2w_n\,, \tag{1.119}$$

where all the integrals on the right hand side are now UV finite. Consider the OPE decomposition :

$$V(z)V(0) = \sum_i \mathbf{C}^k_{VV}(z)A_k(0)\,, \tag{1.120}$$

where $\{A_k\}$ is a complete set of renormalized fields, that we will take listed by increasing dimension ($\Delta_0 \leq \Delta_1 \leq \Delta_2 \leq ...$). The expression (1.120) represents the splitting between the analytic (structure constants) and the non-analytic (one point functions) contributions to the two points functions. Since the structure constants are local quantities, they are analytic functions of the coupling constant λ and can be expanded in power series thereof :

$$\mathbf{C}^k_{VV}(z) = z^{\Delta_k-2\Delta_V}\bar{z}^{\overline{\Delta}_k-2\overline{\Delta}_V}\sum_{n=0}^{\infty}C^{k(n)}_{VV}(\lambda r^{2-2\Delta})^n\,, \tag{1.121}$$

where here and in the following

$$r = |z|\,, \tag{1.122}$$

and the coefficient $C_{VV}^{k(0)} = C_{VV}^{k}$ is the CFT (the theory for $\lambda = 0$) three points function coefficient. The perturbed theory is not conformally invariant anymore, and hence non-vanishing one point functions of operators can appear. The one point functions are by definition *non-local* quantities, hence they admit a non-analytic dependence on λ. On dimensional grounds we also have :

$$\langle A_k(0)\rangle_{\rm QFT} = \lambda^{\frac{\Delta_k}{1-\Delta}} Q_k\,, \quad Q_k \in \mathbb{C}\,, \tag{1.123}$$

since

$$[\lambda^{\frac{\Delta_k}{1-\Delta}}] = \frac{\Delta_k}{1-\Delta}(2-2\Delta) = 2\Delta_k\,, \tag{1.124}$$

which is what we need. All the non-analytic behavior of the OPE expansion (1.120) is therefore hidden inside the one point functions, whereas the OPE coefficients can be calculated by perturbative methods, which is not so surprising since they are purely UV data. We explain this in the next paragraph.

Perturbative calculation of the coefficients. We would like to explicitly compute the first terms of the perturbative expansion (1.121). We closely follow [23] and first define the matrix elements :

$$\begin{aligned}\widetilde{I}_l^k(\lambda, R, \epsilon) &= \langle \widetilde{A}^k(\infty)\widetilde{A}_l(0)\rangle_{\rm QFT} \\ &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_{\epsilon<|w|<R} \langle \tilde{A}^k(\infty)\widetilde{A}_l(0)\varphi(w_1)\,...\,\varphi(w_n)\rangle_{\rm CFT}\, d^2w_1 ... d^2w_n\,. \end{aligned} \tag{1.125}$$

In this expression $\widetilde{A}_k$ are the non-renormalised versions of the fields A_k. Use the conformal normalization (see (1.37)) at zero order :

$$\widetilde{I}_l^k(\lambda, R, \epsilon) = \delta_l^k + O(\lambda)\,. \tag{1.126}$$

Since the UV and IR singularities have completely different nature, one can assume the following factorization property :

$$\widetilde{I}_l^k(\lambda, R, \epsilon) = \sum_p U_p^k(\lambda, \epsilon) I_l^p(\lambda, R)\,. \tag{1.127}$$

In the previous formula $I_l^p(\lambda, R)$ are renormalized matrix elements (they are therefore independent of ϵ) :

$$I_l^k(\lambda, R) = \langle A^k(\infty) A_l(0)\rangle_{\rm QFT}\,. \tag{1.128}$$

The $U_p^k(\lambda, \epsilon)$ is the UV cut-off renormalization matrix. The terms of U can be expanded in a power series in ϵ, that from dimensional considerations reads :

$$U_l^k(\lambda, \epsilon) = \sum_{n=0}^{\infty} \frac{U_l^{k(n)}}{\epsilon^{2(\Delta_l - \Delta_k)}} \left(\lambda \epsilon^{2(1-\Delta)}\right)^n\,. \tag{1.129}$$

When taking the limit $\epsilon \to 0$, we should focus only on terms with negative powers of ϵ. Since the perturbing field φ is relevant ($\Delta < 1$), there are only a finite number of such terms, for each matrix element. In particular it means that :

$$U_l^k(\lambda, \epsilon) = 0\,, \quad \Delta_k > \Delta_l\,, \tag{1.130}$$

and from the ordering of the fields in $\{A_k(0)\}$ defined above, it is clear that the matrix U_l^k together with its inverse $(U^{-1})_k^l$ has a triangular structure. The elements of the renormalized matrix are also normalized using the conformal prescription :

$$U_l^k(\lambda, \epsilon) = \delta_l^k + O(\lambda)\,. \tag{1.131}$$

Now we can define the renormalized perturbative fields as :

$$A_k = \sum_l (U^{-1})_k^l \widetilde{A}_l\,. \tag{1.132}$$

From the normalization (1.131) we recover the formula (1.118) :

$$A_k = \widetilde{A}_k + \dots , \tag{1.133}$$

where the dots mean a *finite* number of terms, given by fields with dimensions lower than Δ_k. Remark, that in the above discussion, we supposed for simplicity that there are no resonances, in other words that $\Delta_k - \Delta_l$ is not an integer multiple of $1 - \Delta$.

For the renormalized matrix I, we assume a similar power series expansion as the one of U, but depending on the cut-off R :

$$I_l^k(\lambda, R) = \sum_{n=0}^{\infty} \frac{I_l^{k(n)}}{R^{2(\Delta_l - \Delta_k)}} \left(\lambda R^{2(1-\Delta)}\right)^n . \tag{1.134}$$

We can now proceed to the perturbative calculation of the three point function. Define the quantity

$$\begin{aligned} G_{VV}^k(\lambda, z, R) &= \langle \widetilde{A}^k(\infty) V(z) V(0) \rangle_{\mathrm{QFT}} , \\ &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_{|w|<R} \langle \widetilde{A}^k(\infty) V(z) V(0) \varphi(w_1) \dots \varphi(w_n) \rangle_{\mathrm{CFT}} \, d^2 w_1 \dots d^2 w_n . \end{aligned} \tag{1.135}$$

Since V is already renormalized, there is no ϵ dependence (no UV singularities) in the integrals. The use of the OPE expansion (1.120) implies :

$$\mathbf{C}_{VV}^k(z) = \sum_l G_{VV}^l(\lambda, z, R) (I^{-1})_l^k(\lambda, R) . \tag{1.136}$$

One will need the following three-point functions at the CFT level (1.38) :

$$\langle \widetilde{A}^k(\infty) V(z) V(0) \rangle_{\mathrm{CFT}} = C_{VV}^k |z|^{2(\Delta_k - 2\Delta_V)} , \tag{1.137}$$

$$\langle \widetilde{A}^k(\infty) \varphi(z) V(0) \rangle_{\mathrm{CFT}} = C_{\varphi V}^k |z|^{2(\Delta_k - \Delta - \Delta_V)} , \tag{1.138}$$

$$\langle \widetilde{A}^k(\infty) \varphi(z) A_l(0) \rangle_{\mathrm{CFT}} = C_{\varphi l}^k |z|^{2(\Delta_k - \Delta_l - \Delta)} . \tag{1.139}$$

Inserting (1.137) in the first order term of (1.125) and using (1.116) one has :

$$\widetilde{I}_l^k(\lambda, R, \epsilon) = \delta_l^k - \lambda\pi \, C_{\varphi l}^k \frac{R^{2(\Delta_k - \Delta_l - \Delta + 1)} - \epsilon^{2(\Delta_k - \Delta_l - \Delta + 1)}}{\Delta_k - \Delta_l - \Delta + 1} . \tag{1.140}$$

Splitting the UV and IR contributions at the same order in λ we get :

$$I_l^k(\lambda, R) = \delta_l^k - \lambda\pi \, C_{\varphi l}^k \frac{R^{2(\Delta_k - \Delta_l - \Delta + 1)}}{\Delta_k - \Delta_l - \Delta + 1} , \tag{1.141}$$

$$U_l^k(\lambda, \epsilon) = \delta_l^k + \lambda\pi \, C_{\varphi l}^k \frac{\epsilon^{2(\Delta_k - \Delta_l - \Delta + 1)}}{\Delta_k - \Delta_l - \Delta + 1} . \tag{1.142}$$

At the first order the OPE coefficients are then :

$$\mathbf{C}_{VV}^k(z) = C_{VV}^k |z|^{2(\Delta_k - 2\Delta_V)} - \lambda \int_{|w|<R} \langle \widetilde{A}(\infty) \varphi(w) V(z) V(0) \rangle \, d^2 w \tag{1.143}$$

$$+ \lambda\pi \sum_l \frac{C_{V\varphi}^l C_{\varphi l}^k R^{2(\Delta_k - \Delta_l - \Delta + 1)}}{\Delta_k - \Delta_l - \Delta + 1} . \tag{1.144}$$

The last sum cancels exactly the IR divergences of the integral. Using the OPE (1.120) we obtain the first order term of the expansion (1.121) :

$$\begin{aligned} \mathbf{C}_{VV}^k(z) &= r^{2(\Delta_k - 2\Delta_V)} \\ &\times \left(C_{VV}^k + \lambda\pi r^{2-2\Delta} \sum_l \left(\frac{C_{VV}^l C_{\varphi l}^k}{\Delta_k - \Delta_l - \Delta + 1} - \frac{C_{\varphi V}^l C_{Vl}^k}{\Delta_l - \Delta_V - \Delta + 1} \right) + O(\lambda^2) \right) . \end{aligned} \tag{1.145}$$

Partial conclusion. The above analysis showed that the only quantities that we cannot access from standard perturbative methods are the one point functions $\langle V_k(0)\rangle$ of the theory. These are the principal quantities of interest and their calculation, in the case of the Super sine-Gordon model, will be the main subject of this dissertation. However we can already claim that a large number of them is equal to zero due to QFT symmetries. These are :

1. One point functions of fields with non zero spin.
2. One point functions of fields which are spatial derivatives of other fields.
3. One point functions of fields generated by the action of any local integral of motion.

The first two points represent the invariance of one point functions under rotations and spatial translations. The third point is related to an outstanding feature of many perturbed CFT : under certain conditions the perturbed field theories happen to be *integrable models*. This will be the topic of the next section.

1.4 Deformation of conformal conservation laws

A crucial observation is that the above deformations of CFT often lead to *integrable* Quantum Field Theories. Integrable QFTs are characterized by an infinite number of *integrals of motions* (IoM), and thus an infinite number of conservation laws. The integrals of motions occupy a central place in our work. First of all, they are already present in the CFT, and we will see why they are crucial to display the integrable structure of conformal field theories in section 2.6. As we will explain in this section, the integrals of motion can under certain conditions be deformed from CFT to the massive QFT, and are then responsible of the integrability of the latter theory. Let us first start with the conformal case.

Integrals of motion in CFT. Recall that in CFT, the field T is conserved :

$$\partial_{\bar{z}} T(z,\bar{z}) = 0\,. \tag{1.146}$$

From this equation we can try to build conserved quantities as combinations of the field T (and its derivatives). To manipulate products of fields we first define the normal ordering between two fields V and W on the plane :

$$:VW(u): := \oint_{c_u} \frac{dz}{2\pi i}\frac{1}{z-u} V(z)W(u)\,. \tag{1.147}$$

We will also employ the notation $(VW) =: VW :$ to denote products of multiple normal ordered operators. More precisely, we can construct conserved densities h by considering polynomials of normal ordered products of T itself and of its derivatives. From the conservation law (1.146), any such h will also satisfy

$$\partial_{\bar{z}} h(z,\bar{z}) = 0\,, \tag{1.148}$$

and is therefore a conserved quantity. A natural question is to understand how one could classify these densities. Recall that the stress energy tensor is a descendant of the identity field $T = L_{-2}I$. It means that all such h belong to the conformal family of the identity $[I]$, that is they are built from I by applying the modes L_n with $n < 0$. Hence, one can try to characterize the h's by their spin. Actually for each spin s, we can define a finite dimensional vector space $\mathfrak{T}_s$ containing the different "candidate" densities with this given value of the spin. Let us list the first possible densities :

$$s = 0 \quad h_0 = I\,, \tag{1.149}$$
$$s = 1 \quad h_1 = L_{-1}I = \partial_z I = 0\,, \tag{1.150}$$
$$s = 2 \quad h_2 = L_{-2}I = T\,, \quad h_2' = L_{-1}^2 I = 0 \tag{1.151}$$
$$s = 3 \quad h_3 = L_{-1}L_{-2}I = \partial_z T\,, \quad h_3' = L_{-1}^3 I = 0 \tag{1.152}$$
$$s = 4 \quad h_4 = L_{-2}^2 T = (TT)\,, \quad h_4' = \partial_z^2 T\,. \tag{1.153}$$

The prime indicate other fields, that might be considered for constructing densities (at each value of the spin). Generally speaking, all density of odd spin will be set to zero, they are either equal to zero, or total derivatives that means that their one point function is zero, and are of no interest for us.

Therefore, we concentrate on even spin densities. At spin 6, we have two non trivial contributions that can appear :

$$h_6^{(1)} = ((TT)T)\,, \quad h_6^{(2)} =: \partial_z T \partial_z T : \,. \tag{1.154}$$

The outcome of the previous analysis is the following : one has an infinite family of conserved fields h_{2k}, parametrized by their spin $2k$ that are expressed as *homogeneous (in the spin)* polynomials of T and its derivatives.
But which density keep at spin 6 ? As we will see, in certain cases, specially constructed integrals of motion can "survive" when the perturbation theory is applied, which means that such densities still satisfy a conservation equation in the massive theory. Before presenting the procedure, we give the densities that one should keep :

$$h_2(z) = T(z)\,, \tag{1.155}$$

$$h_4(z) =: T(z)^2 :\,, \tag{1.156}$$

$$h_6(z) =: T(z)^3 : + \frac{c+2}{12} : (\partial_z T(z))^2 : \,. \tag{1.157}$$

Let us emphasize again that in the spin 4 case, we dropped the contribution of the field $\partial_z^2 T$ because being a total derivative, its one point function is zero.

Action on fields. In the same spirit that we defined the action of the modes $\mathbf{l}_n$ of the stress energy tensor on fields on the cylinder (1.56), we define the local action of the integrals of motion. Recall that on the cylinder we use circular contractible contours to define the local actions :

$$(\mathbf{i}_{2k-1} V)(y) = \oint_{c_y} \frac{dx}{2\pi i} \mathcal{T}(h_{2k}(x) V(y))\,, \tag{1.158}$$

where c_y is a small circle around the point y. One should think about this formula as a local definition, since c_y is homotopical to point. This implies that one could also compute $\mathbf{i}_{2k-1}$ from the residue of $h_{2k}(z)$ (in the plane geometry) at the origin :

$$\mathbf{i}_{2k-1} = \mathrm{Res}_{z=0}(h_{2k}(z))\,. \tag{1.159}$$

Let us list some properties of the modes $\mathbf{i}_{2k-1}$.

1. Since the h_{2k} are constructed out of T's, $\mathbf{i}_{2k-1}$ can be rewritten through the "local" modes $\mathbf{l}_n$.
2. Because h_{2k} is a field of spin $2k$, the operator $\mathbf{i}_{2k-1}$ is of spin $2k-1$. This is equivalent to the equation :
$$[\mathbf{l}_0, \mathbf{i}_{2k-1}] = (2k-1)\mathbf{i}_{2k-1}\,. \tag{1.160}$$
3. Since we are dealing with conserved quantities, all $\mathbf{i}_{2k-1}$ are in involution :
$$[\mathbf{i}_{2k-1}, \mathbf{i}_{2l-1}] = 0\,. \tag{1.161}$$

Now we explicitly calculate the first actions. We have

$$\begin{aligned}(\mathbf{i}_1 V)(y) &= \int_{c_y} \frac{dx}{2\pi i} \mathcal{T}(h_2(x) V(y)) = \int_{c_y} \frac{dx}{2\pi i} \mathcal{T}(T(x) V(y)) = (\mathbf{l}_{-1} V)(y)\,, \\ &\Longrightarrow \ \mathbf{i}_1 = \mathbf{l}_{-1}\,.\end{aligned} \tag{1.162}$$

To get $\mathbf{i}_3$, let us calculate the residue of (1.156). We have the normal ordered expressions [9] :

$$(TT)(z) =: \sum_n \mathbf{l}_n z^{-n-2} \sum_m \mathbf{l}_m z^{-m-2} := \sum_{n,m} : \mathbf{l}_n \mathbf{l}_m : z^{-n-m-4} = \sum_q \left(\sum_m : \mathbf{l}_m \mathbf{l}_{q-m} : \right) z^{-q-4}\,. \tag{1.163}$$

This leads to :

$$\mathbf{i}_3 = \mathrm{Res}_{z=0}\big[(TT)(z)\big] = \sum_m : \mathbf{l}_m \mathbf{l}_{-3-m} := 2 \sum_{m=-1}^{\infty} \mathbf{l}_{-3-m} \mathbf{l}_m\,. \tag{1.164}$$

Finally, by a similar computation, one obtain the expression of $\mathbf{i}_5$ [31] :

$$\mathbf{i}_5 = 3 \left(\sum_{k=-1}^{\infty} \sum_{l=-1}^{\infty} \mathbf{l}_{-5-k-l} \mathbf{l}_l \mathbf{l}_k + \sum_{k=-\infty}^{-2} \sum_{l=-\infty}^{-2} \mathbf{l}_l \mathbf{l}_k \mathbf{l}_{-5-k-l} \right) + \frac{c+2}{6} \sum_{k=-1}^{\infty} (k+2)(k+3) \mathbf{l}_{-5-k} \mathbf{l}_k\,. \tag{1.165}$$

[9] In terms of modes, the normal order corresponds to placing operators with the bigger index to the right.

Integrals of motion in Perturbed CFT. Let us now move to the consideration of perturbed CFT. The key article where the deformations of integrals of motion were studied is [33]. It has been shown that a unitary minimal CFT M_p perturbed by the fields $V_{1,3}$ and $V_{1,2}$ possesses higher order integrals of motion.

How do the conformal conservation laws (1.148) change in the massive context ? Recall the formula (1.9) giving the continuity equation on a general stress-energy tensor and compare it with the conservation law for the densities in the CFT $\partial_{\bar{z}} h_{2k} = 0$. It is natural to suppose that in the massive theory, the previous equation for h_{2k} should develop a non zero right hand side taking the form of a total holomorphic derivative :

$$\partial_{\bar{z}} h_{2k} = \partial_z \Theta_{2k-2} \,, \tag{1.166}$$

for some field Θ_{2k-2} to be specified. To balance the dimensions on both sides of (1.166), we can claim that Θ_{2k-2} has spin $2k-2$. Also from this equation we can directly construct a conserved charge of spin $2k-1$ (in the same spirit as (1.11)):

$$\mathcal{Q}_{2k-1} = \frac{1}{2\pi i} \oint_{c_0} \left(h_{2k} dz + \Theta_{2k-2} d\bar{z} \right) \,. \tag{1.167}$$

In what sense are these charges conserved ? A naive application of the Stokes theorem would give :

$$\begin{aligned} \mathcal{Q}_{2k-1} &= \frac{1}{2\pi i} \oint_{c_0} \left(dz h_{2k} + d\bar{z} \Theta_{2k-2} \right) \\ &= -\frac{1}{2\pi i} \int_{\Omega} (\partial_{\bar{z}} h_{2k} - \partial_z \Theta_{2k-2}) dz \wedge d\bar{z} = 0 \,, \end{aligned} \tag{1.168}$$

where Ω is the interior of c_0. But we should handle this relation only inside correlation functions, where singularities coming from products with other fields are present, and hence give a non vanishing result. Moreover the set of these charges should commute :

$$[\mathcal{Q}_{2k-1}, \mathcal{Q}_{2l-1}] = 0 \,. \tag{1.169}$$

Recalling (1.12), we see that $\mathcal{Q}_1$ is simply the momentum

$$\mathcal{Q}_1 = \mathcal{P} = \partial_z \,. \tag{1.170}$$

Let us make another remark. In the section 4.1 we will use a similar construction in the context of *Supersymmetric* CFT to create conserved densities, but instead of working with the stress-energy tensor we will focus on the super current field. This will in particular yield the usual super charges of a SUSY QFT. Now that we made explicit the general form of the conservation laws in the context of a massive field theory, we need a procedure to select the CFT densities h_{2k} that will effectively give rise to a deformed equation (1.166). This criterion is explained in the next paragraph.

Criterion for the deformation of conservation laws. This rule is based on the OPE between h and the perturbing field, and reads :
For a density h to be an integral of motion in the QFT context it is necessary for the residue of the OPE of h with the perturbation field to be proportional to a total derivative. Let us prove this. First recall a formula for distribution theory :

$$\delta(z-w) = \frac{1}{\pi} \partial_{\bar{z}} \left(\frac{1}{z-w} \right) \,. \tag{1.171}$$

Let h_{2k} be a CFT density, we want to arrive at the conservation law in the context of QFT :

$$\partial_{\bar{z}} h_{2k} = \partial_z \Theta_{2k-2} \,. \tag{1.172}$$

As we said, this is to be understood as an equality inside correlation functions. In a general CFT perturbed by a field φ one can write the expansion of a correlation function of the field h_{2k} :

$$\begin{aligned} \langle h_{2k}(z,\bar{z}) ... \rangle_{\mathrm{QFT}} &= \int \mathcal{D}\phi e^{-\mathcal{A}[\phi]} h_{2k}(z,\bar{z}) ... = \int \mathcal{D}\varphi e^{-S_{\mathrm{CFT}} - \lambda \int d^2 w \varphi(w)} h_{2k}(z,\bar{z}) ... \\ &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int dw_1^2 ... dw_n^2 \, \langle h_{2k}(z,\bar{z}) \varphi(w_1) ... \varphi(w_n) ... \rangle_{\mathrm{CFT}} \\ &= \langle h_{2k}(z) ... \rangle_{\mathrm{CFT}} - \lambda \int dw^2 \, \langle h_{2k}(z,\bar{z}) \varphi(w) ... \rangle_{\mathrm{CFT}} + O(\lambda) \end{aligned} \tag{1.173}$$

As said above, it is enough to do the calculation in the first order of perturbation theory. Take the derivative $\partial_{\bar{z}}$ of (1.173). To do this, we assume that the strongest singularity in the OPE between h_{2k} and φ is of order 2 (the general case is obtained in the same way) :

$$h_{2k}(z,\bar{z})\varphi(w,\bar{w}) = \frac{A^{(2)}}{(z-w)^2}\varphi^{(2)}(w,\bar{w}) + \frac{1}{z-w}\varphi^{(1)}(w,\bar{w}) + ... \tag{1.174}$$

$$h_{2k}(z,\bar{z})\varphi(w,\bar{w}) = \frac{A^{(2)}}{(z-w)^2}\varphi^{(2)}(z,\bar{z}) + \frac{1}{z-w}\left(\varphi^{(1)}(z,\bar{z}) - A^{(2)}\partial_z\varphi^{(2)}(z,\bar{z})\right) + ... \tag{1.175}$$

We do this change because we want the variable of integration w to appear only in the denominator. Performing the OPE (1.175) in the integral of (1.173), and using (1.171) we obtain at first order :

$$\partial_{\bar{z}}h_{2k}(z) = -\lambda\pi\left(\varphi^{(1)}(z,\bar{z}) - A^{(2)}\partial_z\varphi^{(2)}(z,\bar{z})\right) . \tag{1.176}$$

Hence

$$\partial_z\Theta_{2k-2}(z,\bar{z}) = -\lambda\pi\left(\varphi^{(1)}(z,\bar{z}) - A^{(2)}\partial_z\varphi^{(2)}(z,\bar{z})\right) , \tag{1.177}$$

which implies the necessity for the field $\varphi^{(1)}(z,\bar{z})$ to be a total holomorphic derivative.

Example. Let us illustrate this method for the first density $h_2(z) = T(z)$. If the perturbation is done by a primary field $\varphi(z,\bar{z})$ of conformal dimension Δ, we recall the OPE (1.20) :

$$h_2(z)\varphi(w,\bar{w}) = \frac{\Delta}{(z-w)^2}\varphi(w,\bar{w}) + \frac{1}{z-w}\partial_w\varphi(w,\bar{w}) + O(1) \tag{1.178}$$

Hence the density h_2 gives rise to the perturbed conservation law (we already knew it from (1.9)) :

$$\partial_{\bar{z}}T = \partial_z\Theta . \tag{1.179}$$

with

$$\Theta(z,\bar{z}) = \lambda\pi(\Delta-1)\varphi(z,\bar{z}) \tag{1.180}$$

This is a well known result : we have shown that the trace of the stress energy tensor is proportional to the perturbing field, in the first order of perturbation theory. The structure of the spin 6 density h_6 (1.157) is chosen exactly so that it satisfies this criterion and gives an integral of motion in the perturbed theory.

There exists also other, more algebraic methods, that allow to determine the explicit form of the right hand side term Θ_{2k-2}. For example *the counting argument* of A. Zamolodchikov [32].

Application. In the paper [33], A. Zamolodchikov used the above criterion to show that the perturbation of any unitary minimal model by the fields $V_{1,3}$ or $V_{1,2}$, deform the conformal conservation laws to the massive QFT. The main argument of the proof relies on the fact that since these two fields are degenerate primary fields, there exist a dependence relation between their descendants at level 3 and 2 respectively. This relation can be used to show that the residue of the OPE between $V_{1,3}$ or $V_{1,2}$ with any conformal density h_{2k} is a total derivative.

Hence the perturbation of unitary minimal models by $V_{1,3}$ or $V_{1,2}$ gives massive theories with an infinite number of local integrals of motion and conserved charges (1.167). This is what we will call an *Integrable Quantum Field Theory*, they will be discussed in the next section.

As an example of such a theory, let us now consider the following perturbation of the Liouville CFT, termed the *sinh-Gordon* model (shG). The shG is given by the Lagrangian :

$$\mathcal{A}_{\text{shG}} = \int\left(\frac{1}{4\pi}\partial_z\varphi(z,\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z}) + 2\mu\,\text{sh}(b\varphi(z,\bar{z}))\right)\frac{idz\wedge d\bar{z}}{2} . \tag{1.181}$$

This is a perturbation of the Liouville theory $\mathcal{A}_{\rm L}$ by the field $V_{-b}(z,\bar{z}) = e^{-b\varphi(z,\bar{z})}$:

$$\mathcal{A}_{\rm shG} = \int \left(\frac{1}{4\pi} \partial_z \varphi(z,\bar{z}) \partial_{\bar{z}} \varphi(z,\bar{z}) + \mu e^{b\varphi(z,\bar{z})} + \mu e^{-b\varphi(z,\bar{z})} \right) \frac{i dz \wedge d\bar{z}}{2} \,. \tag{1.182}$$

Recall that the stress energy tensor of the Liouville CFT is :

$$T = -\frac{1}{4}(\partial_z \varphi)^2 + \frac{Q}{2}\partial_z^2 \varphi \,. \tag{1.183}$$

The perturbing field V_{-b} has conformal dimension $\Delta = -b(Q+b) = -(1+2b^2)$.

Remember the correspondence between a minimal model M_p and the Liouville CFT parametrized by b (1.107) :

$$b^2 = -\frac{p}{p+1} \,. \tag{1.184}$$

One can then interpret the shG model as a perturbation of a CFT by the field $V_{1,3}$, and hence from the above considerations claim that shG is an Integrable QFT. Of course the integrability of this model was proven without any reference to CFT, but this gives us a rather heuristic, extra argument. The same considerations will hold for our case of interest, that is for the Super sine-Gordon (sinh-Gordon) model.

The two facets of integrals of motion. To close this section, we would like to highlight once again the two different aspects of the integrals of motion in CFT and in QFT, by recapping the two applications that we can make of the densities h_{2k}.

1. We can use the densities to build operators acting on states of Verma modules. More precisely, in the terminology that will be developed in the next chapter, we employ the densities in order to act on the *Matsubara Hilbert space*. This will be made clearer later, for the moment consider that this amounts to construct the operators I_{2k-1} in the Universal Enveloping Algebra of the Virasoro algebra, out of the h_{2k} by integrating the densities along a non-trivial cycle C of the cylinder :

$$I_{2k-1} = \oint_C h_{2k}(y) \frac{dy}{2\pi i} = \int_{u-i\pi}^{u+i\pi} h_{2k}(y) \frac{dy}{2\pi i} \,. \tag{1.185}$$

The construction procedure for the densities described above implies the commutativity of the operators I_{2k-1} :

$$[I_{2k-1}, I_{2l-1}] = 0 \,. \tag{1.186}$$

An explicit calculation gives the expressions of the I_{2k-1} in terms of the modes L_n :

$$I_1 = L_0 - \frac{c}{24} \,, \tag{1.187}$$

$$I_3 = 2\sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} \,. \tag{1.188}$$

Recall that we computed I_3 in (1.73), (1.75). A more complicated expression for I_5 can be found from [34] :

$$I_5 = \sum_{n_1+n_2+n_3=0} : L_{n_1} L_{n_2} L_{n_3} : + \sum_{n=1}^{\infty} \left(\frac{c+11}{6} n^2 - 1 - \frac{c}{4} \right) L_{-n} L_n + \\ \frac{3}{2} \sum_{r=1}^{\infty} L_{1-2r} L_{2r-1} - \frac{c+4}{8} L_0^2 - \frac{c(3c+14)(7c+68))}{290304} \,. \tag{1.189}$$

We have the evident action on the primary state $|\Delta\rangle$:

$$I_{2k-1} |\Delta\rangle = i_{2k-1} |\Delta\rangle \,. \tag{1.190}$$

For the operators (1.187),(1.188),(1.189) these are :

$$i_1 = \Delta - \frac{c}{24} \,, \tag{1.191}$$

$$i_3 = \Delta^2 - \frac{c+2}{12}\Delta + \frac{c(5c+22)}{2880} \,, \tag{1.192}$$

$$i_5 = \Delta^3 - \frac{c+4}{8}\Delta^2 + \frac{1}{576}(c+2)(3c+20)\Delta - \frac{c(14+3c)(68+7c)}{290304} \,. \tag{1.193}$$

2. As local densities, we can use them to define the action of the integrals of motion on other fields V, by the formula (1.158) :

$$(\mathbf{i}_{2k-1}V)(y) = \int_{c_y} \frac{dx}{2\pi i}\mathcal{T}(h_{2k}(x)V(y)) \tag{1.194}$$

Grouping the above results we have :

$$\mathbf{i}_1 = \mathbf{l}_{-1}\,, \tag{1.195}$$

$$\mathbf{i}_3 = 2\sum_{n=0}^{\infty}\mathbf{l}_{-n-2}\mathbf{l}_{n-1}\,, \tag{1.196}$$

$$\mathbf{i}_5 = 3\left(\sum_{k=-1}^{\infty}\sum_{l=-1}^{\infty}\mathbf{l}_{-5-k-l}\mathbf{l}_l\mathbf{l}_k + \sum_{k=-\infty}^{-2}\sum_{l=-\infty}^{-2}\mathbf{l}_l\mathbf{l}_k\mathbf{l}_{-5-k-l}\right) + \frac{c+2}{6}\sum_{k=-1}^{\infty}(k+2)(k+3)\mathbf{l}_{-5-k}\mathbf{l}_k\,. \tag{1.197}$$

To obtain the relation between the two descriptions we calculate the one point functions on the cylinder (remember that we work on the cylinder with primary fields of dimensions $\Delta_\pm$ located at $\pm\infty$ respectively). Again, splitting the contour $c_y = C_+ - C_-$ one gets :

$$\langle(\mathbf{i}_{2k-1}V)(y)\rangle_{\Delta_+,\Delta_-} = -\int_{C_-}\frac{dx}{2\pi i}\langle h_{2k}(x)V(y)\rangle_{\Delta_+,\Delta_-} + \int_{C_+}\frac{dx}{2\pi i}\langle V(y)h_{2k}(x)\rangle_{\Delta_+,\Delta_-}\,, \tag{1.198}$$

$$= (i^+_{2n-1} - i^-_{2n-1})\langle V(y)\rangle_{\Delta_+,\Delta_-}\,. \tag{1.199}$$

To get to the second line, we move the contours $C_\pm$ to infinities, where the density h_{2k} hits the primary states $\Delta_\pm$. This gives the relation between the two descriptions. More importantly, we have shown that when the boundary conditions are equal $\Delta_+ = \Delta_-$, *the one point functions of fields obtained by the action of integrals of motion are zero.*

1.5 Integrable Quantum Field Theory

We have seen that the perturbation of a unitary minimal model M_p by the field $V_{1,3}$ leads to a QFT owning an infinite number of conserved, commuting charges. This is our definition of an Integrable Quantum Field Theory. But how are they connected to general integrable systems ? One needs to say that there is no universal definition for such models, but usually one agrees to consider that integrability means that all physically relevant quantities can be computed analytically. This behavior was first observed in the context of Analytical Mechanics (see [37] for a review) for theories possessing enough conserved quantities, that is at least the same number as of degrees of freedom. This is the result of the famous Arnold-Liouville theorem for classical integrable systems. In the context of QFT one typically has infinitely many degrees of freedom, and as we have seen above infinitely many conserved quantities for integrable QFTs. Why and in what sense do this charges lead to an exact solution of the quantum theory ?

This question has been answered in [38], we shall briefly recall the main steps from this paper (also using [24] as a valuable reference). Hence, for a moment we come back to the relativistic non-euclidean, massive theory. The characteristics of particles are then their masses m and *rapidities* θ. The essential contribution of [38] is to demonstrate that integrability in QFT implies that the spectrum and the scattering data (equivalently the S matrix of the theory) can be computed exactly. We will in short explain why it is so, and how it leads naturally to the *Yang-Baxter equation*, the cornerstone of any quantum integrable system.

Let us start from the last results of the previous section, we have a QFT with infinitely many conserved charges $\mathcal{Q}_s$ (or integrals of motion), labeled by their spins, in involution :

$$[\mathcal{Q}_u, \mathcal{Q}_v] = 0\,. \tag{1.200}$$

Assume that the particles in the theory can be classified by their species, a state of the theory with n particles of species a_i will then be denoted by

$$|A_{a_1}(\theta_1)...A_{a_n}(\theta_n)\rangle\,, \tag{1.201}$$

where θ_i is the rapidity of the particle a_i. Since we are working in only 2 dimensions, we can formally think of $A_{a_i}(\theta_i)$ as non commuting symbols, whose order is associated to the spacial order of the particles that they depict. Then the state (1.201) represent an asymptotic state with $\theta_1 \geq \theta_2 \geq ... \geq \theta_n$. The set of objects defined above (1.201) will be called the *Faddeev-Zamolodchikov algebra*, we will soon explain its multiplication rule. Since the charges commutes, we can take a basis of such states that diagonalize all conserved charges :

$$\mathcal{Q}_u \left|A_a(\theta)\right\rangle = \omega_u^a(\theta) \left|A_a(\theta)\right\rangle \ . \tag{1.202}$$

Then from the transformation properties of $\mathcal{Q}_u$ under rotations, it is possible to explicit the dependence of the eigenvalue ω_u^a on the rapidity θ :

$$\omega_u^a(\theta) = \chi_u^a e^{u\theta} \, . \tag{1.203}$$

The presence of an infinite number of conserved charges has therefore two tremendous consequences :

1. *The scattering processes are purely elastic.* This means that there is no particle production or annihilation, and the sets of initial and final momenta are coinciding. Indeed, let us act with $\mathcal{Q}_u$ on (1.201) and obtain :

$$\mathcal{Q}_u \left|A_{a_1}(\theta_1)...A_{a_n}(\theta_n)\right\rangle = \sum_{k=1}^{n} \chi_u^{a_i} e^{u\theta_i} \left|A_{a_1}(\theta_1)...A_{a_n}(\theta_n)\right\rangle \ . \tag{1.204}$$

   But $\mathcal{Q}_u$ is a conserved quantity $\frac{d\mathcal{Q}_u}{dt} = 0$. This means that there is an infinite series of constraints that involve the sum on the initial and final particles, and that can have any number of terms :

$$\sum_{k\in \text{in}} \chi_u^{a_k} e^{u\theta_k} = \sum_{l\in \text{out}} \chi_u^{a_l} e^{u\theta_l} \, . \tag{1.205}$$

   The only solution to these constraints corresponds to the case where both sums have the same number of terms, and where they are pairwise equal. Hence an infinite number of conserved charges implies the elasticity of the scattering.

2. *The scattering is factorizable.* This means that the scattering process of any number of particles is reduced to 2-particles scattering.

Hence all the scattering processes are encoded in the two particles scattering. The S matrix is therefore defined as the exchange operator for 2-particles states :

$$\left|A_{a_i}(\theta_1)A_{a_j}(\theta_2)\right\rangle = S_{ij}^{kl}(\theta_{12}) \left|A_{a_k}(\theta_2)A_{a_l}(\theta_1)\right\rangle \ , \tag{1.206}$$

where the notation $\theta_{ij} = \theta_i - \theta_j$ is used. This relation fixes the multiplication rules between the elements of the Faddeev-Zamolodchikov algebra. S depends only on the difference of the rapidity θ_{ij} because of relativistic invariance. Since the 3-particles process is factorizable in two different ways (see the picture 1.2) we have the following :

$$S_{ij}^{ab}(\theta_{12}) S_{bk}^{cl}(\theta_{13}) S_{ac}^{nm}(\theta_{23}) = S_{jk}^{ab}(\theta_{23}) S_{ia}^{nc}(\theta_{13}) S_{cb}^{ml}(\theta_{12}) \, , \tag{1.207}$$

this is the celebrated Yang-Baxter equation for the S matrix.

The appearance of the Yang-Baxter equation is a common feature of quantum integrable systems. Indeed, from this equation it is possible to calculate many physically relevant quantities explicitly [40]. Hence the fact that the S matrix satisfies the Yang-Baxter is a strong evidence towards the integrability of the theory. From a slightly different point of view, it can be easily seen that the Yang-Baxter equation is equivalent to the associativity of the composition rule (1.206) in the Faddeev-Zamolodchikov algebra.

There are additional constraints on the S matrix, such that the crossing symmetry and several analytical conditions, we will not discuss them here. Together with the so called Bootstrap principle they allow to find solutions of (1.207) for different theories and hence access the spectrum and the masses of the particles. The key point is that the specificity (for example the spectrum) of each theory will be hidden in the set of spins of the conserved charges, the description of this fact is out of the scope of this presentation (see [24, 32]).

Now that we have explained how to calculated the S matrix, we shall describe how one can apply it in thermodynamical considerations.

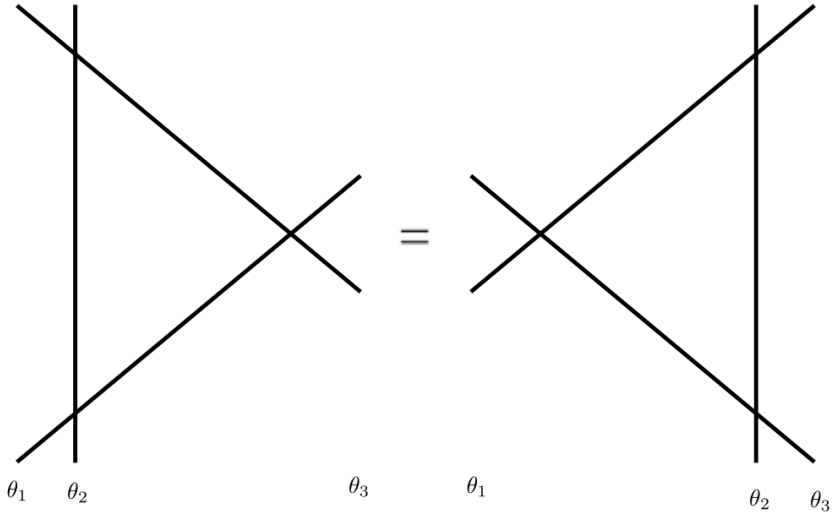

Figure 1.2: Factorization of the scattering process of 3 particles : the Yang-Baxter equation. Each line of the picture is interpreted as a trajectory of one of the particles (with rapidities $\theta_{1,2,3}$) in the two dimensional space-time. An intersection between the lines correspond to the occurrence of a scattering process described by the corresponding entry of the S matrix.

1.6 Thermodynamic Bethe Ansatz

In the future study of the Super sine-Gordon model our main tool will be thermodynamical equations of non-linear, integral type. They allow to study numerically a massive QFT, and obtain results in its short or long distance limits (UV or IR). The first type of thermodynamical equations were derived in [41] for quantum system of bosonic particles. They were afterwards generalized to the relativistic set-up in [19] and applied in the QFT context for the simplest massive perturbations of Conformal Field Theories : the Scaling 3-state Potts model and the Lee-Yang model. Since then, the equations constructed by the methodology of [19] are termed *Thermodynamic Bethe Ansatz* equations (TBA). The TBA equations describe exactly the ground state energy $E_0(R)$ (1.83) of an integrable field theory living on a finite circle of length R with periodic (or twisted) boundary conditions. Usually TBA is derived starting from a relativistic factorized scattering theory, that is from the data of the S matrix of theory, which is supposed to be known for the model under consideration. The TBA technology is one of the main tools that we have at our disposition to study the integrable massive QFT, and some of its applications include :

1. The validation of the assumption that the proposed scattering theory does indeed describe the deformation of the given CFT. One can in fact build a scaling function that tends towards the central charge of the CFT in the UV limit.

2. The calculation of the bulk energy of the QFT.

3. The derivation of the exact relation between the mass of the lightest particle and the coupling constant (1.112) in PCFT.

In addition to provide, as we are about to present, the above exact result for many QFTs, the TBA equations are also intrinsically related to the integrable structure of the underlying CFT [34], this will be discussed in more details in the section 2.6. Even if in our work we shall use different thermodynamical equations (the so-called *Suzuki equations*, in particular to make an even more explicit contact with the construction of [34]), their form and the spirit of their applications are similar to the TBA. This is the reason for us to concisely recall the "historical" TBA equations focusing on two applications :

1. The calculation of the UV CFT central charge.

2. The derivation of the mass - coupling constant formula.

Ground state energy and free energy of a QFT. First we introduce several thermodynamical quantities in the context of field theory. Consider a Quantum Field Theory, characterized by its stress energy tensor T, on a cylinder of radius R and length L (with coordinates x and y respectively). There are two ways to treat this theory, this is pictured in the two different possibilities to write the partition function Z :

$$Z(R,L) = \mathrm{Tr}\, e^{-LH_R}\,, \quad Z(R,L) = \mathrm{Tr}\, e^{-RH_L}\,. \tag{1.208}$$

The two Hamiltonians are expressed in terms of the components of the stress energy tensor :

$$H_R = \int T_{yy}\frac{dx}{2\pi}\,, \quad H_L = \int T_{xx}\frac{dy}{2\pi}\,. \tag{1.209}$$

In the infinite length limit $L \to \infty$, the ground state energy $E_0(R)$ of the Hamiltonian H_R gives the main contribution to the partition function :

$$Z(R,L) \simeq e^{-LE_0(R)}\,. \tag{1.210}$$

But on the other hand this is equivalent [10] to consider the thermodynamics of a one dimensional quantum system with Hamiltonian H_L defined along the L axis at temperature $1/R$. The partition function is then :

$$Z(R,L) \simeq e^{-RLf(R)}\,, \tag{1.211}$$

where $f(R)$ is the free energy per unit length at temperature $1/R$. This gives the relation between the ground-state energy and the free energy of the system :

$$E_0(R) = Rf(R)\,. \tag{1.212}$$

Let m_0 be the mass of the lightest particle in the QFT, one can then define the dimensionless scaling length $r = m_0 R$. It is then natural from dimensional arguments to define the scaling function F in terms of the ground state energy

$$E_0(R) = \frac{2\pi}{R}F(r)\,. \tag{1.213}$$

The knowledge of the ground state energy is important because it allows to determine the one point functions of the components of the stress energy tensor. From (1.209) :

$$\langle T_{xx}\rangle = 2\pi\frac{E_0(R)}{R}\,, \tag{1.214}$$

$$\langle T_{yy}\rangle = 2\pi\frac{d}{dR}E_0(R)\,, \tag{1.215}$$

$$\langle T^a_a\rangle = \langle T_{xx}\rangle + \langle T_{yy}\rangle = \frac{2\pi}{R}\frac{d}{dr}\left(RE_0(R)\right)\,. \tag{1.216}$$

In particular, (1.216) together with the formula (1.180) implies the knowledge of the ground state energy and gives access to the one point function of the perturbing field in PCFT. Furthermore, mimicking (1.83) define the function $\widetilde{c}$ of the scaling length r, related to the ground state energy :

$$E_0(R) = -\frac{\pi\widetilde{c}(r)}{6R}\,. \tag{1.217}$$

In the ultraviolet limit $r \to 0$, one must recover the results of the CFT :

$$E_0(R) = \frac{2\pi}{R}(\Delta_{\min} + \overline{\Delta}_{\min} - \frac{c}{12})\,. \tag{1.218}$$

One has therefore the relation between the function $\widetilde{c}$ and the central charge c (assuming that the field of minimal dimension is spinless) :

$$\lim_{r\to 0}\widetilde{c}(r) = c - 24\Delta_{\min}\,. \tag{1.219}$$

[10] More details on the Quantum-Statistical correspondence will be given at the end of section 2.1.

The equations. We can now present the TBA equations, for a generic massive theory (a perturbation of a CFT by a primary field φ with conformal dimension Δ), generalizing the equations of [19] for the scaling 3-state Potts and Lee-Yang models. Consider that the scattering in the above QFT is detailed by a S matrix S_{ab}, where the Latin letter a ($b, ...$) indexes the particles of mass m_a ($m_b, ...$). Define :

$$\varphi_{ab}(\theta) = -i\frac{d}{d\theta}\log S_{ab}(\theta)\,. \tag{1.220}$$

The TBA equations are non-linear integral constraints on the pseudo-energies ϵ_a which read (in the case of *diagonal scattering*) [11] :

$$m_a R\cosh(\theta) = \epsilon_a(\theta) + \sum_{b=1}^{n}\int \varphi_{ab}(\theta-\theta')\log(1+e^{-\epsilon_b(\theta')})\frac{d\theta'}{2\pi}\,. \tag{1.221}$$

From the pseudo-energies one can reconstruct the free energy by :

$$f(R) = -\frac{1}{R}\sum_{a=1}^{n}\int_{-\infty}^{\infty} m_a\cosh(\theta)\log(1+e^{-\epsilon_a(\theta)})\frac{d\theta}{2\pi}\,, \tag{1.222}$$

and the partition function :

$$Z(L,R) = \exp\left(L\sum_{a=1}^{n}\int_{-\infty}^{\infty} m_a\cosh(\theta)\log(1+e^{-\epsilon_a(\theta)})\frac{d\theta}{2\pi}\right)\,. \tag{1.223}$$

TBA equations and CFT central charge. We aim to take the UV limit $r \to 0$ and make contact with CFT, in particular through the formula (1.219). For $r \to 0$, an analysis of the TBA equations (1.221) shows that the pseudo-energies $\epsilon_a(\theta)$ flatten and tend to constants ϵ_a^0 in the region $-\log\frac{2}{r} \ll \theta \ll \log\frac{2}{r}$. The functions $\log(1+e^{-\epsilon_a(\theta)})$ have then a plateau shape, whose edges are controlled by the so-called kink solutions of the TBA equations. The UV limit of the scaling function F is then given in terms of these kink solutions which can be remarkably expressed using the Rogers dilogarithm function. Without entering all the technical details, the calculation of the UV central charges goes as follows. First, one should find the constant values ϵ_a^0 that satisfy a set of transcendental equations :

$$\epsilon_a^0 = \sum_{b=1}^{n} N_{ab}\log(1+e^{-\epsilon_b^0})\,, \quad N_{ab} = -\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\varphi_{ab}(\theta)\,. \tag{1.224}$$

Then the central charge is related to the scaling function by

$$\widetilde{c}(0) = -12F(0)\,, \tag{1.225}$$

and simplifies drastically to the nice expression :

$$\widetilde{c}(0) = \frac{6}{\pi^2}\sum_{a=1}^{n} L\left(\frac{1}{1+e^{\epsilon_a^0}}\right)\,, \tag{1.226}$$

where L is the dilogarithm function

$$L(z) = -\frac{1}{2}\int_0^z dt\left(\frac{\log t}{1-t} + \frac{\log(1-t)}{t}\right)\,. \tag{1.227}$$

This formula allows to calculate the central charge of the UV CFT form the scattering S matrix ! It constitutes a very strong verification of the identification between the UV CFT and the IR scattering theory.

[11] We consider only the so called fermionic case, in the terminology of [19].

Mass - coupling constant relation. The next natural step in the analysis of TBA equations is to try to expand the scaling function F in a power series of r. Actually, it is possible to determine the exact value of the next-to-leading term T_0 :

$$T_0 = \frac{(2\pi)^2}{r}\frac{dF}{dr}\bigg|_{r=0}, \tag{1.228}$$

by using again the kink solutions. Furthermore, from numerical investigations of (1.222), the scaling function can be expanded in a power series of the variable $G = r^{2(1-\Delta)}$:

$$F(r) - \frac{T_0}{8\pi^2}r^2 = -\frac{\widetilde{c}}{12} + \sum_{n=1}^{\infty} F_n G^n\,, \tag{1.229}$$

with finite radius of convergence. Let us emphasis that one can get exact values for $\widetilde{c}$ and T_0, whereas the coefficients F_n can be estimated numerically. From a different perspective, the series $\sum_{n=1}^{\infty} F_n G^n$ is interpreted as the perturbative expansion of the free energy in the coupling constant. Recalling that $r = m_0 R$, we should therefore expand perturbatively for example the ground state energy E_0 to operate the matching between m_0 and λ. The perturbative corrections to the ground state energy are [19, 42] :

$$E_0^{(\mathrm{pert})} = -\frac{\pi\widetilde{c}}{6R} - R\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int \langle\varphi(X_1)...\varphi(X_n)\rangle\, d^2X_1...d^2X_n\,. \tag{1.230}$$

Here the correlation functions are computed for a CFT on the cylinder. Mapping the correlation functions to the plane by the transformation $z = e^{-\frac{2\pi i X}{R}}$ one gets :

$$\begin{aligned} E_0^{(\mathrm{pert})}(R) = &-\frac{\pi\widetilde{c}}{6R} - R\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\left(\frac{2\pi}{R}\right)^{2(\Delta-1)n+2} \\ &\times\int \langle V(0)\varphi(z_1,\bar{z}_1)...\varphi(z_1,\bar{z}_1)V(\infty)\rangle \prod_{i=1}^{n}(z_i\bar{z}_i)^{\Delta-1} d^2z_1...d^2z_n\,, \end{aligned} \tag{1.231}$$

where V corresponds to the CFT field with lowest dimension. This means that we have a power series expansion of the form :

$$E_0^{(\mathrm{pert})}(R) = \frac{2\pi}{R}\left(-\frac{\widetilde{c}}{12} + \mathcal{E}_1\lambda R^{2(1-\Delta)} + \mathcal{E}_2\lambda^2 R^{2(1-\Delta)2} + \mathcal{E}_3\lambda^3 R^{2(1-\Delta)3} + ...\right), \tag{1.232}$$

that should converge in some finite region near the point $\lambda R^{2(1-\Delta)} = 0$, and whose coefficients $\mathcal{E}_k$ can be computed as integrals of CFT correlation functions. On the other hand, recall that from thermodynamical considerations, the large R behavior of the ground state energy is given in terms of the infinite volume bulk free energy $\mathcal{E}_0$:

$$E_0^{(\mathrm{pert})}(R) \sim \mathcal{E}_0 R\,, \quad R\to\infty\,. \tag{1.233}$$

Normalizing the ground state energy to zero in infinite volume, we have the relation :

$$E_0(R) = E_0^{(\mathrm{pert})}(R) - \mathcal{E}_0 R\,. \tag{1.234}$$

One can now compare the two expansions (1.229) and (1.232) :

$$-\mathcal{E}_0 R + \frac{2\pi}{R}\left(-\frac{\widetilde{c}}{12} + \mathcal{E}_1\lambda R^{2(1-\Delta)} + \mathcal{E}_2\lambda^2 R^{2(1-\Delta)2} + \mathcal{E}_3\lambda^3 R^{2(1-\Delta)3} + ...\right) \tag{1.235}$$

$$= \frac{2\pi}{R}\left(-\frac{\widetilde{c}}{12} - \frac{T_0}{8\pi^2}r^2 + \sum_{n=1}^{\infty} F_n\left(m_0 R\right)^{2(1-\Delta)n}\right). \tag{1.236}$$

First of all, this implies that we can access the bulk free energy from the TBA data :

$$\mathcal{E}_0 = \frac{m_0^2}{4\pi}T_0\,. \tag{1.237}$$

Second, comparing the terms of the series one arrives at :

$$\left(\frac{\lambda}{m_0^{2(1-\Delta)}}\right)^n \mathcal{E}_n = F_n\,. \tag{1.238}$$

From the CFT computation of the coefficient $\mathcal{E}_1$ and the numerical result for F_1, one therefore gets the exact relation between the coupling constant and the mass. These relations have been established in [43] for a very large range of perturbed CFT. In the paper [44] this has been done for the sine-Gordon (sG) model, and can be applied to obtain a similar relation for the sinh-Gordon [12] model (1.181) :

$$m = \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{2+2b^2})\Gamma(1+\frac{b^2}{2+2b^2})}\left(-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)}\right)^{\frac{1}{2+2b^2}} , \tag{1.239}$$

An analogous formula will be given for the Super sine-Gordon QFT in the chapter 4.

Comparison between TBA and Suzuki equations. The TBA equations are derived from the knowledge of the scattering data, that is of the S matrix of the theory. In our case, for the study of the Super sine-Gordon model, we will use scaling equations based on functional relations in the underlying integrable lattice model, and on the position of the Bethe roots. We will called them *Suzuki equations*, in reference to [45]. The advantage of this second approach is that it will be much easier to make contact with the integrable structure of the field theory, in particular with the **T** and **Q** operators of the ssG theory. This will be explained in details in sections 2.6 and 4.4.

1.7 One point functions from reflection relations

1.7.1 Primary fields

In this section we address the problem of providing explicit formulae for the one point functions of local fields on the plane, thanks to the reflection relations. The first step in this direction was done in the paper [46] where the calculation of one point functions of primary fields in the sine-Gordon theory have been considered. The proposed expression was mainly an interpolation between several known results, rather than a derivation from first principles. In the work [47] a much more fundamental approach to this problem has been elaborated. Considering instead the sinh-Gordon model (1.182), and handling it as a perturbation of the Liouville CFT (1.84), the authors conjectured that the shG QFT should inherit the remarkable reflection properties of the latter model. This was then applied to the computation of the one point functions. Recall that the shG model is defined as a perturbation of the Liouville CFT by the primary field $e^{-b\varphi(z,\bar{z})}$:

$$\mathcal{A}_{\rm shG} = \int \left(\frac{1}{4\pi}\partial_z\varphi(z,\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z}) + \mu e^{b\varphi(z,\bar{z})} + \mu e^{-b\varphi(z,\bar{z})}\right)\frac{idz\wedge d\bar{z}}{2} . \tag{1.240}$$

We would like to compute the one point functions :

$$F(a) = \langle e^{a\varphi}\rangle_{\rm shG} . \tag{1.241}$$

The shG model theory inherits the typical reflection property of the Liouville CFT (1.99) :

$$F(a) = R(a)F(Q-a) , \tag{1.242}$$

where R is the reflection amplitude given in (1.95) :

$$R(a) = \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2}\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2}-1) . \tag{1.243}$$

The one point functions of primary fields verify the natural symmetry :

$$F(a) = F(-a) . \tag{1.244}$$

Both symmetries can be interpreted as the transformation rules of the one point functions under the reflections σ_1 and σ_2 of the weight a :

$$\sigma_1 : a \to -a , \quad \sigma_2 : a \to Q-a . \tag{1.245}$$

[12] A remark is in order. This formula is obtained from the result of the sine-Gordon QFT by analytic continuation with respect to the coupling constant b. Moreover, the conformal dimension in the CFT are taken in the Heisenberg convention $\Delta = -a^2$, this is equivalent to consider the shG model as a perturbation of the free boson CFT rather than of the Liouville model. This explains the correctness of the power exponent $\frac{1}{2+2b^2}$ if we compare with (1.112).

The one point functions $F(a)$ are then calculated as the minimal, meromorphic solutions of the reflection equations (1.242),(1.244). The result reads [46]

$$F(a) = \langle e^{a\varphi} \rangle_{\rm shG} = \left(-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)} \right)^{-\frac{a^2}{1+b^2}} \exp\left(-\int_0^\infty \frac{dt}{t} \left(\frac{\sinh^2(2abt)}{2\sinh(b^2t)\sinh(t)\cosh((1+b^2)t)} - 2a^2 e^{-2t} \right) \right) . \tag{1.246}$$

Let us recall some heuristic argument for the validity of the conjecture (1.242) from [47] : expanding the one point function in a power series in the coupling parameter μ one obtains :

$$\langle e^{a\varphi(z,\bar{z})} \rangle_{\rm ShG} = \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \int dw_1^2 ... dw_n^2 \, \langle e^{a\varphi(z,\bar{z})} e^{-b\varphi(w_1,\bar{w_1})} ... e^{-b\varphi(w_n,\bar{w_n})} \rangle_{\rm L} \, , \tag{1.247}$$

where the sum is expressed in terms of correlation functions written in the Liouville CFT. We can then formally use the reflection relation (1.99) of the Liouville theory in the integrals :

$$\langle e^{a\varphi(z,\bar{z})} e^{-b\varphi(w_1,\bar{w_1})} ... e^{-b\varphi(w_n,\bar{w_n})} \rangle = R(a) \, \langle e^{(Q-a)\varphi(z,\bar{z})} e^{-b\varphi(w_1,\bar{w_1})} ... e^{-b\varphi(w_n,\bar{w_n})} \rangle \, , \tag{1.248}$$

which gives the expected result. Let us make some remarks :

1. The above derivation is loose since the convergence of the integrals is hard to show. It is possible to make this argument more rigorous by working with a theory on a curved surface of a certain area, that is then integrated out. Still the convergence of the above series is hard to demonstrate. This is why the relations above for the one point functions were rather conjectures, when they were established.

2. The result (1.246) can be transferred to the case of the sine-Gordon model, described by the action :
$$\mathcal{A}_{\rm sG} = \int \left(\frac{1}{4\pi} \partial_z \varphi(z,\bar{z}) \partial_{\bar{z}} \varphi(z,\bar{z}) - 2\mu \cos(\beta\varphi) \right) \frac{idz \wedge d\bar{z}}{2} \, , \tag{1.249}$$
and which primary fields are of the form $V_\alpha = e^{i\alpha\varphi}$. The result for $\langle e^{i\alpha\varphi} \rangle_{\rm sG}$ is simply obtained by continuing the formula (1.246) to the complex domain :
$$\beta \to ib \, , \quad \alpha \to ia \, , \tag{1.250}$$
and is given by
$$\langle e^{i\alpha\varphi} \rangle_{\rm sG} = \left(\frac{\mu\pi\Gamma(1-\beta^2)}{\Gamma(\beta^2)} \right)^{\frac{\alpha^2}{1-\beta^2}} \exp\left(\int_0^\infty \frac{dt}{t} \left(\frac{\sinh^2(2\alpha\beta t)}{2\sinh(\beta^2 t)\sinh(t)\cosh((1-\beta^2)t)} - 2\alpha^2 e^{-2t} \right) \right) . \tag{1.251}$$

3. Finally, let us briefly mention as an application of the formula (1.251), that it is possible to compute the one point functions of the primary fields in the massive QFTs constructed as perturbations of Minimal Models $M_{pp'}$ by the field $V_{1,3}$. This can be done using the fact that the sG model has a symmetry with respect to the quantum group $U_q(\mathfrak{sl}_2)$ with $q = e^{\frac{i\pi}{\beta^2}}$. Then one can calculate the one point functions in the perturbed CFT by applying the Quantum Group restriction procedure [49], with the relation $\beta^2 = \frac{p}{p'}$ where $p \wedge p' = 1$. Other applications (for the boundary sine-Gordon theory) of the reflection relations method can be found in [50].

Ratio of one point functions for primary fields. Before going further we would like to explain how one can compute the following specific ratio of one point functions

$$f(a) = \frac{\langle e^{(a-b)\varphi} \rangle_{\rm shG}}{\langle e^{a\varphi} \rangle_{\rm shG}} \, , \tag{1.252}$$

without using the explicit formula (1.246). In this expression the numerator consist in the normal order product of the primary field V_a with the perturbation field $V_{-b} = e^{-b\varphi}$ of the Liouville CFT. We have

$$f(a) = \frac{F(a-b)}{F(a)} \,. \tag{1.253}$$

Since $F(a) = F(-a)$ one rewrites (1.242) as $F(a) = F(a-Q)R(a)$. It implies the following functional relation on f :

$$f(a-Q) = \frac{F(a-Q-b)}{F(a-Q)} = \frac{R(a)}{R(a-b)} f(a) \,. \tag{1.254}$$

Compute using (1.95)

$$\begin{aligned}
\frac{R(a)}{R(a-b)} &= \left(\pi\mu\gamma(b^2)\right)^{-2} \frac{\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2}-1)}{\gamma(2(a-b)b-b^2)\gamma(2ab^{-1}-b^{-2}-3)} \\
&= \left(\pi\mu\gamma(b^2)\right)^{-2} \frac{\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2}-1)}{\gamma(2(a-Q)b-b^2+2)\gamma(2ab^{-1}-b^{-2}-3)} \\
&= \left(\pi\mu\gamma(b^2)\right)^{-2} \frac{\gamma(2ab-b^2)}{\gamma(2(a-Q)b-b^2)} \frac{(2ab^{-1}-b^{-2}-2)^2(2ab^{-1}-b^{-2}-3)^2}{(2(a-Q)b-b^2)^2(2(a-Q)b-b^2+1)^2} \\
&= \left(\pi\mu\gamma(b^2)b^4\right)^{-2} \frac{\gamma(2ab-b^2)}{\gamma(2(a-Q)b-b^2)} \frac{(2a-2b-b^{-1})^2(2a-3b-b^{-1})^2}{(2a-3b-2b^{-1})^2(2a-3b-b^{-1})^2} \\
&= \left(\pi\mu\gamma(b^2)b^4\right)^{-2} \frac{\gamma((2a-b)b)}{\gamma((2(a-Q)-b)b)} \frac{\gamma\left(\frac{1}{2Q}(2(a-Q)-b)\right)}{\gamma\left(\frac{1}{2Q}(2a-b)\right)} \frac{\gamma\left(\frac{1}{2}+\frac{1}{2Q}(2a-b)\right)}{\gamma\left(\frac{1}{2}+\frac{1}{2Q}(2(a-Q)-b)\right)} \,.
\end{aligned}$$

The last line is written in a suitable form for solving the equation (1.254). We infer

$$f(a) = \left(\pi\mu\gamma(b^2)b^4\right)^{\frac{2a-b}{Q}} \frac{\gamma\left(\frac{1}{2Q}(2a-b)\right)}{\gamma\left(\frac{1}{2Q}(2a+b^{-1})\right)\gamma((2a-b)b)} \,. \tag{1.255}$$

One can recover this formula from the integral expression (1.246). The calculation of f will be important when we will study the fermionic basis, indeed the explicit formula (1.255) of f will permit the identification of particular fermionic elements with the primary field $V_{a-b} = e^{(a-b)\varphi}$. This will be more precisely described in the chapter 4.

1.7.2 Descendants fields and Riemann-Hilbert problem

The next problem to tackle is the computation of the one point functions of descendant fields in the shG (sG) theory. This problem was considered in the paper [48] and interpreted in a much more fundamental way. The idea is the following : if for a primary field the effect of both reflections on the one point functions is immediately understood (as consequences of the Liouville case), it is not so for the descendants. In particular, if a Virasoro descendant should have a manifest symmetry under the reflection σ_2 (1.245), its transformation rule under σ_1 is unclear. To understand the latter, it is necessary to establish the correspondence between the Virasoro and the Heisenberg descriptions of the Liouville CFT, by the means of a passage matrix $U(a)$. Let us state this in terms of a Riemann-Hilbert problem : define V_N to be the vector containing all the one point functions of Virasoro descendants of a primary field V_a at level N, and define similarly $H_N(a)$ for the Heisenberg descendants. One has straightaway the symmetries :

$$V_N(a) = V_N(Q-a) \,, \quad H_N(a) = H_N(-a) \,. \tag{1.256}$$

The relation between the two is given by :

$$V_N(a) = U(a) H_N(a) \,. \tag{1.257}$$

The Riemann-Hilbert problem can be stated as (presented in such form in [31]) :

$$V(a+Q) = S(a)V(a) \,, \quad S(a) = U(-a)U(a)^{-1} \,. \tag{1.258}$$

It was first uncovered in [48], where the situation was analyzed for the simplest descendant at level two : $L_{-2}\bar{L}_{-2}e^{a\varphi}$. This field was related to its Heisenberg counterpart by :

$$L_{-2}\bar{L}_{-2}e^{a\varphi} = \left(-\frac{1}{4}(\partial\varphi)^2 + \left(\frac{Q}{2}+a\right)\partial^2\varphi\right)\left(-\frac{1}{4}(\bar{\partial}\varphi)^2 + \left(\frac{Q}{2}+a\right)\bar{\partial}^2\varphi\right)e^{a\varphi} \,. \tag{1.259}$$

Now the left (right) hand side of (1.259) has a definite transformation rule under σ_2 (σ_1), which conveys to :

$$(1+2a(Q+2a))^2 \left\langle (\partial\varphi)^2(\bar\partial\varphi)^2 e^{a\varphi}\right\rangle_{\rm shG} = R(a)(1+2(Q-a)(3Q-2))^2 \left\langle (\partial\varphi)^2(\bar\partial\varphi)^2 e^{(Q-a)\varphi}\right\rangle_{\rm shG} . \tag{1.260}$$

It is then possible to compute the one point function of the descendant as the minimal solution of the above equation :

$$\frac{\left\langle (\partial\varphi)^2(\bar\partial\varphi)^2 e^{a\varphi}\right\rangle_{\rm shG}}{\left\langle e^{a\varphi}\right\rangle_{\rm shG}} = -\left(\frac{m\Gamma(\frac{b}{2Q})\Gamma(\frac{1}{2bQ})}{8Q^2\sqrt{\pi}}\right)^4 \times \gamma\left(\frac{a}{Q}-\frac{b}{2Q}\right)\gamma\left(-\frac{a}{Q}-\frac{b}{2Q}\right)\gamma\left(\frac{a}{Q}-\frac{b^{-1}}{2Q}\right)\gamma\left(-\frac{a}{Q}-\frac{b^{-1}}{2Q}\right) , \tag{1.261}$$

where we used (1.239)

$$m = \frac{4\sqrt{\pi}}{\Gamma(\frac{1}{2+2b^2})\Gamma(1+\frac{b^2}{2+2b^2})}\left(-\frac{\mu\pi\Gamma(1+b^2)}{\Gamma(-b^2)}\right)^{\frac{1}{2+2b^2}} , \tag{1.262}$$

the mass of the shG particle.

In fact, the problem was solvable because there exists only *one* non trivial descendant at level 2, both on the Virasoro and on the Heisenberg sides. Going at higher levels is difficult : the problem gets much more involved and it is impossible to compute the one point functions by such a direct approach, unless one possesses extra information on the relation between the two descriptions. As we are going to demonstrate later in the chapter 4, this data is precisely provided by the fermionic basis.

Chapter 2

Integrable Structure of Quantum Field Theories

We have seen that integrability plays an important role in the Quantum Field Theory : in CFT it is possible to construct a series of densities in involution that can survive some types ($V_{1,3}$ for example) of perturbations. In particular the models of QFT that we are interested in are integrable. This is one of the motivations to describe in this chapter the *integrable systems* from a more general point of view. Actually, the integrable systems can be very different in their natures : they range from models of classical mechanics, to specific Quantum Field Theories, and to statistical lattice models. Therefore, as we have already mentioned, there is no unified definition of what an integrable system is. However, one can gather the main recurrent features of this type of models :

- There are many conserved quantities (as least the same number as of degrees of freedom, this means an infinity in the case of QFT) that are in involution, including the Hamiltonian of the system.
- For quantum systems, there exists a quantity that satisfies the Yang-Baxter equation.
- The system is considered to be exactly solvable, in the sense that physically relevant quantities can be in principle calculated analytically.

In this section we shall explain how the integrability in QFT can be used to compute the one point functions. To achieve this goal, we must first establish a description of the space of local operators of the QFT in terms of some "integrable quantities", we will refer to this description of the fields as the integrable structure of the QFT. This was first done for the sine-Gordon model [1, 2, 3, 4, 5], and goes by the name of the *fermionic basis*. The investigation of the fermionic basis in this context started with a deep study of the vacuum expectation values of local operators in the *six vertex model*, the lattice regularization of the sine-Gordon theory. In this chapter, we aim to show how one can use the fermionic basis to describe the UV limit of the sG model, that is its underlying CFT. To do so, we will have first to recollect some information about general integrable systems (throughout this chapter we shall use notations compatible with [3]), and will proceed as follows :

1. Start by recalling some elementary facts about the integrability of the six vertex model (among other we will use the reference [51]). This model is not only the first step towards the fermionic basis but is more generally speaking a prototypical example of lattice integrable systems.
2. Define the main protagonists of the integrable models : the transfer matrix, the Bethe Ansatz equations and the Yang-Baxter equation.
3. Reinterpret the latter in terms of Quantum Groups, and use this general mathematical framework to construct the Q operator. The Quantum Groups will be relevant to go further and illustrate how integrable structures emerge in the context of QFT.
4. Present the Bazhanov-Lukyanov-Zamolodchikov (BLZ) construction that provided the description of the integrable structure in CFT. Remark that the integrable lattice models have continuous limits, in which they give integrable Quantum Field Theories. Thanks to the BLZ construction, it is possible to define the integrable data directly in field theory, without any references to the lattice.

5. Introduce the fermionic basis on the lattice and then explain how it can be extended to CFT using the scaling limit. Finally describe the connection between the fermionic and the Virasoro descriptions. This will give us a very powerful tool for the calculation of the vacuum expectation values at the lattice level and of the one point functions at the QFT level.

2.1 The six vertex model

We consider a statistical system on a rectangular lattice. The lattice has N vertical lines and $\mathbf{n}$ horizontal lines, giving in total $N \times \mathbf{n}$ nodes. Take without restriction of generality both N and $\mathbf{n}$ even. We declare the following boundary conditions : consider that the system is periodic in both directions and think about N as very large, aiming in the following to take it to infinity. Then, the system appears wrapped around a cylinder (more precisely a torus when N is still finite), the compact vertical direction will be termed the *Matsubara* space (or chain), and the horizontal direction will be called the *Space* chain. The lattice in this geometry is represented on the figure 2.1.

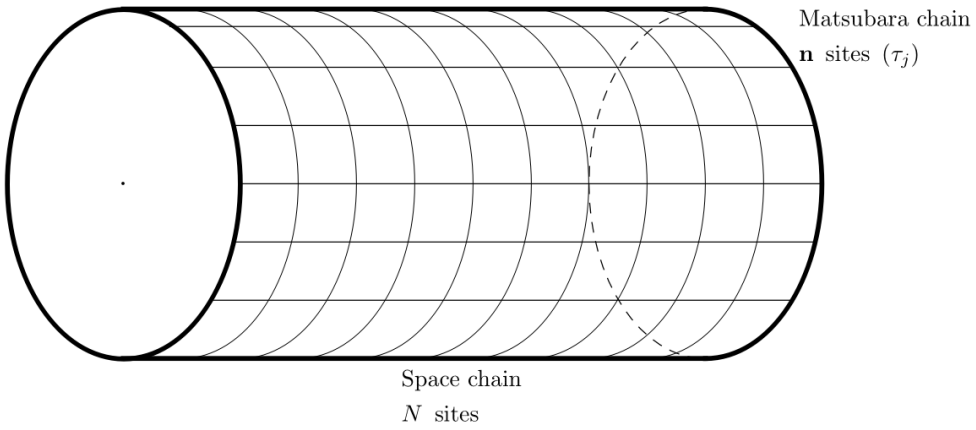

Figure 2.1: 6V model on the cylinder

To each edge of the lattice one can associate an arrow, pointing to the left or to the right for an horizontal edge, and for an vertical edge up or down. For each vertex, the orientation of the neighboring arrows specifies the weight W that will be associated to this vertex in this configuration. For a general configuration of a vertex, we will use the convention of figure 2.2.

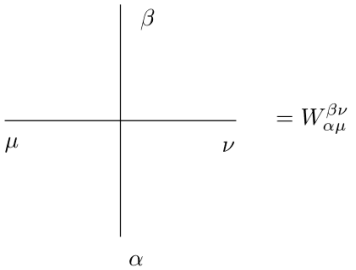

Figure 2.2: Conventions for the weight of a vertex

Notice, that in this picture the line $(\mu - \nu)$ is a vertical line on the cylinder of figure 2.1, and the line $(\alpha - \beta)$ is horizontal.

In the case of interest for us, only vertices that have the same number of incoming and outcoming arrows will carry a non zero weight. This gives a total of six configurations that coined the

name of the six vertex model. Moreover we will also assume that the weights are unchanged under a simultaneous reversal of all arrows of the corresponding vertex, this fact leaves us with only three quantities a, b, c that correspond to the following configurations :

$$a = W_{++}^{++} = W_{--}^{--}\,, \tag{2.1}$$
$$b = W_{+-}^{+-} = W_{-+}^{-+}\,, \tag{2.2}$$
$$c = W_{+-}^{-+} = W_{-+}^{+-}\,. \tag{2.3}$$

The allowed configurations of arrows are represented in the figure 2.3.

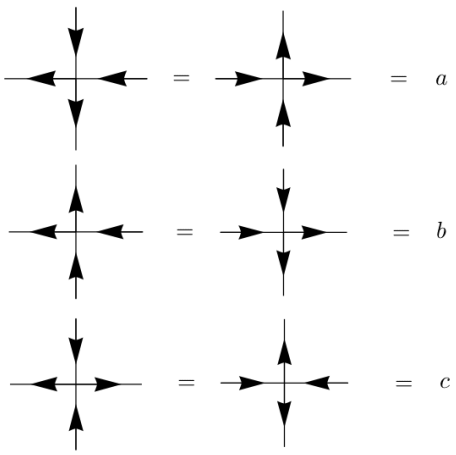

Figure 2.3: Configurations of the six-vertex model

Explicitly, the weights a, b, c that ensure the integrability of the model are :

$$a = \sin(\pi(\theta+\nu))\,, \quad b = \sin(\pi\theta)\,, \quad c = \sin(\pi\nu)\,. \tag{2.4}$$

In writing the weights of the 6 vertex model, we used the usual "trigonometric" notation, that is the most standard when we deal with this system. The number ν is really the parameter of the model, whereas θ is to be thought as a variable on which the thermodynamical functions will depend. This dichotomy will be clearer when we will relate θ to the spectral parameter in (2.30).

As in any problem in statistical physics, our first priority is to calculate the partition function Z_{6V} of the system. The total weight of a particular configuration of arrows x is given by the product of the weights of each vertex :

$$\prod_x W_{\alpha_x\mu_x}^{\beta_x\nu_x}\,, \tag{2.5}$$

and the partition function reads

$$Z_{6V} = \sum_{\alpha_x,\mu_x,\beta_x,\nu_x} \prod_x W_{\alpha_x\mu_x}^{\beta_x\nu_x}\,. \tag{2.6}$$

Let us try to simplify this expression. Each edge of the lattice can support two states : that is an arrow in one direction, or in the opposite. Hence one can associate to every edge the vector space $\mathbb{C}^2$. Take $\{e_+, e_-\}$ the canonical basis of $\mathbb{C}^2$:

$$e_+ = \begin{pmatrix}1\\0\end{pmatrix}\,, \quad e_- = \begin{pmatrix}0\\1\end{pmatrix}\,. \tag{2.7}$$

We declare the following correspondence (in the orientation of figure 2.2) : e_+ represents an arrow pointing to the top or to the right and e_- an arrow pointing to the bottom or to the left. At each vertex we have thus an operator L which acts from a tensor product of two 2-dimensional spaces $V(\alpha)\otimes V(\mu)$ to $V(\beta)\otimes V(\nu)$ (where the letters refer to the position of the spaces in the figure 2.2 and

each $V \simeq \mathbb{C}^2$). Define the 2×2 elementary matrices E_{ij} that satisfy the property $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Then we set :

$$L = W^{\beta\nu}_{\alpha\mu} E_{\beta\alpha} \otimes E_{\nu\mu} \,. \tag{2.8}$$

In the basis $\{e_+ \otimes e_+, e_+ \otimes e_-, e_- \otimes e_+, e_- \otimes e_-\}$ the operator L is given by the matrix :

$$L = \begin{pmatrix} W^{++}_{++} & W^{++}_{+-} & W^{++}_{-+} & W^{++}_{--} \\ W^{+-}_{++} & W^{+-}_{+-} & W^{+-}_{-+} & W^{+-}_{--} \\ W^{-+}_{++} & W^{-+}_{+-} & W^{-+}_{-+} & W^{-+}_{--} \\ W^{--}_{++} & W^{--}_{+-} & W^{--}_{-+} & W^{--}_{--} \end{pmatrix} = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} . \tag{2.9}$$

This interpretation in terms of linear operators is important. From now on, we will call the vector spaces of the Matsubara direction, the "Matsubara spaces", and those that are in the horizontal direction will be termed *auxiliary spaces* or *spaces in the Space chain*. Define the operator T_a :

$$T_a = L_{a,\mathbf{n}} L_{a,\mathbf{n}-1} ... L_{a,1} \in \mathrm{End}(V_a \otimes V_1 ... \otimes V_{\mathbf{n}}) \,, \tag{2.10}$$

where $L_{a,\mathbf{k}}$ acts like L but only on the auxiliary space a and on one of the Matsubara spaces $\mathbf{k}$. To include the periodicity condition in the vertical direction we need to take the trace of (2.10) in the auxiliary space :

$$\widetilde{T} = \mathrm{Tr}_{V_a}(T_a) \,. \tag{2.11}$$

Explicitly, $\widetilde{T}$ acts then on the total Matsubara space, and its components are given by :

$$\widetilde{T}^{\beta_1 ... \beta_n}_{\alpha_1 ... \alpha_n} = W^{\beta_1\nu_2}_{\alpha_1\nu_1} W^{\beta_2\nu_3}_{\alpha_2\nu_2} ... W^{\beta_n\nu_1}_{\alpha_n\nu_n} \,. \tag{2.12}$$

This formula allows to rewrite the partition function (2.6) in a much more illuminating way. Because of the periodicity in the Space direction, Z_{6V} is the trace over the Matsubara space of a product of $\widetilde{T}$ operators :

$$Z_{6V} = \mathrm{Tr}(\widetilde{T}^N) \,, \tag{2.13}$$

The understanding of $\widetilde{T}$, and in particular of its eigenvalues (denote them ψ_k), gives access to the thermodynamical quantities of the model. First we get the partition function :

$$\widetilde{T}\psi_k = \Lambda_k \psi_k \,, \quad Z_{6V} = \sum_{k=1}^{2^{\mathbf{n}}} \Lambda_k^N \,. \tag{2.14}$$

As usual in statistical physics, the knowledge of Z_{6V} leads to all other physical quantities. For example the free energy per lattice site is :

$$f = -\frac{1}{\mathbf{n}N} \log(Z_{6V}) \sim -\frac{1}{\mathbf{n}} \log \Lambda_{\max} \,, \quad N \to \infty \,. \tag{2.15}$$

This formula is of course reminiscent of the expression for the free energy in Quantum Field Theory on a cylinder of radius R (1.212), (1.83). Recall that the Casimir energy E_0 of a CFT is simply related to the (effective) central charge $\widetilde{c}$:

$$E_0 = -\frac{\pi \widetilde{c}}{6R} \,. \tag{2.16}$$

When studying the maximal eigenvalue $\Lambda_{\max}$ it is therefore possible to match the lattice integrable models with their continuum CFT counterparts, by explicitly calculating the central charges. This was successfully done in [52] for the six and nineteen vertex models.

Quantum - Statistical correspondence. There exists a well known equivalence in physics between statistical models in d dimensions and quantum systems in $d-1$ dimensions. We met it already in the discussion on the TBA in section 1.6, where a QFT on a cylinder of radius R was interpreted as a 2d statistical model and at the same time as a quantum system at finite temperature $\frac{1}{R}$ (see (1.209)). Here, we shall illustrate this correspondence in the case of the six vertex model, and show how it is equivalent to the *XXZ spin chain*. To be consistent with our further discussion we shall slightly change our point of view. If in the computation of the partition function Z_{6V} the

emphasis was done on the Matsubara space, we concentrate now on the Space direction. This is the vector space :

$$\mathfrak{H}_{\mathbf{S}} = \bigotimes_{k=1}^{N} V_{a_k} , \quad V_{a_k} \simeq \mathbb{C}^2 . \tag{2.17}$$

The crucial point is that the operator L (2.9) evaluated at $\theta = 0$ is (up to a multiplicative factor) the permutation operator P_{perm} :

$$L(0) = \sin(\pi\nu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \sin(\pi\nu) P_{\text{perm}} . \tag{2.18}$$

In terms of indices this reads :

$$L^{\beta\nu}_{\alpha\mu}(0) = \sin(\pi\nu)\delta^{\nu}_{\alpha}\delta^{\beta}_{\mu} . \tag{2.19}$$

Now we construct a "Space chain T operator" $\mathcal{T}$, analogue of (2.11) but that acts now on $\mathfrak{H}_{\mathbf{S}}$, and that corresponds to a shift operator along the Space direction for $\theta = 0$:

$$\mathcal{T}^{\eta_1 \dots \eta_N}_{\mu_1 \dots \mu_N}(0) \; : \; \mathfrak{H}_{\mathbf{S}} \to \mathfrak{H}_{\mathbf{S}} , \tag{2.20}$$

$$\mathcal{T}^{\eta_1 \dots \eta_N}_{\mu_1 \dots \mu_N}(0) = (\sin(\pi\nu))^N \delta^{\eta_2}_{\mu_1}\delta^{\eta_3}_{\mu_2} \dots \delta^{\eta_1}_{\mu_N} . \tag{2.21}$$

It is possible to find the inverse of this operator as the inversed shift operator :

$$\left(\mathcal{T}^{\eta_1 \dots \eta_N}_{\mu_1 \dots \mu_N}(0)\right)^{-1} = (\sin(\pi\nu))^{-N} \delta^{\eta_N}_{\mu_1}\delta^{\eta_1}_{\mu_2} \dots \delta^{\eta_{N-1}}_{\mu_N} . \tag{2.22}$$

Now consider the derivative of L with respect to $\pi\theta$ evaluated at $\theta = 0$:

$$\frac{1}{\pi}\frac{d}{d\theta} L\Big|_{\theta=0} = \begin{pmatrix} \cos(\pi\nu) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cos(\pi\nu) \end{pmatrix} . \tag{2.23}$$

Applying the same derivative to $\mathcal{T}$ we get :

$$\frac{1}{\pi}\frac{d}{d\theta} \mathcal{T}^{\eta_1 \dots \eta_N}_{\mu_1 \dots \mu_N}\Big|_{\theta=0} = (\sin(\pi\nu))^N \sum_{k=1}^{N} \delta^{\eta_2}_{\mu_1} \dots \delta^{\eta_{k-1}}_{\mu_{k-2}} \left(\frac{1}{\pi}\frac{d}{d\theta} L^{\eta_k \eta_{k+1}}_{\mu_k \mu_{k-1}}\Big|_{\theta=0} \right) \delta^{\eta_{k+2}}_{\mu_{k+1}} \dots \delta^{\eta_1}_{\mu_N} . \tag{2.24}$$

Now using (2.23) and (2.22) in the previous equation one obtains the logarithmic derivative of $\mathcal{T}$:

$$\left(\mathcal{T}^{-1}(0) \frac{1}{\pi}\frac{d}{d\theta} \mathcal{T}(0) \right)^{\eta_1 \dots \eta_N}_{\mu_1 \dots \mu_N} = \sum_{k=1}^{N} \left(L^{-1}(0) \frac{1}{\pi}\frac{d}{d\theta} L(0) \right)^{\eta_k \eta_{k+1}}_{\mu_k \mu_{k+1}} . \tag{2.25}$$

Recall now the definition of the Pauli matrices :

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.26}$$

One can see that the sum in (2.25) is exactly :

$$\frac{1}{2} \sum_{k=1}^{N} \left(\sigma^1_k \sigma^1_{k+1} + \sigma^2_k \sigma^2_{k+1} + \cos(\pi\nu) \sigma^3_k \sigma^3_{k+1} \right) + \frac{1}{2} \sum_{k=1}^{N} \mathrm{Id}_k \, \mathrm{Id}_{k+1} . \tag{2.27}$$

Dropping the last trivial term, we arrive at the Hamiltonian of the XXZ chain :

$$H_{XXZ} = \frac{1}{2} \sum_{k=1}^{N} \left(\sigma^1_k \sigma^1_{k+1} + \sigma^2_k \sigma^2_{k+1} + \cos(\pi\nu) \sigma^3_k \sigma^3_{k+1} \right) . \tag{2.28}$$

We have then shown that :

$$\frac{1}{\pi}\frac{d}{d\theta} \log(\mathcal{T}(\theta))\Big|_{\theta=0} = \sin(\pi\nu) H_{XXZ} . \tag{2.29}$$

We can now make more concrete the correspondence that we stated before. From the formula above it is explicit that H_{XXZ} commutes with the matrix $\mathcal{T}$ *for any value of its variable* θ. Finding the eigenvectors of the $\mathcal{T}(\theta)$ operator is then the same problem as diagonalizing the Hamiltonian of the XXZ chain.

2.2 The Transfer matrix

After this preparatory discussion on the six vertex model we shall take a more general point of view on the subject, and introduce the terminology that will be used all along the rest of this thesis. We will rely on [3]. For our following discussion (in particular to make connection with the Quantum Groups), it will be more convenient to change variables to the *spectral parameter* ζ, and to the *Quantum Group parameter* q :

$$\zeta = e^{i\pi\nu\theta}\,, \quad q = e^{i\pi\nu}\,. \tag{2.30}$$

We still consider a lattice in the cylinder geometry of 2.1, however we allow the following generalization : the spaces in the Matsubara direction can carry any spin $s_{\mathbf{m}}$ and a inhomogeneity $\tau_{\mathbf{m}}$. The Matsubara space is then :

$$\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{m}=1}^{\mathbf{n}} \mathbb{C}^{2s_{\mathbf{m}}+1}\,. \tag{2.31}$$

For the space chain we still have the same structure

$$\mathfrak{H}_{\mathbf{S}} = \bigotimes_{k=1}^{N} \mathbb{C}^2\,. \tag{2.32}$$

Now, instead of dealing with the weight matrix (2.9) we consider the L *operator* [1] :

$$L_{j,\mathbf{m}}(\zeta) = q^{\frac{1}{2}} \begin{pmatrix} \zeta^2 q^{\frac{H+1}{2}} - q^{-\frac{H+1}{2}} & (q-q^{-1})\zeta F q^{\frac{H-1}{2}} \\ (q-q^{-1})\zeta q^{-\frac{H-1}{2}} E & \zeta^2 q^{-\frac{H-1}{2}} - q^{\frac{H-1}{2}} \end{pmatrix}_j\,. \tag{2.33}$$

$L_{j,\mathbf{m}}$ acts on the site j of the Space chain and on the site $\mathbf{m}$ of the Matsubara direction. E, F, H are the images of the generating elements of the algebra $\mathfrak{sl}_2$ in the irreducible representation of spin $s_{\mathbf{m}}$ (that we denote $V(2s_{\mathbf{m}}+1)$). The case that will be of most interest for us is to take spin $\frac{1}{2}$ Matsubara spaces. In this situation the natural formula holds

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{2.34}$$

and the L operator is :

$$L_{j,\mathbf{m}}(\zeta) = q^{\frac{1}{2}} \begin{pmatrix} \zeta^2 q - q^{-1} & & & \\ & \zeta^2 - 1 & (q-q^{-1})\zeta & \\ & (q-q^{-1})\zeta & \zeta^2 - 1 & \\ & & & \zeta^2 q - q^{-1} \end{pmatrix}\,. \tag{2.35}$$

Taking into account the change of variables (2.30), we recover in this case exactly the weight matrix (2.9) (up to some multiplicative prefactor). From the knowledge of the L operator, one can build the monodromy matrix in the same way as it was done for (2.10). Bearing in mind the presence of inhomogeneities for each Matsubara space, we have :

$$T_{j,\mathbf{M}}(\zeta) = L_{j,\mathbf{n}}(\tfrac{\zeta}{\tau_{\mathbf{n}}}) L_{j,\mathbf{n}-1}(\tfrac{\zeta}{\tau_{\mathbf{n}-1}}) \ldots L_{j,\mathbf{1}}(\tfrac{\zeta}{\tau_{\mathbf{1}}})\,. \tag{2.36}$$

The monodromy matrix is a linear operator acting in :

$$T_{j,\mathbf{M}}(\zeta) \in \mathrm{End}\left(V_j \otimes \mathfrak{H}_{\mathbf{M}}\right)\,. \tag{2.37}$$

Introduce a twist κ that will change the periodic boundary condition in the Matsubara direction, and define the twisted transfer matrix $T_{\mathbf{M}}$:

$$T_{j,\mathbf{M}}(\zeta,\kappa) = T_{j,\mathbf{M}}(\zeta) q^{\kappa\sigma_j^3}\,, \tag{2.38}$$

$$T_{\mathbf{M}}(\zeta,\kappa) = \mathrm{Tr}_j(T_{j,\mathbf{M}}(\zeta,\kappa))\,. \tag{2.39}$$

The previous formula paves the way to the calculation of the (twisted by κ) partition function of the six vertex model :

$$Z_{6V}^{\kappa} = \mathrm{Tr}_{\mathbf{M}}(T_{\mathbf{M}}(\zeta,\kappa)^N)\,, \tag{2.40}$$

[1] The subscript j indicates that the matrix $L_{j,\mathbf{m}}$ is written in a basis of the space j, that is the entries of $L_{j,\mathbf{m}}$ are *operators* acting on the space $\mathbb{C}^{2s_{\mathbf{m}}+1}$ at the site $\mathbf{m}$.

and will be the starting point of our calculations with the fermionic basis.

To close this section, we emphasize the critical property that the L operator (2.33) satisfies the Yang-Baxter equation (which is equivalent to the integrability of the model) :

$$L_{12}(\zeta/\mu)L_{13}(\zeta)L_{23}(\mu) = L_{23}(\mu)L_{13}(\zeta)L_{12}(\zeta/\mu)\,, \tag{2.41}$$

which is holding in the space

$$V_1 \otimes V_2 \otimes V_3\,. \tag{2.42}$$

Here V_i $(i = 1, 2, 3)$ can refer to any space in the Space or Matsubara chains. Once we have taken all Matsubara spaces to be of spin $\frac{1}{2}$ we have $V_i \simeq \mathbb{C}^2$. Then the formula (2.35) for $L(\zeta)$ can be decomposed as follows :

$$L(\zeta) = L^{\beta\nu}_{\alpha\mu}(\zeta)E_{\beta\alpha} \otimes E_{\nu\mu} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)\,. \tag{2.43}$$

The operators $L_{i\,j}$ of (2.41) are defined to be :

$$L_{12}(\zeta) = L(\zeta) \otimes \mathrm{Id}_2\,, \quad L_{23}(\zeta) = \mathrm{Id}_2 \otimes L(\zeta)\,, \quad L_{13}(\zeta) = L^{\beta\nu}_{\alpha\mu}(\zeta)E_{\beta\alpha} \otimes \delta_{\rho\sigma}E_{\rho\sigma} \otimes E_{\nu\mu}\,. \tag{2.44}$$

The equation (2.41) implies the Yang-Baxter equation for monodromy matrices (2.38) :

$$L_{jk}(\zeta/\mu)T_{j\mathbf{M}}(\zeta,\kappa)T_{k\mathbf{M}}(\mu,\kappa) = T_{k\mathbf{M}}(\mu,\kappa)T_{j\mathbf{M}}(\zeta,\kappa)L_{jk}(\zeta/\mu)\,, \tag{2.45}$$

that holds in the space $V_j \otimes V_k \otimes \mathfrak{H}_{\mathbf{M}}$ (the first two spaces $V_j \otimes V_k$ are in the Space chain). The immediate consequence of the Yang-Baxter equation (2.45) is the commutativity of the Matsubara transfer matrix at different values of the spectral parameter :

$$[T_{\mathbf{M}}(\zeta,\kappa), T_{\mathbf{M}}(\lambda,\kappa)] = 0\,. \tag{2.46}$$

The Yang-Baxter equation is of primary importance, we will see how it appears naturally if we consider the derivation of the L operators from the Quantum Group $U_q(\widehat{\mathfrak{sl}_2})$. Moreover it is the starting point of a powerful method to diagonalize the transfer matrix, which is termed the *Algebraic Bethe Ansatz*. This is the topic of our next section.

2.3 Algebraic Bethe Ansatz

In this section we describe the Algebraic Bethe Ansatz (ABA), the most traditional method to diagonalize the transfer matrix. It was first introduce in [39], a very complete reference is [40]. Let us start by rewriting the twisted transfer matrix (2.38) in terms of operators acting on the Matsubara space

$$T_{j,\mathbf{M}}(\zeta,\kappa) = \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix}_j \begin{pmatrix} q^\kappa & 0 \\ 0 & q^{-\kappa} \end{pmatrix}_j\,. \tag{2.47}$$

where A, B, C, D are elements of $\mathrm{End}(\mathfrak{H}_{\mathbf{M}})$. The equation (2.39) gives :

$$T_{\mathbf{M}}(\zeta,\kappa) = A(\zeta)q^\kappa + D(\zeta)q^{-\kappa}\,. \tag{2.48}$$

Our goal is therefore to diagonalize this operator. The main point of the ABA is to make an educated guess for the eigenvalues of $T_{\mathbf{M}}(\zeta,\kappa)$. First, consider the "vacuum" vector

$$\Omega = e_- \otimes ... \otimes e_- \in \mathfrak{H}_{\mathbf{M}}\,. \tag{2.49}$$

The action of $L_{j,\mathbf{m}}$ (2.35) on e_- implies the relations :

$$A(\zeta)\Omega = q^{\frac{\mathbf{n}}{2}}a(\zeta)\Omega\,, \quad D(\zeta)\Omega = q^{-\frac{\mathbf{n}}{2}}d(\zeta)\Omega\,, \quad B(\zeta)\Omega = 0\,, \tag{2.50}$$

where the functions $a(\zeta), d(\zeta)$ are :

$$a(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left((\zeta/\tau_{\mathbf{m}})^2 - 1\right)\,, \quad d(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left((\zeta q/\tau_{\mathbf{m}})^2 - 1\right) \quad . \tag{2.51}$$

The vector Ω can hence be considered as a highest weight vector, that is annihilated by the operator B. It is then natural to look for eigenvectors of $T_{\mathbf{M}}$ by acting on Ω with the "creation" operators C. Define

$$\Phi(\{\lambda_i\}) = C(\lambda_1)...C(\lambda_l)\Omega\,. \tag{2.52}$$

One can show that $\Phi(\{\lambda_i\})$ is indeed an eigenvector of $T_{\mathbf{M}}$ if the set of $\{\lambda_i\}_{i=1}^l$ does satisfy certain equations that are referred to as the *Bethe Ansatz Equation* (BAE). To demonstrate this, we should establish the commutation relations between the operators A, B, C, D. They are obtained by expliciting the Yang-Baxter equation (2.45). Omitting arguments (A, B, C, D are function of ζ, A', B', C', D' of μ and a, b, c of $\frac{\zeta}{\mu}$) we have :

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} AA' & AB' & BA' & BB' \\ AC' & AD' & BC' & BD' \\ CA' & CB' & DA' & DB' \\ CC' & CD' & DC' & DD' \end{pmatrix} = \begin{pmatrix} A'A & B'A & A'B & B'B \\ C'A & D'A & C'B & D'B \\ A'C & B'C & A'D & B'D \\ C'C & D'C & C'D & D'D \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \tag{2.53}$$

where we have used a, b, c as a short hand notation for the entries of the L operator (2.35) (not to be mistaken with the particular values of the weights) :

$$L = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}. \tag{2.54}$$

Then one can obtain the algebra of operators A, B, C, D. For example :

$$[C(\zeta), C(\lambda)] = 0\,, \tag{2.55}$$

$$A(\zeta)C(\lambda) = \frac{a(\zeta/\mu)}{b(\zeta/\mu)}C(\lambda)A(\zeta) - \frac{c(\zeta/\mu)}{b(\zeta/\mu)}C(\zeta)A(\lambda)\,, \tag{2.56}$$

$$D(\zeta)C(\lambda) = \frac{a(\mu/\zeta)}{b(\mu/\zeta)}C(\mu)D(\zeta) - \frac{c(\mu/\zeta)}{b(\mu/\zeta)}C(\zeta)D(\mu)\,. \tag{2.57}$$

Define the quotient functions :

$$u(\zeta) = \frac{a(\zeta)}{b(\zeta)} = \frac{\zeta q - \zeta^{-1}q^{-1}}{\zeta - \zeta^{-1}}\,, \quad v(\zeta) = \frac{c(\zeta)}{b(\zeta)} = \frac{q - q^{-1}}{\zeta - \zeta^{-1}}\,. \tag{2.58}$$

Iterating the relations (2.56),(2.57), one can compute $A(\zeta)C(\lambda_1)...C(\lambda_l)\Omega$ and $D(\zeta)C(\lambda_1)...C(\lambda_l)\Omega$ and obtain the action of A and D on the candidate eigenvector $\Phi(\{\lambda_i\})$. This implies the following eigenvector condition :

$$T_{\mathbf{M}}(\zeta, \kappa)\Phi(\{\lambda_i\}) = (q^{\kappa}A(\zeta) + q^{-\kappa}D(\zeta))\Phi(\{\lambda_i\}) = \Lambda(\zeta, \{\lambda_i\})\Phi(\{\lambda_i\})\,, \tag{2.59}$$

where the eigenvalue $\Lambda(\zeta, \{\lambda_i\})$ is expressed through the first terms of the right hand sides of (2.56) and (2.57) :

$$\Lambda(\zeta, \{\lambda_i\}) = q^{\kappa+\frac{\mathbf{n}}{2}}a(\zeta)\prod_{j=1}^{l}u(\zeta/\lambda_j) + q^{-\kappa-\frac{\mathbf{n}}{2}}d(\zeta)\prod_{j=1}^{l}u(\lambda_j/\zeta)\,, \tag{2.60}$$

given that the set $\{\lambda_i\}_{i=1}^l$ satisfies the Bethe Ansatz Equations :

$$q^{\kappa+\frac{\mathbf{n}}{2}}a(\lambda_i)\prod_{k=1}^{l}u(\lambda_j/\lambda_k) - q^{-\kappa-\frac{\mathbf{n}}{2}}d(\lambda_i)\prod_{k=1}^{l}u(\lambda_k/\lambda_j) = 0\,, \quad 1 \leq j \leq l\,. \tag{2.61}$$

The BAE can be rewritten is the following compact way (use the fact that $\frac{u(\lambda_j/\lambda_k)}{u(\lambda_k/\lambda_j)} = q^{-2}\frac{\lambda_k^2 - q^2\lambda_j^2}{\lambda_k^2 - q^{-2}\lambda_j^2}$) :

$$q^{2\kappa+\mathbf{n}-2l}\frac{a(\lambda_j)}{d(\lambda_j)}\prod_{k\neq j}^{l}\frac{\lambda_k - q^2\lambda_j^2}{\lambda_k - q^{-2}\lambda_j^2} = 1\,, \quad 1 \leq j \leq l\,. \tag{2.62}$$

Notice that the terms $q^{\mathbf{n}-2l}$ represents the spin of the vector $\Phi(\{\lambda_i\})$. Indeed, define the spin operator in the Matsubara direction to be :

$$S_{\mathbf{M}} = \frac{1}{2}\sum_{\mathbf{m}=1}^{\mathbf{n}} \sigma^3_{\mathbf{m}}\,, \tag{2.63}$$

then

$$S_{\mathbf{M}}\Phi(\{\lambda_i\}) = \left(l - \frac{\mathbf{n}}{2}\right)\Phi(\{\lambda_i\})\,. \tag{2.64}$$

This will be important to properly define the Q operator latter in section 2.5. Let us now explain how the L operator can be recovered from the more fundamental Quantum Group approach.

2.4 Quantum Groups

There is a more mathematical way to look at integrable systems, and it deals with the technology of Quantum Groups. As we will see, in the Quantum Group set-up it is possible to construct an R matrix, that automatically satisfies the Yang-Baxter equation. One of the advantages of this approach is that it will allow us to define the so-called Q operator, on the same footing as was done for the transfer matrix, and show the similarities between these two objects. Without going into the generalities of all Quantum Groups, after a brief recall, we shall concentrate in the case that will be of most interest for us : the Quantum Group $U_q(\widehat{\mathfrak{sl}_2})$ [2]. For this section we use the references [2, 53, 54, 55].

Generalities

The definition of a quantum group A is the following : it is a *Hopf algebra* that satisfies the *quasi-triangularity* property. It means that A is an associative bialgebra over $\mathbb{C}$, endowed with a *coproduct* homomorphism $\Delta : A \to A \otimes A$, a *co-unit map* homomorphism $\epsilon : A \to \mathbb{C}$ and an *antipode* anti-homomorphism $S : A \to A$, that satisfy natural compatibility conditions. Notice that if we define σ to be the permutation of $A \otimes A$

$$\sigma(a \otimes b) = b \otimes a\,, \quad a, b \in A\,, \tag{2.66}$$

then one can set another coproduct Δ' :

$$\Delta' = \sigma \circ \Delta\,. \tag{2.67}$$

The quasi-triangularity property is the existence of an element $\mathcal{R} \in A \otimes A$, the *universal R matrix*, that satisfies the three axioms :

(i)

$$\mathcal{R}\Delta(x) = \Delta'(x)\mathcal{R}\,, \quad \forall x \in A\,, \tag{2.68}$$

(ii)

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}\,, \tag{2.69}$$
$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\,. \tag{2.70}$$

(iii)

$$(\epsilon \otimes \mathrm{Id})\mathcal{R} = 1 = (\mathrm{Id} \otimes \epsilon)\mathcal{R}\,, \tag{2.71}$$
$$(S \otimes \mathrm{Id})\mathcal{R} = \mathcal{R}^{-1} = (\mathrm{Id} \otimes S)\mathcal{R}\,. \tag{2.72}$$

[2] The algebra $\mathfrak{sl}_2$ is in some sense the fundamental symmetry algebra of the sequence of theories (both lattice and QFT) that we are considering. It governs the integrable 6 vertex model as well as the 19 vertex model, which is a higher spin version (in the representation theoretic language) of the former. In addition to describe the lattice models, the quantum version of $\mathfrak{sl}_2$ further allows to exhibit the integrable structure of the series of related continuum QFTs. We recall therefore again the relation between the Quantum Group parameter q and the "physical" parameter ν :

$$q = e^{i\pi\nu}\,. \tag{2.65}$$

Here again we used the convention $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ and similar formulae for $\mathcal{R}_{13}$ and $\mathcal{R}_{23}$. Remark, that the R matrix is an invertible element of $A \otimes A$, and the antipode S can be thought of as an "inverse" in the algebra A. This will be also more transparent when we will see examples of the action of S on basis elements of $U_q(\widehat{\mathfrak{sl}_2})$.

From the equations (2.68) as well as (2.69), (2.70) one can derive the Yang-Baxter equation for the universal R matrix, that holds in $A \otimes A \otimes A$:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,. \tag{2.73}$$

Let us next analyze the implications of the Quantum Group structure on the representations of the algebra A. This is natural since we would like to apply the Yang-Baxter equation to intertwine different representation spaces. Consider (π^a, V_a) :

$$\pi^a \,:\, A \to \mathrm{End}(V_a)\,, \tag{2.74}$$

to be a representation of the algebra A. The key fact is that the Hopf algebra maps Δ, ϵ, S have direct interpretations in the representation theoretic language. Indeed, we can use Δ to define a representation on the tensor product of two representations (which is not usually possible for a generic algebra). Furthermore ϵ provides us with the unit representation of A :

$$\Delta\text{: tensor product of reps.} \qquad A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi^1 \otimes \pi^2} \mathrm{End}(V_1) \otimes \mathrm{End}(V_2)\,, \tag{2.75}$$

$$\epsilon\text{: unit rep.} \quad \epsilon \,:\, A \to \mathbb{C}\,. \tag{2.76}$$

The coproduct is useful for defining the dual representation of a representation (π, V). Set $V^* = \mathrm{Hom}(V, \mathbb{C})$ and define ${}^t\pi$ as a map $\mathrm{End}(V) \to \mathrm{End}(V^*)$:

$$f \in \mathrm{End}(V)\,,\, \varphi \in V^*\,,\, v \in V\,, \quad {}^t f(\varphi)(v) = \varphi(f(v))\,, \tag{2.77}$$

where we denoted ${}^t f = {}^t\pi(f)$. Then one accesses the dual representation (π^*, V^*) :

$$S\text{: dual rep.} \quad A \xrightarrow{S} A \xrightarrow{{}^t\pi} \mathrm{End}(V^*)\,, \quad \pi^* = {}^t\pi \circ S\,. \tag{2.78}$$

The natural question that arises is the following : is the representation $\pi^a \otimes \pi^b$ isomorphic to the representation $\pi^b \otimes \pi^a$? In view of the equation (2.68) the universal R matrix is the natural candidate for such an isomorphism. Let us apply this and consider the image of the universal R matrix $\mathcal{R} \in A \otimes A$ under two representations :

$$R_{a,b} = (\pi^a \otimes \pi^b)\mathcal{R} \,\in\, \mathrm{End}(V_a \otimes V_b)\,. \tag{2.79}$$

The compatibility between the tensor product operation and the representations (2.75) implies that the Yang-Baxter equation (2.73) mutes into

$$R_{a,b}R_{a,c}R_{b,c} = R_{b,c}R_{a,c}R_{a,b}\,. \tag{2.80}$$

This equations is crucial, the knowledge of a universal R matrix for a specific Quantum Group allows to translate it to any two representations, and hence describe a variety of different physical systems having the same background symmetry.

The Quantum Group $U_q(\mathfrak{sl}_2)$.

We now discuss particular examples of Quantum Groups and start with the simple case of $U_q(\mathfrak{sl}_2)$. This algebra is generated by 4 elements $\{e, f, t, t^{-1}\}$, modulo the relations :

$$te = q^2 et\,, \quad tf = q^{-2} ft\,, \quad [e, f] = \frac{t - t^{-1}}{q - q^{-1}}\,, \tag{2.81}$$

$$tt^{-1} = t^{-1}t = 1\,. \tag{2.82}$$

This defines a structure of an algebra over $\mathbb{C}$. We can endow $U_q(\mathfrak{sl}_2)$ with a Hopf algebra structure by defining the maps Δ, ϵ, S :

$$\Delta(e) = e \otimes 1 + t \otimes e\,, \quad \Delta(f) = f \otimes t^{-1} + 1 \otimes f\,, \quad \Delta(t^{\pm}) = t^{\pm} \otimes t^{\pm}\,, \tag{2.83}$$

$$\epsilon(e) = 0\,, \quad \epsilon(f) = 0\,, \quad \epsilon(t^{\pm}) = 1\,, \tag{2.84}$$

$$S(e) = -t^{-1}e\,, \quad S(f) = -ft\,, \quad S(t^{\pm}) = t^{\mp}\,. \tag{2.85}$$

We shall also meet another set of more "formal" generators of this algebra, : $\{E, F, q^H, q^{-H}\}$ related to the former by :

$$E = e\,, \quad F = f\,, \quad q^{\pm H} = t^{\pm}\,. \tag{2.86}$$

Taking the semi-classical limit $q \to 1$, it is then possible to write the commutation relations between E, F, H and $q^{\pm H}$:

$$[H, E] = 2E\,, \quad [H, F] = -2F\,, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}\,. \tag{2.87}$$

From these relations, it is manifest that the construction of $U_q(\mathfrak{sl}_2)$ is a deformation of the universal enveloping algebra $U(\mathfrak{sl}_2)$ by the parameter q. Generalizing this procedure one can construct the Quantum Groups related to any complex semi-simple Lie algebra $\mathfrak{g}$, and even to the associated Kac-Moody algebra $\widehat{\mathfrak{g}}$. This is explained for example in [55] : one should consider the Cartan matrix of $\mathfrak{g}$ (or $\widehat{\mathfrak{g}}$), and quantize the Serre relations as it is done for $U_q(\mathfrak{sl}_2)$. This is how we are going to proceed to define $U_q(\widehat{\mathfrak{sl}_2})$.

Let us now turn to the representation theory of $U_q(\mathfrak{sl}_2)$. We will be interested in finite dimensional irreducible representations, and we will assume only generic values of q (when q is not a root of unity). In this situation, the representation theory of the algebra $U_q(\mathfrak{sl}_2)$ is similar to that of $\mathfrak{sl}_2$: any irreducible module of $U_q(\mathfrak{sl}_2)$ is isomorphic to some module $V(m)$, characterized by an integer m (the highest weight) and such that $\dim V(m) = m+1$ [3]. One denotes this representation $(\pi^m, V(m))$. The action of the $U_q(\mathfrak{sl}_2)$ on $V(m)$ is given by the deformation of the action of $\mathfrak{sl}_2$. Let $\{v_i\}_{i=0}^m$ be a basis of $V(m)$, then :

$$e \cdot v_k = [m-k+1]v_{k-1}\,, \quad f \cdot v_k = [k+1]v_{k-1}\,, \quad t \cdot v_k = q^{m-2k}v_k\,, \quad t^{-1} \cdot v_k = q^{-m+2k}v_k\,, \tag{2.88}$$

where we used the q-number notation :

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}\,. \tag{2.89}$$

This indeed defines a representation, thanks to the relation $[l-k][k+1] - [l-k+1][k] = [l-2k]$. Remark that $(\pi^0, V(0)) \simeq (\epsilon, \mathbb{C})$ and the simplest non trivial module is $V(1)$ where the representation matrices are given by :

$$\pi^1(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \quad \pi^1(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad \pi^1(t^{\pm}) = \begin{pmatrix} q^{\pm} & 0 \\ 0 & q^{\mp} \end{pmatrix}\,. \tag{2.90}$$

The case of $U_q(\widehat{\mathfrak{sl}_2})$.

As we have seen, the R matrix of the six vertex model depends on a spectral parameter, a feature that is absent in the representation theory of $U_q(\mathfrak{sl}_2)$. To make it appear naturally we should consider instead the Quantum Group $U_q(\widehat{\mathfrak{sl}_2})$ [4]. The affine Kac-Moody algebra $\widehat{\mathfrak{sl}_2}$ has 8 generators

[3] More precisely, the number of representations is doubled since there exists a non trivial automorphism η of $U_q(\mathfrak{sl}_2)$:

$$\eta(e) = -e\,, \quad \eta(f) = f\,, \quad \eta(t) = -t\,.$$

and from a representation (π, V) we can construct a non isomorphic representation $(\pi \circ \eta, V)$.

[4] One way to define $\widehat{\mathfrak{sl}_2}$ is to complete the central extension of the loop algebra of $\mathfrak{sl}_2$:

$$\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[X, X^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d\,, \tag{2.91}$$

with the following centrally extended commutation relations :

$$[v \otimes X^m, u \otimes X^n] = [v, u] \otimes X^{m+n} + c\,\delta_{m+n,0}\,m\,\mathrm{tr}(uv)\,, \tag{2.92}$$

$$[c, \#] = 0\,, \tag{2.93}$$

$$[d, v \otimes X^m] = mv \otimes X^m \qquad d = X\frac{d}{dX}\,. \tag{2.94}$$

Observe, that if (π, V) is a representation of $\mathfrak{sl}_2$, then given any $\lambda \in \mathbb{C}^*$, we can obtain a representation (π_λ, V) of $\widehat{\mathfrak{sl}_2}'$ (where we drop the action of d) :

$$\pi_\lambda(v \otimes X^n) = \lambda^n \pi(v)\,, \quad \pi_\lambda(c) = 0\,. \tag{2.95}$$

This explains the appearance of the spectral parameter in the context of $\widehat{\mathfrak{sl}_2}$.

$\{e_i\,,\ f_i\,,\ t_i^{\pm}\,;\ i=0\,,\ 1\}$ whose commutation relations are encoded in the Cartan matrix :

$$C=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}=(a_{ij})_{i,j}\,. \tag{2.96}$$

According to the remark in the previous section we define $U_q(\widehat{\mathfrak{sl}_2})$ by appropriately deforming the Serre relations to :

$$t_ie_j=q^{a_{ij}}e_jt_i\,,\quad t_if_j=q^{-a_{ij}}f_jt_i\,, \tag{2.97}$$

$$[e_i,f_j]=\delta_{ij}\frac{t_i-t_i^{-1}}{q-q^{-1}}\,, \tag{2.98}$$

$$[e_i,[e_i,[e_i,e_j]_{q^2}]_{q^0}]_{q^{-2}}=0\,,\quad i\neq j\,, \tag{2.99}$$

$$[f_i,[f_i,[f_i,f_j]_{q^2}]_{q^0}]_{q^{-2}}=0\,,\quad i\neq j\,, \tag{2.100}$$

where we used the deformed commutator :

$$[a,b]_r=ab-rba\,. \tag{2.101}$$

We further endow $U_q(\widehat{\mathfrak{sl}_2})$ with a Hopf algebra structure by defining the coproduct, counit and antipode :

$$\Delta(e_i)=e_i\otimes 1+t_i\otimes e_i\,,\quad \Delta(f_i)=f_i\otimes t_i^{-1}+1\otimes f_i\,,\quad \Delta(t_i)=t_i\otimes t_i\,. \tag{2.102}$$

$$\epsilon(e_i)=0\,,\quad \epsilon(f_i)=0\,,\quad \epsilon(t_i)=1\,, \tag{2.103}$$

$$S(e_i)=-t_i^{-1}e_i\,,\quad S(f_i)=-f_it_i\,,\quad S(t_i^{\pm})=t_i^{\mp}\,. \tag{2.104}$$

From now we would like to apply our knowledge about the representation theory of $U_q(\mathfrak{sl}_2)$ to the affine case. This can be done with the use of the homomorphism $\varphi_\lambda\,:\,U_q(\widehat{\mathfrak{sl}_2})\to U_q(\mathfrak{sl}_2)$ defined for any $\lambda\in\mathbb{C}^*$ by :

$$\varphi_\lambda(e_0)=\lambda f\,,\quad \varphi_\lambda(f_0)=\lambda^{-1}e\,,\quad \varphi_\lambda(t_0)=t^{-1}\,, \tag{2.105}$$

$$\varphi_\lambda(e_1)=\lambda e\,,\quad \varphi_\lambda(f_1)=\lambda^{-1}f\,,\quad \varphi_\lambda(t_1)=t\,. \tag{2.106}$$

If (π,V) is a representation of $U_q(\mathfrak{sl}_2)$ the composition map $\pi\circ\varphi_\lambda$ gives a representation of $U_q(\widehat{\mathfrak{sl}_2})$ in V :

$$U_q(\widehat{\mathfrak{sl}_2})\overset{\varphi_\lambda}{\to}U_q(\mathfrak{sl}_2)\overset{\pi}{\to}\mathrm{End}(V)\,. \tag{2.107}$$

In the following we will denote π_λ^m the representation of $U_q(\widehat{\mathfrak{sl}_2})$ given by $(\pi^m\circ\varphi_\lambda,V(m))$:

$$\pi_\lambda^m=\pi^m\circ\varphi_\lambda\,. \tag{2.108}$$

It is possible to show that the representation theory of $U_q(\widehat{\mathfrak{sl}_2})$ and $U_q(\mathfrak{sl}_2)$ are in fact quite different. In particular, the tensor products such that $\pi_\zeta^m\otimes\pi_\lambda^n$ can be irreducible. The general result [53] is that any irreducible, finite dimensional representation of $U_q(\widehat{\mathfrak{sl}_2})$ is of the form :

$$\pi_{\lambda_1}^{m_1}\otimes\ldots\otimes\pi_{\lambda_r}^{m_r}\,. \tag{2.109}$$

The universal R matrix for a wide class of algebras was calculated in [56]. In the case of $U_q(\widehat{\mathfrak{sl}_2})$, the key observation is that $\mathcal{R}$ has a factorized form :

$$\mathcal{R}\in U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{b}^-)\,, \tag{2.110}$$

where $U_q(\mathfrak{b}^+),U_q(\mathfrak{b}^-)$ are the two Borel subalgebras of $U_q(\widehat{\mathfrak{sl}_2})$ generated respectively by $\{e_i\,,\ t_i^{\pm}\,,\ i=0,1\}$ and $\{f_i\,,\ t_i^{\pm}\,,\ i=0,1\}$. This factorization is of crucial importance, since it allows to construct the Baxter Q operator from the data of the R matrix, by a similar procedure to the one that is used

to obtain the L operator. We shall not reproduce the construction of $\mathcal{R}$ here because it is beyond the scope of this work [5] , presenting only the image of the universal R matrix under $(\varphi_\lambda \otimes \pi^1_\mu)$:

$$(\varphi_\lambda \otimes \pi^1_\mu)(\mathcal{R}) = q^{\frac{1}{2}} \begin{pmatrix} \zeta^2 q^{\frac{H+1}{2}} - q^{-\frac{H+1}{2}} & (q-q^{-1})\zeta F q^{\frac{H-1}{2}} \\ (q-q^{-1})\zeta q^{-\frac{H-1}{2}} E & \zeta^2 q^{-\frac{H-1}{2}} - q^{\frac{H-1}{2}} \end{pmatrix} , \tag{2.114}$$

up to some multiplicative factor. This is an element of $U_q(\mathfrak{sl}_2) \otimes \mathrm{End}(V(1))$. Finally, evaluating E, H, F in any representation $V(2s_{\mathbf{m}}+1)$ one recovers the result (2.33).

Nevertheless, let us still give a more "pedestrian" way of deriving the R matrix (in some specific representations) in the case of $U_q(\widehat{\mathfrak{sl}_2})$. The idea, already mentioned, is to reinterpret (2.68) as a intertwiner condition on R. More precisely (see [53]), we claim that if we are given two representations π^m_λ and π^n_ξ of $U_q(\widehat{\mathfrak{sl}_2})$ defined in (2.108), then the R matrix (or some of its simple modification) should realize the isomorphism :

$$\pi^m_\lambda \otimes \pi^n_\mu \simeq \pi^m_\mu \otimes \pi^n_\lambda \, . \tag{2.115}$$

It implies the identity :

$$R(\lambda,\mu)\Delta(v) = \Delta(v)R(\lambda,\mu) \, , \quad \forall v \in U_q(\widehat{\mathfrak{sl}_2}) \, . \tag{2.116}$$

Evaluating this for $v = e_0, f_0$ one obtains :

$$R(\lambda,\mu)(\lambda f \otimes 1 + t^{-1} \otimes \mu f) = (\mu f \otimes 1 + t^{-1} \otimes \lambda f)R(\lambda,\mu) \, , \tag{2.117}$$

$$R(\lambda,\mu)(\lambda^{-1} e \otimes t + 1 \otimes \mu^{-1} e) = (\mu^{-1} e \otimes t + 1 \otimes \lambda^{-1} e)R(\lambda,\mu) \, . \tag{2.118}$$

Using the explicit realization of e, f we can find in the case $m = n = 1$:

$$R_{1,1}(\lambda,\mu) = \begin{pmatrix} \lambda q - \mu q^{-1} & & & \\ & \lambda(q-q^{-1}) & \lambda - \mu & \\ & \lambda - \mu & \mu(q-q^{-1}) & \\ & & & \lambda q - \mu q^{-1} \end{pmatrix} \tag{2.119}$$

This is the same R matrix as we considered for the six vertex model, up to the multiplication by the permutation matrix and the redefinition of the variables. Now, let us write the R matrix for $V(m) \otimes V(m)$. We know that it in the case of $\mathfrak{sl}_2$ this representation is reducible to :

$$V(m) \otimes V(m) = V(2m) \oplus V(2m-2) \oplus ... \oplus V(0) \, . \tag{2.120}$$

Since R intertwines the actions, it must be a linear combination of the projectors P_k from $V(m) \otimes V(m)$ to $V(2m-2k)$. Hence we have :

$$R_{m,m}(\lambda,\mu) = \sum_{k=0}^{m} \rho_k P_k \, , \tag{2.121}$$

where the factors ρ_k are explicitly [53] :

$$R_{m,m}(\lambda,\mu) = \sum_{k=0}^{m} \prod_{r=1}^{k} \frac{\lambda - \mu q^{2m-2r+2}}{\mu - \lambda q^{2m-2r+2}} P_k \, . \tag{2.122}$$

[5] To get a flavor of the computations we present the explicit result for $U_q(\mathfrak{sl}_2)$. Define first the q-exponential :

$$e_q(z) = \sum_{m=0}^{\infty} \frac{q^{-\frac{m(m-1)}{2}}}{[m]!} z^m \, . \tag{2.111}$$

Then the universal R matrix of $U_q(\mathfrak{sl}_2)$ reads [56] :

$$\mathcal{R} = e_q(-(q-q^{-1})e \otimes f) q^{-\frac{h \otimes h}{2}} \, . \tag{2.112}$$

Evaluating this formula for the representation $V(1)$ (2.90) of $U_q(\mathfrak{sl}_2)$ we obtain :

$$R_{1,1} = q^{-\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.113}$$

2.5 The Baxter Q operator

In this section we shall construct the Baxter Q operator, in the same spirit as it was done in [2]. It first appeared a long time ago [57] in the Baxter's solution of the eight-vertex model, and it is in the papers [34, 35, 36] that it was given a more modern interpretation. As we have seen, the T operator is constructed starting with the universal $U_q(\widehat{\mathfrak{sl}_2})$ R matrix $\mathcal{R} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$ by evaluating the factor in $U_q(\mathfrak{b}^+)$ in a finite dimensional irreducible representation of $U_q(\widehat{\mathfrak{sl}_2})$ (of the type $V(m)$) through the use of the homomorphism φ_λ. The key point of [35] is to define Q in a similar way to T, but adopting a different algebra : the q-Oscillator algebra Osc.

The q-Oscillator algebra Osc is generated by 4 elements $\{\mathbf{a}, \mathbf{a}^*, q^D, q^{-D}\}$ which are subject to the commutation relations (that we take from [2]) [6] :

$$q^D \mathbf{a} q^{-D} = q^{-1}\,\mathbf{a}\,, \quad q^D \mathbf{a}^* q^{-D} = q\,\mathbf{a}^*\,. \tag{2.125}$$

$$\mathbf{a}\,\mathbf{a}^* = 1 - q^{2D+2}\,, \quad \mathbf{a}^*\mathbf{a} = q^{2D}\,. \tag{2.126}$$

By analogy with the quantum harmonic oscillator, we can construct two infinite dimensional representations $(\rho^\pm, W^\pm)$:

$$W^+ = \bigoplus_{k\geq 0} \mathbb{C}\,|k\rangle\,, \quad W^- = \bigoplus_{k<0} \mathbb{C}\,|k\rangle\,, \tag{2.127}$$

with the following action of Osc :

$$q^D \cdot |k\rangle = q^k\,|k\rangle\,, \quad \mathbf{a}\cdot|k\rangle = (1-q^{2k})\,|k-1\rangle\,, \quad \mathbf{a}^*\cdot|k\rangle = (1-\delta_{k,-1})\,|k+1\rangle\,. \tag{2.128}$$

Define the homomorphism ψ_λ that embed the Borel subalgebra $U_q(\mathfrak{b}^+)$ into Osc :

$$\psi_\lambda\,:\,U_q(\mathfrak{b}^+) \to Osc\,, \tag{2.129}$$

$$\psi_\lambda(e_0) = \frac{\lambda}{q-q^{-1}}\mathbf{a}\,, \quad \psi_\lambda(e_1) = \frac{\lambda}{q-q^{-1}}\mathbf{a}^*\,, \quad \psi_\lambda(t_0) = q^{-2D}\,, \quad \psi_\lambda(t_1) = q^{2D}\,. \tag{2.130}$$

Now, one can evaluate the universal R matrix of $U_q(\widehat{\mathfrak{sl}_2})$ on the algebra Osc. This is done in the following way. By analoy with (2.108), define two representations $\pi^\pm_\lambda$ of $U_q(\mathfrak{b}^+)$ in $W^\pm$:

$$\pi^\pm_\lambda = \psi_\lambda \circ \rho^\pm\,:\,U_q(\mathfrak{b}^+)\,\to \mathrm{End}(W^\pm)\,. \tag{2.131}$$

The use of the algebra Osc in the evaluation of the universal R matrix will be depicted by the index "A", as well as the superscripts $\pm$ to specify which choice of $W^\pm$ is understood. Then, working with the explicit expression of the universal R matrix of $U_q(\widehat{\mathfrak{sl}_2})$ (2.110) one has [7] :

$$L^\pm_{A,\mathbf{m}}(\zeta) = (\pi^\pm_\zeta \otimes \pi^1_1)(\mathcal{R}) = \begin{pmatrix} q^{-D} - \zeta^2 q^{2D+1} & -\zeta \mathbf{a} q^{-D} \\ \zeta \mathbf{a}^* q^D & q^D \end{pmatrix}_{\mathbf{m}} \in \mathrm{End}(W^\pm) \otimes \mathrm{End}(V_{\mathbf{m}})\,, \tag{2.132}$$

where $V_{\mathbf{m}}$ is a two dimensional vector space (to be thought on the Matsubara chain).

Similarly to the definition of the monodromy matrix (2.36), define the "Q-monodromy" matrix by multiplying different $L^\pm_{A,\mathbf{m}}$ matrices along the Matsubara space, taking into account the presence of inhomogeneities and eventually of the twist factor :

$$L^\pm_{A,\mathbf{M}}(\zeta,\kappa) = L^\pm_{A,\mathbf{n}}\left(\frac{\zeta}{\tau_{\mathbf{n}}}\right) L^\pm_{A,\mathbf{n}-1}\left(\frac{\zeta}{\tau_{\mathbf{n}-1}}\right) ... L^\pm_{A,\mathbf{1}}\left(\frac{\zeta}{\tau_{\mathbf{1}}}\right) q^{\pm 2\kappa D}\,. \tag{2.133}$$

Finally, tracing out the representation A we obtain the wished Q operator ($Q^\pm_{\mathbf{M}}(\zeta,\kappa) \in \mathrm{End}(\mathfrak{H}_{\mathbf{M}})$) :

$$Q^\pm_{\mathbf{M}}(\zeta,\kappa) = \zeta^{\pm(\kappa - S_{\mathbf{M}})} \mathrm{Tr}_{A^\pm}(L^\pm_{A,\mathbf{M}}(\zeta,\kappa))\,, \tag{2.134}$$

[6] We can think about this algebra as the "quantized" version of the creation-annihilation commutation relations for the quantum harmonic oscillator :

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\,, \quad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\,. \tag{2.123}$$

$$[a, a^\dagger] = 1\,, \quad N = a^\dagger a\,. \tag{2.124}$$

[7] Up to some multiplicative factor.

where we have introduced an extra term carrying the twist κ and the total spin operator of the Matsubara space $S_{\mathbf{M}}$ (recall (2.63)) :

$$S_{\mathbf{M}} = \frac{1}{2}\sum_{\mathbf{m}=1}^{\mathbf{n}} \sigma_{\mathbf{m}}^3 \,. \tag{2.135}$$

From the universal Yang-Baxter equation (2.73) it is possible to demonstrate the commutation relations :

$$[T_{\mathbf{M}}(\zeta,\kappa), Q_{\mathbf{M}}(\lambda,\kappa)] = 0\,, \quad [Q_{\mathbf{M}}(\zeta,\kappa), Q_{\mathbf{M}}(\lambda,\kappa)] = 0\,. \tag{2.136}$$

In particular, this means that $T_{\mathbf{M}}$ and $Q_{\mathbf{M}}$ can be simultaneously diagonalized. Moreover, the key identity between $T_{\mathbf{M}}$ and $Q_{\mathbf{M}}$ is the *Baxter equation* (or "TQ equation") [3, 57] :

$$T_{\mathbf{M}}(\zeta,\kappa)Q^{\pm}_{\mathbf{M}}(\zeta,\kappa) = a(\zeta)Q^{\pm}_{\mathbf{M}}(q\zeta,\kappa) + d(\lambda)Q^{\pm}_{\mathbf{M}}(q^{-1}\zeta,\kappa)\,, \tag{2.137}$$

where a and d are the Bethe Ansatz functions (2.51) :

$$a(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left((\zeta/\tau_{\mathbf{m}})^2 - 1\right)\,, \quad d(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \left((\zeta q/\tau_{\mathbf{m}})^2 - 1\right)\,. \tag{2.138}$$

Applying the equation (2.137) on a common eigenvector of $T_{\mathbf{M}}(\zeta,\kappa)$ and $Q^{\pm}_{\mathbf{M}}(\zeta,\kappa)$ we obtain a functional equation on the corresponding eigenvalues $T(\zeta,\kappa)\,, Q^{\pm}(\zeta,\kappa)$. Let us consider the analytical structure of $Q^{\pm}(\zeta,\kappa)$. It is assumed to be a polynomial in ζ^2, and if the common eigenvectors carries a spin s one can write :

$$Q^{\pm}(\zeta,\kappa) = q^{\pm(\kappa-s)}\prod_{k=1}^{l}(\zeta^2 - \lambda_k^2)\,. \tag{2.139}$$

Evaluating the functional version of (2.137) at one of the roots λ_k of $Q^{\pm}$, one recovers exactly the Bethe Ansatz Equations (2.62) :

$$\frac{a(\lambda)Q^{\pm}(q\lambda_k,\kappa)}{d(\lambda)Q^{\pm}(q^{-1}\lambda_k,\kappa)} = -1\,. \tag{2.140}$$

Hence the roots of $Q^{\pm}(\zeta,\kappa)$ are exactly the Bethe roots. Let us make a last comment to conclude this section. Manifestly, Q^+ and Q^- seem to play a pretty similar role. This is natural, indeed Q is a solution of the second order difference equation (2.137), which a priori has two different solutions : Q^+ and Q^-.

2.6 Integrable structures in Conformal Field Theory

We are now ready to discuss how the previous constructions can be generalized to the field theoretic context and reveal the integrable structures of CFT. This was achieved in the paper [34] and its sequels [35], [36] by Bazhanov, Lukyanov and Zamolodchikov, and is termed after the authors the "BLZ construction". Recall that for the lattice six-vertex model (based on $U_q(\widehat{\mathfrak{sl}_2})$), the universal R matrix was written as (2.110) :

$$\mathcal{R} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)\,. \tag{2.141}$$

Depending on the representations that we chose for the Borel subalgebra $U_q(\mathfrak{b}^+)$, we obtained either the transfer matrix or the Baxter operator, whereas $U_q(\mathfrak{b}^-)$ was merely an observer, represented ultimately in a finite dimensional vector space in the Matsubara direction. To make connection with CFT, one should substitute this last piece $U_q(\mathfrak{b}^-)$ by a more "field theoretic", infinite dimensional algebra. The farsighted idea of BLZ was to consider for such a role the Heisenberg algebra $\mathfrak{H}$. Our explanation here is, I hope more natural, but slightly anachronistic. Indeed one must emphasis that the introduction of the Baxter Q operator on the lattice presented before in the section 2.5 is entirely due to the BLZ advances.

In the original BLZ papers, the problematic was rather different. As we have show, any CFT possesses an infinite number of conserved integrals of motion built as descendants of the identity operator, that can be described by local densities ((1.155),(1.156),(1.157) for the first ones). In turns, these densities give rise to operators I_{2k-1} that act on the Matsubara Hilbert space, and commute among themselves. The fundamental achievement of the work [34] was to consistently organize these objects, by the introduction of a CFT transfer matrix $\mathbf{T}$, which acts on a $\mathfrak{V}$-Verma module and is the generating function of the integrals of motions I_{2k-1}. This breakthrough was followed by several important consequences :

1. The introduction of the transfer matrix $\mathbf{T}$ preceded the definition of a CFT Baxter operator $\mathbf{Q}$ [35]. Then the full set of identities that are essentially verified by these operators (such that the Yang-Baxter equation, the Baxter TQ relation, the fusion equations), were shown to be valid also in the CFT context [36]. It means more conceptually, that $\mathbf{T}$ and $\mathbf{Q}$ realize the Yang-Baxter equation in the Hilbert space of the CFT.

2. The description of CFT in terms of the "massless S matrix" scattering theory, proposed in [28], was naturally related to the above construction : the TBA equations obtained from this S matrix theory where shown to be particular truncations of the fusion rules satisfied by the transfer matrices $\mathbf{T}$.

3. As we have seen the integrable structures of CFT remains essentially intact when we perturb the conformal theory by the primary field $V_{1,3}$, the outcome being a massive integrable QFT. Working with IQFT, it is of course possible to first study the integrable lattice system (that is build the operators T and Q) that are described by the CFT at criticality, and then take an appropriate scaling limit towards the massive QFT. Then one would recover the wanted T and Q operators at the field theoretic level. However, in many cases the associated lattice models are not known. It is then very interesting to possess an intrinsic description of these quantities, directly in terms of fields of the theory and without any reference to the lattice model.

In this section, we therefore propose a brief overview of the construction of $\mathbf{T}$ from [34], since it will be crucial for our further analysis of the scaling equations [7]. Let us examine a CFT on a cylinder (to stick to the original notations we take $a = -i$ in (1.52), and drop the extra factor -1 in front of T) whose (chiral) space of states is given by :

$$\mathcal{S} = \bigoplus_{\Delta} \mathcal{V}_{\Delta} \,. \tag{2.142}$$

One has the usual relation between the Matsubara integrals of motion and the densities :

$$I_{2k-1} = \int_0^{2\pi i} \frac{du}{2\pi} h_{2k}(u) \,, \tag{2.143}$$

the first expressions are recalled in (1.187), (1.188),(1.189). As stated at the beginning of this section, one should introduce the Heisenberg algebra $\mathfrak{H}$ [8]. This is done by the means of a bosonic free field φ :

$$\varphi(u) = iQ + iPu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu} \,, \tag{2.146}$$

whose relation to the CFT stress energy tensor is :

$$-\beta^2 T(u) =: \varphi'(u)^2 : + (1-\beta^2)\varphi''(u) + \frac{\beta^2}{24} \,. \tag{2.147}$$

By convention, the densities are normalized such that $h_{2k}(u) =: T(u)^k : + ...$, which amounts to take

$$I_{2k-1} = (-1)^k \beta^{-2k} \int_0^{2\pi} \frac{du}{2\pi} \left(: (\varphi'(u))^{2k} : + ... \right) \,. \tag{2.148}$$

[8] The key original idea of BLZ is to take inspiration from the classical case. It is well known that in the classical limit $c \to \infty$, the Virasoro algebra reduces to the KdV problem described by the Poisson algebra (see [58, 59, 60]) :

$$T(u) \to -\frac{c}{6} U(u) \,, \quad [\,,\,] \to \frac{6\pi}{ic} \{\,,\,\} \,, \tag{2.144}$$

$$\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u-v) + \delta'''(u-v) \,. \tag{2.145}$$

The integrals I_{2k-1} are quantum counterparts of the classical conserved integrals of motion of the KdV hierarchy. To solve the classical KdV problem, one constructs a classical monodromy matrix based on the algebra $\mathfrak{sl}_2$, the important step being in simplifying the problem by the application of a Miura transformation to the classical variable U.

The big progress of [34] is to transpose this classical considerations to the quantum level : the quantum version of a Miura transformation corresponds to the Feigin-Fuchs free field representation of the algebra $\mathfrak{V}$, and instead of working with the algebra $\mathfrak{sl}_2$ the Quantum Group $U_q(\widehat{\mathfrak{sl}_2})$ is considered.

The elements P, Q, a_n satisfy the Heisenberg algebra $\mathfrak{H}$:

$$[Q,P] = \frac{i}{2}\beta^2\,, \quad [a_n, a_m] = \frac{n}{2}\beta^2\delta_{n+m,0}\,, \quad [Q,a_n] = [P,a_n] = 0\,. \tag{2.149}$$

With these definitions, the modes L_n of T satisfy the Virasoro algebra (1.26) with the central charge given by

$$c = 1 - 6\left(\beta - \frac{1}{\beta}\right)^2\,. \tag{2.150}$$

The Fock space representation $\mathcal{F}_p$ for the algebra $\mathfrak{H}$ is constructed as follows. Consider a highest weight vector $|p\rangle$ for $\mathfrak{H}$:

$$P\,|p\rangle = p\,|p\rangle\,, \quad a_n\,|p\rangle = 0\,, \quad n > 0\,. \tag{2.151}$$

Then $\mathcal{F}_p$ is generated by the action of negatively indexed a_n on the vector $|p\rangle$, and is isomorphic to the Verma modules $\mathcal{V}_\Delta$ of $\mathfrak{V}$ with the following correspondence between the highest weights (1.88) :

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}\,. \tag{2.152}$$

The space $\mathcal{F}_p$ naturally decomposes into level subspaces. After these preparations, one can define the vertex fields :

$$V_\pm(u) =: e^{\pm 2\varphi(u)} := \exp\left(\pm 2\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu}\right) e^{\pm 2i(Q+Pu)} \exp\left(\mp 2\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu}\right)\,, \tag{2.153}$$

that act as

$$V_\pm(u) : \mathcal{F}_p \to \mathcal{F}_{p\pm\beta^2}\,. \tag{2.154}$$

We use the generators $\{E, F, q^H, q^{-H}\}$ of $U_q(\mathfrak{sl}_2)$ that satisfy (2.87) :

$$[H,E] = 2E\,, \quad [H,F] = -2F\,, \quad [E,F] = \frac{q^H - q^{-H}}{q - q^{-1}}\,. \tag{2.155}$$

The relation between the Quantum Group parameter q and the CFT parameter β is given by :

$$q = e^{i\pi\beta^2}\,. \tag{2.156}$$

Finally, the monodromy matrix of the CFT is defined to be the element of $U_q(\mathfrak{sl}_2)\otimes\mathfrak{H}$ given by :

$$\mathbf{L}(\lambda) \in U_q(\mathfrak{sl}_2)\otimes\mathfrak{H}\,, \tag{2.157}$$

$$\mathbf{L}(\lambda) = e^{i\pi PH}\mathcal{P}\exp\left(\lambda\int_0^{2\pi} du(V_-(u)q^{\frac{H}{2}}E + V_+(u)q^{-\frac{H}{2}}E)\right)\,. \tag{2.158}$$

In this definition λ is the spectral parameter, and $\mathcal{P}$ denotes the usual path ordering operator. This formula is a direct CFT analog of the lattice monodromy matrices (2.36), (2.38). Notice that the momentum operator P plays the same role as the twist κ, which suggests that these two quantities should be related when we go from the lattice to the continuum. Now consider π^j to be the $(2j+1)$ dimensional irreducible representation $V(2j)$ of $U_q(\mathfrak{sl}_2)$ of highest weight $2j$. We define $\mathbf{L}_j(\lambda)$:

$$\mathbf{L}_j(\lambda) = \pi^j\left(\mathbf{L}(\lambda)\right)\,. \tag{2.159}$$

Tracing out the representation π^j in $\mathbf{L}_j(\lambda)$, we get the wanted transfer matrix $\mathbf{T}_j$ of the CFT :

$$\mathbf{T}_j(\lambda) = \mathrm{Tr}_{\pi^j}\left(\mathbf{L}_j(\lambda)\right)\,. \tag{2.160}$$

By construction $\mathbf{T}_j(\lambda)$ is a CFT field acting as

$$\mathbf{T}_j(\lambda)\,:\,\mathcal{V}_\Delta \to \mathcal{V}_\Delta\,. \tag{2.161}$$

In the paper [36] it is shown that $\mathbf{L}_j(\lambda)$ satisfies the Yang-Baxter equation (2.41) :

$$\mathbf{R}_{jj'}(\lambda\mu^{-1})(\mathbf{L}_j(\lambda)\otimes 1)(1\otimes\mathbf{L}_{j'}(\mu)) = (1\otimes\mathbf{L}_{j'}(\mu))(\mathbf{L}_j(\lambda)\otimes 1)\mathbf{R}_{jj'}(\lambda\mu^{-1})\,. \tag{2.162}$$

with $\mathbf{R}_{jj'}(\zeta)$ the R matrix of $U_q(\widehat{\mathfrak{sl}_2})$ in the tensor product representation $\pi^j \otimes \pi^{j'}$. In the case of spins $\frac{1}{2}$ (highest weights 1), one recovers the usual L operator (2.35) :

$$\mathbf{R}_{\frac{1}{2},\frac{1}{2}}(\zeta) = \zeta^{-1} q^{-\frac{1}{2}} L(\zeta)\,. \tag{2.163}$$

The equation (2.162) implies the commutativity of the transfer matrices :

$$[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')] = 0\,. \tag{2.164}$$

The operators $\mathbf{T}_j$ commute also with P, this implies that they act invariantly in $\mathcal{F}_p$. Moreover the operators $\mathbf{T}_j$ satisfy

$$[\mathbf{T}_j(\lambda), I_{2k-1}] = 0\,. \tag{2.165}$$

It means that the level subspaces $\mathcal{F}_p^{(l)}$ are eigenspaces of $\mathbf{T}_j(\lambda)$. A very important point of [34], is that $\mathbf{T}_{\frac{1}{2}}$ admits specific analytical properties. It is an analytical function of λ^2 that has an essential singularity at the point $\lambda \to \infty$. The asymptotic expansion is given by :

$$\log \mathbf{T}_{\frac{1}{2}}(\lambda) \simeq m\lambda^{\frac{1}{1-\beta^2}} - \sum_{n=1}^{\infty} C_n \lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1}\,. \tag{2.166}$$

where the I_{2n-1} are exactly the integrals of motion (2.143) on the cylinder. This asymptotic expansion will be of crucial importance when we will derive the scaling equations for the Super sine-Gordon model. A remark is in order : why is $\lambda^{\frac{1}{1-\beta^2}}$ the variable of the series ? This holds since $\lambda^{\frac{1}{1-\beta^2}}$ is the truly dimensionless quantity : recall that we used the variable θ related to λ (2.30) by $\lambda = e^{i\pi\nu\theta}$ and we claim that the correspondence between lattice and QFT parameters is

$$\nu = 1 - \beta^2\,. \tag{2.167}$$

The various coefficients of the expansion of $\mathbf{T}_{\frac{1}{2}}$ are given by :

$$m = 2\sqrt{\pi}\frac{\Gamma(\frac{1}{2}\frac{1-2\beta^2}{1-\beta^2})}{\Gamma(\frac{1}{2}\frac{2-3\beta^2}{1-\beta^2})}\left(\Gamma(1-\beta^2)\right)^{\frac{1}{1-\beta^2}}\,, \tag{2.168}$$

$$C_n = \frac{1}{n!(1-\beta^2)}\left(\pi\beta^2\right)^n \left(\frac{2}{m}\frac{\Gamma(\frac{1}{2}\frac{1-2\beta^2}{1-\beta^2})}{\Gamma(\frac{1}{2}\frac{2-3\beta^2}{1-\beta^2})}\right)^{2n-1} \frac{\Gamma((n-\frac{1}{2})\frac{1}{1-\beta^2})}{\Gamma(1+(n-\frac{1}{2})\frac{\beta^2}{1-\beta^2}))}\,. \tag{2.169}$$

In the paper [35] the investigations on the integrable structure of CFT were continued, and the Baxter $\mathbf{Q}$ operator was defined in the CFT context. Exactly like in the lattice case, $\mathbf{Q}$ is built from the q-oscillator algebra.

2.7 The fermionic basis.

In this section we will briefly recall the main features of the fermionic basis. It was developed and successfully applied in the papers [1, 2, 3, 4, 5, 61]. The fermionic basis construction was established in the context of the six-vertex model on the cylinder and progressed through the following steps :

1. It provided a basis of local operators on the lattice, for which the vacuum expectation values are calculated by rather simple formulae involving determinants (see (2.191)). The vacuum expectation values are given by determinants involving a single function of two variables ω, multiplied by a factor depending on a certain function ρ (of one variable). Both these function are given only in terms of Matsubara data [1, 2, 3].

2. In addition to the simplicity of the correlation functions on the lattice, these fermionic operators have very nice scaling properties towards QFT. This is the second very important feature of the fermionic basis, and means that each of its operators has a QFT counterpart. Therefore, when taking the scaling limit, one can construct creation operators for the QFT. This was done in [4] in the CFT limit of the six-vertex model, and provided an alternative (to the usual Virasoro one) basis for fields in the relativistic quantum theory. Considering the scaling limits of the functions ρ and ω, one therefore has very simple expressions for the one point functions of CFT operators generated by the fermionic basis. These one point functions will be presented in (2.229)-(2.232).

3. Finally, the fermionic basis is compatible with Conformal Perturbation Theory. This was used in [5, 61] to obtain the one point functions in the sine-Gordon model. Indeed, we have seen that the fields in the PCFT can be classified in the same way as in the UV conformal theory. The general idea is then to take the scaling limit of the fermionic construction towards the massive QFT, and using the correspondence between the fermionic and the Virasoro description get the wished one point functions in the massive case.

In this section the aim is to describe the derivation of the correspondence between the fermionic and the Virasoro description at the CFT level (the formulae (2.235),(2.236),(2.237),(2.238) below), since this is the work that will be accomplished later for the case of the Super sine-Gordon model in the chapter 4 (see also [9]). Hence we will not discuss the application of the fermionic basis to the computation of one point functions in the sine-Gordon QFT (the point 3 above). Let us start with the lattice considerations.

2.7.1 The fermionic basis on the lattice.

General definitions.

We work here in the setting of section 2.2, in particular we use the terminology of the Matsubara and Space chains. Recall that our goal is to find a simple way to compute vacuum expectation values of local operators on the lattice. We shall first give our definition for the vacuum expectation values and then explain the fermionic basis.

Consider the cylinder to be infinite in length (take $N \to \infty$). The Space direction becomes :

$$\mathfrak{H}_{\mathbf{S}} = \bigotimes_{k=-\infty}^{\infty} \mathbb{C}^2 \,. \tag{2.170}$$

Recall from the above discussion that on the Space chain we have the following spin $\frac{1}{2}$ XXZ Hamiltonian (2.28) :

$$H_{XXZ} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left(\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3 \right) , \quad \Delta = \frac{q+q^{-1}}{2} \,. \tag{2.171}$$

Another operator what will be important is the spin in the Space direction :

$$S(k) = \frac{1}{2} \sum_{j=-\infty}^{k} \sigma_j^3 , \quad S = S(\infty) = \frac{1}{2} \sum_{j=-\infty}^{\infty} \sigma_j^3 \,. \tag{2.172}$$

To obtain the transfer matrix of the model we set :

$$T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1) \,, \tag{2.173}$$

in the notations of (2.36), and define the total monodromy matrix $T_{\mathbf{S},\mathbf{M}}$:

$$T_{[-\frac{N}{2}+1,\frac{N}{2}],\mathbf{M}} = T_{-\frac{N}{2}+1,\mathbf{M}} \ldots T_{\frac{N}{2},\mathbf{M}} \,, \tag{2.174}$$

$$T_{\mathbf{S},\mathbf{M}} = \lim_{N\to\infty} T_{[-\frac{N}{2}+1,\frac{N}{2}],\mathbf{M}} \,. \tag{2.175}$$

Now we should define the operators with which we are going to work. The spirit of the construction is close to CFT considerations. We consider operators $\mathcal{O}$ that are local, in the sense that their action is non trivial only on a finite interval of the space direction. Then we can take products with the lattice "primary field" $q^{\alpha S(0)}$. This is a quasi-local operator with tail α since it stabilizes outside some finite interval to be the action by $q^{\frac{1}{2}\alpha\sigma_j}$. Denote by $\mathcal{W}_\alpha$ the space of quasi-local operators with tail α, and by $\mathcal{W}_{\alpha,s}$ its subspace of operators of spin s (the spin of $\mathcal{O}$ is the eigenvalue of the operator $\mathrm{ad}(S)(.) = [S,.]$). We will work in the space :

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s} \,. \tag{2.176}$$

On $\mathcal{W}^{(\alpha)}$ we define the following vacuum expectation value of the local operator $\mathcal{O}$:

$$Z_{\mathbf{n}}^{\kappa}\left\{ q^{2\alpha S(0)} \mathcal{O} \right\} = \frac{\mathrm{Tr}_{\mathbf{S}} \mathrm{Tr}_{\mathbf{M}} \left(T_{\mathbf{S},\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \mathcal{O} \right)}{\mathrm{Tr}_{\mathbf{S}} \mathrm{Tr}_{\mathbf{M}} \left(T_{\mathbf{S},\mathbf{M}} q^{2\kappa S + 2\alpha S(0)} \right)} \,. \tag{2.177}$$

This formula is in fact natural. Indeed, the double trace of the total transfer matrix is nothing else than the partition function of the six vertex model (2.13) (in the case of a cylinder of infinite length) :

$$Z_{6V} = \mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathbf{S},\mathbf{M}}\right) . \tag{2.178}$$

Then (2.177) appears roughly speaking to be the partition function with the insertion of the operator $\mathcal{O}$, normalized by the partition function. This is a meaningful definition for a vacuum expectation value from a QFT point of view.

Let us now go back to the general case (2.177). We should be worried about the fact that since we are working on an infinite cylinder, (2.177) might be a divergent quantity. This problem is in fact avoided since we are dealing with quasi-local operators. Indeed, expliciting the large N limit we have with evident notations :

$$Z^{\kappa}_{\mathbf{n}}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \lim_{N\to\infty} \frac{\mathrm{Tr}_{\mathbf{M}}\mathrm{Tr}_{[-\frac{N}{2}+1,\frac{N}{2}]}\left(T_{[-\frac{N}{2}+1,\frac{N}{2}],\mathbf{M}}q^{2(\kappa+S_{[-\frac{N}{2}+1,\frac{N}{2}]}+\alpha S_{[-\frac{N}{2}+1,0]})}\mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{M}}\mathrm{Tr}_{[-\frac{N}{2}+1,\frac{N}{2}]}\left(T_{[-\frac{N}{2}+1,\frac{N}{2}],\mathbf{M}}q^{2(\kappa+S_{[-\frac{N}{2}+1,\frac{N}{2}]}+\alpha S_{[-\frac{N}{2}+1,0]})}\right)} . \tag{2.179}$$

Then, consider that the Matsubara transfer matrix (2.39) $T_{\mathbf{M}}(1,\kappa)$ has a unique eigenvector $|\kappa\rangle$ with maximal eigenvalue $T(1,\kappa)$. This holds when the twist parameter κ is not to large. Making the additional assumption that

$$\langle\kappa+\alpha|\alpha\rangle \neq 0 , \tag{2.180}$$

the trace on the Matsubara space in the right hand side of (2.179) reduces to (exactly like in (2.15)) :

$$Z^{\kappa}_{\mathbf{n}}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \lim_{N\to\infty} \frac{\langle\kappa+\alpha|\mathrm{Tr}_{[-\frac{N}{2}+1,\frac{N}{2}]}\left(T_{[-\frac{N}{2}+1,\frac{N}{2}],\mathbf{M}}q^{2(\kappa+S_{[-\frac{N}{2}+1,\frac{N}{2}]}+\alpha S_{[-\frac{N}{2}+1,0]})}\mathcal{O}\right)|\kappa\rangle}{\langle\kappa+\alpha|\mathrm{Tr}_{[-\frac{N}{2}+1,\frac{N}{2}]}\left(T_{[-\frac{N}{2}+1,\frac{N}{2}]\mathbf{M}}q^{2(\kappa+S_{[-\frac{N}{2}+1,\frac{N}{2}]}+\alpha S_{[-\frac{N}{2}+1,0]})}\right)|\kappa\rangle} , \tag{2.181}$$

$$= \lim_{N\to\infty} \rho(1)^{k-1}\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}\left(T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\right)|\kappa\rangle}{T(1,\kappa)^{m-k+1}\langle\kappa+\alpha|\kappa\rangle} , \tag{2.182}$$

where we defined the above mentioned function ρ as the ratio of the eigenvalues of the Matsubara transfer matrices :

$$\rho(\zeta) = \frac{T(\zeta,\kappa+\alpha)}{T(\zeta,\kappa)} . \tag{2.183}$$

We have taken $[k,m]$ to be the interval where the operator $\mathcal{O}$ acts non trivially, and denoted the restriction to this interval of $q^{2\alpha S(0)}\mathcal{O}$ by $X_{[k,m]}$.

Special limit of the functional $Z^{\kappa}_{\mathbf{n}}$. Let us make a comment on the usefulness of the general Matsubara space (2.31) :

$$\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{m}=1}^{\mathbf{n}} \mathbb{C}^{2s_{\mathbf{m}}+1} . \tag{2.184}$$

The beginning of the fermionic construction started in [1] with the study a much simpler quantity than $Z^{\kappa}_{\mathbf{n}}$:

$$Z^{0}_{\infty}(\mathcal{O}) = \frac{\langle vac|q^{2\alpha S(0)}\mathcal{O}|vac\rangle}{\langle vac|q^{2\alpha S(0)}|vac\rangle} , \tag{2.185}$$

which is defined for operators living only on the Space chain and without reference to any additional direction. The above definition of the Matsubara space $\mathfrak{H}_{\mathbf{M}}$ appears to be the right extension to perform. Indeed, taking alternating inhomogeneities $\tau_{\mathbf{m}} = \zeta^{-1}q^{-\frac{1}{2}}$ for $\mathbf{m}$ even and $\tau_{\mathbf{m}} = \zeta q^{\frac{1}{2}}$ for $\mathbf{m}$ odd one has the following expansion :

$$\mathrm{Tr}_{\mathbf{M}}(T_{\mathbf{S},\mathbf{M}}) = C^{\mathbf{n}N}\exp\left(\mathbf{n}\sum_{p=1}^{\infty} z^{2p-1}I_{2p-1}\right) , \tag{2.186}$$

where the variable z is related in some complicated way to ζ and I_{2p-1} are the lattice integrals of motion (in particular $I_1 = H_{\mathrm{XXZ}}$), C is a numerical constant. Considering the limit $\mathbf{n}\to\infty$ and

introducing at the same time the temperature β by setting $z = -\frac{\beta}{\mathbf{n}}$, the above sum is reduced to the first term:

$$\lim_{\mathbf{n}\to\infty} C^{-\mathbf{n}N}\mathrm{Tr}_{\mathbf{M}}(T_{\mathbf{S},\mathbf{M}}) = e^{-\beta H_{XXZ}} . \tag{2.187}$$

This implies that the functional $Z^{\kappa}_{\mathbf{n}}$ boils down in the limit $\mathbf{n} \to \infty$ to

$$Z^{\kappa}_{\infty} = \frac{\mathrm{Tr}_{\mathbf{S}}(e^{-\beta(H_{XXZ}+hS)}q^{2\alpha S(0)}\mathcal{O})}{\mathrm{Tr}_{\mathbf{S}}(e^{-\beta(H_{XXZ}+hS)}q^{2\alpha S(0)})} . \tag{2.188}$$

This quantity has been studied in the paper [62] that provided the important result that the fermionic basis is compatible with the presence of a non zero temperature and magnetic field. Taking the zero temperature limit $\beta \to \infty$ together with $\kappa = 0$ (that is $h = 0$) we recover (2.185). This motivates the study of the general Matsubara space (2.31), since it allows the use of any combination of integrals of motion inside the trace on the Space chain, thanks to the formula (2.186).

A remark about the correlation functions. In the previous definition (2.179) of the vacuum expectation value, the fact that we used a local operator together with the limit $N \to \infty$ automatically singled out the maximal eigenvalue of the transfer matrix in (2.182). However, it is natural to define a similar quantity for any eigenvector $|\Phi\rangle$ of the Matsubara transfer matrix (recall that their general form is given by the ABA procedure (2.52)) :

$$Z_{|\Phi\rangle}(\mathcal{O}) = \frac{\langle\Phi|\,\mathrm{Tr}_{[1,p]}(T_{[1,p],\mathbf{M}}\mathcal{O})\,|\Phi\rangle}{T^p\,\langle\Phi|\Phi\rangle} . \tag{2.189}$$

We are going to use this definition when we will work with the fermion-current basis in the chapter 3 (see (3.36) as well as [8]).

Fermionic basis and main theorem.

Let us now invoke the main results of [3]. In this paper, the authors introduced creation operators b^*, c^*, t^* that depend on spectral parameters in the space direction and create local operators out of the lattice primary field $q^{2\alpha S(0)}$. These operators act in the following way :

$$b^*\,,\, c^*\,,\, t^* \quad : \quad \mathcal{W}^{(\alpha)} \to \mathcal{W}^{(\alpha)} . \tag{2.190}$$

We shall not discuss the detailed definition of these operators, and consider the results of [3] as an existence theorem. On the other hand the completeness of the basis of operators created thanks to b^*, c^*, t^* was shown in [63]. As we already emphasized, the crucial feature of this basis is that one has a very simple expression for the vacuum expectation values :

$$\begin{aligned} &Z^{\kappa}_{\mathbf{n}}\Big\{t^*(\zeta_1^0)...t^*(\zeta_p^0)b^*(\zeta_1^+)...b^*(\zeta_r^+)c^*(\zeta_r^-)...c^*(\zeta_1^-)\left(q^{2\alpha S(0)}\right)\Big\} \\ &= \prod_{i=1}^{p} 2\rho(\zeta_i^0) \times \det\left(\omega(\zeta_i^+,\zeta_j^-)\right)_{1\le i,j\le r} , \end{aligned} \tag{2.191}$$

where ρ is defined in (2.183) and together with ω depends only on the Matsubara data. It was shown in [3] that ω has to satisfy several specific properties. We will recollect them in the case of the general Matsubara situation (2.31) :

$$\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{m}=1}^{\mathbf{n}} \mathbb{C}^{2s_{\mathbf{m}}+1} , \tag{2.192}$$

with the L operator given in (2.33). The general case is important to the future application of the fermionic construction to the 19-vertex model ($s_{\mathbf{m}} = 1$). Of course, to use the result (2.191) for the 6-vertex model we only need the case $s_{\mathbf{m}} = \frac{1}{2}$ discussed above. Hence, in full generality we have :

1. Analytical properties. From this point of view the function ω splits in two parts :

$$\omega(\zeta,\xi) = \omega_{\mathrm{hol}}(\zeta,\xi) + \omega_{\mathrm{sing}}(\zeta,\xi) , \tag{2.193}$$

where ω_{sing} carries all relevant singularities :

$$\zeta^{-\alpha}T(\zeta,\kappa)\left(\omega(\zeta,\xi) - \omega_{\mathrm{sing}}(\zeta,\xi)\right) , \tag{2.194}$$

is a polynomial in ζ^2 of degree $\mathbf{n}$. The singularities of ω_{hol} are located at the zeroes of the Matsubara transfer matrix eigenvalue $T(\zeta,\kappa)$.

2. Normalization condition. First introduce the function φ

$$\varphi(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \prod_{k=0}^{2s_{\mathbf{m}}} \frac{1}{(\zeta/\tau_{\mathbf{m}})^2 q^{-2s_{\mathbf{m}}+2k+1} - 1} \,. \tag{2.195}$$

It satisfies the important relation :

$$a(\zeta)\varphi(\zeta) = d(\zeta q)\varphi(\zeta q) \,, \tag{2.196}$$

where a and d are Bethe Ansatz functions defined in (2.51) (for spin $\frac{1}{2}$), whose definition for any spin is :

$$a(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} ((\zeta/\tau_{\mathbf{m}})^2 q^{-2s_{\mathbf{m}}+1} - 1) \,, \quad d(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} ((\zeta/\tau_{\mathbf{m}})^2 q^{2s_{\mathbf{m}}+1} - 1) \,. \tag{2.197}$$

The function φ is used to specify the normalization of ω. Consider $\mathbf{n}+1$ contours $\Gamma_{\mathbf{m}}$ (several small circles) going around the following points : Γ_0 goes around the point 0, and $\Gamma_{\mathbf{m}}$ around the poles $\zeta^2 = \tau_{\mathbf{m}}^2 q^{2s_{\mathbf{m}}-2k-1}$ with $(k = 0, ..., 2s_{\mathbf{m}})$ of $\prod_{k=0}^{2s_{\mathbf{m}}} \frac{1}{(\zeta/\tau_{\mathbf{m}})^2 q^{-2s_{\mathbf{m}}+2k+1}-1}$. The normalization condition for ω is then :

$$\int_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)\omega(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2} = 0 \,. \tag{2.198}$$

for each $\mathbf{m} \in [0, \mathbf{n}]$.

3. Symmetry condition :

$$\omega(\xi,\zeta|-\kappa,-\alpha) = \omega(\zeta,\xi|\kappa,\alpha) \,. \tag{2.199}$$

4. The singular part is known in closed form :

$$\begin{aligned} \omega_{\text{sing}}(\zeta,\xi) = & \frac{1}{T(\zeta,\kappa)T(\xi,\kappa)} \Big((4d(\xi)a(\zeta) - T(\zeta,\kappa)T(\xi,\kappa)\psi(q\zeta/\xi,\alpha)) \\ & - (4a(\xi)d(\zeta) - T(\zeta,\kappa)T(\xi,\kappa)\psi(q^{-1}\zeta/\xi,\alpha)) \\ & - 2\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)T(\xi,\kappa+\alpha) - T(\xi,\kappa)T(\zeta,\kappa+\alpha)) \Big) \,. \end{aligned} \tag{2.200}$$

where the function ψ is given by :

$$\psi(\zeta,\alpha) = \zeta^\alpha \frac{\zeta^2+1}{2(\zeta^2-1)} \,. \tag{2.201}$$

This closes the part of the discussion of the function ω on the lattice. Let us now present the scaling limit and the one point functions of fermionic operators in the CFT.

2.7.2 The scaling limit towards Conformal Field Theory

In this subsection we are going to explain how to take the scaling limit towards the continuum field theory, in the same way as it is performed in [4]. The scaling is done according to the ideas of the Destri-De Vega equations [64] : knowing the behavior of the Bethe roots for the ground state in the limit of large Matsubara space $\mathbf{n} \to \infty$, one should form Non-Linear Integral Equations on some particular function whose dependence on these roots is traceable. For that purpose define the function $\mathfrak{a}$, constructed out of the eigenvalue Q of the Baxter operator (2.139) :

$$\mathfrak{a}(\zeta,\kappa) = \frac{a(\zeta)Q^-(\zeta q)}{d(\zeta)Q^-(\zeta q^{-1})} \,. \tag{2.202}$$

The Bethe equations (2.62) can be rewritten :

$$\mathfrak{a}(\zeta,\kappa) + 1 = 0 \,. \tag{2.203}$$

Take γ to be a contour around the Bethe roots, then the logarithm of (2.202) becomes after the application of the Residue theorem :

$$\log \mathfrak{a}(\zeta,\kappa) = -2\pi i\nu\kappa + \log\left(\frac{d(\zeta)}{a(\zeta)}\right) - \int_\gamma K\left(\frac{\zeta}{\xi}\right) \log(1+\mathfrak{a}(\xi,\kappa))\frac{d\xi^2}{\xi^2} \,, \tag{2.204}$$

where the kernel K is constructed from the function ψ :

$$K(\zeta,\alpha)=\frac{1}{2\pi i}\left(\psi(\zeta q,\alpha)-\psi(\zeta q^{-1},\alpha)\right)\,. \tag{2.205}$$

The resolvent R is a solution of the following equation :

$$R(\zeta,\xi)-\int_\gamma K(\zeta/\eta,\alpha)R(\eta,\xi)dm(\eta)=K(\zeta/\xi,\alpha)\,. \tag{2.206}$$

The dependence of the resolvent on the function $\mathfrak{a}$ is hidden in the measure dm :

$$dm(\eta)=\frac{d\eta^2}{\eta^2\rho(\eta,\kappa)(1+\mathfrak{a}(\eta,\kappa))}\,. \tag{2.207}$$

Now we introduce the shift operators :

$$\Delta_\zeta f(\zeta)=f(\zeta q)-f(\zeta q^{-1})\,,\quad \delta_\zeta^- f(\zeta)=f(\zeta q)-\rho(\zeta)f(\zeta)\,, \tag{2.208}$$

and build thanks to them two more functions :

$$f_{\text{left}}(\zeta,\xi)=\delta_\zeta^-\psi(\zeta/\xi,\alpha)\,,\quad f_{\text{right}}(\zeta,\xi)=\delta_\xi^-\psi(\zeta/\xi,\alpha)\,. \tag{2.209}$$

Then one can write a scaling equation for the redefined version (according to [4]) of the function ω. This equation was proposed in [65] and reads

$$\frac{1}{4}\omega(\zeta,\xi|,\alpha,\kappa)=\left(f_{\text{left}}\star f_{\text{right}}+f_{\text{left}}\star R\star f_{\text{right}}\right)(\zeta,\xi)+\delta_\zeta^-\delta_\xi^-\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\,, \tag{2.210}$$

with

$$\Delta_\zeta^{-1}\psi(\zeta,\alpha)=VP\int_0^\infty\frac{1}{2\nu(1+(\zeta/\eta)^{\frac{1}{\nu}})}\psi(\eta,\eta)\frac{d\eta^2}{2\pi\eta^2}\,. \tag{2.211}$$

The symbol $\star$ means the convolution product with measure dm :

$$(f\star g)(\zeta)=\int_\gamma f(\zeta/\eta)g(\eta)dm(\eta)\,. \tag{2.212}$$

Recall that the behavior of the ground state Bethe roots in the limit $\mathbf{n}\to\infty$ is the following : the roots are real and concentrate on a certain finite interval $[a,b]$ on the real axis. Moreover, one can show that the function $\mathfrak{a}$ is negligible in the upper half plane and very large in the lower one. Then, after some manipulation of (2.204) we arrive at the scaling equation for $\mathfrak{a}$:

$$\log\mathfrak{a}(\zeta,\kappa)=f(\zeta,\kappa)-2i\int_a^b R(\zeta/\eta)\mathrm{Im}(\log(1+\mathfrak{a}(\eta+i0)))\frac{d\eta^2}{\eta^2}\,, \tag{2.213}$$

with $f(\zeta,\kappa)=(1+R)\cdot(-2\pi\nu\kappa+\log(a(\zeta)/d(\zeta)))$. This equation can be solved by iterations in the conformal regime.

Now we are ready to take the scaling limit (define a to be the lattice step, it will be used to rescale the spectral parameter) :

$$\text{scaling limit}\ :\quad \mathbf{n}\to\infty\,,\quad a\to 0\,,\quad \mathbf{n}\,a=2\pi R\quad\text{fixed}\,, \tag{2.214}$$

where R is the radius of the cylinder on which the model is considered. In the continuum limit, the six-vertex model scales towards to a Conformal Field Theory [9]. The field theory depends on a parameter β related to the lattice parameter ν is the following way :

$$\nu=1-\beta^2\,. \tag{2.215}$$

The limit CFT has central charge :

$$c=1-6(\beta^{-1}-\beta)^2=1-6\frac{\nu^2}{1-\nu}\,. \tag{2.216}$$

[9] The scaling limit performed in [4] is actually rather complicated, and involves the use of a screening procedure as well as the introduction of a second twist κ'. For reasons that will be clear soon we will not go into the details of the screening procedure, referring to [4] for more information on this topic.

The CFT primary fields are obtained as appropriate limits of the lattice primary fields, and take the form :

$$\lim_{\text{scaling}} q^{2\alpha S(0)} = \Phi_\alpha \,, \tag{2.217}$$

$$\Phi_\alpha(z,\bar{z}) = e^{\frac{1}{2}(\beta^{-1}-\beta)\alpha i\varphi(z,\bar{z})} \,. \tag{2.218}$$

This parametrization becomes clear in [5] where the fermionic basis is used to obtain the one point functions in the sG model. The latter is considered as a perturbation of a complex Liouville CFT with central charge (2.216). The conformal dimension of the primary field Φ_α is therefore :

$$\Delta_\alpha = \frac{(\beta^{-1}-\beta)^2}{4}\left((\alpha-1)^2-1\right) \,. \tag{2.219}$$

The scaling limit implies the following limits for the lattice functions ω and ρ :

$$\lim_{\text{scaling}} \rho(\lambda a^\nu) = \rho^{\text{sc}}(\lambda) \,, \tag{2.220}$$

$$\lim_{\text{scaling}} \omega(\lambda a^\nu, \mu a^\nu) = \omega^{\text{sc}}(\lambda,\mu) \,, \tag{2.221}$$

as well as the conjectured limits of the operators of the fermionic basis :

$$\boldsymbol{\tau}^*(\lambda) = \lim_{\text{scaling}} t^*(\lambda a^\nu) \,, \quad \boldsymbol{\beta}^*(\lambda) = \lim_{\text{scaling}} b^*(\lambda a^\nu) \,, \quad \boldsymbol{\gamma}^*(\lambda) = \lim_{\text{scaling}} c^*(\lambda a^\nu) \,. \tag{2.222}$$

Similarly to the lattice case, one has also a power series expansion for the operators $\boldsymbol{\beta}^*, \boldsymbol{\gamma}^*$:

$$\boldsymbol{\beta}^*(\lambda) = \sum_{k=1}^{\infty} \lambda^{-\frac{2k-1}{\nu}} \boldsymbol{\beta}^*_{2k-1} \,, \quad \boldsymbol{\gamma}^*(\lambda) = \sum_{k=1}^{\infty} \lambda^{-\frac{2k-1}{\nu}} \boldsymbol{\gamma}^*_{2k-1} \,. \tag{2.223}$$

According to the BLZ result on the asymptotics of the CFT transfer matrix (recalled in (2.166)), the asymptotics of the scaling limit of ρ^{sc} is simply given by :

$$\log \rho^{\text{sc}}(\lambda) \simeq \sum_{n=1}^{\infty} \lambda^{-\frac{2n-1}{\nu}} C_n \left(i_{2n-1}(\kappa) - i_{2n-1}(\kappa')\right) \,, \quad \lambda^2 \to \infty \,. \tag{2.224}$$

In the previous formula the twists κ, κ' parametrize the conformal dimensions of the boundary fields $(\Delta_\pm)$ [10] , the exact relation is given thanks to the general formula (2.219) :

$$\Delta_+ = \Delta_{\kappa+1} = \frac{(\beta^{-1}-\beta)^2}{4}(\kappa^2-1) \,. \tag{2.225}$$

With the help of the equation (2.213), is it possible to study the properties of $\mathfrak{a}^{\text{sc}}$. Using this information, the authors of [4] calculated the asymptotic expansion of the function ω^{sc} :

$$\omega^{\text{sc}}(\lambda,\mu) \simeq -\sum_{r,s=1}^{\infty} \frac{1}{r+s-1} D_{2r-1}(\alpha) D_{2s-1}(2-\alpha) \times \lambda^{\frac{-2r-1}{\nu}} \mu^{-\frac{2s-1}{\nu}} \omega_{2r-1,2s-1}(\kappa,\alpha) \,, \tag{2.226}$$

$$\lambda, \mu \to \infty \,, \tag{2.227}$$

and the coefficients $D_{2m-1}(\alpha)$ were found to be :

$$D_{2m-1}(\alpha) = \sqrt{\frac{i}{1-\beta^2}} \Gamma(1-\beta^2)^{-\frac{2n-1}{1-\beta^2}} \beta^{2n-1} \frac{1}{(n-1)!} \frac{\Gamma(\frac{\alpha}{2} + \frac{1}{2(1-\beta^2)}(2n-1))}{\Gamma(\frac{\alpha}{2} + \frac{\beta^2}{2(1-\beta^2)}(2n-1))} \,. \tag{2.228}$$

[10] In the formula (2.224), κ' is taken in [4] according to the screening procedure. However, for us the important result of [4] are the Taylor coefficients of the function ω that are obtained in the case $\kappa = \kappa'$, that is when the two boundary fields have the same conformal dimension and also $\rho^{\text{sc}} = 1$. In addition, $\Delta_+ = \Delta_-$ is the situation in which we will perform the calculations in the ssG model in section 4.6.

Conclusively, a procedure permitting to extract the coefficients ω out of the scaling equation (2.210) was elaborated in [4]. The final results take the form :

$$\omega_{1,1} = \frac{i_1}{R} - \frac{\Delta_\alpha}{12R^2}\,, \tag{2.229}$$

$$\omega_{\substack{1,3\\3,1}} = \frac{i_3}{R} - \frac{\Delta_\alpha}{6R^3} i_1 + \frac{\Delta_\alpha^2}{144R^4} + \frac{c+5}{1080R^4}\Delta_\alpha \mp d_\alpha \frac{\Delta_\alpha}{360R^4}\,, \tag{2.230}$$

$$\begin{aligned}\omega_{\substack{1,5\\5,1}} = {} & \frac{i_5}{R} - \frac{\Delta_\alpha}{4R^3} i_3 + \left(\frac{\Delta_\alpha^2}{48R^5} + \frac{c+11}{360}\Delta_\alpha\right) i_1 \\ & - \frac{\Delta_\alpha^3}{1728R^6} - \frac{13(c+35)}{90720R^6}\Delta_\alpha^2 - \frac{2c^2+21c+70}{60480R^6}\Delta_\alpha \\ & \mp d_\alpha \left(\frac{\Delta_\alpha}{120R^5} i_1 - \frac{1}{1440R^6}\Delta_\alpha^2 - \frac{c+7}{7560R^6}\Delta_\alpha\right)\,,\end{aligned} \tag{2.231}$$

$$\begin{aligned}\omega_{3,3} = {} & \frac{i_5}{R} - \frac{\Delta_\alpha}{4R^3} i_3 + \left(\frac{\Delta_\alpha^2}{48R^5} + \frac{c+2}{360R^5}\Delta_\alpha + \frac{c+2}{1440R^5}\right) i_1 \\ & - \frac{1}{1728R^6}\Delta_\alpha^3 - \frac{5c-14}{18144R^6}\Delta_\alpha^2 - \frac{10c^2+37c+70}{362880R^6}\Delta_\alpha - \frac{\frac{1}{2}c^2+c}{36288R^6}\,,\end{aligned} \tag{2.232}$$

where the function d_α is given by :

$$d_\alpha = (\beta^{-2} - \beta^2)(\alpha - 1)\,, \tag{2.233}$$

and the integrals of motion i_1, i_3, i_5 have been calculated in (1.191),(1.192),(1.193). The explicit results for the coefficients $\omega_{2i-1,2j-1}$ allow to establish a correspondence between the Virasoro description of local operators and the fermionic basis. First, one has to properly normalize the fermions to construct purely CFT objects :

$$\boldsymbol{\beta}^*_{2m-1} = D_{2m-1}(\alpha)\boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1}\,, \quad \boldsymbol{\gamma}^*_{2m-1} = D_{2m-1}(2-\alpha)\boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1}\,. \tag{2.234}$$

Then, recalling the Ward-Takahashi equations on the cylinder (1.78), one can compare the values of $\omega_{2i-1,2j-1}$, with the corresponding one point functions of Virasoro descendant fields, working level by level. The outcome is given in [4] :

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_1 \boldsymbol{\gamma}^{\mathrm{CFT}*}_1 = \mathbf{l}_{-2}\,, \tag{2.235}$$

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{\substack{1\\3}} \boldsymbol{\gamma}^{\mathrm{CFT}*}_{\substack{3\\1}} = \mathbf{l}^2_{-2} + \frac{2c-32}{9}\mathbf{l}_{-4} \mp d_\alpha \frac{2}{3}\mathbf{l}_{-4}\,, \tag{2.236}$$

$$\begin{aligned}\boldsymbol{\beta}^{\mathrm{CFT}*}_{\substack{1\\5}} \boldsymbol{\gamma}^{\mathrm{CFT}*}_{\substack{5\\1}} = {} & \mathbf{l}^3_{-2} + \frac{c-2-20\Delta_\alpha+2c\Delta_\alpha}{3(\Delta_\alpha+2)}\mathbf{l}_{-4}\mathbf{l}_{-2} \\ & + \frac{-5600\Delta_\alpha + 428c\Delta_\alpha - 6c^2\Delta_\alpha + 2352\Delta_\alpha^2 - 300c\Delta_\alpha^2 + 12c^2\Delta_\alpha^2 + 893\Delta_\alpha^3 - 32c\Delta_\alpha^3}{60\Delta_\alpha(\Delta_\alpha+2)}\mathbf{l}_6 \\ & \mp d_\alpha\left(\frac{2\Delta_\alpha}{\Delta_\alpha+2}\mathbf{l}_{-4}\mathbf{l}_{-2} + \frac{56-52\Delta_\alpha-2c+4c\Delta_\alpha}{5(\Delta_\alpha+2)}\mathbf{l}_{-6}\right)\,,\end{aligned} \tag{2.237}$$

$$\begin{aligned}\boldsymbol{\beta}^{\mathrm{CFT}*}_3 \boldsymbol{\gamma}^{\mathrm{CFT}*}_3 = {} & \mathbf{l}^3_{-2} + \frac{6+3c-76\Delta_\alpha+4c\Delta_\alpha}{6(\Delta_\alpha+2)}\mathbf{l}_{-2}\mathbf{l}_{-4} \\ & \frac{-6544\Delta_\alpha + 498c\Delta_\alpha - 5c^2\Delta_\alpha + 2152\Delta_\alpha^2 - 314c\Delta_\alpha^2 + 10c^2\Delta_\alpha^2 - 448\Delta_\alpha^3 + 16c\Delta_\alpha^3}{60\Delta_\alpha(\Delta_\alpha+2)}\mathbf{l}_{-6}\,.\end{aligned} \tag{2.238}$$

These equations do not depend on R, which is logical since the expansion of the fermionic basis on the Virasoro basis should not depend on the radius of the cylinder, and have the general structure :

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}*}_{I^-} = C_{I^+,I^-}\left(P^E_{I^+,I^-}(\{\mathbf{l}_{-2k},\Delta_\alpha,c\}) + d_\alpha\, P^O_{I^+,I^-}(\{\mathbf{l}_{-2k},\Delta_\alpha,c\})\right)\,, \tag{2.239}$$

where I^+, I^- are multi-indices, C_{I^+,I^-} is the Cauchy determinant, $P^E_{I^+,I^-}(\{\mathbf{l}_{-2k},\Delta_\alpha,c\})$ and $P^O_{I^+,I^-}(\{\mathbf{l}_{-2k},\Delta_\alpha,c\})$ are polynomial in even indexed Virasoro modes, defined up to the integrals of motion $\mathbf{i}_{2k-1}$. These two polynomials (E stands for even, O for odd) satisfy the following symmetry relations under the exchange of the sets $I^\pm$ (see (2.236) and (2.237)) :

$$P^E_{I^+,I^-} = P^E_{I^-,I^+}\,, \quad P^O_{I^+,I^-} = -P^O_{I^-,I^+}\,. \tag{2.240}$$

It is possible to apply this very nice decomposition structure to further check the results (2.235), (2.236), (2.237), (2.238) against reflection relations. This was done in the paper [31], and we will discuss it in the case of the Super sine-Gordon QFT in the chapter 4.

The results above illustrate the duality between the Virasoro algebra and the fermionic basis, and hold at the CFT level. Recall that when perturbing such a theory, even if the conformal invariance is broken, it is still possible to classify fields in terms of Virasoro descendants. On the other hand, the fermionic basis itself is well adapted to the perturbation theory, and thanks to the scaling equations it is possible to compute the one point functions of fermionic operators in the massive case. Together with the above fermions/Virasoro correspondence, it gives us the one point functions for the sine-Gordon theory.

Conclusion. In this introduction we have explained why the one point functions are the crucial data in Quantum Field Theories, and described the different approaches that exists to calculate them (reflection relations and fermionic basis). Our challenge in this PhD work is to further apply the fermionic basis constructions to the Super sine-Gordon QFT : one of the main outcome will be the analogues of the formulae (2.229),(2.230),(2.231),(2.232) for the ssG theory. This last section about the spin $\frac{1}{2}$ basis, motivates the steps that we have to take in order to calculate the one point functions in the ssG model.

1. The generalization of the fermionic basis to the spin 1 chain has been already achieved in [6]. In this more general case it will be called the *fermion-current basis*, because it involves not only fermionic operators but also $\widehat{\mathfrak{sl}_2}$ Kac-Moody currents.

2. Then in [8], we further checked the well foundedness of the fermion-current basis by applying it to the calculations of vacuum expectation values in the isotropic spin 1 chain. These results will be presented in the next chapter 3.

3. The Suzuki equations for the ssG models have been derived in [7]. In particular we recovered the BLZ expansion of the ground state eigenvalue.

4. Finally, in [9], the one point functions of fermionic operators were calculated for the Super sine-Gordon model and checked against reflection relations.

Chapter 3

The Fermion-Current Basis

We now move to the investigation of the higher spin models : the nineteen vertex model and the XXZ spin 1 chain at the lattice level and the Super sine-Gordon model at the field theoretic level. In this chapter our goal is to describe the "spin 1 fermionic basis" on the lattice. This new basis is more complicated than the original one, and for reasons that will soon become clear, it will be referred to as the *fermion-current basis*. The fermion-current basis was introduced on the lattice in [6], and loosely speaking can be built out of the initial spin $\frac{1}{2}$ basis by a set of fusion relations, in the same spirit as the nineteen vertex model is associated to the six vertex model.

In this chapter we will present some new applications of the fermion-current basis from our work [8], related to the explicit decomposition on the latter of local invariant operators. This is connected to the notion of "lattice OPE" that has been recently introduced in the context of the usual fermionic basis in the papers [66, 67, 68]. As a byproduct of the decomposition we will be able to access the density matrix and the entanglement entropy of the model.

In the papers [66, 67] a method was described which allows to compute expectation values of local operators (up to 11 sites long) for the spin $\frac{1}{2}$ isotropic spin chain. This method is based on the results of the paper [3] in which for the six-vertex model (possibly inhomogeneous one) the expectation values of local operators in the fermionic basis are computed in terms of the function ω defined by the Matsubara data. Let us describe briefly the method of [66, 67].

Every local operator allows a decomposition on the fermionic basis with the coefficients depending only on the operator in question. For sufficiently simple Matsubara data the expectation value of the operator can be computed in two ways: directly with the help of the Algebraic Bethe Ansatz or using the decomposition on the fermionic basis and the function ω. This provides equations for the coefficients of the decomposition for any given Matsubara data. Repeating this procedure for sufficiently large number of Matsubara data one obtains a system of equations for the coefficients which allows to find them.

Let us make this description more precise : consider $\mathcal{O}$ to be a local operator acting on a finite, homogeneous subchain in the Space direction (that we will simply denote by $[1, n]$), and invariant under the action of the algebra $\mathfrak{sl}_2$. We should make here an important remark : in order to be able to reuse the set-up of [66, 67] on the one hand, and to simplify the lattice fermion-current basis on the other hand, we will work *in this chapter* in the isotropic limit $\nu \to 0$ of the six-vertex model or equivalently of the XXZ spin chain model (2.28). For our needs, it is then enough to deal with the algebra $\mathfrak{sl}_2$, instead of working with the whole Quantum Group $U_q(\widehat{\mathfrak{sl}_2})$.

As explained above, one expects that $\mathcal{O}$ can be written as the action of fermionic basis operators on the identity operator I :

$$\mathcal{O} = \sum_{I,J} C_{I,J}\, b_I^* c_J^* \cdot \mathrm{I}\,, \tag{3.1}$$

where $I = \{i_1, ..., i_p\}$ and $J = \{j_1, ..., j_q\}$ are multi-indices and the above notation means :

$$b_I^* = b_{i_1}^* ... b_{i_p}^*\,, \quad c_J^* = c_{i_1}^* ... c_{i_q}^*\,, \tag{3.2}$$

and $C_{I,J}$ are coefficients. The structure of the sets I and J for given $\mathcal{O}$ has been analyzed in [66, 67], it will be detailed to some extent later (see (3.32), (3.33)). Of course the formula (3.1) should be interpreted as an identity between vacuum expectation values with specific Matsubara data Md :

$$\langle \mathcal{O} \rangle_{\mathrm{Md}} = \sum_{I,J} C_{I,J}\, \langle b_I^* c_J^* \cdot \mathrm{I} \rangle_{\mathrm{Md}}\,, \tag{3.3}$$

where the exact definition of $\langle \cdot \rangle_{\mathrm{Md}}$ will be explained soon in (3.26). This formula looks pretty much like the usual OPE in QFT or CFT (1.4) : the coefficients $C_{I,J}$ are some *universal*, local ("Ultra-Violet") data, that should not depend on the Matsubara space, whereas the terms $\langle b_I^* c_J^* \cdot \mathrm{I} \rangle_{\mathrm{Md}}$ are the lattice analogs of the one point functions of local fields in QFT that are manifestly Matsubara data ("Infra-Red") dependent. One of the accomplishments of the papers [66, 67, 68] is to explicitly show that the expansion (3.1) indeed hold for specific operators and to calculate the values of $C_{I,J}$. Since the coefficients $C_{I,J}$ are independent of the Matsubara data, they can be used as such to study the model for particular limits of the Matsubara chain (for example the zero temperature limit). As we said, if we can calculate both $\langle \mathcal{O} \rangle_{\mathrm{Md}}$ and $\langle b_I^* c_J^* \cdot \mathrm{I} \rangle_{\mathrm{Md}}$ for a sufficiently large amount of Matsubara data, one can hope that the system (3.3) is invertible and obtain the coefficients.

In the present chapter we apply a similar method to the much more complicated case of the integrable isotropic spin-1 chain described by the Hamiltonian (3.25). Therefore, we will progress in the following order :

1. Recall the alternative definition of ω from [66, 67] that is particularly suited to perform calculations on the lattice. Then explain how one can use efficiently the fermionic basis to calculate the density matrix.

2. Give some general definitions in the spin 1 lattice model.

3. Define the fermion-current basis for the spin 1 XXX chain, this is the main part of this chapter.

4. Explain the computational procedure of the coefficients and present the results on the expansion of the operator $\sum_{a=1}^{3} S_1^a S_n^a$ on the fermion-current basis. In addition provide the values of the entanglement entropy for small number of sites.

5. In the appendix, detail the inhomogeneous results on the expansion of the operator $\sum_{a=1}^{3} S_1^a S_n^a$, and explain the computational procedure for the density matrix. Finally display the explicit density matrices at zero temperature.

3.1 Alternative formula for ω

First we recall some results on the function ω (2.193). In fact, in the paper [65] an alternative construction of this function was proposed, that we shall recall from [67]. This definition hold for the XXX spin chain, that is the isotropic limit $\nu \to 0$ of the XXZ spin chain model (2.28), where many simplifications occur. To emphasize the fact that we work now towards applications to the spin 1 case, and more importantly to use notations compatible with our papers [7, 8, 9] we will denote L the length of the Matsubara space (instead of $\mathbf{n}$ before). The Bethe Ansatz functions (2.51) become simply polynomials :

$$a(\lambda) = \lambda^L + \sum_{j=1}^{L} a_j \lambda^{L-j} \,, \quad d(\lambda) = \lambda^L + \sum_{j=1}^{L} d_j \lambda^{L-j} \,. \tag{3.4}$$

The Baxter operator in this case is :

$$Q(\lambda) = \prod_{j=1}^{m} (\lambda - \beta_j) \,, \tag{3.5}$$

where β_j are the Bethe roots. One defines also two auxiliary functions K and H :

$$K(\lambda) = \frac{2}{\lambda^2 - 1} \,, \quad H(\lambda) = \frac{1}{(\lambda - 1)\lambda} \,. \tag{3.6}$$

Recall the definition of $\mathfrak{a}$ and of the measure dm :

$$\mathfrak{a}(\lambda) = \frac{a(\lambda) Q(\lambda + 1)}{d(\lambda) Q(\lambda - 1)} \,, \quad dm(\lambda) = \frac{d\lambda}{1 + \mathfrak{a}(\lambda)} \,. \tag{3.7}$$

This allow to rewrite the Bethe equations as in (2.140) :

$$\mathfrak{a}(\beta_j) = -1 \,, \quad 1 \leq j \leq m \,. \tag{3.8}$$

To define ω we first construct the function G which satisfies :

$$G(\eta,\mu) = H(\eta-\mu) + \frac{1}{2\pi i}\oint_{\Gamma} K(\eta-\sigma)G(\sigma,\mu)dm(\sigma)\,, \tag{3.9}$$

where Γ goes around the Bethe roots and the point $\sigma=\mu$. Using the Residue theorem, the above integral equation for G actually reduces to a linear system for the quantities $G(\beta_j,\mu)$, that can be solved knowing the roots β_j. Finally ω is defined by :

$$\omega(\lambda,\mu) = \frac{1}{2\pi i}\oint_{\Gamma'} H(\eta-\lambda)G(\eta,\mu)dm(\eta) + \frac{1}{4}K(\lambda-\mu)\,, \tag{3.10}$$

where Γ' encircles the same points as Γ as well as the additional point $\eta=\lambda$. Let us make an important remark. As we said, our goal is to generate a lot of Matsubara data, which is equivalent to solve many times the Bethe equations. But solving the Bethe equations for β_j is hard, so we should work differently, and in some sense reverse the logic of (3.8). We will consider the starting, input data to be the parameters $\{\beta_1,...,\beta_m,a_{m+1},...,a_L,d_1,...d_L\}$, and the equations (3.8) will be used to obtain the remaining unknown $\{a_1,...,a_m\}$ for which these equations are *linear*. This procedure allows to compute efficiently a very large quantity of sets $\{\beta_1,...,\beta_m,a_1,...,a_L,d_1,...d_L\}$ that satisfies the Bethe equations. From them, one can compute ω thanks to the formulae (3.9), (3.10). The vacuum expectation values of the operators $b_i^*c_j^*$ are encoded in the Taylor decomposition of $\omega(\lambda,\mu)$:

$$\omega(\lambda,\mu) = \sum_{i,j=1}^{\infty} \lambda^{i-1}\mu^{j-1}\omega_{i,j}\,, \tag{3.11}$$

and the application of the general formula [1]

$$\langle b^*(\lambda_1)...b^*(\lambda_p)c^*(\mu_1)...c^*(\mu_q)\cdot I\rangle = \det\left[\,(\omega(\lambda_i,\mu_j))_{1\le i\le p,1\le j\le q}\,\right], \tag{3.12}$$

to the homogeneous case leaves :

$$\langle b_I^*c_J^*\cdot I\rangle_{\rm Md} = \det\big[(\omega_{i,j})_{i\in I,j\in J}\big]\,. \tag{3.13}$$

This formula is the starting point of all calculations that aim to decompose the operator $\mathcal{O}$ on the fermionic basis.

The function ω at zero temperature. In addition to the fact that (3.10) gives a simple way to compute the function ω, this equation is useful to describe this function in the zero temperature limit ($L\to\infty$, for the ground state the Bethe roots concentrate on a finite interval on the real axis). In this case, ω is given by an explicit function of a single argument :

$$\omega(\lambda,\mu)=\omega_1(\lambda-\mu)\quad \omega_1(\lambda) = -\frac{1}{2}+2\log 2+\sum_{k=1}^{\infty}\left(2\zeta(2k+1)(1-2^{-2k})-\frac{1}{2}\right)\,. \tag{3.14}$$

3.2 Density matrix

Definition. The density matrix $D(n)$ (on n sites) is a local operator that allows to access all vacuum expectation values of local invariant operators by a simple trace computation. Let $\mathcal{O}$ be a local operator that acts on the Space chain. Then its vacuum expectation values in terms of the density matrix are given by :

$$\langle\mathcal{O}\rangle_{\rm Md} = {\rm Tr}_{[1,n]}\left(D(n)\mathcal{O}\right)\,. \tag{3.15}$$

Obviously, the dependence on the Matsubara data in the right hand side is hidden inside the density matrix. We would like to have an intrinsic definition of $D(n)$, and a very useful expression can be obtained in terms of the operators of the fermionic basis. On sites $[1,n]$, we first decompose a complete set of invariant operators $(O^a)_a$ (that is operators that commute with the $\mathfrak{sl}_2$ action, they will be described in the Appendix 3.9) on the fermionic basis :

$$O^a = \sum_{IJ} C^a_{IJ}b_I^*c_J^*\,. \tag{3.16}$$

[1] By convention if $p\neq q$ the determinant is zero.

Second, we calculate the dual invariant operators $(\tilde{O}^a)_a$ that satisfy :

$$\mathrm{Tr}_{[1,n]}(\tilde{O}^a O^b) = \delta_{ab}\,. \tag{3.17}$$

The density matrix is then given in terms of the fermionic basis by :

$$D(n) = \sum_a \left(\sum_{IJ} C^a_{IJ} \det\left[(\omega_{i,j})_{i\in I, j\in J}\right] \right) \tilde{O}^a\,, \tag{3.18}$$

which is easy to calculate for any Matsubara data. Let us show that this definition is consistent. Take Y to be an operator that we can decompose on the basis of invariant operators $Y = \sum_a p_a O^a$. Then

$$\mathrm{Tr}_{[1,n]}\left(D(n)Y\right) = \mathrm{Tr}_{[1,n]}\left(\sum_a \left(\sum_{IJ} C^a_{IJ} \left\langle b^*_I c^*_J \right\rangle_{\mathrm{Md}} \right) \tilde{O}^a Y \right) \tag{3.19}$$

$$= \mathrm{Tr}_{[1,n]}\left(\sum_{a,b} \left(\sum_{IJ} C^a_{IJ} \left\langle b^*_I c^*_J \right\rangle_{\mathrm{Md}} \right) \tilde{O}^a p_b O^b \right) \tag{3.20}$$

$$= \sum_a p_a \left(\sum_{IJ} C^a_{IJ} \left\langle b^*_I c^*_J \right\rangle_{\mathrm{Md}} \right) = \langle Y \rangle_{\mathrm{Md}}\,. \tag{3.21}$$

Entanglement entropy an comparison with CFT. From the density matrix, it is straightforward to calculate the entanglement entropy by the formula :

$$s(n) = -\mathrm{Tr}_{[1,n]}(D(n)\log(D(n)))\,. \tag{3.22}$$

In the paper [69], it was shown that $s(n)$ has the following thermodynamical limit $n \to \infty$:

$$s(n) \simeq \frac{c}{3}\log n\,, \tag{3.23}$$

where c is the value of the central charge of the CFT that describes the lattice model in the continuum. In the paper [67], $D(n)$ was calculated thanks to the fermionic basis for the XXX spin $\frac{1}{2}$ chain, for lattice sites up to 11 and showed very good agreement with the CFT with $c = 1$:

$$s(n) \simeq \frac{1}{3}\log n\,. \tag{3.24}$$

In the rest of this chapter we generalize these results to the spin 1 XXX spin chain, and show the consistency of the fermion-current basis from this "lattice OPE" point of view. The results presented were obtained in the paper [8].

3.3 General spin 1 definitions

Let us start by defining the integrable isotropic spin-1 chain described by the Hamiltonian :

$$H = \sum_{j=-\infty}^{\infty} \left(S^a_j S^a_{j+1} - (S^a_j S^a_{j+1})^2 \right)\,, \tag{3.25}$$

where the summation over a is implied, S^a are generators of the spin-1 representation of $\mathfrak{sl}_2$, whose detailed expression will be given below. The infinite chain (the Space chain) is understood as the limit of finite chains with periodic boundary conditions.

The correlation functions for the model (3.25) were studied in [70]. Later in [6] the problem was considered in the spirit of the fermionic basis construction [3]. The authors of [6] were very much influenced by [70]. In the present discussion we use [70] in two ways: indirectly through [6], and directly, comparing exact results on 2 and 3 sites.

Below we formulate our problem. The exposition is close to that of the paper [66] where some more details can be found.

The integrable models are closely related with Quantum Groups, but in the isotropic model under consideration it is sufficient to work with the $\mathfrak{sl}_2$ algebra. We denote by $\pi_{\mathbf{S}}$ the representation

obtained as the tensor product of the spin-1 representations along the Space. In addition we introduce a finite, possibly inhomogeneous and carrying different spins, Matsubara chain and the corresponding representation $\pi_{\mathbf{M}}$ of $\mathfrak{sl}_2$. We visualize the lattice on an infinite cylinder with the compact direction been the Matsubara space. The fundamental object is the evaluation of the universal R-matrix $\mathcal{R}$:

$$\mathbf{T}_{\mathbf{S},\mathbf{M}} = (\pi_{\mathbf{S}} \otimes \pi_{\mathbf{M}})\,\mathcal{R}\,.$$

The relation with the integrable spin chain is due to the commutativity

$$[H, \mathrm{Tr}_{\mathbf{M}}(\mathbf{T}_{\mathbf{S},\mathbf{M}})] = 0\,,$$

which reflects the fact that H is just one element of a huge commutative algebra generated by the transfer-matrices $\mathrm{Tr}_{\mathbf{M}}(\mathbf{T}_{\mathbf{S},\mathbf{M}})$ computed for all possible Matsubara chains.

Denote by Md the data for a given Matsubara chain (length, spins, inhomogeneities). For a local operators $\mathcal{O}$ localized (acting non-trivially) on a finite subchain of the Space chain, define the expectation value

$$\langle \mathcal{O} \rangle_{\mathrm{Md}} = \frac{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}} \cdot \mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}}\right)}\,. \tag{3.26}$$

Using the results of the paper [6] it can be shown that there exists a basis of the local operators for the spin-1 chain created by the action on the unit operators of two fermions and one Kac-Moody current (details will be given below). We shall call this the fermion-current basis. Denote the elements of the fermion-current basis by v_α. For any $\mathcal{O}$ we have

$$\mathcal{O} = \sum_\alpha X_\alpha v_\alpha\,,$$

where X_α are the wanted coefficients of the decomposition depending on the inhomogeneities of the Space. This implies

$$\langle \mathcal{O} \rangle_{\mathrm{Md}} = \sum_\alpha X_\alpha \langle v_\alpha \rangle_{\mathrm{Md}}\,.$$

For reasonable simple Matsubara data there are independent ways to compute $\langle \mathcal{O} \rangle_{\mathrm{Md}}$ and $\langle v_\alpha \rangle_{\mathrm{Md}}$. This is how we get equations for X_α.

3.4 Fermion-Current Basis

3.4.1 Homogeneous case

We begin this section by making our notations more detailed. Consider the algebra $\mathfrak{sl}_2$. Denote by π^{2s}_λ the $(2s+1)$-dimensional evaluation representation with the evaluation parameter λ. In order to handle $\mathbf{T}_{\mathbf{S},\mathbf{M}}$ in the definitions above, we use the following expression for the R matrix of the spin 1 chain $R(\lambda,\mu) = (\pi^2_\lambda \otimes \pi^2_\mu)\mathcal{R}$ which depends only on the difference of arguments $\zeta = \lambda - \mu$:

$$R(\zeta) = \begin{pmatrix}
(\zeta+1)(\zeta+2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta(\zeta+1) & 0 & 2(\zeta+1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\zeta-1)\zeta & 0 & 4\zeta & 0 & 2 & 0 & 0 \\
0 & 2(\zeta+1) & 0 & \zeta(\zeta+1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta & 0 & \zeta+\zeta^2+2 & 0 & \zeta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta(\zeta+1) & 0 & 2(\zeta+1) & 0 \\
0 & 0 & 2 & 0 & 4\zeta & 0 & (\zeta-1)\zeta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2(\zeta+1) & 0 & \zeta(\zeta+1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\zeta+1)(\zeta+2)
\end{pmatrix}$$

In the homogeneous case

$$\pi_{\mathbf{S}} = \cdots \pi^2_0 \otimes \pi^2_0 \otimes \pi^2_0 \otimes \pi^2_0 \otimes \cdots\,.$$

As has been said we are supposed to begin with a finite, periodic Space chain of length $2N$ and then consider the limit $N \to \infty$. However, it is well-known that in the cylindrical geometry adopted here the limiting procedure is trivial, so we shall consider the Space chain as an infinite one. There is a well-known infinite family of commuting local integrals of motion which includes the Hamiltonian. The adjoint action of these operators is well-defined on the space of local operators. We denote by $\mathcal{V}$ the corresponding quotient space. For the problem considered in this chapter this is the space of interest.

The simplest operator I acts as a unit operator in every tensor component. In [6] several operators were introduced acting on the space of local operators, let us describe them. We start with the

operators $\mathbf{j}^-(\lambda)$, $\mathbf{j}^0(\lambda)$, $\mathbf{j}^+(\lambda)$, $\mathbf{b}^*(\lambda)$, $\mathbf{c}^*(\lambda)$, for which we shall often use the universal notation $\mathbf{x}^{\{1,2\}} = \mathbf{b}^*$, $\mathbf{x}^{\{2,1\}} = \mathbf{c}^*$, $\mathbf{x}^{\{1,3\}} = \mathbf{j}^+$, $\mathbf{x}^{\{2,2\}} = \mathbf{j}^0$, $\mathbf{x}^{\{3,1\}} = \mathbf{j}^-$. The indices $\{1,2\}$ *etc.* are natural in the framework of [6]. All these operators are understood as generating functions

$$\mathbf{x}^\epsilon(\lambda) = \sum_{p=-\infty}^{\infty} \lambda^{p-1} \mathbf{x}^\epsilon_p \,.$$

It is almost correct that the space $\mathcal{V}$ is created by action of $\mathbf{x}^\epsilon_p$ with $p > 0$, but some refinements are needed. The first of them concerns the normal ordering. The operators $\mathbf{j}^-(\lambda)$, $\mathbf{j}^0(\lambda)$, $\mathbf{j}^+(\lambda)$ form an $\widehat{\mathfrak{sl}_2}$ Kac-Moody algebra at level 1. The fermions $\mathbf{b}^*(\lambda)$, $\mathbf{c}^*(\lambda)$ form an $\widehat{\mathfrak{sl}_2}$ doublet. That leads to the natural commutation relations and, most importantly for our goals, to the rules of the normal ordering:

$$:\mathbf{j}^0(\lambda)\mathbf{j}^0(\mu): = \mathbf{j}^0(\lambda)\mathbf{j}^0(\mu) - \frac{2}{(\lambda-\mu)^2}\,, \quad :\mathbf{j}^+(\lambda)\mathbf{j}^-(\mu): = \mathbf{j}^+(\lambda)\mathbf{j}^-(\mu) + \frac{\mathbf{j}^0(\mu)}{\lambda-\mu} + \frac{1}{(\lambda-\mu)^2}\,, \tag{3.27}$$

$$:\mathbf{j}^+(\lambda)\mathbf{j}^0(\mu): = \mathbf{j}^+(\lambda)\mathbf{j}^0(\mu) + \frac{2\mathbf{j}^+(\mu)}{\lambda-\mu}\,, \quad :\mathbf{j}^0(\lambda)\mathbf{j}^-(\mu): = \mathbf{j}^0(\lambda)\mathbf{j}^-(\mu) + \frac{2\mathbf{j}^-(\mu)}{\lambda-\mu}\,, \tag{3.28}$$

$$:\mathbf{b}^*(\lambda)\mathbf{j}^-(\mu): = \mathbf{b}^*(\lambda)\mathbf{j}^-(\mu) - \frac{\mathbf{c}^*(\mu)}{\lambda-\mu}\,, \quad :\mathbf{c}^*(\lambda)\mathbf{j}^+(\mu): = \mathbf{b}^*(\lambda)\mathbf{j}^-(\mu) + \frac{\mathbf{b}^*(\mu)}{\lambda-\mu}\,, \tag{3.29}$$

$$:\mathbf{b}^*(\lambda)\mathbf{j}^0(\mu): = \mathbf{b}^*(\lambda)\mathbf{j}^0(\mu) + \frac{\mathbf{b}^*(\mu)}{\lambda-\mu}\,, \quad :\mathbf{c}^*(\lambda)\mathbf{j}^0(\mu): = \mathbf{c}^*(\lambda)\mathbf{j}^0(\mu) - \frac{\mathbf{c}^*(\mu)}{\lambda-\mu}\,. \tag{3.30}$$

So, the local operators are created by acting on the unit operator by normal ordered products

$$:\mathbf{x}^{\epsilon_1}_{p_1}\cdots\mathbf{x}^{\epsilon_l}_{p_l}: I\,, \qquad p_j > 0\,.$$

Introduce the ordering $\{1,2\} \prec \{2,1\} \prec \{1,3\} \prec \{2,2\} \prec \{3,1\}$. For the sake of definiteness we shall require $\epsilon_1 \preceq \epsilon_2 \leq \cdots \preceq \epsilon_l$. The second problem is that of completeness. Contrary to the case of the spin $1/2$ chain [63] we do not have a formal proof of the completeness in the present situation. On the other hand the "Russian doll" construction discussed below makes the completeness quite plausible.

Let us discuss now the most complicated issue. An important question is that of how the operators located exactly on the interval $[1,n]$ look like in our fermion-current basis. In the spin $1/2$ case we had only fermionic operators b^*_p, c^*_p. For the operators

$$b^*_{p_1}\cdots b^*_{p_k} c^*_{q_1}\cdots c^*_{q_l} I\,, \tag{3.31}$$

to be located on $[1,n]$ one imposes first of all two necessary conditions:

$$1) \qquad k + l \leq n\,, \tag{3.32}$$

$$2) \qquad p_j \leq n,\ q_j \leq n \quad \forall j\,. \tag{3.33}$$

Then there are more subtle necessary conditions explained in details in [66, 67]. Taking into account all the necessary conditions we come to the subspace of the fermionic space, whose elements may be located on $[1,n]$, of rather reasonable size. Notice also that in [66, 67] as well as in the present chapter we are interested in operators invariant under the action of global $\mathfrak{sl}_2$. This requires $k = l$ in (3.31).

For the spin 1 case, let us write the elements of the fermion-current basis in complete notations

$$:\mathbf{b}^*_{p_1}\cdots\mathbf{b}^*_{p_{k_1}}\ \mathbf{c}^*_{q_1}\cdots\mathbf{c}^*_{q_{k_2}}\ \mathbf{j}^+_{r_1}\cdots\mathbf{j}^+_{r_{k_3}}\ \mathbf{j}^0_{s_1}\cdots\mathbf{j}^0_{s_{k_4}}\ \mathbf{j}^-_{t_1}\cdots\mathbf{j}^-_{t_{k_5}}: I\,. \tag{3.34}$$

There is one necessary condition which remains unchanged:

$$k_1 + k_2 + k_3 + k_4 + k_5 \leq n\,. \tag{3.35}$$

The requirement of $\mathfrak{sl}_2$-invariance of the operators is equivalent to

$$k_1 - k_2 + 2k_3 - 2k_5 = 0\,.$$

For the fermions the condition (3.33) and additional conditions from [66, 67] (null-vectors) still hold. However, we were not able to formulate reasonable conditions for the currents. That is why in what follows, we are forced to take much more complicated and less efficient ways to calculate the correlations functions of the fermion-current basis, than in [66, 67].

3.4.2 Introducing Matsubara

The Matsubara chain is inhomogeneous

$$\pi_{\mathbf{M}} = \pi^{2s_1}_{\tau_1} \otimes \pi^{2s_2}_{\tau_2} \otimes \cdots \otimes \pi^{2s_L}_{\tau_L} .$$

Let us introduce the transfer-matrix

$$\mathbf{T}_{\mathbf{M}}(\lambda) = (\mathrm{Tr} \otimes \mathrm{id})\,(\pi^{(2)}_{\lambda} \otimes \pi_{\mathbf{M}})(\mathcal{R}) .$$

This is a commutative family, for generic Matsubara data there is a unique eigenvector with the maximal in absolute value eigenvalue of $\mathbf{T}_{\mathbf{M}}(0)$. We shall denote this eigenvector by $|\Psi\rangle$. The corresponding eigenvalue of the transfer-matrix will be denoted by $\mathbf{T}(\lambda)$.

Clearly for any local operator located on the interval $[1,n]$ we have

$$\lim_{N\to\infty} \frac{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}} \cdot \mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}}\right)} = \frac{\langle\Psi|\mathrm{Tr}_{[1,n]}\left(\mathbf{T}_{[1,n],\mathbf{M}}\mathcal{O}\right)|\Psi\rangle}{\mathbf{T}(0)^n\langle\Psi|\Psi\rangle} , \tag{3.36}$$

where $\mathbf{T}_{[1,n],\mathbf{M}}$ is the restriction of $\mathbf{T}_{\mathbf{S},\mathbf{M}}$ for the Space taken to be the finite interval $[1,n]$, its explicit expression is given below for the inhomogeneous case. Our way of computing the right hand side does not depend on the fact that the eigenvalue is maximal being applicable to any eigenvector of the transfer-matrix.

3.4.3 Inhomogeneous case : the "Russian doll" construction

The "Russian doll" construction is present indirectly already in the paper [2], however, in [6] it becomes really indispensable. The construction requires some definitions which we are going to give.

We shall need an inhomogeneous space chain:

$$\pi_{\mathbf{S}} = \cdots \pi^2_0 \otimes \pi^2_0 \otimes \pi^2_{\lambda_1} \otimes \cdots \otimes \pi^2_{\lambda_n} \otimes \pi^2_0 \otimes \pi^2_0 \otimes \cdots .$$

The inhomogeneity is located on a finite subchain $[1,n]$. Consider the space of all the operators located on this interval. Consider the expectation value (3.26) for the inhomogeneous case assuming that the local operator $\mathcal{O}$ is located on the interval $[1,n]$. Denote the corresponding spaces, isomorphic to $\mathbb{C}^3$, by $V_1, \cdots, V_n$.

In order to describe a suitable for our goals basis in $V_1 \otimes \cdots \otimes V_n$ we introduce nine operators $\mathbf{g}^{\epsilon}(\lambda_k)$ ($\epsilon = \{i,j\}$, $i,j = 1,2,3$) and act by these operators on I consequently:

$$\mathbf{g}^{\epsilon_n}(\lambda_n)\mathbf{g}^{\epsilon_{n-1}}(\lambda_{n-1}) \cdots \mathbf{g}^{\epsilon_1}(\lambda_1) I .$$

For generic $\lambda_1, \cdots, \lambda_n$ this gives a basis of the space of operators localized on the interval $[1,n]$. We have the equality $\mathbf{g}^{\{1,1\}}(\lambda) = \mathrm{id}$. The expectation values considered in the present chapter are such that in the weak sense (holding when considered in correlation functions)

$$\mathbf{g}^{\{3,3\}}(\lambda) \overset{\mathrm{w}}{=} \mathbf{g}^{\{1,1\}}(\lambda) , \quad \mathbf{g}^{\{2,3\}}(\lambda) \overset{\mathrm{w}}{=} \mathbf{g}^{\{1,2\}}(\lambda) , \quad \mathbf{g}^{\{3,2\}}(\lambda) \overset{\mathrm{w}}{=} \mathbf{g}^{\{2,1\}}(\lambda) . \tag{3.37}$$

So, effectively we are left with the same set of indices counting the operators $\mathbf{g}$ as we had before for $\mathbf{x}$.

As usual the monodromy matrix $(\pi^2_{\lambda_j} \otimes \pi_{\mathbf{M}})(\mathcal{R})$ with the first tensor component identified with V_j will be devoted by $\mathbf{T}_{j,\mathbf{M}}(\lambda_j)$. The formula (3.36) remains valid for $\mathcal{O}$ being located on the interval $[1,n]$, and certainly,

$$\mathbf{T}_{[1,n]} = \mathbf{T}_{1,\mathbf{M}}(\lambda_1) \cdots \mathbf{T}_{n,\mathbf{M}}(\lambda_n) .$$

These operators $\mathbf{g}^{\epsilon}$ are in one-to-one correspondence with the $\mathbf{x}$'s. Wanting to pass to the homogeneous case one has to apply the normal ordering, the rules are the same as above. The "Russian doll" construction is based on the identity

$$\begin{aligned} &\lim_{N\to\infty} \frac{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}} : \mathbf{x}^{\epsilon_n}(\lambda_n) \cdots \mathbf{x}^{\epsilon_1}(\lambda_1) : I\right)}{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(\mathbf{T}_{\mathbf{S},\mathbf{M}}\right)} \\ &= \frac{\langle\Psi|\mathrm{Tr}_{[1,n]}\left(\mathbf{T}_{1,\mathbf{M}}(\lambda_1) \cdots \mathbf{T}_{n,\mathbf{M}}(\lambda_n) \ : \mathbf{g}^{\epsilon_n}(\lambda_n) \cdots \mathbf{g}^{\epsilon_1}(\lambda_1) : I\right)|\Psi\rangle}{\prod_{j=1}^n \mathbf{T}(\lambda_j)\langle\Psi|\Psi\rangle} . \end{aligned} \tag{3.38}$$

This formula establishes an identity between the expectation values of a family of local operators of different lengths for the homogeneous case with the expectation values for the operators of length

n in the inhomogeneous case. For our goals, rather complicated reasonings concerning this formula which are given in [6] can be avoided just by saying that the explicit computation of the right hand side (which will be given soon for any Matsubara data), defines the operators $\mathbf{x}$ in the left hand side.

Still there is another way to apply this formula. Suppose one computes the right hand side and then sets all λ_j to zero. In that case the right hand side gives the expectation value of a local operator located on $[1,n]$ for the homogeneous chain, this allows to identify the local operators of length n in the left hand side. We shall explain how to apply this idea in practice later.

3.4.4 Fusion

Consider the tensor product of $2n$ two-dimensional spaces v_j. Introduce the projector $\mathcal{P}_j : v_{2j-1} \otimes v_{2j} \to V_j$ onto the symmetric component. Consider the product $\mathcal{P} = \mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_n$. Denote by $T_{j,\mathbf{M}}(\lambda)$ the monodromy matrix whose first tensor component acts in v_j. We have the fusion

$$T_{1,\mathbf{M}}(\lambda_1 - 1/2)T_{2,\mathbf{M}}(\lambda_1 + 1/2)\cdots T_{2n-1,\mathbf{M}}(\lambda_n - 1/2)T_{2n,\mathbf{M}}(\lambda_n + 1/2)\mathcal{P} = \mathcal{P}\ \mathbf{T}_{1,\mathbf{M}}(\lambda_1)\cdots\mathbf{T}_{n,\mathbf{M}}(\lambda_n)\,.$$

We began to consider the tensor product of $2n$ spaces v_j isomorphic to $\mathbb{C}^2$. In the framework of the present chapter the interest of this consideration is due to the fact that $\pi^2_{\lambda_1}\otimes\cdots\otimes\pi^2_{\lambda_n}$ is a submodule of $\pi^1_{\lambda_1-1/2}\otimes\pi^1_{\lambda_1+1/2}\otimes\cdots\otimes\pi^1_{\lambda_n-1/2}\otimes\pi^1_{\lambda_n+1/2}$. In what follows it will be useful to consider a more general module $\pi^1_{\mu_1}\otimes\cdots\otimes\pi^1_{\mu_{2n}}$ with generic $\mu_1,\cdots\mu_{2n}$ specializing to $\mu_j = \lambda_{[\frac{j+1}{2}]} + \frac{(-1)^j}{2}$ when needed. We have operators $g^\sigma(\mu_j)$ ($\sigma = \{1,2\},\ \{2,1\}$) acting on the latter space. The Matsubara expectation values are computed via a particular case of the main fermionic basis formula [3]:

$$\frac{\langle\Psi|\mathrm{Tr}_{[1,2n]}\left(T_{1,\mathbf{M}}(\mu_1)\cdots T_{2n,\mathbf{M}}(\mu_{2n})\ g^{\sigma_{2n}}(\mu_{2n})\cdots g^{\sigma_1}(\mu_1)I\right)|\Psi\rangle}{\prod_{j=1}^n T(\mu_j)\langle\Psi|\Psi\rangle} = (-1)^{\mathrm{sgn}(\pi)}\det|\omega(\mu_i,\mu_j)|_{i:\sigma_i=\{2,1\},\ j:\sigma_j=\{1,2\}}\,, \tag{3.39}$$

where π is the permutation putting all i such that $\sigma_i = \{2,1\}$ to the left. The functions $\omega(\lambda,\mu)$ depends on the Matsubara data as on parameters. It was defined above in (3.10).

Using the formula above one computes the right hand side of (3.38) using the following formulae

$$\begin{aligned}
\mathbf{g}^{\{1,2\}}(\lambda) &= g^{\{1,2\}}(\lambda+1/2) + g^{\{1,2\}}(\lambda-1/2)\,,\\
\mathbf{g}^{\{2,1\}}(\lambda) &= g^{\{2,1\}}(\lambda+1/2) + g^{\{2,1\}}(\lambda-1/2)\,,\\
\mathbf{g}^{\{1,3\}}(\lambda) &= g^{\{1,2\}}(\lambda+1/2)g^{\{1,2\}}(\lambda-1/2)\,,\\
\mathbf{g}^{\{3,1\}}(\lambda) &= g^{\{2,1\}}(\lambda+1/2)g^{\{2,1\}}(\lambda-1/2)\,,\\
\mathbf{g}^{\{2,2\}}(\lambda) &= g^{\{2,1\}}(\lambda+1/2)g^{\{1,2\}}(\lambda-1/2) + g^{\{1,2\}}(\lambda+1/2)g^{\{2,1\}}(\lambda-1/2)\,.
\end{aligned} \tag{3.40}$$

It is important to notice that $\mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1)I$ in which $\mathbf{g}$ are defined by (3.40) satisfies the identity

$$\mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1)I = \mathcal{P}\mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1)I\,,$$

which provides the self-consistence of the fusion.

This procedure expresses the right hand side of (3.38) in terms of determinants of matrices with the matrix elements being expressed in terms of the function $\omega(\lambda,\mu)$ and the normalization

$$\mathcal{N}(\lambda) = \frac{\mathbf{T}(\lambda)}{T(\lambda+\frac{1}{2})T(\lambda-\frac{1}{2})}\,,$$

as follows

$$\frac{\langle\Psi|\mathrm{Tr}_{[1,n]}\left(\mathbf{T}_{1,\mathbf{M}}(\lambda_1)\cdots\mathbf{T}_{n,\mathbf{M}}(\lambda_n)\ \mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1)I\right)|\Psi\rangle}{\prod_{j=1}^n\mathbf{T}(\lambda_j)\langle\Psi|\Psi\rangle} = \prod_{j=1}^n\frac{1}{\mathcal{N}(\lambda_j)} \tag{3.41}$$

$$\times\,\mathcal{F}^{\epsilon_1,\cdots,\epsilon_n}_{\sigma_1,\cdots,\sigma_{2n}}\frac{\langle\Psi|\mathrm{Tr}_{[1,2n]}\left(T_{1,\mathbf{M}}(\mu_1)\cdots T_{2n,\mathbf{M}}(\mu_{2n})\ g^{\sigma_{2n}}(\mu_{2n})\cdots g^{\sigma_1}(\mu_1)I\right)|\Psi\rangle}{\prod_{j=1}^n T(\mu_j)\langle\Psi|\Psi\rangle}\,,$$

where

$$\{\mu_1,\mu_2,\cdots,\mu_{2n-1},\mu_{2n}\} = \{\lambda_1-\tfrac{1}{2},\lambda_1+\tfrac{1}{2},\cdots,\lambda_n-\tfrac{1}{2},\lambda_n+\tfrac{1}{2}\}\,, \tag{3.42}$$

$\mathcal{F}^{\epsilon_1,\cdots,\epsilon_n}_{\sigma_1,\cdots,\sigma_{2n}}$ is a tensor easily read from (3.40).

3.5 Computational procedure and results

3.5.1 General procedure

In the homogeneous case consider an operator localized on the interval $[1, n]$. As usual we simplify the notations in (3.34) introducing multi-indices:

$$: \mathbf{b}_P^* \ \mathbf{c}_Q^* \ \mathbf{j}_R^+ \ \mathbf{j}_S^0 \ \mathbf{j}_T^- \ I : \ .$$

Consider an operator $\mathcal{O}$ localised on the interval $[1, n]$. Our goal is to find the decomposition

$$\mathcal{O} \equiv \sum_{P,Q,R,S,T} X_{P,Q,R,S,T} \ : \mathbf{b}_P^* \ \mathbf{c}_Q^* \ \mathbf{j}_R^+ \ \mathbf{j}_S^0 \ \mathbf{j}_T^- \ I :, \tag{3.43}$$

where $\equiv$ means equality in the quotient by the action of the local integrals of motion space. We would like to proceed as in [66, 67], namely, to use sufficiently simple Matsubara data in order to obtain equations for the coefficients X by computing independently the expectation values of operators on the right hand side and on the left hand side. However, in the present case there are some complications. The first is the normal ordering. The second is the multiplier containing $\mathcal{N}$ in (3.41), it looks quite innocent, but actually it is not. Also, as has been discussed, we did not find an efficient way (similar to [66, 67]) to restrict the number of terms in the right hand side. With all that in mind we decided to take a simpler way based on the inhomogeneous chain.

In the inhomogeneous case the analogue of (3.43) looks like

$$\mathcal{O} \equiv \sum_{\epsilon_1,\cdots\epsilon_n} \mathcal{X}_{\epsilon_1,\cdots\epsilon_n}(\lambda_1,\cdots,\lambda_n) : \mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1) : I \,, \tag{3.44}$$

having in mind (3.37) we reduce the indices to $\epsilon_p = \{1,1\},\{1,2\},\{2,1\},\{1,3\},\{3,1\},\{2,2\}$ remembering that $\mathbf{g}^{\{1,1\}}(\lambda_j) = \mathrm{id}$, and $\equiv$ stands for equality of the expectation values for all Matsubara data in the geometry accepted in the present chapter, in other words for the case when the left and the right Matsubara states are equal (we denote them by $|\Psi\rangle$). This is the inhomogeneous version of the quotient by the action of the local integrals.

The computation of the expectation value of (3.44) follows closely that explained in [66, 67]. In the left hand side we have

$$\frac{\langle\Psi|\mathrm{Tr}_{[1,n]}\left(\mathbf{T}_{1,\mathbf{M}}(\lambda_1)\cdots\mathbf{T}_{n,\mathbf{M}}(\lambda_n)\ \mathcal{O}\right)|\Psi\rangle}{\prod_{j=1}^n \mathbf{T}(\lambda_j)\langle\Psi|\Psi\rangle} \,. \tag{3.45}$$

The choice of Matsubara data is explained in [66]. The numerator of this expression is a linear combination of terms of the kind

$$\langle\Psi|\mathbf{T}_{i_1,j_1}(\lambda_1)\cdots\mathbf{T}_{i_n,j_n}(\lambda_n)|\Psi\rangle \,,$$

where $\mathbf{T}_{i_k,j_k}(\lambda_k) \in \mathrm{End}(\mathbf{M})$ stands for the coefficient at position i_k, j_k of

$$\mathbf{T}_{k,\mathbf{M}}(\lambda_k) = (\mathbf{T}_{i_k,j_k}(\lambda_k))_{1\le i_k,j_k\le 3} \ . \tag{3.46}$$

Using the fusion procedure, the computations are reduced to the ones explained in details in [67]. The norm $\langle\Psi|\Psi\rangle$ is computed by Gaudin formula, the eigenvalue

$$\mathbf{T}(\lambda) = T(\lambda - \tfrac{1}{2})T(\lambda + \tfrac{1}{2}) - \Delta(\lambda) \,,$$

$\Delta(\lambda)$ being the quantum determinant.

The right hand side of (3.44) is computed applying consequently the rules of the normal ordering, that is the formulae (3.40), then we express the result in terms of the functions $\omega(\lambda,\mu)$ and $\mathcal{N}(\lambda)$.

Notice that ω appear only in expressions of the form $\omega(\lambda \pm \frac{1}{2}, \mu \pm \frac{1}{2})$ and is computed from the formula given in (3.10). However ω need to be made compatible with the definition of the normal order. To this end, we introduce an auxiliary function φ :

$$\varphi(z) = \frac{1}{4}\Big(- \frac{3}{z+1} - \frac{1}{z-1} + \frac{3}{z} + \frac{1}{z+2}\Big) \tag{3.47}$$

and consider the two redefinitions :

$$\widetilde{\omega}(\lambda+\tfrac{1}{2},\mu-\tfrac{1}{2}) = \omega(\lambda+\tfrac{1}{2},\mu-\tfrac{1}{2}) + \varphi(\lambda-\mu) \,, \quad \widetilde{\omega}(\lambda-\tfrac{1}{2},\mu+\tfrac{1}{2}) = \omega(\lambda-\tfrac{1}{2},\mu+\tfrac{1}{2}) + \varphi(\lambda-\mu-1) \,,$$

where ω is taken as such from (3.10).

Below we give some examples of the expressions of the simplest elements of the fermion-current basis in terms of ω, and of how the normal ordering works in practice :

$$\begin{aligned}
&\langle \mathbf{b}^*(\lambda)\mathbf{c}^*(\mu)\rangle \\
&= \mathcal{N}(\lambda)\mathcal{N}(\mu)\big(\widetilde{\omega}(\lambda+\tfrac{1}{2},\mu+\tfrac{1}{2}) + \widetilde{\omega}(\lambda+\tfrac{1}{2},\mu-\tfrac{1}{2}) + \widetilde{\omega}(\lambda-\tfrac{1}{2},\mu+\tfrac{1}{2}) + \widetilde{\omega}(\lambda-\tfrac{1}{2},\mu-\tfrac{1}{2})\big)\,, \\
&\langle \mathbf{j}^+(z)\mathbf{j}^-(w)\rangle = -\mathcal{N}(\lambda)\mathcal{N}(\mu)\begin{vmatrix} \widetilde{\omega}(\lambda+\frac{1}{2},\mu+\frac{1}{2}) & \widetilde{\omega}(\lambda+\frac{1}{2},\mu-\frac{1}{2}) \\ \widetilde{\omega}(\lambda-\frac{1}{2},\mu+\frac{1}{2}) & \widetilde{\omega}(\lambda-\frac{1}{2},\mu-\frac{1}{2}) \end{vmatrix} + \frac{1}{(\lambda-\mu)^2}\,.
\end{aligned}$$

This type of formulae are easy to compute for given small Matsubara data and numerical λ_j. Doing that we find experimentally how many different Matsubara data we need to get the expansion (3.44). Recall that L is the length of the Matsubara chain and denote B the number of Bethe roots. For example, for the most complicated case considered in the paper [8], $n=5$, the following stock of Matsubara data is sufficient: 22 with $L=1, B=0$, 149 with $L=2, B=0$, 25 with $L=3, B=0$, 8 with $L=2, B=1$, 35 with $L=3, B=1$, 1 with $L=4, B=2$.

Up to $n=3$ the computation is simple. The structure of the coefficients is as follows

$$\mathcal{X}(\lambda_1,\cdots,\lambda_n) = \prod_{i<j} \frac{1}{(\lambda_i-\lambda_j)^{d_{i,j}}}\,\frac{P(\lambda_1,\cdots,\lambda_n)}{R(\lambda_1,\cdots,\lambda_n)}\,, \tag{3.48}$$

where $d_{i,j}, P, R$ depend on $\epsilon_1,\cdots,\epsilon_n$, $R(0,\cdots,0)\neq 0$. The degrees $d_{i,j}$ are easy to find: we take all λ's sufficiently distant except λ_i and λ_j for which we consider two separations, say, 10^{-12}, $1+10^{-13}$. Obviously, this allows to define $d_{i,j}$.

Using this as an Ansatz in the general case is difficult mostly because of the denominator $R(\lambda_1,\cdots,\lambda_n)$. On the other hand we are not really interested in all the details of this denominator having in mind further application to the homogeneous case. Let us explain that.

Consider the right hand side of (3.44). The normally ordered expression $:\mathbf{g}^{\epsilon_n}(\lambda_n)\cdots\mathbf{g}^{\epsilon_1}(\lambda_1):I$ is regular at the point $\lambda_1=0,\cdots,\lambda_n=0$. The left hand side of (3.44) does not depend on λ's. So, setting $\lambda_j=\epsilon\lambda'_j$ and sending ϵ to 0 one concludes that in the function

$$F(\lambda_1,\cdots,\lambda_n) = \frac{P(\lambda_1,\cdots,\lambda_n)}{R(\lambda_1,\cdots,\lambda_n)}$$

among the terms with ϵ^D, only those with $D=\sum d_{i,j}$ may contribute. The terms with $D>\sum d_{i,j}$ vanish in the limit. The singular terms with $D<\sum d_{i,j}$ must vanish, this gives rise to null-operators whose expectation values vanish regardless of the choice of the Matsubara data.

Experiments show that $F(\lambda_1,\cdots,\lambda_n)$ is invariant under simultaneous shift of arguments. So we need the expansion

$$F(\lambda_1,\cdots,\lambda_n) = \sum_{\substack{m_2,\cdots m_n \\ \sum m_j\le\sum d_{i,j}}} \prod_{j=2}^{n}(\lambda_j-\lambda_1)^{m_j} F_{m_2,\cdots m_n}\,.$$

Practical computations are easier in this form: we do not need to know the denominator R. The computation of the Taylor series are performed taking sufficiently small λ's and determining the Taylor coefficients F step by step. The coefficients of the Taylor series grow rapidly with the length of the interval n, hence the inconvenience of the present procedure: for $n=5$ we are forced to take λ's of the order of 10^{-30}. This makes computations rather slow.

Having the coefficients F, we arrive after a simple computation at the final formula (3.43).

3.5.2 Examples

The simplest $\mathfrak{sl}_2$-invariant operator of length n is $\sum_{a=1}^{3} S_1^a S_n^a$.

It is defined by :

$$\sum_{a=1}^{3} S_1^a S_n^a = \frac{1}{2} h\otimes I_{n-2}\otimes h + e\otimes I_{n-2}\otimes f + f\otimes I_{n-2}\otimes e\,,$$

where we have the usual $\mathfrak{sl}_2$ spin 1 operators :

$$h=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,,\quad e=\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,,\quad f=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\,.$$

For $n = 2, 3$ we compute

$$\sum_{a=1}^{3} S_1^a S_2^a = -\frac{34}{3} - 4\mathbf{b}_1^*\mathbf{c}_1^* - \frac{8}{3}\mathbf{j}_1^+\mathbf{j}_1^- \,, \tag{3.49}$$

$$\begin{aligned}\sum_{a=1}^{3} S_1^a S_3^a = &-478 + \frac{384}{5}\mathbf{b}_1^*\mathbf{c}_1^* + \frac{176}{3}(\mathbf{b}_2^*\mathbf{c}_2^* - \mathbf{b}_3^*\mathbf{c}_1^*) - \frac{13216}{15}\mathbf{j}_1^+\mathbf{j}_1^- \\ &+ \frac{1024}{15}(\mathbf{j}_2^+\mathbf{j}_4^- - \mathbf{j}_5^+\mathbf{j}_1^- - \mathbf{j}_3^+\mathbf{j}_3^- - \mathbf{j}_3^+\mathbf{j}_2^0\mathbf{j}_1^-) \\ &+ 224(\mathbf{j}_3^+\mathbf{j}_1^- - \mathbf{j}_2^+\mathbf{j}_2^-) + 240\mathbf{b}_1^*\mathbf{b}_2^*\mathbf{j}_1^- + \frac{832}{15}(\mathbf{b}_1^*\mathbf{b}_3^*\mathbf{j}_2^- - \mathbf{b}_2^*\mathbf{b}_3^*\mathbf{j}_1^- - \mathbf{b}_1^*\mathbf{b}_2^*\mathbf{j}_3^-) \,.\end{aligned}$$

The first results are derived from the inhomogeneous formula ($n = 2, 3$) which are presented in the Appendix 3.7. In the case of the infinite volume and zero temperature the function $\omega(\lambda, \mu)$ simplifies a lot, this is explained in more details in Appendix 3.8. First, in this case it depends only on the difference of the arguments: $\omega(\lambda, \mu) = \omega(\lambda - \mu)$. Second, we have the functional equation

$$\omega(\lambda + 1) + \omega(\lambda) = \frac{\pi}{2\sin(\pi\lambda)} - \varphi(\lambda) \,, \tag{3.50}$$

where φ is defined in (3.47). The equation (3.50) is easy to solve, but actually the explicit solution is never needed in our computations: the final results are expressed only through the shifted sum of two ω's in the left hand side of (3.50). This explains why the final results are given by sums of even powers of π with rational coefficients. For two and three sites we have

$$\begin{aligned}\langle\sum_{a=1}^{3} S_1^a S_2^a\rangle &= \frac{8\pi^2}{9} - \frac{34}{3} = -2.560351643 \,, \\ \langle\sum_{a=1}^{3} S_1^a S_3^a\rangle &= -478 + \frac{13216\pi^2}{45} - \frac{224\pi^4}{5} + \frac{4096\pi^6}{2025} = 1.283223553 \,,\end{aligned}$$

in full agreement with [70].

We found expressions similar to (3.49) for $n = 4, 5$ which are unfortunately too long to be presented here. But the results for the infinite volume and zero temperature are of reasonable size:

$$\begin{aligned}\langle\sum_{a=1}^{3} S_1^a S_4^a\rangle = &\frac{74317166}{75} - \frac{54372392\pi^2}{27} + \frac{14677235264\pi^4}{10125} - \frac{6743857664\pi^6}{14175} \\ &+ \frac{238274860288\pi^8}{3189375} - \frac{1509154816\pi^{10}}{273375} + \frac{17291214848\pi^{12}}{111628125} = -1.083843468 \,, \\ \langle\sum_{a=1}^{3} S_1^a S_5^a\rangle = &\frac{30764875058782}{175} - \frac{5889239056193536\pi^2}{6615} + \frac{129766077160539584\pi^4}{70875} \\ &- \frac{1795332485778909184\pi^6}{893025} + \frac{6099426887102689018 88\pi^8}{468838125} \\ &- \frac{6922910606153603072\pi^{10}}{13395375} + \frac{2684747793382087192576\pi^{12}}{21097715625} \\ &- \frac{3399560104110390 64064\pi^{14}}{17722081125} + \frac{7217056126203854848\pi^{16}}{4219543125} \\ &- \frac{2439025898062610432\pi^{18}}{29536801875} + \frac{572648486718144512\pi^{20}}{344596021875} = 0.8330261734 \,.\end{aligned}$$

From the expressions above one conjectures that $\langle\sum_{a=1}^{3} S_1^a S_n^a\rangle$ is a polynomial in π^2 of degree $n(n-1)/2$ with rational coefficients.

Having developed the fermion-current basis it is easy to compute the correlators $\langle\sum_{a=1}^{3} S_1^a S_n^a\rangle$ ($n = 2, 3, 4, 5$) for finite temperature (like in [67]), or for the generalized Gibbs ensemble.

Another interesting application consists in the computation of the density matrix $D(n)$ for the interval of length n in the infinite antiferromagnetic chain and of the entanglement entropy. Our methods of computation are far from perfection, so, we are doing much worse than in the paper [67], namely, only up to $n = 4$. This is not enough to compare the entanglement entropy $s(n) = -\mathrm{Tr}(D(n)\log D(n))$ with the CFT prediction [69]

$$s(n) \simeq \frac{c}{3}\log n + a = \frac{1}{2}\log n + a \,,$$

where a is a non-universal constant. We remind that the scaling limit of the model is described by a CFT with $c = 3/2$. Still some resemblance with the scaling behavior is already observed in the table which present the results of our computations.

n	$s(n)$	$s(n) - \frac{1}{2}\log n$
2	1.5005420731509647	1.153968482870992
3	1.7187172552051159	1.169411110871061
4	1.8681251161018912	1.174977935541946

3.6 Conclusion

We have shown that the fermion-current basis works for small subchains of an infinite spin 1 integrable chain. In particular, the completeness holds at least up to intervals of length 5. We produced exact results for lengths $n = 4, 5$ which were not available previously. However, we are far from the length 11 achieved in [66]. There are two reasons for that. First, there is an objective reason: the model is far more complicated and the fermion-current basis contains much more elements than the fermionic basis for the spin $1/2$ case. Second, there is a subjective reason: our method of computation is not perfect, we did not find how to work with the homogeneous case directly, so, we are forced to mix it with the inhomogeneous one, in a rather involved way which requires a lot of computer memory.

3.7 Appendix 1 : Explicit formula for the coefficients

Here we give the formulae (in a weak sense) for the inhomogeneous case having in mind (3.37). The inhomogeneities are $\lambda_1, \cdots, \lambda_n$. The coefficients do not depend on a simultaneous shift of inhomogeneities, for this reason we shall use

$$\mu_j = \lambda_{j+1} - \lambda_1 \,.$$

For $n = 2$ we have

$$\sum_{a=1}^{3} S_1^a S_2^a \stackrel{\mathrm{w}}{=} \frac{2(17 - 6\mu_1^2 + \mu_1^4)}{3(\mu_1^2 - 1)} + (\mu_1^2 - 4)\mathbf{g}^{1,2}(\lambda_1)\mathbf{g}^{2,1}(\lambda_2) - \frac{2}{3}(\mu_1^2 - 4)(\mu_1^2 - 1)\mathbf{g}^{1,3}(\lambda_1)\mathbf{g}^{3,1}(\lambda_2)\,.$$

For $n = 3$ we have

$$\begin{aligned}
&\sum_{a=1}^{3} S_1^a S_3^a \stackrel{\mathrm{w}}{=} -\frac{2}{45(\mu_1^2-1)((\mu_1-\mu_2)^2-1)(\mu_2^2-1)} \times (-10755 + 4406\mu_1^2 - 943\mu_1^4 + 20\mu_1^6 - 4406\mu_1\mu_2 + 1886\mu_1^3\mu_2 \\
&- 60\mu_1^5\mu_2 + 4241\mu_2^2 - 2499\mu_1^2\mu_2^2 + 342\mu_1^4\mu_2^2 + 4\mu_1^6\mu_2^2 + 1556\mu_1\mu_2^3 - 584\mu_1^3\mu_2^3 - 12\mu_1^5\mu_2^3 - 793\mu_2^4 + 372\mu_1^2\mu_2^4 + \mu_1^4\mu_2^4 \\
&- 90\mu_1\mu_2^5 + 18\mu_1^3\mu_2^5 + 35\mu_2^6 - 11\mu_1^2\mu_2^6) \\
&- \frac{2}{15(\mu_1^2-1)((\mu_1-\mu_2)^2-1)(\mu_1-\mu_2)(\mu_2^2-1)\mu_2} \times (1360 - 416\mu_1^2 - 182\mu_1^4 + 30\mu_1^6 + 618\mu_1\mu_2 + 533\mu_1^3\mu_2 - 155\mu_1^5\mu_2 \\
&- 618\mu_2^2 - 259\mu_1^2\mu_2^2 + 300\mu_1^4\mu_2^2 - 5\mu_1^6\mu_2^2 - 548\mu_1\mu_2^3 - 284\mu_1^3\mu_2^3 + 67\mu_1^5\mu_2^3 + 274\mu_2^4 + 127\mu_1^2\mu_2^4 - 178\mu_1^4\mu_2^4 + 5\mu_1^6\mu_2^4 \\
&+ 18\mu_1\mu_2^5 + 179\mu_1^3\mu_2^5 - 20\mu_1^5\mu_2^5 - 6\mu_2^6 - 69\mu_1^2\mu_2^6 + 30\mu_1^4\mu_2^6 + 8\mu_1\mu_2^7 - 20\mu_1^3\mu_2^7 - 2\mu_2^8 + 5\mu_1^2\mu_2^8))\mathbf{g}^{1,2}(\lambda_1)\mathbf{g}^{2,1}(\lambda_2) \\
&+ \frac{1}{15(\mu_1^2-1)\mu_1((\mu_1-\mu_2)^2-1)(\mu_1-\mu_2)(\mu_2^2-1)} \times (2720 - 1176\mu_1^2 + 428\mu_1^4 + 48\mu_1^6 - 4\mu_1^8 + 1176\mu_1\mu_2 - 856\mu_1^3\mu_2 \\
&- 144\mu_1^5\mu_2 + 16\mu_1^7\mu_2 - 832\mu_2^2 - 773\mu_1^2\mu_2^2 + 584\mu_1^4\mu_2^2 - 213\mu_1^6\mu_2^2 + 10\mu_1^8\mu_2^2 + 1201\mu_1\mu_2^3 - 928\mu_1^3\mu_2^3 + 583\mu_1^5\mu_2^3 \\
&- 40\mu_1^7\mu_2^3 - 364\mu_2^4 + 840\mu_1^2\mu_2^4 - 611\mu_1^4\mu_2^4 + 75\mu_1^6\mu_2^4 - 400\mu_1\mu_2^5 + 269\mu_1^3\mu_2^5 - 85\mu_1^5\mu_2^5 + 60\mu_2^6 - 55\mu_1^2\mu_2^6 \\
&+ 55\mu_1^4\mu_2^6 + 15\mu_1\mu_2^7 - 15\mu_1^3\mu_2^7)\mathbf{g}^{1,2}(\lambda_2)\mathbf{g}^{2,1}(\lambda_3) \\
&- \frac{2}{15(\mu_1^2-1)\mu_1((\mu_1-\mu_2)^2-1)(\mu_2^2-1)\mu_2} \times (1360 - 416\mu_1^2 - 182\mu_1^4 + 30\mu_1^6 + 214\mu_1\mu_2 + 195\mu_1^3\mu_2 - 25\mu_1^5\mu_2 \\
&- 416\mu_2^2 + 248\mu_1^2\mu_2^2 - 25\mu_1^4\mu_2^2 - 5\mu_1^6\mu_2^2 + 195\mu_1\mu_2^3 + 34\mu_1^3\mu_2^3 - 37\mu_1^5\mu_2^3 - 182\mu_2^4 - 25\mu_1^2\mu_2^4 + 82\mu_1^4\mu_2^4 + 5\mu_1^6\mu_2^4 \\
&- 25\mu_1\mu_2^5 - 37\mu_1^3\mu_2^5 - 10\mu_1^5\mu_2^5 + 30\mu_2^6 - 5\mu_1^2\mu_2^6 + 5\mu_1^4\mu_2^6))\mathbf{g}^{2,1}(\lambda_2)\mathbf{g}^{1,2}(\lambda_3) \\
&- \frac{8(\mu_1^2-4)(\mu_1^2-1)}{45(-1+(\mu_1-\mu_2)^2-1)(\mu_1-\mu_2)^2(\mu_2^2-1)\mu_2^2}(-96 + 32\mu_1^2 - 2\mu_1^4 + 67\mu_1\mu_2 - 18\mu_1^3\mu_2 - 67\mu_2^2 - 7\mu_1^2\mu_2^2 + 5\mu_1^4\mu_2^2 \\
&+ 50\mu_1\mu_2^3 - 25\mu_2^4 - 25\mu_1^2\mu_2^4 + 30\mu_1\mu_2^5 - 10\mu_2^6)\mathbf{g}^{1,3}(\lambda_1)\mathbf{g}^{3,1}(\lambda_2) \\
&- \frac{2(\mu_2^2-4)}{45(\mu_1^2-1)\mu_1((\mu_1-\mu_2)^2-1)(\mu_1-\mu_2)^2\mu_2} \times (-384 + 992\mu_1^2 - 872\mu_1^4 + 306\mu_1^6 - 44\mu_1^8 + 2\mu_1^10 + 268\mu_1\mu_2 + 987\mu_1^3\mu_2 \\
&- 602\mu_1^5\mu_2 + 147\mu_1^7\mu_2 - 8\mu_1^9\mu_2 - 748\mu_2^2 + 138\mu_1^2\mu_2^2 + 188\mu_1^4\mu_2^2 - 178\mu_1^6\mu_2^2 + 12\mu_1^8\mu_2^2 - 1097\mu_1\mu_2^3 + 274\mu_1^3\mu_2^3 + 52\mu_1^5\mu_2^3 \\
&- 21\mu_1^7\mu_2^3 + 708\mu_2^4 - 194\mu_1^2\mu_2^4 + 92\mu_1^4\mu_2^4 + 54\mu_1^6\mu_2^4 + 140\mu_1\mu_2^5 - 74\mu_1^3\mu_2^5 - 78\mu_1^5\mu_2^5 - 104\mu_2^6 - 20\mu_1^2\mu_2^6 \\
&+ 52\mu_1^4\mu_2^6 + 25\mu_1\mu_2^7 - 13\mu_1^3\mu_2^7)\mathbf{g}^{1,3}(\lambda_1)\mathbf{g}^{3,1}(\lambda_3) \\
&- \frac{4((\mu_1-\mu_2)^2-4)}{45(\mu_1^2-1)\mu_1(\mu_1-\mu_2)(\mu_2^2-1)\mu_2^2} \times (-192 + 256\mu_1^2 - 68\mu_1^4 + 4\mu_1^6 + 614\mu_1\mu_2 - 402\mu_1^3\mu_2 + 52\mu_1^5\mu_2 - 374\mu_2^2 + 540\mu_1^2\mu_2^2 \\
&- 142\mu_1^4\mu_2^2 - 860\mu_1\mu_2^3 + 561\mu_1^3\mu_2^3 - 55\mu_1^5\mu_2^3 + 354\mu_2^4 - 497\mu_1^2\mu_2^4 + 111\mu_1^4\mu_2^4 + 2\mu_1^6\mu_2^4 + 227\mu_1\mu_2^5 - 128\mu_1^3\mu_2^5 - 3\mu_1^5\mu_2^5 \\
&- 52\mu_2^6 + 55\mu_1^2\mu_2^6 + 3\mu_1^4\mu_2^6 - 5\mu_1\mu_2^7 - \mu_1^3\mu_2^7) \times \mathbf{g}^{3,1}(\lambda_2)\mathbf{g}^{1,3}(\lambda_3) \\
&+ \frac{2(\mu_1^2-4)(\mu_2^2-4)(-26 - 7\mu_1^2 + 12\mu_1\mu_2 + 5\mu_1^3\mu_2 - 7\mu_2^2 - 10\mu_1^2\mu_2^2 + 5\mu_1\mu_2^3)}{15\mu_1((\mu_1-\mu_2)^2-1)(\mu_1-\mu_2)\mu_2)}\mathbf{g}^{1,3}(\lambda_1)\mathbf{g}^{2,1}(\lambda_2)\mathbf{g}^{2,1}(\lambda_3) \\
&+ \frac{2(\mu_1^2-4)((\mu_1-\mu_2)^2-4)(-26 - 2\mu_1^2 + 2\mu_1\mu_2 - 7\mu_2^2 + 5\mu_1^2\mu_2^2 - 5\mu_1\mu_2^3}{15\mu_1(\mu_1-\mu_2)(\mu_2^2-1)\mu_2}\mathbf{g}^{2,1}(\lambda_1)\mathbf{g}^{1,3}(\lambda_2)\mathbf{g}^{2,1}(\lambda_3) \\
&- \frac{2((\mu_1-\mu_2)^2-4)(\mu_2^2-4)(26 + 7\mu_1^2 - 2\mu_1\mu_2 + 5\mu_1^3\mu_2 + 2\mu_2^2 - 5\mu_1^2\mu_2^2)}{15(\mu_1^2-1)\mu_1(\mu_1-\mu_2)\mu_2}\mathbf{g}^{2,1}(\lambda_1)\mathbf{g}^{2,1}(\lambda_2)\mathbf{g}^{1,3}(\lambda_3) \\
&- \frac{4(\mu_1^2-4)((\mu_1-\mu_2)^2-4)(\mu_2^2-4)(-12 + \mu_1^2 - \mu_1\mu_2 + \mu_2^2))}{45\mu_1(\mu_1-\mu_2)\mu_2}\ \mathbf{g}^{1,3}(\lambda_1)\mathbf{g}^{3,1}(\lambda_2)\mathbf{g}^{2,2}(\lambda_3)\,.
\end{aligned}$$

3.8 Appendix 2 : The function ω at zero temperature

In the spin 1 case, it is not so evident to understand the right expressions of the function ω at zero temperature. Indeed, if in order to get the expression (3.14) we should appropriately deform the definition holding for general Matsubara space (3.10), in the spin 1 case we do not have such general result. In the paper [70] the following expression has been proposed for the function ω that corresponds to spin 1 Matsubara spaces :

$$\omega_0(\lambda) = \frac{1}{2} - \frac{1}{8}(\lambda^2+4)\left[\psi\left(-\frac{i\lambda}{4}\right) + \psi\left(\frac{i\lambda}{4}\right) - \psi\left(\frac{1}{2}-\frac{i\lambda}{4}\right) - \psi\left(\frac{1}{2}+\frac{i\lambda}{4}\right) - \frac{\pi\lambda}{2\sinh(\frac{\pi\lambda}{2})}\right], \quad (3.51)$$

where ψ is the logarithmic derivative of the Γ function :

$$\psi(\lambda) = \frac{d}{d\lambda}\log\Gamma(\lambda)\,. \quad (3.52)$$

It is interesting to notice that the expression for ω_0 splits in two parts : one transcendental part that involves the functions ψ, and a much simpler part given mainly by the combination $\frac{\pi\lambda}{2\sinh(\frac{\pi\lambda}{2})}$. Moreover, the following function is defined in [70] :

$$\Omega_0(\lambda) = 2i\frac{\omega_0(\lambda)+\frac{1}{2}}{\lambda^2+4}\,. \quad (3.53)$$

To match with our notations, we compare the value of the simplest density matrix on two sites. Therefore, in order to recover the results for the density matrix on 2 sites from [70] we should take for the definition of ω at zero temperature :

$$\omega\left(\frac{\lambda}{2i}\right) = 2i\Omega_0(\lambda) + \frac{2}{4+\lambda^2}\,. \quad (3.54)$$

It is then possible to show that the function ω defined in this way indeed satisfies the functional relation (3.50), and obtain remarkable simplifications in the calculations of the determinants : the transcendental part is absent from the calculations. Moreover the normalization factor also changes in the zero temperature case. In order to recover the results of [70] we take :

$$\frac{\mathbf{T}(\lambda)}{T(\lambda+\frac{1}{2})T(\lambda-\frac{1}{2})} \longrightarrow \frac{1}{2}\,. \quad (3.55)$$

3.9 Appendix 3 : Density matrices at zero temperature

In this section we present the density matrices at zero temperature that led to the results summarized in the table of section 3.5.2. In order to obtain the density matrix we need to expand all invariant operators $(O^a)_a$ on the fermionic basis, in the same way as we did for the sole spin operator. Hence we shall first describe how we calculated the invariant operators and then provide the results of the computation of the density matrix, up to $n = 4$, since this is the largest length for which we were able to do the explicit calculations.

Invariant operators. The starting point is the decomposition of the tensor product of representations of $\mathfrak{sl}_2$ (we present the calculations for $n = 4$ since it is the most complex case) :

$$V(2)^{\otimes 4} = V(8) \oplus 3V(6) \oplus 6V(4) \oplus 6V(2) \oplus 3V(0)\,. \quad (3.56)$$

This decomposition illustrates the fact that the segment of the Space chain $[1,4] = V(2)^{\otimes 4}$ has two bases. The first one is the "physical", or multiplicative basis $e_{i_1}e_{i_2}e_{i_3}e_{i_4}$ with $1 \leq i_k \leq 3$ and the second "additive" basis, is generated by the highest weight vectors of $\mathfrak{sl}_2$. The invariant operators (they form a vector space that we will denote $\mathrm{End}_{\mathfrak{sl}_2}\left(V(2)^{\otimes 4}\right)$) can be trivially written in this second basis, indeed a general invariant operators takes the form :

$$O = \begin{pmatrix} x_1 I_9 & & & & \\ & x_2 I_7 & x_3 I_7 & x_4 I_7 & \\ & x_5 I_7 & x_6 I_7 & x_7 I_7 & \\ & x_8 I_7 & x_9 I_7 & x_{10} I_7 & \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix}, \quad (3.57)$$

where x_k are complex numbers. A basis of the invariant operators has therefore :

$$\dim \mathrm{End}_{\mathfrak{sl}_2}(V(2)^{\otimes 4}) = \sum_\mu m_\mu^2 = 1+9+36+36+9 = 91\,, \tag{3.58}$$

elements. The number m_μ is the multiplicity of the representation space $V(\mu)$ in the decomposition of $V(2)^{\otimes 4}$. So it is very easy to write down a complete set of independent invariant operators in the additive basis : for the invariant operator O^k we can simply take $x_i = \delta_{ik}\,, \quad i = 1, ..., 91$ in (3.57). They are linearly independent by construction.

What is left is to find an effective numerical procedure that will calculate the passage matrix between the "additive" to the "multiplicative" bases.

The main trick is to represent the $\mathfrak{sl}_2$ generators e, f, h as differential operators acting on a basis of $V(2)$ constructed out of homogeneous polynomials in two variables. In the simplest case $n = 2$ we take :

$$\mathcal{B}_{XY} = (X^2, 2XY, Y^2)\,. \tag{3.59}$$

$$e = X\partial_Y\,, \quad f = Y\partial_X\,, \quad h = X\partial_X - Y\partial_Y\,. \tag{3.60}$$

For the basis B of $V(2)^{\otimes 4}$ we take :

$$B = \mathcal{B}_{XY} \otimes \mathcal{B}_{ZT} \otimes \mathcal{B}_{UW} \otimes \mathcal{B}_{RS}\,. \tag{3.61}$$

Numerically it is easy to decompose any polynomial depending on the variables X, Y, Z, T, U, W, R, S on this basis. Then we have :

$$e = X\partial_Y + Z\partial_T + U\partial_W + R\partial_S\,, \tag{3.62}$$

$$f = Y\partial_X + T\partial_Z + W\partial_U + S\partial_R\,, \tag{3.63}$$

and an analogous formula for the action of h on the polynomials. With this polynomial set-up we can do effective calculations on the computer. In particular, we would like to identify the irreducible submodules of the space $V(2)^{\otimes 4}$. An irreducible sub-modules $V(\lambda)$ (with dimension $\dim V(\lambda) = \lambda+1$ and of multiplicity m_λ in $V(2)^{\otimes 4}$) is characterized by a highest weight vector v_λ that satisfy

$$h \cdot v_\lambda = \lambda v_\lambda\,, \quad e \cdot v_\lambda = 0\,. \tag{3.64}$$

Then we have :

$$V(\lambda) = \mathrm{Vect}\{v_\lambda, f v_\lambda, f^2 v_\lambda, ..., f^\lambda v_\lambda\} \tag{3.65}$$

Suppose that we want to find such a basis of $V(\lambda)$ in $V(2)^{\otimes 4}$. Among all vectors of the basis B, we look for independent highest weight (with weight λ) vectors v_λ^k with $k = 1, ..., m_\lambda$. These are represented by polynomials in our set-up. We can then easily act with the element f on them, using its expression as a differential operator (3.63). This gives other polynomials that we can re-express on our initial basis B. Finally, we arrive at vectors of B generating m_λ copies of the submodule $V(\lambda)$. We do this for all submodules of our decomposition and we obtain the wanted passage matrix. Before conjugating the expressions of the form (3.57) by the passage matrix we first compute the "dual" invariant operators $\widetilde{O}^a$ that satisfy (3.17). They have the same form as (3.57), up to some multiplicative normalization factors. Finally, the last step is to conjugate the dual expression of the invariant operators in the highest weight basis by this passage matrix and check that one indeed obtains independent operators.

The outcome of this procedure, is the set of 91 matrices acting on $V(2)^{\otimes 4}$ and representing all the invariant operators. It can be easily generalized to calculate the invariant operators for lengths $n \geq 5$. Below, we present the explicit expressions for the density matrices at zero temperature for $n = 2, 3, 4$.

Density matrix for $n = 2$

The invariant operators can be taken as the usual identity, permutation and projectors : $(O_i)_{i=1,2,3} = (I_9, P_\sigma, P_r)$ (the case $n = 2$ is somehow special and does not fit in the description above, we take these "physical" invariant operators (I_9, P_σ, P_r) to be like those of [70]). The density matrix is written in the dual basis : $(\widetilde{O}_i)_{i=1,2,3}$:

$$D(2) = \sum_{i=1}^{3} p_a \widetilde{O}_a\,. \tag{3.66}$$

$$p_1 = \frac{1}{54}\left(51 - 4\pi^2\right)\,, \quad p_2 = \frac{1}{54}\left(20\pi^2 - 201\right)\,, \quad p_3 = \frac{1}{81}\left(51 - 4\pi^2\right)\,. \tag{3.67}$$

Density matrix for $n = 3$

In this case the decomposition reads :

$$V(2)^{\otimes 3} = V(6) \oplus 2V(4) \oplus 3V(2) \oplus V(0)\,, \tag{3.68}$$

This gives 15 invariant operators in total. These invariant operators and their dual basis are constructed as above. The density matrix is then given by :

$$D(3) = \sum_{a=1}^{15} p_a \widetilde{O}_a\,, \tag{3.69}$$

where the $\widetilde{O}_a$ are the dual operators of the $n = 3$ analogues of (3.57). The coefficients are given by :

$$\begin{pmatrix} p_1 \\ \dots \\ p_{15} \end{pmatrix} = \begin{pmatrix} -\frac{6349}{27} + \frac{95872\pi^2}{675} - \frac{2912\pi^4}{135} + \frac{88576\pi^6}{91125} \\ \frac{13612}{27} - \frac{41492\pi^2}{135} + \frac{6308\pi^4}{135} - \frac{38336\pi^6}{18225} \\ 0 \\ \frac{2344}{9} - \frac{7202\pi^2}{45} + \frac{1096\pi^4}{45} - \frac{6656\pi^6}{6075} \\ -\frac{452}{27} + \frac{344\pi^2}{27} - \frac{268\pi^4}{135} + \frac{64\pi^6}{729} \\ -\frac{12964}{45} + \frac{39472\pi^2}{225} - \frac{5996\pi^4}{225} + \frac{36416\pi^6}{30375} \\ -\frac{358}{15} + \frac{1064\pi^2}{75} - \frac{54\pi^4}{25} + \frac{992\pi^6}{10125} \\ \frac{1432}{15} - \frac{4256\pi^2}{75} + \frac{216\pi^4}{25} - \frac{3968\pi^6}{10125} \\ -\frac{9376}{15} + \frac{28808\pi^2}{75} - \frac{4384\pi^4}{75} + \frac{26624\pi^6}{10125} \\ -\frac{1048}{45} + \frac{2644\pi^2}{225} - \frac{392\pi^4}{225} + \frac{2432\pi^6}{30375} \\ \frac{2864}{15} - \frac{8512\pi^2}{75} + \frac{432\pi^4}{25} - \frac{7936\pi^6}{10125} \\ -\frac{292}{5} + \frac{334\pi^2}{9} - \frac{424\pi^4}{75} + \frac{512\pi^6}{2025} \\ \frac{44}{15} - \frac{11\pi^2}{5} + \frac{26\pi^4}{75} - \frac{32\pi^6}{2025} \\ \frac{572}{45} - \frac{352\pi^2}{45} + \frac{268\pi^4}{225} - \frac{64\pi^6}{1215} \\ \frac{1280}{27} - \frac{3616\pi^2}{135} + \frac{544\pi^4}{135} - \frac{3328\pi^6}{18225} \end{pmatrix}. \tag{3.70}$$

Density matrix for $n = 4$

From the discussion above we have the decomposition

$$D(4) = \sum_{a=1}^{91} p_a \widetilde{O}_a \,. \tag{3.71}$$

The coefficients $(p_1, ..., p_{45})$ are given by :

$$\begin{pmatrix}
\frac{404232227}{375} - \frac{491712704\pi^2}{225} + \frac{185730778112\pi^4}{118125} - \frac{51207443456\pi^6}{99225} + \frac{4308172728328\pi^8}{5315625} - \frac{133707218944\pi^{10}}{22325625} + \frac{218843316224\pi^{12}}{1302328125} \\
-\frac{9342222274}{3375} + \frac{3786411208\pi^2}{675} - \frac{204291811264\pi^4}{50625} + \frac{8448719598088\pi^6}{637875} - \frac{33171290905288\pi^8}{15946875} + \frac{29414235136\pi^{10}}{1913625} - \frac{240715440128\pi^{12}}{558140625} \\
-\frac{185854018}{225} + \frac{3388178248\pi^2}{2025} - \frac{73113828816\pi^4}{6075} + \frac{1866490624\pi^6}{4725} - \frac{84798819488\pi^8}{1366875} + \frac{1315905443848\pi^{10}}{28704375} - \frac{43075346432\pi^{12}}{334884375} \\
\frac{185854018}{225} - \frac{3388178248\pi^2}{2025} + \frac{73113828816\pi^4}{6075} - \frac{1866490624\pi^6}{4725} + \frac{84798819488\pi^8}{1366875} - \frac{1315905443848\pi^{10}}{28704375} + \frac{43075346432\pi^{12}}{334884375} \\
-\frac{2070117532}{1125} + \frac{838761424\pi^2}{225} - \frac{407255145248\pi^4}{151875} + \frac{37427772416\pi^6}{42525} - \frac{2204247573152\pi^8}{15946875} + \frac{418841712648\pi^{10}}{4100625} - \frac{479869116416\pi^{12}}{1674421875} \\
-\frac{37112908}{75} + \frac{677237456\pi^2}{675} - \frac{21927627104\pi^4}{30375} + \frac{503810260488\pi^6}{212625} - \frac{56515465288\pi^8}{151875} + \frac{7892989648\pi^{10}}{28704375} - \frac{5167489024\pi^{12}}{66976875} \\
\frac{1960753508}{3375} - \frac{2382006128\pi^2}{2025} + \frac{128488421824\pi^4}{151875} - \frac{17712805676\pi^6}{637875} + \frac{2086376641568\pi^8}{47840625} - \frac{264297779\pi^{10}}{820125} + \frac{151402405888\pi^{12}}{1674421875} \\
-\frac{1140847442}{1125} + \frac{4160674568\pi^2}{2025} - \frac{224470574848\pi^4}{151875} + \frac{825174272\pi^6}{1701} - \frac{3644784037376\pi^8}{47840625} + \frac{161598654464\pi^{10}}{28704375} - \frac{264492384256\pi^{12}}{1674421875} \\
-\frac{827056762}{3375} + \frac{201234424\pi^2}{405} - \frac{18098716192\pi^4}{50625} + \frac{74848177664\pi^6}{637875} - \frac{881582040512\pi^8}{47840625} + \frac{13028773888\pi^{10}}{9568125} - \frac{21324775424\pi^{12}}{558140625} \\
\frac{74515294}{225} - \frac{271293176\pi^2}{405} + \frac{4876428992\pi^4}{10125} - \frac{33611052032\pi^6}{212625} + \frac{339349007368\pi^8}{1366875} - \frac{650131456\pi^{10}}{354375} + \frac{1915322368\pi^{12}}{37209375} \\
\frac{78829808}{45} - \frac{10062350912\pi^2}{2835} + \frac{36192085616\pi^4}{14175} - \frac{9978499072\pi^6}{11907} + \frac{17630007711048\pi^8}{13395375} - \frac{130276836352\pi^{10}}{13395375} + \frac{42645397504\pi^{12}}{156279375} \\
\frac{420046888}{675} - \frac{3575252476\pi^2}{2835} + \frac{192899888728\pi^4}{212625} - \frac{26591770716\pi^6}{893025} + \frac{3132114523456\pi^8}{66976875} - \frac{138867961856\pi^{10}}{40186125} + \frac{22728923545\pi^{12}}{2344190625} \\
-\frac{68496272}{675} + \frac{584276464\pi^2}{2835} - \frac{31543135952\pi^4}{212625} + \frac{4348570278\pi^6}{893025} - \frac{512180088544\pi^8}{66976875} + \frac{22707782656\pi^{10}}{40186125} - \frac{371660963844\pi^{12}}{2344190625} \\
-\frac{36764764}{135} + \frac{521329924\pi^2}{945} - \frac{5624977196\pi^4}{14175} + \frac{232628982\pi^6}{178605} - \frac{12528923\pi^8}{6125} + \frac{2024779827\pi^{10}}{1339375} - \frac{662797926\pi^{12}}{156279375} \\
-\frac{183901532}{675} + \frac{58034912\pi^2}{105} - \frac{845647349\pi^4}{212625} + \frac{116571846208\pi^6}{893025} - \frac{152554745216\pi^8}{7441875} + \frac{12174742528\pi^{10}}{8037225} - \frac{9963578982\pi^{12}}{2344190625} \\
\frac{1231813072}{675} - \frac{3492602432\pi^2}{945} + \frac{565233578512\pi^4}{212625} - \frac{77920060697\pi^6}{893025} + \frac{3059344128928\pi^8}{22325625} - \frac{406927299584\pi^{10}}{40186125} + \frac{66602399334\pi^{12}}{2344190625} \\
\frac{617494144}{135} - \frac{1250769248\pi^2}{135} + \frac{283399101952\pi^4}{42525} - \frac{19533809858\pi^6}{893025} + \frac{43825303936\pi^8}{127575} - \frac{8160948224\pi^{10}}{321489} + \frac{333929414656\pi^{12}}{468838125} \\
\frac{1030971008}{675} - \frac{250846216\pi^2}{81} + \frac{473760937088\pi^4}{212625} - \frac{13061912038\pi^6}{178605} + \frac{1098916663808\pi^8}{9568125} - \frac{68211193856\pi^{10}}{8037225} + \frac{558218510336\pi^{12}}{2344190625} \\
-\frac{80483584}{225} + \frac{19547776\pi^2}{27} - \frac{1107054181\pi^4}{212625} + \frac{50869714432\pi^6}{297675} - \frac{855973533448\pi^8}{3189375} + \frac{79697186816\pi^{10}}{40186125} - \frac{13043934822\pi^{12}}{2344190625} \\
-\frac{11432032}{15} + \frac{208415008\pi^2}{135} - \frac{47223509696\pi^4}{42525} + \frac{4340032000\pi^6}{11907} - \frac{365145254088\pi^8}{637875} + \frac{169989407744\pi^{10}}{40186125} - \frac{55644790784\pi^{12}}{468838125} \\
-\frac{18479104}{27} + \frac{562353512\pi^2}{405} - \frac{42489515872\pi^4}{42525} + \frac{2928577402888\pi^6}{893025} - \frac{98551339744\pi^8}{1913625} + \frac{30585930752\pi^{10}}{8037225} - \frac{5006189363\pi^{12}}{468838125} \\
\frac{2909268608}{675} - \frac{3537135904\pi^2}{405} + \frac{445291625216\pi^4}{70875} - \frac{18415609707\pi^6}{893025} + \frac{3098719174528\pi^8}{9568125} - \frac{320571011072\pi^{10}}{13395375} + \frac{17489536614\pi^{12}}{260465625} \\
\frac{144414784}{225} - \frac{1227705368\pi^2}{945} + \frac{66219642464\pi^4}{70875} - \frac{9128548582\pi^6}{297675} + \frac{1075237746848\pi^8}{22325625} - \frac{47673054208\pi^{10}}{13395375} + \frac{7802698956\pi^{12}}{781396875} \\
\frac{129812048}{675} - \frac{221233294\pi^2}{567} + \frac{59700389548\pi^4}{212625} - \frac{8230033657\pi^6}{893025} + \frac{969359460416\pi^8}{66976875} - \frac{42978031616\pi^{10}}{40186125} + \frac{7034395033\pi^{12}}{2344190625} \\
-\frac{46994912}{675} + \frac{79717328\pi^2}{567} - \frac{21483253552\pi^4}{212625} + \frac{2961308339\pi^6}{893025} - \frac{348817706464\pi^8}{66976875} + \frac{15465761792\pi^{10}}{40186125} - \frac{2531243622\pi^{12}}{2344190625} \\
-\frac{79408492}{675} + \frac{135116980\pi^2}{567} - \frac{13499383964\pi^4}{7875} + \frac{502469191048\pi^6}{893025} - \frac{59184992806\pi^8}{66976875} + \frac{8746969088\pi^{10}}{13395375} - \frac{1431629004\pi^{12}}{781396875} \\
-\frac{59992804}{675} + \frac{511203466\pi^2}{2835} - \frac{27588875984\pi^4}{212625} + \frac{3803143078\pi^6}{893025} - \frac{447939490288\pi^8}{66976875} + \frac{19860086272\pi^{10}}{40186125} - \frac{3250615500\pi^{12}}{2344190625} \\
\frac{12746048}{25} - \frac{2930014664\pi^2}{2835} + \frac{158098610336\pi^4}{212625} - \frac{14529763328\pi^6}{59535} + \frac{2567088068192\pi^8}{66976875} - \frac{113816415232\pi^{10}}{40186125} + \frac{18628657971\pi^{12}}{2344190625} \\
\frac{1532695424}{675} - \frac{13039892672\pi^2}{2835} + \frac{703480434304\pi^4}{212625} - \frac{96977517072\pi^6}{893025} + \frac{1142268194892\pi^8}{66976875} - \frac{50644737228\pi^{10}}{40186125} + \frac{8289119764\pi^{12}}{2344190625} \\
\frac{488795168}{675} - \frac{4161680672\pi^2}{2835} + \frac{74851987216\pi^4}{70875} - \frac{6191067161\pi^6}{178605} + \frac{3646031254976\pi^8}{66976875} - \frac{53884260352\pi^{10}}{13395375} + \frac{8819430195\pi^{12}}{781396875} \\
-\frac{46654208}{225} + \frac{397096256\pi^2}{945} - \frac{64279369024\pi^4}{212625} + \frac{2953774489\pi^6}{297675} - \frac{347911246336\pi^8}{22325625} + \frac{46275387392\pi^{10}}{40186125} - \frac{7573916876\pi^{12}}{2344190625} \\
-\frac{52596176}{135} + \frac{446841200\pi^2}{567} - \frac{24096220304\pi^4}{42525} + \frac{6643353344\pi^6}{35721} - \frac{391260615584\pi^8}{13395375} + \frac{17347492864\pi^{10}}{8037225} - \frac{567852236\pi^{12}}{93767625} \\
-\frac{24667376}{75} + \frac{378332144\pi^2}{567} - \frac{11343563936\pi^4}{23625} + \frac{4691067384\pi^6}{297675} - \frac{1657538929888\pi^8}{66976875} + \frac{24496436224\pi^{10}}{13395375} - \frac{4009494937\pi^{12}}{781396875} \\
\frac{1354493312}{675} - \frac{142391232\pi^2}{35} + \frac{62235980019\pi^4}{212625} - \frac{8579551559\pi^6}{893025} + \frac{3368477201408\pi^8}{22325625} - \frac{448041877504\pi^{10}}{40186125} + \frac{73332319846\pi^{12}}{2344190625} \\
\frac{147252608}{135} - \frac{695151616\pi^2}{315} + \frac{22494745472\pi^4}{14175} - \frac{6645008691\pi^6}{127575} + \frac{365268180736\pi^8}{4465125} - \frac{80974979072\pi^{10}}{13395375} + \frac{3786604544\pi^{12}}{22325625} \\
\frac{212055232}{675} - \frac{1806138064\pi^2}{2835} + \frac{97466597392\pi^4}{212625} - \frac{19194930176\pi^6}{127575} + \frac{1582611693184\pi^8}{66976875} - \frac{70167578624\pi^{10}}{40186125} + \frac{1640641331\pi^{12}}{33488475} \\
-\frac{28535168}{225} + \frac{726369088\pi^2}{2835} - \frac{43506584\pi^4}{23625} + \frac{2855828\pi^6}{4725} - \frac{635783665408\pi^8}{66976875} + \frac{9396355072\pi^{10}}{13395375} - \frac{313851904\pi^{12}}{15946875} \\
-\frac{15238256}{75} + \frac{233114320\pi^2}{567} - \frac{6985125296\pi^4}{23625} + \frac{4126909184\pi^6}{42525} - \frac{1020833443808\pi^8}{66976875} + \frac{15087008768\pi^{10}}{13395375} - \frac{352754892}{111628125} \\
-\frac{10920176}{75} + \frac{279070784\pi^2}{945} - \frac{45179406304\pi^4}{212625} + \frac{109842944\pi^6}{1575} - \frac{24452134633\pi^8}{22325625} + \frac{3252370534\pi^{10}}{40186125} - \frac{760469094\pi^{12}}{334884375} \\
\frac{558060928}{675} - \frac{4750161664\pi^2}{2835} + \frac{256300547968\pi^4}{212625} - \frac{10095159808\pi^6}{25515} + \frac{4161772366336\pi^8}{66976875} - \frac{184519442432\pi^{10}}{40186125} + \frac{43143528448\pi^{12}}{334884375} \\
\frac{40440688}{225} - \frac{114205144\pi^2}{315} + \frac{6154972336\pi^4}{23625} - \frac{5091104768\pi^6}{59535} + \frac{299867781376\pi^8}{22325625} - \frac{44319109\pi^{10}}{4465125} + \frac{7253491712\pi^{12}}{260465625} \\
\frac{27763856}{675} - \frac{236063822\pi^2}{2835} + \frac{12732590476\pi^4}{212625} - \frac{1755217302\pi^6}{893025} + \frac{206745415232\pi^8}{66976875} - \frac{18333153288\pi^{10}}{8037225} + \frac{1500312371\pi^{12}}{2344190625} \\
-\frac{17693456}{675} + \frac{49958624\pi^2}{945} - \frac{8077097696\pi^4}{212625} + \frac{1113511206\pi^6}{893025} - \frac{48582642568\pi^8}{2480625} + \frac{232640512\pi^{10}}{1607445} - \frac{95186616\pi^{12}}{2344190625} \\
-\frac{24084992}{675} + \frac{203398936\pi^2}{2835} - \frac{36510494\pi^4}{70875} + \frac{1509872867\pi^6}{893025} - \frac{177869116384\pi^8}{66976875} + \frac{52577075\pi^{10}}{267975} - \frac{4302430208\pi^{12}}{781396875} \\
-\frac{12905536}{675} + \frac{36636266\pi^2}{945} - \frac{5929821236\pi^4}{212625} + \frac{817440684\pi^6}{893025} - \frac{32094583904\pi^8}{22325625} + \frac{4268972032\pi^{10}}{40186125} - \frac{698721075\pi^{12}}{2344190625}
\end{pmatrix},$$

and the coefficients $(p_{46}, ..., p_{91})$ are :

$$\left(\begin{array}{c}
\frac{71956112}{675} - \frac{612389912\pi^2}{2835} + \frac{33042456752\pi^4}{212625} - \frac{45553394944\pi^6}{893025} + \frac{536574812864\pi^8}{66976875} - \frac{4758081536\pi^{10}}{8037225} + \frac{38938394624\pi^{12}}{2344190625} \\
-\frac{593790232}{375} + \frac{722418584\pi^2}{225} - \frac{38982621896\pi^4}{16875} + \frac{537392927\pi^6}{70875} - \frac{63296451872\pi^8}{5315625} + \frac{2806367948\pi^{10}}{3189375} - \frac{45932797952\pi^{12}}{186046875} \\
\frac{12050168}{225} - \frac{14490296\pi^2}{135} + \frac{779435944\pi^4}{10125} - \frac{5374658568\pi^6}{212625} + \frac{12667207328\pi^8}{3189375} - \frac{401210368\pi^{10}}{1366875} + \frac{919134208\pi^{12}}{111628125} \\
\frac{106333592}{1125} - \frac{129821432\pi^2}{675} + \frac{7009724456\pi^4}{50625} - \frac{9662166592\pi^6}{212625} + \frac{113791773152\pi^8}{15946875} - \frac{5044968448\pi^{10}}{9568125} + \frac{8257544192\pi^{12}}{558140625} \\
\frac{123217936}{375} - \frac{149949008\pi^2}{225} + \frac{8091888368\pi^4}{16875} - \frac{1239420608\pi^6}{7875} + \frac{131384124496\pi^8}{5315625} - \frac{5825138176\pi^{10}}{3189375} + \frac{9534304256\pi^{12}}{186046875} \\
\frac{22536256}{125} - \frac{82435232\pi^2}{225} + \frac{4451224864\pi^4}{16875} - \frac{409053952\pi^6}{4725} + \frac{72262573568\pi^8}{5315625} - \frac{3203772416\pi^{10}}{3189375} + \frac{5243895808\pi^{12}}{186046875} \\
\frac{22696344}{125} - \frac{82731176\pi^2}{225} + \frac{13390420648\pi^4}{50625} - \frac{6153566912\pi^6}{70875} + \frac{72483217952\pi^8}{5315625} - \frac{1928203264\pi^{10}}{1913625} + \frac{15779209216\pi^{12}}{558140625} \\
-\frac{1060806376}{1125} + \frac{1290922696\pi^2}{675} - \frac{69661898048\pi^4}{50625} + \frac{96031019072\pi^6}{212625} - \frac{113108936376\pi^8}{15946875} + \frac{10029753344\pi^{10}}{1913625} - \frac{82080505856\pi^{12}}{558140625} \\
\frac{106306232}{1125} - \frac{128692888\pi^2}{675} + \frac{770271304\pi^4}{5625} - \frac{9556560064\pi^6}{212625} + \frac{112580816328\pi^8}{15946875} - \frac{16639068816\pi^{10}}{3189375} + \frac{2723160064\pi^{12}}{186046875} \\
\frac{94426616}{1125} - \frac{12834488\pi^2}{75} + \frac{693302632\pi^4}{5625} - \frac{8601391552\pi^6}{212625} + \frac{375182924\pi^8}{590625} - \frac{499013632\pi^{10}}{1063125} + \frac{2450358272\pi^{12}}{186046875} \\
\frac{224436704}{1125} - \frac{91044272\pi^2}{225} + \frac{14739203312\pi^4}{50625} - \frac{20318070208\pi^6}{212625} + \frac{79770438128\pi^8}{5315625} - \frac{10610269696\pi^{10}}{9568125} + \frac{17366380544\pi^{12}}{558140625} \\
\frac{24207424}{375} - \frac{1973216\pi^2}{15} + \frac{177771488\pi^4}{1875} - \frac{2205552128\pi^6}{70875} + \frac{1236804704\pi^8}{253125} - \frac{18277376\pi^{10}}{50625} + \frac{1884790784\pi^{12}}{186046875} \\
\frac{78433016}{1125} - \frac{950814642\pi^2}{675} + \frac{5128119208\pi^4}{50625} - \frac{7070122432\pi^6}{212625} + \frac{83282718976\pi^8}{15946875} - \frac{527501312\pi^{10}}{1366875} + \frac{6043303936\pi^{12}}{558140625} \\
-\frac{188952152}{225} + \frac{1148767384\pi^2}{675} - \frac{12395923816\pi^4}{10125} + \frac{85441731008\pi^6}{212625} - \frac{201277596992\pi^8}{3189375} + \frac{1274863616\pi^{10}}{273375} - \frac{14606221312\pi^{12}}{11162815} \\
\frac{1422984}{25} - \frac{1035032\pi^2}{9} + \frac{837080312\pi^4}{10125} - \frac{1923822272\pi^6}{70875} + \frac{647579392\pi^8}{151875} - \frac{430696448\pi^{10}}{1366875} + \frac{197361664\pi^{12}}{22325625} \\
\frac{66927752}{1125} - \frac{81151592\pi^2}{675} + \frac{1457853992\pi^4}{16875} - \frac{6027410752\pi^6}{212625} + \frac{10141491296\pi^8}{2278125} - \frac{1850368\pi^{10}}{5625} + \frac{1717182464\pi^{12}}{186046875} \\
\frac{197620832}{1125} - \frac{24034816\pi^2}{675} + \frac{432252892\pi^4}{16875} - \frac{17875923136\pi^6}{212625} + \frac{2105510640\pi^8}{15946875} - \frac{1037244416\pi^{10}}{1063125} + \frac{1697701888\pi^{12}}{62015625} \\
\frac{86187232}{1125} - \frac{21028832\pi^2}{135} + \frac{1892486752\pi^4}{16875} - \frac{1565153536\pi^6}{42525} + \frac{92163322912\pi^8}{15946875} - \frac{90801152\pi^{10}}{212625} + \frac{2229346304\pi^{12}}{186046875} \\
\frac{30470488}{375} - \frac{37069304\pi^2}{225} + \frac{2000809384\pi^4}{16875} - \frac{394097216\pi^6}{10125} + \frac{4642145504\pi^8}{759375} - \frac{205816832\pi^{10}}{455625} + \frac{2357977088\pi^{12}}{186046875} \\
-\frac{702233632}{375} + \frac{853980832\pi^2}{225} - \frac{138225399968\pi^4}{50625} + \frac{2352468736\pi^6}{2625} - \frac{748140224512\pi^8}{5315625} + \frac{14215831552\pi^{10}}{1366875} - \frac{162871771136\pi^{12}}{558140625} \\
\frac{264284704}{1125} - \frac{320539808\pi^2}{675} + \frac{1919925728\pi^4}{5625} - \frac{23821561088\pi^6}{212625} + \frac{2806182141444\pi^8}{15946875} - \frac{4147380224\pi^{10}}{3189375} + \frac{6787760128\pi^{12}}{186046875} \\
\frac{202926496}{1125} - \frac{82385248\pi^2}{225} + \frac{4444792096\pi^4}{16875} - \frac{18378849536\pi^6}{212625} + \frac{72152947072\pi^8}{5315625} - \frac{31989800096\pi^{10}}{3189375} + \frac{35618816\pi^{12}}{1265625} \\
\frac{445010176}{1125} - \frac{60128192\pi^2}{75} + \frac{29196160448\pi^4}{50625} - \frac{8049469696\pi^6}{42525} + \frac{1580185233392\pi^8}{5315625} - \frac{3002601472\pi^{10}}{1366875} + \frac{34401222656\pi^{12}}{558140625} \\
\frac{34262912}{375} - \frac{13971328\pi^2}{75} + \frac{2265977216\pi^4}{16875} - \frac{3123632128\pi^6}{70875} + \frac{583862144\pi^8}{84375} - \frac{372736\pi^{10}}{729} + \frac{14123008\pi^{12}}{984375} \\
\frac{43944416}{375} - \frac{160027168\pi^2}{675} + \frac{8635528544\pi^4}{50625} - \frac{3969228544\pi^6}{70875} + \frac{140271745408\pi^8}{15946875} - \frac{6219206656\pi^{10}}{9568125} + \frac{10178428928\pi^{12}}{558140625} \\
-\frac{68626672}{375} + \frac{1855376\pi^2}{5} - \frac{4504723376\pi^4}{16875} + \frac{6210017024\pi^6}{70875} - \frac{2709118976\pi^8}{196875} + \frac{3243096064\pi^{10}}{3189375} - \frac{5308055552\pi^{12}}{186046875} \\
\frac{51291632}{1125} - \frac{4156592\pi^2}{45} + \frac{3362374576\pi^4}{50625} - \frac{4634801152\pi^6}{212625} + \frac{18197348864\pi^8}{5315625} - \frac{2420467712\pi^{10}}{9568125} + \frac{3961618432\pi^{12}}{558140625} \\
\frac{30746224}{1125} - \frac{12567728\pi^2}{225} + \frac{2263059684\pi^4}{5625} - \frac{2807456768\pi^6}{212625} + \frac{3673659296\pi^8}{1771875} - \frac{32574464\pi^{10}}{212625} + \frac{38084608\pi^{12}}{8859375} \\
\frac{43239952}{1125} - \frac{17521088\pi^2}{225} + \frac{2835558656\pi^4}{50625} - \frac{3908947712\pi^6}{212625} + \frac{153476416648\pi^8}{5315625} - \frac{2041421824\pi^{10}}{9568125} + \frac{3341238272\pi^{12}}{558140625} \\
-\frac{47616}{5} + \frac{483008\pi^2}{25} - \frac{140551808\pi^4}{10125} + \frac{107604736\pi^6}{23625} - \frac{760429984\pi^8}{1063125} + \frac{505736192\pi^{10}}{9568125} - \frac{23650304\pi^{12}}{15946875} \\
-\frac{369136}{375} + \frac{1534832\pi^2}{675} - \frac{84103504\pi^4}{50625} + \frac{38514176\pi^6}{70875} - \frac{1356958688\pi^8}{15946875} + \frac{60130304\pi^{10}}{9568125} - \frac{98541568\pi^{12}}{558140625} \\
-\frac{26900152}{225} + \frac{163297304\pi^2}{675} - \frac{1761366824\pi^4}{10125} + \frac{121405156\pi^6}{212625} - \frac{4085838592\pi^8}{455625} + \frac{253620224\pi^{10}}{382725} - \frac{4151050248\pi^{12}}{22325625} \\
\frac{20393992}{1125} - \frac{915992\pi^2}{25} + \frac{1333855256\pi^4}{50625} - \frac{73568192\pi^6}{8505} + \frac{7222280384\pi^8}{5315625} - \frac{960673792\pi^{10}}{9568125} + \frac{224608256\pi^{12}}{79734375} \\
\frac{1481704}{125} - \frac{15998056\pi^2}{675} + \frac{286614056\pi^4}{16875} - \frac{18805312\pi^6}{3375} + \frac{13955500096\pi^8}{15946875} - \frac{206256128\pi^{10}}{3189375} + \frac{337559552\pi^{12}}{186046875} \\
\frac{28398608}{1125} - \frac{11494448\pi^2}{225} + \frac{1859580304\pi^4}{50625} - \frac{2563409344\pi^6}{212625} + \frac{10064820896\pi^8}{5315625} - \frac{1338761216\pi^{10}}{9568125} + \frac{2191187968\pi^{12}}{558140625} \\
\frac{798688}{225} - \frac{4927712\pi^2}{675} + \frac{53341024\pi^4}{10125} - \frac{52493056\pi^6}{30375} + \frac{865076608\pi^8}{3189375} - \frac{191746048\pi^{10}}{9568125} + \frac{62783488\pi^{12}}{111628125} \\
\frac{7262824}{1125} - \frac{2979448\pi^2}{225} + \frac{483845752\pi^4}{50625} - \frac{667364288\pi^6}{212625} + \frac{26202254088\pi^8}{5315625} - \frac{9957376\pi^{10}}{273375} + \frac{570425344\pi^{12}}{558140625} \\
\frac{124884184}{1125} - \frac{50867648\pi^2}{225} + \frac{24737332016\pi^4}{151875} - \frac{2273474176\pi^6}{42525} + \frac{191247809928\pi^8}{2278125} - \frac{17805838336\pi^{10}}{2870437\,5} + \frac{29145055232\pi^{12}}{1674421875} \\
\frac{588704}{1125} - \frac{1951552\pi^2}{2025} + \frac{34238432\pi^4}{50625} - \frac{1896064\pi^6}{8505} + \frac{1683152992\pi^8}{47840625} - \frac{24906752\pi^{10}}{9568125} + \frac{40689664\pi^{12}}{558140625} \\
-\frac{101060648}{3375} + \frac{123216448\pi^2}{2025} - \frac{2218134928\pi^4}{50625} + \frac{1834896512\pi^6}{127575} - \frac{108058999168\pi^8}{47840625} + \frac{532320256\pi^{10}}{3189375} - \frac{290422784\pi^{12}}{62015625} \\
\frac{610765088}{3375} - \frac{745114112\pi^2}{2025} + \frac{4471388096\pi^4}{16875} - \frac{55477754368\pi^6}{637875} + \frac{93342558464\pi^8}{6834375} - \frac{9656311808\pi^{10}}{9568125} + \frac{2257911808\pi^{12}}{79734375} \\
\frac{4137728}{3375} - \frac{540928\pi^2}{225} + \frac{262144384\pi^4}{151875} - \frac{363386368\pi^6}{637875} + \frac{1431389696\pi^8}{15946875} - \frac{190521344\pi^{10}}{28704375} + \frac{6356992\pi^{12}}{34171875} \\
-\frac{55373024}{1125} + \frac{202354688\pi^2}{2025} - \frac{3642251072\pi^4}{50625} + \frac{5022022144\pi^6}{212625} - \frac{25351721216\pi^8}{6834375} + \frac{874225664\pi^{10}}{3189375} - \frac{204406784\pi^{12}}{26578125} \\
\frac{9786344}{1125} - \frac{7139008\pi^2}{405} + \frac{1925706896\pi^4}{151875} - \frac{884875904\pi^6}{212625} + \frac{31268420672\pi^8}{47840625} - \frac{198055936\pi^{10}}{4100625} + \frac{2269233152\pi^{12}}{1674421875} \\
\frac{357632}{3375} - \frac{90304\pi^2}{405} + \frac{2762144\pi^4}{16875} - \frac{34494592\pi^6}{637875} + \frac{407012672\pi^8}{47840625} - \frac{6017024\pi^{10}}{9568125} + \frac{98467\,84\pi^{12}}{558140625} \\
-\frac{2718536}{1125} + \frac{661696\pi^2}{135} - \frac{535867024\pi^4}{151875} + \frac{246318976\pi^6}{212625} - \frac{414524608\pi^8}{2278125} + \frac{55138304\pi^{10}}{4100625} - \frac{631717888\pi^{12}}{1674421875}
\end{array}\right)$$

Chapter 4

One point functions in the Super sine-Gordon model

In this chapter, we describe the integrable structure of the space of local operators for the Supersymmetric sine-Gordon model. This is based on the papers [8, 9]. Namely, we conjecture that this space is created by acting on the primary fields by fermions and a Kac-Moody current. We proceed with the computation of the one point functions. In the UV limit they are shown to be consistent with the alternative results obtained by solving the reflection relations.

Before moving towards the field theoretic fermion-current basis, let us recap what we learned so far on the calculations of the one point functions in integrable QFTs, and demonstrate why it is natural to apply the latter methods to the Super sine-Gordon model.

The importance of the one point functions for the computation of correlation functions in the framework of the Perturbed Conformal Field Theory has been recalled in the chapter 1, and is mainly based on the breakthrough of the paper [23]. As we explained previously, for the sine-Gordon model at finite temperature the one point functions were computed in [5] using the fermionic basis of the space of local operators. This basis was found first on the lattice for the (inhomogeneous) six-vertex model [2]. Since the expectation values in the fermionic basis are rather simple the scaling limit is not very difficult to consider. One of the main achievements is the exact relation between the local operators in the fermionic basis and their counterparts in the UV Conformal Field Theory. This was illustrated by the formulae (2.235),(2.236),(2.237),(2.238) in the section 2.7.

An alternative approach to the one point functions uses the reflection relations which are based on two reflections (Heisenberg and Virasoro). We recalled some details in the section 1.7 and a more modern approach to this problem was presented in [31]. The reflection relation "methodology" of getting one point functions includes certain subtleties with the analytical continuation with respect to the coupling constant. However, if the final goal is restricted to finding a basis in the CFT, invariant under the two reflections, one should not worry because the problem can be considered as a purely algebraic one. The reflection relations are equivalent to a certain Riemann-Hilbert problem, and for a long time it was unclear how to solve it. The synthesis of the two methods, the fermionic basis on the one hand and the reflection relations on the other, was made in [31]. In this paper it was shown that the known examples of the fermionic basis (up to level 8) solve the reflection relations. Moreover, making a qualitative assumption of the existence of the fermionic basis one can use the reflection relations in order to compute the fermionic basis quantitatively.

It is consequently interesting to apply a similar procedure to other integrable models. For the models related to higher ranks the problem does not look very realistic for the moment (advances in the rank 2 "fermionic basis" were done in [71]). However, the $\widehat{\mathfrak{sl}_2}$ (or rather $U_q(\widehat{\mathfrak{sl}_2})$) symmetric case allows a highly nontrivial extension to the Fateev model, symmetric under the exceptional algebra $U_q(\widehat{D}(2|1;\alpha))$ [72]. This model deserves the most profound study. It allows numerous particular cases and restrictions. The simplest of them is the sine-Gordon model and the next in complexity is the Supersymmetric sine-Gordon model. The latter is therefore the subject of the present chapter.

Similarly to the sG case we should begin the study of the ssG model by considering its lattice regularization which is the inhomogeneous 19-vertex model introduced by Fateev and Zamolodchikov [73] in other words the model based on the spin-1 evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$ (see the chapter 3 for the isotropic version). By the method close to that of the fermionic basis this model

was considered in [6] (this paper relies on the previous research [70]). Namely, as we showed in the chapter 3, the space of (quasi)-local operators allows a basis created by fermions and a Kac-Moody (KM) current on level one. It is easy to guess that for the integrable lattice models related to higher representations of spin s of $U_q(\widehat{\mathfrak{sl}_2})$ the space of (quasi)-local operators is generated by currents with all half-integer spins up to s. In the scaling limit these models produce the parafermionic sine-Gordon models. If we learn how to treat them in their totality it will bring us very close to the general case of the Fateev model.

As has been said, in the present chapter we consider the ssG model starting with the 19 vertex lattice model. The first indispensable step towards the one point functions, consists in finding the corresponding description of the local fields in the conformal case like in the paper [4]. The generalization is already not quite trivial. For example, in the computations of the ground state eigenvalues of the local integrals of motion the paper [4] follows the procedure proposed in [35], namely it uses the Destri-DeVega equations on a half-infinite interval. This allows to develop an analytical procedure for the computation of the eigenvalues in question. Then the procedure is generalized in order to compute the expectation values on a cylinder of the CFT operators in the fermionic basis. Unfortunately, a similar procedure for the Super CFT case is unknown to us, and we are forced to proceed with numerical computations based on equations which for the 19-vertex model were proposed by J. Suzuki [45]. It should be said that Suzuki equations have been used already for the ssG model and its conformal limit in [74]. In the present chapter we shall apply the Suzuki equations to the ssG model : in the high temperature limit we will compute numerically the eigenvalues of the first three local integrals of motion. We will get the exact general formulae by interpolation and check them against results alternatively obtained by the ODE-CFT correspondence [75, 76] following Lukyanov [77] as will be explained.

The Suzuki equations pave the way to the scaling limit for the (quasi)-local operators created by fermions and a KM current at the QFT level. These operators provide the basis of local fields for the ssG model. Our consideration relies heavily on the numerical study of scaling equations for a certain function $\Omega(\theta, \theta')$. The equation for this function is not rigorously derived, so, it is considered as a conjecture and should be checked against alternative data. Using the function $\Omega(\theta, \theta')$ it is straightforward to compute the one point functions on the cylinder of radius R (at finite temperature) for the purely fermionic part of the basis. We restrict our attention to these (fermionic) operators leaving the KM contributions for future study. We consider the UV limit $R \to 0$ in order to find agreement with the corresponding CFT.

The UV limit is studied using the numerical data and interpolating with respect to the coupling constant, the quasi-momentum and the parameter of the primary field. There is a difference with the sG case for which this kind of data allowed to obtain exact relation to the Virasoro descendants up to the level 6. Then, an important check of the entire procedure consisted in verifying that the results satisfy the reflection relations. In the ssG case only level 2 is available by this mean. This case agrees with the reflection relations but we would like to proceed a little further. We reverse the procedure following [31], namely, assuming that there are local operators created by fermions which transform simply under the reflection, and compute the elements of the fermionic basis up to the level 6. Needless to say that the reflection relations are considered as a conjecture which is hard to justify rigorously.

Finally, it is possible to compare with the results obtained by the interpolation of numerical data finding a perfect agreement. This is the main result of the present chapter: two kind of formulae whose derivations are based on completely different conjectures agree.

This chapter is organized as follows :

1. We first provide in section 4.1 a brief description of Super Conformal Field Theory, focusing on such theories on the cylinder and on the calculation of correlation functions. We also discuss the Super Liouville CFT.

2. Then we give the definition of the Super sine-Gordon QFT and a short account on its basic properties. This is done in section 4.2.

3. In section 4.3 we describe the fermion-current basis for the 19 vertex model, and present important definitions on the lattice level.

4. The Suzuki equations are derived in section 4.4 and checked against former results coming from the ODE-CFT correspondence.

5. The function Ω on the lattice is defined in section 4.5.

6. We proceed to the derivation of the scaling equations for this function and to the numerical investigation of these equations in section 4.6. The results obtained out of this numerical study are one of the main outcomes of this chapter.

7. Finally in 4.7, we check the above results by comparing them with those obtained from reflection relations.

4.1 Super Conformal Field Theory

In this section we shall give a very brief recall on the $N = 1$ *Supersymmetric Conformal Field Theory* (SCFT). This is a natural extension of usual Conformal Field Theory that is particular in the fact that it is endowed with SUSY. Examples of Super Conformal Field Theories have been studied in [78, 79, 80, 81]. This section is constructed in the same spirit as the section 1.2 : we shall first recollect some general definitions and then focus on the calculation of correlation functions on the cylinder, mostly thanks to the Supersymmetric Ward-Takahashi identities. Finally, we will discuss the $N = 1$ Super Liouville CFT and its three-point functions.

4.1.1 The Super Virasoro algebra.

Let us start with Super Conformal Field Theory on the plane. The novel feature is that SCFT possesses, besides the stress energy tensor T, a fermionic primary field with conformal dimension $\frac{3}{2}$, the *super current* S which is the generator of the supersymmetry. It means that the following OPE between T and S holds :

$$T(z)S(w) = \frac{\frac{3}{2}S(w)}{(z-w)^2} + \frac{S'(w)}{z-w} + O(1)\,. \tag{4.1}$$

We will work in the Neveu-Schwarz sector, since this sector is closed under the OPE procedure. In particular it implies that the super current S is single valued and has the expansion :

$$S(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} S_r z^{-r-\frac{3}{2}}\,. \tag{4.2}$$

The most general OPE between S and itself is [81] :

$$S(z)S(w) = \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T(w)}{z-w} + O(1)\,. \tag{4.3}$$

The expansions (4.1) and (4.3) induce that the Virasoro algebra (1.26) has to be extended to the *Super Virasoro algebra* :

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,, \tag{4.4}$$

$$\{S_r, S_s\} = 2S_{r+s} + \frac{c}{3}(r^2-\frac{1}{4})\delta_{r,-s}\,, \tag{4.5}$$

$$[L_m, S_r] = \left(\frac{m}{2}-n\right)S_{m+n}\,. \tag{4.6}$$

The last equation illustrates the fact that S is a primary field. We also introduce the *reduced central charge* that will be of some utility later :

$$\widehat{c} = \frac{2}{3}c\,. \tag{4.7}$$

Of course, the antichiral part of the Virasoro algebra contains an antiholomorphic super current $\overline{S}$, analogous to $\overline{T}$, with conformal dimensions $(\Delta, \overline{\Delta}) = (0, \frac{3}{2})$, it will be of some utility in Appendix 4.9 in order to realize the SUSY algebra in terms of fields.

Exactly as we defined the action of the modes of T on a generic field V (1.17), we can define the action of S. This is given by the formula :

$$(S_r V)(w) = \oint_{c_w} \frac{du}{2\pi i}(u-w)^{r+\frac{3}{2}-1}S(u)V(w)\,, \tag{4.8}$$

where c_w is a small circular contour around the point w.

Let us now discuss the primary fields in SCFT. To a Virasoro primary field V_a [1] one can associate the fermionic field W_a defined by :

$$W_a = S_{-\frac{1}{2}} V_a \,. \tag{4.9}$$

The field W_a is also a Virasoro primary : indeed it is easy to check that is satisfies the property (1.18), and its conformal dimension is easily obtained to be $\Delta_a + \frac{1}{2}$. Therefore, we obtain the typical OPE between V_a , W_a and T (1.20) :

$$T(z)V_a(w) = \frac{\Delta_a}{(z-w)^2} V_a(w) + \frac{1}{z-w} V_a'(w) + O(1)\,, \tag{4.10}$$

$$T(z)W_a(w) = \frac{\Delta_a + \frac{1}{2}}{(z-w)^2} W_a(w) + \frac{1}{z-w} W_a'(w) + O(1)\,. \tag{4.11}$$

Finally, we give the last important OPE between the super current and the primary fields :

$$S(z)V_a(w) = \frac{1}{z-w} W_a(w) + O(1)\,, \tag{4.12}$$

$$S(z)W_a(w) = \frac{2\Delta_a}{(z-w)^2} V_a + \frac{1}{z-w} V_a'(w) + O(1)\,. \tag{4.13}$$

Characters in Super Conformal Field Theory. In our further application of the reflection relations, it will be important to know how to count the number of states in Super Conformal Field Theory, at each level. This information is exactly given by the characters of theory, and is obtained by the multiplication of the bosonic and fermionic contributions. The Hilbert space of a SCFT has the same decomposition in terms of Verma module as in (1.49). Recall that the usual "bosonic" character (1.50) of a Verma module $\mathcal{V}_\Delta$ (which is irreducible for generic Δ and central charge) is :

$$\chi_\Delta(t) = \mathrm{Tr}_{\mathcal{V}_\Delta}\left(t^{L_0 - \frac{c}{24}}\right) = \sum_{n=0}^{\infty} \dim(n+\Delta) t^{n+\Delta-\frac{c}{24}} = \frac{t^{\Delta-\frac{c}{24}}}{\varphi(t)}\,, \tag{4.14}$$

where $\varphi(t) = \prod_{n=1}^{\infty}(1-t^n)$ [2]. The character of the free fermionic ($c = \frac{1}{2}$) theory in the Neveu-Schwarz sector is given by :

$$\chi_{\mathrm{NS}}(t) = \mathrm{Tr}_F\left(t^{L_0 - \frac{c}{24}}\right) = t^{-\frac{1}{48}} \prod_{k=1}^{\infty}(1 + t^{k-\frac{1}{2}})\,. \tag{4.17}$$

Form these equations, the number of state $P_{\mathrm{NS}}(k)$ in a SCFT at each level k is then given by the product of the two characters :

$$\mathrm{ch}_{\Delta,\mathrm{NS}}(t) = \chi_{\mathrm{NS}}(t)\chi_{\mathcal{V}_\Delta}(t) = t^{\Delta-\frac{c}{24}-\frac{1}{48}} \sum_{k=0}^{\infty} t^k P_{NS}(k) = t^{\Delta-\frac{c}{24}-\frac{1}{48}} \tag{4.18}$$

$$\times \left(1 + \sqrt{t} + t + 2t^{3/2} + 3t^2 + 4t^{5/2} + 5t^3 + 7t^{7/2} + 10t^4 + 13t^{9/2} + 16t^5 + 21t^{11/2} + 28t^6 + ...\right)\,. \tag{4.19}$$

In particular, we obtain $3, 10, 28$ states at levels $2, 4, 6$ respectively. This will be used in section 4.7.

4.1.2 SCFT on the cylinder

We now progress to the main arena of interest for our work : the cylinder. We will take the same conventions as in section 1.2.2 : two contours C and c and a field $\mathcal{O}$ located at the origin. This is recalled on the picture 4.1 :

[1] Here we anticipate the fact the UV CFT of the ssG model is the complex Super Liouville theory, and we therefore use from the start the notations V_a for the primary field with conformal dimension Δ_a (instead of V_Δ as in the chapter 1).

[2] For a generic Verma module one has

$$\dim(n+\Delta) = p(n)\,, \tag{4.15}$$

where $p(n)$ is the number of partitions of the integer n. Its generating function is exactly $\frac{1}{\varphi}$:

$$\sum_{n=0}^{\infty} p(n) t^n = \frac{1}{\varphi(t)}\,. \tag{4.16}$$

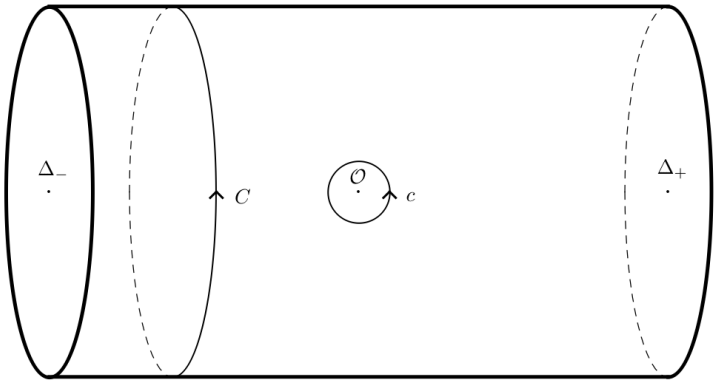

Figure 4.1: SCFT on a cylinder with the insertion of a local operator $\mathcal{O}$ and boundary conditions $\Delta_\pm$.

Recall that in this geometry we assume that two primary fields are located at the infinities $\pm\infty$ of the cylinder, we take them of dimension $\Delta_\pm$.

To verify the results on the one point functions of the fermion-current basis obtained by a numerical study of the scaling equations, we will have to compute the one point functions of descendant operators, created from the action of the stress energy tensor and of the super current, and from the mixed action of both.

As previously explained, we map the plane to the cylinder by the transformation [3]:

$$z = e^{-x}\,. \tag{4.20}$$

On the cylinder the super current can be split as :

$$S(x) = S_+(x) + S_-(x)\,, \tag{4.21}$$

where

$$S_+(x) = \sum_{n=0}^{\infty} S_{n+\frac{1}{2}} e^{(n+\frac{1}{2})x}\,, \quad S_-(x) = \sum_{n=0}^{\infty} S_{-n-\frac{1}{2}} e^{-(n+\frac{1}{2})x}\,. \tag{4.22}$$

The action on any highest weight vectors $|\Delta\rangle$ by these fields is given by :

$$S_+(x)\,|\Delta\rangle = 0\,, \quad \langle\Delta|\,S_-(x) = 0\,. \tag{4.23}$$

The presence of the primary fields $|\Delta_\pm\rangle$ at the infinities and the equation (4.23) implies (see also (1.55)) :

$$\lim_{\mathrm{Re}(x)\to\pm\infty} T(x) = \Delta_\pm - \frac{\hat{c}}{16}\,, \quad \lim_{\mathrm{Re}(x)\to\pm\infty} S(x) = 0\,. \tag{4.24}$$

These boundary conditions on T and S will be of first importance when we will compute the one point functions of descendant fields thanks to the Ward-Takahashi identities.

In the same way as we defined the *local* action of the stress energy tensor (1.56) :

$$(\mathbf{l}_n V)(y) = \oint_{c_y} \frac{dx}{2\pi i}(x-y)^{n+1}\mathcal{T}\left(T(x)V(y)\right)\,, \tag{4.25}$$

we can act locally with S on a field V:

$$(\mathbf{s}_r V)(y) = \oint_{c_y} \frac{dx}{2\pi i}(x-y)^{r+\frac{1}{2}}\mathcal{T}(S(x)V(y))\,. \tag{4.26}$$

where c_y is a small circle around the point y on the cylinder. Here we remark an important convention about the ordering of fermionic fields, that has to be taken as :

$$\mathcal{T}(\psi(x)\theta(y)) = \begin{cases} -\psi(x)\theta(y), & \text{if } x < y\,, \\ \theta(y)\psi(x), & \text{if } y < x\,, \end{cases} \tag{4.27}$$

[3] Here we take the radius $R = 1$ from the start for simplicity. It is possible to rewrite all formulae below for generic radius by applying the appropriate conformal mapping.

where ψ, θ are two fermionic fields and x, y denote the coordinate along the non-compact direction of the cylinder.

As we mentioned above, our leading goal is to to compute the one point functions of the descendant fields on the cylinder. We will use the same notation (recall (1.67)) :

$$\langle \mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a\rangle = \frac{\langle \Delta_-|\mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a|\Delta_+\rangle}{\langle \Delta_-|V_a|\Delta_+\rangle}\,. \tag{4.28}$$

The multiple action of $\mathbf{l}_k$ and $\mathbf{s}_j$ is obtained by a recursive application of the formulae (4.25) and (4.26). We now present two methods for the calculation of (4.28). The first one relies on the commutations relations between the modes of the fields T, S, V_a. The second one is more efficient and uses the Ward-Takahashi identities, this is a generalization of the chiral case presented in section 1.2.2.

Commutation relations on the cylinder. As usual we obtain the commutation relations on the cylinder from the Super Virasoro commutation relations (4.4),(4.5),(4.6). For this purpose we define the function :

$$\xi(z) = \frac{1}{2}\frac{1}{\mathrm{sh}\left(\frac{z}{2}\right)}\,, \tag{4.29}$$

chosen to be compatible with our geometry and with the NS (anti)-periodicity conditions. Then we have the same type of equations as in 1.2.2. For example the commutation relations between the super current and the primary field V_a are :

$$[S_+(x), V_a(y)] = -\xi(x-y)W_a(y)\,, \quad x<y\,, \tag{4.30}$$
$$[V_a(y), S_-(x)] = -\xi(x-y)W_a(y)\,, \quad y<x\,. \tag{4.31}$$

The anti-commutation relation between S and itself is obtained from (4.5) :

$$\{S_+(x), S(y)\} = -2T(y)\xi(x-y) - \frac{c}{3}\xi''(x-y)\,, \quad x<y\,, \tag{4.32}$$
$$\{S(y), S_-(x)\} = 2T(y)\xi(x-y) + \frac{c}{3}\xi''(x-y)\,, \quad y<x\,. \tag{4.33}$$

Finally, the commutation relations between the S and W_a read :

$$\{S_+(x), W_a(y)\} = V_a'(y)\xi(x-y) - 2\Delta_a V_a(y)\xi'(x-y)\,, \quad x<y\,, \tag{4.34}$$
$$\{W_a(y), S_-(x)\} = -V_a'(y)\xi(x-y) + 2\Delta V_a(y)\xi'(x-y)\,, \quad y<x\,. \tag{4.35}$$

Then one can start to compute the action of the modes, using the customary splitting prescription for the contours. Let us begin with the simplest case $(\mathbf{s}_{-\frac{1}{2}}V_a)(y)$. Since V_a is a bosonic field :

$$(\mathbf{s}_{-\frac{1}{2}}V_a)(y) = \oint_{c_y}\frac{dx}{2\pi i}\mathcal{T}(S(x)V_a(y)) = \oint_{c_y^-}\frac{dx}{2\pi i}S(x)V_a(y) + \oint_{c_y^+}\frac{dx}{2\pi i}V_a(y)S(x)\,. \tag{4.36}$$

Then, for each term we have :

$$\oint_{c_y^-}\frac{dx}{2\pi i}S(x)V_a(y) = \oint_{c_y^-}\frac{dx}{2\pi i}\left([S_+(x), V_a(y)] + V_a(y)S_+(x) + S_-(x)V_a(y)\right)\,, \tag{4.37}$$
$$\oint_{c_y^+}\frac{dx}{2\pi i}V_a(y)S(x) = \oint_{c_y^+}\frac{dx}{2\pi i}\left([V_a(y), S_-(y)] + V_a(y)S_+(x) + S_-(x)V_a(y)\right)\,. \tag{4.38}$$

It is possible to recollect the terms that are not inside commutators, their integrals vanish by application of the Residue theorem. The commutators can be brought together similarly. We are left with :

$$(\mathbf{s}_{-\frac{1}{2}}V_a)(y) = \oint_{c_y}\frac{dx}{2\pi i}\left(-\xi(x-y)\right)W_a(y) = -W_a(y)\,. \tag{4.39}$$

The sign in this identity might look paradoxical in view of (4.9), but the result will be confirmed when we will calculate the one point functions of fields like $\mathbf{s}_{-r}\mathbf{s}_{-\frac{1}{2}}V_a$. In addition, from the definition of $\mathbf{s}_r$ it is clear that the one point function on the cylinder of W_a is vanishing :

$$\langle \mathbf{s}_{-\frac{1}{2}}V_a\rangle = 0\,, \quad \langle W_a\rangle = 0\,. \tag{4.40}$$

The computations above can be recast in a more general case. Consider $\mathbf{s}_{-r}V_a$ with generic $r \in \mathbb{Z}+\frac{1}{2}$. We obtain from a similar computation :

$$(\mathbf{s}_{-r}V_a)(y) = \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{r-\frac{1}{2}}} \left(-\xi(x-y)\right) W_a(y) + \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{(x-y)^{r-\frac{1}{2}}} \left(V_a(y)S_+(x) + S_-(x)V_a(y)\right) . \tag{4.41}$$

A particular case that will be useful later is $r = \frac{3}{2}$, we get :

$$(\mathbf{s}_{-\frac{3}{2}}V_a)(y) = V_a(y)S_+(y) + S_-(y)V_a(y) . \tag{4.42}$$

Let us now proceed to a more interesting example :

$$\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a = -\mathbf{s}_{-\frac{3}{2}}W_a . \tag{4.43}$$

We will employ the following fermionic commutation relations :

$$S(x)W_a(y) = \{S_+(x), W_a(y)\} - W_a(y)S_+(x) + S_-(x)W_a(y) , \tag{4.44}$$
$$W_a(y)S(x) = \{W_a(y), S_-(x)\} + W_a(y)S_+(x) - S_-(x)W_a(y) . \tag{4.45}$$

Then, by definition one has :

$$\begin{aligned}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a = -\mathbf{s}_{-\frac{3}{2}}W_a &= -\oint_{c_y} \frac{dx}{2\pi i(x-y)} \mathcal{T}(S(x)W_a(y)) \\ &= -\left(-\int_{c_y^-} \frac{dx}{2\pi i(x-y)} S(x)W_a(y) + \int_{c_y^+} \frac{dx}{2\pi i(x-y)} W_a(y)S(x) \right) .\end{aligned} \tag{4.46}$$

Now we apply the formulae (4.44) and (4.45), as well as (4.34),(4.35) and obtain :

$$\begin{aligned}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a &= -\Big[-\int_{c_y^-} \frac{dx}{2\pi i} \frac{1}{x-y} \Big(\{S_+(x), W_a(y)\} + S_-(x)W_a(y) - W_a(y)S_+(x) \Big) \\ &\quad + \int_{c_y^+} \frac{dx}{2\pi i} \frac{1}{x-y} \Big(\{W_a(y), S_-(x)\} + W_a(y)S_+(x) - S_-(x)W_a(y) \Big) \Big] \\ &= -W_a(y)S_+(y) + S_-(y)W_a(y) \\ &\quad + \oint_{c_y} \frac{dx}{2\pi i} \frac{1}{x-y} \Big(V_a'(y)\xi(x-y) - 2\Delta V_a(y)\xi'(x-y) \Big) .\end{aligned} \tag{4.47}$$

Taking into account the simple residue calculation :

$$\oint_{c_0} \frac{dz}{2\pi i} \frac{2\Delta_a}{z} \left(-\xi'(z)\right) = \frac{\Delta_a}{12} , \tag{4.48}$$

we arrive at the result :

$$\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a = -W_a(y)S_+(y) + S_-(y)W_a(y) + \frac{\Delta_a}{12}V_a , \tag{4.49}$$

which implies :

$$\langle \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a \rangle = \frac{\Delta_a}{12} , \tag{4.50}$$

and is the wished formula at level 2. Moreover, it is straightforward to generalize the above calculation to higher levels :

$$\langle \mathbf{s}_{-r}\mathbf{s}_{-\frac{1}{2}}V_a \rangle = \oint_{c_0} \frac{dz}{2\pi i} \frac{2\Delta_a}{z^{r-\frac{1}{2}}} \left(-\xi'(z)\right) . \tag{4.51}$$

This gives in particular :

$$\langle \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}V_a \rangle = -\frac{7\Delta_a}{960} , \quad \langle \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}}V_a \rangle = \frac{31\Delta_a}{96768} . \tag{4.52}$$

The advantage of this approach is that it allows in principle to compute explicitly the *fields* $\mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a$ and not only their one point functions. More examples of this type of calculations are presented in the appendix 4.8. However, we are mostly interested in the one point functions, and as we already mentioned in the first chapter, this approach is not really efficient. We prefer to use instead the Ward-Takahashi identities.

Super Ward-Takahashi identities on the cylinder

In this section we present the Super Ward-Takahashi identities. First we need to rewrite the superconformal OPE (4.1),(4.3),(4.10),(4.11) on the cylinder. Recall the definitions of the functions χ (with $R=1$) and ξ (1.60), (4.29).

With them in hand, we conclude that the superconformal algebra is generated by the operators $T(z)$, $S(z)$ with the OPE's

$$\begin{aligned} T(z)T(w) &= -\frac{c}{12}\chi'''(z-w) - 2\chi'(z-w)T(w) + \chi(z-w)T'(w) + O(1)\,, \\ T(z)S(w) &= -\frac{3}{2}\chi'(z-w)S(w) + \chi(z-w)S'(w) + O(1)\,, \\ S(z)S(w) &= -2\xi(z-w)T(w) - \frac{c}{3}\xi''(z-w) + O(1)\,. \end{aligned} \tag{4.53}$$

We will also need the OPE's:

$$\begin{aligned} T(z)V_a(w) &= -\Delta_a\chi'(z-w)V_a(w) + \chi(z-w)V_a'(w) + O(1)\,, \\ T(z)W_a(w) &= -(\Delta_a+\tfrac{1}{2})\chi'(z-w)W_a(w) + \chi(z-w)W_a'(w) + O(1)\,, \\ S(z)V_a(w) &= -\xi(z-w)W_a(w) + O(1)\,, \\ S(z)W_a(w) &= 2\Delta_a\xi'(z-w)V_a(w) - \xi(z-w)V_a'(w) + O(1)\,, \end{aligned} \tag{4.54}$$

Recall that we are interested in the calculation of (4.28) :

$$\langle \mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a(y)\rangle = \frac{\langle\Delta_-|\mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a(y)|\Delta_+\rangle}{\langle\Delta_-|V_a(y)|\Delta_+\rangle}\,, \tag{4.55}$$

The main idea is to follow the route of [4], where Ward-Takahashi identities have been used to obtain the values of the same kind of correlation functions but containing purely Virasoro generators. Using Ward-Takahashi identities to express the correlation functions $\langle T(z_1)...T(z_p)S(w_1)...S(w_q)V_a(y)\rangle$ we can then obtain (4.28) by a successive application of (4.25),(4.26) :

$$\begin{aligned} \langle \mathbf{l}_{-n_1}...\mathbf{l}_{-n_p}\mathbf{s}_{-r_1}...\mathbf{s}_{-r_q}V_a\rangle = &\oint_{c_{z_1}} \boldsymbol{dz_1}... \oint_{c_{z_p}} \boldsymbol{dz_p} \oint_{c_{w_1}} \boldsymbol{dw_1}... \oint_{c_{w_q}} \boldsymbol{dw_q} \\ &\times \langle T(z_1)...T(z_p)S(w_1)...S(w_q)V_a(y)\rangle\,, \end{aligned} \tag{4.56}$$

with the notation :

$$\oint_{c_{z_k}} \boldsymbol{dz_k} = \oint_{c_{z_k}} \frac{dz_k}{2\pi i(z_k-y)^{n_k-1}}\,, \qquad \oint_{c_{w_j}} \boldsymbol{dw_j} = \oint_{c_{w_j}} \frac{dw_j}{2\pi i(w_j-y)^{r_j-\frac{1}{2}}}\,, \tag{4.57}$$

and the contours being small concentric circles around the point y : $c_{z_1} \subset ... \subset c_{w_q}$. Using the OPEs (4.53), (4.54), we can deduce the following simple correlation functions between the fields (we present here only the specific identities that we shall need later) :

$$\begin{aligned} \langle S(x)V_a(y)\rangle =& 0\,, \\ \langle S(x)W_a(y)\rangle =& \left(2\Delta_a\xi'(x-y) - \xi(x-y)(\Delta_+-\Delta_-)\right)\langle V_a(y)\rangle\,, \\ \langle S(x_2)S(x_1)V_a(y)\rangle =& -\xi(x_1-y)\langle S(x_2)W_a(y)\rangle - 2\xi(x_1-x_2)\langle T(x_2)V_a(y)\rangle + \\ & -\frac{c}{3}\xi''(x_1-x_2)\langle V_a(y)\rangle\,. \end{aligned}$$

And the more complicated :

$$
\begin{aligned}
&\langle T(x_3)S(x_2)S(x_1)V_a(y)\rangle = \\
&\left(-\frac{3}{2}\chi'(x_3-x_2)+(\chi(x_3-x_2)-\chi(x_3-y))\frac{\partial}{\partial x_2}\right)\langle S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-\frac{3}{2}\chi'(x_3-x_1)+(\chi(x_3-x_1)-\chi(x_3-y))\frac{\partial}{\partial x_1}\right)\langle S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-\Delta_a\chi'(x_3-y)+\chi(x_3-y)(\Delta_+-\Delta_-)+\left(\frac{\Delta_++\Delta_-}{2}-\frac{c}{24}\right)\right)\langle S(x_2)S(x_1)V_a(y)\rangle\ , \\
&\langle T(x_4)T(x_3)S(x_2)S(x_1)V_a(y)\rangle = \\
&-\frac{c}{12}\chi'''(x_4-x_3)\langle S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-2\chi'(x_4-x_3)+(\chi(x_4-x_3)-\chi(x_4-y))\frac{\partial}{\partial x_3}\right)\langle T(x_3)S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-\frac{3}{2}\chi'(x_4-x_2)+(\chi(x_4-x_2)-\chi(x_4-y))\frac{\partial}{\partial x_2}\right)\langle T(x_3)S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-\frac{3}{2}\chi'(x_4-x_1)+(\chi(x_4-x_1)-\chi(x_4-y))\frac{\partial}{\partial x_1}\right)\langle T(x_3)S(x_2)S(x_1)V_a(y)\rangle \\
&+\left(-\Delta_a\chi'(x_4-y)+\chi(x_4-y)(\Delta_+-\Delta_-)+\left(\frac{\Delta_++\Delta_-}{2}-\frac{c}{24}\right)\right)\langle T(x_3)S(x_2)S(x_1)V_a(y)\rangle\ .
\end{aligned}
$$

These formulae echo the general formula (1.78) obtained for the non-supersymmetric case. As will be explained in the next section 4.2, for applications to the Super sine-Gordon model we should consider the case of equal boundary conditions

$$\Delta_+ = \Delta_- = \delta_P\,, \tag{4.58}$$

where the precise definition of δ_P will be given soon.

The calculation of one point functions of descendants on the cylinder are then given by the application of (4.56). As examples we recover the previously obtained results at level 2

$$\langle \mathbf{l}_{-2}V_a\rangle = \delta_P - \frac{c}{24} - \frac{\Delta_a}{12}\,, \qquad \langle \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a\rangle = \frac{\Delta_a}{12}\,, \tag{4.59}$$

and new ones at level 4 :

$$
\begin{aligned}
&\langle \mathbf{l}^2_{-2}V_a\rangle = \left(\delta_P-\frac{c}{24}\right)^2 - \frac{1}{6}\left(\delta_P-\frac{c}{24}\right) - \frac{\Delta_a}{6}\left(\delta_P-\frac{c}{24}\right) + \frac{\Delta_a^2}{144} + \frac{7}{360}\Delta_a\,, \\
&\langle \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}V_a\rangle = -\Delta_a\frac{7}{960}\,, \qquad \langle \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}V_a\rangle = -\frac{1}{12}\left(\delta_P-\frac{c}{24}\right) + \Delta_a\frac{17}{960} + \frac{7c}{2880}\,, \\
&\langle \mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\rangle = \frac{\Delta_a}{12}\left(\delta_P-\frac{c}{24}-\frac{\delta_a}{12}\right) - \frac{1}{144}\Delta_a\,, \qquad \langle \mathbf{l}_{-4}V_a\rangle = \frac{\Delta_a}{240}\,.
\end{aligned}
\tag{4.60}
$$

We also will need the one point functions at level 6. Since the results are quite long, we prefer to display them in due time.

4.1.3 Super Liouville CFT

In this section we will describe the Super CFT that will be the most important for us : the $N=1$ *Super Liouville CFT* (Super LCFT) (see [16] for more references), the conventions are taken from [82] [4]. The Super Liouville CFT is given by the supersymmetric generalization of the Liouville Lagrangian (1.84) with coupling constant b and cosmological constant μ [5] :

$$\mathcal{A}_{\rm SL} = \int\Big(\frac{1}{4\pi}\partial_z\varphi\partial_{\bar z}\varphi + \frac{1}{2\pi}\left(\psi\partial_{\bar z}\psi + \bar\psi\partial_z\bar\psi\right) - \mu\bar\psi\psi e^{\frac{b}{\sqrt{2}}\varphi}\Big)d^2z\,. \tag{4.61}$$

[4] With the correspondence that $\widetilde{\varphi} = \frac{\varphi}{\sqrt{2}}$ and $\widetilde{\mu} = \frac{\mu b^{-2}}{2}$, where tilded quantities are those used in [82].

[5] Actually this is not the "complete" Super Liouville Lagrangian, that can be found for example in [82]. One of the terms assuring SUSY invariance has been dropped. Indeed this term is not relevant for the applications to the ssG model, this will be explained later.

The central charge of this theory is :

$$c = \frac{3}{2}\left(1 + 2Q^2\right) , \quad Q = b + b^{-1} . \tag{4.62}$$

The primary fields take again the form of vertex operators $V_a = e^{\frac{a}{\sqrt{2}}\varphi}$, with conformal dimensions :

$$\Delta_a = \frac{1}{2}a(Q-a) . \tag{4.63}$$

In addition, it is also possible to construct primary fields as normal ordered products of vertex operators and fermionic fields : $W_a = \bar{\psi}\psi e^{\frac{a}{\sqrt{2}}\varphi}$ whose conformal dimension is naturally :

$$\Delta(W_a) = \frac{1}{2}a(Q-a) + \frac{1}{2} . \tag{4.64}$$

The fields V_a and W_a will be the main building blocks for us.

Notice that for primary fields, the three-point function on a sphere on the one hand, and the one point function on the cylinder (with our usual asymptotical conditions) on the other hand coincide in the CFT. In the paper [82] the important three points functions involving both fields V_a and W_a with two other bosonic primary operators have been calculated. In a rather formal way, we define them as :

$$C(a_1, a_2, a_3) = \langle V_{a_1} V_{a_2} V_{a_3} \rangle , \tag{4.65}$$

$$\widetilde{C}(a_1, a_2, a_3) = \langle W_{a_1} V_{a_2} V_{a_3} \rangle . \tag{4.66}$$

Before presenting the results we should mention that they involve two generalizations $\Upsilon_{\rm NS}$ and $\Upsilon_{\rm R}$ of the function Υ (1.92) that are given by :

$$\Upsilon_{\rm NS}(x) = \Upsilon\left(\frac{x}{2}\right)\Upsilon\left(\frac{x+Q}{2}\right) , \quad \Upsilon_{\rm R}(x) = \Upsilon\left(\frac{x+b}{2}\right)\Upsilon\left(\frac{x+b^{-1}}{2}\right) . \tag{4.67}$$

From the symmetries of Υ it is possible to show that they also satisfy the fundamental property :

$$\Upsilon_{\rm NS}(x) = \Upsilon_{\rm NS}(Q-x) , \quad \Upsilon_{\rm R}(x) = \Upsilon_{\rm R}(Q-x) . \tag{4.68}$$

The results from [82] are then given by :

$$\begin{aligned} C(a_1, a_2, a_3) = &\left(\tfrac{1}{2}\pi\mu\gamma(\tfrac{1}{2}bQ)b^{-1-b^2}\right)^{\frac{Q-a}{b}} \Upsilon_{\rm NS}(2a_1) \\ &\times \frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2a_2)\Upsilon_{\rm NS}(2a_3)}{\Upsilon_{\rm NS}(a-Q)\Upsilon_{\rm NS}(a_{1+2-3})\Upsilon_{\rm NS}(a_{2+3-1})\Upsilon_{\rm NS}(a_{3+1-2})} , \end{aligned} \tag{4.69}$$

where $a = a_1 + a_2 + a_3$, $a_{1+2-3} = a_1 + a_2 - a_3$ *etc.*

The function $\Upsilon_{\rm R}(x)$ was introduced in order to be able to write down the three-point function $\widetilde{C}(a_1, a_2, a_3)$:

$$\begin{aligned} \widetilde{C}(a_1, a_2, a_3) = &\left(\tfrac{1}{2}\pi\mu\gamma(\tfrac{1}{2}bQ)b^{-1-b^2}\right)^{\frac{Q-a}{b}} \Upsilon_{\rm NS}(2a_1) \\ &\times \frac{2i\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2a_2)\Upsilon_{\rm NS}(2a_3)}{\Upsilon_{\rm R}(a-Q)\Upsilon_{\rm R}(a_{1+2-3})\Upsilon_{\rm R}(a_{2+3-1})\Upsilon_{\rm R}(a_{3+1-2})} . \end{aligned} \tag{4.70}$$

In the formulae (4.69), (4.70) we separated the multiplier in the first line from the rest because this is the only one which is not invariant under $a_1 \to Q - a_1$. This gives the possibility to compute the reflection coefficients, which happen to be the same for V_a and W_a :

$$V_a = R(a)V_{Q-a} , \quad W_a = R(a)W_{Q-a} , \tag{4.71}$$

and are given by :

$$R(a) = \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2}\gamma(2ab - b^2)\gamma(2ab^{-1} - b^{-2} - 1) .$$

Applications to the integrability of the ssG model.

It is also possible to introduce other Super CFT, for example the unitary Super Minimal Models SM_p [79], which are (like non supersymmetric ones) also parametrized by an integer number p. In the NS sector, the central charge and the conformal dimensions for such models are :

$$c = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right), \tag{4.72}$$

$$\Delta_{r,s} = \frac{(r(p+2) - sp)^2 - 4}{8p(p+2)} + \frac{1}{32}(1 - (-1)^{r-s}). \tag{4.73}$$

Among the finite number of primaries, the important field is again $V_{1,3}$ with lowest (without taking into account the identity field) conformal dimension :

$$\Delta_{1,3} = \frac{1}{2}\left(1 - \frac{4}{p+2}\right). \tag{4.74}$$

Again it is possible starting from the Super Liouville CFT, to recover Super unitary Minimal Models by taking specific values of the central charge. Select for b the particular value :

$$b^2 = -\frac{p}{p+2}, \tag{4.75}$$

then :

$$c = \frac{3}{2} + 3Q^2 = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right). \tag{4.76}$$

We have formally reduced the Liouville SCFT parameters to those of a Super Minimal Model. The Super Liouville field V_{-b} has the conformal dimension :

$$\Delta_{-b} = -\frac{1}{2}(1 + 2b^2) = -\frac{1}{2}\left(1 - 2\frac{p}{p+2}\right) = \Delta_{1,3}. \tag{4.77}$$

Jumping slightly forward, and assuming that the conformal conservation laws can be deformed in the same way for Super Minimal Models as for usual Minimal models, this discussion can be taken as an heuristic argument in favor of the integrability of the Super sine-Gordon model [6]. This theory will be the topic of the next section.

4.2 Supersymmetric sine-Gordon model

In this section we give a very brief description of the $N = 1$ Supersymmetric sine-Gordon field theory, more details can be found in [83]. In the framework of Perturbed CFT, the ssG model can be *at first* considered as a perturbation of of the $c = 3/2$ CFT (one boson+one Majorana fermion) by the relevant operator $V = -\mu\bar{\psi}\psi\cos\left(\frac{\beta\varphi}{\sqrt{2}}\right)$:

$$\mathcal{A}_{\text{ssG}} = \int\Big(\frac{1}{16\pi}\partial_z\varphi\partial_{\bar{z}}\varphi + \frac{1}{2\pi}\left(\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi}\right) - 2\mu\bar{\psi}\psi\cos\left(\frac{\beta\varphi}{\sqrt{2}}\right)\Big)d^2z. \tag{4.78}$$

The dimensional coupling constant μ is of dimension $[\text{mass}]^{1-\beta^2}$. The scaling dimension of this operator $\Delta_{\text{pert}} = \frac{1}{2}(1+\beta^2)$ is greater than $\frac{1}{2}$, so the UV regularization is needed. The OPE

$$V(z,\bar{z})V(0) = \frac{1}{(z\bar{z})^{1+\beta^2}} + C\cdot\frac{1}{(z\bar{z})^{1-\beta^2}}\cos\left(\sqrt{2}\beta\varphi\right) + \cdots, \tag{4.79}$$

shows that the UV regularization is simple: the first non-trivial contribution comes with integrable singularity. The model is shown to be integrable, and as mentioned in the introduction actually is the simplest example of perturbations of parafermionic models whose integrals of motion are obtained in [72]. The factorizable S-matrix is known, it coincides with the S-matrix for the spin-1 integrable magnetic [84], in the context of relativistic field theory it was discussed in [85]. The S-matrix is compatible with the $N = 1$ Supersymmetry.

The formula for the action (4.78) may contradict the reader's intuition because the supersymmetric classical action contains the additional term $V_1 = -\frac{\pi\mu^2}{\beta^2}\cos\left(\sqrt{2}\beta\varphi\right)$ which we have seen

[6] More precisely, this is an argument in favor of the integrability of the Super *sinh* Gordon model.

already in the OPE (4.79). In the frame work of the PCFT this term, as it is written, cannot be added to the action for dimensional reasons, at least it needs a new dimensional coupling constant. In the classical limit $\beta \to 0$ the situation becomes more complicated. That is why, when proceeding in the opposite direction, i.e. quantizing the classical model by more traditional methods of QFT, one should indeed begin with the supersymmetric action which includes V_1 and take care of preserving the Supersymmetry. This was done in [86], the result is exactly as expected from our dimensional considerations: the dimensional coupling constants for the two terms of the interaction are renormalized differently, the term with V_1 containing vanishing power of the cutoff.

The subject of the present thesis are the one point functions, this corresponds to the geometry of the cylinder (that we take of radius R) with a local insertion. Correspondingly we consider two types of contours: the contour c encircling the local insertion and a contour C which goes around the cylinder. We will write $C_\pm$ to denote those which are to the right and to the left of the insertion respectively, and use the notation C talking about any of $C_\pm$.

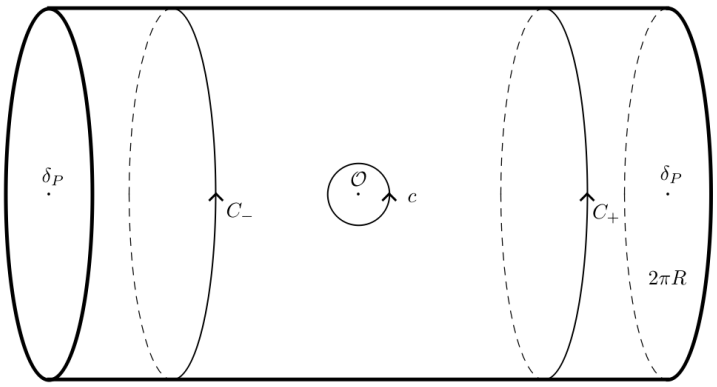

Figure 4.2: Super sine-Gordon model on a cylinder with the insertion of a local operator $\mathcal{O}$ and boundary conditions δ_P.

The cylinder is infinite, in the terminology of the chapter 2 its generatrix is called the Space direction, its directrix is called the Matsubara direction. In the present context by the Matsubara transfer-matrix we understand an operator acting from the Matsubara Hilbert space to itself which is graphically represented as a slice of our cylinder of small Space length ϵ. Since the cylinder is infinite, both transfer-matrices to the left and to the right of the insertion are replaced by the one-dimensional projectors on the same eigenvector with maximal eigenvalue. Since the potential is invariant under $\varphi \to \varphi + 2\sqrt{2}/\beta$ we can introduce additional parameter P which is the Floquet index of the Matsubara wave-function. The one point function (partition function with insertion) is denoted by

$$\langle \mathcal{O}(0) \rangle_{P,R} \,. \tag{4.80}$$

From a different point of view, the ssG model can be formally considered as the perturbation of the conformal *complex* Super Liouville model

$$\mathcal{A}_{\mathrm{SL}'} = \int \Big[\Big(\frac{1}{4\pi} \partial_z \varphi \partial_{\bar z} \varphi + \frac{1}{2\pi} \left(\psi \partial_{\bar z} \psi + \bar\psi \partial_z \bar\psi \right) - \mu \bar\psi \psi e^{-i \frac{\beta}{\sqrt{2}} \varphi} \Big) \Big] d^2 z \,, \tag{4.81}$$

by the relevant operator

$$W = \mu \bar\psi \psi e^{i \frac{\beta}{\sqrt{2}} \varphi} \,,$$

whose scaling dimension is $\Delta = \beta^2$.

Let us for a moment concentrate on this conformal theory. The central charge of the complex Super Liouville model is

$$c = \tfrac{3}{2}\, \hat c \,, \quad \hat c = 1 - 2\big(\beta^{-1} - \beta\big)^2 \,. \tag{4.82}$$

The comparison with the real Super Liouville CFT will be disclosed soon, see (4.157). We will consider only the NS sector. According to the usual argument the operator $\mathcal{O}$ with scaling dimensions

$(\Delta_{\mathcal{O}}, \overline{\Delta}_{\mathcal{O}})$ in generic position has a uniquely defined counterpart in the perturbed theory. We do not distinguish the two notationally, the UV limit is

$$\lim_{r\to 0} r^{\Delta_{\mathcal{O}}+\overline{\Delta}_{\mathcal{O}}} \langle \mathcal{O}(0)\rangle_{P,R} = \langle \mathcal{O}(0)\rangle_P \,, \tag{4.83}$$

where $r \sim R$ is a dimensionless quantity proportional to R, see (4.116) for details. As usual, in the conformal case we can easily change the scale to have $R = 1$. By a change of variables from the cylinder to the sphere the CFT one point function $\langle \mathcal{O}(0)\rangle_P$ is mapped to the three point function for the image of the operator $\mathcal{O}$ (for descendants this image can have rather complicated expression in terms of $\mathcal{O}$) and two primary fields with dimensions (compare with (1.88))

$$\delta_P = P^2 + \frac{\hat{c}-1}{16} \,.$$

The superconformal algebra has been discussed in the previous section. Among its elements we will single out two kinds of primary fields parametrized by $\alpha \in \mathbb{C}$

$$V_\alpha = e^{i\alpha(\beta^{-1}-\beta)\frac{1}{2\sqrt{2}}\varphi} \,, \quad W_\alpha = \bar{\psi}\psi e^{i\alpha(\beta^{-1}-\beta)\frac{1}{2\sqrt{2}}\varphi} \,. \tag{4.84}$$

The scaling dimension of V_α is

$$\Delta_\alpha = \frac{1}{8}(\beta^{-1}-\beta)^2\alpha(\alpha-2) \,.$$

The scaling dimension of W_α equals $\Delta_\alpha + 1/2$, their OPE with T and S are given in (4.54).

From the discussion in the last section, it is clear that the convenient way of finding the CFT one point functions consists in using the OPE and the asymptotical conditions

$$\lim_{\mathrm{Re}(z)\to\pm\infty} T(z) = \delta_P - \frac{\hat{c}}{16} \,, \quad \lim_{\mathrm{Re}(z)\to\pm\infty} S(z) = 0 \,. \tag{4.85}$$

In Section 4.1.2 we already saw how to apply the above conditions to the computations of one point functions of descendants.

The Super Liouville model, in addition to the super conformal symmetry, possesses the structure of an integrable model, namely, it allows an infinite number of local integrals of motion with chiral local densities $h_{2j}(z), \bar{h}_{2j}(\bar{z})$. In our geometry there are two facets of the local integrals of motion: they act either on the Matsubara Hilbert space or on the local operators inserted at the point $z = 0$, and are respectively

$$I_{2j-1} = \oint_C h_{2j}(z)\frac{dz}{2\pi i} \,, \quad (\mathbf{i}_{2j-1}\mathcal{O})(0) = \oint_c h_{2j}(z)\mathcal{O}(0)\frac{dz}{2\pi i} \,, \tag{4.86}$$

and similarly for the other chirality. Let us write explicitly the first two densities (more on this can be found in the Appendix 4.9) :

$$h_2(z) = T(z) \,, \quad h_4(z) = T(z)^2 - \frac{1}{4}S(z)S'(z) \,. \tag{4.87}$$

The formula for $h_2(z)$ means simply that the light cone component of the energy-momentum tensor is the first integral, the formula for $h_4(z)$ is the most important: it is well-known that higher local integrals of motion are completely defined by the requirement of commutativity with the density $h_4(z)$.

Let us return to the perturbed model. It has been said that at least for irrational α the local operator V_α and its Super Virasoro descendants (W_α in particular) have uniquely defined counterparts in the perturbed theory, which we do not distinguish notationally. The local integrals of motion survive the perturbation, and give rise to an infinite series of pairs of operators $(h_{2j}(z,\bar{z}), \Theta_{2j-2}(z,\bar{z}))$, $(\bar{h}_{2j}(z,\bar{z}), \bar{\Theta}_{2j-2}(z,\bar{z}))$ satisfying the continuity equations

$$\partial_{\bar{z}} h_{2j}(z,\bar{z}) = \partial_z \Theta_{2j-2}(z,\bar{z}) \,, \quad \partial_z \bar{h}_{2j}(z,\bar{z}) = \partial_{\bar{z}} \bar{\Theta}_{2j-2}(z,\bar{z}) \,. \tag{4.88}$$

the other pair being treated quite similarly. The action on the local operators is

$$I_{2j-1} = \oint_C h_{2j}(z)\frac{dz}{2\pi i} + \Theta_{2j-2}(z)\frac{d\bar{z}}{2\pi i} \,, \quad (\mathbf{i}_{2j-1}\mathcal{O})(0) = \oint_c h_{2j}(z)\mathcal{O}(0)\frac{dz}{2\pi i} + \Theta_{2j-2}(z)\mathcal{O}(0)\frac{d\bar{z}}{2\pi i} \,,$$

The operators I_1, $\bar{I}_1$ are the light-cone components of the energy-momentum tensor.

Together with $c = C_+ - C_-$ this implies that the one point function of the local operators obtained by the action of $\mathbf{i}_{2j-1}, \bar{\mathbf{i}}_{2j-1}$ vanish. So, like in [4] we work with the quotient space $\mathcal{V}_\alpha^{\rm quo} \otimes \overline{\mathcal{V}}_\alpha^{\rm quo}$ obtained from the tensor product of two super conformal Verma modules by factoring out the descendants of the integrals of motion. The quotient space $\mathcal{V}_\alpha^{\rm quo}$ will be realized as the one obtained by the action on V_α of modes $\mathbf{s}_r$ of the super current and of the Virasoro generators $\mathbf{l}_m$.

The particle content of the ssG model consists of solitons and, for $\beta^2 < 1/2$, their bound states. There is an exact formula relating the mass of the soliton M to the dimensional coupling constant μ:

$$M = \frac{4(1-\beta^2)}{\pi\beta^2}\left(\frac{\pi}{2}\mu\gamma\Big(\frac{1-\beta^2}{2}\Big)\right)^{\frac{1}{1-\beta^2}}, \tag{4.89}$$

here and later we use again the conventional notation $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$.

After these preparations we are now ready to give the description of the integrable structure of the ssG model by the fermion-current basis. As usual we start with lattice considerations.

4.3 Expectation values in the 19 vertex model

In this section we present the fermion-current basis in the case of the anisotropic lattice model.

4.3.1 General structure

In the lattice case we historically use for the coupling constant

$$\nu = \frac{1-\beta^2}{2}\,.$$

The paper [6] considers an (inhomogeneous) 19 vertex Fateev-Zamolodchikov model on a cylinder (or equivalently in an arbitrary generalized Gibbs ensemble) for the (inhomogeneous) spin-1 integrable spin chain. In what follows we closely follow the notations of [3, 6] with one exception: we switch from the multiplicative spectral parameter to the additive one. Let us present some basic formulae. As usual we combine the 19 vertices of the model into the L-operator

$$\mathcal{L}(\theta) = \left(\begin{array}{ccc|ccc|ccc}
a(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b(\theta) & 0 & c(\theta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f(\theta) & 0 & d(\theta) & 0 & h(\theta) & 0 & 0 \\ \hline
0 & c(\theta) & 0 & b(\theta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d(\theta) & 0 & e(\theta) & 0 & d(\theta) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(\theta) & 0 & c(\theta) & 0 \\ \hline
0 & 0 & h(\theta) & 0 & d(\theta) & 0 & f(\theta) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c(\theta) & 0 & b(\theta) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\theta)
\end{array}\right),$$

where

$$\begin{aligned}
&a(\theta) = \sinh\nu(\theta+\tfrac{\pi i}{2})\sinh\nu\theta\,, \quad b(\theta) = \sinh\nu\pi(\theta-\tfrac{\pi i}{2})\sinh\nu\theta\,, \quad c(\theta) = \sinh\nu\pi i\sinh\nu\theta\,,\\
&d(\theta) = \sinh\nu(\theta-\tfrac{\pi i}{2})\sinh\nu\pi i\,, \quad f(\theta) = \sinh\nu(\theta-\tfrac{\pi i}{2})\sinh\nu(\theta-\pi i)\,,\\
&e(\theta) = \cosh\nu(\theta+\tfrac{\pi i}{2})\cosh\nu(\theta-\pi i) - \cosh\tfrac{\nu\pi i}{2}\,, \quad h(\theta) = \sinh\tfrac{\nu\pi i}{2}\sinh\nu\pi i\,.
\end{aligned}$$

We consider an ought to be infinite Space chain of length N and a Matsubara chain of length L. Introduce the rectangular monodromy matrix

$$T_{\mathbf{S},\mathbf{M}} = \prod_{j=-N/2+1}^{\overset{\curvearrowright}{N/2}} T_{j,\mathbf{M}}\,, \qquad T_{j,\mathbf{M}} = \prod_{m=1}^{\overset{\curvearrowleft}{L}} \mathcal{L}_{j,m}\,,$$

where both Space and Matsubara chains can be inhomogeneous,

$$\mathcal{L}_{j,m} = \mathcal{L}_{j,m}(\xi_j - \tau_m)\,, \tag{4.90}$$

ξ_j, (τ_m) are Space (Matsubara) inhomogeneities. The indices j, m in the right hand side have double meaning: they count inhomogeneities and the copies in the tensor product. These notations are standard. Eventually we take the limit $N \to \infty$, the Space inhomogeneities are supposed to follow some regular pattern in the limit.

Introduce the operators

$$H(j) = \sum_{k=-N/2+1}^{j} H_j\,, \quad H = H(N/2)\,,$$

with H_j being the Cartan generator of $U_q(\mathfrak{sl}_2)$ acting on j-th Space site. Consider the "primary field" $q^{\alpha H(0)}$, and an operator $\mathcal{O}$ acting non trivially on a finite number of Space sites. The operators $q^{\alpha H(0)}\mathcal{O}$ are called quasi-local. The main object of our study is (2.177)

$$\mathcal{Z}_L^\kappa\left\{q^{\alpha H(0)}\mathcal{O}\right\} = \lim_{N\to\infty} \frac{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathbf{S},\mathbf{M}} q^{\kappa H+\alpha H(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathbf{S},\mathbf{M}} q^{\kappa H+\alpha H(0)}\right)}\,, \tag{4.91}$$

with κ being a parameter. Graphically this is represented on the figure 4.3.

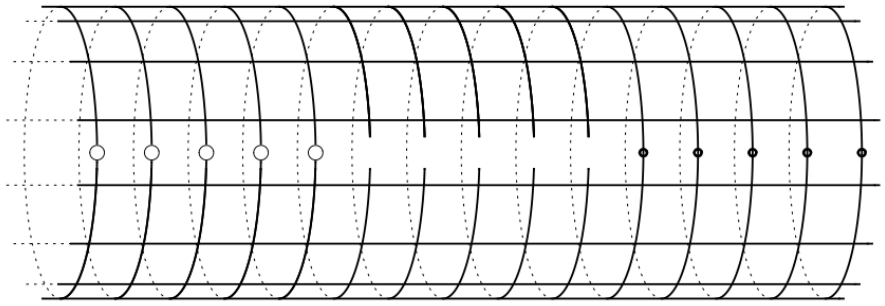

Figure 4.3: 19 vertex model on a cylinder with quasi-local insertion. The broken lines represent the spaces where the action of $\mathcal{O}$ is non trivial, full circles represent the operator $q^{\kappa H_j}$, empty circles represent the operator $q^{(\kappa+\alpha)H_j}$.

The main result of [6] is that an effective way of computation goes through the introduction of eight families of creation operators acting on the space of quasi-local operators. These families are fermions $\mathbf{b}^*(\theta), \mathbf{c}^*(\theta), \tilde{\mathbf{b}}^*(\theta), \tilde{\mathbf{c}}^*(\theta)$[7], level 1 Kac-Moody currents $\mathbf{j}^+(\theta), \mathbf{j}^-(\theta), \mathbf{j}^0(\theta)$, and an operator lying in the center of the entire algebra $\mathbf{t}^*(\theta)$. To be more precise the generating functions of the quasi-local operators are produced by normally ordered products of fermions and Kac-Moody currents (the central operator $\mathbf{t}^*(\theta)$ does not need normal ordering). This is explained in [6, 8] and recalled in chapter 3 (3.27). Since the most significant results of the present thesis concern the quasi-local operators created by fermions only, in which case the normal ordering is not needed, we shall not go into the details.

In the case of homogeneous Space ($\xi_j = 0, \forall j$) the creation operators are understood as power series in θ. We shall be interested in the case when the Space inhomogeneities are staggering: ξ at even sites and $-\xi$ at odd one. In that case every of above operators give rise to two "chiral" families defined as power series in $\theta - \xi$, $\theta + \xi$. All that is absolutely parallel to [5] so we do not go into much details.

As we have seen in the two previous chapters, the main advantage of our creation operators is that on the descendants which they create acting on the "primary field", the functional $\mathcal{Z}_L^\kappa$ takes a simple form. We shall describe a formal prescription for the computation, detailed explanations being given in [6]. Introduce the creation operators $b^*(\theta)$, $c^*(\theta)$, $t^*(\theta)$, $n(\theta)$ (the first two are fermions, the last two are bosons) which (anti)-commute among themselves. Prescribe the following values of the functional $\mathcal{Z}_L^\kappa$:

$$\begin{aligned}&\mathcal{Z}_L^\kappa\{b^*(\theta_1^+)\cdots b^*(\theta_k^+)c^*(\theta_k^-)\cdots b^*(\theta_1^-)t^*(\theta_1^0)\cdots t^*(\theta_m^0)n(\sigma_1)\cdots n(\sigma_n)q^{\alpha H(0)}\}\\ &= \prod_{j=1}^{n}\frac{1}{\mathcal{N}(\sigma_j)}\prod_{j=1}^{m}\rho(\theta_j^0)\det\left(\omega(\theta_i^+,\theta_j^-)\right)_{i,j=1,\cdots k}\,,\end{aligned}$$

[7] These operators were denoted by $\bar{\mathbf{b}}^*(\theta), \bar{\mathbf{c}}^*(\theta)$ in [6], but we prefer to keep the "bars" for a different, more important, use.

where the functions $\mathcal{N}(\theta)$, $\rho(\theta)$, $\omega(\theta,\theta')$ depending on the Matsubara data will be defined soon. The expectation values of the operators created by $\mathbf{j}^+,\mathbf{j}^0,\mathbf{j}^-$, $\mathbf{b}^*,\mathbf{c}^*$, $\tilde{\mathbf{b}}^*,\tilde{\mathbf{c}}^*$ are computed using the identification

$$\begin{aligned}
\mathbf{j}^+(\theta) &= n(\theta)\ b^*(\theta+\tfrac{\pi i}{2})b^*(\theta-\tfrac{\pi i}{2})\,, \\
\mathbf{j}^-(\theta) &= n(\theta)\ c^*(\theta-\tfrac{\pi i}{2})c^*(\theta+\tfrac{\pi i}{2})\,, \\
\mathbf{j}^0(\theta) &= n(\theta)\ \big(b^*(\theta+\tfrac{\pi i}{2})c^*(\theta-\tfrac{\pi i}{2})+c^*(\theta+\tfrac{\pi i}{2})b^*(\theta-\tfrac{\pi i}{2})\big)\,, \\
\mathbf{b}^*(\theta) &= n(\theta)\ \big(b^*(\theta+\tfrac{\pi i}{2})t^*(\theta-\tfrac{\pi i}{2})+b^*(\theta-\tfrac{\pi i}{2})\big)\,, \\
\mathbf{c}^*(\theta) &= n(\theta)\ \big(c^*(\theta+\tfrac{\pi i}{2})t^*(\theta-\tfrac{\pi i}{2})+c^*(\theta-\tfrac{\pi i}{2})\big)\,, \\
\tilde{\mathbf{b}}^*(\theta) &= n(\theta)\ \big(b^*(\theta+\tfrac{\pi i}{2})+t^*(\theta+\tfrac{\pi i}{2})b^*(\theta-\tfrac{\pi i}{2})\big)\,, \\
\tilde{\mathbf{c}}^*(\theta) &= n(\theta)\ \big(c^*(\theta+\tfrac{\pi i}{2})+t^*(\theta+\tfrac{\pi i}{2})c^*(\theta-\tfrac{\pi i}{2})\big)\,.
\end{aligned} \tag{4.92}$$

We had one more operator: $\mathbf{t}^*(\theta)$, it is similar to $t^*(\theta)$ with $\rho(\theta)$ being replaced by $\mathrm{P}(\theta)$, this function will be given soon. The operator $\mathbf{t}^*(\theta)$ is in the center, so, we manipulate it as a $\mathbb{C}$-number.

4.3.2 Basic functions

The functions $\omega(\theta,\theta')$, $\rho(\theta)$, $\mathrm{P}(\theta)$ are defined by the Matsubara data. The latter consists of the length L chain, with inhomogeneities τ_j, right and left twists κ, $\kappa+\alpha$, and the eigenvectors with maximal eigenvalues of the right and left transfer-matrices:

$$T_{\mathbf{M}}(\theta,\kappa)=\mathrm{Tr}_j\left(T_{j,\mathbf{M}}(\theta)q^{\kappa H_j}\right)\,,\quad T_{\mathbf{M}}(\theta,\kappa+\alpha)=\mathrm{Tr}_j\left(T_{j,\mathbf{M}}(\theta)q^{(\kappa+\alpha)H_j}\right)\,. \tag{4.93}$$

The corresponding ground state eigenvalues will be denoted respectively by $T_2(\theta,\kappa)$, $T_2(\theta,\kappa+\alpha)$. Then we are ready to define the first of our functions:

$$\mathrm{P}(\theta)=\frac{T_2(\theta,\kappa+\alpha)}{T_2(\theta,\kappa)}\,. \tag{4.94}$$

We shall need the eigenvalues of the two Baxter Q-operators [35]

$$Q^{\pm}(\theta,\kappa)=e^{\pm\nu\kappa\theta}\prod_{j=1}^{m}\sinh\nu(\theta-\sigma_j(\kappa))\,, \tag{4.95}$$

and similarly for $\kappa+\alpha$. The Bethe roots are denoted by $\sigma_j(\kappa)$. If κ is not too large the maximal eigenvalue corresponds to $m=L/2$. We shall also use the eigenvalues of the transfer-matrix with the two-dimensional auxiliary space $T_1(\theta,\kappa)$, for which

$$T_1(\theta,\kappa)Q^{\pm}(\theta,\kappa)=a(\theta)Q^{\pm}(\theta+\pi i,\kappa)+d(\theta)Q^{\pm}(\theta-\pi i,\kappa)\,, \tag{4.96}$$

$$T_2(\theta,\kappa)=T_1(\theta-\pi i/2,\kappa)T_1(\theta+\pi i/2,\kappa)-f(\theta)\,, \tag{4.97}$$

$$f(\theta)=a(\theta-\pi i/2)d(\theta+\pi i/2)\,, \tag{4.98}$$

where

$$a(\theta)=s(\theta-\pi i),\quad d(\theta)=s(\theta+\pi i),\quad s(\theta)=\prod_{j=1}^{L}\sinh\nu(\theta-\tau_j)\,.$$

We have the relation between T_2 and $Q^{\pm}$:

$$\begin{aligned}
T_2(\theta,\kappa) &= a(\theta+\pi i/2)a(\theta-\pi i/2)\frac{Q^{\pm}(\theta+3\pi i/2,\kappa)}{Q^{\pm}(\theta-\pi i/2,\kappa)} \\
&+ a(\theta+\pi i/2)d(\theta-\pi i/2)\frac{Q^{\pm}(\theta-3\pi i/2,\kappa)Q^{\pm}(\theta+3\pi i/2,\kappa)}{Q^{\pm}(\theta-\pi i/2,\kappa)Q^{\pm}(\theta+\pi i/2,\kappa)} \\
&+ d(\theta+\pi i/2)d(\theta-\pi i/2)\frac{Q^{\pm}(\theta-3\pi i/2,\kappa)}{Q^{\pm}(\theta+\pi i/2,\kappa)}\,.
\end{aligned}$$

Denote y the quantity that has to be interpreted as the free energy of the system :

$$y(\theta)=\frac{T_2(\theta,\kappa)}{f(\theta)}\,. \tag{4.99}$$

We do not explicitly indicate the dependence of $y(\theta)$ on κ because it will be never used for another value of twist.

Now we are ready to define two more functions

$$\rho(\theta) = \frac{T_1(\theta, \kappa + \alpha)}{T_1(\theta, \kappa)}, \quad \mathcal{N}(\theta) = \frac{y(\theta)}{1 + y(\theta)}. \tag{4.100}$$

4.4 Suzuki equations for the free energy

In this section we will prepare the ground for the calculation of scaling equations for the free energy y. This is the first step in order to derive similar equations for the one point functions of fermionic elements of the fermion-current basis, that is for the function Ω. In this section, we will denote the Baxter operator simply by [8]

$$Q(\theta) = e^{\nu\kappa\theta} \prod_{j=1}^{m} \sinh \nu(\theta - \sigma_j(\kappa)), \tag{4.101}$$

and T_1 and T_2 depend only on the twist κ. We shall be interested in the case of real τ_j and κ which implies

$$\overline{a(\theta)} = d(\bar{\theta}). \tag{4.102}$$

For large L and sufficiently small κ the Bethe roots are close to the two-strings: $\sigma_{2j-1} \simeq \eta_j - \pi i/2$, $\sigma_{2j} \simeq \eta_j + \pi i/2$ for certain real η_j.

Let us introduce the auxiliary function Y :

$$Y(\theta) = 1 + y(\theta) = 1 + \frac{T_2(\theta)}{f(\theta)}. \tag{4.103}$$

The function $\log(T_2(\theta))$ grows for $\mathrm{Re}(\theta) \to \pm\infty$ slowly (as $\pm 2L\theta$). This allows to derive from (4.97) the first important relation:

$$\log T_1(\theta) = (L * \log(fY))(\theta), \tag{4.104}$$

where we introduced the kernel which will be often used:

$$L(\theta) = \frac{1}{2\pi \cosh \theta},$$

and $*$ means the usual convolution product.

We have

$$T_2(\theta) = \lambda_1(\theta) + \lambda_2(\theta) + \lambda_3(\theta),$$

where

$$\begin{aligned}
\lambda_1(\theta) &= a(\theta + \pi i/2) a(\theta - \pi i/2) \frac{Q(\theta + 3\pi i/2)}{Q(\theta - \pi i/2)}, \\
\lambda_2(\theta) &= a(\theta + \pi i/2) d(\theta - \pi i/2) \frac{Q(\theta - 3\pi i/2) Q(\theta + 3\pi i/2)}{Q(\theta - \pi i/2) Q(\theta + \pi i/2)}, \\
\lambda_3(\theta) &= d(\theta + \pi i/2) d(\theta - \pi i/2) \frac{Q(\theta - 3\pi i/2)}{Q(\theta + \pi i/2)}.
\end{aligned}$$

The second auxiliary function is defined by

$$b(\theta) = \frac{\lambda_1(\theta + \pi i/2) + \lambda_2(\theta + \pi i/2)}{\lambda_3(\theta + \pi i/2)}, \quad B(\theta) = 1 + b(\theta). \tag{4.105}$$

Using the Baxter equation we derive

$$b(\theta) = T_1(\theta) \frac{Q(\theta + 2\pi i)}{Q(\theta - \pi i)} \frac{a(\theta + \pi i)}{d(\theta) d(\theta + \pi i)}. \tag{4.106}$$

[8] A priori one has two Baxter operators $Q^{\pm}$, that differ by $e^{\pm\nu\kappa\theta}$. However, as we will see soon, the results obtained in this section depend only on κ^2 so we can work with Q^{+} for simplicity.

On the other hand it is obvious from the definition that

$$T_2(\theta+\pi i/2) = B(\theta)d(\theta+\pi i)d(\theta)\frac{Q(\theta-\pi i)}{Q(\theta+\pi i)}\,. \tag{4.107}$$

Multiplying the latter equation by the conjugated one for real θ one easily derives the second important equation

$$\log y(\theta) = (L * \log(B\overline{B}))(\theta)\,. \tag{4.108}$$

Now comes the main of Suzuki's tricks. Consider a function $G(\theta)$ which is regular in the strip $0 < \mathrm{Im}(\theta) < \pi$, and which decrease sufficiently fast at $\pm\infty$. Then having in mind the structure of zeros of $T_2(\theta)$ described above we have

$$\int_{-\infty}^{\infty} \left(G(\theta-\theta')\log T_2(\theta'+\pi i/2) - G(\theta-\theta'+\pi i)\log T_2(\theta'-\pi i/2)\right)d\theta' = 0\,. \tag{4.109}$$

Using (4.107) we rewrite this as follows

$$\begin{aligned}
&\int_{-\infty}^{\infty} (G(\theta-\theta') + G(\theta-\theta'+\pi i))\log\frac{Q(\theta'+\pi i)}{Q(\theta'-\pi i)}d\theta' \\
&= \int_{-\infty}^{\infty} \left(G(\theta-\theta')\log(d(\theta')d(\theta'+\pi i)) - G(\theta-\theta'+\pi i)\log(a(\theta')a(\theta'-\pi i))\right)d\theta' \\
&+ \int_{-\infty}^{\infty} \left(G(\theta-\theta')\log(B(\theta')) - G(\theta-\theta'+\pi i)\log(\overline{B}(\theta'))\right)d\theta'\,.
\end{aligned}$$

The goal now is to rewrite the left hand side in terms of the auxiliary functions $y(\theta), b(\theta)$. From (4.106) and (4.104) one derives

$$\log b(\theta) = \log\left(\frac{Q(\theta+2\pi i)}{Q(\theta-\pi i)}\right) + \log\left(\frac{a(\theta+\pi i - i0)}{d(\theta)d(\theta+\pi i)}\right) + \int_{-\infty}^{\infty} L(\theta-\theta')\log\left(f(\theta')Y(\theta')\right)d\theta'\,.$$

So, our goal will be achieved if we find such $G(\theta)$ that

$$\begin{aligned}
\int_{-\infty}^{\infty} (G(\theta-\theta') + G(\theta-\theta'+\pi i))\log\frac{Q(\theta'+\pi i)}{Q(\theta'-\pi i)} &= \log\left(\frac{Q(\theta+2\pi i)}{Q(\theta-\pi i)}\right) \\
&+ \pi i\nu\kappa(4\overline{G}-3)\,,
\end{aligned} \tag{4.110}$$

where the last term takes account of the multiplier $e^{\nu\kappa\theta}$ in $Q(\theta)$, $\overline{G}$ being the average of G over the real line. Recalling that in the formula for $Q(\theta)$ (4.95) the Bethe roots are approximately two-string one finds $G(\theta)$ by Fourier transform (more details on this derivation are given in the appendix 4.10):

$$G(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\sinh\left(\frac{\pi k}{2\nu}(1-3\nu)\right)}{\sinh\left(\frac{\pi k}{2\nu}(1-2\nu)\right)\cosh\left(\frac{\pi k}{2}\right)}e^{-ik\theta}dk\,. \tag{4.111}$$

Notice that $\overline{G} = \frac{1-3\nu}{2(1-2\nu)}$.

Finally, after some computation we arrive at

$$\begin{aligned}
\log b(\theta) &= 2\sum_j \log\Big(\tanh\frac{1}{2}(\theta-\tau_j-i0)\Big) - \frac{\pi i\nu\kappa}{1-2\nu} \\
&+ (L*\log Y)(\theta) + (G*\log B)(\theta) - (G*\log\overline{B})(\theta+\pi i)\,.
\end{aligned} \tag{4.112}$$

We obtain the massive relativistic model from the inhomogeneous lattice one by the usual prescription: set $\tau_j = (-1)^j\tau$ and consider the *scaling limit*

$$\tau\to\infty\,,\quad L\to\infty,\quad 2Le^{-\tau}\to 2\pi MR \ \text{ finite}\,.$$

In this situation

$$2\sum_j \log\Big(\tanh\frac{1}{2}(\theta-\tau_j)\Big) \to -2\pi MR\cosh(\theta)\,.$$

The idea is that in this limit we should obtain the eigenvalue of the transfer-matrix corresponding to the NS ground state with the twist defined by

$$\sqrt{2}\beta P = \nu\kappa\,. \tag{4.113}$$

Here $\sqrt{2}$ comes from the normalization of the topological charge consistent with (4.78). The normalization of this twist is explained by the requirement that in the high temperature limit $R\to 0$ the eigenvalue of the first integral of motion, I_1, which is nothing but $L_0 - c/24$ is given by

$$i_1 = P^2 - \frac{1}{16}\,.$$

4.4.1 Numerical work

The function $b(\theta)$ rapidly decreases when $\mathrm{Re}(\theta)\to\pm\infty$, $0 > \mathrm{Im}\theta > -\pi/2$. Introducing the shift $0 < \pi\gamma < \pi/2$ and moving the contours of integration we arrive at the system which allows a numerical investigation:

$$\begin{aligned}\log b(\theta-\pi i\gamma) &= -2\pi MR\cosh(\theta-\pi i\gamma) - \frac{\pi i\sqrt{2}}{\beta}P + \tfrac{1}{2}\log 2 \\ &+ \int_{-\infty}^{\infty} L(\theta-\theta'+\pi i\gamma)\log\left(\tfrac{1}{2}Y(\theta')\right)d\theta' \\ &+ \int_{-\infty}^{\infty}\Big[G(\theta-\theta')\log B(\theta'-\pi i\gamma) - G(\theta-\theta'+\pi i(1-2\gamma))\log\overline{B(\theta'-\pi i\gamma)}\Big]d\theta'\,.\end{aligned} \tag{4.114}$$

$$\log y(\theta) = \int_{-\infty}^{\infty} 2\mathrm{Re}\Big[L(\theta-\theta'+\pi i\gamma)\log B(\theta'-\pi i\gamma)\Big]d\theta'\,. \tag{4.115}$$

The integrals containing $\log B$ converge at infinities very rapidly because the absolute value of the integrand is estimated as $\exp(-Const\cdot e^{|\theta|})$ with positive $Const$. The integral with $\log(\frac{1}{2}Y)$ converges much more slowly because $y(\theta)$ behaves as $1+O(e^{-|\theta|})$. In the numerical computations we replace integrals by finite sums, and the above estimates mean that the number of points needed for the approximation of the integral containing $\log(\frac{1}{2}Y)$ should be bigger than that for the integrals containing $\log B$.

Our goal is to consider the high temperature limit $R\to 0$. The previous formulae are simplified if we use the parametrization:

$$R = \frac{\beta}{\sqrt{2}}\left(\frac{\pi}{2}\mu\gamma\Big(\frac{1-\beta^2}{2}\Big)\right)^{-\frac{1}{1-\beta^2}} e^{-\theta_0}\,, \tag{4.116}$$

with θ_0 being a dimensionless parameter. Now the driving term in the equation (4.114) becomes

$$-4\sqrt{2}\frac{1-\beta^2}{\beta}e^{-\theta_0}\cosh(\theta-i\gamma)\,.$$

The local integrals of motion are extracted form $y(\theta)$ (recall that y represents the normalized transfer matrix of auxiliary spin 1 (4.99)). Namely, for $\theta\to\infty$ the asymptotical formula holds:

$$\log y(\theta) \simeq \sum_{k=1}^{\infty} C_{2k-1} i_{2k-1}(\theta_0) e^{-(2k-1)\theta}\,, \tag{4.117}$$

similarly the asymptotic for $\theta\to-\infty$ is related to $\bar{i}_{2k-1}(\theta_0)$. The constants C_m are given by

$$C_m = -\frac{\beta}{\sqrt{2}(1-\beta^2)}\frac{\sqrt{\pi}\;\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{1-\beta^2}m\right)}{(m-1)!\left(\frac{m+1}{2}\right)!\Gamma\left(1+\frac{\beta^2}{1-\beta^2}m\right)}\,. \tag{4.118}$$

This normalization is chosen for the sake of the conformal limit, the appearance of this kind of coefficients is not surprising for a reader familiar with [35], we shall give more explanation in the next section.

The main advantage of the above normalization is that in the high temperature limit we have

$$e^{-(2k-1)\theta_0} i_{2k-1}(\theta_0) \underset{\theta_0\to\infty}{\to} \quad i_{2k-1}\,,$$

with i_{2k-1} being the local integrals of motion for the CFT case normalized as follows:

$$i_{2k-1} = P^{2k} + \cdots\,.$$

Now we start the numerical work. Our goal is to obtain the formulae for i_1, i_3, i_5 by interpolation in P and ν. This may sound as a purely academic exercise having in mind that these formulae can be obtained analytically as explained in the next section. However, in our further study we shall need to guess the formulae for the one point functions in the integrable basis of Supersymmetric CFT, which are unknown. That is why we want to be sure that our numerical methods are sufficiently precise.

The twist P cannot be too large, we restrict ourselves to $P \le 0.2$, we take β sufficiently close to 1. For given β we interpolate in P from the solutions to (4.114), (4.115) for $\theta_0 = 18$. Integrals are replaced by sums with step 0.1, the shift is $\gamma = 0.1$, the limits in the integrals containing $\log B(\theta - \pi i\gamma)$ are $[-24, 24]$, the limits of the integral containing $\log(Y(\theta)/2)$ are $[-72, 72]$.

We normalize by the leading coefficient which is later compared with C_{2k-1}. Doing that for a sufficient number of different β's and assuming that due to the general structure of CFT the local integrals must be polynomials in

$$Q^2 = -\frac{(1-\beta^2)^2}{\beta^2}\,,$$

we were able to interpolate further:

$$\begin{aligned} i_1 &= P^2 - \frac{1}{16} \\ i_3 &= P^4 - \frac{5}{16}P^2 + \frac{1}{512}(9 + 2Q^2)\,, \\ i_5 &= P^6 - \frac{35}{48}P^4 + \frac{537 + 46Q^2}{3072}P^2 - \frac{475 + 190Q^2 + 24Q^4}{49152}\,. \end{aligned} \tag{4.119}$$

We shall not go into the details of the interpolation restricting ourselves to two examples in which we compare the results of the numerical computations using the equations (4.114), (4.115) with the analytical formulae (4.118), (4.119).

It is more direct to compare the computational results with

$$j_m = C_m i_m\,.$$

Here are the results for $\beta^2 = \frac{1}{2}$:

P	j_1 comp.	j_3 comp.	j_5 comp.	j_1 analyt.	j_3 analyt.	j_5 analyt.
0.02	0.195092899	-0.121737971	0.385270717	0.195092904	-0.121737972	0.385270720
0.04	0.191322988	-0.118811577	0.375422434	0.191322993	-0.118811578	0.375422438
0.06	0.185039803	-0.113984520	0.359237764	0.185039807	-0.113984521	0.359237767
0.08	0.176243343	-0.107332198	0.337056416	0.176243348	-0.107332199	0.337056419
0.1	0.164933610	-0.0989601675	0.309346006	0.164933614	-0.0989601686	0.309346008
0.12	0.151110603	-0.0890041464	0.276694070	0.151110607	-0.0890041473	0.276694072
0.14	0.134774321	-0.0776300103	0.239797812	0.134774325	-0.0776300111	0.239797814
0.16	0.115924766	-0.0650337947	0.199451558	0.115924769	-0.0650337954	0.199451559
0.18	0.0945619364	-0.0514416943	0.156531934	0.0945619389	-0.0514416947	0.156531935
0.2	0.0706858328	-0.0371100629	0.111980775	0.0706858347	-0.0371100632	0.111980775

Here are the results for $\beta^2 = \frac{3}{5}$:

P	j_1 comp.	j_3 comp.	j_5 comp.	j_1 analyt.	j_3 analyt.	j_5 analyt.
0.02	0.267141860	-0.315491660	1.87869822	0.267141961	-0.315491728	1.87869854
0.04	0.261979700	-0.308328920	1.83430033	0.261979797	-0.308328984	1.83430063
0.06	0.253376100	-0.296514050	1.76131551	0.253376191	-0.296514109	1.76131579
0.08	0.241331061	-0.280231598	1.66124365	0.241331143	-0.280231651	1.66124390
0.1	0.225844581	-0.259739931	1.53614935	0.225844653	-0.259739975	1.53614955
0.12	0.206916661	-0.235371233	1.38862669	0.206916720	-0.235371267	1.38862685
0.14	0.184547302	-0.207531507	1.22175395	0.184547345	-0.207531531	1.22175405
0.16	0.158736503	-0.176700578	1.03903821	0.158736527	-0.176700590	1.03903826
0.18	0.129484265	-0.143432085	0.844349942	0.129484268	-0.143432087	0.844349948
0.2	0.0967905868	-0.108353490	0.641847507	0.0967905654	-0.108353481	0.641847468

It is clear from these tables that the agreement is quite good. It can be made better by choosing bigger θ_0, using finer discretization *etc.* But this is not needed for our goals since our precision was sufficient for a successful interpolation.

4.4.2 Eigenvalues of integrals from ODE - CFT correspondence

The ODE - CFT correspondence is the statement that in the conformal case the vacuum eigenvalues of the operator $Q(\theta)$ coincide with the determinants of certain ordinary differential equations. The eigenvalues of the transfer-matrices $T_j(\theta)$ coincide with certain Stokes multipliers for the corresponding equation. In the case of $c < 1$ CFT this statement goes back to a remarkable observation due to Dorey ans Tateo [75], which was later essentially clarified and generalized by Bazhanov, Lukyanov, Zamolodchikov [76]. We shall not go into details of further generalization of the ODE-CFT correspondence and its generalization to the massive case, restricting ourselves to the case of Supersymmetric CFT which is considered in the present thesis. It is useful to consider a more general situation of a parafermion Ψ_k interacting with a free boson because there is certain difference between k even or odd. The $c = 1$ CFT corresponds to $k = 1$, and the $c = 3/2$ case, considered in our situation, corresponds to $k = 2$. In general case Lukyanov [77] proved that the operator $Q(\theta)$ is related to the following ODE:

$$\psi''(z) - \Big((z^{2\alpha} - E)^k + \frac{l(l+1)}{z^2}\Big)\psi(z) = 0\,, \tag{4.120}$$

the relation of E, α, l to parameters θ, β^2, k, P is as follows

$$\alpha = \frac{1-\beta^2}{k\beta^2}\,, \quad E = \frac{\beta}{\sqrt{k}} e^{\frac{1-\beta^2}{k}(\theta-\theta_0)}\,, \quad l = \frac{\sqrt{k}}{\beta}P - \frac{1}{2}\,. \tag{4.121}$$

and θ_0 is defined by a formula analogous to (4.116). The parameter α is positive, so, we are dealing with a self-adjoint operator on the positive half-line. Then $Q(E)$ is just its determinant (here and later we allow ourselves to use both $Q(\theta)$ and $Q(E)$ having in mind the identification (4.121)).

The eigenvalues $Q(E)$ and $T_j(E)$ are entire functions of E. We are interested in their large E asymptotics. It is known that for $\log Q(E)$ and for $\log T_j(E)$ with j up to $k-1$ the asymptotics go in two kinds of exponents: $E^{-\frac{2j-1}{2k(1-\beta^2)}}$ and $E^{\frac{j}{k\beta^2}}$, $(j \geq 1)$, the coefficients being proportional to the eigenvalues of local and non-local integrals of motion. The latter are of no interest for us, that is why we shall deal directly with $\log T_k(E)$ which possesses an exceptional property of containing in its asymptotics $E^{-\frac{2j-1}{2k(1-\beta^2)}}$ only. In order to explain that we have to consider (4.120) as an equation of a complex variable.

Let $z = |z|e^{i\varphi}$. Since the parameter α is generally irrational we are dealing with an infinite covering of the plane: $-\infty < \varphi < \infty$.

The main property allowing to investigate the determinant and the Stokes multipliers is the fact that for any solution $\psi(z, E)$ the function

$$(\Omega\psi)(z, E) = q^{1/2}\psi(pz, q^2E)\,, \quad p = e^{\pi i\beta^2}\,, \quad q = e^{\pi i\frac{1-\beta^2}{k}}\,,$$

is also a solution.

Consider the solution $\chi(z, E)$ characterized by the following asymptotics for real $z \to +\infty$:

$$\chi(z, E) \simeq z^{-\frac{\alpha k}{2}} \exp\Big(-\frac{z^{\alpha k+1}}{\alpha k+1}\Big)\,.$$

Following the [76, 77] and using the fusion relations it is not hard to derive for any j the relation between the three solutions:

$$(\Omega^{j+1}\chi)(z,E) = -T_{j-1}(Eq^{j+1})\chi(z,E) + T_j(Eq^j)(\Omega\chi)(z,E)\,.$$

The asymptotic behavior at $E \to \infty$ is investigated by the WKB method, where the important role is played by the the function $\sqrt{(z^\alpha - E)^k + \frac{l(l+1)}{z^2}}$.

One rescales z for large E so that the term $\frac{l(l+1)}{z^2}$ is small. It is clear that exactly for $j = k$ the function $T_k(Eq^k)$ can be considered as the Stokes multiplier between growing solutions $(\Omega\chi)(z,E)$ and $(\Omega^{k+1}\chi)(z,E)$ for two neighboring sectors which are semi-classically separated by the cut of the square root. This implies a simple formula for the asymptotics of $\log T_k(Eq^k)$ given below.

Let us change variables rewriting (4.120) as

$$a^2\psi''(x) - \Big((x^{2\alpha}-1)^k + a^2\frac{l(l+1)}{x^2}\Big)\psi(x) = 0\,, \tag{4.122}$$

where $a^2 = E^{-\frac{k}{(1-\beta^2)}}$.

We prefer to write the WKB formulae in a somewhat XIX century way in order to avoid some total derivatives. Namely, we present the solution to (4.122) in the form

$$\psi(x,x_0) = S(x,a)^{\frac{1}{2}}\exp\Big(\frac{1}{a}\int\limits_{x_0}^{x}\frac{dy}{S(y,a)}\Big)\,,$$

where $S(x,a)$ satisfies the Riccati equation (we omit arguments)

$$\frac{4}{a^2}\left(1 - FS^2\right) - S'^2 + 2S''S + x^{-2}S^2 = 0\,,$$

with

$$F(x,a,b) = (x^{2\alpha}-1)^k + \frac{b^2}{x^2}\,,$$

where we introduced $b = a(l+1/2)$ (not to be confused with the Liouville parameter), in spite of the fact that $b \ll 1$ it is convenient to develop S into a series in this parameter only at the final stage. The ansatz for ψ is different from usual quantum mechanical formulae, and it allows to avoid the appearance of redundant total derivatives. Using Riccati equation we find for $S(x,a)$ the power series

$$S(x,a,b) = \sum_{k=0}^{\infty} a^{2k}S_k(x,b)\,. \tag{4.123}$$

In particular,

$$\frac{1}{S_0(x,b)} = \sqrt{F(x,a,b)}\,.$$

According to our reasoning concerning the Stokes multiplier, we have for the asymptotics

$$\log T_k(Eq^k) \simeq \frac{1}{a}\int_C \frac{dy}{S(y,a)}\,, \tag{4.124}$$

where the contour C goes from $\infty\cdot e^{+i0}$ to $\infty\cdot e^{-i0}$ around the cut of $\sqrt{F(x,a,b)}$. Let us consider the contribution from $S_0(x,b)$. Recalling that $b \ll 1$ we develop

$$\frac{1}{S_0(x,b)} = \sum_{p=0}^{\infty}\binom{1/2}{p}(x^{2\alpha}-1)^{\frac{k(1-2p)}{2}}b^{2p}x^{-2p}\,.$$

Now the difference between k odd or even becomes clear. We have to evaluate the integral

$$\int\limits_C (y^{2\alpha}-1)^{\frac{k(1-2p)}{2}}y^{-2p}dy\,.$$

By the change of variables $w = y^{2\alpha}$ this integral reduces for odd k to a beta-function and for even k to a binomial coefficient. In spite of this computational difference the final result does not depend on the parity of k, after some simplification we get

$$\int_C (y^{2\alpha}-1)^{\frac{k(1-2p)}{2}} y^{-2p} dy = \frac{\pi i k \beta^2}{1-\beta^2} e^{-\frac{\pi i}{2}k(2p-1)} \frac{\Gamma\left(\frac{k(2p-1)}{2(1-\beta^2)}\right)}{\Gamma\left(1+\frac{k\beta^2(2p-1)}{2(1-\beta^2)}\right)\Gamma\left(\frac{k(2p-1)}{2}\right)} .$$

Plugging this into (4.124) we find the constants C_m. Higher corrections in a^2 following from (4.123) are considered similarly. For $k=2$ one finds exactly the expressions (4.119).

From the above perspective, this method allows to compute the eigenvalue i_7 that is hardly accessible by interpolation, and was not given in [7] :

$$\begin{aligned} i_7 = P^8 - \frac{21}{16}P^6 + \frac{7(3881+202Q^2)}{38400}P^4 - \frac{301877+57734Q^2+3800Q^4}{1843200}P^2 \\ + \frac{1089809+622748Q^2+149660Q^4+12240Q^6}{117964800} . \end{aligned} \tag{4.125}$$

4.5 Lattice ω function

Now that we obtained the scaling equations for the free energy, we can move towards the computation of one point functions. We start by considering the main piece ω on the lattice.

4.5.1 Definitions

Recall the definition of the function ω from [3] and given also in the section 2.7 . This function splits in two parts:

$$\omega(\theta,\theta') = \omega_{\rm hol}(\theta,\theta') + \omega_{\rm sing}(\theta,\theta') , \tag{4.126}$$

where $\omega_{\rm hol}(\theta,\theta')$ as a function of θ has no other singularities but simple poles at the zeros of $T_1(\theta,\kappa)$, and $\omega_{\rm sing}$ is its singular part given by :

$$\begin{aligned} \omega_{\rm sing}(\theta,\theta') = \frac{1}{T_1(\theta,\kappa)T_1(\theta',\kappa)}\Big(a(\theta)d(\theta')\psi(\theta-\theta'+\pi i,\alpha) - d(\theta)a(\theta')\psi(\theta-\theta'-\pi i,\alpha)\Big) \\ + (1+\rho(\theta)\rho(\theta'))\phi(\theta-\theta',\alpha) - \rho(\theta)\phi(\theta-\theta'+\pi i,\alpha) - \rho(\theta')\phi(\theta-\theta'-\pi i,\alpha) , \end{aligned} \tag{4.127}$$

where

$$\psi(\theta,\alpha) = 2\nu \frac{e^{\alpha\nu\theta}}{e^{2\nu\theta}-1} , \tag{4.128}$$

and ϕ is defined as a solution of the difference equation:

$$\Delta_\theta \phi(\theta,\alpha) = \phi(\theta+i\pi,\alpha) - \phi(\theta-i\pi,\alpha) = \psi(\theta,\alpha) . \tag{4.129}$$

We shall remind the normalization conditions for the function ω. Start by defining the function φ:

$$\varphi(\theta) = \prod_{j=1}^{L} \frac{1}{\sinh\nu(\theta-\tau_j-\pi i)\sinh\nu(\theta-\tau_j)\sinh\nu(\theta-\tau_j+\pi i)} ,$$

satisfying

$$d(\theta+\pi i)\varphi(\theta+\pi i) = a(\theta)\varphi(\theta) ,$$

and the measure

$$d\mu^{\pm}(\theta) = Q^{\mp}(\theta,\kappa+\alpha)Q^{\pm}(\theta,\kappa)\varphi(\theta)d\theta . \tag{4.130}$$

The poles of φ come in triplets reflecting the fact that the Matsubara chain consists of spin-1 representations. Let the contour Γ_j go around the three points $\tau_j, \tau_j \pm \pi i$. The normalization conditions on the function $\omega(\theta,\eta)$ from [3] are given by :

$$\int_{\Gamma_j} T_1(\theta,\kappa)\omega(\theta,\eta)d\mu^+(\theta) = 0 . \tag{4.131}$$

The equations (4.127) , (4.131) define $\omega(\theta,\theta')$ completely. Due to the deformed Riemann bilinear identity the following relation is automatic:

$$\int\limits_{\Gamma_j} T_1(\theta,\kappa)\omega(\theta,\eta)d\mu^-(\theta)=0\,.$$

In order to make the further formulae more readable we shall denote by τ without index any of the inhomogeneities τ_j.

For future use we rewrite the normalization condition as

$$\omega(\tau+\pi i,\eta)+Y(\tau)\omega(\tau,\eta)+X(\tau)\omega(\tau-\pi i,\eta)=0\,, \tag{4.132}$$

with

$$X(\theta)=\frac{T_2(\theta+\pi i/2,\kappa+\alpha)}{a(\theta)d(\theta+\pi i)}\,,\qquad Y(\theta)=\frac{1}{\rho(\theta)}\left(1+X(\theta)\right)\,.$$

Similarly,

$$\omega(\eta,\tau+\pi i)+Y(\tau)\omega(\eta,\tau)+X(\tau)\omega(\eta,\tau-\pi i)=0\,. \tag{4.133}$$

4.5.2 Rewriting normalization conditions

Introduce

$$\begin{aligned}
F^+(\theta,\eta)&=\langle \mathbf{b}^*(\theta)c^*(\eta)\rangle=\frac{1}{\mathcal{N}(\theta)}\Big(\omega(\theta+\pi i/2,\eta)\rho(\theta-\pi i/2)+\omega(\theta-\pi i/2,\eta)\Big)\,,\\
\tilde{F}^+(\theta,\eta)&=\langle \tilde{\mathbf{b}}^*(\theta)c^*(\eta)\rangle=\frac{1}{\mathcal{N}(\theta)}\Big(\omega(\theta+\pi i/2,\eta)+\rho(\theta+\pi i/2)\omega(\theta-\pi i/2,\eta)\Big)\,,\\
F^-(\eta,\theta)&=\langle b^*(\eta)\mathbf{c}^*(\theta)\rangle=\frac{1}{\mathcal{N}(\theta)}\Big(\omega(\eta,\theta+\pi i/2)\rho(\theta-\pi i/2)+\omega(\eta,\theta-\pi i/2)\Big)\,,\\
\tilde{F}^-(\eta,\theta)&=\langle b^*(\eta)\tilde{\mathbf{c}}^*(\theta)\rangle=\frac{1}{\mathcal{N}(\theta)}\Big(\omega(\eta,\theta+\pi i/2)+\rho(\theta+\pi i/2)\omega(\eta,\theta-\pi i/2)\Big)\,.
\end{aligned}$$

These functions describe the pairings between the fused operators $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$ with not fused ones c^*, b^*. Clearly the knowledge of these pairings is sufficient to compute any expectation value containing $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$. So, the analytical properties of $F^+(\theta,\eta)$, *etc* characterize in the weak sense the analytical properties of $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$.

Similarly, in order to understand the analytical properties of $\mathbf{j}^+(\theta),\mathbf{j}^0(\theta),\mathbf{j}^-(\theta)$ we introduce

$$\begin{aligned}
G^+(\theta,\eta_1,\eta_2)&=\langle \mathbf{j}^+(\theta)c^*(\eta_2)c^*(\eta_1)\rangle=\frac{1}{\mathcal{N}(\theta)}\begin{vmatrix}\omega(\theta+\pi i/2,\eta_1) & \omega(\theta+\pi i/2,\eta_2)\\ \omega(\theta-\pi i/2,\eta_1) & \omega(\theta-\pi i/2,\eta_2)\end{vmatrix}\,,\\
G^-(\eta_1,\eta_2,\theta)&=\langle b^*(\eta_1)b^*(\eta_2)\mathbf{j}^-(\theta)\rangle=\frac{1}{\mathcal{N}(\theta)}\begin{vmatrix}\omega(\eta_1,\theta+\pi i/2) & \omega(\eta_2,\theta+\pi i/2)\\ \omega(\eta_1,\theta-\pi i/2) & \omega(\eta_2,\theta-\pi i/2)\end{vmatrix}\,,\\
G^0(\theta,\eta_1,\eta_2)&=\langle \mathbf{j}^0(\theta)b^*(\eta_1)c^*(\eta_2)\rangle\\
&=\frac{1}{\mathcal{N}(\theta)}\Big((\omega(\theta+\pi i/2,\theta-\pi i/2)-\omega(\theta-\pi i/2,\theta+\pi i/2))\omega(\eta_1,\eta_2)\\
&+\omega(\theta-\pi i/2,\eta_2)\omega(\eta_1,\theta+\pi i/2)-\omega(\theta+\pi i/2,\eta_2)\omega(\eta_1,\theta-\pi i/2)\Big)\,,
\end{aligned}$$

where in the last line we imply

$$\omega(\theta+\pi i/2,\theta-\pi i/2)-\omega(\theta-\pi i/2,\theta+\pi i/2)=\lim_{\theta'\to\theta}\Big(\omega(\theta+\pi i/2,\theta'-\pi i/2)-\omega(\theta-\pi i/2,\theta'+\pi i/2)\Big)\,.$$

We want to rewrite the normalization conditions in terms of these functions and $\mathrm{P}(\theta)$ only. As

before let τ be any inhomogeneity. Then we claim that

$$\begin{aligned}
&F^+(\tau+\pi i/2,\eta)+\mathrm{P}(\tau+\pi i/2)\tilde F^+(\tau-\pi i/2,\eta)=0\,,\\
&\tilde F^+(\tau+\pi i/2,\eta)+\mathrm{P}(\tau+\pi i/2)F^+(\tau-\pi i/2,\eta)=0\,,\\
&F^-(\eta,\tau+\pi i/2)+\mathrm{P}(\tau+\pi i/2)\tilde F^-(\eta,\tau-\pi i/2)=0\,,\\
&\tilde F^-(\eta,\tau+\pi i/2)+\mathrm{P}(\tau+\pi i/2)F^-(\eta,\tau-\pi i/2)=0\,,\\
&G^+(\tau+\pi i/2,\eta_1,\eta_2)-\mathrm{P}(\tau+\pi i/2)G^+(\tau-\pi i/2,\eta_1,\eta_2)=0\,,\\
&G^-(\tau+\pi i/2,\eta_1,\eta_2)-\mathrm{P}(\tau+\pi i/2)G^-(\tau-\pi i/2,\eta_1,\eta_2)=0\,,\\
&G^0(\tau+\pi i/2,\eta_1,\eta_2)-\mathrm{P}(\tau+\pi i/2)G^0(\tau-\pi i/2,\eta_1,\eta_2)=0\,.
\end{aligned}\tag{4.134}$$

Let us prove the first of these identities, others are checked similarly.

We begin with some useful identities. Using

$$a(\tau+\pi i)=0\,,\qquad d(\tau-\pi i)=0\,,$$

we find

$$\begin{aligned}
&T_1(\tau+\pi i,\kappa)=d(\tau+\pi i)\frac{Q^\pm(\tau,\kappa)}{Q^\pm(\tau+\pi i,\kappa)}\,,\\
&T_1(\tau-\pi i,\kappa)=a(\tau-\pi i)\frac{Q^\pm(\tau,\kappa)}{Q^\pm(\tau-\pi i,\kappa)}\,,\\
&T_2(\tau+\pi i/2,\kappa)=d(\tau+\pi i)d(\tau)\frac{Q^\pm(\tau-\pi i,\kappa)}{Q^\pm(\tau+\pi i,\kappa)}\,,\\
&T_2(\tau-\pi i/2,\kappa)=a(\tau-\pi i)a(\tau)\frac{Q^\pm(\tau+\pi i,\kappa)}{Q^\pm(\tau-\pi i,\kappa)}\,.
\end{aligned}\tag{4.135}$$

For (4.134) we have

$$\begin{aligned}
&F^+(\tau+\pi i/2,\eta)+\mathrm{P}(\tau+\pi i/2)\tilde F^+(\tau-\pi i/2,\eta)\\
&=\frac{1}{\mathcal{N}(\tau+\pi i/2)}\Big(\omega(\tau+\pi i,\eta)\rho(\tau)+\omega(\tau,\eta)\Big(1+\frac{\mathcal{N}(\tau+\pi i/2)}{\mathcal{N}(\tau-\pi i/2)}\mathrm{P}(\tau+\pi i/2)\Big)\\
&+\omega(\tau-\pi i,\eta)\rho(\tau)\frac{\mathcal{N}(\tau+\pi i/2)}{\mathcal{N}(\tau-\pi i/2)}\mathrm{P}(\tau+\pi i/2)\Big)\,.
\end{aligned}$$

Using (4.135) we compute

$$\frac{\mathcal{N}(\tau+\pi i/2)}{\mathcal{N}(\tau-\pi i/2)}\mathrm{P}(\tau+\pi i/2)=\frac{d(\tau)}{a(\tau)}\frac{Q^-(\tau-\pi i,\kappa+\alpha)}{Q^-(\tau+\pi i,\kappa+\alpha)}=X(\tau)\,.$$

Using the latter identity we evaluate

$$\begin{aligned}
&F^+(\tau+\pi i/2,\eta)+\mathrm{P}(\tau+\pi i/2)\tilde F^+(\tau-\pi i/2,\eta)\\
&=\frac{\rho(\tau)}{\mathcal{N}(\tau+\pi i/2)}\Big(\omega(\tau+\pi i,\eta)+\omega(\tau,\eta)\frac{Q^-(\tau,\kappa+\alpha)T_1(\tau,\kappa+\alpha)}{a(\tau)\rho(\tau)Q^-(\tau+\pi i,\kappa+\alpha)}\\
&+X(\tau)\omega(\tau-\pi i,\eta)\Big)\\
&=\frac{\rho(\tau)}{\mathcal{N}(\tau+\pi i/2)}\Big(\omega(\tau+\pi i,\eta)+Y(\tau)\omega(\tau,\eta)+X(\tau)\omega(\tau-\pi i,\eta)\Big)=0\,,
\end{aligned}$$

due to (4.131).

4.5.3 The case $\alpha=0$

In the case $\alpha=0$ the left and right eigenstates coincide, hence $\rho(\theta)=1$ and in the weak sense there is no difference between $\mathbf{b}^*,\mathbf{c}^*$ on the one hand and $\tilde{\mathbf{b}}^*,\tilde{\mathbf{c}}^*$ on the other. So, all the expectation values containing only fermions are expressed via one function

$$\begin{aligned}
&\Omega(\theta,\theta')=\frac{1}{\mathcal{N}(\theta)\mathcal{N}(\theta')}\\
&\times\Big(\omega(\theta+\tfrac{\pi i}{2},\theta'+\tfrac{\pi i}{2})+\omega(\theta+\tfrac{\pi i}{2},\theta'-\tfrac{\pi i}{2})+\omega(\theta-\tfrac{\pi i}{2},\theta'+\tfrac{\pi i}{2})+\omega(\theta-\tfrac{\pi i}{2},\theta'-\tfrac{\pi i}{2})\Big)\,.
\end{aligned}\tag{4.136}$$

We want to find an independent way of defining this function. As explained in [3] for $\alpha = 0$ there is an important analogy between the function $\omega(\theta, \theta')$ and the normalized second kind differential on a hyperelliptic Riemann surface. The normalization condition (4.131) is the analogue of the requirement of vanishing of the a-periods.

We set

$$\tau = \tau_j \,.$$

Consider the function

$$\tilde{\omega}(\theta) = \frac{\delta}{\delta\tau}\Big\{\log t(\theta) - \log\Big(\frac{s(\theta - \pi i)s(\theta + \pi i)}{s(\theta)}\Big)\Big\}\,.$$

Notice that

$$\frac{\delta}{\delta\tau}\log\Big(\frac{s(\theta - \pi i)s(\theta + \pi i)}{s(\theta)}\Big) = (\delta_\theta^+)^{-1}\frac{\delta}{\delta\tau}\log\big(s(\theta - 2\pi i)s(\theta + \pi i)\big)\,,$$

where $\delta_\theta^+ f(\theta) = f(\theta) + f(\theta - \pi i)$.

We want to show that $\tilde{\omega}(\theta)$ is a normalized differential. First we prove that

$$\int_{\Gamma_k} T_1(\theta)\tilde{\omega}(\theta)d\mu^{\pm}(\theta) = 0\,, \quad k \neq j\,.$$

The case $k = j$ is special, instead of a direct computation for this case we consider $\Gamma_{\pm\infty} = [\pm\Lambda, \pm\Lambda + \pi i/\nu]$ for $|\lambda| > \max(|\tau_k|)$. For $\Gamma_{\pm\infty}$ the computation is exactly the same as for Γ_k, $k \neq j$.

Recall that (in the case $\alpha = 0$ we have $d\mu^+ = d\mu^- = d\mu$) :

$$d\mu(\theta) = Q^+(\theta)Q^-(\theta)\varphi(\theta)d\theta\,.$$

We have two identities [3]:

$$\begin{aligned}\int_{\Gamma_k} T_1(\theta)(\delta_\theta^+)^{-1}f(\theta)d\mu(\theta) &= \int_{\Gamma_k} d(\theta)f(\theta)Q^+(\theta - \pi i)Q^-(\theta)\varphi(\theta)d\theta\,,\\ &= \int_{\Gamma_k} a(\theta)f(\theta + \pi i)Q^+(\theta + \pi i)Q^-(\theta)\varphi(\theta)d\theta\,.\end{aligned}$$

Using these identities we derive

$$\begin{aligned}&\int_{\Gamma_k} T_1(\theta)\tilde{\omega}(\theta)d\mu(\theta)\\ &= \int_{\Gamma_k}\Big\{Q^+(\theta)\frac{\delta}{\delta\tau}t(\theta) - Q^+(\theta + \pi i)\frac{\delta}{\delta\tau}a(\theta) - Q^+(\theta - \pi i)\frac{\delta}{\delta\tau}d(\theta)\Big\}Q^-(\theta)\varphi(\theta)d\theta\\ &= \int_{\Gamma_k}\Big\{a(\theta)\frac{\delta}{\delta\tau}Q^+(\theta + \pi i) + d(\theta)\frac{\delta}{\delta\tau}Q^+(\theta - \pi i) - t(\theta)\frac{\delta}{\delta\tau}Q^+(\theta)\Big\}Q^-(\theta)\varphi(\theta)d\theta\\ &= \int_{\Gamma_k}\Big\{a(\theta)Q^-(\theta)\frac{\delta}{\delta\tau}Q^+(\theta + \pi i) - d(\theta)Q^-(\theta - \pi i)\frac{\delta}{\delta\tau}Q^+(\theta)\Big\}\varphi(\theta)d\theta\\ &+ \int_{\Gamma_k}\Big\{d(\theta)Q^-(\theta)\frac{\delta}{\delta\tau}Q^+(\theta - \pi i) - a(\theta)Q^-(\theta + \pi i)\frac{\delta}{\delta\tau}Q^+(\theta)\Big\}\varphi(\theta)d\theta = 0\,.\end{aligned}$$

As a normalized differential $\tilde{\omega}(\theta)$ must be expressible as a linear combination of $\omega(\theta, \eta_j)$ for some set $\{\eta_j\}$. The structure of singularities of $\tilde{\omega}(\theta)$ suggests that this set is just $\tau, \tau + \pi i$. To be precise we claim that

$$\tilde{\omega}(\zeta) = \frac{1}{\mathcal{N}(\tau + \frac{\pi i}{2})}(\omega(\zeta, \tau) + \omega(\zeta, \tau + \pi i))\,. \tag{4.137}$$

Let us prove this. We have

$$\omega(\theta, \tau) + \omega(\theta, \tau + \pi i) = \omega_{\rm hol}(\theta, \tau) + \omega_{\rm hol}(\theta, \tau + \pi i) + \omega_{\rm sing}(\theta, \tau) + \omega_{\rm sing}(\theta, \tau + \pi i)\,,$$

where $\omega_{\rm hol}(\theta, \eta)$ as a function of θ has no other singularities but simple poles at zeros of $T_1(\theta)$,

$$\begin{aligned}\omega_{\rm sing}(\theta, \eta) &= \delta_\theta^-\delta_\eta^-\Delta_\theta^{-1}(\nu\coth\nu(\theta - \eta))\\ &+ \frac{1}{T_1(\theta)T_1(\eta)}\big(a(\theta)d(\eta)\nu\coth\nu(\theta - \eta + \pi i) - d(\theta)a(\eta)\nu\coth\nu(\theta - \eta - \pi i)\big)\,,\end{aligned} \tag{4.138}$$

which implies

$$\begin{aligned}
&\omega_{\rm sing}(\theta,\tau)+\omega_{\rm sing}(\theta,\tau+\pi i)=\nu\coth\nu(\theta-\tau-\pi i)-\nu\coth\nu(\theta-\tau)\\
&+\frac{a(\theta)d(\tau)}{T_1(\theta)T_1(\tau)}\nu\coth\nu(\theta-\tau+\pi i)-\frac{d(\theta)a(\tau)}{T_1(\theta)T_1(\tau)}\nu\coth\nu(\theta-\tau-\pi i)\\
&+\frac{a(\theta)d(\tau+\pi i)}{T_1(\theta)T_1(\tau+\pi i)}\nu\coth\nu(\theta-\tau)\,.\\
&=\frac{a(\theta)d(\tau)}{T_1(\theta)T_1(\tau)}\nu\coth\nu(\theta-\tau+\pi i)+\frac{T_1(\theta)T_1(\tau)-d(\theta)a(\tau)}{T_1(\theta)T_1(\tau)}\nu\coth\nu(\theta-\tau-\pi i)\\
&-\frac{T_1(\theta)T_1(\tau+\pi i)-a(\theta)d(\tau+\pi i)}{T_1(\theta)T_1(\tau+\pi i)}\nu\coth\nu(\theta-\tau)\,.
\end{aligned}$$

Using this identity one finds

$$\begin{aligned}
&{\rm res}_{\theta=\tau-\pi i}\big(\omega_{\rm sing}(\theta,\tau)+\omega_{\rm sing}(\theta,\tau+\pi i)\big)=\frac{a(\tau-\pi i)d(\tau)}{T_1(\tau-\pi i)T_1(\tau)}=\mathcal{N}(\tau+\frac{\pi i}{2})\,,\\
&{\rm res}_{\theta=\tau+\pi i}\big(\omega_{\rm sing}(\theta,\tau)+\omega_{\rm sing}(\theta,\tau+\pi i)\big)=\frac{T_2(\tau+\frac{\pi i}{2})}{T_1(\tau+\pi i)T_1(\tau)}=\mathcal{N}(\tau+\frac{\pi i}{2})\,,\\
&{\rm res}_{\theta=\tau}\big(\omega_{\rm sing}(\theta,\tau)+\omega_{\rm sing}(\theta,\tau+\pi i)\big)=-\frac{T_2(\tau+\frac{\pi i}{2})}{T_1(\tau+\pi i)T_1(\tau)}=-\mathcal{N}(\tau+\frac{\pi i}{2})\,.
\end{aligned}$$

This finishes the proof.

Now we obtain the most important relation of this section :

$$\begin{aligned}
&\frac{\delta}{\delta\tau}\log\left(\frac{T_2(\theta)}{f(\theta)}\right) \qquad (4.139)\\
&=\frac{1}{\mathcal{N}(\theta)}\left(\frac{\delta}{\delta\tau}\log T_1(\theta+\pi i/2)+\frac{\delta}{\delta\tau}\log T_1(\theta-\pi i/2)\right)-\left(\frac{f(\theta)}{T_2(\theta)}+1\right)\frac{\delta}{\delta\tau}\log f(\theta)\\
&=\frac{1}{\mathcal{N}(\theta)}\left(\frac{\delta}{\delta\tau}\log T_1(\theta+\pi i/2)+\frac{\delta}{\delta\tau}\log T_1(\theta-\pi i/2)-\frac{\delta}{\delta\tau}\log f(\theta)\right)\\
&=\frac{1}{\mathcal{N}(\theta)\mathcal{N}(\tau+\frac{\pi i}{2})}\left(\omega(\theta+\frac{\pi i}{2},\tau)+\omega(\theta+\frac{\pi i}{2},\tau+\pi i)+\omega(\theta-\frac{\pi i}{2},\tau)+\omega(\theta-\frac{\pi i}{2},\tau+\pi i)\right)\,.\\
&=\Omega(\theta,\tau+\frac{\pi i}{2})\,.
\end{aligned}$$

4.6 Scaling limit of the function Ω

In considering the scaling limit, we want, similarly to [4, 5], to combine two seemingly inconsistent requirements: $\alpha\neq 0$ and $\rho(\theta)=\mathrm{P}(\theta)=1$. In fact this can be achieved for a discrete set of α's introducing the fermionic screening operators [4], and then invoking the analytical continuation. As will be clear later our definition is consistent rather with the understanding of the model in terms of the action (4.81).

Remember that the scaling limit presented in section 4.4 consists in taking in both Space and Matsubara directions staggering inhomogeneities $\tau_j=(-1)^j\tau$, and considering

$$\tau\to\infty\,,\quad L\to\infty,\quad 2Le^{-\tau}\to 2\pi MR\ \text{ finite}\,,$$

where R is the radius of the cylinder, M is the mass of the soliton (4.89).

For $\rho(\theta)=\mathrm{P}(\theta)=1$ in the weak sense the operators $\tilde{\mathbf{b}}^*(\zeta)$, $\tilde{\mathbf{c}}^*(\zeta)$ coincide with the operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$. Similarly to [4, 5] the relations (4.134) hint that the asymptotics for $\theta\to\pm\infty$ of the fermions (KM currents) are anti-periodic (periodic) in θ. Explicitly we assume

$$\begin{aligned}
&\mathbf{b}^*(\theta)\underset{\theta\to\pm\infty}{\simeq}\sum_{j=1}^{\infty}e^{\mp(2j-1)\theta}\mathbf{b}^*_{2j-1}\,,\quad \mathbf{c}^*(\theta)\underset{\theta\to\pm\infty}{\simeq}\sum_{j=1}^{\infty}e^{\mp(2j-1)\theta}\mathbf{c}^*_{2j-1}\,,\\
&\mathbf{j}^\sigma(\theta)\underset{\theta\to\pm\infty}{\simeq}\sum_{j=1}^{\infty}e^{\mp 2j\theta}\mathbf{j}^\sigma_{2j}\,,\quad \sigma=0,\pm\,.
\end{aligned}$$

As we have seen, the Suzuki equations (4.114) are obtained by this procedure from the corresponding lattice equations. The latter have the same structure as (4.114), but differ only by the driving term. In the case of the lattice it is given by :

$$D(\theta) = 2\sum_j \log\Big(\tanh\frac{1}{2}(\theta - \tau_j - i0)\Big) - \frac{\pi i\nu\kappa}{1-2\nu}\,, \tag{4.140}$$

for which we have in the scaling limit

$$D(\theta) \to -2\pi MR\cosh(\theta - \pi i\gamma) - \frac{\pi i\sqrt{2}}{\beta}P\,,$$

where P was given in (4.113).

4.6.1 Equations for Ω

Now we shall present a conjecture for the scaling limit of $\Omega(\theta,\theta')$ in the case $\alpha \neq 0$ and $\rho(\theta) = \mathrm{P}(\theta) = 1$ and provide some justifications for it :

$$\begin{aligned}\Omega(\theta,\theta') = &\int_{-\infty}^{\infty} L(\theta-\eta+\pi i\gamma)\mathcal{G}(\eta-\pi i\gamma,\theta')dm_b(\eta-\pi i\gamma) \\ &+ \int_{-\infty}^{\infty} L(\theta-\eta-\pi i\gamma)\overline{\mathcal{G}}(\eta+\pi i\gamma,\theta')d\overline{m}_b(\eta+\pi i\gamma)\,,\end{aligned} \tag{4.141}$$

where for the auxiliary functions we have the linear equations

$$\begin{aligned}\mathcal{G}(\theta-\pi i\gamma,\theta') = &L(\theta-\theta'-\pi i\gamma) + \int_{-\infty}^{\infty} L(\theta-\eta-\pi i\gamma)\Omega(\eta,\theta')dm_y(\eta) \\ &+ \int_{-\infty}^{\infty} G_\alpha(\theta-\eta)\mathcal{G}(\eta-\pi i\gamma,\theta')dm_b(\eta-\pi i\gamma) \\ &- \int_{-\infty}^{\infty} G_\alpha(\theta-\eta+\pi i(1-2\gamma))\overline{\mathcal{G}}(\eta+\pi i\gamma,\theta')d\overline{m}_b(\eta+\pi i\gamma)\,,\end{aligned} \tag{4.142}$$

$$\begin{aligned}\overline{\mathcal{G}}(\theta+\pi i\gamma,\theta') = &L(\theta-\theta'+\pi i\gamma) + \int_{-\infty}^{\infty} L(\theta-\eta+\pi i\gamma)\Omega(\eta,\theta')dm_y(\eta) \\ &+ \int_{-\infty}^{\infty} G_\alpha(\theta-\eta-\pi i(1-2\gamma))\mathcal{G}(\eta-\pi i\gamma,\theta')dm_b(\eta-\pi i\gamma) \\ &- \int_{-\infty}^{\infty} G_\alpha(\theta-\eta)\overline{\mathcal{G}}(\eta+\pi i\gamma,\theta')d\overline{m}_b(\eta+\pi i\gamma)\,,\end{aligned} \tag{4.143}$$

and we defined [9]

$$\begin{aligned}&dm_y(\theta) = \frac{y(\theta)}{1+y(\theta)}\,, \quad dm_b(\theta) = \frac{b(\theta)}{1+b(\theta)}\,, \quad d\overline{m}_b(\theta) = \frac{\overline{b}(\theta)}{1+\overline{b}(\theta)}\,, \\ &L(\theta) = \frac{1}{2\pi\cosh\theta}\,, \quad G_\alpha(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\sinh\left(\frac{3\beta^2-1}{2(1-\beta^2)}\pi k + \frac{\pi i\alpha}{2}\right)}{\sinh\left(\frac{\beta^2}{1-\beta^2}\pi k + \frac{\pi i\alpha}{2}\right)\cosh\left(\frac{1}{2}\pi k\right)}e^{ik\theta}dk\,.\end{aligned} \tag{4.144}$$

The shift γ is an arbitrary real number from the interval $(0,\pi/2)$.

[9] Of course we have $G_{\alpha=0}(\theta) = G(\theta)$ defined in (4.111).

The most important support for this definition is provided by the case $\alpha = 0$ for which the requirements $\rho(\theta) = \mathrm{P}(\theta) = 1$ are automatic and do not demand additional work even on the lattice. In that case we have (4.139)

$$\frac{\delta}{\delta\tau}\log y(\theta) = \Omega(\theta, \tau + \tfrac{\pi i}{2})\,. \tag{4.145}$$

Consider therefore the Suzuki equations for $y(\theta)$ at the lattice level, that is (4.114) with the driving term replaced by $D(\theta)$ (4.140). One can readily compute the variation of $y(\theta)$ with respect to any τ_j, finding agreement with (4.141) after scaling.

Strictly speaking even for $\alpha = 0$ to combine the equations (4.145) for all τ_j, we do not have enough conditions to assert (4.141) for all θ', but this is a very natural conjecture to make.

The next question is how did we incorporate α into the equations (4.141), (4.142), (4.143). This was done due to the experience with equations of this kind [4, 5]. Our choice is supported by the computation of the residue at $\theta = \theta + \pi i$. After some rather tedious computation we obtain the following result

$$\operatorname*{res}_{\theta=\theta'+\pi i}\Omega(\theta,\theta') = \frac{1}{2\pi i}\frac{y(\theta')y(\theta'+\pi i)-1}{y(\theta')y(\theta'+\pi i)}\,,$$

which coincides with the expected result from the definition (4.136) and known singularities of $\omega(\theta,\theta')$ [3].

4.6.2 Numerical results by interpolation

Our method of numerical investigation of the equations (4.114) was explained in the section 4.4. With these results at hand the numerical solution to the linear equations (4.141), (4.142), (4.143) is rather straightforward. The most interesting thing to study is the limit $\theta_0 \to \infty$ where we make contact with the UV CFT. We begin with the case $\theta \to \infty, \theta' \to \infty$ for which we assume

$$\Omega(\theta,\theta') \simeq \sum_{i,j=1}^{\infty} e^{-(2i-1)\theta}e^{-(2j-1)\theta'}D_{2i-1}(\alpha)D_{2j-1}(2-\alpha)\Omega_{2i-1,2j-1}(\theta_0)\,. \tag{4.146}$$

The coefficients $D_{2i-1}(\alpha)$ are not hard to guess from (4.118) and by analogy with [4]:

$$D_m(\alpha) = i^m\sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{1-\beta^2}m+\frac{\alpha}{2}\right)}{(m-1)!\left(\frac{m-1}{2}\right)!\Gamma\left(\frac{\beta^2}{(1-\beta^2)}m+\frac{\alpha}{2}\right)}\,. \tag{4.147}$$

Additional support for this formula will be given below by considering the reflection relations. We have further

$$\lim_{\theta_0\to\infty} e^{-2(i+j-1)\theta_0}\Omega_{2i-1,2j-1}(\theta_0) \to \Omega_{2i-1,2j-1}\,,$$

$\Omega_{2i-1,2j-1}$ is a polynomial in P of degree $2i+2j-2$ with the leading coefficient equal to $1/(i+j-1)$.

We proceed with numerical checks of these assumptions. For $\theta_0 = 15$ we obtain already perfect agreement with the scaling behavior. The values of P should not be to large, we take $P \le 0.2$. Considering an important amount of numerical data with different P, α, ν we come with the following conjectures for the exact forms of the first several $\Omega_{2i-1,2j-1}$:

$$\Omega_{1,1} = P^2 - \frac{1}{16} - \frac{1}{8}\Delta_\alpha\,. \tag{4.148}$$

$$2\cdot\Omega_{\substack{1,3\\3,1}} = P^4 - P^2\frac{5}{48}(2\Delta_\alpha+3) + \frac{\hat{c}+8}{1536}(4\Delta_\alpha+3) + \frac{1}{128}\Delta_\alpha^2 \mp \frac{1}{96}d_\alpha\Delta_\alpha\,. \tag{4.149}$$

$$3\cdot\Omega_{3,3} = P^6 - P^4\frac{1}{64}(18\Delta_\alpha+47) + P^2\left(\frac{27}{1280}\Delta_\alpha^2 + \frac{23\hat{c}+378}{3840}\Delta_\alpha + \frac{46\hat{c}+881}{5120}\right) \tag{4.150}$$
$$- \frac{1}{2048}\Delta_\alpha^3 - \frac{40\hat{c}+21}{61440}\Delta_\alpha^2 - \frac{5\hat{c}^2+52\hat{c}+222}{81920}(2\Delta_\alpha+3)\,.$$

$$3\cdot\Omega_{\substack{1,5\\5,1}} = P^6 - P^4\left(\frac{35}{48}+\frac{7}{24}\Delta_\alpha\right) + P^2\left(\frac{89}{3840}\Delta_\alpha^2 + \frac{23\hat{c}+514}{15360}(4\Delta_\alpha+5)\right) \tag{4.151}$$
$$- \frac{1}{2048}\Delta_\alpha^3 - \frac{10\hat{c}+479}{61440}\Delta_\alpha^2 - \frac{6\hat{c}^2+83\hat{c}+386}{245760}(6\Delta+5)$$
$$\mp d_\alpha\Delta_\alpha\left(P^2\frac{23}{960} - \frac{1}{512}\Delta_\alpha - \frac{83+12\hat{c}}{15360}\right)\,.$$

where

$$d_\alpha = \frac{1}{4}(\beta^{-2} - \beta^2)(\alpha - 1)\,. \tag{4.152}$$

Below we give some examples of comparison between numerical results and the analytical conjectures above.

Coefficient $\Omega_{1,1}$ and $\beta^2 = \frac{1}{2}$

	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$	
P	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.
0.02	-0.059287494	-0.0592875	-0.057099995	-0.0571	-0.055537495	-0.0555375
0.04	-0.058087495	-0.0580875	-0.055899995	-0.0559	-0.054337495	-0.0543375
0.06	-0.056087495	-0.0560875	-0.053899995	-0.0539	-0.052337495	-0.0523375
0.08	-0.053287495	-0.0532875	-0.051099995	-0.0511	-0.049537496	-0.0495375
0.1	-0.049687495	-0.0496875	-0.047499996	-0.0475	-0.045937496	-0.0459375
0.12	-0.045287496	-0.0452875	-0.043099996	-0.0431	-0.041537497	-0.0415375
0.14	-0.04008745	-0.0400875	-0.037899997	-0.0379	-0.03633745	-0.0363375
0.16	-0.034087497	-0.0340875	-0.031899998	-0.0319	-0.030337498	-0.0303375
0.18	-0.027287498	-0.0272875	-0.025099998	-0.0251	-0.023537499	-0.0235375
0.2	-0.019687499	-0.0196875	-0.017499999	-0.0175	-0.015937400	-0.0159375

Coefficient $\Omega_{1,3}$ and $\beta^2 = \frac{3}{5}$

	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$	
P	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.
0.02	0.01612247	0.01612249	0.01591101	0.01591102	0.01577159	0.01577160
0.04	0.01575287	0.01575289	0.01554374	0.01554376	0.01540599	0.01540600
0.06	0.01514328	0.01514329	0.01493803	0.01493804	0.01480306	0.01480307
0.08	0.01430328	0.01430329	0.0141035	0.01410349	0.01397239	0.01397240
0.1	0.01324632	0.01324633	0.01305352	0.01305353	0.01292743	0.01292744
0.12	0.01198969	0.01198969	0.01180544	0.01180545	0.01168546	0.01168547
0.14	0.01055449	0.01055449	0.01038035	0.01038036	0.01026760	0.01026760
0.16	0.008965691	0.008965693	0.008803224	0.008803226	0.008698802	0.008698804
0.18	0.007252093	0.007252093	0.007102848	0.007102848	0.007007872	0.007007871
0.2	0.005446336	0.005446333	0.005311868	0.005311867	0.005227447	0.005227444

Coefficient $\Omega_{3,3}$ and $\beta^2 = \frac{1}{2}$

	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$	
P	$\Omega_{3,3}$ comp.	$\Omega_{3,3}$ analyt.	$\Omega_{3,3}$ comp.	$\Omega_{3,3}$ analyt.	$\Omega_{3,3}$ comp.	$\Omega_{3,3}$ analyt.
0.02	-0.0079402716	-0.0079402720	-0.0078464501	-0.0078464506	-0.0077795388	-0.0077795392
0.04	-0.0077381755	-0.0077381759	-0.0076463818	-0.0076463822	-0.0075809093	-0.0075809097
0.06	-0.0074059724	-0.0074059727	-0.0073175266	-0.0073175270	-0.0072544297	-0.0072544302
0.08	-0.0069505170	-0.0069505174	-0.0068666923	-0.0068666926	-0.0068068739	-0.0068068743
0.1	-0.0063812449	-0.0063812453	-0.0063032481	-0.0063032484	-0.006247564	-0.0062475644
0.12	-0.0057100111	-0.0057100113	-0.0056389640	-0.0056389643	-0.0055882093	-0.0055882096
0.14	-0.0049508823	-0.0049508825	-0.0048878029	-0.0048878031	-0.0048426982	-0.0048426985
0.16	-0.0041198841	-0.0041198842	-0.0040656674	-0.0040656675	-0.0040268458	-0.0040268459
0.18	-0.0032347010	-0.0032347011	-0.0031901003	-0.0031901004	-0.0031580934	-0.0031580936
0.2	-0.0023143311	-0.0023143311	-0.0022799390	-0.0022799391	-0.0022551639	-0.002255164

The scaling limit of (4.91) is supposed to give the ratio

$$\frac{\langle \mathcal{O}_\alpha(0)\rangle_{P,R}}{\langle V_\alpha(0)\rangle_{P,R}}\,,$$

for some operator $\mathcal{O}_\alpha$. In the case under consideration this operator is supposed to be a chiral descendant of V_α (recall that we do not distinguish between the CFT operators and their perturbed counterparts). To be more precise $\Omega_{2i-1,2j-1}(\theta_0)$ should be related to a descendant on the level $2i+2j-2$. The determinants made of $\Omega_{2i-1,2j-1}(\theta_0)$ correspond to other descendants but we shall not discuss them here restricting ourselves to the simplest cases.

All together we must have

$$\lim_{\theta_0\to\infty} e^{-2(i+i-1)\theta_0}\Omega_{2i-1,2j-1}(\theta_0) = \frac{\langle \mathcal{P}_{2i-1,2j-1}(\{\mathbf{s}_k,\mathbf{l}_m\})V_\alpha\rangle_P}{\langle V_\alpha(0)\rangle_P}\,, \tag{4.153}$$

where $\mathcal{P}_{2i-1,2j-1}(\{\mathbf{s}_k,\mathbf{l}_m\})V_\alpha$ is an element of the Verma module generated by V_α quotiented by the action of local integrals of motion, this will be discussed in Section 4.7.

The expressions like the one in the right hand side of (4.153) can be computed for any $\mathcal{P}_{2i-1,2j-1}$, some examples have been given in the previous section 4.1.2. However, trying to find $\mathcal{P}_{2i-1,2j-1}$ from (4.153) we encounter more problems than in the usual Virasoro case [4]. The point is that the universal enveloping algebra of the super conformal algebra contains much more elements than that of the Virasoro algebra. The coefficients of the polynomials $\mathcal{P}_{2i-1,2j-1}$ do not depend on P, and actually the appearance of different degrees of P is the source (the only one) of different equations. When the level grows the number of coefficients of $\mathcal{P}_{2i-1,2j-1}$ grows much faster than the degree of the left hand side in P. For the Virasoro case we still could define the coefficients up to the level 6, and for levels 2 and 4 the systems of equations were even overdetermined, the fact that they allowed solutions was considered as an important check of our procedure. In the super conformal case the only possibility to find the coefficients occurs on the level 2: we have two descendants created by $\mathbf{l}_{-2}$ and $\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}$ and two coefficients of the polynomial in P in the left hand side. Starting from the level 4 we do not have enough equations.

One way out of this difficulty would be to allow descendants in the asymptotic states like it was done in [87] for the level 8 in the Virasoro case. This would be too hard, and not necessary: we have another, similar to that of [31], way of fixing the polynomials $\mathcal{P}_{2i-1,2j-1}$ based on the reflection relations [47, 48, 50]. We shall explain this in the next section. When the polynomials $\mathcal{P}_{2i-1,2j-1}$ are defined from the reflection relations, the formulae (4.153), (4.148), (4.149), (4.151), (4.150) can be used for checks. Since both our equation for $\Omega(\theta,\theta')$ and the reflection relations have the status of conjectures the fact that the results of their application are in agreement provides a very solid support for both.

4.6.3 Primary fields

Let us now consider the asymptotics $\theta \to -\infty$, $\theta' \to \infty$. We have

$$\Omega(\theta,\theta') \simeq \sum_{i,j=1}^{\infty} e^{(2i-1)\theta} e^{-(2j-1)\theta'} \Omega_{-(2i-1),2j-1}(\theta_0)\,.$$

We suspect that similarly to [5] the $\Omega_{-1,1}(\theta_0)$ is related to the ratio of the expectation values of two shifted primary fields. The question is: which primary fields exactly? Now we have two of them: V_α, W_α. Solving numerically our equations we find that for fixed β, α, P

$$\log \Omega_{-1,1}(\theta_0) \simeq 2\theta_0 \Big(\Delta_{\alpha+\frac{2\beta^2}{1-\beta^2}} + 1/2 - \Delta_\alpha \Big)\,.$$

Let us give an example. Consider the normalized expression:

$$R(\theta_0) = \exp\Big\{ -2\theta_0 \Big(\Delta_{\alpha+\frac{2\beta^2}{1-\beta^2}} + 1/2 - \Delta_\alpha \Big) \Big\} \Omega_{-1,1}(\theta_0)\,.$$

For $\alpha = 1/2, \beta^2 = 1/2, P = 0.1$ we have

θ_0	12	13	14	15	16
$R(\theta_0)$	0.16825979	0.16825580	0.16825433	0.16825379	0.16825359

So, we see that the scaling is achieved with great precision.

This suggests that $\Omega_{-1,1}(\theta_0)$ is proportional to the ratio of the expectation values of $W_{\alpha+\frac{2\beta^2}{1-\beta^2}}$ and V_α. Let us check the limiting value against the CFT. First, we have to normalize the primary fields

$$\widehat{V}_\alpha = \frac{1}{F(\alpha)} V_\alpha\,, \quad \widehat{W}_\alpha = \frac{1}{F(\alpha)} W_\alpha\,,$$

where $F(\alpha)$ is the one point function of the operator V_α on the plane (for $R = \infty$) [88]. For the operator W_α the one point function on the plane vanishes since this operator is a super Poincaré descendant of V_α and the vacuum is super Poincaré invariant. Nevertheless we normalize W_α by the same function $F(\alpha)$. The reason for that is in the reflection relations as explained in the next section. Denote by $c(\alpha,P)$ ($\tilde{c}(\alpha,P)$) the CFT one point functions of the normalized operator $\widehat{V}_\alpha$ ($\widehat{W}_\alpha$) on the cylinder with our usual asymptotic conditions. In the next section we will find that

$$\begin{aligned} \frac{\tilde{c}(\alpha+\frac{2\beta^2}{1-\beta^2},P)}{c(\alpha,P)} &= \frac{\pi^2}{1-\beta^2} \beta^{\frac{1}{2}(\alpha\beta^2-2\beta^2-\alpha)} \frac{\gamma(\frac{1}{2}(1-\beta^2)(2-\alpha))}{\gamma(\frac{1}{4}(1-\beta^2)(2-\alpha))^2} \\ &\times \gamma(\tfrac{1}{2}(1+\beta^2)+(1-\beta^2)\alpha-\beta P)\gamma(\tfrac{1}{2}(1+\beta^2)+(1-\beta^2)\alpha+\beta P)\,. \end{aligned} \tag{4.154}$$

Consider the ratio

$$R_1(\theta_0) = R(\theta_0)\frac{c(\alpha,P)}{\tilde{c}(\alpha+\frac{2\beta^2}{1-\beta^2},P)}\,. \tag{4.155}$$

For $\theta_0 = 15$ and the choice of ν, α, P presented below, we have

data	$\beta^2=\frac{1}{2},\alpha=\frac{2}{5},P=0.2$	$\beta^2=\frac{3}{5},\alpha=\frac{2}{3},P=0.1$	$\beta^2=\frac{1}{3},\alpha=\frac{1}{2},P=0.15$
$R_1(15)$	1.00000211	1.00009870	0.99999998

The agreement is very good.

4.7 Reflection relations and three-point functions in Super CFT

Long ago Al. Zamolodchikov did a remarkable observation that the one point functions for sine-Gordon and sinh-Gordon model are related by analytical continuation. This is very different from other properties of these models, for example the particle content is quite different. Nevertheless the Al. Zamolodchikov's observation proved to be correct in many other models. Here we shall apply it to the ssG model relating it to the Super sinh-Gordon theory (sshG) with the action

$$\mathcal{A}_{\rm sshG} = \int\Big[\Big(\frac{1}{4\pi}\partial_z\varphi\partial_{\bar z}\varphi + \frac{1}{2\pi}\left(\psi\partial_{\bar z}\psi + \bar\psi\partial_z\bar\psi\right) - \mu\bar\psi\psi e^{\frac{b}{\sqrt{2}}\varphi}\Big) - \mu\bar\psi\psi e^{-\frac{b}{\sqrt{2}}\varphi}\Big]d^2z\,. \tag{4.156}$$

We shall use the habitual notation

$$Q = b + b^{-1}\,.$$

The analytical continuation to the ssG case corresponds to

$$\beta = ib\,, \qquad \alpha = \frac{2a}{Q}\,. \tag{4.157}$$

Slightly abusing the notation we will write the primary fields defined (4.84) as V_a and W_a. The idea behind the reflection relations is that the physical quantities must be invariant under the two reflections:

$$\sigma_1 : a \to -a\,, \qquad \sigma_2 : a \to Q - a\,. \tag{4.158}$$

The first of them reflects simply the C-reflection of the action (4.156) while the second one is inherited from the symmetry of the Super Liouville model. The reflection relations can be applied to the calculation of one point functions. For the primary fields it is rather direct, since their one point functions are invariant under σ_1 and their transformation rule under σ_2 is inherited from a remarkable property of the (Super) Liouville three point function. This will be explained in more details in Section 4.7.1. The situation is more complicated for descendants fields : a Virasoro descendants has a manifest σ_2 symmetry, but its behavior for σ_1 is unclear. This explains the necessity to construct a passage matrix $U(a)$ that relates the Virasoro and Heisenberg descendants in order to use the action of the two reflections simultaneously. Recall that $\mathcal{V}_a^{\rm quo}$ is the quotient of the Verma module by the action of the local integrals of motion. Consider $V(a) \in (\mathcal{V}_a^{\rm quo})^*$. The reflection relations [48] can be presented as the following Riemann-Hilbert problem (see [31] for more details):

$$V(a+Q) = S(a)V(a)\,, \quad S(a) = U(-a)U(a)^{-1}\,. \tag{4.159}$$

Let us apply this idea, and first explain how the reflection relations can be used to recover the results of the previous section.

4.7.1 Primary fields

Let us now explain the values of the one point functions of primary fields in infinite volume . Recall that we computed in section 4.1.3 the reflection coefficient relating

$$V_a = R(a)V_{Q-a}\,, \quad W_a = R(a)W_{Q-a}\,,$$
$$R(a) = \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2}\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2}-1)\,.$$

The one point function $F(a)$ of V_a in infinite volume for the Super sinh-Gordon model has the expected transformation properties under both reflections (4.158)

$$F(a) = F(a-Q)R(a)\,.$$

The operators

$$\widehat{V}_a = \frac{1}{F(a)}V_a\,, \qquad \widehat{W}_a = \frac{1}{F(a)}W_a\,,$$

are invariant under both reflections. For our goals we do not need $F(a)$ but rather the ratio $f(a) = \frac{F(a-b)}{F(a)}$ for which

$$f(a-Q) = f(a)\frac{R(a)}{R(a-b)}\,.$$

We compute and rewrite the result in a useful for us way

$$\frac{R(a)}{R(a-b)} = \left(\tfrac{1}{2}\pi\mu\gamma(\tfrac{1}{2}bQ)\right)^{-2}\frac{\gamma\left(\frac{1}{2}+\frac{1}{2}b(2a-b)\right)}{\gamma\left(\frac{1}{2}+\frac{1}{2}b(2(a-Q)-b)\right)}\,.$$

This equality implies

$$f(a) = C(b)\left(\tfrac{1}{2}\pi\mu\gamma(\tfrac{1}{2}bQ)\right)^{\frac{2}{bQ}(\Delta_{a-b}+\frac{1}{2}-\Delta_a)}\gamma\left(\tfrac{1}{2}b(Q-2a)\right)\,, \tag{4.160}$$

where $C(b)$ is a constant depending on b only. To finish the consideration of primary fields let us give the expression for the ratio

$$\begin{aligned}\frac{\widetilde{C}(a-b,Q/2+k,Q/2-k)}{C(a,Q/2+k,Q/2-k)} &= \left(\tfrac{1}{2}\pi\mu\gamma(\tfrac{1}{2}bQ)b^{-1}\right)\frac{\gamma^2(\frac{1}{2}(1+ab-b^2))\gamma(\frac{1}{2}b(Q-2a))}{\gamma(ab-b^2)} \\ &\times\gamma(\tfrac{1}{2}(1-b^2+ab)+bk)\gamma(\tfrac{1}{2}(1-b^2+ab)-bk)\,.\end{aligned} \tag{4.161}$$

Divide (4.161) by $f(a)$ (4.160) (there is an important cancellation) and change the variables by (4.157) and

$$bk = \beta P\,,$$

after some simplification this gives (4.154).

4.7.2 Super Virasoro and Super Heisenberg algebras

We would like to have an independent check of the results (4.148) - (4.151). In order to do so, we should interpret the expressions obtained for $\boldsymbol{\beta}^*_{2m-1}\boldsymbol{\gamma}^*_{2m-1}V_a$ as decompositions of the fermionic operators on the Super Virasoro basis, and check that this decomposition is compatible with the reflection relations. As has been explained above and is clear from the interpretation of the reflections, it is first important to make the connection between the Super-Virasoro and the Super-Heisenberg algebras, that is to construct the passage matrix $U(a)$. This is our goal in this subsection.

The expression of the stress energy tensor and the super current in terms of the fields in the action (4.156) are given by :

$$\begin{aligned}T(z) &= -\frac{1}{4}(\partial_z\varphi)^2 + \frac{Q}{2\sqrt{2}}\partial^2\varphi - \frac{1}{2}\psi\partial\psi\,,\\ S(z) &= i\left(\frac{1}{\sqrt{2}}\psi\partial\varphi - Q\partial\psi\right)\,.\end{aligned}$$

In order to exhibit the Heisenberg basis, we split the field $\varphi(z,\bar{z}) = \phi(z)+\phi(\bar{z})$ in chiral parts and expand in modes :

$$\phi(z) = \phi_0 - 2i\pi_0 + i\sum_{k\in\mathbb{Z}^*}\frac{a_k}{k}z^{-k}\,,$$

where the Heisenberg algebra is :

$$[a_k,a_l] = 2k\delta_{k,-l}\,, \qquad [\phi_0,\pi_0] = i\,. \tag{4.162}$$

The same analysis holds for the fermionic field

$$\psi(z) = \sum_{r\in\mathbb{Z}}b_{r+\frac{1}{2}}z^{-r-1}\,,$$

with the fermionic algebra defined by :

$$\{b_r, b_s\} = \delta_{r,-s}\,. \tag{4.163}$$

We will call the combination of (4.162) and (4.163) the Super Heisenberg algebra (together with the commutation relation $[a_k, b_r] = 0$). The primary field $e^{\frac{a}{\sqrt{2}}\phi(0)}$ is identified with the highest weight vector of the Super Heisenberg algebra :

$$e^{\frac{a}{\sqrt{2}}\phi(0)} \Longleftrightarrow e^{\frac{a}{\sqrt{2}}\phi_0}\,|0\rangle\,, \quad a_k\,|0\rangle = b_r\,|0\rangle = 0\,, \quad k, r > 0\,.$$

In the general case we should then take :

$$V_a = e^{\frac{a}{\sqrt{2}}(\phi_0 + \overline{\phi_0})}\,|0\rangle \otimes \overline{|0\rangle}\,. \tag{4.164}$$

The calculation for the two chiralities being independent, we will work only with the holomorphic one. We can now introduce the generators of the Super Virasoro algebra :

$$\begin{aligned}\mathbf{l}_m =& \frac{1}{4}\sum_{k\neq 0,m} :a_k a_{m-k}: + (\pi_0^2 + i\pi_0\frac{Q}{\sqrt{2}})\delta_{m,0} + \\ & (\pi_0 + i\frac{Q}{2\sqrt{2}}(m+1))a_m(1-\delta_{m,0}) + \frac{1}{2}\sum_{k\in\widetilde{\mathbb{Z}}} :b_{m-k}b_k: (k+\frac{1}{2})\,,\end{aligned}$$

and the modes of the super current :

$$\mathbf{s}_r = \frac{1}{\sqrt{2}}\sum_{k\in\widetilde{\mathbb{Z}}'} b_k a_{r-k} + \left(\sqrt{2}\pi_0 + iQ(r+\frac{1}{2})\right) b_r\,.$$

Here the symbol : ... : means normal order. These generators satisfy the Super Virasoro algebra

$$\begin{aligned}[\mathbf{l}_m, \mathbf{l}_n] &= (m-n)\mathbf{l}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \\ \{\mathbf{s}_r, \mathbf{s}_s\} &= 2\mathbf{l}_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r,-s},\end{aligned}$$

with $c = \frac{3}{2}(1+2Q^2)$, and since S is a primary field of conformal dimension $\Delta = \frac{3}{2}$ we also have the relation :

$$[\mathbf{l}_m, \mathbf{s}_r] = \left(\frac{m}{2} - r\right)\mathbf{s}_{m+r}\,.$$

Finally, the natural identity holds :

$$\mathbf{l}_0 V_a = \Delta_a V_a\,, \qquad \Delta_a = \frac{1}{2}a(Q-a)\,.$$

We are now ready to compute the passage matrix between the Super Virasoro and the Super Heisenberg bases. Recall that we work modulo the action of local integrals of motion. For our calculations (up to level 6), the integrals of motion that will be involved are just the first two given by the densities (4.87). Explicitly :

$$\mathbf{i}_1 = \mathbf{l}_{-1}\,, \tag{4.165}$$

$$\mathbf{i}_3 = 2\sum_{k=-1}^{\infty}\mathbf{l}_{-3-k}\mathbf{l}_k + \frac{1}{2}\sum_{k=-\frac{1}{2}}^{\infty}\mathbf{s}_{-3-k}\mathbf{s}_k\left(k+\frac{3}{2}\right)\,. \tag{4.166}$$

Level 2.

At level 2 there is only one integral of motion to take into account :

$$\mathbf{i}_1^2 V_a = \mathbf{l}_{-1}^2 V_a = 0\,.$$

We define $U^{(2)}$ to be the passage matrix between the base $\{\mathbf{l}_{-2}, \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\}$ and $\{a_{-1}^2, b_{-\frac{3}{2}}b_{-\frac{1}{2}}\}$ which is found to be :

$$U^{(2)} = \begin{pmatrix} \frac{1}{4}\left(2a^2+Qa+1\right) & \frac{1}{2} \\ \frac{a^2}{2} & -a(a+Q)\end{pmatrix}\,. \tag{4.167}$$

Its determinant factorizes and gives as expected the null-vector conditions :

$$\det(U^{(2)}) = -\frac{1}{4}a\left(2a+b+b^{-1}\right)(a+b)(a+b^{-1})\,.$$

Level 4.

At this level there are 10 operators in total, but working modulo integrals of motion (in this case also only $\mathbf{i}_1$) we need to keep only 5 of them, that we choose to be

$$\mathbf{l}_{-2}^2\,,\qquad \mathbf{l}_{-4}\,,\qquad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}\,,\qquad \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}\,,\qquad \mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\,.$$

On the other hand, we select the following operators to describe the states at level 4 from the Super Heisenberg algebra point of view :

$$a_{-2}^2\,,\quad a_{-3}a_{-1}\,,\qquad b_{-\frac{7}{2}}b_{-\frac{1}{2}}\,,\quad b_{-\frac{5}{2}}b_{-\frac{3}{2}}\,,\quad a_{-1}^2b_{-\frac{3}{2}}b_{-\frac{1}{2}}\,.$$

We find for the matrix $U^{(4)}$:

$$U^{(4)}=\begin{pmatrix} U_{11}^{(4)} & U_{12}^{(4)} & \frac{29a+12Q}{4a} & \frac{5a+4Q}{4a} & \frac{1}{4}\left(2a^2+Qa+1\right)\\ \frac{1}{4} & \frac{1}{12}\left(2a^2+3Qa+6\right) & \frac{3}{2} & \frac{1}{2} & 0\\ 0 & \frac{a^2}{6} & -a^2-3Qa+3 & 1 & \frac{a^2}{2}\\ 0 & \frac{1}{6}\left(a^2+Qa+3\right) & -\frac{4(2a+3Q)}{a} & U_{44}^{(4)} & \frac{1}{2}\left(-a^2-Qa+1\right)\\ \frac{1}{4}\left(-2a^2-Qa-1\right) & \frac{1}{6}\left(2a^2-3\right) & U_{53}^{(4)} & U_{54}^{(4)} & U_{55}^{(4)} \end{pmatrix}, \tag{4.168}$$

where the lengthiest coefficients are :

$$\begin{aligned} U_{11}^{(4)}&=-\frac{4a^4+4Qa^3+Q^2a^2+4a^2+2Qa+3}{8a^2}\,,\quad U_{12}^{(4)}=\frac{4a^4+2Qa^3-6a^2-6Qa-9}{12a^2}\,,\\ U_{44}^{(4)}&=-\frac{a^3+3Qa^2+2Q^2a+3a+4Q}{a}\,,\quad U_{53}^{(4)}=\frac{1}{2}\left(-17a^2-23Qa-6Q^2+6\right)\,,\\ U_{54}^{(4)}&=\frac{1}{2}\left(-a^2-3Qa-2Q^2+2\right)\,,\quad U_{55}^{(4)}=-\frac{1}{4}a\left(2a^3+3Qa^2+Q^2a+Q\right)\,. \end{aligned}$$

Its determinant can be factorized :

$$\det(U^{(4)})=\frac{1}{384}\frac{D_V^{(4)}(\Delta,c)}{D_H^{(4)}(a^2,Q^2)}N^{(4)}(a,b)\,.$$

The contribution from the null vectors is :

$$\begin{aligned} N^{(4)}(a,b)=a^4(a+b)^2(a+b^{-1})^2(a+2b)(a+2b^{-1})(a+3b)(a+3b^{-1})\\ \times(2a+b+b^{-1})(2a+b+3b^{-1})(2a+3b+b^{-1})\,, \end{aligned} \tag{4.169}$$

and we have :

$$D_V^{(4)}(\Delta,c)=1\,,\qquad D_H^{(4)}(a^2,Q^2)=a^2\,. \tag{4.170}$$

Level 6.

We proceed through the same analysis. At level 6 we will need to factor out the action of both $\mathbf{i}_1$ and $\mathbf{i}_3$. There are 28 Virasoro operators at level 6, but the factorization of the action of the integrals of motion leaves only 10, that we choose to be :

$$\begin{aligned} &\mathbf{l}_{-2}^3\,,\quad \mathbf{l}_{-6}\,,\quad \mathbf{l}_{-3}^2\,,\quad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}}\,,\quad \mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}}\,,\quad \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}}\,,\\ &\mathbf{l}_{-2}^2\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\,,\quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}\,,\quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}\,,\quad \mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}\,. \end{aligned}$$

These are expressed on the Super Heisenberg basis :

$$\begin{aligned} &a_{-1}^6\,,\quad a_{-1}^4a_{-2}\,,\quad a_{-3}^2\,,\quad b_{-\frac{7}{2}}b_{-\frac{5}{2}}\,,\quad b_{-\frac{9}{2}}b_{-\frac{3}{2}}\,,\quad b_{-\frac{11}{2}}b_{-\frac{1}{2}}\,,\\ &a_{-1}a_{-2}b_{-\frac{5}{2}}b_{-\frac{1}{2}}\,,\quad a_{-1}^2b_{-\frac{7}{2}}b_{-\frac{1}{2}}\,,\quad a_{-1}a_{-3}b_{-\frac{3}{2}}b_{-\frac{1}{2}}\,,\quad a_{-1}^2b_{-\frac{5}{2}}b_{-\frac{1}{2}}\,. \end{aligned}$$

The passage matrix $U^{(6)}$ is unfortunately too large to be presented here, but we can give its determinant :

$$\det(U^{(6)})=-\frac{1}{212336640}N^{(6)}(a,b)\frac{D_V^{(6)}(\Delta,c)}{D_H^{(6)}(a^2,Q^2)}\,. \tag{4.171}$$

with :

$$\begin{aligned} N^{(6)}(a,b) &= a^2(a+b)^5(a+b^{-1})^5(a+2b)^2(a+3b)^2(a+2b^{-1})^2(a+3b^{-1})^2 \\ &\times (a+4b)(a+5b)(a+4b^{-1})(a+5b^{-1}) \\ &\times \left(a+b+b^{-1}\right)\left(2a+b+b^{-1}\right)^5\left(2a+b+3b^{-1}\right)^2\left(2a+3b+b^{-1}\right)^2 \\ &\times \left(2a+5b+b^{-1}\right)\left(2a+b+5b^{-1}\right)\,, \end{aligned} \tag{4.172}$$

the null vector contribution, and

$$D_H^{(6)}(a^2,Q^2) = a^2(-15+3a^2-10Q^2)\,, \qquad D_V^{(6)}(\Delta,c) = 1\,. \tag{4.173}$$

4.7.3 Reflections relations

We claim that similarly to [31], the action of both reflections σ_1 and σ_2 implies that the fermions transform as :

$$\boldsymbol{\beta}^*_{2j-1} \to \boldsymbol{\gamma}^*_{2j-1}\,, \quad \boldsymbol{\gamma}^*_{2j-1} \to \boldsymbol{\beta}^*_{2j-1}\,. \tag{4.174}$$

This means that we can use the coefficients (4.147) to redefine the elements of the fermionic basis and obtain purely CFT objects :

$$\boldsymbol{\beta}^*_{2m-1} = D_{2m-1}(a)\boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1}\,, \qquad \boldsymbol{\gamma}^*_{2m-1} = D_{2m-1}(Q-a)\boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1}\,. \tag{4.175}$$

For $\boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1}$ and $\boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1}$ we have clear transformation rules under $\sigma_{1,2}$. As in the non-supersymmetric case for σ_2

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1} \to \boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1}\,, \qquad \boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1} \to \boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1}\,. \tag{4.176}$$

For σ_1 we must consider an additional term coming from the change in the passage from $D_{2m-1}(a)$ to $D_{2m-1}(Q-a)$:

$$D_{2m-1}(Q-a) = D_{2m-1}(-a)\left(\frac{a-(2m-1)b^{-1}}{a+(2m-1)b}\right)\,, \tag{4.177}$$

which implies

$$\begin{aligned} \boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1} &\to \left(\frac{a-(2m-1)b}{a+(2m-1)b^{-1}}\right)\boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1}\,, \\ \boldsymbol{\gamma}^{\mathrm{CFT}*}_{2m-1} &\to \left(\frac{a-(2m-1)b^{-1}}{a+(2m-1)b}\right)\boldsymbol{\beta}^{\mathrm{CFT}*}_{2m-1}\,. \end{aligned} \tag{4.178}$$

The main conclusion drawn from Section 4.6.2, is that the fermionic basis should be decomposable on the Super Virasoro basis in the following way :

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}*}_{I^-}V_a = C_{I^+,I^-}\left(P^E_{I^+,I^-}(\{\mathbf{l}_{-k},\mathbf{s}_{-r}\},\Delta_a,c) + d_a P^O_{I^+,I^-}(\{\mathbf{l}_{-k},\mathbf{s}_{-r}\},\Delta_a,c)\right)V_a\,, \tag{4.179}$$

where C_{I^+,I^-} is the Cauchy determinant and d_a is the function (4.152) rewritten in the variables a,b :

$$d_a = \frac{1}{8}\sqrt{(9-\widehat{c})\,(16\Delta_\alpha+1-\widehat{c})} = \frac{1}{4}(b-b^{-1})(Q-2a)\,. \tag{4.180}$$

The functions $P^E_{I^+,I^-}$ and $P^O_{I^+,I^-}$ (E,O superscripts stand respectively for even and odd) are polynomials in the modes of the Super Virasoro algebra, depending rationally on the parameters Δ_a, c. They are defined modulo the local integrals $\mathbf{i}_{2k-1}$ and satisfy the symmetry relations :

$$P^E_{I^+,I^-} = P^E_{I^-,I^+}\,, \qquad P^O_{I^+,I^-} = -P^O_{I^-,I^+}\,. \tag{4.181}$$

The decomposition (4.179), as well as the transformation rules (4.176) and (4.178), imply a relation of the type

$$\begin{aligned} \boldsymbol{\beta}^{\mathrm{CFT}*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}*}_{I^-}V_a &= C_{I^+,I^-}\prod_{2j-1\in I^+}(a+(2j-1)b^{-1})\prod_{2j-1\in I^-}(a+(2j-1)b) \\ &\times\left(Q^E_{I^+,I^-}(\{a_{-k},b_{-r}\},b_r\},a^2,Q^2) + g_a Q^O_{I^+,I^-}(\{a_{-k},b_{-r}\},a^2,Q^2)\right)V_a, \end{aligned} \tag{4.182}$$

with

$$g_a = a(b-b^{-1})\,,$$

and $Q^E_{I^+,I^-}$, $Q^O_{I^+,I^-}$ polynomials in the Super Heisenberg algebra, depending rationally on a^2 and Q^2. In the following we are going to verify this conjecture level by level.

Level 2

Let us start with the simplest case of level 2 :

$$\langle \boldsymbol{\beta}_1^{\mathrm{CFT}*}\boldsymbol{\gamma}_1^{\mathrm{CFT}*}V_a\rangle = \Omega_{1,1} = P^2 - \frac{1}{16} - \frac{\Delta_a}{8}\,. \tag{4.183}$$

On this level only two operators $\mathbf{l}_{-2}$ and $\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}$ are present. The calculation of one point functions on the cylinder was explained in Subsection 4.1.2 and gave in this case (4.59):

$$\langle \mathbf{l}_{-2}V_a\rangle = \delta_P - \frac{c}{24} - \frac{\Delta_a}{12}\,, \qquad \langle \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\rangle = \frac{\Delta_a}{12}\,. \tag{4.184}$$

Hence it is not difficult to compare with (4.183) and obtain :

$$\boldsymbol{\beta}_1^{\mathrm{CFT}*}\boldsymbol{\gamma}_1^{\mathrm{CFT}*}V_a = \left(\mathbf{l}_{-2} - \frac{1}{2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\right)V_a\,. \tag{4.185}$$

Using (4.167), one can rewrite the combination (4.185) as :

$$\boldsymbol{\beta}_1^{\mathrm{CFT}*}\boldsymbol{\gamma}_1^{\mathrm{CFT}*}V_a = \frac{1}{4}(a+b)(a+b^{-1})\left(a_{-1}^2 + 2b_{-\frac{3}{2}}b_{-\frac{1}{2}}\right)V_a\,. \tag{4.186}$$

This neat factorization of the term $(a+b)(a+b^{-1})$ is a check of our conjecture, and the above shows that : $Q^E_{\{1,1\}} = \frac{1}{4}\left((a_{-1})^2 + 2b_{-\frac{3}{2}}b_{-\frac{1}{2}}\right)$.

The main difference with the usual Liouville case, is that at higher levels, we do not know a priori the decompositions of the type (4.179) (recall the discussion at the end of the Section 4.6.2). To overcome this difficulty, we shall proceed as in [31] and obtain the decomposition by solving the reflection constraints implied by (4.182). Let us briefly recall the main steps.

Consider that at an (even) level $k = |I^+| + |I^-|$ we have a basis of Super Virasoro generators $\{\mathbf{v}_1^{(k)}, ..., \mathbf{v}_d^{(k)}\}$ (by convention we consider that $\mathbf{v}_1^{(k)} = \mathbf{l}_{-2}^{\frac{k}{2}}$) that are related to the Super Heisenberg basis $\{\mathbf{h}_1^{(k)}, ..., \mathbf{h}_d^{(k)}\}$ modulo the action of integrals of motions by :

$$\mathbf{v}_i^{(k)} = \sum_{k=1}^{d} U_{i,j}^{(k)}(a)\mathbf{h}_j^{(k)}\,,$$

with $U^{(k)}(a)$ the passage matrix, whose determinant can be factorized :

$$\det(U^{(k)}(a)) = C^{(k)}N^{(k)}(a,b)\frac{D_V^{(k)}(\Delta_a, c)}{D_H^{(k)}(a^2, Q^2)}\,, \tag{4.187}$$

where $N^{(k)}(a,b)$ is the null vector contribution. We look for $P^E_{I^+,I^-}$, $P^O_{I^+,I^-}$ in the form :

$$P^E_{I^+,I^-} = \mathbf{v}_1 + \frac{1}{D_V^{(k)}(\Delta, c)}\sum_{i=2}^{d} X_{I^+,I^-,i}(\Delta_a, c)\mathbf{v}_i\,,$$
$$P^O_{I^+,I^-} = \frac{1}{D_V^{(k)}(\Delta, c)}\sum_{i=2}^{d} Y_{I^+,I^-,i}(\Delta_a, c)\mathbf{v}_i\,,$$

where $X_{I^+,I^-,i}(\Delta_a, c)$, $Y_{I^+,I^-,i}(\Delta_a, c)$ are polynomials of some degree D to be determined. Also introduce the polynomials :

$$T^+_{I^+I^-}(a) = \frac{1}{2}\left(\prod_{j\in I^+}(a+jb^{-1})\prod_{j\in I^-}(a+jb) + \prod_{j\in I^+}(a+jb)\prod_{j\in I^-}(a+jb^{-1})\right)\,, \tag{4.188}$$

$$T^-_{I^+I^-}(a) = \frac{1}{2(b-b^{-1})}\left(\prod_{j\in I^+}(a+jb^{-1})\prod_{j\in I^-}(a+jb) - \prod_{j\in I^+}(a+jb)\prod_{j\in I^-}(a+jb^{-1})\right)\,. \tag{4.189}$$

Then (4.182) gives strong conditions on the structure of $X_{I^+,I^-,i}(\Delta_a,c)$, $Y_{I^+,I^-,i}(\Delta_a,c)$ (see [31] as well as the appendix 4.11 for details). For any $1 \leq j \leq d$ we must have

$$\begin{aligned} & D_V^{(k)}(\Delta(-a),c) D_H^{(k)}(a^2,Q^2) \\ & \times \{ T^+_{I^+I^-}(-a) \left(D_V^{(k)}(\Delta_a,c) U^{(k)}_{1,j}(a) + \sum_{i=2}^{d} X_{I^+,I^-,i} U^{(k)}_{i,j}(a) \right) \\ & - (Q^2-4)(Q-2a) T^-_{I^+I^-}(-a) \sum_{i=2}^{d} Y_{I^+,I^-,i} U^{(k)}_{i,j}(a) \} \quad \text{even in } a, \end{aligned} \tag{4.190}$$

and

$$\begin{aligned} & D_V^{(k)}(\Delta(-a),c) D_H^{(k)}(a^2,Q^2) \\ & \times \{ -T^-_{I^+I^-}(-a) \left(D_V^{(k)}(\Delta_a,c) U^{(k)}_{1,j}(a) + \sum_{i=2}^{d} X_{I^+,I^-,i} U^{(k)}_{i,j}(a) \right) \\ & + (Q^2-4)(Q-2a) T^+_{I^+I^-}(-a) \sum_{i=2}^{d} Y_{I^+,I^-,i} U^{(k)}_{i,j}(a) \} \quad \text{odd in } a, \end{aligned} \tag{4.191}$$

Taking the degree D appropriately large, we obtain enough linear equations on the coefficients of $X_{I^+,I^-,i}(\Delta_a,c)$, $Y_{I^+,I^-,i}(\Delta_a,c)$. Now we demonstrate how this procedure works at higher levels.

Level 4

Consider the set up described in 4.7.2. Recall that at this level there are 5 operators in total (modulo the action of $\mathbf{i}_1$), that are :

$$\mathbf{l}^2_{-2}\,, \qquad \mathbf{l}_{-4}\,, \qquad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \qquad \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}\,, \qquad \mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\,.$$

We solve the constraints (4.190) and (4.191) with the use of (4.168) and (4.170), and obtain the following expressions :

$$\begin{aligned} P^E_{\{1,3\}} &= \mathbf{l}^2_{-2} + \left(\frac{-45+4c}{18} - \frac{\Delta_a}{3} \right) \mathbf{l}_{-4} + \left(\frac{45-4c}{36} + \frac{\Delta_a}{6} \right) \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} + \\ & \frac{1}{4}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}} - \frac{1}{2}\mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \\ P^O_{\{1,3\}} &= \frac{1}{3}\mathbf{l}_{-4} - \frac{1}{6}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \end{aligned}$$

as well as the mirror polynomials $P^E_{\{3,1\}}, P^O_{\{3,1\}}$. One can now compute the one point function of $\left(P^E_{\substack{\{1,3\}\\\{3,1\}}} \mp d_a P^O_{\substack{\{1,3\}\\\{3,1\}}} \right) V_a$, all the individual contributions of descendants at level 4 are given in (4.60). One recovers exactly the values of $\Omega_{\substack{1,3\\3,1}}$ obtained in (4.149) by interpolation. Summarizing :

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{\substack{1\\3}} \boldsymbol{\gamma}^{\mathrm{CFT}*}_{\substack{3\\1}} V_a = \frac{1}{2} \left(P^E_{\substack{\{1,3\}\\\{3,1\}}} (\{\mathbf{l}_{-k}, \mathbf{s}_{-r}\}, \Delta, c) \mp d_a P^O_{\substack{\{1,3\}\\\{3,1\}}} (\{\mathbf{l}_{-k}, \mathbf{s}_{-r}\}, \Delta, c) \right) V_a\,. \tag{4.192}$$

This is an independent argument in favor of (4.149).

Level 6

We proceed through the same analysis. Recall that we had (modulo the action of $\mathbf{i}_1$ and $\mathbf{i}_3$) 10 Virasoro operators, that we took to be

$$\begin{aligned} & \mathbf{l}^3_{-2}\,, \quad \mathbf{l}_{-6}\,, \quad \mathbf{l}^2_{-3}\,, \quad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}}\,, \quad \mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}}\,, \quad \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \\ & \mathbf{l}^2_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}\,, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}\,, \quad \mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}\,. \end{aligned}$$

Using the explicit value of $U^{(6)}$ and the factors (4.173), the reflection constraints bring the following results :

$$
\begin{aligned}
&P^E_{\{3,3\}} = \mathbf{l}^3_{-2} + \frac{1}{480}\left(572\Delta_a^2 + 1976\Delta_a - 80c^2 - 96c\Delta_a + 2076c - 18381\right)\mathbf{l}_{-6} + \\
&\frac{1}{96}\left(12\Delta_a^2 + 228\Delta_a - 16c\Delta_a - 12c - 27\right)\mathbf{l}^2_{-3} + \\
&\frac{1}{192}\left(-28\Delta_a^2 + 192\Delta_a - 16c\Delta_a - 20c + 117\right)\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}} + \\
&\frac{1}{64}\left(-4\Delta_a^2 + 92\Delta_a - 8c\Delta_a - 8c + 105\right)\mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}} + \\
&\frac{1}{960}\left(28\Delta_a^2 + 404\Delta_a + 56c\Delta_a - 136c + 6021\right)\mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}} + \\
&-\frac{1}{2}\mathbf{l}^2_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}} + \frac{1}{12}\left(4\Delta_a - 2c + 27\right)\mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} + \\
&\frac{1}{16}\left(9 - 2\Delta_a\right)\mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}} + \frac{1}{192}\left(4\Delta_a^2 - 68\Delta_a + 8c\Delta_a - 8c + 291\right)\mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&P^E_{\{1,5\}} = \mathbf{l}^3_{-2} + \frac{1}{90}\left(79\Delta_a^2 + 1004\Delta_a - 12c^2 - 98c\Delta_a + 322c - 2855\right)\mathbf{l}_{-6} + \\
&\frac{1}{12}\left(2\Delta_a^2 + 12\Delta_a - 2c\Delta_a - c - 2\right)\mathbf{l}^2_{-3} + \frac{1}{36}\left(\Delta_a^2 + 7\Delta_a - 2c\Delta_a - 3c + 18\right)\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}} + \\
&\frac{1}{8}\left(2\Delta_a - c + 11\right)\mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}} + \frac{1}{180}\left(-19\Delta_a^2 - 804\Delta_a + 2c^2 + 88c\Delta_a - 72c + 1315\right)\mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}} + \\
&-\frac{1}{2}\mathbf{l}^2_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}} + \frac{1}{9}\left(2\Delta_a - c + 14\right)\mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} + \\
&\frac{1}{2}\mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}} + \frac{1}{36}\left(-\Delta_a^2 + \Delta_a + 2c\Delta_a - c + 38\right)\mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}\,.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&P^O_{\{1,5\}} = \frac{1}{30}\left(-136\Delta_a - 12c + 335\right)\mathbf{l}_{-6} - \Delta_a\mathbf{l}^2_{-3} + \frac{1}{12}\left(-4\Delta_a - 3\right)\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}} + \\
&-\frac{3}{4}\mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}} + \frac{1}{60}\left(76\Delta_a - 8c - 115\right)\mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}} + -\frac{2}{3}\mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} + \frac{1}{12}\left(4\Delta_a - 5\right)\mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}\,.
\end{aligned}
$$

We also find the same expressions for the polynomials $P^E_{\{5,1\}}$ and $P^O_{\{5,1\}}$ (up to an overall relevant minus sign for $P^O_{\{5,1\}}$). Then one can proceed and calculate the relevant one point functions of descendants on the cylinder (see Subsection 4.1.2). We summarize here the results :

$$
\langle \mathbf{l}_{-6} V_a\rangle = -\frac{\Delta_a}{6048}\,, \qquad \langle \mathbf{l}^2_{-3} V_a\rangle = \frac{72\Delta_a + 31c - 504\delta_P}{30240}\,,
$$

$$
\begin{aligned}
\langle \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}} V_a\rangle &= \frac{604\Delta_a + 457c - 3528\delta_P}{483840}\,, \qquad \langle \mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}} V_a\rangle = \frac{-1371\Delta_a - 457c + 3528\delta_P}{1451520}\,, \\
\langle \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}} V_a\rangle &= \frac{31\Delta_a}{96768}\,, \qquad \langle \mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} V_a\rangle = \frac{294\Delta_a^2 + 1252\Delta_a + 147c\Delta_a - 3528\Delta_a\delta_P}{483840}\,, \\
\langle \mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}} V_a\rangle &= \frac{17\Delta_a}{60480}\,,
\end{aligned}
$$

as well as the most complex results :

$$\langle \mathbf{l}_{-2}^3 V_a\rangle = \frac{1}{483840}\Big(-280\Delta_a^3 - 2352\Delta_a^2 - 3968\Delta_a - 35c^3 - 210c^2\Delta_a + 2520c^2$$
$$\delta_P - 462c^2 - 420c\Delta_a^2 - 2100c\Delta_\alpha + 10080c\Delta_a\delta_P - 60480c\delta_P^2 + 21168c\delta_P - 1504c +$$
$$10080\Delta_a^2\delta_P - 120960\Delta_a\delta_P^2 + 48384\Delta_a\delta_P + 483840\delta_P^3 - 241920\delta_P^2 + 32256\delta_P\Big)$$

$$\langle \mathbf{l}_{-2}^2 \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}} V_a\rangle = \frac{\Delta_a}{241920}\Big(140\Delta_a^2 + 672\Delta_a + 35c^2 + 140c\Delta_a - 1680c\delta_P +$$
$$294c - 3360\Delta_a\delta_P + 20160\delta_P^2 - 6720\delta_P + 544\Big)\,,$$

$$\langle \mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}} V_a\rangle = \frac{1}{483840}\Big(-714\Delta_a^2 - 3588\Delta_a - 119c^2 -$$
$$595c\Delta_a + 4536c\delta_P - 1196c + 11928\Delta_a\delta_P - 40320\delta_P^2 + 18144\delta_P\Big)\,.$$

Using these values for the one point functions, we recover exactly the expressions (4.150) and (4.151). That is we check that :

$$\boldsymbol{\beta}_{\substack{1\\5}}^{\mathrm{CFT}^*}\boldsymbol{\gamma}_{\substack{5\\1}}^{\mathrm{CFT}^*} V_a = \frac{1}{3}\left(P^E_{\substack{\{1,5\}\\\{5,1\}}}(\{\mathbf{l}_{-k},\mathbf{s}_{-r}\},\Delta_a,c) \mp d_a P^O_{\substack{\{1,5\}\\\{5,1\}}}(\{\mathbf{l}_{-k},\mathbf{s}_{-r}\},\Delta_a,c)\right) V_a \tag{4.193}$$

$$\boldsymbol{\beta}_3^{\mathrm{CFT}^*}\boldsymbol{\gamma}_3^{\mathrm{CFT}^*} V_a = \frac{1}{3}P^E_{\{3,3\}}(\{\mathbf{l}_{-k},\mathbf{s}_{-r}\},\Delta_a,c)V_a\,. \tag{4.194}$$

This strongly confirms the results obtained by interpolation.

4.8 Appendix 1 : One point functions on the cylinder

Here we present two more examples of direct calculations of one point functions.

Example 1. Let us now compute the general field $\mathbf{s}_{-r}W_a$. We have by definition and using the relations (4.44),(4.45) :

$$\begin{aligned}\mathbf{s}_{-r}W_a(y) &= \oint_{c_y}\frac{dx}{2\pi i(x-y)^{r-\frac{1}{2}}}\mathcal{T}(S(x)W_a(y)) = \\ &- \oint_{c_y^-}\frac{dx}{2\pi i(x-y)^{r-\frac{1}{2}}}\Big[\{S_+(x),W_a(y)\} - W_a(y)S_+(x) + S_-(x)W_a(y)\Big] \\ &+ \oint_{c_y^+}\frac{dx}{2\pi i(x-y)^{r-\frac{1}{2}}}\Big[\{W_a(y),S_-(x)\} - S_-(x)W_a(y) + W_a(y)S_+(x)\Big]\,.\end{aligned}$$

Using the commutation relations (4.34),(4.35) we arrive at :

$$\begin{aligned}\mathbf{s}_{-r}W_a(y) &= -\oint_{c_y}\frac{dx}{2\pi i(x-y)^{r-\frac{1}{2}}}\Big[V_a'(y)\xi(x-y) - 2\Delta_a V_a(y)\xi'(x-y)\Big] \\ &- \oint_{c_y}\frac{dx}{2\pi i(x-y)^{r-\frac{1}{2}}}\Big[S_-(x)W_a(y) - W_a(y)S_+(x)\Big]\,.\end{aligned} \tag{4.195}$$

In particular for $r=\frac{1}{2}$ we recover that :

$$\mathbf{s}_{-\frac{1}{2}}W_a(y) = -V_a'(y) \implies \mathbf{s}^2_{-\frac{1}{2}}V_a(y) = V_a'(y)\,, \tag{4.196}$$

which is natural considering the relation $\{S_{-\frac{1}{2}},S_{-\frac{1}{2}}\} = 2L_{-1}$ holding in the Super Virasoro algebra.

Example 2. Let us compute the one point function of $\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}V_a$. We have already seen that :

$$\mathbf{s}_{-\frac{3}{2}}V_a = V_a(y)S_+(y) + S_-(y)V_a(y)\,, \tag{4.197}$$

and we will denote this field by $\Psi(y)$. Then we have

$$\begin{aligned}\mathbf{s}_{-\frac{5}{2}}\Psi(y) &= -\int_{c_y^-}\frac{dx}{2\pi i(x-y)^2}\Big[\{S_+(x),\Psi(y)\} - \Psi(y)S_+(x) + S_-(x)\Psi(y)\Big] \\ &+ \int_{c_y^+}\frac{dx}{2\pi i(x-y)^2}\Big[\{\Psi(y),S_-(x)\} + \Psi(y)S_+(x) - S_-(x)\Psi(y)\Big]\,.\end{aligned} \tag{4.198}$$

When applying the boundary states the non-bracketed terms will disappear. Therefore, we only need to compute :

$$\mathbf{s}_{-\frac{5}{2}}\Psi(y) = -\int_{c_y^-}\frac{dx}{2\pi i(x-y)^2}\{S_+(x),\Psi(y)\} + \int_{c_y^+}\frac{dx}{2\pi i(x-y)^2}\{\Psi(y),S_-(x)\}\,. \tag{4.199}$$

Working modulo terms that vanish when $\langle\Delta_-|$ and $|\Delta_+\rangle$ are applied, we have :

$$\begin{aligned}\{S_+(x),\Psi(y)\} &= [S_+(x),V_a(y)]S_+(y) + V_a(y)\{S_+(x),S_+(y)\} \\ &+ \{S_+(x),S_-(y)\}V_a(y) + S_-(y)[V_a(y),S_+(x)] \\ &= \{S_+(x),S_-(y)\}V_a(y) = \{S(x),S_-(y)\}V_a(y)\,.\end{aligned} \tag{4.200}$$

And similarly :

$$\begin{aligned}\{\Psi(y),S_-(x)\} &= [S_-(x),V_a(y)]S_+(y) + V_a(y)\{S_-(x),S_+(y)\} \\ &+ \{S_-(x),S_-(y)\}V_a(y) + S_-(y)[S_-(x),V_a(y)] \\ &= V_a(y)\{S_-(x),S_+(y)\} = V_a(y)\{S(x),S_+(y)\}\,.\end{aligned} \tag{4.201}$$

Using the explicit expressions for $\{S(x),S_-(y)\}$ and $\{S(x),S_+(y)\}$ (4.32) and (4.33) in the equation for $\mathbf{s}_{-\frac{5}{2}}\Psi(y)$ (be careful the variables are in opposite order compared to (4.32),(4.33) which produces an extra minus sign) :

$$\begin{aligned}\mathbf{s}_{-\frac{5}{2}}\Psi(y) &= \int_{c_y^-}\frac{dx}{2\pi i(x-y)^2}\left(2T(x)\xi(x-y) + \frac{c}{3}\xi''(x-y)\right)V_a(y) \\ &+ \int_{c_y^+}\frac{dx}{2\pi i(x-y)^2}V_a(y)\left(2T(x)\xi(x-y) + \frac{c}{3}\xi''(x-y)\right)\,.\end{aligned} \tag{4.202}$$

This gives :

$$\mathbf{s}_{-\frac{5}{2}}\Psi(y) = \oint_{c_y} \frac{dx}{2\pi i(x-y)^2}\frac{c}{3}\xi''(x-y)V_a(y) + \int_{c_y^-} \frac{2\xi(x-y)dx}{2\pi i(x-y)^2}T(x)V_a(y) + \int_{c_y^+} \frac{2\xi(x-y)dx}{2\pi i(x-y)^2}V_a(y)T(x)\,. \tag{4.203}$$

Using the commutation relation between T and V_a (1.61),(1.62) and the following residue calculations :

$$\oint_{c_0} \frac{dz}{2\pi i z^2}\frac{c}{3}\xi''(z) = \frac{7c}{2880}\,,$$
$$\oint_{c_0} \frac{2\xi(z)dz}{2\pi i z^2} = -\frac{1}{12}\,,$$
$$\oint_{c_0} \frac{2\chi(z)\xi(z)dz}{2\pi i z^2} = 0\,,$$
$$\oint_{c_0} \frac{2\chi'(z)\xi(z)dz}{2\pi i z^2} = -\frac{17}{960}\,,$$

we arrive at the result :

$$\langle \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}V_a\rangle = -\frac{1}{12}\left(\delta_P - \frac{c}{24}\right) + \Delta_a\frac{17}{960} + \frac{7c}{2880}\,. \tag{4.204}$$

4.9 Appendix 2 : Conserved charges in Super Conformal Field Theory

The SUSY charge in a perturbed SCFT.

Recall how above we have derived the conserved charges created from the stress energy tensor in 1.4. In the same way we can construct the *conserved super charge* : this is done by considering the residue term of the OPE between the super current and the perturbing field (recall that previously all the discussion was based on the stress energy tensor). Recall that in our case the perturbing field is $W_{-b} = \psi\overline{\psi}e^{-\frac{b}{\sqrt{2}}\varphi}$ and the needed OPE reads :

$$S(z)W_{-b}(w) = \frac{2\Delta_{-b}}{(z-w)^2}V_{-b}(w) + \frac{1}{z-w}V'_{-b}(w) + O(1)\,. \tag{4.205}$$

Then we set $\Psi = -\pi\lambda(1-2\Delta_{-b})V_{-b}$. The conserved super charge is therefore :

$$\mathcal{Q}_{\frac{1}{2}} = \int (Sdz + \Psi d\bar{z})\ . \tag{4.206}$$

The charge $\mathcal{Q}_{\frac{1}{2}}$ has manifestly spin $\frac{1}{2}$ and it is possible to show that

$$\{\mathcal{Q}_{\frac{1}{2}}, \mathcal{Q}_{\frac{1}{2}}\} = 2\mathcal{P}\,, \tag{4.207}$$

where the momentum $\mathcal{P}$ has been defined in (1.11). Moreover, using the anti-holomorphic part $\overline{T}, \overline{S}$ of the Super Virasoro algebra one can construct by the same procedure the charge $\overline{\mathcal{Q}}_{\frac{1}{2}}$ and recover the full SUSY algebra.

The deformation of integrals of motion to the super case.

Let us now return to the integrals of motion. In chiral CFT, it is known that the densities h_{2k} are homogeneous polynomials in the stress energy tensor T and its derivatives, of total spin $2k$. In Supersymmetric CFT we should extend this definition by considering also polynomials in the super current S and its derivatives, remembering that S has a spin $\frac{3}{2}$.

For the first element of the series of integrals of motion, we have only one possibility :

$$h_2(y) = T(y)\,, \tag{4.208}$$

which gives the known result $\mathbf{i}_1 = \mathbf{l}_{-1}$. The situation starts to be more complicated at spin 4, where the general form is $h_4(y) = (TT)(y) + \kappa(S\partial S)(y)$. Recall that the conservation law $\partial_{\bar{z}}h_{2k} = \partial_z\Theta_{2k-2}$ that holds for any density, is equivalent to the fact that the residue term in the OPE of h_{2k} with the perturbing field has to be a total derivative. The coefficient κ is determined to satisfy this prescription. Since the perturbing field is $W_{-b} = \psi\overline{\psi}e^{-\frac{b}{\sqrt{2}}\varphi}$, we find explicitly that :

$$h_4(y) = (TT)(y) - \frac{1}{4}(S\partial S)(y)\,, \tag{4.209}$$

which gives

$$\mathbf{i}_3 = 2\sum_{k=-1}^{\infty}\mathbf{l}_{-3-k}\mathbf{l}_k + \frac{1}{2}\sum_{k=-\frac{1}{2}}^{\infty}\mathbf{s}_{-3-k}\mathbf{s}_k\left(k+\frac{3}{2}\right)\ . \tag{4.210}$$

From the value of h_4 it is possible to compute the operator acting on the Matsubara Hilbert space :

$$I_3 = \int_C \frac{dy}{2\pi i}h_4(y)\,, \tag{4.211}$$

and to recover its eigenvalue i_3 from (4.119).

4.10 Appendix 3 : The kernel G_α

In this appendix we give more information about the derivation of the kernels G_α and G. In particular, we demonstrate that G solves the equations (4.110), and explain how this equation can be used to define the kernel G_α. Also, we will calculate the residue of the function G_α. First, remark that the functions G and G_α are defined as integrals (4.111),(4.144), but it is possible to show that they can be continued to meromorphic functions on the entire complex plane. In particular, this implies that the functions $b, \bar{b}, \mathcal{G}, \bar{\mathcal{G}}$ can be defined everywhere.

Functional equation for G_α.

The main idea is that G has to satisfy a functional integral equation with the Baxter eigenvalue Q (4.95), or more precisely with its logarithmic derivative :

$$\frac{d}{d\theta}\log Q(\theta) = \sum_{j=1}^{m} \coth(\nu(\theta-\sigma_j))\,, \tag{4.212}$$

where the σ_j are the Bethe roots and the term $e^{\nu\kappa\theta}$ can be treated separately. Define the function A :

$$A(x) = G(x) + G(x+i\pi)\,. \tag{4.213}$$

The defining property of G was the fact that it solves the equation (4.110) :

$$\sum_{j=1}^{m}\int_{\mathbb{R}} A(x-y)\left(f(y+i\pi-\sigma_k)-f(y-i\pi-\sigma_k)\right)dy = \sum_{j=1}^{m}\left[f(x+2i\pi-\sigma_k)-f(x-i\pi-\sigma_k)\right], \tag{4.214}$$

where we introduced the basic function

$$f(\theta) = \coth(\nu\theta)\,. \tag{4.215}$$

In going to the case $\alpha \neq 0$, we should appropriately deform the function f, in the same way when going from the function $\psi(\theta,0)$ to $\psi(\theta,\alpha)$. The basic function has to be changed to :

$$g(\theta) = e^{-\alpha\nu\theta}\coth(\nu\theta)\,. \tag{4.216}$$

The equation (4.214) with the new function g can be solved term by term, applying the Fourier transform defined by

$$\widehat{h}(q) = \int_{\mathbb{R}} h(y)e^{iyq}dy\,, \quad h(y) = \int_{\mathbb{R}} \widehat{h}(q)e^{-iqy}\frac{dy}{2\pi}\,. \tag{4.217}$$

First one has

$$\int_{\mathbb{R}} dy\, e^{iqy}g(y+ic) = e^{-\alpha\nu ic}\frac{i\pi}{\nu}\frac{e^{-(q+i\alpha\nu)(\frac{\pi}{2\nu}-c)}}{\mathrm{sh}(\frac{\pi}{2\nu}(q+i\alpha\nu))}\,. \tag{4.218}$$

Indeed, since

$$\int_{-\infty}^{\infty} e^{iax}\mathrm{cth}(\beta x)\mathrm{d}x = \frac{i\pi}{\beta}\mathrm{cth}\left(\frac{\pi a}{2\beta}\right)\,, \tag{4.219}$$

one can use the Sokhotsky relations :

$$\mathrm{cth}(\beta(x\pm i0)) = \mathrm{cth}(\beta x) \mp \frac{i\pi}{\beta}\delta(x)\,, \tag{4.220}$$

to obtain the Fourier transform

$$\int_{-\infty}^{\infty} e^{iax}\mathrm{cth}(\beta(x\pm i0))\mathrm{d}x = \frac{i\pi}{\beta}\frac{e^{\mp\frac{\pi a}{2\beta}}}{\mathrm{sh}(\frac{\pi a}{2\beta})}\,. \tag{4.221}$$

The analytic continuation of the previous formula reads :

$$\int_{-\infty}^{\infty} e^{iax}\mathrm{cth}(\beta(x+ic))\mathrm{d}x = \frac{i\pi}{\beta}\frac{e^{-a(\frac{\pi}{2\beta}-c)}}{\mathrm{sh}(\frac{\pi a}{2\beta})}\,. \tag{4.222}$$

Finally, calculating the Fourier transform of the left hand side and of the right hand side of (4.214) one gets :

$$\widehat{A}(q) = \frac{e^{\frac{q'\pi}{2}} e^{-\frac{\pi}{2}\alpha i\nu} \mathrm{sh}\left(\frac{q'\pi}{2\nu}(1-3\nu) + \frac{3}{2}\alpha i\nu\pi\right)}{\mathrm{sh}\left(\frac{q'\pi}{2\nu}(1-2\nu) + \alpha i\nu\pi\right)} = \frac{e^{\frac{q\pi}{2}} \mathrm{sh}\left(\frac{\pi}{2\nu}(q(1-3\nu) + \alpha i\pi)\right)}{\mathrm{sh}\left(\frac{\pi}{2\nu}(q(1-2\nu) + \alpha i\pi)\right)}\,, \tag{4.223}$$

where $q' = q + \alpha i\nu$. With this result in hand, applying the relation :

$$\widehat{A}(q) = e^{\frac{q\pi}{2}} 2\mathrm{ch}\left(\frac{q\pi}{2}\right)\widehat{G}(q)\,, \tag{4.224}$$

we recover the expected formula for G_α (remember that $\nu = \frac{1-\beta^2}{2}$) :

$$G_\alpha(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\mathrm{sh}\left(\frac{\pi}{2\nu}(q(1-3\nu) + \alpha i\pi)\right)}{\mathrm{sh}\left(\frac{\pi}{2\nu}(q(1-2\nu) + \alpha i\pi)\right)\mathrm{ch}\left(\frac{\pi q}{2}\right)} e^{iq\theta} dq\,. \tag{4.225}$$

Singularities of G.

As we said G is a meromorphic function on $\mathbb{C}$, which has poles at the points $\theta = \pm\pi i$. To see this, let us derive a functional equation on G :

$$F(\theta) = G(\theta + i\tfrac{\pi}{2}) + G(\theta - i\tfrac{\pi}{2}) = \frac{1}{2\pi}\int_{\mathbb{R}} dk \frac{\mathrm{sh}(\frac{\pi}{2\nu}(1-3\nu)k)}{\mathrm{sh}(\frac{\pi}{2\nu}(1-2\nu)k)} e^{ik\theta}\,. \tag{4.226}$$

The function F can be given an explicit formula (see the result 3.511.5 in [89]) :

$$\int_{\mathbb{R}} dk e^{iak} \frac{\mathrm{sh}(\beta k)}{\mathrm{sh}(\gamma k)} = 2\int_{\mathbb{R}} dk \cos(ka) \frac{\mathrm{sh}(\beta k)}{\mathrm{sh}(\gamma k)} = 2\frac{\pi}{2\gamma} \frac{\sin(\frac{\pi\gamma}{2})}{\mathrm{ch}(\frac{a\pi}{\gamma}) + \cos(\frac{\beta\pi}{\gamma})}\,. \tag{4.227}$$

Replacing : $a = \theta\,, \beta = \frac{\pi}{2\nu}(1-3\mu)\,, \gamma = \frac{\pi}{2\nu}(1-2\nu)$, we obtain :

$$F(\theta) = \frac{1}{2\pi}\frac{2\nu}{1-2\nu} \frac{\sin(\pi\frac{1-3\nu}{1-2\nu})}{\mathrm{ch}(\frac{2\nu\theta}{1-2\nu}) + \cos(\pi\frac{1-3\nu}{1-2\nu})}\,. \tag{4.228}$$

The Taylor expansion of the denominator of F gives :

$$\mathrm{ch}\left(\frac{2\nu\theta}{1-2\nu}\right) + \cos\left(\pi\frac{1-3\nu}{1-2\nu}\right) = \frac{2i\nu\sin\left(\frac{\pi\nu}{1-2\nu}\right)}{1-2\nu}\left(\theta - \frac{i\pi}{2}\right) + O\left(\theta - \frac{i\pi}{2}\right)\,. \tag{4.229}$$

From this relation one can deduce the value of the residue :

$$\mathrm{Res}_{\theta=\pm i\frac{\pi}{2}} F(\theta) = \mathrm{Res}_{\theta=\pm i\frac{\pi}{2}} G(\theta) = \frac{1}{2\pi i}\,. \tag{4.230}$$

4.11 Appendix 4 : Constraints from reflection relations

Here we explain how to obtain the constraints from the reflection relations (4.190) and (4.191), that were first established in [31]. Recall that we expect the following form for the two polynomials $P^E_{I^+,I^-}$ and $P^O_{I^+,I^-}$:

$$P^E_{I^+,I^-} = \mathbf{v}_1 + \frac{1}{D^{(k)}_V(\Delta_a,c)} \sum_{i=2}^{d} X_{I^+,I^-,i}(\Delta_a,c)\mathbf{v}_i \,, \tag{4.231}$$

$$P^O_{I^+,I^-} = \frac{1}{D^{(k)}_V(\Delta_a,c)} \sum_{i=2}^{d} Y_{I^+,I^-,i}(\Delta_a,c)\mathbf{v}_i \,. \tag{4.232}$$

They provide the decomposition of a fermionic element of the fermion-current basis on the Super Virasoro basis :

$$\boldsymbol{\beta}^{\mathrm{CFT}*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}*}_{I^-}V_a = C_{I^+,I^-}\left(P^E_{I^+,I^-}(\{\mathbf{l}_{-k},\mathbf{s}_r\},\Delta_a,c) + d_a P^O_{I^+,I^-}(\{\mathbf{l}_{-k},\mathbf{s}_r\},\Delta_a,c)\right)V_a \,. \tag{4.233}$$

This decomposition has to be compatible with the expansion on the Super Heisenberg algebra

$$\begin{aligned}\boldsymbol{\beta}^{\mathrm{CFT}*}_{I^+}\boldsymbol{\gamma}^{\mathrm{CFT}*}_{I^-}V_a &= C_{I^+,I^-}\prod_{2j-1\in I^+}(a+(2j-1)b^{-1})\prod_{2j-1\in I^-}(a+(2j-1)b)\\ &\times\left(Q^E_{I^+,I^-}(\{a_{-k},b_r\},b_r\},a^2,Q^2) + g_a Q^O_{I^+,I^-}(\{a_{-k},b_r\},a^2,Q^2)\right)V_a.\end{aligned} \tag{4.234}$$

Recall that U is the passage matrix between the Super Virasoro $\{\mathbf{v}^{(k)}_i\}_{i=1}^d$ and the Super Heisenberg $\{\mathbf{h}^{(k)}_i\}_{i=1}^d$ bases, at level k :

$$\mathbf{v}^{(k)}_i = \sum_{k=1}^{d} U^{(k)}_{i,j}(a)\mathbf{h}^{(k)}_j \,, \tag{4.235}$$

Let us work with an arbitrary basis element indexed by j. We first start by plugging the polynomial expressions (4.231),(4.232) in the decomposition (4.233) and by factorizing the denominators :

$$\begin{aligned}&\frac{1}{D^{(k)}_V(\Delta_a,c)}\left(U_{1j} + \sum_i X_{I^+,I^-,i}(\Delta_a,c)U_{ij} + d_a\sum_i Y_{I^+,I^-,i}(\Delta_a,c)U_{ij}\right)\\ &= \frac{1}{D^{(k)}_H(a^2)}\left(Q^E_{I^+,I^-,j}(a^2,Q^2) + g_a Q^O_{I^+,I^-,j}(a^2,Q^2)\right)\\ &\times\prod_{2k-1\in I^+}(a+(2k-1)b^{-1})\prod_{2k-1\in I^-}(a+(2k-1)b)\,.\end{aligned} \tag{4.236}$$

Then multiply both sides by $D^{(k)}_V(\Delta(a))D^{(k)}_V(\Delta(-a))D^{(k)}_H(a^2)$ and obtain :

$$\begin{aligned}&D^{(k)}_V(\Delta(-a))D^{(k)}_H(a^2)\left(U_{1j} + \sum_i X_{I^+,I^-,i}(\Delta_a,c)U_{ij} + d_a\sum_i Y_{I^+,I^-,i}(\Delta_a,c)U_{ij}\right)\\ &= D^{(k)}_V(\Delta(a))D^{(k)}_V(\Delta(-a))\left(Q^E_{I^+,I^-,j}(a^2,Q^2) + g_a Q^O_{I^+,I^-,j}(a^2,Q^2)\right)\\ &\times\prod_{2k-1\in I^+}(a+(2k-1)b^{-1})\prod_{2k-1\in I^-}(a+(2k-1)b)\,.\end{aligned} \tag{4.237}$$

Define the product $L_{I^+I^-}(a)$:

$$L_{I^+I^-}(a) = \prod_{2k-1\in I^+}(a+(2k-1)b^{-1})\prod_{2k-1\in I^-}(a+(2k-1)b)\,,$$

and notice that it implies the relations (recall (4.188),(4.189)) :

$$T^+_{I^+I^-}(a) = \frac{1}{2}\left(L_{I^+I^-}(a) + L_{I^-I^+}(a)\right)\,, \tag{4.238}$$

$$T^-_{I^+I^-}(a) = \frac{1}{2(b-b^{-1})}\left(L_{I^+I^-}(a) - L_{I^-I^+}(a)\right)\,. \tag{4.239}$$

We rewrite the relation (4.237) using the natural notations for the left hand side :

$$A_{I^+,I^-,j} + d_a B_{I^+,I^-,j} = L_{I^+I^-}(a)\left(Q^E_{I^+,I^-,j} + g_a Q^O_{I^+,I^-,j}\right) , \tag{4.240}$$

where we have redefined the functions Q, by multiplying them by the *even in* a factor $D_V^{(k)}(\Delta(a))D_V^{(k)}(\Delta(-a))$ independent of I^+, I^-. Now, performing the replacement $I^+ \leftrightarrow I^-$ in the previous equation (4.240) we obtain :

$$A_{I^+,I^-,j} - d_a B_{I^+,I^-,j} = L_{I^-I^+}(a)\left(Q^E_{I^+,I^-,j} - g_a Q^O_{I^+,I^-,j}\right) . \tag{4.241}$$

The equations (4.240) and (4.241) allow to express the terms $A_{I^+,I^-,j}$ and $B_{I^+,I^-,j}$ in terms of $Q^E_{I^+,I^-,j}$ and $Q^O_{I^+,I^-,j}$. We can now check the validity of the constraints (4.190),(4.191). The first one can be rewritten as :

$$\begin{aligned} &T^+_{I^+I^-}(-a)A_{I^+,I^-,j} - d_a(b-b^{-1})T^-_{I^+I^-}(-a)B_{I^+,I^-,j} = \\ &\frac{1}{2}\left(L_{I^+I^-}(-a)L_{I^-I^+}(a)\left(Q^E_{I^+,I^-,j} - g_a Q^O_{I^+,I^-,j}\right) + L_{I^-I^+}(-a)L_{I^+I^-}(a)\left(Q^E_{I^+,I^-,j} + g_a Q^O_{I^+,I^-,j}\right)\right) . \end{aligned} \tag{4.242}$$

The right hand side of this expression is indeed an even function (because of the fact that $Q^E_{I^+,I^-,j}$ and $Q^O_{I^+,I^-,j}$ are even functions of a). Now we write out the second constraint :

$$\begin{aligned} &- T^-_{I^+I^-}(-a)A_{I^+,I^-,j} + \frac{d_a}{b-b^{-1}}T^+_{I^+I^-}(-a)B_{I^+,I^-,j} = \\ &\frac{1}{2(b-b^{-1})}\Big(- L_{I^+I^-}(-a)L_{I^-I^+}(a)\big(Q^E_{I^+,I^-,j} - g_a Q^O_{I^+,I^-,j}\big) + \\ &L_{I^-I^+}(-a)L_{I^+I^-}(a)\left(Q^E_{I^+,I^-,j} + g_a Q^O_{I^+,I^-,j}\right)\Big) . \end{aligned} \tag{4.243}$$

Similarly, the right hand side is manifestly an odd function of a.

Conclusion

In this conclusion, we highlight the main results of this work and propose some further directions of investigation.

The ultimate achievement of this PhD research is the computation of the one point functions of fermionic operators in the Super sine-Gordon model. They are constructed out of a single function Ω, defined by a set of scaling equations, and which origin is traced to the computation of vacuum expectation values of lattice operators on the underlying 19-vertex model. On one hand, the analysis of the scaling equations in the conformal regime allowed to compute the one point functions of specific fermionic operators in the UV limit, and to establish the connection between the usual Virasoro description of CFT and the fermionic part of the fermion-current description. On the other hand, these results have been checked by an alternative method that relies on the reflection symmetry of the ssG model. We emphasize again that both techniques completely differ in their nature and are both based on conjectures. The matching of the results from both sides is a very strong assertion for both of them.

In this field theoretic context, notice that for what concerns the primary fields, we have obtained the most important quantity for applications. Indeed, we argued that the simplest non-chiral fermionic descendant provides the ratio of one point functions of the operators $W_{\alpha+\frac{2\beta^2}{1-\beta^2}}$ and V_α. The former operator is exactly the most relevant contribution occurring in the OPE of the latter one with the perturbing operator $W_{\frac{2\beta^2}{1-\beta^2}}$. In other words the ratio of one point functions in question provides the most important contribution to the conformal perturbation theory.

A second, just as important result produced in this thesis, is the verification of the decomposition of specific operators on the fermion-current basis, on the lattice. From a more conceptual point of view, this can be taken as an argument in favor of the completeness of the spin 1 basis. A research direction that would be interesting to consider is to study the structure of the set of elements in the fermion-current decomposition for a given operator, that is conditions such that (3.35). However, in the study of this problem (as well as in the continuation of our work concerning the density matrices for example) one has to cope with extremely involved computer calculations.

From the point of view of QFT, the natural direction to pursue is to consider the entire space of local operators adding those created by the KM currents. The one point functions of the latter include the function $\omega(\theta,\theta')$. Recall the equation (4.136). Using this equation and the known $\Omega(\theta,\theta')$ one can, in principle, reconstruct $\omega(\theta,\theta')$. The result is not unique, one has to find a way of fixing the quasi-constants (anti-periodic with period πi functions of θ,θ'). When doing this numerically, it is hard to achieve a good precision which makes it difficult to put forward a conjecture based on the interpolation. This is a technical difficulty which we hope to overcome in the future.

Having in hand the scaling equations for the KM currents, one could extract (provided that such quantities exist) the analogs of the coefficients $D_m(a)$ (4.147) that are actually the starting point of the work on the reflection relations. The understanding of the properties of the currents under reflections represents an important advance since it would a priori provide an independent, and purely algebraic path for the calculation of the one point functions. In particular the KM currents are necessary to establish the full correspondence between the Super Virasoro and the fermion-current basis.

Furthermore, the solution to the "reflection problem" for currents would bring more understanding of the integrable structure of the higher spin models. As we said, we do not expect the appearance of any new types of operators for spin $s \geq \frac{3}{2}$, and at higher spins all basis elements should be consistently organized in the following matrix :

$$\begin{pmatrix} \mathbf{1} & \mathbf{b}^* & \mathbf{j}^+ & \dots \\ \mathbf{c}^* & \mathbf{j}^0 & \dots & \tilde{\mathbf{j}}^+ \\ \mathbf{j}^- & \dots & \tilde{\mathbf{j}}^0 & \tilde{\mathbf{b}}^* \\ \dots & \tilde{\mathbf{j}}^- & \tilde{\mathbf{c}}^* & \mathbf{t}^* \end{pmatrix},$$

where the dots represent higher spin currents. This picture is a generalization of a similar matrix expression for the spin 1 fermion-current basis given in [6]. The knowledge of the current reflection transformation rules could give an independent approach for the study of such models, short-cutting the general Suzuki equations [45].

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